Case I: The quads have order (q, q)

. The generalized octagon has order $\left(q^2, 1\right)$.

```
> restart;
> with(LinearAlgebra) :
> q := q;
```

$$q := q \tag{1}$$

```
> s := q; t := 2·q; t2 := q;
```

$$s := q$$
$$t := 2\,q$$
$$t2 := q \tag{2}$$

```
> v := (s + 1)·(1 + s·t + s^2·t·(t − t2) + s^3·t·(t − t2)^2 + s^4·t2·(t − t2)^3);
```

$$v := (q + 1)\left(q^8 + 2\,q^6 + 2\,q^4 + 2\,q^2 + 1\right) \tag{3}$$

```
> M := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [s, s − 1, 0, 1, 0, 0, 0, 0, 0, 0], [s·t, 0, s − 1, t2, 1,
    0, 0, 0, 0, 0], [0, s·t, s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0, 0, 0], [0, 0, s·(t − t2), 0, s − 1, t2,
    1, 0, 0, 0], [0, 0, 0, s·(t − t2), s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0], [0, 0, 0, 0, s·(t − t2), 0, s
    − 1, t2, t/t2, 0], [0, 0, 0, 0, 0, s·(t − t2), s·t2, (s − 1)·(t2 + 1), 0, t/t2], [0, 0, 0, 0, 0, 0, s
    ·(t − t2), 0, t/t2·(s − 1), t + 1 − t/t2], [0, 0, 0, 0, 0, 0, 0, s·(t − t2), s·(t + 1 − t/t2), (s
    − 1)·(t + 1)]]);
```

$$M := \begin{bmatrix} [0, 1, 1, 0, 0, 0, 0, 0, 0, 0] \\ [q, q-1, 0, 1, 0, 0, 0, 0, 0, 0] \\ [2\,q^2, 0, q-1, q, 1, 0, 0, 0, 0, 0] \\ [0, 2\,q^2, q^2, (q+1)\,(q-1), 0, 1, 0, 0, 0, 0] \\ [0, 0, q^2, 0, q-1, q, 1, 0, 0, 0] \\ [0, 0, 0, q^2, q^2, (q+1)\,(q-1), 0, 1, 0, 0] \\ [0, 0, 0, 0, q^2, 0, q-1, q, 2, 0] \\ [0, 0, 0, 0, 0, q^2, q^2, (q+1)\,(q-1), 0, 2] \\ [0, 0, 0, 0, 0, 0, q^2, 0, 2\,q-2, 2\,q-1] \\ [0, 0, 0, 0, 0, 0, 0, q^2, q\,(2\,q-1), (q-1)\,(2\,q+1)] \end{bmatrix} \tag{4}$$

```
> factor(CharacteristicPolynomial(M, x));
```

$$-(x + 1)\left(2\,q^2 + q - x\right)\left(q^2 + q - x - 1\right)(2\,q + 1 + x)\,(2\,q - 1 - x)\,(q - 2 - x)\left(2\,q^2\right. \tag{5}$$

$$-x^2-2x-1)\,(q^4+2q^3-2q^2x-3q^2-2qx+x^2-2q+2x+1)$$

```
> k := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0]]):
> a1 := v:
> a2 := v·k[1][1]:
> a3 := v·Multiply(k, M)[1][1]:
> a4 := v·Multiply(k, M^2)[1][1]:
> a5 := v·Multiply(k, M^3)[1][1]:
> a6 := v·Multiply(k, M^4)[1][1]:
> a7 := v·Multiply(k, M^5)[1][1]:
> a8 := v·Multiply(k, M^6)[1][1]:
> a9 := v·Multiply(k, M^7)[1][1]:
> a10 := v·Multiply(k, M^8)[1][1]:
> A := Matrix([[a1], [a2], [a3], [a4], [a5], [a6], [a7], [a8], [a9], [a10]]):
> V := Transpose(VandermondeMatrix([[-1, 2q^2+q, q^2+q-1, -(2·q+1), 2·q-1, q-2,
      -1+q·2^(1/2), -1-q·2^(1/2), q^2+q-1+q·2^(1/2), q^2+q-1-q·2^(1/2)]])):
> J := Multiply(V^(-1), A):
> A1 := simplify(simplify(J[1][1])):
> B1 := simplify(expand(numer(A1))):
> C1 := simplify(expand(denom(A1))):
> D1 := simplify(B1/C1);
```

$$D1 := \frac{1}{4}\,(q^4+1)\,q\,(q^2+1)^2 \tag{6}$$

```
> A2 := simplify(simplify(J[2][1])):
> B2 := simplify(expand(numer(A2))):
> C2 := simplify(expand(denom(A2))):
> D2 := simplify(B2/C2);
```

$$D2 := 1 \tag{7}$$

```
> A3 := simplify(simplify(J[3][1])):
> B3 := simplify(expand(numer(A3))):
> C3 := simplify(expand(denom(A3))):
> D3 := simplify(B3/C3);
```

$$D3 := (q^4+1)\,q^2 \tag{8}$$

```
> A4 := simplify(simplify(J[4][1])):
> B4 := simplify(expand(numer(A4))):
> C4 := simplify(expand(denom(A4))):
```

```
> D4 := simplify(B4/C4);
```

$$D4 := \frac{1}{8}\left(q^4+1\right) q \left(q^2+1\right)^2 \tag{9}$$

```
> A5 := simplify(simplify(J[5][1])):
> B5 := simplify(expand(numer(A5))):
> C5 := simplify(expand(denom(A5))):
> D5 := simplify(B5/C5);
```

$$D5 := \frac{1}{8}\left(q^4+1\right) q \left(q^2+1\right)^2 \tag{10}$$

```
> A6 := simplify(simplify(J[6][1])):
> B6 := simplify(expand(numer(A6))):
> C6 := simplify(expand(denom(A6))):
> D6 := simplify(B6/C6);
```

$$D6 := q^8 \tag{11}$$

```
> A7 := simplify(simplify(J[7][1])):
> B7 := simplify(expand(numer(A7))):
> C7 := simplify(expand(denom(A7))):
> D7 := simplify(B7/C7);
```

$$D7 := \frac{1}{4}\left(q^4+1\right) q \left(q^2+1\right)^2 \tag{12}$$

```
> A8 := simplify(simplify(J[8][1])):
> B8 := simplify(expand(numer(A8))):
> C8 := simplify(expand(denom(A8))):
> D8 := simplify(B8/C8);
```

$$D8 := \frac{1}{4}\left(q^4+1\right) q \left(q^2+1\right)^2 \tag{13}$$

```
> A9 := simplify(simplify(J[9][1])):
> B9 := simplify(expand(numer(A9))):
> C9 := simplify(expand(denom(A9))):
> D9 := simplify(B9/C9);
```

$$D9 := \frac{1}{2}\left(q^2+1\right)^2 q^2 \tag{14}$$

```
> A10 := simplify(simplify(J[10][1])):
> B10 := simplify(expand(numer(A10))):
> C10 := simplify(expand(denom(A10))):
```

```
> D10 := simplify( B10 / C10 );
```

$$D10 := \frac{1}{2}\left(q^2+1\right)^2 q^2 \tag{15}$$

The multiplicities are integral if and only if q is either odd or divisible by 8.

Case II: The quads have order (q, q) *. The generalized octagon has order* $\left(q^2, q\right)$.

```
> restart;
> with(LinearAlgebra) :
> assume(r, positive);
> q := 2·r^2;
```

$$q := 2\,r\sim^2 \tag{16}$$

```
> s := q; t := (q + 1)·q; t2 := q;
```

$$s := 2\,r\sim^2$$

$$t := 2\left(2\,r\sim^2+1\right)r\sim^2$$

$$t2 := 2\,r\sim^2 \tag{17}$$

```
> v := (s + 1)·(1 + s·t + s^2·t·(t − t2) + s^3·t·(t − t2)^2 + s^4·t2·(t − t2)^3);
```

$$v := \left(2\,r\sim^2+1\right)\left(1+4\,r\sim^4\left(2\,r\sim^2+1\right)+8\,r\sim^6\left(2\,r\sim^2+1\right)\left(2\left(2\,r\sim^2+1\right)r\sim^2-2\,r\sim^2\right)\right. \tag{18}$$
$$\left.+16\,r\sim^8\left(2\,r\sim^2+1\right)\left(2\left(2\,r\sim^2+1\right)r\sim^2-2\,r\sim^2\right)^2+32\,r\sim^{10}\left(2\left(2\,r\sim^2+1\right)r\sim^2\right.\right.$$
$$\left.\left.-2\,r\sim^2\right)^3\right)$$

```
> M := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [s, s − 1, 0, 1, 0, 0, 0, 0, 0, 0], [s·t, 0, s − 1, t2, 1,
    0, 0, 0, 0, 0], [0, s·t, s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0, 0, 0], [0, 0, s·(t − t2), 0, s − 1, t2,
    1, 0, 0, 0], [0, 0, 0, s·(t − t2), s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0], [0, 0, 0, 0, s·(t − t2), 0, s
    − 1, t2, t/t2, 0], [0, 0, 0, 0, 0, s·(t − t2), s·t2, (s − 1)·(t2 + 1), 0, t/t2], [0, 0, 0, 0, 0, 0, s
    ·(t − t2), 0, t/t2·(s − 1), t + 1 − t/t2], [0, 0, 0, 0, 0, 0, 0, s·(t − t2), s·(t + 1 − t/t2), (s
    − 1)·(t + 1)]]);
```

$$M := \left[\left[0, 1, 1, 0, 0, 0, 0, 0, 0, 0\right],\right. \tag{19}$$
$$\left[2\,r\sim^2, 2\,r\sim^2 - 1, 0, 1, 0, 0, 0, 0, 0, 0\right],$$
$$\left[4\,r\sim^4\left(2\,r\sim^2 + 1\right), 0, 2\,r\sim^2 - 1, 2\,r\sim^2, 1, 0, 0, 0, 0, 0\right],$$
$$\left[0, 4\,r\sim^4\left(2\,r\sim^2 + 1\right), 4\,r\sim^4, \left(2\,r\sim^2 + 1\right)\left(2\,r\sim^2 - 1\right), 0, 1, 0, 0, 0, 0\right],$$
$$\left[0, 0, 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), 0, 2\,r\sim^2 - 1, 2\,r\sim^2, 1, 0, 0, 0\right],$$
$$\left[0, 0, 0, 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), 4\,r\sim^4, \left(2\,r\sim^2 + 1\right)\left(2\,r\sim^2 - 1\right), 0, 1, 0, 0\right],$$
$$\left[0, 0, 0, 0, 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), 0, 2\,r\sim^2 - 1, 2\,r\sim^2, 2\,r\sim^2 + 1, 0\right],$$
$$\left[0, 0, 0, 0, 0, 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), 4\,r\sim^4, \left(2\,r\sim^2 + 1\right)\left(2\,r\sim^2 - 1\right), 0, 2\,r\sim^2 + 1\right],$$
$$\left[0, 0, 0, 0, 0, 0, 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), 0, \left(2\,r\sim^2 + 1\right)\left(2\,r\sim^2 - 1\right), 2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right],$$
$$\left[0, 0, 0, 0, 0, 0, 0, 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), 2\,r\sim^2\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 - 2\,r\sim^2\right), \left(2\,r\sim^2 - 1\right)\left(2\left(2\,r\sim^2 + 1\right)r\sim^2 + 1\right)\right]\right]$$

```
> factor(CharacteristicPolynomial(M, x));
```

