A characterization of a class of hyperplanes of $DW(2n-1,\mathbb{K})$

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Abstract

A hyperplane of the symplectic dual polar space $DW(2n-1,\mathbb{K})$, $n \geq 2$, is called of subspace-type if it consists of all maximal singular subspaces of $W(2n-1,\mathbb{K})$ which meet a given $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$. We show that a hyperplane H of $DW(2n-1,\mathbb{K})$ is of subspace-type if and only if for every hex F of $DW(2n-1,\mathbb{K})$, $H \cap F$ is either F or a hyperplane of subspace-type of F. In the case $\mathbb{K}$ is a perfect field of characteristic 2, we prove a stronger result, namely a hyperplane H of $DW(2n-1,\mathbb{K})$ is of subspace-type or arises from the spin-embedding of $DW(2n-1,\mathbb{K}) \cong DQ(2n,\mathbb{K})$ if and only if for every hex F of $DW(2n-1,\mathbb{K})$, $H \cap F$ is either F, a hexagonal hyperplane of F or a hyperplane of subspace-type of F.

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1 Introduction

1.1 Basic definitions and properties

Let Π be a non-degenerate polar space (Tits [15], Veldkamp [16]) of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment.

The geometry Δ is called a *dual polar space of rank n* (Cameron [2]). The distance $\mathrm{d}(x, y)$ between two points x and y of Δ will be measured in the collinearity graph of Δ. The maximal distance between two points of Δ is equal to n. The dual polar space Δ is a near polygon (Shult and Yanushka [14]; De Bruyn [5]) which means that for every point p and every line L, there exists a unique point on L nearest to p. For every point x of Δ and every $i \in \mathbb{N}$, let $\Delta_i(x)$, respectively $\Delta_i^*(x)$, denote the set of points at distance i, respectively distance at most i, from x. We denote $\Delta_1^*(x)$ also by $x^\perp$. The diameter of a set X of points of Δ is the maximal distance between two of its points and is denoted by $\mathrm{diam}(X)$.

If α is a singular subspace of Π, then the set of all maximal singular subspaces of Π containing α is a convex subspace of diameter $n-1-\mathrm{dim}(\alpha)$ of Δ. Conversely, every non-empty convex subspace of Δ is obtained in this way. The convex subspaces of diameter 2, 3, respectively $n-1$, are called the *quads*, *hexes*, respectively *maxes*, of Δ. The convex subspaces through a given point x of Δ determine a projective space of dimension $n-1$ which we will denote by $Res_\Delta(x)$. If $*_1$ and $*_2$ are two objects of Δ (like points or non-empty sets of points), then $\langle *_1, *_2 \rangle$ denotes the smallest convex subspace of Δ containing $*_1$ and $*_2$. If x is a point and A is a non-empty convex subspace of Δ, then A contains a unique point $\pi_A(x)$ nearest to x and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_A(x)) + \mathrm{d}(\pi_A(x), y)$ for every point y of A. We call $\pi_A(x)$ the *projection* of x onto A. If M is a max, then every point not contained in M is collinear with a unique point of M. If M is a max and A is a convex subspace of diameter k meeting M, then either $A \subseteq M$ or $A \cap M$ is a convex subspace of diameter $k-1$.

A *hyperplane* of Δ is a proper subspace of Δ which meets every line. For every point x of Δ, the set H_x of points at distance at most $n-1$ from x is a hyperplane of Δ, called the *singular hyperplane* of Δ with *deepest point* x. If H_F is a hyperplane of a convex subspace F of diameter $\delta \geq 1$ of Δ, then the set H of points of Δ at distance at most $n-\delta$ from a point of H_F is a hyperplane of Δ, called the *extension* of H_F. If $\delta = n$, then $F = \Delta$, $H = H_F$ and the extension is called *trivial*. A point x of a hyperplane H of Δ is called *deep* with respect to H if $x^\perp \subseteq H$. A convex subspace A of diameter at least 1 is called *deep* with respect to H if $A \subseteq H$.

Suppose now that the dual polar space Δ is thick. Then every hyperplane of Δ is a maximal proper subspace by Shult [12, Lemma 6.1]. If H is a hyperplane of Δ and if Q is a quad of Δ, then either $Q \subseteq H$ or $Q \cap H$ is a hyperplane of Q. By Payne and Thas [9, 2.3.1], one of the following cases then occurs: (1) $Q \subseteq H$; (2) there exists a point x in Q such that $x^\perp \cap Q = H \cap Q$; (3) $Q \cap H$ is a subquadrangle of Q; (4) $Q \cap H$ is an ovoid of Q. (Recall that an *ovoid* is a set of points meeting each line in a unique point.) If case (1), (2), (3), respectively (4) occurs, then we say that Q is *deep*, *singular* (with *deep point* x), *subquadrangular*, respectively *ovoidal* with respect to H. A hyperplane H of Δ is called *locally singular*, *locally ovoidal*, respectively *locally subquadrangular*, if every non-deep quad of Δ is *singular*, *ovoidal*, respectively *subquadrangular*, with respect to H.

A *full (projective) embedding* of Δ is an injective mapping e from the point-set $\mathcal{P}$ of Δ to the point-set of a projective space Σ satisfying: (i) $\langle e(\mathcal{P})\rangle_\Sigma = \Sigma$, (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Σ for every line L of Δ. If e is a full embedding of Δ into the projective space Σ, then for every hyperplane α of Σ, the set $H(\alpha) := e^{-1}(e(\mathcal{P}) \cap \alpha)$ is a hyperplane of Δ. We say that the hyperplane $H(\alpha)$ *arises from the embedding* e.

In this paper, we will meet two classes of dual polar spaces: the dual polar space $DQ(2n, \mathbb{K})$ related to a nonsingular quadric of Witt-index n in $\mathrm{PG}(2n, \mathbb{K})$ and the dual polar space $DW(2n-1, \mathbb{K})$ related to the polar space $W(2n-1, \mathbb{K})$ of the subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$ which are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(2n-1, \mathbb{K})$. We have $DW(2n-1, \mathbb{K}) \cong DQ(2n, \mathbb{K})$ if and only if $\mathbb{K}$ is a perfect field of characteristic 2 (see e.g. [8]). If F is a convex subspace of diameter $\delta \geq 2$ of $DW(2n-1, \mathbb{K})$, then the points and lines contained in F determine a point-line geometry isomorphic to $DW(2\delta - 1, \mathbb{K})$. In particular, the points and lines contained in a quad determine a generalized quadrangle isomorphic to $DW(3, \mathbb{K}) \cong Q(4, \mathbb{K})$. Since every proper subquadrangle of $Q(4, \mathbb{K})$ is a full subgrid, the following holds for every hyperplane H and every quad Q of $DW(2n-1, \mathbb{K})$: if Q is subquadrangular with respect to H, then $Q \cap H$ is a full subgrid of Q.

