

On minimum size blocking sets of the outer tangents to a hyperbolic quadric in $\text{PG}(3, q)$

Bart De Bruyn Binod Kumar Sahoo

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Abstract

Let $Q^+(3, q)$ be a hyperbolic quadric in $\text{PG}(3, q)$ and \mathcal{T}_1 be the set of all lines of $\text{PG}(3, q)$ meeting $Q^+(3, q)$ in singletons (the so-called outer tangents). If k is the minimum size of a \mathcal{T}_1 -blocking set in $\text{PG}(3, q)$, then we prove that $k \geq q^2 - 1$. It is known that there is no \mathcal{T}_1 -blocking set of size $q^2 - 1$ for $q > 2$ even and that there is a unique (up to isomorphism) \mathcal{T}_1 -blocking set of size 3 for $q = 2$. For $q = 3$, we prove as well that there is a unique \mathcal{T}_1 -blocking set of size 8. Using a computer, we also classify all \mathcal{T}_1 -blocking sets of size $q^2 - 1$ for each prime power $q \leq 13$. On basis of some structural similarities we are subsequently able to recognize three families of blocking sets whose further study shows that they can be constructed from certain objects related to finite fields (like nice subsets or permutations of the latter). This connection with finite fields allows us to obtain some computer free descriptions.

Keywords: Projective space, Blocking set, Conic, Ovoid, Hyperbolic quadric

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1 Introduction

1.1 On the hyperbolic quadric $Q^+(3, q)$ in $\text{PG}(3, q)$

Let $Q^+(3, q)$ be a hyperbolic quadric in $\text{PG}(3, q)$. We denote by \mathcal{E} , \mathcal{T}_1 , \mathcal{S} and \mathcal{T}_0 the set of lines of $\text{PG}(3, q)$ that intersect $Q^+(3, q)$ in respectively 0, 1, 2 and $q + 1$ points. The elements of \mathcal{E} are called *external lines*, those of \mathcal{S} *secant lines* and those of $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_1$ *tangent lines*. The tangent lines belonging to \mathcal{T}_0 are called the *inner tangents* and those of \mathcal{T}_1 the *outer tangents*. The inner tangents are precisely the lines contained in $Q^+(3, q)$, and so we have $|\mathcal{T}_0| = 2(q + 1)$. As every point of $Q^+(3, q)$ is contained in $q - 1$ outer tangents, we have $|\mathcal{T}_1| = (q + 1)^2(q - 1)$ and hence $|\mathcal{T}| = (q + 1)(q^2 + 1)$. We also have $|\mathcal{S}| = \frac{1}{2}q^2(q + 1)^2$ and $|\mathcal{E}| = (q^2 + 1)(q^2 + q + 1) - \frac{1}{2}q^2(q + 1)^2 - (q + 1)(q^2 + 1) = \frac{1}{2}q^2(q - 1)^2$.

With the quadric $Q^+(3, q)$, there is naturally associated a polarity ζ which is symplectic if q is even and orthogonal if q is odd. For every point x of $Q^+(3, q)$, x^ζ is a plane which is

tangent to $Q^+(3, q)$ at the point x and intersects $Q^+(3, q)$ in the union of two lines through x . The $q + 1$ tangent lines through x are precisely the lines through x contained in x^ζ .

Suppose q is even. With the symplectic polarity ζ , there is associated a symplectic generalized quadrangle $W(q)$, whose points are the points of $\text{PG}(3, q)$ and whose lines are the lines of $\text{PG}(3, q)$ that are totally isotropic with respect to ζ , with incidence being containment. The lines of $W(q)$ are precisely the elements of \mathcal{T} .

For every point x of $\text{PG}(3, q) \setminus Q^+(3, q)$, x^ζ is a plane through the point x , and the tangent lines contained in x^ζ are precisely the lines of x^ζ through x . The plane x^ζ intersects $Q^+(3, q)$ in a conic O_x with kernel x . This conic O_x is also an ovoid¹ of $Q^+(3, q)$ as it intersects each inner tangent in a unique point. The map $x \mapsto O_x$ defines a bijection between $\text{PG}(3, q) \setminus Q^+(3, q)$ and the set of conics contained in $Q^+(3, q)$.

Suppose q is odd. For every point x of $\text{PG}(3, q) \setminus Q^+(3, q)$, x^ζ is a nontangent plane with $x \notin x^\zeta$ and it meets $Q^+(3, q)$ in a conic O_x of x^ζ . The tangent lines through x are precisely the lines through x meeting the conic O_x . Again, the conic O_x is an ovoid of $Q^+(3, q)$ and the map $x \mapsto O_x$ defines a bijection between $\text{PG}(3, q) \setminus Q^+(3, q)$ and the set of conics contained in $Q^+(3, q)$.

1.2 Blocking sets in $\text{PG}(3, q)$

For a given nonempty set \mathcal{L} of lines of $\text{PG}(3, q)$, a set X of points of $\text{PG}(3, q)$ is called an \mathcal{L} -blocking set if each line of \mathcal{L} meets X . Continuing with the notation of Section 1.1, we call two \mathcal{L} -blocking sets X_1 and X_2 with $\mathcal{L} \in \{\mathcal{E}, \mathcal{S}, \mathcal{T}, \mathcal{T}_0, \mathcal{T}_1\}$ *isomorphic* if there exists an automorphism of $\text{PG}(3, q)$ stabilizing $Q^+(3, q)$ and mapping X_1 to X_2 .

The first step in the study of blocking sets has been to determine the smallest cardinality of a blocking set and to characterize, if possible, all blocking sets of that cardinality.

If \mathcal{L} is the set of all lines of $\text{PG}(3, q)$ and X is an \mathcal{L} -blocking set, then $|X| \geq q^2 + q + 1$ and equality holds if and only if X is a plane of $\text{PG}(3, q)$. This follows from a more general result by Bose and Burton [4, Theorem 1]. Biondi et al. characterized the minimum size \mathcal{E} -blocking sets in [3, Theorem 2.4] for $q \geq 9$ odd and in [2, Theorem 1.1] for $q \geq 8$ even (also see [12, Section 3] and [9, Section 2] for a different proof which works for all q). When $q > 2$ is even, the minimum size $(\mathcal{E} \cup \mathcal{S})$ -blocking sets were determined in [14, Theorem 1.3] using properties of generalized quadrangles. For $\mathcal{L} \in \{\mathcal{S}, \mathcal{T} \cup \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$, the minimum size \mathcal{L} -blocking sets are described in [13, Theorems 1.7, 1.8, 1.9] for all q . The minimum size $(\mathcal{E} \cup \mathcal{T})$ -blocking sets are characterized in [12, Proposition 1.5] for q even and in [9, Theorem 1.1] for all $q \geq 3$.

If k is the minimum size of a \mathcal{T} -blocking set in $\text{PG}(3, q)$, then $q^2 + 1 \leq k \leq q^2 + q$ by [8, Lemmas 2.1, 2.2]. If q is even, then the \mathcal{T} -blocking sets of size $q^2 + 1$ are precisely the ovoids of the generalized quadrangle $W(q)$ associated with $Q^+(3, q)$. There are two known types of such ovoids: ovoids which are *elliptic quadrics* of the ambient projective space $\text{PG}(3, q)$ and the so-called *Suzuki-Tits ovoids*. The former ovoids exist for each power of 2 and the latter only for the prime powers of the form 2^{2e+1} with $e \in \mathbb{N} \setminus \{0\}$.

¹An *ovoid* of a point-line geometry is a set of points meeting each line in a singleton.

For $q \leq 32$, it has been proved that every ovoid of $W(q)$, q even, is either an elliptic quadric or a Suzuki-Tits ovoid. In the q odd case, not much is known for the minimum size \mathcal{T} -blocking sets. In $\text{PG}(3,3)$, by [8, Theorem 1.1], there is no \mathcal{T} -blocking set of size 10 and there are exactly two \mathcal{T} -blocking sets of size 11 up to isomorphism. By means of the computer algebra systems GAP [16] and Sage [11], it was proved that there exist no \mathcal{T} -blocking sets of size $q^2 + 1$ for each odd prime power $q \leq 13$, see [8, Theorem 1.2].

1.3 \mathcal{T}_1 -blocking sets

The minimum size \mathcal{T}_0 -blocking sets are well known as these are precisely the ovoids of $Q^+(3, q)$. In this paper, we shall study the minimum size \mathcal{T}_1 -blocking sets. In the following lemma, we derive a lower bound for the size of such blocking sets.

Lemma 1.1 *If X is a \mathcal{T}_1 -blocking set, then $|X| \geq q^2 - 1$, with equality if and only if $X \cap Q^+(3, q) = \emptyset$ and every outer tangent contains a unique point of X .*

Proof. We count in two ways the number of pairs (x, L) , where $x \in X$ and L is an outer tangent containing x . The number N of such pairs is equal to $|X \cap Q^+(3, q)| \cdot (q - 1) + |X \setminus Q^+(3, q)| \cdot (q + 1)$. We thus have $N \leq |X| \cdot (q + 1)$, with equality if and only if $X \cap Q^+(3, q) = \emptyset$. As each of the $(q + 1)^2(q - 1)$ outer tangents contains at least one point of X , we have $(q + 1)^2(q - 1) \leq N$ with equality if and only if every outer tangent contains a unique point of X . Combining both inequalities for N , we find $(q + 1)^2(q - 1) \leq |X| \cdot (q + 1)$, that is, $q^2 - 1 \leq |X|$. By the above, the necessary and sufficient conditions for equality are as stated in the lemma. ■

The following lemma is proved in the same way as Lemma 1.1.

Lemma 1.2 *Let Ω be a set of points of $\text{PG}(3, q)$ disjoint from $Q^+(3, q)$ such that every outer tangent meets Ω in at most one point. Then $|\Omega| \leq q^2 - 1$, with equality if and only if every outer tangent meets Ω in precisely one point.*

In the case q even, every ovoid of $W(q)$ is a \mathcal{T}_1 -blocking set, showing that the minimum size k of a \mathcal{T}_1 -blocking set satisfies $q^2 - 1 \leq k \leq q^2 + 1$. \mathcal{T}_1 -blocking sets of size $q^2 - 1$ are also maximal partial ovoids of $W(q)$, and by [5, Corollary 1] such partial ovoids cannot exist if $q > 2$. For $q = 2$, there is up to isomorphism a unique partial ovoid of size 3 of $W(2)$ that is also a \mathcal{T}_1 -blocking set, see e.g. [15, Theorem 3.9]. Nothing seems to be known about \mathcal{T}_1 -blocking sets of size q^2 . It is also not known whether \mathcal{T}_1 -blocking sets of size $q^2 + 1$ are always ovoids of $W(q)$.

Let us therefore consider the case where q is odd. For $q \in \{3, 5, 7, 11\}$, we are able to construct a number of \mathcal{T}_1 -blocking sets of size $q^2 - 1$. We have also used the computer algebra systems GAP [16] and Sage [11] to classify all \mathcal{T}_1 -blocking sets of minimum size $q^2 - 1$ for each odd prime power $q \leq 13$, see [7]. Our results are as follows.

