On a class of thin near polygons admitting a polygonal triple

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Abstract

We introduce and study a family of thin near polygons, each member of which has a line spread S and a set $\mathcal{Q}$ of quads such that the point-line geometry formed by the lines of S and the quads of $\mathcal{Q}$ is itself also a near polygon. We study the automorphism groups of these thin near polygons and classify all convex subspaces. Special attention will be given to a class of thin near polygons related to quadrics of projective spaces, of which we show that all its members have a regular set of convex subspaces.

Keywords: near polygon, generalized quadrangle, line spread, polygonal triple, convex subspaces
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1 Introduction

The paper [6] arose from the observation that many near polygons $\mathcal{A}$ (including some recently discovered ones) have a common feature, namely they posses a line spread S and a set $\mathcal{Q}$ of quads such that the point-line geometry $\mathcal{S}$ defined by the lines of S and the quads of $\mathcal{Q}$ is itself also a near polygon. In [6], such triples $(\mathcal{A}, S, \mathcal{Q})$ were called *polygonal triples* and the corresponding near polygons $\mathcal{S}$ their *associated near polygons*. In [6], many constructions, properties and characterization results for polygonal triples were obtained.

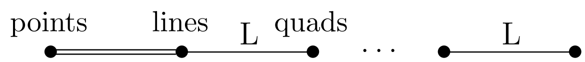
The research of the present paper arose from the study of polygonal triples $(\mathcal{A}, S, \mathcal{Q})$ for which the near polygon $\mathcal{A}$ is *thin*, meaning that all lines of $\mathcal{A}$ are incident with precisely two points. For every near polygon $\mathcal{S}$, we construct a thin near polygon $\overline{\mathcal{S}}$, a line spread S in $\overline{\mathcal{S}}$ and a set $\mathcal{Q}$ of quads of $\overline{\mathcal{S}}$ such that $(\overline{\mathcal{S}}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is isomorphic to $\mathcal{S}$. We moreover show that if $(\mathcal{A}, S', \mathcal{Q}')$ is a polygonal triple with $\mathcal{A}$ thin whose associated near polygon is isomorphic to $\mathcal{S}$, then there is an isomorphism from $\mathcal{A}$ to $\overline{\mathcal{S}}$ mapping S' to S and $\mathcal{Q}'$ to $\mathcal{Q}$.

We perform a study of the family of thin near polygons $\overline{\mathcal{S}}$ that arise from near polygons $\mathcal{S}$. In particular, we determine all convex subspaces of these thin near polygons and study their automorphism groups. If $\mathcal{S}$ is isomorphic to the parabolic dual polar space $DQ(2n, \mathbb{F})$, then $\overline{\mathcal{S}}$ is isomorphic to the hyperbolic dual polar space $DQ^+(2n+1, \mathbb{F})$. The hyperbolic dual polar spaces seem to be the standard examples of the family, but we will construct many other examples, several of which satisfy the following two properties:

(i) every two points are contained in a unique convex subspace whose diameter is equal to the distance between these points;

(ii) for every convex subspace F and every $x \in F$, there exists a point $y \in F$ whose distance to x is equal to the diameter of F.

Near polygons satisfying the properties (i) and (ii) are said to have a *regular set of convex subspaces*. Besides dual polar spaces and dense near polygons, not so many near polygons were known to have such a regular set of convex subspaces (one had the folded n-cubes with n even, incidence graphs of biplanes, the coset graph of the extended binary Golay code, and some near polygons that arise from these via direct products and glueing). The properties (i) and (ii) are of interest as they imply that the convex subspaces define a diagram geometry [2, 7] of type



which is a very useful fact for their study. The convex subspace of diameter 2 are called the *quads* and the points and lines contained in it define a generalized quadrangle [8]. In the diagram, L stands for the class of linear spaces. In the case of dual polar spaces, these linear spaces are even projective planes.

There are no known existence results for convex subspaces in general thin near polygons. Even under the assumption that every two points at distance 2 have at least two common neighbours, it is in general false that two points at distance 2 are contained in a quad. A quite different situation occurs for dense near polygons. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. In a dense near polygon, Yanushka's lemma [9, Proposition 2.5] implies that every two points at distance 2 are contained in a unique quad. Sometimes the mere fact that quads exist through any two points at distance 2 is sufficient to show that any two points at distance $\delta \geq 3$ are also contained in a unique convex subspace of diameter δ. This is the case if lines have at least three points (Brouwer and Wilbrink [1, Theorem 4]) or when quads are assumed to be classical (Cameron [3]). A similar situation does no longer hold for general (thin) near polygons, but only a few counter examples are known to exist. The present paper shows that also the near octagons $\overline{\mathbb{E}_1}$ and $\overline{\mathbb{E}_2}$ have the property that every two points at distance 2 are contained in a unique quad, but that not every two points at distance 3 are contained in a convex subspace of diameter 3. These near octagons arise from the near hexagons $\mathbb{E}_1$ and $\mathbb{E}_2$ that are respectively associated with the extended ternary Golay code and the Witt design $S(5, 8, 24)$.

