GAP code for “The uniqueness of a certain generalized octagon of order $(2,4)$”

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Abstract

We provide more information about the GAP code that we used in [1] to obtain the classification result.

1 Introduction

2 Polygonal valuations

3 The valuation geometry of $\mathrm{GO}(2,1)$

3.1 A computer model of $\mathrm{GO}(2,1)$

3.2 Computing the valuation geometry of $\mathrm{GO}(2,1)$

The GAP code to determine the valuation geometry of $\mathrm{GO}(2,1)$ has been implemented in ReeTits1.g. Valuations have been implemented there as arrays where the i-th entry is the value of the point i. Valuations have also been scaled such that their maximal values are equal to 0. In ReeTits1.g, the information of Table 1 (except for the last column) has been stored in arrays:

- Second column: `NumberValuations`
- Third column: `MaximalValues`
- Fourth column: `SizeZeroSets`
- Fifth column: `SizeMinSets`
- Sixth column: `SizeHyperplanes`
- Seventh column: `StructureStabilizers`
- Eighth column: `OrbitsStabilizer`

The number of lines of the valuation geometry incident with a given point, and the types of these lines can be found with the function `ThroughPoint1`. For instance, if we give the following commands:

```
CompType(complements[12]);
ThroughPoint1(complements[12]);
```

then the output is

```
"C7"
[ [ [ "B1", "C5", "C7" ], [ "C7", "D2", "D2" ] ], 7 ]
```

which means that there are precisely 7 $\mathcal{V}$-lines through each valuation of Type C_7, and that each of these lines has Type $B_1C_5C_7$ or Type $C_7D_2D_2$.

If we apply `ThroughPoint1` to each of the 12 elements of `complements`, then we can determine the set `AllLineTypes`, which contains all the 52 types for the lines of the valuation geometry. These are the types mentioned in Tables 2 and 3. The information mentioned in Table 3 can be obtained from the function `ThroughPoint2`. For instance, in order to find the entry "8;4" that has been mentioned in row $C_3C_5D_2$ and column C_5, we just give the command

```
ThroughPoint2("C5","C3","D2");
```

and the following output arises

```
[ 8, 4 ]
```

The numbers in each entry of Table 3 have been ordered such that the largest number always comes first. The information given in Table 2 can easily be extracted from the information of Tables 1 and 3.

3.3 Example: The Ree-Tits octagon of order (2,4)

The Ree-Tits octagon $\mathrm{RT}(2,4)$ has been implemented in ReeTits2.g. We have also implemented a suboctagon of order $(2,1)$ inside $\mathrm{RT}(2,4)$. Let us discuss the theoretical background that allowed us to compute such a suboctagon.

A set X of points of a partial linear space $\mathcal{S}$ is called *convex of depth* $i \in \mathbb{N} \setminus \{0,1\}$ if for any two distinct points x and y of X whose mutual distance $j := \mathrm{d}(x,y)$ is at most i, the set $\Gamma_1(y) \cap \Gamma_{j-1}(x)$ is contained in X. The whole point set is a convex subspace of depth i and the intersection of convex subspaces of depth i is again a convex subspace of depth i. So, it makes sense to talk about the smallest convex subspace of depth i containing a given nonempty set X. The following can be proved.

Lemma 3.1 *Let $\mathcal{S}$ be a generalized $2d$-gon with $d \in \mathbb{N} \setminus \{0,1\}$ and X a nonempty set of points of $\mathcal{S}$.*

(1) *Suppose $\mathcal{S}'$ is a full sub-$2d$-gon of $\mathcal{S}$. Then the point set of $\mathcal{S}'$ is a convex subspace of depth $d-1$.*

(2) *Suppose X is convex of depth $d-1$. If $x_1, x_2 \in X$ with $d(x_1,x_2) \in \{1,2,\dots,d-1\}$, then x_2 has a unique neighbor in X at distance $d(x_1,x_2)-1$ from x_1.*

(3) *Suppose X is a convex subspace of depth $d-1$ containing a line L, and let $\widetilde{X}$ denote the subgeometry of $\mathcal{S}$ induced on the point set X. Then: (a) $\widetilde{X}$ is a connected geometry; (b) if $x_1, x_2 \in X$, then $\mathrm{d}_{\widetilde{X}}(x_1, x_2) = \mathrm{d}_{\mathcal{S}}(x_1, x_2)$.*

(4) *Suppose X is a convex subspace of depth $d-1$ containing two points x_1 and x_2 at distance d from each other and two distinct lines through x_2. Then $\widetilde{X}$ is a sub-$2d$-gon of $\mathcal{S}$.*

Proof. (1) As $\mathcal{S}'$ is a full sub-$2d$-gon, we have that $\mathrm{d}_{\mathcal{S}'}(x_1, x_2) = \mathrm{d}_{\mathcal{S}}(x_1, x_2)$ for any two points x_1 and x_2 of $\mathcal{S}'$. Moreover, the point set $\mathcal{P}'$ of $\mathcal{S}'$ is a subspace of $\mathcal{S}$. Now, take two points x and y of $\mathcal{S}'$ at distance $i \in \{1, 2, \ldots, d-1\}$ from each other. In the generalized polygon $\mathcal{S}'$, there is a unique neighbor of y at distance $i-1$ from x. This is also the unique neighbor of y at distance $i-1$ from x in the generalized polygon $\mathcal{S}$. This implies that $\mathcal{P}'$ is convex of depth $d-1$.

