Case I: The case where (s, t', t) *is equal to* $(r, r, 2r)$.

> `s := r; t2 := r; alpha := r;`

$$s := r$$
$$t2 := r$$
$$\alpha := r \qquad (1)$$

> $Eig1 := \alpha^2 + s \cdot t2 + s;\ Mult1 := 1;$

$$Eig1 := 2r^2 + r$$
$$Mult1 := 1 \qquad (2)$$

> $Eig2 := s + \alpha - t2 - 1;\ Mult2 := simplify\left(\frac{1}{2}\,\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)(\alpha^2+s\,t2)\,\alpha^2\,s}{(\alpha^2+\alpha\,s+s^2)(\alpha^2-\alpha\,t2+t2^2)}\right);$

$$Eig2 := r - 1$$
$$Mult2 := \frac{1}{3}(r^2+r+1)(r^2-r+1)(r^2+1)\,r \qquad (3)$$

> $Eig3 := s - \alpha - t2 - 1;\ Mult3 := simplify\left(\frac{1}{2}\,\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)(\alpha^2+s\,t2)\,\alpha^2\,s}{(\alpha^2-\alpha\,s+s^2)(\alpha^2+\alpha\,t2+t2^2)}\right);$

$$Eig3 := -r - 1$$
$$Mult3 := \frac{1}{3}(r^2+r+1)(r^2-r+1)(r^2+1)\,r \qquad (4)$$

> $Eig4 := s \cdot t2 + s - \alpha - 1;\ Mult4 := simplify\left(\frac{(\alpha^2-\alpha+1)(s\,t2+1)(\alpha^2+s\,t2)\,\alpha^2}{2\,s^2\,t2^2 - 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig4 := r^2 - 1$$
$$Mult4 := r^2(r^2+1) \qquad (5)$$

> $Eig5 := s \cdot t2 + s + \alpha - 1;\ Mult5 := simplify\left(\frac{(\alpha^2+\alpha+1)(s\,t2+1)(\alpha^2+s\,t2)\,\alpha^2}{2\,s^2\,t2^2 + 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig5 := r^2 + 2r - 1$$
$$Mult5 := r^2(r^2+1) \qquad (6)$$

> $Eig6 := simplify\left(-\frac{(\alpha^2 + s \cdot t2 + s)}{s}\right);\ Mult6 := simplify\left(\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)\,s^6}{(\alpha^4+\alpha^2\,s^2+s^4)(s+t2)}\right);$

$$Eig6 := -2r - 1$$

(7)

$$Mult6 := \frac{1}{6}\left(r^2+r+1\right)\left(r^2-r+1\right)\left(r^2+1\right) r \quad \mathbf{(7)}$$

> $Eig7 := simplify\left(\frac{\left(\alpha^2+s\cdot t2-t2\right)}{t2}\right); Mult7 := simplify\left(\frac{t2^5\, s\left(\alpha^2+\alpha+1\right)\left(\alpha^2-\alpha+1\right)\left(s\, t2+1\right)}{\left(\alpha^2+\alpha\, t2+t2^2\right)\left(\alpha^2-\alpha\, t2+t2^2\right)\left(s+t2\right)}\right);$

$$Eig7 := 2\, r-1$$

$$Mult7 := \frac{1}{6}\left(r^2+r+1\right)\left(r^2-r+1\right)\left(r^2+1\right) r \quad \mathbf{(8)}$$

> $Eig8 := simplify\left(\frac{\left(s^2\cdot t2-s\cdot t2-\alpha^2\right)}{s\cdot t2}\right); Mult8 := simplify\left(\frac{\left(\alpha^4+\alpha^2+1\right) s^5\, t2^5}{s^4\, t2^4+\alpha^2\, s^2\, t2^2+\alpha^4}\right);$