$$(x+1)^2\left(4\,r\sim^4 + 2\,r\sim^2 + x + 1\right)\left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2 - x\right)\left(4\,r\sim^3 + x + 1\right)\left(4\,r\sim^3 - x - 1\right)\left(4\,r\sim^4 - 4\,r\sim^3 + 2\,r\sim^2 - x - 1\right)\left(4\,r\sim^4 + 4\,r\sim^3 + 2\,r\sim^2 - x - 1\right)\left(4\,r\sim^4 + 2\,r\sim^2 - x - 1\right)^2 \tag{20}$$

```
> k := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0]]) :
> a1 := v :
> a2 := v· k[1][1] :
> a3 := v· Multiply( k, M)[1][1] :
> a4 := v· Multiply( k, M^2)[1][1] :
> a5 := v· Multiply( k, M^3)[1][1] :
> a6 := v· Multiply( k, M^4)[1][1] :
> a7 := v· Multiply( k, M^5)[1][1] :
> a8 := v· Multiply( k, M^6)[1][1] :
> A := Matrix([[a1], [a2], [a3], [a4], [a5], [a6], [a7], [a8]]) :
> V := Transpose(VandermondeMatrix([[-1, -(q^2 + q + 1), q^3 + q^2 + q, q^2 + q - 1, -1 + q·(2·q)^(1/2), -1 - q·(2·q)^(1/2), q^2 + q - 1 + q·(2·q)^(1/2), q^2 + q - 1 - q·(2·q)^(1/2)]])):
```

$$V := \left[\left[1, 1, 1, 1, 1, 1, 1, 1\right],\right. \tag{21}$$
$$\left[-1, -4\,r\sim^4 - 2\,r\sim^2 - 1, 8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2, 4\,r\sim^4 + 2\,r\sim^2 - 1, -1 + 4\,r\sim^2\sqrt{r\sim^2}, -1 - 4\,r\sim^2\sqrt{r\sim^2}, 4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}, 4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right],$$

$$\left[1, \left(-4\,r\sim^4 - 2\,r\sim^2 - 1\right)^2, \left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2\right)^2, \left(4\,r\sim^4 + 2\,r\sim^2 - 1\right)^2, \left(-1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^2, \left(-1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^2, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^2, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^2\right],$$

$$\left[-1, \left(-4\,r\sim^4 - 2\,r\sim^2 - 1\right)^3, \left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2\right)^3, \left(4\,r\sim^4 + 2\,r\sim^2 - 1\right)^3, \left(-1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^3, \left(-1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^3, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^3, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^3\right],$$

$$\left[1, \left(-4\,r\sim^4 - 2\,r\sim^2 - 1\right)^4, \left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2\right)^4, \left(4\,r\sim^4 + 2\,r\sim^2 - 1\right)^4, \left(-1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^4, \left(-1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^4, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^4, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^4\right],$$

$$\left[-1, \left(-4\,r\sim^4 - 2\,r\sim^2 - 1\right)^5, \left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2\right)^5, \left(4\,r\sim^4 + 2\,r\sim^2 - 1\right)^5, \left(-1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^5, \left(-1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^5, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^5, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^5\right],$$

$$\left[1, \left(-4\,r\sim^4 - 2\,r\sim^2 - 1\right)^6, \left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2\right)^6, \left(4\,r\sim^4 + 2\,r\sim^2 - 1\right)^6, \left(-1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^6, \left(-1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^6, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^6, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^6\right],$$

$$\left[-1, \left(-4\,r\sim^4 - 2\,r\sim^2 - 1\right)^7, \left(8\,r\sim^6 + 4\,r\sim^4 + 2\,r\sim^2\right)^7, \left(4\,r\sim^4 + 2\,r\sim^2 - 1\right)^7, \left(-1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^7, \left(-1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^7, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 + 4\,r\sim^2\sqrt{r\sim^2}\right)^7, \left(4\,r\sim^4 + 2\,r\sim^2 - 1 - 4\,r\sim^2\sqrt{r\sim^2}\right)^7\right]\Big]$$

```
> J := Multiply(V^(-1), A):
> A1 := simplify(simplify(J[1][1])):
> B1 := simplify(expand(numer(A1))):
> C1 := simplify(expand(denom(A1))):
> D1 := simplify(B1/C1);
```

$$D1 := 2048\,r\sim^{24} + 1024\,r\sim^{22} - 256\,r\sim^{18} + 384\,r\sim^{16} - 32\,r\sim^{12} + 16\,r\sim^{10} + 16\,r\sim^8 - 4\,r\sim^6 + 2\,r\sim^4 \quad \textbf{(22)}$$

```
> A2 := simplify(simplify(J[2][1])):
> B2 := simplify(expand(numer(A2))):
```

```
> C2 := simplify(expand(denom(A2))):
> D2 := simplify(B2/C2);
```

$$D2 := 256\,r\sim^{18} - 128\,r\sim^{16} + 64\,r\sim^{14} + 4\,r\sim^{6} - 2\,r\sim^{4} + r\sim^{2} \tag{23}$$

```
> A3 := simplify(simplify(J[3][1])):
> B3 := simplify(expand(numer(A3))):
> C3 := simplify(expand(denom(A3))):
> D3 := simplify(B3/C3);
```

$$D3 := 1 \tag{24}$$

```
> A4 := simplify(simplify(J[4][1])):
> B4 := simplify(expand(numer(A4))):
> C4 := simplify(expand(denom(A4))):
> D4 := simplify(B4/C4);
```

$$D4 := 512\,r\sim^{20} + 256\,r\sim^{18} + 64\,r\sim^{14} + 8\,r\sim^{8} + 4\,r\sim^{6} + r\sim^{2} \tag{25}$$

```
> A5 := simplify(simplify(J[5][1])):
> B5 := simplify(expand(numer(A5))):
> C5 := simplify(expand(denom(A5))):
> D5 := simplify(B5/C5);
```

$$D5 := 1024\,r\sim^{24} + 512\,r\sim^{22} + 128\,r\sim^{18} + 64\,r\sim^{16} + 16\,r\sim^{12} + 8\,r\sim^{10} + 2\,r\sim^{6} + r\sim^{4} \tag{26}$$

```
> A6 := simplify(simplify(J[6][1])):
> B6 := simplify(expand(numer(A6))):
> C6 := simplify(expand(denom(A6))):
> D6 := simplify(B6/C6);
```

$$D6 := 1024\,r\sim^{24} + 512\,r\sim^{22} + 128\,r\sim^{18} + 64\,r\sim^{16} + 16\,r\sim^{12} + 8\,r\sim^{10} + 2\,r\sim^{6} + r\sim^{4} \tag{27}$$

```
> A7 := simplify(simplify(J[7][1])):
> B7 := simplify(expand(numer(A7))):
> C7 := simplify(expand(denom(A7))):
> D7 := simplify(simplify(B7/C7));
```

$$D7 := r\sim^{4}\left(4\,r\sim^{4} - 2\,r\sim^{2} + 1\right)\left(2\,r\sim^{2} - 2\,r\sim + 1\right)^{2}\left(2\,r\sim^{2} + 1\right)^{2}\left(4\,r\sim^{4} + 4\,r\sim^{3} + 2\,r\sim^{2} + 2\,r\sim + 1\right) \tag{28}$$

```
> A8 := simplify(simplify(J[8][1])):
> B8 := simplify(expand(numer(A8))):
> C8 := simplify(expand(denom(A8))):
> D8 := simplify(B8/C8);
```

$$D8 := r\sim^4 \left(2\,r\sim^2 + 2\,r\sim + 1\right)^2 \left(4\,r\sim^4 - 2\,r\sim^2 + 1\right) \left(4\,r\sim^4 - 4\,r\sim^3 + 2\,r\sim^2 - 2\,r\sim + 1\right) \left(2\,r\sim^2 + 1\right)^2 \tag{29}$$

All multiplicities are integral.

Case III: The quads have order $\left(q, q^2\right)$. The generalized octagon has order $\left(q^3, 1\right)$.

```
> restart;
> with(LinearAlgebra) :
> q := q;
```

$$q := q \tag{30}$$

```
> s := q; t := 2·q^2; t2 := q^2;
```

$$s := q$$
$$t := 2\,q^2$$
$$t2 := q^2 \tag{31}$$

```
> v := (s + 1)·(1 + s·t + s^2·t·(t − t2) + s^3·t·(t − t2)^2 + s^4·t2·(t − t2)^3);
```

$$v := (q+1)\left(q^{12} + 2\,q^9 + 2\,q^6 + 2\,q^3 + 1\right) \tag{32}$$

```
> M := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [s, s − 1, 0, 1, 0, 0, 0, 0, 0, 0], [s·t, 0, s − 1, t2, 1,
    0, 0, 0, 0, 0], [0, s·t, s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0, 0, 0], [0, 0, s·(t − t2), 0, s − 1, t2,
    1, 0, 0, 0], [0, 0, 0, s·(t − t2), s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0], [0, 0, 0, 0, s·(t − t2), 0, s
    − 1, t2, t/t2, 0], [0, 0, 0, 0, 0, s·(t − t2), s·t2, (s − 1)·(t2 + 1), 0, t/t2], [0, 0, 0, 0, 0, 0, s
    ·(t − t2), 0, t/t2·(s − 1), t + 1 − t/t2], [0, 0, 0, 0, 0, 0, 0, s·(t − t2), s·(t + 1 − t/t2), (s
    − 1)·(t + 1)]]);
```

$$M := \begin{bmatrix} [0, 1, 1, 0, 0, 0, 0, 0, 0, 0] \\ [q, q-1, 0, 1, 0, 0, 0, 0, 0, 0] \\ [2\,q^3, 0, q-1, q^2, 1, 0, 0, 0, 0, 0] \\ [0, 2\,q^3, q^3, \left(q^2+1\right)(q-1), 0, 1, 0, 0, 0, 0] \\ [0, 0, q^3, 0, q-1, q^2, 1, 0, 0, 0] \\ [0, 0, 0, q^3, q^3, \left(q^2+1\right)(q-1), 0, 1, 0, 0] \end{bmatrix} \tag{33}$$

$$[0, 0, 0, 0, q^3, 0, q-1, q^2, 2, 0],$$
$$[0, 0, 0, 0, 0, q^3, q^3, (q^2+1)\,(q-1), 0, 2],$$
$$[0, 0, 0, 0, 0, 0, q^3, 0, 2q-2, 2q^2-1],$$
$$[0, 0, 0, 0, 0, 0, 0, q^3, q\,(2q^2-1), (q-1)\,(2q^2+1)]]$$

```
> factor(CharacteristicPolynomial(M, x));
```

$$(2q-1-x)\,(2q^2+x+1)\,(2q^3+q-x)\,(q^3+q-x-1)\,(q-2-x)\,(q^2-q+x+1)\,(q^4-4q^3+2q^2x+3q^2-2qx+x^2-2q+2x+1)\,(q^6+2q^4-2q^3x-4q^3+q^2-2qx+x^2-2q+2x+1) \quad (34)$$

```
> k := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0]]):
> a1 := v:
> a2 := v·k[1][1]:
> a3 := v·Multiply(k, M)[1][1]:
> a4 := v·Multiply(k, M^2)[1][1]:
> a5 := v·Multiply(k, M^3)[1][1]:
> a6 := v·Multiply(k, M^4)[1][1]:
> a7 := v·Multiply(k, M^5)[1][1]:
> a8 := v·Multiply(k, M^6)[1][1]:
> a9 := v·Multiply(k, M^7)[1][1]:
> a10 := v·Multiply(k, M^8)[1][1]:
> A := Matrix([[a1], [a2], [a3], [a4], [a5], [a6], [a7], [a8], [a9], [a10]]):
> V := Transpose(VandermondeMatrix([[2·q-1, -(2·q^2+1), 2·q^3+q, q^3+q-1, q-2,
      -(q^2-q+1), -(q^2-q+1)+q·(2·q)^(1/2), -(q^2-q+1)-q·(2·q)^(1/2), q^3+q-1
      +q·(2·q)^(1/2), q^3+q-1-q·(2·q)^(1/2)]])):
> J := Multiply(V^(-1), A):
> A1 := simplify(simplify(J[1][1])):
> B1 := simplify(expand(numer(A1))):
> C1 := simplify(expand(denom(A1))):
> D1 := simplify(B1/C1);
```

$$D1 := (q^4-q^2+1)\,(q^2-q+1)^2\,q^5 \quad (35)$$

```
> A2 := simplify(simplify(J[2][1])):
> B2 := simplify(expand(numer(A2))):
> C2 := simplify(expand(denom(A2))):
> D2 := simplify(B2/C2);
```