(2) This follows from the fact that $\mathcal{S}$ is a generalized $2d$-gon and the fact that X is convex of depth $d-1$.

(3) In order to show that $\widetilde{X}$ is connected, it suffices to show that every point $x \in X$ is connected via a path in X to some point of L. Since $\mathcal{S}$ is a near $2d$-gon, there exists a unique point x' on L nearest to x. Since $\mathrm{d}_{\mathcal{S}}(x, x') \leq d-1$ and X is convex of depth $d-1$, there is a path of length $\mathrm{d}_{\mathcal{S}}(x, x')$ in X connecting x and x'.

If x_1 and x_2 are two points of $\widetilde{X}$, then $\mathrm{d}_{\widetilde{X}}(x_1, x_2) \geq \mathrm{d}_{\mathcal{S}}(x_1, x_2)$. If $\mathrm{d}_{\mathcal{S}}(x_1, x_2) \leq d-1$, then the fact that X is convex of depth $d-1$ implies that $\mathrm{d}_{\widetilde{X}}(x_1, x_2) = \mathrm{d}_{\mathcal{S}}(x_1, x_2)$. Suppose now that $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = d$. Then $\mathrm{d}_{\widetilde{X}}(x_1, x_2) \geq d$. Since $\widetilde{X}$ is connected, there exists a line K through x_2 contained in X. This line K contains a point x_2' at $\mathcal{S}$-distance $d-1$ from x_1 and hence also at $\widetilde{X}$-distance $d-1$ from x_1. It follows that $\mathrm{d}_{\widetilde{X}}(x_1, x_2) = d$.

(4) The geometry $\widetilde{X}$ is a near polygon (as it is connected and distances are preserved). It is a near $2d$-gon as it has two points at maximal distance d from each other. Any two non-opposite points of $\widetilde{X}$ are connected by a unique shortest path. These properties imply that $\mathcal{S}$ is a generalized $2d$-gon if there are two opposite points having the property that at least one of them is incident with at least two lines. This is indeed the case by the assumption on X that we have made. ∎

Lemma 3.1 implies that if x and z are two opposite points of a generalized octagon $\mathcal{S}$ of order (s, t) and y_1, y_2 two points of $\Gamma_3(x) \cap \Gamma_1(z)$, then:

> $(*)$ Any full suboctagon of order $(s, 1)$ of $\mathcal{S}$ containing $\{x, z, y_1, y_2\}$ coincides with the smallest convex subspace of depth 3 containing $\{x, z, y_1, y_2\}$.

In case s and t are finite, this fact in combination with a straightforward double counting argument yields that there are at most $\frac{(s+1)(st+1)(s^2t^2+1)\cdot s^4t^3\cdot(t+1)t}{(s+1)(s+1)(s^2+1)\cdot s^4\cdot 2} = \frac{t^4(t+1)(st+1)(s^2t^2+1)}{2(s+1)(s^2+1)}$ full suboctagons of order $(s, 1)$. In particular, any generalized octagon of order $(2, 4)$ contains at most 24960 suboctagons of order $(2, 1)$. Based on $(*)$, we have implemented in

ReeTits2.g a suboctagon `octa` of order $(2,1)$ inside our implemented computer model of $\mathrm{RT}(2,4)$.

We denote the stabilizer of `octa` in the full automorphism group of $\mathrm{RT}(2,4)$ by `g2`. It can be verified that `g2` has size 1440 by means of the following command:

```
Size(g2)=1440;
```

The centralizer inside `g2` of `octa` is trivial, as can be verified by means of the following command:

```
Stabilizer(g2,octa,OnTuples)=Group(());
```

This means that `g2` induces the full group of automorphisms of `octa`. The number of suboctagons isomorphic to `octa` is 24960 (= the maximal number), as can be computed with the following code.

```
Index(g,g2);
```

The group `g2` has 8 orbits on the set of points of $\mathrm{RT}(2,4)$. The sizes of the orbits and the types of the involved points can be computed with the commands:

```
orbs1:=Orbits(g2,[1..1755]);;
info1:=List(orbs1,x-> [ Size(x) , PointType(x[1]) ] );
```

resulting in the following output

```
[ [45,"A"], [180,"B1"], [720,"C2"], [360,"C6"], [90,"B1"], [180,"C5"], [36,"C7"], [144,"D1"] ]
```

