Weighted $\{\delta(q+1), \delta; k-1, q\}$-minihypers

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Abstract

In [5, 6, 7, 8, 9], weighted $\{\delta v_{\mu+1}, \delta v_\mu; N, q\}$-minihypers were classified. This class of minihypers is, next to being interesting for classifying linear codes meeting the Griesmer bound, a very important geometrical structure for solving problems in finite projective spaces. In [5, 6, 7, 8], there were restrictions on the weights of the points of the minihypers; in [9], there were no restrictions on the weights of the points, but the results were only valid for $\delta \le \epsilon$, with $q+1+\epsilon$ the size of the smallest non-trivial blocking sets in $PG(2,q)$. In this article, we improve this latter result for weighted $\{\delta(q+1), \delta; N, q\}$-minihypers, without restrictions on the weights of the points. The largest improvements are obtained for $q = p^2$, p prime, where we increase the upper bound to $\delta \le (q-1)/4$.

1 Introduction

Let $PG(N,q)$ be the N-dimensional projective space over the finite field of order q.

Definition 1.1 (Hamada and Tamari [15]) *An* $\{f, m; N, q\}$-minihyper *is a pair* (F, w)*, where* F *is a subset of the point set of* $PG(N,q)$ *and where* w *is a weight function* $w : PG(N,q) \to \mathbb{N} : x \mapsto w(x)$*, satisfying:*
(1) $w(x) > 0 \Leftrightarrow x \in F$,
(2) $\sum_{x \in F} w(x) = f$*, and*
(3) $\min(|F \cap H| = \sum_{x \in H} w(x) || H \in \mathcal{H}) = m$*; where* $\mathcal{H}$ *denotes the set of hyperplanes of* $PG(N,q)$.

In the case that w *is a mapping onto* $\{0,1\}$*, the minihyper* (F,w) *can be identified with the set* F *and is simply denoted by* F.

The excess e *of a minihyper* (F,w) *is the number* $\sum_{x \in F}(w(x)-1)$.

Let $v_s = (q^s - 1)/(q-1)$.

Minihypers in finite projective spaces were first introduced to study *linear codes meeting the Griesmer bound.* The Griesmer bound states that if there

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exists an $[n,k,d;q]$ code for given values of k,d and q, then

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k,d),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x [10, 22].

Suppose that there exists a linear $[n,k,d;q]$ code meeting the Griesmer bound ($d \geq 1, k \geq 3$), then we can write d in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i$ such that $\theta \geq 1$ and $0 \leq \epsilon_i < q$.

Using this expression for d, the Griesmer bound for an $[n,k,d;q]$ code can be expressed as: $n \geq \theta v_k - \sum_{i=0}^{k-2} \epsilon_i v_{i+1}$.

Let $E(t,q)$ denote the set of all ordered tuples $(\zeta_0,\ldots,\zeta_{t-1})$ of integers ζ_i such that $(\zeta_0,\ldots,\zeta_{t-1}) \neq (0,\ldots,0)$ and $0 \leq \zeta_0,\ldots,\zeta_{t-1} \leq q-1$.

From now on, we suppose that $(\epsilon_0,\ldots,\epsilon_{k-2})$ belongs to $E(k-1,q)$.

Hamada and Helleseth [14] showed that there is a one-to-one correspondence between the set of all non-equivalent $[n,k,d;q]$ codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihypers (F,w), such that $1 \leq w(p) \leq \theta$ for every point $p \in F$.

More precisely, the link is described in the following way. Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n,k,d;q]$ code, meeting the Griesmer bound. We look at a column of G as being the coordinates of a point in $PG(k-1,q)$. Let the point set of $PG(k-1,q)$ be $\{s_1,\ldots,s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $m(G) = \max\{m_i(G) || i = 1,2,\ldots,v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code C and we call it the *maximum multiplicity* of the code. Define the weight function $w : PG(k-1,q) \to \mathbb{N}$ as $w(s_i) = \theta - m_i(G)$, $i = 1,2,\ldots,v_k$. Let $F = \{s_i \in PG(k-1,q) || w(s_i) > 0\}$, then (F,w) is a $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper with weight function w.

The easiest way to construct weighted minihypers is to construct a *sum* of certain geometrical objects.

Consider a number of geometrical objects, such as subspaces $PG(d, q = p^h)$ of $PG(N, q = p^h)$, subgeometries $PG(d, p^t)$ of $PG(N, q = p^h)$, where $t|h$, and projected subgeometries $PG(d, p^t)$ in $PG(N, q = p^h)$, where $t|h$. In the first two cases, a point of $PG(d,q)$ or $PG(d,p^t)$ has weight one, while all the other points not belonging to respectively $PG(d,q)$ or $PG(d,p^t)$ have weight zero. In the latter case, let Π be a projected $PG(d,p^t)$. The *weight* of a point $s \in \Pi$ is the number of points s' of $PG(d,p^t)$ that are projected onto s; all points not belonging to Π have weight zero.

Then the *sum* of these subspaces and (projected) subgeometries is the weighted set (F,w), where the weight $w(s)$ of a point s of (F,w) is the sum of all the weights of s in the subspaces and (projected) subgeometries of (F,w).

Minihypers also have many applications in finite geometries [2, 4, 7, 8, 9]. A class of minihypers which is crucial in the study of *maximal partial t-spreads* and *minimal t-covers* in finite projective spaces $PG(N,q)$, where $(t+1)|(N+1)$, is the class of $\{\delta v_{t+1}, \delta v_t; N, q\}$-minihypers. These have been used by Govaerts and Storme [7, 9] to study the extendability of maximal partial t-spreads in $PG(N,q)$, $(t+1)|(N+1)$, of small deficiency δ; by Ferret and Storme [4] to study the extendability of maximal partial 1-spreads in $PG(3,q)$ of small deficiency δ; by Eisfeld, Storme and Sziklai to study the smallest $(n-1)$-covers of the hyperbolic quadric $Q^+(2n+1,q)$ [2]; and by Govaerts, Storme and Van Maldeghem [8] to obtain results on other types of substructures in finite incidence structures.

This article improves the results of [5, 6, 7, 8] for weighted $\{\delta(q+1), \delta; N, q\}$-minihypers.

Presently, the following results on weighted $\{\delta(q+1), \delta; N, q\}$-minihypers are known:

Theorem 1.2 (Govaerts and Storme [9]) *Let (F,w) be a weighted $\{\delta(q+1), \delta; N, q\}$-minihyper, where $\delta \leq \epsilon$ with $q+1+\epsilon$ the size of the smallest non-trivial blocking sets in $PG(2,q)$, then (F,w) is a sum of lines.*

Theorem 1.3 (Govaerts and Storme [7]) *If F is a $\{\delta(q+1), \delta; N, q\}$-minihyper, $q > 16$ a square, $\delta < q^{5/8}/\sqrt{2}+1$, $N \geq 3$, then F is a unique union of pairwise disjoint lines and Baer subgeometries $PG(3,\sqrt{q})$.*

Theorem 1.4 (Ferret and Storme [5, 6]) *A $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper (F,w), $p = p_0^h$, p_0 prime, $p \geq 9$, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, with excess $e \leq p^3$ if $N = 3$ and with excess $e \leq p^3 - 4p$ if $N > 3$, is either:*
(1) a sum of lines, (projected) $PG(3,p^{3/2})$ if p is a square, and of at most one projected $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines, (projected) $PG(3,p^{3/2})$ if p is a square, and of a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega, w) \setminus R$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where R is the line contained in Ω.

We will improve these results for arbitrary weighted $\{\delta(q+1), \delta; N, q\}$-minihypers (F,w), with no restrictions on the weight function w. The upper bounds on δ are improved for q square, but not a cube. The largest improvements are obtained for $q = p^2$, p prime, $p \geq 11$, where we improve the upper bound to $\delta \leq (q-1)/4$.

Remark 1.5 Sometimes, we will intersect the minihyper (F,w) with a set of points (for example, the point set of a plane) α, and briefly write $(F,w) \cap \alpha$. With this, we mean the point set $F \cap \alpha$ with as weight function the restriction of w to the points of α. If we take an element or a point of the minihyper (F,w), then we mean a point of F.

Crucial in our classification results are the recent classification results on non-trivial minimal blocking sets in $PG(2,q)$.

Definition 1.6 *A* blocking set *of $PG(2,q)$ is a set of points intersecting every line of $PG(2,q)$ in at least one point.*

A blocking set is called minimal *when no proper subset of it is still a blocking set; and we call a blocking set* non-trivial *when it does not contain a line.*

A blocking set of $PG(2,q)$ is called small *when it has less than $3(q+1)/2$ points.*

If $q = p^h$, p prime, we call the exponent *E of the minimal blocking set B the maximal integer E such that every line intersects B in $1 \pmod{p^E}$ points.*

From results of Szőnyi [26] and Sziklai [25], it follows that $E \geq 1$ for every small non-trivial minimal blocking set in $PG(2,q)$, $q = p^h$, p prime, $h \geq 1$, and that E divides h.

