Two-character sets as subsets of parabolic quadrics

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Abstract

A two-character set is a set of points of a finite projective space that has two intersection numbers with respect to hyperplanes. Two-character sets are related to strongly regular graphs and two-weight codes. In the literature, there are plenty of constructions for (non-trivial) two-character sets by considering suitable subsets of quadrics and Hermitian varieties. Such constructions exist for the quadrics $Q^+(2n-1,q) \subseteq \mathrm{PG}(2n-1,q)$, $Q^-(2n+1,q) \subseteq \mathrm{PG}(2n+1,q)$ and the Hermitian varieties $H(2n-1,q^2) \subseteq \mathrm{PG}(2n-1,q^2)$, $H(2n,q^2) \subseteq \mathrm{PG}(2n,q^2)$. In this note we show that every two-character set of $\mathrm{PG}(2n,q)$ that is contained in a given nonsingular parabolic quadric $Q(2n,q) \subseteq \mathrm{PG}(2n,q)$ is a subspace of $\mathrm{PG}(2n,q)$. This offers some explanation for the absence of the parabolic quadrics in the above-mentioned constructions.

Keywords: Two-character set, parabolic quadric
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1 Motivation and main result

A set X of points of the projective space $\mathrm{PG}(k-1,q)$ is called a *two-character set* with *intersection numbers* h_1 and h_2 if every hyperplane of $\mathrm{PG}(k-1,q)$ intersects X in either h_1 or h_2 points. With every two-character set, there is associated a two-weight code and a strongly regular graph, see Delsarte [12] and Calderbank & Kantor [6]. A nonempty proper subspace of $\mathrm{PG}(k-1,q)$ is an example of a (trivial) two-character set.

Many of the known constructions for two-character sets are related to finite polar spaces. In this note, we are interested in sets of points of finite polar spaces that are two-character sets of their ambient projective spaces. In the literature, there are plenty of constructions for obtaining such two-character sets. In the overview we give below, $\mathcal{P}$ denotes one of the following polar spaces of rank $n \geq 2$: • $Q(2n, q)$; • $Q^+(2n-1, q)$; • $Q^-(2n+1, q)$; • $H(2n-1, q)$ with q a square; • $H(2n, q)$ with q a square.

An *m-ovoid* of $\mathcal{P}$ is a set of points intersecting each maximal subspace of $\mathcal{P}$ in precisely m points. Examples of m-ovoids arise from the so-called m'-systems of $\mathcal{P}$. These nice sets of m'-dimensional subspaces of $\mathcal{P}$ were introduced by Shult and Thas [23]. In [23], it was proved, among other things, that the union of the subspaces of an m'-ovoid of $\mathcal{P}$ is a $\frac{q^{m'+1}-1}{q-1}$-ovoid of $\mathcal{P}$. If $\mathcal{P}$ is one of the polar spaces $H(2n, q)$, $Q^-(2n+1, q)$, then every m-ovoid of $\mathcal{P}$ is a two-character set of the ambient projective space Σ, see Bamberg, Kelly, Law & Penttila [1] and Bamberg, Law & Penttila [2]. In particular, the union of the subspaces of an m'-system of $\mathcal{P}$ is a two-character set of Σ.

The $\frac{q+1}{2}$-ovoids of $Q^-(5, q)$, q odd, are also known as *hemisystems*. They were first studied by Segre [22]. Constructions of hemisystems can be found in the papers [10, 11, 22].

If X is a set of points of $\mathcal{P}$, then the total number of ordered pairs of distinct collinear points of $\mathcal{P}$ is at most $\frac{\lambda-1}{q} \cdot |X| \cdot (\frac{|X|}{\lambda} + (q-1))$, with λ denoting the total number of points contained in a given maximal subspace of $\mathcal{P}$. If equality holds, then X is called a *tight*. Tight sets were introduced by Payne [21] for generalized quadrangles and by Drudge [13] for arbitrary polar spaces. If $\mathcal{P}$ is one of the polar spaces $H(2n-1, q)$, $Q^+(2n-1, q)$, then every tight set of $\mathcal{P}$ is a two-character set of the ambient projective space, see Bamberg, Kelly, Law & Penttila [1] and Bamberg, Law & Penttila [2].