The dual polar space $\Delta = DQ(2n, \mathbb{K})$ has a nice full embedding into $\mathrm{PG}(2^n - 1, \mathbb{K})$, which is called the *spin-embedding* of $DQ(2n, \mathbb{K})$, see Chevalley [4] or Buekenhout and Cameron [1]. The following proposition characterizes the hyperplanes of $DQ(2n, \mathbb{K})$ which arise from its spin-embedding.

Proposition 1.1 ([6], [13]) *The hyperplanes of $DQ(2n, \mathbb{K})$, $n \geq 2$, which arise from its spin-embedding are precisely the locally singular hyperplanes of $DQ(2n, \mathbb{K})$.*

By Pralle [10], see also Shult [11] for the finite case, the dual polar space

$DQ(6, \mathbb{K})$ has two types of locally singular hyperplanes, the singular hyperplanes and the so-called hexagonal hyperplanes. The points and lines contained in a hexagonal hyperplane define a split Cayley hexagon $H(\mathbb{K})$. If H is a hexagonal hyperplane of $DQ(6, \mathbb{K})$, then every quad of $DQ(6, \mathbb{K})$ is singular with respect to H.

1.2 The main results

Let ζ be a symplectic polarity of $\mathrm{PG}(2n-1, \mathbb{K})$, $n \geq 2$, and let π be an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1, \mathbb{K})$. Then the set H_π of all maximal totally isotropic subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$ (w.r.t. ζ) meeting π is a hyperplane of the dual polar space $DW(2n-1, \mathbb{K})$ associated with ζ, see De Bruyn [7]. We will call any hyperplane which can be obtained in this way a *hyperplane of subspace-type*. If n is even and if π is nonsingular, then we will denote the hyperplane H_π also by $\mathrm{Hyp}(2n-1, \mathbb{K})$. By [7], the hyperplane $\mathrm{Hyp}(3, \mathbb{K})$ is a full subgrid of $DW(3, \mathbb{K}) \cong Q(4, \mathbb{K})$. Also by [7], a hyperplane of subspace-type of $DW(2n-1, \mathbb{K})$ is either a singular hyperplane or the (possibly trivial) extension of a hyperplane of type $\mathrm{Hyp}(4m-1, \mathbb{K})$ where $2 \leq 2m \leq n$.

The aim of this paper is to give a characterization of the hyperplanes of subspace-type. We will prove the following two theorems.

Theorem 1.2 *The following are equivalent for a hyperplane H of $DW(2n-1, \mathbb{K})$, $n \geq 3$:*

(1) *H is a hyperplane of subspace-type;*

(2) *For every hex F of $DW(2n-1, \mathbb{K})$, $F \cap H$ is either F, a singular hyperplane of F or the extension of a full subgrid of a quad of F.*

Theorem 1.3 *Let $n \geq 3$ and let $\mathbb{K}$ be a perfect field of characteristic 2. Then the following are equivalent for a hyperplane H of $DW(2n-1, \mathbb{K})$:*

(1) *H is either a hyperplane of subspace-type or arises from the spin-embedding of $DW(2n-1, \mathbb{K}) \cong DQ(2n, \mathbb{K})$;*

(2) *For every hex F of $DW(2n-1, \mathbb{K})$, $F \cap H$ is either F, a singular hyperplane of F, a hexagonal hyperplane of F or the extension of a full subgrid in a quad of F.*

1.3 Regarding the proof of the main results

Suppose H is a hyperplane of subspace-type of $DW(2n-1,\mathbb{K})$, $n \geq 3$. By Proposition 2.9 of [7], we know that for every convex subspace F of diameter at least 2 of $DW(2n-1,\mathbb{K})$, either $F \subseteq H$ or $F \cap H$ is a hyperplane of subspace-type of F. In particular, if F is a hex of $DW(2n-1,\mathbb{K})$, then $F \cap H$ is either F, a singular hyperplane of F or the extension of a full subgrid of a quad of $DW(2n-1,\mathbb{K})$.

Suppose $\mathbb{K}$ is a perfect field of characteristic 2 and that H is a hyperplane of $DW(2n-1,\mathbb{K})$, $n \geq 3$, arising from the spin-embedding of $DW(2n-1,\mathbb{K})$. Then by Proposition 1.1, H is locally singular. If F is a hex of $DW(2n-1,\mathbb{K})$, then either $F \subseteq H$ or $F \cap H$ is a locally singular hyperplane of F. Hence, $F \cap H$ is either F, a singular hyperplane of F or a hexagonal hyperplane of F.

This proves the implications (1) $\Rightarrow$ (2) in Theorems 1.2 and 1.3. We still need to prove the implications (2) $\Rightarrow$ (1) in these theorems. In Section 3, we will prove the following proposition by induction on n.

Proposition 1.4 *Let H be a hyperplane of $DW(2n-1,\mathbb{K})$, $n \geq 3$, such that for any hex F of $DW(2n-1,\mathbb{K})$, $F \cap H$ is one of the following: (i) F; (ii) a singular hyperplane of F; (iii) the extension of a full subgrid of a quad of F; (iv) (only possible when $\mathbb{K}$ is a perfect field of characteristic 2) a hexagonal hyperplane of F. Then the following holds:*

(1) *If there are no subquadrangular quads, then H is a locally singular hyperplane.*

(2) *If there exists at least one subquadrangular quad, then H is a hyperplane of subspace-type.*

Theorems 1.2 and 1.3 easily follow from Proposition 1.4. Obviously, this is the case for Theorem 1.3 (Recall Proposition 1.1). We will now also show that the implication (2) $\Rightarrow$ (1) in Theorem 1.2 is a consequence of Proposition 1.4.

