The uniqueness of a certain generalized octagon of order $(2,4)$

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: `bdb@cage.ugent.be`

Abstract

In the theory of generalized polygons, the question whether there exists a unique generalized octagon of order $(2,4)$ is still open. In this paper, we show the uniqueness of such a generalized octagon under an extra assumption. We give a computer-assisted proof for the fact that the Ree-Tits octagon of order $(2,4)$ is, up to isomorphism, the unique generalized octagon of order $(2,4)$ containing at least one suboctagon of order $(2,1)$.

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1 Introduction

Let $d \in \mathbb{N} \setminus \{0,1\}$. A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *generalized* $2d$*-gon* if it satisfies the following three properties:

(GP1) $\mathcal{S}$ is a *partial linear space*, i.e. every two distinct points of $\mathcal{S}$ are incident with at most one line;

(GP2) if $\{A_1, A_2\} \subseteq \mathcal{P} \cup \mathcal{L}$, then there exists a subgeometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ of $\mathcal{S}$ isomorphic to an ordinary $2d$-gon for which $\{A_1, A_2\} \subseteq \mathcal{P}' \cup \mathcal{L}'$;

(GP3) $\mathcal{S}$ has no subgeometries that are ordinary m-gons with $m \in \{3, 4, \ldots, 2d-1\}$.

Recall that a point-line geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is called a *subgeometry of* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathcal{P} \times \mathcal{L})$. If $\{x \in \mathcal{P} \,|\, x \ \mathrm{I}\ L\} = \{x \in \mathcal{P}' \,|\, x \ \mathrm{I}'\ L\}$ for every line L of $\mathcal{L}'$, then the subgeometry $\mathcal{S}'$ of $\mathcal{S}$ is called *full*.

Generalized $2d$-gons were introduced by Tits in [18]. The point-line dual of a generalized $2d$-gon is again a generalized $2d$-gon. A generalized $2d$-gon is called *thick* if every line is incident with at least three points and if every point is incident with at least three lines.

A generalized $2d$-gon is said to have *order* (s,t) if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines.

There are plenty of constructions known for finite thick generalized quadrangles. This situation is no longer true for generalized $2d$-gons with $d \geq 3$. There are, up to duality, only two classes of finite thick generalized hexagons and one class of finite thick generalized octagons known. There is moreover a result due to Feit & Higman [8] which states that finite thick generalized $2d$-gons can only exist if $d \in \{2,3,4\}$.

The two classes of generalized hexagons mentioned above were already described in Tits' paper [18]. The class of generalized octagons alluded to above was first constructed in Tits [19] using a new family of simple groups discovered by Ree [15]. More precisely, Tits showed that with every field $\mathbb{F}$ of characteristic two having an endomorphism σ satisfying $x^{\sigma^2} = x^2$, $\forall x \in \mathbb{F}$, there corresponds a generalized octagon $O(\mathbb{F},\sigma)$ of order $(|\mathbb{F}|,|\mathbb{F}|^2)$. This generalized octagon $O(\mathbb{F},\sigma)$ is called a *Ree-Tits octagon*. An alternative construction for the Ree-Tits octagons using coordinates can be found in Joswig & Van Maldeghem [12], see also Van Maldeghem [21, Section 3.6]. It is known that the Ree-Tits octagon $O(\mathbb{F},\sigma)$ has full suboctagons of order $(|\mathbb{F}|,1)$, see Joswig & Van Maldeghem [12, Section 5.1], Sarli [16, 6.1.3], Tits [20, 3.17] or Van Maldeghem [21, Theorem 3.6.3].

It is a well-known open problem whether finite thick generalized hexagons or octagons exist besides the known ones described in the papers [18, 19]. If $\mathcal{S}$ is a thick generalized $2d$-gon of order (s,t) with $d \in \{2,3,4\}$ and s,t finite, then the numbers s and t are known to satisfy certain parameter restrictions, see Feit & Higman [8], Haemers & Roos [10] and Higman [11] (see also Van Maldeghem [21, Section 1.7]). If $s,t \geq 2$ satisfy all these conditions, then one can ask for a classification of all generalized $2d$-gons of order (s,t). As one can expect, such a classification can probably only be performed for small values of s and t. For generalized quadrangles, this task has been completed if either (s,t) or (t,s) belongs to the set $\{(2,2),(2,4),(3,3),(3,5),(3,6),(3,9),(4,4)\}$, see Payne & Thas [14, Chapter 6]. Cohen & Tits [2] classified all generalized hexagons of order $(2,2)$ and $(2,8)$. For generalized octagons however, no classification result in this direction is known, not even for the smallest case $(s,t)=(2,4)$. In fact, very few characterization results for generalized octagons are known up to present. For the Ree-Tits octagons, there are the geometrical characterization due to Van Maldeghem [22] and the group theoretical characterizations due to Tits [20] and Cohen, O'Brien & Shpectorov [1].

The aim of the present paper is to give a partial solution to the classification problem of the generalized octagons of order $(2,4)$. The following is our main result.

Main Theorem. *The Ree-Tits octagon* $\mathrm{RT}(2,4)$ *of order* $(2,4)$ *is, up to isomorphism, the unique generalized octagon of order* $(2,4)$ *containing a suboctagon of order* $(2,1)$.

A result, similar to our Main Theorem, can be found in De Medts & Van Maldeghem [7], where it was shown that the split Cayley hexagon $H(3)$ is, up to isomorphism, the unique generalized hexagon of order $(3,3)$ containing a subhexagon of order $(3,1)$.

There exists, up to isomorphism, a unique generalized octagon $\mathrm{GO}(2,1)$ of order $(2,1)$. It is related to the generalized quadrangle $W(2)$ whose points and lines are the points and lines of $\mathrm{PG}(3,2)$ that are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(3,2)$ (natural incidence). The points of $\mathrm{GO}(2,1)$ are the flags of $W(2)$ (i.e. the unordered point-line pairs $\{p, L\}$ with $p \in L$) and the lines of $\mathrm{GO}(2,1)$ are the points and lines of $W(2)$, with incidence being reverse containment.

The generalized octagon $\mathrm{GO}(2,1)$ contains 45 points and 30 lines, while any generalized octagon of order $(2,4)$ contains 1755 points and 2925 lines. So, the condition that a generalized octagon $\mathcal{S}$ of order $(2,4)$ contains a suboctagon $\mathcal{S}' \cong \mathrm{GO}(2,1)$ only seems to reveal a small part of the structure of $\mathcal{S}$. Despite this fact, we will be able to completely reconstruct $\mathcal{S}$ from its suboctagon $\mathcal{S}'$. As the number of points and lines of $\mathcal{S}$ is quite large, we believe that the completion of $\mathcal{S}'$ to the whole octagon $\mathcal{S}$ cannot happen via a standard backtrack algorithm.

The proof of the main theorem relies on the theory of polygonal valuations which was developed in De Bruyn [3]. We recall the basics of this theory in Section 2. An important notion from this theory is the notion of the valuation geometry of a generalized $2d$-gon. If we know the valuation geometry of a particular generalized $2d$-gon, then we also know some information of how this generalized polygon can be fully embedded as a subpolygon in a larger generalized $2d$-gon.

In Section 3, we give a detailed description of the valuation geometry of the generalized octagon $\mathrm{GO}(2,1)$. To achieve this goal, we need to invoke the help of a computer since computations get so extensive that they can hardly be done by a human. All our computations will be done with the aid of the computer algebra system GAP [9].

In Sections 4 and 5, we use the information on the valuation geometry of $\mathrm{GO}(2,1)$ gathered in Section 3 to explicitly describe the structure of any generalized octagon $\mathcal{S}$ of order $(2,4)$ containing a suboctagon $\mathcal{S}'$ of order $(2,1)$. In Section 4, we collect those results that remain valid for any $\mathrm{GO}(2,t)$ (with t possibly infinite) containing a suboctagon of order $(2,1)$. In Section 5, we show that there is essentially one way to reconstruct $\mathcal{S}$ from $\mathcal{S}'$, finishing the proof of our Main Theorem. Also during that process we will often rely on GAP to perform certain computations and verifications.

We have decided not to include all used GAP code here since this would overload the paper. We will still include the GAP code that implements the geometries $\mathrm{GO}(2,1)$ and $\mathrm{RT}(2,4)$, together with the GAP code that determines all hyperplane complements of $\mathrm{GO}(2,1)$. All used GAP code can be found in [4] which is available online. The GAP code is available in separate files, arranged in such a way that they can immediately be processed by the GAP computer algebra system.

The main tool used in this paper to obtain the desired classification result is that of valuations. Valuations are very useful for studying near polygons (in particular, generalized polygons) that contain full subgeometries that are themselves near polygons (generalized polygons). Valuations have been used to obtain several classification results regarding near polygons, and also to construct new near polygons. The first result where valuations have effectively been used to obtain classification results for near polygons was in [6] where a complete classification of all dense near octagons with three points per line was obtained (24 examples; computer free).

2 Polygonal valuations

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized $2d$-gon with $d \in \mathbb{N} \setminus \{0, 1\}$. Distances $\mathrm{d}(\cdot, \cdot)$ between points of $\mathcal{S}$ will always be measured in the collinearity graph of $\mathcal{S}$. If $x \in \mathcal{P}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x. If $\emptyset \neq X \subseteq \mathcal{P}$ and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of points at distance i from X, i.e. the set of all points y for which $\min\{\mathrm{d}(y, x) \,|\, x \in X\} = i$. The generalized $2d$-gon $\mathcal{S}$ belongs to the class of the *near polygons* introduced in Shult & Yanushka [17]. This means that for every point x and every line L of $\mathcal{S}$ there exists a unique point on L nearest to x. If x and y are two points of $\mathcal{S}$ at distance $i \in \{1, 2, \ldots, d-1\}$ from each other, then there is a unique line through y containing a point at distance $i-1$ from x. A subgeometry of $\mathcal{S}$ isomorphic to a generalized $2d$-gon will shortly be called a *sub-$2d$-gon* of $\mathcal{S}$.

A map $f : \mathcal{P} \to \mathbb{N}$ is called a *polygonal valuation* if the following three conditions are satisfied.

(PV1) There exists at least one point with f-value 0.

(PV2) Every line L of $\mathcal{S}$ contains a unique point x_L such that $f(x) = f(x_L) + 1$ for every point $x_L \neq x$ of L.

(PV3) If x is a point of $\mathcal{S}$ for which $f(x) < \max\{f(y) \,|\, y \in \mathcal{P}\}$, then there is at most one line through x containing a (necessary unique) point with f-value $f(x) - 1$.

In De Bruyn [3] a theory of polygonal valuations of generalized polygons was developed. This theory has some similarities with the theory of valuations of dense near polygons introduced in De Bruyn & Vandecasteele [5]. In the sequel of this paper, we will use the word "valuation" as a shortening of "polygonal valuation".

Two valuations f_1 and f_2 of $\mathcal{S}$ are called *isomorphic* if there exists an automorphism θ of $\mathcal{S}$ such that $f_2 = f_1 \circ \theta$. If f is a valuation of $\mathcal{S}$, then we denote by $\mathcal{O}_f$ the set of points with f-value 0 and by $\mathcal{M}_f$ the set of all points x of $\mathcal{S}$ that are not collinear with a point having f-value $f(x) - 1$. We denote by M_f the maximal value attained by f. Clearly, $\mathcal{O}_f \subseteq \mathcal{M}_f$ and $M_f \in \{1, 2, \ldots, d\}$. By Property (PV2), the set H_f of all points of $\mathcal{S}$ with non-maximal f-value is a hyperplane of $\mathcal{S}$, that is, H_f is a proper subset of $\mathcal{P}$ having the property that every line has either one or all its points in H_f. A hyperplane of $\mathcal{S}$ is said to be of *valuation type* if it is of the form H_f for some valuation f of $\mathcal{S}$.

We now describe some classes of valuations that were introduced in [3].

(1) Let x be a given point of $\mathcal{S}$ and put $f(y) := \mathrm{d}(x, y)$ for every $y \in \mathcal{P}$. Then f is a so-called *classical valuation* of $\mathcal{S}$.

(2) Suppose $x \in \mathcal{P}$ and $O \subseteq \Gamma_d(x)$ such that every line of $\mathcal{S}$ at distance $d-1$ from x has a unique point in common with O. If y is a point of $\mathcal{S}$ at distance at most $d-1$ from x, then we define $f(y) := \mathrm{d}(x, y)$. If y is a point of $\mathcal{S}$ at distance d from x, then $f(y) := d-2$ if $y \in O$ and $f(y) := d-1$ otherwise. Then f is a so-called *semi-classical valuation* of $\mathcal{S}$.

(3) Let $j \in \{2, 3, \ldots, d\}$ and let X be a set of points of $\mathcal{S}$ satisfying: (i) $|X| \geq 2$; (ii) $j = \min\{\mathrm{d}(x, y) \,|\, x, y \in X \text{ with } x \neq y\}$; (iii) for every point a, there exists a point

$x \in X$ such that $\mathrm{d}(a,x) \leq \frac{j}{2}$; (iv) for every line L, there exists a point $x \in X$ such that $\mathrm{d}(L,x) \leq \frac{j-1}{2}$. Using the terminology of Offer & Van Maldeghem [13], X is a so-called *distance-j-ovoid* of $\mathcal{S}$. If j is even, then the map $\mathcal{P} \to \mathbb{N}; x \mapsto \mathrm{d}(x,X)$ is a so-called *distance-j-ovoidal valuation* of $\mathcal{S}$. Distance-2-ovoidal valuations are also called *ovoidal valuations*.

The following propositions were proved in De Bruyn [3].

Proposition 2.1 ([3]) *Suppose f is a valuation of $\mathcal{S}$. Then $M_f = 1$ if and only if f is ovoidal, $M_f = d-1$ if and only if f is semi-classical and $M_f = d$ if and only if f is classical. Also, $\mathcal{O}_f = \mathcal{M}_f$ if and only if f is either classical or distance-j-ovoidal for some even j.*

Proposition 2.2 ([3]) *Let f be a valuation and H a hyperplane of $\mathcal{S}$. Let M denote the maximal distance from a point of $\mathcal{S}$ to $\mathcal{P} \setminus H$ and put $f_H(x) := M - d(x, \mathcal{P} \setminus H)$ for every $x \in \mathcal{P}$. Then:*

(1) *We have $H = H_f$ if and only if $f = f_H$.*

(2) *If every point of $\mathcal{S}$ is incident with precisely two lines, then f_H is a valuation of $\mathcal{S}$ if and only if f_H satisfies Property (PV2).*

Let f_i, $i \in I$, be a collection of mutually distinct valuations of $\mathcal{S}$, where I is some index set of size at least two. We say that the set $\mathcal{F} = \{f_i \,|\, i \in I\}$ is an *L-set* if for every point x of $\mathcal{S}$, there exists a (necessarily unique) $i \in I$ such that $f_j(x) - M_{f_j} = f_i(x) - M_{f_i} + 1$ for every $j \in I \setminus \{i\}$. If $\mathcal{F}$ is an L-set, then by [3], there exists a line of $\mathcal{S}$ containing precisely $|\mathcal{F}| = |I|$ points. The set $\mathcal{F}$ is called *admissible* if the following holds for all $i_1, i_2 \in I$ with $i_1 \neq i_2$, for every $x \in \mathcal{M}_{f_{i_1}}$ and every $y \in \mathcal{M}_{f_{i_2}}$:

- if f_{i_1} and f_{i_2} are classical, then $\mathrm{d}(x,y) = 1$;
- if $x = y$, then $(f_{i_1}(x) - M_{f_{i_1}}) - (f_{i_2}(x) - M_{f_{i_2}}) \in \{-1, 0, 1\}$;
- if $x \neq y$ and at least one of f_{i_1}, f_{i_2} is not classical, then $\mathrm{d}(x,y) + f_{i_1}(x) + f_{i_2}(y) - M_{f_{i_1}} - M_{f_{i_2}} + 1 \geq 0$.