The group `g2` has 9 orbits on the set of lines of $\mathrm{RT}(2,4)$. The sizes of the orbits and the types of the involved lines can be computed with the commands:

```
orbs2:=Orbits(g2,lines,OnSets);;
info2:=List(orbs2,x-> [ Size(x) , LineType2(x[1]) ] );
```

resulting in the following output

```
[ [ 30, [ "A", "A", "A" ] ], [ 90, [ "A", "B1a", "B1a" ] ], [ 45, [ "A", "B1b", "B1b" ] ],
  [ 720, [ "B1a", "C2", "C2" ] ], [ 720, [ "C2", "C2", "C6" ] ], [ 720, [ "C2", "C5", "D1" ] ],
  [ 180, [ "B1b", "C6", "C6" ] ], [ 240, [ "C6", "C6", "C6" ] ], [ 180, [ "B1b", "C5", "C7" ] ] ]
```

The properties mentioned in (1), (2) and (5) of Section 3.3 follow from that (B1a means B_1' and B1b means B_1''). The (constant) numbers in the last table of Section 3.3 can be derived from the sizes of the various orbits by means of double counting. Property (3) follows from the fact that there are as many points of Type T in $\mathrm{RT}(2,4)$ as there are valuations of Type T, also taking into account that `g2` induces the full group of automorphisms of the implemented suboctagon. For similar reasons, Property (4) will be true.

4 Generalized octagons of order $(2,t)$ containing a suboctagon of order $(2,1)$

Lemmas 4.3, 4.4, 4.5 and 4.6 of the paper can also be proved directly in the generalized octagon $\mathcal{S}$.

Lemma 4.1 *Let xy_1y_2 be an $\mathcal{S}$-line of Type BCC and let x' be the unique point of $\mathcal{P}'$ collinear with x. Then:*

(1) $(\Gamma_2(y_1)\cap\mathcal{P}')\cap(\Gamma_2(y_2)\cap\mathcal{P}')=\Gamma_1(x)\cap\mathcal{P}'=\mathcal{O}_{f_x}=\{x'\}$;

(2) $\Gamma_2(y_i)\cap\mathcal{P}'\subseteq\mathcal{M}_{f_x}$ *and* $(\Gamma_2(y_i)\cap\mathcal{P}')\setminus\{x'\}\subseteq\mathcal{M}_{f_x}\setminus\mathcal{O}_{f_x}$ *for every* $i\in\{1,2\}$;

(3) $\Gamma_2(y_i)\cap\mathcal{P}'\subseteq\mathcal{M}_{f_{y_j}}$ *and* $(\Gamma_2(y_i)\cap\mathcal{P}')\setminus\{x'\}\subseteq\mathcal{M}_{f_{y_j}}\setminus\mathcal{O}_{f_{y_j}}$ *for all* $i,j\in\{1,2\}$ *with* $i\neq j$.

Proof. (1) Clearly, $x'\in\Gamma_2(y_i)\cap\mathcal{P}'$ for every $i\in\{1,2\}$. On the other hand, suppose $z\in(\Gamma_2(y_1)\cap\mathcal{P}')\cap(\Gamma_2(y_2)\cap\mathcal{P}')$. Then since $\mathrm{d}(z,y_1)=\mathrm{d}(z,y_2)=2$, we necessarily have $\mathrm{d}(z,x)=1$, i.e. $z=x'$.

(2) + (3) Without loss of generality, we may suppose that $i=1$. Let z be an arbitrary point of $\Gamma_2(y_1)\cap\mathcal{P}'$. If $z=x'$, then $z\in\mathcal{O}_{f_x}\cap\mathcal{O}_{f_{y_2}}\subseteq\mathcal{M}_{f_x}\cap\mathcal{M}_{f_{y_2}}$ and we are done. So, we may suppose that $z\neq x'$. Put $\{z'\}=\Gamma_1(z)\cap\Gamma_1(y_1)$. Then x,y_1,z',z and y_2,y_1,z',z are two paths of length three with $xy_1=y_2y_1\neq y_1z'\neq z'z$. (If $y_2y_1=y_1z'$, then the fact that $\mathrm{d}(z',\mathcal{P}')=1$ would imply that $z'=x$ and $z\in\Gamma_1(x)\cap\mathcal{P}'=\{x'\}$, in contradiction with the fact that $z\neq x'$.) So, $\mathrm{d}(x,z)=\mathrm{d}(y_2,z)=3$. The line $z'z$ is thus the unique line of $\mathcal{S}$ through z containing a point at distance 2 from y_2 (or from x). Since the line zz' is not contained in $\mathcal{P}'$, we have $z\in(\mathcal{M}_{f_x}\cap\mathcal{M}_{f_{y_2}})\setminus(\mathcal{O}_{f_x}\cup\mathcal{O}_{f_{y_2}})$. ■

Lemma 4.2 *Let $x_1x_2x_3$ be an $\mathcal{S}$-line of Type CCC. Then:*

(1) $\Gamma_2(x_1)\cap\mathcal{P}'$, $\Gamma_2(x_2)\cap\mathcal{P}'$ *and* $\Gamma_2(x_3)\cap\mathcal{P}'$ *are mutually disjoint;*

(2) $\Gamma_2(x_i)\cap\mathcal{P}'\subseteq\mathcal{M}_{f_{x_j}}\setminus\mathcal{O}_{f_{x_j}}$ *for all* $i,j\in\{1,2,3\}$ *with* $i\neq j$.

