Minimal blocking sets in $PG(2, 9)$

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Abstract

We classify the minimal blocking sets of size 15 in $PG(2, 9)$. We show that the only examples are the projective triangle and the sporadic example arising from the secants to the unique complete 6-arc in $PG(2, 9)$. This classification was used to solve the open problem of the existence of maximal partial spreads of size 76 in $PG(3, 9)$. No such maximal partial spreads exist [13]. In [14], also the non-existence of maximal partial spreads of size 75 in $PG(3, 9)$ has been proven. So, the result presented here contributes to the proof that the largest maximal partial spreads in $PG(3, q = 9)$ have size $q^2 - q + 2 = 74$.

1 Introduction

A *spread* of $PG(3, q)$ is a set of $q^2 + 1$ lines partitioning the point set of $PG(3, q)$. A *partial spread* of $PG(3, q)$ is a set of pairwise disjoint lines of $PG(3, q)$ not forming a spread. A partial spread is called *maximal* when it is not contained in a larger partial spread. Let $\mathcal{S}$ be a maximal partial spread of size $q^2 + 1 - \delta$, then δ is called the *deficiency* of $\mathcal{S}$.

A lot of attention has been paid to the construction of maximal partial spreads. Until recently, the largest known maximal partial spreads in $PG(3, q)$, $q > 3$, were constructed by Bruen [6], Bruen and Thas [7], Freeman [9] and Jungnickel [19], and were maximal partial spreads of size $q^2 - q + 2$.

This led to the conjecture that $q^2 - q + 2$ is the largest size for a maximal partial spread.

However, Heden recently found a maximal partial spread in $PG(3, 7)$ of size $(q^2 - q + 3 =)45$ [12].

The validity of this conjecture for $q = 8$ was recently proved by Barát, Del Fra, Innamorati and Storme [1].

Concentrating on $q = 9$, presently, it is known that the deficiency of a maximal partial spread in $PG(3, 9)$ satisfies $\delta \geq 6$.

So the first open case is whether there exists a maximal partial spread with deficiency $\delta = 6$.

The standard technique to study this problem is to rely on the link between maximal partial spreads of $PG(3, q)$ and blocking sets of $PG(2, q)$.

A plane of $PG(3, q)$ containing one line of a maximal partial spread $\mathcal{S}$ is called a *rich* plane of $\mathcal{S}$. In the other case, this plane is called *poor*. A point not lying on a line of $\mathcal{S}$ is called a *hole* of $\mathcal{S}$.

Let $\mathcal{S}$ be a maximal partial spread of deficiency δ. Then a rich plane contains δ holes and a poor plane contains $q+\delta$ holes. Moreover, the holes in a poor plane Π form a *blocking set* in Π. This means that every line of Π contains at least one hole. For proofs, we refer to [21, Lemma 2.1]. A *trivial* blocking set is a blocking set containing a line.

When $\mathcal{S}$ is maximal, no line consists entirely of holes. This means that the holes in Π form a *non-trivial* blocking set in Π.

Hence, lower bounds on the cardinality of non-trivial blocking sets in $PG(2,q)$, and information on the structure of minimal blocking sets in $PG(2,q)$, yield information on maximal partial spreads in $PG(3,q)$.

Presently, the following results are known on non-trivial blocking sets in $PG(2,q)$, which have led to the following results on maximal partial spreads in $PG(3,q)$.

Theorem 1.1 (1) (Bruen [5]) *The smallest non-trivial blocking sets in $PG(2,q)$, q square, have cardinality $q+\sqrt{q}+1$ and are equal to Baer subplanes $PG(2,\sqrt{q})$.*
(2) (Blokhuis, Storme, Szőnyi [4]) *In $PG(2,q)$, q non-square, $q=p^h, h>2, p\geq 5$, p prime, $|B|\geq q+q^{2/3}+1$ for every non-trivial blocking set B.*
(3) (Blokhuis [2]) *In $PG(2,q)$, q prime, $q>2$, $|B|\geq 3(q+1)/2$ for every non-trivial blocking set B.*
(4) (Blokhuis, Storme, Szőnyi [4]) *In $PG(2,q)$, q square, $q=p^h, h>2, p\geq 5$, p prime, every non-trivial blocking set B of cardinality $|B|<q+q^{2/3}+1$ contains a Baer subplane.*
(5) (Szőnyi [26]) *In $PG(2,q)$, $q=p^2$, p prime, every non-trivial blocking set B of cardinality $|B|<3(q+1)/2$ contains a Baer subplane.*

