On tetrahedrally closed line systems and a generalization of the Haemers-Roos inequality

Bart De Bruyn*

Dedicated to the memory of Ernest Shult

Abstract

This paper addresses and partially solves a 35 year open problem in the theory of tetrahedrally closed line systems of Euclidean spaces. The study of those line systems was initiated by Shult and Yanushka in [26], who not only constructed several interesting examples of them but also showed that they were related to so-called near hexagons with three points per line. Shult and Yanushka also showed how tetrahedrally closed line systems can be constructed from such near hexagons, but it was very soon clear that their proof contained an essential error. The present paper fixes the original result for a large class of near hexagons, namely for those that have an order. The used techniques will allow us to improve other results from the literature. In particular, we will generalize the well-known Haemers-Roos inequality for generalized hexagons of order (s, t) to arbitrary near hexagons with an order.

1 Introduction

Consider the n-dimensional Euclidean space $\mathbb{R}^n$ with origin o, and suppose $\mathcal{L}$ is a nonempty set of lines through o. If $L_1, L_2 \in \mathcal{L}$, then $\theta_{L_1, L_2} \in [0, \frac{\pi}{2}]$ denotes the angle between the lines L_1 and L_2. The set $\mathcal{L}$ is called a *line system of type* $(a_1, a_2, \ldots, a_k)$, where $0 \le a_1, a_2, \ldots, a_k \le 1$, whenever $\cos(\theta_{L_1, L_2}) \in \{a_1, a_2, \ldots, a_k\}$ for any two distinct lines L_1 and L_2 of $\mathcal{L}$. Line systems of type $(a_1, a_2, \ldots, a_k)$ of $\mathbb{R}^n$ are equivalent with *systems of vectors of type* $(a_1, a_2, \ldots, a_k)$ of $\mathbb{R}^n$. Such systems of vectors of $\mathbb{R}^n$ are nonempty sets Σ of vectors of $\mathbb{R}^n$ that satisfy the following:

- $-\bar{v} \in \Sigma$ for every $\bar{v} \in \Sigma$;

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*Ghent University, Department of Mathematics, Krijgslaan 281 (S25), 9000 Gent (Belgium); Bart.DeBruyn@Ugent.be

- the (squared) norm $\bar{v} \cdot \bar{v}$ of any $\bar{v} \in \Sigma$ is a nonzero constant c;

- if $\bar{v}_1, \bar{v}_2 \in \Sigma$ and $\bar{v}_2 \notin \{\bar{v}_1, -\bar{v}_1\}$, then $\frac{|\bar{v}_1 \cdot \bar{v}_2|}{c} \in \{a_1, a_2, \dots, a_k\}$.

Line systems of type (a_1) are also known as *equiangular line systems*. Such line systems were first studied by Haantjes [14] and van Lint & Seidel [20], and have ever since been investigated by several other researchers, see e.g. [13, 25] and the references therein. They are also related to other combinatorial objects, like two-graphs [24].

Line systems of type $(0, \frac{1}{2})$ were first studied by Cameron et al. [5] and those of type $(0, \frac{1}{3})$ by Shult and Yanushka [26], and subsequently by Neumaier [22] and Cuypers [9]. Many interesting examples of such line systems exist. The line system spanned by the root systems of the Lie algebras of type A_n, D_n, E_6, E_7 or E_8 are examples of line systems of type $(0, \frac{1}{2})$. Constructions of line systems of type $(0, \frac{1}{3})$ can be found in Shult and Yanushka [26] and Cuypers [9], with many of the described examples arising from the well known Leech lattice [19, 8].

The angle $\theta \in [0, \pi]$ for which $\cos(\theta)$ equals $-\frac{1}{2}$ (respectively, $-\frac{1}{3}$) is the angle subtended by two chords drawn from the barycenter of an equilateral triangle (respectively, tetrahedron) to two of its corner vertices. In [5], the notion of a *star-closed line system* was introduced, meaning that if two lines inclined at $\frac{\pi}{3}$ are present in the line system, then the unique third line in the plane they generate inclined at $\frac{\pi}{3}$ with both of them is also present. A similar notion was introduced in [26]. A line system (of type $(0, \frac{1}{3})$) of $\mathbb{R}^n$ is said to be *tetrahedrally closed* whenever it has 0, 1, 2 or 4 lines in common with every set of four lines obtained by connecting o with the four corners of some tetrahedron centered at o. A system Σ of vectors of type $(0, \frac{1}{3})$ of Σ is called tetrahedrally closed whenever the corresponding line system is tetrahedrally closed. Assuming that all norms of the vectors of Σ equal a constant $c > 0$, then this is equivalent to the condition that if $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in \Sigma$ with $\bar{v}_1 \cdot \bar{v}_2 = \bar{v}_1 \cdot \bar{v}_3 = \bar{v}_2 \cdot \bar{v}_3 = -\frac{c}{3}$, then also $\bar{v}_1 + \bar{v}_2 + \bar{v}_3 \in \Sigma$.

The study of tetrahedrally closed line systems of type $(0, \frac{1}{3})$ was initiated in [26]. Besides classification results, Shult and Yanushka showed that these line systems were closely related to star-closed line systems of type $(0, \frac{1}{2}, \frac{1}{4})$ and to a certain class of point-line geometries which they called near polygons.

Suppose Σ is a tetrahedrally closed system of norm 3 vectors of type $(0, \frac{1}{3})$ of $\mathbb{R}^n$. Then $\bar{v}_1 \cdot \bar{v}_2 \in \{0, 1, -1, 3, -3\}$ for any two vectors $\bar{v}_1, \bar{v}_2 \in \Sigma$. Let $\bar{v} \in \Sigma$ and put $\mathcal{P}_{\Sigma, \bar{v}} := \{\bar{u} \in \Sigma \mid \bar{v} \cdot \bar{u} = -1\}$. Let $\mathcal{L}_{\Sigma, \bar{v}}$ denote the set of all triples $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \subseteq \mathcal{P}_{\Sigma, \bar{v}}$ such that $\{\bar{v}, \bar{v}_1, \bar{v}_2, \bar{v}_3\}$ defines a tetrahedron centered at o. We say that Σ satisfies Property $(P_{\bar{v}})$ if $\mathcal{L}_{\Sigma, \bar{v}} \neq \emptyset$ and for any two vectors $\bar{u}_1, \bar{u}_2 \in \mathcal{P}_{\Sigma, \bar{v}}$ with $\bar{u}_1 \cdot \bar{u}_2 = 1$ there exists a vector $\bar{w} \in \mathcal{P}_{\Sigma, \bar{v}}$ such that $\bar{u}_1 \cdot \bar{w} = \bar{u}_2 \cdot \bar{w} = -1$.

Proposition 1.1 ([26, Proposition 3.10]) *If Σ satisfies Property $(P_{\bar{v}})$ for some $\bar{v} \in \Sigma$, then the point-line geometry $\mathcal{S}_{\Sigma, \bar{v}}$ with point set $\mathcal{P}_{\Sigma, \bar{v}}$ and line set $\mathcal{L}_{\Sigma, \bar{v}}$ satisfies the following property:*

(NP) *for every point p and every line L, there exists a unique point on L that is nearest to p (with respect to the distance in the collinearity graph).*

Moreover, the diameter of $\mathcal{S}_{\Sigma,\bar{v}}$ is either 1, 2 or 3, and every line of $\mathcal{S}_{\Sigma,\bar{v}}$ is incident with precisely three points. Two points $\bar{w}_1$ and $\bar{w}_2$ of $\mathcal{S}_{\Sigma,\bar{v}}$ lie at distance 0, 1, 2 or 3 from each other depending on whether $\bar{w}_1 \cdot \bar{w}_2$ is equal to 3, -1, 1 or 0.

