Blocking sets of tangent lines to a hyperbolic quadric in PG(3, 3)

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September 4, 2017

Abstract

Let $Q^+(3,q)$ be a hyperbolic quadric in $\mathrm{PG}(3,q)$ and $\mathcal{T}$ be the set of all lines of $\mathrm{PG}(3,q)$ which are tangent to $Q^+(3,q)$. If k is the minimum size of a $\mathcal{T}$-blocking set in $\mathrm{PG}(3,q)$, then we prove that $q^2+1 \leq k \leq q^2+q$. When $q=3$, we show that: (i) there is no $\mathcal{T}$-blocking set of size 10, and (ii) there are exactly two $\mathcal{T}$-blocking sets of size 11 up to isomorphism. By means of the computer algebra systems GAP [13] and Sage [9], we find that there exist no $\mathcal{T}$-blocking sets of size q^2+1 for each odd prime power $q \leq 13$.

Keywords: Projective space, Blocking set, Conic, Ovoid, Hyperbolic quadric

AMS 2010 subject classification: 05B25, 51E21

1 Introduction

Throughout, q is a prime power. Let $\mathrm{PG}(3,q)$ be the three dimensional projective space defined over a finite field of order q and $Q^+(3,q)$ be a hyperbolic quadric in $\mathrm{PG}(3,q)$. One can refer to [6] for the basic properties of the points, lines and planes of $\mathrm{PG}(3,q)$ with respect to the quadric $Q^+(3,q)$. Every line of $\mathrm{PG}(3,q)$ meets $Q^+(3,q)$ in 0, 1, 2 or $q+1$ points. We denote by $\mathcal{E}$ (respectively, $\mathcal{T}_1$, $\mathcal{S}$, $\mathcal{T}_0$) the set of lines of $\mathrm{PG}(3,q)$ that intersect $Q^+(3,q)$ in 0 (respectively, 1, 2, $q+1$) points. The elements of $\mathcal{E}$ are called *external lines*, those of $\mathcal{S}$ *secant lines* and those of $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_1$ *tangent lines*. If $L \in \mathcal{T}_i$ with $i \in \{0,1\}$, then L is also called a $\mathcal{T}_i$-line. The $\mathcal{T}_0$-lines are precisely the lines contained in $Q^+(3,q)$, and so we have $|\mathcal{T}_0| = 2(q+1)$. As every point of $Q^+(3,q)$ is contained in $q-1$ $\mathcal{T}_1$-lines, we have $|\mathcal{T}_1| = (q+1)^2(q-1)$ and hence $|\mathcal{T}| = (q+1)(q^2+1)$. We also have $|\mathcal{S}| = \frac{1}{2}q^2(q+1)^2$ and $|\mathcal{E}| = (q^2+1)(q^2+q+1) - (q+1)(q^2+1) - \frac{1}{2}q^2(q+1)^2 = \frac{1}{2}q^2(q-1)^2$.

For a given nonempty set $\mathcal{L}$ of lines of $\mathrm{PG}(3,q)$, a set X of points of $\mathrm{PG}(3,q)$ is called an *$\mathcal{L}$-blocking set* if each line of $\mathcal{L}$ meets X. The first step in the study of blocking sets has been to determine the smallest cardinality of a blocking set and to characterize, if possible, all blocking sets of that cardinality. If $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(3,q)$ and X

is an $\mathcal{L}$-blocking set, then $|X| \geq q^2+q+1$ and equality holds if and only if X is a plane of $\mathrm{PG}(3,q)$. This follows from a more general result by Bose and Burton [4, Theorem 1]. Biondi et al. characterized the minimum size $\mathcal{E}$-blocking sets in [2, Theorem 2.4] for $q \geq 9$ odd and in [1, Theorem 1.1] for $q \geq 8$ even (also see [10, Section 3] for a different proof which works for all even q). When $q > 2$ is even, the minimum size $(\mathcal{E} \cup \mathcal{S})$-blocking sets were determined in [12, Theorem 1.3] using the properties of generalized quadrangles. For $\mathcal{L} \in \{\mathcal{S}, \mathcal{T} \cup \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$, the minimum size $\mathcal{L}$-blocking sets are described in [11] for all q. When q is even, the minimum size $(\mathcal{E} \cup \mathcal{T})$-blocking sets are characterized in [10, Proposition 1.5].

Suppose q is even and let ζ denote the symplectic polarity of $\mathrm{PG}(3,q)$ associated with the quadric $Q^+(3,q)$. With the symplectic polarity ζ, there is associated a symplectic generalized quadrangle $W(q)$, whose points are the points of $\mathrm{PG}(3,q)$ and whose lines are the lines of $\mathrm{PG}(3,q)$ that are totally isotropic with respect to ζ, with incidence being containment (see [8] for more on generalized quadrangles). The lines of $W(q)$ are precisely the elements of $\mathcal{T}$. If X is a $\mathcal{T}$-blocking set in $\mathrm{PG}(3,q)$, then $|X| \geq q^2+1$ and equality holds if and only if X is an ovoid[1] of $W(q)$. There are two families of ovoids known, namely the classical ovoids (being elliptic quadrics of the ambient projective space $\mathrm{PG}(3,q)$) and the Ree-Tits ovoids (which exist only when $q > 2$ is a nonsquare).

In the q odd case, nothing seemed to be known for the minimum size $\mathcal{T}$-blocking sets. If k is the minimum size of such a blocking set, then the following bounds hold by Lemmas 2.1 and 2.2 in the next section:

$$q^2+1 \leq k \leq q^2+q.$$

Calling two $\mathcal{T}$-blocking sets X_1 and X_2 *isomorphic* if there is an automorphism of $\mathrm{PG}(3,q)$ stabilizing $Q^+(3,q)$ and mapping X_1 to X_2, we prove the following (without the aid of a computer) for the case $q=3$.

Theorem 1.1. *Suppose that $q=3$. Then there is no $\mathcal{T}$-blocking set of size 10 in $\mathrm{PG}(3,3)$. Up to isomorphism, there are two $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$.*

In Lemma 2.1 of the next section, we show that a $\mathcal{T}$-blocking set of size q^2+1 is an ovoid of the subgeometry of $\mathrm{PG}(3,q)$ defined by the tangent lines. In Section 4 of [3], computer code in Sage [9] can be found for classifying ovoids of point-line geometries. With the aid of this code and some computations in GAP [13], we were able to show the nonexistence of $\mathcal{T}$-blocking sets of size q^2+1 for certain small values of q, see [5].

Theorem 1.2. *There exist no $\mathcal{T}$-blocking sets of size q^2+1 in $\mathrm{PG}(3,q)$ for each odd prime power $q \leq 13$.*

In Section 2, we prove a few basic results. In Section 3, we construct two nonisomorphic $\mathcal{T}$-blocking sets in $\mathrm{PG}(3,3)$ each of size 11. Finally, in Section 4, we prove the nonexistence of $\mathcal{T}$-blocking sets of size 10 and classify the $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$.

