A 16-dimensional module for $Sp(4, \mathbb{F})$ and projective embeddings of certain generalized octagons

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Abstract

Let V be a 4-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear form f, and let $Sp(V, f) \cong Sp(4, \mathbb{F})$ denote the symplectic group associated with (V, f). We consider a 16-dimensional submodule W_{16} of the 24-dimensional $Sp(V, f)$-module $V \otimes \bigwedge^2 V$, and show that this $Sp(V, f)$-module is irreducible if and only if $\mathrm{char}(\mathbb{F}) \neq 5$. If $\mathrm{char}(\mathbb{F}) = 5$, then there is a unique nontrivial submodule, and the dimension of this submodule is equal to 4. These results will have some consequences to full projective embeddings of generalized octagons. The projective space $\mathrm{PG}(W_{16})$ admits a full projective embedding for the generalized octagon which arises as flag geometry of the symplectic quadrangle associated with (V, f). We show that this embedding is polarized and also homogeneous, unless $|\mathbb{F}| > 2$ and $\mathbb{F}$ is a perfect field of characteristic 2. Other properties of this embedding will also be investigated.

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1 Introduction

Let V be a 4-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear from f. An ordered basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ of V is called a *hyperbolic basis* of (V, f) if $f(\bar{e}_1, \bar{f}_1) = f(\bar{e}_2, \bar{f}_2) = 1$ and $f(\bar{e}_1, \bar{e}_2) = f(\bar{e}_1, \bar{f}_2) = f(\bar{f}_1, \bar{e}_2) = f(\bar{f}_1, \bar{f}_2) = 0$. We denote by $Sp(V, f) \cong Sp(4, \mathbb{F})$ the *symplectic group* associated with (V, f), i.e. the subgroup of $GL(V)$ consisting of all $\theta \in GL(V)$ such that $f(\bar{v}_1^\theta, \bar{v}_2^\theta) = f(\bar{v}_1, \bar{v}_2)$ for all $\bar{v}_1, \bar{v}_2 \in V$. The group $Sp(V, f)$ consists of precisely those elements of $GL(V)$ that map hyperbolic basis of (V, f) to hyperbolic basis of (V, f). Now, put

$$W_{24} := V \otimes \bigwedge^2 V,$$

where $\bigwedge^2 V$ is the second exterior power of V. Then W_{24} is a 24-dimensional vector space over $\mathbb{F}$. For every $\theta \in GL(V)$, there exists a unique $\widetilde{\theta} \in GL(W_{24})$ such that

$(\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3)^{\widetilde{\theta}} = \bar{v}_1^\theta \otimes \bar{v}_2^\theta \wedge \bar{v}_3^\theta$ for all $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$. By abuse of notation, we will denote $\widetilde{\theta}$ also by θ. By looking at the subgroup $Sp(V, f)$ of $GL(V)$, we thus see that the vector space W_{24} can be regarded as an $Sp(V, f)$-module.

Let $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ be a hyperbolic basis of (V, f) and let W_{16} denote the 16-dimensional subspace of W_{24} generated by the following 16 vectors:

$$\chi_1 := \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \qquad \chi_2 := \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_2, \qquad \chi_3 := \bar{f}_1 \otimes \bar{f}_1 \wedge \bar{e}_2, \qquad \chi_4 := \bar{f}_1 \otimes \bar{f}_1 \wedge \bar{f}_2,$$

$$\chi_5 := \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{e}_1, \qquad \chi_6 := \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{f}_1, \qquad \chi_7 := \bar{f}_2 \otimes \bar{f}_2 \wedge \bar{e}_1, \qquad \chi_8 := \bar{f}_2 \otimes \bar{f}_2 \wedge \bar{f}_1,$$

$$\chi_9 := \bar{e}_2 \otimes \bar{f}_2 \wedge \bar{e}_1 + \bar{f}_2 \otimes \bar{e}_2 \wedge \bar{e}_1, \qquad \chi_{10} := \bar{e}_2 \otimes \bar{f}_2 \wedge \bar{f}_1 + \bar{f}_2 \otimes \bar{e}_2 \wedge \bar{f}_1,$$

$$\chi_{11} := \bar{e}_1 \otimes \bar{f}_1 \wedge \bar{e}_2 + \bar{f}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \qquad \chi_{12} := \bar{e}_1 \otimes \bar{f}_1 \wedge \bar{f}_2 + \bar{f}_1 \otimes \bar{e}_1 \wedge \bar{f}_2,$$

$$\chi_{13} := \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{f}_2 - \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{f}_2,$$

$$\chi_{14} := \bar{f}_1 \otimes \bar{e}_2 \wedge \bar{f}_2 - \bar{f}_1 \otimes \bar{e}_1 \wedge \bar{f}_1 - \bar{f}_2 \otimes \bar{f}_1 \wedge \bar{e}_2,$$

$$\chi_{15} := \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{f}_1,$$

$$\chi_{16} := \bar{f}_2 \otimes \bar{e}_1 \wedge \bar{f}_1 - \bar{f}_2 \otimes \bar{e}_2 \wedge \bar{f}_2 - \bar{f}_1 \otimes \bar{f}_2 \wedge \bar{e}_1.$$

We will show that W_{16} is stabilized by $Sp(V, f)$, implying that the subspace W_{16} is independent of the chosen hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$. We will prove the following.

Theorem 1.1 *The $Sp(V, f)$-module W_{16} is irreducible if and only if* $\mathrm{char}(\mathbb{F}) \neq 5$. *If* $\mathrm{char}(\mathbb{F}) = 5$*, then W_{16} has a unique nontrivial submodule, namely the subspace*

$$\langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle.$$

Define now the following point-line geometry $W(\mathbb{F})$:

- the points of $W(\mathbb{F})$ are the 1-dimensional subspaces of V;
- the lines of $W(\mathbb{F})$ are the 2-dimensional subspaces of V which are totally isotropic with respect to f;
- incidence is containment.

Then $W(\mathbb{F})$ is a *generalized quadrangle* ([5]) meaning that for every line L and every point x not incident with L, there exists a unique point on L collinear with x. The generalized quadrangle $W(\mathbb{F})$ is called *symplectic*. The *flag-geometry* $\mathcal{F}(W(\mathbb{F}))$ of $W(\mathbb{F})$ is the following point-line geometry:

- the points of $\mathcal{F}(W(\mathbb{F}))$ are the *flags* of $W(\mathbb{F})$, that is the unordered point-line pairs $\{x, L\}$, where L is a line of $W(\mathbb{F})$ and x is a point of $W(\mathbb{F})$ incident with L;
- the lines of $\mathcal{F}(W(\mathbb{F}))$ are of two types, the points of $W(\mathbb{F})$ on the one hand and the lines of $W(\mathbb{F})$ on the other hand;
- incidence is reverse containment.

The geometry $\mathcal{F}(W(\mathbb{F}))$ is a so-called generalized octagon of order $(|\mathbb{F}|, 1)$, see [12].

For every flag $F = \{\langle \bar{v}_1 \rangle, \langle \bar{v}_1, \bar{v}_2 \rangle\}$ of $W(\mathbb{F})$, let $e^*(F)$ be the point $\langle \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 \rangle$ of $\mathrm{PG}(W_{24})$. The point $e^*(F)$ is well-defined. Indeed, if $\bar{v}_1', \bar{v}_2'$ are other vectors of V such that $\langle \bar{v}_1 \rangle = \langle \bar{v}_1' \rangle$ and $\langle \bar{v}_1, \bar{v}_2 \rangle = \langle \bar{v}_1', \bar{v}_2' \rangle$, then $\langle \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 \rangle = \langle \bar{v}_1' \otimes \bar{v}_1' \wedge \bar{v}_2' \rangle$. We will prove the following (see Theorem 4.3).

Theorem 1.2 *The map e^* defined a full projective embedding of $\mathcal{F}(W(\mathbb{F}))$ into $\mathrm{PG}(W_{16})$.*

With a *full projective embedding* of a point-line geometry into a projective space $\mathrm{PG}(W)$, we mean an injective mapping e from its point set to the point set of $\mathrm{PG}(W)$, mapping lines to full lines of $\mathrm{PG}(W)$ such that the image of e generates the whole projective space $\mathrm{PG}(W)$. K. Coolsaet (unpublished) also observed that the flag geometry $\mathcal{F}(W(\mathbb{F}))$ admits a full projective embedding into a 15-dimensional projective space. Projective embeddings of generalized octagons and flag geometries of projective planes have already been studied in the literature, see [2, 3] and [7, 8, 9, 10, 11]. Projective embeddings of $\mathcal{F}(W(\mathbb{F}))$ in a 15-dimensional and a 24 dimensional projective space were already described in [2, 3], in case the underlying field $\mathbb{F}$ satisfies additional restrictions. The description of the 15-dimensional embedding described here is essentially different from the one given in [2]. In [2], the embedding space is of the form $\mathrm{PG}(Q \otimes Q')$, where Q and Q' are two 4-dimensional vector spaces, while here the embedding space is a subspace of $\mathrm{PG}(V \otimes U)$ where $\dim(U) > 4$. (Note that we can take for U the 5-dimensional subspace $\langle \bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \wedge \bar{f}_2, \bar{f}_1 \wedge \bar{e}_2, \bar{f}_1 \wedge \bar{f}_2, \bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2 \rangle$ of $\bigwedge^2 V$.)

In this paper, distances between points of $\mathcal{F}(W(\mathbb{F}))$ will always be measured in the collinearity graph of $\mathcal{F}(W(\mathbb{F}))$. The maximal distance between two points of $\mathcal{F}(W(\mathbb{F}))$ is equal to 4. The set of points at distance i (at most i) from a given point x will be denoted by $\Gamma_i(x)$ ($\Gamma_{\leq i}(x)$). For every point p of $\mathcal{F}(W(\mathbb{F}))$, we define $H_p := \Gamma_{\leq 3}(p)$. A full projective embedding e of $\mathcal{F}(W(\mathbb{F}))$ into $\mathrm{PG}(W)$ is called *polarized* if for every point p, there exists a hyperplane Π_p of $\mathrm{PG}(W)$ such that $H_p = e^{-1}(e(\mathcal{P}_{\mathbb{F}}) \cap \Pi_p)$, where $\mathcal{P}_{\mathbb{F}}$ denotes the point set of $\mathcal{F}(W(\mathbb{F}))$. We will also show the following.

Theorem 1.3 *For every point p of $\mathcal{F}(W(\mathbb{F}))$, there exists a unique hyperplane Π_p of $\mathrm{PG}(W_{16})$ such that $H_p = e^{*-1}(e^*(\mathcal{P}_{\mathbb{F}}) \cap \Pi_p)$. As a consequence, the embedding e^* is polarized.*

We will also determine the dimensions of all subspaces $\langle e^*(\Gamma_{\leq i}(p)) \rangle$, where $p \in \mathcal{P}_{\mathbb{F}}$ and $i \in \{0, 1, 2, 3, 4\}$.

Suppose $e : \mathcal{F}(W(\mathbb{F})) \to \mathrm{PG}(W)$ is a full projective embedding of $\mathcal{F}(W(\mathbb{F}))$ and G is a group of automorphisms of $\mathcal{F}(W(\mathbb{F}))$. Then e is called *G-homogeneous* if for every $g \in G$ there exists a (necessarily unique) automorphism $\overline{g}$ of $\mathrm{PG}(W)$ such that $e \circ g = \overline{g} \circ e$. A G-homogeneous full projective embedding where G is the full automorphism group is also called a *homogeneous full projective embedding*. We will show the following.

Theorem 1.4 *If G is the group of automorphisms of $\mathcal{F}(W(\mathbb{F}))$ preserving the line types, then e^* is a G-homogeneous embedding. The embedding e^* is homogeneous, unless $|\mathbb{F}| > 2$ and $\mathbb{F}$ is a perfect field of characteristic 2.*

Suppose $e : \mathcal{F}(W(\mathbb{F})) \to \mathrm{PG}(W)$ is a full projective embedding of $\mathcal{F}(W(\mathbb{F}))$ and π is a subspace of $\mathrm{PG}(W)$ disjoint from the image of e. We denote by $\mathrm{PG}(W)/\pi$ the quotient projective space whose points are those subspaces of $\mathrm{PG}(W)$ that contain π as a hyperplane. The function e/π which maps each point p of $\mathcal{F}(W(\mathbb{F}))$ to the point $\langle e(p), \pi\rangle$ of $\mathrm{PG}(W)/\pi$ is then a full projective embedding of $\mathcal{F}(W(\mathbb{F}))$ into $\mathrm{PG}(W)/\pi$. We call e/π a *quotient* of e.

The subspace $\langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16}\rangle$ determines a subspace α of $\mathrm{PG}(W_{16})$ which is disjoint from the image of e^*, implying that the embedding $\bar{e} := e^*/\alpha$ is well-defined. We will show the following.

Theorem 1.5 *The embedding $\bar{e}$ is polarized and homogeneous.*

2 Preliminaries

We continue with the notation introduced in Section 1. If $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ is a hyperbolic basis of (V, f), then

(1) for every $\lambda \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$, also $(\lambda\bar{e}_1, \frac{\bar{f}_1}{\lambda}, \bar{e}_2, \bar{f}_2)$ is a hyperbolic basis of (V, f);

(2) for every $\lambda \in \mathbb{F}$, also $(\bar{e}_1 + \lambda\bar{e}_2, \bar{f}_1, \bar{e}_2, -\lambda\bar{f}_1 + \bar{f}_2)$ is a hyperbolic basis of (V, f);

(3) for every $\lambda \in \mathbb{F}$, also $(\bar{e}_1, \bar{f}_1, \bar{e}_2 + \lambda\bar{f}_2, \bar{f}_2)$ is a hyperbolic basis of (V, f);

(4) for every $\lambda \in \mathbb{F}$, also $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2 + \lambda\bar{e}_2)$ is a hyperbolic basis of (V, f);

(5) also $(\bar{e}_2, \bar{f}_2, \bar{e}_1, \bar{f}_1)$ is a hyperbolic basis of (V, f);

(6) also $(-\bar{f}_1, \bar{e}_1, \bar{e}_2, \bar{f}_2)$ is a hyperbolic basis of (V, f);

(7) also $(\bar{e}_1 + \bar{e}_2, \bar{f}_1, \bar{e}_2, \bar{f}_2 - \bar{f}_1)$ is a hyperbolic basis of (V, f).