Theorem 1.2 (Polverino, Polverino and Storme [22, 23, 24]) *The smallest minimal blocking sets in $PG(2,p^3)$, $p=p_0^h$, p_0 prime, $p_0\geq 7$, with exponent $e\geq h$, are:*
(1) *a line,*
(2) *a Baer subplane of cardinality $p^3+p^{3/2}+1$, when p is a square,*
(3) *a set of cardinality p^3+p^2+1, equivalent to*

$$\{(x,T(x),1)||x\in GF(p^3)\}\cup\{(x,T(x),0)||x\in GF(p^3)\setminus\{0\}\},$$

with T the trace function from $GF(p^3)$ to $GF(p)$,
(4) *a set of cardinality p^3+p^2+p+1, equivalent to*

$$\{(x,x^p,1)||x\in GF(p^3)\}\cup\{(x,x^p,0)||x\in GF(p^3)\setminus\{0\}\}.$$

Corollary 1.3 *Let $\mathcal{S}$ be a maximal partial spread of $PG(3,q)$ of deficiency δ. Then*
(1) *$\delta\geq\sqrt{q}+1$ when q is square,*
(2) *$\delta\geq q^{2/3}+1$ when q is non-square, $q=p^h, h>2, p\geq 5$, p prime,*
(3) *$\delta\geq(q+3)/2$ when q is an odd prime.*

Corollary 1.4 (Metsch and Storme [21]) (a) *Suppose that* δ *is an integer and* q *square,* $q = p^h, h > 2, p \geq 5$, p *prime, such that* $0 < \delta < q^{2/3} + 1$.

If $\mathcal{S}$ *is a maximal partial spread of* $PG(3, q)$ *with* $q^2 + 1 - \delta$ *lines, then* $\delta = s(\sqrt{q} + 1)$ *for an integer* $s \geq 2$ *and the set of holes is the disjoint union of* s *Baer subgeometries* $PG(3, \sqrt{q})$.

(b) *Suppose that* δ *is an integer and* $q = p^2$, p *prime,* $q > 4$, *such that* $0 < 2\delta \leq q + 1$.

If $\mathcal{S}$ *is a maximal partial spread of* $PG(3, q)$ *with* $q^2 + 1 - \delta$ *lines, then* $\delta = s(\sqrt{q} + 1)$ *for an integer* $s \geq 2$ *and the set of holes is the disjoint union of* s *Baer subgeometries.*

Theorem 1.5 (Metsch and Storme [21]) *Let* $\mathcal{S}$ *be a maximal partial spread of* $PG(3, q^3)$, q *non-square,* $q = p^h$, $h \geq 1$, p *prime,* $p \geq 7$, *of deficiency* $\delta \leq q^2 + q + 1$. *Then* $\delta = q^2 + q + 1$ *and the set of holes forms a projected subgeometry* $PG(5, q)$ *in* $PG(3, q^3)$.

Theorem 1.6 (Metsch and Storme [21]) *Let* $\mathcal{S}$ *be a maximal partial spread of* $PG(3, q^3)$, $q = p^h$, $h \geq 2$, h *even,* p *prime,* $p \geq 7$, *of deficiency* $\delta \leq q^2 + q + 1$.

Then, (1) $\delta \equiv 0 \pmod{q^{3/2} + 1}$, $\delta \geq 2(q^{3/2} + 1)$, *and the set of holes is the union of disjoint subgeometries* $PG(3, q^{3/2})$, *or* (2) $\delta = q^2 + q + 1$ *and the set of holes forms a projected subgeometry* $PG(5, q)$ *in* $PG(3, q^3)$.

