The near hexagon $\mathbb{H}_3$

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 12 & 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 8 & 0 & 3 & 4 \\ 0 & 0 & 8 & 6 & 6 \end{bmatrix}$$

$$B_{2a} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 \\ 12 & 2 & 5 & 0 & 1 \\ 0 & 0 & 0 & 6 & 4 \\ 0 & 8 & 4 & 6 & 5 \end{bmatrix}, \quad B_{2b} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 8 & 0 & 3 & 4 \\ 0 & 0 & 0 & 6 & 4 \\ 32 & 8 & 16 & 10 & 8 \\ 0 & 16 & 16 & 12 & 16 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 8 & 6 & 6 \\ 0 & 8 & 4 & 6 & 5 \\ 0 & 16 & 16 & 12 & 16 \\ 48 & 24 & 20 & 24 & 20 \end{bmatrix}$$

$$\boxed{B_* = A_*^{\theta}}$$

$$d_{H_3} = \frac{s^5 v}{(s+1)^2(s-1)(s^2+1)+st(s-1)(s+1)^2+v} = \frac{32\cdot 105}{9\cdot 5+2\cdot 5\cdot 9+105} = 14$$

Eigenspace	Dimension	I	A_1	A_{2a}	A_{2b}	A_3	J
E_1	14	1	-6	3	8	-6	0
E_2	56	1	-1	-2	-2	4	0
E_3	14	1	2	7	-8	-2	0
E_4	20	1	5	-2	4	-8	0
E_5	1	1	12	12	32	48	105

$A_{2a} = p_{2a}(A)$, where

$$p_{2a}(x) = \frac{2}{143}x^4 - \frac{3}{22}x^3 - \frac{92}{143}x^2 + \frac{1139}{286}x + \frac{354}{143}$$

$A_{2b} = p_{2b}(A)$, where

$$p_{2b}(x) = -\frac{4}{429}x^4 + \frac{1}{11}x^3 + \frac{109}{143}x^2 - \frac{1282}{429}x - \frac{808}{143}$$

$A_2 = p_2(A)$, where

$$p_2(x) = \frac{2}{429}x^4 - \frac{1}{22}x^3 + \frac{17}{143}x^2 + \frac{853}{858}x - \frac{454}{143}$$

$A_3 = p_3(A)$, where

$$p_3(x) = \frac{1}{572}x^4 + \frac{1}{22}x^3 - \frac{189}{572}x^2 - \frac{519}{286}x + \frac{366}{143}$$

The near hexagon $\mathbb{E}_3$

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 30 & 1 & 3 & 5 & 0 \\ 0 & 12 & 3 & 0 & 9 \\ 0 & 16 & 0 & 5 & 6 \\ 0 & 0 & 24 & 20 & 15 \end{bmatrix}$$

$$B_{2a} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 12 & 3 & 0 & 9 \\ 120 & 12 & 36 & 30 & 21 \\ 0 & 0 & 24 & 30 & 18 \\ 0 & 96 & 56 & 60 & 72 \end{bmatrix}, \quad B_{2b} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 16 & 0 & 5 & 6 \\ 0 & 0 & 24 & 30 & 18 \\ 96 & 16 & 24 & 20 & 12 \\ 0 & 64 & 48 & 40 & 60 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 24 & 20 & 15 \\ 0 & 96 & 56 & 60 & 72 \\ 0 & 64 & 48 & 40 & 60 \\ 320 & 160 & 192 & 200 & 172 \end{bmatrix}$$

$$\boxed{B_* = A_*^{\emptyset}}$$

$$d_{E_3} = \frac{s^5 v}{(s+1)^2(s-1)(s^2+1) + sk(s-1)(s+1)^2 + v} = \frac{32 \cdot 567}{9 \cdot 5 + 2 \cdot 14 \cdot 9 + 567} = 21$$

Eigenspace	Dimension	I	A_1	A_{2a}	A_{2b}	A_3	J
E_1	21	1	-15	30	24	-40	0
E_2	315	1	-3	-6	0	8	0
E_3	140	1	3	12	-12	-4	0
E_4	90	1	9	-6	12	-16	0
E_5	1	1	30	120	96	320	567

$A_2 = p_2(A_1)$, where

$$p_2(x) = \frac{19}{35640}x^4 - \frac{127}{11880}x^3 + \frac{191}{3960}x^2 + \frac{1447}{1320}x - \frac{153}{44}$$