A point-line geometry of diameter d satisfying Property (NP) is called a *near $2d$-gon* or a *near polygon* if the value of d is not specified. A near polygon is said to have *order* (s,t) if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines. Near polygons are related to the generalized polygons of Tits [27], to the polar spaces of Tits and Veldkamp [28, 30], to distance-regular graphs [3] and to finite simple groups. While dual polar spaces [4] and generalized polygons (which can be seen as special classes of near polygons) are related to classical groups and exceptional groups of Lie type, there are also several near polygons corresponding to sporadic simple groups, like the $U_4(3)$ or Aschbacher near hexagon [1], the Hall-Janko or Cohen-Tits near octagon [6, 7] and the $G_2(4)$ near octagon [2].

The point-line geometry $\mathcal{S}_{\bar{v}}$ defined in Proposition 1.1 is called a *local near polygon* of Σ. In [26, Proposition 3.14], Shult and Yanushka proved that every near $2k$-gon, $k \in \{1, 2, 3\}$, with three points per line occurs as a local near polygon of some tetrahedrally closed system of norm 3 vectors of type $(0, \frac{1}{3})$. However, soon after their paper was published, it turned out that their arguments contained an essential error. The nature of this error is explained by Neumaier [22, p. 566] who also observed that the result remains valid for those near hexagons for which the collinearity graph is distance-regular (there are nine such geometries). In the present paper we will fill this 35 year old gap in the original proof, at least for those near hexagons that have an order:

Theorem 1.2 *Let $\mathcal{S}$ be a finite near $2k$-gon of order $(2, t)$ with $k \in \{1, 2, 3\}$. Then $\mathcal{S}$ occurs as a local near polygon of some tetrahedrally closed system of norm 3 vectors of type $(0, \frac{1}{3})$.*

In Section 3 we will be able to compute certain intersection numbers in a finite near hexagon $\mathcal{S}$ of order (s,t). This information will be used in Section 4 where we will show that a certain matrix associated with $\mathcal{S}$ is idempotent and thus also positive semidefinite. In Section 5, Theorem 1.2 will be proved from the fact that this matrix is positive semidefinite.

The fact that the matrix is positive semidefinite has additional consequences. It allows to improve a result on generalized polygons obtained by Haemers and Roos [16]. Tits [27] showed that for every prime power q there exists a generalized hexagon of order $(s,t) = (q, q^3)$, the so-called dual twisted triality hexagon associated with the simple group $^3D_4(q^3)$. These dual twisted triality hexagons are extremal in the sense that t attains its maximal possible value for given s. Indeed, Haemers and Roos [16] proved that the inequality $t \leq s^3$ holds in any generalized hexagon of order (s,t) with $s > 1$. The question whether the dual twisted triality hexagons are the only generalized hexagons of order (s, s^3), $s > 1$, remains as of today one of the most important problems in the

theory of generalized polygons. In Section 6, we generalize the Haemers-Roos bound for generalized hexagons to arbitrary near hexagons with an order, by proving the following:

Theorem 1.3 *Let $\mathcal{S}$ be a finite near hexagon of order (s,t) with $s \neq 1$. Let x and y be two opposite points of $\mathcal{S}$. Let N denote the number of geodesics connecting x and y, and put $\overline{t_2} := \frac{N}{t+1} - 1$. Then*

$$\overline{t_2} \cdot (s^2 + s + 1) - s^3 \leq t \leq s^3 + \overline{t_2} \cdot (s^2 - s + 1).$$

If $\mathcal{S}$ is a generalized hexagon, then $\overline{t_2} = 0$ for any choice of the points x and y, and in this case, Theorem 1.3 reduces to the Haemers-Roos inequality $t \leq s^3$. In case $\mathcal{S}$ is a regular near hexagon, i.e. a finite near hexagon with an order (s,t) such that any two points at distance 2 have precisely $t_2 + 1$ common neighbours for a certain constant t_2, then $\overline{t_2}$ equals t_2 for any choice of the points x and y. In this case, the upper bound in Theorem 1.3 becomes $t \leq s^3 + t_2(s^2 - s + 1)$. This upper bound is known as the *Mathon* or *Haemers-Mathon bound*. It can be derived using the theory of distance-regular graphs, see [15, 21, 23]. This theory of distance-regular graphs can no longer be used here as the collinearity graph of the near hexagon is not necessarily distance-regular. However, besides geometric and counting arguments, the proof of the present paper still relies on methods from algebraic combinatorics.

For $s > 1$, we will see that $\overline{t_2} \leq s^2$. Taking into account Theorem 1.3, we thus find that $t \leq s^4 + s^2$. So, for any given $s > 1$ there can only be a finite number of finite near hexagons of order $(s, *)$. In Section 6, we will also be able to treat the case where t attains the upper bound $s^4 + s^2$. We will prove the following.

Theorem 1.4 *Let $\mathcal{S}$ be a finite near hexagon of order (s,t) with $s \neq 1$. Then $t \leq s^4 + s^2$, with equality if and only if $\mathcal{S}$ is a Hermitian dual polar space of rank 3.*

In particular, Theorem 1.4 says that finite near hexagons of order $(2,t)$ can only exist for $t \leq 20$. This improves the upper bound $t \leq 33$ which was achieved in [10, page 120] by means of other techniques.

2 Preliminaries

In the present paper, we will always consider a point-line geometry as a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is the nonempty point set and the line set $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$. If x and y are two points of $\mathcal{S}$, then $\mathrm{d}(x,y)$ denotes the distance between x and y in the collinearity graph. If x is a point and Y a nonempty set of points of $\mathcal{S}$, then we define $\mathrm{d}(x,Y) := \min\{\mathrm{d}(x,y) \mid y \in Y\}$. If x is a point of $\mathcal{S}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x.

Recall that a point-line geometry with finite diameter d is called a *near $2d$-gon* if for every point x and every line L of $\mathcal{S}$, there exists a unique point on L nearest to x. A near

0-gon in this context is just a point (no lines) and a near 2-gon is a line. The near quadrangles are precisely the generalized quadrangles and the degenerate generalized quadrangles (consisting of a number of lines through a point). More generally, a *generalized 2d-gon* is a near $2d$-gon that satisfies the following additional properties:

(GP1) every point is incident with at least two lines;

(GP2) if x and y are two points of $\mathcal{S}$ at distance $i \in \{1, 2, \ldots, d-1\}$ from each other, then there exists a unique neighbour of y at distance $i - 1$ from x.

The near polygons with only lines of size 2 are precisely the bipartite graphs of finite diameter.

Suppose $\mathcal{S}$ is a near polygon. A *quad* of $\mathcal{S}$ is a set Q of points satisfying the following:

- if a line of $\mathcal{S}$ has two points in Q, then all its points are in Q;

- Q is *convex*, meaning that every point on a shortest path (or geodesic) between two points of Q is also contained in Q;

- the subgeometry of $\mathcal{S}$ determined by those points and lines of $\mathcal{S}$ that are contained in Q is a (nondegenerate) generalized quadrangle.

If every line of $\mathcal{S}$ is incident with at least three points, then there exists a constant t_Q such that every point of Q is incident with precisely $t_Q + 1$ lines of Q. The following two propositions are taken from Shult and Yanushka [26].