In the following theorems, for $q = p^3$, p prime, $p \geq 17$, δ_0 is the largest integer smaller than $(3p^3 + 27p^2 - 5p + 25)/25$. For $p = 7, 11, 13$, $\delta_0 = 90, \delta_0 = 285$ and $\delta_0 = 441$ respectively. For $q = p^3$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h > 1$, δ_0 is defined as the largest integer smaller than $(3p^3 + 27p^2 - 5p + 25)/25$ and smaller than the value δ' for which $p^3 + \delta'$ is the cardinality of the smallest non-trivial minimal blocking set in $PG(2, p^3)$ of cardinality larger than $p^3 + p^2 + p + 1$.

Theorem 1.7 (Ferret and Storme [8]) *Let* $p = p_0^h$, $p_0 \geq 7$ *a prime,* $h \geq 1$ *odd. The set of holes of a maximal partial spread in* $PG(3, p^3)$ *of deficiency* $\delta \leq \delta_0$ *is the disjoint union of projected* $PG(5, p)$*'s of cardinality* $p^5 + p^4 + p^3 + p^2 + p + 1$, *and so* $\delta = s(p^2 + p + 1)$ *for some integer* s.

Theorem 1.8 (Ferret and Storme [8]) *Let* $p = p_0^h$, $p_0 \geq 7$ *a prime,* $h > 1$ *even. The set of holes of a maximal partial spread in* $PG(3, p^3)$ *of deficiency* $\delta \leq \delta_0$ *is the disjoint union of* $PG(3, p^{3/2})$*'s and of projected* $PG(5, p)$*'s of cardinality* $p^5 + p^4 + p^3 + p^2 + p + 1$ *and so the deficiency* δ *of a maximal partial spread in* $PG(3, p^3)$ *can be written as* $\delta = r(p^{3/2} + 1) + s(p^2 + p + 1)$ *for some integers* r *and* s.

In $PG(2, 8)$, the following results on the smallest non-trivial blocking sets are known.

Theorem 1.9 (Innamorati and Zuanni [17]) *Let* $\mathcal{B}$ *be a non-trivial minimal blocking set of size* 13 *in* $PG(2, 8)$, *then* $\mathcal{B}$ *is projectively equivalent to the set*

$$\{(t, t + t^2 + t^4, 1) || t \in GF(8)\} \cup \{(t, t + t^2 + t^4, 0) || t \in GF(8) \setminus \{0\}\}.$$

Theorem 1.10 (Barát, Del Fra, Innamorati and Storme [1]) *There do not exist minimal blocking sets of size* 14 *in* $PG(2,8)$.

The two preceding results led to the following sharp result on the size of the largest maximal partial spreads in $PG(3,8)$.

Theorem 1.11 (Barát, Del Fra, Innamorati and Storme [1]) *The largest maximal partial spreads in* $PG(3,8)$ *have size* q^2-q+2.

In all of the preceding results on maximal partial spreads in $PG(3,q)$ of deficiency δ, information on minimal blocking sets of size $q+\delta$ in $PG(2,q)$ was of crucial importance.

To prove the non-existence of maximal partial spreads of deficiency $\delta=6$ in $PG(2,9)$ in [13], we will classify the non-trivial blocking sets of size $15=q+\delta$ in $PG(2,q=9)$. We will show that next to the classical example of the projective triangle, there is a unique second example.

The minimal blocking sets of size 15 in $PG(2,q=9)$ are minimal blocking sets of size $3(q+1)/2$.

Regarding their classification in other planes $PG(2,q)$, for small odd values of q, we note that also in $PG(2,7)$ and in $PG(2,13)$, there is a unique example different from the projective triangle. But in $PG(2,q)$, $q=11$, or q an odd prime number satisfying $17\leq q\leq 37$, the projective triangles are the only examples of minimal blocking sets of size $3(q+1)/2$ (see Blokhuis, Brouwer and Wilbrink [3]).

Regarding the classification of the largest maximal partial spreads in $PG(3,9)$, we note that also the non-existence of maximal partial spreads of size 75 in $PG(3,9)$ has been proven [14]. This altogether proves that the largest maximal partial spreads in $PG(3,q=9)$ have size $q^2-q+2=74$.

2 The known minimal blocking sets of size 15

Presently, there are two known examples of minimal blocking sets of size 15 in $PG(2,9)$.