The *valuation geometry* $\mathcal{V}_{\mathcal{S}}$ of $\mathcal{S}$ is the point-line geometry whose points are the valuations of $\mathcal{S}$ and whose lines are the admissible L-sets of valuations of $\mathcal{S}$, with incidence being containment. The following two propositions were also proved in [3].

Proposition 2.3 ([3]) *Suppose $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a full sub-$2d$-gon of $\mathcal{S}$. Let x be a point of $\mathcal{S}$ and put $m := \min\{d(x,y) \,|\, y \in \mathcal{P}'\}$. For every point $y \in \mathcal{P}'$, we define $f_x(y) := d(x,y) - m$. Then:*

(1) *f_x is a valuation of $\mathcal{S}'$ with $M_{f_x} = d - m$.*

(2) *If x_1 and x_2 are two distinct collinear points of $\mathcal{S}$, then the valuations f_{x_1} and f_{x_2} are distinct.*

(3) *The map $\theta : x \mapsto f_x$ between the point-sets of $\mathcal{S}$ and $\mathcal{V}_{\mathcal{S}'}$ maps every line of $\mathcal{S}$ to a full line of $\mathcal{V}_{\mathcal{S}'}$.*

Proposition 2.4 ([3]) *Let $\mathcal{S}$ be a generalized $2d$-gon with three points per line. Then:*

(1) *If $\{f_1, f_2, f_3\}$ is an admissible L-set of valuations of $\mathcal{S}$, then H_{f_3} equals the complement $\overline{H_{f_1} \Delta H_{f_2}}$ of the symmetric difference $H_{f_1} \Delta H_{f_2}$ of H_{f_1} and H_{f_2}.*

(2) *Let H_1, H_2 and H_3 be three mutual distinct hyperplanes of valuation type of $\mathcal{S}$ such that $H_3 = \overline{H_1 \Delta H_2}$. Let f_i, $i \in \{1,2,3\}$, be the unique valuation of $\mathcal{S}$ for which $H_{f_i} = H_i$. If $\{f_1, f_2, f_3\}$ is admissible, then $\{f_1, f_2, f_3\}$ is an admissible L-set of $\mathcal{S}$.*

(3) *The valuation geometry of $\mathcal{S}$ is a partial linear space.*

Proposition 2.5 *Let $\mathcal{F}$ be an admissible L-set of valuations of $\mathcal{S}$ and let f_1, f_2 be two distinct elements of $\mathcal{F}$. Then the following hold:*

(1) $|M_{f_1} - M_{f_2}| \in \{0, 1\}$.

(2) *If $M_{f_2} = M_{f_1} + 1$, then $\mathcal{O}_{f_2} \subseteq \mathcal{O}_{f_1}$.*

(3) *If f_2 is not ovoidal and not both f_1, f_2 are classical, then $\mathcal{O}_{f_2} \subseteq \mathcal{M}_{f_1}$.*

(4) *If $M_{f_1} = M_{f_2} \geq M_f$ for every $f \in \mathcal{F} \setminus \{f_1, f_2\}$, then $\mathcal{O}_{f_1} \cap \mathcal{O}_{f_2} = \emptyset$.*

(5) *If $\mathcal{F} = \{f_1, f_2, f_3\}$ and $M_{f_1} = M_{f_2} = M_{f_3} - 1$, then $\mathcal{O}_{f_1} \cap \mathcal{O}_{f_2} = \mathcal{O}_{f_3}$.*

Proof. (1) The fact that $\mathcal{F}$ is an L-set implies that $|(f_1(x) - M_{f_1}) - (f_2(x) - M_{f_2})| \leq 1$ for every point x of $\mathcal{S}$. If we take $x \in \mathcal{O}_{f_1}$, then the fact that $|M_{f_2} - M_{f_1} - f_2(x)| \leq 1$ implies that $M_{f_2} - M_{f_1} \geq -1$. Reversing the roles of f_1 and f_2, we also see that $M_{f_1} - M_{f_2} \geq -1$. It follows that $|M_{f_1} - M_{f_2}| \in \{0, 1\}$.

(2) Let x be a point of $\mathcal{O}_{f_2}$. Then from $|(f_1(x) - M_{f_1}) - (f_2(x) - M_{f_2})| \leq 1$, it follows that $|f_1(x) + 1| \leq 1$. Hence, $f_1(x) = 0$ and $x \in \mathcal{O}_{f_1}$.

(3) Since f_2 is not ovoidal, we have $M_{f_2} \geq 2$ by Proposition 2.1. Suppose x is a point of $\mathcal{O}_{f_2}$ not contained in $\mathcal{M}_{f_1}$. There exists then a point $y \in \mathcal{M}_{f_1}$ at a certain distance $\delta \geq 1$ from x such that $f_1(y) = f_1(x) - \delta$. Then $\mathrm{d}(x, y) + f_1(y) - M_{f_1} + f_2(x) - M_{f_2} + 1 \geq (f_1(x) - M_{f_1}) - M_{f_2} + 1 \leq 0 + (-2) + 1 \leq -1$. But this is impossible. The facts that $x \neq y$, that $\mathcal{F}$ is admissible and that at least one of f_1, f_2 is nonclassical implies that $\mathrm{d}(x, y) + f_1(y) + f_2(x) - M_{f_1} - M_{f_2} + 1 \geq 0$.

(4) Suppose $x \in \mathcal{O}_{f_1} \cap \mathcal{O}_{f_2}$. Since $f(x) \geq 0$ and $M_f \leq M_{f_1} = M_{f_2}$ for every $f \in \mathcal{F} \setminus \{f_1, f_2\}$, the number $f(x) - M_f$ is at least equal to $f_1(x) - M_{f_1} = -M_{f_1}$ and $f_2(x) - M_{f_2} = -M_{f_2}$. Since $\mathcal{F}$ is an L-set, the smallest among the numbers $f_1(x) - M_{f_1}$, $f_2(x) - M_{f_2}$, $f(x) - M_f$ ($f \in \mathcal{F} \setminus \{f_1, f_2\}$) should be attained precisely once, in contradiction with the fact that $M_{f_1} = M_{f_2}$.

(5) From (2), we already know that $\mathcal{O}_{f_3} \subseteq \mathcal{O}_{f_1} \cap \mathcal{O}_{f_2}$. Suppose now that $x \in \mathcal{O}_{f_1} \cap \mathcal{O}_{f_2}$. Since $f_1(x) - M_{f_1} = f_2(x) - M_{f_2} = -M_{f_1} = -M_{f_2}$, the fact that $\{f_1, f_2, f_3\}$ is an admissible L-set implies that $f_3(x) - M_{f_3} = -M_{f_1} - 1$ and hence that $x \in \mathcal{O}_{f_3}$. ∎

3 The valuation geometry of $\mathrm{GO}(2, 1)$

The aim of this section is to determine, up to isomorphism, all valuations and admissible L-sets of valuations of the generalized octagon $\mathrm{GO}(2, 1)$. We will compute these objects with the aid of the computer algebra system GAP. Before we can compute these objects, we first need to implement a computer model of $\mathrm{GO}(2,1)$.

3.1 A computer model of $\mathrm{GO}(2,1)$

Recall that the generalized octagon $\mathrm{GO}(2,1)$ is related to the generalized quadrangle $W(2)$. Its points are the flags of $W(2)$ and its lines are the points and lines of $W(2)$, with incidence being reverse containment. Since $\mathrm{Aut}(W(2)) \cong S_6$ and $W(2)$ is isomorphic to its point-line dual, the full automorphism group G of $\mathrm{GO}(2,1)$ has order 1440. This automorphism group acts primitively on the point set of the octagon.

The following GAP code implements a model of $\mathrm{GO}(2,1)$ with point set $\{1,2,\dots,45\}$, line set `lines`, automorphism group `g` and distance function `dist`.

```
g:=AllPrimitiveGroups(DegreeOperation,45,Size,1440)[1];
orbs:=Orbits(Stabilizer(g,1),[1..45]);
dist1:=Filtered(orbs,x->Size(x)=4)[1];
dist2:=Filtered(orbs,x->Size(x)=8)[1];
perp:=Union([1],dist1);
perp2:=OnSets(perp,RepresentativeAction(g,1,dist2[1]));
dist3:=Filtered(orbs,x->Size(x)=16 and Intersection(x,perp2)<>[])[1];
dist4:=Filtered(orbs,x->Size(x)=16 and Intersection(x,perp2)=[])[1];
line:=Intersection(perp,OnSets(perp,RepresentativeAction(g,1,dist1[1])));
lines:=Orbit(g,line,OnSets);
partition:=[[1],dist1,dist2,dist3,dist4];
DistMat:=NullMat(45,45);
for x in [1..45] do
 r:=RepresentativeAction(g,x,1);
 for y in [1..45] do
  z:=y^r;
  i:=1; while not(z in partition[i]) do i:=i+1; od;
  DistMat[x][y]:=i-1;
 od;
od;
dist:=function(x,y)
 return DistMat[x][y];
end;
```

3.2 Computing the valuation geometry of $\mathrm{GO}(2,1)$

Now that we have a computer model of $\mathrm{GO}(2,1)$, we can search for valuations inside $\mathrm{GO}(2,1)$. We have implemented two algorithms to find the valuations, an algorithm equivalent with a backtrack and an algorithm based on the connection between valuations and hyperplanes. Among the two implemented algorithms, the one using the connection with hyperplanes was the fastest.

Let us first discuss the backtrack approach we implemented. We could assign to each of the 45 points one of the five possible values (0 till 4) and check which of the 5^{45} maps that arise this way are valuations. Since the number $5^{45} > 10^{31}$ is already huge, we will not proceed this way and make use of some completion process that relies on the fact that if you have a line $L = \{x_1, x_2, x_3\}$, then the value of x_3 is uniquely determined by the values of x_1 and x_2 (Property (PV2)). During such a completion process an inconsistency can occur if the values of x_1 and x_2 differ by at least 2. As the automorphism group of $\mathrm{GO}(2,1)$ is point-transitive, we are allowed to give one specific point x of $\mathrm{GO}(2,1)$ the value 0 and its neighbors the value 1. In this way, we obtain a "partial valuation", where some of the points of $\mathrm{GO}(2,1)$ have been given a value, while others might not have received a value. Suppose that we have a list of partial valuations (containing for instance the partial valuation just described), then we can proceed as follows:

(1) Take a partial valuation f from the list which is not yet complete and take a point which has not yet been assigned a value. By assigning one of the five possible values (0 till 4) to this point we obtain a collection $f_0', f_1', \ldots, f_4'$ of five partial valuations.

(2) For each f_i', try to assign values to additional points by performing the completion process described above. We collect those completed versions of $f_0', f_1', \ldots, f_4'$ for which no inconsistency has occurred during this completion process, and replace f in the list by this collection.

(3) Do Steps (1) and (2) till our list only contains members for which all points have been given a value.

(4) Select those members from the list that are valuations by verifying Properties (PV1), (PV2) and (PV3).

In this way, we found that $\mathrm{GO}(2,1)$ has up to isomorphism 12 valuations.

We have implemented another algorithm, which is based on the fact that for geometries with three points per line, it is computationally very easy to generate hyperplanes (or hyperplane complements). Once we know representatives for the various isomorphism classes of hyperplane complements, we can then proceed to find all valuations by making use of Proposition 2.2.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with three points on each line. Then a set $X \neq \emptyset$ of points of $\mathcal{S}$ is a hyperplane complement if and only if its characteristic vector is orthogonal (over $\mathbb{F}_2$) with the characteristic vector of all lines of $\mathcal{S}$. It is this property that makes it often possible for a computer to generate in a rather easy, fast and straightforward way all hyperplanes of a given point-line geometry with three points per line. Once we have implemented the vector space consisting of those vectors orthogonal with the characteristic vectors of all lines, we just need to take vectors from this vector space and compute the corresponding hyperplane complements. Based on this principle, the following GAP code determines a set (called `hypcomplements`) of representatives for the various isomorphism classes of hyperplane complements of $\mathrm{GO}(2,1)$. To speed up the computation, the code is implemented in such a way that the search stops as soon as there is a representative of each class. Each time a new hyperplane complement has been found, a certain number (called `balance`) is updated which keeps track of the number of missing hyperplane complements.

```
M:=NullMat(45,30,GF(2));
for i in [1..45] do for j in [1..30] do
 if i in lines[j] then M[i][j]:=Z(2); fi;
od; od;
Y:=NullspaceMat(M);
U:=Subspace(GF(2)^(45),Y);
hypcomplements:=[Set(dist4)];
balance:=2^(Dimension(U))-46;
VectorToSet:=function(X) return Filtered([1..45],i->X[i]=One(GF(2))); end;
for w in U do
 if w<>Zero(U) then
  compl:=VectorToSet(w);
  new:=true;
  for i in [1..Length(hypcomplements)] do
   if RepresentativeAction(g,compl,hypcomplements[i],OnSets) <> fail then
      new:=false; break; fi;
  od;
  if new then
   Append(hypcomplements,[compl]);
   balance:=balance-Index(g,Stabilizer(g,compl,OnSets));
  fi;
  if balance=0 then break; fi;
 fi;
od;
```

In this way, we found that $\mathrm{GO}(2,1)$ has up to isomorphism 92 hyperplanes. If f is a valuation of $\mathrm{GO}(2,1)$, then H_f is a hyperplane of $\mathrm{GO}(2,1)$, and Proposition 2.2 tells us how f can be reconstructed from H_f. Based on the method exposed in Proposition 2.2, we have implemented an algorithm ([4]) to determine for each of the 92 hyperplane complements a map that is a possible candidate for a valuation. After checking the conditions (PV1), (PV2) and (PV3), it turned out that only 12 from the 92 hyperplane complements were associated with valuations, confirming our earlier computation.

We say that a valuation of $\mathrm{GO}(2,1)$ is of *Type* A, B, C, respectively D, if its maximal value is equal to 4, 3, 2, respectively 1. It turns out that there are up to isomorphism two valuations of Type D (D_1 and D_2), seven of Type C (C_1 till C_7), two of Type B (B_1 and B_2) and a unique one of Type A. In Table 1, one can find the list of twelve valuations together with some properties that make it possible to distinguish between them. The information provided in the last column can easily be extracted from the values M_f, $|\mathcal{O}_f|$ and $|\mathcal{M}_f|$, taking into account Proposition 2.1. In the second last column of Table 1 we

Type	#	M_f	$\lvert\mathcal{O}_f\rvert$	$\lvert\mathcal{M}_f\rvert$	$\lvert H_f\rvert$	Stabilizer G_f	Orbits	Type
A	45	4	1	1	29	$(C_2 \times D_8) \rtimes C_2$	$4 + 1$	classical
B_1	90	3	1	9	21	$(C_4 \times C_2) \rtimes C_2$	$4 + 2$	semi-classical
B_2	90	3	1	9	21	D_{16}	$4 + 2$	semi-classical
C_1	720	2	1	13	17	C_2	$10 + 15$	–
C_2	720	2	2	11	19	C_2	$10 + 15$	–
C_3	720	2	2	11	19	C_2	$11 + 14$	–
C_4	360	2	3	9	21	$C_2 \times C_2$	$8 + 7$	–
C_5	180	2	1	13	17	$C_4 \times C_2$	$4 + 4$	–
C_6	180	2	1	13	17	D_8	$4 + 5$	–
C_7	36	2	5	5	25	$C_2 \times (C5 \rtimes C_4)$	$2 + 1$	distance-4-ovoidal
D_1	144	1	15	15	15	D_{10}	$2 + 5$	ovoidal
D_2	144	1	15	15	15	D_{10}	$3 + 4$	ovoidal

Table 1: The valuations of $\mathrm{GO}(2,1)$

have listed the number of orbits of the stabilizer G_f of H_f on the hyperplane H_f and on the complement $\overline{H_f}$ of H_f. For instance, the entry "8+7" means that if f is a valuation of Type C_4, then G_f has 8 orbits on H_f and 7 on $\overline{H_f}$.

