# Characterizations of the $G_2(4)$ and $L_3(4)$ near octagons

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January 24, 2019

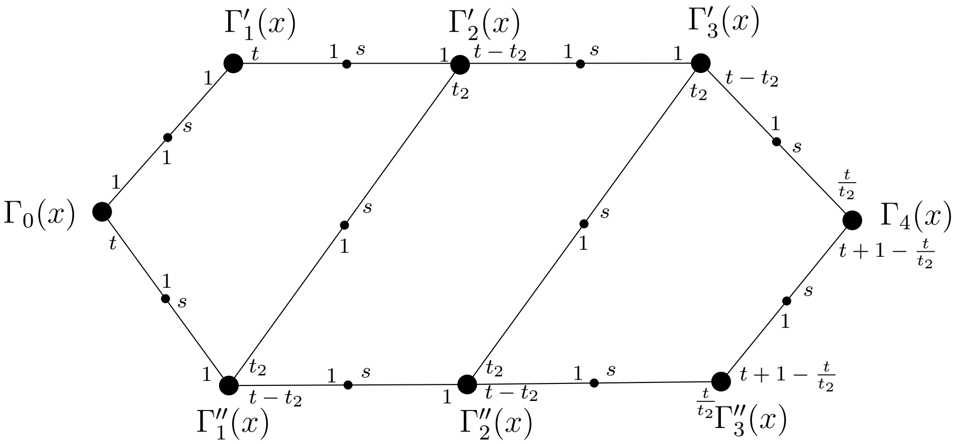
**Abstract**

A triple $(\mathcal{S}, S, \mathcal{Q})$ consisting of a near polygon $\mathcal{S}$, a line spread $S$ of $\mathcal{S}$ and a set $\mathcal{Q}$ of quads of $\mathcal{S}$ is called a *polygonal triple* if certain nice properties are satisfied, among which there is the requirement that the point-line geometry $\mathcal{S}'$ formed by the lines of $S$ and the quads of $\mathcal{Q}$ is itself also a near polygon. This paper addresses the problem of classifying all near polygons $\mathcal{S}$ that admit a polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ for which a given generalized polygon $\mathcal{S}'$ is the associated near polygon. We obtain several nonexistence results and show that the $G_2(4)$ and $L_3(4)$ near octagons are the unique near octagons that admit polygonal triples whose quads are isomorphic to the generalized quadrangle $W(2)$ and whose associated near polygons are respectively isomorphic to the dual split Cayley hexagon $H^D(4)$ and the unique generalized hexagon of order $(4, 1)$.

**Keywords:** near polygon, generalized polygon, polygonal triple, $G_2(4)$ near octagon, $L_3(4)$ near octagon
**MSC2010:** 05B25, 51E12, 51E25

## 1 Introduction

In [1] and [2], Anurag Bishnoi and the author constructed two new near octagons related to the respective simple groups $G_2(4)$ and $L_3(4)$. These $G_2(4)$ and $L_3(4)$ near octagons share many common properties. In fact, in [2], we described a family $\mathcal{F}$ of near octagons to which both belong. If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near octagon belonging to this family $\mathcal{F}$, then the structure of $\mathcal{S}$ with respect to any of its points $x$ can be described by a diagram



where $s$, $t$ and $t_2$ are positive integers such that $t_2 \mid t$. In the diagram, the edges denote lines and the big nodes denote points (belonging to the mentioned point sets). The number of lines through a given point meeting the various sets are mentioned around the big nodes, while the number of points on a given line contained in the various sets are mentioned around the small nodes. The set $L_x := \{x\} \cup \Gamma'_1(x)$ is a line through $x$, and the collection of all lines $L_x$, $x \in \mathcal{P}$, is a line spread $S$ of $\mathcal{S}$. Moreover, the point-line geometry $\mathcal{S}'$ whose points are the lines of $S$ and whose lines are all the quads of $\mathcal{S}$ is a generalized hexagon of order $(st_2, \frac{t}{t_2} - 1)$, if we take containment as incidence relation.

In the case of the $G_2(4)$ near octagon, we have $(s, t_2, t) = (2, 2, 10)$, all quads are isomorphic to $W(2)$ and the associated generalized hexagon $\mathcal{S}'$ is isomorphic to the dual split Cayley generalized hexagon $H^D(4)$. In the case of the $L_3(4)$ near octagon, we have $(s, t_2, t) = (2, 2, 4)$, all quads are isomorphic to $W(2)$ and the associated generalized hexagon $\mathcal{S}'$ is the unique generalized hexagon $GH(4, 1)$ of order $(4, 1)$.

The research of the present paper resulted from investigations whether it is possible to characterize the $G_2(4)$ and $L_3(4)$ near octagons as the unique members of the family $\mathcal{F}$ for which $(s, t_2) = (2, 2)$ and whose associated generalized hexagons are respectively isomorphic to $H^D(4)$ and $GH(4, 1)$. To that end, we developed some algorithms which in combination with computer computations allowed to verify this. The methods can be applied to other generalized hexagons as well, and our hope was that new near octagons could be found in this way. Unfortunately, this was not the case. Our computations indeed show that the $G_2(4)$ and $L_3(4)$ near octagons are the unique members of $\mathcal{F}$ with $(s, t_2) = (2, 2)$ whose associated generalized hexagons are isomorphic to $H^D(4)$ and $GH(4, 1)$, respectively. We also show that there are no members in $\mathcal{F}$ with $(s, t_2) = (2, 2)$ whose associated generalized hexagon is the split Cayley hexagon $H(4)$. In principle, the techniques can also be applied to other generalized hexagons, but we found the hexagons to be too big for our computer computations. However, we are able to exclude some generalized hexagons by means of a number of divisibility conditions involving the parameters $s, t, t_2$ that we derive by means of algebraic combinatorial techniques.

The family $\mathcal{F}$ of near octagons belongs to a larger class of near polygons, namely those that admit a so-called polygonal triple, and it will be more natural to derive our results in this more general context.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near $(2d+2)$-gon, $d \geq 1$, having a line spread $S$ and a family $\mathcal{Q}$ of quads for which the following hold:

(PT1) For every point $x$ of $\mathcal{S}$, the quads of $\mathcal{Q}$ through $x$ all contain the unique line $L_x$ of $S$ through $x$ and partition the set of lines through $x$ distinct from $L_x$.

(PT2) The point-line geometry $\mathcal{S}'$ with point set $S$, line set $\mathcal{Q}$ and natural incidence (i.e. containment) is a near $2d$-gon.

Then we call $(\mathcal{S}, S, \mathcal{Q})$ a *polygonal triple* with *associated near polygon* $\mathcal{S}'$. The elements of $\mathcal{Q}$ are called the *quads of the polygonal triple.* For polygonal triples, $\mathcal{Q}$ is uniquely determined by $\mathcal{S}$ and $S$ as its consists of all quads of $\mathcal{S}$ containing a line of $S$. Polygonal triples were introduced and studied in [4].

The polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ for which all lines of $\mathcal{S}$ are thin or for which all quads of $\mathcal{Q}$ are grids are easy to describe (see Section 2) and will therefore be called *trivial.* In Section 5, we develop algorithms to classify (with the aid of a computer) polygonal triples for which the associated near polygons are generalized polygons. This allows us to prove the following results in Section 6.

**Theorem 1.1** *Suppose* $(\mathcal{S}, S, \mathcal{Q})$ *is a polygonal triple with associated near polygon* $\mathcal{S}'$ *such that all quads of* $\mathcal{Q}$ *are isomorphic to* $W(2)$*. Then the following hold.*

- *If* $\mathcal{S}' \cong H^D(4)$*, then* $\mathcal{S}$ *is isomorphic to the* $G_2(4)$ *near octagon.*
- *If* $\mathcal{S}' \cong GH(4,1)$*, then* $\mathcal{S}$ *is isomorphic to the* $L_3(4)$ *near octagon.*
- $\mathcal{S}'$ *cannot be isomorphic to* $H(4)$*,* $T(4,64)$*,* $GO(4,1)$*,* $RT(4,2)$*,* $\mathcal{F}(H(4))$ *and* $\mathcal{F}(H^D(4))$*.*

In Theorem 1.1, the generalized polygons $H(4)$, $T(4,64)$, $GO(4,1)$, $RT(4,2)$, $\mathcal{F}(H(4))$ and $\mathcal{F}(H^D(4))$ are respectively isomorphic to the split Cayley generalized hexagon of order $(4,4)$, the dual twisted triality hexagon of order $(4,64)$, the unique generalized octagon of order $(4,1)$, the Ree-Tits octagon of order $(4,2)$, the flag geometry of $H(4)$ and the flag geometry of $H^D(4)$.

Theorem 1.1 covers precisely those cases where the quads of $\mathcal{Q}$ are isomorphic to $W(2)$ and $\mathcal{S}'$ is a known finite generalized $2d$-gon[1] with $d \geq 3$. Note that as a line spread of $W(2)$ contains five lines, the generalized polygon $\mathcal{S}'$ should have five points per line, and thus have order $(4,t)$ for some $t \in \mathbb{N} \setminus \{0\}$.

In Section 4, we show the nonexistence of certain polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ by computing the multiplicities of the eigenvalues of $\mathcal{S}$ and expressing that these are all nonnegative integers. These nonexistence results already cover certain of the cases mentioned in Theorem 1.1, namely the cases when $\mathcal{S}'$ is isomorphic to $T(4,64)$ or $GO(4,1)$.

We wish to note here that our characterization of the $G_2(4)$ near octagon is basically computer free. Indeed, our algorithms already imply (without additional computer computations) that there is at most one polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ (up to some obvious form of

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[1] The case $d = 2$ has been treated more generally in Theorem 5.5 of [4].

isomorphism) for which al quads are isomorphic to $W(2)$ and for which the corresponding generalized hexagon $\mathcal{S}'$ is isomorphic to $H^D(4)$. However, also in this case, as a verification of our methods, we have used our algorithms to reconstruct $\mathcal{S}$ from $\mathcal{S}' \cong H^D(4)$ and to check that $\mathcal{S}$ is indeed a near octagon with similar properties as the $G_2(4)$ near octagon.