[1] An *ovoid* of a point-line geometry is a set of points meeting each line in a singleton.

Acknowledgement: The first author would like to thank the National Institute of Science Education and Research, Bhubaneswar for the kind hospitality provided during his visit to the School of Mathematical Sciences in March-April 2017.

2 Preliminaries

As in Section 1, consider a hyperbolic quadric $Q^+(3,q)$ in $\mathrm{PG}(3,q)$. A lower bound for the sizes of $\mathcal{T}$-blocking sets is easily derived from the fact that there are $(q+1)(q^2+1)$ tangent lines in total and $q+1$ tangent lines through a given point.

Lemma 2.1. *Let X be a $\mathcal{T}$-blocking set in $\mathrm{PG}(3,q)$. Then $|X| \geq q^2+1$, with equality if and only if every tangent line contains a unique point of X.*

Proof. Each of the $(q+1)(q^2+1)$ tangent lines contains at least one point of X. As every point of $\mathrm{PG}(3,q)$ is contained in precisely $q+1$ tangent lines, we have $|X| \geq \frac{(q+1)(q^2+1)\cdot 1}{q+1} = q^2+1$. Equality holds if and only if every tangent line contains a unique point of X. □

With the quadric $Q^+(3,q)$, there is naturally associated a polarity ζ which is symplectic if q is even and orthogonal if q is odd. For every point x of $Q^+(3,q)$, x^ζ is a plane which is *tangent to* $Q^+(3,q)$ *at the point* x and intersects $Q^+(3,q)$ in the union of two lines through x. The $q+1$ tangent lines through x are precisely the lines through x contained in x^ζ. By the following lemma, the size of a $\mathcal{T}$-blocking set is bounded above by q^2+q.

Lemma 2.2. *Let π be a plane of $\mathrm{PG}(3,q)$ which is tangent to $Q^+(3,q)$ at the point x. Then $\pi \setminus \{x\}$ is a $\mathcal{T}$-blocking set of size q^2+q.*

Proof. We have $|\pi \setminus \{x\}| = q^2+q$. As every line meets π, every tangent line not containing x meets $\pi \setminus \{x\}$. If L is a tangent line containing x, then L is contained in $x^\zeta = \pi$ and hence contains points of $\pi \setminus \{x\}$. So, $\pi \setminus \{x\}$ is a $\mathcal{T}$-blocking set. □

Suppose q is odd. For every point x of $\mathrm{PG}(3,q) \setminus Q^+(3,q)$, x^ζ is a nontangent plane with $x \notin x^\zeta$ and the set $O_x := x^\zeta \cap Q^+(3,q)$ is a conic of x^ζ. The $q+1$ tangent lines through x are precisely the lines through x meeting O_x. The conic O_x is an *ovoid* of $Q^+(3,q)$, that is, a set of points intersecting each $\mathcal{T}_0$-line in a unique point. The map $x \mapsto O_x$ defines a bijection between $\mathrm{PG}(3,q) \setminus Q^+(3,q)$ and the set of conics contained in $Q^+(3,q)$. When $q=3$, we note that the set of conics contained in $Q^+(3,3)$ coincides with the set of ovoids of $Q^+(3,3)$. If $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$, then the number of secant lines through x is equal to $\frac{|Q^+(3,q)\setminus O_x|}{2} = \frac{(q+1)q}{2}$ and the number of external lines through x is equal to $(q^2+q+1)-(q+1)-\frac{(q+1)q}{2} = \frac{(q-1)q}{2}$.

Since q is odd, every point of $x^\zeta \setminus O_x$ lies on 0 or 2 $\mathcal{T}_1$-lines contained in x^ζ. Such a point is called *interior* to O_x in the first case and *exterior* to O_x in the latter. There are $q(q-1)/2$ interior points and $q(q+1)/2$ exterior points in x^ζ with respect to O_x. Every interior point lies on $(q+1)/2$ external lines and $(q+1)/2$ secant lines contained in x^ζ. Every exterior point lies on $(q-1)/2$ external lines and $(q-1)/2$ secant lines contained in x^ζ. Every external line contained in x^ζ contains $(q+1)/2$ interior points and $(q+1)/2$ exterior points. Every secant line contained in x^ζ contains $(q-1)/2$ interior points and $(q-1)/2$ exterior points. One can refer to [7] for these basic properties.

Lemma 2.3. *Suppose $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$ with q odd. Then each line of $\mathrm{PG}(3,q)$ through x, which is external to $Q^+(3,q)$, meets x^ζ in a point interior to O_x.*

Proof. Let L be an external line through x. Since $x \notin x^\zeta$, L contains exactly one point of x^ζ. Denote this point by z. We show that z is interior to O_x.

Suppose this is not true. Then z is exterior to O_x. Let M be a $\mathcal{T}_1$-line through z in x^ζ and π be the plane generated by L and M. Then π is a nontangent plane, as it contains the external line L. On the other hand, if y is the unique point of the intersection $M \cap O_x$, then the $\mathcal{T}_1$-line $M_1 := xy$ is contained in π. So π is also the plane generated by the tangent lines M and M_1. It follows that π is the plane which is tangent to $Q^+(3,q)$ at the point y, a contradiction. □

Again under the assumption that $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$ with q odd, we denote by $\mathcal{E}_x$ the set of lines in $\mathrm{PG}(3,q)$ through x that are external to $Q^+(3,q)$, and by I_x the set of interior points in x^ζ with respect to the conic O_x. We have $|\mathcal{E}_x| = q(q-1)/2 = |I_x|$. As a consequence of Lemma 2.3, we have the following.

Corollary 2.4. *Suppose $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$ with q odd. Then the map from $\mathcal{E}_x$ to I_x, sending each line in $\mathcal{E}_x$ to its point of intersection with I_x, is bijective.*

Proof. By Lemma 2.3, the map is well-defined and is injective. Since $|\mathcal{E}_x| = |I_x|$, the map is surjective also. □

In the special case that $q = 3$, the following can be said.

Lemma 2.5. *Suppose $q = 3$. Let π_1 be a nontangent plane and O_1 be the conic $\pi_1 \cap Q^+(3,3)$ in π_1. Fix a line L in π_1 which is external to O_1. Then there exists exactly one more nontangent plane π_2 satisfying the following:*

1. *L is an external line in π_2 with respect to the conic $O_2 := \pi_2 \cap Q^+(3,3)$.*
2. *If $a \in L$ is exterior (respectively, interior) to O_1 in π_1, then it is also exterior (respectively, interior) to O_2 in π_2.*

In fact, if $a \in L$ is exterior to O_1 in π_1, then the two $\mathcal{T}_1$-lines through a not in π_1 are contained in π_2.

Proof. Let x be the point in $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ such that $O_x = O_1$. Such a point x exists, since the map $\alpha \mapsto O_\alpha := \alpha^\zeta \cap Q^+(3,3)$ is a bijection between $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ and the set of conics contained in $Q^+(3,3)$. We have $\pi_1 = x^\zeta$. Write $L = \{a, b, z_1, z_2\}$, where a, b (respectively, z_1, z_2) are exterior (respectively, interior) to O_1 in π_1. By Corollary 2.4, the lines $T_1 := xz_1$ and $T_2 := xz_2$ are external lines.

