Polygonal triples

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Abstract

This paper arose from the observation that several (families of) near polygons $\mathcal{S}$, including the recently discovered $G_2(4)$ and $L_3(4)$ near octagons, share similar properties. They all have a line spread S and a set $\mathcal{Q}$ of quads that behave very nicely. In particular, S and $\mathcal{Q}$ define a near polygon $\mathcal{S}'$ whose diameter is one less than the one of $\mathcal{S}$. In this paper, we derive several properties of such "polygonal triples" $(\mathcal{S}, S, \mathcal{Q})$ and obtain some classification results.

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1 Introduction

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near polygon* if every two distinct points are incident with at most one line, if the collinearity graph Γ of $\mathcal{S}$ has finite diameter and if for every point-line pair (x, L), there exists a unique point on L that is nearest to x with respect to the distance in Γ. If d is the diameter of Γ, then $\mathcal{S}$ is called a *near $2d$-gon.* A near 0-gon is a point (no lines) and a near 2-gon is a line. A near quadrangle with a pair of disjoint lines is also called a *generalized quadrangle* [12].

A set X of points of a near polygon $\mathcal{S}$ is called a *subspace* if every line that has two points in X has all its points in X. If X is a nonempty subspace, then the subgeometry of $\mathcal{S}$ defined by the points contained in X and the lines that have all their points in X will be denoted by $\widetilde{X}$. A subspace X will be called *convex* if every point on a shortest path between two points of X is also contained in X. If X is a nonempty convex subspace of a near polygon, then $\widetilde{X}$ itself is also a near polygon. Such a nonempty convex subspace X is called a *quad* if $\widetilde{X}$ is a generalized quadrangle.

Suppose now that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near $(2d+2)$-gon with $d \geq 1$ having some *line spread* S, i.e. a set of lines partitioning the point set, and suppose $\mathcal{Q}$ is a family of quads of $\mathcal{S}$. Then we call $(\mathcal{S}, S, \mathcal{Q})$ a *polygonal triple* if the following two properties are satisfied:

(P1) For every point x of $\mathcal{S}$, the quads of $\mathcal{Q}$ through x all contain the unique line L_x of S through x, and partition the set of lines through x distinct from L_x.

(P2) The point-line geometry $\mathcal{S}'$ with point set S, line set $\mathcal{Q}$ and natural incidence (i.e. containment) is a near $2d$-gon.

We will also say that $\mathcal{S}'$ is the *near polygon associated with the polygonal triple* $(\mathcal{S}, S, \mathcal{Q})$. If $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple, then $\mathcal{Q}$ is uniquely determined by $\mathcal{S}$ and S as it consists of all quads of $\mathcal{S}$ containing a line of S. (If two distinct intersecting lines of a near polygon are contained in a quad, then this quad must be unique.) If $d = 1$, then $\mathcal{S}$ is a generalized quadrangle, S is a line spread of $\mathcal{S}$ and $\mathcal{Q} = \{\mathcal{S}\}$.

The present paper arose from the observation that several (families of) near polygons $\mathcal{S}$, including the recently discovered $G_2(4)$ and $L_3(4)$ near octagons [1, 2], have line spreads S and families $\mathcal{Q}$ of quads for which (P1) and (P2) are satisfied. This led us to the idea to develop a theory for polygonal triples.

Several known examples of polygonal triples will be listed in Section 3. In Section 4, we derive several properties of near polygons $\mathcal{S}$ that admit a polygonal triple $(\mathcal{S}, S, \mathcal{Q})$, and use these in Section 5 to obtain classification results for polygonal triples. Among other results, we will classify there all polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ for which $\mathcal{S}$ is a finite near hexagon with only thick lines (i.e. lines with at least three points) and for which the associated near polygon is a (nondegenerate) generalized quadrangle. The results obtained in Sections 3, 4 and 5 rely on several technical definitions and facts from the theory of near polygons, in particular on results of product and glued near polygons. These will be surveyed in Section 2.

Consider now the following problem.

> Given a near $2d$-gon $\mathcal{S}'$. Determine all polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ for which $\mathcal{S}'$ is the associated near polygon.

The present paper contains some results in this direction. In Theorem 5.4 for instance, we solve this problem in case $\mathcal{S}'$ is a grid. In [3], we will develop an algorithm for settling this problem for certain near polygons $\mathcal{S}'$ and apply it to some particular cases. Our hope is that this algorithm can be used to construct new interesting examples of near polygons by means of computer. We were not (yet) successful in doing that, but instead we have used it to obtain characterization results for the $G_2(4)$ and $L_3(4)$ near octagons, and to prove the nonexistence of certain polygonal triples. While computer results and algorithms play an important role in [3], the current paper is merely devoted to a theoretical treatment of near polygons admitting a polygonal triple.

2 Preliminaries

2.1 Quads in near polygons

If x and y are two points of a near polygon $\mathcal{S}$, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the collinearity graph Γ of $\mathcal{S}$ and $\Gamma_i(x)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from x. Recall that a quad of a near polygon is a nonempty convex subspace Q for which $\widetilde{Q}$ is a generalized quadrangle. Any two points at distance 2, or equivalently any two distinct intersecting lines, are contained in at most one quad. The following two propositions are taken from Shult and Yanushka [13].

Proposition 2.1 ([13, Proposition 2.5]) *Suppose a and b are two points of a near polygon at distance 2 from each other, and suppose a and b have two common neighbours c and d such that at least one of the lines ac, cb, bd, da contains at least three points. Then a and b are contained in a unique quad.*

Proposition 2.2 ([13, Proposition 2.6]) *Suppose $\mathcal{S}$ is a near polygon having the property that every line is incident with at least three points. Then one of the following cases occurs for a point-quad pair (x, Q) of $\mathcal{S}$:*

(1) *There exists a unique point $x' \in Q$ nearest to x. In this case, $d(x, y) = d(x, x') + d(x', y)$ for every point $y \in Q$.*

(2) *The points in Q nearest to x form an* ovoid *of $\widetilde{Q}$, i.e. a set of points of $\widetilde{Q}$ having a unique point in common with every line of $\widetilde{Q}$.*

If case (1) occurs in Proposition 2.2, then the point x is called *classical* with respect to Q. If case (2) occurs, then the point x is called *ovoidal* with respect to Q. The following property was proved in Brouwer and Wilbrink [4, Lemma 8], see also [9, Theorem 1.23].

Proposition 2.3 ([4]) *Suppose $\mathcal{S}$ is a near polygon having at least three points on each line. Let Q be a quad of $\mathcal{S}$ and L a line containing points at distance i and $i + 1$ from Q. Then L has a unique point at distance i from Q.*

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two neighbors. Proposition 2.1 tells us that every two points of a dense near polygon at distance 2 from each other are contained in a unique quad. The existence of quads can be used to prove the following, see Brouwer and Wilbrink [4, Lemma 19] and [9, Theorem 2.2].

Proposition 2.4 ([4]) *In a dense near polygon $\mathcal{S}$, there exists a constant t such that every point of $\mathcal{S}$ is incident with precisely $t + 1$ lines.*

A near polygon is said to have *order* (s, t) if every line is incident with precisely $s + 1$ points and every point is incident with exactly $t + 1$ lines. If $\mathcal{Q}$ is a finite generalized quadrangle of order (s, t), then $\mathcal{Q}$ contains $(s + 1)(st + 1)$ points, and every ovoid of $\mathcal{Q}$ contains precisely $st + 1$ points. If $s, t \geq 2$, then an inequality of Higman [11, (6.4)] states that $t \leq s^2$ and $s \leq t^2$.

If all points of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ are classical with respect to some quad Q, then Q is called *classical*. More generally, a convex subspace F of $\mathcal{S}$ is called *classical* if for every point x of $\mathcal{S}$ there exists a unique point $x' \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, x') + \mathrm{d}(x', y)$ for every point $y \in F$. The point x' is called the *projection* of x on F and will often be denoted by $\pi_F(x)$. Every line of a near polygon is a classical convex subspace. A convex subspace $F \neq \mathcal{P}$ is called *big* if every point outside F is collinear with a (necessarily unique) point of F. Every big convex subspace is also classical. In a near hexagon, a quad is classical if and only if it is big. The following is a special case of Theorem 1.7 of [9].

Proposition 2.5 ([9]) *If Q_1 and Q_2 are two quads of a near hexagon $\mathcal{S}$ intersecting in a singleton, then none of Q_1, Q_2 can be big in $\mathcal{S}$.*

2.2 Dual polar spaces

Suppose Π is a polar space of rank $n \geq 1$ in the sense[1] of Tits [14, Chapter 7]. Then with Π, there is associated a near $2n$-gon Δ, called a *dual polar space of rank n* ([5, 13]). The points of Δ are the maximal singular subspaces of Π (those of projective dimension $n-1$), the lines of Δ are the next-to-maximal singular subspaces of Π (those of projective dimension $n-2$), and incidence is reverse containment. A dual polar space of rank 1 is a line and a dual polar space of rank 2 is a generalized quadrangle. By definition, a dual polar space of rank 0 is a point.