For every $i \in \{1, 2, \ldots, 7\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of hyperbolic bases of (V, f) such that B_2 can be obtained from B_1 as described in (i) above. The following was proved in [4, Lemma 2.1]:

Lemma 2.1 *If B and B' are two hyperbolic bases of (V, f), then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ of (V, f) for some $k \geq 0$ such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_5$ for every $i \in \{1, 2, \ldots, k\}$.*

We shall make use of the following improved version of Lemma 2.1.

Lemma 2.2 *If B and B' are two hyperbolic bases of (V,f), then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ of (V,f) for some $k \geq 0$ such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_4 \cup \Omega_5 \cup \Omega_6 \cup \Omega_7$ for every $i \in \{1,2,\ldots,k\}$.*

Proof. In view of Lemma 2.1, it suffices to prove this in the case where $(B,B') \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$. We leave the verification in each of these five cases as a straightforward exercise to the reader. ■

Lemma 2.2 implies the following.

Proposition 2.3 *Let $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ be a hyperbolic basis of (V,f). Let θ_1 be the element of $Sp(V,f)$ mapping B to $(\bar{e}_2, \bar{f}_2, \bar{e}_1, \bar{f}_1)$, θ_2 the element of $Sp(V,f)$ mapping B to $(-\bar{f}_1, \bar{e}_1, \bar{e}_2, \bar{f}_2)$, $\theta_3(\lambda)$ with $\lambda \in \mathbb{F}^*$ the element of $Sp(V,f)$ mapping B to $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2 + \lambda\bar{e}_2)$, and θ_4 the element of $Sp(V,f)$ mapping B to $(\bar{e}_1 + \bar{e}_2, \bar{f}_1, \bar{e}_2, \bar{f}_2 - \bar{f}_1)$. Then the group $G = \langle \theta_1, \theta_2, \theta_3(\lambda), \theta_4 \,|\, \lambda \in \mathbb{F}^* \rangle$ coincides with $Sp(V,f)$.*

Proof. Let θ be an arbitrary element of $Sp(V,f)$. By Lemma 2.2, there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ of (V,f) for some $k \geq 0$ such that $B_0 = B$, $B_k = B^\theta$ and $(B_{i-1}, B_i) \in \Omega_4 \cup \Omega_5 \cup \Omega_6 \cup \Omega_7$ for every $i \in \{1,2,\ldots,k\}$. We prove by induction on k that $\theta \in G$. This clearly holds if $k \in \{0,1\}$. So, we will suppose that $k \geq 2$ and that the proposition holds for smaller values of k. Let θ' be the element of $Sp(V,f)$ mapping the hyperbolic B to the hyperbolic basis B_{k-1}. By the induction hypothesis, $\theta' \in G$. Now, there exists a $\theta'' \in G$ mapping the hyperbolic basis $B = B_{k-1}^{\theta'^{-1}}$ to the hyperbolic basis $B_k^{\theta'^{-1}}$. Then $\theta' \circ \theta''$ maps B to B_k and hence coincides with θ. Since $\theta', \theta'' \in G$, also $\theta \in G$. ■

Now, let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*)$ be a fixed hyperbolic basis of (V,f). For every $h \in \mathbb{F}^*$, let θ_h^* be the element of $GL(V)$ mapping the ordered basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*)$ of V to the ordered basis $(h\bar{e}_1^*, \bar{f}_1^*, h\bar{e}_2^*, \bar{f}_2^*)$ of V, and for every automorphism α of $\mathbb{F}$, let θ_α^* be the element of $\Gamma L(V)$ defined by

$$\lambda_1 \bar{e}_1^* + \mu_1 \bar{f}_1^* + \lambda_2 \bar{e}_2^* + \mu_2 \bar{f}_2^* \mapsto \lambda_1^\alpha \bar{e}_1^* + \mu_1^\alpha \bar{f}_1^* + \lambda_2^\alpha \bar{e}_2^* + \mu_2^\alpha \bar{f}_2^*.$$

Every $\theta \in Sp(V,f)$ will induce an automorphism A_θ of $W(\mathbb{F})$, every θ_h^* with $h \in \mathbb{F}^*$ will induce an automorphism A_h of $W(\mathbb{F})$, and every θ_α^* with $\alpha \in \mathrm{Aut}(\mathbb{F})$ will induce an automorphism A_α of $W(\mathbb{F})$. In fact the following holds.

Proposition 2.4 *Every automorphism of $W(\mathbb{F})$ is induced by an element of $\Gamma L(V)$ of the form $\theta_h^* \circ \theta \circ \theta_\alpha^*$, where $\theta \in Sp(V,f)$, $h \in \mathbb{F}^*$ and $\alpha \in \mathrm{Aut}(\mathbb{F})$.*

The following result is also known.

Proposition 2.5 *The generalized quadrangle $W(\mathbb{F})$ is isomorphic to its point-line dual $W^D(\mathbb{F})$ if and only if $\mathbb{F}$ is a perfect field of characteristic 2.*

Every automorphism A of $W(\mathbb{F})$ induces an automorphism $\overline{A}$ of $\mathcal{F}(W(\mathbb{F}))$ that does not alter the types of the lines. If $\mathbb{F}$ is a perfect field of characteristic 2, then every duality D of $W(\mathbb{F})$ will induce an automorphism $\overline{D}$ of $\mathcal{F}(W(\mathbb{F}))$ which interchanges the line types. In fact, we have the following:

Proposition 2.6 • *If $\mathbb{F}$ is not a perfect field of characteristic 2, then every automorphism of $\mathcal{F}(W(\mathbb{F}))$ is induced by an automorphism of $W(\mathbb{F})$.*

- *If $\mathbb{F}$ is a perfect field of characteristic 2, then every automorphism of $\mathcal{F}(W(\mathbb{F}))$ is induced by an automorphism or a duality of $W(\mathbb{F})$.*

We follow the convention that distances in $W(\mathbb{F})$, $W^D(\mathbb{F})$ and $\mathcal{F}(W(\mathbb{F}))$ are measured in their respective collinearity graphs. We denote by $\mathrm{d}(\cdot,\cdot)$, $\delta(\cdot,\cdot)$ and $\delta^D(\cdot,\cdot)$ the respective distance functions in $\mathcal{F}(W(\mathbb{F}))$, $W(\mathbb{F})$ and $W^D(\mathbb{F})$. We have the following:

Proposition 2.7 *If $\{x_1, L_1\}$ and $\{x_2, L_2\}$ are two flags of $W(\mathbb{F})$, then $d(\{x_1, L_1\}, \{x_2, L_2\})$ $= \delta(x_1, x_2) + \delta^D(L_1, L_2)$.*

Two points $p_1 = \{x_1, L_1\}$ and $p_2 = \{x_2, L_2\}$ of $\mathcal{F}(W(\mathbb{F}))$ are said to be *opposite* if they lie at maximal distance 4 from each other, i.e. if x_1 and x_2 are two noncollinear points of $W(\mathbb{F})$ and if L_1, L_2 are two nonintersecting lines of $W(\mathbb{F})$.

3 The embedding and generating ranks of $\mathcal{F}(W(2))$

If $\mathbb{F}$ is a finite field with q elements, then we denote $W(\mathbb{F})$ and $\mathcal{F}(W(\mathbb{F}))$ also by $W(q)$ and $\mathcal{F}(W(q))$. The generalized octagon $\mathcal{F}(W(2))$ is, up to isomorphism, the unique octagon of order $(2,1)$ and for this reason, we will also denote it by $\mathrm{GO}(2,1)$. By [6, Corollary 4, p.184], the geometry $\mathrm{GO}(2,1)$ has full projective embeddings and hence admits an absolutely universal embedding $\widetilde{e} : \mathrm{GO}(2,1) \to \mathrm{PG}(\widetilde{W})$ (meaning that every full embedding of $\mathrm{GO}(2,1)$ is isomorphic to a quotient of $\widetilde{e}$). The dimension $\dim(\widetilde{W})$ of $\widetilde{W}$ is called the *embedding rank* of $\mathrm{GO}(2,1)$ and is equal to $v - \mathrm{rank}_{\mathbb{F}_2}(N)$, where $v = 45$ is the total number of points of $\mathrm{GO}(2,1)$ and N is an *incidence matrix* of $\mathrm{GO}(2,1)$, that is a 0-1 matrix whose rows are indexed by the points and whose columns are indexed by the lines, where an entry equals 1 if and only if the corresponding point-line pair is incident. We will now determine $\mathrm{rank}_{\mathbb{R}}(N)$. To achieve this goal, we will make use of the known spectrum of the collinearity graph of $\mathrm{GO}(2,1)$. This spectrum can easily be derived from Table 6.4 on page 203 of [1].

Lemma 3.1 *The collinearity graph of* $\mathrm{GO}(2,1)$ *has spectrum* $(-2)^{16}(-1)^9 1^{10} 3^9 4^1$.

Lemma 3.2 *We have* $\mathrm{rank}_{\mathbb{R}}(N) = 29$.

Proof. Let A denote the adjacency matrix of $\mathrm{GO}(2,1)$, where the ordering of the points used to label the rows and columns of A is the same as the ordering of the points we used to label the rows of N. We then have $N \cdot N^T = A + 2I$, where I is the (45×45)-identity matrix. So, by Lemma 3.1 we have

$$\mathrm{rank}_{\mathbb{R}}(N) = \mathrm{rank}_{\mathbb{R}}(N \cdot N^T) = \mathrm{rank}_{\mathbb{R}}(A + 2I) = 9 + 10 + 9 + 1 = 29.$$

■

Lemma 3.2 allows us to determine a lower bound for the embedding rank of $\mathrm{GO}(2,1)$.

Lemma 3.3 *We have* $\dim(\widetilde{W}) \geq 16$.

Proof. We have $\dim(\widetilde{W}) = 45 - \mathrm{rank}_{\mathbb{F}_2}(N) \geq 45 - \mathrm{rank}_{\mathbb{R}}(N) = 16$. ■

We will later show that for any field $\mathbb{F}$, the generalized octagon $\mathcal{F}(W(\mathbb{F}))$ admits a full projective embedding into a 15-dimensional projective space over $\mathbb{F}$. The next goal in this section will be to show that $\dim(\widetilde{W}) = 16$. To achieve this goal, we will make use of the notion of generating rank and of another model of $W(2)$.

A *subspace* of a point-line geometry $\mathcal{S}$ is a set X of points containing all the points of a line if this line has at least two of its points in X. Obviously, the whole point set is an example of a subspace. If X is a nonempty set of points, then the intersection of all subspaces containing X is the smallest subspace that contains X and is denoted by $\langle X \rangle$. If $\langle X \rangle$ coincides with the whole point set, then X is called a *generating set* of $\mathcal{S}$. In general, $\langle X \rangle$ is called the *subspace generated* by X. The smallest size of a generating set of $\mathcal{S}$ is called the *generating rank* of $\mathcal{S}$ and denoted by $gr(\mathcal{S})$. If $e : \mathcal{S} \to \mathrm{PG}(W)$ is a full projective embedding of $\mathcal{S}$, then $\dim(W) \leq gr(\mathcal{S})$.

In particular, we thus have that $\dim(\widetilde{W}) \leq gr(\mathrm{GO}(2,1))$. In order to show that $\dim(\widetilde{W}) = 16$, it thus suffices to show that $\mathrm{GO}(2,1)$ has a generating set of size 16. We achieve this goal by using another model of $W(2)$. The generalized quadrangle $W(2)$ is isomorphic to the point-line geometry

- whose points are the subsets of size 2 of $\{1,2,3,4,5,6\}$,
- whose lines are the partitions of $\{1,2,3,4,5,6\}$ into three subsets of size 2,
- whose incidence relation is containment.

This model is called *Sylvester's model* of $W(2)$.

If x is a point of $\mathrm{GO}(2,1)$, then:

- $|\Gamma_0(x)| = 1$, $|\Gamma_1(x)| = 4$, $|\Gamma_2(x)| = 8$, $|\Gamma_3(x)| = 16$ and $|\Gamma_4(x)| = 16$;
- there are two lines containing x, four lines meeting $\Gamma_1(x)$ and $\Gamma_2(x)$, eight lines meeting $\Gamma_2(x)$ and $\Gamma_3(x)$, and sixteen lines meeting $\Gamma_3(x)$ and $\Gamma_4(x)$.