Now that we have determined all points of the valuation geometry $\mathcal{V}$ of $\mathrm{GO}(2,1)$, we can also determine its lines. The generation of admissible L-sets can be speed up by relying on the fact that for an admissible L-set $\{f_1, f_2, f_3\}$, the valuation f_3 is uniquely determined by f_1 and f_2. This is most easily expressed in terms of the associated hyperplane complements: the hyperplane complement associated with f_3 should be the symmetric difference of the hyperplane complements associated with f_1 and f_2 (Proposition 2.4(1)). For each of the 12 possible valuations f_1, we have determined all admissible L-sets $\{f_1, f_2, f_3\}$ containing f_1 and we have also computed the types of f_2 and f_3 ([4]). In this way, we could find all types for the lines of $\mathcal{V}$, where the *type of a line* is defined as the array that consists of the types of the three involved valuations, where these types have been lexicographically ordered.

After computation ([4]), we found that there are 52 line types. The 52 types can be found in Table 2. With every automorphism θ of $\mathrm{GO}(2,1)$, there corresponds an automorphism $\widetilde{\theta}$ of $\mathcal{V}$ in the following way: for every valuation f of $\mathrm{GO}(2,1)$, we define $f^{\widetilde{\theta}} := f \circ \theta^{-1}$. Put $\widetilde{G} := \{\widetilde{\theta} \mid \theta \in G\}$. In Table 2, we have also listed the orbits of $\widetilde{G}$ on the set of lines of $\mathcal{V}$. There are in total 58 such orbits. Six of the 52 classes of lines of $\mathcal{V}$ with a given type split in two $\widetilde{G}$-orbits.

In Table 3, we list how many lines a given point of $\mathcal{V}$ is incident with. The entry "4;2" in row $C_1C_1D_2$ and column C_1 means that every point of Type C_1 is incident with precisely six lines of Type $C_1C_1D_2$. Four of these lines belong to the $\widetilde{G}$-orbit of size 1440 and two of these lines belong to the $\widetilde{G}$-orbit of size 720 (recall Table 2).

Type	# lines	$\widetilde{G}$-orbits	Type	# lines	$\widetilde{G}$-orbits
AAA	30	30	$C_1C_2D_2$	2160	1440 + 720
AB_1B_1	45	45	$C_1C_3C_6$	2880	1440 + 1440
AB_2B_2	45	45	$C_1C_3D_1$	2880	1440 + 1440
$B_1C_1C_4$	720	720	$C_1C_4D_2$	720	720
$B_1C_2C_2$	720	720	$C_1C_5C_6$	720	720
$B_1C_3C_3$	360	360	$C_1C_5D_1$	1440	1440
$B_1C_4C_4$	180	180	$C_1D_1D_2$	720	720
$B_1C_5C_5$	90	90	$C_2C_2C_2$	480	480
$B_1C_5C_7$	180	180	$C_2C_2C_6$	720	720
$B_1C_6C_6$	90	90	$C_2C_2D_2$	720	720
$B_2C_1C_3$	720	720	$C_2C_3D_1$	1440	1440
$B_2C_2C_2$	720	720	$C_2C_4D_2$	720	720
$B_2C_4C_5$	360	360	$C_2C_5D_1$	720	720
$B_2C_6C_6$	90	90	$C_2D_1D_2$	1440	720 + 720
$C_1C_1C_1$	480	480	$C_3C_3C_5$	720	720
$C_1C_1C_2$	1440	1440	$C_3C_5D_2$	2160	1440 + 720
$C_1C_1C_3$	1440	1440	$C_3C_6D_1$	1440	1440
$C_1C_1C_4$	720	720	$C_3D_2D_2$	720	720
$C_1C_1C_5$	1440	1440	$C_4C_6D_2$	720	720
$C_1C_1C_6$	720	720	$C_4D_1D_1$	720	720
$C_1C_1D_1$	1440	1440	$C_5C_5C_5$	240	240
$C_1C_1D_2$	2160	1440 + 720	$C_5D_2D_2$	360	360
$C_1C_2C_3$	1440	1440	$C_6C_6C_6$	240	240
$C_1C_2C_5$	1440	1440	$C_6D_1D_2$	720	720
$C_1C_2C_6$	1440	1440	$C_7D_2D_2$	72	72
$C_1C_2D_1$	1440	1440	$D_1D_1D_2$	144	144

Table 2: The lines of the valuation geometry of $\mathrm{GO}(2,1)$.

Type	A	B_1	B_2	C_1	C_2	C_3	C_4	C_5	C_6	C_7	D_1	D_2
AAA	2	–	–	–	–	–	–	–	–	–	–	–
AB_1B_1	1	1	–	–	–	–	–	–	–	–	–	–
AB_2B_2	1	–	1	–	–	–	–	–	–	–	–	–
$B_1C_1C_4$	–	8	–	1	–	–	2	–	–	–	–	–
$B_1C_2C_2$	–	8	–	–	2	–	–	–	–	–	–	–
$B_1C_3C_3$	–	4	–	–	–	1	–	–	–	–	–	–
$B_1C_4C_4$	–	2	–	–	–	–	1	–	–	–	–	–
$B_1C_5C_5$	–	1	–	–	–	–	–	1	–	–	–	–
$B_1C_5C_7$	–	2	–	–	–	–	–	1	–	5	–	–
$B_1C_6C_6$	–	1	–	–	–	–	–	–	1	–	–	–
$B_2C_1C_3$	–	–	8	1	–	1	–	–	–	–	–	–
$B_2C_2C_2$	–	–	8	–	2	–	–	–	–	–	–	–
$B_2C_4C_5$	–	–	4	–	–	–	1	2	–	–	–	–
$B_2C_6C_6$	–	–	1	–	–	–	–	–	1	–	–	–
$C_1C_1C_1$	–	–	–	2	–	–	–	–	–	–	–	–
$C_1C_1C_2$	–	–	–	4	2	–	–	–	–	–	–	–
$C_1C_1C_3$	–	–	–	4	–	2	–	–	–	–	–	–
$C_1C_1C_4$	–	–	–	2	–	–	2	–	–	–	–	–
$C_1C_1C_5$	–	–	–	4	–	–	–	8	–	–	–	–
$C_1C_1C_6$	–	–	–	2	–	–	–	–	4	–	–	–
$C_1C_1D_1$	–	–	–	4	–	–	–	–	–	–	10	–
$C_1C_1D_2$	–	–	–	4;2	–	–	–	–	–	–	–	10;5
$C_1C_2C_3$	–	–	–	2	2	2	–	–	–	–	–	–
$C_1C_2C_5$	–	–	–	2	2	–	–	8	–	–	–	–
$C_1C_2C_6$	–	–	–	2	2	–	–	–	8	–	–	–
$C_1C_2D_1$	–	–	–	2	2	–	–	–	–	–	10	–
$C_1C_2D_2$	–	–	–	2;1	2;1	–	–	–	–	–	–	10;5
$C_1C_3C_6$	–	–	–	2;2	–	2;2	–	–	8;8	–	–	–
$C_1C_3D_1$	–	–	–	2;2	–	2;2	–	–	–	–	10;10	–
$C_1C_4D_2$	–	–	–	1	–	–	2	–	–	–	–	5
$C_1C_5C_6$	–	–	–	1	–	–	–	4	4	–	–	–
$C_1C_5D_1$	–	–	–	2	–	–	–	8	–	–	10	–
$C_1D_1D_2$	–	–	–	1	–	–	–	–	–	–	5	5
$C_2C_2C_2$	–	–	–	–	2	–	–	–	–	–	–	–
$C_2C_2C_6$	–	–	–	–	2	–	–	–	4	–	–	–
$C_2C_2D_2$	–	–	–	–	2	–	–	–	–	–	–	5
$C_2C_3D_1$	–	–	–	–	2	2	–	–	–	–	10	–
$C_2C_4D_2$	–	–	–	–	1	–	2	–	–	–	–	5
$C_2C_5D_1$	–	–	–	–	1	–	–	4	–	–	5	–
$C_2D_1D_2$	–	–	–	–	1;1	–	–	–	–	–	5;5	5;5
$C_3C_3C_5$	–	–	–	–	–	2	–	4	–	–	–	–
$C_3C_5D_2$	–	–	–	–	–	2;1	–	8;4	–	–	–	10;5
$C_3C_6D_1$	–	–	–	–	–	2	–	–	8	–	10	–
$C_3D_2D_2$	–	–	–	–	–	1	–	–	–	–	–	10
$C_4C_6D_2$	–	–	–	–	–	–	2	–	4	–	–	5
$C_4D_1D_1$	–	–	–	–	–	–	2	–	–	–	10	–
$C_5C_5C_5$	–	–	–	–	–	–	–	4	–	–	–	–
$C_5D_2D_2$	–	–	–	–	–	–	–	2	–	–	–	5
$C_6C_6C_6$	–	–	–	–	–	–	–	–	4	–	–	–
$C_6D_1D_2$	–	–	–	–	–	–	–	–	4	–	5	5
$C_7D_2D_2$	–	–	–	–	–	–	–	–	–	2	–	1
$D_1D_1D_2$	–	–	–	–	–	–	–	–	–	–	2	1
Total	4	27	22	54	29	24	14	58	58	7	107	102

Table 3: Description of the valuation geometry of $\mathrm{GO}(2,1)$: the number of lines incident with a given point.

All the GAP code that allowed to determine the information of Tables 1 and 3 can be found in [4]. The information given in Table 2 can easily be extracted from the ones given in Tables 1 and 3.

3.3 Example: The Ree-Tits octagon of order (2,4)

Only one generalized octagon of order $(2, 4)$ is currently known (up to isomorphism), namely the Ree-Tits octagon $\mathrm{RT}(2, 4)$. This generalized octagon is known to have suboctagons of order $(2, 1)$. Let $\mathrm{GO}(2, 1)$ denote one of these suboctagons. By Proposition 2.3, every point x of $\mathrm{RT}(2, 4)$ will induce a valuation f_x of $\mathrm{GO}(2, 1)$ and every line $\{x_1, x_2, x_3\}$ of $\mathrm{RT}(2, 4)$ will induce a line $\{f_{x_1}, f_{x_2}, f_{x_3}\}$ of the valuation geometry $\mathcal{V}$ of $\mathrm{GO}(2, 1)$. We can now attach types to the points and lines of $\mathrm{RT}(2, 4)$ according to the types of the points and lines of $\mathcal{V}$ they induce. At this stage, the curious reader might already be interested in knowing which of the 12 possible point types and which of the 52 possible line types can actually occur for the points and lines of $\mathrm{RT}(2, 4)$. To find out, we have implemented a computer model for $\mathrm{RT}(2, 4)$ and one of its suboctagons. The generalized octagon $\mathrm{RT}(2, 4)$ has 24960 suboctagons of order $(2, 1)$ and its automorphism group acts transitively on these suboctagons ([4]).

A computer model for $\mathrm{RT}(2, 4)$ can be implemented in a similar way as a computer model for $\mathrm{GO}(2, 1)$. The first lines of such a code would now read as follows:

```
g:=AllPrimitiveGroups(DegreeOperation,1755)[2];
orbs:=Orbits(Stabilizer(g,1),[1..1755]);
dist1:=Filtered(orbs,x->Size(x)=10)[1];
dist2:=Filtered(orbs,x->Size(x)=80)[1];
dist3:=Filtered(orbs,x->Size(x)=640)[1];
dist4:=Filtered(orbs,x->Size(x)=1024)[1];
```

A set X of points of $\mathrm{RT}(2, 4)$ is called *convex of depth 3* if for any two distinct points x and y of X at distance $j \in \{1, 2, 3\}$ from each other, the singleton $\Gamma_1(y) \cap \Gamma_{j-1}(x)$ is contained in X. Now, any suboctagon of order $(2, 1)$ of $\mathrm{RT}(2, 4)$ can be found as the smallest convex subspace of depth 3 containing a set $\{a, b, c_1, c_2\}$ where $\mathrm{d}(a, b) = 4$ and c_1, c_2 are two distinct points of $\Gamma_1(b) \cap \Gamma_3(a)$. Based on this observation, a suboctagon of order $(2, 1)$ can be implemented in our computer model of $\mathrm{RT}(2, 4)$.

Based on the information provided by Table 1, we have implemented functions in [4] to determine the types of the points and lines of $\mathrm{RT}(2, 4)$ (with respect to the implemented suboctagon). With the aid of these functions, the following information about the structure of $\mathrm{RT}(2, 4)$ can be verified.

(1) Among the 1755 points of $\mathrm{RT}(2, 4)$, 45 have Type A, 270 have Type B_1, 720 have Type C_2, 180 have Type C_5, 360 have Type C_6, 36 have Type C_7 and 144 have Type D_1.

(2) Among the 2925 lines, 30 have Type AAA, 135 have Type AB_1B_1, 720 have Type $B_1C_2C_2$, 180 have Type $B_1C_5C_7$, 180 have Type $B_1C_6C_6$, 720 have Type $C_2C_2C_6$, 720 have Type $C_2C_5D_1$ and 240 have Type $C_6C_6C_6$.

(3) For each $T \in \{A, C_2, C_5, C_7, D_1\}$, the map $x \mapsto f_x$ defines a bijection between the set of points of Type T of $\mathrm{RT}(2,4)$ and the set of points of Type T of $\mathcal{V}$.

(4) For each $\mathcal{T} \in \{AAA, B_1C_2C_2, B_1C_5C_7, C_2C_2C_6, C_2C_5D_1, C_6C_6C_6\}$, the map $\{x_1, x_2, x_3\} \mapsto \{f_{x_1}, f_{x_2}, f_{x_3}\}$ defines a bijection between the set of lines of Type $\mathcal{T}$ of $\mathrm{RT}(2,4)$ and the set of lines of Type $\mathcal{T}$ of $\mathcal{V}$.

(5) The stabilizer of the implemented octagon has two orbits on the set of points of Type B_1, one orbit of size 180 (Type B_1') and one orbit of size 90 (Type B_1'').

The following table provides information about the number of lines of each type that are incident with a given point of Type $T \in \{A, B_1', B_1'', C_2, C_5, C_6, C_7, D_1\}$.