There exists a bijective correspondence between the possibly empty singular subspaces α of Π and the nonempty convex subspaces F_α of Δ: if α is a singular subspace of dimension $n-1-k$ of Π with $k \in \{0, 1, \ldots, n\}$, then the set F_α of all maximal singular subspaces of Π containing α is a convex subspace of diameter k of Δ. Obviously, if α_1 and α_2 are singular subspaces of Π, then $\alpha_1 \subseteq \alpha_2$ if and only if $F_{\alpha_2} \subseteq F_{\alpha_1}$. As the subspaces of a projective space Σ, ordered by reverse containment, also determine a projective space (the dual of Σ), we thus see that the following should hold.

Proposition 2.6 *The system of convex subspaces through a given point of Δ is isomorphic to the system of subspaces of a certain $(n-1)$-dimensional projective space.*

Every two points of Δ at distance k from each other are contained in a unique convex subspace of diameter k. If F is a nonempty convex subspace of Δ, then $\widetilde{F}$ itself is also a dual polar space. The convex subspaces of diameter 2 are precisely the quads. The convex subspaces of Δ (of diameter $n-1$) corresponding to the points of Π are called the *maxes* of Δ. Every convex subspace of Δ is classical, in particular every max is big. The following can be proved, see Cameron [5].

Proposition 2.7 ([5]) *The dual polar spaces of rank n are precisely the near $2n$-gons having the property that every two points at distance 2 are contained in a unique classical quad.*

We shall also need the following property.

[1] The singular subspaces are allowed to be reducible projective spaces (having lines of size 2).

Proposition 2.8 *Let F be a max of Δ, x a point not contained in F and L the unique line through x meeting F. Then every convex subspace through x not containing L is disjoint from F.*

Proof. Let x' denote the unique point of F on the line L. Suppose F' is a convex subspace through x containing a point y of F. Then every shortest path between x and y is contained in F'. Now, since F is classical, there exists a shortest path between x and y containing the point x'. So, $x' \in F'$ and L is contained in F'. We conclude that every convex subspace through x not containing L is disjoint from F. ∎

A survey of the basic properties of dual polar spaces can be found in Chapter 8 of [10]. This book also contains proofs of all properties of dual polar spaces that we have mentioned above.

2.3 Admissible line spreads

Two nonempty classical convex subspaces (in particular, two lines) F_1 and F_2 of a near polygon $\mathcal{S}$ are called *parallel* if for every point $x_1 \in F_1$, there exists a unique point $\Pi_{F_1,F_2}(x_1) \in F_2$ at distance $\mathrm{d}(F_1, F_2)$ from x_1, and for every point $x_2 \in F_2$, there exists a unique point $\pi_{F_2,F_1}(x_2) \in F_1$ at distance $\mathrm{d}(F_1, F_2)$ from x_2. If F_1 and F_2 are such convex subspaces, then by [9, Theorem 1.10], $\Pi_{F_1,F_2} : F_1 \to F_2$ and $\Pi_{F_2,F_1} : F_2 \to F_1$ define isomorphisms between $\widetilde{F_1}$ and $\widetilde{F_2}$ (and moreover they are each other's inverses).

A line spread of a near polygon is called *admissible* if every two of its lines are parallel.

Proposition 2.9 *Suppose $\mathcal{S}$ is a near polygon having at least three points on each line, and S is an admissible line spread of $\mathcal{S}$. Then any two lines L_1 and L_2 of S at distance 1 from each other are contained in a (necessarily unique) quad.*

Proof. Let x_1 and x_1' be two distinct points of L_1, and let x_2 and x_2' be the unique points of L_2 collinear with respectively x_1 and x_1'. The points x_1 and x_2' then lie at distance 2 from each other and x_2, x_1' are two common neighbours of these points. By Proposition 2.1, we know that x_1 and x_2' are contained in a unique quad. This quad necessarily contains the lines L_1 and L_2. ∎

Proposition 2.10 *Let S be an admissible line spread of a near $(2d+2)$-gon $\mathcal{S}$, and let Γ denote the graph whose vertices are the element of S, where two elements are adjacent whenever they lie at distance 1 from each other regarded as lines of $\mathcal{S}$. Then $d_{\mathcal{S}}(L_1, L_2) = d_\Gamma(L_1, L_2)$ for all lines $L_1, L_2 \in S$. Moreover, the diameter of the graph Γ is equal to d.*

Proof. Put $\delta := \mathrm{d}_\Gamma(L_1, L_2)$ and let $L_1 = K_0, K_1, \ldots, K_\delta = L_2$ be a (shortest) path of length δ in Γ connecting L_1 and L_2. Let $x_0 \in K_0$ and for every $i \in \{1, 2, \ldots, \delta\}$, let x_i denote the unique point of K_i collinear with x_{i-1}. Such a point exists since the lines K_{i-1} and K_i are parallel and at distance 1 from each other. Since $x_0 \in L_1$, $x_\delta \in L_2$ and $\mathrm{d}(x_0, x_\delta) \leq \delta$, we have $\mathrm{d}_{\mathcal{S}}(L_1, L_2) \leq \delta = \mathrm{d}_\Gamma(L_1, L_2)$.

Put $\delta' = \mathrm{d}_{\mathcal{S}}(L_1, L_2)$. Let $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = \delta'$. Let $x_1 = y_0, y_1, \ldots, y_{\delta'} = x_2$ be a (shortest) path of length δ' in $\mathcal{S}$ connecting x_1 and x_2. For every $i \in \{0, 1, \ldots, \delta'\}$, let K_i denote the unique line of S containing y_i. Then for every $i \in \{1, 2, \ldots, \delta'\}$, either $K_{i-1} = K_i$ or $\mathrm{d}_\Gamma(K_{i-1}, K_i) = 1$. We conclude that $\mathrm{d}_\Gamma(L_1, L_2) \leq \delta' = \mathrm{d}_{\mathcal{S}}(L_1, L_2)$.

Hence, $\mathrm{d}_{\mathcal{S}}(L_1, L_2) = \mathrm{d}_\Gamma(L_1, L_2)$.

Since $\mathcal{S}$ is a near $(2d+2)$-gon, we have $\mathrm{d}(x, L) \leq d$ for any point-line pair (x, L), implying that $\mathrm{d}(K_1, K_2) \leq d$ for every two lines K_1 and K_2 of S. Suppose x_1 and x_2 are two points of $\mathcal{S}$ at maximal distance $d + 1$ from each other, and denote by K_i with $i \in \{1, 2\}$ the unique line of S containing x_i. Since $d(x_1, x_2) = d + 1$ and K_1, K_2 are parallel lines, we necessarily have that $\mathrm{d}(K_1, K_2) = d$. This shows that the diameter of Γ is indeed d. ■

2.4 Product near polygons

From any two near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$, a new near polygon $\mathcal{S}_1 \times \mathcal{S}_2$ can be derived which is called the *cartesian* or *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$. The collinearity graph Γ of $\mathcal{S}_1 \times \mathcal{S}_2$ is the cartesian product of the collinearity graphs Γ_1 and Γ_2 of respectively $\mathcal{S}_1$ and $\mathcal{S}_2$. So, the vertices of Γ are the pairs (x_1, x_2) where x_i with $i \in \{1, 2\}$ is a point of $\mathcal{S}_i$. If (x_1, x_2) and (y_1, y_2) are two distinct vertices of Γ, then $(x_1, x_2) \sim_\Gamma (y_1, y_2)$ if and only if $x_i \sim_{\Gamma_i} y_i$ and $x_{3-i} = y_{3-i}$ for an $i \in \{1, 2\}$. A near polygon (like for instance a grid) isomorphic to the cartesian product of two near polygons of diameters at least 1 is called a *product near polygon.*

The product near polygon $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ has two partitions $\mathcal{T}_1$ and $\mathcal{T}_2$ of its point set in mutually parallel classical convex subspaces such that the following hold:

(1) if $F \in \mathcal{T}_i$ with $i \in \{1, 2\}$, then $\widetilde{F} \cong \mathcal{S}_i$;

(2) every element of $\mathcal{T}_1$ intersects every element of $\mathcal{T}_2$ in a point;

(3) every line of $\mathcal{S}$ is contained in a unique element of $\mathcal{T}_1 \cup \mathcal{T}_2$.

A proof of the following proposition can be found in [10, Theorem 8.18].

Proposition 2.11 ([10]) *The cartesian product of two dual polar spaces is again a dual polar space.*

The following was shown by Brouwer and Wilbrink [4, Theorem 1], see also [9, Theorem 1.12].

Proposition 2.12 ([4]) *Every near polygon with at least two line sizes having the property that every two points at distance* 2 *have at least two common neighbours is a product near polygon.*

The following property of generalized quadrangles is well known, but could also be regarded as a consequence of Proposition 2.12.