Lemma 3.4 *If x is a point of* $\mathrm{GO}(2,1)$*, then:*

(1) *The graph defined on $\Gamma_4(x)$ by the collinearity relation has two connected components. Each connected component is a cycle of length 8.*

(2) *If $y_0, y_1, \ldots, y_8 = y_0$ is a cycle of length 8 contained in $\Gamma_4(x)$, then each line $y_{i-1}y_i$ with $i \in \{1,2,\ldots,8\}$ meets a unique line L_i of $\mathrm{GO}(2,1)$ for which $L_i \cap \Gamma_2(x) \neq \emptyset \neq L_i \cap \Gamma_3(x)$. The eight lines $L_1, L_2, \ldots, L_8$ are mutually distinct and are all the lines meeting $\Gamma_2(x)$ and $\Gamma_3(x)$.*

Proof. We will use Sylvester's model of $W(2)$. Without loss of generality, we may suppose that

$$x = \{\{1,2\},\{\{1,2\},\{3,4\},\{5,6\}\}.$$

$\Gamma_4(x)$ is a regular graph of degree 2 and hence is the disjoint union of a number of cycles. Since the stabilizer of x acts transitively on the set of points opposite to x, all these cycles will have the same length k. We now determine the length of such a cycle starting from a point $y_0 \in \Gamma_4(x)$. Without loss of generality, we may suppose that

$$y_0 = \{\{1,3\},\{\{1,3\},\{2,6\},\{4,5\}\}.$$

The first step we take is along the line $\{1,3\}$. The cycle which then arises will be denoted by $y_0, y_1, \ldots, y_k = y_0$. We find:

$$\begin{aligned}
y_1 &= \{\{1,3\},\{\{1,3\},\{2,5\},\{4,6\}\}\}, & y_2 &= \{\{2,5\},\{\{1,3\},\{2,5\},\{4,6\}\}\}, \\
y_3 &= \{\{2,5\},\{\{1,4\},\{2,5\},\{3,6\}\}\}, & y_4 &= \{\{1,4\},\{\{1,4\},\{2,5\},\{3,6\}\}\}, \\
y_5 &= \{\{1,4\},\{\{1,4\},\{2,6\},\{3,5\}\}\}, & y_6 &= \{\{2,6\},\{\{1,4\},\{2,6\},\{3,5\}\}\}, \\
y_7 &= \{\{2,6\},\{\{1,3\},\{2,6\},\{4,5\}\}\}, & y_8 &= \{\{1,3\},\{\{1,3\},\{2,6\},\{4,5\}\}\}.
\end{aligned}$$

Thus $k = 8$. Now, for every $i \in \{1, 2, \ldots, 8\}$, put $\{z_i\} := y_{i-1}y_i \cap \Gamma_3(x)$, and let L_i denote the unique line through z_i containing a point at distance 2 from x. Then one easily computes:

$z_1 = \{\{1,3\},\{\{1,3\},\{2,4\},\{5,6\}\}\}$	$L_1 = \{\{1,3\},\{2,4\},\{5,6\}\}$
$z_2 = \{\{4,6\},\{\{1,3\},\{2,5\},\{4,6\}\}\}$	$L_2 = \{4,6\}$
$z_3 = \{\{2,5\},\{\{1,6\},\{2,5\},\{3,4\}\}\}$	$L_3 = \{\{1,6\},\{2,5\},\{3,4\}\}$
$z_4 = \{\{3,6\},\{\{1,4\},\{2,5\},\{3,6\}\}\}$	$L_4 = \{3,6\}$
$z_5 = \{\{1,4\},\{\{1,4\},\{2,3\},\{5,6\}\}\}$	$L_5 = \{\{1,4\},\{2,3\},\{5,6\}\}$
$z_6 = \{\{3,5\},\{\{1,4\},\{2,6\},\{3,5\}\}\}$	$L_6 = \{3,5\}$
$z_7 = \{\{2,6\},\{\{1,5\},\{2,6\},\{3,4\}\}\}$	$L_7 = \{\{1,5\},\{2,6\},\{3,4\}\}$
$z_8 = \{\{4,5\},\{\{1,3\},\{2,6\},\{4,5\}\}\}$	$L_8 = \{4,5\}$

Note that $L_1, L_2, \ldots, L_8$ are precisely the eight lines meeting $\Gamma_2(x)$ and $\Gamma_3(x)$. This shows the validity of the lemma. ∎

Lemma 3.4 has the following corollary.

Corollary 3.5 *Let L be a line meeting $\Gamma_2(x)$ and $\Gamma_3(x)$. Then the number of connected components of $\Gamma_4(x)$ defined by the lines meeting $\Gamma_4(x)$ and $\Gamma_3(x) \setminus L$ is also equal to 2.*

Proposition 3.6 *The generalized octagon* GO(2, 1) *can be generated by* 16 *points.*

Proof. Let x be a fixed point of $\mathrm{GO}(2,1)$. Put $u_1 := x$, and let $u_2, u_3 \in \Gamma_1(x)$ such that xu_2 and xu_3 are the two lines through x.

There are four lines meeting $\Gamma_1(x)$ and $\Gamma_2(x)$. On each of these four lines, we take a point not contained in $\Gamma_1(x)$. In this way, we obtain four points which we will denote by u_4, u_5, u_6 and u_7.

There are eight lines meeting $\Gamma_2(x)$ and $\Gamma_3(x)$. We denote by L one of these eight lines. On each of the seven other lines, we take a point not contained in $\Gamma_2(x)$. In this way, we obtain seven points which we will denote by $u_8, u_9, \ldots, u_{14}$.

By Lemma 3.4(1), we know that $\Gamma_4(x)$ has two connected components C_1 and C_2. We take an arbitrary point $u_{15} \in C_1$ and an arbitrary point $u_{16} \in C_2$.

We now show that $\{u_1, u_2, \ldots, u_{16}\}$ is a generating set of $\mathrm{GO}(2,1)$. Obviously, the following hold:

$$\langle u_1 \rangle = \{x\}, \quad \langle u_1, u_2, u_3 \rangle = \Gamma_{\leq 1}(x), \quad \langle u_1, u_2, \ldots, u_7 \rangle = \Gamma_{\leq 2}(x),$$

$$\langle u_1, u_2, \ldots, u_{14} \rangle = \Gamma_{\leq 3}(x) \setminus (L \cap \Gamma_3(x)).$$

By Corollary 3.5, we also know that C_1 and C_2 are the two connected components of $\Gamma_4(x)$ defined by the lines meeting $\Gamma_4(x)$ and $\Gamma_3(x) \setminus L$. So, the smallest subspace of $\mathrm{GO}(2,1)$ containing $\Gamma_{\leq 3}(x) \setminus (L \cap \Gamma_3(x))$ and u_{15} contains C_1 and the smallest subspace of $\mathrm{GO}(2,1)$ containing $\Gamma_{\leq 3}(x) \setminus L$ and u_{16} contains C_2. We conclude that $\langle u_1, u_2, \ldots, u_{16} \rangle$ also contains $C_1 \cup C_2 = \Gamma_4(x)$ and hence the whole point set of $\mathrm{GO}(2,1)$. Indeed, every point of $L \cap \Gamma_3(x)$ is contained in a line that contains two points of $\Gamma_4(x)$. ■

By Lemma 3.3 and Proposition 3.6, we have:

Corollary 3.7 *The embedding and generating ranks of* $\mathrm{GO}(2,1)$ *are equal to* 16.

We also have the following:

Proposition 3.8 *Let* $\widetilde{e} : \mathrm{GO}(2,1) \to \mathrm{PG}(\widetilde{W})$ *denote the universal embedding of* $\mathrm{GO}(2,1)$. *Then:*

(1) $\widetilde{e}$ *is polarized. For every point* x *of* $\mathrm{GO}(2,1)$, *there is a unique hyperplane* Π_x *of* $\mathrm{PG}(\widetilde{W})$ *such that* $H_x = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi_x)$. *Here,* $\mathcal{P}$ *is the point set of* $\mathrm{GO}(2,1)$.

(2) *For every point* x *of* $\mathrm{GO}(2,1)$, *the subspace of* $\mathrm{PG}(\widetilde{W})$ *generated by* $\widetilde{e}(H_x)$ *is a subspace of co-dimension* 2 *of* $\mathrm{PG}(\widetilde{W})$.

Proof. If Π is a hyperplane of $\mathrm{PG}(\widetilde{W})$, then $H_\Pi := \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi)$ is a *hyperplane* of $\mathrm{GO}(2,1)$, i.e. a set of points distinct from $\mathcal{P}$ meeting each line in either 1 or 3 points. By [6, Corollary 2, p.180], every hyperplane H of $\mathrm{GO}(2,1)$ is equal to H_Π for a (necessarily unique) hyperplane Π of $\mathrm{PG}(\widetilde{W})$.

For every point x of $\mathrm{GO}(2,1)$, the set H_x is a hyperplane of $\mathrm{GO}(2,1)$. Such hyperplanes are called *singular*. Applying the previous paragraph to the singular hyperplanes

	θ_1	θ_2	$\theta_3(\lambda)$	θ_4
χ_1	χ_5	χ_3	χ_1	$\chi_1 - \chi_5$
χ_2	χ_6	χ_4	$\chi_2 + \lambda\chi_1$	$\chi_2 - \chi_6 + \chi_{13} - \chi_{15}$
χ_3	χ_7	χ_1	χ_3	χ_3
χ_4	χ_8	χ_2	$\chi_4 + \lambda\chi_3$	χ_4
χ_5	χ_1	$-\chi_6$	χ_5	χ_5
χ_6	χ_2	χ_5	χ_6	χ_6
χ_7	χ_3	$-\chi_8$	$\chi_7 + \lambda\chi_9 + \lambda^2\chi_5$	$\chi_3 + \chi_7 + \chi_{14} + \chi_{16}$
χ_8	χ_4	χ_7	$\chi_8 + \lambda\chi_{10} + \lambda^2\chi_6$	$\chi_4 + \chi_8$
χ_9	χ_{11}	$-\chi_{10}$	$\chi_9 + 2\lambda\chi_5$	$\chi_9 + \chi_6 + \chi_{15} + \chi_{11}$
χ_{10}	χ_{12}	χ_9	$\chi_{10} + 2\lambda\chi_6$	$\chi_3 + \chi_{10}$
χ_{11}	χ_9	$-\chi_{11}$	χ_{11}	$\chi_{11} - \chi_6$
χ_{12}	χ_{10}	$-\chi_{12}$	$\chi_{12} + \lambda\chi_{11}$	$\chi_{12} + \chi_3 + \chi_{14} - \chi_{10}$
χ_{13}	χ_{15}	$-\chi_{14} + \chi_{10}$	$\chi_{13} - \lambda\chi_5$	$\chi_{13} - 3\chi_6 - 2\chi_{15}$
χ_{14}	χ_{16}	$\chi_{13} + \chi_9$	$\chi_{14} + \lambda\chi_6$	$\chi_{14} + 3\chi_3$
χ_{15}	χ_{13}	$\chi_{15} + \chi_{11}$	χ_{15}	$\chi_{15} + 3\chi_6$
χ_{16}	χ_{14}	$\chi_{16} - \chi_{12}$	$\chi_{16} + \lambda\chi_{15} + \lambda\chi_{11}$	$\chi_{16} + 2\chi_{14} + 3\chi_3$

Table 1: The actions of θ_1, θ_2, $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}^*$), θ_4 on W_{16}.

of GO(2, 1), we see that Claim (1) of the proposition is valid. Applying the previous paragraph to the hyperplanes containing a given hyperplane H_x, $x \in \mathcal{P}$, we see that the number of hyperplanes of GO(2, 1) containing H_x is equal to $2^\delta - 1$, where δ is the co-dimension (in PG($\widetilde{W}$)) of the subspace generated by $\widetilde{e}(H_x)$. By Lemma 3.4(1), we know that there are three hyperplanes containing H_x, namely H_x, $H_x \cup C_1$ and $H_x \cup C_2$, where C_1 and C_2 are the two connected components of $\Gamma_4(x)$. It follows that $\delta = 2$. ■

4 A 16-dimensional $Sp(V, f)$-module hosting a full projective embedding of $\mathcal{F}(W(\mathbb{F}))$

We continue with the notation introduced in Section 1.

Proposition 4.1 *The subspace W_{16} is stabilized by $Sp(V, f)$.*

Proof. With the notation of Proposition 2.3, we have that

$$Sp(V, f) = \langle \theta_1, \theta_2, \theta_3(\lambda), \theta_4 \,|\, \lambda \in \mathbb{F}^* \rangle.$$

So, it suffices to show that each of θ_1, θ_2, $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}^*$), θ_4 stabilizes W_{16}. The actions of θ_1, θ_2, $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}^*$) and θ_4 on W_{16} are summarized in Table 1, and from this information it indeed follows that each of θ_1, θ_2, $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}^*$), θ_4 stabilizes W_{16}. ■

Lemma 4.2 *Suppose U is a subspace of W_{16} containing χ_1 that is stabilized by $Sp(V,f)$. Then $U = W_{16}$.*

Proof. Since $\chi_1 \in U$, the following vectors also belong to U:

$$\chi_5 = \chi_1^{\theta_1}, \quad \chi_3 = \chi_1^{\theta_2}, \quad \chi_6 = -\chi_5^{\theta_2}, \quad \chi_7 = \chi_3^{\theta_1}, \quad \chi_2 = \chi_6^{\theta_1}, \quad \chi_4 = \chi_2^{\theta_2}, \quad \chi_8 = -\chi_7^{\theta_2},$$

$$\chi_9 = \chi_7^{\theta_3(1)} - \chi_7 - \chi_5, \quad \chi_{10} = \chi_8^{\theta_3(1)} - \chi_8 - \chi_6, \quad \chi_{11} = \chi_9^{\theta_1}, \quad \chi_{12} = \chi_{10}^{\theta_1},$$

$$\chi_{14} = \chi_{12}^{\theta_4} - \chi_{12} + \chi_{10} - \chi_3, \quad \chi_{15} = \chi_9^{\theta_4} - \chi_9 - \chi_6 - \chi_{11}, \quad \chi_{13} = \chi_{15}^{\theta_1}, \quad \chi_{16} = \chi_{14}^{\theta_1}.$$

Hence, $W_{16} = \langle \chi_1, \chi_2, \ldots, \chi_{16} \rangle \subseteq U$, i.e. $U = W_{16}$. ■

Theorem 4.3 *The map e^* defines a full projective embedding of $\mathcal{F}(W(\mathbb{F}))$ into $\mathrm{PG}(W_{16})$.*

Proof. We first show that $e^*(F)$ is a point of $\mathrm{PG}(W_{16})$ for every flag $F = \{\langle \bar{v}_1 \rangle, \langle \bar{v}_1, \bar{v}_2 \rangle\}$ of $W(\mathbb{F})$. Let θ be an element of $Sp(V,f)$ mapping $\bar{e}_1$ to $\bar{v}_1$ and $\bar{e}_2$ to $\bar{v}_2$. Then $\bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 = (\bar{e}_1 \otimes \bar{e}_2 \wedge \bar{e}_2)^\theta = \chi_1^\theta \in W_{16}$.

The latter also implies that the subspace of $\mathrm{PG}(W_{16})$ generated by the image of e^* coincides with the subspace of $\mathrm{PG}(W_{16})$ generated by all points $\langle \chi_1^\theta \rangle$ where $\theta \in Sp(V,f)$. By Lemma 4.2, we then know that this subspace coincides with $\mathrm{PG}(W_{16})$.