In [26], it is also proven that if E is the exponent of a small non-trivial minimal blocking set in $PG(2,q)$, $q = p^h$, p prime, then the size of the blocking set must lie in certain intervals depending on p^E. We note that the bounds given in [26] are improved in [18] and in [20].

The results of [26] have been used to classify all non-trivial small minimal blocking sets of $PG(2,q)$, $q = p^h$, of exponent $E \geq h/3$.

Theorem 1.7 (Polverino, Polverino and Storme [19, 20, 21]) *The smallest minimal blocking sets in $PG(2,p^3)$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, with exponent $E \geq h$, are:*
(1) *a line,*
(2) *a Baer subplane of cardinality $p^3 + p^{3/2} + 1$, when p is a square,*
(3) *a set of cardinality $p^3 + p^2 + 1$, equivalent to*

$$\{(x, T(x), 1) || x \in GF(p^3)\} \cup \{(x, T(x), 0) || x \in GF(p^3) \setminus \{0\}\},$$

with T the trace function from $GF(p^3)$ to $GF(p)$,
(4) *a set of cardinality $p^3 + p^2 + p + 1$, equivalent to*

$$\{(x, x^p, 1) || x \in GF(p^3)\} \cup \{(x, x^p, 0) || x \in GF(p^3) \setminus \{0\}\}.$$

Theorem 1.8 (Szőnyi [26]) *A small minimal non-trivial blocking set in $PG(2,q)$, $q = p^h$, p prime, h even, of exponent $E = h/2$, is a Baer subplane of $PG(2,q)$.*

These results are also the complete classifications of all small minimal non-trivial blocking sets in $PG(2,p^3)$, p prime, $p \geq 7$, and in $PG(2,q)$, $q = p^2$, p prime.

Corollary 1.9 *Every small minimal blocking set in $PG(2,p^2)$, p prime, is equal to a line or to a Baer subplane.*

Every small minimal blocking set in $PG(2,p^3)$, p prime, $p \geq 7$, is projectively equivalent to one of the blocking sets described in Theorem 1.7.

From the intervals for the sizes of minimal non-trivial blocking sets in $PG(2,p^3)$ [19], the following result follows.

Theorem 1.10 *In $PG(2,p^3)$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h \geq 1$, every non-trivial blocking set B of size at most $p^3 + 2p^2$ contains a minimal blocking set of one of the types described in Theorem* 1.7.

Remark 1.11 (1) The minimal blocking set of size $p^3 + p^2 + 1$ (Theorem 1.7 (3)) has a unique point, called the *vertex*, lying on exactly $p+1$ lines containing p^2+1 points of the blocking set. We will call such an intersection *a* (p^2+1)*-set*. These $p+1$ lines form a dual $PG(1,p)$. All other lines intersect the blocking set in 1 or in $p+1$ points.

Furthermore, these (p^2+1)-sets are equivalent to the set $\{\infty\} \cup \{x \in GF(p^3) || x + x^p + x^{p^2} = 0\}$, with ∞ corresponding to the vertex of the blocking set.

Later on, we will refer to the point corresponding to ∞ as being the *special point* of this (p^2+1)-set.

The lines sharing $p+1$ points with this blocking set intersect the blocking set in a subline $PG(1,p)$.

(2) The minimal blocking set of size $p^3 + p^2 + p + 1$ (Theorem 1.7 (4)) has $p^2 + p + 1$ points in common with exactly one line; all other lines intersect the blocking set in 1 or in $p+1$ points.
The intersection of the blocking set with the (p^2+p+1)-secant will be called *a* (p^2+p+1)*-set*. This (p^2+p+1)-set is equivalent to $\{x \in GF(p^3) || x^{p^2+p+1} = 1\}$. The $(p+1)$-secants intersect the blocking set in a subline $PG(1,p)$.

(3) These two latter blocking sets are also characterized [16] as being a projected subgeometry $PG(3,p)$ in the plane $PG(2,p^3)$. Namely, embed the plane $PG(2,p^3)$ in a 3-dimensional space $PG(3,p^3)$. Consider a subgeometry $PG(3,p)$ of $PG(3,p^3)$, and a point r not belonging to this subgeometry $PG(3,p)$ and not belonging to the plane $PG(2,p^3)$.

Project $PG(3,p)$ from r onto $PG(2,p^3)$.

If the point r belongs to a line of the subgeometry $PG(3,p)$, then this $PG(3,p)$ is projected onto the blocking set of size $p^3 + p^2 + 1$; else we obtain the blocking set of size $p^3 + p^2 + p + 1$.

(4) In this article, every set of p^2+1 collinear points projectively equivalent to the set $\{\infty\} \cup \{x || x^{p^2} + x^p + x = 0\}$ will be called a (p^2+1)*-set*, and every set of p^2+p+1 collinear points projectively equivalent to the set $\{x || x^{p^2+p+1} = 1\}$ will be called a (p^2+p+1)*-set*.

We will use the following result of Hamada and Helleseth.

Theorem 1.12 ([3, 13]) *Let (F,w) be a $\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=1}^{t-1} \epsilon_i v_i; t, q\}$-minihyper where $t \geq 2$, $h \geq 2$, $q - 1 \geq h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{t-1} \epsilon_i = h$.*
(1) *If there exists a hyperplane H of $PG(t,q)$ such that $|(F,w) \cap H| = \sum_{i=1}^{t} m_i v_i$ for some $(m_1, \ldots, m_t) \in E(t,q)$, then $((F,w) \cap H, w)$ is a $\{\sum_{i=1}^{t} m_i v_i$, $\sum_{i=1}^{t} m_i v_{i-1}; t-1, q\}$-minihyper in H.*
(2) *There does not exist a hyperplane H in $PG(t,q)$ such that $|(F,w) \cap H| = \sum_{i=1}^{t} m_i v_i$ for any $(m_1, \ldots, m_t) \in E(t,q)$ such that $\sum_{i=1}^{t} m_i > h$.*
(3) *In the case $\epsilon_0 = 0$ and $q \geq 2h - 1$, there is no hyperplane H in $PG(t,q)$*

such that $|(F,w) \cap H| = \sum_{i=1}^{t} m_i v_i$ *for any* $(m_1, \ldots, m_t) \in E(t,q)$ *such that* $\sum_{i=1}^{t} m_i < h$.

Corollary 1.13 *Let* (F,w) *be a* $\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=1}^{t-1} \epsilon_i v_i; t, q\}$*-minihyper where* $t \geq 2$, $h \geq 2$, $q-1 \geq h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{t-1} \epsilon_i = h$.

Then every plane, not contained in F*, intersects* (F,w) *in an* $\{m_1(q+1)+m_0, m_1; 2, q\}$*-minihyper, with* $m_0 + m_1 \leq h$.

The following theorem is a special case of these general results of Hamada and Helleseth.

Theorem 1.14 (Hamada and Helleseth [13]) *Let* (F,w) *be a* $\{\delta(q+1), \delta; 3, q\}$*-minihyper, with* $\delta \leq (q+1)/2$.

Then a plane intersects (F,w) *in an* $\{m_0 + m_1(q+1), m_1; 2, q\}$*-minihyper, with* $m_0 + m_1 = \delta$.

For a plane intersecting a $\{\delta(q+1), \delta; 3, q\}$-minihyper (F,w) in an $\{m_0 + m_1(q+1), m_1; 2, q\}$-minihyper, we will call m_1 the *multiplicity* of that plane with respect to (F,w). If $m_1 \geq 1$, then we call the plane a *blocking plane* or *rich plane* of (F,w).

Theorem 1.15 (Govaerts and Storme [7]) *Let* (F,w) *be a* $\{\delta(q+1), \delta; 3, q\}$*-minihyper.*

A point of (F,w) *having weight* α *is contained in exactly* $\alpha q + \delta$ *planes* π*, where the planes* π *are counted with multiplicity* $m_1(\pi)$.

A point having weight zero is contained in exactly δ *planes* π*, counted with multiplicities* $m_1(\pi)$.

Lemma 1.16 (Govaerts and Storme [7]) *Let* (F,w) *be a* $\{\delta(q+1), \delta; 3, q\}$*-minihyper, with* $\delta \leq (q+1)/2$.

A line L *contains* α *points of* (F,w) *if and only if there are exactly* α *planes of* (F,w)*, counted with multiplicities* $m_1(\pi)$*, through* L.

Corollary 1.17 *A line* L *containing a point* r *not in* (F,w)*, contains at most* δ *points of* (F,w).