Let π_2 be the plane generated by the line L and the point x. Then π_2 is a nontangent plane in which L is external to the conic $O_2 := \pi_2 \cap Q^+(3,3)$. The lines T_1 and T_2 in π_2

are external to O_2. Thus, for $i \in \{1, 2\}$, L and T_i are two external lines in π_2 through z_i. It follows that both z_1 and z_2 are interior to O_2 in π_2. This implies that both a and b must be exterior to O_2 in π_2. Hence π_2 satisfies the conditions (1) and (2).

Out of the four $\mathcal{T}_1$-lines through a (respectively, through b), two are contained in π_1 and the other two are in π_2 (as $\pi_1 \cap \pi_2 = L$ is not a $\mathcal{T}_1$-line). This must hold for any nontangent plane satisfying the conditions (1) and (2). This fact implies the uniqueness of π_2 satisfying (1) and (2). □

3 Two constructions of $\mathcal{T}$-blocking sets

In this section, we construct two nonisomorphic $\mathcal{T}$-blocking sets of size 11 each in PG(3, 3).

3.1 First construction

Consider a point $x \in \mathrm{PG}(3, 3) \setminus Q^+(3, 3)$ and let $I_x = \{z_1, z_2, z_3\}$. Fix a line L in the plane x^ζ which is external to O_x. Then L contains exactly two points of I_x, say z_2 and z_3. Let $\overline{L}$ be the unique line in $\mathcal{E}_x$ such that $\overline{L}$ meets x^ζ in z_1, see Corollary 2.4. Define the following set:

$$B_1 := O_x \cup L \cup \left(\overline{L} \setminus \{x\}\right).$$

We prove the following.

Proposition 3.1. *B_1 is a $\mathcal{T}$-blocking set of size* 11 *in* PG(3, 3).

Proof. Clearly, $|B_1| = 11$. Let $A = x^\zeta \setminus B_1$. Then A consists of four exterior points, each of which is different from the two exterior points contained in L. Since every tangent line meets x^ζ, it is enough to prove that each $\mathcal{T}_1$-line through a point of A meets B_1.

Take a point $a \in A$ and a $\mathcal{T}_1$-line T through a. If T is contained in x^ζ, then observe that T meets B_1 in two points, one from O_x and the other one is an exterior point contained in L. So assume that T is not contained in x^ζ. We show that T contains a point of $B_1 \setminus x^\zeta = \overline{L} \setminus \{x, z_1\}$.

Let M be the line in x^ζ through a and z_1. Then M is either external or secant to O_x in x^ζ, as it contains the interior point z_1. Since M has to intersect the external line L in x^ζ in a point different from a and z_1, it follows that M can not be secant to O_x. So M is external to O_x in x^ζ and hence contains an interior point different from z_1. Without loss, we may assume that M contains z_2 as the second interior point.

Setting $\pi_1 = x^\zeta$ and taking the external line M of π_1 in Lemma 2.5, we get a nontangent plane π_2 containing M such that z_1, z_2 are interior points and a is an exterior point in π_2 with respect to the conic $O_2 := \pi_2 \cap Q^+(3, 3)$. Note that T is a $\mathcal{T}_1$-line through a in π_2.

Let $\overline{M}$ ($\neq M$) be the second line in π_2 through z_1 which is external to O_2. Out of the three lines through z_1 external to $Q^+(3, 3)$, the line M is common to both the planes $\pi_1 = x^\zeta$ and π_2. The plane x^ζ contains one more external line through z_1. So $\overline{M}$ must be the external line through x which corresponds to the point z_1 under the map defined in Corollary 2.4. It follows that $\overline{M} = \overline{L}$. As xz_1 and xz_2 are external lines in π_2 (by

Corollary 2.4), x must be interior to O_2 in π_2. Since the $\mathcal{T}_1$-line T and the external line $\overline{L}$ in π_2 meet in a point exterior to O_2, it follows that T contains a point of $\overline{L} \setminus \{x, z_1\}$. This completes the proof. □

3.2 Second construction

Fix a point $x \in \mathrm{PG}(3,3) \setminus Q^+(3,3)$ and let $I_x = \{z_1, z_2, z_3\}$. Let y be a point in x^ζ exterior to O_x. Let L_1 and L_2 be the two $\mathcal{T}_1$-lines through y which are not contained in x^ζ. For $i \in \{1, 2\}$, let w_i be the tangency point of L_i in $Q^+(3,3)$. Define the following set:

$$B_2 := O_x \cup I_x \cup \Big(L_1 \setminus \{y, w_1\}\Big) \cup \Big(L_2 \setminus \{y, w_2\}\Big).$$

We prove the following:

Proposition 3.2. *B_2 is a $\mathcal{T}$-blocking set of size 11 in $\mathrm{PG}(3,3)$.*

Proof. Clearly, $|B_2| = 11$. Let $D = x^\zeta \setminus B_2$. Then D consists of the six exterior points in x^ζ with respect to O_x. Since every tangent line meets x^ζ, it is enough to prove that each $\mathcal{T}_1$-line through a point of D meets B_2.

Take a point $a \in D$ and a $\mathcal{T}_1$-line T through a. If T is contained in x^ζ, then T meets B_2 in the unique point of $T \cap O_x$. So assume that T is not contained in x^ζ. If $a = y$, then T is either L_1 or L_2 and hence meets B_2 at two points. Assume that $a \neq y$. Since both a and y are exterior to O_x, the line $M := ay$ in x^ζ is either tangent or external to O_x.

Case I: M is tangent to O_x. Let π be the nontangent plane generated by the lines T and M. Denote by O_π the conic $\pi \cap Q^+(3,3)$ in π. The point y in π is exterior to O_π. So there exists one more $\mathcal{T}_1$-line in π (different from M) containing y. Since $\pi \cap x^\zeta = M$, it follows that either L_1 or L_2 is a line in π. Without loss, we may assume that L_1 is a line in π. The lines T and L_1 intersect in π in a point different from y and w_1. So T meets B_2 at a point of $L_1 \setminus \{y, w_1\}$.

Case II: M is external to O_x. Setting $\pi_1 = x^\zeta$ and taking the external line M of π_1 in Lemma 2.5, we get a nontangent plane π_2 through M containing the lines T, L_1 and L_2. Now it can be seen that T intersects L_1 (respectively, L_2) in π_2 at a point different from y and w_1 (respectively, w_2). So T meets B_2 at two points, one from $L_1 \setminus \{y, w_1\}$ and one from $L_2 \setminus \{y, w_2\}$.