$$Eig8 := r-2$$

$$Mult8 := r^6 \quad \mathbf{(9)}$$

 # All multiplicities are integral.

 # Case II: The case where (s, t', t) is equal to $\left(r, r^2, 2\, r^2\right)$. We put $r = u^2$ with $u > 1$.

> $s := u^2; t2 := u^4;$ alpha $:= u^3;$

$$s := u^2$$

$$t2 := u^4$$

$$\alpha := u^3 \quad \mathbf{(10)}$$

> $Eig1 := \alpha^2+s\cdot t2+s; Mult1 := 1;$

$$Eig1 := 2\, u^6+u^2$$

$$Mult1 := 1 \quad \mathbf{(11)}$$

> $Eig2 := s+\alpha-t2-1; Mult2 := simplify\left(\frac{1}{2}\,\frac{\left(\alpha^2+\alpha+1\right)\left(\alpha^2-\alpha+1\right)\left(s\, t2+1\right)\left(\alpha^2+s\, t2\right)\alpha^2\, s}{\left(\alpha^2+\alpha\, s+s^2\right)\left(\alpha^2-\alpha\, t2+t2^2\right)}\right);$

$$Eig2 := -u^4+u^3+u^2-1$$

$$Mult2 := \frac{u^{22}+2\, u^{16}+2\, u^{10}+u^4}{\left(u^2+u+1\right)\left(u^2-u+1\right)} \quad \mathbf{(12)}$$

> $Eig3 := s - \alpha - t2 - 1;$ $Mult3 :=$
$simplify\left(\frac{1}{2}\,\frac{(\alpha^2+\alpha+1)\,(\alpha^2-\alpha+1)\,(s\,t2+1)\,(\alpha^2+s\,t2)\,\alpha^2\,s}{(\alpha^2-\alpha\,s+s^2)\,(\alpha^2+\alpha\,t2+t2^2)}\right);$

$$Eig3 := -u^4 - u^3 + u^2 - 1$$

$$Mult3 := \frac{u^{22} + 2\,u^{16} + 2\,u^{10} + u^4}{(u^2+u+1)\,(u^2-u+1)} \quad \textbf{(13)}$$

> $Eig4 := s\cdot t2 + s - \alpha - 1;$ $Mult4 := simplify\left(\frac{(\alpha^2-\alpha+1)\,(s\,t2+1)\,(\alpha^2+s\,t2)\,\alpha^2}{2\,s^2\,t2^2 - 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig4 := u^6 - u^3 + u^2 - 1$$

$$Mult4 := u^{12} + u^6 \quad \textbf{(14)}$$

> $Eig5 := s\cdot t2 + s + \alpha - 1;$ $Mult5 := simplify\left(\frac{(\alpha^2+\alpha+1)\,(s\,t2+1)\,(\alpha^2+s\,t2)\,\alpha^2}{2\,s^2\,t2^2 + 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig5 := u^6 + u^3 + u^2 - 1$$

$$Mult5 := u^{12} + u^6 \quad \textbf{(15)}$$

> $Eig6 := simplify\left(-\frac{(\alpha^2 + s\cdot t2 + s)}{s}\right);$ $Mult6 :=$
$simplify\left(\frac{(\alpha^2+\alpha+1)\,(\alpha^2-\alpha+1)\,(s\,t2+1)\,s^6}{(\alpha^4+\alpha^2\,s^2+s^4)\,(s+t2)}\right);$

$$Eig6 := -2\,u^4 - 1$$

$$Mult6 := \frac{u^2\,(u^4-u^2+1)\,(u^6-u^3+1)\,(u^6+u^3+1)}{u^4+u^2+1} \quad \textbf{(16)}$$

> $Eig7 := simplify\left(\frac{(\alpha^2 + s\cdot t2 - t2)}{t2}\right);$ $Mult7 :=$
$simplify\left(\frac{t2^5\,s\,(\alpha^2+\alpha+1)\,(\alpha^2-\alpha+1)\,(s\,t2+1)}{(\alpha^2+\alpha\,t2+t2^2)\,(\alpha^2-\alpha\,t2+t2^2)\,(s+t2)}\right);$

$$Eig7 := 2\,u^2 - 1$$

$$Mult7 := \frac{(u^4-u^2+1)\,(u^6-u^3+1)\,(u^6+u^3+1)\,u^8}{(u^2-u+1)\,(u^2+u+1)} \quad \textbf{(17)}$$