## 2 Preliminaries

A connected partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near polygon* if for every point $x$ and every line $L$, there exists a unique point $\pi_L(x)$ on $L$ nearest to $x$. Here, distances are measured in the collinearity graph $\Gamma$. If $d$ is the diameter of $\Gamma$, then the near polygon is called a *near* $2d$*-gon*. Very often, we regard the lines of a near polygon as sets of points; incidence is then containment. A near polygon can be reconstructed from its collinearity graph, since the lines are the maximal cliques of this graph.

If $x$ and $y$ are two points of a near polygon $\mathcal{S}$, then we denote by $\mathrm{d}(x, y)$ the distance between $x$ and $y$ in the collinearity graph. If $x$ is a point of $\mathcal{S}$, then $\Gamma_i(x)$ with $i \in \mathbb{N}$ denotes the set of points at distance $i$ from $x$.

If $L_1$ and $L_2$ are two lines of a near $2d$-gon $\mathcal{S}$, then $\mathrm{d}(L_1, L_2)$ denotes the minimal distance between a point of $L_1$ and a point of $L_2$. If $\mathrm{d}(L_1, L_2) = d - 1$, then the lines $L_1$ and $L_2$ are called *opposite*. The lines $L_1$ and $L_2$ are called *parallel* if for every $i \in \{1, 2\}$ and every point $x_i \in L_i$, there exists a unique point $x_{3-i} \in L_{3-i}$ at distance $\mathrm{d}(L_1, L_2)$ from $x_i$. Two opposite lines are always parallel. A set $S$ of lines of $\mathcal{S}$ is called a *line spread* if every point is contained in a unique line of $S$.

A near $2d$-gon with $d \geq 2$ is called a *generalized* $2d$*-gon* if every point is incident with at least two lines and if for every two points $x$ and $y$ at distance $i \in \{1, 2, \dots, d-1\}$, there is a unique neighbour of $y$ at distance $i - 1$ from $x$. For the definition of the symplectic generalized quadrangle $W(q)$ (with $q$ a prime power), we refer to [8]. For the definitions of the generalized polygons $H^D(4)$, $H(4)$, $T(4, 64)$ and $RT(4, 2)$ mentioned in Theorem 1.1, we refer to [12]. If $\mathcal{S}$ is a point-line geometry, then with the *flag-geometry* $\mathcal{F}(\mathcal{S})$ of $\mathcal{S}$, we mean the geometry whose points are the flags of $\mathcal{S}$ (i.e. the unordered incident point-line pairs of $\mathcal{S}$) and whose lines are the points and lines of $\mathcal{S}$, with incidence being reverse containment. The flag-geometries of the projective plane of order 4 and the symplectic generalized quadrangle $W(4)$ are precisely the generalized polygons $GH(4, 1)$ and $GO(4, 1)$.

A set $Q$ of points of a near polygon $\mathcal{S}$ is called a *quad* if the following three properties hold:

- every line of $\mathcal{S}$ that has two points in $Q$ has all its points in $Q$;
- if $x$ and $y$ are two points of $Q$ at distance 2, then every common neighbour of $x$ and $y$ is also contained in $Q$;

- the point-line geometry $\widetilde{Q}$ defined on $Q$ by those lines that have all their points in $Q$ is a generalized quadrangle.

A quad $Q$ of $\mathcal{S}$ is *classical* if for every point $x$ of $\mathcal{S}$, there exists a unique point $\pi_Q(x) \in Q$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_Q(x)) + \mathrm{d}(\pi_Q(x),y)$ for every point $y \in Q$.

We recall from [4] some properties of polygonal triples which will be useful later. Suppose $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon $\mathcal{S}'$. Then any two lines $L_1$ and $L_2$ of $S$ are parallel, and the distance in $\mathcal{S}'$ between them is equal to $\mathrm{d}(L_1, L_2)$. Every line of $\mathcal{S}$ not belonging to $S$ is contained in a unique quad of $\mathcal{Q}$. Regarding the quads of $\mathcal{Q}$, the following additional properties hold. For every quad $Q \in \mathcal{Q}$, the set of lines of $S$ contained in $Q$ is a line spread of $\widetilde{Q}$. Every quad $Q \in \mathcal{Q}$ is classical in $\mathcal{S}$, and for every line $L \in S$, also $\pi_Q(L) := \{\pi_Q(x) \,|\, x \in L\}$ is a line belonging to $S$.

A near polygon is said to have *order* $(s,t)$ if every point is incident with precisely $t+1$ lines and if every line is incident with precisely $s+1$ points. A finite near polygon $\mathcal{S}$ is said to be *regular with parameters* $s, t, t_i$, $i \in \{0, 1, \ldots, d\}$, if $\mathcal{S}$ has order $(s,t)$ and if for every two points $x$ and $y$ at distance $i$, there are precisely $t_i + 1$ lines through $y$ containing a point at distance $i-1$ from $x$. We have $t_0 = -1$, $t_1 = 0$ and $t_d = t$. A finite generalized $2d$-gon of order $(s,t)$ is regular with parameters $t_i$ equal to 0, for every $i \in \{1, 2, \ldots, d-1\}$.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near polygon. Let $\overline{\Gamma}$ be the bipartite graph defined on the set $\overline{\mathcal{P}} := \mathcal{P} \times \{+,-\}$ by calling two distinct vertices $(x_1, \epsilon_1)$ and $(x_2, \epsilon_2)$ adjacent whenever $\mathrm{d}(x_1, x_2) \leq 1$ and $\epsilon_2 = -\epsilon_1$. As $\overline{\Gamma}$ is bipartite, it is the collinearity graph of a near polygon $\overline{\mathcal{S}}$. If we put $L_x := \{(x,+),(x,-)\}$ for every $x \in \mathcal{P}$, then the set $S := \{L_x \,|\, x \in \mathcal{P}\}$ is a line spread of $\mathcal{S}$ and the set $\mathcal{Q} := \{L \times \{+,-\} \,|\, L \in \mathcal{L}\}$ is a set of quads. By [5], $(\overline{\mathcal{S}}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon isomorphic to $\mathcal{S}$, and every near polygon with two points on each line admitting a polygonal triple is obtained in the above described way.

Suppose $\mathcal{S}$ is a near polygon and $s \in \mathbb{N} \setminus \{0\}$. By considering $s+1$ isomorphic copies of $\mathcal{S}$ and joining the corresponding points to form lines of size $s+1$, we obtain a new near polygon which we denote by $\mathcal{S} \times \mathbb{L}_{s+1}$. The set $S$ of all lines of $\mathcal{S} \times \mathbb{L}_{s+1}$ joining corresponding points is a line spread $S$ of $\mathcal{S} \times \mathbb{L}_{s+1}$. If $\mathcal{Q}$ is the set of all quads containing a line of $S$, then $(\mathcal{S} \times \mathbb{L}_{s+1}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is isomorphic to $\mathcal{S}$ and for which all quads of $\mathcal{Q}$ are grids. In fact, we are able to prove the following.

**Proposition 2.1** *Let $(\mathcal{S}, S, \mathcal{Q})$ be a polygonal triple for which every quad of $\mathcal{Q}$ is a grid. Then $\mathcal{S}$ is isomorphic to $\mathcal{S}' \times \mathbb{L}$, where $\mathcal{S}'$ is the near polygon associated with $(\mathcal{S}, S, \mathcal{Q})$ and $\mathbb{L}$ is a line.*

**Proof.** Consider a fixed line $L \in S$. For every point $x \in L$, let $A_x$ denote the set of all points $y$ for which $x$ is the unique point of $L$ nearest to $y$. The point set $\mathcal{P}$ of $\mathcal{S}$ is then the disjoint union $\bigcup_{x \in L} A_x$.

Suppose $K$ is a line of $\mathcal{S}$ not belonging to $S$. There is a unique quad $Q \in \mathcal{Q}$ containing $K$. This quad is classical in $\mathcal{S}$ and $L' := \pi_Q(L)$ is a line of $\widetilde{Q}$ belonging to $S$. The intersection $L' \cap K$ is a singleton $\{y\}$, and as $Q$ is classical we know that $K$ is contained in $A_x$, where $x \in L$ is the unique point of $L$ nearest to $y$.

Now, each line of $S$ intersects each $A_x$ in a unique point and each quad of $\mathcal{Q}$ intersects each $A_x$ in a line, showing that the subgeometry induced on each $A_x$, $x \in L$, is isomorphic to $\mathcal{S}'$. It is now also clear that $\mathcal{S} \cong \mathcal{S}' \times \mathbb{L}$, where $\mathbb{L}$ is any line of the same size as $L$. ■

A polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ is called *trivial* if every line of $\mathcal{S}$ is thin or if every quad of $\mathcal{Q}$ is a grid. By the above discussion, it is easy to describe all such polygonal triples.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a finite near octagon of order $(s,t)$ and $S$ is a line spread of $\mathcal{S}$. For every point $x \in \mathcal{P}$, let $L_x$ denote the unique line of $S$ containing $x$. We define the following sets of points of $\mathcal{S}$:

- For every point $x$ of $\mathcal{S}$, we define $\Gamma_1'(x) := L_x \setminus \{x\}$ and $\Gamma_1''(x) := \Gamma_1(x) \setminus \Gamma_1'(x)$.
- For every point $x$ of $\mathcal{S}$ and every $i \in \{2,3\}$, $\Gamma_i'(x)$ denotes the set of points of $\Gamma_i(x)$ that are collinear with a point of $\Gamma_{i-1}'(x)$, and we put $\Gamma_i''(x) := \Gamma_i(x) \setminus \Gamma_i'(x)$.

Suppose moreover that there exists a positive divisor $t_2 \neq t$ of $t$ such that the following hold for every point $x$ of $\mathcal{S}$:

(P1) Every point of $\Gamma_2'(x)$ is incident with $t_2$ lines meeting $\Gamma_1''(x)$.

(P2) Every point of $\Gamma_2''(x)$ is incident with a unique line meeting $\Gamma_1''(x)$.

(P3) Every point of $\Gamma_3'(x)$ is incident with $t_2$ lines meeting $\Gamma_2''(x)$.

(P4) Every point of $\Gamma_3''(x)$ is incident with $\frac{t}{t_2}$ lines meeting $\Gamma_2''(x)$.

If these properties hold, then we say that $(\mathcal{S}, S)$ is an *octagonal pair with parameters* $(s,t,t_2)$. The family $\mathcal{F}$ of near octagons introduced and studied in [2] are precisely the near octagons that admit an octagonal pair. The following was proved there.