The G_3 near hexagon

$$B_1 = B_{1a} + B_{1b} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 0 & 0 \\ 18 & 0 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 6 & 2 & 2 & 0 & 0 & 0 & 9 & 3 & 3 \\ 0 & 0 & 12 & 0 & 3 & 0 & 0 & 0 & 6 & 6 \\ 0 & 4 & 2 & 0 & 0 & 2 & 1 & 3 & 2 & 0 \\ 0 & 12 & 6 & 0 & 0 & 3 & 4 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 6 & 6 & 12 & 2 & 9 & 5 & 3 \\ 0 & 0 & 0 & 12 & 12 & 0 & 12 & 0 & 6 & 9 \end{bmatrix}$$

$$B_2 = B_{2a} + B_{2b} + B_{2c} + B_{2d} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 6 & 1 & 0 & 2 & 2 & 3 & 3 & 3 \\ 0 & 18 & 16 & 1 & 3 & 3 & 3 & 9 & 9 & 9 \\ 36 & 6 & 2 & 17 & 18 & 18 & 18 & 18 & 12 & 12 \\ 72 & 0 & 12 & 36 & 36 & 36 & 36 & 18 & 24 & 24 \\ 12 & 4 & 2 & 6 & 6 & 5 & 7 & 6 & 5 & 3 \\ 36 & 12 & 6 & 18 & 18 & 21 & 19 & 9 & 10 & 12 \\ 0 & 4 & 4 & 4 & 2 & 4 & 2 & 3 & 4 & 3 \\ 0 & 36 & 36 & 24 & 24 & 30 & 20 & 36 & 29 & 30 \\ 0 & 72 & 72 & 48 & 48 & 36 & 48 & 54 & 60 & 60 \end{bmatrix}$$

$$d_{\Phi_3^*} = \frac{s^5 v}{(s+1)^2(s-1)(s^2+1)+sk(s-1)(s+1)^2+v} = \frac{32\cdot 405}{9\cdot 5+2\cdot 11\cdot 9+405} = 20$$

Eigenvalues A_1	-12	-3	0	3	6	9	24
Eigenvalues A_2	39	-6	-3	0	3	6	156
Multiplicities	20	175	60	90	20	39	1

$A_2 = p(A_1)$, where

$$p(x) = \frac{17}{4898880}x^6 - \frac{23}{233280}x^5 + \frac{31}{36288}x^4 - \frac{59}{36288}x^3 - \frac{241}{30240}x^2 + \frac{859}{840}x - 3$$

The product near hexagons

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the direct product of a $GQ(1,k_2)$, $k_2 \neq 1$, with a line of size $s+1$. Let S denote the set of all lines of $\mathcal{S}$ not contained in any quad of order $(1,k_2)$. Then S is a line spread of $\mathcal{S}$. Consider the following relations on the set $\mathcal{P}$:

- $R_0 = \{(x,x) \mid x \in \mathcal{P}\}$;
- $R_{1a} = \{(x,y) \in \mathcal{P} \times \mathcal{P} \mid d(x,y) = 1 \text{ and } xy \in S\}$;
- $R_{1b} = \{(x,y) \in \mathcal{P} \times \mathcal{P} \mid d(x,y) = 1 \text{ and } xy \notin S\}$;
- $R_{2a} = \{(x,y) \in \mathcal{P} \times \mathcal{P} \mid d(x,y) = 2 \text{ and } \langle x,y \rangle \text{ is a quad of order } (1,k_2)\}$;
- $R_{2b} = \{(x,y) \in \mathcal{P} \times \mathcal{P} \mid d(x,y) = 2 \text{ and } \langle x,y \rangle \text{ is grid}\}$;
- $R_3 = \{(x,y) \in \mathcal{P} \times \mathcal{P} \mid d(x,y) = 3\}$

$$
\begin{array}{lllllll}
A\cdot I = & - & + A_{1a} & + A_{1b} & + - & + - & + - \\
A\cdot A_{1a} = & s\cdot I & + (s-1)A_{1a} & + - & + - & + A_{2b} & + - \\
A\cdot A_{1b} = & s(t_2+1)\cdot I & + - & + (s-1)A_{1b} & + (t_2+1)A_{2a} & + A_{2b} & + - \\
A\cdot A_{2a} = & - & + - & + s t_2\cdot A_{1b} & + (s-1)(t_2+1)A_{2a} & + - & + A_3 \\
A\cdot A_{2b} = & - & + s(t_2+1)A_{1a} & + s\cdot A_{1b} & + - & + 2(s-1)A_{2b} & + (k_2+1)A_3 \\
A\cdot A_3 = & - & + - & + - & + s\cdot A_{2a} & + s k_2\cdot A_{2b} & + (s-1)(t_2+2)A_3
\end{array}
$$

$$
d_g = \frac{s^5 v}{(s+1)^2(s-1)(s^2+1) + s t(s-1)(s+1)^2 + v} = \frac{s^5 \cancel{(s+1)^2}(s k_2+1)}{\cancel{(s+1)^2}(s-1)(s^2+1) + s(k_2+1)(s-1)\cancel{(s+1)^2} + \cancel{(s+1)^2}(s k_2+1)}
$$

$$
= \frac{s^5(s k_2+1)}{s^3 \cancel{-s^2} \cancel{+s} \cancel{-1} + \cancel{s^2} \cancel{-s} \cancel{+1} + s(s-1)k_2 + s k_2} = \frac{s^5(s k_2+1)}{s^3+s^2 k_2} = \frac{s^3(s k_2+1)}{s+k_2}
$$