2 Preliminaries

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near 2d-gon* if the following properties hold:

(1) For every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x with respect to the distance in the collinearity graph Γ of $\mathcal{S}$.

(2) The diameter of Γ is equal to d.

A near 0-gon is a point and a near 2-gon is a line. A near quadrangle with at least two lines through each point is called a *generalized quadrangle*.

The distance between two points x and y of a near polygon $\mathcal{S}$ will be denoted by $\mathrm{d}_{\mathcal{S}}(x, y)$ or shortly by $\mathrm{d}(x, y)$ if no confusion can arise. By [1, Lemma 1], precisely one of the following two cases occurs for two lines K and L of a near polygon $\mathcal{S}$:

(1) There exist unique points $k^* \in K$ and $l^* \in L$ such that $\mathrm{d}(k, l) = \mathrm{d}(k, k^*) + \mathrm{d}(k^*, l^*) + \mathrm{d}(l^*, l)$ for all $k \in K$ and all $l \in L$.

(2) If $m = \mathrm{d}(K, L) := \min\{\mathrm{d}(k, l) \,|\, k \in K \text{ and } l \in L\}$, then every point of K has distance m to precisely one point of L and every point of L has distance m to precisely one point of K.

If case (2) occurs then the lines K and L are called *parallel*.

A set S of lines of a near polygon $\mathcal{S}$ is called a *line spread* if every point of $\mathcal{S}$ is incident with a unique line of S. If the line spread consists of mutually parallel lines, then it is called *admissible*.

A set X of points of a near polygon $\mathcal{S}$ is called a *subspace* if every line that has two points in X has all its points in X. For every nonempty subspace X, we denote by $\widetilde{X}$ the point-line geometry whose points are the elements of X and whose lines are the lines of $\mathcal{S}$ that have all their points in X (natural incidence). A set X of points of $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of X is also contained in X. If X is a nonempty convex subspace of a near polygon $\mathcal{S}$, then $\widetilde{X}$ itself is also a near polygon. If $\widetilde{X}$ is a generalized quadrangle then the convex subspace X will be called a *quad*.

If x and y are two points of $\mathcal{S}$ at distance 2 from each other having two neighbours z_1 and z_2 such that at least one of the lines xz_1, xz_2, z_1y, z_2y has at least three points, then Yanushka's lemma [9, Proposition 2.5] implies that x and y are contained in a unique quad. We note that if there is a quad through two points at distance 2, then this quad has to be unique.

A convex subspace F of a near polygon $\mathcal{S}$ is called *classical (in $\mathcal{S}$)* if for every point x of $\mathcal{S}$, there exists a unique point $\pi_F(x) \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point $y \in F$. Every line of near polygon is classical.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near $(2d + 2)$-gon, $d \geq 1$, having a line spread S and a family $\mathcal{Q}$ of quads for which the following hold:

(PT1) For every point x of $\mathcal{S}$, the quads of $\mathcal{Q}$ through x all contain the unique line L_x of S through x and partition the set of lines through x distinct from L_x.

(PT2) The point-line geometry $\mathcal{S}'$ with point set S, line set $\mathcal{Q}$ and natural incidence (i.e. containment) if a near $2d$-gon.

We also say that $\mathcal{S}'$ is the near polygon *associated* with the polygonal triple $(\mathcal{S}, S, \mathcal{Q})$. If $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple, then $\mathcal{Q}$ is uniquely determined by $\mathcal{S}$ and S as its consists of all quads of $\mathcal{S}$ containing a line of S. Polygonal triples were introduced and studied in [6]. We mention here three properties of polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ that we need later:

(i) every line of $\mathcal{S}$ not belonging to S is contained in a unique element of $\mathcal{Q}$;

(ii) every quad of $\mathcal{Q}$ is classical in $\mathcal{S}$.

(iii) If $L_1, L_2 \in S$, then the distance between L_1 and L_2 in the associated near polygon is equal to $\mathrm{d}_{\mathcal{S}}(L_1, L_2)$.