Corollary 2.13 *Suppose $\mathcal{Q}$ is a generalized quadrangle that is not a grid. Then every line of $\mathcal{Q}$ is incident with a constant number of points.*

Suppose $\mathcal{S}$ is the product near polygon $\mathcal{S}_1 \times \mathcal{S}_2$, where $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ is a near polygon and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ is a line. For every point x of $\mathcal{S}_1$, let L_x denote the line of $\mathcal{S}$ consisting of all points (x, y) where $y \in \mathcal{P}_2$. The set $S = \{L_x \,|\, x \in \mathcal{P}_1\}$ is then an (admissible) line spread of $\mathcal{S}$. Any line spread of a near polygon which can be obtained in this way is called *trivial.* In fact, with $\mathcal{S}_1$ and $\mathcal{S}_2$ as above, we will say that S is a trivial line spread of $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ with *associated near polygon* $\mathcal{S}_1$. The isomorphism class of $\mathcal{S}_1$ is indeed uniquely determined by the line spread S: if we fix a line $L \in S$ and a point $x \in L$, then the set X of points of $\mathcal{S}$ for which x is the nearest point on L is a convex subspace for which $\widetilde{X} \cong \mathcal{S}_1$. Alternatively, the collinearity graph of $\mathcal{S}_1$ (which uniquely determines $\mathcal{S}_1$) is isomorphic to the graph with the elements of S as vertices, where two lines of S are adjacent whenever they lie at distance 1 from each other.

2.5 Glued near polygons

The cartesian product construction allows to create new near polygons from other near polygons. It is also possible to create new near polygons from other near polygons by a construction known as glueing. The theory of glueing was developed in a series of papers by the author. A survey of the main results can be found in Chapter 4 of [9]. We only mention here the notions and properties that we will need later.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two near polygons and suppose S_i with $i \in \{1, 2\}$ is an admissible line spread of $\mathcal{S}_i$. If certain nice properties are satisfied, then according to [8], it is possible to construct new near polygons of diameter d_1+d_2-1, where d_i with $i \in \{1, 2\}$ is the diameter of $\mathcal{S}_i$. Any such near polygon is called a *glued near polygon of type* $\mathcal{S}_1 \otimes \mathcal{S}_2$. In case S_2 is a trivial line spread of $\mathcal{S}_2$ with associated near polygon $\mathcal{S}_2'$, then the corresponding glued near polygons are all isomorphic to the product near polygon $\mathcal{S}_1 \times \mathcal{S}_2'$. We will not discuss the precise way how glued near polygons of type $\mathcal{S}_1 \otimes \mathcal{S}_2$ are constructed from the near polygons $\mathcal{S}_1$, $\mathcal{S}_2$ and their respective admissible line spreads S_1 and S_2. We only mention here that if $\mathcal{S}$ is a glued near polygon of type $\mathcal{S}_1 \otimes \mathcal{S}_2$, then there exist two partitions $\mathcal{T}_1$ and $\mathcal{T}_2$ of the point set $\mathcal{P}$ of $\mathcal{S}$ in mutually parallel classical convex subspaces such that the following hold:

(I) For every $i \in \{1, 2\}$ and every $F_i \in \mathcal{T}_i$, we have $\widetilde{F_i} \cong \mathcal{S}_i$.

(II) Every element of $\mathcal{T}_1$ intersects every element of $\mathcal{T}_2$ in a line.

(III) Every line of $\mathcal{S}$ is contained in an element of $\mathcal{T}_1 \cup \mathcal{T}_2$.

In case $\mathcal{S}$ is a glued near hexagon of type $\mathcal{S}_1 \otimes \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are two generalized quadrangles, then every two points of $\mathcal{S}$ at distance 2 are contained in a unique quad, which either belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$ or is a grid. Glued near hexagons can be characterized as follows, see [7, Section 7].

Proposition 2.14 ([7]) *Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near hexagon and $\mathcal{T}_1$, $\mathcal{T}_2$ are two partitions of $\mathcal{P}$ in big quads such that every element of $\mathcal{T}_1$ intersects every element of $\mathcal{T}_2$ in a line and every line is contained in an element of $\mathcal{T}_1 \cup \mathcal{T}_2$. Then the quads in a given $\mathcal{T}_i$ are all isomorphic, and $\mathcal{S}$ is a glued near hexagon of type $\widetilde{Q_1} \otimes \widetilde{Q_2}$, where Q_i with $i \in \{1, 2\}$ is an arbitrary element of $\mathcal{T}_i$.*

In the following proposition, we prove some facts about glued near polygons.

Proposition 2.15 *Suppose $\mathcal{S}$ is a glued near polygon of type $\mathcal{S}_1 \otimes \mathcal{S}_2$ and suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two partitions of the point set $\mathcal{P}$ of $\mathcal{S}$ in mutually parallel classical convex subspaces satisfying the properties (I), (II) and (III) above. Then the following hold:*

(1) *The lines of the form $F_1 \cap F_2$ where $F_1 \in \mathcal{T}_1$ and $F_2 \in \mathcal{T}_2$ form a line spread S^* of $\mathcal{S}$.*

(2) *For every $F \in \mathcal{T}_1 \cup \mathcal{T}_2$, the set of lines of S^* contained in F is a line spread S_F of $\widetilde{F}$.*

(3) *If F_1 and F_2 are two elements belonging to the same $\mathcal{T}_i$ with $i \in \{1, 2\}$, then $\Pi_{F_1,F_2}(S_{F_1}) = S_{F_2}$. Specifically, if $L \in S_{F_1}$ and G is the unique element of $\mathcal{T}_{3-i}$ containing L, then $\Pi_{F_1,F_2}(L) = F_2 \cap G$.*

(4) *For each $i \in \{1, 2\}$, let F_i be an arbitrary element of $\mathcal{T}_i$, and put $S_i := S_{F_i}$. Let Γ_i be the graph whose vertices are the elements of S_i, with two elements of S_i being adjacent whenever they lie at distance 1 from each other regarded as lines of $\widetilde{F_i}$. Let Γ be the graph whose vertices are the elements of S^*, with two elements of S^* being adjacent whenever they lie at distance 1 from each other regarded as lines of $\mathcal{S}$. Then Γ is isomorphic to the cartesian product of the graphs Γ_1 and Γ_2.*

Proof. (1) This follows from the fact that every point of $\mathcal{S}$ is contained in a unique element of $\mathcal{T}_1$ and a unique element of $\mathcal{T}_2$, and hence also in a unique element of the form $F_1 \cap F_2$, where $F_1 \in \mathcal{T}_1$ and $F_2 \in \mathcal{T}_2$.

(2) Suppose $F \in \mathcal{T}_i$ for some $i \in \{1, 2\}$. Then the lines of S^* contained in F are the lines $F \cap F'$ where $F' \in \mathcal{T}_{3-i}$. Each point x of F is contained in one such line, obtained by putting F' equal to the unique element of $\mathcal{T}_{3-i}$ containing x.

(3) Suppose $F \in \mathcal{T}_i$ where $i \in \{1, 2\}$. For every point x of $\mathcal{S}$, let $\pi(x)$ denote the unique point of F nearest to x. We prove by induction on $\mathrm{d}(x, F)$ that $\pi(x) \in F \cap F'$ where F' is the unique element of $\mathcal{T}_{3-i}$ containing x. Claim (3) of the proposition is an immediate consequence of this fact.

The mentioned property clearly holds if $\mathrm{d}(x, F) = 0$, i.e. if $x \in F$. So, suppose $\mathrm{d}(x, F) > 0$ and let y be a point collinear with x for which $\mathrm{d}(y, F) = \mathrm{d}(x, F) - 1$. Then $\pi(y) = \pi(x)$. The unique element of $\mathcal{T}_i$ through x is parallel and at distance $\mathrm{d}(x, F)$ from F, implying that y should be contained in the unique element F' of $\mathcal{T}_{3-i}$ containing x.

By the induction hypothesis, we know that $\pi(y)$ is contained in $F' \cap F$. Hence, also $\pi(x)$ is contained in $F' \cap F$.

(4) Let (L_1, L_2) be an arbitrary vertex of $\Gamma_1 \times \Gamma_2$. The unique element of $\mathcal{T}_2$ containing L_1 intersects the unique element of $\mathcal{T}_1$ containing L_2 in a line which we will denote by $(L_1, L_2)^\theta$. This map θ defines a bijection between the vertex sets of $\Gamma_1 \times \Gamma_2$ and Γ.

Let (L_1, L_2) and (L_1', L_2) be two adjacent vertices of $\Gamma_1 \times \Gamma_2$. Let G (respectively G') denote the unique element of $\mathcal{T}_2$ containing L_1 (respectively L_1'). Let H denote the unique element of $\mathcal{T}_1$ containing L_2. Then $(L_1, L_2)^\theta = G \cap H$ and $(L_1', L_2)^\theta = G' \cap H$. By (3), $(L_1, L_2)^\theta$ and $(L_1', L_2)^\theta$ are the projections of $L_1 \subseteq F_1$ and $L_1' \subseteq F_1$ on H. Since L_1 and L_1' lie at distance 1 from each other, the same should be true for their projections. So, $(L_1, L_2)^\theta$ and $(L_1', L_2)^\theta$ are adjacent vertices of Γ.

Similarly, as above we can show that if (L_1, L_2) and (L_1, L_2') are two adjacent vertices of $\Gamma_1 \times \Gamma_2$, then also $(L_1, L_2)^\theta$ and $(L_1, L_2')^\theta$ are adjacent vertices of Γ.

We conclude that θ maps adjacent vertices to adjacent vertices. We now show that if L_1 and L_2 are adjacent vertices of Γ, then $L_1^{\theta^{-1}}$ and $L_2^{\theta^{-1}}$ are adjacent vertices of $\Gamma_1 \times \Gamma_2$.