Lemma 1.18 (Govaerts and Storme [9]) *Let* (F,w) *be a* $\{\delta v_{\mu+1}, \delta v_{\mu}; N, q\}$*-minihyper satisfying* $0 \leq \delta \leq (q+1)/2$, $0 \leq \mu \leq N-1$*, and containing a* μ*-dimensional space* π_μ*. Then the minihyper* (F', w') *defined by the weight function* w'*, where*

- $w'(p) = w(p) - 1$*, for* $p \in \pi_\mu$*, and*
- $w'(p) = w(p)$*, for* $p \in PG(N,q) \setminus \pi_\mu$*,*

is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_{\mu}; N, q\}$*-minihyper.*

A *1-fold blocking set* in $PG(N,q)$ is a set of points intersecting every hyperplane in at least one point. A 1-fold blocking set of $PG(N,q)$ is called *minimal* when no proper subset of it still is a 1-fold blocking set.

The following result characterizes the smallest 1-fold blocking sets in $PG(N,q)$, $N \geq 3$. It is based on results of Storme and Weiner [23].

Theorem 1.19 (1) *Let $q+1+\epsilon$ be the size of the smallest blocking set in $PG(2,q)$, q square, $q=p^h$, $p>3$ prime, $h \geq 2$ even, not containing a line or Baer subplane.*

Then every minimal 1-fold blocking set of size at most $q+1+\epsilon$ in $PG(n,q)$, $n \geq 3$, is a planar minimal blocking set.

(2) *Let B be a minimal 1-fold blocking set in $PG(n,p^3)$, $n \geq 3$, $p=p_0^h$, $h \geq 1$, p_0 prime, $p_0 \geq 7$, of size at most p^3+2p^2.*

Then B is either a line, a Baer subplane $PG(3,p^{3/2})$ if p is a square, a minimal planar blocking set of size $p^3+p^2(+p)+1$, or a subgeometry $PG(3,p)$.

Proof Part (1) follows from [23].

For Part (2), we proceed as follows. It is known that every minimal 1-fold blocking set B in $PG(2,p^3)$, of size at most p^3+2p^2, intersects every plane in 1 (mod p) points [27]. It is possible to find a point r not in B only lying on tangent lines to B. If we project B from this point r onto a plane, a minimal planar blocking set in $PG(2,p^3)$ is obtained [27, Corollary 3.2]. Such a minimal blocking set in $PG(2,p^3)$ of size at most p^3+2p^2 is either a line, Baer subplane, or minimal blocking set of size $p^3+p^2(+p)+1$ (Theorem 1.10). Then also $|B| \leq p^3+p^2+p+1$.

We are now reduced to the setting of [23], leading to the proof of Part (2). □

2 Projected $PG(5,p)$ in $PG(3,p^3)$

In the classification results on $\{\delta(p^3+1), \delta; N, p^3\}$-minihypers (F,w) that will be obtained, it is possible that such minihypers contain a projected subgeometry $PG(5,p) \equiv \Omega$. The techniques for proving that such a projected subgeometry $PG(5,p)$ is contained in (F,w) were developed in [5, 6]. We will be able to use the ideas of [5, 6], so we will refer a lot to these two articles. To make the notations and descriptions in this article clear to the readers, we repeat the descriptions of the projected subgeometries $PG(5,p)$ in $PG(3,p^3)$.

Consider a subgeometry $\Lambda = PG(5,p)$ naturally embedded in $PG(5,p^3)$. Let L be a line of $PG(5,p^3)$ skew to Λ. Then the line L has two conjugate lines with respect to Λ. We will always denote these conjugate lines by L^p and L^{p^2}. We project Λ from L onto a solid not passing through L.

Case 1. Suppose that Ω is the projection of $PG(5,p) \equiv \Lambda$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Then every projected point s in Ω has weight one. Every point $s \in \Omega$ lies on exactly one (p^2+p+1)-set of Ω, on $p^4+p^3+p^2$ $(p+1)$-secants to Ω, and lies in p^3+p^2+p+1 planes of $PG(3,p^3)$ sharing a minimal 1-fold blocking set of size p^3+p^2+p+1 with Ω.

In general, a plane of $PG(3,p^3)$ intersects Ω in either a subplane $PG(2,p)$, a (p^2+p+1)-set, or in a minimal blocking set of size p^3+p^2+p+1.

Case 2. Suppose that Ω is the projection of $PG(5,p) \equiv \Lambda$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Then the 4-dimensional space $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ is called the *special* 4-space of Λ, and similarly, its projection is called the *special* projected 4-space of Ω. We will denote this special 4-space $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ by $\mathcal{P}$.

Then for exactly one point r of L, $\dim\langle r, r^p, r^{p^2}\rangle = 1$. This line $M = \langle L, L^p\rangle \cap \langle L^p, L^{p^2}\rangle \cap \langle L, L^{p^2}\rangle = \langle r, r^p, r^{p^2}\rangle$ is projected from L onto a point m of Ω of weight $p+1$. The other p^3 points r of L satisfy $\dim\langle r, r^p, r^{p^2}\rangle = 2$. These latter planes are projected onto (p^2+p+1)-sets of Ω.

Every plane π of Λ passing through M and not lying in $\mathcal{P}$ is projected from L onto a (p^2+1)-set with special point m. Each such plane π lies in p^2+p+1 solids of Λ which are projected onto planar minimal blocking sets of size p^3+p^2+1; thus implying that m lies in $p^4+p^3+p^2$ planes of $PG(3,p^3)$ sharing a 1-fold blocking set of size p^3+p^2+1 with Ω.

Let s be a point of Ω different from m and not lying in the special 4-space of Ω. Assume that s is the projection of $s' \in \Lambda$. Then each solid $\langle r, r^p, r^{p^2}, s'\rangle \cap \Lambda$, with $r \in L \setminus M$, is projected onto a planar minimal blocking set of size p^3+p^2+p+1; hence, s lies in p^3 such planes. Every solid of Λ passing through M and s' is projected onto a planar minimal blocking set of size p^3+p^2+1 passing through s; thus giving p^2+p+1 extra planes through s intersecting Ω in a projected $PG(3,p)$.

Let s be a point of weight one of Ω which is the projection of a point s' of $\mathcal{P}$. Then the plane $\langle M, s'\rangle$ lies in p^2 distinct 3-spaces of Λ not contained in $\mathcal{P}$ which are projected onto planar blocking sets of size p^3+p^2+1 through s.

Case 3. Suppose that Ω is the projection of $PG(5,p) \equiv \Lambda$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

Let $\mathcal{P} = \langle L, L^p, L^{p^2}\rangle \cap \Lambda$.

Every plane α through L in $\langle L, L^p, L^{p^2}\rangle$ has two conjugate planes α^p, α^{p^2} with respect to Λ, and these three planes intersect in at least one point of $\mathcal{P}$. Hence, every plane through L in $\langle L, L^p, L^{p^2}\rangle$ contains at least one point of $\mathcal{P}$ and the projection of $\mathcal{P}$ is a line N of $PG(3,p^3)$. There are $p+1$ skew lines $L_1, \ldots, L_{p+1}$ in $\mathcal{P}$ which are projected onto points of weight $p+1$, and the remaining p^3-p points of $\mathcal{P}$ are projected onto points of weight one of the line N.

We call the 3-dimensional space $\mathcal{P}$ the *special* 3-space of Λ, and its projection will always be denoted by the line N.

A point s' of $\Lambda \setminus \mathcal{P}$ is projected onto a point s lying on $p+1$ (p^2+1)-secants to Ω, which are the projections of $\langle s', L_i\rangle \cap \Lambda$, $i = 1, \ldots, p+1$. Each such (p^2+1)-secant through s lies in p^2 planes of $PG(3, p^3)$ containing a projected $PG(3, p)$ of Λ, which is a minimal blocking set of size $p^3 + p^2 + 1$; hence, s' lies in $p^3 + p^2$ such planes. Considering these subspaces $PG(3, p)$ in Λ; these are the subspaces $PG(3, p)$ through a plane $\langle s', L_i\rangle$ only intersecting $\mathcal{P}$ in L_i.

Furthermore, $\mathcal{P}$ is projected on the line N through which there are $p+1$ planes of $PG(3, p^3)$ containing $p^4 + p^3 + p^2 + p + 1$ projected points of Λ. The other planes through N contain $p^3 + p^2 + p + 1$ projected weighted points; the points with weights larger than one all lie on N.

Hence, this projection forms a $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper (Ω, w) containing the line N. Reducing the weight of every point on N by one yields a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega, w) \setminus N$.

Case 4. Suppose that Ω is the projection of $PG(5, p) \equiv \Lambda$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

Then this projection is a cone of $p^2 + p + 1$ lines; the vertex of the cone is a point having weight $p^2 + p + 1$ arising from the projection of the points of the plane $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$, and the base of the cone is a subplane $PG(2, p)$.