	A	B_1'	B_1''	C_2	C_5	C_6	C_7	D_1
AAA	2	–	–	–	–	–	–	–
$AB_1'B_1'$	2	1	–	–	–	–	–	–
$AB_1''B_1''$	1	–	1	–	–	–	–	–
$B_1'C_2C_2$	–	4	–	2	–	–	–	–
$B_1''C_6C_6$	–	–	2	–	–	1	–	–
$B_1''C_5C_7$	–	–	2	–	1	–	5	–
$C_2C_2C_6$	–	–	–	2	–	2	–	–
$C_2C_5D_1$	–	–	–	1	4	–	–	5
$C_6C_6C_6$	–	–	–	–	–	2	–	–

We now define two graphs. The graph $\widetilde{\Gamma}_0$ is the subgraph of the collinearity graph $\widetilde{\Gamma}$ of $\mathrm{RT}(2,4)$ induced on the points of Type C_2, C_5, C_7, D_1, and $\widetilde{\Gamma}_1$ is the subgraph of $\widetilde{\Gamma}$ induced on the points of Type C_2, C_5, C_6, C_7 and D_1.

4 Generalized octagons of order $(2,t)$ containing a suboctagon of order $(2,1)$

We suppose here that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized octagon of order $(2,t)$ with t possibly infinite containing a suboctagon $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ of order $(2,1)$. Recall that there exists up to isomorphism a unique generalized octagon of order $(2,1)$ and that it is related to the generalized quadrangle $W(2)$.

Let $\mathcal{V}$ denote the valuation geometry of $\mathcal{S}'$. In order to distinguish between points or lines of $\mathcal{S}$ and $\mathcal{V}$, we will often talk about $\mathcal{S}$-points, $\mathcal{V}$-points, $\mathcal{S}$-lines and $\mathcal{V}$-lines.

For every point x of $\mathcal{S}$ and every point y of $\mathcal{P}'$, we define $f_x(y) := \mathrm{d}(x,y) - \mathrm{d}(x,\mathcal{P}')$. By Proposition 2.3, f_x is a valuation of $\mathcal{S}'$ and hence a point of the valuation geometry $\mathcal{V}$.

The valuation f_x has Type A, B, C, or D, depending on whether $\mathrm{d}(x, \mathcal{P}')$ is equal to 0, 1, 2 or 3. We have $\mathcal{O}_{f_x} = \Gamma_i(x) \cap \mathcal{P}'$ where $i = \mathrm{d}(x, \mathcal{P}')$. A point x of $\mathcal{S}$ is said to be of *Type* $T \in \{A, B, C, D, B_1, B_2, C_1, C_2, C_3, C_4, C_5, C_6, C_7, D_1, D_2\}$ if its corresponding valuation f_x has Type T. If $x_1, x_2, \ldots, x_k$ is a nonempty finite collection of points such that x_i has Type T_i, then we will also say that $x_1x_2 \ldots x_k$ has Type $T_1T_2 \ldots T_k$. A line $\{x_1, x_2, x_3\}$ of $\mathcal{S}$ is said to be of Type $T_1T_2T_3$ if $x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ has Type $T_1T_2T_3$ for some permutation σ of $\{1, 2, 3\}$. If $\{x_1, x_2, x_3\}$ is a line of $\mathcal{S}$ such that x_i has Type T_i, then by abuse of notation we will also say that the line $x_1x_2x_3$ of $\mathcal{S}$ has Type $T_1T_2T_3$. We will follow a similar convention for the lines of $\mathcal{V}$. By Proposition 2.3(3), we immediately have:

Lemma 4.1 *Every line of $\mathcal{S}$ has a Type XYZ, where XYZ is one of the* 52 *possibilities occurring in the first column of Table 3.*

Lemma 4.1 will be crucial for the rest of our discussion. In the following lemma, we collect some obvious facts.

Lemma 4.2 (1) *The $\mathcal{S}$-lines of Type AAA are precisely the lines of $\mathcal{S}$ contained in $\mathcal{P}'$.*

(2) *Every $\mathcal{S}$-point x of Type A is contained in precisely two $\mathcal{S}$-lines of Type AAA and every other $\mathcal{S}$-line through x has Type ABB.*

(3) *Every $\mathcal{S}$-point x of Type B is contained in a unique $\mathcal{S}$-line of Type ABB and every other $\mathcal{S}$-line through x has Type BCC. The unique $\mathcal{S}$-line of Type ABB through x is equal to xy, where $\{y\} = \mathcal{O}_{f_x}$.*

The lemmas 4.3, 4.4, 4.5 and 4.6 below are consequences of Proposition 2.5, taking into account that $\mathcal{O}_{f_x} = \Gamma_i(x) \cap \mathcal{P}'$ for every point x of $\mathcal{S}$ for which $\mathrm{d}(x, \mathcal{P}') = i$. These lemmas can also be proved by means of a direct reasoning inside the generalized octagon $\mathcal{S}$.

Lemma 4.3 *Let xy_1y_2 be an $\mathcal{S}$-line of Type BCC and let x' be the unique point of $\mathcal{P}'$ collinear with x. Then:*

(1) $(\Gamma_2(y_1) \cap \mathcal{P}') \cap (\Gamma_2(y_2) \cap \mathcal{P}') = \Gamma_1(x) \cap \mathcal{P}' = \mathcal{O}_{f_x} = \{x'\}$;

(2) $\Gamma_2(y_i) \cap \mathcal{P}' \subseteq \mathcal{M}_{f_x}$ *and* $(\Gamma_2(y_i) \cap \mathcal{P}') \setminus \{x'\} \subseteq \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$ *for every* $i \in \{1, 2\}$;

(3) $\Gamma_2(y_i) \cap \mathcal{P}' \subseteq \mathcal{M}_{f_{y_j}}$ *and* $(\Gamma_2(y_i) \cap \mathcal{P}') \setminus \{x'\} \subseteq \mathcal{M}_{f_{y_j}} \setminus \mathcal{O}_{f_{y_j}}$ *for all* $i, j \in \{1, 2\}$ *with* $i \neq j$.

Lemma 4.4 *Let $x_1x_2x_3$ be an $\mathcal{S}$-line of Type CCC. Then:*

(1) $\Gamma_2(x_1) \cap \mathcal{P}'$, $\Gamma_2(x_2) \cap \mathcal{P}'$ *and* $\Gamma_2(x_3) \cap \mathcal{P}'$ *are mutually disjoint;*

(2) $\Gamma_2(x_i) \cap \mathcal{P}' \subseteq \mathcal{M}_{f_{x_j}} \setminus \mathcal{O}_{f_{x_j}}$ *for all* $i, j \in \{1, 2, 3\}$ *with* $i \neq j$.

Lemma 4.5 *Let x_1x_2y be an $\mathcal{S}$-line of Type CCD. Then:*

(1) $\Gamma_2(x_i) \cap \mathcal{P}' \subseteq \Gamma_3(y) \cap \mathcal{P}'$ *for every* $i \in \{1, 2\}$*;*

(2) $(\Gamma_2(x_1) \cap \mathcal{P}') \cap (\Gamma_2(x_2) \cap \mathcal{P}') = \emptyset$*;*

(3) $\Gamma_2(x_i) \cap \mathcal{P}' \subseteq \mathcal{M}_{f_{x_j}} \setminus \mathcal{O}_{f_{x_j}}$ *for all* $i, j \in \{1, 2\}$ *with* $i \neq j$*.*

Lemma 4.6 *Let* xy_1y_2 *be an* $\mathcal{S}$*-line of Type* CDD*. Then* $(\Gamma_3(y_1) \cap \mathcal{P}') \cap (\Gamma_3(y_2) \cap \mathcal{P}') = \Gamma_2(x) \cap \mathcal{P}'$*.*

Lemma 4.7 *Let* $x_1x_2x_3$ *be an* $\mathcal{S}$*-line of Type* DDD*. Then* $\{\Gamma_3(x_1) \cap \mathcal{P}', \Gamma_3(x_2) \cap \mathcal{P}', \Gamma_3(x_3) \cap \mathcal{P}'\}$ *is a partition of* $\mathcal{P}'$ *into distance-2-ovoids of* $\mathcal{S}'$*.*

Proof. Since f_{x_i} is ovoidal, $\mathcal{O}_{f_{x_i}} = \Gamma_3(x_i) \cap \mathcal{P}'$ is a distance-2-ovoid of $\mathcal{S}'$ for every $i \in \{1, 2, 3\}$. Let y be an arbitrary point of $\mathcal{P}'$. Then $\mathrm{d}(y, x_i) \in \{3, 4\}$ for every $i \in \{1, 2, 3\}$. Since $\{x_1, x_2, x_3\}$ contains a unique point nearest to y, there exists a unique $i \in \{1, 2, 3\}$ such that $y \in \Gamma_3(x_i) \cap \mathcal{P}'$. So, $\{\Gamma_3(x_1) \cap \mathcal{P}', \Gamma_3(x_2) \cap \mathcal{P}', \Gamma_3(x_3) \cap \mathcal{P}'\}$ is a partition of $\mathcal{P}'$ into distance-2-ovoids of $\mathcal{S}'$. ∎

Lemma 4.8 *Let* x *be a point of* $\mathcal{S}$ *not contained in* $\mathcal{P}'$*, and let* $y_1, y_2, \ldots, y_k$ *be the (necessarily finite) collection of mutually distinct points of* $\mathcal{S}$ *collinear with* x *at distance* $d(x, \mathcal{P}') - 1$ *from* $\mathcal{P}'$*. Then* $\mathcal{O}_{f_x} = \bigcup_{i=1}^{k} \mathcal{O}_{f_{y_i}}$ *and* $\mathcal{O}_{f_{y_{i_1}}} \cap \mathcal{O}_{f_{y_{i_2}}} = \emptyset$ *for any two distinct* $i_1, i_2 \in \{1, 2, \ldots, k\}$*.*

Proof. If $z \in \mathcal{O}_{f_x}$, then $\mathrm{d}(x, z) \leq 3$ and $z \in \mathcal{O}_{f_y}$, where $y \in \{y_1, \ldots, y_k\}$ is the unique point collinear with x at distance $\mathrm{d}(x, \mathcal{P}') - 1$ from z. Conversely, if $z \in \mathcal{O}_{f_y}$ for some $y \in \{y_1, \ldots, y_k\}$ then z has distance at most and hence precisely $\mathrm{d}(z, y) + 1 = \mathrm{d}(x, \mathcal{P}')$ from x, showing that $z \in \mathcal{O}_{f_x}$. Hence, $\mathcal{O}_{f_x} = \bigcup_{i=1}^{k} \mathcal{O}_{f_{y_i}}$. Suppose $z \in \mathcal{O}_{f_{y_{i_1}}} \cap \mathcal{O}_{f_{y_{i_2}}}$ where i_1 and i_2 are two distinct elements of $\{1, 2, \ldots, k\}$. Then both y_{i_1} and y_{i_2} would coincide with the unique point collinear with x at distance $\mathrm{d}(x, z) - 1 = \mathrm{d}(x, \mathcal{P}') - 1$ from z, which is impossible. ∎

If x is a point of $\mathcal{S}$, then we define

- $\mathcal{C}(x) := \Gamma_2(x) \cap \mathcal{P}'$ if x has Type C and $\mathcal{C}(x) := \emptyset$ otherwise;
- $\mathcal{N}(x) := \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$ is x has Type A, B or C, and $\mathcal{N}(x) := \mathcal{O}_{f_x}$ if x has Type D.

The sizes $c(x) := |\mathcal{C}(x)|$ and $N(x) := |\mathcal{N}(x)|$ of these sets are given in the following table:

Type x	A	B_1, B_2	C_1, C_5, C_6	C_2, C_3	C_4	C_7	D_1, D_2
$c(x)$	0	0	1	2	3	5	0
$N(x)$	0	8	12	9	6	0	15

Clearly, $\mathcal{N}(x) = \emptyset$ if x has Type A.

Lemma 4.9 $\mathcal{N}(x)$ *consists of all points of* $\mathcal{P}'$ *at distance* 3 *from* x *that are not collinear with a point of* $\mathcal{P}'$ *at distance* 2 *from* x*.*

Proof. This is clear for the Type A and Type D points. Suppose therefore that x has Type B or C, and let $y \in \mathcal{N}(x) = \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$.

If x has Type C, then f_x assumes the values 0, 1 and 2. The point y cannot assume the value 2 since every point with maximal value is collinear with points whose value is smaller (one on each line through that point). The point y cannot assume the value 0 either since it does not belong to $\mathcal{O}_{f_x}$. Hence, $f_x(y) = 1$.

Suppose x has Type B. Then f_x is a semi-classical valuation having a unique point z with value 0. By the definition of semi-classical valuation we know that the points of $\mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$ lie at distance 4 from z and have value 2. In particular, $f_x(y) = 2$.

In any case, we have that $\mathrm{d}(x, y) = \mathrm{d}(x, \mathcal{P}') + f_x(y) = 3$. ■

Since $c(x) = |\Gamma_2(x) \cap \mathcal{P}'|$ for every $\mathcal{S}$-point of Type C, Lemma 4.2(3) and the fact that every two points at distance 2 have a unique common neighbor imply the following:

Lemma 4.10 *Let x be an $\mathcal{S}$-point of Type C. Then there are precisely $c(x)$ lines of $\mathcal{S}$ through x containing a (necessarily unique) $\mathcal{S}$-point of Type B.*

For every line $L = \{x, y, z\}$ of $\mathcal{S}$, we define $c(x, L) := c(y) + c(z) - \epsilon$, where

- $\epsilon = 2$ if xyz has Type BCC;
- $\epsilon = 1$ if x is a point of Type C and precisely one of the points y, z has Type B;
- $\epsilon = 0$ otherwise.

For every line $L = \{x, y, z\}$ of $\mathcal{S}$, we define

$$\mathcal{C}(x, L) := \Big(\mathcal{C}(y) \Delta \mathcal{C}(z)\Big) \setminus \mathcal{C}(x),$$

where $\mathcal{C}(y)\Delta\mathcal{C}(z)$ denotes the symmetric difference of the sets $\mathcal{C}(y)$ and $\mathcal{C}(z)$.

Lemma 4.11 *For every line $L = \{x, y, z\}$ of $\mathcal{S}$, we have $c(x, L) = |\mathcal{C}(x, L)|$ and $\mathcal{C}(x, L) \subseteq \mathcal{N}(x)$.*

Proof. We distinguish between the distinct cases.

If xyz has Type AAA, ABB or BBA, then $\mathcal{C}(x) = \mathcal{C}(y) = \mathcal{C}(z) = \emptyset$, and hence $\mathcal{C}(x, L) = \emptyset$. The claims of the lemma obviously hold in this case.

Suppose xyz has Type BCC. Let x' denote the unique point of $\mathcal{P}'$ collinear with x. Then $\mathcal{C}(x) = \emptyset$ and $\mathcal{C}(y) \cap \mathcal{C}(z) = \{x'\}$ by Lemma 4.3(1). Hence, $|\mathcal{C}(x, L)| = |\mathcal{C}(y)| + |\mathcal{C}(z)| - 2 = c(y) + c(z) - 2$. By Lemma 4.3(2), $\mathcal{C}(x, L) = \Big((\Gamma_2(y) \cap \mathcal{P}') \cup (\Gamma_2(z) \cap \mathcal{P}')\Big) \setminus \{x'\}$ is contained in $\mathcal{N}(x) = \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$.

Suppose xyz has Type CCB, and denote by z' the unique point of $\mathcal{P}'$ collinear with z. Then $\mathcal{C}(z) = \emptyset$ and $\mathcal{C}(x) \cap \mathcal{C}(y) = \{z'\}$ (again by Lemma 4.3(1)). Hence, $\mathcal{C}(x, L) = \mathcal{C}(y) \setminus \{z'\}$ and thus $|\mathcal{C}(x, L)| = c(y) - 1 = c(y) + c(z) - 1$. By Lemma 4.3(3), the set $\mathcal{C}(x, L) = \Big(\Gamma_2(y) \cap \mathcal{P}'\Big) \setminus \{z'\}$ is contained in $\mathcal{N}(x) = \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$.