Let H be a hyperplane of $DW(2n-1,\mathbb{K})$, $n \geq 3$, such that for every hex F of $DW(2n-1,\mathbb{K})$, $F \cap H$ is either F, a singular hyperplane of F or the extension of a full subgrid in a quad of F. If there exists a subquadrangular quad, then H is a hyperplane of subspace-type by Proposition 1.4. Suppose therefore that there are no subquadrangular quads. Then by Proposition 1.4, H is a locally singular hyperplane of $DW(2n-1,\mathbb{K})$. If F is a hex of

$DW(2n-1,\mathbb{K})$, then $F\cap H$ is either F or a locally singular hyperplane of F. Hence, either $F \cap H = F$ or $F \cap H$ is a singular hyperplane of F. Now, by Cardinali, De Bruyn and Pasini [3, Lemma 3.4], a hyperplane H' of a thick dual polar space is singular if and only if for every hex F' not contained in H', $F'\cap H'$ is a singular hyperplane of F'. Applying this here we see that H is a singular hyperplane of $DW(2n-1,\mathbb{K})$. Denote by π the $(n-1)$-dimensional singular subspace of $W(2n-1,\mathbb{K})$ corresponding to the deepest point of H. Then the hyperplane H_π of subspace-type coincides with H. So, in any case we have that H is a hyperplane of subspace-type.

2 Preliminary properties

2.1 Some properties of the hyperplanes of subspace-type

Let ζ be a symplectic polarity of $\mathrm{PG}(2n-1,\mathbb{K})$, $n\geq 2$, and let $W(2n-1,\mathbb{K})$ and $\Delta := DW(2n-1,\mathbb{K})$ denote the corresponding polar and dual polar spaces. Let π be an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{K})$ and let H_π denote the corresponding hyperplane of subspace-type of $DW(2n-1,\mathbb{K})$. By De Bruyn [7, Corollary 2.11], H_π does not admit ovoidal quads. For a proof of the following lemma, see [7, Proposition 2.6].

Lemma 2.1 *A max M of $DW(2n-1,\mathbb{K})$ is contained in H_π if and only if the point of $W(2n-1,\mathbb{K})$ corresponding to M belongs to $\pi\cup\pi^\zeta$.*

Lemma 2.2 *Let x be a point of $DW(2n-1,\mathbb{K})$, $n\geq 3$, and let M be a max through x. If every max through x distinct from M is contained in H_π, then also M is contained in H_π and H_π is the singular hyperplane of $DW(2n-1,\mathbb{K})$ with deepest point x.*

Proof. Let α be the maximal singular subspace of $W(2n-1,\mathbb{K})$ corresponding to x. By Lemma 2.1, there is at most 1 point in α which is not covered by $\pi\cup\pi^\zeta$. It follows that every point of α is covered by $\pi\cup\pi^\zeta$. Hence, also M is contained in H_π. So, the singular hyperplane H_x of $DW(2n-1,\mathbb{K})$ with deepest point x is contained in H_π. Since H_x is a maximal proper subspace of $DW(2n-1,\mathbb{K})$, $H_x = H_\pi$. ■

Lemma 2.3 *If α is a maximal totally isotropic subspace of $W(2n-1,\mathbb{K})$, then* $\dim(\pi\cap\alpha)=\dim(\pi^\zeta\cap\alpha)$.

Proof. Put $\beta = \pi \cap \alpha$ and $k = \dim(\beta)$. The space β^ζ has dimension $2n-2-k$ and contains the $(n-1)$-dimensional subspaces π^ζ and α. Hence, $\dim(\pi^\zeta \cap \alpha) \geq k = \dim(\pi \cap \alpha)$. By symmetry, also $\dim(\pi \cap \alpha) \geq \dim(\pi^\zeta \cap \alpha)$. ■

Corollary 2.4 $H_\pi = H_{\pi^\zeta}$. ■

Consider now the following three types of points in H_π:

(1) maximal singular subspaces α for which $\dim(\alpha \cap \pi) = \dim(\alpha \cap \pi^\zeta) = 0$ and $\alpha \cap \pi = \alpha \cap \pi^\zeta$;

(2) maximal singular subspaces α for which $\dim(\alpha \cap \pi) = \dim(\alpha \cap \pi^\zeta) = 0$ and $\alpha \cap \pi \neq \alpha \cap \pi^\zeta$;

(3) maximal singular subspaces α for which $\dim(\alpha \cap \pi) = \dim(\alpha \cap \pi^\zeta) \geq 1$.

Lemma 2.5 *Let α be a point of H_π.*

(i) If α is a point of type (1), then there exists a unique deep max $A(\alpha)$ through α and the lines of Δ through α which are contained in H_π are precisely the lines of $A(\alpha)$ through α.

(ii) If α is a point of type (2), then there are two distinct deep maxes $A_1(\alpha)$ and $A_2(\alpha)$ through α such that the lines through x contained in H_π are precisely the lines through x which are contained in $A_1(\alpha) \cup A_2(\alpha)$.

(iii) If α is a point of type (3), then α is a deep point.

Proof. If α is a point of type (3) of Δ, then every $(n-2)$-dimensional subspace of α contains a point of π. It follows that every line through α is contained in H_π. So, α is a deep point of Δ.

Let α be a point of type (1) of Δ and let x denote the unique point of $W(2n-1, \mathbb{K})$ contained in $\alpha \cap \pi = \alpha \cap \pi^\zeta$. Let β be an $(n-2)$-dimensional subspace of α. If β contains the point x, then β obviously is a deep line. If β does not contain the point x, then $\beta^\zeta \cap \pi = \{x\}$, and it follows that α is the unique point of the line β contained in H_π. [If $\beta^\zeta \cap \pi$ would be a line L, then L must be a singular line and $\langle \beta, L \rangle$ would be a singular subspace of dimension n, which is impossible.] Hence, there exists a unique deep max $A(\alpha)$ through α and the lines of Δ through α which are contained in H_π are precisely the lines of $A(\alpha)$ through α.

Let α be a point of type (2) of Δ and let x_1 and x_2 be the points of $W(2n-1, \mathbb{K})$ contained in $\alpha \cap \pi$ and $\alpha \cap \pi^\zeta$, respectively. Let β be an $(n-2)$-dimensional subspace of α. If β contains at least one of the points

x_1 and x_2, then by Lemma 2.3, every maximal singular subspace through β meets π, proving that β is a deep line. Suppose now that $\beta \cap \{x_1, x_2\} = \emptyset$. If $\alpha' \neq \alpha$ is a maximal singular subspace through β meeting π in a point $x \neq x_1$, then $\beta = x^\perp \cap \alpha$ contains the point x_2, a contradiction. So, if $\beta \cap \{x_1, x_2\} = \emptyset$, then α is the unique point on the line β which is contained in H_π. It follows that there are two deep maxes $A_1(\alpha)$ and $A_2(\alpha)$ through α such that the lines through x contained in H_π are precisely the lines through x which are contained in $A_1(\alpha) \cup A_2(\alpha)$. ■

2.2 On a certain class of hyperplanes of $DW(2n-1, \mathbb{K})$

In this subsection, we suppose that H is a hyperplane of $\Delta := DW(2n-1, \mathbb{K})$, $n \geq 3$, such that for any hex F of Δ, $F \cap H$ is one of the following: (i) F; (ii) a singular hyperplane of F; (iii) the extension of a full subgrid of a quad of F; (iv) (only possible when $\mathbb{K}$ is a perfect field of characteristic 2) a hexagonal hyperplane of F.