> $Eig8 := simplify\left(\frac{(s^2\cdot t2 - s\cdot t2 - \alpha^2)}{s\cdot t2}\right);$ $Mult8 := simplify\left(\frac{(\alpha^4+\alpha^2+1)\,s^5\,t2^5}{s^4\,t2^4 + \alpha^2\,s^2\,t2^2 + \alpha^4}\right);$

$$Eig8 := u^2 - 2$$

$$Mult8 := u^{18} \quad \textbf{(18)}$$

> $rem(u^{18} + 2\,u^{12} + 2\,u^6 + 1,\, u^4 + u^2 + 1,\, u);$

$$6 \quad \textbf{(19)}$$

All multiplicities can be written as a rational function in the variable $r = u^2$ (*not true for the eigenvalues*)*. The GCD of*
u^4 *and* $u^4 + u^2 + 1$ *is* 1*. So, Mult2 is integral if and only if* $u^{18} + 2u^{12} + 2u^6 + 1$ *is divisible by* $u^4 + u^2 + 1$*. The*
latter is however not true as the remainder of the division is equal to $6 < u^4 + u^2 + 1$.

Case III: The case where (s, t', t) *is equal to* $\left(r, r, r\left(r^2 + 1\right)\right)$.

> $s := r;\ t2 := r;\ \text{alpha} := r^2;$

$$s := r$$
$$t2 := r$$
$$\alpha := r^2 \qquad \textbf{(20)}$$

> $Eig1 := \alpha^2 + s \cdot t2 + s;\ Mult1 := 1;$

$$Eig1 := r^4 + r^2 + r$$
$$Mult1 := 1 \qquad \textbf{(21)}$$

> $Eig2 := s + \alpha - t2 - 1;\ Mult2 := \text{simplify}\left(\frac{1}{2}\,\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)(\alpha^2+s\,t2)\,\alpha^2\,s}{(\alpha^2+\alpha\,s+s^2)(\alpha^2-\alpha\,t2+t2^2)}\right);$

$$Eig2 := r^2 - 1$$
$$Mult2 := \frac{1}{2}\,(r^4 - r^2 + 1)\,(r^2+1)^2\,r^3 \qquad \textbf{(22)}$$

> $Eig3 := s - \alpha - t2 - 1;\ Mult3 := \text{simplify}\left(\frac{1}{2}\,\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)(\alpha^2+s\,t2)\,\alpha^2\,s}{(\alpha^2-\alpha\,s+s^2)(\alpha^2+\alpha\,t2+t2^2)}\right);$

$$Eig3 := -r^2 - 1$$

$$Mult3 := \frac{1}{2}\left(r^4 - r^2 + 1\right)\left(r^2 + 1\right)^2 r^3 \quad (23)$$

> $Eig4 := s \cdot t2 + s - \alpha - 1; Mult4 := simplify\left(\frac{\left(\alpha^2 - \alpha + 1\right)\left(s\ t2 + 1\right)\left(\alpha^2 + s\ t2\right)\alpha^2}{2\ s^2\ t2^2 - 2\ \alpha\ s\ t2 + 2\ \alpha^2}\right);$

$$Eig4 := r - 1$$

$$Mult4 := \frac{1}{2}\left(r^4 - r^2 + 1\right)\left(r^2 + 1\right)^2 r^2 \quad (24)$$

> $Eig5 := s \cdot t2 + s + \alpha - 1; Mult5 := simplify\left(\frac{\left(\alpha^2 + \alpha + 1\right)\left(s\ t2 + 1\right)\left(\alpha^2 + s\ t2\right)\alpha^2}{2\ s^2\ t2^2 + 2\ \alpha\ s\ t2 + 2\ \alpha^2}\right);$

$$Eig5 := 2\ r^2 + r - 1$$

$$Mult5 := \frac{1}{6}\left(r^4 + r^2 + 1\right)\left(r^2 + 1\right)^2 r^2 \quad (25)$$

> $Eig6 := simplify\left(-\frac{\left(\alpha^2 + s \cdot t2 + s\right)}{s}\right); Mult6 := simplify\left(\frac{\left(\alpha^2 + \alpha + 1\right)\left(\alpha^2 - \alpha + 1\right)\left(s\ t2 + 1\right) s^6}{\left(\alpha^4 + \alpha^2\ s^2 + s^4\right)\left(s + t2\right)}\right);$

$$Eig6 := -r^3 - r - 1$$

$$Mult6 := \frac{1}{2}\ r\left(r^2 + 1\right)\left(r^4 - r^2 + 1\right) \quad (26)$$