Suppose xyz has Type $T_1T_2T_3$, where $T_1T_2T_3 \in \{C, D\}$. Then $\mathcal{C}(x)$, $\mathcal{C}(y)$ and $\mathcal{C}(z)$ are mutually disjoint by Lemmas 4.4(1) and 4.5(2), implying that $\mathcal{C}(x, L) = \mathcal{C}(y) \cup \mathcal{C}(z)$ and $|\mathcal{C}(x, L)| = |\mathcal{C}(y)| + |\mathcal{C}(z)| = c(y) + c(z)$. If x has Type C, then $\mathcal{C}(x, L) = \mathcal{C}(y) \cup \mathcal{C}(z)$ is contained in $\mathcal{N}(x) = \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$ by Lemmas 4.4(2) and 4.5(3). If x has Type D, then $\mathcal{C}(x, L) = \mathcal{C}(y) \cup \mathcal{C}(z)$ is contained in $\mathcal{N}(x) = \mathcal{O}_{f_x}$ by Lemmas 4.5(1) and 4.6. ■

Lemma 4.12 *Let x be a point of $\mathcal{S}$ and $y \in \mathcal{N}(x)$. Then the following hold:*

(1) *There exists a unique $\mathcal{S}$-point z collinear with x at distance 2 from y. This point z has Type C.*

(2) *$y \in \mathcal{C}(x, xz)$.*

Proof. Observe that x cannot have Type A since $\mathcal{N}(x) \neq \emptyset$.

(1) We already know that $\mathrm{d}(x, y) = 3$. So, there exists a unique $\mathcal{S}$-point z collinear with x at distance 2 from y. If z' is the unique common neighbor of z and y, then $z, z' \notin \mathcal{P}'$ (since $y \in \mathcal{M}_{f_x}$), $zz' \neq z'y$ and hence zz' has Type BCC by Lemma 4.2(3). In particular, z has Type C.

(2) Put $xz = \{x, z, \tilde{z}\}$. We need to prove that $y \in \Big(\mathcal{C}(z) \Delta \mathcal{C}(\tilde{z})\Big) \setminus \mathcal{C}(x)$. Since z has Type C and $\mathrm{d}(y, z) = 2$, we have $y \in \mathcal{C}(z)$. Since $\mathrm{d}(x, y) = 3$, we cannot have $y \in \mathcal{C}(x)$. It remains to show that $y \notin \mathcal{C}(\tilde{z})$. If $y \in \mathcal{C}(\tilde{z})$, then $y \in \mathcal{C}(z) \cap \mathcal{C}(\tilde{z})$ would imply that $xz\tilde{z}$ has Type BCC and $\{y\} = \mathcal{O}_{f_x}$, in contradiction with $y \in \mathcal{N}(x) = \mathcal{M}_{f_x} \setminus \mathcal{O}_{f_x}$. ■

Lemma 4.13 *Let x be a point of $\mathcal{S}$ and let $\mathcal{L}_x$ denote the set of lines of $\mathcal{S}$ through x. If L_1 and L_2 are two distinct $\mathcal{S}$-lines through x, then $\mathcal{C}(x, L_1)$ and $\mathcal{C}(x, L_2)$ are disjoint. Moreover, $\bigcup_{L \in \mathcal{L}_x} \mathcal{C}(x, L) = \mathcal{N}(x)$. Hence, $\sum_{L \in \mathcal{L}_x} c(x, L) = N(x)$.*

Proof. This is trivial if x is a point of Type A and is a direct consequence of Lemmas 4.11 and 4.12 if x has Type B, C or D. ■

Lemma 4.14 *Let x be an $\mathcal{S}$-point and let $\{f_x, f_1, f_2\}$ be a $\mathcal{V}$-line such that x has Type B, C or D and f_1 has Type C. Suppose moreover that if x has Type B, then f_1 has Type C_j with $j \in \{2, 3, 4, 7\}$. Then there is at most one $\mathcal{S}$-line $\{x, y_1, y_2\}$ through x such that $\{f_x, f_{y_1}, f_{y_2}\} = \{f_x, f_1, f_2\}$.*

Proof. The valuation f_2 has Type B, C or D.

Suppose first that f_2 has Type B. Then $f_x f_1 f_2$ necessarily has Type CCB. Let y be the unique point of $\mathcal{O}_{f_2}$. If $\{x, y_1, y_2\}$ is an $\mathcal{S}$-line through x such that $\{f_x, f_{y_1}, f_{y_2}\} = \{f_x, f_1, f_2\}$, then $y \in \mathcal{O}_{f_x}$, $\mathrm{d}(x, y) = 2$ and $\{x, y_1, y_2\}$ necessarily is the unique $\mathcal{S}$-line through x containing a point collinear with y.

Suppose next that f_2 has Type C or D. Then $f_x f_1 f_2$ has Type BCC, CCC, CCD, DCC or DCD. Let y be an arbitrary point of $\mathcal{O}_{f_1}$ such that $y \notin \mathcal{O}_{f_x}$ if f_x has Type B. It is possible to choose y in such a way. Indeed, if x has Type B, then f_1 has Type C_j for a certain $j \in \{2, 3, 4, 7\}$, $|\mathcal{O}_{f_x}| = 1$ and $|\mathcal{O}_{f_1}| \geq 2$. Suppose now that $\{x, y_1, y_2\}$ is an $\mathcal{S}$-line through x such that $f_{y_1} = f_1$ and $f_{y_2} = f_2$. Then the point y lies at distance 2 from y_1

and the unique point y' in $\Gamma_1(y_1) \cap \Gamma_1(y)$ cannot be contained in the line $\{x, y_1, y_2\}$ since otherwise we would have $y' = x$, x has Type B and $y \in \mathcal{O}_{f_x}$, which is impossible. So, $\mathrm{d}(x, y) = 3$ and $\{x, y_1, y_2\}$ necessarily is the unique $\mathcal{S}$-line through x containing a point at distance 2 from y. ■

If f is a valuation of $\mathcal{S}'$, then we define $\mathcal{C}(f) := \mathcal{O}_f$ if f has Type C and $\mathcal{C}(f) := \emptyset$ otherwise. We also define $\mathcal{N}_f := \mathcal{M}_f \setminus \mathcal{O}_f$ if f has Type A, B or C, and $\mathcal{N}_f := \mathcal{O}_f$ if f has Type D. For every line $\mathcal{F} = \{f_1, f_2, f_3\}$ of $\mathcal{V}$, we define $\mathcal{C}(f_1, \mathcal{F}) := (\mathcal{C}(f_2) \Delta \mathcal{C}(f_3)) \setminus \mathcal{C}(f_1)$.

Suppose now that $\mathcal{F}_i = \{f, g_i, h_i\}$, $i \in I$, is a collection of (not necessarily distinct) $\mathcal{V}$-lines through the same valuation f. For every $i \in I$, put

- $X_i := \mathcal{O}_{g_i}$ if $M_{g_i} = M_f + 1$ and $X_i := \emptyset$ otherwise;
- $Y_i := \mathcal{O}_{h_i}$ if $M_{h_i} = M_f + 1$ and $Y_i := \emptyset$ otherwise.

We say that the above collection of $\mathcal{V}$-lines is *compatible with respect to f* if the following two properties hold:

- $\mathcal{O}_f$ is the disjoint union of the sets $X_i \cup Y_i$, $i \in I$;
- $\mathcal{N}_f$ is the disjoint union of the sets $\mathcal{C}(f, \mathcal{F}_i)$, $i \in I$.

The reason why we have introduced the notion of compatibility for such a collection is because of the following lemma.

Lemma 4.15 *Let x be an $\mathcal{S}$-point not contained in $\mathcal{P}'$ and let $\{x, y_i, z_i\}$, $i \in I$, denote all the $\mathcal{S}$-lines through x. Then the collection $\{f_x, f_{y_i}, f_{z_i}\}$, $i \in I$, of $\mathcal{V}$-lines is compatible with respect to f_x.*

Proof. This is a consequence of Lemmas 4.8 and 4.13. ■

5 Generalized octagons of order $(2, 4)$ containing a suboctagon of order $(2, 1)$

We continue with the notation introduced in Section 4. But we consider here the special case $t = 4$. So, $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized octagon of order $(2, 4)$ containing a suboctagon $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ of order $(2, 1)$. We denote by Γ the collinearity graph of $\mathcal{S}$. Since the lines of $\mathcal{S}$ correspond to the maximal cliques of Γ, the generalized octagon $\mathcal{S}$ is uniquely determined by its collinearity graph Γ. So, in order to prove that $\mathcal{S}$ is unique (up to isomorphism), it suffices to prove that Γ is isomorphic to a certain specific graph Γ^*. This graph Γ^* will be defined here in terms of objects of the valuation geometry $\mathcal{V}$. To prove the isomorphism $\Gamma \cong \Gamma^*$, we will proceed as follows.

(1) We show that the subgraph Γ_0 of Γ induced on the $\mathcal{S}$-points of Type C_2, C_5, C_7, D_1 is isomorphic to some specific graph Γ_0^*.

(2) From the isomorphism $\Gamma_0 \cong \Gamma_0^*$, we then derive an isomorphism $\Gamma \cong \Gamma^*$.

The graph Γ_0^* will be the collinearity graph of the geometry $\mathcal{G}$ whose points are the $\mathcal{V}$-points (i.e., the valuations of $\mathcal{S}'$) of Type C_2, C_5, C_7, D_1, whose lines are the $\mathcal{V}$-lines of Type $B_1C_2C_2$, $C_2C_5D_1$, $C_2C_2C_6$, $B_1C_5C_7$, with incidence being containment.

5.1 Proof of the isomorphism $\Gamma_0 \cong \Gamma_0^*$

We will say that a point x of $\mathcal{S}$ has *line distribution* $X_1Y_1Z_1 + X_2Y_2Z_2 + \cdots + X_5Y_5Z_5$ if the five $\mathcal{S}$-lines $L_1, L_2, \ldots, L_5$ incident with x can be labeled such that L_i has Type $X_iY_iZ_i$. We will often shorten line distributions to expressions of the form $n_1 \times X_1Y_1Z_1 + n_2 \times X_2Y_2Z_2 + \cdots + n_k \times X_kY_kZ_k$, where $k \in \mathbb{N} \setminus \{0\}$ and $n_1, n_2, \ldots, n_k \in \{1, 2, \ldots, 5\}$ such that $n_1 + n_2 + \cdots + n_k = 5$.

In this subsection, we prove that the graphs Γ_0 and Γ_0^* are isomorphic. We achieve this goal in the following three steps:

Step 1: We reduce the set of 52 possible types for the lines of $\mathcal{S}$ to a set of only 9 possible line types. From this set of 9 possible line types, it will follow that there can only be 7 possible types for the points of $\mathcal{S}$, namely A, B_1, C_2, C_5, C_6, C_7 and D_1.

Step 2: For each $\mathcal{S}$-point x of Type $T \in \{A, B_1, C_2, C_5, C_6, C_7, D_1\}$, we determine the possible line distributions, taking into account that there can now only be 9 possible line types. From this information, we are able to prove the nonexistence of one additional line type.

Step 3: From the information about the point types, line types and line distributions, we show that the map $x \mapsto f_x$ defines an isomorphism between Γ_0 and Γ_0^*.

Step 1. From the information gathered so-far, we are now already able to exclude several line types without further computer computations. Take for instance an $\mathcal{S}$-point of Type C_7. From Lemma 4.10, we know that there are $c(x) = 5$ lines of Type BCC through x. Since these are all the $\mathcal{S}$-lines through x, we can conclude that there cannot be $\mathcal{S}$-lines of Type $C_7D_2D_2$. In fact, the equality $\sum_{L \in \mathcal{L}_x} c(x, L) = N(x)$ and Lemmas 4.10, 4.14 (successively) applied to particular $\mathcal{S}$-points allow to exclude several additional line types. Since we were not able to exclude all lines types without additional help from a computer, we have implemented a computer algorithm to exclude line types in an automatic fashion.

Suppose you have a set $\mathcal{T}$ of possible line types, for instance the 52 possible line types mentioned in Table 2. To show the nonexistence of certain line types, we can proceed as follows.

(1) Take a line type $T_1T_2T_3$ contained in $\mathcal{T}$.

(2) Consider a set R of representatives for the distinct $\widetilde{G}$-orbits of $\mathcal{V}$-lines of Type $T_1T_2T_3$. By Table 2, we know that $|R| \in \{1, 2\}$.

(3) For each $L_1 = \{f_1, f_2, f_3\} \in R$ and for each $i \in \{1,2,3\}$, consider all collections L_2, L_3, L_4, L_5 of four not necessarily distinct $\mathcal{V}$-lines through f_i all whose types belong to $\mathcal{T}$.

(4) Verify whether at least one of the considered collections L_1, L_2, L_3, L_4, L_5 is compatible with respect to f_i. If this turns out not to be the case for at least one $i \in \{1,2,3\}$ for which f_i is not classical, then by Lemma 4.15 we know that there cannot exist an $\mathcal{S}$-line whose corresponding $\mathcal{V}$-line is equal to $\{f_1, f_2, f_3\}$.

(5) Suppose Step 4 has shown that for each $\{f_1, f_2, f_3\} \in R$, there cannot exist an $\mathcal{S}$-line whose corresponding $\mathcal{V}$-line is $\{f_1, f_2, f_3\}$. Then there cannot exist $\mathcal{S}$-lines of Type $T_1T_2T_3$.

(6) Perform operations (2), (3), (4), (5) for each line type $T_1T_2T_3$ contained in $\mathcal{T}$. Collect all line types whose nonexistence as an $\mathcal{S}$-line type was obtained in Step 5. If the found collection is empty, then we stop the procedure. If the collection is nonempty, then we remove all "nonexisting line types" from the set $\mathcal{T}$ and go back to Step 1.

We have implemented the above reduction process [4] and applied it to the set of 52 possible line types. The original set of 52 possible line types could in this way be reduced to only 9 possible line types (in the following sequence: $52 \mapsto 21 \mapsto 13 \mapsto 10 \mapsto 9 \mapsto 9$). Our conclusion was the following ([4]):

Lemma 5.1 *Every line of $\mathcal{S}$ has Type AAA, AB_1B_1, $B_1C_2C_2$, $B_1C_5C_7$, $B_1C_6C_6$, $C_2C_2C_6$, $C_2C_5D_1$, $C_5C_5C_5$ or $C_6C_6C_6$.*

Now, we put $\mathcal{T}^* := \{AAA, AB_1B_1, B_1C_2C_2, B_1C_5C_7, B_1C_6C_6, C_2C_2C_6, C_2C_5D_1, C_5C_5C_5, C_6C_6C_6\}$. From Lemma 5.1, we immediately have:

Corollary 5.2 *Every point of $\mathcal{S}$ has Type A, B_1, C_2, C_5, C_6, C_7 or D_1.*

Step 2. Suppose T is one of the types $B_1, C_2, C_5, C_6, C_7, D_1$. To determine the possible line distributions of the $\mathcal{S}$-points of Type T, we can proceed as follows.

> Take a valuation f of Type T and consider all collections L_1, L_2, L_3, L_4, L_5 of admissible L-sets through f that are compatible with respect to f and all whose types belong to $\mathcal{T}^*$. If $L_i = \{f, g_i, h_i\}$, $i \in \{1,2,3,4,5\}$, and if U_i and V_i denote the respective types of g_i and h_i, then $TU_1V_1 + \cdots + TU_5V_5$ is a possible line distribution (in the sense of Lemma 4.15).