Proof. Let $i,j\in\{1,2,3\}$ with $i\neq j$.

(1) If y is a point of $(\Gamma_2(x_i)\cap\mathcal{P}')\cap(\Gamma_2(x_j)\cap\mathcal{P}')$, then the fact that $\mathrm{d}(y,x_i)=\mathrm{d}(y,x_j)=2$ would imply that y has distance 1 from the unique point in $\{x_1,x_2,x_3\}\setminus\{x_i,x_j\}$, clearly a contradiction.

(2) Let $y\in\Gamma_2(x_i)\cap\mathcal{P}'$ and let y' be the unique common neighbor of x_i and y. Then y,y',x_i,x_j is a path of length 3 with $yy'\neq y'x_i\neq x_ix_j$. So, $\mathrm{d}(y,x_j)=3$. The line yy' is the unique line of $\mathcal{S}$ through y containing a point at distance 2 from x_j. Since yy' is not contained in $\mathcal{P}'$, we have $y\in\mathcal{M}_{f_{x_j}}$. Since $\mathrm{d}(y,x_j)=3$, we also have $y\notin\mathcal{O}_{f_{x_j}}$. ■

Lemma 4.3 *Let x_1x_2y be an $\mathcal{S}$-line of Type CCD. Then:*

(1) $\Gamma_2(x_i)\cap\mathcal{P}'\subseteq\Gamma_3(y)\cap\mathcal{P}'$ *for every* $i\in\{1,2\}$;

(2) $(\Gamma_2(x_1) \cap \mathcal{P}') \cap (\Gamma_2(x_2) \cap \mathcal{P}') = \emptyset$;

(3) $\Gamma_2(x_i) \cap \mathcal{P}' \subseteq \mathcal{M}_{f_{x_j}} \setminus \mathcal{O}_{f_{x_j}}$ *for all* $i, j \in \{1, 2\}$ *with* $i \neq j$.

Proof. Claim (1) is obvious. The proofs of (2) and (3) are completely similar to the proofs given in Lemma 4.2. ∎

Lemma 4.4 *Let* xy_1y_2 *be an* $\mathcal{S}$*-line of Type* CDD. *Then* $(\Gamma_3(y_1) \cap \mathcal{P}') \cap (\Gamma_3(y_2) \cap \mathcal{P}') = \Gamma_2(x) \cap \mathcal{P}'$.

Proof. The proof is similar to the proof of Lemma 4.1(1). ∎

The values for $c(x, L)$ and $N(x)$ have been mentioned in Table 1. The entry "2" in column B_1 and row $B_1C_1C_4$ means that if x is a point of Type B_1 incident with a line of Type $B_1C_1C_4$, then $c(x, L) = 2$. The condition "$\sum_{L \in \mathcal{L}_x} c(x, L) = N(x)$" thus means that in the column corresponding to the type of x the sum of $|\mathcal{L}_x|$ entries (not necessarily in distinct rows!) is equal to the entry $N(x)$ occurring in the last row.

The GAP code that allows to compute the contents of Table 1 is mentioned below. If x is an $\mathcal{S}$-point of Type T1 and L is an $\mathcal{S}$-line of Type T1,T2,T3 incident with x, then `CValue1(T1)` is equal to $c(x)$ and `CValue2(T1,T2,T3)` is equal to the $c(x, L)$.

```
BB:=["B1","B2"];
CC:=["C1","C2","C3","C4","C5","C6","C7"];
DD:=["D1","D2"];
CValue1:=function(T)
 if T in ["C1","C5","C6"] then return 1; fi;
 if T in ["C2","C3"] then return 2; fi;
 if T in ["C4"] then return 3; fi;
 if T in ["C7"] then return 5; fi;
 if T in ["A","B1","B2","D1","D2"] then return 0; fi;
end;
CValue2:=function(T1,T2,T3)
 local help;
 help := CValue1(T2) + CValue1(T3);
 if T1 in BB and T2 in CC and T3 in CC then help := help-2; fi;
 if T1 in CC and (T2 in BB or T3 in BB) then help := help-1; fi;
 return help;
end;
```