Remark 2.1 (1) In the remaining part of this article, the symbols Ω, Λ and N will always have the following meaning. The symbol Ω will always denote the projection of a $PG(5, p) \equiv \Lambda$ from a line L, and if $\dim\langle L, L^p, L^{p^2}\rangle = 3$, then N will always denote the line contained in Ω.

(2) In the latter case, when Ω contains a line N, then $(\Omega, w) \setminus N$ denotes the $\{(p^2 + p)(p^3 + 1), p^2 + p; 3, p^3\}$-minihyper obtained by reducing the weight of every point of N by one.

3 Weighted $\{\delta(q+1), \delta; 3, q\}$-minihypers, q square

In this section, we classify weighted $\{\delta(q+1), \delta; 3, q\}$-minihypers, q square, but not a cube. We assume that $\delta \leq (q-1)/4$ when $q = p^2$, p prime, and, when $q = p^{2h}$, p prime, $h > 1$, that $\delta \leq \epsilon$, with $q + 1 + \epsilon$ the size of the smallest minimal blocking set in $PG(2, q)$ different from a line and Baer subplane.

We present the proofs for $q = p^2$, p prime. By Lemma 1.18, we can assume that (F, w) does not contain any lines.

If r is a point of $PG(3, q)$, then a *Baer cone* with vertex r is a set of points that is the union of $q + \sqrt{q} + 1$ lines on r that form a Baer subplane in the quotient space on r. The *planes of this cone* are the $q + \sqrt{q} + 1$ planes on r that contain $\sqrt{q} + 1$ of these lines.

Lemma 3.1 (Govaerts and Storme [7]) *Suppose that* (F, w) *is a* $\{\delta(q+1), \delta; 3, q\}$*-minihyper with* δ *satisfying*

(i) $\delta \leq \frac{q+1}{2}$ *and,*
(ii) *every blocking set with size at most* $q+\delta$ *contains a line or a Baer subplane.*

Suppose furthermore that (F, w) *does not contain a line. If* r *is a point of* (F, w) *with minimal weight, then the rich planes through* r *contain a Baer cone* $\mathcal{B}$ *with vertex* r*.*

Lemma 3.2 *Suppose that* $r \in F$ *has minimal weight* α*, that* $\delta \leq (q+1)/2$*,* $q = p^2$*,* $p > 2$ *prime, and let* $\mathcal{B}$ *be such a Baer cone with vertex* r*, as described in Lemma* 3.1*. Then any plane* E_i *of* $\mathcal{B}$ *with* $m_1(E_i) = \alpha$ *contains a unique Baer subplane* $B(E_i)$ *consisting of points of* (F, w)*, and this Baer subplane is contained in* $\mathcal{B}$*.*

Proof The plane E_i intersects F in a set $E_i \cap F$ which is at least a 1-fold blocking set in E_i.

Now $|E_i \cap F| \leq (\alpha q + \delta)/\alpha \leq q + \delta$, so $E_i \cap F$ contains a Baer subplane since $\delta \leq (q+1)/2$.

The point r has weight α. The lines of the cone $\mathcal{B}$ through r lie in $\sqrt{q}+1$ planes π of the cone $\mathcal{B}$, all satisfying $m_1(\pi) \geq \alpha$, so these lines contain at least $\alpha(\sqrt{q}+1)$ points of $E_i \cap (F, w)$ (Lemma 1.16). This implies that in total, these $\sqrt{q}+1$ lines of $\mathcal{B}$ through r contain at least $\alpha(\sqrt{q}+1)\sqrt{q} + \alpha = \alpha(q + \sqrt{q} + 1)$ points of (F, w).

There remain at most $\delta - \alpha\sqrt{q} - \alpha < q - \sqrt{q}$ points of (F, w) on the remaining $q - \sqrt{q}$ lines of E_i through r. So r belongs to this Baer subplane in $E_i \cap F$.

Since $\delta - \alpha\sqrt{q} - \alpha < q - \sqrt{q}$, more than $2\sqrt{q}+1$ points of the Baer subplane in $E_i \cap (F, w)$ lie in $\mathcal{B}$, so this Baer subplane lies completely in $\mathcal{B}$. □

Lemma 3.3 *Suppose that* (F, w) *is a* $\{\delta(q+1), \delta; 3, q\}$*-minihyper,* $q = p^2$*,* p *prime, with* $\delta \leq (q-1)/4$*.*

Through every point r *of minimal weight* α *of* (F, w)*, there exists a Baer subgeometry* $D := PG(3, \sqrt{q})$ *consisting entirely of points of* (F, w)*.*

Proof The point r lies in $\alpha q + \delta$ rich planes of (F, w) (Theorem 1.15). All $q + \sqrt{q} + 1$ planes π of $\mathcal{B}$ satisfy $m_1(\pi) \geq \alpha$. So r lies in at most $\delta - \alpha\sqrt{q} - \alpha$ planes π for which $m_1(\pi) > \alpha$.

Let $E_0, E_1, \ldots, E_{s-1}$ be the planes of $\mathcal{B}$ satisfying $m_1(E_i) = \alpha$. Note that $s \geq q + \sqrt{q} + 1 - (\delta - \alpha\sqrt{q} - \alpha) \geq q - \delta + (\alpha+1)(\sqrt{q}+1) \geq q - \delta + 2\sqrt{q} + 2$.

Let $\pi \in \{E_0, E_1, \ldots, E_{s-1}\}$ and let $\pi \cap \mathcal{B} = \{L_0, L_1, \ldots, L_{\sqrt{q}}\}$. Suppose that β of these lines contain more than one Baer subline consisting of points of (F, w). Then $|\pi \cap (F, w)| = \alpha q + \delta \geq \alpha(q + \sqrt{q} + 1) + \beta(\sqrt{q} - 1)\alpha$, such that $\delta \geq \alpha\sqrt{q} + \alpha + \beta(\sqrt{q} - 1)\alpha$. Here, we used the fact that α is the minimal weight of the points of (F, w) and that two distinct Baer sublines share at most two points.

Call the lines containing exactly one Baer subline consisting of points of (F, w) *good* lines.

Let π and π' be two distinct elements of $\{E_0, E_1, \ldots, E_{s-1}\}$ intersecting in a good line. Denote by B (B') the Baer subplane of π (π') consisting of points of (F, w). Define D as the $PG(3, \sqrt{q})$ spanned by B and B'.

Since $\delta \leq (q-1)/4$, $\beta \leq \eta = (\sqrt{q}+1)/4$. So at least $\sqrt{q}+1-\eta$ lines L_i contain exactly one Baer subline of $\mathcal{B}$.

The good lines of π and π' define at least $(\sqrt{q}-\eta)^2$ planes of $\mathcal{B}$ intersecting π as well as π' in a good line. Thus, using $\delta \leq (q-1)/4 = \eta(\sqrt{q}-1)$, there are at least $q - 2\eta\sqrt{q} + \eta^2 - \delta + \alpha\sqrt{q} + \alpha \geq q - 3\eta\sqrt{q} + \eta^2 + \eta + \sqrt{q} + 1$ planes E_i of $\mathcal{B}$, with $m_1(E_i) = \alpha$, that intersect π as well as π' in a good line. Since the Baer subplanes of (F, w) in these planes have two Baer sublines in common with D, they are contained in D.

So there exists a line of π on r that is contained in at least $(q - 3\eta\sqrt{q} + \eta^2 + \eta + \sqrt{q} + 1)/\sqrt{q} \geq \sqrt{q} - 3\eta + (\eta^2 + \eta + 1)/\sqrt{q} + 1$ of those planes.

Therefore, including the plane π, at least $(\sqrt{q} - 3\eta + (\eta^2+\eta+1)/\sqrt{q} + 2)\sqrt{q} + 1 = q - 3\eta\sqrt{q} + 2\sqrt{q} + \eta^2 + \eta + 2$ lines of $\mathcal{B}$ have a Baer subline, consisting of points of F, that is contained in D. Denote these lines by $M_0, M_1, \ldots$.

Suppose that there exists a point r' of D that does not belong to (F, w). Then r' lies in δ rich planes (Theorem 1.15). The q planes of D through r' but not through r, intersect each of the lines $M_0, M_1, \ldots,$ in a point of (F, w). Therefore they contain at least $q - 3\eta\sqrt{q} + 2\sqrt{q} + \eta^2 + \eta + 2$ points of (F, w). Since this number is larger than δ, these planes are rich. So, there are more than δ rich planes through r', implying that $r' \in (F, w)$, such that all points of D belong to (F, w). $\square$

Theorem 3.4 *Let (F, w) be a weighted $\{\delta(q+1), \delta; 3, q\}$-minihyper, with $\delta \leq (q-1)/4$ if $q = p^2$, p prime, and with $\delta \leq \epsilon$, where $q + 1 + \epsilon$ is the size of the smallest blocking set in $PG(2, q)$, $q = p^{2h}$, $h > 1$, p prime, not containing a line or a Baer subplane.*

Then F is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$.