> $Eig7 := simplify\left(\frac{\left(\alpha^2 + s \cdot t2 - t2\right)}{t2}\right); Mult7 := simplify\left(\frac{t2^5\ s\left(\alpha^2 + \alpha + 1\right)\left(\alpha^2 - \alpha + 1\right)\left(s\ t2 + 1\right)}{\left(\alpha^2 + \alpha\ t2 + t2^2\right)\left(\alpha^2 - \alpha\ t2 + t2^2\right)\left(s + t2\right)}\right);$

$$Eig7 := r^3 + r - 1$$

$$Mult7 := \frac{1}{2}\ r\left(r^2 + 1\right)\left(r^4 - r^2 + 1\right) \quad (27)$$

> $Eig8 := simplify\left(\frac{\left(s^2 \cdot t2 - s \cdot t2 - \alpha^2\right)}{s \cdot t2}\right); Mult8 := simplify\left(\frac{\left(\alpha^4 + \alpha^2 + 1\right) s^5\ t2^5}{s^4\ t2^4 + \alpha^2\ s^2\ t2^2 + \alpha^4}\right);$

$$Eig8 := -r^2 + r - 1$$

$$Mult8 := \frac{1}{3}\left(r^8 + r^4 + 1\right) r^2 \quad (28)$$

 # All multiplicities are integral.

Case IV: The case where (s, t', t) *is equal to* $\left(r, r^2, r^2\left(r^3 + 1\right)\right)$.

> $s := r;\ t2 := r^2;\ \text{alpha} := r^3;$

$$s := r$$
$$t2 := r^2$$
$$\alpha := r^3 \tag{29}$$

> $Eig1 := \alpha^2 + s \cdot t2 + s;\ Mult1 := 1;$

$$Eig1 := r^6 + r^3 + r$$
$$Mult1 := 1 \tag{30}$$

> $Eig2 := s + \alpha - t2 - 1;\ Mult2 := simplify\left(\frac{1}{2}\,\frac{\left(\alpha^2 + \alpha + 1\right)\left(\alpha^2 - \alpha + 1\right)(s\,t2 + 1)\left(\alpha^2 + s\,t2\right)\alpha^2\,s}{\left(\alpha^2 + \alpha\,s + s^2\right)\left(\alpha^2 - \alpha\,t2 + t2^2\right)}\right);$

$$Eig2 := r^3 - r^2 + r - 1$$
$$Mult2 := \frac{\left(r^6 + r^3 + 1\right)\left(r^6 - r^3 + 1\right)(r + 1)^2\,r^4}{2\,r^2 + 2\,r + 2} \tag{31}$$

> $Eig3 := s - \alpha - t2 - 1;\ Mult3 := simplify\left(\frac{1}{2}\,\frac{\left(\alpha^2 + \alpha + 1\right)\left(\alpha^2 - \alpha + 1\right)(s\,t2 + 1)\left(\alpha^2 + s\,t2\right)\alpha^2\,s}{\left(\alpha^2 - \alpha\,s + s^2\right)\left(\alpha^2 + \alpha\,t2 + t2^2\right)}\right);$

$$Eig3 := -r^3 - r^2 + r - 1$$
$$Mult3 := \frac{1}{2}\,\frac{r^{22} + 2\,r^{19} + 2\,r^{16} + 2\,r^{13} + 2\,r^{10} + 2\,r^7 + r^4}{\left(r^4 - r^2 + 1\right)\left(r^2 + r + 1\right)} \tag{32}$$

> $Eig4 := s \cdot t2 + s - \alpha - 1;\ Mult4 := simplify\left(\frac{\left(\alpha^2 - \alpha + 1\right)(s\,t2 + 1)\left(\alpha^2 + s\,t2\right)\alpha^2}{2\,s^2\,t2^2 - 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig4 := r - 1$$
$$Mult4 := \frac{1}{2}\,r^3\left(r^{12} + r^9 + r^3 + 1\right) \tag{33}$$