	A	B_1	B_2	C_1	C_2	C_3	C_4	C_5	C_6	C_7	D_1	D_2
AAA	0	–	–	–	–	–	–	–	–	–	–	–
AB_1B_1	0	0	–	–	–	–	–	–	–	–	–	–
AB_2B_2	0	–	0	–	–	–	–	–	–	–	–	–
$B_1C_1C_4$	–	2	–	2	–	–	0	–	–	–	–	–
$B_1C_2C_2$	–	2	–	–	1	–	–	–	–	–	–	–
$B_1C_3C_3$	–	2	–	–	–	1	–	–	–	–	–	–
$B_1C_4C_4$	–	4	–	–	–	–	2	–	–	–	–	–
$B_1C_5C_5$	–	0	–	–	–	–	–	0	–	–	–	–
$B_1C_5C_7$	–	4	–	–	–	–	–	4	–	0	–	–
$B_1C_6C_6$	–	0	–	–	–	–	–	–	0	–	–	–
$B_2C_1C_3$	–	–	1	1	–	0	–	–	–	–	–	–
$B_2C_2C_2$	–	–	2	–	1	–	–	–	–	–	–	–
$B_2C_4C_5$	–	–	2	–	–	–	0	2	–	–	–	–
$B_2C_6C_6$	–	–	0	–	–	–	–	–	0	–	–	–
$C_1C_1C_1$	–	–	–	2	–	–	–	–	–	–	–	–
$C_1C_1C_2$	–	–	–	3	2	–	–	–	–	–	–	–
$C_1C_1C_3$	–	–	–	3	–	2	–	–	–	–	–	–
$C_1C_1C_4$	–	–	–	4	–	–	2	–	–	–	–	–
$C_1C_1C_5$	–	–	–	2	–	–	–	2	–	–	–	–
$C_1C_1C_6$	–	–	–	2	–	–	–	–	2	–	–	–
$C_1C_1D_1$	–	–	–	1	–	–	–	–	–	–	2	–
$C_1C_1D_2$	–	–	–	1	–	–	–	–	–	–	–	2
$C_1C_2C_3$	–	–	–	4	3	3	–	–	–	–	–	–
$C_1C_2C_5$	–	–	–	3	2	–	–	3	–	–	–	–
$C_1C_2C_6$	–	–	–	3	2	–	–	–	3	–	–	–
$C_1C_2D_1$	–	–	–	2	1	–	–	–	–	–	3	–
$C_1C_2D_2$	–	–	–	2	1	–	–	–	–	–	–	3
$C_1C_3C_6$	–	–	–	3	–	2	–	–	3	–	–	–
$C_1C_3D_1$	–	–	–	2	–	1	–	–	–	–	3	–
$C_1C_4D_2$	–	–	–	3	–	–	1	–	–	–	–	4
$C_1C_5C_6$	–	–	–	2	–	–	–	2	2	–	–	–
$C_1C_5D_1$	–	–	–	1	–	–	–	1	–	–	2	–
$C_1D_1D_2$	–	–	–	0	–	–	–	–	–	–	1	1
$C_2C_2C_2$	–	–	–	–	4	–	–	–	–	–	–	–
$C_2C_2C_6$	–	–	–	–	3	–	–	–	4	–	–	–
$C_2C_2D_2$	–	–	–	–	2	–	–	–	–	–	–	4
$C_2C_3D_1$	–	–	–	–	2	2	–	–	–	–	4	–
$C_2C_4D_2$	–	–	–	–	3	–	2	–	–	–	–	5
$C_2C_5D_1$	–	–	–	–	1	–	–	2	–	–	3	–
$C_2D_1D_2$	–	–	–	–	0	–	–	–	–	–	2	2
$C_3C_3C_5$	–	–	–	–	–	3	–	4	–	–	–	–
$C_3C_5D_2$	–	–	–	–	–	1	–	2	–	–	–	3
$C_3C_6D_1$	–	–	–	–	–	1	–	–	2	–	3	–
$C_3D_2D_2$	–	–	–	–	–	0	–	–	–	–	–	2
$C_4C_6D_2$	–	–	–	–	–	–	1	–	3	–	–	4
$C_4D_1D_1$	–	–	–	–	–	–	0	–	–	–	3	–
$C_5C_5C_5$	–	–	–	–	–	–	–	2	–	–	–	–
$C_5D_2D_2$	–	–	–	–	–	–	–	0	–	–	–	1
$C_6C_6C_6$	–	–	–	–	–	–	–	–	2	–	–	–
$C_6D_1D_2$	–	–	–	–	–	–	–	–	0	–	1	1
$C_7D_2D_2$	–	–	–	–	–	–	–	–	–	0	–	5
$D_1D_1D_2$	–	–	–	–	–	–	–	–	–	–	0	0
$N(x)$	0	8	8	12	9	9	6	12	12	0	15	15

Table 1: The values $c(x, L)$ and $N(x)$.

5 Generalized octagons of order $(2,4)$ containing a suboctagon of order $(2,1)$

5.1 Proof of the isomorphism $\Gamma_0 \cong \Gamma_0^*$

Step 1

Having a set of possible line types, we can reduce it to a possible smaller set by means of the implemented function `Reduction` from ReeTits1.g. If we apply this function to the set `AllLineTypes` of all 52 possible line types, then we find the following set of 21 line types:

$$AAA,\ AB_1B_1,\ AB_2B_2,\ B_1C_1C_4,\ B_1C_2C_2,\ B_1C_4C_4,\ B_1C_5C_5,\ B_1C_5C_7,\ B_1C_6C_6,\ B_2C_2C_2,\ C_1C_1C_2,\ C_1C_1C_4,\ C_1C_1C_5,\ C_1C_1C_6,\ C_1C_2D_2,\ C_2C_2C_2,\ C_2C_2C_6,\ C_2C_5D_1,\ C_5C_5C_5,\ C_5D_2D_2,\ C_6C_6C_6.$$