The tight sets of the hyperbolic quadric $Q^+(5, q)$ are related to the so-called *Cameron-Liebler line classes* of $\mathrm{PG}(3, q)$. Recall that a *spread* of $\mathrm{PG}(3, q)$ is a set of lines partitioning its point set. A set $\mathcal{L}$ of lines of $\mathrm{PG}(3, q)$ is said to be a *Cameron-Liebler line class* if the number $|\mathcal{L} \cap S|$ is independent of the spread S of $\mathrm{PG}(3, q)$. Sets of lines satisfying this property were first studied by Cameron and Liebler in [7]. By Drudge [13], the Cameron-Liebler line classes correspond via the Klein correspondence to the tight sets of $Q^+(5, q)$. Nontrivial examples of Cameron-Liebler line classes of $\mathrm{PG}(3, q)$ can be found in [5, 14, 15].

Further constructions of tight sets of the polar spaces $Q^+(2n-1, q)$, $H(2n-1, q)$, and of m-ovoids or m'-systems of the polar spaces $H(2n, q)$, $Q^-(2n+1, q)$ can be found in the papers [1, 4, 9, 16, 17, 18, 23]. By a procedure referred to as "field-reduction" in Kelly [20], tight sets and m-ovoids of these polar spaces will give rise to further examples of tight sets and m-ovoids and (hence) also to further examples of two-character sets.

The above discussion shows that there are plenty of constructions for two-character sets that appear as subsets of quadrics or Hermitian varieties of finite projective spaces. An attentive reader might have noticed that none of the above constructions involves a parabolic quadric $Q(2n,q) \subseteq \mathrm{PG}(2n,q)$. This is not a coincidence. The following theorem, which is the main result of this note, gives an explanation for this fact.

Theorem 1.1 *Let X be a set of points of $Q(2n,q)$, $n \geq 2$, which is a two-character set of the ambient projective space $\mathrm{PG}(2n,q)$ of $Q(2n,q)$. Then X is the set of points of a subspace of $Q(2n,q)$.*

So, in future attempts to construct new two-character sets related to polar spaces, one should not spent any energy in those sets of points that can occur as subsets of nonsingular parabolic quadrics. Other nonexistence results for certain classes of two-character sets can be found in the literature, see e.g. the papers [3] and [8].

2 Proof of Theorem 1.1

Let $Q(2n,q)$ be a nonsingular parabolic quadric of $\mathrm{PG}(2n,q)$, $n \geq 2$. The quadric $Q(2n,q)$ contains $\psi(2n,q) = \frac{q^{2n}-1}{q-1}$ points. There are three possibilities for a hyperplane α of $\mathrm{PG}(2n,q)$, see e.g. Hirschfeld and Thas [19].

(1) α is a tangent hyperplane. If α is tangent to $Q(2n,q)$ at the point x, then $\alpha \cap Q(2n,q)$ is a cone of the form $xQ(2n-2,q)$, where $Q(2n-2,q)$ is a nonsingular parabolic quadric of a hyperplane of α not containing x. Observe that $|\alpha \cap Q(2n,q)| = |xQ(2n-2,q)| = \frac{q^{2n-1}-1}{q-1}$.

(2) α is a non-tangent hyperplane of type $Q^+(2n-1,q)$, or shortly a $Q^+(2n-1,q)$-hyperplane. This means that $\alpha \cap Q(2n,q)$ is a nonsingular hyperbolic quadric of α. Observe that $|\alpha \cap Q(2n,q)|$ is equal to $\psi^+(2n-1,q) := |Q^+(2n-1,q)| = \frac{q^{2n-1}-1}{q-1} + q^{n-1} = \frac{(q^n-1)(q^{n-1}+1)}{q-1}$.

(3) α is a non-tangent hyperplane of type $Q^-(2n-1,q)$, or shortly a $Q^-(2n-1,q)$-hyperplane. This means that $\alpha \cap Q(2n,q)$ is a nonsingular elliptic quadric of α. Observe that $|\alpha \cap Q(2n,q)|$ is equal to $\psi^-(2n-1,q) := |Q^-(2n-1,q)| = \frac{q^{2n-1}-1}{q-1} - q^{n-1} = \frac{(q^n+1)(q^{n-1}-1)}{q-1}$.