Proposition 2.1 ([26, Proposition 2.5]) *Suppose x_1 and x_2 are two points at distance 2 in a near polygon $\mathcal{S}$ and y_1 and y_2 are two distinct common neighbours of x_1 and x_2. If at least one of the lines x_1y_1, y_1x_2, x_2y_2, y_2x_2 contains at least three points, then x_1 and x_2 are contained in a unique quad.*

Proposition 2.2 ([26, Proposition 2.6]) *Suppose $\mathcal{S}$ is a near polygon having at least three points on each line. Then precisely one of the following two cases occurs for a point-quad pair (p, Q):*

(a) *There exists a unique point x' in Q nearest to x. In this case, $d(x, y) = d(x, x') + d(x', y)$ for any point $y \in Q$.*

(b) *The points in Q nearest to x from an ovoid, i.e. a set of points of Q intersecting each line of Q in a singleton.*

If case (a) of Proposition 2.2 occurs, then x is called *classical* with respect to Q. If case (b) occurs, then x is called *ovoidal* with respect to Q.

Proposition 2.3 *Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite near hexagon of order (s,t) having v points. Put*

$$n_0 := 1, \; n_1 := s(t+1), \; n_2 := \frac{v}{s+1} - 1 + s^2 t - st, \; n_3 := \frac{sv}{s+1} - s - s^2 t.$$

Then $|\Gamma_i(x)| = n_i$ for every point x and every $i \in \{0, 1, 2, 3\}$.

Proof. For every point x of $\mathcal{S}$ and every $i \in \{0, 1, 2, 3\}$, we define $n_i(x) := |\Gamma_i(x)|$. Since every line L of $\mathcal{S}$ contains a unique point nearest to x, the sum $\sum_{y \in L}(-\frac{1}{s})^{\mathrm{d}(x,y)}$ is equal to 0. So, $0 = \sum_{L \in \mathcal{L}} \sum_{y \in L}(-\frac{1}{s})^{\mathrm{d}(x,y)} = \sum_{y \in \mathcal{P}} \sum_{L \ni y}(-\frac{1}{s})^{\mathrm{d}(x,y)} = (t+1) \cdot \sum_{y \in \mathcal{P}}(-\frac{1}{s})^{\mathrm{d}(x,y)}$, implying that $n_0(x) - \frac{n_1(x)}{s} + \frac{n_2(x)}{s^2} - \frac{n_3(x)}{s^3} = 0$. Since $n_0(x) = 1$, $n_1(x) = s(t+1)$ and $n_0(x) + n_1(x) + n_2(x) + n_3(x) = v$, we derive that $n_i(x) = n_i$ for every $i \in \{0, 1, 2, 3\}$. ∎

Proposition 2.4 *Suppose $\mathcal{S}$ is a finite near hexagon of order (s,t) with $s > 1$ and x is a point of $\mathcal{S}$. If $\mathcal{R}$ denote the set of quads through x, then*

$$\sum_{R \in \mathcal{R}} t_R^2 = t(t+1) - \frac{n_2}{s^2},$$

where n_2 is the constant mentioned in Proposition 2.3.

Proof. If $R \in \mathcal{R}$, then $|\Gamma_2(x) \cap R| = s^2 t_R$, and so the number of points of $\Gamma_2(x)$ that are contained in a quad together with x is equal to $\sum_{R \in \mathcal{R}} s^2 t_R$, and the number of other points of $\Gamma_2(x)$ is equal to $n_2 - \sum_{R \in \mathcal{R}} s^2 t_R$. Now, counting in two different ways the number of connections between $\Gamma_1(x)$ and $\Gamma_2(x)$, we find

$$s(t+1) \cdot st = \sum_{R \in \mathcal{R}} s^2 t_R (t_R + 1) + n_2 - \sum_{R \in \mathcal{R}} s^2 t_R,$$

from which it readily follows that $\sum_{R \in \mathcal{R}} t_R^2 = t(t+1) - \frac{n_2}{s^2}$. ∎

3 Calculation of some intersection numbers

In this section, we suppose that $\mathcal{S}$ is a finite near hexagon of order (s,t) having v points. By Proposition 2.3, there exist constants n_i, $i \in \{0, 1, 2, 3\}$, such that $n_i = |\Gamma_i(x)|$ for every point x of $\mathcal{S}$ and every $i \in \{0, 1, 2, 3\}$. These constants are equal to

$$n_0 = 1, \; n_1 = s(t+1), \; n_2 = \frac{v}{s+1} - 1 + s^2 t - st, \; n_3 = \frac{sv}{s+1} - s - s^2 t.$$

Suppose x and y are two points of $\mathcal{S}$ at distance δ from each other. For all $i, j \in \{0, 1, 2, 3\}$, let a_{ij} be the intersection number $|\Gamma_i(x) \cap \Gamma_j(y)|$. We determine here all a_{ij}'s for $0 \le i, j \le 2$. We deal separately with the various possibilities for $\delta \in \{0, 1, 2, 3\}$.

First, suppose $\delta = 0$. Then $a_{ij} = n_i \cdot \delta_{ij}$ for all $i, j \in \{0, 1, 2\}$, where δ_{ij} denotes the Kronecker delta.

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Suppose $\delta = 1$. By the triangle inequality, we have $a_{00} = a_{02} = a_{20} = 0$. Since $\Gamma_1(x) \cap \Gamma_0(y) = \{y\}$ and $\Gamma_0(x) \cap \Gamma_1(y) = \{x\}$, we have $a_{01} = a_{10} = 1$. By (NP), the points at distance 1 from x and y are precisely the points of the line xy distinct from x and y implying that $a_{11} = s - 1$. By (NP), the points of $\Gamma_1(x) \cap \Gamma_2(y)$ are precisely the points collinear with x not contained on the line xy, showing that $a_{12} = st$. By symmetry, also $a_{21} = st$. By (NP), the points of $\Gamma_2(x) \cap \Gamma_2(y)$ are precisely the points not on the line xy that are collinear with one of the $s - 1$ points of $xy \setminus \{x, y\}$, implying that $a_{22} = (s-1)st$.

Suppose $\delta = 2$, and put $a_{11} := |\Gamma_1(x) \cap \Gamma_1(y)| = t_2 + 1$. Obviously, $a_{02} = a_{20} = 1$ and by the triangle inequality we have $a_{00} = a_{10} = a_{01} = 0$. If $z \in \Gamma_1(x) \cap \Gamma_2(y)$, then by (NP), the line xz contains a point collinear with y, implying that xz is one of the $t_2 + 1$ lines through x meeting $\Gamma_1(y)$. Conversely, each line through x meeting $\Gamma_1(y)$ contains $s - 1$ points of $\Gamma_1(x) \cap \Gamma_2(y)$. It follows that $a_{12} = (s-1)(t_2 + 1)$. By symmetry, also $a_{21} = (s-1)(t_2 + 1)$. We will now compute a_{22}.

If $s = 1$, then $\mathcal{S}$ is a bipartite graph implying that $\Gamma_2(x) \cap \Gamma_2(y) = P \setminus \{x, y\}$, where P is the part of the bipartite graph containing x and y. In this case $\mathcal{S}$ has order $(1, t)$, and so both parts of $\mathcal{S}$ have equal size, implying that $a_{22} = |\Gamma_2(x) \cap \Gamma_2(y)| = \frac{v}{2} - 2$. Assume therefore that $s \geq 2$.