(36)

$$D2 := \left(q^4 - q^2 + 1\right)\left(q^2 - q + 1\right)^2 q \tag{36}$$

```
> A3 := simplify(simplify(J[3][1])):
> B3 := simplify(expand(numer(A3))):
> C3 := simplify(expand(denom(A3))):
> D3 := simplify(B3/C3);
```

$$D3 := 1 \tag{37}$$

```
> A4 := simplify(simplify(J[4][1])):
> B4 := simplify(expand(numer(A4))):
> C4 := simplify(expand(denom(A4))):
> D4 := simplify(B4/C4);
```

$$D4 := q^9 + q^3 \tag{38}$$

```
> A5 := simplify(simplify(J[5][1])):
> B5 := simplify(expand(numer(A5))):
> C5 := simplify(expand(denom(A5))):
> D5 := simplify(B5/C5);
```

$$D5 := q^{12} \tag{39}$$

```
> A6 := simplify(simplify(J[6][1])):
> B6 := simplify(expand(numer(A6))):
> C6 := simplify(expand(denom(A6))):
> D6 := simplify(B6/C6);
```

$$D6 := \left(q^2 + 1\right)\left(q^4 - q^2 + 1\right)\left(q^2 - q + 1\right)^2 q^2 \tag{40}$$

```
> A7 := simplify(simplify(J[7][1])):
> B7 := simplify(expand(numer(A7))):
> C7 := simplify(expand(denom(A7))):
> D7 := simplify(B7/C7);
```

$$D7 := \frac{1}{2}\left(q^4 - q^2 + 1\right)(q+1)^2\left(q^2 - q + 1\right)^2 q^2 \tag{41}$$

```
> A8 := simplify(simplify(J[8][1])):
> B8 := simplify(expand(numer(A8))):
> C8 := simplify(expand(denom(A8))):
> D8 := simplify(B8/C8);
```

$$D8 := \frac{1}{2}\left(q^4 - q^2 + 1\right)(q+1)^2\left(q^2 - q + 1\right)^2 q^2 \tag{42}$$

```
> A9 := simplify(simplify(J[9][1])):
```

```
> B9 := simplify(expand(numer(A9))):
> C9 := simplify(expand(denom(A9))):
> D9 := simplify(B9/C9);
```

$$D9 := \frac{1}{2} q^9 + q^6 + \frac{1}{2} q^3 \tag{43}$$

```
> A10 := simplify(simplify(J[10][1])):
> B10 := simplify(expand(numer(A10))):
> C10 := simplify(expand(denom(A10))):
> D10 := simplify(B10/C10);
```

$$D10 := \frac{1}{2} q^9 + q^6 + \frac{1}{2} q^3 \tag{44}$$

All multiplicities are integral.

Case IV: The quads have order $\left(q, q^2\right)$ *. The generalized octagon has order* $\left(q^3, q^6\right)$.

```
> restart;
> with(LinearAlgebra):
> q := 2·r^2;
```

$$q := 2 r^2 \tag{45}$$

```
> s := q; t := (q^6 + 1)·q^2; t2 := q^2;
```

$$s := 2 r^2$$

$$t := 4 \left(64 r^{12} + 1\right) r^4$$

$$t2 := 4 r^4 \tag{46}$$

```
> v := (s + 1)·(1 + s·t + s^2·t·(t − t2) + s^3·t·(t − t2)^2 + s^4·t2·(t − t2)^3);
```

$$v := \left(2 r^2 + 1\right) \left(1 + 8 r^6 \left(64 r^{12} + 1\right) + 16 r^8 \left(64 r^{12} + 1\right) \left(4 \left(64 r^{12} + 1\right) r^4 - 4 r^4\right) + 32 r^{10} \left(64 r^{12} + 1\right) \left(4 \left(64 r^{12} + 1\right) r^4 - 4 r^4\right)^2 + 64 r^{12} \left(4 \left(64 r^{12} + 1\right) r^4 - 4 r^4\right)^3\right) \tag{47}$$

```
> M := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [s, s − 1, 0, 1, 0, 0, 0, 0, 0, 0], [s·t, 0, s − 1, t2, 1,
    0, 0, 0, 0, 0], [0, s·t, s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0, 0, 0], [0, 0, s·(t − t2), 0, s − 1, t2,
    1, 0, 0, 0], [0, 0, 0, s·(t − t2), s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0], [0, 0, 0, 0, s·(t − t2), 0, s
```

$$-1, t2, \frac{t}{t2}, 0\Big], \Big[0, 0, 0, 0, 0, s\cdot(t-t2), s\cdot t2, (s-1)\cdot(t2+1), 0, \frac{t}{t2}\Big], \Big[0, 0, 0, 0, 0, 0, s \cdot(t-t2), 0, \frac{t}{t2}\cdot(s-1), t+1-\frac{t}{t2}\Big], \Big[0, 0, 0, 0, 0, 0, 0, s\cdot(t-t2), s\cdot\Big(t+1-\frac{t}{t2}\Big), (s-1)\cdot(t+1)\Big]\Big]\Big);$$

$$M := [[0, 1, 1, 0, 0, 0, 0, 0, 0, 0], \quad (48)$$
$$[2r^2, 2r^2-1, 0, 1, 0, 0, 0, 0, 0, 0],$$
$$[8r^6(64r^{12}+1), 0, 2r^2-1, 4r^4, 1, 0, 0, 0, 0],$$
$$[0, 8r^6(64r^{12}+1), 8r^6, (4r^4+1)(2r^2-1), 0, 1, 0, 0, 0, 0],$$
$$[0, 0, 2r^2(4(64r^{12}+1)r^4-4r^4), 0, 2r^2-1, 4r^4, 1, 0, 0, 0],$$
$$[0, 0, 0, 2r^2(4(64r^{12}+1)r^4-4r^4), 8r^6, (4r^4+1)(2r^2-1), 0, 1, 0, 0],$$
$$[0, 0, 0, 0, 2r^2(4(64r^{12}+1)r^4-4r^4), 0, 2r^2-1, 4r^4, 64r^{12}+1, 0],$$
$$[0, 0, 0, 0, 0, 2r^2(4(64r^{12}+1)r^4-4r^4), 8r^6, (4r^4+1)(2r^2-1), 0, 64r^{12}+1],$$
$$[0, 0, 0, 0, 0, 0, 2r^2(4(64r^{12}+1)r^4-4r^4), 0, (64r^{12}+1)(2r^2-1), 4(64r^{12}+1)r^4-64r^{12}],$$
$$[0, 0, 0, 0, 0, 0, 0, 2r^2(4(64r^{12}+1)r^4-4r^4), 2r^2(4(64r^{12}+1)r^4-64r^{12}), (2r^2-1)(4(64r^{12}+1)r^4+1)]]$$

```
> factor(CharacteristicPolynomial(M, x));
```

$$-(128r^{14}+2r^2-x-1)(256r^{16}+4r^4+x+1)(512r^{18}+8r^6+2r^2-x)(4r^4-2r^2+x+1)(8r^6+2r^2-x-1)(64r^{12}-2r^2+x+1)(32r^9+4r^4-2r^2+x+1)(32r^9-4r^4+2r^2-x-1)(32r^9-8r^6-2r^2+x+1)(32r^9+8r^6+2r^2-x-1) \quad (49)$$

```
> k := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0]]) :
> a1 := v :
> a2 := v· k[1][1] :
> a3 := v· Multiply( k, M )[1][1] :
> a4 := v· Multiply( k, M^2 )[1][1] :
> a5 := v· Multiply( k, M^3 )[1][1] :
> a6 := v· Multiply( k, M^4 )[1][1] :
> a7 := v· Multiply( k, M^5 )[1][1] :
> a8 := v· Multiply( k, M^6 )[1][1] :
> a9 := v· Multiply( k, M^7 )[1][1] :
> a10 := v· Multiply( k, M^8 )[1][1] :
> A := Matrix([[a1], [a2], [a3], [a4], [a5], [a6], [a7], [a8], [a9], [a10]]) :
```

```
> V := Transpose(VandermondeMatrix([[q^9 + q^3 + q, -(q^2 - q + 1), -(q^8 + q^2 + 1), -(q^6
      - q + 1), q^3 + q - 1, q^7 + q - 1, -(q^2 - q + 1) + q^4·(2·q)^(1/2), -(q^2 - q + 1)
      - q^4·(2·q)^(1/2), q^3 + q - 1 + q^4·(2·q)^(1/2), q^3 + q - 1 - q^4·(2·q)^(1/2)]])) :
> J := Multiply(V^(-1), A) :
> A1 := simplify(simplify(J[1][1])) :
> B1 := simplify(expand(numer(A1))) :
> C1 := simplify(expand(denom(A1))) :
> D1 := simplify(B1/C1);
```

$$D1 := 1 \tag{50}$$

```
> A2 := simplify(simplify(J[2][1])) :
> B2 := simplify(expand(numer(A2))) :
> C2 := simplify(expand(denom(A2))) :
> D2 := simplify(B2/C2);
```

$$D2 := \left(64\left(4r^4 - 2r^2 + 1\right)^2\left(2r^2 - 2r + 1\right)^2\left(4r^4 + 4r^3 + 2r^2 + 2r + 1\right)^2 r^{14}\left(64r^{12} - 32r^9 + 8r^6 - 4r^3 + 1\right)\left(64r^{12} - 8r^6 + 1\right)\left(4r^4 - 4r^3 + 2r^2 - 2r + 1\right)^2\left(2r^2 + 2r + 1\right)^2\left(64r^{12} + 32r^9 + 8r^6 + 4r^3 + 1\right)\right) \Big/ \left(\left(64r^{12} - 32r^{10} + 16r^8 - 8r^6 + 4r^4 - 2r^2 + 1\right)\left(16r^8 - 8r^6 + 4r^4 - 2r^2 + 1\right)\right) \tag{51}$$

```
> A3 := simplify(simplify(J[3][1])) :
> B3 := simplify(expand(numer(A3))) :
> C3 := simplify(expand(denom(A3))) :
> D3 := simplify(B3/C3);
```

$$D3 := \left(2\left(4r^4 - 2r^2 + 1\right)^2\left(4r^4 + 4r^3 + 2r^2 + 2r + 1\right)r^2\left(64r^{12} - 32r^9 + 8r^6 - 4r^3 + 1\right)\left(64r^{12} - 8r^6 + 1\right)\left(4r^4 - 4r^3 + 2r^2 - 2r + 1\right)\left(64r^{12} + 32r^9 + 8r^6 + 4r^3 + 1\right)\right) \Big/ \left(262144r^{36} - 131072r^{34} + 16384r^{28} - 8192r^{26} + 2048r^{22} - 512r^{18} + 128r^{14} - 32r^{10} + 16r^8 - 2r^2 + 1\right) \tag{52}$$

```
> A4 := simplify(simplify(J[4][1])) :
> B4 := simplify(expand(numer(A4))) :
> C4 := simplify(expand(denom(A4))) :
> D4 := simplify(B4/C4);
```

$$D4 := 2097152r^{42} - 262144r^{36} + 4096r^{24} - 64r^{12} + 8r^6 \tag{53}$$

```
> A5 := simplify(simplify(J[5][1])) :
> B5 := simplify(expand(numer(A5))) :
```

```
> C5 := simplify(expand(denom(A5))):
> D5 := simplify(B5/C5);
```

$$D5 := 536870912\,r^{60} + 8388608\,r^{48} + 2048\,r^{24} + 32\,r^{12} \tag{54}$$

```
> A6 := simplify(simplify(J[6][1])):
> B6 := simplify(expand(numer(A6))):
> C6 := simplify(expand(denom(A6))):
> D6 := simplify(B6/C6);
```

$$D6 := \left(4\left(64\,r^{12} + 32\,r^{9} + 8\,r^{6} + 4\,r^{3} + 1\right)\left(4\,r^{4} - 4\,r^{3} + 2\,r^{2} - 2\,r + 1\right)\left(64\,r^{12} - 8\,r^{6} + 1\right)\left(64\,r^{12} - 32\,r^{9} + 8\,r^{6} - 4\,r^{3} + 1\right)r^{4}\left(4\,r^{4} + 4\,r^{3} + 2\,r^{2} + 2\,r + 1\right)\left(4\,r^{4} - 2\,r^{2} + 1\right)^{2}\right) \Big/ \left(4096\,r^{24} - 2048\,r^{22} + 256\,r^{16} - 64\,r^{12} + 16\,r^{8} - 2\,r^{2} + 1\right) \tag{55}$$

```
> A7 := simplify(simplify(J[7][1])):
> B7 := simplify(expand(numer(A7))):
> C7 := simplify(expand(denom(A7))):
> D7 := simplify(B7/C7);
```