Lemma 2.6 *No quad of Δ is ovoidal with respect to H.*

Proof. Let Q be an arbitrary quad of Δ and let F be an arbitrary hex containing Q. If $F \subseteq H$, then also $Q \subseteq H$. If $F \cap H$ is a singular hyperplane of F, then either $Q \subseteq H$ or $Q \cap H$ is a singular hyperplane of Q. If $F \cap H$ is the extension of a full subgrid of a quad of F, then $Q \cap H$ is either Q, a singular hyperplane of Q or a full subgrid of Q. If $F \cap H$ is a hexagonal hyperplane of F, then $Q \cap H$ is a singular hyperplane of Q. ■

Definition. Let x be a point of Δ. The convex subspaces through x define a projective space $Res_\Delta(x)$ isomorphic to $\mathrm{PG}(n-1, \mathbb{K})$. If $x \in H$, then $\Lambda_H(x)$ (or $\Lambda(x)$ when no confusion is possible) denotes the set of lines through x contained in H. The set $\Lambda(x)$ is a set of points of $Res_\Delta(x)$.

Lemma 2.7 *For every point x of H, $\Lambda(x)$ is one of the following sets of points of the projective space $Res_\Delta(x)$:*
(1) *a hyperplane;*
(2) *the union of two distinct hyperplanes;*
(3) *the whole space.*

Proof. Let α be a subspace of $Res_\Delta(x)$ of dimension at least 1. We will show the following by induction on $\dim(\alpha)$:

(∗) $\alpha \cap \Lambda(x)$ is equal to either α, a hyperplane of α or the union of two distinct hyperplanes of α.

Case I. Suppose $\dim(\alpha) = 1$. Property $(*)$ then follows from the fact that every quad is either deep, singular or subquadrangular.

Case II. Suppose $\dim(\alpha) = 2$. Let F denote the hex through x corresponding to α. If $F \subseteq H$, then $\alpha \cap \Lambda(x)$ is equal to α. If $F \cap H$ is a singular hyperplane of F, then $\alpha \cap \Lambda(x)$ is either α or a hyperplane of α. If $F \cap H$ is the extension of a full subgrid of a quad of F, then $\alpha \cap \Lambda(x)$ is either α, a hyperplane of α or the union of two hyperplanes of α. If $F \cap H$ is a hexagonal hyperplane of F, then $\alpha \cap \Lambda(x)$ is a hyperplane of α. In any case, we see that property $(*)$ holds.

Case III. Suppose $\dim(\alpha) = 3$. By the induction hypothesis, property $(*)$ holds for any line or plane of α. If every line of α intersects $\Lambda(x)$ in the whole line or a singleton, then $\alpha \cap \Lambda(x)$ is either α itself or a hyperplane of α. So, we may suppose that there exists a line L in α which intersects $\Lambda(x)$ in two points x_1 and x_2. Every plane of α through L intersects $\Lambda(x)$ in the union of a line through x_1 and a line through x_2. Now, let $\beta_1, \beta_2, \beta_3$ be three distinct planes of α through L. For every $i \in \{1, 2, 3\}$, let L_i, respectively M_i, denote the unique line through x_1, respectively x_2, contained in $\beta_i \cap \Lambda(x)$. Put $\gamma_1 := \langle L_1, L_2 \rangle$, $\gamma_2 := \langle M_1, M_2 \rangle$, $\{u_1\} = \gamma_1 \cap M_3$ and $\{v_1\} = \gamma_2 \cap L_3$. Since $L_1 \cup L_2 \cup \{u_1\} \subseteq \Lambda(x)$ and $u_1 \notin L_1 \cup L_2$, we have $\gamma_1 \subseteq \Lambda(x)$ by the induction hypothesis applied to the subspace γ_1. In a similar way, one shows that $\gamma_2 \subseteq \Lambda(x)$. Now, every plane of α through L intersects $\gamma_1 \cup \gamma_2$ in the union of a line through x_1 and a line through x_2. This forces $\Lambda(x) \cap \alpha$ to be equal to $\gamma_1 \cup \gamma_2$.

Case IV. Suppose that $\dim(\alpha) \geq 4$ and that property $(*)$ holds for any subspace of dimension less than $\dim(\alpha)$. If every line of α intersects $\Lambda(x)$ in the whole line or a singleton, then $\alpha \cap \Lambda(x)$ is either α itself or a hyperplane of α. So, we may suppose that there exists a line L in α which intersects $\Lambda(x)$ in two points x_1 and x_2. For every plane $\beta \subseteq \alpha$ through L, let $k(\beta)$ denote the unique point of β such that $\beta \cap \Lambda(x)$ is the union of the two lines $k(\beta)x_1$ and $k(\beta)x_2$.

Claim. *The set $K = \{k(\beta) \,|\, \beta$ a plane through L contained in $\alpha\}$ is a subspace of α with* $\dim(K) = \dim(\alpha) - 2$.

Let β_1 and β_2 be two distinct planes of α through L. By the induction hypothesis, the three-space $\langle \beta_1, \beta_2 \rangle$ intersects $\Lambda(x)$ in the union of two planes δ_1 and δ_2. The line $\delta_1 \cap \delta_2$ coincides with the line through $k(\beta_1)$ and $k(\beta_2)$, and every point of $\delta_1 \cap \delta_2$ is of the form $k(\beta)$ for some plane β of $\langle \beta_1, \beta_2 \rangle$ through L. This proves that K is a subspace. Since L is disjoint from K, $\dim(K) \leq \dim(\alpha) - 2$. Since every plane of α through L meets K, $\dim(K) = \dim(\alpha) - 2$.

It is now obvious that $\alpha \cap \Lambda(x) = \langle K, x_1 \rangle \cup \langle K, x_2 \rangle$. ■

Definition. If x is a point of H such that case (1), (2), respectively (3) of Lemma 2.7 occurs, then we say that x has *type* (1), (2), respectively (3) (with respect to H).

Lemma 2.8 *The set of points of type* (3) *of* H *forms a subspace of* Δ.