2.1 The projective triangle

The first example is the projective triangle [15, Lemma 13.6]. This is the set of points projectively equivalent to the set

$$\{(0,1,a_0),(1,0,a_1),(-a_2,1,0)||a_0,a_1,a_2 \text{ squares of } GF(9)\}.$$

There are exactly three non-concurrent 6-secants to the projective triangle. The intersection points of two of these 6-secants are called the *vertices* of the projective triangle.

A vertex lies on two 6-secants, four 2-secants and four tangents to the projective triangle. A non-vertex point of the projective triangle lies on one 6-secant, four 3-secants, one 2-secant and four tangents.

2.2 The sporadic blocking set

In $PG(2,9)$, there is a unique complete 6-arc [15, p. 386]. The 15 bisecants to this complete 6-arc form a minimal blocking set in the dual projective plane.

So, dualizing this situation, a sporadic example of a minimal blocking set of size 15 arises.

The characteristic properties of this sporadic example are:

1. There are exactly six 5-secants to this blocking set which form a complete 6-arc of lines.

2. There are ten 3-secants to the blocking set. These ten 3-secants form a dual conic.

3. And furthermore, there are fifteen 2-secants to the blocking set. These fifteen 2-secants are the secants to a complete 6-arc in $PG(2,9)$.

3 The classification of the minimal blocking sets of size 15

From now on, let B be a minimal blocking set of size 15 in $PG(2,9)$. Since B is non-trivial, a line L intersects B in at most 6 points. Namely, for a fixed point $p \in L \setminus B$, the nine lines through p which are different from L all contain at least one point of B, so L contains at most 6 points of B. Blocking sets of size 15 in $PG(2,9)$ having at least one 6-secant are called *blocking sets of Rédei-type* [25].

3.1 Introductory results

Lemma 3.1 *Every point of B lies on at least four tangents to B.*

Proof: Let $p \in B$ and let L be a tangent line to B at p. Consider $PG(2,9) \setminus L$ and call this $AG(2,9)$. Then a set $B \setminus L$ of size 14 remains.

A minimal blocking set in $AG(2,9)$ contains at least 17 points [18]. This means that we need to add at least three points to $B \setminus L$ to get a blocking set in $AG(2,9)$.

The only external lines to $B \setminus L$ in $AG(2,9)$ are the tangents to B at p (different from L). Since at least three points need to be added to $B \setminus L$ to obtain a blocking set in $AG(2,9)$, there are at least three external lines to $B \setminus L$ in $AG(2,9)$; so p lies already on at least three tangents to B, different from L. Also L is a tangent line to B. Hence p lies on at least four tangents to B. $\square$

Lemma 3.2 *B has at least one secant with at least four points.*

Proof: Suppose there are only 1-, 2- and 3-secants. Let the number of them be denoted by a, b and c respectively. Then the following equations must hold by standard counting arguments.

$$a + b + c = 91$$
$$a + 2b + 3c = 150$$
$$2b + 6c = 210$$

From these equations, $b = -33$, which is a contradiction. □

3.2 There are at least 5- and/or 6-secants

Suppose that there are only 1-, 2-, 3- and 4-secants. Let the respective numbers be a, b, c, d. Then the standard counting arguments imply that

$$\begin{aligned} b &= -3a + 201 \\ c &= 3a - 188 \\ d &= -a + 78 \end{aligned}$$

So $a \geq 63$.

It is impossible that there is a point lying on at least 9 tangents. Namely, if a point p of B lies on at least 9 tangents, then the 14 other points of B lie on the tenth line through p, which is impossible. If a point p not belonging to B lies on 9 tangents, then the tenth line contains the 6 remaining points of B, but this contradicts the fact that there are at most 4-secants to B. So, the tangents form a $(k, 8)$-arc in the dual plane of $PG(2, 9)$. Table 5.4 of [16] shows us that a $(k, 8)$-arc in $PG(2, 9)$ contains at most 65 elements, so there are at most 65 tangents to B.