Theorem 1.3 *There are no \mathcal{T}_1 -blocking sets of size $q^2 - 1$ for $q \in \{9, 13\}$. The number of nonisomorphic \mathcal{T}_1 -blocking sets of size $q^2 - 1$ is equal to 1 if $q = 3$, equal to 3 if $q = 5$ and equal to 4 if $q \in \{7, 11\}$.*

In the following table, we provide some information about these \mathcal{T}_1 -blocking sets. Also this information was obtained by means of computer computations.

Set	q	Number	IP secants	IP externals	\mathcal{T} -block. sets
B_1	3	9	$0^{32}1^{32}2^8$	$0^82^84^2$	1
B_2	5	200	$0^{198}1^{144}2^{108}$	$0^{64}1^{72}2^{36}3^{24}6^4$	1
B_3	5	400	$0^{180}1^{198}2^{54}3^{18}$	$0^{60}1^{90}2^{18}3^{14}4^{18}$	2
B_4	5	25	$0^{144}1^{288}4^{18}$	$0^{96}2^{72}3^{32}$	1
B_5	7	3528	$0^{576}1^{736}2^{192}3^{32}4^{32}$	$0^{294}1^{320}2^{168}3^{64}4^{28}6^8$	2
B_6	7	784	$0^{576}1^{720}2^{216}3^{48}6^8$	$0^{306}1^{288}2^{216}4^{72}$	1
B_7	7	4704	$0^{592}1^{696}2^{216}3^{40}4^{24}$	$0^{276}1^{360}2^{156}3^{48}4^{18}5^{24}$	2
B_8	7	588	$0^{640}1^{576}2^{288}3^{64}$	$0^{324}1^{192}2^{336}4^{24}8^6$	1
B_9	11	2904	$0^{2880}1^{4200}2^{1500}5^{120}10^{12}$	$0^{2500}1^{1200}2^{1800}3^{400}4^{150}$	1
B_{10}	11	29040	$0^{2928}1^{4140}2^{1500}5^{84}6^{60}$	$0^{2140}1^{2100}2^{1200}3^{400}4^{150}5^{60}$	2
B_{11}	11	24200	$0^{2880}1^{4248}2^{1440}5^{72}6^{72}$	$0^{2112}1^{2304}2^{792}3^{672}4^{162}6^8$	2
B_{12}	11	2420	$0^{3168}1^{3600}2^{1800}5^{144}$	$0^{2040}1^{2160}2^{1440}3^{240}4^{90}6^{80}$	1

Given a particular \mathcal{T}_1 -blocking set B and a set $\mathcal{L} \in \{\mathcal{S}, \mathcal{E}\}$, the *intersection pattern (IP)* of B with respect to the lines of \mathcal{L} is defined as the sequence $n_1^{e_1}n_2^{e_2} \cdots n_k^{e_k}$, where

- $k \in \mathbb{N} \setminus \{0\}$;
- $n_1, n_2, \dots, n_k \in \mathbb{N}$ with $n_1 < n_2 < \cdots < n_k$;
- $e_1, e_2, \dots, e_k \in \mathbb{N} \setminus \{0\}$;
- for every $i \in \{1, \dots, k\}$, there are e_i lines in \mathcal{L} that intersect B in n_i points.

The intersection pattern of the various blocking sets with respect to \mathcal{S} and \mathcal{E} have been mentioned in columns 4 and 5 of the above table. The number of \mathcal{T}_1 -blocking sets isomorphic to a given blocking set B_i is given in the third column. On basis of computer data of the various blocking sets, we have been able to describe three families of blocking sets. These will respectively be studied in Sections 2, 3 and 4.

The blocking sets B_1, B_2, B_8 belong to the first family, all whose members B have the special property that they can be partitioned by $q - 1$ external lines. These external lines are moreover contained in a regular spread together with one of the two reguli of $Q^+(3, q)$. We show in Section 2 that these blocking sets arise from subsets $\{\alpha_1, \alpha_2, \dots, \alpha_{q-1}\}$ of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ that satisfy the following properties:

- $\alpha_i - \alpha_j \notin \mathbb{F}_q$ for all $i, j \in \{1, 2, \dots, q - 1\}$ with $i \neq j$;
- $\frac{1}{\alpha_i+l} - \frac{1}{\alpha_j+l} \notin \mathbb{F}_q$ for all $l \in \mathbb{F}_q$ and all $i, j \in \{1, 2, \dots, q - 1\}$ with $i \neq j$.

The blocking sets B_1, B_4, B_6 and B_9 belong to the second family, all whose members B have the special property that there exist two disjoint lines L_1 and L_2 of $Q^+(3, q)$ and $q + 1$ mutually disjoint lines K_1, K_2, \dots, K_{q+1} meeting L_1 and L_2 such that $B = (K_1 \cup K_2 \cup \cdots \cup K_{q+1}) \setminus (L_1 \cup L_2)$. We show in Section 3 that these blocking sets arise from permutations σ of $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$ that satisfy the following properties:

- (P1) σ has no fixed points;
- (P2) $\sigma^2 = 1$;
- (P3) $(x - x^\sigma)x$ is a nonsquare for all $x \in \mathbb{F}_q^*$;
- (P4) $(x^\sigma - x)(y^\sigma - y)(y - x)(y^\sigma - x^\sigma)$ is a nonsquare for all $x, y \in \mathbb{F}_q^*$ with $y \notin \{x, x^\sigma\}$.

The blocking sets B_1 , B_3 and B_5 belong to the third family, all whose members B have the special property that there exist two mutually disjoint external lines L_1 and L_2 and $q + 1$ mutually disjoint external lines K_1, K_2, \dots, K_{q+1} meeting L_1 and L_2 such that $B = (K_1 \cup K_2 \cup \dots \cup K_{q+1}) \setminus (L_1 \cup L_2)$.

The following proposition says that every \mathcal{T}_1 -blocking set of size $q^2 - 1$ will give rise to a \mathcal{T} -blocking set of size $q^2 + q - 1$.

Proposition 1.4 *Suppose X is a \mathcal{T}_1 -blocking set of size $q^2 - 1$ in $\text{PG}(3, q)$. Let x be a point of X and let Y be the set of $q + 1$ points of $Q^+(3, q)$ that are the tangency points of the outer tangents through x . Then $(X \setminus \{x\}) \cup Y$ is a \mathcal{T} -blocking set of size $q^2 + q - 1$.*

Proof. Note that $X \cap Q^+(3, q) = \emptyset$ by Lemma 1.1. As Y is an ovoid of $Q^+(3, q)$, every inner tangent meets Y . Suppose L is an outer tangent. Then L meets X . If L would not meet $X \setminus \{x\}$, then it contains x and hence must contain a point of Y . So, $(X \setminus \{x\}) \cup Y$ is a \mathcal{T} -blocking set of size $q^2 + q - 1$. ■

Note that it is possible that a \mathcal{T}_1 -blocking set of size $q^2 - 1$ gives rise to several non-isomorphic \mathcal{T} -blocking sets of size $q^2 + q - 1$. The last column of the table mentions how many nonisomorphic \mathcal{T} -blocking sets of size $q^2 + q - 1$ arise from the \mathcal{T}_1 -blocking set B_i , $i \in \{1, \dots, 12\}$. Using Theorem 1.3 and Proposition 1.4, we can thus construct \mathcal{T} -blocking sets whose size is strictly smaller than $q^2 + q$ for $q \in \{3, 5, 7, 11\}$. The number of nonisomorphic \mathcal{T} -blocking sets of size $q^2 + q - 1$ that arise in this fashion is one if $q = 3$, four if $q = 5$ and six if $q \in \{7, 11\}$. Besides these seventeen \mathcal{T} -blocking sets, one additional \mathcal{T} -blocking set is known whose size is strictly smaller than $q^2 + q$, namely a particular \mathcal{T} -blocking set of size 11 in $\text{PG}(3, 3)$, see [8, Section 3]. Note that \mathcal{T} -blocking sets of size $q^2 + q$ always exist, namely tangent planes minus their tangency points.

2 A first family of \mathcal{T}_1 -blocking sets

The first family of \mathcal{T}_1 -blocking sets that we are going to define and investigate is related to regular spreads of $\text{PG}(3, q)$.

2.1 Regular spreads of $\text{PG}(3, q)$

Let \tilde{V} be a 4-dimensional vector space over \mathbb{F}_{q^2} with basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$. The set V consisting of all linear combinations of $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ with coefficients in \mathbb{F}_q can naturally

be given the structure of a 4-dimensional vector space over \mathbb{F}_q . We denote by $\text{PG}(3, q^2) = \text{PG}(\tilde{V})$ and $\text{PG}(3, q) = \text{PG}(V)$ the projective spaces associated with the respective vector spaces \tilde{V} and V . Since every 1-dimensional subspace of V is contained in a unique 1-dimensional subspace of \tilde{V} , we can and will regard the points of $\text{PG}(V)$ as points of $\text{PG}(\tilde{V})$.

For every vector $\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3 + \alpha_4\bar{v}_4$ of \tilde{V} with all α_i 's belonging to \mathbb{F}_{q^2} , we define $\bar{v}^q := \alpha_1^q\bar{v}_1 + \alpha_2^q\bar{v}_2 + \alpha_3^q\bar{v}_3 + \alpha_4^q\bar{v}_4$, for every point $x = \langle \bar{v} \rangle$ of $\text{PG}(\tilde{V})$, we define $x^q := \langle \bar{v}^q \rangle$, and for every subspace Σ of $\text{PG}(\tilde{V})$ we define $\Sigma^q := \{x^q \mid x \in \Sigma\}$. The vectors $\bar{v} \in \tilde{V}$ for which $\bar{v}^q = \bar{v}$ are precisely the vectors of V , and the points x of $\text{PG}(\tilde{V})$ satisfying $x^q = x$ are precisely the points of $\text{PG}(V)$.

Now, let L^* be a line of $\text{PG}(\tilde{V})$ for which $L^* \cap (L^*)^q = \emptyset$. For every point x of L^* , the line xx^q of $\text{PG}(\tilde{V})$ intersects $\text{PG}(V)$ in a line L_x of $\text{PG}(V)$. In this way, we get a collection $S = \{L_x \mid x \in L^*\}$ of $q^2 + 1$ mutually disjoint lines of $\text{PG}(V)$ forming a *regular spread* of $\text{PG}(3, q)$. The following fact is known, see e.g. Theorem 5.3 of Bruck [6].

Suppose x_1, x_2, x_3 are three mutually distinct points of L^* and let B be the unique Baer subline of L^* containing $\{x_1, x_2, x_3\}$. Then the set $\mathcal{R}_B := \{L_x \mid x \in B\}$ is the unique regulus of $\text{PG}(3, q)$ containing the lines L_{x_1}, L_{x_2} and L_{x_3} .