Thus B_2 is a $\mathcal{T}$-blocking set of $\mathrm{PG}(3,3)$ of size 11. This completes the proof. □

3.3 The blocking sets B_1 and B_2 are nonisomorphic

Proposition 3.3. *The two blocking sets B_1 and B_2 are nonisomorphic.*

Proof. Write B_2 as a disjoint union $B_2 = (B_2 \cap x^\zeta) \cup (B_2 \setminus x^\zeta)$. Observe that any line meets $B_2 \setminus x^\zeta$ in at most two points. Let R be a line external to $Q^+(3,3)$. If R is a line in x^ζ, then R meets B_2 at exactly two points of $B_2 \cap x^\zeta$ (which come from I_x) and is disjoint from $B_2 \setminus x^\zeta$. Suppose that R is not a line in x^ζ. Then R contains at most one point from

$B_2 \cap x^\zeta$ and at most two points from $B_2 \setminus x^\zeta$. So R is not contained in B_2. Thus every external line meets B_2 in at most three points.

However, from the construction of B_1, it is clear that B_1 contains a line external to $Q^+(3,3)$. So B_1 and B_2 are nonisomorphic. □

4 $\mathcal{T}$-blocking sets of sizes 10 and 11 in $\mathrm{PG}(3,3)$

Consider a hyperbolic quadric $Q^+(3,3)$ in $\mathrm{PG}(3,3)$. We label the points of $Q^+(3,3)$ by x_{ij} where $i,j \in \{1,2,3,4\}$ such that two distinct points x_{ij} and $x_{i'j'}$ of $Q^+(3,3)$ are incident with a $\mathcal{T}_0$-line if either $i = i'$ or $j = j'$.

We denote by O^* the ovoid $\{x_{11}, x_{22}, x_{33}, x_{44}\}$ of $Q^+(3,3)$. There are nine ovoids of $Q^+(3,3)$ that are disjoint from O^*. These are:

$$O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\},\ O_2 = \{x_{13}, x_{31}, x_{24}, x_{42}\},\ O_3 = \{x_{14}, x_{41}, x_{23}, x_{32}\},$$
$$O_4 = \{x_{12}, x_{24}, x_{43}, x_{31}\},\ O_5 = \{x_{12}, x_{23}, x_{34}, x_{41}\},\ O_6 = \{x_{13}, x_{24}, x_{32}, x_{41}\},$$
$$O_7 = \{x_{13}, x_{21}, x_{34}, x_{42}\},\ O_8 = \{x_{14}, x_{21}, x_{32}, x_{43}\},\ O_9 = \{x_{14}, x_{23}, x_{31}, x_{42}\}.$$

Lemma 4.1. *There are four collections, each of six ovoids from $\{O_1, O_2, \dots, O_9\}$, such that every point of $Q^+(3,3) \setminus O^*$ is contained in precisely two ovoids of a given collection. These four collections are $\mathcal{C}^* = \{O_4, O_5, O_6, O_7, O_8, O_9\}$, $\{O_1, O_2, O_5, O_6, O_8, O_9\}$, $\{O_1, O_3, O_4, O_6, O_7, O_9\}$ and $\{O_2, O_3, O_4, O_5, O_7, O_8\}$.*

Proof. It is easily verified that each of these four collections satisfies the required condition. Conversely, suppose that $\mathcal{C} \neq \mathcal{C}^*$ is a collection of six ovoids satisfying the condition of the lemma. As $\mathcal{C} \neq \mathcal{C}^*$, at least one of O_1, O_2, O_3 is contained in $\mathcal{C}$. Now, any partition of $Q^+(3,3) \setminus O^*$ in three ovoids must contain either one or three ovoids of the set $\{O_1, O_2, O_3\}$, implying that at least one of O_1, O_2, O_3 is not contained in $\mathcal{C}$.

Suppose $O_1 \in \mathcal{C}$ and $O_2 \notin \mathcal{C}$. As each of x_{13}, x_{31} should be contained in two ovoids of $\mathcal{C}$, we then must have $O_4, O_6, O_7, O_9 \in \mathcal{C}$. At this stage, x_{12} and x_{21} are already contained in two ovoids of the collection $\mathcal{C}$, implying that O_5 and O_8 do not belong to $\mathcal{C}$. So, $\mathcal{C}$ is necessarily equal to $\{O_1, O_3, O_4, O_6, O_7, O_9\}$.

By symmetry we then see that $\mathcal{C}$ always contains precisely two ovoids of the set $\{O_1, O_2, O_3\}$. If $O_1, O_2 \in \mathcal{C}$ and $O_3 \notin \mathcal{C}$, then a similar reasoning as above shows that $\mathcal{C} = \{O_1, O_2, O_5, O_6, O_8, O_9\}$. Similarly, if $O_2, O_3 \in \mathcal{C}$ and $O_1 \notin \mathcal{C}$, then $\mathcal{C} = \{O_2, O_3, O_4, O_5, O_7, O_8\}$. □

Invoking Lemma 4.1, the verification of the following lemma is straightforward.

Lemma 4.2. *Suppose $\mathcal{C}$ is a collection of six ovoids from $\{O_1, O_2, \dots, O_9\}$ such that every point of $Q^+(3,3) \setminus O^*$ is contained in precisely two ovoids of $\mathcal{C}$. Let S denote the set of all points $x \in Q^+(3,3) \setminus O^*$ such that $\{x\}$ is the intersection of two distinct ovoids of $\mathcal{C}$. Then $S = Q^+(3,3) \setminus O^*$ if $\mathcal{C} = \mathcal{C}^*$, and $S = O$ if $\mathcal{C} \neq \mathcal{C}^*$, where O is the unique element of $\{O_1, O_2, O_3\}$ not contained in $\mathcal{C}$.*

Lemma 4.3. *Let x be a point of $Q^+(3,3)$ and let $L_1 = \{x, y_1, y_2, y_3\}$ and $L_2 = \{x, z_1, z_2, z_3\}$ be the two $\mathcal{T}_1$-lines through x. Then the following hold:*

(1) *$\{O_{y_1}, O_{y_2}, O_{y_3}\}$ (resp. $\{O_{z_1}, O_{z_2}, O_{z_3}\}$) is a set of ovoids of $Q^+(3,3)$ through x partitioning the set of points of $Q^+(3,3)$ noncollinear with x.*

(2) *If $i, j \in \{1,2,3\}$, then $O_{y_i} \cap O_{z_j}$ contains precisely two points (one of which is x).*

Proof. (1) As L_1 is a $\mathcal{T}_1$-line, we see that $x \in O_{y_i}$ for every $i \in \{1,2,3\}$. Now, take an arbitrary point $u \in Q^+(3,3)$ noncollinear with x. Then u^ζ does not contain x and so intersects L_1 in a unique point y_i. The point y_i is the unique point v of $L_1 \setminus \{x\}$ for which $u \in v^\zeta$. So, $\{O_{y_1}, O_{y_2}, O_{y_3}\}$ partitions the set of points of $Q^+(3,3)$ noncollinear with x. A similar argument holds for the line L_2.