So, $Q(2n,q)$ itself is not a two-character set since there are three possible intersection sizes with hyperplanes. In the proof of Theorem 1.1, we have to make use of the following lemma.

Lemma 2.1 (1) *There are precisely* $\frac{q^{2n}+q^n}{2}$ *non-tangent hyperplanes of type* $Q^+(2n-1,q)$ *and* $\frac{q^{2n}-q^n}{2}$ *non-tangent hyperplanes of type* $Q^-(2n-1,q)$.

(2) *Through every point of* $Q(2n,q)$, *there are precisely* $\frac{q^n(q^{n-1}+1)}{2}$ *non-tangents hyperplanes of type* $Q^+(2n-1,q)$ *and* $\frac{q^n(q^{n-1}-1)}{2}$ *non-tangent hyperplanes of type* $Q^-(2n-1,q)$.

(3) *Through every two distinct collinear points of* $Q(2n,q)$, *there are precisely* $\frac{q^n(q^{n-2}+1)}{2}$ *non-tangent hyperplanes of type* $Q^+(2n-1,q)$ *and* $\frac{q^n(q^{n-2}-1)}{2}$ *non-tangent hyperplanes of type* $Q^-(2n-1,q)$.

(4) *Through every two non-collinear points of* $Q(2n,q)$, *there are precisely* $\frac{q^{n-1}(q^{n-1}+1)}{2}$ *non-tangent hyperplanes of type* $Q^+(2n-1,q)$ *and* $\frac{q^{n-1}(q^{n-1}-1)}{2}$ *non-tangent hyperplanes of type* $Q^-(2n-1,q)$.

Proof. This can easily be verified by means of double counting, taking into account some elementary properties of quadrics and the precise values of the numbers $\psi(2n,q)$, $\psi^+(2n-1,q)$ and $\psi^-(2n-1,q)$. For Claim (1), see e.g. Hirschfeld and Thas [19, Section 22.8]. Claims (2), (3) and (4) follow from Claim (1) and the fact that the group of collineations of $\mathrm{PG}(2n,q)$ stabilizing $Q(2n,q)$ acts transitively on the points of $Q(2n,q)$, the ordered pairs of distinct collinear points of $Q(2n,q)$ and the ordered pairs of noncollinear points of $Q(2n,q)$. ■

Now, let X be a set of points of $Q(2n,q)$ that is a two-character set of the projective space $\mathrm{PG}(2n,q)$. Let h_1 and h_2 denote the two intersection numbers. Let N_1 denote the total number of ordered pairs of distinct collinear points of X. For every hyperplane α of $\mathrm{PG}(2n,q)$, we define

$$t_\alpha := |X \cap \alpha|.$$

Summing over all hyperplanes α of $\mathrm{PG}(2n,q)$, we find by Lemma 2.1 that

$$\begin{aligned}
\sum_\alpha 1 &= \frac{q^{2n+1}-1}{q-1}, \\
\sum_\alpha t_\alpha &= |X| \cdot \frac{q^{2n}-1}{q-1}, \\
\sum_\alpha t_\alpha(t_\alpha - 1) &= |X| \cdot \big(|X|-1\big) \cdot \frac{q^{2n-1}-1}{q-1}, \\
\sum_\alpha t_\alpha^2 &= |X|^2 \cdot \frac{q^{2n-1}-1}{q-1} + |X| \cdot q^{2n-1}.
\end{aligned}$$

Putting $\sum_\alpha (t_\alpha - h_1)(t_\alpha - h_2)$ equal to 0, we find

$$|X|^2 \cdot \frac{q^{2n-1}-1}{q-1} + |X| \cdot q^{2n-1} - |X| \cdot \frac{q^{2n}-1}{q-1} \cdot (h_1+h_2) + \frac{q^{2n+1}-1}{q-1} \cdot h_1 h_2 = 0. \tag{1}$$