Proof. Let x_1 and x_2 be two distinct collinear points of type (3) and let x_3 denote a point on the line x_1x_2 distinct from x_1 and x_2. We must show that every line L through x_3 is contained in H. Obviously, this holds if $L = x_1x_2$. So, suppose $L \neq x_1x_2$ and let Q be the quad $\langle L, x_1x_2 \rangle$. Every point of $x_1^\perp \cap Q$ and $x_2^\perp \cap Q$ is contained in Q. So, Q is a deep quad and $L \subseteq H$. ■

Lemma 2.9 *A point x of H is of type* (2) *if and only if it is contained in a subquadrangular quad.*

Proof. A point x of H is of type (2) if and only if there exists a line in $Res_\Delta(x)$ intersecting $\Lambda(x)$ in precisely two points. Obviously, the lines of $Res_\Delta(x)$ which intersect $\Lambda(x)$ in precisely two points correspond to the subquadrangular quads through x. ■

The following is an immediate consequence of Lemmas 2.6 and 2.9.

Corollary 2.10 *If there are no points of type* (2)*, then H is locally singular.*

3 Proof of Proposition 1.4

We will now prove Proposition 1.4 by induction on n. Obviously, Proposition 1.4 holds if $n = 3$. So, we will suppose that $n \geq 4$, that H is a hyperplane of $\Delta := DW(2n-1, \mathbb{K})$ satisfying the conditions of Proposition 1.4 and that Proposition 1.4 holds for symplectic dual polar spaces of rank smaller than n.

Recall that by Lemma 2.7, there are three types of points in H. If there are no points of type (2), then Proposition 1.4 holds by Lemma 2.9 and Corollary 2.10. In the sequel, we will assume that there exists at least 1 point of type (2).

Definition. Let P_1, P_2, respectively P_3 denote the set of points of H which have type (1), (2), respectively (3) with respect to H (cf. Lemma 2.7). Notice that P_3 is the set of deep points. For every point $x \in P_2$, let $A_1^H(x)$ and $A_2^H(x)$ denote the two maxes through x such that $(A_1^H(x) \cup A_2^H(x)) \cap x^\perp \subseteq H$. Put

$$I^H(x) := A_1^H(x) \cap A_2^H(x).$$

Then $I^H(x)$ has diameter $n-2$. If no confusion is possible, we will write $I(x)$, $A_1(x)$ and $A_2(x)$ instead of $I^H(x)$, $A_1^H(x)$ and $A_2^H(x)$.

Lemma 3.1 *The following holds for every point $x \in P_2$.*
(i) If $y \in I(x)$ with $d(y,x) \leq n-3$, then $y \in H$.
(ii) If $y' \in (A_1(x) \cup A_2(x)) \setminus I(x)$ with $d(y',x) \leq n-2$, then $y' \in H$.

Proof. In (i), let y' be a point of $A_1(x) \setminus I(x)$ collinear with y. In (ii), let y be the unique point of $I(x)$ collinear with y'. In both cases, we have $\mathrm{d}(x,y) \leq n-3$ and $\mathrm{d}(x,y') \leq n-2$. Now, let M denote a max through x, y and y' such that $\mathrm{diam}(M \cap I(x)) = n-3$, $\mathrm{diam}(M \cap A_1(x)) = n-2$ and $\mathrm{diam}(M \cap A_2(x)) = n-2$. The point x of $M \cap H$ has type (2) with respect to the hyperplane $H' := M \cap H$ of M. So, the hyperplane H' cannot be locally singular. By the induction hypothesis, H' is a hyperplane of subspace-type. By Lemma 2.5, $I^{H'}(x)$, $A_1^{H'}(x)$ and $A_2^{H'}(x)$ are contained in H'. Now, $I^{H'}(x) = I^H(x) \cap M$ and $\{A_1^{H'}(x), A_2^{H'}(x)\} = \{A_1^H(x) \cap M, A_2^H(x) \cap M\}$. It follows that y and y' are contained in H. ■

Lemma 3.2 *For every point $x \in P_2$, $I(x) \subseteq H$.*

Proof. By the induction hypothesis, either $A_1(x) \subseteq H$, $A_1(x) \cap H$ is a hyperplane of subspace-type of $A_1(x)$ or $A_1(x) \cap H$ is a locally singular hyperplane of $A_1(x)$. If $A_1(x) \subseteq H$, then also $I(x) \subseteq H$.

Suppose $A_1(x) \cap H$ is a locally singular hyperplane of $A_1(x)$. Let y be a point of $I(x)$ at distance at most $n-3$ from x. Since there are no subquadrangular quads in $A_1(x)$ through y, y has type (1) or (3) with respect to the hyperplane $A_1(x) \cap H$ of $A_1(x)$. Since every line of $A_1(x)$ through y not contained in $I(x)$ is contained in H (recall Lemma 3.1), y necessarily has type (3) with respect to the hyperplane $A_1(x) \cap H$ of $A_1(x)$. Since this holds for every point y of $I(x)$ at distance at most $n-3$ from x, $I(x) \subseteq H$.

Suppose $A_1(x) \cap H$ is a hyperplane of subspace-type of $A_1(x)$. By Lemma 3.1, every max of $A_1(x)$ through x distinct from $I(x)$ is contained in $A_1(x) \cap H$. Hence, by Lemma 2.2 also $I(x)$ is contained in $A_1(x) \cap H$. ■

Corollary 3.3 *Let x be a point of P_2. Then for every $i \in \{1, 2\}$, $\Delta^*_{n-2}(x) \cap A_i(x) \subseteq H$.* ■

Lemma 3.4 *Let x be a point of P_2. Then on every line L through x contained in $I(x)$, there is a unique point x_L belonging to P_3. Moreover, $L \setminus \{x_L\} \subseteq P_2$ and for every $x' \in L \setminus \{x_L\}$, we have $I(x) = I(x')$ and $\{A_1(x), A_2(x)\} = \{A_1(x'), A_2(x')\}$.*

Proof. By Corollary 3.3, $y^\perp \cap (A_1(x) \cup A_2(x)) \subseteq H$ for every point $y \in L$. So, every point of L belongs to $P_2 \cup P_3$. Now, let Q denote a quad through L not contained in $A_1(x) \cup A_2(x)$. Then Q is singular with respect to H, since $L \subseteq H$ and $(x^\perp \cap H) \cap Q = L$. If x_L is the deep point of Q with respect to H, then ${x_L}^\perp \cap Q \subseteq H$ and hence $x_L \in P_3$. By Lemma 2.8, $L \setminus \{x_L\} \subseteq P_2$. For every $x' \in L \setminus \{x_L\}$, $\{A_1(x), A_2(x)\} = \{A_1(x'), A_2(x')\}$ and $I(x) = I(x')$ since $x'^\perp \cap (A_1(x) \cup A_2(x)) \subseteq H$ by Corollary 3.3. ■