It remains to show that e^* maps every line L of $\mathcal{F}(W(\mathbb{F}))$ to some line of $\mathrm{PG}(W_{16})$. There are two cases to consider for such a line L.

Suppose there exist linearly independent vectors $\bar{v}_1, \bar{v}_2, \bar{v}_2'$ such that L consists of all flags of the form $\{\langle \bar{v}_1 \rangle, \langle \bar{v}_1, \lambda_2 \bar{v}_2 + \lambda_2' \bar{v}_2' \rangle\}$ where $\lambda_2, \lambda_2' \in \mathbb{F}$ with $(\lambda_2, \lambda_2') \neq (0,0)$. Then $e^*(L)$ consists of all points of the form $\langle \bar{v}_1 \otimes \bar{v}_1 \wedge (\lambda_2 \bar{v}_2 + \lambda_2' \bar{v}_2') \rangle = \langle \lambda_2 \cdot \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 + \lambda_2' \cdot \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2' \rangle$, where $\lambda_2, \lambda_2' \in \mathbb{F}$ with $(\lambda_2, \lambda_2') \neq (0,0)$, i.e. $e^*(L)$ is a line of $\mathrm{PG}(W_{16})$.

Suppose there exist linearly independent vectors $\bar{v}_1$ and $\bar{v}_2$ such that L consists of all flags of the form $\{\langle \lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 \rangle, \langle \bar{v}_1, \bar{v}_2 \rangle\}$ where $\lambda_1, \lambda_2 \in \mathbb{F}$ such that $(\lambda_1, \lambda_2) \neq (0,0)$. Then $e^*(L)$ consists of all points of the form $\langle (\lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2) \otimes \bar{v}_1 \wedge \bar{v}_2 \rangle = \langle \lambda_1 \cdot \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 + \lambda_2 \cdot \bar{v}_2 \otimes \bar{v}_1 \wedge \bar{v}_2 \rangle$, where $\lambda_1, \lambda_2 \in \mathbb{F}$ with $(\lambda_1, \lambda_2) \neq (0,0)$, i.e. $e^*(L)$ is a line of $\mathrm{PG}(W_{16})$. ■

Theorem 4.4 *If $|\mathbb{F}| = 2$, then the embedding e^* is absolutely universal.*

Proof. This follows from the fact that $\dim(W_{16}) = 16$ equals the embedding rank of $\mathcal{F}(W(2))$, see Corollary 3.7. ■

5 The (ir)reducibility of the $Sp(V,f)$-module W_{16}

We continue with the notation introduced in the previous sections.

Proposition 5.1 *If $\mathrm{char}(\mathbb{F}) = 5$, then the subspace*

$$\langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle$$

is stabilized by $Sp(V,f)$.

	θ_1	θ_2	$\theta_3(\lambda)$	θ_4
$\chi_9 + 2\chi_{13}$	$\chi_{11} + 2\chi_{15}$	$-2(2\chi_{10} + \chi_{14})$	$\chi_9 + 2\chi_{13}$	$(\chi_9 + 2\chi_{13}) + (\chi_{11} + 2\chi_{15})$
$2\chi_{10} + \chi_{14}$	$2\chi_{12} + \chi_{16}$	$3(\chi_9 + 2\chi_{13})$	$2\chi_{10} + \chi_{14}$	$2\chi_{10} + \chi_{14}$
$\chi_{11} + 2\chi_{15}$	$\chi_9 + 2\chi_{13}$	$\chi_{11} + 2\chi_{15}$	$\chi_{11} + 2\chi_{15}$	$\chi_{11} + 2\chi_{15}$
$2\chi_{12} + \chi_{16}$	$2\chi_{10} + \chi_{14}$	$2\chi_{12} + \chi_{16}$	$(2\chi_{12} + \chi_{16}) + 3\lambda \cdot (\chi_{11} + 2\chi_{15})$	$(2\chi_{12} + \chi_{16}) - (2\chi_{10} + \chi_{14})$

Table 2: The actions of θ_1, θ_2, $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}^*$), θ_4 on the subspace $\langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle$.

Proof. This follows from Table 2, where the actions of θ_1, θ_2, $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}^*$), θ_4 on the subspace $\langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle$ have been described. ∎

Lemma 5.2 *Let U be a subspace of W_{16} stabilized by $Sp(V, f)$, and let $i, j \in \{1, 2, \dots, 8\}$. If U contains a vector having a nonzero component in χ_i, then U also contains a vector having a nonzero component in χ_j.*

Proof. The property mentioned in Lemma 5.2 that we need to prove is called Property (P_{ij}) here. Let Γ be the graph with vertex set $\{1, 2, \dots, 8\}$, where two distinct vertices i and j are adjacent whenever Property (P_{ij}) holds. By using the fact that $\chi \in U$ if and only if $\chi^{\theta_1} \in U$, we see that $\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$ and $\{4, 8\}$ are edges of Γ. By using the fact that $\chi \in U$ if and only if $\chi^{\theta_2} \in U$, we also see that $\{1, 3\}$, $\{2, 4\}$, $\{5, 6\}$ and $\{7, 8\}$ are edges of Γ. These edges already turn Γ into a connected graph, proving the validity of the lemma. ∎

Lemma 5.3 *Let U be a subspace of W_{16} stabilized by $Sp(V, f)$. If U contains a vector χ_i with $i \in \{1, 2, \dots, 8\}$, then $U = W_{16}$.*

Proof. As there exists a $\theta \in Sp(V, f)$ such that $\chi_i^\theta = \chi_1$, we must have $U = W_{16}$ by Lemma 4.2. ∎

Lemma 5.4 *Let $U \neq W_{16}$ be a subspace of W_{16} stabilized by $Sp(V, f)$. Then $U \subseteq \langle \chi_9, \chi_{10}, \dots, \chi_{16} \rangle$.*

Proof. We first deal with the case $|\mathbb{F}| = 2$. Let $\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_{16}\chi_{16}$ denote an arbitrary vector of U. Then also the following vectors belong to U (with $\theta_3 = \theta_3(1)$):

$$\chi^{(1)} := \chi^{\theta_3} - \chi, \quad \chi^{(2)} := (\chi^{(1)})^{\theta_1}, \quad \chi^{(3)} := (\chi^{(2)})^{\theta_3} - \chi^{(2)}, \quad \chi^{(4)} := (\chi^{(3)})^{\theta_4} - \chi^{(3)},$$

$$\chi^{(5)} := (\chi^{(4)})^{\theta_1}, \quad \chi^{(6)} := (\chi^{(5)})^{\theta_3} - \chi^{(5)}, \quad \chi^{(7)} := (\chi^{(6)})^{\theta_4} - \chi^{(6)}.$$

One computes that

- $\chi^{(1)} = a_2\chi_1 + a_4\chi_3 + (a_7 + a_{13})\chi_5 + (a_8 + a_{14})\chi_6 + a_7\chi_9 + a_8\chi_{10} + (a_{12} + a_{16})\chi_{11} + a_{16}\chi_{15}$,
- $\chi^{(2)} = a_2\chi_5 + a_4\chi_7 + (a_7 + a_{13})\chi_1 + (a_8 + a_{14})\chi_2 + a_7\chi_{11} + a_8\chi_{12} + (a_{12} + a_{16})\chi_9 + a_{16}\chi_{13}$,

- $\chi^{(3)} = (a_8 + a_{14})\chi_1 + (a_4 + a_{16})\chi_5 + a_4\chi_9 + a_8\chi_{11}$,
- $\chi^{(4)} = (a_8 + a_{14})\chi_5 + (a_4 + a_8)\chi_6 + a_4\chi_{11} + a_4\chi_{15}$,
- $\chi^{(5)} = (a_8 + a_{14})\chi_1 + (a_4 + a_8)\chi_2 + a_4\chi_9 + a_4\chi_{13}$,
- $\chi^{(6)} = (a_4 + a_8)\chi_1 + a_4\chi_5$,
- $\chi^{(7)} = (a_4 + a_8)\chi_5$.

Since $U \neq W_{16}$, we have by Lemma 5.3 that none of χ_1, χ_5 belongs to U. Since $\chi^{(6)}, \chi^{(7)} \in U$, we then have that $a_4 + a_8 = a_4 = 0$, i.e. $a_4 = a_8 = 0$. Since χ was an arbitrary vector of U, Lemma 5.2 then implies that $a_1 = a_2 = \cdots = a_8 = 0$, i.e. $U \subseteq \langle \chi_9, \chi_{10}, \ldots, \chi_{16}\rangle$.

From now on, we assume that $|\mathbb{F}| \geq 3$. Again, let $\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_{16}\chi_{16}$ denote an arbitrary vector of U. Since $|\mathbb{F}| \geq 3$, we can take two distinct elements $\lambda_1, \lambda_2 \in \mathbb{F}^*$. The fact that $\lambda_1(\chi^{\theta_3(\lambda_2)} - \chi) - \lambda_2(\chi^{\theta_3(\lambda_1)} - \chi) \in U$ then implies that $\lambda_1\lambda_2(\lambda_2 - \lambda_1)(a_7\chi_5 + a_8\chi_6) \in U$, i.e. $a_7\chi_5 + a_8\chi_6 \in U$. By Lemma 5.3, we also know that χ_5 and χ_6 do not belong to U.

Suppose $|\mathbb{F}| > 3$. Then we can take $\lambda \in \mathbb{F}^*$ such that $\lambda^2 \neq 1$. Let θ be the element of $Sp(V, f)$ mapping $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ to $(\lambda\bar{e}_1, \frac{\bar{f}_1}{\lambda}, \bar{e}_2, \bar{f}_2)$. The facts that $\chi_5 \notin U$, $\chi_6 \notin U$, $a_7\chi_5 + a_8\chi_6 \in U$ and $\lambda a_7\chi_5 + \frac{a_8}{\lambda}\chi_6 = (a_7\chi_5 + a_8\chi_6)^\theta \in U$ then imply that $a_7 = a_8 = 0$. Since χ was an arbitrary vector of U, Lemma 5.2 then implies that $a_1 = a_2 = \cdots = a_8 = 0$, i.e. $U \subseteq \langle \chi_9, \chi_{10}, \ldots, \chi_{16}\rangle$.

Suppose $\mathbb{F} = \mathbb{F}_3$. The facts that $\chi_5 \notin W$, $\chi_6 \notin W$, $a_7\chi_5 + a_8\chi_6 \in W$ and $a_8\chi_5 - a_7\chi_6 = (a_7\chi_5 + a_8\chi_6)^{\theta_2} \in U$, then imply that $a_7^2 + a_8^2 = 0$, i.e. $a_7 = a_8 = 0$. Since χ was an arbitrary vector of U, Lemma 5.2 then again implies that $a_1 = a_2 = \cdots = a_8 = 0$, i.e. $U \subseteq \langle \chi_9, \chi_{10}, \ldots, \chi_{16}\rangle$. ∎

Proposition 5.5 *Let U be a subspace of W_{16} stabilized by $Sp(V, f)$. If $\{\bar{o}\} \neq U \neq W_{16}$, then* $\mathrm{char}(\mathbb{F}) = 5$ *and* $U = \langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16}\rangle$.

Proof. By Lemma 5.4, we know that $U \subseteq \langle \chi_9, \chi_{10}, \ldots, \chi_{16}\rangle$. Let $\chi = \sum_{i=9}^{16} a_i\chi_i$ be an arbitrary vector of U.

The fact that $\chi^{\theta_3(1)} - \chi \in U$ has no components in χ_5 and χ_6 imply that $2a_9 = a_{13}$ and $2a_{10} = -a_{14}$. These facts and the fact that $\chi^{\theta_1} \in U$ imply that $2a_{11} = a_{15}$ and $2a_{12} = -a_{16}$. So, we have that U is a subpace of

$$\langle \chi_9 + 2\chi_{13}, \chi_{10} - 2\chi_{14}, \chi_{11} + 2\chi_{15}, \chi_{12} - 2\chi_{16}\rangle.$$

Now,

- $(\chi_9 + 2\chi_{13})^{\theta_4} = (\chi_9 + \chi_6 + \chi_{15} + \chi_{11}) + (2\chi_{13} - 6\chi_6 - 4\chi_{15}) = (\chi_9 + 2\chi_{13}) + (\chi_{11} + 2\chi_{15}) - 5(\chi_6 + \chi_{15})$,
- $(\chi_{10} - 2\chi_{14})^{\theta_4} = (\chi_{10} + \chi_3) - 2\chi_{14} - 6\chi_3 = (\chi_{10} - 2\chi_{14}) - 5\chi_3$,
- $(\chi_{11} + 2\chi_{15})^{\theta_4} = (\chi_{11} - \chi_6) + 2\chi_{15} + 6\chi_6 = (\chi_{11} + 2\chi_{15}) + 5\chi_6$,

- $(\chi_{12}-2\chi_{16})^{\theta_4} = (\chi_{12}+\chi_3+\chi_{14}-\chi_{10}) - 2\chi_{16} - 4\chi_{14} - 6\chi_3 = (\chi_{12}-2\chi_{16}) - (\chi_{10} - 2\chi_{14}) - 5(\chi_3+\chi_{14})$.

Since $U^{\theta_4} = U \subseteq \langle \chi_9+2\chi_{13}, \chi_{10}-2\chi_{14}, \chi_{11}+2\chi_{15}, \chi_{12}-2\chi_{16}\rangle$, $\langle \chi_3, \chi_6, \chi_6+\chi_{15}, \chi_3 + \chi_{14}\rangle \cap \langle \chi_9+2\chi_{13}, \chi_{10}-2\chi_{14}, \chi_{11}+2\chi_{15}, \chi_{12}-2\chi_{16}\rangle = \{\bar{o}\}$ and the vectors $\chi_6+\chi_{15}$, χ_3, χ_6, $\chi_3+\chi_{14}$ are linearly independent, we necessarily have that $\mathrm{char}(\mathbb{F}) = 5$. In this case, we also have

$$U \subseteq \langle \chi_9+2\chi_{13}, 2\chi_{10}+\chi_{14}, \chi_{11}+2\chi_{15}, 2\chi_{12}+\chi_{16}\rangle.$$

An arbitrary vector χ can thus be written as a linear combination of the vectors $\chi_9+2\chi_{13}$, $2\chi_{10}+\chi_{14}$, $\chi_{11}+2\chi_{15}$ and $2\chi_{12}+\chi_{16}$. By considering the linear transformation θ_1, we see the following:

(a) U has a vector having a nonzero component in $\chi_9+2\chi_{13}$ if and only if U has a vector having a nonzero component in $\chi_{11}+2\chi_{15}$;

(b) U has a vector having a nonzero component in $2\chi_{10}+\chi_{14}$ if and only if U has a vector having a nonzero component in $2\chi_{12}+\chi_{16}$.