If we apply `Reduction` to the above collection of 21 possible line types, the following 13 line types remain:

$$AAA,\ AB_1B_1,\ AB_2B_2,\ B_1C_2C_2,\ B_1C_4C_4,\ B_1C_5C_7,\ B_1C_6C_6,\ B_2C_2C_2,\ C_2C_2C_2,\ C_2C_2C_6,\ C_2C_5D_1,\ C_5C_5C_5,\ C_6C_6C_6.$$

If we apply `Reduction` to the above collection of 13 possible line types, the following ten line types remain:

$$AAA,\ AB_1B_1,\ AB_2B_2,\ B_1C_2C_2,\ B_1C_5C_7,\ B_1C_6C_6,\ C_2C_2C_6,\ C_2C_5D_1,\ C_5C_5C_5,\ C_6C_6C_6.$$

If we apply `Reduction` to the above collection of ten possible line types, the following nine line types remain:

$$AAA,\ AB_1B_1,\ B_1C_2C_2,\ B_1C_5C_7,\ B_1C_6C_6,\ C_2C_2C_6,\ C_2C_5D_1,\ C_5C_5C_5,\ C_6C_6C_6.$$

Applying the procedure another time, we found that no further line types can be excluded.

Some of the line types that we have killed in the reduction process, can also be killed without use of computer computations. We give some examples.

Lemma 5.1 *There are no $\mathcal{S}$-lines of Type $B_2C_1C_3$, nor of Type $B_2C_6C_6$.*

Proof. Suppose x is an $\mathcal{S}$-point of Type B_2. Then we know that $\sum_{L\in\mathcal{L}'_x} c(x,L) = N(x) = 8$, where $\mathcal{L}'_x$ denotes the set of four lines of $\mathcal{S}$ through x not meeting $\mathcal{P}'$. Now, any line $L \in \mathcal{L}'_x$ has Type $B_2C_1C_3$, $B_2C_2C_2$, $B_2C_4C_5$ or $B_2C_6C_6$ and hence $c(x,L) \leq 2$ by Table 1. It follows that $c(x,L) = 2$ for every $L \in \mathcal{L}'_x$. This excludes the possibility that $\mathcal{L}'_x$ contains lines of Type $B_2C_1C_3$ or $B_2C_6C_6$. ∎

Lemma 5.2 *There are no $\mathcal{S}$-lines of Type $C_7D_2D_2$.*

Proof. Let x be an $\mathcal{S}$-point of Type C_7. Then there are precisely $c(x) = 5$ lines of $\mathcal{S}$ through x containing an $\mathcal{S}$-point of Type B. These are all the lines of $\mathcal{S}$ through x. So, there cannot exist $\mathcal{S}$-lines of Type $C_7D_2D_2$ through x. ■

Lemma 5.3 *There are no $\mathcal{S}$-points of Type C_3. As a consequence, there are no $\mathcal{S}$-lines of Type $B_1C_3C_3$, $C_1C_1C_3$, $C_1C_2C_3$, $C_1C_3C_6$, $C_1C_3D_1$, $C_2C_3D_1$, $C_3C_3C_5$, $C_3C_5D_2$, $C_3C_6D_1$ and $C_3D_2D_2$.*

Proof. Let x be an $\mathcal{S}$-point of Type C_3. Then there are precisely $c(x) = 2$ $\mathcal{S}$-lines of Type BCC through x. Each such line has Type $B_1C_3C_3$ or $B_2C_1C_3$. Lemma 5.1 then implies that the two $\mathcal{S}$-lines of Type BCC through x have Type $B_1C_3C_3$. But this contradicts Lemma 4.14 of the paper and the fact that there exists a unique $\mathcal{V}$-line of Type $B_1C_3C_3$ through f_x. ■

Lemma 5.4 *There are no $\mathcal{S}$-lines of Type $C_1C_1D_1$, $C_1C_5D_1$, $C_1D_1D_2$, $C_2D_1D_2$, $C_6D_1D_2$ and $D_1D_1D_2$.*

Proof. Let x be an $\mathcal{S}$-point of Type D_1. By Lemma 5.3, any $\mathcal{S}$-line L through x has Type $C_1C_1D_1$, $C_1C_2D_1$, $C_1C_5D_1$, $C_1D_1D_2$, $C_2C_5D_1$, $C_2D_1D_2$, $C_4D_1D_1$, $C_6D_1D_2$ or $D_1D_1D_2$. For such a line L, we have $c(x, L) \leq 3$ by Table 1. The equality $\sum_{L\in\mathcal{L}_x} c(x, L) = N(x) = 15$ then implies that $c(x, L) = 3$ for every $L \in \mathcal{L}_x$. We conclude that there are no $\mathcal{S}$-lines of Type $C_1C_1D_1$, $C_1C_5D_1$, $C_1D_1D_2$, $C_2D_1D_2$, $C_6D_1D_2$ and $D_1D_1D_2$ through x. ■

Step 2

The information in Lemma 5.3 can be obtained by means of the function `ThroughPoint4`. For instance, if we give the command

```
ThroughPoint4("C5");
```

then the following output arises

```
[ [ [ "B1", "C5", "C7" ], [ "C2", "C5", "D1" ], [ "C2", "C5", "D1" ],
    [ "C2", "C5", "D1" ], [ "C2", "C5", "D1" ] ],
  [ [ "B1", "C5", "C7" ], [ "C2", "C5", "D1" ], [ "C2", "C5", "D1" ],
    [ "C5", "C5", "C5" ], [ "C5", "C5", "C5" ] ],
  [ [ "B1", "C5", "C7" ], [ "C5", "C5", "C5" ], [ "C5", "C5", "C5" ],
    [ "C5", "C5", "C5" ], [ "C5", "C5", "C5" ] ] ]
```

From this we can immediately see the validity of Claim (4) of Lemma 5.3.