Lemma 3.5 *If $x \in P_2$, then $A_1(x) \cup A_2(x) \subseteq H$.*

Proof. Let $i \in \{1, 2\}$. By Lemma 3.4, there exists an $x' \in \Delta_1(x) \cap P_2$ such that $I(x) = I(x')$ and $\{A_1(x), A_2(x)\} = \{A_1(x'), A_2(x')\}$. Without loss of generality, we may suppose that $A_1(x) = A_1(x')$ and $A_2(x) = A_2(x')$. By Corollary 3.3, the subspace $A_i(x) \cap H$ of $A_i(x)$ contains $\Delta^*_{n-2}(x) \cap A_i(x)$ and $\Delta^*_{n-2}(x') \cap A_i(x)$. Now, $\Delta^*_{n-2}(x) \cap A_i(x) \subseteq H$ and $\Delta^*_{n-2}(x') \cap A_i(x) \subseteq H$ are maximal proper subspaces of $A_i(x)$. It follows that $A_i(x) \cap H = A_i(x)$, in other words $A_i(x) \subseteq H$. ■

Corollary 3.6 *Every point of P_2 is contained in precisely two deep maxes.* ■

Lemma 3.7 *Let x be a point of P_2. Then*

(i) $I(y) = I(x)$ for every point $y \in I(x) \cap P_2$;

(ii) $I(x) \subseteq P_2 \cup P_3$ and $H(x) := I(x) \cap P_3$ is a hyperplane of $I(x)$.

Proof. (i) Let M_1 and M_2 denote the two deep maxes through x. Then $I(x) = M_1 \cap M_2$. Hence, M_1 and M_2 are also the two deep maxes through y and $I(y) = M_1 \cap M_2 = I(x)$.

(ii) Since $A_1(x) \subseteq H$ and $A_2(x) \subseteq H$, every point of $I(x)$ belongs to $P_2 \cup P_3$. The claim follows from Lemma 3.4 and part (i). ■

Lemma 3.8 *Let M be a deep max, let $x \in M \cap P_2$ and let x' be a point of $\Delta_1(x) \cap M$ not contained in $I(x)$. Then*

(i) $L = xx'$ only contains points of P_2;

(ii) $I(x')$ is disjoint from $I(x)$.

Proof. (i) Let M' denote the other deep max through x. Then $I(x) = M \cap M'$. Let L' denote a line of M' through x not contained in $I(x)$. Then the quad $\langle L, L' \rangle$ is subquadrangular, since $x \in P_2$. Hence, $L \cup L' \subseteq P_2$ by Lemma 2.9.

(ii) By Lemma 3.7 and (i), $xx' \not\subseteq I(x')$. This implies that $I(x)$ and $I(x')$ are disjoint. ■

Lemma 3.9 *No point of P_1 is collinear with a point of P_2.*

Proof. Suppose that the point $x \in P_1$ is collinear with the point $y \in P_2$. Without loss of generality, we may suppose that $x \in A_1(y)$. Now, $I(y) \subseteq A_1(y)$ and $I(y)$ does not contain the point x by Lemma 3.7. But then by Lemma 3.8, the line xy must be contained in P_2 which is clearly not the case. ■

The following is an immediate corollary of Lemmas 2.8 and 3.9.

Corollary 3.10 *If a line $L \subseteq H$ contains a point of P_1, then either all points of L belong to P_1, or precisely one point of L belongs to P_3 and the remaining points belong to P_1.*

Lemma 3.11 *If there exists a deep max M containing a point of P_1, then H is a hyperplane of subspace-type which extends a hyperplane of M.*

Proof. We first prove that $M \subseteq P_1 \cup P_3$. Suppose $M \cap P_2 \neq \emptyset$. Let $u \in P_1$ and $v \in P_2$ be points of M with $\mathrm{d}(u, v)$ as small as possible. By Lemma 3.9, $\mathrm{d}(u, v) \geq 2$. The convex subspace $I(v)$ is contained in M and only contains points of $P_2 \cup P_3$ by Lemma 3.7. Hence, $u \notin I(v)$. Put $u' := \pi_{I(v)}(u)$ and let L denote an arbitrary line of M through u different from uu' such that

$L' := \pi_{I(v)}(L) \subseteq \langle v, u' \rangle$. By Corollary 3.10, at most one point of L does not belong to P_1. By Lemma 3.9 and the fact that $I(v) \subseteq P_2 \cup P_3$, it follows that at most one point of L' does not belong to P_3. It follows that $L' \subseteq P_3$ since P_3 is a subspace. So, $v \notin L'$ and $\mathrm{d}(v, u') \geq 2$. Let y denote the unique point of L' nearest to v. Let L'' denote a line of $\langle v, u' \rangle$ through v not contained in $\langle v, y \rangle$. Then L'' contains a unique point z of P_3 by Lemma 3.7. Now, every point of L'' has distance $\mathrm{d}(v, u') - 1$ to a (unique) point of L'. Since $|L'|, |L''| \geq 3$, there exists a point $v_1 \in L'' \setminus \{z\}$ and a point $u_1 \in L \cap P_1$ such that $\mathrm{d}(v_1, \pi_{I(v)}(u_1)) = \mathrm{d}(v, u') - 1$. Hence, $u_1 \in P_1$, $v_1 \in P_2$ and $\mathrm{d}(u_1, v_1) = \mathrm{d}(u, v) - 1$, a contradiction. Hence, $M \subseteq P_1 \cup P_3$.

Next, we prove that $G := M \cap P_3$ is a hyperplane of M. Let L be an arbitrary line of M containing a point x of P_1 and let Q be a quad through L not contained in M. Since $x^\perp \cap Q \cap H = L$, Q is singular. The deep point y of Q necessarily is contained in P_3. Since P_3 is a subspace, $L \cap P_3 = \{y\}$. This proves that $M \cap P_3$ is a hyperplane of M.