Proof If F contains a line L, reducing the weights of the points of L by one, a new $\{(\delta-1)(q+1), \delta-1; 3, q\}$-minihyper (F', w') is obtained (Lemma 1.18).

So assume that (F, w) does not contain any lines.

Let r be a point of minimal weight of (F, w). The preceding lemma shows that F contains a Baer subgeometry $PG(3, \sqrt{q})$. The arguments of [7, Theorem 2.1] show that if we reduce the weights of the points of this latter subgeometry $PG(3, \sqrt{q})$ by one, a new $\{(\delta - \sqrt{q} - 1)(q+1), \delta - \sqrt{q} - 1; 3, q\}$-minihyper (F', w') is obtained.

Repeating the arguments for (F', w'), this shows that (F, w) is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$. $\square$

4 Weighted $\{\delta(q+1), \delta; k-1, q\}$-minihypers, q square

In this section, we classify weighted $\{\delta(q+1), \delta; k-1, q\}$-minihypers, $k > 4$, q square, but not a cube. We assume that $\delta \leq (q-1)/4$ when $q = p^2$, p prime,

$p \geq 11$, and, when $q = p^{2h}$, p prime, $h > 1$, that $\delta \leq (q-1)/4$ and that

$$\delta + \frac{\delta^2}{q} + \frac{2\delta^2 - \delta}{q^2} + \frac{\delta^2 - \delta}{q^3} < 1 + \epsilon,$$

with $q + 1 + \epsilon$ the size of the smallest minimal blocking set in $PG(2, q)$ different from a line and Baer subplane.

We present the proofs for $q = p^2$, p prime. Again by Lemma 1.18, we can assume that (F, w) does not contain any lines.

Lemma 4.1 *Let (F, w) be a weighted $\{\delta(q+1), \delta; 4, q\}$-minihyper, $q = p^2$, p prime, with $\delta \leq (q-1)/4$.*

Then F can be projected from a point r not in F onto a solid resulting in a new $\{\delta(q+1), \delta; 3, q\}$-minihyper (F', w'). It is possible to select the point r in such a way that r lies on at most $q/32$ secants to F, containing at most $q/16$ distinct points of (F, w).

For such a point r, there is a bijective relation between the lines contained in F and the lines contained in F'.

Proof The number of secants to F is at most $((q^2-1)/4) \cdot ((q^2-5)/4)/2$, containing at most $(q^5 + q^4 - 6q^3 - 6q^2 + 5q + 5)/32$ points of $PG(4, q)$. So there is a point r not in F lying on at most $q/32$ distinct secants to F. These latter secants contain at most $q/16$ distinct points of F [7, Lemma 2.2]. In this counting argument, secants through r containing m distinct points of F are counted $m(m-1)/2$ times in the upper bound on the number of secants through r and they are counted $m(m-1)$ times in the upper bound on the number of points of F on these secants through r.

Suppose that F' contains a line L. Then the plane $\langle L, r\rangle$ contains at most $q+1+q/16$ distinct points of F. So this plane intersects F in a 1-fold blocking set (Corollary 1.13) containing a line or a Baer subplane, contained in F (Corollary 1.9).

If $\langle L, r\rangle$ contains a Baer subplane π_0 contained in F, then r lies on a Baer subline to π_0. This latter Baer subline is a $(\sqrt{q}+1)$-secant to F, so it contributes $(\sqrt{q}+1)\sqrt{q}$ to the upper bound $q/16$ on the number of points of F on the secants through r. This is false.

So a line L contained in F' is the projection, from r, of a line contained in F. $\square$

Theorem 4.2 *Let (F, w) be a weighted $\{\delta(q+1), \delta; 4, q\}$-minihyper, with $q = p^2$, p prime, $p \geq 11$, and with $\delta \leq (q-1)/4$.*

Then F is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$.

Proof Consider again the point r of the preceding lemma. This point r projects (F, w) onto a weighted $\{\delta(q+1), \delta; 3, q\}$-minihyper (F', w'), which is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$.

There is a bijective relation between the set of lines contained in (F, w) and the set of lines contained in (F', w'). By Lemma 1.18, these lines can be removed from (F,w) to obtain a new $\{\delta'(q+1),\delta';4,q\}$-minihyper (F'',w'') not containing any lines.

So we can assume that (F',w') is a sum of Baer subgeometries $PG(3,\sqrt{q})$. We show that these Baer subgeometries $PG(3,\sqrt{q})$ contained in (F',w') arise from Baer subgeometries $PG(3,\sqrt{q})$ contained in (F,w). Since $\delta \le (q-1)/4$, (F',w') is a sum of at most $\delta/(\sqrt{q}+1) \le (\sqrt{q}-1)/4$ Baer subgeometries $PG(3,\sqrt{q})$.

Part 1. Consider a point s' of (F',w') of minimal weight, lying in the Baer subgeometry $\pi_3 = PG(3,\sqrt{q})$ of (F',w'). Then at least $q\sqrt{q}+q+\sqrt{q}+1-\sqrt{q}(q+\sqrt{q}+2)/4 = 3q\sqrt{q}/4+3q/4+\sqrt{q}/2+1$ points of π_3 have the same minimal weight since (F',w') is a sum of at most $\sqrt{q}/4$ Baer subgeometries $PG(3,\sqrt{q})$ and since two distinct Baer subgeometries share at most $q+\sqrt{q}+2$ distinct points [24].

Since $3q\sqrt{q}/4+3q/4+\sqrt{q}/2+1 > q/32$, it is possible to select the point s' of minimal weight of $F'\cap\pi_3$ in such a way that it lies on a tangent line to F through r. Let s be the point of F projected from r onto the point s' of (F',w').

Part 2. This point s' lies on $q+\sqrt{q}+1$ secant lines to π_3. So it is possible to select a secant L' through s' to π_3 containing $\sqrt{q}+1$ points of $F'\cap\pi_3$ which lie on tangent lines to F through r.

Since (F',w') contains less than $\sqrt{q}/4$ distinct Baer subgeometries $PG(3,\sqrt{q})$, it is even possible to select L' in such a way that it contains no Baer sublines of the Baer subgeometries $PG(3,\sqrt{q})$, different from π_3, contained in (F',w'). Then L' contains at most $\sqrt{q}+1+\sqrt{q}/4$ distinct points of F'.

Part 3. We consider the $\sqrt{q}+1$ planes of π_3 through L'. Since L' is not a secant line to the Baer subgeometries $\pi = PG(3,\sqrt{q})$, different from π_3, contained in F', at most $\sqrt{q}/4$ of those planes intersect such a Baer subgeometry $\pi \ne \pi_3$, contained in (F',w'), in a Baer subplane.

So at least $3\sqrt{q}/4$ planes Π through L' only intersect the Baer subgeometry π_3 of (F',w') in a Baer subplane. They then intersect F' in a 1-fold blocking set, if we do not consider the weights of the points of F'.

If there would be a plane Δ in $\langle \Pi, r\rangle$ skew to F, then all solids through Δ would contain δ points of (F,w), but $\langle \Pi, r\rangle$ contains more than δ points of (F,w).

So the corresponding solids $\langle \Pi, r\rangle$ intersect F in a 1-fold blocking set since they contain at least $q+\sqrt{q}+1$ and at most $q+\delta+q/16 < 3q/2$ distinct points of F.

It then follows from Theorem 1.19 that $\langle \Pi, r\rangle \cap F$ contains a line or a Baer subplane. So all these solids $\langle \Pi, r\rangle$ intersect F in a set containing a Baer subplane since the projection of the minimal blocking set in $\langle \Pi, r\rangle \cap F$ from r is a Baer subplane.

These latter Baer subplanes, denoted by $\pi'_0,\ldots$, all pass through the Baer subline L of F which is projected from r onto the Baer subline of L' contained in F'.

So, we find at least $3\sqrt{q}/4$ distinct Baer subplanes $PG(2,\sqrt{q})$ of F through a common Baer subline L; these latter Baer subplanes contain at least $3\sqrt{q}q/4+\sqrt{q}+1$ points of F projected from r onto π_3.

Part 4. From the preceding part, we know that there are at least $3\sqrt{q}\sqrt{q}/4$ Baer sublines through s in these $3\sqrt{q}/4$ Baer subplanes $\pi'_0, \ldots$, of F through L.

Consider the projections from r of these latter Baer subplanes $\pi'_0, \ldots$, of F through L.