Step 3

The validity of Lemma 5.5 was verified in ReeTits1.g. If this lemma is true, then `Status1` (as defined in ReeTits1.g) will be true. It is also possible to give another proof of Lemma 5.5 that does not require extra computer computations.

Lemma 5.5 *The geometry $\mathcal{G}$ is connected.*

Proof. Let $\mathcal{S}^*$ be the Ree-Tits octagon of order $(2,4)$ and let $\mathcal{S}_1^*$ be a suboctagon of order $(2,1)$ of $\mathcal{S}^*$. We will identify $\mathcal{V}$ with the valuation geometry of $\mathcal{S}_1^*$. Every point x of $\mathcal{S}^*$ will induce a valuation f_x^* of $\mathcal{S}_1^*$. We call a point x of $\mathcal{S}^*$ of Type $T \in \{A, B_1, C_2, C_5, C_6, C_7, D_1\}$ if the valuation f_x^* of $\mathcal{S}_1^*$ has Type T. We say that a line $\{x_1, x_2, x_3\}$ of $\mathcal{S}^*$ has Type $T_1T_2T_3$ if the line $\{f_{x_1}^*, f_{x_2}^*, f_{x_3}^*\}$ of $\mathcal{V}$ is of Type $T_1T_2T_3$. Now, let $\mathcal{G}^*$ be the point-line geometry whose points are the points of $\mathcal{S}^*$ of Type C_2, C_5, C_7 and D_1, and whose lines are the lines of $\mathcal{S}^*$ of Type $B_1C_2C_2$, $C_2C_5D_1$, $C_2C_2C_6$ and $B_1C_5C_7$, with incidence being containment. Then the map $x \mapsto f_x^*$ defines an isomorphism between $\mathcal{G}^*$ and $\mathcal{G}$ (recall paragraph 3.3). So, it suffices to prove that $\mathcal{G}^*$ is connected. Since $\mathcal{G}^*$ contains 1080 points, it thus suffices to prove that each connected component C of $\mathcal{G}^*$ contains more than 540 points. By the last table of paragraph 3.3 (see also Lemma 5.3(3)+(6)+(7) and Lemma 5.4(3)) of the paper, C contains a line L of Type $C_2C_5D_1$. For every point x of C whose $\mathcal{G}^*$-distance to L is at most 3, we have $\mathrm{d}_{\mathcal{G}^*}(x, L) = \mathrm{d}_{\mathcal{S}^*}(x, L)$ since $\mathcal{S}^*$ has no ordinary m-gons with $m \in \{3, 4, \dots, 7\}$ as subgeometries. Also, we denote by L_x the unique line through x nearest to L, with $L_x = L$ if $x \in L$. For every $i \in \{1, 2, \dots, 7\}$ and every $j \in \{0, 1, 2, 3\}$, we define the following numbers:

- $N_1^{(j)}$: number of points $x \in C$ of Type C_2 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ and L_x has Type $B_1C_2C_2$;
- $N_2^{(j)}$: number of points $x \in C$ of Type C_2 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ and L_x has Type $C_2C_2C_6$;
- $N_3^{(j)}$: number of points $x \in C$ of Type C_2 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ and L_x has Type $C_2C_5D_1$;
- $N_4^{(j)}$: number of points $x \in C$ of Type C_5 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ and L_x has Type $B_1C_5C_7$;
- $N_5^{(j)}$: number of points $x \in C$ of Type C_5 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ and L_x has Type $C_2C_5D_1$;
- $N_6^{(j)}$: number of points $x \in C$ of Type C_7 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ (and L_x has Type $B_1C_5C_7$);
- $N_7^{(j)}$: number of points $x \in C$ of Type D_1 for which $\mathrm{d}_{\mathcal{G}^*}(x, L) = j$ (and L_x has Type $C_2C_5D_1$).