If $t_2 \geq 1$, then x and y are contained in a unique quad Q by Proposition 2.1. If $t_2 = 0$, then x and y have a unique neighbour u, and we define $Q := xu \cup uy$. Also in the latter case, Q is convex. Let $\mathcal{R}_1$ denote the set of all quads through x meeting Q in a line and let $\mathcal{R}_2$ denote the set of all quads through x meeting Q in the singleton $\{x\}$. Then $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ is the set of all quads through x. Regardless of whether $t_2 = 0$ or $t_2 \geq 1$, we always have

$$\sum_{R \in \mathcal{R}} t_R^2 = t(t+1) - t_2^2 - \frac{n_2}{s^2}$$

by Proposition 2.4. We count the number N_1 of points in $\Gamma_2(x) \cap \Gamma_2(y) \cap Q$. If $t_2 = 0$, then obviously $N_1 = 0$. If $t_2 \geq 1$, then Q is a generalized quadrangle of order (s, t_2) containing $(s+1)(st_2 + 1)$ points and hence $N_1 = (s+1)(st_2 + 1) - 2 - 2s(t_2 + 1) + t_2 + 1 = s^2 t_2 - st_2 - s + t_2$.

We count in two different ways the number N_2 of pairs (u, z), where $z \in (\Gamma_2(x) \cap \Gamma_2(y)) \setminus Q$ and $u \in \Gamma_1(x) \cap \Gamma_1(z)$.

If $u \in \Gamma_1(x) \cap \Gamma_2(y)$, then by (NP) the line xu contains a unique point collinear with y, implying that $u \in Q$. If u have been chosen in such a way and $z \in \Gamma_1(u) \cap \Gamma_2(y)$, then uz contains a point collinear with y, implying that $z \in Q$ since it is on a shortest path between $u \in Q$ and $y \in Q$. So, there are no such pairs (u, z) with $u \in \Gamma_1(x) \cap \Gamma_2(y)$.

For each of the $s(t - t_2)$ points u of $\Gamma_1(x) \cap \Gamma_3(y)$, there are t possibilities for z, namely one on each line through u distinct from ux. Note that if u and z have been chosen in the above way, then $z \notin Q$. Indeed, if z would belong to Q, then as Q is convex, we would also have $u \in Q$, which is impossible as $y \in Q$ and $d(u, y) = 3$.

For each of the $t_2 + 1$ points u of $\Gamma_1(x) \cap \Gamma_1(y)$, there are $s(t - t_2)$ possibilities for z if $t_2 \geq 1$ and $s(t - 1)$ possibilities if $t_2 = 0$. Indeed, the number of lines through u not contained in Q is equal to $t - t_2$ (if $t_2 \geq 1$) or $t - 1$ (if $t_2 = 0$), and on each such line,

there are s points in $(\Gamma_2(x) \cap \Gamma_2(y)) \setminus Q$. So, the total number N_2 of pairs is equal to

$$N_2 = \begin{cases} s(t - t_2)t + (t_2 + 1)s(t - t_2) & \text{if } t_2 \geq 1, \\ s(t - t_2)t + (t_2 + 1)s(t - 1) = st^2 + st - s & \text{if } t_2 = 0. \end{cases}$$

Remark that $N_1 + N_2 = s^2 t_2 - st_2 - s + t_2 + s(t - t_2)t + (t_2 + 1)s(t - t_2)$, regardless of whether $t_2 = 0$ or $t_2 \geq 1$.

We now count the number of pairs in another way. If $R \in \mathcal{R}_1$, then R contains a unique point y' collinear with y and so y is classical with respect to Q. Since $d(x, y) = 2$, the point y' is also collinear with x. In R, there are precisely st_R points of $(\Gamma_2(x) \cap \Gamma_2(y)) \setminus Q$, namely the st_R neighbours of y' in R that are not contained on the line xy'. If $R \in \mathcal{R}_2$, then R does not contain a point of $\Gamma_1(y)$, implying that y is ovoidal with respect to R, i.e. $\Gamma_2(y) \cap R$ is an ovoid of R containing $st_R + 1$ points. In R, there are then precisely st_R points of $(\Gamma_2(x) \cap \Gamma_2(y)) \setminus Q$, namely the st_R points of $(\Gamma_2(y) \cap R) \setminus \{x\}$. We thus see that among the $a_{22} - N_1$ points of $(\Gamma_2(x) \cap \Gamma_2(y)) \setminus Q$, there are $\sum_{R \in \mathcal{R}} st_R$ points z_1 that are contained in a quad together with x and $a_{22} - N_1 - \sum_{R \in \mathcal{R}} st_R$ points z_2 that are not contained in a quad together with x. For each such $z_1 \in R \in \mathcal{R}$, there are $t_R + 1$ possibilities for u and for each such z_2 there is a unique possibility for u. Hence, the total number of pairs is also equal to $N_2 = \sum_{R \in \mathcal{R}} st_R(t_R + 1) + a_{22} - N_1 - \sum_{R \in \mathcal{R}} st_R$. This implies that

$$a_{22} = N_1 + N_2 - \sum_{R \in \mathcal{R}} st_R^2.$$

Taking into account that

$$\begin{aligned} N_1 + N_2 &= s^2 t_2 - st_2 - s + t_2 + s(t - t_2)t + (t_2 + 1)s(t - t_2), \\ \sum_{R \in \mathcal{R}} t_R^2 &= t(t + 1) - t_2^2 - \frac{n_2}{s^2}, \\ n_2 &= \frac{v}{s + 1} - 1 + s^2 t - st, \end{aligned}$$

we can compute that

$$a_{22} = s^2 t_2 - 2st_2 - s + t_2 + st - t - \frac{1}{s} + \frac{v}{s(s + 1)}.$$

Observe that the latter number remains valid if $s = 1$. Indeed, above we already showed that $a_{22} = \frac{v}{2} - 2$ if $s = 1$.

Finally, suppose that $\delta = 3$. By the triangle inequality, we have $a_{00} = a_{01} = a_{02} = a_{10} = a_{20} = a_{11} = 0$. On each line through x, there is a unique point at distance 2 from y, implying that $a_{12} = |\Gamma_1(x) \cap \Gamma_2(y)| = t + 1$. Similarly, we have that $a_{21} = t + 1$. We will now compute a_{22}.

If $s = 1$, then since $\mathcal{S}$ is a bipartite graph, we have $a_{22} = |\Gamma_2(x) \cap \Gamma_2(y)| = 0$. Assume therefore that $s \geq 2$.

Let $\mathcal{R}_1$ denote the set of quads through x containing a point of $\Gamma_1(y)$, and let $\mathcal{R}_2$ denote the set of all remaining quads through x. Again, $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ is the set of all quads through x.

Lemma 3.1 *The number of shortest paths between x and y is equal to $t + 1 + \sum_{R \in \mathcal{R}_1} t_R$.*

Proof. For every point $u \in \Gamma_2(x) \cap \Gamma_1(y)$, let u' denote a common neighbour of u and x. Then there are $t + 1$ shortest paths between x and y of the form $xu'uy$. For every $R \in \mathcal{R}_1$, the point x and the unique point $u \in \Gamma_1(y) \cap R$ have t_R common neighbours distinct from u', giving rise to t_R extra shortest paths between x and y. By Proposition 2.1, we know that we have accounted for all shortest paths between x and y now. So, the total number of them should be $t + 1 + \sum_{R \in \mathcal{R}_1} t_R$. ∎

Lemma 3.2 *We have $a_{22} = (s - 1)\left((t + 1)^2 - \sum_{R \in \mathcal{R}} t_R^2\right)$.*

Proof. We count in two different ways the number of pairs (u, z) satisfying $z \in \Gamma_2(x) \cap \Gamma_2(y)$ and $u \in \Gamma_1(x) \cap \Gamma_1(z)$.