In this way, we can collect all possible line distributions for $\mathcal{S}$-points of Type $T \neq A$. The implementation in GAP ([4]) to determine the possible line distributions for the points of $\mathcal{S}$ produced the following results (hereby also taking into account Lemma 4.2(2)):

Lemma 5.3 (1) *Every $\mathcal{S}$-point of Type A has line distribution* $2 \times AAA + 3 \times AB_1B_1$.

(2) *Every $\mathcal{S}$-point of Type B_1 has line distribution $AB_1B_1 + 4 \times B_1C_2C_2$ or $AB_1B_1 + 2 \times B_1C_6C_6 + 2 \times B_1C_5C_7$.*

(3) *Every $\mathcal{S}$-point of Type C_2 has line distribution $2 \times B_1C_2C_2 + 2 \times C_2C_2C_6 + C_2C_5D_1$.*

(4) *Every $\mathcal{S}$-point of Type C_5 has line distribution $B_1C_5C_7 + 4 \times C_2C_5D_1$, $B_1C_5C_7 + 2 \times C_5C_5C_5 + 2 \times C_2C_5D_1$ or $B_1C_5C_7 + 4 \times C_5C_5C_5$.*

(5) *Every $\mathcal{S}$-point of Type C_6 has line distribution $B_1C_6C_6 + 2 \times C_2C_2C_6 + 2 \times C_6C_6C_6$.*

(6) *Every $\mathcal{S}$-point of Type C_7 has line distribution $5 \times B_1C_5C_7$.*

(7) *Every $\mathcal{S}$-point of Type D_1 has line distribution $5 \times C_2C_5D_1$.*

From Lemma 5.3, we are now able to determine the number of points and lines of each type, and to show the nonexistence of one additional line type. For every $T \in \{A, B_1, C_2, C_5, C_6, C_7, D_1\}$, let N_T denote the total number of $\mathcal{S}$-points of Type T. Let N'_{B_1} (N''_{B_1}) denote the total number of $\mathcal{S}$-points of Type B_1 that have line distribution $AB_1B_1 + 4 \times B_1C_2C_2$ ($AB_1B_1 + 2 \times B_1C_6C_6 + 2 \times B_1C_5C_7$). Let N'_{C_5} (N''_{C_5}; N'''_{C_5}) denote the total number of $\mathcal{S}$-points of Type C_5 having line distribution $B_1C_5C_7 + 4 \times C_2C_5D_1$ ($B_1C_5C_7 + 2 \times C_5C_5C_5 + 2 \times C_2C_5D_1$; $B_1C_5C_7 + 4 \times C_5C_5C_5$).

Lemma 5.4 (1) *There are no $\mathcal{S}$-lines of Type $C_5C_5C_5$.*

(2) *We have $N_A = 45$, $N_{B_1} = 270$, $N_{C_2} = 720$, $N_{C_5} = 180$, $N_{C_6} = 360$, $N_{C_7} = 36$, $N_{D_1} = 144$ and $N'_{B_1} = 180$, $N''_{B_1} = 90$, $N'_{C_5} = 180$, $N''_{C_5} = 0$, $N'''_{C_5} = 0$.*

(3) *Every $\mathcal{S}$-point of Type C_5 has line distribution $B_1C_5C_7 + 4 \times C_2C_5D_1$.*

(4) *Among the 2925 lines of $\mathcal{S}$, 30 have Type AAA, $|\mathcal{P}'| \cdot 3 = 135$ have Type AB_1B_1, $N'_{B_1} \cdot 4 = 720$ have Type $B_1C_2C_2$, $N''_{B_1} \cdot 2 = 180$ have Type $B_1C_5C_7$, $N''_{B_1} \cdot 2 = 180$ have Type $B_1C_6C_6$, $N_{C_6} \cdot 2 = 720$ have Type $C_2C_2C_6$, $N_{D_1} \cdot 5 = 720$ have Type $C_2C_5D_1$ and $\frac{1}{3} \cdot N_{C_6} \cdot 2 = 240$ have Type $C_6C_6C_6$.*

Proof. We have $N_A = |\mathcal{P}'| = 45$ and $N_{B_1} = |\Gamma_1(\mathcal{P}')| = |\mathcal{P}'| \cdot 3 \cdot 2 = 270$. We also have

$$N'_{B_1} + N''_{B_1} = N_{B_1} = 270, \qquad 8 \cdot N'_{B_1} = 2 \cdot N_{C_2}, \qquad 4 \cdot N''_{B_1} = N_{C_6}, \qquad 2 \cdot N_{C_2} = 4 \cdot N_{C_6},$$

where the first equation is trivial and the last three are obtained by means of double counting relying on Claims (2), (3) and (5) of Lemma 5.3. The above system of equations implies that $N'_{B_1} = 180$, $N''_{B_1} = 90$, $N_{C_2} = 720$ and $N_{C_6} = 360$. Claims (3) and (7) of Lemma 5.3 imply that $N_{C_2} = 5 \cdot N_{D_1}$ and hence that $N_{D_1} = 144$. Claims (2) and (6) of Lemma 5.3 imply that $2 \cdot N''_{B_1} = 5 \cdot N_{C_7}$ and hence that $N_{C_7} = 36$. The fact that $N_A + N_{B_1} + N_{C_2} + N_{C_5} + N_{C_6} + N_{C_7} + N_{D_1} = |\mathcal{P}| = 1755$ then implies that $N_{C_5} = 180$. Claims (4) and (7) of Lemma 5.3 imply that $720 = 5 \cdot N_{D_1} = 4 \cdot N'_{C_5} + 2 \cdot N''_{C_5} + 0 \cdot N'''_{C_5} = 4 \cdot N_{C_5}$, implying that $N''_{C_5} = N'''_{C_5} = 0$ and hence that $N'_{C_5} = N_{C_5} = 180$. So, we have proved Claim

(2) of the lemma. Claims (1) and (3) follow directly from the fact that $N''_{C_5} = N'''_{C_5} = 0$ and Claim (4) follows from a straightforward double counting. ■

Step 3. From the information gathered so-far, we are now going to establish an isomorphism between Γ_0 and Γ_0^*. The graph Γ_0^* can be implemented in GAP (package GRAPE) and the following can then be verified ([4]):

Lemma 5.5 Γ_0^* *is a connected graph.*

Lemma 5.6 (1) *Let f be a $\mathcal{V}$-point whose type belongs to $\{A, C_2, C_5, C_7, D_1\}$. Then there exists a unique $\mathcal{S}$-point x such that $f = f_x$.*

(2) *If $\{f_1, f_2, f_3\}$ is a $\mathcal{V}$-line of Type AAA, $B_1C_2C_2$, $B_1C_5C_7$, $C_2C_2C_6$ or $C_2C_5D_1$, then there exists a unique $\mathcal{S}$-line $\{x_1, x_2, x_3\}$ such that $\{f_1, f_2, f_3\} = \{f_{x_1}, f_{x_2}, f_{x_3}\}$.*

(3) *If f is a $\mathcal{V}$-point of Type B_1, then there are precisely three $\mathcal{S}$-points x such that $f = f_x$. These three points are collinear with the same point of $\mathcal{S}'$.*

(4) *If f is a $\mathcal{V}$-point of Type C_6, then there are precisely two $\mathcal{S}$-points x such that $f = f_x$.*

Proof. (1) If f is a $\mathcal{V}$-point of Type A, then there exists a unique $\mathcal{S}$-point x of Type A such that $f = f_x$. This point x is the unique element in the singleton $\mathcal{O}_f$.

Let $\mathcal{A}'$ denote the point set of $\mathcal{G}$ (i.e. the vertex set of Γ_0^*) and let $\mathcal{A}$ denote the set of those points of $\mathcal{S}$ whose type is either C_2, C_5, C_7 or D_1. Consider the following map $\theta : \mathcal{A} \to \mathcal{A}'; x \mapsto f_x$. By Table 3 and Lemmas 4.14, 5.3 and 5.4(3), θ defines a bijection between $x^\perp$ and $\theta(x)^\perp$ for every point $x \in \mathcal{A}$. So, if x_1 and x_2 are two $\mathcal{G}$-collinear points of $\mathcal{A}'$, then $|\theta^{-1}(x_1)| = |\theta^{-1}(x_2)|$ (as this is equal to the number of pairs (y_1, y_2) of collinear points of $\mathcal{S}$ for which $f_{y_1} = x_1$ and $f_{y_2} = x_2$). Since $\mathcal{G}$ is connected, this implies that $|\theta^{-1}(x)|$ is independent of the chosen point $x \in \mathcal{A}'$. By Lemma 5.4(2) and Table 1, we then find that if f is a valuation of $\mathcal{S}'$ of Type C_2, C_5, C_7 or D_1, then there exists a unique point x of $\mathcal{S}$ such that $f = f_x$.

(2) Claim (2) is a consequence of Claim (1), Lemmas 4.14, 5.3, 5.4(3) and Table 3.

(3) Let y be an arbitrary point of $\mathcal{S}'$. Since there are precisely 90 $\mathcal{V}$-points of Type B_1, there are precisely two $\mathcal{V}$-points f of Type B_1 for which $\mathcal{O}_f = \{y\}$. If $\{y, z_1, z_2\}$ is one of the three $\mathcal{S}$-lines through y not contained in $\mathcal{P}'$, then f_{z_1} and f_{z_2} are these two $\mathcal{V}$-points of Type B_1. This implies that for every $\mathcal{V}$-point f of Type B_1, there are precisely three $\mathcal{S}$-points x such that $f = f_x$. These three $\mathcal{S}$-points are collinear with the unique point in $\mathcal{O}_f$.

(4) The map $x \mapsto f_x$ defines a bijection between the $\mathcal{S}$-lines of Type $C_2C_2C_6$ and the $\mathcal{V}$-lines of Type $C_2C_2C_6$. Since there are precisely four $\mathcal{V}$-lines of Type $C_2C_2C_6$ through a given $\mathcal{V}$-point of Type C_6, but only two $\mathcal{S}$-lines of Type $C_2C_2C_6$ through a given $\mathcal{S}$-point of Type C_6, there exists for every valuation f of Type C_6 of $\mathcal{S}'$ precisely two $\mathcal{S}$-points x of Type C_6 such that $f = f_x$. ■

The following is an immediate consequence of Lemma 5.3, Lemma 5.4(3) and Lemma 5.6(1)+(2).

Corollary 5.7 *The map $\theta_0 : x \mapsto f_x$ defines an isomorphism between Γ_0 and Γ_0^*.*

Remark. In the special case that $\mathcal{S} \cong \mathrm{RT}(2,4)$, we have denoted Γ_0 earlier by $\widetilde{\Gamma_0}$. Corollary 5.7 thus implies that $\widetilde{\Gamma_0} \cong \Gamma_0^*$. In fact, this fact was already clear from the discussion in Subsection 3.3 (see Properties (3) and (4) there).

5.2 Proof of the isomorphism $\Gamma \cong \Gamma^*$

Above, we defined the graph Γ_0 as the subgraph of Γ induced on the set of $\mathcal{S}$-points of Type C_2, C_5, C_7, D_1. We now define a number of additional subgraphs of Γ:

- Γ_1 is the subgraph of Γ induced on the set of $\mathcal{S}$-points of Type C_2, C_5, C_6, C_7, D_1.
- Γ_2 is the subgraph of Γ induced on the set of $\mathcal{S}$-points of Type B_1, C_2, C_5, C_6, C_7, D_1, but with all edges between B_1-vertices removed.
- Γ_3 is the subgraph of Γ induced on the set of $\mathcal{S}$-points of Type B_1, C_2, C_5, C_6, C_7, D_1.

The graph Γ_0 is thus the first graph in the sequence $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma$ of graphs, each of which is a subgraph of the next one in the chain. Our intention now is to construct a suitable sequence $\Gamma_0^*, \Gamma_1^*, \Gamma_2^*, \Gamma_3^*, \Gamma^*$ of graphs, only defined in terms of objects of $\mathcal{V}$ (and not of $\mathcal{S}$), such that each graph is a subgraph of the graph that comes next in the chain. These new graphs will be defined in such a way that the initial isomorphism $\theta_0 : \Gamma_0 \to \Gamma_0^*$ can (subsequently) be extended to isomorphisms $\theta_1 : \Gamma_1 \to \Gamma_1^*$, $\theta_2 : \Gamma_2 \to \Gamma_2^*$, $\theta_3 : \Gamma_3 \to \Gamma_3^*$ and $\theta^* : \Gamma \to \Gamma^*$. The final conclusion will then be that we have established an isomorphism between the collinearity graph Γ of $\mathcal{S}$ and a certain graph Γ^* which is only defined in terms of objects of $\mathcal{V}$. This will allow us to conclude that $\mathcal{S}$ is unique as a generalized octagon of order $(2,4)$ containing a suboctagon of order $(2,1)$.

Step 1: Definition of Γ_1^* and the isomorphism $\theta_1 : \Gamma_1 \to \Gamma_1^*$

The definition of the map θ_0 between the vertex sets of Γ_0 and Γ_0^* was rather easy, because of the fact that there exists a bijective correspondence between the set of $\mathcal{S}$-points of Type $T \in \{C_2, C_5, C_7, D_1\}$ and the set of valuations of Type T. A similar bijective correspondence no longer holds for the $\mathcal{S}$-points of Type C_6, see Lemma 5.6(4). This will make the definition of the graph Γ_1^* and the isomorphism $\theta_1 : \Gamma_1 \to \Gamma_1^*$ more complicated. Still we will be able to establish a bijective correspondence between the set of $\mathcal{S}$-points of Type C_6 and certain objects inside $\mathcal{V}$, called C_6-sets. Every $\mathcal{S}$-point x of Type C_6 will induce such a C_6-set S_x, see Lemma 5.11 below. Moreover, the bijective correspondence $x \mapsto S_x$ will be bijective. In order to achieve these goals, we first need some further information about $\mathcal{S}$-points of Type B_1 and C_6.

An $\mathcal{S}$-point of Type B_1 is said to be of *Type* B_1' if it has line distribution $AB_1B_1 + 4 \times B_1C_2C_2$ and it is said to have *Type* B_1'' if it has line distribution $AB_1B_1 + 2 \times B_1C_6C_6 + 2 \times B_1C_5C_7$. By Lemma 5.4, there are 180 points of Type B_1' and 90 points of Type B_1''.