Now, let B^* be a fixed Baer subline of L^* and put $\mathcal{R}^* := \mathcal{R}_{B^*}$. Then the union of all lines of \mathcal{R}^* is a hyperbolic quadric $Q^+(3, q)$ of $\text{PG}(3, q)$.

If L is a line of $\text{PG}(3, q)$ not belonging to S , then the set of lines of S meeting L is a regulus \mathcal{R}_L of $\text{PG}(3, q)$. We denote by B_L the unique Baer subline B of L^* for which $\mathcal{R}_B = \mathcal{R}_L$.

2.2 Description and investigation of the family

Lemma 2.1 *If L is an outer tangent to $Q^+(3, q)$, then B_L intersects B^* in a singleton. Conversely, every Baer subline of L^* intersecting B^* in a singleton is of the form B_L for some outer tangent line L .*

Proof. Suppose L is an outer tangent. Let x denote the unique point of $L \cap Q^+(3, q)$, let U denote the unique line of S through x and let y denote the unique point of L^* for which $L_y = U$. Since $L \cap Q^+(3, q) = \{x\}$, we have $\mathcal{R}_L \cap \mathcal{R}^* = \{U\}$ and $B_L \cap B^* = \{y\}$.

Conversely, suppose that B is a Baer subline of L^* intersecting B^* in a singleton $\{y\}$. Then $\mathcal{R}_B \cap \mathcal{R}^* = \mathcal{R}_B \cap \mathcal{R}_{B^*} = \{L_y\}$. Let x be an arbitrary point of L_y and let L be the unique line through x meeting each line of \mathcal{R}_B . As $\mathcal{R}_B \cap \mathcal{R}_{B^*} = \{L_y\}$, L_y is the only line of \mathcal{R}_{B^*} that meets L , implying that $\{x\} = L_y \cap L$ is the only point of $Q^+(3, q)$ contained in L . So, L is an outer tangent. Obviously, we have $\mathcal{R}_L = \mathcal{R}_B$ and so $B_L = B$. ■

Now, suppose that X' is a set of points of L^* which is disjoint from B^* . Put $S' = \{L_x \mid x \in X'\} \subseteq S \setminus \mathcal{R}^*$, and $\Omega := \bigcup_{L \in S'} L$.

Lemma 2.2 *Let L be an outer tangent. Then $|L \cap \Omega|$ equals the number of points of X' that are contained in B_L .*

Proof. As L intersects one of the lines of $\mathcal{R}^* \subseteq S$ in a unique point, it will intersect each line of S in at most one point. So, $|L \cap \Omega|$ equals the number of lines of S' that L meets, that is, $|L \cap \Omega| = |\mathcal{R}_L \cap S'|$. The latter number is also equal to $|B_L \cap X'|$. ■

The following is an immediate consequence of Lemmas 1.2, 2.1 and 2.2.

Proposition 2.3 *Suppose X' is a set of $q - 1$ points. Then Ω is a \mathcal{T}_1 -blocking set of size $q^2 - 1$ if and only if one of the following two equivalent conditions hold:*

- (1) *Every Baer subline of L^* meeting B^* in a singleton contains a unique point of X' .*
- (2) *Every Baer subline of L^* meeting B^* in a singleton contains at most one point of X' .*

Now, we choose a reference system $\{\bar{e}_1, \bar{e}_2\}$ for L^* in such a way that B^* consists of all points $\langle k_1 \bar{e}_1 + k_2 \bar{e}_2 \rangle$, where $k_1, k_2 \in \mathbb{F}_q \times \mathbb{F}_q \setminus \{(0, 0)\}$. Suppose $X' = \{x_1, x_2, \dots, x_{q-1}\}$ is a set of $q - 1$ points of $L^* \setminus B^*$. Then there exists unique $\alpha_1, \alpha_2, \dots, \alpha_{q-1} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $x_i = \langle \alpha_i \bar{e}_1 + \bar{e}_2 \rangle$ for every $i \in \{1, 2, \dots, q - 1\}$. We now determine necessary and sufficient conditions that must be satisfied by $\alpha_1, \alpha_2, \dots, \alpha_{q-1}$ such that every Baer subline of L^* meeting B^* in a singleton contains at most one point of X' . The following observation will be useful to that end.

Lemma 2.4 *For every $x \in B^*$ and every $y \in L^* \setminus B^*$, there exists a unique Baer subline of L^* through y intersecting B^* in the singleton $\{x\}$.*

Proof. Since any three distinct points of L^* are contained in a unique Baer subline, the number of Baer sublines containing $\{x, y\}$ is equal to $\frac{|L^* \setminus \{x, y\}|}{q-1} = \frac{q^2-1}{q-1} = q + 1$, and any of these Baer sublines intersects B^* in either 1 or 2 points. In fact, the number of these Baer sublines intersecting B^* in two points is equal to $|B^* \setminus \{x\}| = q$, showing that there exists a unique such Baer subline intersecting B^* in the singleton $\{x\}$.

Another proof goes as follows. Let κ be the Klein correspondence from the set of lines of $\text{PG}(3, q)$ to the points of the hyperbolic quadric $Q^+(5, q)$ of $\text{PG}(5, q)$. The regular spread S is then mapped by κ to an elliptic quadric $Q^-(3, q) \subseteq Q^+(5, q)$ and the reguli contained in S are mapped to conics contained in $Q^-(3, q)$, see e.g. [10, §15.4 and Lemma 17.1.2]. Let C^* denote the conic corresponding to B^* , denote by $\tilde{x} \in C^*$ the point of $\text{PG}(5, q)$ corresponding to the line L_x and by $\tilde{y} \in Q^-(3, q) \setminus C^*$ the point of $\text{PG}(5, q)$ corresponding to the line L_y . The Baer sublines through y meeting B^* in the singleton $\{x\}$ correspond (via their associated reguli) to conics in $Q^-(3, q)$ through \tilde{y} meeting C^* in the singleton $\{\tilde{x}\}$. There is a unique conic like that. Its carrying plane is $\langle \tilde{y}, \tilde{L} \rangle$, where \tilde{L} is the line of the plane $\langle C^* \rangle$ that is tangent to C^* at the point \tilde{x} . ■

Now, let $i, j \in \{1, 2, \dots, q - 1\}$ with $i \neq j$, and let $p \in B^*$. We determine necessary and sufficient conditions to be satisfied by α_i and α_j in order that the unique Baer subline B_p through x_i intersecting B^* in $\{p\}$ would not contain x_j .

Consider first the case that $p = \langle \bar{e}_1 \rangle$. Then B_p contains the point $\langle \bar{e}_1 \rangle$ as well as all points of the form $\langle (\alpha_i + k)\bar{e}_1 + \bar{e}_2 \rangle$, where $k \in \mathbb{F}_q$. So, a necessary and sufficient condition is that $\alpha_i - \alpha_j \notin \mathbb{F}_q$.

Consider next the case where $p = \langle \bar{e}_2 - l\bar{e}_1 \rangle$ for some $l \in \mathbb{F}_q$. We have $x_i = \left\langle \frac{\bar{e}_2 - l\bar{e}_1}{\alpha_i + l} + \bar{e}_1 \right\rangle$ and $x_j = \left\langle \frac{\bar{e}_2 - l\bar{e}_1}{\alpha_j + l} + \bar{e}_1 \right\rangle$. The unique Baer subline through x_i intersecting B^* in $p = \langle \bar{e}_2 - l\bar{e}_1 \rangle$ consists of the point p together with all points of the form $\left\langle \left(k + \frac{1}{\alpha_i + l}\right)(\bar{e}_2 - l\bar{e}_1) + \bar{e}_1 \right\rangle$, where $k \in \mathbb{F}_q$. So, a necessary and sufficient condition is that $\frac{1}{\alpha_i + l} - \frac{1}{\alpha_j + l} \notin \mathbb{F}_q$.

So, we can conclude the following:

Proposition 2.5 *Suppose $A = \{\alpha_1, \alpha_2, \dots, \alpha_{q-1}\}$ is a set of $q - 1$ elements of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$, and put $X' = \{\langle \alpha_i \bar{e}_1 + \bar{e}_2 \rangle \mid i \in \{1, 2, \dots, q - 1\}\}$. Then every Baer subline of L^* meeting B^* in a singleton contains a unique point of X' if and only if the following properties are satisfied:*

- (a) $\alpha_i - \alpha_j \notin \mathbb{F}_q$ for all $i, j \in \{1, 2, \dots, q - 1\}$ with $i \neq j$;
- (b) $\frac{1}{\alpha_i + l} - \frac{1}{\alpha_j + l} \notin \mathbb{F}_q$ for all $l \in \mathbb{F}_q$ and all $i, j \in \{1, 2, \dots, q - 1\}$ with $i \neq j$.

We call a subset $A = \{\alpha_1, \alpha_2, \dots, \alpha_{q-1}\}$ of size $q - 1$ of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ satisfying the properties (a) and (b) of Proposition 2.5 a *subset of blocking type* of \mathbb{F}_{q^2} . By Propositions 2.3 and 2.5, we then have

Proposition 2.6 *The set Ω is a \mathcal{T}_1 -blocking set if and only if the set $\{\alpha_1, \alpha_2, \dots, \alpha_{q-1}\}$ is a subset of blocking type of \mathbb{F}_{q^2} .*

A \mathcal{T}_1 -blocking set of size $q^2 - 1$ that is obtained from a subset of blocking type of \mathbb{F}_{q^2} has the property that it contains at least $q - 1$ external lines. In our computer classification of all \mathcal{T}_1 -blocking sets of size $q^2 - 1$ with $q \leq 13$, there are three blocking sets that have this property, namely B_1 , B_2 and B_8 (see the table in Section 1). For each $q \in \{3, 5, 7\}$, we now show that there exist subsets of blocking type of \mathbb{F}_{q^2} . This then shows that the blocking sets B_1 , B_2 and B_8 belong to the family of blocking sets under consideration in this section.

Lemma 2.7 *Let $a, b, c, d \in \mathbb{F}_q$ with $ad - bc \neq 0$ and let τ be an automorphism of \mathbb{F}_{q^2} . Then $\phi : \mathbb{F}_{q^2} \setminus \mathbb{F}_q \rightarrow \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ defined by $x \mapsto \frac{ax^\tau + b}{cx^\tau + d}$ is a bijection which maps every subset of blocking type of \mathbb{F}_{q^2} to a subset of blocking type of \mathbb{F}_{q^2} .*

Proof. Let A_1 be a subset of blocking type of \mathbb{F}_{q^2} and let X_1 denote the corresponding set of points of $L^* \setminus B^*$ (see Proposition 2.5). Let θ be the automorphism of L^* determined by $\langle u\bar{e}_1 + v\bar{e}_2 \rangle \mapsto \langle (au^\tau + bv^\tau)\bar{e}_1 + (cu^\tau + dv^\tau)\bar{e}_2 \rangle$, and put $X_2 = X_1^\theta$. Note that $(B^*)^\theta = B^*$. Since every Baer subline meeting B^* in a singleton contains a unique point of X_1 , the same property holds for X_2 . So, the subset A_2 of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ corresponding to X_2 (in the sense of Proposition 2.5) is a subset of blocking type of \mathbb{F}_{q^2} . Obviously, $A_2 = A_1^\phi$. ■

Let A_1 and A_2 be two subsets of blocking type of \mathbb{F}_{q^2} . Then A_1 and A_2 are called *equivalent* if there exist $a, b, c, d \in \mathbb{F}_q$ with $ad - bc \neq 0$ and an automorphism τ of \mathbb{F}_{q^2} such that $A_2 = A_1^\phi$, where $\phi : \mathbb{F}_{q^2} \setminus \mathbb{F}_q \rightarrow \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ is the map defined by $x \mapsto \frac{ax^\tau + b}{cx^\tau + d}$.