We can select a second line M' through s' in one of those $3\sqrt{q}/4$ distinct Baer subplanes of F' through L' (Part 3), where the points of F' on M' lie on tangent lines to r, and where M' does not intersect any of the other Baer subgeometries $PG(3, \sqrt{q}) \neq \pi_3$ contained in F' in a Baer subline, since $3q/4 > q/32 + \sqrt{q}/4$. Here, $q/32$ is the upper bound on the number of secants to F through r and $\sqrt{q}/4$ is the upper bound on the number of distinct Baer subgeometries $PG(3, \sqrt{q})$ contained in F'.

Repeating the arguments as for L, this then leads to a second Baer subline M through s lying in at least $3\sqrt{q}/4$ Baer subplanes $\pi''_0, \ldots$, contained in F, and which are projected from r onto Baer subplanes of π_3. Again, these latter $3\sqrt{q}/4$ Baer subplanes all contain at least $3q\sqrt{q}/4 + \sqrt{q} + 1$ points of F.

Part 5. Consider the Baer subplanes $\pi'_0, \ldots$, through L, and the Baer subplanes $\pi''_0, \ldots$, through M, which are contained in (F, w). These Baer subplanes are projected from r onto Baer subplanes of the same Baer subgeometry π_3 contained in F'. If we include the, at most, $q/16$ points of F on secants through r, we are considering in $\pi'_0, \ldots$, and $\pi''_0, \ldots$, at most $q\sqrt{q} + q + \sqrt{q} + 1 + q/16$ distinct points of F.

So the two sets of $3\sqrt{q}/4$ planes $\pi'_0, \ldots$, and $\pi''_0, \ldots$, through L or M intersect in at least $x = 2 \cdot 3q\sqrt{q}/4 + 2\sqrt{q} + 2 - q\sqrt{q} - q - \sqrt{q} - 1 - q/16 = q\sqrt{q}/2 - 17q/16 + \sqrt{q} + 1$ points of F, which are projected from r onto points of π_3.

Part 6. We do not consider the $q + \sqrt{q} + 1$ points of the Baer subplane in $\langle L, M\rangle$ which are projected from r onto π_3, and we also do not consider the, at most $q/16$, points of F on secants through r.

Then at least $x - q - \sqrt{q} - 1 - q/16 = q\sqrt{q}/2 - 17q/8$ distinct points of F, lying on tangent lines to F through r, lie in Baer subplanes of F in planes through L and in planes through M.

We considered at most $\sqrt{q}+1$ planes through L or M, one of which is $\langle L, M\rangle$, but this plane was already excluded. So at least one of those planes Π' through L contains at least $(q\sqrt{q}/2 - 17q/8)/\sqrt{q} > q/2 - 17\sqrt{q}/8$ points of F, projected from r onto points of π_3. Let π_0 be the Baer subplane of Π', contained in F, and projected from r onto a Baer subplane of π_3.

Part 7. This Baer subplane π_0 and the Baer subline of F on M define a unique Baer subgeometry $D \equiv PG(3, \sqrt{q})$. We show that D is contained in F.

We use the planes $\pi''_0, \ldots$, through M. There are at most $\sqrt{q}$ of them, if we do not consider the plane $\langle L, M\rangle$. They intersect π_0 either in s, in two points including s, or in a Baer subline through s.

We want to find a lower bound on the number of planes $\pi''_0, \ldots$, intersecting π_0 in a Baer subline. They all contain s. We first subtract $\sqrt{q}$ from $q/2 - 17\sqrt{q}/8$ to express that they might contain a second point of π_0. If they contain at least one extra point of π_0, then they contain $\sqrt{q} - 1$ other points of π_0. So, at least $(q/2 - 25\sqrt{q}/8)/(\sqrt{q} - 1) > \sqrt{q}/2 - 29/8$ planes $\pi''_0, \ldots$, through M share a Baer subline with π_0.

The Baer subplanes of these latter planes $\pi_0'', \ldots$, sharing a Baer subline with M and a Baer subline with π_0, are completely contained in D. Also the Baer subplane of F in $\langle L, M\rangle$ is contained in D. So D contains already at least $\sqrt{q}q/2 - 21q/8 + \sqrt{q} + 1$ points of F. Moreover, at least $1 + \sqrt{q}(\sqrt{q}/2 - 21/8) = q/2 - 21\sqrt{q}/8 + 1$ Baer sublines of D through s lie in F.

Part 8. The preceding result implies that every plane Π' of D, not passing through s, contains at least $q/2 - 21\sqrt{q}/8 + 1$ points of F. This implies that they intersect F in at least a 1-fold blocking set since this number is larger than δ (Corollary 1.13). If the Baer subplane $\Pi' \cap D$ is not contained in F, let t be a point of $\Pi' \cap D$ not belonging to F.

We consider the q planes of D through t, but not through s. They each contain $q - \sqrt{q}$ lines through t, which are not lines of D. None of these latter lines is doubly counted, and contains at least one point of F.

So $|F| \geq (q - \sqrt{q})q$, which is false.

So D is contained in (F, w).

Part 9. The arguments of the proof of [7, Theorem 2.1] imply that, by reducing the weight of the points of D by one, a new $\{(\delta - \sqrt{q} - 1)(q + 1), \delta - \sqrt{q} - 1; 4, q\}$-minihyper is obtained.

Proceeding as in the preceding parts, it follows that (F, w) is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$. $\square$

Theorem 4.3 *Let (F, w) be a weighted $\{\delta(q + 1), \delta; k - 1, q = p^2\}$-minihyper, with p prime, $p \geq 11$, $k \geq 4$, and $\delta \leq (q - 1)/4$.*

Then (F, w) is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$.

Proof This is proven by induction on k, using the cases $k = 4$ and $k = 5$ as induction hypothesis.

The arguments for $k > 5$ are easier than for $k = 5$ since for $k > 5$, it is possible to find a point r only lying on tangent lines to F. $\square$

Remark 4.4 As indicated in the beginning of Section 4, the proofs were given for the case $q = p^2$, p prime, since in such planes $PG(2, q)$, the smallest minimal blocking sets different from a line and different from a Baer subplane have size $3(q + 1)/2$ (Corollary 1.9).

In planes $PG(2, q = p^h)$, p prime, h even, $h > 2$, they have size at most $q + q/p + 1$.

All the arguments in the preceding lemmas are still valid if we impose the condition

$$\delta + \frac{\delta^2}{q} + \frac{2\delta^2 - \delta}{q^2} + \frac{\delta^2 - \delta}{q^3} < 1 + \epsilon,$$

with $q + 1 + \epsilon$ the size of the smallest minimal blocking set in $PG(2, q)$ different from a line and Baer subplane.

Namely, a crucial calculation is done in Part 3 of the proof of Theorem 4.2.

We consider a point r of $PG(4, q)$ lying on the smallest possible number of secants to F. The secants to a set of $\delta(q + 1)$ points contain at most

$$\frac{\delta^2 q^3 + (3\delta^2 - \delta)q^2 + (3\delta^2 - 2\delta)q + \delta^2 - \delta}{2}$$

points.

Since $|PG(4,q) \setminus F| > q^4 + q^3$, there is a point r in $PG(4,q) \setminus F$ lying on less than $(\delta^2/q + (2\delta^2 - \delta)/q^2 + (\delta^2 - \delta)/q^3)/2$ secants to F. These secants contain less than $\delta^2/q + (2\delta^2 - \delta)/q^2 + (\delta^2 - \delta)/q^3$ points of F.

We needed this number in Part 3 where we consider a solid through r containing at most $q + \delta + \delta^2/q + (2\delta^2 - \delta)/q^2 + (\delta^2 - \delta)/q^3$ distinct points of F. We must be sure that this solid, which intersects F in a 1-fold blocking set, contains a Baer subplane contained in F. We are sure of this if the size of this intersection is smaller than the size $q + 1 + \epsilon$ of the smallest blocking set in $PG(2,q)$ not containing a line or a Baer subplane (Theorem 1.19).

A second crucial calculation is done in Part 8 of the proof of Theorem 4.2. We wish to be sure that this plane Π shares more than δ points with F. To be sure of this, we also impose that $\delta \leq (q-1)/4$ and that $q \geq 11^2$.

This leads to the following theorem.

Theorem 4.5 *Let (F,w) be a weighted $\{\delta(q+1), \delta; k-1, q\}$-minihyper, with $q = p^{2h}$, p prime, $h > 1$, $k \geq 4$, $\delta \leq (q-1)/4$, $q \geq 11^2$, and*

$$\delta + \frac{\delta^2}{q} + \frac{2\delta^2 - \delta}{q^2} + \frac{\delta^2 - \delta}{q^3} < 1 + \epsilon,$$

with $q + 1 + \epsilon$ the size of the smallest minimal blocking set in $PG(2,q)$ different from a line and Baer subplane.

Then (F,w) is a sum of lines and of Baer subgeometries $PG(3, \sqrt{q})$.