The glued near hexagons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite glued near hexagon. Then there exist two partitions T_a and T_b of $\mathcal{P}$ in quads such that the following hold:

(1) all quads of T_a are isomorphic;

(2) all quads of T_b are isomorphic;

(3) every quad of T_a intersects every quad of T_b in a line;

We suppose that all quads of T_a have order (s, t_a) and all quads of T_b have order (s, t_b), where $t_a, t_b \geq 2$. The set S^* of lines of $\mathcal{S}$ that are contained in a quad of T_a and a quad of T_b is a line spread of $\mathcal{S}$.

We define the following relations on the point set $\mathcal{P}$:

$R_0 = \{(x, x) \mid x \in \mathcal{P}\};$

$R_{1a} = \{(x,y) \in \mathcal{P} \times \mathcal{P} \mid d(x,y) = 1,\ xy \notin S^* \text{ and } xy \text{ is contained in a quad of } T_a\};$

$R_{1b} = \{(x,y) \in P \times P \mid d(x,y) = 1,\ xy \notin S^*$ and xy is contained in a quad of $T_b\}$;

$R_{1c} = \{(x,y) \in P \times P \mid d(x,y) = 1$ and $xy \in S^*\}$;

$R_{2a} = \{(x,y) \in P \times P \mid d(x,y) = 2$ and $\langle x,y \rangle \in T_a\}$;

$R_{2b} = \{(x,y) \in P \times P \mid d(x,y) = 2$ and $\langle x,y \rangle \in T_b\}$;

$R_{2c} = \{(x,y) \in P \times P \mid d(x,y) = 2$ and $\langle x,y \rangle$ is grid-quad$\}$;

$R_3 = \{(x,y) \in P \times P \mid d(x,y) = 3\}$

Let $A_0, A_{1a}, A_{1b}, A_{1c}, A_{2a}, A_{2b}, A_{2c}, A_3$ denote the $|P| \times |P|$-matrices associated with the relations $R_0, R_{1a}, R_{1b}, R_{1c}, R_{2a}, R_{2b}, R_{2c}$ and R_3.

$$A\,I = - + A_{1a} + A_{1b} + A_{1c} + - + - + - + -$$

$$A\,A_{1a} = s k_a \cdot I + (s-1)A_{1a} + - + - + k_a \cdot A_{2a} + - + A_{2c} + -$$

$$A \cdot A_{1b} = s k_b \cdot I + - + (s-1)A_{1b} + - + - + k_b \cdot A_{2b} + A_{2c} + -$$

$$A \cdot A_{1c} = s \cdot I + - + - + (s-1)A_{1c} + A_{2a} + A_{2b} + - + -$$

$$A \cdot A_{2a} = - + s k_a \cdot A_{1a} + - + s k_a \cdot A_{1c} + (s-1)(k_a+1)A_{2a} + - + - + A_3$$

$$A \cdot A_{2b} = - + - + s k_b \cdot A_{1b} + s k_b \cdot A_{1c} + - + (s-1)(k_b+1)A_{2b} + - + A_3$$

$$A \cdot A_{2c} = - + s k_b \cdot A_{1a} + s k_a \cdot A_{1b} + - + - + - + 2(s-1)A_{2c} + (k_a+k_b-1)A_3$$

$$A \cdot A_3 = - + - + - + - + s k_b A_{2a} + s k_a \cdot A_{2b} + s(k_a+k_b-1)A_{2c} + (k_a+k_b+1)(s-1)A_3$$

$$d_g = \frac{s^5 V}{(s+1)^2(s-1)(s^2+1) + sk(s-1)(s+1)^2 + V}$$

$$= \frac{s^5 \cancel{(s+1)} (s k_a + 1)(s k_b + 1)}{\cancel{(s+1)}\left(s^4 \cancel{-1} + (s^3 \cancel{-s})(k_a+k_b) + s^2 k_a k_b + \cancel{s k_a + s k_b + 1}\right)}$$

$$= \frac{s^5 (s k_a + 1)(s k_b + 1)}{s^4 + s^3(k_a+k_b) + k_a k_b} = \frac{s^3 (s k_a + 1)(s k_b + 1)}{(s+k_a)(s+k_b)}$$