We show that there are $(s - 1)(t + 1)t$ such pairs (u, z) where $u \in \Gamma_1(x) \cap \Gamma_3(y)$. Indeed, there are $(s - 1)(t + 1)$ points $u \in \Gamma_1(x) \cap \Gamma_3(y)$, and for a given u there are t possibilities for z, namely one on each line through u distinct from ux.

We show that there are $(s - 1)(t + 1 + \sum_{R \in \mathcal{R}_1} t_R)$ such pairs (u, z) with $u \in \Gamma_1(x) \cap \Gamma_2(y)$. If (u, z) is such a pair, then the fact that $d(u, y) = d(z, y) = 2$ implies that the line uz contains a point u' collinear with y, implying that $xuu'y$ is a shortest path from x to y. Conversely, if $xuu'y$ is a shortest path from x to y and z is one of the $s - 1$ points of $uu' \setminus \{u, u'\}$, then (u, z) is a suitable pair. From Lemma 3.1, it now follows that the number of suitable pairs (u, z) with $u \in \Gamma_1(x) \cap \Gamma_2(y)$ is equal to $(s - 1)(t + 1 + \sum_{R \in \mathcal{R}_1} t_R)$.

So, the total number of suitable pairs is equal to $(s - 1)\left((t + 1)^2 + \sum_{R \in \mathcal{R}_1} t_R\right)$. We now count the number of pairs (u, z) in a different way.

If $R \in \mathcal{R}_1$, then y has distance 1 from a unique point y' of R, implying that y is classical with respect to R. Since $d(x, y) = 3$, we have $d(x, y') = 2$. The quad R contains $(s - 1)(t_R + 1)$ points z of $\Gamma_2(x) \cap \Gamma_2(y)$ (namely the points of $R \setminus \{y'\}$ collinear with y', but not with x) and each of these points is collinear with $t_R + 1$ points $u \in \Gamma_1(x)$.

If $R \in \mathcal{R}_2$, then y is ovoidal with respect to R, and so R contains $st_R + 1$ points of $\Gamma_2(y)$, $t_R + 1$ of which are collinear with x, implying that there are $(s - 1)t_R$ points $z \in R \cap \Gamma_2(x) \cap \Gamma_2(y)$. Moreover, each such point z is collinear with $t_R + 1$ points $u \in \Gamma_1(x)$.

Since each of the $a_{22} - (s - 1) \cdot \sum_{R \in \mathcal{R}_1}(t_R + 1) - (s - 1) \cdot \sum_{R \in \mathcal{R}_2} t_R$ points of $\Gamma_2(x) \cap \Gamma_2(y)$ not contained in a quad together with x is collinear with a unique point $u \in \Gamma_1(x)$, we see that the number of suitable pairs is also equal to

$$a_{22} - (s - 1) \sum_{R \in \mathcal{R}_1} (t_R + 1) - (s - 1) \sum_{R \in \mathcal{R}_2} t_R + (s - 1) \sum_{R \in \mathcal{R}_1} (t_R + 1)^2$$

$$+ (s - 1) \sum_{R \in \mathcal{R}_2} t_R(t_R + 1) = a_{22} + (s - 1) \sum_{R \in \mathcal{R}_1} t_R(t_R + 1) + (s - 1) \sum_{R \in \mathcal{R}_2} t_R^2.$$

Since this number is equal to $(s - 1)\left((t + 1)^2 + \sum_{R \in \mathcal{R}_1} t_R\right)$, we see that $a_{22} = (s - 1)\left((t + 1)^2 - \sum_{R \in \mathcal{R}} t_R^2\right)$. ∎

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$\delta=0$	0	1	2	$\delta=2$	0	1	2
0	1	0	0	0	0	0	1
1	0	$s(t+1)$	0	1	0	t_2+1	$(s-1)(t_2+1)$
2	0	0	$\frac{v}{s+1}-1+s^2t-st$	2	1	$(s-1)(t_2+1)$	$s^2t_2-2st_2-s+t_2+st-t-\frac{1}{s}+\frac{v}{s(s+1)}$
$\delta=1$	0	1	2	$\delta=3$	0	1	2
0	0	1	0	0	0	0	0
1	1	$s-1$	st	1	0	0	$t+1$
2	0	st	$(s-1)st$	2	0	$t+1$	$\frac{(s-1)(2s-1)}{s}t+\frac{(s^2-1)(s-1)}{s^2}+\frac{v(s-1)}{s^2(s+1)}$

Table 1: The intersection numbers a_{ij} for $\delta\in\{0,1,2,3\}$

Since $\sum_{R\in\mathcal{R}}t_R^2=t(t+1)-\frac{n_2}{s^2}$, we find that $a_{22}=(s-1)(t+1+\frac{n_2}{s^2})$. If we replace in the latter expression n_2 by $\frac{v}{s+1}-1+s^2t-st$, then we find

$$a_{22}=\frac{(s-1)(2s-1)}{s}t+\frac{(s^2-1)(s-1)}{s^2}+\frac{v(s-1)}{s^2(s+1)}.$$

Observe that the latter number remains valid if $s=1$. Indeed, above we have already shown that $a_{22}=0$ if $s=1$.

We summarize the values of a_{ij} in Table 1.

4   An idempotent matrix

In this section, $\mathcal{S}=(\mathcal{P},\mathcal{L})$ denotes a finite near hexagon of order (s,t) having v points. We will consider a particular ordering $p_1,p_2,\ldots,p_v$ of the points, and use this ordering to define real matrices A_i, $i\in\{0,1,2,3\}$, whose rows and columns are indexed by the points. For every $i\in\{0,1,2,3\}$ and all $x,y\in\mathcal{P}$, we define

$$(A_i)_{xy}=\begin{cases}1 & \text{if } \mathrm{d}(x,y)=i,\\ 0 & \text{otherwise.}\end{cases}$$

Then A_1 equals the collinearity matrix A of $\mathcal{S}$ (with respect to the ordering) and A_0 equals I, the $v\times v$ identity matrix. We also define

$$N:=\sum_{i=0}^{3}(-\frac{1}{s})^iA_i$$

and

$$\alpha:=n_0+\frac{n_1}{s^2}+\frac{n_2}{s^4}+\frac{n_3}{s^6}=\frac{(s+1)^2(s-1)(s^2+1)+st(s-1)(s+1)^2+v}{s^5}>0.$$

In this section, we prove the following result.