So, there are only the following three possibilities:

a	b	c	d
63	12	1	15
64	9	4	14
65	6	7	13

Lemma 3.3 *Only the case* $(a, b, c, d) = (65, 6, 7, 13)$ *occurs.*

Proof: Otherwise, the number of 4-secants is at least 14. Two 4-secants always intersect in a point of B. For assume they intersect in a point p not in B. Then since the eight other lines through p all contain at least one point of B, $|B| \geq 2 \times 4 + 8 = 16$, which is false.

Consider a 4-secant L. The (at least) 13 other 4-secants intersect L in a point of B, so some point p of $L \cap B$ lies on at least five 4-secants, the line L included. But then $|B| \geq 1 + 5 \times 3 = 16$ when counting the number of points of B on the lines through p, which is false. □

Let L be a 4-secant. Let $L : Z = 0$ where the coordinates of a point are (x, y, z). Let $r_1 = (0, 1, 0), r_2 = (1, 0, 0)$ be points of L not belonging to B. Let

r_3, r_4, r_5, r_6 be the other points of $S = L \setminus B$. We identify r_1 with (∞), r_2 with (0), and the points $(1, y, 0)$ with (y). We also identify the affine points $(x, y, 1)$ with (x, y). Let $(a_i, b_i), i = 1, \ldots, 11$, be the points of $B \setminus L$.

Then the following result is valid.

Lemma 3.4 *At most two of the points $r_i, i = 1, \ldots, 6$, lie on a 3-secant.*

Proof: Suppose the points r_1 and r_2 lie on 3-secants to B. Let $p = (0, 0)$ be the intersection of these 3-secants. Since r_1 and r_2 are the points at infinity of respectively the vertical and horizontal line through the origin p, the vertical and horizontal line through the origin contain three affine points of B, and this implies $\{a_i || i = 1, \ldots, 11\} = \{b_i || i = 1, \ldots, 11\} = GF(9)$ where every non-zero element appears once and where zero appears three times in the sequence of elements a_i, respectively b_i.

This shows that $\prod_{i=1}^{11}(X - a_i) = \prod_{i=1}^{11}(X - b_i) = X^{11} - X^3$.

Let

$$\sigma_{k,l}(a_1, \ldots, a_{11}; b_1, \ldots, b_{11}) = \sum a_{i_1} \cdots a_{i_k} \cdot b_{j_1} \cdots b_{j_l}$$

where the sum is over all index sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_l\}$ being disjoint subsets of $\{1, \ldots, 11\}$ of cardinality k and l, respectively.

Then $\prod_i (X - a_i) = \prod_i (X - b_i) = X^{11} - X^3$ implies $\sigma_{1,0} = \sigma_{0,1} = \sigma_{2,0} = \sigma_{0,2} = 0$.

We now use the lacunary polynomial associated with the set $\{(a_i, b_i) || i = 1, \ldots, 11\}$. This is the polynomial

$$H(X, Y) = \prod_{i=1}^{11}(X + a_i Y - b_i) = X^{11} + a(Y)X^{10} + b(Y)X^9 + \cdots,$$

where $a(Y) = \sigma_{1,0}Y - \sigma_{0,1}$ and where $b(Y) = \sigma_{2,0}Y^2 - \sigma_{1,1}Y + \sigma_{0,2}$.

Since $\sigma_{1,0} = \sigma_{0,1} = \sigma_{2,0} = \sigma_{0,2} = 0$, $a(Y)$ is identically zero and $b(Y) = -cY$, for some constant c.

So, $H(X, y) = (X^9 - X)(X^2 - cy)$ for all $(\infty) \neq (y) \in S = L \setminus B$ since all affine lines through such a point must contain a point of B.

If $c \neq 0$, then $X^2 - cy$ cannot have a double root for a fixed value $y \neq 0$, so these points (y) lie on two 2-secants to the affine part. On the other hand, $c = 0$ would imply that all lines through p and a point of S are 3-secants. If $p \notin B$, then $|B| \geq 1 + 3 \times 6$, which is false. So $p \in B$, but then B is not minimal. $\square$

Lemma 3.5 *It is impossible that B has at most 4-secants.*

Proof: The preceding lemma shows that there are at least eight 2-secants to B since we know that there are at least four points r_i lying on two 2-secants to B. But the number b of 2-secants is $b = 6$ (Lemma 3.3). So we have a contradiction. $\square$

3.3 The computer search for a minimal blocking set of size 15 of Rédei-type

A minimal blocking set of size 15 of Rédei-type has at least one 6-secant L.