Proposition 2.8 *If $q > 2$ is even, then no subsets of blocking type of \mathbb{F}_{q^2} do exist.*

Proof. By Proposition 2.5, it suffices to prove that there does not exist a set X of $q - 1$ points in $L^* \setminus B^*$ such that every Baer subline intersecting B^* in a singleton contains a unique point of X . Suppose to the contrary that X is such a set. Since $q > 2$, we can take two distinct points $x_1, x_2 \in X$. For every point $y \in B^*$, the unique Baer subline of L^* containing x_1, x_2 and y cannot intersect B^* in 1 or more than 2 points and so must intersect in precisely 2 points. So, the Baer sublines containing x_1, x_2 determine a partition of B^* in pairs, proving that $q + 1$ is even, that is, q is odd. ■

Proposition 2.9 *Up to equivalence, \mathbb{F}_4 has a unique subset of blocking type.*

Proof. Suppose $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$ is a root of $x^2 + x + 1$. Then both $\{\alpha\}$ and $\{\alpha + 1\}$ are subsets of blocking type. The map $\phi : x \mapsto x + 1$ shows that they are equivalent. ■

Proposition 2.10 *Up to equivalence, \mathbb{F}_9 has a unique subset of blocking type.*

Proof. Suppose $\beta \in \mathbb{F}_9 \setminus \mathbb{F}_3$ is a root of $x^2 + 1$. Suppose A is a subset of blocking type of \mathbb{F}_9 . Then property (a) of Proposition 2.5 implies that A has the form $\{\beta + \epsilon_1, -\beta + \epsilon_2\}$, where $\epsilon_1, \epsilon_2 \in \mathbb{F}_3$. By considering the map $\phi : x \mapsto x - \epsilon_1$, we may suppose that $\epsilon_1 = 0$ and $\epsilon := \epsilon_2 \in \mathbb{F}_3$. Since $\frac{1}{\beta} - \frac{1}{-\beta+1} = -\beta - (-\beta - 1) = 1 \in \mathbb{F}_3$ and $\frac{1}{\beta} - \frac{1}{-\beta-1} = -\beta - (-\beta + 1) = -1 \in \mathbb{F}_3$, we must have $\epsilon = 0$ by property (b) of Proposition 2.5. Then the facts that

- $\frac{1}{\beta} - \frac{1}{-\beta} = (-\beta) - \beta = \beta \notin \mathbb{F}_3$,
- $\frac{1}{\beta+1} - \frac{1}{-\beta+1} = (\beta - 1) - (-\beta - 1) = -\beta \notin \mathbb{F}_3$,
- $\frac{1}{\beta-1} - \frac{1}{-\beta-1} = (\beta + 1) - (-\beta + 1) = -\beta \notin \mathbb{F}_3$,

imply that $\{\beta, -\beta\}$ is indeed a subset of blocking type of \mathbb{F}_9 . ■

We have verified the following with GAP and Sage, see [7].

Proposition 2.11 • *Up to equivalence, \mathbb{F}_{25} has a unique subset of blocking type. If $\beta \in \mathbb{F}_{25} \setminus \mathbb{F}_5$ is a root of $x^2 - x + 2$, then $\{\beta, \beta^8, \beta^9, \beta^{14}\}$ is a subset of blocking type.*

- *Up to equivalence, \mathbb{F}_{49} has a unique subset of blocking type. If $\beta \in \mathbb{F}_{49} \setminus \mathbb{F}_7$ is a root of $x^2 - x + 3$, then $\{\beta, \beta^4, \beta^7, \beta^{19}, \beta^{28}, \beta^{37}\}$ is a subset of blocking type.*
- *Suppose q is an odd prime power for which $9 \leq q \leq 49$. Then \mathbb{F}_{q^2} has no subsets of blocking type.*

The following lemma might be well known, but we found no suitable reference. Therefore a proof has been added.

Lemma 2.12 (1) *For every automorphism θ of the projective line L^* , there exists an automorphism θ' of $\text{PG}(3, q)$ stabilizing the spread S such that $L_x^{\theta'} = L_{x^\theta}$ for every point x of L^* .*

(2) *Suppose θ' is an automorphism of $\text{PG}(3, q)$ stabilizing the spread S . Then there exists an automorphism θ of L^* such that $(L_x)^{\theta'} = L_{x^\theta}$ for every point x of L^* .*

Proof. (1) Let W denote the 2-dimensional subspace of \tilde{V} such that $L^* = \text{PG}(W)$. The q^4 vectors of V are precisely the q^4 vectors of the form $\bar{w} + \bar{w}^q$, where $\bar{w} \in W$. Let ϕ be the semi-linear map of W that induces the automorphism θ of $\text{PG}(W) = L^*$. Then the map $V \rightarrow V : \bar{w} + \bar{w}^q \mapsto \bar{w}^\phi + (\bar{w}^\phi)^q$ is a semi-linear map inducing an automorphism θ' of $\text{PG}(3, q)$. The automorphism θ' stabilizes the spread S and satisfies $L_x^{\theta'} = L_{x^\theta}$ for every point x of L^* .

(2) Suppose that θ' is an automorphism of $\text{PG}(3, q)$ stabilizing the spread S . The automorphism θ' is induced by an automorphism $\bar{\theta}'$ of $\text{PG}(3, q^2)$. Every line $L \in S$ is contained in a unique line \bar{L} of $\text{PG}(3, q^2)$, and we define $\bar{S} := \{\bar{L} \mid L \in S\}$. It is known, see e.g. Theorem 3.9 of Beutelspacher and Ueberberg [1], that there are two lines in $\text{PG}(3, q^2)$ meeting every line of \bar{S} . These are the lines L^* and $(L^*)^q$. As θ' stabilizes the spread S , we must have $(L^*)^{\theta'} \in \{L^*, (L^*)^q\}$. If θ^* is the automorphism of $\text{PG}(3, q^2)$ induced by the map $\bar{v} \mapsto \bar{v}^q$, then both $\bar{\theta}'$ and $\bar{\theta}' \circ \theta^*$ induce θ' . So, without loss of generality, we may suppose that $(L^*)^{\theta'} = L^*$. Then $\bar{\theta}'$ induces an automorphism θ of L^* . Obviously, $L_x^{\theta'} = L_{x^\theta}$ for every point x of L^* . ■

Proposition 2.13 *Suppose A_1 and A_2 are two subsets of blocking type of \mathbb{F}_{q^2} . Let X'_i , $i \in \{1, 2\}$, denote the set of points of L^* corresponding to A_i (as in Proposition 2.5). If A_1 and A_2 are equivalent, then there exists an automorphism of $\text{PG}(3, q)$ stabilising $Q^+(3, q)$ and mapping $\Omega_1 = \bigcup_{x \in X'_1} L_x$ to $\Omega_2 = \bigcup_{x \in X'_2} L_x$.*

Proof. Since A_1 and A_2 are equivalent, there exist $a, b, c, d \in \mathbb{F}_q$ with $ad - bc \neq 0$ and an automorphism τ of \mathbb{F}_{q^2} such that $A_2 = A_1^\phi$, where ϕ is the permutation of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ determined by $x \mapsto \frac{ax^\tau + b}{cx^\tau + d}$. Let θ be the automorphism of L^* determined by the map $\langle u\bar{e}_1 + v\bar{e}_2 \rangle \mapsto \langle (au^\tau + bv^\tau)\bar{e}_1 + (cu^\tau + dv^\tau)\bar{e}_2 \rangle$. Then $(X'_1)^\theta = X'_2$ and $(B^*)^\theta = B^*$. If θ' is the automorphism of $\text{PG}(3, q)$ as in Lemma 2.12(1), then $\Omega_1^{\theta'} = \Omega_2$ and $\mathcal{R}_{B^*}^{\theta'} = \mathcal{R}_{B^*}$. The latter implies that θ' stabilizes the hyperbolic quadric $Q^+(3, q)$ of $\text{PG}(3, q)$. ■

Our intention is now to prove the converse of Proposition 2.13. To that end the following lemma will be useful.

Lemma 2.14 *Suppose $S' \subseteq S$ is a set of $q - 1$ lines disjoint from $Q^+(3, q)$ such that $\Omega = \bigcup_{L \in S'} L$ is a \mathcal{T}_1 -blocking set of size $q^2 - 1$. If $q > 5$, then S is the unique regular spread containing the lines of S' .*

Proof. By Propositions 2.6 and 2.8, we know that $q \geq 7$ is odd. Suppose by way of contradiction that S is not the unique regular spread containing the lines of S' . Then

S' is contained² in a regulus $\mathcal{R} \subset S$. We denote by Q' the unique hyperbolic quadric containing the lines of \mathcal{R} .

The quadrics Q' and $Q^+(3, q)$ have at most $2(q+1)$ common points (the union of 0, 1 or 2 lines of S). Let $x \in Q^+(3, q) \setminus Q'$ and denote by π the plane of $\text{PG}(3, q)$ that is tangent to $Q^+(3, q)$ in the point x . Let $L_1 \subseteq S$ and L_2 be the two lines of $Q^+(3, q)$ through x . As $x \notin Q'$, the line L_1 is not contained in \mathcal{R} . As any two lines of π meet, L_1 is the unique line of S in π . In particular, no line of \mathcal{R} is contained in π . The plane π can therefore not be tangent to the quadric Q' .

The intersection $\pi \cap Q'$ is thus a conic of π containing $q+1$ points. The set $\Omega \cap \pi$ is a set of $q-1$ points. Since Ω is a \mathcal{T}_1 -blocking set of size q^2-1 , each line of π through x distinct from L_1 and L_2 contains a unique point of $\Omega \cap \pi$. So there are at least $q-3 > 2$ tangent lines to $Q' \cap \pi$ among the $q-1$ lines of π through x distinct from L_1 and L_2 . This is impossible. Since q is odd, there are at most 2 such tangent lines through x . ■

We now prove the converse of Proposition 2.13.

