On the non-existence of a maximal partial spread of size 76 in $PG(3,9)$

O. Heden, S. Marcugini, F. Pambianco and L. Storme

September 13, 2005

Abstract

We prove the non-existence of maximal partial spreads of size 76 in $PG(3,9)$. Relying on the classification of the minimal blocking sets of size 15 in $PG(2,9)$ [22], we show that there are only two possibilities for the set of holes of such a maximal partial spread. The weight argument of Blokhuis and Metsch [3] then shows that these sets cannot be the set of holes of a maximal partial spread of size 76. In [17], the non-existence of maximal partial spreads of size 75 in $PG(3,9)$ is proven. This altogether proves that the largest maximal partial spreads, different from a spread, in $PG(3,q=9)$ have size $q^2-q+2=74$.

1 Introduction

A *maximal partial spread* in $PG(3,q)$ is a set S of mutually skew lines such that any line of $PG(3,q)$ intersects at least one of the lines of S. The *deficiency* δ of a maximal partial spread in $PG(3,q)$ of size n is the integer $\delta = q^2+1-n$. A *spread* is a set of q^2+1 mutually skew lines in $PG(3,q)$. Maximal partial spreads were first studied by Mesner in 1967 [21]. He observed that if you pick a line ℓ_1 in $PG(3,q)$, and then a second line ℓ_2 skew to the first line, and then a third line ℓ_3 skew to these two lines, and so on, then this process either terminates before a certain bound, or can be continued until you get a spread.

Bruen extended Mesner's result. He showed in 1971 [5] that $q+\sqrt{q} < |S| \leq q^2+1-\sqrt{q}$ for a maximal strictly partial spread S in $PG(3,q)$.

Many constructions of maximal partial spreads of size q^2-q+2 in $PG(3,q)$ are known [5, 6, 7, 20].

There have been several improvements to these results, see [3]. The best upper bound for maximal strictly partial spreads in $PG(3,q)$ is now given by Blokhuis. It follows from his results on blocking sets [2] that $|S| < p^2+1-\frac{p+1}{2}$ for a maximal strictly partial spread S in $PG(3,p)$, p prime.

In [15] it was shown that this bound cannot be improved in general. An example of a maximal partial spread in $PG(3,q)$, for $q=7$, of size $45 = q^2-q+3 = q^2-\frac{q+1}{2}$ was found.

Maximal partial spreads in $PG(3,8)$ with deficiency $\delta \leq q-2$ have been studied by Barát, Del Fra, Innamorati and Storme [1]. Here it was shown that the largest strictly maximal partial spreads of $PG(3,8)$ have size q^2-q+2.

The next open problem to settle is the question of the existence of a maximal partial spread of size 75 or 76 in $PG(3,9)$.

In the construction of a maximal partial spread of size 45 in $PG(3,7)$ [15], the set of points not lying on a line of the maximal partial spread was first constructed. Such points are called *holes* of the maximal partial spread. It can be proved, see the next section or [9], that the set of holes must satisfy certain conditions. The study of the set of holes has been a useful tool in proving (non-)existence of maximal partial spreads, see e.g. [9], [4] and [3].

We will show that, in case of a maximal partial spread of size 76 in $PG(3,9)$, two non-isomorphic candidate sets for the set of holes satisfying all these conditions exist.

The weight argument of Blokhuis and Metsch [3] then can be used to eliminate the existence of maximal partial spreads of size 76 in $PG(3,9)$.

In [17], the non-existence of maximal partial spreads of size 75 in $PG(3,9)$ is proven. This altogether proves that the largest maximal partial spreads, different from a spread, in $PG(3,q=9)$ have size $q^2-q+2=74$.

Let us also mention that Glynn proved in 1981 that no maximal partial spread in $PG(3,q)$ has a size smaller than $2q$ [8] and that several maximal partial spreads of size at most q^2-q+2 have been constructed, see [10], [11], [12], [13] and [14].

Acknowledgement

This research was carried out with the support of the Swedish NFR, of the Italian MIUR (progetto "Strutture geometriche, combinatorie e loro applicazioni") and of GNSAGA.

2 Preliminaries

For a general introduction to this subject, see e.g. [18].

To any partial spread S of $PG(3,q)$ of deficiency δ corresponds a colouring of the points and the planes of $PG(3,q)$ in the colours white and black. A *white plane* is a plane not containing a line of S, while a *black plane* contains a line of S. Similarly, a *white point* is a point not lying on a line of S, while a *black point* lies on a line of S. In the literature, the white points are also called the *holes* of S.