> $Eig5 := s \cdot t2 + s + \alpha - 1;\ Mult5 := simplify\left(\frac{\left(\alpha^2 + \alpha + 1\right)(s\,t2 + 1)\left(\alpha^2 + s\,t2\right)\alpha^2}{2\,s^2\,t2^2 + 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig5 := 2\,r^3 + r - 1$$
$$Mult5 := \frac{1}{6}\left(r^6 + r^3 + 1\right)\left(r^3 + 1\right)^2 r^3 \tag{34}$$

> $Eig6 := simplify\left(-\frac{(\alpha^2 + s\cdot t2 + s)}{s}\right)$; $Mult6 := simplify\left(\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)\,s^6}{(\alpha^4+\alpha^2 s^2+s^4)(s+t2)}\right)$;

$$Eig6 := -r^5 - r^2 - 1$$

$$Mult6 := \frac{(r^6+r^3+1)(r^6-r^3+1)\,r}{r^6+r^5-r^3+r+1} \quad (35)$$

> $Eig7 := simplify\left(\frac{(\alpha^2 + s\cdot t2 - t2)}{t2}\right)$; $Mult7 := simplify\left(\frac{t2^5\,s\,(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)}{(\alpha^2+\alpha\,t2+t2^2)(\alpha^2-\alpha\,t2+t2^2)(s+t2)}\right)$;

$$Eig7 := r^4 + r - 1$$

$$Mult7 := \frac{(r^6-r^3+1)(r^6+r^3+1)\,r^2}{r^2+r+1} \quad (36)$$

> $Eig8 := simplify\left(\frac{(s^2\cdot t2 - s\cdot t2 - \alpha^2)}{s\cdot t2}\right)$; $Mult8 := simplify\left(\frac{(\alpha^4+\alpha^2+1)\,s^5\,t2^5}{s^4\,t2^4+\alpha^2 s^2\,t2^2+\alpha^4}\right)$;

$$Eig8 := -r^3 + r - 1$$

$$Mult8 := \frac{1}{3}(r^{12}+r^6+1)\,r^3 \quad (37)$$

> $rem(r^{12} + r^6 + 1, r^2 + r + 1, r)$;

$$3 \quad (38)$$

 # The GCD of r^2 and $r^2 + r + 1$ is 1. So, Mult 7 is integral if and only if $r^{12} + r^6 + 1$ is divisible by $r^2 + r + 1$.
 # The latter is however not true as the remainder of the division is equal to $3 < r^2 + r + 1$.

 # Note that Eig5=Eig7 for r=2. Since Mult5 is always integral, the above conclusion remains valid for r=2.

Case V: The case where (s, t', t) is equal to $\left(r, r, r\left(r^6 + 1\right)\right)$.

> $s := r$; $t2 := r$; alpha $:= r^4$;

$$s := r$$
$$t2 := r$$
$$\alpha := r^4 \quad (39)$$

> $Eig1 := \alpha^2 + s \cdot t2 + s$; $Mult1 := 1$;

$$Eig1 := r^8 + r^2 + r$$
$$Mult1 := 1 \quad (40)$$

> $Eig2 := s + \alpha - t2 - 1$; $Mult2 := simplify\left(\frac{1}{2} \frac{(\alpha^2 + \alpha + 1)(\alpha^2 - \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\alpha^2 s}{(\alpha^2 + \alpha s + s^2)(\alpha^2 - \alpha\,t2 + t2^2)}\right)$;

$$Eig2 := r^4 - 1$$
$$Mult2 := \frac{1}{2} \frac{(r^8 + r^4 + 1)(r^8 - r^4 + 1)(r^2 + 1) r^7 (r^6 + 1)}{(r^6 + r^3 + 1)(r^6 - r^3 + 1)} \quad (41)$$

> $Eig3 := s - \alpha - t2 - 1$; $Mult3 := simplify\left(\frac{1}{2} \frac{(\alpha^2 + \alpha + 1)(\alpha^2 - \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\alpha^2 s}{(\alpha^2 - \alpha s + s^2)(\alpha^2 + \alpha\,t2 + t2^2)}\right)$;

$$Eig3 := -r^4 - 1$$
$$Mult3 := \frac{1}{2} \frac{(r^8 + r^4 + 1)(r^8 - r^4 + 1)(r^2 + 1) r^7 (r^6 + 1)}{(r^6 + r^3 + 1)(r^6 - r^3 + 1)} \quad (42)$$