Proposition 2.15 *Suppose A_1 and A_2 are two subsets of blocking type of \mathbb{F}_{q^2} . Let X'_i , $i \in \{1, 2\}$, denote the set of points of L^* corresponding to A_i (as in Proposition 2.5). If there exists an automorphism of $\text{PG}(3, q)$ stabilising $Q^+(3, q)$ and mapping $\Omega_1 = \bigcup_{x \in X'_1} L_x$ to $\Omega_2 = \bigcup_{x \in X'_2} L_x$, then A_1 and A_2 are equivalent.*

Proof. By Propositions 2.8, 2.9, 2.10 and 2.11, we may assume $q > 5$. Let θ' denote the automorphism of $\text{PG}(3, q)$ mentioned in the statement of the proposition. For every $i \in \{1, 2\}$, $S'_i := \{L_x \mid x \in X'_i\}$ is the set of lines contained in Ω_i . As $\Omega_1^{\theta'} = \Omega_2$, we have $S_1^{\theta'} = S_2$. By Lemma 2.14, we then know that θ' stabilizes the spread S . By Lemma 2.12(2) and its proof, we know that θ' is induced by an automorphism $\bar{\theta}'$ of $\text{PG}(3, q^2)$ such that $(L^*)^{\bar{\theta}'} = L^*$. Let θ be the restriction of $\bar{\theta}'$ to L^* . Since θ' stabilizes $Q^+(3, q)$, we have $(B^*)^\theta = B^*$. Since $\Omega_1^\theta = \Omega_2$, we have $(X'_1)^\theta = X'_2$. If the automorphism θ of L^* is described by $\langle u\bar{e}_1 + v\bar{e}_2 \rangle \mapsto \langle (au^\tau + bv^\tau)\bar{e}_1 + (cu^\tau + dv^\tau)\bar{e}_2 \rangle$, where τ is an automorphism of \mathbb{F}_{q^2} and $a, b, c, d \in \mathbb{F}_q$ such that $ad - bc \neq 0$, then $A_2 = A_1^\phi$ where ϕ is the permutation $x \mapsto \frac{ax^\tau + b}{cx^\tau + d}$ of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. So, A_1 and A_2 are equivalent. ■

By Propositions 2.13 and 2.15, the classification of the isomorphism classes of \mathcal{T}_1 -blocking sets arising from subsets of blocking type of \mathbb{F}_{q^2} is thus equivalent with the classification, up to equivalence, of these subsets of blocking type.

3 A second family of \mathcal{T}_1 -blocking sets

Suppose that L_1 and L_2 are two disjoint lines of $Q^+(3, q)$ and let \mathcal{L} denote a set of $q+1$ secant lines meeting L_1 and L_2 . Put $X := \left(\bigcup_{L \in \mathcal{L}} L \right) \setminus (L_1 \cup L_2)$.

²This follows for instance from the Klein correspondence which maps reguli to conics and regular spreads to elliptic quadrics.

Lemma 3.1 *We have $|X| = q^2 - 1$.*

Proof. For every point $x \in \text{PG}(3, q) \setminus Q^+(3, q)$, there is a unique (secant) line through x meeting both L_1 and L_2 . It follows that any two distinct lines of \mathcal{L} cannot meet at a point of $\text{PG}(3, q) \setminus Q^+(3, q)$. As $|\mathcal{L}| = q + 1$, we have $|X| = (q + 1)(q - 1) = q^2 - 1$. ■

We now investigate when X is a \mathcal{T}_1 -blocking set of size $q^2 - 1$.

Lemma 3.2 *If X is a \mathcal{T}_1 -blocking set of size $q^2 - 1$, then any two lines of \mathcal{L} are disjoint.*

Proof. Suppose K_1 and K_2 are two distinct lines of \mathcal{L} sharing a common point x , necessarily contained in $L_1 \cup L_2$. Without loss of generality, we may suppose that $x \in L_1$. The plane $\langle x, L_2 \rangle$ is then a tangent plane. If $y \in L_2$ is the tangency point of this plane, then any outer tangent through y contained in $\langle x, L_2 \rangle$ meets K_1 and K_2 (and hence X) in two points, in contradiction with Lemma 1.1. ■

We take a reference system $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ for $\text{PG}(3, q)$ such that $Q^+(3, q)$ has equation $X_1X_4 = X_2X_3$, $L_1 = \text{PG}(\langle \bar{e}_1, \bar{e}_2 \rangle)$ and $L_2 = \text{PG}(\langle \bar{e}_3, \bar{e}_4 \rangle)$.

Now, take two distinct lines P and R of \mathcal{L} , and for every $i \in \{1, 2\}$, put $P \cap L_i = \{p_i\}$ and $R \cap L_i = \{r_i\}$. By Lemma 3.2, we may suppose that $p_1 \neq r_1$ and $p_2 \neq r_2$. Put

$$p_1 = (a, b, 0, 0), \quad p_2 = (0, 0, c, d), \quad r_1 = (\alpha, \beta, 0, 0), \quad r_2 = (0, 0, \gamma, \delta),$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{F}_q$. The fact that the lines $P = p_1p_2$ and $R = r_1r_2$ are not contained in $Q^+(3, q)$ implies that $ad - bc \neq 0$ and $\alpha\delta - \beta\gamma \neq 0$.

Now, let p be an arbitrary point of $P \setminus \{p_1, p_2\}$ and let r be an arbitrary point of $R \setminus \{r_1, r_2\}$. Then

$$p = (a, b, cs, ds), \quad r = (\alpha, \beta, \gamma t, \delta t),$$

for certain $s, t \in \mathbb{F}_q^*$. An arbitrary point of $pr \setminus \{r\}$ has the form

$$(a + \alpha x, b + \beta x, cs + \gamma tx, ds + \delta tx),$$

with $x \in \mathbb{F}_q$. The intersection of $pr \setminus \{r\}$ with $Q^+(3, q)$ is determined by the equation

$$(ad - bc)s + \left((\alpha\delta - \beta\gamma)t + (\alpha d - \beta c)s \right)x + (\alpha\delta - \beta\gamma)tx^2 = 0.$$

By Lemma 1.2, X will be a \mathcal{T}_1 -blocking set if and only if pr is not a tangent line for all possibilities for $P, R \in \mathcal{L}$ with $P \neq R$, $p \in P \setminus \{p_1, p_2\}$ and $r \in R \setminus \{r_1, r_2\}$. Now, the line pr is not a tangent line if and only if

$$\left((\alpha\delta - \beta\gamma)t + (\alpha d - \beta c)s \right)^2 - 4(ad - bc)(\alpha\delta - \beta\gamma)st \neq 0,$$

that is,

$$(ad - bc)^2t^2 + (\alpha d - \beta c)^2s^2 + 2(ad - bc)(\alpha\delta - \beta\gamma)st - 4(ad - bc)(\alpha\delta - \beta\gamma)st \neq 0. \quad (*)$$

The fact that $(*)$ holds for all $s, t \in \mathbb{F}_q^*$ implies that $a\delta - b\gamma = 0$ if and only if $\alpha d - \beta c = 0$, or equivalently, $p_2 r_1$ is a line of $Q^+(3, q)$ if and only if $p_1 r_2$ is a line of $Q^+(3, q)$. So, we may suppose that we have chosen the reference system in such a way that $\text{PG}(\langle \bar{e}_1, \bar{e}_4 \rangle)$ and $\text{PG}(\langle \bar{e}_2, \bar{e}_3 \rangle)$ are lines of \mathcal{L} .

If $a\delta - b\gamma = \alpha d - \beta c = 0$, then $(*)$ holds for all $s, t \in \mathbb{F}_q^*$. Suppose therefore that $a\delta - b\gamma \neq 0 \neq \alpha d - \beta c$. The fact that $(*)$ holds for all $s, t \in \mathbb{F}_q^*$ is equivalent with the fact that

$$\Delta := \left((a\delta - b\gamma)(\alpha d - \beta c) - 2(ad - bc)(\alpha\delta - \beta\gamma) \right)^2 - (a\delta - b\gamma)^2(\alpha d - \beta c)^2$$

is a nonsquare. One computes that

$$\begin{aligned} \Delta &= 4(ad - bc)(\alpha\delta - \beta\gamma) \left((ad - bc)(\alpha\delta - \beta\gamma) - (a\delta - b\gamma)(\alpha d - \beta c) \right) \\ &= 4(ad - bc)(\alpha\delta - \beta\gamma)(a\beta - b\alpha)(c\delta - d\gamma). \end{aligned}$$

The affine coordinates of p_1, p_2, r_1 and r_2 are as follows:

$$x = \frac{b}{a} \in \mathbb{F}_q \cup \{\infty\}, \quad x' = \frac{d}{c} \in \mathbb{F}_q \cup \{\infty\}, \quad y = \frac{\beta}{\alpha} \in \mathbb{F}_q \cup \{\infty\}, \quad y' = \frac{\delta}{\gamma} \in \mathbb{F}_q \cup \{\infty\}.$$

Since $p_1 \neq r_1$ and $p_2 \neq r_2$, we have $x \neq y$ and $x' \neq y'$. The facts that $ad - bc \neq 0$ and $\alpha\delta - \beta\gamma \neq 0$ imply that $x \neq x'$ and $y \neq y'$. The facts that $a\delta - b\gamma \neq 0 \neq \alpha d - \beta c$ imply that $x \neq y'$ and $y \neq x'$. So, x, x', y and y' are mutually distinct. Since $\text{PG}(\langle \bar{e}_1, \bar{e}_4 \rangle)$ and $\text{PG}(\langle \bar{e}_2, \bar{e}_3 \rangle)$ are lines of \mathcal{L} , we have $0' = \infty$ and $\infty' = 0$.

If $a = 0$, then $x = \infty, x' = 0$ and $d = 0$. So, $y, y' \notin \{0, \infty\}$ and the fact that $\frac{\Delta}{4} = -bc(\alpha\delta - \beta\gamma)(-b\alpha)(c\delta)$ is a nonsquare implies that $(y' - y)y'$ is a nonsquare.

If $c = 0$, then $x' = \infty, x = 0$ and $b = 0$. So, $y, y' \notin \{0, \infty\}$ and the fact that $\frac{\Delta}{4} = ad(\alpha\delta - \beta\gamma)a\beta(-d\gamma)$ is a nonsquare implies that $(y - y')y$ is a nonsquare.

By supposing that one of α, γ is zero, we find by symmetry that $x, x' \notin \{0, \infty\}$ and that also $(x' - x)x'$ and $(x - x')x$ are nonsquares.

If a, c, α, γ are distinct from 0, then $x, x', y, y' \notin \{0, \infty\}$ and the fact that Δ is a nonsquare implies that $(x' - x)(y' - y)(y - x)(y' - x')$ is a nonsquare.

As $0' = \infty$ and $\infty' = 0$, Lemma 3.2 implies that the map $x \mapsto x'$ defines a permutation σ of the set \mathbb{F}_q^* . The fact that $a\delta - b\gamma = 0$ if and only if $\alpha d - \beta c = 0$ implies that $x = y' \Leftrightarrow y = x'$ for $x, y \in \mathbb{F}_q^*$, i.e. that $\sigma^2 = 1$. The fact that $x \neq x'$ implies that σ has no fixed points.