(2) There are six ovoids through the point x. One coincides with O_{y_i}, two (O_{y_r}, O_{y_s}) intersect O_{y_i} in $\{x\}$ where $\{i, r, s\} = \{1,2,3\}$, and the remaining three (necessarily O_{z_1}, O_{z_2}, O_{z_3}) intersect O_{y_i} in two points (one of which is x). □

4.1 Nonexistence of $\mathcal{T}$-blocking sets of size 10

The following result proves the nonexistence of $\mathcal{T}$-blocking sets of size 10 in $\mathrm{PG}(3,3)$.

Proposition 4.4. *There are no $\mathcal{T}$-blocking sets of size* 10 *in* $\mathrm{PG}(3,3)$.

Proof. Suppose X is a $\mathcal{T}$-blocking sets of size 10 in $\mathrm{PG}(3,3)$. By Lemma 2.1, we then know that each tangent line contains a unique point of X. In particular, $O := X \cap Q^+(3,3)$ is an ovoid of $Q^+(3,3)$ and $Y := X \setminus Q^+(3,3)$ is a set of 6 points outside $Q^+(3,3)$ intersecting each $\mathcal{T}_1$-line in a unique point. Without loss of generality, we may suppose that $O = O^*$. We show the following properties for the collection $\mathcal{C} = \{O_y \,|\, y \in Y\}$:

(a) all ovoids of $\mathcal{C}$ are disjoint from O;

(b) any two ovoids of $\mathcal{C}$ cannot intersect in a singleton;

(c) every point of $Q^+(3,3) \setminus O$ is contained in precisely two ovoids of $\mathcal{C}$.

If O_y with $y \in Y$ contains a point $x \in O$, then the tangent line xy would contain two points of $X = O \cup Y$, namely x and y, a contradiction. If $O_{y_1} \cap O_{y_2}$ is a singleton $\{x\}$, where $y_1, y_2 \in Y$ with $y_1 \neq y_2$, then Lemma 4.3 would imply that there is a $\mathcal{T}_1$-line through x containing y_1 and y_2, a contradiction. Finally, every point $x \in Q^+(3,3) \setminus O$ is contained in two $\mathcal{T}_1$-lines, each containing exactly one point of Y, showing that x is contained in precisely two ovoids of $\mathcal{C}$.

By Lemmas 4.1 and 4.2, we however know that there are no collections $\mathcal{C}$ of six ovoids that satisfy the above properties (a), (b) and (c). □

4.2 Classification of the $\mathcal{T}$-blocking sets of size 11

In the rest of the paper, we classify the $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$. We show that there are only two such $\mathcal{T}$-blocking sets up to isomorphism, necessarily isomorphic to the blocking sets B_1 and B_2 constructed in Section 3.

Lemma 4.5. *If X is a $\mathcal{T}$-blocking set of size 11 in $\mathrm{PG}(3,3)$, then $|X \setminus Q^+(3,3)| \in \{6,7\}$ and $|X \cap Q^+(3,3)| \in \{4,5\}$.*

Proof. Since $|X \cap Q^+(3,3)| \leq |X| < 12$, there exists a line L in $Q^+(3,3)$ meeting X in either 1 or 2 points. Suppose every line of $Q^+(3,3)$ meets X in 2 points. Then $|X \cap Q^+(3,3)| = 8$. If L is a line of $Q^+(3,3)$ and $L \setminus X = \{a,b\}$, then each of the four $\mathcal{T}_1$-lines meeting $\{a,b\}$ contains at least one point of $X \setminus Q^+(3,3)$. Any collection of four points of $X \setminus Q^+(3,3)$ that arise in this way are mutually distinct, implying that $|X| = |X \cap Q^+(3,3)| + |X \setminus Q^+(3,3)| \geq 8+4 = 12$, which is a contradiction.

Hence, there exists a line L in $Q^+(3,3)$ meeting X in a unique point. If $L \setminus X = \{a,b,c\}$, then there are six $\mathcal{T}_1$-lines meeting $\{a,b,c\}$ and each of these six $\mathcal{T}_1$-lines contains at least one point of $X \setminus Q^+(3,3)$. Any collection of six points of $X \setminus Q^+(3,3)$ that arise in this way are mutually distinct, implying that $|X \setminus Q^+(3,3)| \geq 6$. As $|X \cap Q^+(3,3)| \geq 4$, we thus have that $|X \setminus Q^+(3,3)| \in \{6,7\}$ and $|X \cap Q^+(3,3)| \in \{4,5\}$. $\square$

Proposition 4.6. *If X is a $\mathcal{T}$-blocking set of size 11 in $\mathrm{PG}(3,3)$, then $|X \cap Q^+(3,3)| = 4$ and $|X \setminus Q^+(3,3)| = 7$.*

Proof. Suppose that this is not the case. Then $|X \cap Q^+(3,3)| = 5$ and $|X \setminus Q^+(3,3)| = 6$ by Lemma 4.5. As each $\mathcal{T}_0$-line contains a point of X, there are precisely two $\mathcal{T}_0$-lines L_1 and L_2 that contain exactly two points of X (while every other $\mathcal{T}_0$-line intersects X in a singleton). The lines L_1 and L_2 belong to distinct parallel classes of lines of $Q^+(3,3)$. We distinguish two cases.

Case I. The singleton $L_1 \cap L_2$ is not contained in X. Without loss of generality, we may suppose that $X \cap Q^+(3,3) = \{x_{12}, x_{13}, x_{21}, x_{31}, x_{44}\}$. The reasoning in Lemma 4.5 leading to the inequality $|X \setminus Q^+(3,3)| \geq 6$ shows that if L is a $\mathcal{T}_0$-line meeting X in a singleton, then any $\mathcal{T}_1$-line meeting $L \setminus X$ cannot contain more than one point of X, and any $\mathcal{T}_1$-line meeting $L \cap X$ cannot contain a point of $X \setminus Q^+(3,3)$. As any point of $Q^+(3,3) \setminus \{x_{11}\}$ is contained in a $\mathcal{T}_0$-line intersecting X in a singleton, we thus see from Lemma 4.3 that any two ovoids O_{y_1} and O_{y_2}, where $y_1, y_2 \in X \setminus Q^+(3,3)$, cannot intersect in a singleton distinct from $\{x_{11}\}$. Also, no ovoid O_y with $y \in X \setminus Q^+(3,3)$ can contain a point of $X \cap Q^+(3,3)$. It can be seen that there are exactly six ovoids of $Q^+(3,3)$ disjoint from $X \cap Q^+(3,3)$ and so these ovoids are precisely the six ovoids O_y, where $y \in X \setminus Q^+(3,3)$. But that is impossible as two of these ovoids, namely $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ and $\{x_{14}, x_{23}, x_{32}, x_{41}\}$, intersect in the singleton $\{x_{23}\} \neq \{x_{11}\}$.