> $Eig4 := s \cdot t2 + s - \alpha - 1$; $Mult4 := simplify\left(\frac{(\alpha^2 - \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\alpha^2}{2 s^2 t2^2 - 2 \alpha s\,t2 + 2 \alpha^2}\right)$;

$$Eig4 := -r^4 + r^2 + r - 1$$
$$Mult4 := \frac{1}{2} (r^2 + 1)^2 r^6 (r^8 - r^4 + 1) \quad (43)$$

> $Eig5 := s \cdot t2 + s + \alpha - 1$; $Mult5 := simplify\left(\frac{(\alpha^2 + \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\alpha^2}{2 s^2 t2^2 + 2 \alpha s\,t2 + 2 \alpha^2}\right)$;

$$Eig5 := r^4 + r^2 + r - 1$$
$$Mult5 := \frac{1}{2} (r^2 + 1)^2 (r^4 - r^2 + 1)^2 r^6 \quad (44)$$

> $Eig6 := simplify\left(-\frac{(\alpha^2 + s\cdot t2 + s)}{s}\right); Mult6 := simplify\left(\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)\,s^6}{(\alpha^4+\alpha^2 s^2+s^4)(s+t2)}\right);$

$$Eig6 := -r^7 - r - 1$$

$$Mult6 := \frac{r^{19}+r^{17}+r^{11}+r^9+r^3+r}{2\,r^{12}+2\,r^6+2} \tag{45}$$

> $Eig7 := simplify\left(\frac{(\alpha^2 + s\cdot t2 - t2)}{t2}\right); Mult7 := simplify\left(\frac{t2^5\, s\,(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)}{(\alpha^2+\alpha\,t2+t2^2)(\alpha^2-\alpha\,t2+t2^2)(s+t2)}\right);$

$$Eig7 := r^7 + r - 1$$

$$Mult7 := \frac{1}{2}\,\frac{r^{19}+r^{17}+r^{11}+r^9+r^3+r}{(r^6+r^3+1)(r^6-r^3+1)} \tag{46}$$

> $Eig8 := simplify\left(\frac{(s^2\cdot t2 - s\cdot t2 - \alpha^2)}{s\cdot t2}\right); Mult8 := simplify\left(\frac{(\alpha^4+\alpha^2+1)\,s^5\,t2^5}{s^4\,t2^4+\alpha^2 s^2\,t2^2+\alpha^4}\right);$

$$Eig8 := -r^6 + r - 1$$

$$Mult8 := r^2\,(r^8 - r^4 + 1) \tag{47}$$

> $rem(r^{18}+r^{16}+r^{10}+r^8+r^2+1, r^6-r^3+1, r);$

$$r^5 - r^4 + 2 \tag{48}$$

The GCD of r and $r^6 - r^3 + 1$ *is 1. So, if Mult 7 is integral then* $r^{18} + r^{16} + r^{10} + r^8 + r^2 + 1$ *is divisible by* $r^6 - r^3 + 1$.

The latter is however not true as the remainder of the division is equal to $r^5 - r^4 + 2 < r^6 - r^3 + 1$.

Case VI: The case where (s, t', t) *is equal to* $\left(r, r^2, r^2\left(r^9\right.\right.$

$+ 1))$

.

> $s := r; t2 := r^2; \text{alpha} := r^6;$

$$s := r$$
$$t2 := r^2$$
$$\alpha := r^6 \quad \textbf{(49)}$$

> $Eig1 := \alpha^2 + s \cdot t2 + s; Mult1 := 1;$

$$Eig1 := r^{12} + r^3 + r$$
$$Mult1 := 1 \quad \textbf{(50)}$$

> $Eig2 := s + \alpha - t2 - 1; Mult2 := simplify\left(\frac{1}{2} \frac{(\alpha^2 + \alpha + 1)(\alpha^2 - \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\,\alpha^2\,s}{(\alpha^2 + \alpha\,s + s^2)(\alpha^2 - \alpha\,t2 + t2^2)}\right);$

$$Eig2 := r^6 - r^2 + r - 1$$
$$Mult2 := \frac{1}{2} \frac{(r^{12} + r^6 + 1)(r^{12} - r^6 + 1)(r^3 + 1)\,r^{10}\,(r^9 + 1)}{(r^{10} + r^5 + 1)(r^8 - r^4 + 1)} \quad \textbf{(51)}$$