By the previous two paragraphs, H consists of those points of Δ which have distance at most 1 from G. So, the hyperplane H is the extension of G. Let M' denote a max disjoint from M. By the induction hypothesis, the hyperplane $M' \cap H$ of M' is either locally singular or a hyperplane of subspace-type. Hence, also the hyperplane G of M (which is isomorphic to the hyperplane $H \cap M'$ of M') is either locally singular or a hyperplane of subspace-type. Now, the extension of a locally singular hyperplane is again locally singular and the extension of a hyperplane of subspace-type is again a hyperplane of subspace-type. Since we assumed $P_2 \neq \emptyset$, the hyperplane H cannot be locally singular. Hence, H is a hyperplane of subspace-type. ■

In view of Lemma 3.11, we may now suppose that no point of P_1 is contained in a deep max. Since H is a proper set of points of Δ, every deep max contains at least one point of P_2. We now define a relation R on the set $\mathcal{M}$ of all deep maxes. For $M_1, M_2 \in \mathcal{M}$, we say that $(M_1, M_2) \in R$ if and only if either $M_1 = M_2$ or $M_1 \cap M_2 \subseteq P_3$.

Lemma 3.12 *The relation R is an equivalence relation.*

Proof. Obviously, R is reflexive and symmetric. We prove that R is also transitive. Let M_1, M_2 and M_3 be three maxes such that $(M_1, M_2) \in R$ and $(M_2, M_3) \in R$. We will show that $(M_1, M_3) \in R$. We may suppose that M_1, M_2 and M_3 are mutually different.

Case I: $M_1 \cap M_2 = \emptyset$ and $\emptyset \neq M_2 \cap M_3 \subseteq P_3$.
If M_3 is disjoint from M_1, then we are done. So, suppose $M_1 \cap M_3 \neq \emptyset$. Let y denote an arbitrary point of $M_1 \cap M_3$, let L_y denote the unique line through y meeting M_2 in a point z and let Q denote an arbitrary quad through L_y. Then $z^\perp \cap Q \subseteq H$ and $Q \cap M_1 \subseteq H$. It follows that Q is deep. Since Q was an arbitrary quad through L_y, $y \in P_3$. This proves that $M_1 \cap M_3 \subseteq P_3$ and that $(M_1, M_3) \in R$.

Case II: $M_1 \cap M_2 = \emptyset$ and $M_2 \cap M_3 = \emptyset$.
Suppose that $(M_1, M_3) \not\in R$. Then $M_1 \cap M_3 \neq \emptyset$ and there exists a point $y \in M_1 \cap M_3 \cap P_2$. Let L_y denote the unique line through y meeting M_2. Then $L_y \subseteq H$. Now, $\{A_1(y), A_2(y)\} = \{M_1, \langle I(y), L_y \rangle\}$. Hence, $M_3 = \langle I(y), L_y \rangle$, contradicting the fact that M_2 and M_3 are disjoint.

Case III: $M_2 \cap M_3 = \emptyset$ and $\emptyset \neq M_1 \cap M_2 \subseteq P_3$.
Similarly as in Case I, one shows that $(M_1, M_3) \in R$.

Case IV: $\emptyset \neq M_1 \cap M_2 \subseteq P_3$ and $\emptyset \neq M_2 \cap M_3 \subseteq P_3$.
If $M_1 \cap M_2 = M_2 \cap M_3$, then $M_1 \cap M_3 \neq \emptyset$ and $M_1 \cap M_3 = M_1 \cap M_2 \subseteq P_3$, proving that $(M_1, M_3) \in R$. If $M_1 \cap M_2$ and $M_2 \cap M_3$ are disjoint, then also M_1 and M_3 are disjoint and we are done. So, we may suppose that $\mathrm{diam}(M_1 \cap M_2) = \mathrm{diam}(M_1 \cap M_3) = \mathrm{diam}(M_2 \cap M_3) = n-2$, $\mathrm{diam}(M_1 \cap M_2 \cap M_3) = n-3$, $M_1 \cap M_2 \subseteq P_3$ and $M_2 \cap M_3 \subseteq P_3$.

Suppose that there exists a point y in $(M_1 \cap M_3 \cap P_2) \setminus M_2$, let y' denote the unique point of M_2 collinear with y (so, $y' \in M_1 \cap M_2 \cap M_3$), let Q denote a quad through yy' not contained in $M_1 \cup M_3$ and let z be an arbitrary point of $Q \cap M_2 \setminus \{y'\}$. Since $y \in P_2$, Q is not deep. Now, ${y'}^\perp \cap Q \subseteq H$ since $y' \in P_3$. So, Q is singular with deep point y'. It follows that $z \in P_2$. By Lemma 3.8, $y'z \subseteq I(z)$. So, $I(z) \cap M_1 \cap M_2$ and $I(z) \cap M_3 \cap M_2$ have diameter $n-3$. Now, let R be a quad of M_2 through zy' not contained in $I(z) \cup \langle z, M_1 \cap M_2 \cap M_3 \rangle$. Then R intersects M_i, $i \in \{1,3\}$, in a line L_i which is not contained in $M_1 \cap M_2 \cap M_3$. Let A be the hex $\langle R, yy' \rangle$. Since $y \in P_2$, A contains a subquadrangular quad through y. It follows that the hyperplane $A \cap H$ is the extension of a full subgrid G^* in a quad Q^*. Since $L_1, L_3 \subseteq P_3$, $L_1, L_3 \subseteq G^*$ and hence $Q^* = R$. Now, let M_2' denote the unique deep max through z different from M_2. Then $M_2' \cap A$ is a deep quad of A through z which necessarily coincides with $Q^* = R$ (notice that $z \in Q^* \setminus G^*$). But this would imply that $R \subseteq I(z)$, a contradiction. Hence, $M_1 \cap M_3 \subseteq P_3$ and $(M_1, M_3) \in R$. ∎

Definition. For every point x of P_2 and every $i \in \{1,2\}$, let $\mathcal{C}_i(x)$ denote the equivalence class of R containing the element $A_i(x)$. Obviously, $\mathcal{C}_1(x) \neq \mathcal{C}_2(x)$.

Lemma 3.13 *If x and y are two collinear points of P_2, then $\{\mathcal{C}_1(x), \mathcal{C}_2(x)\} = \{\mathcal{C}_1(y), \mathcal{C}_2(y)\}$.*

Proof. If $I(x) = I(y)$, then $\{A_1(x), A_2(x)\} = \{A_1(y), A_2(y)\}$ and hence $\{\mathcal{C}_1(x), \mathcal{C}_2(x)\} = \{\mathcal{C}_1(y), \mathcal{C}_2(y)\}$. Suppose therefore that $I(x) \neq I(y)$. Then $\langle y, I(x)\rangle$ is a deep max. Without loss of generality, we may suppose that $A_1(x) = A_1(y) = \langle y, I(x)\rangle$. So, $\mathcal{C}_1(x) = \mathcal{C}_1(y)$. By Lemma 3.8, $I(y)$ is disjoint from $I(x)$. Hence, also $A_2(y)$ is disjoint from $A_2(x)$. Hence, $\mathcal{C}_2(x) = \mathcal{C}_2(y)$. ■

Lemma 3.14 *The equivalence relation R has precisely two classes.*

Proof. Let x be an arbitrary point of P_2 and let M be an arbitrary element of $\mathcal{M}$. We will prove that either $M \in \mathcal{C}_1(x)$ or $M \in \mathcal{C}_2(x)$.