Case II. The singleton $L_1 \cap L_2$ is contained in X. Without loss of generality, we may suppose that $X \cap Q^+(3,3) = O^* \cup \{x_{12}\}$. The reasoning in Lemma 4.5 leading to the inequality $|X \setminus Q^+(3,3)| \geq 6$ shows that if L is a $\mathcal{T}_0$-line meeting $Q^+(3,3) \cap X$ in a singleton, then each of the $\mathcal{T}_1$-lines meeting $L \setminus X$ cannot contain more than one point of X. As any point of $Q^+(3,3) \setminus \{x_{12}\}$ is contained in a line of $Q^+(3,3)$ intersecting X in a singleton, we thus see from Lemma 4.3 that any two ovoids O_{y_1} and O_{y_2}, where $y_1, y_2 \in X \setminus Q^+(3,3)$, cannot intersect in a singleton distinct from $\{x_{12}\}$.

Put $\mathcal{C} = \{O_y \,|\, y \in X \setminus Q^+(3,3)\}$. Then $\mathcal{C}$ is a set of six ovoids of $Q^+(3,3)$, no two of which intersect in a singleton distinct from $\{x_{12}\}$. Moreover, each point $x \in Q^+(3,3) \setminus X$ is contained in precisely two $\mathcal{T}_1$-lines and hence in precisely two ovoids of $\mathcal{C}$.

We count the number of pairs (L, x), where L is a $\mathcal{T}_1$-line disjoint from $X \cap Q^+(3,3)$ and $x \in L \cap X$. There are $|Q^+(3,3) \setminus X| \cdot 2 = 22$ possibilities for L, and each such L contains a unique point of X, implying that there are 22 such pairs. On the other hand, there are 6 possibilities for $x \in X \setminus Q^+(3,3)$.

Since $6 \cdot 3 = 18$, there are at least $22 - 18 = 4$ points of $X \setminus Q^+(3,3)$ whose induced ovoids are disjoint from $Q^+(3,3) \cap X$. There are six ovoids of $Q^+(3,3)$ that are disjoint from $X \cap Q^+(3,3)$:

$$A_1 = \{x_{13}, x_{24}, x_{31}, x_{42}\}, \qquad A_2 = \{x_{14}, x_{23}, x_{32}, x_{41}\}, \qquad A_3 = \{x_{13}, x_{21}, x_{34}, x_{42}\},$$

$$A_4 = \{x_{13}, x_{24}, x_{32}, x_{41}\}, \qquad A_5 = \{x_{14}, x_{23}, x_{31}, x_{42}\}, \qquad A_6 = \{x_{14}, x_{21}, x_{32}, x_{43}\}.$$

Among the six ovoids that we have to choose for the set $\mathcal{C}$, at least four come from the collection $\{A_1, A_2, \ldots, A_6\}$. As exactly two of the six ovoids of $\mathcal{C}$ contain x_{13}, at most two of A_1, A_3, A_4 can occur in $\mathcal{C}$. Similarly, by considering the point x_{14}, we see that at most two of A_2, A_5, A_6 can occur in $\mathcal{C}$. We can conclude that precisely two of A_1, A_3, A_4, as well as precisely two of A_2, A_5, A_6 belong to $\mathcal{C}$. As $A_3 \cap A_4$ and $A_5 \cap A_6$ are singletons distinct from $\{x_{12}\}$, the ovoids A_1 and A_2 must belong to $\mathcal{C}$. Then the fact that $A_3 \cap A_5$, $A_3 \cap A_6$ and $A_4 \cap A_6$ are singletons distinct from $\{x_{12}\}$ implies that A_3 and A_6 cannot belong to $\mathcal{C}$. So, $\mathcal{C}$ certainly contains the ovoids A_1, A_2, A_4 and A_5.

We still need to find two additional ovoids for $\mathcal{C}$. As the points x_{21}, x_{34} and x_{43} are not contained in $A_1 \cup A_2 \cup A_4 \cup A_5$ and need to be covered twice, each of these two ovoids should contain these points. But that is impossible as there is only one ovoid containing these three points, namely $\{x_{12}, x_{21}, x_{34}, x_{43}\}$. □

In the sequel, we suppose that X is a set of 11 points of PG(3, 3) that is a $\mathcal{T}$-blocking set. Then $|X \cap Q^+(3,3)| = 4$ and $|X \setminus Q^+(3,3)| = 7$ by Proposition 4.6. In fact, $U_1 := X \cap Q^+(3,3)$ is an ovoid of $Q^+(3,3)$. Denote by U_2 the subset of $Q^+(3,3)$ consisting of the following points:

- points of $X \cap Q^+(3,3)$ contained in a $\mathcal{T}_1$-line that contains points of $X \setminus Q^+(3,3)$,
- points of $Q^+(3,3) \setminus X$ contained in a $\mathcal{T}_1$-line that contains at least two points of $X \setminus Q^+(3,3)$.

Lemma 4.7. *The set U_2 is an ovoid of $Q^+(3,3)$.*

Proof. Let L be a line of $Q^+(3,3)$ and put $\{x_L\} := L \cap U_1$. For every $y \in X \setminus Q^+(3,3)$ denote by y' the unique point of $L \cap O_y$, that is, the unique point y' of L for which yy' is a $\mathcal{T}_1$-line. Each $\mathcal{T}_1$-line meeting $L \setminus \{x_L\}$ contains at least one point of $X \setminus Q^+(3,3)$, and so each point of $L \setminus \{x_L\}$ is the image of at least two points of $X \setminus Q^+(3,3)$ under the map $y \mapsto y'$. So, precisely one of the following two cases occurs:

(a) The point x_L is the image of precisely one point of $X \setminus Q^+(3,3)$ and each of the three points of $L \setminus \{x_L\}$ is the image of precisely two points of $X \setminus Q^+(3,3)$.

(b) There exists a unique point x'_L on $L \setminus \{x_L\}$ which is the image of precisely three points of $X \setminus Q^+(3,3)$, each of the two remaining points of $L \setminus \{x_L\}$ is the image of precisely two points of $X \setminus Q^+(3,3)$. In this case, the point x_L itself is not the image of any point of $X \setminus Q^+(3,3)$.

In case (a), we see that x_L is the unique point of U_2 on L. In case (b), we see that x'_L is the unique point of U_2 on L. Since $L \cap U_2$ is always a singleton, we conclude that U_2 must be an ovoid of $Q^+(3,3)$. □

Now, if $\mathcal{C}$ is the collection of the seven ovoids O_y, where $y \in X \setminus Q^+(3,3)$, then the following properties hold:

(P1) No point of $U_1 \setminus U_2$ is contained in an ovoid of $\mathcal{C}$.

(P2) Every point of $U_1 \cap U_2$ is contained in precisely one ovoid of $\mathcal{C}$.

(P3) Every point of $Q^+(3,3) \setminus (U_1 \cup U_2)$ is contained in precisely two ovoids of $\mathcal{C}$.

(P4) Every point of $U_2 \setminus U_1$ is contained in precisely three ovoids of $\mathcal{C}$.

(P5) No two ovoids of $\mathcal{C}$ intersect in a singleton $\{x\}$, where $x \in Q^+(3,3) \setminus (U_1 \cup U_2)$.