Notice also that the set \mathcal{L} of lines is completely determined by the map $\sigma : \mathbb{F}_q^* \rightarrow \mathbb{F}_q^*$. Summarizing, we thus have:

Proposition 3.3 *X is a \mathcal{T}_1 -blocking set of size $q^2 - 1$ if and only if the permutation σ of \mathbb{F}_q^* satisfies the following properties:*

(P1) σ has no fixed points;

(P2) $\sigma^2 = 1$;

(P3) $(x - x^\sigma)x$ is a nonsquare for all $x \in \mathbb{F}_q^*$;

(P4) $(x^\sigma - x)(y^\sigma - y)(y - x)(y^\sigma - x^\sigma)$ is a nonsquare for all $x, y \in \mathbb{F}_q^*$ with $y \notin \{x, x^\sigma\}$.

Note that the property that $(x^\sigma - x)x^\sigma$ is a nonsquare for all $x \in \mathbb{F}_q^*$ is implied by properties (P2) and (P3). Indeed, it is obtained from (P3) by replacing x by x^σ and using the fact that $\sigma^2 = 1$. In fact, also Property (P1) is implied by (P3), but we have decided to keep it for clarity reasons. The facts that $(x - x^\sigma)x$ and $(x^\sigma - x)x^\sigma$ are nonsquares imply that $-xx^\sigma$ itself should be a square. So, we have:

Proposition 3.4 *If a permutation σ of \mathbb{F}_q^* satisfies the properties (P2) and (P3) of Proposition 3.3, then $(x^\sigma - x)x^\sigma$ is a nonsquare and $-xx^\sigma$ is a square for every $x \in \mathbb{F}_q^*$.*

A \mathcal{T}_1 -blocking set of size $q^2 - 1$ that is obtained from a permutation σ of \mathbb{F}_q^* that satisfies the properties (P1), (P2), (P3) and (P4) has the property that there are $q + 1$ secant lines meeting the blocking set in exactly $q - 1$ points. In our computer classification of all \mathcal{T}_1 -blocking sets of size $q^2 - 1$ with $q \leq 13$, there are four blocking sets that have this property, namely B_1, B_4, B_6 and B_9 (see the table in Section 1). For each $q \in \{3, 5, 7, 11\}$, we now prove that there exists a permutation σ satisfying (P1), (P2), (P3) and (P4). This then shows that the blocking sets B_1, B_4, B_6 and B_9 belong to the family of blocking sets under consideration in this section.

Proposition 3.5 *Let τ be the permutation of \mathbb{F}_q^* mapping $x \in \mathbb{F}_q^*$ to x^{-1} . If σ is a permutation of \mathbb{F}_q^* satisfying the properties (P1), (P2), (P3) and (P4), then $\tau^{-1}\sigma\tau$ also satisfies the same properties. Moreover, the associated \mathcal{T}_1 -blocking sets of size $q^2 - 1$ are isomorphic.*

Proof. The verification that $\tau^{-1}\sigma\tau$ satisfies the properties (P1), (P2), (P3) and (P4) is straightforward, but also follows from the discussion below.

Let V be the 4-dimensional vector space over \mathbb{F}_q for which $\text{PG}(3, q) = \text{PG}(V)$. The linear map of V mapping $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ to $(\bar{e}_2, \bar{e}_1, \bar{e}_4, \bar{e}_3)$ defines an automorphism of $\text{PG}(3, q)$ stabilizing $Q^+(3, q)$, L_1 and L_2 . The set \mathcal{L} is transformed into another set \mathcal{L}' of $q + 1$ secant lines. The affine coordinates of the images of points of $L_1 \cup L_2$ are the multiplicative inverses of the original affine coordinates. The involution of \mathbb{F}_q^* corresponding to \mathcal{L}' must thus be the map $\tau^{-1}\sigma\tau$. ■

Proposition 3.6 *Let σ be a permutation of \mathbb{F}_q^* satisfying the properties (P1), (P2), (P3) and (P4). The following hold:*

- For $q = 3$, there is only one possibility for σ , namely $(1, 2)$.
- For $q = 5$, there is only one possibility for σ , namely $(1, 4)(2, 3)$.
- For $q = 7$, there are two possibilities for σ , namely $(1, 3)(2, 6)(4, 5)$ and $(1, 5)(4, 6)(2, 3)$. They are conjugate under the permutation τ .

- For $q = 11$, there are two possibilities for σ , namely $(1, 2)(4, 8)(3, 6)(5, 10)(9, 7)$ and $(1, 6)(4, 2)(3, 7)(5, 8)(9, 10)$. They are conjugate under the permutation τ .

Proof. We only work out the case $q = 11$. The other (easier) cases can be treated with a similar reasoning. If $q = 11$, then -1 is a nonsquare. Since xx^σ and $(x - x^\sigma)x$ are nonsquares for all $x \in \mathbb{F}_{11}^*$, we have that $1^\sigma \in \{2, 6, 10\}$, $3^\sigma \in \{6, 7, 8\}$, $4^\sigma \in \{2, 7, 8\}$, $5^\sigma \in \{6, 8, 10\}$ and $9^\sigma \in \{2, 7, 10\}$. This leads to the following 13 possibilities for the involution σ :

$(1, 2)(4, 7)(9, 10)(3, 8)(5, 6)$; $(1, 2)(4, 7)(9, 10)(3, 6)(5, 8)$; $(1, 2)(4, 8)(3, 6)(5, 10)(9, 7)$;
 $(1, 2)(4, 8)(3, 7)(5, 6)(9, 10)$; $(1, 6)(4, 2)(3, 7)(5, 8)(9, 10)$; $(1, 6)(4, 2)(3, 8)(5, 10)(9, 7)$;
 $(1, 6)(9, 2)(5, 10)(3, 7)(4, 8)$; $(1, 6)(9, 2)(5, 10)(3, 8)(4, 7)$; $(1, 10)(4, 2)(9, 7)(3, 6)(5, 8)$;
 $(1, 10)(4, 2)(9, 7)(3, 8)(5, 6)$; $(1, 10)(9, 2)(4, 7)(3, 6)(5, 8)$; $(1, 10)(9, 2)(4, 7)(3, 8)(5, 6)$;
 $(1, 10)(9, 2)(4, 8)(5, 6)(3, 7)$.

Only the involutions $(1, 2)(4, 8)(3, 6)(5, 10)(9, 7)$ and $(1, 6)(4, 2)(3, 7)(5, 8)(9, 10)$ satisfy the required properties. ■

We have verified the following with GAP and Sage, see [7].

Proposition 3.7 *If $q \leq 25$ is an odd prime power, and σ is a permutation of \mathbb{F}_q^* satisfying the properties (P1), (P2), (P3) and (P4), then $q \in \{3, 5, 7, 11\}$.*

Proposition 3.8 *Suppose $q \equiv 3 \pmod{4}$. Let $\alpha, \beta \in \mathbb{F}_q^*$ and suppose $\sigma : \mathbb{F}_q^* \rightarrow \mathbb{F}_q^*$ is defined as follows: $x^\sigma = \alpha x$ if $x \in \mathbb{F}_q^*$ is a square and $x^\sigma = \beta x$ if $x \in \mathbb{F}_q^*$ is a nonsquare. Then σ satisfies the properties (P1), (P2), (P3) and (P4) if and only if*

- (i) $\beta = \alpha^{-1}$;
- (ii) α is a nonsquare;
- (iii) $\alpha - 1$ is a square;
- (iv) for every nonsquare $y \in \mathbb{F}_q^*$ distinct from α , $(y - 1)(y - \alpha^2)$ is a nonsquare.

Proof. As $q \equiv 3 \pmod{4}$, we know that -1 is not a square. As $1^\sigma = \alpha$ and $-xx^\sigma$ is a square for all $x \in \mathbb{F}_q^*$, we see that α is a nonsquare. Property (P1) implies that $\alpha \neq 1 \neq \beta$. Property (P2) implies that $\beta = \alpha^{-1}$ (evaluate σ^2 at 1). Property (P3) applied to the case where x is a square yields that $(1 - \alpha)x^2$ is a nonsquare, that is, $\alpha - 1$ is a square. Property (P3) applied to the case where x is a nonsquare yields that $x^2(1 - \frac{1}{\alpha})$ is a nonsquare, which is certainly the case since $\alpha - 1$ is a square and α is a nonsquare. So, properties (P1), (P2) and (P3) hold if and only if $\beta = \alpha^{-1}$, α is a nonsquare and $\alpha - 1$ is a square.

We now determine when property (P4) holds, under the assumption that (i), (ii) and (iii) already hold.

If x and y are distinct nonzero squares, then $f(x, y) := (x^\sigma - x)(y^\sigma - y)(y - x)(y^\sigma - x^\sigma) = (\alpha - 1)^2 xy \alpha (y - x)^2$ is a nonsquare.

If x and y are distinct nonsquares, then $f(x, y) = (\frac{1}{\alpha} - 1)^2 xy \frac{1}{\alpha} (y - x)^2$ is a nonsquare.

As $f(x, y) = f(y, x)$, the only remaining case to examine is the case where x is a square and y is a nonsquare. In this case, $f(x, y) = (\alpha - 1)(\frac{1}{\alpha} - 1)xyx^2(\frac{y}{x} - 1)(\frac{y}{x} \cdot \frac{1}{\alpha} - \alpha) = -\frac{1}{\alpha^2}(\alpha - 1)^2 xyx^2(\frac{y}{x} - 1)(\frac{y}{x} - \alpha^2)$. So, $f(x, y)$ is a nonsquare if and only if $(\frac{y}{x} - 1)(\frac{y}{x} - \alpha^2)$ is a nonsquare. As $y \neq x^\sigma = \alpha x$, $\frac{y}{x}$ can reach all nonsquares distinct from α . So, besides (i), (ii) and (iii), we should also require that (iv) holds. ■

By Proposition 3.6, we know that if $q \in \{3, 7, 11\}$, then all known examples of permutations σ satisfying the properties (P1), (P2), (P3) and (P4) are obtained as in Proposition 3.8, with $\alpha = 2$ if $q = 3$, $\alpha \in \{3, 5\}$ if $q = 7$ and $\alpha \in \{2, 6\}$ if $q = 11$. This led us to the question whether more examples of elements $\alpha \in \mathbb{F}_q^*$ with q congruent to 3 modulo 4 exist that satisfy the conditions (ii), (iii) and (iv) of Proposition 3.8. A computer computation however showed the following, see [7].

Proposition 3.9 *If $q \leq 5000$ is a prime power which is congruent to 3 modulo 4 and $\alpha \in \mathbb{F}_q^*$ satisfies the conditions (ii), (iii) and (iv) of Proposition 3.8, then $q \in \{3, 7, 11\}$. Moreover, $\alpha = 2$ if $q = 3$, $\alpha \in \{3, 5\}$ if $q = 7$ and $\alpha \in \{2, 6\}$ if $q = 11$.*

Note that if (α, β) satisfies the conditions (i), (ii), (iii) and (iv) of Proposition 3.8, then also $(\alpha^{-1}, \beta^{-1})$ satisfies these conditions.