$$\begin{aligned} D7 := {} & \left(3868562622766813359059763 2\left(\left(\frac{1}{4398046511104}\,r^{2} + \frac{9}{64}\,r^{72} - \frac{7}{512}\,r^{70}\right.\right.\right. \\ & - \frac{1}{128}\,r^{68} - \frac{3}{512}\,r^{66} - \frac{79}{4096}\,r^{64} + \frac{83}{8192}\,r^{62} - \frac{61}{16384}\,r^{60} + \frac{27}{262144}\,r^{52} + r^{86} \\ & - \frac{3}{16}\,r^{80} - \frac{7}{16}\,r^{78} + \frac{11}{64}\,r^{76} - \frac{5}{64}\,r^{74} + \frac{39}{8192}\,r^{58} - \frac{13}{16384}\,r^{56} + \frac{25}{131072}\,r^{54} \\ & - \frac{55}{262144}\,r^{50} - \frac{21}{8589934592}\,r^{22} + \frac{15}{17179869184}\,r^{18} + \frac{181}{536870912}\,r^{30} \\ & - \frac{83}{8388608}\,r^{42} + \frac{71}{16777216}\,r^{40} + \frac{63}{33554432}\,r^{38} + \frac{69}{67108864}\,r^{34} - \frac{15}{33554432}\,r^{32} \\ & + \frac{195}{1048576}\,r^{48} - \frac{33}{524288}\,r^{46} + \frac{259}{4194304}\,r^{44} - \frac{103}{67108864}\,r^{36} - \frac{15}{1073741824}\,r^{26} \\ & + \frac{97}{17179869184}\,r^{20} - \frac{65}{1073741824}\,r^{28} + \frac{9}{1073741824}\,r^{24} + \frac{23}{68719476736}\,r^{14} \\ & + \frac{23}{549755813888}\,r^{10} + \frac{5}{1099511627776}\,r^{8} + \frac{43}{34359738368}\,r^{16} \\ & + \frac{3}{1099511627776}\,r^{6} + \frac{1}{17592186044416} + \frac{1}{17179869184}\,r^{12} \\ & \left.\left.\left. + \frac{3}{4398046511104}\,r^{4}\right)\operatorname{csgn}(r) + \frac{1}{2}\left(r^{2} + \frac{1}{2}\right)r^{5}\left(-\frac{1}{137438953472}\,r^{2} + r^{72}\right.\right.\right. \end{aligned} \tag{56}$$

$$
\begin{aligned}
&- \frac{7}{8} r^{70} + \frac{17}{32} r^{68} - \frac{1}{32} r^{66} - \frac{5}{128} r^{64} + \frac{17}{128} r^{62} - \frac{5}{32} r^{60} - \frac{57}{4096} r^{52} + r^{76} \\
&- \frac{3}{2} r^{74} + \frac{49}{512} r^{58} - \frac{137}{2048} r^{56} + \frac{79}{2048} r^{54} + \frac{1}{256} r^{50} + \frac{73}{134217728} r^{22} \\
&- \frac{5}{536870912} r^{18} + \frac{7}{1048576} r^{30} - \frac{171}{131072} r^{42} + \frac{363}{524288} r^{40} - \frac{165}{524288} r^{38} \\
&- \frac{77}{2097152} r^{34} - \frac{13}{4194304} r^{32} + \frac{9}{32768} r^{48} - \frac{81}{32768} r^{46} + \frac{233}{131072} r^{44} \\
&+ \frac{117}{1048576} r^{36} + \frac{123}{33554432} r^{26} - \frac{47}{268435456} r^{20} - \frac{113}{16777216} r^{28} \\
&- \frac{173}{134217728} r^{24} - \frac{9}{536870912} r^{14} - \frac{9}{4294967296} r^{10} - \frac{1}{34359738368} r^{8} \\
&+ \frac{23}{2147483648} r^{16} - \frac{1}{4294967296} r^{6} - \frac{5}{549755813888} + \frac{31}{8589934592} r^{12} \\
&- \frac{11}{137438953472} r^{4} \Bigg) \Bigg) \left(r^4 - \frac{1}{2} r^2 + \frac{1}{4} \right)^2 \left(r^2 + \frac{1}{2} \right)^2 \left(r^2 - r + \frac{1}{2} \right)^2 \left(r^4 \right. \\
&\left. + r^3 + \frac{1}{2} r^2 + \frac{1}{2} r + \frac{1}{4} \right)^2 r^{14} \left(r^{12} - \frac{1}{2} r^9 + \frac{1}{8} r^6 - \frac{1}{16} r^3 + \frac{1}{64} \right) \left(r^{12} - \frac{1}{8} r^6 \right. \\
&\left. + \frac{1}{64} \right) \left(r^4 - r^3 + \frac{1}{2} r^2 - \frac{1}{2} r + \frac{1}{4} \right)^2 \left(r^2 + r + \frac{1}{2} \right)^2 \left(r^{12} + \frac{1}{2} r^9 + \frac{1}{8} r^6 \right. \\
&\left. + \frac{1}{16} r^3 + \frac{1}{64} \right) \Bigg) \Bigg/ \Bigg(\left(72057594037927936\, r^{110} - 13510798882111488\, r^{104} \right. \\
&- 40532396646334464\, r^{102} + 28147497671065600\, r^{100} - 14636698788954112\, r^{98} \\
&+ 15199648742375424\, r^{96} - 3940649673949184\, r^{94} - 2814749767106560\, r^{92} \\
&+ 2533274790395904\, r^{90} - 3676766883282944\, r^{88} + 2761973208973312\, r^{86} \\
&- 1293025674264576\, r^{84} + 725677674332160\, r^{82} - 169324790677504\, r^{80} \\
&- 78065325572096\, r^{78} + 144860656959488\, r^{76} - 112974819753984\, r^{74} \\
&+ 80539226734592\, r^{72} - 38689065402368\, r^{70} + 17454747090944\, r^{68} \\
&- 3298534883328\, r^{66} - 1541893259264\, r^{64} + 2345052143616\, r^{62} \\
&- 1717986918400\, r^{60} + 1063004405760\, r^{58} - 482378514432\, r^{56} + 182804545536\, r^{54}
\end{aligned}
$$

$$
\begin{aligned}
&- 29796335616\, r^{52} - 22213033984\, r^{50} + 19964887040\, r^{48} - 11156848640\, r^{46} \\
&+ 7377780736\, r^{44} - 2587885568\, r^{42} + 870318080\, r^{40} + 120586240\, r^{38} \\
&- 126353408\, r^{36} + 73138176\, r^{34} - 42729472\, r^{32} + 18481152\, r^{30} - 5079040\, r^{28} \\
&- 114688\, r^{26} + 581632\, r^{24} - 245760\, r^{22} + 159744\, r^{20} + 3072\, r^{18} + 29696\, r^{16} \\
&+ 7424\, r^{14} + 1280\, r^{12} + 1088\, r^{10} + 80\, r^{8} + 48\, r^{6} + 12\, r^{4} + 4\, r^{2} + 1\Big)\, \mathrm{csgn}(r) \\
&+ 36028797018963968 \left(r^{50} - \frac{1}{8} r^{44} - \frac{3}{16} r^{42} + \frac{3}{32} r^{40} - \frac{3}{64} r^{38} + \frac{5}{128} r^{36}\right. \\
&- \frac{1}{64} r^{34} - \frac{1}{512} r^{30} - \frac{1}{256} r^{28} + \frac{1}{512} r^{26} - \frac{5}{4096} r^{24} + \frac{3}{8192} r^{22} - \frac{7}{32768} r^{20} \\
&+ \frac{3}{32768} r^{18} + \frac{1}{65536} r^{16} - \frac{1}{32768} r^{14} + \frac{1}{65536} r^{12} - \frac{3}{1048576} r^{10} \\
&\left. + \frac{1}{1048576} r^{8} - \frac{5}{4194304} r^{6} - \frac{3}{8388608} r^{4} - \frac{1}{16777216} r^{2} - \frac{3}{33554432}\right) r^{5} \left(r^{50}\right. \\
&- \frac{7}{8192} r^{22} + \frac{3}{65536} r^{18} - \frac{31}{1024} r^{30} + \frac{11}{16} r^{42} - \frac{7}{32} r^{40} + \frac{3}{64} r^{38} - \frac{1}{16} r^{34} \\
&+ \frac{13}{256} r^{32} - 2\, r^{48} + \frac{3}{2} r^{46} - r^{44} + \frac{3}{64} r^{36} - \frac{19}{2048} r^{26} + \frac{1}{32768} r^{20} + \frac{41}{2048} r^{28} \\
&+ \frac{7}{2048} r^{24} + \frac{19}{262144} r^{14} + \frac{7}{1048576} r^{10} - \frac{1}{2097152} r^{8} - \frac{13}{131072} r^{16} \\
&\left. + \frac{1}{4194304} r^{6} - \frac{11}{524288} r^{12} + \frac{1}{8388608} r^{4} + \frac{1}{33554432}\right) \left(r^{2} + \frac{1}{2}\right)\Big)
\end{aligned}
$$

```
> A8 := simplify(simplify(J[8][1])):
> B8 := simplify(expand(numer(A8))):
> C8 := simplify(expand(denom(A8))):
> D8 := simplify(B8/C8);
```

$$
\begin{aligned}
D8 := {} & \Big(32 \left(4 r^{4} - 2 r^{2} + 1\right)^{2} \left(2 r^{2} + 1\right)^{2} \left(4 r^{4} + 4 r^{3} + 2 r^{2} + 2 r + 1\right)^{2} r^{14} \left(64 r^{12}\right. \\
& - 32 r^{9} + 8 r^{6} - 4 r^{3} + 1\Big) \left(64 r^{12} - 8 r^{6} + 1\right) \left(1 + 2 r^{2} + 274877906944\, r^{72}\right. \\
& - 68719476736\, r^{70} + 85899345920\, r^{68} + 77309411328\, r^{66} - 90194313216\, r^{64} \\
& + 68719476736\, r^{62} - 39728447488\, r^{60} + 3288334336\, r^{52} + 274877906944\, r^{76} \\
& - 687194767360\, r^{74} + 19864223744\, r^{58} - 5100273664\, r^{56} + 134217728\, r^{54} \\
& \left. - 2986344448\, r^{50} - 116736\, r^{22} - 13824\, r^{18} + 2686976\, r^{30} - 121634816\, r^{42}\right.
\end{aligned}
\qquad \textbf{(57)}
$$