By considering the linear transformation θ_2, we see the following:

(c) U has a vector having a nonzero component in $\chi_9+2\chi_{13}$ if and only if U has a vector having a nonzero component in $2\chi_{10}+\chi_{14}$.

From (a), (b) and (c), we can then see that U contains a vector having a nonzero component in $2\chi_{12}+\chi_{16}$. Since $\chi^{\theta_3(1)} - \chi \in U$, we then see that $\chi_{11}+2\chi_{15} \in U$. Hence, also $\chi_9+2\chi_{13} = (\chi_{11}+2\chi_{15})^{\theta_1} \in U$, $2\chi_{10}+\chi_{14} = -\frac{1}{2}(\chi_9+2\chi_{13})^{\theta_2} \in U$ and $2\chi_{12}+\chi_{16} = (2\chi_{10}+\chi_{14})^{\theta_1} \in U$. It follows that $U = \langle \chi_9+2\chi_{13}, 2\chi_{10}+\chi_{14}, \chi_{11}+2\chi_{15}, 2\chi_{12}+\chi_{16}\rangle$. ■

By Propositions 5.1 and 5.5, we have

Corollary 5.6 *The $Sp(V,f)$-module W_{16} is reducible if and only if* $\mathrm{char}(\mathbb{F}) = 5$, *in which case there is a unique nontrivial submodule. This submodule has dimension* 4.

6 The embedding e^* is polarized

Consider again the 24-dimensional subspace W_{24}. The following 24 vectors determine a basis of W_{24}:

$$\bar{b}_1 := \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \quad \bar{b}_2 := \bar{f}_1 \otimes \bar{f}_1 \wedge \bar{f}_2, \quad \bar{b}_3 := \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_1, \quad \bar{b}_4 := \bar{f}_1 \otimes \bar{e}_2 \wedge \bar{f}_2,$$

$$\bar{b}_5 := \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_2, \quad \bar{b}_6 := \bar{f}_1 \otimes \bar{e}_2 \wedge \bar{f}_1, \quad \bar{b}_7 := \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{f}_1, \quad \bar{b}_8 := \bar{f}_1 \otimes \bar{e}_1 \wedge \bar{f}_2,$$

$$\bar{b}_9 := \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{f}_2, \quad \bar{b}_{10} := \bar{f}_1 \otimes \bar{e}_1 \wedge \bar{f}_1, \quad \bar{b}_{11} := \bar{e}_1 \otimes \bar{f}_1 \wedge \bar{f}_2, \quad \bar{b}_{12} := \bar{f}_1 \otimes \bar{e}_1 \wedge \bar{e}_2,$$

$$\bar{b}_{13} := \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{e}_2, \quad \bar{b}_{14} := \bar{f}_2 \otimes \bar{f}_1 \wedge \bar{f}_2, \quad \bar{b}_{15} := \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{f}_1, \quad \bar{b}_{16} := \bar{f}_2 \otimes \bar{e}_2 \wedge \bar{f}_2,$$

$$\bar{b}_{17} := \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{f}_2, \quad \bar{b}_{18} := \bar{f}_2 \otimes \bar{e}_2 \wedge \bar{f}_1, \quad \bar{b}_{19} := \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{f}_1, \quad \bar{b}_{20} := \bar{f}_2 \otimes \bar{e}_1 \wedge \bar{f}_2,$$

$\bar{b}_{21} := \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{f}_2, \qquad \bar{b}_{22} := \bar{f}_2 \otimes \bar{e}_1 \wedge \bar{f}_1, \qquad \bar{b}_{23} := \bar{e}_2 \otimes \bar{f}_1 \wedge \bar{f}_2, \qquad \bar{b}_{24} := \bar{f}_2 \otimes \bar{e}_1 \wedge \bar{e}_2.$

Consider now the nondegenerate alternating bilinear form $\widetilde{f}$ of W_{24} for which the following ordered basis is a hyperbolic basis:

$$(\bar{b}_1, -\bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5, -\bar{b}_6, \bar{b}_7, -\bar{b}_8, \bar{b}_9, \bar{b}_{10}, \bar{b}_{11}, -\bar{b}_{12}, \bar{b}_{13}, -\bar{b}_{14}, \bar{b}_{15}, \bar{b}_{16}, \bar{b}_{17}, -\bar{b}_{18}, \bar{b}_{19}, -\bar{b}_{20}, \bar{b}_{21}, \bar{b}_{22}, \bar{b}_{23}, -\bar{b}_{24}).$$

Then the following holds.

Lemma 6.1 *Let $i, j \in \{1, 2, \ldots, 24\}$. Then*

$$\bar{b}_i = \bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3 \quad \text{and} \quad \bar{b}_j = \bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3,$$

where $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3 \in \{\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3\}$. Put $a := f(\bar{v}_1, \bar{w}_1)$ and let $b \in \mathbb{F}$ such that $b \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3$. Then $\widetilde{f}(\bar{b}_i, \bar{b}_j) = a \cdot b$.

Proof. The above-mentioned hyperbolic basis has been defined in such a way for this to be true. ■

Lemma 6.2 *If $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3 \in V$. Then*

$$\widetilde{f}(\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3) = ab,$$

where $a := f(\bar{v}_1, \bar{w}_1)$ and $b \in \mathbb{F}$ such that $b \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3$.

Proof. We first show that the number ab is well-defined. Suppose $\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3 = \bar{v}_1' \otimes \bar{v}_2' \wedge \bar{v}_3'$ and $\bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3 = \bar{w}_1' \otimes \bar{w}_2' \wedge \bar{w}_3'$ for certain vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3' \in V$. We may suppose that the vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3'$ are distinct from $\bar{o}$. Then there exist unique $\alpha_1, \alpha_2 \in \mathbb{F}^*$ such that $\bar{v}_1 = \alpha_1 \bar{v}_1'$, $\bar{w}_1 = \alpha_2 \bar{w}_1'$, $\bar{v}_2 \wedge \bar{v}_3 = \frac{1}{\alpha_1} \bar{v}_2' \wedge \bar{v}_3'$ and $\bar{w}_2 \wedge \bar{w}_3 = \frac{1}{\alpha_2} \bar{w}_2' \wedge \bar{w}_3'$. Then $a' := f(\bar{v}_1', \bar{w}_1') = \frac{1}{\alpha_1 \alpha_2} f(\bar{v}_1, \bar{w}_1) = \frac{a}{\alpha_1 \alpha_2}$. Since $b \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3 = \frac{1}{\alpha_1 \alpha_2} \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_2' \wedge \bar{w}_3' = \frac{b'}{\alpha_1 \alpha_2} \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$, we have $a'b' = \frac{a}{\alpha_1 \alpha_2} \cdot \alpha_1 \alpha_2 b = ab$.

Now, let $\mathcal{A}$ denote the set of all 6-tuples $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) \in V^6$ such that $\widetilde{f}(\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3) = ab$, where $a := f(\bar{v}_1, \bar{w}_1)$ and $b \in \mathbb{F}$ such that $b \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3$. If $\bar{v}, \bar{v}' \in V$, $k, k' \in \mathbb{F}$ and γ_1, γ_2 are two (possibly empty) sequences of vectors of V whose lengths add up to five, then the facts that $\widetilde{f}$ and $\mathbb{F} \times \mathbb{F} \mapsto \mathbb{F} : (a, b) \mapsto ab$ are bilinear imply the following:

(∗) If $(\gamma_1, \bar{v}, \gamma_2)$ and $(\gamma_1, \bar{v}', \gamma_2)$ belong to $\mathcal{A}$, then also $(\gamma_1, k\bar{v} + k'\bar{v}', \gamma_2)$ belongs to $\mathcal{A}$.

Lemma 6.2 now follows from Lemma 6.1 and Property (∗). ■

Lemma 6.3 *Every $\theta \in Sp(V, f)$ leaves the form $\widetilde{f}$ invariant.*

	χ_1	χ_4	χ_2	χ_3	χ_5	χ_8	χ_6	χ_7	χ_9	χ_{10}	χ_{13}	χ_{14}	χ_{11}	χ_{12}	χ_{15}	χ_{16}
χ_1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_3	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_5	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0
χ_8	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
χ_6	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
χ_7	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0
χ_9	0	0	0	0	0	0	0	0	0	2	0	1	0	0	0	0
χ_{10}	0	0	0	0	0	0	0	0	-2	0	1	0	0	0	0	0
χ_{13}	0	0	0	0	0	0	0	0	0	-1	0	-3	0	0	0	0
χ_{14}	0	0	0	0	0	0	0	0	-1	0	3	0	0	0	0	0
χ_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	1
χ_{12}	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	1	0
χ_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-3
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	3	0

Table 3: The values $\bar{f}(\chi_i, \chi_j)$, $i, j \in \{1, 2, \ldots, 16\}$.

Proof. Since $\widetilde{f}$ is bilinear, it suffices to prove that

$$\widetilde{f}((\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3)^\theta, (\bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3)^\theta) = \widetilde{f}(\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3),$$

i.e., that

$$\widetilde{f}(\bar{v}_1^\theta \otimes \bar{v}_2^\theta \wedge \bar{v}_3^\theta, \bar{w}_1^\theta \otimes \bar{w}_2^\theta \wedge \bar{w}_3^\theta) = \widetilde{f}(\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3).$$

Put $a := f(\bar{v}_1, \bar{w}_1)$ and let $b \in \mathbb{F}$ such that $\bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3 = b \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$. Then $f(\bar{v}_1^\theta, \bar{w}_1^\theta) = f(\bar{v}_1, \bar{w}_1) = a$ and $\bar{v}_2^\theta \wedge \bar{v}_3^\theta \wedge \bar{w}_2^\theta \wedge \bar{w}_3^\theta = \det(\theta) \cdot \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3 = \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3 = b \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$. Indeed, every $\theta \in Sp(V, f)$ has determinant 1. It follows that

$$\widetilde{f}(\bar{v}_1^\theta \otimes \bar{v}_2^\theta \wedge \bar{v}_3^\theta, \bar{w}_1^\theta \otimes \bar{w}_2^\theta \wedge \bar{w}_3^\theta) = ab = \widetilde{f}(\bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \otimes \bar{w}_2 \wedge \bar{w}_3).$$

$\blacksquare$

Definition. Let $\bar{f}$ denote the restriction of f to $W_{16} \times W_{16}$.

From

$$\chi_1 = \bar{b}_1, \quad \chi_2 = \bar{b}_5, \quad \chi_3 = -\bar{b}_6, \quad \chi_4 = \bar{b}_2, \quad \chi_5 = -\bar{b}_{13}, \quad \chi_6 = \bar{b}_{19}, \quad \chi_7 = -\bar{b}_{20},$$

$$\chi_8 = -\bar{b}_{14}, \quad \chi_9 = -\bar{b}_{17} - \bar{b}_{24}, \quad \chi_{10} = -\bar{b}_{23} + \bar{b}_{18}, \quad \chi_{11} = -\bar{b}_7 + \bar{b}_{12}, \quad \chi_{12} = \bar{b}_{11} + \bar{b}_8,$$

$$\chi_{13} = \bar{b}_9 - \bar{b}_3 + \bar{b}_{17}, \quad \chi_{14} = \bar{b}_4 - \bar{b}_{10} + \bar{b}_{18}, \quad \chi_{15} = \bar{b}_{15} - \bar{b}_{21} + \bar{b}_7, \quad \chi_{16} = \bar{b}_{22} - \bar{b}_{16} + \bar{b}_8,$$

all the values $\bar{f}(\chi_i, \chi_j)$, $i, j \in \{1, 2, \ldots, 16\}$, can easily be computed. They have been listed in Table 3. From this table, we easily deduce the following:

Proposition 6.4 *The alternating bilinear form $\bar{f}$ of W_{16} is nondegenerate if and only if* $\mathrm{char}(\mathbb{F}) \neq 5$. *If* $\mathrm{char}(\mathbb{F}) = 5$, *then*

$$Rad(\bar{f}) = \langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle.$$

Proof. From Table 3, we see that $\bar{f}$ is nondegenerate if and only if $\det(M) \neq 0$, where

$$M = \begin{bmatrix} 0 & 2 & 0 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -3 \\ -1 & 0 & 3 & 0 \end{bmatrix},$$

i.e., if and only if $\mathrm{char}(\mathbb{F}) \neq 5$. If $\mathrm{char}(\mathbb{F}) = 5$, then M has rank 2, implying that $Rad(\bar{f})$ is 4-dimensional. A straightforward calculation shows that

$$Rad(\bar{f}) = \langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle.$$

■

Let ζ denote the possibly degenerate symplectic polarity of $\mathrm{PG}(W_{16})$ induced by $\bar{f}$.