(P6) No three ovoids of $\mathcal{C}$ can mutually intersect in the same singleton $\{x\}$, where $x \in U_2 \setminus U_1$.

Proposition 4.8. *Suppose that U_1 and U_2 are two (not necessarily distinct) ovoids of $Q^+(3,3)$. Let Y be a set of seven points of* $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ *and put $\mathcal{C} := \{O_y \,|\, y \in Y\}$. If $\mathcal{C}$ satisfies the properties (P1) – (P6) above, then $U_1 \cup Y$ is a $\mathcal{T}$-blocking set of size* 11.

Proof. We have $|U_1 \cup Y| = 11$. Since U_1 is an ovoid of $Q^+(3,3)$, every $\mathcal{T}_0$-line meets U_1 at a unique point. Every $\mathcal{T}_1$-line through a point of U_1 obviously meets U_1. By (P4) and (P6), every $\mathcal{T}_1$-line through a point of $U_2 \setminus U_1$ contains a point of Y. By (P3) and (P5), every $\mathcal{T}_1$-line through a point of $Q^+(3,3) \setminus (U_1 \cup U_2)$ contains a point of Y. □

We now use the above result to classify the $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$. We assume that U_1 and U_2 are two ovoids of $Q^+(3,3)$ and that $\mathcal{C}$ is a collection of seven ovoids of $Q^+(3,3)$ satisfying the properties (P1) – (P6) above. If Y is the set of seven points of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which the collection $\{O_y \,|\, y \in Y\}$ coincides with $\mathcal{C}$, then $X = U_1 \cup Y$ is a $\mathcal{T}$-blocking set of size 11 by Proposition 4.8. Without loss of generality, we may suppose that $U_1 = O^* = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. Then the nine ovoids disjoint from $U_1 = \{x_{11}, x_{22}, x_{33}, x_{44}\}$ are $O_1, O_2, \ldots, O_9$ as defined in the beginning of this section.

The ovoid U_2 can have five positions with respect to U_1 (up to isomorphism):

I: $U_2 = \{x_{11}, x_{22}, x_{33}, x_{44}\} = U_1$,

II: $U_2 = \{x_{11}, x_{22}, x_{34}, x_{43}\}$,

III: $U_2 = \{x_{11}, x_{23}, x_{34}, x_{42}\}$,

IV: $U_2 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$,

V: $U_2 = \{x_{12}, x_{23}, x_{34}, x_{41}\}$.

Treatment of Case I

In this case, (P2) implies that the points of $U_1 \cap U_2 = U_1 = U_2$ are partitioned by certain ovoids of $\mathcal{C}$. The partition has shape 4, $2+2$, $2+1+1$ or $1+1+1+1$, leading to four subcases.

(Ia) Suppose the mentioned partition has shape 4. Then $U_1 = U_2 \in \mathcal{C}$. Again (P2) implies that every ovoid of $\mathcal{C} \setminus \{U_1\}$ is disjoint from $U_1 = U_2$. By (P3), $\mathcal{C} \setminus \{U_1\}$ is a collection of six ovoids as in Lemma 4.1. A contradiction is then readily obtained from Lemma 4.2 and property (P5).

(Ib) Suppose the mentioned partition has shape $2+2$. Without loss of generality, we may suppose that $\{x_{11}, x_{22}, x_{34}, x_{43}\}$ and $\{x_{33}, x_{44}, x_{12}, x_{21}\}$ belong to $\mathcal{C}$. By (P2), each of the remaining five ovoids of $\mathcal{C}$ is disjoint from $U_1 = U_2$. So we need to find five additional ovoids from the collection $\{O_1, O_2, \ldots, O_9\}$. By (P3) and (P5), the second ovoid of $\mathcal{C}$ through x_{12} must contain x_{21} and therefore be equal to $O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. As x_{12}, x_{21}, x_{34} and x_{43} have already been covered twice, the remaining four ovoids should be contained in $\{x_{13}, x_{14}, x_{23}, x_{24}, x_{31}, x_{32}, x_{41}, x_{42}\}$ and hence equal to O_2, O_3, O_6 and O_9. One readily verifies that the collection consisting of the seven ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, O_1, O_2, O_3, O_6 and O_9 satisfies the properties (P1) – (P6).

(Ic) Suppose the mentioned partition has shape 2+1+1. Without loss of generality, we may suppose that $\{x_{11}, x_{22}, x_{34}, x_{43}\}$ is present in $\mathcal{C}$. Then the ovoid $\{x_{12}, x_{21}, x_{33}, x_{44}\}$ is not in $\mathcal{C}$. By (P3) and (P5), the second ovoid of $\mathcal{C}$ through x_{34} must contain x_{43} and hence coincides with $O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. Note that each of x_{34}, x_{43} has now been covered twice, while each of x_{12} and x_{21} only once. Therefore the second ovoid of $\mathcal{C}$ through x_{12}, which cannot intersect $\{x_{12}, x_{21}, x_{34}, x_{43}\}$ in a singleton, must also contain x_{21}. But that is impossible as the two ovoids through $\{x_{12}, x_{21}\}$, namely O_1 and $\{x_{12}, x_{21}, x_{33}, x_{44}\}$ are already forbidden.

(Id) Suppose the mentioned partition has shape 1+1+1+1. Without loss of generality, we may suppose that $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ belongs to $\mathcal{C}$. Each $y \in \{x_{23}, x_{34}, x_{42}\}$ is contained in a second ovoid of $\mathcal{C}$ which meets $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ in a second point $y' \in \{x_{23}, x_{34}, x_{42}\}$. But then the pairs $\{y, y'\}$ would partition $\{x_{23}, x_{34}, x_{42}\}$, an obvious contradiction.

Treatment of Case II

We have $U_2 = \{x_{11}, x_{22}, x_{34}, x_{43}\}$. If $U_2 \in \mathcal{C}$, then by (P1) – (P4), $\mathcal{C} \setminus \{U_2\}$ is a collection of six ovoids as in Lemma 4.1. A contradiction is then readily obtained from Lemma 4.2 and property (P5). So, $U_2 \notin \mathcal{C}$. By (P1) and (P2), it follows that the unique ovoid of $\mathcal{C}$ containing x_{11} is either $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ or $\{x_{11}, x_{24}, x_{32}, x_{43}\}$. In view of the symmetry $3 \leftrightarrow 4$, we may without loss of generality suppose that $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ is the unique ovoid of $\mathcal{C}$ containing x_{11}. There are still six ovoids to choose for $\mathcal{C}$, one of them contains x_{22} and the other five are contained in the collection $\{O_1, O_2, \ldots, O_9\}$. None of these six ovoids can intersect $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ in the singleton $\{x_{23}\}$ or the singleton $\{x_{42}\}$, implying that O_2 and O_3 do not belong to $\mathcal{C}$. So, we need to take five ovoids among the seven ovoids $O_1, O_4, O_5, O_6, O_7, O_8, O_9$. Since $O_4 \cap O_5 = \{x_{12}\}$, $O_5 \cap O_6 = \{x_{41}\}$, $O_4 \cap O_6 = \{x_{24}\}$ and $O_7 \cap O_9 = \{x_{42}\}$, (P5) implies that none of the pairs $\{O_4, O_5\}$, $\{O_5, O_6\}$, $\{O_4, O_6\}$, $\{O_7, O_9\}$ can be contained in $\mathcal{C}$. So, two among O_4, O_5, O_6 cannot be in $\mathcal{C}$, as well as one among O_7, O_9. So, it is impossible to find the five required ovoids from the collection $\{O_1, O_4, O_5, \cdots, O_9\}$.