Using MAGMA [20], it was determined that there are two orbits of the group PΓL(2,9) on the subsets of size 4 of a line L. This gives two possibilities for the orbits of sets of 6 points on such a line. So there are two possibilities for $L \cap B$. The stabilizer group of the first 6-set acts transitively on the 6 points; the stabilizer group of the other 6-set has two orbits on the 6-set.

Consider the affine plane $PG(2,9) \setminus L$. This shares 9 points with B. Every secant M to $B \setminus L$ intersects L in a point of B. For let p be a point of $L \setminus B$. Since L contains already 6 points of B, there only remain 9 other points in B, and since every one of the nine lines through p different from L must contain at least one point of B, these nine points of $B \setminus L$ must lie one by one on the nine lines through p different from L. So a point of $L \setminus B$ does not lie on a secant to $B \setminus L$; secants to $B \setminus L$ intersect L in a point of $L \cap B$.

Suppose the 9 points of $B \setminus L$ form a 9-arc, then the four points of $L \setminus B$ extend this 9-arc to a 10-arc since they only lie on tangents to $B \setminus L$. A 9-arc in $PG(2,9)$ consists of 9 points of a conic [16, p. 386], so can only be extended by the tenth point of this conic to a 10-arc.

So there are at least three collinear points in $B \setminus L$. The line containing these collinear points intersects L in a point of B. Using the preceding results on the stabilizer groups of the two possibilities for the 6-sets $B \cap L$, there are in total three possibilities for this intersection point.

So it is possible to determine 9 points of B, without having too many possibilities.

The computer search showed that the projective triangles are the only examples.

Theorem 3.6 *The projective triangles are the only minimal blocking sets of size* 15 *in* $PG(2,9)$ *that are of Rédei-type.*

3.4 The computer search for a minimal blocking set of size 15 having no 6-secants, but at least one 5-secant

First of all, MAGMA showed that the group PΓL(2,9) has two orbits on the 5-sets of a projective line. So, for the 5-secant L to B, there are two possibilities for $L \cap B$.

Consider now the affine part $B \setminus L$ of size 10. Here, the following result of Gács gives crucial information on the structure of this affine part.

Theorem 3.7 (Gács [10]) *In* $PG(2,q)$*, let* B *be a minimal blocking set of size* $q+k$*, and suppose there is a line* L *intersecting* B *in exactly* $k-1$ *points. Then there is a point* $p \notin B$ *such that every line joining* p *to a point of* $L \setminus B$ *contains two points of* B*. Hence* $k \geq (q+3)/2$.

Using this result, we see that there is a point p not in B such that the five lines joining p to the points of $L \setminus B$ each contain two points of B; so these lines contain the 10 points of $B \setminus L$.

This information was used to conduct a computer search. The computer search showed that the only example that satisfies this condition is the sporadic example coming from the complete 6-arc in $PG(2, 9)$.

Theorem 3.8 *Every minimal blocking set in $PG(2, 9)$ of size 15 having at least one 5-secant, but no 6-secant, is projectively equivalent to the minimal blocking set arising from the complete 6-arc in $PG(2, 9)$.*

4 Application

As indicated in the introduction, this classification of the minimal blocking sets of size 15 in $PG(2, 9)$ was used in [13] to prove the non-existence of maximal partial spreads of size 76 (deficiency 6) in $PG(3, 9)$.

Theorem 4.1 *There do not exist maximal partial spreads of size 76 in $PG(3, 9)$.*

In [14], the non-existence of maximal partial spreads of size 75 in $PG(3, 9)$ has been proven. There exist in $PG(3, q = 9)$ maximal partial spreads of size $q^2 - q + 2 = 74$. So the size of the largest maximal partial spreads is now also known in $PG(3, 9)$.

Theorem 4.2 *The largest maximal partial spreads in $PG(3, q = 9)$ have size $q^2 - q + 2 = 74$.*

References

[1] J. Barát, A. Del Fra, S. Innamorati and L. Storme, Minimal blocking sets in $PG(2, 8)$ and maximal partial spreads in $PG(3, 8)$. *Des. Codes Cryptogr.* **31** (2004), 15-26.