We show that the lines L_1 and L_2 of S^* are contained in precisely one element of $\mathcal{T}_1 \cup \mathcal{T}_2$. Choose $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x_1, x_2) = 1$. Every element of $\mathcal{T}_1 \cup \mathcal{T}_2$ containing L_1 and L_2 also contains the line x_1x_2. As $x_1x_2 \notin S^*$, there exists a unique element of $\mathcal{T}_1 \cup \mathcal{T}_2$ containing x_1x_2, and this element of $\mathcal{T}_1 \cup \mathcal{T}_2$ also contains the lines L_1 and L_2.

Suppose L_1 and L_2 are contained in an element F_i' of $\mathcal{T}_i$, where $i \in \{1, 2\}$. Let G_1 be the unique element of $\mathcal{T}_{3-i}$ containing L_1 and G_2 be the unique element of $\mathcal{T}_{3-i}$ containing L_2. Then $L_1^{\theta^{-1}}$ and $L_2^{\theta^{-1}}$ have a coordinate in common, namely $F_i' \cap F_{3-i}$. The other coordinates are the lines $G_1 \cap F_i$ and $G_2 \cap F_i$ and by (3) these are the projections of $L_1 \subseteq F_i'$ and $L_2 \subseteq F_i'$ on F_i. Since $\mathrm{d}(L_1, L_2) = 1$, also their projections lie at distance 1 from each other, implying that $L_1^{\theta^{-1}}$ and $L_2^{\theta^{-1}}$ are adjacent in the graph $\Gamma_1 \times \Gamma_2$.

We conclude that θ defines an isomorphism between $\Gamma_1 \times \Gamma_2$ and Γ. ■

3 Examples of polygonal triples

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near $(2d+2)$-gon with $d \geq 1$ having a line spread S and a family $\mathcal{Q}$ of quads. Recall that $(\mathcal{S}, S, \mathcal{Q})$ is a *polygonal triple* if the conditions (P1) and (P2) of Section 1 are satisfied.

Lemma 3.1 *Suppose the triple $(\mathcal{S}, S, \mathcal{Q})$ satisfies condition (P1) and let $\mathcal{S}'$ be the point-line geometry as defined in (P2). Then the following are equivalent for two distinct lines L_1 and L_2 of S:*

(1) *L_1 and L_2 are collinear in $\mathcal{S}'$;*

(2) *L_1 and L_2 are parallel lines of $\mathcal{S}$ at distance 1 from each other in $\mathcal{S}$.*

(3) *L_1 and L_2 lie at distance 1 from each other in $\mathcal{S}$.*

Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

If L_1 and L_2 are collinear in $\mathcal{S}'$, then there is a quad of $\mathcal{Q}$ containing L_1 and L_2, implying that L_1 and L_2 are parallel lines at distance 1 from each other in the near polygon $\mathcal{S}$. Hence, (1) $\Rightarrow$ (2).

Obviously, (2) $\Rightarrow$ (3). It remains to show that (3) $\Rightarrow$ (1). Suppose therefore that L_1 and L_2 lie at distance 1 from each other, and let $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x_1, x_2) = 1$. By (P1), there exists a unique quad of $\mathcal{Q}$ containing L_1 and x_1x_2. Since this quad also contains x_2, it should by (P1) also contain the unique line L_2 of S through x_2. So, L_1 and L_2 are contained in a quad of $\mathcal{Q}$, implying that L_1 and L_2 are collinear regarded as points of $\mathcal{S}'$. ∎

Lemma 3.2 *Suppose the triple $(\mathcal{S}, S, \mathcal{Q})$ satisfies condition (P1) and let $\mathcal{S}'$ be the point-line geometry as defined in (P2). Suppose L_1, L_2 and L_3 are three mutually collinear points of $\mathcal{S}'$. Then there is a line of $\mathcal{S}'$ containing them.*

Proof. We may suppose that L_1, L_2 and L_3 are mutually distinct. Then there exists a unique quad $Q_1 \in \mathcal{Q}$ containing L_2 and L_3, a unique quad $Q_2 \in \mathcal{Q}$ containing L_1 and L_3, and a unique quad $Q_3 \in \mathcal{Q}$ containing L_1 and L_2. Suppose L_3 is not contained in Q_3. Then by Lemma 3.1 any point of $L_3 \setminus Q_3$ is collinear with two distinct points of Q_3, namely one of L_1 and one on L_2. This is clearly not possible as Q_3 is a convex subspace. ∎

The following is an immediate consequence of Lemma 3.2.

Corollary 3.3 *Suppose the triple $(\mathcal{S}, S, \mathcal{Q})$ satisfies condition (P1) and let $\mathcal{S}'$ be the point-line geometry as defined in (P2). Then the lines of $\mathcal{S}'$ bijectively correspond to the maximal cliques of its collinearity graph.*

We now list several examples of polygonal triples.

Example 1. Suppose $\mathcal{S}$ is a generalized quadrangle having a line spread S and put $\mathcal{Q} = \{\mathcal{S}\}$. Then $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is a line.

Example 2. Suppose $\mathcal{S}$ is a glued near hexagon. Then there exist two partitions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of its point set into big quads such that:

- every quad of $\mathcal{Q}_1$ intersects every quad of $\mathcal{Q}_2$ in a line;
- every line is contained in an element of $\mathcal{Q}_1 \cup \mathcal{Q}_2$.

If we put $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2$ and $S := \{Q_1 \cap Q_2 \,|\, Q_1 \in \mathcal{Q}_1 \text{ and } Q_2 \in \mathcal{Q}_2\}$, then $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple for which the associated near polygon is a grid.

Example 3. Suppose S is a trivial line spread of a near polygon $\mathcal{S}$ with associated near polygon $\mathcal{S}'$. If $\mathcal{Q}$ denotes the set of quads of $\mathcal{S}$ containing a line of S, then $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is isomorphic to $\mathcal{S}'$.

Example 4. Suppose Π is a polar space of rank $d+1 \geq 3$ and X is a *hyperplane* of Π, i.e. a proper set[2] of points of Π such that every line of Π has either 1 or all its points in X. Suppose also that the singular subspaces of Π contained in X define a (nondegenerate) polar space Π' of rank d. Let Δ and Δ' denote the dual polar spaces associated with respectively Π and Π'. Let S denote the set of lines of Δ that correspond to the maximal singular subspaces of Π' and denote by $\mathcal{Q}$ the set of quads of Δ that correspond to the next-to-maximal singular subspaces of Π'.

Theorem 3.4 *$(\Delta, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is isomorphic to Δ'.*

Proof. Let α be a point of Δ, i.e. a singular subspace of dimension d of Π. Since the rank of Π' is equal to d, α cannot be contained in X and so $\alpha \cap X$ is a hyperplane of α, i.e. $\alpha \cap X$ is a maximal singular subspace of Π'. The line of Δ corresponding to $\alpha \cap X$ is then the unique element of S containing α. This shows that S is a line spread of Δ.

Let α be a point of Δ, let L_α denote the unique element of S containing α and let M denote another line of Δ through α. Denote by $\beta \subseteq \alpha$ the $(d-1)$-dimensional subspace of Π corresponding to M. Recall that $\alpha \cap X$ is the $(d-1)$-dimensional subspace corresponding to L_α. As $\alpha \cap X$ and β are two distinct hyperplanes of α, they intersect in a next-to-maximal singular subspace of Π', showing that L_α and M are contained in a unique quad of $\mathcal{Q}$. So, we see that Property (P1) is satisfied.

By the definition of the dual polar space Δ', it must be isomorphic to the point-line geometry with point set S and line set $\mathcal{Q}$, where incidence is containment. So, we see that also Property (P2) is satisfied with associated near polygon isomorphic to Δ'. ■

In the case Π is fully embeddable in a finite Desarguesian projective space, the following examples arise (with $n = d+1 \geq 3$ and q a prime power).

Π	$Q(2n,q)$	$Q^+(2n-1,q)$	$H(2n-1,q^2)$
Π'	$Q^-(2n-1,q)$	$Q(2n-2,q)$	$H(2n-2,q^2)$

Example 5. Suppose $\mathcal{S}$ is the $G_2(4)$ near octagon as described in [1]. Let $\mathcal{Q}$ denote the set of all quads of $\mathcal{S}$ and S the set of lines that are contained in at least two elements of $\mathcal{Q}$. The results in [1] then show that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is isomorphic to the dual split Cayley hexagon $H(4)^D$.

Example 6. Suppose $\mathcal{S}$ is the $L_3(4)$ near octagon described in [1, 2], which occurs as a full subgeometry of the $G_2(4)$ near octagon. Again, let $\mathcal{Q}$ denote the set of all quads of $\mathcal{S}$ and denote by S the set of all lines that are contained in at least two elements of $\mathcal{Q}$. Again from results of [1, 2], it follows that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is the unique generalized hexagon of order $(4,1)$.

[2] In case Π is fully embeddable in a Desarguesian projective space Σ, then we know from Cohen and Shult [6, Theorem 5.2] that X arises by intersecting Π with a hyperplane of Σ.