Proposition 4.1 *We have $N\cdot N=\alpha\cdot N$. As a consequence, $\frac{N}{\alpha}$ is an idempotent matrix.*

Proof. For every point x of $\mathcal{S}$, let χ_x be the row vector whose columns are indexed by the v points of $\mathcal{S}$ (with respect to the above ordering), such that $\chi_x(z) := (-\frac{1}{s})^{\mathrm{d}(x,z)}$ for every point z of $\mathcal{S}$. If x and y are two points of $\mathcal{S}$, then the standard inner product of the two row vectors χ_x and χ_y is equal to

$$\chi_x \cdot \chi_y = \sum_{0 \leq i,j \leq 3} a_{ij} \left(-\frac{1}{s}\right)^{i+j},$$

where $a_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$ for all $i, j \in \{0, 1, 2, 3\}$. Taking into account that

$$a_{i3} = n_i - a_{i0} - a_{i1} - a_{i2}, \quad a_{3i} = n_i - a_{0i} - a_{1i} - a_{2i}, \quad a_{33} = 2n_3 - v + \sum_{i,j=0}^{2} a_{ij},$$

for every $i \in \{0, 1, 2\}$ and making use of Proposition 2.3, we see that

$$\chi_x \cdot \chi_y = \left(\sum_{i,j=0}^{2} k_{ij} a_{ij} \right) - \frac{v}{s^6} + \left(2n_0 \left(-\frac{1}{s}\right)^3 + 2n_1 \left(\frac{1}{s}\right)^4 + 2n_2 \left(-\frac{1}{s}\right)^5 + 2n_3 \left(\frac{1}{s}\right)^6 \right)$$

$$= -\frac{v}{s^6} + \sum_{i,j=0}^{2} k_{ij} a_{ij},$$

where $k_{ij} = (-\frac{1}{s})^{i+j} - (-\frac{1}{s})^{i+3} - (-\frac{1}{s})^{j+3} + \frac{1}{s^6}$ for all $i, j \in \{0, 1, 2\}$, i.e.

$$k_{00} = \frac{(s^3+1)^2}{s^6}, \quad k_{01} = k_{10} = -\frac{(s^2-1)(s^3+1)}{s^6}, \quad k_{02} = k_{20} = \frac{(s+1)(s^3+1)}{s^6},$$

$$k_{11} = \frac{(s^2-1)^2}{s^6}, \quad k_{12} = k_{21} = -\frac{(s+1)(s^2-1)}{s^6}, \quad k_{22} = \frac{(s+1)^2}{s^6}.$$

Using the contents of Table 1, it is straightforward (but somewhat tedious) to compute that $\chi_x \cdot \chi_y = \alpha \cdot (-\frac{1}{s})^{\mathrm{d}(x,y)}$ for every two points x and y of $\mathcal{S}$. If x is a point of $\mathcal{S}$, then the row x of the matrix N coincides with χ_x. As N is a symmetric matrix, we thus have that $N \cdot N^T = N^2 = \alpha \cdot N$. ∎

Remark. For a finite near $2d$-gon of order (s, t), we can define similarly as above matrices A_i, $i \in \{0, 1, \ldots, d\}$, and put $N := \sum_{i=0}^{d} (-\frac{1}{s})^i A_i$. For several known examples of such near $2d$-gons, the author verified that $N^2 = \alpha N$ for some $\alpha \in \mathbb{Q}$. The explanation of this fact seems to come from the following result.

Lemma 4.2 *If there exists a polynomial $p^*(x) \in \mathbb{Q}[x]$ such that $N = p^*(A)$, then $N^2 = \alpha N$ for some $\alpha \in \mathbb{Q}$.*

Proof. It can be shown, see Proposition 1 of [11], that $AN = -(t+1)N$. This implies that $N^2 = p^*(A)N = p^*(-(t+1))N = \alpha N$, where $\alpha = p^*(-(t+1)) \in \mathbb{Q}$. ∎

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For many near $2d$-gons, e.g. those with a distance-regular collinearity graph, it can be shown that there exist polynomials $p_i(x) \in \mathbb{Q}[x]$ ($i \in \{0, 1, \ldots, d\}$) such that $A_i = p_i(A)$, so that one can take $p^*(x) = \sum_{i=0}^{d}(-\frac{1}{s})^i p_i(x)$. However, to prove that these polynomials exist seems to be a hard problem for general finite near polygons with an order (the author does not have a clue on how to approach this problem). Despite the fact that the existence of these polynomials $p_i(x)$ was not proved here, it was still possible (for the near hexagon case) to show that $N^2 = \alpha N$ for some $\alpha \in \mathbb{Q}$.

We also note that if there exists a polynomial $p^*(x) \in \mathbb{Q}[x]$ such that $N = p^*(A)$, then the idempotent matrix $\frac{N}{\alpha}$ with $\alpha = p^*(-(t+1))$ can be regarded as the projection matrix on the eigenspace of the smallest eigenvalue $-(t+1)$ of A.

5 Application to tetrahedrally closed line systems

Every tetrahedron T centered at the origin o of the Euclidean 3-space $\mathbb{R}^3$ determines a tetrahedrally closed system Σ_T of norm 3 vectors of type $(0, \frac{1}{3})$ for which all four local near polygons are lines of size 3. There are up to isomorphism three generalized quadrangles of order $(2, *)$ (namely, the (3×3)-grid, $W(2)$ and $Q(5, 2)$), and by Shult and Yanushka [26] or Cuypers [9] we already know that each of them occurs as a local near polygon of a suitable tetrahedrally closed system of norm 3 vectors of type $(0, \frac{1}{3})$. So, it suffices to prove Theorem 1.2 in the case of near hexagons.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite near hexagon with three points per line and denote by $\mathrm{d}(\cdot, \cdot)$ the distance function in $\mathcal{S}$. Put $\mathcal{P} = \{p_1, p_2, \ldots, p_v\}$ with $v = |\mathcal{P}|$ and denote by ∞ a symbol that is not contained in $\mathcal{P}$. Similarly as in Section 4, we can define real $v \times v$ matrices A_0, A_1, A_2, A_3, N with respect to the ordering $(p_1, p_2, \ldots, p_v)$ of the points. We now define an additional matrix M which has dimensions $(v + 1) \times (v + 1)$ and whose rows and columns are indexed by the set $\{\infty\} \cup \mathcal{P}$ using the ordering $(\infty, p_1, p_2, \ldots, p_v)$. The matrix M is defined as follows:

- $M_{\infty,\infty} := 3$;

- $M_{\infty,x} := -1$ and $M_{x,\infty} := -1$ for every point $x \in \mathcal{P}$;

- $M_{x,x} := 3$ if $x \in \mathcal{P}$, $M_{x,y} := -1$ if $x, y \in \mathcal{P}$ with $\mathrm{d}(x, y) = 1$, $M_{x,y} := 1$ if $x, y \in \mathcal{P}$ with $\mathrm{d}(x, y) = 2$, and $M_{x,y} := 0$ if $x, y \in \mathcal{P}$ with $\mathrm{d}(x, y) = 3$.

Let P denote the following real $(v + 1) \times (v + 1)$-matrix:

$$P := \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \cdots & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is straightforward to verify that $P^T \cdot M \cdot P$ is the following block matrix:

$$\begin{pmatrix} 3 & 0 \\ 0 & \frac{8}{3} \cdot N \end{pmatrix}$$

So, the matrix M is positive-semidefinite if and only if N is positive semidefinite.

Proposition 5.1 *The following are equivalent:*

(1) *$\mathcal{S}$ is some local near polygon of a tetrahedrally closed system of norm 3 vectors of type $(0, \frac{1}{3})$.*

(2) *The matrix N is positive semidefinite.*

Proof. We use some of the notation introduced in Proposition 1.1. Suppose $\mathcal{S} = \mathcal{S}_{\Sigma, \bar{v}}$, where Σ is a tetrahedrally closed system of norm 3 vectors of type $(0, \frac{1}{3})$ of $\mathbb{R}^n$ satisfying Property $(P_{\bar{v}})$ where $\bar{v}$ is some vector of Σ. By Proposition 1.1, we then know that M is a Gram matrix of the set $\{\bar{v}\} \cup \mathcal{P}_{\Sigma, \bar{v}}$, implying (see e.g. [18, Section 7.2]) that M and hence also N are positive semidefinite.

Conversely, suppose that N and hence also M are positive semidefinite matrices. If n is the rank of M, then by [18, Section 7.2], there exists a collection Σ of vectors in $\mathbb{R}^n$ for which M is a Gram matrix. The remainder of the proof follows the arguments of Shult and Yanushka exposed in Proposition 3.14 of their paper [26] and is added here for reasons of completeness.