Proposition 6.5 *For every point x of $\mathcal{F}(W(\mathbb{F}))$, $e^*(x)^\zeta$ is a hyperplane of* $\mathrm{PG}(W_{16})$ *containing all points $e^*(y)$, where y is a point at distance at most* 3 *from x, and none of the points $e^*(z)$, where z is a point of $\mathcal{F}(W(\mathbb{F}))$ opposite to x.*

Proof. Choose vectors $\bar{v}_1, \bar{v}_2, \bar{v}'_1, \bar{v}'_2, \bar{v}''_1, \bar{v}''_2 \in V$ such that

$$x = \{\langle \bar{v}_1 \rangle, \langle \bar{v}_1, \bar{v}_2 \rangle\}, \quad y = \{\langle \bar{v}'_1 \rangle, \langle \bar{v}'_1, \bar{v}'_2 \rangle\}, \quad z = \{\langle \bar{v}''_1 \rangle, \langle \bar{v}''_1, \bar{v}''_2 \rangle\}.$$

If $\mathrm{d}(x, y) \leq 3$, then $f(\bar{v}_1, \bar{v}'_1) = 0$ or $\langle \bar{v}_1, \bar{v}_2 \rangle \cap \langle \bar{v}'_1, \bar{v}'_2 \rangle \neq \{\bar{o}\}$. In any case, we have $\bar{f}(\bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2, \bar{v}'_1 \otimes \bar{v}'_1 \wedge \bar{v}'_2) = 0$ by Lemma 6.2.

If $\mathrm{d}(x, z) = 4$, then $f(\bar{v}_1, \bar{v}''_1) \neq 0$ and $\langle \bar{v}_1, \bar{v}_2 \rangle \cap \langle \bar{v}''_1, \bar{v}''_2 \rangle = \{\bar{o}\}$, i.e. $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}''_1 \wedge \bar{v}''_2 \neq 0$. In this case, we have $\bar{f}(\bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2, \bar{v}''_1 \otimes \bar{v}''_1 \wedge \bar{v}''_2) \neq 0$ by Lemma 6.2.

The claims of the proposition follow. ■

The following is an immediate corollary of Proposition 6.5.

Corollary 6.6 *The projective embedding e^* is polarized.*

If $\mathrm{char}(\mathbb{F}) = 5$, then we denote by $\bar{e}$ the embedding e^*/α, where α is the subspace of $\mathrm{PG}(W_{16})$ corresponding to $Rad(\bar{f}) = \langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle$. Note that α is indeed disjoint from the image of e^*. Indeed, for every point x there exists a point y of $\mathcal{F}(W(\mathbb{F}))$ opposite to x, and for each such point y the hyperplane $e^*(y)^\zeta$ contains α but not $e^*(x)$, implying that $e^*(x)$ cannot be contained in α.

Proposition 6.7 *If* $\mathrm{char}(\mathbb{F}) = 5$, *then the embedding $\bar{e}$ is polarized.*

Proof. By Proposition 6.5, we know that for every point x of $\mathcal{F}(W(\mathbb{F}))$, $H_x = e^{*-1}(e^*(\mathcal{P}_\mathbb{F}) \cap \Pi_x)$, where Π_x is the hyperplane $e^*(x)^\zeta$ of $\mathrm{PG}(W_{16})$. The proposition now follows from the fact that $\alpha \subseteq \Pi_x$. ■

7 Further properties of the embedding e^*

Let us define the following subspaces of W_{16}:

$$Z_0 := \langle \chi_1 \rangle, \qquad Z_1 := \langle \chi_1, \chi_2, \chi_5 \rangle, \qquad Z_4 := W_{16}.$$

If $|\mathbb{F}| \geq 3$, then we define

$$Z_2 := \langle \chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_9, \chi_{13}, \chi_{15} \rangle,$$

$$Z_3 := \langle \{\chi_1, \chi_2, \ldots, \chi_{16}\} \setminus \{\chi_4\} \rangle.$$

If $|\mathbb{F}| = 2$, then we define

$$Z_2 := \langle \chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_9, \chi_{13} + \chi_{15} \rangle,$$

$$Z_3 := \langle \{\chi_1, \chi_2, \ldots, \chi_{16}, \chi_{14} + \chi_{16}, \chi_{12} + \chi_{16}\} \setminus \{\chi_4, \chi_{12}, \chi_{14}, \chi_{16}\} \rangle.$$

Proposition 7.1 *If x is the point $\{\langle \bar{e}_1 \rangle, \langle \bar{e}_1, \bar{e}_2 \rangle\}$ of $\mathcal{F}(W(\mathbb{F}))$ and $i \in \{0, 1, 2, 3, 4\}$, then $\langle e^*(\Gamma_{\leq i}(x)) \rangle = \mathrm{PG}(Z_i)$.*

Proof. We identify the subspaces $\langle e^*(\Gamma_{\leq i}(x)) \rangle$ here with their corresponding subspaces of W_{16}. Obviously, $\langle e^*(\Gamma_{\leq 0}(x)) \rangle = \langle e^*(x) \rangle = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle = \langle \chi_1 \rangle = \mathrm{PG}(Z_0)$. There are two lines through the point $x = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_1, \bar{e}_2 \rangle\}$. The line $\langle \bar{e}_1 \rangle$ contains the points $x = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_1, \bar{e}_2 \rangle\}$ and $x_1 = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_1, \bar{f}_2 \rangle\}$ and the line $\langle \bar{e}_1, \bar{e}_2 \rangle$ contains the two points $x = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_1, \bar{e}_2 \rangle\}$ and $x_2 = \{\langle \bar{e}_2 \rangle, \langle \bar{e}_1, \bar{e}_2 \rangle\}$. This implies that $\langle e^*(\Gamma_{\leq 1}(x)) \rangle = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \otimes \bar{e}_1 \otimes \bar{f}_2, \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle = \langle \chi_1, \chi_2, \chi_5 \rangle = \mathrm{PG}(Z_1)$.

We now determine a generating set of $\langle e^*(\Gamma_{\leq 2}(x)) \rangle$. Points at distance 2 from x are of one of the following two types:

(i) $\{\langle \alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \bar{f}_2 \rangle, \langle \alpha_2 \bar{e}_2 + \bar{f}_2, \bar{e}_1 \rangle\}$ with $\alpha_1, \alpha_2 \in \mathbb{F}$;

(ii) $\{\langle \alpha \bar{e}_1 + \bar{e}_2 \rangle, \langle \alpha \bar{e}_1 + \bar{e}_2, \beta \bar{e}_1 + \bar{f}_1 - \alpha \bar{f}_2 \rangle\}$ with $\alpha, \beta \in \mathbb{F}$.

We determine the contribution of both parts to $\langle e^*(\Gamma_{\leq 2}(x)) \rangle$. We compute that

$$(\alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \bar{f}_2) \otimes (\alpha_2 \bar{e}_2 + \bar{f}_2) \wedge \bar{e}_1$$

$$= \alpha_1 \alpha_2 \cdot \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{e}_1 + \alpha_1 \cdot \bar{e}_1 \otimes \bar{f}_2 \wedge \bar{e}_1 + \alpha_2^2 \cdot \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{e}_1 + \alpha_2 \cdot \bar{e}_2 \otimes \bar{f}_2 \wedge \bar{e}_1 + \alpha_2 \cdot \bar{f}_2 \otimes \bar{e}_2 \wedge \bar{e}_1 + \bar{f}_2 \otimes \bar{f}_2 \wedge \bar{e}_1$$

$$= -\alpha_1 \alpha_2 \chi_1 - \alpha_1 \chi_2 + \alpha_2^2 \chi_5 + \chi_7 + \alpha_2 \chi_9.$$

Besides χ_1, χ_2 and χ_5 which are already present in $\langle e^*(\Gamma_{\leq 1}(x)) \rangle$, we also add the vectors χ_7 and χ_9 to the generating set for $\langle e^*(\Gamma_{\leq 2}(x)) \rangle$.

We compute $(\alpha \bar{e}_1 + \bar{e}_2) \otimes (\alpha \bar{e}_1 + \bar{e}_2) \wedge (\beta \bar{e}_1 + \bar{f}_1 - \alpha \bar{f}_2)$. The part $(\alpha \bar{e}_1 + \bar{e}_2) \otimes (\alpha \bar{e}_1 + \bar{e}_2) \wedge \bar{e}_1 = -\alpha \chi_1 + \chi_5$ contributes no extra vectors to the generating set for $\langle e^*(\Gamma_{\leq 2}(x)) \rangle$. We therefore compute the part $(\alpha \bar{e}_1 + \bar{e}_2) \otimes (\alpha \bar{e}_1 + \bar{e}_2) \wedge (\bar{f}_1 - \alpha \bar{f}_2)$. This is equal to

$$\alpha^2 \cdot \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_1 + \alpha \cdot \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{f}_1 - \alpha^3 \cdot \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_2 - \alpha^2 \cdot \bar{e}_1 \otimes \bar{e}_2 \wedge \bar{f}_2$$

$$+\alpha\cdot\bar e_2\otimes\bar e_1\wedge\bar f_1+\bar e_2\otimes\bar e_2\wedge\bar f_1-\alpha^2\cdot\bar e_2\otimes\bar e_1\wedge\bar f_2-\alpha\cdot\bar e_2\otimes\bar e_2\wedge\bar f_2$$
$$=-\alpha^3\cdot\bar e_1\otimes\bar e_1\wedge\bar f_2+\bar e_2\otimes\bar e_2\wedge\bar f_1+\alpha\cdot(\bar e_2\otimes\bar e_1\wedge\bar f_1-\bar e_2\otimes\bar e_2\wedge\bar f_2$$
$$+\bar e_1\otimes\bar e_2\wedge\bar f_1)-\alpha^2(\bar e_1\otimes\bar e_2\wedge\bar f_2-\bar e_1\otimes\bar e_1\wedge\bar f_1+\bar e_2\otimes\bar e_1\wedge\bar f_2)$$
$$=-\alpha^3\cdot\chi_2+\chi_6+\alpha\cdot\chi_{15}-\alpha^2\cdot\chi_{13}.$$

So, we conclude that:

- if $|\mathbb{F}|\geq 3$, then $\langle e^*(\Gamma_{\leq 2}(x))\rangle=\langle\chi_1,\chi_2,\chi_5,\chi_6,\chi_7,\chi_9,\chi_{13},\chi_{15}\rangle$;
- if $|\mathbb{F}|=2$, then $\langle e^*(\Gamma_{\leq 2}(x))\rangle=\langle\chi_1,\chi_2,\chi_5,\chi_6,\chi_7,\chi_9,\chi_{13}+\chi_{15}\rangle$.

We now determine a generating set of $\langle e^*(\Gamma_{\leq 3}(x))\rangle$ by enlarging and modifying the generating set for $\langle e^*(\Gamma_{\leq 2}(x))\rangle$ that we just determined. Points at distance 3 from x are of one of the following types:

(i) $\{\langle\bar f_2+\alpha\bar e_1+\beta\bar e_2\rangle,\langle\bar f_2+\alpha\bar e_1+\beta\bar e_2,\bar f_1+\gamma\bar e_1+\alpha\bar e_2\rangle\}$ for some $\alpha,\beta,\gamma\in\mathbb{F}$;

(ii) $\{\langle\bar f_1+\alpha\bar e_1+\beta\bar e_2+\gamma\bar f_2\rangle,\langle\bar f_1+\alpha\bar e_1+\beta\bar e_2+\gamma\bar f_2,\gamma\bar e_1-\bar e_2\rangle\}$ for some $\alpha,\beta,\gamma\in\mathbb{F}$.

We first consider $(\bar f_2+\alpha\bar e_1+\beta\bar e_2)\otimes(\bar f_2+\alpha\bar e_1+\beta\bar e_2)\wedge(\bar f_1+\gamma\bar e_1+\alpha\bar e_2)$, where $\alpha,\beta,\gamma\in\mathbb{F}$. By the above, we know that the part $(\bar f_2+\alpha\bar e_1+\beta\bar e_2)\otimes(\bar f_2+\alpha\bar e_1+\beta\bar e_2)\wedge\bar e_1=\bar f_2\otimes\bar f_2\wedge\bar e_1+\beta\cdot\bar f_2\otimes\bar e_2\wedge\bar e_1+\alpha\cdot\bar e_1\otimes\bar f_2\wedge\bar e_1+\alpha\beta\cdot\bar e_1\otimes\bar e_2\wedge\bar e_1+\beta\cdot\bar e_2\otimes\bar f_2\wedge\bar e_1+\beta^2\cdot\bar e_2\otimes\bar e_2\wedge\bar e_1=\chi_7+\beta\chi_9-\alpha\chi_2-\alpha\beta\chi_1+\beta^2\chi_5$ is contained in $\langle e^*(\Gamma_{\leq 2}(x))\rangle$. The part $(\bar f_2+\alpha\bar e_1+\beta\bar e_2)\otimes(\bar f_2+\alpha\bar e_1+\beta\bar e_2)\wedge(\bar f_1+\alpha\bar e_2)$ is equal to

$$\chi_8+\alpha\cdot(\chi_{16}-\chi_{12})+\beta\cdot\chi_{10}-\alpha^2\cdot(\chi_9+\chi_{13})+\beta^2\cdot\chi_6+\alpha\beta\cdot\chi_{15}+\alpha^3\cdot\chi_1-\alpha^2\beta\cdot\chi_5.$$

We already know that χ_1, χ_5, χ_6 and χ_9 belong to $\langle e^*(\Gamma_{\leq 2}(x))\rangle$. So, we know that for all $\alpha,\beta\in\mathbb{F}$, the vector

$$\chi_8+(\chi_{16}-\chi_{12})\alpha+\chi_{10}\beta-\chi_{13}\alpha^2+\alpha\beta\chi_{15}$$

belongs to $\langle e^*(\Gamma_{\leq 3}(x))\rangle$. This implies that χ_8,χ_{10} belong to $\langle e^*(\Gamma_{\leq 3}(x))\rangle$ and hence also the vector $(\chi_{16}-\chi_{12})-\alpha\chi_{13}+\beta\chi_{15}$ for all $\alpha,\beta\in\mathbb{F}$ with $\alpha\neq 0$. This implies that

- χ_{15} and hence also $\chi_{13}=(\chi_{13}+\chi_{15})-\chi_{15}$ belong to $\langle e^*(\Gamma_{\leq 3}(x))\rangle$;
- $\chi_{16}-\chi_{12}$ belongs to $\langle e^*(\Gamma_{\leq 3}(x))\rangle$.