Example 7. Here we show how polygonal triples can be constructed from other polygonal triples. Suppose $(\mathcal{S}_1, S_1, \mathcal{Q}_1)$ and $(\mathcal{S}_2, S_2, \mathcal{Q}_2)$ are two polygonal triples with associated near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$. We denote the diameter of $\mathcal{S}_i$, $i \in \{1,2\}$, by $d_i + 1$. Suppose $\mathcal{S}$ is a glued near polygon of type $\mathcal{S}_1 \otimes \mathcal{S}_2$, necessarily having diameter $d_1 + d_2 + 1$. Then there exist two partitions $\mathcal{T}_1$ and $\mathcal{T}_2$ of the point set of $\mathcal{S}$ into mutually parallel classical convex subspaces such that:

- For every $i \in \{1,2\}$ and every $F \in \mathcal{T}_i$, we have $\widetilde{F} \cong \mathcal{S}_i$.
- Every element of $\mathcal{T}_1$ intersects every element of $\mathcal{T}_2$ in a line. Moreover, the set S of all lines that are obtained in this way is a line spread of $\mathcal{S}$.
- Every line of $\mathcal{S}$ is contained in an element of $\mathcal{T}_1 \cup \mathcal{T}_2$.

Let F_i, $i \in \{1,2\}$, be a specific element of $\mathcal{T}_i$ and let S_i' be the line spread of $\mathcal{S}_i' := \widetilde{F_i}$ consisting of all lines of S that have all their points in F_i. Suppose moreover that the following holds:

(∗) For every $i \in \{1,2\}$, there exists an isomorphism from $\mathcal{S}_i'$ to $\mathcal{S}_i$ mapping S_i' to S_i.

By Proposition 2.15(3), we know that if the latter property holds for F_1 and F_2, then it holds for any convex subspace in $\mathcal{T}_1 \cup \mathcal{T}_2$. Now, denote by $\mathcal{Q}$ the set of all quads of $\mathcal{S}$ that contain a line of S.

Theorem 3.5 $(\mathcal{S}, S, \mathcal{Q})$ *is a polygonal triple for which the associated near polygon is isomorphic to the cartesian product of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*.*

Proof. Let x be an arbitrary point of $\mathcal{S}$. Let G_i with $i \in \{1,2\}$ denote the unique element of $\mathcal{T}_i$ containing x. Then the unique line $L_x \in S$ through x is contained in G_1 and G_2. Let S_i'' with $i \in \{1,2\}$ denote the line spread of $\widetilde{G_i}$ formed by those lines of S that have all their points in G_i. The elements of $\mathcal{Q}$ contained in G_i are precisely the quads of $\widetilde{G_i}$ containing some line of S_i'', and the elements of $\mathcal{Q}_i$ are precisely the quads of $\mathcal{S}_i$ containing some line of S_i. By (∗) and Proposition 2.15(3), there exists an isomorphism θ_i from $\widetilde{G_i}$ to $\mathcal{S}_i$ mapping S_i'' to S_i. As every line through x distinct from L_x is contained in either G_1 or G_2, we can now easily see that Property (P1) is satisfied. Let $\mathcal{A}$ be the point-line geometry associated with $(\mathcal{S}, S, \mathcal{Q})$ as defined in (P2).

Let Γ_i' with $i \in \{1,2\}$ be the graph whose vertices are the elements of S_i' with two lines of S_i' being adjacent whenever they lie at distance 1 from each other. Let Γ be the graph whose vertices are the elements of S, with two lines being adjacent whenever they lie at distance 1 from each other. By Lemma 3.1, we then know that Γ is the collinearity graph of the point-line geometry $\mathcal{A}$. By Proposition 2.15(4), we moreover know that Γ is isomorphic to the cartesian product of the graphs Γ_1' and Γ_2'.

Now, let Γ_i with $i \in \{1,2\}$ be the graph whose vertices are the elements of S_i with two vertices being adjacent whenever the lines lie at distance 1 from each other in the near polygon $\mathcal{S}_i$. Then Γ_i is the collinearity graph of the near polygon $\mathcal{A}_i$ by Lemma 3.1. By (∗), the graphs Γ_i and Γ_i' are isomorphic. So, the collinearity graph of $\mathcal{A}$ is isomorphic to the cartesian product of the collinearity graphs of the near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e. to the collinearity graph of the product near polygon $\mathcal{A}_1 \times \mathcal{A}_2$. By Corollary 3.3, we then know that $\mathcal{A}$ is isomorphic to the cartesian product $\mathcal{A}_1 \times \mathcal{A}_2$. ∎

4 Properties of polygonal triples

Throughout this section (with exception of Proposition 4.8), we suppose that S is a line spread of a near $(2d+2)$-gon $\mathcal{S}$ and $\mathcal{Q}$ is a set of quads of $\mathcal{S}$ such that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple with associated near $2d$-gon $\mathcal{S}'$. For every point x of $\mathcal{S}$, let L_x denote the unique line of S containing x.

Proposition 4.1 *Every line L of $\mathcal{S}$ not belonging to S is contained in a unique quad of $\mathcal{Q}$.*

Proof. Let x be an arbitrary point of L. Then $L_x \neq L$. Every quad containing L also contains the point x. Since the quads of $\mathcal{Q}$ through x all contain the line L_x and partition the set of lines through x distinct from L_x, we see that there exists a unique member of $\mathcal{Q}$ containing L. ∎

Proposition 4.2 *For every quad $Q \in \mathcal{Q}$, the lines of S contained in Q form a line spread of $\widetilde{Q}$.*

Proof. This follows from the fact that for every point x and every quad $Q \in \mathcal{Q}$ through x, the quad Q contains the line L_x. ∎

Proposition 4.3 *Every two lines L_1 and L_2 of S are parallel, i.e. S is an admissible line spread. The distance between L_1 and L_2 in $\mathcal{S}'$ is equal to the distance between L_1 and L_2 in $\mathcal{S}$.*

Proof. Put $\delta := \mathrm{d}_{\mathcal{S}}(L_1, L_2)$. If $\delta = 0$, then $L_1 = L_2$ and so the proposition obviously holds then. Suppose therefore that $\delta \geq 1$. In order to show that L_1 and L_2 are parallel, it suffices to show that $\mathrm{d}(x, L_2) \leq \delta$ for every point x of L_1. Choose $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x_1, x_2) = \delta$, and let $y_0, y_1, \ldots, y_\delta$ be a path of length δ in $\mathcal{S}$ with $y_0 = x_1$ and $y_\delta = x_2$. If y_{i-1} and y_i are two consecutive points of this path such that $y_{i-1}y_i \notin S$, then $L_{y_{i-1}}$ and L_{y_i} are two disjoint lines of the unique quad of $\mathcal{Q}$ containing $y_{i-1}y_i$, showing that every point of $L_{y_{i-1}}$ is collinear with a unique point of L_{y_i}. If y_{i-1} and y_i are two consecutive points of this path such that $y_{i-1}y_i \in S$, then $L_{y_{i-1}} = L_{y_i}$. We thus see that every point $x \in L_1$ is connected to some point of L_2 by means of a path of length at most δ. As said before, this shows that the lines L_1 and L_2 are parallel. The second claim follows from Proposition 2.10 and Lemma 3.1. ∎

Proposition 4.4 *Every quad of $\mathcal{Q}$ is classical in $\mathcal{S}$.*

Proof. Let x be a point of $\mathcal{S}$ and Q a quad of $\mathcal{Q}$. Let $\mathcal{L}_Q$ denote the set of lines of S contained in Q. By Proposition 4.2, $\mathcal{L}_Q$ is a line spread of $\widetilde{Q}$. If $L \in \mathcal{L}_Q$, then

$\mathrm{d}(x, L) = \mathrm{d}(L_x, L)$ as L_x and L are parallel. By Property (P2), there must exist a unique line $L^* \in \mathcal{L}_Q$ nearest to L_x in $\mathcal{S}'$ and hence also in $\mathcal{S}$ (by Proposition 4.3). If y is the unique point of L^* nearest to x, then y must be the unique point of Q nearest to x. Moreover, $\pi_L(x) = \pi_L(y)$ for every $L \in \mathcal{L}_Q$. In fact, if $z \in Q$, then $\mathrm{d}(x, z) = \mathrm{d}(L_x, L_z) + \mathrm{d}(\pi_{L_z}(x), z) = \mathrm{d}(L_x, L_y) + \mathrm{d}(L_y, L_z) + \mathrm{d}(\pi_{L_z}(y), z) = \mathrm{d}(x, y) + \mathrm{d}(y, \pi_{L_z}(y)) + \mathrm{d}(\pi_{L_z}(y), z) = \mathrm{d}(x, y) + \mathrm{d}(y, z)$, showing that x is classical with respect to Q. ■

Proposition 4.5 *One of the following cases occurs for a convex subspace F of $\mathcal{S}$:*