Treatment of Case III

We have $U_2 = \{x_{11}, x_{23}, x_{34}, x_{42}\}$. If $U_2 \in \mathcal{C}$, then by (P1) – (P4), $\mathcal{C} \setminus \{U_2\}$ is a collection of six ovoids as in Lemma 4.1. A contradiction is then readily obtained from Lemma 4.2 and property (P5). So, $U_2 \notin \mathcal{C}$. Then, using (P1) and (P2), the unique ovoid of $\mathcal{C}$ containing x_{11} must be $\{x_{11}, x_{24}, x_{32}, x_{43}\}$. Each point $y \in \{x_{24}, x_{32}, x_{43}\}$ is contained in a second ovoid of the collection $\mathcal{C}$ which meets $\{x_{11}, x_{24}, x_{32}, x_{43}\}$ in a second point $y' \in \{x_{24}, x_{32}, x_{43}\}$. Then the pairs $\{y, y'\}$ would partition $\{x_{24}, x_{32}, x_{43}\}$, an obvious contradiction.

Treatment of Case IV

We have $U_2 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. By (P1), all ovoids of $\mathcal{C}$ are disjoint from U_1. So we have to choose seven ovoids for $\mathcal{C}$ among the nine ovoids $O_1, O_2, \ldots, O_9$. By (P4), there are three ovoids of $\mathcal{C}$ containing x_{12}. So the ovoids O_1, O_4 and O_5 belong to $\mathcal{C}$. As $O_4 \cap O_6 = \{x_{24}\}$ and $O_4 \cap O_9 = \{x_{31}\}$, the ovoids O_6 and O_9 are not in $\mathcal{C}$ by (P5). Hence, $\mathcal{C} = \{O_1, O_2, O_3, O_4, O_5, O_7, O_8\}$. One readily verifies that this collection of ovoids satisfies the properties (P1) – (P6).

Treatment of Case V

Here $U_2 = \{x_{12}, x_{23}, x_{34}, x_{41}\}$. By (P1), all ovoids of $\mathcal{C}$ are disjoint from U_1. So we have to choose seven ovoids for $\mathcal{C}$ among the nine ovoids $O_1, O_2, \ldots, O_9$. Since $O_4 \cap O_6 = \{x_{24}\}$, $O_4 \cap O_8 = \{x_{43}\}$ and $O_4 \cap O_9 = \{x_{31}\}$, O_4 cannot occur in $\mathcal{C}$ by (P5). Since $O_6 \cap O_7 = \{x_{13}\}$ and $O_6 \cap O_8 = \{x_{32}\}$, we then know that also O_6 cannot occur in $\mathcal{C}$. So, we should have that $\mathcal{C} = \{O_1, O_2, O_3, O_5, O_7, O_8, O_9\}$. But that is impossible again by (P5) as $O_7 \cap O_8 = \{x_{21}\}$.

Let $X_1 = U_1 \cup Y_1 = O^* \cup Y_1$, where Y_1 is the set of seven points of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which the collection $\{O_y \,|\, y \in Y_1\}$ consists of the ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, O_1, O_2, O_3, O_6 and O_9 of $Q^+(3,3)$. Similarly, let $X_2 = U_1 \cup Y_2 = O^* \cup Y_2$, where Y_2 is the set of seven points of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which the collection $\{O_y \,|\, y \in Y_2\}$ coincides with $\{O_1, O_2, O_3, O_4, O_5, O_7, O_8\}$. Note that X_1 is associated with the seven ovoids corresponding to subcase (Ib) in the treatment of Case I and X_2 is associated with the seven ovoids in the treatment of Case IV.

By the above discussion, we thus know:

Proposition 4.9. *Up to isomorphism, X_1 and X_2 are the two $\mathcal{T}$-blocking sets of size* 11 *in* PG(3, 3).

Our intention is now to identify the two blocking sets X_1 and X_2 with that of B_1 and B_2 constructed, respectively, in Sections 3.1 and 3.2. We shall rely on the following lemma.

Lemma 4.10. *Every ovoid O of $Q^+(3,3)$ is contained in precisely four partitions of $Q^+(3,3)$ into ovoids. Three of these are induced by external lines.*

Proof. Without loss of generality, we may suppose that $O = O^*$. The partitions then have the form $\{O^*, O_i, O_j, O_k\}$, where $i, j, k \in \{1, 2, \ldots, 9\}$ with $i < j < k$. It is straightforward to verify that these partitions are $\{O^*, O_1, O_2, O_3\}$, $\{O^*, O_1, O_6, O_9\}$, $\{O^*, O_2, O_5, O_8\}$ and $\{O^*, O_3, O_4, O_7\}$. Now, let x denote the unique point of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which $O_x = O = O^*$. There are three external lines through x. If $\{x, u_1, u_2, u_3\}$, $\{x, u_4, u_5, u_6\}$ and $\{x, u_7, u_8, u_9\}$ are these external lines, then the nine ovoids $\{O_{u_1}, O_{u_2}, \ldots, O_{u_9}\}$ are mutually distinct. So, $\{O^*, O_1, O_6, O_9\}$, $\{O^*, O_2, O_5, O_8\}$ and $\{O^*, O_3, O_4, O_7\}$ must be the partitions among the four that are induced by external lines. □

Proposition 4.11. *There exist two mutually disjoint external lines K, L and a point $x \in K$ such that $X_1 = O_x \cup (K \setminus \{x\}) \cup L$.*

Proof. Let K denote the external line determined by the ovoids O^*, O_1, O_6, O_9, and denote by x the unique point of K for which $O_x = O^*$. Among the four partitions of $Q^+(3,3)$ into ovoids containing O_2, $\{O^*, O_1, O_2, O_3\}$ is not induced by any external line (see the proof of Lemma 4.10). So, again by Lemma 4.10, the partition of $Q^+(3,3)$ by the ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, O_2, O_3 is induced by some external line, say L. Then we have $K \cap L = \emptyset$ and $X_1 = O_x \cup \Big(K \setminus \{x\}\Big) \cup L$. □

By Proposition 3.3, we know that the two blocking sets B_1 and B_2 constructed in Sections 3.1 and 3.2 are nonisomorphic. In fact, by the proof of that proposition, we know that B_2 does not contain any external line, while B_1 does. Comparing this with Propositions 4.9 and 4.11, we then conclude that the blocking set X_1 is isomorphic to B_1 and that the blocking set X_2 is isomorphic to B_2.

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