The following properties are well-known:

(i) *Any white plane contains $\delta+q$ white points;*

(ii) *Any black plane contains δ white points.*

Since the definition of a partial spread is self-dual, also the dual properties are valid:

(iii) *Any white point is contained in $q+\delta$ white planes;*

(iv) *Any black point is contained in δ white planes.*

By simple counting arguments, see [9], it is easy to prove that

(v) *The number of white points is* $\delta(q+1)$*;*

(vi) *The number of white planes is* $\delta(q+1)$*.*

A line with exactly α white points will be called *a line of weight* α or an α*-line.* In [9], it was proved that

(vii) *Any* α*-line is contained in exactly* α *white planes;*

(viii) *The weight* α *of a line either equals* $q+1$ *or is less than or equal to* δ*;*

(ix) *The partial spread* S *is maximal if and only if there is no* $(q+1)$*-line.*

There are a number of almost trivial consequences of the conditions (i), (ii), (iii) and (iv). We will use the following lemmas. To make this paper more self-contained we give the proofs of them. Similar proofs can also be found in [16].

Lemma 1 *The set of white points of any white plane constitutes a blocking set of the plane.*

Proof. The intersection line of a white plane π and any other plane, black or white, is a ν-line with $\nu \geq 1$. Hence, π cannot contain any 0-line. □

Lemma 2 *The intersection point of a* δ*-line,* $\delta \leq q$*, and any white plane* π *is a white point.*

Proof. We may assume that the δ-line ℓ is not contained in the white plane π.

Assume that the intersection point of ℓ and π is a black point P. Consider a black plane π_B through ℓ. The plane π_B intersects π at a line containing the point P. By the previous lemma any line through P in π contains at least one white point. Consequently, the black plane π_B will contain at least $\delta+1$ white points, which is impossible. □

We will use the following description of the points in $PG(3,q)$ and $PG(2,q)$.

We consider the affine geometry $AG(3,q)$. The points of this geometry are described by 3-tuples (x,y,z), where $x,y,z \in GF(q)$. We extend $AG(3,q)$ to the projective geometry $PG(3,q)$ by adjoining the slopes of the lines of $AG(3,q)$.

The point (x,y), $x,y \in GF(q)$, of the *plane at infinity* will be the slope of the lines:

$$\{(x_1,x_2,x_3) \mid (x_1,x_2,x_3) = (a,b,0) + t(x,y,1), \quad t \in GF(q)\}, \quad (a,b,0) \in AG(3,q).$$

The point $\overline{x}$, $x \in GF(q)$, of the *line at infinity* will be the slope of the line

$$\{(x_1,x_2,x_3) \mid (x_1,x_2,x_3) = (a,b,0) + t(1,x,0), \quad t \in GF(q)\}, \quad (a,b,0) \in AG(3,q).$$

The point $\overline{\infty}$ of the line at infinity will be the slope of the line

$$\{(x_1,x_2,x_3) \mid (x_1,x_2,x_3) = (a,b,0) + t(0,1,0), \quad t \in GF(q)\}, \quad (a,b,0) \in AG(3,q).$$

We will say that the first set of lines are the *vertical lines*, the remaining set of lines the *horizontal lines*, and the line $\overline{0}\overline{\infty}$ is the *line at infinity.*

The description of the points of the projective plane $PG(2,q)$ is similar. The points are 2-tuples (x,y), $x,y \in GF(q)$, of the affine plane $AG(2,q)$ and the points of the line at infinity are the slopes of the lines of the affine plane.

The finite field $GF(9)$ will be considered. We will let $GF(9) = \{a\iota + b \mid a,b \in \mathbb{Z}_3\}$, where $\iota^2 = 2\iota + 1$.

3 Proof of the results

Throughout this section we assume that there is a maximal partial spread in $PG(3,9)$ of size 76 and therefore of deficiency $\delta = 6$. In a white plane, the white points form a non-trivial blocking set of size 15. The non-trivial blocking sets of size 15 were classified by Pambianco and Storme [22].

(1) The first example is a non-trivial blocking set which consists of a Baer subplane $PG(2,3)$ plus two extra points.

(2) The second example is the projective triangle [19, Lemma 13.6]. This is the set of points projectively equivalent to the set $\{\overline{\infty}\} \cup \{(x^2,0) \mid x \in GF(9)\} \cup \{(0,y^2) \mid y \in GF(9)\} \cup \{\overline{d} \mid d = -x^2,\ \ x \in GF(9)\}$.

There are exactly three non-concurrent 6-secants to the projective triangle. The intersection points of two of these 6-secants are called the *vertices* of the projective triangle.

A vertex lies on two 6-secants, four 2-secants and four tangents to the projective triangle. A non-vertex point of the projective triangle lies on one 6-secant, four 3-secants, one 2-secant and four tangents.

(3) There is also a third sporadic example.

In $PG(2,9)$, there is a unique complete 6-arc [19, p. 386]. The 15 bisecants to this complete 6-arc form a minimal blocking set in the dual projective plane.

So, dualizing, a sporadic example of a minimal blocking set of size 15 arises.

The characteristic properties of this sporadic example are:

1. There are exactly six 5-secants to this blocking set which form a complete 6-arc of lines.
2. There are ten 3-secants to the blocking set. These ten 3-secants form a dual conic.
3. And furthermore, there are fifteen 2-secants to the blocking set. These fifteen 2-secants are the secants to a complete 6-arc in $PG(2,9)$.