(1) *No line of S is contained in F.*

(2) *The lines of S contained in F form a line spread of $\widetilde{F}$.*

Proof. Suppose L is a line of S contained in F. Let x be an arbitrary point of F. We show by induction on $\mathrm{d}(x, L)$ that the line L_x is contained in F. Obviously this is the case when $\mathrm{d}(x, L) = 0$ since $L_x = L$ in that case. Suppose therefore that $\mathrm{d}(x, L) > 0$. Let y be a point collinear with x at distance $\mathrm{d}(x, L) - 1$ from L. Then $y \in F$ since F is convex. By Proposition 4.3, every point of L_y has distance $\mathrm{d}(x, L) - 1$ from L. So, $x \notin L_y$. By the induction hypothesis, the line L_y is contained in F. So, the smallest convex subspace containing xy and L_y is also contained in F. By Property (P1), this smallest convex subspace is a quad of $\mathcal{Q}$ that contains the lines L_x and L_y. Hence, $L_x \subseteq F$. ■

Proposition 4.6 *Suppose $\mathcal{S}$ is a near hexagon every line of which is incident with at least three points and that $\mathcal{S}'$ is a (nondegenerate) generalized quadrangle. Then every two points at distance 2 from each other are contained in a unique quad, i.e. $\mathcal{S}$ is a dense near hexagon.*

Proof. Let x and y be two points of $\mathcal{S}$ at distance 2 not contained in a quad of $\mathcal{Q}$. In order to show that x and y are contained in a unique quad, it suffices by Proposition 2.1 to show that x and y have at least two common neighbours. Let Q_1 and Q_2 be two distinct quads of $\mathcal{Q}$ through L_x. Since Q_1 and Q_2 are classical, they are also big and so y is collinear with a unique point $y_1 \in Q_1$ and a unique point $y_2 \in Q_2$. Then y_1 and y_2 are distinct common neighbours of x and y. Indeed, if y_1 and y_2 were equal, then $y_1 = y_2 \in Q_1 \cap Q_2 = L_x$, but then y is collinear with a point of L_x and thus contained in a quad of $\mathcal{Q}$ together with L_x, a contradiction. ■

Proposition 4.7 *Let $\mathcal{S}_1'$ be a full isometrically embedded subgeometry of $\mathcal{S}'$ that is a near 2δ-gon with $\delta \geq 1$. Then the point set S_1 of $\mathcal{S}_1'$ is a subset of S and the line set $\mathcal{Q}_1$ of $\mathcal{S}_1'$ is a subset of $\mathcal{Q}$. Consider the following sets:*

- *$\mathcal{P}_1$ is the set of points of $\mathcal{S}$ that are contained in some line of S_1;*
- *$\mathcal{L}_1$ is the set of lines of $\mathcal{S}$ that are contained in some quad of $\mathcal{Q}_1$.*

Then:

(1) *$\mathcal{P}_1$ is a subspace of $\mathcal{S}$;*

(2) *The point-line geometry* $\mathcal{S}_1 := \widetilde{\mathcal{P}_1}$ *has* $\mathcal{L}_1$ *as line set and is isometrically embedded into* $\mathcal{S}$*.*

(3) $\mathcal{S}_1$ *is a near* $(2\delta+2)$*-gon.*

(4) $(\mathcal{S}_1, S_1, \mathcal{Q}_1)$ *is a polygonal triple with associated near polygon* $\mathcal{S}_1'$*.*

Proof. We show that $\mathcal{P}_1$ is a subspace of $\mathcal{S}$. To that end consider two distinct collinear points x and y of $\mathcal{S}$ that are contained in $\mathcal{P}_1$. We distinguish two cases.

If $L_x = L_y \in S_1$, then obviously every point of the line $xy = L_x = L_y$ belongs to $\mathcal{P}_1$.

Suppose next that $L_x \neq L_y$. As $\mathrm{d}(x,y) = 1$, the unique element Q of $\mathcal{Q}$ containing the line xy also contains L_x and L_y. The lines L_x and L_y are two collinear points of $\mathcal{S}'$ belonging to S_1, and since $\mathcal{S}_1'$ is a full isometrically embedded subgeometry of $\mathcal{S}'$, we know that L_x and L_y should be two collinear points in $\mathcal{S}_1'$, implying that $Q \in \mathcal{Q}_1$. So, all lines of S contained in Q belong to S_1. As these lines cover all points of xy, we see that the line xy is completely contained in $\mathcal{P}_1$.

So, we see that $\mathcal{P}_1$ is a subspace of $\mathcal{S}$. We denote the line set of $\mathcal{S}_1 = \widetilde{\mathcal{P}_1}$ by $\mathcal{L}_1'$. We show that $\mathcal{L}_1 = \mathcal{L}_1'$.

We first prove that $\mathcal{L}_1 \subseteq \mathcal{L}_1'$. Let L be an arbitrary line of $\mathcal{L}_1$. Then there exists a quad $Q \in \mathcal{Q}_1$ containing L. The lines of S contained in Q define a line spread of $\widetilde{Q}$ and all these lines belong to S_1 as $\mathcal{S}_1'$ is a full subgeometry of $\mathcal{S}'$. These lines cover all points of L, implying that all points of L belong to $\mathcal{P}_1$ and that L itself belongs to $\mathcal{L}_1'$.

Next, we prove that $\mathcal{L}_1' \subseteq \mathcal{L}_1$. Suppose that $L \in \mathcal{L}_1'$, or equivalently, that all points of L belong to $\mathcal{P}_1$. If $L \in S$, then $L \in S_1$ and the fact that a near δ-gon is connected then implies that L is contained in some element of $\mathcal{Q}_1$, i.e. $L \in \mathcal{L}_1$. If $L \notin S$, then take two distinct points x and y on L. The lines L_x and L_y belong to S_1 and are collinear in $\mathcal{S}'$, implying that they are also collinear in $\mathcal{S}_1'$. The unique quad Q containing xy then also contains L_x, L_y and must belong to $\mathcal{Q}_1$, implying that $L = xy \in \mathcal{L}_1$.

We show that $\mathcal{S}_1$ is isometrically embedded into $\mathcal{S}$. Suppose x and y are two points of $\mathcal{S}_1$ at distance δ from each other in the near polygon $\mathcal{S}$. There are then two possibilities.

Suppose the lines L_x and L_y lie at distance $\delta - 1$ from each other in $\mathcal{S}$ and hence also in $\mathcal{S}'$ by Proposition 4.3. Then the unique point on L_y nearest to x is distinct from y. In $\mathcal{S}'$ there exists a path of length $\delta - 1$ connecting L_x and L_y, implying that there also exists such a path in $\mathcal{S}_1'$ (as $\mathcal{S}_1'$ is isometrically embedded in $\mathcal{S}'$). This implies that there exists a path in $\mathcal{S}_1$ of length δ that connects x with y.

Suppose the lines L_x and L_y lie at distance δ from each other. Then the unique point of L_y nearest to x coincides with y. In $\mathcal{S}'$ there exists a path of length δ connecting L_x and L_y, implying (again) that there also exists such a path in $\mathcal{S}_1'$. This implies that there exists a path in $\mathcal{S}_1$ of length δ connecting x with a point of L_y. The latter point necessarily coincides with y.

Since $\mathcal{S}_1$ is isometrically embedded into $\mathcal{S}$, it must be a near polygon. Since δ is the maximal distance between two lines of S_1, $\mathcal{S}_1$ must be a near $(2\delta+2)$-gon.

We prove that $(\mathcal{S}_1, S_1, \mathcal{Q}_1)$ is a polygonal triple. The points of $\mathcal{S}_1$ are precisely the points that are contained in some line of S_1, showing that S_1 is indeed a line spread of $\mathcal{S}_1$.

Let x be an arbitrary point of $\mathcal{S}_1$. Then $L_x \in S_1$ has all its points in $\mathcal{P}_1$. If M is a line of $\mathcal{S}_1$ through x distinct from L_x, then there exists a (necessarily unique) quad $Q \in \mathcal{Q}_1$ containing L_x and M, showing that Property (P1) of Section 1 is satisfied. The point-line geometry with point set S_1, line set $\mathcal{Q}_1$ and natural incidence is precisely the near polygon $\mathcal{S}_1'$, completing the proof that $(\mathcal{S}_1, S_1, \mathcal{Q}_1)$ is a polygonal triple with associated near polygon $\mathcal{S}_1'$. ■

Remark. If $(\mathcal{S}, S, \mathcal{Q})$ is the polygonal triple associated with the $G_2(4)$ near octagon as described in Example 5 of Section 3, and $\mathcal{S}_1'$ is a sub-generalized-hexagon of order $(4,1)$ of $\mathcal{S}' \cong H(4)^D$, then from [1, 2] we know that $\mathcal{S}_1$ is the $L_3(4)$ near octagon.

The following proposition provides an alternative definition of the notion "polygonal triple".

Proposition 4.8 *Suppose S is a line spread of a near $(2d+2)$-gon $\mathcal{S}$ with $d \geq 1$ and let $\mathcal{Q}$ denote the set of all quads of $\mathcal{S}$ containing a line of S. Then $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple if and only if the following conditions[3] are satisfied:*

(1) *S is an admissible line spread.*

(2) *Every two lines of S at distance 1 from each other are contained in a (necessarily unique) quad.*

(3) *Every quad containing two lines of S is classical.*

Proof. For every point x of $\mathcal{S}$, we denote by L_x the unique line of S containing x.