**Proposition 2.2** *Suppose $(\mathcal{S}, S)$ is an octagonal pair with parameters $(s,t,t_2)$, $s \geq 2$, and denote by $\mathcal{Q}$ the set of quads of $\mathcal{S}$. Then $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple for which all quads have order $(s,t_2)$ and whose associated near polygon is a generalized hexagon of order $(st_2, \frac{t}{t_2}-1)$.*

In Section 3, we show that every polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ for which the associated near polygon is a finite generalized hexagon with an order and for which all quads of $\mathcal{Q}$ are finite with the same order arises from an octagonal pair.

As the symplectic generalized quadrangle $W(2)$ is the unique generalized quadrangle of order $(2,2)$, we thus see that classifying octagonal pairs with parameters $(s,t,t_2) = (2,t,2)$ is equivalent with classifying all polygonal triples whose quads are isomorphic to $W(2)$ and whose associated near polygons are generalized hexagons of order $(4, \frac{t}{2}-1)$. The classification results for the family $\mathcal{F}$ mentioned in Section 1 are thus consequences of Theorem 1.1.

# 3 On polygonal triples for which the associated near polygon is regular

Let $(\mathcal{S}, S, \mathcal{Q})$ be a polygonal triple for which the corresponding near polygon $\mathcal{S}'$ has diameter $d$ and is regular with parameters $s, t, t_i$, $i \in \{0, 1, \ldots, d\}$. Recall then that $t_0 = -1$, $t_1 = 0$, $t_d = t$ and that $\mathcal{S}$ has diameter $d+1$. Suppose every quad of $\mathcal{Q}$ has order $(\widetilde{s}, \widetilde{t_2})$. Then $\mathcal{S}$ has order $(\widetilde{s}, \widetilde{t_2}(t+1))$. As a line spread of a quad of $\mathcal{Q}$ contains $1 + \widetilde{s} \cdot \widetilde{t_2}$ lines, we have $s = \widetilde{s} \cdot \widetilde{t_2}$. For every point $u$ of $\mathcal{S}$, we denote by $L_u$ the unique line of $S$ containing $u$. For every point $x$ and every $i \in \{0, 1, \ldots, d+1\}$, we denote by $\Gamma''_i(x)$ the set of all points $y \in \Gamma_i(x)$ for which $y$ is the unique point of $L_y$ nearest to $x \in L_x$, and we define $\Gamma'_i(x) := \Gamma_i(x) \setminus \Gamma''_i(x)$. Then $\Gamma'_0(x) = \Gamma''_{d+1}(x) = \emptyset$ and $\Gamma'_1(x) = L_x \setminus \{x\}$. If $L$ is a line of $S$ at distance $i$ from $L_x$, then $L$ contains a unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.

**Lemma 3.1** *Let $x$ be a point of $\mathcal{S}$ and $i \in \{1, 2, \ldots, d\}$. Then no point of $\Gamma''_i(x)$ is collinear with a point of $\Gamma'_{i-1}(x)$.*

**Proof.** Suppose $y \in \Gamma''_i(x)$ is collinear with a point $z \in \Gamma'_{i-1}(x)$. As $y \in \Gamma''_i(x)$, we have $\mathrm{d}(L_x, L_y) = i$ and hence $\mathrm{d}(L_x, L_z) \geq i-1$. As $z \in \Gamma_{i-1}(x)$, the point $z$ must be the unique point of $L_z$ nearest to $x$, i.e. $z \in \Gamma''_{i-1}(x)$, in contradiction with the fact that $z \in \Gamma'_{i-1}(x)$. ■

**Lemma 3.2** *Let $x$ be a point of $\mathcal{S}$ and $i \in \{1, 2, \ldots, d\}$. Then no point of $\Gamma''_i(x)$ is collinear with a point of $\Gamma'_i(x)$.*

**Proof.** Suppose $y \in \Gamma''_i(x)$ is collinear with a point $z \in \Gamma'_i(x)$. Then $yz \notin S$ and so there exists a unique quad $Q \in \mathcal{Q}$ containing $yz$. The line $L_z \subseteq Q$ contains a point $u \in \Gamma''_{i-1}(x)$. The line $L_y \subseteq Q$ contains points of $\Gamma'_{i+1}(x)$. As $x$ is classical with respect to $Q$, $u$ is the unique point of $Q$ nearest to $x$ and $\mathrm{d}(u, y) = 1$, a contradiction, since $z$ is the only point of $yz$ collinear with $u$. ■

**Lemma 3.3** *Let $Q$ be a quad of $\mathcal{S}$. Put $i := d(x, Q)$. Let $u$ denote the unique point of $Q$ nearest to $x$. Then $u \in \Gamma''_i(x)$, $L_u \setminus \{u\} \subseteq \Gamma'_{i+1}(x)$, $u^\perp \setminus L_u \subseteq \Gamma''_{i+1}(x)$ and $\Gamma_2(u) \cap Q \subseteq \Gamma'_{i+2}(x)$.*

**Proof.** As $u$ is the unique point of $L_u \subseteq Q$ nearest to $x \in L_x$, we have $u \in \Gamma''_i(x)$, $L_u \setminus \{u\} \subseteq \Gamma'_{i+1}(x)$ and $\mathrm{d}(L_x, L_u) = i$. Every line $L \in S \setminus \{L_u\}$ contained in $Q$ lies at distance $i+1$ from $L_x$ and its unique point nearest to $x$ lies at distance 1 from $u$ as $x$ is classical with respect to $Q$. Every other point of $L$ lies at distance $i+2$ from $x$ and must belong to $\Gamma'_{i+2}(x)$, proving the remaining claims of the lemma. ■

We now use Lemma 3.3 to prove the following two lemmas.

**Lemma 3.4** *Let $y \in \Gamma''_i(x)$ with $i \in \{0, 1, \ldots, d\}$. Then:*

- *$y$ is incident with $t_i + 1$ lines containing a unique point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma''_i(x)$;*

- *$y$ is incident with $(t - t_i)\widetilde{t_2}$ lines containing $y$ as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma''_{i+1}(x)$;*

- *$y$ is incident with $1 + (t_i + 1)(\widetilde{t_2} - 1)$ lines containing $y$ as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.*

**Proof.** The line $L_y$ contains $y$ as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$. Note that $\mathrm{d}(L_x, L_y) = i$.

Let $Q$ be one of the $t_i + 1$ quads through $L_y$ containing a line $L \in S$ at distance $i - 1$ from $L_x$. Let $u$ denote the unique point of $Q$ nearest to $x$. Then $u \in L$ and $\mathrm{d}(u, y) = 1$. By Lemma 3.3, the line $yu$ contains a unique point of $\Gamma''_{i-1}(x)$ (namely $u$) and $\widetilde{s}$ points of $\Gamma''_i(x)$. Also by Lemma 3.3, each of the $\widetilde{t_2} - 1$ lines of $Q$ through $y$ distinct from $yu$ and $L_y$ contains $y$ as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.

Let $Q$ be one of the $t - t_i$ quads through $L_y$ not containing a line of $S$ at distance $i - 1$ from $L_x$. By Lemma 3.3, each of the $\widetilde{t_2}$ lines of $Q$ through $y$ distinct from $L_y$ contains $y$ as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma''_{i+1}(x)$. ■

**Lemma 3.5** *Let $y \in \Gamma'_i(x)$ with $i \in \{1, 2, \ldots, d + 1\}$. Then:*

- *$y$ is incident with $t_{i-1} + 1$ lines containing a unique point of $\Gamma'_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$;*

- *$y$ is incident with $(t_{i-1} + 1)(\widetilde{t_2} - 1) + 1$ lines containing a unique point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$;*

- *$y$ is incident with $\widetilde{t_2}(t - t_{i-1})$ lines containing $y$ as unique point of $\Gamma'_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.*

**Proof.** The line $L_y$ contains a point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$. Note that $\mathrm{d}(L_x, L_y) = i - 1$.

Let $Q$ be one of the $t_{i-1} + 1$ quads through $L_y$ containing a line of $S$ at distance $i - 2$ from $L_x$. Denote by $u$ the unique point of $Q$ nearest to $x$. Then $u \in \Gamma''_{i-2}(x)$ and $L_u$ is the unique line of $Q$ nearest to $L_x$. By Lemma 3.3, the unique line $U$ through $y$ meeting $L_u$ contains a unique point of $\Gamma'_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$. Also by Lemma 3.3, each of the $\widetilde{t_2} - 1$ lines of $Q$ through $y$ distinct from $U$ and $L_y$ contains a unique point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$.

Let $Q$ be one of the $t - t_{i-1}$ quads through $L_y$ not containing a line of $S$ at distance $i - 2$ from $L_x$. By Lemma 3.3, each line of $Q$ through $y$ distinct from $L_y$ contains $y$ as unique point of $\Gamma'_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$. ■

**Example.** Consider now the special case that $\mathcal{S}'$ is a generalized hexagon of order $(s, t)$. Then $t_0 = -1$, $t_1 = t_2 = 0$, $t_3 = t$ and $\mathcal{S}$ is a near octagon. Above we already remarked that

(A1) $\Gamma_1'(x) = L_x \setminus \{x\}$.

By Lemmas 3.1 and 3.5, we know the following:

(A2) $\Gamma_i'(x)$ with $i \in \{2, 3\}$ consists of all points of $\Gamma_i(x)$ that are collinear with a point of $\Gamma_{i-1}'(x)$.

By the above, we also know that

(A3) $\Gamma_i''(x) = \Gamma_i(x) \setminus \Gamma_i'(x)$ for every $i \in \{1, 2, 3\}$.

By Lemmas 3.4 and 3.5, we also know:

(P1) Every point of $\Gamma_2'(x)$ is incident with $(t_1 + 1)(\widetilde{t_2} - 1) + 1 = \widetilde{t_2}$ lines meeting $\Gamma_1''(x)$.

(P2) Every point of $\Gamma_2''(x)$ is incident with $t_2 + 1 = 1$ lines meeting $\Gamma_1''(x)$.

(P3) Every point of $\Gamma_3'(x)$ is incident with $(t_2 + 1)(\widetilde{t_2} - 1) + 1 = \widetilde{t_2}$ lines meeting $\Gamma_2''(x)$.