$$
\begin{aligned}
&- 8388608\,r^{40} + 55574528\,r^{38} + 14942208\,r^{34} - 4718592\,r^{32} + 1795162112\,r^{48} \\
&- 838860800\,r^{46} + 297795584\,r^{44} - 34865152\,r^{36} - 385024\,r^{26} + 22528\,r^{20} \\
&- 360448\,r^{28} + 299008\,r^{24} + 1536\,r^{14} + 416\,r^{10} + 2816\,r^{16} + 16\,r^{6} - 576\,r^{12} + 4\,r^{4} \\
&- 32\,\mathrm{csgn}(r)\,r^{5} + 12288\,\mathrm{csgn}(r)\,r^{17} + 29884416\,\mathrm{csgn}(r)\,r^{37} - 12320768\,\mathrm{csgn}(r)\,r^{35} \\
&- 4194304\,\mathrm{csgn}(r)\,r^{33} + 4521984\,\mathrm{csgn}(r)\,r^{31} - 2523136\,\mathrm{csgn}(r)\,r^{29} \\
&+ 737280\,\mathrm{csgn}(r)\,r^{27} - 286720\,\mathrm{csgn}(r)\,r^{25} + 131072\,\mathrm{csgn}(r)\,r^{23} + 8192\,\mathrm{csgn}(r)\,r^{21} \\
&- 24576\,\mathrm{csgn}(r)\,r^{19} - 3072\,\mathrm{csgn}(r)\,r^{15} + 512\,\mathrm{csgn}(r)\,r^{13} - 384\,\mathrm{csgn}(r)\,r^{11} \\
&- 16\,\mathrm{csgn}(r)\,r^{7} + 2617245696\,r^{51}\,\mathrm{csgn}(r) + 197132288\,r^{41}\,\mathrm{csgn}(r) \\
&+ 274877906944\,r^{69}\,\mathrm{csgn}(r) + 16106127360\,r^{59}\,\mathrm{csgn}(r) - 96\,\mathrm{csgn}(r)\,r^{9} \\
&- 274877906944\,\mathrm{csgn}(r)\,r^{71} - 171798691840\,\mathrm{csgn}(r)\,r^{67} + 85899345920\,\mathrm{csgn}(r)\,r^{65} \\
&- 42949672960\,\mathrm{csgn}(r)\,r^{63} - 4294967296\,\mathrm{csgn}(r)\,r^{61} - 19864223744\,\mathrm{csgn}(r)\,r^{57} \\
&+ 12616466432\,\mathrm{csgn}(r)\,r^{55} - 6845104128\,\mathrm{csgn}(r)\,r^{53} - 939524096\,\mathrm{csgn}(r)\,r^{49} \\
&- 33554432\,\mathrm{csgn}(r)\,r^{47} + 444596224\,\mathrm{csgn}(r)\,r^{45} - 369098752\,\mathrm{csgn}(r)\,r^{43} \\
&- 71303168\,\mathrm{csgn}(r)\,r^{39} - 137438953472\,\mathrm{csgn}(r)\,r^{73} + 1099511627776\,\mathrm{csgn}(r)\,r^{79}\Big) \\
&\Big(4\,r^{4} - 4\,r^{3} + 2\,r^{2} - 2\,r + 1\Big)^{2}\Big(64\,r^{12} + 32\,r^{9} + 8\,r^{6} + 4\,r^{3} + 1\Big)\Big) \Big/ \Big(\Big(1 + 2\,r^{2} \\
&+ 67108864\,r^{52} - 100663296\,r^{50} - 26624\,r^{22} - 5120\,r^{18} + 327680\,r^{30} + 8388608\,r^{42} \\
&- 4194304\,r^{40} + 4718592\,r^{38} + 1310720\,r^{34} - 327680\,r^{32} + 33554432\,r^{48} \\
&- 16777216\,r^{46} + 12582912\,r^{44} - 2621440\,r^{36} - 81920\,r^{26} + 4096\,r^{20} + 49152\,r^{28} \\
&+ 57344\,r^{24} + 1024\,r^{14} + 192\,r^{10} + 1536\,r^{16} + 16\,r^{6} - 256\,r^{12} + 4\,r^{4} - 24\,\mathrm{csgn}(r)\,r^{5} \\
&+ 4096\,\mathrm{csgn}(r)\,r^{17} - 524288\,\mathrm{csgn}(r)\,r^{35} - 1048576\,\mathrm{csgn}(r)\,r^{33} + 524288\,\mathrm{csgn}(r)\,r^{31} \\
&- 327680\,\mathrm{csgn}(r)\,r^{29} + 98304\,\mathrm{csgn}(r)\,r^{27} - 57344\,\mathrm{csgn}(r)\,r^{25} + 24576\,\mathrm{csgn}(r)\,r^{23} \\
&+ 4096\,\mathrm{csgn}(r)\,r^{21} - 8192\,\mathrm{csgn}(r)\,r^{19} - 768\,\mathrm{csgn}(r)\,r^{15} + 256\,\mathrm{csgn}(r)\,r^{13} \\
&- 320\,\mathrm{csgn}(r)\,r^{11} - 16\,\mathrm{csgn}(r)\,r^{7} + 10485760\,r^{41}\,\mathrm{csgn}(r) - 96\,\mathrm{csgn}(r)\,r^{9} \\
&+ 268435456\,\mathrm{csgn}(r)\,r^{55} - 33554432\,\mathrm{csgn}(r)\,r^{49} - 50331648\,\mathrm{csgn}(r)\,r^{47} \\
&+ 25165824\,\mathrm{csgn}(r)\,r^{45} - 12582912\,\mathrm{csgn}(r)\,r^{43} - 4194304\,\mathrm{csgn}(r)\,r^{39}\Big)\Big(64\,r^{12} \\
&+ 64\,r^{11} + 32\,r^{10} - 16\,r^{8} - 16\,r^{7} - 8\,r^{6} - 8\,r^{5} - 4\,r^{4} + 2\,r^{2} + 2\,r + 1\Big)\Big(64\,r^{12} \\
&- 64\,r^{11} + 32\,r^{10} - 16\,r^{8} + 16\,r^{7} - 8\,r^{6} + 8\,r^{5} - 4\,r^{4} + 2\,r^{2} - 2\,r + 1\Big)\Big(16\,r^{8} \\
&- 16\,r^{7} + 8\,r^{6} - 4\,r^{4} + 2\,r^{2} - 2\,r + 1\Big)\Big(16\,r^{8} + 16\,r^{7} + 8\,r^{6} - 4\,r^{4} + 2\,r^{2} + 2\,r \\
&+ 1\Big)\Big)
\end{aligned}
$$

```
> A9 := simplify(simplify(J[9][1])) :
```

```
> B9 := simplify(expand(numer(A9))):
> C9 := simplify(expand(denom(A9))):
> D9 := simplify(B9/C9);
```

$$
\begin{aligned}
D9 := {} & \left(72057594037927936\left(r^4-\frac{1}{2}r^2+\frac{1}{4}\right)^2\left(r^2+\frac{1}{2}\right)^2\left(r^3\left(-\frac{1}{34359738368}r^2\right.\right.\right. \\
& -\frac{1}{8}r^{70}+\frac{1}{32}r^{66}-\frac{3}{64}r^{64}-\frac{1}{32}r^{62}-\frac{3}{256}r^{60}-\frac{3}{4096}r^{52}+r^{76}+\frac{3}{512}r^{58} \\
& -\frac{1}{1024}r^{54}-\frac{5}{8192}r^{50}+\frac{1}{33554432}r^{22}+\frac{3}{134217728}r^{18}-\frac{7}{4194304}r^{30} \\
& +\frac{5}{131072}r^{42}-\frac{1}{65536}r^{40}+\frac{9}{524288}r^{38}-\frac{5}{1048576}r^{34}-\frac{15}{2097152}r^{32} \\
& -\frac{9}{16384}r^{48}+\frac{1}{16384}r^{46}+\frac{13}{65536}r^{44}+\frac{19}{1048576}r^{36}-\frac{3}{16777216}r^{26} \\
& +\frac{3}{33554432}r^{20}+\frac{1}{8388608}r^{28}-\frac{19}{67108864}r^{24}+\frac{1}{1073741824}r^{14} \\
& -\frac{1}{2147483648}r^{10}-\frac{7}{17179869184}r^8-\frac{3}{1073741824}r^{16}-\frac{1}{8589934592}r^6 \\
& \left.\left.+\frac{1}{4294967296}r^{12}-\frac{1}{34359738368}r^4-\frac{1}{274877906944}\right)\operatorname{csgn}(r)-\frac{1}{4}\left(r^2\right.\right. \\
& \left.\left.+\frac{1}{2}\right)\left(r^{72}-r^{70}+\frac{1}{8}r^{68}-\frac{7}{16}r^{66}+\frac{7}{32}r^{64}-\frac{5}{64}r^{62}+\frac{3}{128}r^{60}+\frac{1}{2048}r^{52}\right.\right. \\
& +r^{74}-\frac{7}{256}r^{58}+\frac{3}{256}r^{56}-\frac{9}{1024}r^{54}-\frac{1}{4096}r^{50}-\frac{5}{33554432}r^{22} \\
& +\frac{9}{268435456}r^{18}-\frac{35}{4194304}r^{30}+\frac{5}{16384}r^{42}+\frac{1}{65536}r^{40}-\frac{13}{262144}r^{38} \\
& +\frac{7}{1048576}r^{34}-\frac{1}{262144}r^{32}+\frac{1}{4096}r^{48}-\frac{5}{8192}r^{46}+\frac{5}{16384}r^{44}-\frac{3}{524288}r^{36} \\
& +\frac{5}{8388608}r^{26}+\frac{5}{67108864}r^{20}+\frac{1}{8388608}r^{28}+\frac{11}{33554432}r^{24}-\frac{3}{268435456}r^{14} \\
& -\frac{1}{1073741824}r^{10}-\frac{5}{8589934592}r^8+\frac{1}{67108864}r^{16}-\frac{1}{8589934592}r^6 \\
& \left.\left.\left.-\frac{1}{137438953472}+\frac{1}{1073741824}r^{12}-\frac{1}{34359738368}r^4\right)\right)r^{12}\left(r^{12}-\frac{1}{8}r^6\right. \\
& \left.\left.+\frac{1}{64}\right)\right)\Big/\left(1+2r^2+67108864r^{52}-100663296r^{50}-26624r^{22}-5120r^{18}\right. \\
& +327680r^{30}+8388608r^{42}-4194304r^{40}+4718592r^{38}+1310720r^{34}-327680r^{32} \\
& \left.+33554432r^{48}-16777216r^{46}+12582912r^{44}-2621440r^{36}-81920r^{26}+4096r^{20}\right. \\
& \left.+49152r^{28}+57344r^{24}+1024r^{14}+192r^{10}+1536r^{16}+16r^6-256r^{12}+4r^4\right)
\end{aligned}
\tag{58}
$$

$$+\left(268435456\,r^{55}-33554432\,r^{49}-50331648\,r^{47}+25165824\,r^{45}-12582912\,r^{43}\right.$$
$$+10485760\,r^{41}-4194304\,r^{39}-524288\,r^{35}-1048576\,r^{33}+524288\,r^{31}-327680\,r^{29}$$
$$+98304\,r^{27}-57344\,r^{25}+24576\,r^{23}+4096\,r^{21}-8192\,r^{19}+4096\,r^{17}-768\,r^{15}$$
$$\left.+256\,r^{13}-320\,r^{11}-96\,r^{9}-16\,r^{7}-24\,r^{5}\right)\operatorname{csgn}(r)\Bigg)$$

```
> A10 := simplify(simplify(J[10][1])):
> B10 := simplify(expand(numer(A10))):
> C10 := simplify(expand(denom(A10))):
> D10 := simplify(B10/C10);
```