4 A third family of \mathcal{T}_1 -blocking sets

Let $Q^+(3, q)$ with q odd be a hyperbolic quadric in $\text{PG}(3, q)$. Let \mathcal{L}_1 and \mathcal{L}_2 denote the two reguli of $\text{PG}(3, q)$ defined by $Q^+(3, q)$. Let κ be the Klein correspondence between the set of lines of $\text{PG}(3, q)$ and the Klein quadric $Q^+(5, q)$ of $\text{PG}(5, q)$. Let ζ be the orthogonal polarity of $\text{PG}(5, q)$ associated with $Q^+(5, q)$. For every $i \in \{1, 2\}$, $\mathcal{C}_i := \kappa(\mathcal{L}_i)$ is then a nondegenerate conic of the plane $\pi_i := \langle \mathcal{C}_i \rangle$ of $\text{PG}(5, q)$, see e.g. [10, Section 15.4]. The lines of \mathcal{L}_2 are precisely those lines that meet each line of \mathcal{L}_1 , implying that $\pi_2 = \pi_1^\zeta$. Let L_1 and L_2 be two disjoint lines of $\text{PG}(3, q)$ which themselves are also disjoint from $Q^+(3, q)$. Put $p_i := \kappa(L_i)$, $i \in \{1, 2\}$. Then p_1 and p_2 do not belong to $\mathcal{C}_1 \cup \mathcal{C}_2$ (and hence also not to $\pi_1 \cup \pi_2$) and are not $Q^+(5, q)$ -collinear. Suppose that the following conditions hold:

- (a) $\Pi := \langle p_1, p_2, \pi_1 \rangle$ is a 3-dimensional subspace of $\text{PG}(5, q)$ meeting $Q^+(5, q)$ in an elliptic quadric $Q^-(3, q)$;
- (b) if p_3 is the unique point in $p_1 p_2 \cap \pi_1$, then p_3 is an internal point to the conic \mathcal{C}_1 .

Note that the 3-dimensional space $(p_1 p_2)^\zeta$ intersects $Q^+(5, q)$ in a $(q+1) \times (q+1)$ -grid \mathcal{G} .

Now, let K_1, K_2, \dots, K_{q+1} be a collection of $q+1$ mutually disjoint lines meeting L_1 and L_2 , and put $k_i := \kappa(K_i)$, $i \in \{1, 2, \dots, q+1\}$.

Lemma 4.1 *The set $\{k_1, k_2, \dots, k_{q+1}\}$ is an ovoid of \mathcal{G} . Conversely, every ovoid of \mathcal{G} is the image under the Klein correspondence of a set of $q+1$ mutually disjoint lines meeting L_1 and L_2 .*

Proof. Since K_i with $i \in \{1, 2, \dots, q+1\}$ meets L_1 and L_2 , the point $k_i = \kappa(K_i)$ is $Q^+(5, q)$ -collinear with both $p_1 = \kappa(L_1)$ and $p_2 = \kappa(L_2)$, that is, k_i is a point of \mathcal{G} . Since no two lines of the collection K_1, K_2, \dots, K_{q+1} meet, no two points of the collection k_1, k_2, \dots, k_{q+1} are $Q^+(5, q)$ -collinear, and so $\{k_1, k_2, \dots, k_{q+1}\}$ is an ovoid of \mathcal{G} .

Conversely, if O is an ovoid of \mathcal{G} , then $\kappa^{-1}(O)$ is a set of $q+1$ lines meeting L_1 and L_2 (as $O \subseteq (p_1 p_2)^\zeta$). As no two points of O are $Q^+(5, q)$ -collinear, these $q+1$ lines are mutually disjoint. ■

Let P denote the set of all points p contained in a (unique) set $u_1 u_2 \setminus \{u_1, u_2\}$, where $u_1 \in \mathcal{C}_1$ and $u_2 \in \mathcal{C}_2$.

Lemma 4.2 *The set P is precisely the image under the Klein correspondence of the set of outer tangents.*

Proof. For every incident point-plane pair (p, π) of $\text{PG}(3, q)$, let $\mathcal{L}(p, \pi)$ denote the set of lines of π through p . The images of these sets under the Klein correspondence are precisely the lines of $Q^+(5, q)$. The outer tangents are precisely the lines contained in a set $\mathcal{L}(p, \pi) \setminus \{M_1, M_2\}$, where $p = M_1 \cap M_2$ and $\pi = \langle M_1, M_2 \rangle$ for some $M_1 \in \mathcal{L}_1$ and $M_2 \in \mathcal{L}_2$. Using the Klein correspondence κ , we see that the image of the set of outer tangents under κ coincides with the set P . ■

Lemma 4.3 *No point of P can be $Q^+(5, q)$ -collinear with both p_1 and p_2 .*

Proof. Suppose $p \in P$ is $Q^+(5, q)$ -collinear with both p_1 and p_2 . Also assume that $p \in u_1 u_2 \setminus \{u_1, u_2\}$, where $u_1 \in \mathcal{C}_1$ and $u_2 \in \mathcal{C}_2$. The planes $\langle p_1, p_2, u_1 \rangle$ and π_1 of the 3-space Π meet in a line U . As $p_3 \in U$ is an internal point to \mathcal{C}_1 , this line U contains besides u_1 still one other point u'_1 of \mathcal{C}_1 . The point p is $Q^+(5, q)$ -collinear with p_1, p_2, u_1 and hence with every point of the set $\langle p_1, p_2, u_1 \rangle \cap Q^+(5, q)$. In particular, p is $Q^+(5, q)$ -collinear with u'_1 . Since also $u_2 \in \mathcal{C}_2$ is $Q^+(5, q)$ -collinear with $u'_1 \in \mathcal{C}_1$, we have that $u_1 \in pu_2$ is $Q^+(5, q)$ -collinear with u'_1 , a contradiction. ■

Proposition 4.4 *The set $(K_1 \cup K_2 \cup \dots \cup K_{q+1}) \setminus (L_1 \cup L_2)$ is a \mathcal{T}_1 -blocking set of size $q^2 - 1$ if and only if the following holds for every point $p \in P$:*

The number of points in the collection k_1, k_2, \dots, k_{q+1} that are $Q^+(5, q)$ -collinear with p is equal to 1 if p is $Q^+(5, q)$ -collinear with neither of p_1, p_2 and equal to 2 if p is $Q^+(5, q)$ -collinear with (precisely) one of p_1, p_2 .

Proof. By Lemmas 4.2 and 4.3, an outer tangent can never meet both L_1 and L_2 . In particular, none of the lines K_1, K_2, \dots, K_{q+1} is an outer tangent. For the set $(K_1 \cup K_2 \cup \dots \cup K_{q+1}) \setminus (L_1 \cup L_2)$ to be a \mathcal{T}_1 -blocking set (of size $q^2 - 1$), the following should hold for an arbitrary outer tangent line U .

The number of $i \in \{1, 2, \dots, q+1\}$ such that U and K_i meet is equal to 1 if U is disjoint from $L_1 \cup L_2$, and equal to 2 if U meets (precisely) one of L_1, L_2 .

Translating this condition via the Klein correspondence and taking into account Lemma 4.2, we see that $(K_1 \cup K_2 \cup \dots \cup K_{q+1}) \setminus (L_1 \cup L_2)$ is a \mathcal{T}_1 -blocking set if and only if the condition mentioned in the proposition is satisfied. \blacksquare

A \mathcal{T}_1 -blocking set of size $q^2 - 1$ that is obtained as described in Proposition 4.4 must have (at least) $q+1$ external lines, each of which meets the blocking set in $q-1$ points. In our computer classification of all \mathcal{T}_1 -blocking sets of size $q^2 - 1$ with $q \leq 13$, there are only three blocking sets that have this property, namely B_1, B_3 and B_5 (see the table in Section 1). Using the criterion mentioned in Proposition 4.4, we have constructed (with the aid of a computer) a \mathcal{T}_1 -blocking set for each $q \in \{3, 5, 7\}$. This then shows that the blocking sets B_1, B_3 and B_5 indeed belong to the family of \mathcal{T}_1 -blocking sets given by Proposition 4.4. Moreover, they are the unique members of this family with q respectively equal to 3, 5 and 7.

In order to use the criterion mentioned in Proposition 4.4, we shall introduce coordinates. We can choose a reference system in $\text{PG}(5, q)$ such that $p_1 = (1, 0, 0, 0, 0, 0)$, $p_2 = (0, 1, 0, 0, 0, 0)$ and the Klein quadric $Q^+(5, q)$ has equation $X_1X_2 + X_3X_4 + X_5X_6 = 0$. The grid \mathcal{G} determined by the points of $(p_1p_2)^\zeta \cap Q^+(5, q)$ is then described by the equations $X_1 = X_2 = 0, X_3X_4 + X_5X_6 = 0$. The points of \mathcal{G} are

- $M(\alpha, \beta) := (0, 0, 1, -\alpha\beta, \alpha, \beta), \alpha, \beta \in \mathbb{F}_q;$
- $M(\alpha, \infty) := (0, 0, 0, -\alpha, 0, 1), \alpha \in \mathbb{F}_q;$
- $M(\infty, \beta) := (0, 0, 0, -\beta, 1, 0), \beta \in \mathbb{F}_q;$
- $M(\infty, \infty) := (0, 0, 0, 1, 0, 0).$

The lines of \mathcal{G} are

- $K_\infty := \{(0, 0, 0, 1, 0, 0)\} \cup \{(0, 0, 0, -\beta, 1, 0) \mid \beta \in \mathbb{F}_q\};$
- $K_\alpha := \{(0, 0, 0, -\alpha, 0, 1)\} \cup \{(0, 0, 1, -\alpha\beta, \alpha, \beta) \mid \beta \in \mathbb{F}_q\}, \alpha \in \mathbb{F}_q;$
- $L_\infty := \{(0, 0, 0, 1, 0, 0)\} \cup \{(0, 0, 0, -\alpha, 0, 1) \mid \alpha \in \mathbb{F}_q\};$
- $L_\beta := \{(0, 0, 0, -\beta, 1, 0)\} \cup \{(0, 0, 1, -\alpha\beta, \alpha, \beta) \mid \alpha \in \mathbb{F}_q\}, \beta \in \mathbb{F}_q.$

Then $K_x \cap L_y = M(x, y)$ for all $x, y \in \overline{\mathbb{F}_q} := \mathbb{F}_q \cup \{\infty\}$. There is a one-to-one correspondence between the permutations τ of $\overline{\mathbb{F}_q}$ and the ovoids O_τ of \mathcal{G} , where

$$O_\tau := \{M(x, x^\tau) \mid x \in \overline{\mathbb{F}_q}\}.$$

We suppose that Π is the subspace of co-dimension 2 of $\text{PG}(5, q)$ having equations $X_4 = X_3$ and $X_6 = -\alpha X_5$, with α a given nonsquare. Then $\Pi \cap Q^+(5, q)$ is an elliptic quadric $Q^-(3, q)$ with equations

$$X_4 = X_3, \quad X_6 = -\alpha X_5, \quad X_1X_2 + X_3^2 - \alpha X_5^2 = 0.$$

Note that $p_1, p_2 \in Q^-(3, q)$.