(P4) Every point of $\Gamma_3''(x)$ is incident with $t_3 + 1 = t + 1 = \frac{\widetilde{t_2}(t+1)}{\widetilde{t_2}}$ lines meeting $\Gamma_2''(x)$.

Noting that $\mathcal{S}$ has order $(\widetilde{s}, \widetilde{t_2}(t+1))$, we thus see $(\mathcal{S}, S)$ is an octagonal pair with parameters $(\widetilde{s}, \widetilde{t_2}(t+1), \widetilde{t_2})$.

## 4 Eigenvalues and multiplicities of near polygons admitting a polygonal triple

Suppose $\mathcal{T} = (\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple for which $\mathcal{S}$ is a finite near octagon of order $(s, t)$ and such that every quad of $\mathcal{Q}$ has order $(s, t_2)$. Suppose also that the near hexagon $\mathcal{S}'$ associated with $\mathcal{T}$ is a generalized hexagon, necessarily of order $(st_2, \frac{t}{t_2} - 1)$. Since $t > t_2$, we can put $\frac{t}{t_2} - 1 = \frac{\alpha^2}{st_2}$, where $\alpha$ is some real positive number.

Let $x$ be a point of $\mathcal{S}$. By the example at the end of Section 3, we know that $(\mathcal{S}, S)$ is an octagonal pair and so $\mathcal{S}$ belongs to the family $\mathcal{F}$ discussed in [2]. With respect to the line spread $S$, the point set $\mathcal{P}$ of $\mathcal{S}$ can thus be written as a disjoint union $\Gamma_0(x) \cup \Gamma_1'(x) \cup \Gamma_1''(x) \cup \Gamma_2'(x) \cup \Gamma_2''(x) \cup \Gamma_3'(x) \cup \Gamma_3''(x) \cup \Gamma_4(x)$. This expression naturally gives rise to relations $R_0, R_1', \ldots, R_3'', R_4$ on $\mathcal{P}$ that partition $\mathcal{P} \times \mathcal{P}$ (e.g., $(x, y) \in R_3'' \Leftrightarrow y \in \Gamma_3''(x)$). These relations are symmetric. Indeed, if $i \in \{1, 2, 3\}$, then $(x, y) \in R_i'' \Leftrightarrow \mathrm{d}(x, y) = \mathrm{d}(L_x, L_y) = i$, with $L_u$, $u \in \mathcal{P}$, again denoting the unique line of $S$ containing $u$. With each relation $R \in \{R_0, R_1', R_1'', \ldots, R_4\}$, there is associated a symmetric matrix $U$ whose rows and column are indexed by the points. Specifically, we put $U_{xy}$ equal to 1 if $(x, y) \in R$ and equal to 0 otherwise. In this way, we obtain symmetric $v \times v$-matrices $A_0, A_1', A_1'', \ldots, A_4$, where $v$ is the total number of points. Here, $A_0$ is the $v \times v$ identity matrix $I_v$ and

$A := A_1' + A_1''$ is the collinearity matrix of $\mathcal{S}$. From Lemmas 3.4 and 3.5, we easily deduce

$$\begin{aligned}
A \cdot A_0 &= A_1' + A_1'', \\
A \cdot A_1' &= sA_0 + (s-1)A_1' + A_2', \\
A \cdot A_1'' &= stA_0 + (s-1)A_1'' + t_2A_2' + A_2'', \\
A \cdot A_2' &= stA_1' + st_2A_1'' + (s-1)(t_2+1)A_2' + A_3', \\
A \cdot A_2'' &= s(t-t_2)A_1'' + (s-1)A_2'' + t_2A_3' + \frac{t}{t_2}A_3'', \\
A \cdot A_3' &= s(t-t_2)A_2' + st_2A_2'' + (s-1)(t_2+1)A_3' + \frac{t}{t_2}A_4, \\
A \cdot A_3'' &= s(t-t_2)A_2'' + \frac{(s-1)t}{t_2}A_3'' + (t+1-\frac{t}{t_2})A_4, \\
A \cdot A_4 &= s(t-t_2)A_3' + s(t+1-\frac{t}{t_2})A_3'' + (s-1)(t+1)A_4.
\end{aligned} \tag{1}$$

As all involved matrices are symmetric, these equations can be written as

$$[A_0\ A_1'\ A_1''\ \cdots\ A_4]^T \cdot A = (B \otimes I_v) \cdot [A_0\ A_1'\ A_1''\ \cdots\ A_4]^T,$$

where $B \otimes I_v$ denotes the Kronecker product [7, Section 4.2] of

$$B = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
s & s-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
st & 0 & s-1 & t_2 & 1 & 0 & 0 & 0 \\
0 & st & st_2 & (s-1)(t_2+1) & 0 & 1 & 0 & 0 \\
0 & 0 & s(t-t_2) & 0 & s-1 & t_2 & \frac{t}{t_2} & 0 \\
0 & 0 & 0 & s(t-t_2) & st_2 & (s-1)(t_2+1) & 0 & \frac{t}{t_2} \\
0 & 0 & 0 & 0 & s(t-t_2) & 0 & \frac{(s-1)t}{t_2} & t+1-\frac{t}{t_2} \\
0 & 0 & 0 & 0 & 0 & s(t-t_2) & s(t+1-\frac{t}{t_2}) & (s-1)(t+1)
\end{bmatrix}$$

and $I_v$. Let $\mathcal{A}$ be the subalgebra of $\mathbb{R}^{v \times v}$ generated by $A$, and let $\mathcal{B}$ be the subalgebra of $\mathbb{R}^{8 \times 8}$ generated by $B$. For every $M \in \mathcal{A}$, there exists a unique $M^\theta \in \mathbb{R}^{8 \times 8}$ such that

$$[A_0\ A_1'\ A_1''\ \cdots\ A_4]^T \cdot M = (M^\theta \otimes I_v) \cdot [A_0\ A_1'\ A_1''\ \cdots\ A_4]^T.$$

In fact, if $M = p(A)$ for a certain polynomial $p(X) \in \mathbb{R}[X]$, then we can take $M^\theta = p(B)$ by [7, Lemma 4.2.10]. The uniqueness of $M^\theta \in \mathbb{R}^{8 \times 8}$ follows from the fact that the matrices $A_0, A_1', A_1'', \ldots, A_4$ are linearly independent in $\mathbb{R}^{v \times v}$. We conclude that $\theta$ defines an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Taken into account that $t = \frac{\alpha^2}{s} + t_2$, we can compute that the eigenvalues of $B$ are equal to

$$\lambda_1 = \alpha^2 + st_2 + s,\ \lambda_2 = s + \alpha - t_2 - 1,\ \lambda_3 = s - \alpha - t_2 - 1,\ \lambda_4 = st_2 + s - \alpha - 1,$$

$$\lambda_5 = st_2 + s + \alpha - 1,\ \lambda_6 = -\frac{\alpha^2 + st_2 + s}{s}, \lambda_7 = \frac{\alpha^2 + st_2 - t_2}{t_2},\ \lambda_8 = \frac{s^2t_2 - st_2 - \alpha^2}{st_2}.$$

Hence, these are also the eigenvalues of $A$. Let $m_i$ with $i \in \{1, 2, \ldots, 8\}$ denote the multiplicity of the eigenvalue $\lambda_i$ of $A$.

If $j \in \mathbb{N}$, then by (1), we can write

$$A^j = a_j A_0 + b_j A_1' + c_j A_1'' + d_j A_2' + e_j A_2'' + f_j A_3' + g_j A_3'' + h_j A_4 \tag{2}$$

for certain (necessarily unique) $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j \in \mathbb{N}$. We then have $Tr(A^j) = va_j$.

Suppose now that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_8$ are mutually distinct. Then we have $\sum_{i=1}^{8} m_i \lambda_i^j = Tr(A^j) = va_j$ for every $j \in \{0, 1, \ldots, 7\}$. These eight equations determine a system of linear equations which can be solved for the unknowns $m_1, m_2, \ldots, m_8$ as the matrix of the system is a nonsingular Vandermonde matrix. We find (e.g. with Maple, see [6]) that

$$\begin{aligned}
m_1 &= 1,\\
m_2 &= \frac{\alpha^2 s(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(\alpha^2+\alpha s+s^2)(\alpha^2-\alpha t_2+t_2^2)},\\
m_3 &= \frac{\alpha^2 s(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(\alpha^2-\alpha s+s^2)(\alpha^2+\alpha t_2+t_2^2)},\\
m_4 &= \frac{\alpha^2(\alpha^2-\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(s^2t_2^2-\alpha s t_2+\alpha^2)},\\
m_5 &= \frac{\alpha^2(\alpha^2+\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(s^2t_2^2+\alpha s t_2+\alpha^2)},\\
m_6 &= \frac{s^6(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)}{(\alpha^2+\alpha s+s^2)(\alpha^2-\alpha s+s^2)(s+t_2)},\\
m_7 &= \frac{st_2^5(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)}{(\alpha^2+\alpha t_2+t_2^2)(\alpha^2-\alpha t_2+t_2^2)(s+t_2)},\\
m_8 &= \frac{s^5t_2^5(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)}{(s^2t_2^2+\alpha s t_2+\alpha^2)(s^2t_2^2-\alpha s t_2+\alpha^2)}.
\end{aligned}$$

These multiplicities need to be integers. In case $\alpha \in \mathbb{N}$, this leads to a number of divisibility conditions that need to be satisfied by the parameters $s$, $t_2$ and $\alpha$.

If a particular eigenvalue $\lambda$ occurs more than once in the collection $\lambda_1, \lambda_2, \cdots, \lambda_8$, then its multiplicity is equal to $\sum m_i$, where the summation ranges over all $i \in \{1, 2, \ldots, 8\}$ for which $\lambda_i = \lambda$.

Recall that $\mathcal{S}'$ is a generalized hexagon of order $(st_2, \frac{t}{t_2} - 1)$. By Section 2, we know that if $t_2 = 1$, then $\mathcal{S} \cong \mathcal{S}' \times \mathbb{L}_{s+1}$, where $\mathbb{L}_{s+1}$ is some line of size $s+1$, and if $s = 1$, then $\mathcal{S} \cong \overline{\mathcal{S}'}$. In the sequel, we may therefore assume that $s, t_2 > 1$.