$$D10 := \left(72057594037927936\left(r^4-\frac{1}{2}r^2+\frac{1}{4}\right)^2\left(r^2+\frac{1}{2}\right)^2\left(r^2-r+\frac{1}{2}\right)^2 r^{12}\left(r^3\left(\right.\right.\right. \tag{59}$$
$$-\frac{1}{4294967296}r^2+r^{68}-\frac{1}{2}r^{64}+\frac{1}{8}r^{62}-\frac{3}{16}r^{60}+\frac{3}{128}r^{52}+\frac{3}{32}r^{58}-\frac{1}{64}r^{56}$$
$$+\frac{1}{128}r^{54}-\frac{1}{64}r^{50}-\frac{7}{4194304}r^{22}-\frac{9}{16777216}r^{18}+\frac{3}{131072}r^{30}+\frac{1}{8192}r^{42}$$
$$-\frac{1}{8192}r^{40}+\frac{9}{32768}r^{38}+\frac{11}{65536}r^{34}-\frac{15}{131072}r^{32}+\frac{7}{1024}r^{48}-\frac{19}{2048}r^{46}$$
$$+\frac{11}{4096}r^{44}-\frac{1}{4096}r^{36}-\frac{1}{524288}r^{26}+\frac{19}{16777216}r^{20}+\frac{7}{1048576}r^{28}$$
$$+\frac{1}{1048576}r^{24}+\frac{3}{67108864}r^{14}+\frac{1}{134217728}r^{10}-\frac{3}{536870912}r^8$$
$$-\frac{1}{67108864}r^{16}+\frac{1}{1073741824}r^6+\frac{1}{17179869184}+\frac{1}{268435456}r^{12}$$
$$\left.-\frac{1}{2147483648}r^4\right)\operatorname{csgn}(r)+\frac{3}{4}\left(r^2+\frac{1}{2}\right)\left(r^{66}-r^{64}+\frac{1}{6}r^{62}-\frac{1}{24}r^{60}\right.$$
$$+\frac{7}{96}r^{52}+\frac{13}{96}r^{56}-\frac{7}{64}r^{54}-\frac{35}{768}r^{50}-\frac{19}{6291456}r^{22}-\frac{29}{50331648}r^{18}$$
$$+\frac{13}{262144}r^{30}+\frac{1}{512}r^{42}-\frac{53}{24576}r^{40}+\frac{13}{8192}r^{38}+\frac{5}{16384}r^{34}-\frac{55}{393216}r^{32}$$
$$+\frac{29}{1536}r^{48}-\frac{5}{768}r^{46}+\frac{5}{6144}r^{44}-\frac{77}{98304}r^{36}-\frac{37}{3145728}r^{26}+\frac{31}{25165824}r^{20}$$
$$+\frac{1}{262144}r^{28}+\frac{43}{6291456}r^{24}+\frac{1}{33554432}r^{14}+\frac{1}{100663296}r^{10}-\frac{1}{268435456}r^8$$
$$+\frac{1}{6291456}r^{16}+\frac{1}{1610612736}r^6-\frac{3}{134217728}r^{12}-\frac{1}{6442450944}r^4$$
$$\left.\left.+\frac{1}{25769803776}\right)\right)\left(r^{12}-\frac{1}{8}r^6+\frac{1}{64}\right)\left(r^2+r+\frac{1}{2}\right)^2\Bigg)\Big/\left(1+2\,r^2\right.$$

$$+ 67108864\,r^{52} - 100663296\,r^{50} - 26624\,r^{22} - 5120\,r^{18} + 327680\,r^{30} + 8388608\,r^{42}$$
$$- 4194304\,r^{40} + 4718592\,r^{38} + 1310720\,r^{34} - 327680\,r^{32} + 33554432\,r^{48}$$
$$- 16777216\,r^{46} + 12582912\,r^{44} - 2621440\,r^{36} - 81920\,r^{26} + 4096\,r^{20} + 49152\,r^{28}$$
$$+ 57344\,r^{24} + 1024\,r^{14} + 192\,r^{10} + 1536\,r^{16} + 16\,r^{6} - 256\,r^{12} + 4\,r^{4}$$
$$+ \left(268435456\,r^{55} - 33554432\,r^{49} - 50331648\,r^{47} + 25165824\,r^{45} - 12582912\,r^{43}\right.$$
$$+ 10485760\,r^{41} - 4194304\,r^{39} - 524288\,r^{35} - 1048576\,r^{33} + 524288\,r^{31} - 327680\,r^{29}$$
$$+ 98304\,r^{27} - 57344\,r^{25} + 24576\,r^{23} + 4096\,r^{21} - 8192\,r^{19} + 4096\,r^{17} - 768\,r^{15}$$
$$\left.+ 256\,r^{13} - 320\,r^{11} - 96\,r^{9} - 16\,r^{7} - 24\,r^{5}\right)\,\mathrm{csgn}(r)\Big)$$

```
> q1 := numer(D2);
```

$$q1 := 64\left(4r^4 - 2r^2 + 1\right)^2\left(2r^2 - 2r + 1\right)^2\left(4r^4 + 4r^3 + 2r^2 + 2r + 1\right)^2 r^{14}\left(64r^{12}\right. \quad (60)$$
$$\left. - 32r^9 + 8r^6 - 4r^3 + 1\right)\left(64r^{12} - 8r^6 + 1\right)\left(4r^4 - 4r^3 + 2r^2 - 2r\right.$$
$$\left. + 1\right)^2\left(2r^2 + 2r + 1\right)^2\left(64r^{12} + 32r^9 + 8r^6 + 4r^3 + 1\right)$$

```
> q2 := denom(D2);
```

$$q2 := \left(64r^{12} - 32r^{10} + 16r^8 - 8r^6 + 4r^4 - 2r^2 + 1\right)\left(16r^8 - 8r^6 + 4r^4 - 2r^2 + 1\right) \quad (61)$$

```
> q3 := 16*r^8 - 8*r^6 + 4*r^4 - 2*r^2 + 1;
```

$$q3 := 16r^8 - 8r^6 + 4r^4 - 2r^2 + 1 \quad (62)$$

```
> rem(q1, q3, r);
```

$$2r^4 - r^2 \quad (63)$$

 # The fact that the multiplicity of the second eigenvalue is integral implies that q3=16r^8 - 8 r^6 + 4 r^4 - 2 r^2 + 1 is a divisor of q1.

 # This is however impossible as the remainder of the division is equal to 2 r^4 - r^2 < 16 r^8 - 8 r^6 + 4 r^4 - 2 r^2 + 1.

 # Case V: The quads have order (q^2, q^4). The generalized octagon has order (q^6, q^3).

```
> restart;
> with(LinearAlgebra) :
> q := 2·r^2;
```

$$q := 2r^2 \quad (64)$$

```
> s := q^2; t := (q^3 + 1)·q^4; t2 := q^4;
```

$$s := 4\,r^4$$

$$t := 16\,\left(8\,r^6 + 1\right)\,r^8$$

$$t2 := 16\,r^8 \tag{65}$$

```
> v := (s + 1)·(1 + s·t + s^2·t·(t − t2) + s^3·t·(t − t2)^2 + s^4·t2·(t − t2)^3);
```

$$v := \left(4\,r^4 + 1\right)\,\left(1 + 64\,r^{12}\,\left(8\,r^6 + 1\right) + 256\,r^{16}\,\left(8\,r^6 + 1\right)\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right)\right. \tag{66}$$
$$\left. + 1024\,r^{20}\,\left(8\,r^6 + 1\right)\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right)^2 + 4096\,r^{24}\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right)^3\right)$$

```
> M := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [s, s − 1, 0, 1, 0, 0, 0, 0, 0, 0], [s·t, 0, s − 1, t2, 1,
    0, 0, 0, 0, 0], [0, s·t, s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0, 0, 0], [0, 0, s·(t − t2), 0, s − 1, t2,
    1, 0, 0, 0], [0, 0, 0, s·(t − t2), s·t2, (t2 + 1)·(s − 1), 0, 1, 0, 0], [0, 0, 0, 0, s·(t − t2), 0, s
    − 1, t2, t/t2, 0], [0, 0, 0, 0, 0, s·(t − t2), s·t2, (s − 1)·(t2 + 1), 0, t/t2], [0, 0, 0, 0, 0, 0, s
    ·(t − t2), 0, t/t2·(s − 1), t + 1 − t/t2], [0, 0, 0, 0, 0, 0, 0, s·(t − t2), s·(t + 1 − t/t2), (s
    − 1)·(t + 1)]]);
```

$$M := \left[\left[0, 1, 1, 0, 0, 0, 0, 0, 0, 0\right],\right. \tag{67}$$
$$\left[4\,r^4, 4\,r^4 - 1, 0, 1, 0, 0, 0, 0, 0, 0\right],$$
$$\left[64\,r^{12}\,\left(8\,r^6 + 1\right), 0, 4\,r^4 - 1, 16\,r^8, 1, 0, 0, 0, 0, 0\right],$$
$$\left[0, 64\,r^{12}\,\left(8\,r^6 + 1\right), 64\,r^{12}, \left(16\,r^8 + 1\right)\,\left(4\,r^4 - 1\right), 0, 1, 0, 0, 0, 0\right],$$
$$\left[0, 0, 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right), 0, 4\,r^4 - 1, 16\,r^8, 1, 0, 0, 0\right],$$
$$\left[0, 0, 0, 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right), 64\,r^{12}, \left(16\,r^8 + 1\right)\,\left(4\,r^4 - 1\right), 0, 1, 0, 0\right],$$
$$\left[0, 0, 0, 0, 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right), 0, 4\,r^4 - 1, 16\,r^8, 8\,r^6 + 1, 0\right],$$
$$\left[0, 0, 0, 0, 0, 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right), 64\,r^{12}, \left(16\,r^8 + 1\right)\,\left(4\,r^4 - 1\right), 0, 8\,r^6 + 1\right],$$
$$\left[0, 0, 0, 0, 0, 0, 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right), 0, \left(8\,r^6 + 1\right)\,\left(4\,r^4 - 1\right), 16\,\left(8\,r^6 + 1\right)\,r^8 - 8\,r^6\right],$$
$$\left[0, 0, 0, 0, 0, 0, 0, 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 16\,r^8\right), 4\,r^4\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 - 8\,r^6\right), \left(4\,r^4 - 1\right)\,\left(16\,\left(8\,r^6 + 1\right)\,r^8 + 1\right)\right]\right]$$

```
> factor(CharacteristicPolynomial(M, x));
```

$$\left(512\,r^{18} + 64\,r^{12} + 4\,r^4 - x\right)\,\left(8\,r^6 - 4\,r^4 + x + 1\right)\,\left(32\,r^{10} + 4\,r^4 - x - 1\right)\,\left(64\,r^{12} + 4\,r^4\right. \tag{68}$$
$$\left. - x - 1\right)\,\left(16\,r^8 - 4\,r^4 + x + 1\right)\,\left(128\,r^{14} + 16\,r^8 + x + 1\right)\,\left(32\,r^9 + 16\,r^8 - 4\,r^4 + x\right.$$

$$+1)\left(32r^9-16r^8+4r^4-x-1\right)\left(64r^{12}-32r^9+4r^4-x-1\right)\left(64r^{12}+32r^9+4r^4-x-1\right)$$

```
> k := Matrix([[0, 1, 1, 0, 0, 0, 0, 0, 0, 0]]):
> a1 := v:
> a2 := v·k[1][1]:
> a3 := v·Multiply(k, M)[1][1]:
> a4 := v·Multiply(k, M^2)[1][1]:
> a5 := v·Multiply(k, M^3)[1][1]:
> a6 := v·Multiply(k, M^4)[1][1]:
> a7 := v·Multiply(k, M^5)[1][1]:
> a8 := v·Multiply(k, M^6)[1][1]:
> a9 := v·Multiply(k, M^7)[1][1]:
> a10 := v·Multiply(k, M^8)[1][1]:
> A := Matrix([[a1], [a2], [a3], [a4], [a5], [a6], [a7], [a8], [a9], [a10]]):
> V := Transpose(VandermondeMatrix([[q^6+q^2-1, q^5+q^2-1, -(q^7+q^4+1), q^9+q^6+q^2, -(q^4-q^2+1), -(q^3-q^2+1), -(q^4-q^2+1)+q^4·(2·q)^(1/2), -(q^4-q^2+1)-q^4·(2·q)^(1/2), q^6+q^2-1+q^4·(2·q)^(1/2), q^6+q^2-1-q^4·(2·q)^(1/2)]])):
> J := Multiply(V^(-1), A):
> A1 := simplify(simplify(J[1][1])):
> B1 := simplify(expand(numer(A1))):
> C1 := simplify(expand(denom(A1))):
> D1 := simplify(B1/C1);
```

$$D1 := 536870912\,r^{60}+8388608\,r^{48}+2048\,r^{24}+32\,r^{12} \tag{69}$$

```
> A2 := simplify(simplify(J[2][1])):
> B2 := simplify(expand(numer(A2))):
> C2 := simplify(expand(denom(A2))):
> D2 := simplify(B2/C2);
```

$$D2 := 16\left(4r^4-4r^3+2r^2-2r+1\right)^2\left(64r^{12}+32r^9+8r^6+4r^3+1\right)\left(64r^{12}-32r^9+8r^6-4r^3+1\right)\left(4r^4+4r^3+2r^2+2r+1\right)^2 r^8\left(4r^4-2r^2+1\right)\left(64r^{12}-8r^6+1\right) \tag{70}$$

```
> A3 := simplify(simplify(J[3][1])):
> B3 := simplify(expand(numer(A3))):
> C3 := simplify(expand(denom(A3))):
```

```
> D3 := simplify(B3/C3);
```