Suppose $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple. Then S is an admissible line spread by Proposition 4.3. Assume L_1 and L_2 are two lines of S at distance 1 from each other. Let $x_1 \in L_1$ and $x_2 \in L_2$ such that $\mathrm{d}(x_1, x_2) = 1$. By Property (P1), there exists a quad of $\mathcal{Q}$ containing x_1x_2 and $L_1 = L_{x_1}$. By Proposition 4.2, this quad also contains $L_2 = L_{x_2}$ and hence is the unique quad of $\mathcal{S}$ containing L_1 and L_2. By Proposition 4.2, the quads containing two lines of S are precisely the quads of $\mathcal{Q}$ and by Proposition 4.4 we know that each of these quads is classical.

Conversely, suppose that properties (1), (2) and (3) of the proposition are satisfied. We show that for every $Q \in \mathcal{Q}$, the lines of S contained in Q define a line spread of $\widetilde{Q}$. As $Q \in \mathcal{Q}$, there is a line L of S that is contained in Q. Let x denote an arbitrary point of Q not contained in L. Then Q is the unique quad containing x and L. So, the unique quad containing L_x and L must coincide with Q, implying that $L_x \subseteq Q$. As this holds for every point $x \in Q$ not contained in L, we thus indeed see that the lines of S contained in Q define a line spread of $\widetilde{Q}$. This implies that the quads containing two lines of S are precisely the quads of $\mathcal{Q}$. It follows that every quad of $\mathcal{Q}$ is classical.

We show that $(\mathcal{S}, S, \mathcal{Q})$ satisfies Property (P1). Let x be a point and K a line through x distinct from L_x. The lines K and L_x are then contained in a (necessarily unique) quad,

[3] If every line of $\mathcal{S}$ is incident with at least three points, then condition (2) is superfluous as it is implied by (1), see Proposition 2.1.

namely the unique quad through the lines L_x and L_y, where y is an arbitrary point of K distinct from x.

Let $\mathcal{S}'$ be the point-line geometry with point set S, line set $\mathcal{Q}$ and natural incidence relation (i.e. containment). Consider a line $L \in S$ and a quad $Q \in \mathcal{Q}$. Let x be any point of L. As Q is classical in $\mathcal{S}$, there exists a unique point x' in Q nearest to x. As S is admissible, we know from Proposition 2.10 that the distance between the lines L_x and $L_{x'}$ in the geometry $\mathcal{S}'$ is equal to $\mathrm{d}(x, x')$. If K is a line of S contained in Q but distinct from $L_{x'}$, then the unique point of K nearest to x has distance $\mathrm{d}(x, x') + 1$ from x, implying that L_x and K have distance $\mathrm{d}(x, x') + 1$ from each other in $\mathcal{S}'$. This implies that the point-line geometry $\mathcal{S}'$ is a near polygon. By Proposition 2.10, $\mathcal{S}'$ is a near $2d$-gon. ∎

5 Classification results

Theorem 5.1 *Suppose Π is a polar space of rank $d+1 \geq 3$. Suppose S is a line spread of the dual polar space Δ associated with Π and $\mathcal{Q}$ is a set of quads of Δ such that $(\Delta, S, \mathcal{Q})$ is a polygonal triple. Then S and $\mathcal{Q}$ are obtained as in Example 4 of Section 3.*

Proof. Let $\mathcal{M}$ denote the set of maxes of Δ that contain a line of S. If $M \in \mathcal{M}$, then we know by Proposition 4.5 that the lines of S contained in M form a line spread of $\widetilde{M}$. Let X denote the set of points of Π corresponding to the elements of $\mathcal{M}$.

We show that X does not coincide with the whole point set of Π. Let x denote a point of Δ and let M denote a max through x not containing the line L_x (by Proposition 2.6 we know that such an M exists). Then M does not belong to $\mathcal{M}$ and so the point of Π corresponding to M does not belong to X.

We show that X is a hyperplane of Π. In view of the previous paragraph, it suffices to show that a line L of Π has either one or all its points in X. Let F denote the convex subspace of diameter $d-1$ corresponding to L. We distinguish two cases:

(a) Suppose F contains a line of S. Then every max through F belongs to $\mathcal{M}$, showing that all points of L belong to X.

(b) Suppose no line of S is contained in F. Let x be an arbitrary point of F. By Proposition 2.6, there exists a unique max M through F and L_x. This M is the unique max of $\mathcal{M}$ containing F. The point of Π corresponding to M is then the unique point of X on L.

We show that no point x of X is collinear on Π with all remaining points of X. Let M denote the max of $\mathcal{M}$ corresponding to x. Let L denote a line of S not contained in M. By Propositions 2.6 and 2.8, there exists a max M' through L disjoint from M. Let $x' \in X$ be the point of Π corresponding to $M' \in \mathcal{M}$. Then x and x' are not collinear on Π (otherwise the convex subspace corresponding to the line xx' would be contained in $M \cap M' = \emptyset$).

We show that the nondegenerate polar space Π' defined by the set X has rank d. Let L be a line of S and A the singular subspace (of projective dimension $d-1$) of Π corresponding to L. As any max through L belongs to $\mathcal{M}$, any point of A should belong to X. This shows that the rank of Π' is at least d. Suppose the rank is $d+1$. Then there exists a singular subspace of projective dimension d contained in X. This implies that there exists a point x in Δ having the property that every max through it belongs to $\mathcal{M}$. However, that is not possible. By Proposition 2.6, there exists a max through x not containing the line L_x, and any such max does not belong to $\mathcal{M}$.

By definition, every max containing a line of S belongs to $\mathcal{M}$. If L is a line not belonging to S and $x \in L$, then Proposition 2.6 implies that there exists a max through L not containing L_x, and such a max cannot belong to $\mathcal{M}$. So, the lines of $\mathcal{S}$ with only maxes of $\mathcal{M}$ through them are precisely the lines of S. This implies that the maximal singular subspaces of Π' are precisely the $(d-1)$-dimensional subspaces of Π corresponding to the lines of S. So, S is obtained as described in Example 4 of Section 3. As $\mathcal{Q}$ is uniquely determined by $\mathcal{S}$ and S, the set $\mathcal{Q}$ is also obtained as described in Example 4 of Section 3. ∎

Theorem 5.2 *Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a glued near hexagon that is not a product near hexagon. Let S be a line spread of $\mathcal{S}$ and $\mathcal{Q}$ a set of quads of $\mathcal{S}$ such that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple. Then S and $\mathcal{Q}$ are obtained as in Example 2 of Section 3.*

Proof. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two partitions of $\mathcal{P}$ into big quads such that the following hold:

- Every element of $\mathcal{Q}_1$ intersects every element of $\mathcal{Q}_2$ in a line. Moreover, the collection S^* of all these lines is a line spread of $\mathcal{S}$.
- Every line is contained in a quad of $\mathcal{Q}_1 \cup \mathcal{Q}_2$

Recall from Section 2.5 that every two points of $\mathcal{S}$ at distance 2 from each other are contained in a unique quad, and that each such quad either belongs to $\mathcal{Q}_1 \cup \mathcal{Q}_2$ or is a grid. As $\mathcal{S}$ is not a product near hexagon, the quads of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ should have nontrivial line spreads, implying that none of them is a grid. By Corollary 2.13 applied to all members of $\mathcal{Q}_1 \cup \mathcal{Q}_2$, we then see that all lines of $\mathcal{S}$ are incidence with the same number of points, say $s+1$. If $s+1 \geq 3$, then every quad $Q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$ should have an order (s, t_Q) (see e.g. Proposition 2.4) and the fact that Q is not a grid-quad implies that $t_Q \geq 2$. If $s+1=2$, then the fact that $Q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$ has a line spread implies that the dual grid Q has some order $(1, t_Q)$ and again we have that $t_Q \geq 2$ since Q is not a grid-quad.

We now show that no grid-quad G is classical (or equivalently, big). Let $x \in G$ and let Q_i with $i \in \{1,2\}$ be the unique quad of $\mathcal{Q}_i$ containing x. Then $L_x := Q_1 \cap Q_2 \in S^*$ is not contained in G. For every $i \in \{1,2\}$, put $L_i := G \cap Q_i$ and let L_i' be a line of Q_i through x distinct from L_x and L_i. Such a line exists as $t_{Q_i} \geq 2$. Let G' denote the unique grid-quad containing L_1' and L_2'. As $G \cap G'$ is convex and $\Gamma_1(x) \cap G \cap G' = \{x\}$, we have $G \cap G' = \{x\}$. Let $y \in G' \cap \Gamma_2(x)$. If G were big, then the unique point $z \in G$ collinear with y would lie on a shortest path from $y \in G'$ to $x \in G \cap G'$, implying that $z \in G \cap G' \setminus \{x\}$, a contradiction. So, G cannot be big.