Our first goal is to prove that there is no white plane in which the white points form a Baer subplane plus two points. Such a blocking set always has 4-lines.

3.1 There are no 4-lines

Proposition 1. *If there is a maximal partial spread in $PG(3,9)$ of deficiency $\delta = 6$, then there will be no 4-line.*

Proof. Assume that there is a 4-line ℓ. Let $\pi_1, \pi_2, \ldots, \pi_6$ be the black planes of ℓ. There will be two white points P_{i1} and P_{i2} in each of the sets $\pi_i \setminus \ell$, $i = 1,2,\ldots,6$.

Let P be any black point of ℓ. There is, by (vii) of Section 2, at least one white plane that contains P and the point P_{11}. The line ℓ contains six black points. Hence there will be at least six white planes through P_{11} that meet ℓ at a black point.

Each of these six white planes contains, by (vii) of Section 2, at least one white point of each of the black planes π_2, π_3, ..., π_6. It follows that at least three of these six white planes share the same white point P_{21} with $\pi_2 \setminus \ell$. Denote these three planes by π'_{i_1}, π'_{i_2} and π'_{i_3}. These planes must share a white point with each of the planes π_3, π_4, ..., π_6. If there is no white point of π_3 on the common intersection line $P_{11}P_{21}$ of the planes π'_{i_1}, π'_{i_2} and π'_{i_3}, then these three planes must share three distinct white points with $\pi_3 \setminus \ell$. As $\pi_3 \setminus \ell$ only contains two white points, this is impossible. So the line $P_{11}P_{21}$ is an α-line with $\alpha \geq 6$. As there is no α-line with $\alpha > 6$, we get that $\alpha = 6$.

From above we know that there are three of the white planes through the line $P_{11}P_{21}$ such that each one of them contains a black point of ℓ. Hence at most three of the six white planes through $P_{11}P_{21}$ contain a white point of ℓ. It follows that at least one of the black planes through the line $P_{11}P_{21}$ will contain a white point of ℓ. This plane will have at least seven white points. This is impossible for a black plane. □

3.2 Two distinct cases

The preceding result implies that the white points in a white plane form a minimal blocking set; either the projective triangle or the sporadic example. The projective triangle has three non-concurrent 6-lines and the sporadic example has six 5-lines, but no 6-lines.

Since the definition of maximal partial spreads is self-dual, also the dual result is valid.

Corollary 1. *If there is a maximal partial spread of deficiency $\delta = 6$ in $PG(3, 9)$, then any white point is contained in three 6-lines or six 5-lines.*

Presently, it is still possible that both types of minimal blocking sets occur in the distinct white planes. We show this possibility does not occur.

Proposition 2. *Assume that there is a maximal partial spread of deficiency $\delta = 6$ in $PG(3, 9)$. If there is a 6-line in $PG(3, 9)$, then there will be no 5-line.*

Proof. Let ℓ be a 6-line and consider the white planes $\pi_1, \pi_2, \dots, \pi_6$ through ℓ. These planes contain all the white points. Any white point of the white planes π_i, $i = 1, 2, \dots, 6$, will, by the description of the projective triangle, lie on at least one 6-line. By Corollary 1, no white point will be contained in a 5-line. □

The preceding proposition implies that all the white planes either contain a projective triangle of white points, or a sporadic minimal blocking set of white points. So, either all white planes contain exactly three 6-lines or all white planes contain six 5-lines.

We discuss both cases below.

3.3 The case of three 6-lines in all white planes

We will show that if there is no 5-line, then we may form tetrahedra consisting of 6-lines.

Lemma 3. *Let the* 6*-lines* ℓ_1, ℓ_2 *and* ℓ_3 *be the sides of a projective triangle. Let* $Q = \ell_1 \cap \ell_2$, *and let* ℓ *be the third* 6*-line through* Q. *Then there is a white point* P *on* ℓ *such that through* P *there are two* 6*-lines that meet the line* ℓ_3.

Proof. Let π denote the white plane containing the lines ℓ_1, ℓ_2 and ℓ_3. Through any white point P of ℓ there are two other 6-lines. These two 6-lines meet, by Lemma 2, the plane π at white points. These white points might be on the lines ℓ_1 or ℓ_2. The white planes π_1 and π_2 containing the lines ℓ and ℓ_1, respectively ℓ and ℓ_2, contain only three 6-lines each. As the line ℓ contains six white points, we may thus choose P such that the two 6-lines through P do not meet neither the line ℓ_1 nor the line ℓ_2. $\square$

Lemma 4. *Let the* 6*-lines* ℓ_1, ℓ_2 *and* ℓ_3 *be the sides of a projective triangle. Let* $Q = \ell_1 \cap \ell_2$, *and let* ℓ *be the third* 6*-line through* Q. *Through any white point* P *on* ℓ *there are two* 6*-lines that meet the line* ℓ_3.