$$D3 := \left(4\left(4r^4-4r^3+2r^2-2r+1\right)^2\left(64r^{12}+32r^9+8r^6+4r^3+1\right)\left(64r^{12}-32r^9+8r^6-4r^3+1\right)\left(4r^4+4r^3+2r^2+2r+1\right)^2 r^4\left(4r^4-2r^2+1\right)\left(64r^{12}-8r^6+1\right)\right)\Big/\left(4096r^{24}-2048r^{22}+256r^{16}-64r^{12}+16r^8-2r^2+1\right) \tag{71}$$

```
> A4 := simplify(simplify(J[4][1])):
> B4 := simplify(expand(numer(A4))):
> C4 := simplify(expand(denom(A4))):
> D4 := simplify(B4/C4);
```

$$D4 := 1 \tag{72}$$

```
> A5 := simplify(simplify(J[5][1])):
> B5 := simplify(expand(numer(A5))):
> C5 := simplify(expand(denom(A5))):
> D5 := simplify(B5/C5);
```

$$D5 := \frac{1}{16r^8-8r^6+4r^4-2r^2+1}\left(16\left(2r^2-2r+1\right)^2\left(4r^4-4r^3+2r^2-2r+1\right)^2\left(64r^{12}+32r^9+8r^6+4r^3+1\right)\left(2r^2+2r+1\right)^2\left(64r^{12}-32r^9+8r^6-4r^3+1\right)\left(4r^4+4r^3+2r^2+2r+1\right)^2 r^{10}\left(4r^4-2r^2+1\right)^2\left(64r^{12}-8r^6+1\right)\right) \tag{73}$$

```
> A6 := simplify(simplify(J[6][1])):
> B6 := simplify(expand(numer(A6))):
> C6 := simplify(expand(denom(A6))):
> D6 := simplify(B6/C6);
```

$$D6 := 32768\,r^{30}\left(262144\,r^{36}-32768\,r^{30}+512\,r^{18}-8r^6+1\right) \tag{74}$$

```
> A7 := simplify(simplify(J[7][1])):
> B7 := simplify(expand(numer(A7))):
> C7 := simplify(expand(denom(A7))):
> D7 := simplify(B7/C7);
```

$$D7 := \left(2361183241434822606848\left(r^2-r+\frac{1}{2}\right)^2\left(\left(\frac{5}{2147483648}r^2+\frac{181}{131072}r^{30}+\frac{271}{262144}r^{28}+\frac{85}{524288}r^{26}-\frac{209}{1048576}r^{24}-\frac{41}{1048576}r^{22}+\frac{25}{1048576}r^{20}\right.\right.\right. \tag{75}$$

$$
\begin{aligned}
&- \frac{11}{8388608} r^{16} - \frac{19}{8388608} r^{14} + \frac{11}{32} r^{54} + \frac{19}{64} r^{52} + \frac{19}{64} r^{50} + \frac{95}{256} r^{48} \\
&+ \frac{39}{512} r^{46} + \frac{17}{16} r^{56} + r^{62} + \frac{7}{4} r^{60} + \frac{15}{8} r^{58} - \frac{1}{256} r^{44} - \frac{3}{128} r^{42} + \frac{77}{4096} r^{40} \\
&+ \frac{3}{128} r^{38} + \frac{5}{4096} r^{36} - \frac{113}{32768} r^{34} - \frac{19}{8192} r^{32} + \frac{35}{67108864} r^{10} + \frac{53}{2097152} r^{18} \\
&+ \frac{1}{4294967296} - \frac{7}{8388608} r^{12} - \frac{21}{1073741824} r^{4} - \frac{3}{536870912} r^{6} \\
&+ \frac{17}{268435456} r^{8} \Big) \operatorname{csgn}(r) - 2 \Big(- \frac{1}{536870912} r^{2} + \frac{7}{131072} r^{30} + \frac{23}{32768} r^{28} \\
&+ \frac{39}{131072} r^{26} - \frac{71}{1048576} r^{24} - \frac{61}{1048576} r^{22} - \frac{1}{524288} r^{20} + \frac{27}{8388608} r^{16} \\
&- \frac{17}{16777216} r^{14} + \frac{3}{16} r^{54} + \frac{9}{64} r^{52} + \frac{3}{16} r^{50} + \frac{17}{128} r^{48} + \frac{75}{512} r^{46} + r^{56} + r^{60} \\
&+ \frac{5}{8} r^{58} - \frac{3}{512} r^{44} - \frac{21}{2048} r^{42} + \frac{5}{4096} r^{40} + \frac{95}{8192} r^{38} + \frac{109}{16384} r^{36} \\
&- \frac{25}{16384} r^{34} - \frac{123}{65536} r^{32} + \frac{5}{33554432} r^{10} + \frac{79}{4194304} r^{18} - \frac{35}{33554432} r^{12} \\
&- \frac{7}{536870912} r^{4} + \frac{5}{536870912} r^{6} + \frac{17}{134217728} r^{8} + \frac{3}{4294967296} \Big) r \Big) \Big(r^{4} \\
&- r^{3} + \frac{1}{2} r^{2} - \frac{1}{2} r + \frac{1}{4} \Big)^{2} \Big(r^{12} + \frac{1}{2} r^{9} + \frac{1}{8} r^{6} + \frac{1}{16} r^{3} + \frac{1}{64} \Big) \Big(r^{2} + r \\
&+ \frac{1}{2} \Big)^{2} \Big(r^{12} - \frac{1}{2} r^{9} + \frac{1}{8} r^{6} - \frac{1}{16} r^{3} + \frac{1}{64} \Big) \Big(r^{4} + r^{3} + \frac{1}{2} r^{2} + \frac{1}{2} r \\
&+ \frac{1}{4} \Big)^{2} r^{10} \Big(r^{4} - \frac{1}{2} r^{2} + \frac{1}{4} \Big)^{2} \Big(r^{2} + \frac{1}{2} \Big)^{2} \Big(r^{12} - \frac{1}{8} r^{6} + \frac{1}{64} \Big) \Big) \Big/ \\
&\Big((274877906944 r^{74} + 1168231104512 r^{72} + 1099511627776 r^{70} + 893353197568 r^{68} \\
&+ 68719476736 r^{66} - 94489280512 r^{64} + 94489280512 r^{62} + 194347270144 r^{60} \\
&+ 130996502528 r^{58} - 34628173824 r^{56} - 60934848512 r^{54} - 4831838208 r^{52} \\
&+ 24226299904 r^{50} + 14126415872 r^{48} - 4395630592 r^{46} - 5859442688 r^{44} \\
&- 159383552 r^{42} + 1940914176 r^{40} + 856686592 r^{38} - 315359232 r^{36} - 297271296 r^{34} \\
&- 19333120 r^{32} + 73793536 r^{30} + 22773760 r^{28} - 7634944 r^{26} - 6778880 r^{24}
\end{aligned}
$$

$$+ 122880\,r^{22} + 1170432\,r^{20} + 273408\,r^{18} - 119296\,r^{16} - 52224\,r^{14} + 5760\,r^{12}$$

$$+ 7488\,r^{10} + 256\,r^{8} - 480\,r^{6} - 96\,r^{4} + 24\,r^{2} + 1\Big)\,\mathrm{csgn}(r) - 824633720832\left(r^{36}\right.$$

$$+ \frac{1}{3}\,r^{34} + \frac{1}{3}\,r^{32} - \frac{1}{8}\,r^{30} - \frac{1}{16}\,r^{28} + \frac{5}{48}\,r^{26} + \frac{1}{24}\,r^{24} + \frac{1}{384}\,r^{22} - \frac{17}{768}\,r^{20}$$

$$- \frac{3}{512}\,r^{18} + \frac{5}{1024}\,r^{16} + \frac{1}{384}\,r^{14} + \frac{1}{4096}\,r^{12} - \frac{13}{24576}\,r^{10} - \frac{1}{24576}\,r^{8}$$

$$+ \frac{1}{16384}\,r^{6} + \frac{1}{32768}\,r^{4} - \frac{1}{196608}\,r^{2} - \frac{1}{786432}\Big)\left(r^{36} + r^{34} + \frac{3}{4}\,r^{32} - \frac{1}{8}\,r^{28}\right.$$

$$+ \frac{1}{16}\,r^{26} + \frac{7}{64}\,r^{24} + \frac{7}{128}\,r^{22} - \frac{5}{128}\,r^{20} - \frac{1}{64}\,r^{18} + \frac{15}{2048}\,r^{14} + \frac{7}{4096}\,r^{12}$$

$$- \frac{1}{2048}\,r^{10} - \frac{1}{2048}\,r^{8} + \frac{1}{8192}\,r^{6} + \frac{1}{16384}\,r^{4} + \frac{1}{65536}\,r^{2} - \frac{1}{131072}\Big)\,r\Big)$$

> $A8 := simplify(simplify(J[8][1]))$:

> $B8 := simplify(expand(numer(A8)))$:

> $C8 := simplify(expand(denom(A8)))$:

> $D8 := simplify\left(\frac{B8}{C8}\right)$;

$$D8 := \Big(8\left(4\,r^{4} - 4\,r^{3} + 2\,r^{2} - 2\,r + 1\right)^{2}\left(1 + 14\,r^{2} + 3145728\,r^{30} - 344064\,r^{28}\right. \tag{76}$$

$$- 1335296\,r^{26} - 368640\,r^{24} + 233472\,r^{22} + 152576\,r^{20} - 33536\,r^{16} - 6784\,r^{14}$$

$$- 83886080\,r^{48} - 75497472\,r^{46} - 41943040\,r^{44} + 23068672\,r^{42} + 25165824\,r^{40}$$

$$- 1048576\,r^{38} - 13369344\,r^{36} - 5373952\,r^{34} + 4784128\,r^{32} + 1792\,r^{10} - 3072\,r^{18}$$

$$+ 3392\,r^{12} + 100663296\,r^{47}\,\mathrm{csgn}(r) + 33554432\,\mathrm{csgn}(r)\,r^{49} + 58720256\,\mathrm{csgn}(r)\,r^{45}$$

$$+ 8388608\,\mathrm{csgn}(r)\,r^{43} - 29360128\,\mathrm{csgn}(r)\,r^{41} - 14680064\,\mathrm{csgn}(r)\,r^{39}$$

$$+ 12582912\,\mathrm{csgn}(r)\,r^{37} + 10223616\,\mathrm{csgn}(r)\,r^{35} - 786432\,\mathrm{csgn}(r)\,r^{33}$$

$$- 4718592\,\mathrm{csgn}(r)\,r^{31} - 1409024\,\mathrm{csgn}(r)\,r^{29} + 1392640\,\mathrm{csgn}(r)\,r^{27}$$

$$+ 835584\,\mathrm{csgn}(r)\,r^{25} - 8192\,\mathrm{csgn}(r)\,r^{23} - 247808\,\mathrm{csgn}(r)\,r^{21} - 57344\,\mathrm{csgn}(r)\,r^{19}$$

$$+ 31744\,\mathrm{csgn}(r)\,r^{17} + 20736\,\mathrm{csgn}(r)\,r^{15} - 1024\,\mathrm{csgn}(r)\,r^{13} - 3072\,\mathrm{csgn}(r)\,r^{11}$$

$$- 448\,\mathrm{csgn}(r)\,r^{9} + 336\,\mathrm{csgn}(r)\,r^{7} + 104\,\mathrm{csgn}(r)\,r^{5} - 32\,r^{4} - 208\,r^{6} - 256\,r^{8}$$

$$- 8\,\mathrm{csgn}(r)\,r^{3} - 6\,\mathrm{csgn}(r)\,r\Big)\left(64\,r^{12} + 32\,r^{9} + 8\,r^{6} + 4\,r^{3} + 1\right)\left(64\,r^{12} - 32\,r^{9}\right.$$

$$+ 8\,r^{6} - 4\,r^{3} + 1\Big)\left(4\,r^{4} + 4\,r^{3} + 2\,r^{2} + 2\,r + 1\right)^{2} r^{10}\left(4\,r^{4} - 2\,r^{2} + 1\right)^{2}\left(2\,r^{2}\right.$$