So, the only big quads are those of $\mathcal{Q}_1 \cup \mathcal{Q}_2$. By Proposition 4.4 and Property (P1), every quad through a line of S is big. Hence, $S = \{Q_1 \cap Q_2 \,|\, Q_1 \in \mathcal{Q}_1 \text{ and } Q_2 \in \mathcal{Q}_2\} = S^*$, i.e. S and $\mathcal{Q}$ are obtained as in Example 2 of Section 3. ∎

Theorem 5.3 *Suppose $\mathcal{S}$ is a product near hexagon, S is a line spread of $\mathcal{S}$ and $\mathcal{Q}$ is a set of quads such that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple. Then at least one of the following cases occurs:*

(1) *S and $\mathcal{Q}$ arise in the way as described in Example 3 of Section 3.*

(2) *$\mathcal{S}$ is a glued near hexagon and S and $\mathcal{Q}$ arise as described in Example 2 of Section 3.*

Proof. If S is a trivial line spread, then $\mathcal{Q}$ is obtained as described in Example 3 of Section 3. Suppose therefore that S is nontrivial. Suppose $\mathcal{S}$ is the product near hexagon $\mathcal{G} \times \mathbb{L}$, where $\mathcal{G}$ is some generalized quadrangle and $\mathbb{L}$ is some line. Then there exists a partition $\mathcal{T}_1$ of the point set $\mathcal{P}$ of $\mathcal{S}$ in big quads isomorphic to $\mathcal{G}$. Since S is not a trivial line spread, one of the elements of $\mathcal{T}_1$, say Q_1, contains a line of S. By Proposition 4.2, the lines of S contained in Q_1 form a line spread S_1 of $\widetilde{Q_1}$. Every line of S_1 is contained in a unique quad distinct from Q_1. We denote by $\mathcal{T}_2$ the set of all quads that arise in this way. The following then holds:

(1) $\mathcal{T}_1$ is a partition of $\mathcal{P}$ in big quads isomorphic to $\mathcal{G}$.

(2) $\mathcal{T}_2$ is a partition of $\mathcal{P}$ in isomorphic big grids.

(3) Every quad of $\mathcal{T}_1$ intersects every quad of $\mathcal{T}_2$ in a line. The collection S' of all lines obtained in this way is a line spread of $\mathcal{S}$.

(4) Every line is contained in an element of $\mathcal{T}_1 \cup \mathcal{T}_2$.

By Proposition 2.14, we know that $\mathcal{S}$ is a glued near hexagon.

We show that $S' = S$, which allows us then to conclude that S and $\mathcal{Q}$ are obtained as in Example 2 of Section 3. In order to show that $S' = S$, it suffices to prove that $S' \subseteq S$. To that end, consider an arbitrary line L of S', and let $G \in \mathcal{T}_2$ and $Q \in \mathcal{T}_1$ such that $L = G \cap Q$. The quad G intersects Q_1 in a line L' belonging to S, implying that the lines of S contained in G form a line spread of $\widetilde{G}$. Since either $L = L'$ or $L \cap L' = \emptyset$, we should also have $L \in S$. So, $S' \subseteq S$ and hence $S' = S$ as we needed to prove. ∎

Theorem 5.4 *Suppose $\mathcal{S}$ is a near hexagon, S is a line spread of $\mathcal{S}$ and $\mathcal{Q}$ is a set of quads of $\mathcal{S}$. If $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon $\mathcal{S}'$ is a grid, then $\mathcal{S}$ is a glued near hexagon.*

Proof. The grid $\mathcal{S}'$ has two partitions in lines. Let $\mathcal{Q}_1 \subseteq \mathcal{Q}$ and $\mathcal{Q}_2 \subseteq \mathcal{Q}$ denote the set of classical (i.e. big) quads corresponding to these partitions. Then:

(1) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two partitions of the point set $\mathcal{P}$ of $\mathcal{S}$ into quads. (As each point of $\mathcal{S}$ is contained in a unique line of S which itself is contained in a unique quad of $\mathcal{Q}_1$ and a unique quad of $\mathcal{Q}_2$).

(2) Every quad of $\mathcal{Q}_1$ intersects every quad of $\mathcal{Q}_2$ in a line.

(3) Every line of $\mathcal{S}$ is contained in an element of $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ (see Proposition 4.1).

By Proposition 2.14, this is sufficient to conclude that $\mathcal{S}$ is a glued near hexagon of type $\widetilde{Q_1} \otimes \widetilde{Q_2}$, where Q_1 is an arbitrary element of $\mathcal{Q}_1$ and Q_2 is an arbitrary element of $\mathcal{Q}_2$. The sets of lines that arise by intersecting the elements of $\mathcal{Q}_1$ with those of $\mathcal{Q}_2$ is a line spread that coincides with S. So, S and $\mathcal{Q}$ are obtained as described in Example 2 of Section 3. ∎

Theorem 5.5 *Suppose $\mathcal{S}$ is a finite near hexagon with only lines of size at least 3. Suppose S is a line spread of $\mathcal{S}$ and $\mathcal{Q}$ is a set of quads of $\mathcal{S}$ such that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is a (nondegenerate) generalized quadrangle. Then $\mathcal{S}$ is either a glued near hexagon or a dual polar space of rank 3.*

Proof. By Proposition 4.6, we know that $\mathcal{S}$ is a dense near hexagon. If not all lines of $\mathcal{S}$ are incident with the same number of points, then by Proposition 2.12, $\mathcal{S}$ is the cartesian product of a generalized quadrangle and a line and thus a dual polar space of rank 3 by Proposition 2.11. We may thus suppose that all lines of $\mathcal{S}$ are incident with precisely $s+1$ points where $s \in \mathbb{N} \setminus \{0,1\}$. By Proposition 2.4, we know that there exists a constant t such that every point of $\mathcal{S}$ is contained in precisely $t+1$ lines. If the generalized quadrangle $\mathcal{S}'$ associated with the polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ is a grid, then $\mathcal{S}$ is a glued near hexagon by Theorem 5.4. So, we may assume that $\mathcal{S}'$ is not a grid. Then by Corollary 2.13, every line of $\mathcal{S}'$ is incident with the same number of points, say st_2+1. By Proposition 4.2, all quads of $\mathcal{Q}$ then have order (s, t_2), and so by Property (P1) every point of $\mathcal{S}$ is incident with precisely $\frac{t}{t_2}$ quads of $\mathcal{Q}$. We put $t_2' := \frac{t}{t_2} - 1$. Then $\mathcal{S}'$ is a generalized quadrangle of order (st_2, t_2'). Since $\mathcal{S}'$ is not a grid, we have $t_2' \geq 2$.

Suppose x and y are two points at distance 2 from each other that are not contained in a quad of $\mathcal{Q}$. If $z \in \Gamma_1(x) \cap \Gamma_1(y)$, then $xz \notin S$ and the line xz is contained in a unique quad of $\mathcal{Q}$. Conversely, every quad of $\mathcal{Q}$ through x contains a unique point collinear with y since each such quad is big by Proposition 4.4. So, x and y then have precisely $t_2'+1$ common neighbours and hence are contained in a (necessarily unique) quad of order (s, t_2').

By elementary counting, we see that a quad of order (s, t_2') is big if and only if the total number v of points is equal to $(s+1)(st_2'+1)s(t-t_2')$. So, if there exists a big quad of order (s, t_2'), then all quads are big and by Proposition 2.7, we then know that $\mathcal{S}$ is a dual polar space of rank 3. So, we may assume that there exists a quad Q of order (s, t_2') that is not big. Let x be a given point at distance 2 from Q. Then x is ovoidal with respect to Q, implying that $\Gamma_2(x) \cap Q$ is an ovoid of $\widetilde{Q}$ containing $st_2'+1$ points.

We count the number of lines through x containing a point of $\Gamma_1(Q)$, i.e. a point at distance 1 from Q. (Such a point is unique by Proposition 2.3.) Suppose L is a line through x containing a point $y \in \Gamma_1(Q)$ and that z is the unique point of Q collinear with y. Then z is one of the $st_2'+1$ points of the ovoid $\Gamma_2(x) \cap Q$ of $\widetilde{Q}$. As $\mathrm{d}(x,Q)=2$, the unique quad $Q(x,z)$ through x and z cannot intersect Q in more than a line and hence $Q(x,z) \cap Q = \{z\}$. The latter implies by Proposition 2.5 that $Q(x,z)$ itself can also not be big, i.e. $Q(x,z)$ must also have order (s,t_2'). Since every quad through a line of S is big (Proposition 4.4), we see that L must be distinct from the unique line L_x of S through x.

Now, the $st_2'+1$ quads of the form $Q(x,z)$ with $z \in \Gamma_2(x) \cap Q$ determine a collection of $(st_2'+1)(t_2'+1)$ lines through x which are contained in one of these quads. All the lines in this collection meet $\Gamma_1(Q)$ in a (necessarily unique) point and are mutually distinct. Indeed, if L would be contained in the quads $Q(x,z_1)$ and $Q(x,z_2)$, where $z_1 \neq z_2$, then the unique point y of $\Gamma_1(Q)$ on the line L would be collinear with two distinct points of Q (namely z_1 and z_2) and this is impossible. As there are t lines through x distinct from L_x, we should thus have

$$(1+st_2')(1+t_2') \leq t = (t_2'+1)t_2,$$

i.e. $t_2 \geq 1+st_2'$. Hence, $st_2 > s^2t_2'$. From Higman's equality, we know that $s^2 \geq t_2'$. Hence $st_2 > (t_2')^2$. However, since the generalized quadrangle $\mathcal{S}'$ has order (st_2, t_2') with $t_2' \geq 2$, Higman's inequality also implies that $st_2 \leq (t_2')^2$. So, we have found a contradiction.

We conclude that $\mathcal{S}$ is either a glued near hexagon or a dual polar space of rank 3. ■

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