Lemma 5.8 *Let f be a valuation of Type B_1 of $\mathcal{S}'$. Then there exist two $\mathcal{S}$-points x of Type B_1' such that $f = f_x$ and a unique $\mathcal{S}$-point y of Type B_1'' such that $f = f_y$.*

Proof. By Lemma 5.6, there are three $\mathcal{S}$-points x of Type B_1 such that $f = f_x$. Since the number of valuations of Type B_1 equals the number of $\mathcal{S}$-points of Type B_1'', namely 90, it suffices to prove that for every valuation f of Type B_1, there exists at least one $\mathcal{S}$-point y of Type B_1'' such that $f = f_y$. Let fg_1g_2 be a $\mathcal{V}$-line of Type $B_1C_5C_7$ through f. Let z_1 be the unique $\mathcal{S}$-point of Type C_5 such that $f_{z_1} = g_1$, and suppose that yz_1z_2 is the unique $\mathcal{S}$-line of Type $B_1C_5C_7$ through z_1. Since fg_1g_2 is the unique $\mathcal{V}$-line of Type $B_1C_5C_7$ through g_1 (Table 3), we have $f_y = f$ and $f_{z_2} = g_2$. So, y is an $\mathcal{S}$-point of Type B_1'' for which $f_y = f$. ∎

Lemma 5.9 *Let f be a valuation of Type C_6 of $\mathcal{S}'$ and let x_1 and x_2 be the two $\mathcal{S}$-points of Type C_6 such that $f_{x_1} = f_{x_2} = f$. Then x_1 and x_2 lie at distance 2 from each other, and the unique $\mathcal{S}$-point y collinear with x_1 and x_2 has Type B_1''. Moreover, the $\mathcal{S}$-lines yx_1 and yx_2 are precisely the two $\mathcal{S}$-lines of Type $B_1C_6C_6$ through y.*

Proof. For every $i \in \{1,2\}$, let $\{x_i, y_i, z_i\}$ be the unique $\mathcal{S}$-line of Type $B_1C_6C_6$ through x_i and suppose that y_i has Type B_1''. By Table 3, there exists a unique $\mathcal{V}$-line $\{f, g, h\}$ through $f = f_{x_1} = f_{x_2}$ such that g has Type B_1. It follows that $f_{y_1} = f_{y_2} = g$. Hence, $y_1 = y_2$ by Lemma 5.8. The claims of the lemma are now obvious. ∎

Lemma 5.10 *Let $x_1y_1z_1$ and $x_2y_2z_2$ be two distinct $\mathcal{S}$-lines of Type $C_6C_2C_2$ such that $f_{x_1} = f_{x_2}$. Then the following hold.*

(1) *If $x_1 = x_2$, then any path of Γ_0 connecting a vertex of $\{y_1, z_1\}$ with a vertex of $\{y_2, z_2\}$ has length at least 6.*

(2) *If $x_1 \neq x_2$, then any path of Γ_0 connecting a vertex of $\{y_1, z_1\}$ with a vertex of $\{y_2, z_2\}$ has length at least 4.*

Proof. Let u_i, $i \in \{1,2\}$, be an arbitrary element of $\{y_i, z_i\}$.

If $x_1 = x_2$, then the $\mathcal{S}$-points u_1 and u_2 lie at distance 2 from each other and their unique common neighbor $x_1 = x_2$ has Type C_6. Since every cycle of Γ has length at least 8, any path of Γ_0 connecting u_1 and u_2 has length at least 6.

If $x_1 \neq x_2$, then by Lemma 5.9, the $\mathcal{S}$-points u_1 and u_2 lie at distance distance 4 from each other. Indeed, if x' is the unique common neighbor of x_1 and x_2 (of Type B_1''), then u_1, x_1, x', x_2, u_2 is a geodesic path of length 4 connecting u_1 and u_2. ∎

Let f be a given valuation of Type C_6. Suppose we have a collection of five $\mathcal{V}$-lines $\{f, g_i, h_i\}$ ($i \in \{0, 1, \ldots, 4\}$) which are compatible with respect to f and satisfy the following properties: (a) fg_0h_0 has Type $C_6C_6B_1$; (b) fg_1h_1, fg_2h_2 have Type $C_6C_2C_2$; (c)

fg_3h_3, fg_4h_4 have Type $C_6C_6C_6$; (d) the distance between $\{g_1, h_1\}$ and $\{g_2, h_2\}$ in the graph Γ_0^* is at least 6. Then we call $S := \{\{f, g_i, h_i\} \,|\, i \in \{1, 2, 3, 4\}\}$ a C_6-*set*, and we define $\eta(S) := f$. We computed [4] that there are two collections of five $\mathcal{V}$-lines through f that are compatible with respect to f and satisfy (a), (b), (c) and (d) (disregarding the ordering of the elements). Moreover, the two associated C_6-sets partition the set of 8 $\mathcal{V}$-lines through f that have Type $C_6C_2C_2$ or $C_6C_6C_6$ (Table 3). So, the total number of C_6-sets is equal to two times the number of $\mathcal{V}$-points of Type C_6, i.e. equal to 360.

Lemma 5.11 *Let x be an $\mathcal{S}$-point of Type C_6 and let $\{x, y_i, z_i\}$, $i \in \{1, 2, 3, 4\}$, denote the four $\mathcal{S}$-lines through x whose type is either $C_6C_2C_2$ or $C_6C_6C_6$. Then $S_x := \{\{f_x, f_{y_i}, f_{z_i}\} \,|\, i \in \{1, 2, 3, 4\}\}$ is a C_6-set. Moreover, the map $x \mapsto S_x$ defines a bijection between the set of* 360 *$\mathcal{S}$-points of Type C_6 and the* 360 *C_6-sets.*

Proof. The fact that S_x is a C_6-set is a consequence of Lemma 4.15, Lemma 5.3(5), Corollary 5.7 and Lemma 5.10. Since there are as many $\mathcal{S}$-points of Type C_6 as C_6-sets, namely 360, in order to show that the map $x \mapsto S_x$ is bijective, it suffices to show that it is surjective. So, consider an arbitrary C_6-set S and let fg_1h_1 be a $\mathcal{V}$-line of Type $C_6C_2C_2$ contained in S. By Lemma 5.6(2), there exists a unique $\mathcal{S}$-line xy_1z_1 such that $(f_x, f_{y_1}, f_{z_1}) = (f, g_1, h_1)$. But then the C_6-set S_x contains $\{f_x, f_{y_1}, f_z\}$ and hence must coincide with the unique C_6-set containing $\{f, g_1, h_1\}$, which equals S. ∎

Since for every $\mathcal{V}$-line $\{f_1, f_2, f_3\}$ of Type $C_6C_6C_6$, there exists a unique C_6-set S containing $\{f_1, f_2, f_3\}$ and satisfying $\eta(S) = f_1$, we know that the following must hold.

Lemma 5.12 *For every $\mathcal{V}$-line $\{f_1, f_2, f_3\}$ of Type $C_6C_6C_6$, there exists a unique $\mathcal{S}$-line $\{x_1, x_2, x_3\}$ such that $\{f_1, f_2, f_3\} = \{f_{x_1}, f_{x_2}, f_{x_3}\}$.*

Since we have established a bijective correspondence between the $\mathcal{S}$-points of Type C_6 and certain objects inside $\mathcal{V}$, namely the C_6-sets, we now understand how the vertex set of the graph Γ_1^* should be defined, namely as the vertex set of Γ_0^*, union the C_6-sets. We define θ_1 then as the map which maps each vertex x of Γ_0 to $\theta_1(x) := \theta_0(x) = f_x$ and each $\mathcal{S}$-point y of Type C_6 to $\theta_1(y) := S_y$, where S_y is the C_6-set as defined in Lemma 5.11. The only thing that remains to be done is to extend the adjacency relation in Γ_0^* to an adjacency relation in Γ_1^* such that $\theta_1 : \Gamma_1 \to \Gamma_1^*$ is an isomorphism. In fact, at this stage we already know how some of the extra adjacencies inside Γ_1^* should be defined.

- Suppose x is an $\mathcal{S}$-point of Type C_6 and y is a $\mathcal{S}$-point of Type C_2. The points x and y are collinear in $\mathcal{S}$ if and only if one of the lines through x contains y. So, the C_6-set S_x and the valuation f_y must be defined as adjacent in Γ_1^* if and only if one of the four elements of S_x contains f_y.

- Suppose x and y are two $\mathcal{S}$-points of Type C_6. These points can be incident with a line of Type $C_6C_6C_6$. In view of Lemma 5.12, the two C_6-sets S_x and S_y should be defined as adjacent if these sets have a $\mathcal{V}$-line of Type $C_6C_6C_6$ in common.

The extra adjacency relations we have defined in Γ_1^* are not yet sufficient for θ_1 to be an isomorphism. Indeed, two distinct $\mathcal{S}$-points x and y of Type C_6 can also be incident with a line of Type $B_1C_6C_6$. With the above conventions, we would not yet have defined an adjacency between S_x and S_y. That is now our intention. For every $\mathcal{S}$-point x of Type C_6, there exists a unique $\mathcal{S}$-point $y \neq x$ which is contained in some line of Type $B_1C_6C_6$ together with x. For this unique point y, we have that the valuations f_x and f_y are contained in a $\mathcal{V}$-line of Type $B_1C_6C_6$. So, for every C_6-set S, we should define one extra adjacency between S and a certain other C_6-set S'. For this unique C_6-set S', we should moreover have that $\eta(S)$ and $\eta(S')$ are incident with some $\mathcal{V}$-line of Type $B_1C_6C_6$. So, the extra adjacencies inside Γ_1^* should defined by a certain symmetric relation $\widetilde{R}$ on the C_6-sets such that for every C_6-set S, there will be a unique C_6-set S' such that $(S, S') \in \widetilde{R}$. The motivation for the definition of $\widetilde{R}$ that we will give follows from the following lemma.

Lemma 5.13 *Let x be a point of Type B_1'' of $\mathcal{S}$, let $\{x, y_1, z_1\}$ and $\{x, y_2, z_2\}$ be two $\mathcal{S}$-lines of Type $B_1C_6C_6$ through x such that $f_{y_1} = f_{y_2}$ and $f_{z_1} = f_{z_2}$. Let X denote the set of $\mathcal{S}$-points of Type C_2 collinear with z_1 and let X_i, $i \in \{1, 2\}$, denote the set of $\mathcal{S}$-points of Type C_2 collinear with y_i. Then the following holds:*

(1) *any path of Γ_0 connecting a vertex of X with a vertex of X_1 has length at least* 5*;*

(2) *any path of Γ_0 connecting a vertex of X with a vertex of X_2 has length at least* 4.

Proof. Let u be an arbitrary element of X and let u_i, $i \in \{1, 2\}$, be an arbitrary element of X_i. The $\mathcal{S}$-points u and u_1 are connected by the path u, z_1, y_1, u_1 and hence lie at distance 3 from each other. The $\mathcal{S}$-points u and u_2 are connected by the path u, z_1, x, y_2, u_2 and hence lie at distance 4 from each other. Observe that none of the points z_1, y_1, x, y_2 is a vertex of the graph Γ_0. The claims of the lemma now follow from the fact that every cycle in Γ has length at least 8. ∎

Suppose now that S is a given C_6-set. Put $f := \eta(S)$ and let $f' \neq f$ be the unique valuation of Type C_6 of $\mathcal{S}'$ such that $\{f, f'\}$ is contained in a $\mathcal{V}$-line of Type $B_1C_6C_6$ (recall Table 3). Let S_1 and $S_2 \neq S_1$ be the two C_6-sets such that $\eta(S_1) = \eta(S_2) = f'$. Let X denote the set of $\mathcal{V}$-points of Type C_2 contained in the elements of S and let X_i, $i \in \{1, 2\}$, denote the set of $\mathcal{V}$-points of Type C_2 contained in an element of S_i. We have implemented the graph Γ_0^* in GAP (package GRAPE), see [4], and verified that there exists a unique $i^* \in \{1, 2\}$ such that:

- The distance between X and X_{i^*} in Γ_0^* is equal to 5.
- The distance between X and X_{3-i^*} in Γ_0^* is equal to 4.

By definition, S_{i^*} is the unique C_6-set S' for which $(S, S') \in \widetilde{R}$. Obviously, $\widetilde{R}$ is a symmetric relation. In view of Lemma 5.13, we have that two distinct $\mathcal{S}$-points x and y of Type C_6 are incident with some line of Type $B_1C_6C_6$ if and only if $(S_x, S_y) \in \widetilde{R}$. So, two C_6-sets S_1 and S_2 should also be defined as adjacent in Γ_1^* if $(S_1, S_2) \in \widetilde{R}$.

Above, we have thus explained how adjacencies in Γ_1^* should be defined in order for θ_1 to be an isomorphism. Summarizing, we thus have:

Lemma 5.14 *The map θ_1 defines an isomorphism between the graphs Γ_1 and Γ_1^*.*

In the special case that $\mathcal{S} \cong \mathrm{RT}(2,4)$, we have denoted Γ_1 earlier by $\widetilde{\Gamma_1}$ (see Subsection 3.3). Lemma 5.14 thus implies that $\widetilde{\Gamma_1} \cong \Gamma_1^*$. The graph $\widetilde{\Gamma_1}$ seems to have a much easier computer implementation than the graph Γ_1^*. It might therefore be more convenient to replace computer computations that involve the graph Γ_1^* by its equivalent computations in $\widetilde{\Gamma_1}$.

Step 2: Definition of Γ_2^* and the isomorphism $\theta_2 : \Gamma_2 \to \Gamma_2^*$

Our next goal will be to define the graph Γ_2^* and to establish an isomorphism θ_2 between Γ_2 and Γ_2^*. To achieve this goal, we shall have to enlarge Γ_1^* with extra vertices that are in bijective correspondence with the $\mathcal{S}$-points of Type B_1, and define some extra adjacencies. The extra vertices that will be added are the 270 elements of a certain set $\overline{\mathcal{C}}$ and the alluded bijective correspondence will be achieved in Lemma 5.16 below. The set $\overline{\mathcal{C}}$ will be defined in terms of certain objects of Γ_1^*. Before we can do that we need to mention some properties of this graph. These properties directly follow from the corresponding properties of the graph Γ_1 in view of the isomorphism $\theta_1 : \Gamma_1 \to \Gamma_1^*$ (if we take $\mathcal{S} \cong \mathrm{RT}(2,4)$, for instance, then $\theta_1 : \widetilde{\Gamma_1} \to \Gamma_1^*$).

If x and y are two distinct adjacent vertices of Γ_1, then x and y are contained in a unique maximal clique of size 2 or 3. There are $1080 = N_{B_1} \cdot 4$ cliques of size 2 in Γ_1. If U is a maximal clique of size 2 of Γ_1, then we denote by $\phi_1(U)$ the unique $\mathcal{S}$-point of Type B_1 contained in the unique $\mathcal{S}$-line through U. Observe that such a line of $\mathcal{S}$ has Type $B_1C_2C_2$, $B_1C_5C_7$ or $B_1C_6C_6$. If x is an $\mathcal{S}$-point of Type B_1, then there are precisely four $\mathcal{S}$-lines of Type BCC through x and hence there are precisely four cliques U of size two of Γ_1 such that $x = \phi_1(U)$.

In view of the isomorphism $\theta_1 : \Gamma_1 \to \Gamma_1^*$, we thus know that if x and y are two distinct adjacent vertices of Γ_1^*, then x and y are contained in a unique maximal clique of size 2 or 3. If $U = \{x, y\}$ is such a maximal clique, then one of the following cases occurs.

(1) x and y are $\mathcal{V}$-points that are collinear with a unique $\mathcal{V}$-line of Type $B_1C_5C_7$ or $B_1C_2C_2$. We denote by $\phi_2(U)$ the unique $\mathcal{V}$-point of Type B_1 contained in this line.

(2) x and y are C_6-sets for which $(x, y) \in \widetilde{R}$. Then the $\mathcal{V}$-points $\eta(x)$ and $\eta(y)$ of Type C_6 are contained in a unique $\mathcal{V}$-line of Type $B_1C_6C_6$. We denote by $\phi_2(U)$ the unique $\mathcal{V}$-point of Type B_1 contained in this line.

Let $\mathcal{C}$ denote the set of all 1080 cliques of size two of Γ_1^*. For every valuation f of Type B_1, we denote by $\mathcal{C}_f$ the set of $\frac{1080}{90} = 12$ cliques $U \in \mathcal{C}$ for which $\phi_2(U) = f$.