> $Eig3 := s - \alpha - t2 - 1; Mult3 := simplify\left(\frac{1}{2} \frac{(\alpha^2 + \alpha + 1)(\alpha^2 - \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\,\alpha^2\,s}{(\alpha^2 - \alpha\,s + s^2)(\alpha^2 + \alpha\,t2 + t2^2)}\right);$

$$Eig3 := -r^6 - r^2 + r - 1$$
$$Mult3 := \frac{1}{2} \frac{(r^6 + r^3 + 1)(r^6 - r^3 + 1)^2(r^{12} - r^6 + 1)(r + 1)^2\,r^{10}}{(r^4 - r^2 + 1)(r^2 + r + 1)(r^8 + r^7 - r^5 - r^4 - r^3 + r + 1)} \quad \textbf{(52)}$$

> $Eig4 := s \cdot t2 + s - \alpha - 1; Mult4 := simplify\left(\frac{(\alpha^2 - \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\,\alpha^2}{2\,s^2\,t2^2 - 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig4 := -r^6 + r^3 + r - 1$$
$$Mult4 := \frac{1}{2}(r^3 + 1)^2\,r^9\,(r^{12} - r^6 + 1) \quad \textbf{(53)}$$

> $Eig5 := s \cdot t2 + s + \alpha - 1; Mult5 := simplify\left(\frac{(\alpha^2 + \alpha + 1)(s\,t2 + 1)(\alpha^2 + s\,t2)\,\alpha^2}{2\,s^2\,t2^2 + 2\,\alpha\,s\,t2 + 2\,\alpha^2}\right);$

$$Eig5 := r^6 + r^3 + r - 1$$
$$Mult5 := \frac{1}{2}(r^9 + 1)\,r^9\,(r^3 + 1)(r^6 - r^3 + 1) \quad \textbf{(54)}$$

> $Eig6 := simplify\left(-\frac{(\alpha^2 + s \cdot t2 + s)}{s}\right); Mult6 :=$

$$simplify\left(\frac{(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)s^6}{(\alpha^4+\alpha^2 s^2+s^4)(s+t2)}\right);$$

$$Eig6 := -r^{11}-r^2-1$$

$$Mult6 := \frac{r^{25}+r^{13}+r}{r^{18}+r^{17}-r^{15}-r^{14}+r^{12}+r^{11}-r^9+r^7+r^6-r^4-r^3+r+1} \quad \textbf{(55)}$$

$$> Eig7 := simplify\left(\frac{(\alpha^2+s\cdot t2-t2)}{t2}\right); Mult7 := simplify\left(\frac{t2^5\, s\,(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(s\,t2+1)}{(\alpha^2+\alpha\, t2+t2^2)(\alpha^2-\alpha\, t2+t2^2)(s+t2)}\right);$$

$$Eig7 := r^{10}+r-1$$

$$Mult7 := \frac{r^{26}+r^{14}+r^2}{(r^8-r^4+1)(r^6+r^5-r^3+r+1)} \quad \textbf{(56)}$$

$$> Eig8 := simplify\left(\frac{(s^2\cdot t2-s\cdot t2-\alpha^2)}{s\cdot t2}\right); Mult8 := simplify\left(\frac{(\alpha^4+\alpha^2+1)s^5\, t2^5}{s^4\, t2^4+\alpha^2 s^2\, t2^2+\alpha^4}\right);$$

$$Eig8 := -r^9+r-1$$

$$Mult8 := r^3(r^{12}-r^6+1) \quad \textbf{(57)}$$

$$> rem(r^{26}+r^{14}+r^2, r^8-r^4+1, r);$$

$$r^2 \quad \textbf{(58)}$$

 # If Mult 7 is integral then $r^{26}+r^{14}+r^2$ *is divisible by* r^8-r^4+1.

 # The latter is however not true as the remainder of the division is equal to $r^2 < r^8-r^4+1$.