If $M \cap A_1(x) \subseteq P_3$, then $M \in \mathcal{C}_1(x)$. Suppose therefore that there exists a point $y \in M \cap A_1(x) \cap P_2$. We prove by induction on $\mathrm{d}(x,y)$ that there exists a path in $A_1(x)$ which connects the points x and y and which entirely consists of points of P_2. Obviously, this holds if $\mathrm{d}(x,y) \leq 1$. So, suppose that $\mathrm{d}(x,y) \geq 2$. Let L_x denote an arbitrary line through x contained in $\langle x,y\rangle$, let z denote the unique point on L_x at distance $\mathrm{d}(x,y)-1$ from y and let L_y be a line of $\langle x,y\rangle$ through y not contained in $\langle x,z\rangle$. Then every point of L_x has distance $\mathrm{d}(x,y)-1$ from a unique point of L_y. Since $|L_x|, |L_y| \geq 3$ and $|L_x \cap P_3|, |L_y \cap P_3| \leq 1$ (recall Lemma 2.8), there exist points $x' \in L_x \setminus P_3$ and $y' \in L_y \setminus P_3$ at at distance $\mathrm{d}(x,y)-1$ from each other. By the induction hypothesis, the points $x' \in P_2$ and $y' \in P_2$ are connected by a path of $A_1(x)$ which entirely consists of points of P_2. Hence, also x and y are connected by such a path.

Now, by Lemma 3.13, we have $\{\mathcal{C}_1(x), \mathcal{C}_2(x)\} = \{\mathcal{C}_1(y), \mathcal{C}_2(y)\}$. Since either $M \in \mathcal{C}_1(y)$ or $M \in \mathcal{C}_2(y)$, we have that either $M \in \mathcal{C}_1(x)$ or $M \in \mathcal{C}_2(x)$. ■

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote the two classes of the equivalence relation R. Let π_i, $i \in \{1,2\}$, be the set of points of $W(2n-1,\mathbb{K})$ corresponding to the maxes of $\mathcal{C}_i$. Every maximal totally isotropic subspace meeting $\pi_1 \cup \pi_2$ belongs to H.

Lemma 3.15 *The set π_i is a subspace of the ambient projective space* $\mathrm{PG}(2n-1,\mathbb{K})$ *of* $W(2n-1,\mathbb{K})$.

Proof. Let L be a line of $\mathrm{PG}(2n-1,\mathbb{K})$ containing two distinct points x and y of π_i. For every point z of L, let M_z denote the max of Δ corresponding to z.

Suppose L is a hyperbolic line of $W(2n-1,\mathbb{K})$. Then the maxes M_z, $z \in L$, are mutually disjoint, and every line meeting M_x and M_y also meets every M_z, $z \in L \setminus \{x, y\}$. It follows that all the maxes M_z, $z \in L$, are deep and belong to the equivalence class $\mathcal{C}_i$ of R. This proves that $L \subseteq \pi_i$.

Suppose L is a line of $W(2n-1,\mathbb{K})$. Then there exists a convex subspace A of diameter $n-2$ in Δ which is contained in all maxes M_z, $z \in L$. We have $A = M_x \cap M_y \subseteq P_3$. Hence, all maxes M_z, $z \in L$, are deep and belong to the equivalence class $\mathcal{C}_i$ of R. This proves that $L \subseteq \pi_i$. ■

Lemma 3.16 *Let M denote an arbitrary deep max of $\mathcal{C}_1$. Then for every point x of M, there exists a max $M_x \in \mathcal{C}_2$ containing x.*

Proof. If $x \in P_2$, then we are done. So, suppose $x \in P_3$. Let y denote a point of $M \cap P_2$ with $\mathrm{d}(x, y)$ as small as possible. Every line of $\langle x, y\rangle$ through y contains a point at distance $\mathrm{d}(x, y) - 1$ from x. This point belongs to P_3 and hence is contained in $I(y)$ by Lemma 3.8. It follows that $\langle x, y\rangle \subseteq I(y)$. So, if M_x denotes the unique deep max through y different from M, then $M_x \in \mathcal{C}_2$ and $x \in M_x$. ■

Corollary 3.17 *Let M denote a deep max of $\mathcal{C}_1$ and let u denote the unique point of π_1 corresponding to M. Then every maximal totally isotropic subspace of $W(2n-1,\mathbb{K})$ containing u meets π_2.* ■

Lemma 3.18 *We have* $\dim(\pi_2) = n - 1$.

Proof. Suppose $\dim(\pi_2) \geq n$. Then every maximal totally isotropic subspace of $W(2n-1,\mathbb{K})$ meets π_2 and hence belongs to H, a contradiction.

Suppose $\dim(\pi_2) \leq n-2$. Let u denote an arbitrary point of π_1. The subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ through u contained in u^ζ define a projective space u^ζ/u of dimension $2n-3$ and the totally isotropic subspaces of $\mathrm{PG}(2n-1,\mathbb{K})$ through u define a polar space $W(2n-3,\mathbb{K})$ in u^ζ/u. The space $\langle \pi_2 \cap u^\zeta, u\rangle$ has dimension at most $n-2$ in u^ζ/u. One easily proves (see e.g. Lemma 2.3 of [7]) that there exists a maximal singular subspace in $W(2n-3,\mathbb{K})$ disjoint from $\langle \pi_2 \cap u^\zeta, u\rangle$ (in u^ζ/u). This implies that there exists a maximal totally isotropic subspace of $W(2n-1,\mathbb{K})$ through u disjoint from π_2, contradicting Corollary 3.17.

Hence, $\dim(\pi_2) = n-1$. ■

The following lemma finishes the proof of Proposition 1.4.

Lemma 3.19 *The hyperplane H is of subspace-type.*

Proof. The hyperplane H_{π_2} of subspace-type is contained in H and is a maximal proper subspace of $DW(2n-1,\mathbb{K})$. It follows that $H = H_{\pi_2}$. ■

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