Proof. Let π, π_1 and π_2 be as in the previous proof. By the preceding lemma, there is a white point P_0 of ℓ through which there are two 6-lines that meet the line ℓ_3. Let π_0 denote the white plane that contains the line ℓ_3 and the point P_0. Let $P' \neq Q, P_0$ be any other white point of ℓ. Consider any of the two 6-lines through P' and distinct from ℓ. Denote this line by ℓ'. By Lemma 2, the line ℓ' intersects the plane π in a white point P_1. If P_1 is not contained in the line ℓ_3, then $P_1 \in \ell_i$, $i = 1$ or 2. The line $\ell' = P'P_1$ intersects the plane π_0 at a white point. This white point must be on one of the two 6-lines from P_0 to the line ℓ_3. This 6-line must then be contained in the same plane as ℓ, ℓ' and the line ℓ_i. It follows that the plane π_i contains four 6-lines, which is impossible. $\square$

Lemma 5. *To any white plane* π *containing three* 6*-lines* ℓ_1, ℓ_2 *and* ℓ_3 *constituting a projective triangle in* π *with vertices* P_1, P_2 *and* P_3, *there is a white point* Q, $Q \notin \pi$, *such that the lines* QP_i, *for* $i = 1, 2, 3$, *are* 6*-lines.*

Proof. Consider the third 6-line ℓ meeting the intersection point P_1 of the lines ℓ_1 and ℓ_2, $\ell \neq \ell_1, \ell_2$. From the previous lemma we deduce that there must be a white point Q on ℓ such that the line QP_2, where P_2 is the intersection point of ℓ_1 and ℓ_3, is a 6-line. Once again using the previous lemma, we get that there is a 6-line $\ell' \neq \ell$ passing through Q and meeting the line ℓ_3. A final use of previous lemma, with the line QP_2 playing the role of the line ℓ in that lemma, there must be a 6-line $\ell'' \neq \ell$ passing through Q and meeting the line ℓ_2. By Corollary 1 there are only three 6-lines that meet the point Q. Hence the line ℓ' equals the line ℓ'' and that line meets the intersection point P_3 of the lines ℓ_2 and ℓ_3. $\square$

Corollary 2. *Consider a tetrahedron in* $PG(3, 9)$ *and the set of white points associated with a maximal partial spread of size* 76. *If five of the lines of the tetrahedron are* 6*-lines then also the sixth is a* 6*-line.*

Proof. Assume that P and Q are the vertices of the tetrahedron of which it is known that all lines of the tetrahedron, with the exception of the line PQ, are 6-lines. The 6-lines of the tetrahedron not meeting the point Q will constitute a projective triangle. We will let the sides of this triangle correspond to the lines ℓ_1, ℓ_2 and ℓ_3 of Lemma 5. The point Q will then correspond to the point Q in that lemma. $\square$

We now show that in case $q = 9$, $\delta = 6$, and if there is no 5-line, then there is up to equivalence one and only one way to colour the points in white and black such that the conditions (i), (ii), (iii) and (iv) of Section 2 are satisfied.

We will use the following lemma several times.

Lemma 6. *Consider any white plane π. If you know*

(a) *which three lines ℓ_1, ℓ_2 and ℓ_3 of π are 6-lines,*

(b) *the six white points of the 6-line ℓ_1,*

(c) *one white point $P_2 \in \ell_2$, $P_2 \notin \ell_1$, $P_2 \notin \ell_3$,*

then the white points of π are uniquely determined.

Proof. From the description of the projective triangle, for any white point W of ℓ_1 that is not a vertex, the line P_2W intersects ℓ_3 at a white point. To find the white points of ℓ_2, consider lines through one of the white points of ℓ_3 and the white points of ℓ_1. $\square$

We will use the following notations: $\ell_x = \{(t,0,0) \mid t \in GF(9)\} \cup \{\overline{0}\}$, $\ell_y = \{(0,t,0) \mid t \in GF(9)\} \cup \{\overline{\infty}\}$, $\ell_z = \{(0,0,t) \mid t \in GF(9)\} \cup \{(0,0)\}$, and $\ell_\infty = \overline{0}\overline{\infty}$. Let $\pi_z = \langle l_\infty, (0,0,z)\rangle$, $z \in GF(9)$, $\pi_{xz} = \langle l_x, l_z\rangle$, $\pi_{yz} = \langle l_y, l_z\rangle$, and π_∞ denotes the plane at infinity. Then $l_x^\infty = \pi_{xz} \cap \pi_\infty$ and $l_y^\infty = \pi_{yz} \cap \pi_\infty$.

Without loss of generality, and using the description of the projective triangle, we may assume that the plane π_0 is a white plane and that the white points of this plane are the points in the union of the sets $\{\overline{\infty}\}$, $\{(x^2,0,0) \mid x \in GF(9)\}$, $\{(0,y^2,0) \mid y \in GF(9)\}$ and $\{\overline{d} \mid d = -x^2,\ x \in GF(9)\}$. (We note that -1 is a square of $GF(9)$.)