Now, consider a plane $\pi_1 \subseteq \Pi$ with equation $X_1 + c_2X_2 + c_3X_3 + c_5X_5 = 0$, where $c_2 \neq 0$. Then p_1 and p_2 do not belong to π_1 . Put $\mathcal{C}_1 := \pi_1 \cap Q^+(5, q)$.

Lemma 4.5 \mathcal{C}_1 is a nondegenerate conic of π_1 if and only if $\alpha(4c_2 + c_3^2) \neq c_5^2$.

Proof. Replacing X_1 by $-c_2X_2 - c_3X_3 - c_5X_5$ in the quadratic equation $X_1X_2 + X_3^2 - \alpha X_5^2 = 0$, we find $-c_2X_2^2 - c_3X_2X_3 - c_5X_2X_5 + X_3^2 - \alpha X_5^2 = 0$. This determines a nondegenerate conic if and only if

$$\begin{vmatrix} -c_2 & -\frac{c_3}{2} & -\frac{c_5}{2} \\ -\frac{c_3}{2} & 1 & 0 \\ -\frac{c_5}{2} & 0 & -\alpha \end{vmatrix} \neq 0,$$

that is, $\alpha(c_2 + \frac{c_3^2}{4}) - \frac{c_5^2}{4} \neq 0$. ■

Now, suppose \mathcal{C}_1 is a nondegenerate conic of π_1 . Then $\mathcal{C}_2 := \mathcal{C}_1^\zeta \cap Q^+(5, q)$ is a nondegenerate conic of $\pi_2 := \pi_1^\zeta$ as well. Put $p_3 := p_1p_2 \cap \pi_1 = (-c_2, 1, 0, 0, 0, 0)$.

Lemma 4.6 The point $p_3 \in \pi_1$ is an internal point to the conic \mathcal{C}_1 if and only if $c_2(\alpha c_3^2 - c_5^2 + 4\alpha c_2)$ is a nonsquare in \mathbb{F}_q .

Proof. In order for the point p_3 to be internal, we need to show that the line $p_3^\zeta \cap \pi_1$ has no points in common with $\mathcal{C}_1 \subset \pi_1$. The hyperplane p_3^ζ has equation $X_1 = c_2X_2$. So, if $p = (X_1, X_2, X_3, X_4, X_5, X_6)$ belongs to $\mathcal{C}_1 \cap p_3^\zeta$, then the conditions $X_1 = c_2X_2$ and $X_1 + c_2X_2 + c_3X_3 + c_5X_5 = 0$ imply that

$$X_1 = -\frac{c_3X_3 + c_5X_5}{2}, \quad X_2 = -\frac{c_3X_3 + c_5X_5}{2c_2},$$

and the condition $X_1X_2 + X_3^2 - \alpha X_5^2 = 0$ implies that

$$\frac{(c_3X_3 + c_5X_5)^2}{4c_2} + X_3^2 - \alpha X_5^2 = 0,$$

that is,

$$(c_3^2 + 4c_2)X_3^2 + (c_5^2 - 4\alpha c_2)X_5^2 + 2c_3c_5X_3X_5 = 0.$$

For the point p_3 to be internal, the latter equation should only have $(X_3, X_5) = (0, 0)$ as solution. So, p_3 is an internal point if and only if $(c_3c_5)^2 - (c_3^2 + 4c_2) \cdot (c_5^2 - 4\alpha c_2)$ is a nonsquare, that is, $c_2(\alpha c_3^2 - c_5^2 + 4\alpha c_2)$ is a nonsquare in \mathbb{F}_q . ■

Remark. The conditions mentioned in Lemmas 4.5 and 4.6 are always satisfied when $c_3 = c_5 = 0$. In this case, the line p_1p_2 not only meets π_1 in an internal point of \mathcal{C}_1 , but also π_2 in an internal point of \mathcal{C}_2 . This follows from a similar reasoning as in the proof of Lemma 4.6 using the fact that the plane π_2 is described by the equations $X_1 = c_2X_2$, $X_4 = -X_3$, $X_6 = \alpha X_5$.

We have written a computer program in GAP (see [7]) to verify for each ovoid $\{k_1, k_2, \dots, k_{q+1}\}$ of \mathcal{G} (determined by a permutation τ of $\overline{\mathbb{F}_q}$) whether the condition mentioned in Proposition 4.4 is satisfied. We have found solutions for $q \in \{3, 5, 7\}$. For each of these values of q , we mention one solution.

- If $q = 3$, then $\overline{\mathbb{F}}_q = \{\infty, 0, 1, 2\}$, $\alpha = 2$, $c_2 = 2$, $c_3 = 0$, $c_5 = 0$ and τ is trivial.
- If $q = 5$, then $\overline{\mathbb{F}}_q = \{\infty, 0, 1, 2, 3, 4\}$, $\alpha = 2$, $c_2 = 2$, $c_3 = 1$, $c_5 = 2$ and $\tau = (031)(24)$.
- If $q = 7$, then $\overline{\mathbb{F}}_q = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$, $\alpha = 3$, $c_2 = 3$, $c_3 = 0$, $c_5 = 0$ and $\tau = (2453)$.

In fact, for each $q \in \{3, 5, 7\}$, we have found several solutions for $(\alpha, c_2, c_3, c_5, \tau)$, but by our earlier remark, they should all give rise to isomorphic \mathcal{T}_1 -blocking sets. We have thus given explicit constructions for the blocking sets B_1 , B_3 and B_5 .

5 \mathcal{T}_1 -blocking sets of size 8 in $\text{PG}(3, 3)$

From Sections 2, 3 and 4, we know that there exist \mathcal{T}_1 -blocking sets of size 8 in $\text{PG}(3, 3)$. All these \mathcal{T}_1 -blocking sets are isomorphic as the following proposition shows.

Proposition 5.1 *Suppose $q = 3$. Then up to isomorphism there is a unique \mathcal{T}_1 -blocking set of size 8 in $\text{PG}(3, 3)$.*

Proof. Suppose B is such a set of points. By Lemma 1.1, $B \cap Q^+(3, 3) = \emptyset$ and every outer tangent contains a unique point of B . We prove a number of additional properties for such \mathcal{T}_1 -blocking sets.

Fact 1: *Every tangent plane to $Q^+(3, 3)$ contains precisely two points of B .*

Proof. Suppose π is a tangent plane with tangent point x . The two lines of π through x contained in $Q^+(3, 3)$ are disjoint from B , and the other two lines of π through x are outer tangents and so each contains a unique point of B . (qed)

Fact 2: *Every nontangent plane to $Q^+(3, 3)$ contains 2, 3, 4 or 5 points of B .*

Proof. Let π be a nontangent plane. Put $\pi \cap Q^+(3, 3) = \{x_1, x_2, x_3, x_4\}$ and let y_1, y_2, y_3 denote the three points of π that are internal to the conic $\pi \cap Q^+(3, 3)$. For every $i \in \{1, 2, 3, 4\}$, let L_i denote the unique tangent line through x_i contained in π . Let a denote the unique point of $L_1 \cap B$. Through a there are two tangents to the conic $\pi \cap Q^+(3, 3)$. Without loss of generality, we may suppose that the second tangent line through a is L_2 . The tangent line L_3 does not contain a and so there exists a unique point b on L_3 distinct from x_3 , $L_3 \cap L_1$ and $L_3 \cap L_2$. The points $L_3 \cap L_1$ and $L_3 \cap L_2$ are not contained in B , implying that $b \in B$. The two lines through b that are tangent to $\pi \cap Q^+(3, 3)$ are the lines L_3 and L_4 . So, the points of B contained in $L_1 \cup L_2 \cup L_3 \cup L_4$ are the points a and b . Every other point of B must be an internal point, and so equal to y_1, y_2 or y_3 . (qed)

Suppose L is a secant line to $Q^+(3, 3)$. Let π_1 and π_2 denote the two nontangent planes through L and denote by E_i , $i \in \{1, 2\}$, the set of three points of $K_i \setminus L$, where K_i is the unique external line of π_i intersecting L in a point that is external to the conic $\pi_i \cap Q^+(3, 3)$. Put $B_L := (L \setminus Q^+(3, 3)) \cup E_1 \cup E_2$. Then $|B_L| = 8$.

Fact 3: *Suppose $(L \setminus Q^+(3, 3)) \subseteq B$ for a certain secant line L . Then $B = B_L$.*

Proof. By Fact 1, each of the two tangent planes through L does not contain extra

points of B besides those of $L \setminus Q^+(3, 3)$. By Fact 2, each of the two secant planes through L contains at most 3 extra points of B besides those of $L \setminus Q^+(3, 3)$. Since $8 = |B| = |L \setminus Q^+(3, 3)| + 3 \cdot 2 = 2 + 6 = 8$, we necessarily have $B = B_L$. Indeed, suppose π is a nontangent plane through L , and x is the unique point of L that is external to the conic $\pi \cap Q^+(3, 3)$ of π . The two tangent lines through x contained in π cannot contain extra points of B (besides x). So, the unique external line through x contained in π must contain three extra points of B (besides x). (qed)

In the sequel we will assume that $L \setminus Q^+(3, 3)$ is not contained in B for every secant line L , and derive a contradiction.

Let $x \in B$. The point x is contained in four tangent lines, six secant lines and three external lines. None of the four tangent lines through x can contain extra points of B besides x . The same holds for the secant lines (by our assumption). Since $|B| = 8$, at least one of the external lines through x , say K , is completely contained in B . Let y be a point of $B \setminus K$. As K is an external line, the plane $\pi = \langle K, y \rangle$ is a nontangent plane. In π , there is a secant line S through y containing two points of B , namely y and $S \cap K$, and this is impossible.

In conclusion, we can say that $B = B_L$ for a certain secant line L . As the automorphism group of $\text{PG}(3, 3)$ stabilising $Q^+(3, 3)$ acts transitively on the set of secant lines, there is up to isomorphism at most one (and hence precisely one) set of 8 points that is a \mathcal{T}_1 -blocking set. ■

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Addresses:

Bart De Bruyn

Department of Mathematics, Ghent University
 Krijgslaan 281 (S25), B-9000 Gent, Belgium
 Email: Bart.DeBruyn@Ugent.be

Binod Kumar Sahoo

School of Mathematical Sciences

National Institute of Science Education and Research, Bhubaneswar (HBNI)

P.O. - Jatni, District- Khurda, Odisha - 752050, India

Email: bksahoo@niser.ac.in