Put $q := st_2 \geq 4$ and as before let $\alpha$ be the positive real number such that $\alpha^2 = q \cdot (\frac{t}{t_2} - 1)$. If $\frac{t}{t_2} - 1 > 1$, then $\alpha \in \mathbb{N}$, see e.g. [12, Theorem 1.7.1]. The eigenvalues and multiplicities can thus be computed with the aid of the above formulas. Note that all the multiplicities must be integral.

Using the information provided in [3, 8, 10, 12], we can easily find all triples $(s, t_2, t)$ of natural numbers distinct from 0 and 1 satisfying the following, with $d = 3$:

(A) There exists a known generalized quadrangle $Q$ of order $(s, t_2)$ that is moreover known to have a line spread.

(B) There exists a known generalized $2d$-gon of order $(st_2, \frac{t}{t_2} - 1)$.

We find that there exists a prime power $r$ such that $(s, t_2, t)$ is equal to either $(r, r, 2r)$, $(r, r^2, 2r^2)$, $(r, r, r(r^2 + 1))$, $(r, r^2, r^2(r^3 + 1))$, $(r, r, r(r^6 + 1))$ or $(r, r^2, r^2(r^9 + 1))$. The multiplicities of the eigenvalues are only integral when $(s, t_2, t) \in \{(r, r, 2r), (r, r, r(r^2 + 1))\}$, see [6].

In case $(s, t_2, t) = (r, r, 2r)$, the associated generalized hexagon $\mathcal{S}'$ has order $(r^2, 1)$ and all quads of $\mathcal{S}$ have order $(r, r)$. One example is known, namely the polygonal triple corresponding to the $L_3(4)$ near octagon ($r = 2$).

In case $(s, t_2, t) = (r, r, r(r^2 + 1))$, the associated generalized hexagon $\mathcal{S}'$ has order $(r^2, r^2)$ and all quads of $\mathcal{S}$ have order $(r, r)$. One example is known, namely the polygonal triple corresponding to the $G_2(4)$ near octagon ($r = 2$).

We can give a similar treatment for polygonal triples $(\mathcal{S}, S, \mathcal{Q})$, where $\mathcal{S}$ is a finite near decagon of order $(s, t)$ with quads of order $(s, t_2)$ such that the associated near polygon $\mathcal{S}'$ is a generalized octagon of order $(st_2, \frac{t}{t_2} - 1)$. Then we must find all triples $(s, t_2, t)$ of natural numbers distinct from 0 and 1 such that properties (A) and (B) hold with $d = 4$. Using the information provided in [3, 8, 10, 12], we find the following possibilities for $(s, t_2, t)$:

(a) $(q, q, 2q)$, where $q$ is some prime power;
(b) $(q, q, (q + 1)q)$, where $q$ is a power of 2 with odd exponent;
(c) $(q, q^2, 2q^2)$, where $q$ is some prime power;
(d) $(q, q^2, (q^6 + 1)q^2)$, where $q$ is a power of 2 with odd exponent;
(e) $(q^2, q^4, (q^3 + 1)q^4)$, where $q$ is a power of 2 with odd exponent.

With similar techniques as above, we can compute all eigenvalues and multiplicities, see [6]. The cases (d) and (e) cannot occur since not all multiplicities are integral. In cases (b) and (c), the multiplicities are always integral. In case (a), all multiplicities are integral if and only if $q$ is odd or a multiple of 8.

The above discussion implies that there are no polygonal triples whose quads are isomorphic to $W(2)$ and whose associated near polygons are isomorphic to either $T(4, 64)$ or $GO(4, 1)$.

# 5 Algorithms to classify polygonal triples

## 5.1 Polygonal data

In this section, $\mathcal{S}$ denotes a near polygon, $S$ is a line spread of $\mathcal{S}$ and $\mathcal{Q}$ is a set of quads of $\mathcal{S}$ such that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon $\mathcal{S}'$. For every point $x$ of $\mathcal{S}$, $L_x$ denotes the unique line of $S$ containing $x$. Recall that every quad $Q \in \mathcal{Q}$ is classical in $\mathcal{S}$.

If $L_1, L_2 \in S$, then $L_1$ and $L_2$ are parallel and we denote by $\Pi_{L_1,L_2}$ the bijection between $L_1$ and $L_2$ that sends each point $x$ of $L_1$ to the unique point of $L_2$ nearest to $x$. If $L_1, L_2, L_3 \in S$, then we define $\Phi_{L_1,L_2,L_3} = \Pi_{L_3,L_1} \circ \Pi_{L_2,L_3} \circ \Pi_{L_1,L_2}$.

Let $L^*$ be some specific line of $S$. For every point $x$ of $\mathcal{S}$, let $\pi(x)$ denote the unique point of $L^*$ nearest to $x$. We coordinatize $\mathcal{S}$ as follows. With each point $x$ of $\mathcal{S}$, we associate the pair $(L_x, \pi(x))$. If $L_1$ and $L_2$ are two distinct lines of $S$, then we denote by $\Phi_{L_1,L_2}$ the permutation $\Phi_{L^*,L_1,L_2}$ of $L^*$.

**Lemma 5.1** (1) *$\Phi_{L^*,L}$ is the identical permutation $1_{L^*}$ of $L^*$ for every line $L$ of $S$.*

(2) *For any two distinct lines $L_1, L_2 \in S$, we have $\Phi_{L_2,L_1} = \Phi^{-1}_{L_1,L_2}$.*

(3) *If $L_1$, $L_2$ and $L_3$ are three lines of $S$ such that $L_2$ is contained on a shortest path from $L_1$ to $L_3$ in the near polygon $\mathcal{S}'$, then $\Phi_{L_1,L_3} = \Phi_{L_2,L_3} \circ \Phi_{L_1,L_2}$.*

**Proof.** (1) We have $\Phi_{L^*,L} = \Pi_{L,L^*} \circ \Pi_{L^*,L} \circ \Pi_{L^*,L^*} = \Pi_{L,L^*} \circ \Pi_{L^*,L} = 1_{L^*}$.

(2) We have $\Phi^{-1}_{L_1,L_2} = \Big(\Pi_{L_2,L^*} \circ \Pi_{L_1,L_2} \circ \Pi_{L^*,L_1}\Big)^{-1} = \Pi^{-1}_{L^*,L_1} \circ \Pi^{-1}_{L_1,L_2} \circ \Pi^{-1}_{L_2,L^*} = \Pi_{L_1,L^*} \circ \Pi_{L_2,L_1} \circ \Pi_{L^*,L_2} = \Phi_{L_2,L_1}$.

(3) If $L_2$ is on a shortest path from $L_1$ to $L_3$, then $\Pi_{L_1,L_3} = \Pi_{L_2,L_3} \circ \Pi_{L_1,L_2}$ and this implies that $\Phi_{L_1,L_3} = \Phi_{L_2,L_3} \circ \Phi_{L_1,L_2}$. ■

**Lemma 5.2** (1) *If $L_1, L_2 \in S$ such that $L_1$ is contained on a shortest path from $L^*$ to $L_2$ in the geometry $\mathcal{S}'$, then $\Phi_{L_1,L_2}$ is the identical permutation of $L^*$. In particular, $\Phi_{L,L}$ is the identical permutation of $L^*$ for every line $L \in S$.*

(2) *Let $Q$ and $Q'$ be two opposite quads of $\mathcal{Q}$ (i.e. two opposite lines of $\mathcal{S}'$). Let $L_1$ and $L_2$ be two lines of $S$ contained in $Q$, and put $L_1' := \pi_{Q'}(L_1) \in S$ and $L_2' := \pi_{Q'}(L_2) \in S$. Then $\Phi_{L_1,L_2'} = \Phi_{L_1',L_2'} \circ \Phi_{L_1,L_1'} = \Phi_{L_2,L_2'} \circ \Phi_{L_1,L_2}$.*

**Proof.** (1) From Lemma 5.1(3), we know that $\Phi_{L^*,L_2} = \Phi_{L_1,L_2} \circ \Phi_{L^*,L_1}$. From Lemma 5.1(1), we know that $\Phi_{L^*,L_1} = \Phi_{L^*,L_2} = 1_{L^*}$. Hence, $\Phi_{L_1,L_2} = 1_{L^*}$.

(2) Since every quad of $\mathcal{Q}$ is classical, we observe that $L_1'$ and $L_2$ are on shortest path from $L_1$ to $L_2'$ in the geometry $\mathcal{S}'$. The remaining part of (2) then follows from Lemma 5.1(3). ■

**Lemma 5.3** *$\mathcal{S}$ can be completely described in terms of $\mathcal{S}'$ and the maps $\Phi_{L_1,L_2}$, where $L_1$ and $L_2$ are two lines of $S$ at distance 1 from each other.*

**Proof.** Consider two points with labels $(K_1, x)$ and $(K_2, y)$. If $K_1 = K_2$, then the points $(K_1, x)$ and $(K_2, y)$ are collinear. If $K_1, K_2$ are distinct collinear points of $\mathcal{S}'$, then $(K_1, x)$ and $(K_2, y)$ are collinear if and only if $y = \Phi_{K_1,K_2}(x)$. If $K_1, K_2$ are distinct noncollinear points in $\mathcal{S}'$, then $(K_1, x)$ and $(K_2, y)$ are not collinear. So, the collinearity graph of $\mathcal{S}$ and hence also $\mathcal{S}$ itself can be completely described in terms of $\mathcal{S}'$ and the maps $\Phi_{L_1,L_2}$, where $L_1$ and $L_2$ are two points of $\mathcal{S}'$ at distance 1 from each other. ■

Suppose $\widetilde{\mathcal{S}}$ is a near polygon isomorphic to $\mathcal{S}'$, $\widetilde{X}$ is a set of the same cardinality of $L^*$, and $\widetilde{\Phi}$ is a map which associates with each pair $(x_1, x_2)$ of distinct collinear points of $\widetilde{\mathcal{S}}$ a permutation $\widetilde{\Phi}(x_1, x_2)$ of $\widetilde{X}$. Suppose $\theta$ is an isomorphism from $\widetilde{\mathcal{S}}$ to $\mathcal{S}'$ and $\phi$ is a bijection of $\widetilde{X}$ to $L^*$ such that $\widetilde{\Phi}(x_1, x_2) = \phi^{-1} \circ \Phi_{x_1^\theta, x_2^\theta} \circ \phi$ for every pair $(x_1, x_2)$ of distinct collinear points of $\widetilde{\mathcal{S}}$. If $\widetilde{p}$ is the unique point of $\widetilde{\mathcal{S}}$ for which $\widetilde{p}^\theta = L^*$, then we call the quadruple $(\widetilde{\mathcal{S}}, \widetilde{X}, \widetilde{p}, \widetilde{\Phi})$ *polygonal data* for the polygonal triple $(\mathcal{S}, S, \mathcal{Q})$. Note that $(\mathcal{S}', L^*, L^*, \Phi')$ is polygonal data for $(\mathcal{S}, S, \mathcal{Q})$, where $\Phi'$ is the restriction of $\Phi$ to distinct collinear points of $\mathcal{S}'$, the former $L^*$ is regarded as set of points of $\mathcal{S}$ and the latter is regarded as point of $\mathcal{S}'$.