Remark 4.6 It is known that every plane $PG(2,q)$, $q = p^{2h}$, $h > 1$, p prime, contains a minimal blocking set of size $q + q/p + 1$, not containing a line or Baer subplane.

Moreover, from the results of Sziklai and Szőnyi [25, 26], the smallest minimal blocking set B in $PG(2,q)$, $q = p^{2h}$, $h > 1$, p prime, not containing a line or a Baer subplane, has exponent E where $E|(2h)$, $E < h$, and it then follows from Blokhuis [1] that its size satisfies

$$|B| \geq q + 1 + p^E \left\lceil \frac{q/p^E + 1}{p^E + 1} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

5 Weighted $\{\delta(p^3 + 1), \delta; 3, p^3\}$-minihypers

We now improve Theorem 1.4 by deleting the upper bound on the excess e.

We immediately present the results for p square since a lot of the techniques of [5] can be repeated.

Let (F,w) be a $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper, p square, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h \geq 2$ even, $\delta \leq 2p^2 - 4p$.

Consider a point r of (F, w) of minimal weight α. If we consider the $\alpha q+\delta$ rich planes through r, then they form a dual blocking set in the quotient geometry of r (Lemma 1.16). As $m_1(\pi) \geq \alpha$ for every rich plane since the minimal weight of the points of (F, w) is equal to α, there are at most $(\alpha q + \delta)/\alpha \leq q + \delta$ distinct rich planes through r, forming this 1-fold dual blocking set, which we will denote by B_r^D. This dual minimal blocking set B_r^D contains a dual minimal blocking set E.

Suppose that there is a point r of (F, w) of minimal weight α for which the dual blocking set B_r^D contains a Baer subplane E. Construct the cone $\mathcal{B}$ with vertex r and with base E. A plane through r containing a line of E is called a *plane* of $\mathcal{B}$.

Through r, there are $\alpha p^3 + \delta$ rich planes. Since every rich plane π satisfies $m_1(\pi) \geq \alpha$, there are at most $\alpha p^3 + \delta - \alpha(p^3 + p^{3/2} + 1) = \delta - \alpha p^{3/2} - \alpha$ planes through r for which $m_1(\pi) > \alpha$. The planes of $\mathcal{B}$ for which $m_1(\pi) = \alpha$ intersect F in a 1-fold blocking set, where we do not consider the weights of the points of (F, w) in this plane (Theorem 1.14).

Lemma 5.1 *A plane π of $\mathcal{B}$ intersecting F in a 1-fold blocking set contains a Baer subplane B_0 completely contained in $\mathcal{B}$.*

Proof The arguments of [4, Lemma 2.9] show that $\pi \cap F$ must contain a Baer subplane B_0. The arguments of the proof of Lemma 3.2 again show that B_0 lies in $\mathcal{B}$. □

Lemma 5.2 *A point r of minimal weight of (F, w) for which the dual blocking set B_r^D contains a Baer subplane is contained in a unique Baer subgeometry $PG(3, p^{3/2})$ completely contained in (F, w).*

Proof The arguments of the proof of Lemma 3.3 can be repeated. □

Lemma 5.3 *Let r be a point of minimal weight α of (F, w) for which the dual blocking set of rich planes through r contains a dual minimal blocking set of size $p^3 + p^2(+p) + 1$, then r has weight one.*

Proof The sum of the weights $m_1(\pi)$ of the rich planes π through r is equal to $\alpha p^3 + \delta$ (Theorem 1.15).

Since every rich plane π satisfies $m_1(\pi) \geq \alpha$, the number of distinct rich planes through r is at most $p^3 + (2p^2 - 4p)/\alpha$.

If $\alpha \geq 2$, then this upper bound is at most $p^3 + p^2 - 2p$, but this is false since it should be at least $p^3 + p^2 + 1$. □

Theorem 5.4 *A $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper (F, w), $p = p_0^h$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, is either:*
(1) a sum of lines, (projected) $PG(3, p^{3/2})$ if p is a square, and of at most one projected $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines, (projected) $PG(3, p^{3/2})$ if p is a square, and of a $\{(p^2 + p)(p^3 + 1), p^2 + p; 3, p^3\}$-minihyper $(\Omega, w) \setminus N$, where Ω is a projected $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Proof Let r be a point of minimal weight α of (F,w). Let B_r^D be the dual blocking set of rich planes through r. If B_r^D contains a dual minimal blocking set equal to a point s, then rs lies in p^3+1 rich planes. Hence, rs is contained in F (Corollary 1.17).

Then Lemma 1.18 implies that $(F,w)\setminus rs$ is a $\{(\delta-1)(p^3+1),\delta-1;3,p^3\}$-minihyper.

If B_r^D contains a Baer subplane, then r lies in a Baer subgeometry $PG(3,p^{3/2})$ completely contained in F. The proof of [7, Theorem 2.1] implies that, by reducing the weights of the points of this subgeometry $PG(3,p^{3/2})$ by one, a new $\{(\delta-p^{3/2}-1)(p^3+1),\delta-p^{3/2}-1;3,p^3\}$-minihyper is obtained.

If B_r^D contains a dual minimal blocking set of size $p^3+p^2(+p)+1$, then r has weight one. Then r lies in p^3+2p^2-4p rich planes, of which at most p^2-4p-1 intersect (F,w) in a multiple blocking set.

The arguments of [5] can now be used to prove that (F,w) contains a weighted minihyper equal to a projected subgeometry $(\Omega,w')\equiv PG(5,p)$ or to a $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihyper $(\Omega,w')\setminus N$, with (Ω,w') a projected subgeometry $PG(5,p)$, projected from a line L for which $\dim\langle L,L^p,L^{p^2}\rangle=3$, and with N the line contained in Ω. These weighted minihypers (Ω,w') are equal to the weighted $\{(p^2+p+1)(p^3+1),p^2+p+1;3,p^3\}$-minihypers and weighted $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihypers described in Section 2.

Reducing the weight of every point s of Ω by $w'(s)$, or by its weight in $(\Omega,w')\setminus N$ in the latter case, a new weighted $\{(\delta-p^2-p(-1))(p^3+1),\delta-p^2-p(-1);3,p^3\}$-minihyper is obtained.

This shows that (F,w) is a sum of the objects described in the statement of the theorem. □

6 Weighted $\{\delta(p^3+1),\delta;k-1,p^3\}$-minihypers

We now present the proof for the following theorem.

Theorem 6.1 *A $\{\delta(p^3+1),\delta;k-1,p^3\}$-minihyper (F,w), $k\geq 4$, $p=p_0^h$, p_0 prime, $h\geq 1$, $p_0\geq 7$, $p\geq 9$, $\delta\leq 2p^2-4p$, is either:*
(1) a sum of lines, (projected) subgeometries $PG(3,p^{3/2})$ if p is a square, and of at most one (projected) subgeometry $PG(5,p)$,
(2) a sum of lines, (projected) subgeometries $PG(3,p^{3/2})$ if p is a square, and of a $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihyper $(\Omega,w)\setminus N$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L,L^p,L^{p^2}\rangle=3$, and where N is the line contained in Ω.

We first prove the theorem for $k=5$. Let (F,w) be a minihyper in $PG(4,p^3)$ satisfying the conditions of the preceding theorem.

It is possible to find a point $r\notin F$ lying on less than $2p$ secants to F [6]. These latter secants to F through r contain less than $4p$ distinct points of F.

We project F from r onto a solid not passing through r. Then this projection is a weighted $\{\delta(p^3+1),\delta;3,p^3\}$-minihyper (F',w').

Lemma 6.2 *There is a bijective relation between the lines contained in* (F, w) *and the lines contained in* (F', w').

Proof The arguments of the proof of [6, Lemma 3.1] can be repeated. $\square$

Since, by Lemma 1.18, it is possible to reduce the weights of the points of a line contained in (F, w) by one, to obtain a weighted $\{(\delta-1)(p^3+1), \delta-1; 4, p^3\}$-minihyper, we now assume that there are no lines contained in (F, w).

This implies that it is possible to assume that (F', w') is a sum of Baer subgeometries $PG(3, p^{3/2})$, and of at most one projected subgeometry $PG(5, p)$ or projected $PG(5, p)$ from which one line N was omitted.

We first prove the following lemma.

Lemma 6.3 *A Baer subgeometry* $PG(3, p^{3/2})$ *and a projected subgeometry* $PG(5, p)$ $\equiv (\Omega, w)$ *in* $PG(3, p^3)$ *share at most* $p^{1/2}p^3 + 2p^3 + 2p^2 + 2p^{3/2} + p + p^{1/2} + 2$ *distinct points.*

Proof In [17], it is shown that a subline $PG(1, p^{3/2})$ and a subline $PG(1, p)$ share at most $p^{1/2} + 1$ points, and it is also shown that a subline $PG(1, p^{3/2})$ and a $(p^2(+p) + 1)$-set share at most $p + p^{1/2} + 1$ points.