Summing over all $Q^+(2n-1,q)$-hyperplanes of $\mathrm{PG}(2n,q)$, we find by Lemma 2.1 that

$$\begin{aligned}
\sum_\alpha 1 &= \frac{q^{2n}+q^n}{2}, \\
\sum_\alpha t_\alpha &= |X| \cdot \frac{q^n(q^{n-1}+1)}{2}, \\
\sum_\alpha t_\alpha(t_\alpha - 1) &= N_1 \cdot \frac{q^n(q^{n-2}+1)}{2} + \Big(|X| \cdot (|X|-1) - N_1\Big) \\
&\qquad \cdot \frac{q^{n-1}(q^{n-1}+1)}{2} \\
&= N_1 \cdot \frac{q^n - q^{n-1}}{2} + |X|^2 \cdot \frac{q^{n-1}(q^{n-1}+1)}{2} \\
&\qquad - |X| \cdot \frac{q^{n-1}(q^{n-1}+1)}{2}, \\
\sum_\alpha t_\alpha^2 &= N_1 \cdot \frac{q^n - q^{n-1}}{2} + |X|^2 \cdot \frac{q^{n-1}(q^{n-1}+1)}{2} \\
&\qquad + |X| \cdot \frac{(q^{n-1}+1)(q^n - q^{n-1})}{2}.
\end{aligned}$$

Putting $\sum_\alpha (t_\alpha - h_1)(t_\alpha - h_2)$ equal to 0, we find

$$\begin{aligned}
&N_1 \cdot (q-1) + |X|^2 \cdot (q^{n-1}+1) + |X| \cdot (q^{n-1}+1)(q-1) \\
&\quad - |X| \cdot (q^n+q) \cdot (h_1+h_2) + (q^{n+1}+q) \cdot h_1 h_2 = 0.
\end{aligned} \tag{2}$$

Summing over all $Q^-(2n-1,q)$-hyperplanes α of $\mathrm{PG}(2n,q)$, we find by Lemma 2.1 that

$$\begin{aligned}
\sum_\alpha 1 &= \frac{q^{2n}-q^n}{2}, \\
\sum_\alpha t_\alpha &= |X| \cdot \frac{q^n(q^{n-1}-1)}{2}, \\
\sum_\alpha t_\alpha(t_\alpha - 1) &= N_1 \cdot \frac{q^n(q^{n-2}-1)}{2} + \Big(|X| \cdot (|X|-1) - N_1\Big) \\
&\qquad \cdot \frac{q^{n-1}(q^{n-1}-1)}{2}
\end{aligned}$$

$$\begin{aligned} &= -N_1 \cdot \frac{q^n - q^{n-1}}{2} + |X|^2 \cdot \frac{q^{n-1}(q^{n-1}-1)}{2} \\ &\qquad - |X| \cdot \frac{q^{n-1}(q^{n-1}-1)}{2}, \\ \sum_\alpha t_\alpha^2 &= -N_1 \cdot \frac{q^n - q^{n-1}}{2} + |X|^2 \cdot \frac{q^{n-1}(q^{n-1}-1)}{2} \\ &\qquad + |X| \cdot \frac{(q^{n-1}-1)(q^n - q^{n-1})}{2}. \end{aligned}$$

Putting $\sum_\alpha (t_\alpha - h_1)(t_\alpha - h_2)$ equal to 0, we find

$$\begin{aligned} &-N_1 \cdot (q-1) + |X|^2 \cdot (q^{n-1}-1) + |X| \cdot (q^{n-1}-1)(q-1) \\ &\qquad - |X| \cdot (q^n - q) \cdot (h_1 + h_2) + (q^{n+1} - q) \cdot h_1 h_2 = 0. \end{aligned} \tag{3}$$

Eliminating N_1 from equations (2) and (3), we find

$$|X|^2 + |X| \cdot (q-1) - |X| \cdot q \cdot (h_1 + h_2) + q^2 \cdot h_1 h_2 = 0. \tag{4}$$