We now use Lemma 5. The lines ℓ_1, ℓ_2 and ℓ_3 in that lemma will correspond to the lines ℓ_x, ℓ_y and ℓ_∞. Without loss of generality we may let Q be the point $(0,0)$. Hence the lines ℓ_z, ℓ_x^∞ and ℓ_y^∞ will be 6-lines.

From the proof of Proposition 2 and using a perspectivity with axis π_0 and center $(0,0)$, we may assume that the points $\{(0,0,z^2) \mid z \in GF(9)\} \cup \{(0,0)\}$ are the white points of the line ℓ_z and, then by Lemma 6, the points $\{\overline{0}\} \cup \{(x^2,0) \mid x \in GF(9)\}$ are the white points of the line ℓ_x^∞ and similarly for the line ℓ_y^∞.

The remaining 32 white points are on the white planes through the line at infinity. As the planes π_{x^2}, $x \in GF(9)$, contain at least seven white points, these planes are the white planes through the line at infinity. Each of the two 6-lines, distinct from the line at infinity, of the white plane π_{1^2} meets, by Lemma 4, the point $(0,0,1)$ and a white point on the line at infinity. That point cannot be neither the point $\overline{0}$ nor the point $\overline{\infty}$. Hence there are only $\binom{4}{2} = 6$ possibilities for the 6-lines of the plane π_{1^2}. Let P_1 and P_2 denote these intersection points with the line at infinity. For the plane π_{ι^2}, let Q_1 and Q_2 be the intersection points on the line at infinity for the two 6-lines of π_{ι^2} that meet the point $(0,0,\iota^2)$. There are, by Corollary 2, only two possibilities for the set of points Q_1 and Q_2, either

$$\{Q_1, Q_2\} = \{P_1, P_2\} \quad \text{or} \quad \{Q_1, Q_2\} = \{\overline{\infty}, \overline{0}, \overline{\iota}^2, \overline{\iota}^4, \overline{\iota}^6, \overline{1}\} \setminus \{P_1, P_2, \overline{\infty}, \overline{0}\}.$$

Similarly for the planes π_{ι^4} and π_{ι^6}. So there are only a few possibilities to consider. Searching through these very few cases we found some possible sets of white points satisfying the conditions (i), (ii), (iii) and (iv) of Section 2. These sets of white points can be shown to be equivalent to the following set of white points:

In $AG(3,9)$: (0,0,0), (0,1,0), (0,ι^6,0), (0,2,0), (0,ι^2,0), (1,0,0), (ι^6,0,0), (2,0,0), (ι^2,0,0), (0,0,1), (1,1,1), (2,2,1), (ι^6,ι^6,1), (ι^2,ι^2,1), (1,2,1), (2,1,1), (ι^6,ι^2,1), (ι^2,ι^6,1), (0,0,2), (1,1,2), (2,2,2), (ι^6,ι^6,2), (ι^2,ι^2,2), (1,2,2), (2,1,2), (ι^6,ι^2,2), (ι^2,ι^6,2), (0,0,ι^6), (1,ι^6,ι^6), (2,ι^2,ι^6), (ι^6,2,ι^6), (ι^2,1,ι^6), (1,ι^2,ι^6), (2,ι^6,ι^6), (ι^6,1,ι^6), (ι^2,2,ι^6), (0,0,ι^2), (1,ι^2,ι^2), (2,ι^6,ι^2), (ι^6,1,ι^2), (ι^2,2,ι^2), (1,ι^6,ι^2), (2,ι^2,ι^2), (ι^6,2,ι^2), (ι^2,1,ι^2), and in π_∞: (0,0), (1,0), (2,0), (ι^6,0), (ι^2,0), (0,1), (0,2), (0,ι^6), (0,ι^2), $\overline{\infty}, \overline{0}, \overline{1}, \overline{2}, \overline{\iota}^6$ and $\overline{\iota}^2$.

The weight argument of Blokhuis and Metsch proves that this set cannot be the set of white points of a maximal partial spread of size 76. We slightly generalize the statement of their weight argument.

Lemma 7. (Blokhuis and Metsch [3, Lemma 2.1]) *Consider the affine space $AG(d,q)$ in which the coordinates of the points are described by the d-tuples $(x_1,\ldots,x_d)$ over $GF(q)$.*

Let $f(x_1,\ldots,x_d) = (\prod_{i=1}^d x_i)^t$, with $1 \le t \le q-2$, define a weight *function on the points of $AG(d,q)$. For any set S of points, we define the* weight *of S to be the sum of the weights of the points of S.*

Then,

(1) *the weight of $AG(d,q)$ is zero,*

(2) *if $dt < q-1$, then the weight of every affine subspace of $AG(d,q)$ is zero.*

Theorem 1 *The preceding set cannot be the set of white points of a maximal partial spread of size* 76 *in $PG(3,9)$.*

Proof. Apply Lemma 7 for $t=2$. Then the weight of $AG(3,9)$ is zero, the weight of each line of the maximal partial spread is zero, and so the set of affine white points also must have weight zero. But the exact calculation of the weight of the set of affine white points gives a non-zero weight. □

3.4 The case of six 5-lines in all white planes

The set of holes in a white plane forms a sporadic minimal blocking set of size 15 in this white plane. The following proposition describes coordinates for such a blocking set.