$$\left.+ 1\right)^{2}\left(64\,r^{12} - 8\,r^{6} + 1\right)\Big)\Big/\Big(\left(1 + 4\,r^{2} + 98304\,r^{30} + 49152\,r^{28} - 81920\,r^{26}\right.$$

$$- 32768\,r^{24} - 2048\,r^{22} + 17408\,r^{20} - 3840\,r^{16} - 2048\,r^{14} - 786432\,r^{36} - 262144\,r^{34}$$

$$
\begin{aligned}
&- 262144\,r^{32} + 416\,r^{10} + 4608\,r^{18} - 192\,r^{12} + 524288\,\mathrm{csgn}(r)\,r^{37} \\
&+ 524288\,\mathrm{csgn}(r)\,r^{35} + 393216\,\mathrm{csgn}(r)\,r^{33} - 65536\,\mathrm{csgn}(r)\,r^{29} + 32768\,\mathrm{csgn}(r)\,r^{27} \\
&+ 57344\,\mathrm{csgn}(r)\,r^{25} + 28672\,\mathrm{csgn}(r)\,r^{23} - 20480\,\mathrm{csgn}(r)\,r^{21} - 8192\,\mathrm{csgn}(r)\,r^{19} \\
&+ 3840\,\mathrm{csgn}(r)\,r^{15} + 896\,\mathrm{csgn}(r)\,r^{13} - 256\,\mathrm{csgn}(r)\,r^{11} - 256\,\mathrm{csgn}(r)\,r^{9} \\
&+ 64\,\mathrm{csgn}(r)\,r^{7} + 32\,\mathrm{csgn}(r)\,r^{5} - 24\,r^{4} - 48\,r^{6} + 32\,r^{8} + 8\,\mathrm{csgn}(r)\,r^{3} - 4\,\mathrm{csgn}(r)\,r\Big) \\
&\left(16\,r^{8} + 16\,r^{7} + 8\,r^{6} - 4\,r^{4} + 2\,r^{2} + 2\,r + 1\right)\left(16\,r^{8} - 16\,r^{7} + 8\,r^{6} - 4\,r^{4} + 2\,r^{2}\right. \\
&\left.- 2\,r + 1\right)\Big)
\end{aligned}
$$

> $A9 := simplify(simplify(J[9][1]))$:

> $B9 := simplify(expand(numer(A9)))$:

> $C9 := simplify(expand(denom(A9)))$:

> $D9 := simplify\left(\frac{B9}{C9}\right)$;

$$
\begin{aligned}
D9 := &\Bigg(140737488355328\left(\frac{1}{33554432}\,r^{2} - \frac{21}{4096}\,r^{30} - \frac{5}{8192}\,r^{28} + \frac{7}{16384}\,r^{26}\right. \\
&- \frac{3}{32768}\,r^{24} - \frac{3}{65536}\,r^{22} + \frac{13}{65536}\,r^{20} - \frac{17}{524288}\,r^{16} - \frac{27}{1048576}\,r^{14} - \frac{3}{2}\,r^{52} \\
&- r^{50} - \frac{1}{4}\,r^{48} + \frac{1}{8}\,r^{46} + \frac{5}{16}\,r^{44} - \frac{3}{32}\,r^{42} - \frac{15}{128}\,r^{40} + \frac{1}{64}\,r^{38} + \frac{21}{512}\,r^{36} \\
&+ \frac{5}{256}\,r^{34} - \frac{23}{2048}\,r^{32} + \frac{19}{4194304}\,r^{10} + \frac{3}{65536}\,r^{18} + \frac{7}{2097152}\,r^{12} - \frac{1}{8388608}\,r^{4} \\
&- \frac{13}{16777216}\,r^{6} + \frac{3}{8388608}\,r^{8} + \frac{1}{134217728} + r\left(\frac{1}{33554432}\,r^{2} + \frac{17}{2048}\,r^{30}\right. \\
&+ \frac{7}{2048}\,r^{28} - \frac{1}{1024}\,r^{26} + \frac{3}{16384}\,r^{24} + \frac{1}{32768}\,r^{22} - \frac{3}{32768}\,r^{20} + \frac{1}{262144}\,r^{16} \\
&+ \frac{21}{524288}\,r^{14} + r^{52} + r^{50} + r^{48} - \frac{1}{8}\,r^{46} - \frac{5}{16}\,r^{44} - \frac{1}{32}\,r^{42} + \frac{3}{32}\,r^{40} + \frac{5}{64}\,r^{38} \\
&- \frac{13}{256}\,r^{36} - \frac{17}{512}\,r^{34} + \frac{1}{512}\,r^{32} - \frac{13}{2097152}\,r^{10} - \frac{9}{65536}\,r^{18} + \frac{1}{131072}\,r^{12} \\
&\left.\left.- \frac{1}{33554432} + \frac{3}{8388608}\,r^{4} + \frac{3}{4194304}\,r^{6} - \frac{9}{4194304}\,r^{8}\right)\mathrm{csgn}(r)\right)\left(r^{2} - r\right. \\
&\left.+ \frac{1}{2}\right)^{2}\left(r^{2} + r + \frac{1}{2}\right)^{2} r^{12}\left(r^{4} - \frac{1}{2}\,r^{2} + \frac{1}{4}\right)^{2}\left(r^{2} + \frac{1}{2}\right)^{2}\left(r^{12} - \frac{1}{8}\,r^{6} + \frac{1}{64}\right)\Bigg) \\
&\Big/\Big(\left(524288\,r^{37} + 524288\,r^{35} + 393216\,r^{33} - 65536\,r^{29} + 32768\,r^{27} + 57344\,r^{25}\right. \\
&\left.+ 28672\,r^{23} - 20480\,r^{21} - 8192\,r^{19} + 3840\,r^{15} + 896\,r^{13} - 256\,r^{11} - 256\,r^{9} + 64\,r^{7}\right. \\
&\left.+ 32\,r^{5} + 8\,r^{3} - 4\,r\right)\mathrm{csgn}(r) - 786432\,r^{36} - 262144\,r^{34} - 262144\,r^{32} + 98304\,r^{30}
\end{aligned}
\tag{77}
$$

$$+ 49152\,r^{28} - 81920\,r^{26} - 32768\,r^{24} - 2048\,r^{22} + 17408\,r^{20} + 4608\,r^{18} - 3840\,r^{16}$$
$$- 2048\,r^{14} - 192\,r^{12} + 416\,r^{10} + 32\,r^{8} - 48\,r^{6} - 24\,r^{4} + 4\,r^{2} + 1\Big)$$

```
> A10 := simplify(simplify(J[10][1])):
> B10 := simplify(expand(numer(A10))):
> C10 := simplify(expand(denom(A10))):
> D10 := simplify(B10/C10);
```

$$D10 := \Bigg(140737488355328\,r^{12}\left(\frac{1}{536870912}\,r^{2} - \frac{27}{32768}\,r^{30} + \frac{63}{131072}\,r^{28}\right. \tag{78}$$
$$+ \frac{73}{262144}\,r^{26} - \frac{7}{131072}\,r^{24} - \frac{45}{524288}\,r^{22} - \frac{1}{2097152}\,r^{20} + \frac{5}{2097152}\,r^{16}$$
$$- \frac{17}{8388608}\,r^{14} + \frac{3}{8}\,r^{54} - \frac{1}{32}\,r^{52} - \frac{11}{32}\,r^{50} - \frac{1}{128}\,r^{48} + \frac{9}{128}\,r^{46} - \frac{3}{2}\,r^{60}$$
$$+ \frac{29}{512}\,r^{44} - \frac{19}{512}\,r^{42} - \frac{13}{512}\,r^{40} + \frac{11}{4096}\,r^{38} + \frac{37}{4096}\,r^{36} + \frac{17}{16384}\,r^{34}$$
$$- \frac{11}{4096}\,r^{32} + \frac{19}{67108864}\,r^{10} + \frac{61}{4194304}\,r^{18} - \frac{31}{33554432}\,r^{12} - \frac{5}{268435456}\,r^{4}$$
$$- \frac{1}{268435456}\,r^{6} + \frac{3}{33554432}\,r^{8} + \frac{1}{2147483648} + r\left(\frac{3}{536870912}\,r^{2} + \frac{33}{16384}\,r^{30}\right.$$
$$- \frac{1}{16384}\,r^{28} - \frac{1}{2048}\,r^{26} - \frac{9}{131072}\,r^{24} + \frac{13}{131072}\,r^{22} + \frac{21}{524288}\,r^{20}$$
$$- \frac{1}{131072}\,r^{16} + \frac{5}{8388608}\,r^{14} - \frac{1}{8}\,r^{54} - \frac{1}{4}\,r^{52} + \frac{11}{32}\,r^{50} + \frac{5}{32}\,r^{48} - \frac{5}{128}\,r^{46}$$
$$+ r^{60} + r^{58} - \frac{11}{128}\,r^{44} - \frac{1}{256}\,r^{42} + \frac{11}{256}\,r^{40} + \frac{15}{2048}\,r^{38} - \frac{33}{4096}\,r^{36} - \frac{45}{8192}\,r^{34}$$
$$+ \frac{31}{16384}\,r^{32} - \frac{33}{2097152}\,r^{18} + \frac{17}{8388608}\,r^{12} + \frac{3}{134217728}\,r^{4} - \frac{1}{33554432}\,r^{6}$$
$$\left.\left.\left.\left.- \frac{13}{67108864}\,r^{8} - \frac{1}{536870912}\right)\operatorname{csgn}(r)\right)\left(r^{4} - \frac{1}{2}\,r^{2} + \frac{1}{4}\right)^{2}\left(r^{2} + \frac{1}{2}\right)^{2}\left(r^{12}\right.\right.\right.$$
$$\left.\left.\left.- \frac{1}{8}\,r^{6} + \frac{1}{64}\right)\right)\Bigg/\Big(\big(524288\,r^{37} + 524288\,r^{35} + 393216\,r^{33} - 65536\,r^{29}$$
$$+ 32768\,r^{27} + 57344\,r^{25} + 28672\,r^{23} - 20480\,r^{21} - 8192\,r^{19} + 3840\,r^{15} + 896\,r^{13}$$
$$- 256\,r^{11} - 256\,r^{9} + 64\,r^{7} + 32\,r^{5} + 8\,r^{3} - 4\,r\big)\operatorname{csgn}(r) - 786432\,r^{36} - 262144\,r^{34}$$
$$- 262144\,r^{32} + 98304\,r^{30} + 49152\,r^{28} - 81920\,r^{26} - 32768\,r^{24} - 2048\,r^{22}$$
$$+ 17408\,r^{20} + 4608\,r^{18} - 3840\,r^{16} - 2048\,r^{14} - 192\,r^{12} + 416\,r^{10} + 32\,r^{8} - 48\,r^{6}$$
$$- 24\,r^{4} + 4\,r^{2} + 1\Big)$$

```
> q1 := numer(D5);
```

$$q1 := 16\left(2r^2-2r+1\right)^2\left(4r^4-4r^3+2r^2-2r+1\right)^2\left(64r^{12}+32r^9+8r^6+4r^3+1\right)\left(2r^2+2r+1\right)^2\left(64r^{12}-32r^9+8r^6-4r^3+1\right)\left(4r^4+4r^3+2r^2+2r+1\right)^2 r^{10}\left(4r^4-2r^2+1\right)^2\left(64r^{12}-8r^6+1\right) \quad (79)$$

```
> q2 := denom(D5);
```

$$q2 := 16r^8-8r^6+4r^4-2r^2+1 \quad (80)$$

```
> rem(q1, q2, r);
```

$$4r^6-2r^4+r^2 \quad (81)$$

 # The fact that the multiplicity of the fifth eigenvalue is integral implies that q2= $16r^8-8r^6+4r^4-2r^2+1$ *is a divisor of q1.*

 # This is however impossible as the remainder of the division is equal to $4r^6-2r^4+r^2 < 16r^8-8r^6+4r^4-2r^2+1$.