From (1) and (4), we find

$$h_1 + h_2 = \frac{(q+1) \cdot |X| - 1}{q}, \qquad h_1 h_2 = \frac{|X|^2 - |X|}{q} \tag{5}$$

and hence

$$\{h_1, h_2\} = \{|X|, \frac{|X|-1}{q}\}. \tag{6}$$

Combining (3) and (5), we can calculate N_1. We find

$$N_1 = |X| \cdot (|X| - 1). \tag{7}$$

By (7), every two distinct points of X are collinear on $Q(2n, q)$. Hence, $< X >$ is a subspace of $Q(2n, q)$. Put $k := \dim < X >$. By (6), every hyperplane of $< X >$ contains precisely $\frac{|X|-1}{q}$ points. Counting in two different ways the number of pairs (x, U), where $x \in X$ and U a hyperplane of $< X >$ containing x, we find

$$\frac{q^{k+1}-1}{q-1} \cdot \frac{|X|-1}{q} = |X| \cdot \frac{q^k - 1}{q-1}.$$

It follows that

$$|X| = \frac{q^{k+1}-1}{q-1}.$$

Hence, X is the whole set of points of the subspace $< X >$ of $Q(2n, q)$.

References

[1] J. Bamberg, S. Kelly, M. Law and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A* 114 (2007), 1293–1314.

[2] J. Bamberg, M. Law and T. Penttila. Tight sets and m-ovoids of generalised quadrangles. *Combinatorica* 29 (2009), 1–17.

[3] L. Berardi and T. Masini. On sets of type $(m, n)_{r-1}$ in $\mathrm{PG}(r, q)$. *Discrete Math.* 309 (2009), 1629–1636.

[4] A. E. Brouwer. Some new two-weight codes and strongly regular graphs. *Discrete Appl. Math.* 10 (1985), 111–114.

[5] A. A. Bruen and K. Drudge. The construction of Cameron-Liebler line classes in $\mathrm{PG}(3, q)$. *Finite Fields Appl.* 5 (1999), 35–45.

[6] R. Calderbank and W. M. Kantor. The geometry of two-weight codes. *Bull. London Math. Soc.* 18 (1986), 97–122.

[7] P. J. Cameron and R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.* 46 (1982), 91–102.

[8] A. C. S. Chan, J. A. Davis and J. Jedwab. On the non-existence of a projective $(75, 4, 12, 5)$ set in $\mathrm{PG}(3, 7)$. *J. Geom.* 97 (2010), 29–44.

[9] A. Cossidente, N. Durante, G. Marino, T. Penttila and A. Siciliano. The geometry of some two-character sets. *Des. Codes Cryptogr.* 46 (2008), 231–241.

[10] A. Cossidente and T. Penttila. Hemisystems on the Hermitian surface. *J. London Math. Soc. (2)* 72 (2005), 731–741.

[11] A. Cossidente and T. Penttila. A new hemisystem on $H(3, 49)$. *Ars Combin.*, to appear.

[12] Ph. Delsarte. Weights of linear codes and strongly regular normed spaces. *Discrete Math.* 3 (1972), 47–64.

[13] K. Drudge. *Extremal sets in projective and polar spaces.* Ph.D. Thesis. The University of Western Ontario (Canada), 1998.

[14] K. Drudge. On a conjecture of Cameron and Liebler. *European J. Combin.* 20 (1999), 263–269.

[15] P. Govaerts and T. Penttila. Cameron-Liebler line classes in $\mathrm{PG}(3, 4)$. *Bull. Belg. Math. Soc. Simon Stevin* 12 (2005), 793–804.

[16] N. Hamilton. Strongly regular graphs from differences of quadrics. *Discrete Math.* 256 (2002), 465–469.

[17] N. Hamilton and R. Mathon. Existence and non-existence of m-systems of polar spaces. *European J. Combin.* 22 (2001), 51–61.

[18] N. Hamilton and C. T. Quinn. m-systems of polar spaces and maximal arcs in projective planes. *Bull. Belg. Math. Soc. Simon Stevin* 7 (2000), 237–248.

[19] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[20] S. Kelly. Constructions of intriguing sets of polar spaces from field reduction and derivation. *Des. Codes Cryptogr.* 43 (2007), 1–8.

[21] S. E. Payne. Tight pointsets in finite generalized quadrangles. Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla., 1987). *Congr. Numer.* 60 (1987), 243–260.

[22] B. Segre. Forme e geometrie hermitiane, con particolare riguardo al caso finito. *Ann. Mat. Pura Appl.* (4) 70 (1965), 1–201.

[23] E. E. Shult and J. A. Thas. m-systems of polar spaces. *J. Combin. Theory Ser. A* 68 (1994), 184–204.