Having this polygonal data $(\widetilde{\mathcal{S}}, \widetilde{X}, \widetilde{p}, \widetilde{\Phi})$, it is possible (by using Lemma 5.3) to reconstruct an isomorphic copy[2] $(\mathcal{S}_1, S_1, \mathcal{Q}_1)$ of $(\mathcal{S}, S, \mathcal{Q})$ in the following way.

- $\mathcal{S}_1$ is the near polygon with points the pairs $(a, x)$, with $a$ a point of $\widetilde{\mathcal{S}}$ and $x \in \widetilde{X}$, where two distinct points $(a_1, x_1)$ and $(a_2, x_2)$ are collinear whenever either $a_1 = a_2$ or ($\mathrm{d}_{\widetilde{\mathcal{S}}}(a_1, a_2) = 1$ and $x_2 = x_1^{\widetilde{\Phi}(a_1,a_2)}$).
- $S_1$ consists of all lines of the form $\{(a, x) \,|\, x \in \widetilde{X}\}$, with $a$ a point of $\widetilde{\mathcal{S}}$.
- $\mathcal{Q}_1$ consists of all quads of $\mathcal{S}_1$ containing a line of $S_1$.

Classifying particular polygonal triples is thus equivalent with determining the corresponding polygonal data. The proof of Theorem 1.1 will make use of this observation.

## 5.2 Properties of polygonal data

The following lemma is a consequence of Lemmas 5.1 and 5.2.

**Lemma 5.4** *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data for a polygonal triple. Then the following hold:*

(*a*) *$\Phi(p, x)$ is the identical permutation $1_X$ of $X$ for every point $x$ of $\mathcal{S}$ at distance 1 from $p$.*

(*b*) *If $x_1$ and $x_2$ are two distinct collinear points of $\mathcal{S}$, then $\Phi(x_1, x_2) = \Phi(x_2, x_1)^{-1}$.*

(*c*) *If $x_1$ and $x_2$ are two distinct collinear points of $\mathcal{S}$ such that the unique point of $x_1x_2$ nearest to $p$ coincides with either $x_1$ or $x_2$, then $\Phi(x_1, x_2) = 1_X$.*

(*d*) *If $u = x_1, x_2, \ldots, x_k = v$ and $u = y_1, y_2, \ldots, y_k = v$ are two shortest paths connecting the points $u$ and $v$, then*

$$\Phi(x_{k-1}, x_k) \circ \Phi(x_{k-2}, x_{k-1}) \circ \cdots \circ \Phi(x_1, x_2) = \Phi(y_{k-1}, y_k) \circ \Phi(y_{k-2}, y_{k-1}) \circ \cdots \circ \Phi(y_1, y_2).$$

**Lemma 5.5** *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data and $L_1$, $L_2$ are two parallel lines of $\mathcal{S}$ such that $p \in L_1$. Then $\Phi(u, v) = \Phi(\pi_{L_1}(u), \pi_{L_1}(v))$ for any two distinct points $u$ and $v$ of $L_2$.*

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[2] This means that there is an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}$ mapping $S_1$ to $S$ and $\mathcal{Q}_1$ to $\mathcal{Q}$.

**Proof.** Put $\pi_{L_1}(u) = u'$ and $\pi_{L_1}(v) = v'$. If $w_1$ and $w_2$ are two consecutive points on a shortest path from $u'$ to $u$ (or from $v'$ to $v$), then $\Phi(w_1, w_2) = 1_X$ by Lemma 5.4(c). Now, consider two shortest paths connecting $u'$ and $v$, one that passes through the point $u$ and another that passes through $v'$. If we apply Lemma 5.4(d) to these two paths, then we find $\Phi(u, v) = \Phi(u', v')$ if we take into account that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points $w_1$ and $w_2$ on these paths for which the line $w_1w_2 \notin \{L_1, L_2\}$. ∎

We now prove a number of useful lemmas. In these lemmas, the following notation is used. If $L$ is a line of a near polygon, then $\mathcal{C}_L$ denotes the set of all ordered pairs of distinct points of $L$.

**Lemma 5.6** *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized hexagon with at least three lines through each point and at least three points on each line. Let $L$ be an arbitrary line through $p$. Then $\Phi$ is uniquely determined by the values it takes on $\mathcal{C}_L$.*

**Proof.** Let $\mathcal{L}$ denote the line set of $\mathcal{S}$. We write $\mathcal{L} \setminus \{L\}$ as a disjoint union $\mathcal{L}_{0,0} \cup \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1} \cup \mathcal{L}_{2,1} \cup \mathcal{L}_{2,2}$, where $\mathcal{L}_{i,j}$ denotes the set of lines of $\mathcal{L} \setminus \{L\}$ at distance $i$ from $p$ and at distance $j$ from $L$. Suppose we know all values of $\Phi$ on the set $\mathcal{C}_L$. We gradually show how these values uniquely determine $\Phi$ on each $\mathcal{C}_K$, $K \in \mathcal{L} \setminus \{L\}$.

**Step 1.** If $K \in \mathcal{L}_{2,2}$, then for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_L(x), \pi_L(y))$ by Lemma 5.5.

**Step 2.** If $K \in \mathcal{L}_{0,0}$, then Lemma 5.5 implies that for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$, where $M$ is a line of $\mathcal{L}_{2,2}$ opposite to $K$ and $L$. Such a line $M$ can be constructed in the following way. Let $M''$ be a line through $p$ distinct from $K$ and $L$, let $M'$ be a line intersecting $M''$ in a singleton distinct from $\{p\}$ and let $M$ be a line meeting $M'$ in a singleton distinct from $M' \cap M''$.

**Step 3.** If $K \in \mathcal{L}_{2,1}$, then Lemma 5.5 implies that for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$, where $M$ is any line of $\mathcal{L}_{0,0}$ opposite to $K$.

**Step 4.** Suppose $K \in \mathcal{L}_{1,1}$. Let $u_1$ denote the unique point of $K$ collinear with $p$ and let $u_2$ denote a point at distance 1 from $pu_1$ such that the unique point of $pu_1$ collinear $u_2$ is distinct from $p$ and $u_1$. Such a point exists as there are at least three points on the line $pu_1$. Let $M$ be a line through $u_2$ opposite to $K$. Let $(x, y) \in \mathcal{C}_K$, put $x'' := \pi_M(x)$, $y'' := \pi_M(y)$, $\{x'\} := \Gamma_1(x) \cap \Gamma_1(x'')$ and $\{y'\} := \Gamma_1(y) \cap \Gamma_1(y'')$. If one of $x, y$ coincides with $u_1$, then $\Phi(x, y) = 1_X$ by Lemma 5.4(c). Suppose therefore that $x \neq u_1 \neq y$. Then $\Phi(x', x) = \Phi(y, y') = 1_X$ by Lemma 5.4(c). Lemma 5.4(d) then implies that $\Phi(x, y) = \Phi(y, y') \circ \Phi(x, y) \circ \Phi(x', x) = \Phi(y'', y') \circ \Phi(x'', y'') \circ \Phi(x', x'')$. Note that the lines $y''y'$, $M = x''y''$ and $x'x''$ belong to $\mathcal{L}_{2,2} \cup \mathcal{L}_{2,1}$ since $y', x' \in \Gamma_3(p)$ and $\mathrm{d}(p, M) = 2$.

**Step 5.** Suppose $K \in \mathcal{L}_{1,0}$. In an ordinary 6-gon containing $p$ and $K$, we can take a line $M$ opposite to $K$. Then $M \in \mathcal{L}_{1,1}$ and for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. Indeed, if $\Gamma_1(x) \cap \Gamma_1(\pi_M(x)) = \{x'\}$ and $\Gamma_1(y) \cap \Gamma_1(\pi_M(y)) = \{y'\}$, then Lemma 5.4(c) implies that $\Phi(x', x) = \Phi(x', \pi_M(x)) = \Phi(y', y) = \Phi(y', \pi_M(y)) = 1_X$. Lemma 5.4(d) then implies that $\Phi(x, y) = \Phi(y, y') \circ \Phi(x, y) \circ \Phi(x', x) = \Phi(\pi_M(y), y') \circ \Phi(\pi_M(x), \pi_M(y)) \circ \Phi(x', \pi_M(x)) = \Phi(\pi_M(x), \pi_M(y))$. ∎

**Lemma 5.7** *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized octagon with at least three lines through each point and at least three points on each line. Let $L$ be an arbitrary line through $p$. Then $\Phi$ is uniquely determined by the values it takes on $\mathcal{C}_L$.*

**Proof.** We write $\mathcal{L} \setminus \{L\}$ as a disjoint union $\mathcal{L}_{0,0} \cup \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1} \cup \mathcal{L}_{2,1} \cup \mathcal{L}_{2,2} \cup \mathcal{L}_{3,2} \cup \mathcal{L}_{3,3}$, where $\mathcal{L}_{i,j}$ denotes the set of lines of $\mathcal{L} \setminus \{L\}$ at distance $i$ from $p$ and at distance $j$ from $L$. Suppose we know all values of $\Phi$ on the set $\mathcal{C}_L$. We gradually show how these values uniquely determine $\Phi$ on each $\mathcal{C}_K$, $K \in \mathcal{L} \setminus \{L\}$.

**Step 1.** If $K \in \mathcal{L}_{3,3}$, then for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_L(x), \pi_L(y))$ by Lemma 5.5.