Let f be one of the 90 valuations of Type B_1. If U_1 and U_2 are two distinct cliques of $\mathcal{C}_f$, then using the computer model $\widetilde{\Gamma}_1$ of Γ_1^* implemented in [4], we found that $\mathrm{d}_{\Gamma_1^*}(U_1, U_2)$ is equal to either 4 or 6. We call U_1 and U_2 *equivalent* if $\mathrm{d}_{\Gamma_1^*}(U_1, U_2) = 6$. In [4], we also found that this defines an equivalence relation on the set $\mathcal{C}_f$. There are 3 equivalence classes of size 4. If $U \in \mathcal{C}_f$, then we denote by $\overline{U}$ the equivalence class of size 4 containing U and we put $f_{\overline{U}} := f$. We also put $\overline{\mathcal{C}} := \{\overline{U} \,|\, U \in \mathcal{C}\}$. Then $|\overline{\mathcal{C}}| = 90 \cdot 3 = 270$.

In Lemma 5.16 below, we establish a bijection between the set of $\mathcal{S}$-points of Type B_1 and the set $\overline{\mathcal{C}}$. The proof of that lemma will rely on the following observation.

Lemma 5.15 (1) *If U_1 and U_2 are two distinct cliques of size 2 of Γ_1 such that $\phi_1(U_1) = \phi_1(U_2)$, then every path in Γ_1 connecting a vertex of U_1 with a vertex of U_2 has length at least 6.*

(2) *Let x and x' be two distinct $\mathcal{S}$-points of Type B_1 such that $f_x = f_{x'}$. Let U and U' be two cliques of size 2 of Γ_1 such that $x = \phi_1(U)$ and $x' = \phi_1(U')$. Then any path of Γ_1 connecting a vertex of U with a vertex of U' has length at least 4.*

Proof. (1) If y_1 is a vertex of U_1 and y_2 is a vertex of U_2, then y_1 and y_2 lie at distance 2 from each other in $\mathcal{S}$ and their unique common neighbor has Type B_1. Claim (2) of the lemma then follows from the fact that every cycle in Γ has length at least 8.

(2) If y is the unique point of $\mathcal{O}_{f_x} = \mathcal{O}_{f_{x'}}$, then yx and yx' are two distinct $\mathcal{S}$-lines through y. So, if z is a vertex of U and z' is a vertex of U', then z, x, y, x', z' is a geodesic path of length 4 in $\mathcal{S}$ and hence z and z' lie at distance 4 from each other (in $\mathcal{S}$). ∎

Lemma 5.16 *Let x be an $\mathcal{S}$-point of Type B_1 and let $\{x, y_1, y_2\}$, $\{x, y_3, y_4\}$, $\{x, y_5, y_6\}$ and $\{x, y_7, y_8\}$ be the four $\mathcal{S}$-lines of Type BCC through x. Put $f := f_x$. Then $U_i := \theta_1(\{y_{2i-1}, y_{2i}\}) \in \mathcal{C}_f$ for every $i \in \{1, 2, 3, 4\}$. Moreover, $S_x := \{U_1, U_2, U_3, U_4\} \in \overline{\mathcal{C}}$, $f_{S_x} = f$ and the map $x \mapsto S_x$ defines a bijection between the set of 270 $\mathcal{S}$-points of Type B_1 and $\overline{\mathcal{C}}$.*

Proof. Since $\{y_{2i-1}, y_{2i}\}$, $i \in \{1, 2, 3, 4\}$, is a clique of Γ_1 and θ_1 is an isomorphism from Γ_1 to Γ_1^*, we have that $U_i \in \mathcal{C}$. Since $\{f_x, f_{y_{2i-1}}, f_{y_{2i}}\}$ is a $\mathcal{V}$-line we have that $U_i \in \mathcal{C}_{f_x} = \mathcal{C}_f$. By Lemma 5.15 and the fact that θ_1 is an isomorphism from Γ_1 to Γ_1^*, we have $\mathrm{d}_{\Gamma_1^*}(U_i, U_j) \geq 6$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. So, we have that $\{U_1, U_2, U_3, U_4\} \in \overline{\mathcal{C}}$. Observe that by Lemma 5.3(2), there exists an $i \in \{1, 2, 3, 4\}$ such that U_i only consists of $\mathcal{V}$-points of Type C_2, C_5 or C_7. In view of Lemma 5.6(1), the point x is thus uniquely determined by the set $\{U_1, U_2, U_3, U_4\}$. In conclusion, the map $x \mapsto S_x$ is injective and thus defines a bijection between the set of 270 $\mathcal{S}$-points of Type B_1 and $\overline{\mathcal{C}}$. ∎

Now, let Γ_2^* denote the graph whose vertex set consists of the vertices of Γ_1^*, plus the 270 elements of $\overline{\mathcal{C}}$. Two vertices of Γ_1^* are adjacent in Γ_2^* whenever they are adjacent in Γ_1^*. Two vertices belonging to $\overline{\mathcal{C}}$ are never adjacent, and a vertex $\gamma \in \overline{\mathcal{C}}$ is Γ_2^*-adjacent with a vertex x of Γ_1^* if there exists a $U \in \gamma$ such that $x \in U$.

Now, define the following map θ_2 between the vertex sets of Γ_2 and Γ_2^*:

- if x is an $\mathcal{S}$-point of Type C_2, C_5, C_6, C_7 and D_1, then $\theta_2(x) := \theta_1(x)$;
- if x is an $\mathcal{S}$-point of Type B_1 of $\mathcal{S}$, then $\theta_2(x) := S_x$, where S_x is the set as defined in Lemma 5.16.

The following must then hold:

Lemma 5.17 *The map θ_2 defines an isomorphism between the graphs Γ_2 and Γ_2^*.*

Step 3: Definition of Γ_3^* and the isomorphism $\theta_3 : \Gamma_3 \to \Gamma_3^*$

In the previous step, we have enlarged the graph Γ_1^* to a graph Γ_2^* by adding objects that bijectively correspond to the $\mathcal{S}$-points of Type B_1, namely the 270 elements of $\overline{\mathcal{C}}$. Subsequently, we have explained how adjacencies between $\mathcal{S}$-points of Type B_1 and $\mathcal{S}$-points of Type $T \in \{C_2, C_6, C_5, C_7\}$ should be recognized inside Γ_2^*. What we have so far failed to do is to give an interpretation of when two distinct $\mathcal{S}$-points of Type B_1 are adjacent. Indeed, each $\mathcal{S}$-point x of Type B_1 is collinear with a unique other $\mathcal{S}$-point y of Type B_1 (on the unique line through x meeting $\mathcal{P}'$). The question for which we need to find an answer is thus how we can obtain the set $S_y \in \overline{\mathcal{C}}$ from the set $S_x \in \overline{\mathcal{C}}$.

As Γ_2 and Γ_3 have the same vertices, the vertex graphs of Γ_2^* and Γ_3^* and the maps θ_2 and θ_3 should remain the same. Γ_3^* is obtained from Γ_2^* by adding extra edges between the elements of $\overline{\mathcal{C}}$. In fact, every vertex belonging to $\overline{\mathcal{C}}$ should be joined to precisely one other vertex of $\overline{\mathcal{C}}$ (later we will call these two vertices conjugate). Notice that if x and y are two distinct collinear $\mathcal{S}$-points of Type B_1, then f_y is the unique valuation f of Type B_1 of $\mathcal{S}'$ for which $\mathcal{O}_f = \mathcal{O}_{f_x}$ and $f \neq f_y$. The motivation for the definition of conjugate elements will follow from the following lemma.

Lemma 5.18 *Let x and y be two $\mathcal{S}$-points of Type B_1 such that $\mathcal{O}_{f_x} = \mathcal{O}_{f_y}$ and $f_x \neq f_y$. Let X (respectively, Y) denote the set of $\mathcal{S}$-points of Type C collinear with x (respectively, y). Then the following hold:*

(1) *If x and y are two collinear points of $\mathcal{S}$, then any path of Γ_1 connecting a vertex of X with a vertex of Y has length at least* 5.

(2) *If x and y are two noncollinear points of $\mathcal{S}$ (and so are collinear with the same point of $\mathcal{P}'$), then any path in Γ_1 connecting a vertex of X with a vertex of Y has length at least* 4.

Proof. Let $x_1 \in X$ and $y_1 \in Y$. If x and y are collinear points of $\mathcal{S}$, then x_1 and y_1 lie at distance 3 from each other in $\mathcal{S}$ since x_1, x, y, y_1 is a path. If x and y are noncollinear points of $\mathcal{S}$, then x and y have a unique neighbor which is an $\mathcal{S}$-point of Type A (namely the unique point in the singleton $\mathcal{O}_{f_x} = \mathcal{O}_{f_y}$), implying that x_1 and y_1 lie at distance 4 from each other. The claims of the lemma now follow from the fact that every cycle in Γ has length at least 8. ∎

Now, let γ be an arbitrary element of $\overline{\mathcal{C}}$. Put $f_1 := f_\gamma$ and let f_2 be the unique valuation of Type B_1 of $\mathcal{S}'$ such that $f_1 \neq f_2$ and $\mathcal{O}_{f_1} = \mathcal{O}_{f_2}$. Then there are precisely three distinct elements $\gamma_1, \gamma_2, \gamma_3 \in \overline{\mathcal{C}}$ such that $f_2 = f_{\gamma_1} = f_{\gamma_2} = f_{\gamma_3}$. Let X denote the set of vertices of Γ_2^* adjacent to γ and for every $i \in \{1, 2, 3\}$, let X_i denote the set of vertices of Γ_2^* adjacent to γ_i. Then each of the sets X, X_1, X_2, X_3 only consist of vertices of Γ_1^*. Using the computer model $\widetilde{\Gamma}_1$ of Γ_1^* implemented in [4], we found that there exists a unique $i \in \{1, 2, 3\}$ such that

- the distance between X and X_i in Γ_1^* is equal to 5,
- for every $j \in \{1, 2, 3\} \setminus \{i\}$, the distance between X and X_j in Γ_1^* is equal to 4.

We will say that γ and γ_i are *conjugate elements* of $\overline{\mathcal{C}}$. By Lemma 5.18 it follows that if x and y are two distinct collinear $\mathcal{S}$-points of Type B_1, then the corresponding elements S_x and S_y of $\overline{\mathcal{C}}$ are conjugate. So, we (should) define Γ_3^* as the graph obtained from Γ_2^* by adding edges between two conjugate vertices of $\overline{\mathcal{C}}$. Our conclusion is then as follows:

Lemma 5.19 *The map $\theta_3 := \theta_2$ defines an isomorphism between Γ_3 and Γ_3^*.*

Step 4: Definition of Γ^* and the isomorphism $\theta^* : \Gamma \to \Gamma^*$

The graph Γ_3 is already a good approximation of Γ. In order to find Γ from Γ_3, we should take the disjoint union of Γ_3 and Γ', the collinearity graph of $\mathcal{S}'$. For every vertex x of Type B_1 of $\mathcal{S}$, we should then draw an extra edge between x and the unique point y in $\mathcal{O}_{f_x}$. From this observation, we can see how Γ^* and θ^* should be defined.

Let Γ_4^* denote the disjoint union of the graphs Γ_3^* and Γ'. Let Γ^* denote the graph obtained from Γ_4^* by drawing for every $\gamma \in \overline{\mathcal{C}}$ the extra edge between the vertices γ and x_γ, where x_γ is the unique point contained in the singleton $\mathcal{O}_{f_\gamma}$. Consider the following map θ^* between the vertices of Γ and Γ^*:

- if x is a point of $\mathcal{S}$ not contained in $\mathcal{P}'$, then we define $\theta^*(x) := \theta_3(x)$;
- if x is a point of $\mathcal{P}'$, then we define $\theta^*(x) := x$.

The following thus holds:

Proposition 5.20 *The map θ^* defines an isomorphism between Γ and Γ^*.*

So, the collinearity graph Γ of $\mathcal{S}$ is uniquely determined. Γ is isomorphic to a graph Γ^* which can be completely described in terms of certain objects of $\mathcal{S}'$. Since the generalized octagon $\mathcal{S}$ is isomorphic to the point-line geometry whose points and lines are the vertices and maximal cliques of Γ^*, with incident being containment, there can be, up to isomorphism, at most one generalized octagon of order $(2, 4)$ that contains a suboctagon of order $(2, 1)$. We conclude:

Theorem 5.21 *The generalized octagon $\mathcal{S}$ is isomorphic to the Ree-Tits octagon* $\mathrm{RT}(2, 4)$ *of order* $(2, 4)$.

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References

[1] A. M. Cohen, E.A. O'Brien and S. Shpectorov. On the uniqueness of the generalized octagon of order $(2, 4)$. *Journal of Algebra*, to appear.

[2] A. M. Cohen and J. Tits. On generalized hexagons and a near octagon whose lines have three points. *European J. Combin.* 6 (1985), 13–27.

[3] B. De Bruyn. Polygonal valuations. *Discrete Math.* 313 (2013), 84–93.

[4] B. De Bruyn. GAP code for "The uniqueness of a certain generalized octagon of order $(2, 4)$" See `http://cage.ugent.be/geometry/preprints.php`.

[5] B. De Bruyn and P. Vandecasteele. Valuations of near polygons. *Glasg. Math. J.* 47 (2005), 347–361.

[6] B. De Bruyn and P. Vandecasteele. The classification of the slim dense near octagons. *European J. Combin.* 28 (2007), 410–428.

[7] T. De Medts and H. Van Maldeghem. The uniqueness of a generalized hexagon of order 3 containing a subhexagon of order $(1, 3)$. *Discrete Math.* 309 (2009), 714–720.

[8] W. Feit and G. Higman. The nonexistence of certain generalized polygons. *J. Algebra* 1 (1964), 114–131.

[9] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.5; 2014. (http://www.gap-system.org)

[10] W. Haemers and C. Roos. An inequality for generalized hexagons. *Geom. Dedicata* 10 (1981), 219–222.

[11] D. G. Higman. Invariant relations, coherent configurations and generalized polygons. In *Combinatorics* (ed. M. Hall and J. H. Van Lint), D. Reichel, Dordrecht (1974), 347–363.

[12] M. Joswig and H. Van Maldeghem. An essay on the Ree octagons. *J. Algebraic Combin.* 4 (1995), 145–164.

[13] A. Offer and H. Van Maldeghem. Distance-j ovoids and related structures in generalized polygons. *Discrete Math.* 294 (2005), 147–160.

[14] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[15] R. Ree. A family of simple groups associated with the simple Lie algebra of type (F_4). *Amer. J. Math.* 83 (1961), 401–420.

[16] J. Sarli. The geometry of root subgroups in Ree groups of type ${}^2F_4$. *Geom. Dedicata* 26 (1988), 1–28.

[17] E. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.

[18] J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Etudes Sci. Publ. Math.* 2 (1959), 13–60.

[19] J. Tits. Les groupes simples de Suzuki et de Ree. *Séminaire Bourbaki* 13 (1960/61), No. 210, (1961), 18 pp.

[20] J. Tits. Moufang octagons and the Ree groups of type $^2F_4$. *Amer. J. Math.* 105 (1983), 539–594.

[21] H. Van Maldeghem. *Generalized polygons*. Birkhäuser, Basel, 1998.

[22] H. Van Maldeghem. A geometric characterization of the perfect Ree-Tits generalized octagons. *Proc. London Math. Soc.* (3) 76 (1998), 203–256.