The expression $(\bar f_1+\alpha\bar e_1+\beta\bar e_2+\gamma\bar f_2)\otimes(\bar f_1+\alpha\bar e_1+\beta\bar e_2+\gamma\bar f_2)\wedge(\gamma\bar e_1-\bar e_2)$ is equal to

$$-\chi_3-\alpha\cdot\chi_{11}+\beta\cdot\chi_6+\gamma\cdot\chi_{14}-\alpha^2\cdot\chi_1-\gamma^2\cdot\chi_{16}-\alpha\beta\cdot\chi_1+\alpha\gamma\cdot(\chi_9+\chi_{13})-\beta\gamma\cdot(\chi_{11}+\chi_{15})+\beta^2\gamma\cdot\chi_5$$
$$+\gamma^3\cdot\chi_7-\alpha\beta\gamma\cdot\chi_1-\alpha\gamma^2\cdot\chi_2+\beta\gamma^2\cdot\chi_9.$$

We already know that the vectors $\chi_1, \chi_2, \chi_5, \chi_6, \chi_7, \chi_9, \chi_{13}, \chi_{15}$ are contained in $\langle e^*(\Gamma_{\leq 3}(x))\rangle$. So, we know that for all α, β, γ, the vector

$$-\chi_3 - \alpha\chi_{11} + \gamma\chi_{14} - \gamma^2\chi_{16} - \beta\gamma\chi_{11}$$

is also contained in $\langle e^*(\Gamma_{\leq 3}(x))\rangle$. Putting $\gamma = 0$, we see that all vectors of the form $-\chi_3 - \alpha\chi_{11}$ are contained in $\langle e^*(\Gamma_{\leq 3}(x))\rangle$, i.e. χ_3 and χ_{11} are contained in $\langle e^*(\Gamma_{\leq 3}(x))\rangle$. This implies that for every $\gamma \in \mathbb{F}^*$, the vector $\chi_{14} - \gamma\chi_{16}$ is contained in $\langle e^*(\Gamma_{\leq 3}(x))\rangle$. If $|\mathbb{F}| \neq 2$, then also the vectors χ_{14} and χ_{16} are contained in $\langle e^*(\Gamma_{\leq 3}(x))\rangle$ and hence also the vector $\chi_{12} = \chi_{16} - (\chi_{16} - \chi_{12})$. If $|\mathbb{F}| = 2$, then we can only conclude that the vector $\chi_{14} + \chi_{16}$ is contained in $\langle e(\Gamma_{\leq 3}(x))\rangle$. The claims of the proposition should now be obvious. ∎

Proposition 7.1 has the following corollary.

Corollary 7.2 (1) *If $|\mathbb{F}| \geq 3$, then for every point x of $\mathcal{F}(W(\mathbb{F}))$, the subspace of* $\mathrm{PG}(W_{16})$ *generated by $e(H_x)$ is a hyperplane.*

(2) *If $|\mathbb{F}| = 2$, then for every point x of $\mathcal{F}(W(\mathbb{F}))$, the subspace of* $\mathrm{PG}(W_{16})$ *generated by $e(H_x)$ is a subspace of co-dimension* 2.

Proof. Since W_{16} is an $Sp(V,f)$-module, we can take the point x to be equal to $\{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}$. The claims then follow from Proposition 7.1. ∎

Note that Corollary 7.2(2) was also proved in Proposition 3.8(2). By Proposition 3.8(1), Proposition 6.5 and Corollary 7.2(1), we have:

Proposition 7.3 *For every point x of $\mathcal{F}(W(\mathbb{F}))$, there exists a unique hyperplane of* $\mathrm{PG}(W_{16})$ *containing all points $e^*(y)$, where y is a point at distance at most* 3 *from x, and none of the points $e^*(z)$, where z is a point of $\mathcal{F}(W(\mathbb{F}))$ opposite to x.*

We finish this section by proving the following result.

Proposition 7.4 *The points and lines contained in the image of e^* define a point-line geometry isomorphic to $\mathcal{F}(W(\mathbb{F}))$.*

Proof. Suppose x and y are two noncollinear points of $\mathcal{F}(W(\mathbb{F}))$. It then suffices to prove that the unique line of $\mathrm{PG}(W_{16})$ through $e^*(x)$ and $e^*(y)$ intersects the image of e^* in precisely two points. We may suppose that the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ has been chosen in such a way that one of the following cases occurs:

(1) $x = \{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}$, $y = \{\langle \bar{e}_2\rangle, \langle \bar{e}_2, \bar{f}_1\rangle\}$,

(2) $x = \{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}$, $y = \{\langle \bar{f}_2\rangle, \langle \bar{f}_1, \bar{f}_2\rangle\}$,

(3) $x = \{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}$, $y = \{\langle \bar{f}_1\rangle, \langle \bar{f}_1, \bar{e}_2\rangle\}$,

(4) $x = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_1, \bar{e}_2 \rangle\}$, $y = \{\langle \bar{f}_1 \rangle, \langle \bar{f}_1, \bar{f}_2 \rangle\}$.

From $\lambda \in \mathbb{F}^*$, it is obvious that none of the points

$$\langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 + \lambda \cdot \bar{e}_2 \otimes \bar{e}_2 \wedge \bar{f}_1 \rangle, \quad \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 + \lambda \cdot \bar{f}_2 \otimes \bar{f}_2 \wedge \bar{f}_1 \rangle,$$

$$\langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 + \lambda \cdot \bar{f}_1 \otimes \bar{f}_1 \wedge \bar{e}_2 \rangle, \quad \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 + \lambda \cdot \bar{f}_1 \otimes \bar{f}_1 \wedge \bar{f}_2 \rangle$$

belong to the image of e^*. So, the unique line of $\mathrm{PG}(W_{16})$ through $e^*(x)$ and $e^*(y)$ intersects the image of e^* in precisely two points. ■

8 The homogeneity of the embedding e^*

Theorem 8.1 *Let A denote the group of automorphisms of $\mathcal{F}(W(\mathbb{F}))$ preserving the line types. Then e^* is A-homogeneous.*

Proof. Let θ be an element of $Sp(V, f)$, an element of the form θ_h^* with $h \in \mathbb{F}^*$ or an element of the form θ_α^* with $\alpha \in Aut(\mathbb{F})$ (as defined in Section 2). Then θ can be regarded as an element of $\Gamma L(W_{16})$ such that $(\bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2)^\theta = \bar{v}_1^\theta \otimes \bar{v}_1^\theta \wedge \bar{v}_2^\theta$ holds for every totally isotropic 2-space $\langle \bar{v}_1, \bar{v}_2 \rangle$ of (V, f). The map $\{\langle \bar{v}_1 \rangle, \langle \bar{v}_1, \bar{v}_2 \rangle\} \mapsto \{\langle \bar{v}_1^\theta \rangle, \langle \bar{v}_1^\theta, \bar{v}_2^\theta \rangle\}$ for totally isotropic 2-spaces $\langle \bar{v}_1, \bar{v}_2 \rangle$ also induces an automorphism of $\mathcal{F}(W(\mathbb{F}))$. Theorem 8.1 then follows from Proposition 2.4. ■

Proposition 2.6 and Theorem 8.1 has the following corollary.

Corollary 8.2 *If $\mathbb{F}$ is not a perfect field of characteristic 2, then e^* is a homogeneous embedding.*

We also have:

Theorem 8.3 *If $\mathrm{char}(\mathbb{F}) = 5$, then $\bar{e}$ (as defined in Section 6) is a homogeneous embedding.*

Proof. The embedding $\bar{e}$ is obtained by taking the quotient of e^* by the subspace of $\mathrm{PG}(W_{16})$ determined by

$$\langle \chi_9 + 2\chi_{13}, 2\chi_{10} + \chi_{14}, \chi_{11} + 2\chi_{15}, 2\chi_{12} + \chi_{16} \rangle.$$

This subspace is stabilized by the induced actions of the elements of $Sp(V, f)$, the elements θ_h^* with $h \in \mathbb{F}^*$ and the elements θ_α^* with $\alpha \in Aut(\mathbb{F})$, showing that $\bar{e}$ is also homogeneous. ■

We already know that e^* is absolutely universal (see Theorem 4.4) and hence homogeneous if $|\mathbb{F}| = 2$. We are now going to show this directly by using our explicit description of e^*. We start from the following element θ^* of $GL(W_{16})$:

$$\chi_1 \mapsto \chi_1,\ \chi_2 \mapsto \chi_5,\ \chi_3 \mapsto \chi_8,\ \chi_4 \mapsto \chi_4,\ \chi_5 \mapsto \chi_2,\ \chi_6 \mapsto \chi_7,\ \chi_7 \mapsto \chi_6,\ \chi_8 \mapsto \chi_3,$$

$\chi_9 \mapsto \chi_{13}+\chi_{15}$, $\chi_{10} \mapsto \chi_{14}+\chi_{16}$, $\chi_{11} \mapsto \chi_9+\chi_{12}+\chi_{13}+\chi_{16}$, $\chi_{12} \mapsto \chi_{10}+\chi_{11}+\chi_{14}+\chi_{15}$,

$$\chi_{13} \mapsto \chi_{13},\ \chi_{14} \mapsto \chi_{14},\ \chi_{15} \mapsto \chi_9+\chi_{13},\ \chi_{16} \mapsto \chi_{10}+\chi_{14}.$$

It is straightforward to verify that $\theta^* \circ \theta^* = 1$, i.e. θ^* is an involution of $GL(W_{16})$. We show the following:

Lemma 8.4 *Suppose $|\mathbb{F}| = 2$. For every $\theta \in Sp(V,f)$, there then exists a $\theta' \in Sp(V,f)$ such that $\theta^*\widetilde{\theta}\theta^* = \widetilde{\theta'}$. Here, $\widetilde{\theta}$ and $\widetilde{\theta'}$ are the induced actions of θ and θ' on W_{16}.*

Proof. With the notation of Proposition 2.3, we have that $Sp(V,f) = \langle \theta_1, \theta_2, \theta_3(1), \theta_4 \rangle$. So, it suffices to prove the proposition in the case $\theta \in \{\theta_1, \theta_2, \theta_3(1), \theta_4\}$. The verification is straightforward if one uses the θ' mentioned in the following table:

$\theta \in Sp(V,f)$	$\theta' \in Sp(V,f)$
θ_1	$(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2) \mapsto (\bar{e}_1, \bar{f}_1, \bar{f}_2, \bar{e}_2)$
θ_2	$(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2) \mapsto (\bar{f}_2, \bar{e}_2, \bar{f}_1, \bar{e}_1)$
$\theta_3(1)$	$(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2) \mapsto (\bar{e}_1, \bar{f}_1+\bar{f}_2, \bar{e}_1+\bar{e}_2, \bar{f}_2)$
θ_4	$(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2) \mapsto (\bar{e}_1, \bar{f}_1, \bar{e}_2+\bar{f}_2, \bar{f}_2)$

■

Proposition 8.5 *Suppose $|\mathbb{F}| = 2$. Then $\theta^*(Im(e^*)) = Im(e^*)$.*

Proof. Since θ^* is an involution, it suffices to show that $\theta^*(Im(e^*)) \subseteq Im(e^*)$. Let $\langle \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 \rangle$ be an arbitrary point of $Im(e^*)$, where $\langle \bar{v}_1, \bar{v}_2 \rangle$ is a totally isotropic subspace of (V,f). We must show that $\langle \bar{v}_1 \otimes \bar{v}_2 \wedge \bar{v}_3 \rangle^{\theta^*} \in Im(e^*)$. Now, there exists a $\theta \in Sp(V,f)$ such that $\bar{v}_1 = \bar{e}_1^\theta$ and $\bar{v}_2 = \bar{e}_2^\theta$. By Lemma 8.4, $\theta^*\widetilde{\theta}\theta^* = \widetilde{\theta'}$ for some $\theta' \in Sp(V,f)$. Now, $Im(e^*)$ contains the point

$$\langle \bar{e}_1^{\theta'} \otimes \bar{e}_1^{\theta'} \wedge \bar{e}_2^{\theta'} \rangle = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle^{\widetilde{\theta'}} = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle^{\theta^*\widetilde{\theta}\theta^*} = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle^{\widetilde{\theta}\theta^*} = \langle \bar{v}_1 \otimes \bar{v}_1 \wedge \bar{v}_2 \rangle^{\theta^*},$$

which is precisely what we needed to prove. ■

Theorem 8.6 *If $|\mathbb{F}| = 2$, then the embedding e^* is homogeneous.*

Proof. Since $\theta^*(Im(e^*)) = Im(e^*)$, Proposition 7.4 implies that $\theta^* \in GL(W_{16})$ is the lifting of an automorphism of $\mathcal{F}(W(2))$. By Proposition 2.6 and Theorem 8.1, it suffices to prove that this automorphism corresponds to a duality of $W(2)$.

Consider the line $\langle \bar{e}_1 \rangle$ of $\mathcal{F}(W(2))$. The image of this line under e^* is equal to $\langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_2 \rangle$. Now, $\langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{f}_2 \rangle^{\theta^*} = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2, \bar{e}_2 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle$, which is the image under e^* of the line $\langle \bar{e}_1, \bar{e}_2 \rangle$ of $\mathcal{F}(W(2))$. Since θ^* interchanges the line types, it must be associated with a duality of $W(2)$. ■

Theorem 8.7 *Suppose $\mathbb{F}$ is a perfect field of characteristic 2. Then e^* is homogeneous if and only if $|\mathbb{F}| = 2$.*

Proof. In view of Theorem 8.6, it suffices to prove that $|\mathbb{F}| = 2$ if e^* is homogeneous.

Since $\mathbb{F}$ is a perfect field of characteristic 2, there exists an automorphism of $\mathcal{F}(W(\mathbb{F}))$ arising from a duality of $W(\mathbb{F})$. Since the automorphism group of $\mathcal{F}(W(\mathbb{F}))$ preserving the line types acts transitively on the set of opposite points of $\mathcal{F}(W(\mathbb{F}))$, there is an automorphism of $\mathcal{F}(W(\mathbb{F}))$ arising from a duality D of $W(\mathbb{F})$ that fixes the points $x_1 = \{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}$ and $x_2 = \{\langle \bar{f}_1\rangle, \langle \bar{f}_1, \bar{f}_2\rangle\}$. Since e^* is homogeneous, the automorphism $\overline{D}$ of $\mathcal{F}(W(\mathbb{F}))$ lifts to an automorphism Δ of $\mathrm{PG}(W_{16})$. Since field automorphisms induce automorphisms of $\mathcal{F}(W(\mathbb{F}))$ fixing x_1, x_2 and preserving the line types, we may without loss of generality suppose that $\Delta \in PGL(W_{16})$.