**Step 2.** If $K \in \mathcal{L}_{0,0}$, then for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$, where $M$ is a line of $\mathcal{L}_{3,3}$ opposite to $K$ and $L$. Similarly as in Step 2 of Lemma 5.6, we can construct such a line opposite to $K$ and $L$, by starting from a line through $p$ distinct from $K$ and $L$.

**Step 3.** If $K \in \mathcal{L}_{3,2}$, then Lemma 5.5 implies that for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$, where $M$ is any line of $\mathcal{L}_{0,0}$ opposite to $K$.

**Step 4.** Suppose $K \in \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1}$. Let $u_1$ denote the unique point of $K$ collinear with $p$ and let $u_2$ denote a point at distance 2 from $pu_1$ such that the unique point of $pu_1$ nearest to $u_2$ is distinct from $p$ and $u_1$. Such a point exists since the line $pu_1$ contains at least three points. Let $M$ be a line through $u_2$ opposite to $K$. Let $(x, y) \in \mathcal{C}_K$, put $x''' := \pi_M(x)$, $y''' := \pi_M(y)$, $\{x'\} := \Gamma_1(x) \cap \Gamma_2(x''')$, $\{y'\} := \Gamma_1(y) \cap \Gamma_2(y''')$, $\{x''\} := \Gamma_2(x) \cap \Gamma_1(x''')$ and $\{y''\} := \Gamma_2(y) \cap \Gamma_1(y''')$. If one of $x$, $y$ coincides with $u_1$, then $\Phi(x, y) = 1_X$ by Lemma 5.4(c). Suppose therefore that $x \neq u_1 \neq y$. By Lemma 5.4(c), we know that $\Phi(y, y') = \Phi(y', y'') = \Phi(x, x') = \Phi(x', x'') = 1_X$. With a similar reasoning as in Step 4 of Lemma 5.6, this allows to conclude that $\Phi(x, y) = \Phi(y''', y'') \circ \Phi(x''', y''') \circ \Phi(x'', x''')$. Note that the lines $y'''y''$, $x'''y''' = M$ and $x''x'''$ belong to $\mathcal{L}_{3,3} \cup \mathcal{L}_{3,2}$ since $y'', x'' \in \Gamma_4(p)$ and $\mathrm{d}(p, M) = 3$.

**Step 5.** Suppose $K \in \mathcal{L}_{2,1} \cup \mathcal{L}_{2,2}$. In an ordinary 8-gon containing $p$, $K$ and $L$, we can take a line $M$ opposite to $K$. Then $M \in \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1}$ for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. The proof of that claim is similar to the proof of Step 5 in Lemma 5.6, taking into account that Lemma 5.4(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points $w_1$ and $w_2$ on a shortest path connecting $x$ with $\pi_M(x)$ (or $y$ with $\pi_M(y)$). ■

**Lemma 5.8** *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized $2d$-gon, $d \in \{3, 4\}$, with exactly two lines through each point and at least three points on each line. Let $L_1$ and $L_2$ be the two lines through $p$. Then $\Phi$ is uniquely determined by the values it takes on $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$.*

**Proof.** We follow the same notational convention as in the proofs of Lemmas 5.6 and 5.7. In particular, we write $\mathcal{L} \setminus \{L\}$ as the disjoint union of the mentioned sets, where $L := L_1$. Note that $\mathcal{L}_{0,0}$ is the singleton $\{L_2\}$. We observe now that the proofs of Lemmas 5.6 and

5.7 only break down at one point, namely in Step 2, where it is no longer possible to choose a line $M \in \mathcal{L}_{d-1,d-1}$ opposite to $K$ and $L$. However, by assuming that the values of $\Phi$ on the set $\mathcal{C}_{L_2}$ are also known, the rest of the proof can remain without any changes. ■

**Lemma 5.9** *Suppose* $(\mathcal{S}, X, p, \Phi)$ *is polygonal data, where* $\mathcal{S}$ *is a generalized dodecagon with exactly two lines through each point and at least three points on each line. Let* $L_1$ *and* $L_2$ *be the two lines through* $p$*. Then* $\Phi$ *is uniquely determined by the values it takes on* $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$*.*

**Proof.** We write $\mathcal{L} \setminus \{L_1, L_2\}$ as a disjoint union $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4 \cup \mathcal{L}_5$, where $\mathcal{L}_i$ with $i \in \{1, 2, \ldots, 5\}$ is the set of lines of $\mathcal{L} \setminus \{L_1, L_2\}$ at distance $i$ from $p$. Suppose we know all values of $\Phi$ on the set $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$. We gradually show how these values uniquely determine $\Phi$ on each $\mathcal{C}_K$, $K \in \mathcal{L} \setminus \{L_1, L_2\}$.

**Step 1.** Suppose $K \in \mathcal{L}_5$. Let $i \in \{1, 2\}$ such that $K$ and $L_i$ are opposite lines. Then $\Phi(x, y) = \Phi(\pi_{L_i}(x), \pi_{L_i}(y))$ by Lemma 5.5.

**Step 2.** Suppose $K \in \mathcal{L}_1$. Then $K$ meets one of $L_1$, $L_2$, say $L_i$, in a point $u_1$. Let $u_2$ denote a point at distance 4 from $pu_1$ such that the unique point of $pu_1$ nearest to $u_2$ is distinct from $p$ and $u_1$. Such a point exists since the line $pu_1 = L_i$ contains at least three points. Let $M$ be a line through $u_2$ opposite to $K$. Let $(x, y) \in \mathcal{C}_K$, put $x'' := \pi_M(x)$, $y'' := \pi_M(y)$, $\{x'\} := \Gamma_1(x'') \cap \Gamma_4(x)$, $\{y'\} := \Gamma_1(y'') \cap \Gamma_4(y)$. If one of $x$, $y$ coincides with $u_1$, then $\Phi(x, y) = 1_X$ by Lemma 5.4(c). Suppose therefore that $x \neq u_1 \neq y$. With a similar reasoning as in the proof of Step 4 in Lemma 5.6, we have $\Phi(x, y) = \Phi(y'', y') \circ \Phi(x'', y'') \circ \Phi(x', x'')$. Indeed, Lemma 5.4(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from $x$ to $x'$ (or from $y$ to $y'$). Note that the lines $y''y'$, $x''y'' = M$ and $x'x''$ belong to $\mathcal{L}_5$ since $y', x' \in \Gamma_6(p)$ and $\mathrm{d}(p, M) = 5$.

**Step 3.** Suppose $K \in \mathcal{L}_4$. In an ordinary 12-gon containing $p$ and $K$, we can take a line $M$ opposite to $K$. Then $M \in \mathcal{L}_1$ and for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. The proof of that claim is similar to the proof of Step 5 in Lemma 5.6, taking into account that Lemma 5.4(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from $x$ to $\pi_M(x)$ (or from $y$ to $\pi_M(y)$).

**Step 4.** Suppose $K \in \mathcal{L}_2$. Then there is a unique line meeting one of $L_1, L_2$ in a point $u_1$ and $K$ in a point $u_1'$. Let $u_2$ denote a point at distance 4 from $u_1u_1'$ such that the unique point of $u_1u_1'$ nearest to $u_2$ is distinct from $u_1$ and $u_1'$. Such a point exists since the line $u_1u_1'$ contains at least three points. Let $M$ be a line through $u_2$ opposite to $K$. Let $(x, y) \in \mathcal{C}_K$. Put $x''' := \pi_M(x)$, $y''' := \pi_M(y)$, $\{x''\} := \Gamma_1(x''') \cap \Gamma_4(x)$, $\{x'\} := \Gamma_2(x''') \cap \Gamma_3(x)$, $\{y''\} := \Gamma_1(y''') \cap \Gamma_4(y)$, $\{y'\} := \Gamma_2(y''') \cap \Gamma_3(y)$. If one of $x$, $y$ coincides with $u_1'$, then $\Phi(x, y) = 1_X$. Suppose therefore that $x \neq u_1' \neq y$. With a similar reasoning as in the proof of Step 4 in Lemma 5.6, we have $\Phi(x, y) = \Phi(y'', y') \circ \Phi(y''', y'') \circ \Phi(x''', y''') \circ \Phi(x'', x''') \circ \Phi(x', x'')$. Indeed, Lemma 5.4(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from $x$ to $x'$ (or from $y$ to $y'$). Note that the lines $y''y'$, $y'''y''$, $x'''y''' = M$, $x''x'''$, $x'x''$ belong to $\mathcal{L}_4 \cup \mathcal{L}_5$ since $u_2, x', y' \in \Gamma_6(p)$ and $x'', y'' \in \Gamma_5(p) \cup \Gamma_6(p)$.

**Step 5.** Suppose $K \in \mathcal{L}_3$. In an ordinary 12-gon containing $p$ and $K$, we can take a line $M$ opposite to $K$. Then $M \in \mathcal{L}_2$ and for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. The proof of that claim is similar to the proof of Step 5 in Lemma 5.6, taking into account that Lemma 5.4(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from $x$ to $\pi_M(x)$ (or from $y$ to $\pi_M(y)$). ■

Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is some generalized polygon. If all points of $\mathcal{S}$ are incident with at least three lines and $L$ is a line through $p$, then the quadruple $(\mathcal{S}, X, p, \Phi')$, where $\Phi'$ is the restriction of $\Phi$ to the set $\mathcal{C}_L$, is called *partial polygonal data.* If every point of $\mathcal{S}$ is incident with precisely two lines and $L_1$, $L_2$ are the two lines through the point $p$, then the quadruple $(\mathcal{S}, X, p, \Phi')$, where $\Phi'$ is the restriction of $\Phi$ to the set $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$, is called *partial polygonal data.*

Suppose $(\mathcal{S}, X, p, \Phi')$ is partial polygonal data, where $\mathcal{S}$ is one of the generalized polygons under consideration in Lemmas 5.6, 5.7, 5.8 and 5.9. Using the algorithms exposed in the proofs of these lemmas, it is possible to reconstruct the whole polygonal data, and hence to construct an isomorphic copy of the polygonal triple from which $(\mathcal{S}, X, p, \Phi')$ arose. This approach will be followed during the proof of Theorem 1.1.