Consider a projected subgeometry $PG(5, p) \equiv (\Omega, w)$ in $PG(3, p^3)$, and let r be a point not belonging to (Ω, w). We use the descriptions of the projected subgeometries $PG(5, p)$ of Section 2.

If $\dim\langle L, L^p, L^{p^2}\rangle = 5$, then r can belong to at most one $(p^2 + p + 1)$-secant since no two (p^2+p+1)-secants to Ω are coplanar. If $\dim\langle L, L^p, L^{p^2}\rangle = 4$, and r does not belong to the plane of $PG(3, p^3)$ containing the projected $PG(4, p) \equiv \mathcal{P}$, then r lies on at most one $(p^2 + 1)$-secant since all $(p^2 + 1)$-secants pass through m. This latter point r does not lie on a $(p^2 + p + 1)$-secant to (Ω, w) since they all lie in the plane containing $\mathcal{P}$.

If $\dim\langle L, L^p, L^{p^2}\rangle = 3$, then r lies on at most $p+1$ different (p^2+1)-secants through $r_0, \ldots, r_p$.

Let r be a point of a Baer subgeometry $PG(3, p^{3/2}) \setminus (\Omega, w)$. Consider all $p^3 + p^{3/2} + 1$ Baer sublines of $PG(3, p^{3/2})$ through r. They all contain at most $p^{1/2} + 1$ points of (Ω, w), except for maybe $p + 1$ lines containing $p + p^{1/2} + 1$ points of (Ω, w). So we find an upper bound of $(p^3+p^{3/2}+1)(p^{1/2}+1)+(p+1)p$.

We also add $p^3 + p^{3/2} + 1$ to exclude the plane containing the projection of the special $PG(4, p) \equiv \mathcal{P}$ when $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

This gives the upper bound $p^{1/2}p^3 + 2p^3 + 2p^2 + 2p^{3/2} + p + p^{1/2} + 2$ on the size for $PG(3, p^{3/2}) \cap (\Omega, w)$. $\square$

Lemma 6.4 *Let* (F', w') *be the weighted minihyper in* $PG(3, p^3)$ *which is the projection of* (F, w) *from* r*. Assume that* (F', w') *contains a Baer subgeometry* $\pi_3 = PG(3, p^{3/2})$*, and a projected subgeometry* $(\Omega, w'') \equiv PG(5, p)$ *or a* $\{(p^2 + p)(p^3+1), p^2+p; 3, p^3\}$*-minihyper* $(\Omega, w'')\setminus N$*, with* (Ω, w'') *a projected* $PG(5, p)$ *in* $PG(3, p^3)$ *containing the line* N.

Then it is possible to find a point $s' \in \pi_3 \setminus (\Omega, w'')$ lying on a line L' containing a Baer subline of π_3 through s', not containing a Baer subline from an other $PG(3, p^{3/2})$ in (F', w'), and sharing at most a subline $PG(1, p)$ with (Ω, w'').

Proof Consider a point s' of π_3 not lying in (Ω, w'') and not lying in an other Baer subgeometry $PG(3, p^{3/2})$ of (F', w').

Since s' does not lie in any other $PG(3, p^{3/2})$ in (F', w'), every other Baer subgeometry $PG(3, p^{3/2})$ in (F', w') is intersected by at most one line through s' in a Baer subline. There are at most $2p$ Baer subgeometries $PG(3, p^{3/2})$ contained in (F', w') since $\delta \leq 2p^2 - 4p$. So we need to exclude at most $2p$ such lines through s'.

Similarly, s' lies on at most $p + 1$ distinct $(p^2(+p) + 1)$-sets to (Ω, w'') in (F', w') if we do not select s' in the plane containing the projected $PG(4, p) \equiv \mathcal{P}$ of Case 2.

So, we need to exclude at most $3p + 1$ Baer sublines of $PG(3, p^{3/2})$ through s'. The desired line through s' exists. $\square$

We now will show that there is a bijective relation between the Baer subgeometries $PG(3, p^{3/2})$ contained in (F', w') and the Baer subgeometries contained in (F, w).

Let α' be the minimal weight of the points of (F', w') lying in a Baer subgeometry $PG(3, p^{3/2})$ contained in (F', w'), and also lying on a tangent line to F passing through r. This latter point exists, as the following arguments prove.

We start by considering a point s' of (F', w') of minimal weight α' lying in a Baer subgeometry $\pi_3 \equiv PG(3, p^{3/2})$ contained in (F', w'). Such a point does not lie in a second Baer subgeometry $PG(3, p^{3/2})$ contained in (F', w').

In π_3, there are at least $p^{3/2}p^3 + p^3 + p^{3/2} + 1 - (p^{1/2}p^3 + 2p^3 + 2p^2 + 2p^{3/2} + p + p^{1/2} + 2) - 2p(p^3 + p^{3/2} + 2)$ points of weight α' since two distinct Baer subgeometries $PG(3, p^{3/2})$ share at most $p^3 + p^{3/2} + 2$ distinct points [24], and since a Baer subgeometry $PG(3, p^{3/2})$ and a projected $PG(5, p)$ share at most $p^{1/2}p^3 + 2p^3 + 2p^2 + 2p^{3/2} + p + p^{1/2} + 2$ distinct points (Lemma 6.3).

Since r lies on at most $2p$ secants to F, the desired point s' of minimal weight α' and lying on a tangent to F through r exists.

Consider such a line L' through s' satisfying the conditions of the preceding lemma. This line L' contains at most $p^{3/2} + 1 + 2p + p + 1 = p^{3/2} + 3p + 2$ distinct points of (F', w').

Consider the $p^{3/2} + 1$ planes through L' intersecting π_3 in a Baer subplane. At most $2p$ of them intersect an other $PG(3, p^{3/2})$ of (F', w') in a Baer subplane.

Since $|L' \cap (\Omega, w)| \leq p + 1$, at most $p + 1$ planes through L' intersect (Ω, w) in at least a 1-fold blocking set.

So at least $p^{3/2} + 1 - 2p - p - 1 = p^{3/2} - 3p$ planes through L' intersect π_3 in a Baer subplane, but do not intersect any other $PG(3, p^{3/2})$ of (F', w') in a Baer subplane, and intersect the projected $PG(5, p)$ in (F', w') in at most $p^2 + p + 1$ points.

So such a plane through L' shares at most $p^3+\delta$ distinct points with (F', w'). So the solid generated by this plane and r contains at most $p^3+\delta+4p$ distinct points of F. Hence, this solid intersects F in a 1-fold blocking set (Corollary 1.13). This latter blocking set must contain a Baer subplane B_0 since its projection from r contains a Baer subplane (Theorem 1.19).

We denote by L the line of B_0 which is projected from r onto the Baer subline of L' contained in (F', w').

Repeat this argument for the $p^{3/2}-3p$ planes through L', described above. These arguments show that there are at least $p^{3/2}-3p$ planes of $PG(4,p^3)$ through a line L all containing a Baer subplane of (F, w). These latter Baer subplanes all share the same Baer subline on L and contain at least $(p^{3/2}-3p)p^3+p^{3/2}+1$ points of F.

Select a second line M' through s' playing the role of L'. This gives a second line M playing the role of L. It is possible to select M in such a way that it lies in one of those $p^{3/2}-3p$ planes through L sharing a Baer subplane with F.

We are now reduced to the situation of the proof of Theorem 4.2. We find a Baer subgeometry $PG(3,p^{3/2})$ completely contained in (F, w). It is possible to reduce the weights of the points of this latter $PG(3,p^{3/2})$ contained in (F, w) by one to obtain a weighted $\{(\delta-p^{3/2}-1)(p^3+1), \delta-p^{3/2}-1; 4, p^3\}$-minihyper [7, Theorem 2.1].

It is now possible to assume that there are no lines and no Baer subgeometries $PG(3,p^{3/2})$ contained in (F, w). This implies that (F', w') is equal to a projected subgeometry $PG(5,p)$ or equal to $(\Omega', w')\setminus N'$, with (Ω', w') a projected $PG(5,p)$ in $PG(3,p^3)$, and with N' the line contained in (Ω', w').

We are now reduced to the situation of [6]. The arguments of [6] show that (F, w) equals a projected subgeometry $PG(5,p)$ in $PG(4,p^3)$, or a projected subgeometry $(\Omega, w)\setminus N$, with (Ω, w) a projected $PG(5,p)$ in $PG(3,p^3)$, and with N the line contained in (Ω, w).

This proves Theorem 6.1 for $k=5$.

By induction on k, the theorem is proven for arbitrary dimension $k>5$.

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