Denote by $\bar{v} \in \Sigma$ the vector corresponding to the first row and column of M. Every vector of $\Sigma \setminus \{\bar{v}\}$ then corresponds to a point of $\mathcal{S}$ which we will denote by $\theta(\bar{v})$. We define

$$\overline{\Sigma} := \Sigma \cup -\Sigma,$$

where $-\Sigma := \{-\bar{u} \,|\, \bar{u} \in \Sigma\}$. Since all diagonal entries of M are equal to 3 and all non-diagonal entries are equal to -1, 1 or 0, we see that $\Sigma \cap (-\Sigma) = \emptyset$ and $\overline{\Sigma}$ is system of norm 3 vectors of type $(0, \frac{1}{3})$. Since all non-diagonal elements in the first row and column of M are equal to -1, we see that $\mathcal{P}_{\overline{\Sigma}, \bar{v}} = \{\bar{u} \in \overline{\Sigma} \,|\, \bar{v} \cdot \bar{u} = -1\} = \Sigma \setminus \{\bar{v}\}$.

We prove that $\overline{\Sigma}$ satisfies Property $(P_{\bar{v}})$. If $\{p_1, p_2, p_3\}$ is a line of $\mathcal{S}$ and $\bar{v}_i = \theta^{-1}(p_i)$ for every $i \in \{1, 2, 3\}$, then $\bar{u}_1 \cdot \bar{u}_2 = -1$ for any two distinct vectors $\bar{u}_1, \bar{u}_2 \in \{\bar{v}, \bar{v}_1, \bar{v}_2, \bar{v}_3\}$, implying that $\{\bar{v}, \bar{v}_1, \bar{v}_2, \bar{v}_3\}$ defines a tetrahedron. This implies that $\mathcal{L}_{\overline{\Sigma}, \bar{v}} \neq \emptyset$. If $\bar{w}_1$ and $\bar{w}_2$ are two vectors of $\mathcal{P}_{\overline{\Sigma}, \bar{v}} = \Sigma \setminus \{\bar{v}\}$ satisfying $\bar{w}_1 \cdot \bar{w}_2 = 1$. Then the points $p_1 = \theta(\bar{w}_1)$ and $p_2 = \theta(\bar{w}_2)$ of $\mathcal{S}$ lie at distance 2 from each other. If p_3 denotes a common neighbour of p_1 and p_2, and $\bar{w}_3 := \theta^{-1}(p_3)$, then $\bar{w}_3 \cdot \bar{w}_1 = \bar{w}_3 \cdot \bar{w}_2 = -1$. We conclude that $\overline{\Sigma}$ indeed satisfies Property $(P_{\bar{v}})$.

Suppose $\{\bar{v}, \bar{v}_1, \bar{v}_2, \bar{v}_3\} \subseteq \overline{\Sigma}$ determines a tetrahedron. Then $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in \Sigma \setminus \{\bar{v}\}$ and $\bar{v}_1 \cdot \bar{v}_2 = \bar{v}_1 \cdot \bar{v}_3 = \bar{v}_2 \cdot \bar{v}_3 = -1$, implying that the points $p_1 = \theta(\bar{v}_1)$, $p_2 = \theta(\bar{v}_2)$ and $p_3 = \theta(\bar{v}_3)$ of $\mathcal{S}$ are mutually collinear, i.e. they form a line of $\mathcal{S}$. Above, we have already shown that every line of $\mathcal{S}$ arises from such a tetrahedron. So, the map θ defines an isomorphism between the point-line geometries $\mathcal{S}_{\overline{\Sigma}, \bar{v}}$ and $\mathcal{S}$.

It remains to show that $\overline{\Sigma}$ is tetrahedrally closed, i.e. we need to show that if $W = \{\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4\}$ is a collection of four vectors defining a tetrahedron such that $\bar{w}_1, \bar{w}_2, \bar{w}_3 \in \overline{\Sigma}$, then also $\bar{w}_4 \in \overline{\Sigma}$. This property holds for W if and only if it holds for $-W := \{-\bar{w}_1, -\bar{w}_2, -\bar{w}_3, -\bar{w}_4\}$. So, we may suppose that a majority of the vectors $\bar{w}_1, \bar{w}_2, \bar{w}_3$ belong to Σ. Without loss of generality, we may suppose that $\bar{w}_1, \bar{w}_2 \in \Sigma$. We then necessarily have $-\bar{v} \notin \{\bar{w}_1, \bar{w}_2, \bar{w}_3\}$. We distinguish three cases.

- Suppose $\bar{v} \in \{\bar{w}_1, \bar{w}_2, \bar{w}_3\}$, say $\bar{w}_1 = \bar{v}$. Then $\bar{w}_2, \bar{w}_3 \in \Sigma \setminus \{\bar{v}\}$ and the points $\theta(\bar{w}_2)$, $\theta(\bar{w}_3)$ of $\mathcal{S}$ are collinear. If $\theta(\bar{w}_4')$ with $\bar{w}_4' \in \Sigma \setminus \{\bar{v}\}$ denotes the third point on the line joining them, then $\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4'\}$ defines a tetrahedron, implying that $\bar{w}_4 = \bar{w}_4' \in \Sigma \subseteq \overline{\Sigma}$.

- Suppose $\{\bar{w}_1, \bar{w}_2, \bar{w}_3\} \subseteq \Sigma \setminus \{\bar{v}\}$. Then $\{\bar{v}, \bar{w}_1, \bar{w}_2, \bar{w}_3\}$ determines a tetrahedron, implying that $\bar{w}_4 = \bar{v} \in \overline{\Sigma}$.

- Suppose $\{\bar{w}_1, \bar{w}_2\} \subseteq \Sigma \setminus \{\bar{v}\}$ and $\bar{w}_3 \in -\Sigma \setminus \{-\bar{v}\}$. Put $p_1 = \theta(\bar{w}_1)$, $p_2 = \theta(\bar{w}_2)$ and $p_3 = \theta(-\bar{w}_3)$. As $\bar{w}_1 \cdot \bar{w}_2 = -1$ and $\bar{w}_1 \cdot (-\bar{w}_3) = \bar{w}_2 \cdot (-\bar{w}_3) = 1$, we have $d(p_1, p_2) = 1$ and $d(p_3, p_1) = d(p_3, p_2) = 2$. If p denotes the third point on the line $p_1 p_2$, then (NP) implies that p_3 is collinear with p. Denote by p_4 the third point on the line $p_3 p$, and let $\bar{w}_4' \in -\Sigma \setminus \{-\bar{v}\}$ such that $\theta(-\bar{w}_4') = p_4$. From $d(p_3, p_4) = 1$ and $d(p_1, p_4) = d(p_2, p_4) = 2$, it follows that $\bar{w}_1 \cdot \bar{w}_4' = \bar{w}_2 \cdot \bar{w}_4' = \bar{w}_3 \cdot \bar{w}_4' = -1$, i.e. $\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4'\}$ determines a tetrahedron. Hence, $\bar{w}_4 = \bar{w}_4' \in \overline{\Sigma}$.

■

Theorem 1.2 of Section 1 is an immediate consequence of Propositions 4.1 and 5.1, taking into account that any idempotent matrix is also positive semidefinite.