Proposition 3. *Assume that there is a maximal partial spread of deficiency* $\delta = 6$ *in* $PG(3,9)$ *and assume that there are no* 6*-lines. Then any white plane is isomorphic to the plane* π_0 *with the white points* $(0,0,0)$, $(\iota^5,0,0)$, $(2,0,0)$, $(\iota^3,0,0)$, $(0,\iota^5,0)$, $(0,2,0)$, $(0,\iota^3,0)$, $(\iota^5,\iota^5,0)$, $(1,\iota^3,0)$, $(\iota^3,1,0)$, $\overline{0}$, $\overline{\iota}^5$, $\overline{2}$, $\overline{\iota}^3$ *and* $\overline{\infty}$.

Furthermore, any of the white points above is contained in exactly two of the 3*-lines of the plane and exactly two* 5*-lines.*

To construct a set of 60 white points satisfying the conditions (i),(ii), (iii), (iv), we will use the following two lemmas.

Lemma 8. *Any white point of the set of white points associated with a maximal partial spread of size* 76 *in* $PG(3,9)$*, is contained in exactly six* 5*-lines.*

Proof. See Corollary 1. □

Lemma 9. *Let* S *denote the set of white points associated with a maximal partial spread of size* 76 *in* $PG(3,9)$*. Consider any* 5*-line* ℓ *and any two white planes* π_1 *and* π_2 *containing* ℓ*. Let* P *and* Q *be any two distinct white points of* ℓ*. If* S_i, $i = 1,2$, *are the intersection points of the two* 5*-lines, distinct from* ℓ *in* π_i, $i = 1,2$, *that pass through* P *and* Q*, then the line* S_1S_2 *will be a* 5*-line.*

Proof. Without loss of generality we may assume that the line ℓ is the line ℓ_x, π_1 is the plane π_{xy}, π_2 is the plane π_{xz}, P the point $(0,0,0)$, Q the point $\overline{0}$, S_1 the point $\overline{\infty}$ and S_2 the point $(0,0)$ of the plane at infinity.

Further, without loss of generality, we may assume that the white points of π_{xy} are the white points described as in Proposition 3.

It is a triviality to see, by going through all possible cases, that we, without loss of generality, may assume that the white points of the line ℓ_z are the points $(0,0,0)$, $(0,0,\iota^3)$, $(0,0,2)$, $(0,0,\iota^5)$ and $(0,0)$.

The plane π_{yz} will be a white plane. The white points of that plane will thus constitute a blocking set. Using this fact, and again by going through all possible cases, it is a triviality to see that the only possibility to make π_{yz} into a white plane, with the white points distributed on the lines ℓ_y and ℓ_z as described above, is to let the line ℓ_y^∞, i.e. the line S_1S_2, be a 5-line. □

We now construct the set of holes.

Without loss of generality we may assume that the line at infinity is a 5-line and that the plane at infinity and the plane containing the points in the set $\pi_0 = \{(x,y,0) \mid x,y \in GF(9)\}$ are white planes. From Proposition 3 we get that we, without loss of generality, may assume that the white points of these planes are the points

(0,0,0), (ι^5,0,0), (2,0,0), (ι^3,0,0),(0,ι^5,0), (0,2,0), (0,ι^3,0), (ι^5,ι^5,0), (1,ι^3,0), (ι^3,1,0),
(0,0), (ι^5,0), (0,ι^5), (2,0), (0,2), (ι^3,0), (0,ι^3), (ι^5,ι^5), (1,ι^3), (ι^3,1), $\overline{0}$, $\overline{\iota}^5$, $\overline{2}$, $\overline{\iota}^3$, $\overline{\infty}$.

Let ℓ_x and ℓ_x^∞ denote the two lines that meet the points $\overline{0}$ and (0,0,0), respectively $\overline{0}$ and $(0,0)$. Through each of the white points of these 5-lines there is another 5-line of the plane π_0 and π_∞. These 5-lines are easily found from the given set of white points of these planes. By using Lemma 9, we get the 5-lines of the white plane π_{xz}. Any two 5-lines of a white plane intersect at a white point and through any white point of the white plane π_{xz} there are two 5-lines of the plane. Hence all white points of the plane π_{xz} will be found by using the 5-lines of π_{xz}. This will give us another set of six white points, the points

$(0,0,\iota^5)$, $(\iota^5,0,\iota^5)$, $(0,0,\iota^3)$, $(1,0,\iota^3)$, $(\iota^3,0,1)$, $(0,0,2)$.