# 6 Proof of Theorem 1.1

The intention of this section is to prove Theorem 1.1. During this proof, we will use the following notation. Suppose $\eta$ is a function on two arguments belonging to the same set $\{x_1, x_2, x_3, x_4, x_5\}$ of size 5. Then $\mathcal{T}(\eta, (x_1, x_2, x_3, x_4, x_5))$ denotes the $5 \times 5$ table whose entry in the $i$-th row and the $j$-th column is equal to $\eta(x_i, x_j)$, where we put a "–" if $\eta(x_i, x_j)$ is not defined.

By Section 4, we already know that there exists no polygonal triple whose quads are isomorphic to $W(2)$ and whose associated near polygon is isomorphic to either $T(4, 64)$ and $GO(4, 1)$.

Using the algorithms involving (partial) polygonal data discussed in Section 5, we now classify all polygonal triples whose quads are isomorphic to $W(2)$ and whose associated near polygons $\mathcal{A}$ are isomorphic to either $H^D(4)$, $GH(4, 1)$, $H(4)$, $GO(4, 1)$, $RT(4, 2)$, $\mathcal{F}(H(4))$ or $\mathcal{F}(H^D(4))$. The case where $\mathcal{A}$ is isomorphic to $T(4, 64)$ will not be treated here as it seems to be out of reach of our computer computations.

For each generalized polygon $\mathcal{S} \in \{H^D(4), H(4), RT(4, 2)\}$, the following lemma in combination with the fact that $\mathcal{S}$ is flag-transitive shows that there is essentially one quadruple that can serve as potential[3] partial polygonal data for the problem.

**Lemma 6.1** *Suppose $(\mathcal{A}, X, p, \Phi)$ is polygonal data for a polygonal triple $(\mathcal{S}, S, \mathcal{Q})$, where each quad of $\mathcal{Q}$ is isomorphic to $W(2)$ and $\mathcal{A} \in \{H^D(4), H(4), RT(4, 2), GH(4, 1), GO(4, 1), \mathcal{F}(H(4)), \mathcal{F}(H^D(4))\}$. If $L = \{p, x_1, x_2, x_3, x_4\}$ is a line of $\mathcal{A}$ through $p$, then the elements of $X$ can be labeled with $a$, $b$ and $c$ such that $\mathcal{T}(\Phi, (p, x_1, x_2, x_3, x_4))$ is the following table:*

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[3] We are not sure in advance whether there is a polygonal triple associated with the quadruple; if there is one, then the quadruple should be polygonal data.

| | | | | |
|---|---|---|---|---|
| – | $()$ | $()$ | $()$ | $()$ |
| $()$ | – | $(a,b)$ | $(a,c)$ | $(b,c)$ |
| $()$ | $(a,b)$ | – | $(b,c)$ | $(a,c)$ |
| $()$ | $(a,c)$ | $(b,c)$ | – | $(a,b)$ |
| $()$ | $(b,c)$ | $(a,c)$ | $(a,b)$ | – |

**Proof.** The point $p$ corresponds to a line $L^* \in S$ and the line $L$ with a quad $Q \in \mathcal{Q}$ through $L^*$. Let $L^*$, $L_1$, $L_2$, $L_3$ and $L_4$ denote the five lines of $S$ contained in $Q$. For two distinct $K, M \in \{L^*, L_1, L_2, L_3, L_4\}$, let $\Phi_{K,M}$ be as defined in Section 5.1. Obviously, $\Phi_{K,M}$ is the trivial permutation if $L^* \in \{K, M\}$.

If $K$ and $M$ are distinct elements of $\{L_1, L_2, L_3, L_4\}$, then there is a unique line meeting $L^*$, $K$ and $M$, implying that $\Phi_{K,M}$ has a unique fixpoint, i.e. $\Phi_{K,M}$ is a transposition. Put $L^* = \{x_1, x_2, x_3\}$ and $K \in \{L_1, L_2, L_3, L_4\}$. Then there is a unique line (of size 3) through each $x_i$ meeting $K$, showing that the permutation $(x_{i+1}, x_{i+2})$ (with subindices taken modulo 3) occurs exactly once in the row and column corresponding to the line $K$. The lemma now follows if we take also into account that $\Phi_{K,M} = \Phi_{M,K}^{-1}$ for two distinct lines $K$ and $M$ of $S$. ■

For each generalized polygon $\mathcal{A} \in \{GH(4,1), GO(4,1), \mathcal{F}(H(4)), \mathcal{F}(H^D(4))\}$, the following lemma, which is an immediate consequence of Lemma 6.1, shows that several quadruples might serve as potential polygonal data for the problem, namely one for each permutation $\tau$ of the set $\{1,2,3,4\}$.

**Lemma 6.2** *Suppose* $(\mathcal{A}, X, p, \Phi)$ *is polygonal data for a polygonal triple* $(\mathcal{S}, S, \mathcal{Q})$, *where each quad of* $\mathcal{Q}$ *is isomorphic to* $W(2)$ *and* $\mathcal{A} \in \{GH(4,1), GO(4,1), \mathcal{F}(H(4)), \mathcal{F}(H^D(4))\}$. *If* $L_1 = \{p, x_1, x_2, x_3, x_4\}$ *and* $L_2 = \{p, y_1, y_2, y_3, y_4\}$ *are the two lines of* $\mathcal{A}$ *through* $p$, *then the elements of* $X$ *can be labeled with* $a$, $b$ *and* $c$ *such that* $\mathcal{T}(\Phi, (p, x_1, x_2, x_3, x_4))$ *and* $\mathcal{T}(\Phi, (p, y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)}))$ *are equal to the table mentioned in Lemma 6.1 for a certain permutation* $\tau$ *of* $\{1,2,3,4\}$.

Now that we know all potential partial polygonal data, we can construct the potential polygonal data, following the algorithms exposed in Lemmas 5.6, 5.7, 5.8 and 5.9. For that purpose, we have implemented a computer program in GAP [11], see [6]. It should be remarked that there is no unique way to reconstruct the potential polygonal data from the potential partial polygonal data. During the reconstruction process, certain choices need to be made which are not unique. E.g., if one needs to take a line opposite a given line, there are usual several choices for that, and the "complete polygonal data" might depend on the choices made during this reconstruction process. However, if the "partial polygonal data" is associated with a polygonal triple, then we know that the complete polygonal data one obtains has to be independent of the choices made during the reconstruction process. As our purpose is to classify polygonal triples, we do not have to worry about this complication, and we can make any choices we like during the reconstruction process.

Once we have obtained the "complete polygonal data", we have followed the algorithm exposed in Lemma 5.3 to build the graph which – in case of an associated polygonal triple $T$ – must be isomorphic to the collinearity graph of the near polygon that occurs as first component of $T$. Subsequently, we have verified whether this graph was indeed the collinearity graph of a near polygon.

This turned out to be the case if $\mathcal{A} = H^D(4)$ (as it should be), but not if $\mathcal{A} = H(4)$ or $\mathcal{A} = RT(4,2)$. In the case $\mathcal{A} = H^D(4)$, the graph must be isomorphic to the collinearity graph of the $G_2(4)$ near octagon. In this case, there is essentially one quadruple that can serve as partial polygonal data and so we already knew in advance, without making any computer computations, that the $G_2(4)$ near octagon is the unique near octagon admitting a polygonal triple all whose quads are isomorphic to $W(2)$ and for which the associated near polygon is isomorphic to $H^D(4)$.

When $\mathcal{A}$ is equal to either $GO(4,1)$, $\mathcal{F}(H(4))$ or $\mathcal{F}(H^D(4))$, the graph was not the collinearity graph of a near polygon for any of the 24 choices of the permutation $\tau$.

In case $\mathcal{A}$ is equal to the unique generalized hexagon $GH(4,1)$ of order $(4,1)$, this graph turned out to be the collinearity graph of a near polygon for precisely 12 of the 24 possible choices of $\tau$. These 12 permutations turn out to have the same parity. The twelve near octagons that arise in this way are all isomorphic since the potential partial polygonal data from which they are derived are all equivalent. The latter follows from the symmetry of the generalized hexagon $GH(4,1)$ exposed in Lemma 6.3(b) below.

**Lemma 6.3** (*a*) *Let* $\{x, L\}$ *be a flag of the projective plane* PG(2, 4) *and let* $H$ *denote the group of automorphisms of* PG(2, 4) *fixing each point of* $L$. *If* $\mathcal{L}$ *denotes the set of four lines through* $x$ *distinct from* $L$, *then each* $h \in H$ *induces a permutation* $\bar{h}$ *of* $\mathcal{L}$. *The group* $\{\bar{h} \mid h \in H\}$ *of permutations of* $\mathcal{L}$ *consists of all even permutations of this set.*

(*b*) *Let* $p$ *be a point of* $GH(4,1)$, *and* $L_1, L_2$ *be the two lines of* $GH(4,1)$ *through* $p$. *Let* $H$ *denote the group of automorphisms of* $GH(4,1)$ *fixing each point of the line* $L_1$. *Then each* $h \in H$ *induces a permutation* $\bar{h}$ *of* $L_2 \setminus \{p\}$. *Then the group* $\{\bar{h} \mid h \in H\}$ *of permutations of* $L_2 \setminus \{p\}$ *consists of all even permutations of this set.*

**Proof.** (a) This is easily verified. Each $h \in H$ is either an elation or a homology. If $h$ is an elation, then $\bar{h}$ is the product of two disjoint transpositions. If $h$ is a homology, then $\bar{h}$ is a cycle of length 3.

(b) Since the automorphism group of $GH(4,1)$ acts transitively on the set of lines of the generalized hexagon, we may without loss of generality suppose that $L_1$ is a line of $L$ of PG(2, 4). Then $p$ is a certain flag $\{x, L\}$ of PG(2, 4). The automorphisms of $GH(4,1)$ that fix each point of $L_1$ bijectively correspond with the automorphisms of PG(2, 4) that fix each point of $L$. The points of $L_2 \setminus \{p\}$ are the four flags $\{x, K\}$ where $K$ is one of the four lines through $x$ distinct from $L$. The lemma then follows from Claim (a). ∎

In the case that $\mathcal{A} = GH(4,1)$, these twelve near octagons thus have to be isomorphic to the $L_3(4)$ near octagon.

# Acknowledgment

The author wishes to thank Anurag Bishnoi for verifying certain of his computations.

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