6 A generalization of the Haemers-Roos inequality

We continue with the notation of Section 4. In particular, $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite near hexagon of order (s, t). If X is a set of points of $\mathcal{S}$, then γ_X denotes the characteristic vector of X, i.e. the row matrix whose columns are indexed by the points of $\mathcal{S}$ (respecting the earlier mentioned ordering of points) such that $\gamma_X(y) = 1$ if $y \in X$ and $\gamma_X(y) = 0$ if $y \in \mathcal{P} \setminus X$. If X is the singleton $\{x\}$, then we denote γ_X also by γ_x.

Consider two opposite points x and y of $\mathcal{S}$, and put $Z_1 := \Gamma_1(x) \cap \Gamma_2(y)$, $Z_2 := \Gamma_2(x) \cap \Gamma_1(y)$. We show that if $z_1 \in Z_1$ and $z_2 \in Z_2$, then $d(z_1, z_2) \in \{1, 3\}$. By the triangle inequality, we know that $d(z_1, z_2) \in \{1, 2, 3\}$. If $d(z_1, z_2) \leq 2$, then (NP) and the fact that $d(z_1, y) = 2$ implies that there exists a unique point z_2' on the line $y z_2$ collinear with z_1. This point lies at distance at most 2 from x and hence coincides with z_2, implying that $d(z_1, z_2) = 1$.

Since the matrix $\frac{N}{\alpha}$ is idempotent (Proposition 4.1), it is positive semidefinite, implying that also N is positive semidefinite (as $\alpha > 0$). It follows that for all $X_1, X_2, X_3, X_4 \in \mathbb{R}$,

$$(X_1 \gamma_x + X_2 \gamma_y + X_3 \gamma_{Z_1} + X_4 \gamma_{Z_2}) \cdot N \cdot (X_1 \gamma_x + X_2 \gamma_y + X_3 \gamma_{Z_1} + X_4 \gamma_{Z_2})^T \geq 0,$$

i.e.,
$$[X_1 \ X_2 \ X_3 \ X_4] \cdot M \cdot [X_1 \ X_2 \ X_3 \ X_4]^T \geq 0,$$

where

$$M = \begin{bmatrix} 1 & -\frac{1}{s^3} & -\frac{t+1}{s} & \frac{t+1}{s^2} \\ -\frac{1}{s^3} & 1 & \frac{t+1}{s^2} & -\frac{t+1}{s} \\ -\frac{t+1}{s} & \frac{t+1}{s^2} & \frac{(t+1)(s^2+t)}{s^2} & \gamma \cdot (-\frac{1}{s}) + ((t+1)^2 - \gamma) \cdot (-\frac{1}{s^3}) \\ \frac{t+1}{s^2} & -\frac{t+1}{s} & \gamma \cdot (-\frac{1}{s}) + ((t+1)^2 - \gamma) \cdot (-\frac{1}{s^3}) & \frac{(t+1)(s^2+t)}{s^2} \end{bmatrix},$$

with γ denoting the number of shortest paths connecting x and y. It follows that M is also positive semidefinite matrix. By Sylvester's criterion for positive semidefinite matrices [18, Theorem 7.2.5], we then know that

$$\det(M(1)) \geq 0, \quad \det(M(2)) \geq 0, \quad \det(M(3)) \geq 0, \quad \det(M(4)) \geq 0,$$

where $M(i)$ is the submatrix of M defined by the first i rows and first i columns of M.

If $s = 1$, then $\det(M(1)) = 1$ and $\det(M(2)) = \det(M(3)) = \det(M(4)) = 0$, and so the inequalities for $\det(M(i))$ are always fulfilled.

Suppose therefore that $s \geq 2$. Then $\det(M(1)) = 1$, $\det(M(2)) = 1 - \frac{1}{s^6} > 0$ and

$$\det(M(3)) = \frac{t+1}{s^8}(s^2 - 1)^2(s^4 + s^2 - t),$$

implying that

$$t \leq s^4 + s^2.$$

If x and y are two points at distance 2 having at least two common neighbours, then Proposition 2.1 implies that x and y are contained in a unique quad of order (s, t_{xy}) for some $t_{xy} \in \mathbb{N} \setminus \{0\}$. By Higman [17, (6.4)], we know that $t_{xy} \leq s^2$. This implies that every two opposite points of $\mathcal{S}$ are connected by at most $(t + 1)(s^2 + 1)$ shortest paths, i.e.

$$\gamma \leq (t + 1)(s^2 + 1).$$

We compute

$$\det(M(4)) = -\frac{1}{s^{12}}(s^2-1)^3\Big((t+1)^2 - (s^2+s+1)(\gamma - s(t+1))\Big) \cdot \Big((t+1)^2 - (s^2-s+1)(\gamma + s(t+1))\Big),$$

$$= -\frac{1}{s^{12}}(s^2 - 1)^3(t + 1)^2 \cdot \Big(t + 1 - (s^2 + s + 1)(\overline{\gamma} - s)\Big) \cdot \Big(t + 1 - (s^2 - s + 1)(\overline{\gamma} + s)\Big),$$

where $\overline{\gamma} := \frac{\gamma}{t+1} \leq s^2 + 1$. The latter inequality implies that $(s^2 + s + 1)(\overline{\gamma} - s) \leq (s^2 - s + 1)(\overline{\gamma} + s)$. The fact that $\det(M(4)) \geq 0$ then implies that

$$(s^2 + s + 1)(\overline{\gamma} - s) \leq t + 1 \leq (s^2 - s + 1)(\overline{\gamma} + s).$$

If we put $\overline{t_2} := \overline{\gamma} - 1$, then we find

$$(s^2 + s + 1)\overline{t_2} - s^3 \leq t \leq s^3 + \overline{t_2}(s^2 - s + 1). \tag{1}$$

As mentioned earlier, the upper bound for t generalizes the Haemers-Roos inequality for generalized polygons [16] and the Haemers-Mathon bound for regular near hexagons [15, 21, 23]. The lower bound for t generalizes a lower bound obtained in [12]. In fact, the techniques we invoked in this section are similar to some of the techniques used in [12] and [29].

We will now deal with the case where t attains the upper bound $s^4 + s^2$ (under the assumption that $s > 1$). If $t = s^4 + s^2$, then (1) implies that $\overline{t_2} = s^2$, i.e. $\overline{\gamma} = s^2 + 1$ and $\gamma = (t+1)(s^2+1)$. By the above we then know that every two points at distance 2 have precisely $s^2 + 1$ common neighbours. This implies that every two points at distance 2 are contained in a unique quad of order (s, s^2), or equivalently, that every two intersecting lines are contained in a unique quad of order (s, s^2). For every point x, consider now the point-line geometry $\mathcal{S}_x$ with point set $\mathcal{L}_x = \{L \in \mathcal{L} \,|\, x \in L\}$ and line set $\mathcal{L}_x := \{L_Q \,|\, Q \text{ a quad through } x\}$, where L_Q is the set consisting of all lines through x contained in Q. The point-line geometry $\mathcal{S}_x$ satisfies the following properties:

- there are $s^4 + s^2 + 1$ points and every line contains $s^2 + 1$ points;

- every two distinct points are incident with a unique line.

So, $\mathcal{S}_x$ is a Steiner system $S(2, s^2 + 1, s^4 + s^2 + 1)$ and hence a projective plane of order s^2. As this holds for all points x of $\mathcal{S}$, the near hexagon $\mathcal{S}$ must be a dual polar space by Shult and Yanushka [26, Lemma 2.21]. By the classification of polar spaces (Veldkamp [30], Tits [28]), we then know that $\mathcal{S}$ must be a Hermitian dual polar space, i.e. s is a prime power and $\mathcal{S}$ is isomorphic to $DH(5, s^2)$.

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