Then

$$N_1^{(0)} = N_2^{(0)} = N_4^{(0)} = N_6^{(0)} = 0, \qquad N_3^{(0)} = N_5^{(0)} = N_7^{(0)} = 1.$$

By the last table of paragraph 3.3 (see also Lemma 5.3(3)+(6)+(7) and Lemma 5.4(3)) of the paper, we have

$$\begin{aligned}
N_1^{(j+1)} &= N_1^{(j)} + 2 \cdot N_2^{(j)} + 2 \cdot N_3^{(j)}, \\
N_2^{(j+1)} &= 2 \cdot N_1^{(j)} + N_2^{(j)} + 2 \cdot N_3^{(j)}, \\
N_3^{(j+1)} &= 4 \cdot N_4^{(j)} + 3 \cdot N_5^{(j)} + 4 \cdot N_7^{(j)}, \\
N_4^{(j+1)} &= 4 \cdot N_6^{(j)}, \\
N_5^{(j+1)} &= N_1^{(j)} + N_2^{(j)} + 4 \cdot N_7^{(j)},
\end{aligned}$$

$$\begin{aligned} N_6^{(j+1)} &= N_5^{(j)}, \\ N_7^{(j+1)} &= N_1^{(j)} + N_2^{(j)} + 4 \cdot N_4^{(j)} + 3 \cdot N_5^{(j)} \end{aligned}$$

for every $j \in \{0, 1, 2\}$. From this it follows that

$$N_1^{(1)} = 2,\ N_2^{(1)} = 2,\ N_3^{(1)} = 7,\ N_4^{(1)} = 0,\ N_5^{(1)} = 4,\ N_6^{(1)} = 1,\ N_7^{(1)} = 3,$$

$$N_1^{(2)} = 20,\ N_2^{(2)} = 20,\ N_3^{(2)} = 24,\ N_4^{(2)} = 4,\ N_5^{(2)} = 16,\ N_6^{(2)} = 4,\ N_7^{(2)} = 16,$$

$$N_1^{(3)} = 108,\ N_2^{(3)} = 108,\ N_3^{(3)} = 128,\ N_4^{(3)} = 16,\ N_5^{(3)} = 104,\ N_6^{(3)} = 16,\ N_7^{(3)} = 104.$$

Hence, $|C| \geq \sum_{i=1}^{7} \sum_{j=0}^{3} N_i^{(j)} = 710$. So, the geometries $\mathcal{G}$ and $\mathcal{G}^*$ are connected. ■

5.2 Proof of the isomorphism $\Gamma \cong \Gamma^*$

Step 1: Definition of Γ_1^* and the isomorphism $\theta_1 : \Gamma_1 \to \Gamma_1^*$

Regarding the following claims:

- There are two collections of five $\mathcal{V}$-lines through f that are compatible with respect to f and satisfy (a), (b), (c) and (d).
- The two associated C_6-sets partition the set of 8 $\mathcal{V}$-lines through f that have Type $C_6C_2C_2$ or $C_6C_6C_6$.

These claims have been verified in ReeTits1.g. If they are valid, then `Status2` (as defined in ReeTits1.g) will be true.

The claim that there exists a unique $i^* \in \{1, 2\}$ such that
- The distance between X and X_{i^*} in Γ_0^* is equal to 5,
- The distance between X and X_{3-i^*} in Γ_0^* is equal to 4,

has been verified in ReeTits1.g. If this claim is true, then `Status3` (as defined in ReeTits1.g) will be true.

Step 2: Definition of Γ_2^* and the isomorphism $\theta_2 : \Gamma_2 \to \Gamma_2^*$

The claim that $\mathrm{d}_{\Gamma_1^*}(U_1, U_2)$ is equal to 4 or 6 has been verified in ReeTits2.g. If this claim is true, then `Status1` (as defined in ReeTits2.g) will be true.

The claim that there exists an equivalence relation on the set $\mathcal{C}_f$ has been verified in ReeTits2.g. If this claim is true, then `Status2` (as defined in ReeTits2.g) will be true.

The claim that there exist three equivalence classes of size 4 has been verified in ReeTits2.g. If this claim is true, then `Status3` (as defined in ReeTits2.g) will be true.

Step 3: Definition of Γ_3^* and the isomorphism $\theta_3 : \Gamma_3 \to \Gamma_3^*$

The claim that there exists a unique $i \in \{1, 2, 3\}$ such that

- the distance between X and X_i in Γ_1^* is equal to 5,
- for every $j \in \{1, 2, 3\} \setminus \{i\}$, the distance between X and X_j in Γ_1^* is equal to 4,

has been verified in ReeTits2.g. If this claim is true, then `Status4` (as defined in ReeTits2.g) will be true.

Step 4: Definition of Γ^* and the isomorphism $\theta^* : \Gamma \to \Gamma^*$

There is no extra information here.

A GAP Code

All used GAP code has been organized in three files such that they can immediately be processed by the GAP computer algebra system. These three files are:

- ReeTits1.g : Contains all GAP code related to a computer implementation of the generalized octagon GO(2, 1).
- ReeTits2.g : Contains all GAP code related to a computer implementation of the Ree-Tits octagon RT(2, 4).
- ReeTits3.g : In this file, an algorithm is implemented that is equivalent to a backtrack for finding all valuations of GO(2, 1).

The present manuscript, along with comments in these files should guide the user on how the various commands in these files should be used. The three files are available online, see `http://cage.ugent.be/geometry/preprints.php`.

References

[1] B. De Bruyn. The uniqueness of a certain generalized octagon of order (2, 4).