Since $x_1^{\overline{D}} = x_1$ and $x_2^{\overline{D}} = x_2$, we are able to determine the action of $\overline{D}$ on additional points of $\mathcal{F}(W(\mathbb{F}))$. Since $x_1^{\overline{D}} = x_1$, the map $\overline{D}$ swaps the lines $\langle \bar{e}_1\rangle$ and $\langle \bar{e}_1, \bar{e}_2\rangle$ and hence should swap the unique points of $\langle \bar{e}_1\rangle$ and $\langle \bar{e}_1, \bar{e}_2\rangle$ at distance 3 from x_2. It follows that

$$\{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{f}_2\rangle\}^{\overline{D}} = \{\langle \bar{e}_2\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}, \tag{1}$$
$$\{\langle \bar{e}_2\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}^{\overline{D}} = \{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{f}_2\rangle\}. \tag{2}$$

Similarly, since $x_2^{\overline{D}} = x_2$, the map $\overline{D}$ swaps the lines $\langle \bar{f}_1\rangle$ and $\langle \bar{f}_1, \bar{f}_2\rangle$, and hence also the unique points on these lines at distance 3 from x_1. It follows that

$$\{\langle \bar{f}_1\rangle, \langle \bar{f}_1, \bar{e}_2\rangle\}^{\overline{D}} = \{\langle \bar{f}_2\rangle, \langle \bar{f}_1, \bar{f}_2\rangle\}, \tag{3}$$
$$\{\langle \bar{f}_2\rangle, \langle \bar{f}_1, \bar{f}_2\rangle\}^{\overline{D}} = \{\langle \bar{f}_1\rangle, \langle \bar{f}_1, \bar{e}_2\rangle\}. \tag{4}$$

Now, put $x_3 = \{\langle \bar{e}_2\rangle, \langle \bar{e}_1, \bar{e}_2\rangle\}$ and $x_4 = \{\langle \bar{f}_2\rangle, \langle \bar{f}_1, \bar{f}_2\rangle\}$. Then x_3 and x_4 are two opposite points, as well as the two points $x_3^{\overline{D}} = \{\langle \bar{e}_1\rangle, \langle \bar{e}_1, \bar{f}_2\rangle\}$ and $x_4^{\overline{D}} = \{\langle \bar{f}_1\rangle, \langle \bar{f}_1, \bar{e}_2\rangle\}$. The line $\langle \bar{e}_2\rangle$ through x_3 must be mapped by $\overline{D}$ to the line $\langle \bar{e}_1, \bar{f}_2\rangle$ through $x_3^{\overline{D}}$. It follows that the unique point of $\langle \bar{e}_2\rangle$ at distance 3 from x_4 must be mapped by $\overline{D}$ to the unique point of $\langle \bar{e}_1, \bar{f}_2\rangle$ at distance 3 from $x_4^{\overline{D}}$. Hence,

$$\{\langle \bar{e}_2\rangle, \langle \bar{e}_2, \bar{f}_1\rangle\}^{\overline{D}} = \{\langle \bar{f}_2\rangle, \langle \bar{e}_1, \bar{f}_2\rangle\}. \tag{5}$$

Similarly, the line $\langle \bar{f}_2\rangle$ through x_4 must be mapped by $\overline{D}$ to the line $\langle \bar{f}_1, \bar{e}_2\rangle$ through $x_4^{\overline{D}}$. It follows that the unique point of $\langle \bar{f}_2\rangle$ at distance 3 from x_3 must be mapped by $\overline{D}$ to the unique point of $\langle \bar{f}_1, \bar{e}_2\rangle$ at distance 3 from $x_3^{\overline{D}}$. Hence,

$$\{\langle \bar{f}_2\rangle, \langle \bar{f}_2, \bar{e}_1\rangle\}^{\overline{D}} = \{\langle \bar{e}_2\rangle, \langle \bar{f}_1, \bar{e}_2\rangle\}. \tag{6}$$

Equations (1)–(6), and the facts that $x_1^{\overline{D}} = x_1$, $x_2^{\overline{D}} = x_2$ then imply that

$$\langle \chi_1\rangle^\Delta = \langle \chi_1\rangle, \quad \langle \chi_4\rangle^\Delta = \langle \chi_4\rangle, \quad \langle \chi_2\rangle^\Delta = \langle \chi_5\rangle, \quad \langle \chi_5\rangle^\Delta = \langle \chi_2\rangle,$$
$$\langle \chi_3\rangle^\Delta = \langle \chi_8\rangle, \quad \langle \chi_8\rangle^\Delta = \langle \chi_3\rangle, \quad \langle \chi_6\rangle^\Delta = \langle \chi_7\rangle, \quad \langle \chi_7\rangle^\Delta = \langle \chi_6\rangle.$$

Suppose θ^* is the element of $GL(W_{16})$ inducing Δ. Let $k_1, k_2, \ldots, k_8 \in \mathbb{F}^*$ such that

$$\chi_1^{\theta^*} = k_1\chi_1, \quad \chi_2^{\theta^*} = k_2\chi_5, \quad \chi_3^{\theta^*} = k_3\chi_8, \quad \chi_4^{\theta^*} = k_4\chi_4,$$

$$\chi_5^{\theta^*} = k_5\chi_2, \quad \chi_6^{\theta^*} = k_6\chi_7, \quad \chi_7^{\theta^*} = k_7\chi_6, \quad \chi_8^{\theta^*} = k_8\chi_3.$$

Now, let λ be an arbitrary element of $\mathbb{F}^*$, and let θ be the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ to the hyperbolic basis $(\frac{\bar{e}_1}{\lambda}, \lambda\bar{f}_1, \bar{e}_2, \bar{f}_2)$. Identifying θ with the induced action on W_{16}, we have

- $\chi_1^\theta = \frac{\bar{e}_1}{\lambda} \otimes \frac{\bar{e}_1}{\lambda} \wedge \bar{e}_2 = \frac{\chi_1}{\lambda^2}$,
- $\chi_2^\theta = \frac{\bar{e}_1}{\lambda} \otimes \frac{\bar{e}_1}{\lambda} \wedge \bar{f}_2 = \frac{\chi_2}{\lambda^2}$,
- $\chi_3^\theta = (\lambda\bar{f}_1) \otimes (\lambda\bar{f}_1) \wedge \bar{e}_2 = \lambda^2\chi_3$,
- $\chi_4^\theta = (\lambda\bar{f}_1) \otimes (\lambda\bar{f}_1) \wedge \bar{f}_2 = \lambda^2\chi_4$,
- $\chi_5^\theta = \bar{e}_2 \otimes \bar{e}_2 \wedge \frac{\bar{e}_1}{\lambda} = \frac{1}{\lambda}\chi_5$,
- $\chi_6^\theta = \bar{e}_2 \otimes \bar{e}_2 \wedge (\lambda\bar{f}_1) = \lambda\chi_6$,
- $\chi_7^\theta = \bar{f}_2 \otimes \bar{f}_2 \wedge \frac{\bar{e}_1}{\lambda} = \frac{\chi_7}{\lambda}$,
- $\chi_8^\theta = \bar{f}_2 \otimes \bar{f}_2 \wedge (\lambda\bar{f}_1) = \lambda\bar{\chi}_8$.

Now, put $\theta' = (\theta^*)^{-1}\theta\theta^* \in GL(W_{16})$. We compute

- $\chi_1^{\theta'} = (\frac{\chi_1}{k_1})^{\theta\theta^*} = (\frac{\chi_1}{\lambda^2 k_1})^{\theta^*} = \frac{\chi_1}{\lambda^2}$,
- $\chi_2^{\theta'} = (\frac{\chi_5}{k_5})^{\theta\theta^*} = (\frac{\chi_5}{\lambda k_5})^{\theta^*} = \frac{\chi_2}{\lambda}$,
- $\chi_3^{\theta'} = (\frac{\chi_8}{k_8})^{\theta\theta^*} = (\frac{\lambda\chi_8}{k_8})^{\theta^*} = \lambda\chi_3$,
- $\chi_4^{\theta'} = (\frac{\chi_4}{k_4})^{\theta\theta^*} = (\frac{\lambda^2\chi_4}{k_4})^{\theta^*} = \lambda^2\chi_4$,
- $\chi_5^{\theta'} = (\frac{\chi_2}{k_2})^{\theta\theta^*} = (\frac{\chi_2}{\lambda^2 k_2})^{\theta^*} = \frac{\chi_5}{\lambda^2}$,
- $\chi_6^{\theta'} = (\frac{\chi_7}{k_7})^{\theta\theta^*} = (\frac{\chi_7}{\lambda k_7})^{\theta^*} = \frac{\chi_6}{\lambda}$,
- $\chi_7^{\theta'} = (\frac{\chi_6}{k_6})^{\theta\theta^*} = (\frac{\lambda\chi_6}{k_6})^{\theta^*} = \lambda\chi_7$,
- $\chi_8^{\theta'} = (\frac{\chi_3}{k_3})^{\theta\theta^*} = (\frac{\lambda^2\chi_3}{k_3})^{\theta^*} = \lambda^2\chi_8$.

Since θ, θ^* stabilize $Im(e^*)$, also θ' stabilizes $Im(e^*)$ and hence by Proposition 7.4 corresponds to an automorphism of $\mathcal{F}(W(\mathbb{F}))$ that does not alter the line types. We denote by θ'' the element of $GL(V)$ that induces this automorphism of $\mathcal{F}(W(\mathbb{F}))$ (note that the field automorphisms corresponding to the actions of θ'' on V and of θ' on W_{16} are the same and so both of them are trivial).

Since $\langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle^{\theta'} = \langle \chi_1 \rangle^{\theta'} = \langle \chi_1 \rangle = \langle \bar{e}_1 \otimes \bar{e}_1 \wedge \bar{e}_2 \rangle$, we have $\langle \bar{e}_1 \rangle^{\theta''} = \langle \bar{e}_1 \rangle$. In a similar way, the facts that $\langle \chi_3 \rangle^{\theta'} = \langle \chi_3 \rangle$, $\langle \chi_5 \rangle^{\theta'} = \langle \chi_5 \rangle$ and $\langle \chi_7 \rangle^{\theta'} = \langle \chi_7 \rangle$ imply that $\langle \bar{f}_1 \rangle^{\theta''} = \langle \bar{f}_1 \rangle$, $\langle \bar{e}_2 \rangle^{\theta''} = \langle \bar{e}_2 \rangle$ and $\langle \bar{f}_2 \rangle^{\theta''} = \langle \bar{f}_2 \rangle$.

This implies that there exist $h, k \in \mathbb{F}^*$ such that $\theta'' \sim \theta_{hk}$, where θ_{hk} is the element of $GL(V)$ that maps the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ to the ordered basis $(\bar{e}_1, h\bar{f}_1, k\bar{e}_2, \frac{h}{k}\bar{f}_2)$. The notation $\theta'' \sim \theta_{hk}$ means here that θ'' and θ_{hk} induce the same element of $PGL(V)$, i.e. they are equal up to a nonzero factor. By looking at the induced actions on W_{16}, we see that there exists an $\eta \in \mathbb{F}^*$ such that:

(I) $\frac{\chi_1}{\lambda^2} = \chi_1^{\theta'} = \eta(\bar{e}_1 \otimes \bar{e}_1 \wedge k\bar{e}_2) = \eta k \chi_1$,

(II) $\frac{\chi_2}{\lambda} = \chi_2^{\theta'} = \eta(\bar{e}_1 \otimes \bar{e}_1 \wedge \frac{h}{k}\bar{f}_2) = \frac{\eta h}{k}\chi_2$,

(III) $\lambda\chi_3 = \chi_3^{\theta'} = \eta(h\bar{f}_1 \otimes h\bar{f}_1 \wedge k\bar{e}_2) = \eta h^2 k \chi_3$,

(IV) $\lambda^2\chi_4 = \chi_4^{\theta'} = \eta(h\bar{f}_1 \otimes h\bar{f}_1 \wedge \frac{h}{k}\bar{f}_2) = \frac{\eta h^3}{k}\chi_4$,

(V) $\frac{\chi_5}{\lambda^2} = \chi_5^{\theta'} = \eta(k\bar{e}_2 \otimes k\bar{e}_2 \wedge \bar{e}_1) = \eta k^2 \chi_5$,

(VI) $\frac{\chi_6}{\lambda} = \chi_6^{\theta'} = \eta(k\bar{e}_2 \otimes k\bar{e}_2 \wedge h\bar{f}_1) = \eta k^2 h \chi_6$,

(VII) $\lambda\chi_7 = \chi_7^{\theta'} = \eta(\frac{h}{k}\bar{f}_2 \otimes \frac{h}{k}\bar{f}_2 \wedge \bar{e}_1) = \frac{\eta h^2}{k^2}\chi_7$,

(VIII) $\lambda^2\chi_8 = \chi_8^{\theta'} = \eta(\frac{h}{k}\bar{f}_2 \otimes \frac{h}{k}\bar{f}_2 \wedge h\bar{f}_1) = \frac{\eta h^3}{k^2}\chi_8$.

From (I) and (V), it follows that $\eta k = \eta k^2$, i.e. $k = 1$. Equation (I) then implies that $\eta = \frac{1}{\lambda^2}$. Combining this with Equation (II), we find $h = \lambda$. By Equation (III), we then know that $\lambda = \frac{1}{\lambda^2} \cdot \lambda^2 \cdot 1 = 1$. Since λ was an arbitrary element of $\mathbb{F}^*$, we must have that $|\mathbb{F}| = 2$, as we needed to prove. ■

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