Similarly, for the plane π_{yz}, we derive another set of three white points

$(0,\iota^5,\iota^5)$, $(0,1,\iota^3)$, $(0,\iota^3,1)$.

Let π_t denote the plane that contains the line at infinity and the point $(0,0,t)$. We get from the white points already found that the planes π_{ι^5}, π_{ι^3} and π_1 will contain more than six white points and hence that they will be white planes.

The point $(0,0,\iota^5)$ is a white point of the plane π_{ι^5}. By Proposition 3, there are two 5-lines ℓ and ℓ' in π_{ι^5} that meet this point. These two 5-lines cannot meet the points $\overline{0}$ or $\overline{\infty}$, as the planes π_{xz} and π_{yz} already contain two 5-lines meeting these points and these 5-lines do not meet the point $(0,0,\iota^5)$. The 5-lines ℓ and ℓ' meet the line at infinity at two of the three other white points distinct from $\overline{0}$ and $\overline{\infty}$. Denote the three white points different from $\overline{0}$ and $\overline{\infty}$ on the line at infinity by p_1, p_2 and p_3, and assume that the lines joining $(0,0,\iota^5)$ with p_1 and p_2 are 5-lines. Since the white point $(0,\iota^5,\iota^5)$ belongs to the line joining $\overline{\infty}$ with $(0,0,\iota^5)$ which is not a 5-line, the 5-lines through the point $(0,\iota^5,\iota^5)$ meet the line at infinity in the point $\overline{0}$ and in p_3. And similarly, since the white point $(\iota^5,0,\iota^5)$ belongs to the line through $(0,0,\iota^5)$ and $\overline{0}$ which is not a 5-line, necessarily, the 5-lines through the point $(\iota^5,0,\iota^5)$ meet the line at infinity in the point $\overline{\infty}$ and in p_3. Since p_3 belongs to exactly two 5-lines in π_{ι^5}, p_3 is necessarily the intersection of the line at infinity with the line joining $(0,\iota^5,\iota^5)$ with $(\iota^5,0,\iota^5)$. So p_3 is uniquely determined and so p_1 and p_2 are determined. The intersection point $(\iota^5,\iota^5,\iota^5)$ of the two 5-lines through respectively $(0,\iota^5,\iota^5)$ and $\overline{0}$, and through $(\iota^5,0,\iota^5)$ and $\overline{\infty}$, must be a white point. We get that the only possibility for the plane π_{ι^5} to be a white plane is that the remaining white points are the points $(1,\iota^3,\iota^5)$, $(\iota^3,1,\iota^5)$, $(\iota^5,1,\iota^5)$, $(1,\iota^5,\iota^5)$, $(\iota^2,\iota^5,\iota^5)$, $(\iota^5,\iota^2,\iota^5)$.

Similarly for the plane π_{ι^3} and π_1. Continuing in the same manner, we finally end up with the following set of white points. They will be: In $AG(3,9)$ the points: $(0,0,0)$, $(\iota^5,0,0)$, $(2,0,0)$, $(\iota^3,0,0)$,$(0,\iota^5,0)$, $(0,2,0)$, $(0,\iota^3,0)$, $(\iota^5,\iota^5,0)$, $(1,\iota^3,0)$, $(\iota^3,1,0)$, $(0,0,\iota^5)$, $(\iota^5,0,\iota^5)$, $(0,\iota^5,\iota^5)$, $(1,\iota^3,\iota^5)$, $(\iota^3,1,\iota^5)$, $(\iota^5,1,\iota^5)$, $(1,\iota^5,\iota^5)$, $(\iota^5,\iota^5,\iota^5)$, $(\iota^2,\iota^5,\iota^5)$, $(\iota^5,\iota^2,\iota^5)$, $(0,0,\iota^3)$, $(1,0,\iota^3)$, $(0,1,\iota^3)$, $(1,1,\iota^3)$, $(\iota^6,\iota^3,\iota^3)$, $(\iota^3,5,\iota^3)$, $(\iota^5,1,\iota^3)$, $(1,\iota^5,\iota^3)$, $(\iota^3,1,\iota^3)$, $(1,\iota^3,\iota^3)$, $(\iota^3,0,1)$, $(0,\iota^3,1)$, $(\iota^3,\iota^3,1)$, $(\iota^5,\iota^5,1)$, $(\iota^3,\iota^5,1)$, $(\iota^5,\iota^3,1)$, $(1,\iota^7,1)$, $(\iota^7,1,1)$ $(1,\iota^3,1)$, $(\iota^3,1,1)$, (ι,ι,ι), $(\iota^5,\iota^5,\iota^2)$, $(\iota^3,\iota^3,\iota^6)$, $(1,1,\iota^7)$, $(0,0,2)$, in π_∞: $(0,0)$, $(\iota^5,0)$, $(0,\iota^5)$, $(2,0)$, $(0,2)$, $(\iota^3,0)$, $(0,\iota^3)$, (ι^5,ι^5), $(1,\iota^3)$, $(\iota^3,1)$, $\overline{0}$, $\overline{\iota}^5$, $\overline{2}$, $\overline{\iota}^3$, $\overline{\infty}$.