[2] A. Blokhuis, On the size of a blocking set in $PG(2, p)$. *Combinatorica* **14** (1994), 111-114.

[3] A. Blokhuis, A.E. Brouwer and H.A. Wilbrink, Blocking sets in PG$(2, p)$ for small p, and partial spreads in PG$(3, 7)$. *Adv. Geom.* (2003), suppl., Special issue dedicated to Adriano Barlotti, S245-S253.

[4] A. Blokhuis, L. Storme and T. Szőnyi, Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc.* (2) **60** (1999), 321-332.

[5] A.A. Bruen, Baer subplanes and blocking sets. *Bull. Amer. Math. Soc.* **76** (1970), 342-344.

[6] A.A. Bruen, Partial spreads and replaceable nets. *Canad. J. Math.* **23** (1971), 381-391.

[7] A.A. Bruen and J.A. Thas, Partial spreads, packings and Hermitian manifolds in $PG(3,q)$. *Math. Z.* **151** (1976), 207-214.

[8] S. Ferret and L. Storme, Results on maximal partial spreads in $PG(3,p^3)$ and on related minihypers. *Des. Codes Cryptogr.* **29** (2003), 105-122.

[9] J.W. Freeman, Reguli and pseudo-reguli in $PG(3,s^2)$. *Geom. Dedicata* **9** (1980), 267-280.

[10] A. Gács, *The Rédei Method Applied to Finite Geometry*, PhD Thesis, Eötvös Loránd University, Budapest (Hungary), 1997.

[11] N. Hamada, A characterization of some $[n,k,d;q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry. *Discr. Math.* **116** (1993), 229-268.

[12] O. Heden, A maximal partial spread of size 45 in $PG(3,7)$. *Des. Codes Cryptogr.* **22** (2001), 331-334.

[13] O. Heden, S. Marcugini, F. Pambianco and L. Storme, On the non-existence of maximal partial spreads of size 76 in $PG(3,9)$. (In preparation).

[14] O. Heden, S. Marcugini and F. Pambianco, The maximum size of a maximal partial spread in $PG(3,9)$. (In preparation).

[15] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields (Second Edition)*. Oxford: Oxford University Press 1998.

[16] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: update 2001. *Developments in Mathematics* Vol. **3**, Kluwer Academic Publishers. *Finite Geometries*, Proceedings of the *Fourth Isle of Thorns Conference* (Chelwood Gate, July 16-21, 2000) (Eds. A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel and J.A. Thas), pp. 201-246.

[17] S. Innamorati and F. Zuanni, Minimum blocking configurations. *J. Geom.* **55** (1996), 86-98.

[18] R.E. Jamison, Covering finite fields with cosets of subspaces. *J. Combin. Theory Ser. A* **22** (1977), 253-266.

[19] D. Jungnickel, Maximal partial spreads and nets of small deficiency. *J. Algebra* **90** (1984), 119-132.

[20] http://www.maths.uysd.edu.au:8000/u/magma/index.html

[21] K. Metsch and L. Storme, Partial t-spreads in $PG(2t+1,q)$. *Des. Codes Cryptogr.* **18** (1999), 199-216.

[22] O. Polverino, Small minimal blocking sets and complete k-arcs in $PG(2, p^3)$. *Discrete Math.* **208/209** (1999), 469-476.

[23] O. Polverino, Small blocking sets in $PG(2, p^3)$. *Des. Codes Cryptogr.* **20** (2000), 319-324.

[24] O. Polverino and L. Storme, Minimal blocking sets in $PG(2, q^3)$. *Europ. J. Combin.* **23** (2002), 83-92.

[25] L. Rédei, *Lückenhafte Polynome über endlichen Körpern,* Birkhäuser Verlag, Basel (1970) (english translation: *Lacunary Polynomials over Finite Fields*, North-Holland, Amsterdam (1973)).

[26] T. Szőnyi, Blocking sets in Desarguesian affine and projective planes. *Finite Fields Appl.* **3** (1997), 187-202.

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