It is easily checked, by using computers, that this set satisfies the conditions (i),(ii),(iii) and (iv) of Section 2, with the parameters $q = 9$ and $\delta = 6$. However we now prove the following theorem:

Theorem 2 *The preceding set cannot be the set of holes of a maximal partial spread of size* 76 *in* $PG(3,9)$.

Proof. Apply Lemma 7 for $t = 2$. Then the weight of the set of affine white points is non-zero, and it should be zero. □

Corollary 3 *There does not exist a maximal partial spread of size* 76 *in* $PG(3, 9)$.

Proof. There were only two possibilities for the set of white points and in both cases, Lemma 7 showed that this set cannot be the set of white points of a maximal partial spread of size 76. □

In [17], the non-existence of maximal partial spreads of size 75 in $PG(3, 9)$ is proven. There exist maximal partial spreads of size $q^2 - q + 2 = 74$ in $PG(3, 9)$, so this altogether proves that the largest maximal partial spreads, different from a spread, in $PG(3, q = 9)$ have size $q^2 - q + 2 = 74$.

Theorem 3 *The largest maximal partial spreads, different from a spread, in* $PG(3, q = 9)$ *have size* $q^2 - q + 2 = 74$.

References

[1] J. Barát, A. Del Fra, S. Innamorati and L. Storme, Minimal blocking sets in $PG(2, 8)$ and maximal partial spreads in $PG(3, 8)$, *Des. Codes Cryptogr.* **31** (2004), 15-26.

[2] A. Blokhuis, On the size of a blocking set in PG$(2, p)$, *Combinatorica* **14** (1994), 111-114.

[3] A. Blokhuis and K. Metsch, On the size of a maximal partial spread, *Des. Codes Cryptogr.* **3** (1993), 187-191.

[4] A. Blokhuis, A. E. Brouwer and H. A. Wilbrink, Heden's bound on maximal partial spreads, *Discrete Math.* **74** (1989), 335-339.

[5] A. A. Bruen, Partial spreads and replaceable nets, *Canad. J. Math.* **23** (1971), 381-392.

[6] A. A. Bruen and J.A. Thas, Partial spreads, Packings and Hermitian manifolds in PG$(3, q)$, *Math. Z.* **151** (1976), 207-214.

[7] J.W. Freeman, Reguli and pseudo-reguli in $PG(3, s^2)$. *Geom. Dedicata* **9** (1980), 267-280.

[8] D. G. Glynn, A lower bound for maximal partial spreads in $PG(3, q)$, *Ars Combin.* **13** (1982), 39-40.

[9] O. Heden, Maximal partial spreads and two-weight codes, *Discrete Math.* **62** (1986), 277-293.

[10] O. Heden, A greedy search for maximal partial spreads in $PG(3,7)$, *Ars Combin.* **32** (1991), 253-255.

[11] O. Heden, Maximal partial spreads and the modular n-queen problem, *Discrete Math.* **120** (1993), 75-91.

[12] O. Heden, Maximal partial spreads and the modular n-queen problem II, *Discrete Math.* **142** (1995), 97-106.

[13] O. Heden, Maximal partial spreads in $PG(3,5)$, *Ars Combin.* **57** (2000), 97–101.

[14] O. Heden, Maximal partial spreads and the modular n-queen problem III, *Discrete Math.* **243** (2002), 135-150.

[15] O. Heden, A maximal partial spread of size 45 in $PG(3,7)$, *Des. Codes Cryptogr.* **22** (2001), 331-334.

[16] O. Heden, No maximal partial spread of size 115 in $PG(3,11)$, *Ars Combin.* **66** (2003), 139-155.

[17] O. Heden, S. Marcugini and F. Pambianco, The maximum size of a maximal partial spread in $PG(3,9)$. (In preparation).

[18] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*. Clarendon Press, Oxford, 1985.

[19] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields (Second Edition)*. Oxford: Oxford University Press 1998.

[20] D. Jungnickel, Maximal partial spreads and nets of small deficiency. *J. Algebra* **90** (1984), 119-132.

[21] D. Mesner, Sets of disjoint lines in $PG(3,q)$, *Canad. J. Math.* **19** (1967), 273-280.

[22] F. Pambianco and L. Storme, Minimal blocking sets in $PG(2,9)$, *Ars Combin.*, submitted.

O. Heden, Department of Mathematics, KTH, S-100 44 Stockholm, Sweden
(olohed@math.kth.se)

S. Marcugini and F. Pambianco, Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli, 1, I-06123 Perugia, Italy.
(gino@dipmat.unipg.it, fernanda@dipmat.unipg.it)

L. Storme, Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281 - S22, 9000 Ghent, Belgium.
(ls@cage.ugent.be, http://cage.ugent.be/∼ls)