If $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ are two near polygons, then another near polygon $\mathcal{S}_1 \times \mathcal{S}_2$ with point set $\mathcal{P}_1 \times \mathcal{P}_2$ can be constructed, see e.g. Section 6.6 of [5]. Here, two distinct points (x_1, x_2) and (y_1, y_2) are adjacent whenever there exists an $i \in \{1, 2\}$ such that $x_i = y_i$ and $x_{3-i} \sim y_{3-i}$ in the near polygon $\mathcal{S}_{3-i}$. The near polygon $\mathcal{S}_1 \times \mathcal{S}_2$ is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$. If the near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$ have admissible spreads S_1 and S_2 that are compatible (in some sense), then by [4, Theorem 1] it is possible to construct other near polygons from it by considering multiple copies of $\mathcal{S}_1$ and $\mathcal{S}_2$, and "glueing" along the lines of S_1 and S_2 (see [4] for more details). Such a *glued near polygon* will be denoted by $\mathcal{S}_1 \otimes \mathcal{S}_2$. In the case that $\mathcal{S}_1$ and $\mathcal{S}_2$ are thin, then any pair of admissible spreads is always compatible.

Suppose Π is a polar space of rank $r \geq 1$ in the sense of Tits [11, Chapter 7]. Then with Π, there is associated a dual polar space Δ. This is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces, with incidence being reverse containment. There exists a bijective correspondence between the possibly empty singular subspaces of Π and the nonempty convex subspaces of Δ: if α s a singular subspace of Π, then the set F_α of all maximal singular subspaces containing α is a convex subspace of diameter $r - 1 - \dim(\alpha)$. Every two points of Δ at distance δ are contained in a unique convex subspace of diameter δ. If F is a convex subspace of diameter $\delta \geq 1$ of Δ, then by Theorem 8.6 of [5], $\widetilde{F}$ is a dual polar space of rank δ. We collect a number of known properties of (dual) polar spaces. For proofs and more background information, we refer to Chapter 8 of [5]. The following lemma is precisely Theorem 8.2 of [5].

Lemma 2.1 *Let Π be a polar space of rank $r \geq 1$, and let α, β be two maximal singular subspaces of Π. Then the distance between α and β in the dual polar space Δ associated with Π is equal to $r - 1 - \dim(\alpha \cap \beta)$.*

Lemma 2.2 *Let Π be a polar space of rank $r \geq 1$, and let α, β, γ be maximal singular subspaces such that γ is on a shortest path from α to β in the dual polar space Δ associated with Π. If $\gamma_1 = \gamma \cap \alpha$ and $\gamma_2 = \gamma \cap \beta$, then $\gamma = \langle \gamma_1, \gamma_2 \rangle$.*

Proof. The set of all maximal singular subspaces containing $\alpha \cap \beta$ is a convex subspace containing α and β, and so we have $\alpha \cap \beta \subseteq \gamma$. Put $\dim(\gamma_1) = \dim(\alpha \cap \beta) + s_1$ and $\dim(\gamma_2) = \dim(\alpha \cap \beta) + s_2$. Then $\dim(\langle \gamma_1, \gamma_2 \rangle) = \dim(\alpha \cap \beta) + s_1 + s_2$. We have $r-1-\dim(\alpha\cap\beta) = \mathrm{d}_\Delta(\alpha,\beta) = \mathrm{d}_\Delta(\alpha,\gamma)+\mathrm{d}_\Delta(\gamma,\beta) = r-1-\dim(\gamma_1)+r-1-\dim(\gamma_2) = 2(r-1)-2\dim(\alpha\cap\beta)-s_1-s_2$, i.e. $\dim(\langle\gamma_1,\gamma_2\rangle) = \dim(\alpha\cap\beta)+s_1+s_2 = r-1 = \dim(\gamma)$. This implies that $\gamma = \langle \gamma_1, \gamma_2 \rangle$. ∎

For a proof of the following lemma, see Theorems 8.4, 8.6 and 8.7 of [5].

Lemma 2.3 *Let F be a convex subspace of diameter δ in a dual polar space Δ. Then*

(1) *F is classical in Δ;*

(2) *If u and v are two points of F at distance at most $\delta - 1$ from each other, then there is a line of F through v containing a point at distance $d(u,v)+1$ from u.*

A (dual) polar space is said to be of *quadratic type* if it arises from a nonsingular quadric in the following sense. Suppose Q is a nonsingular quadric of Witt index $r \geq 1$ in a projective space Σ over a field $\mathbb{F}$. The maximal (projective) dimension of a subspace of Σ contained in Q is then equal to $r-1$. Subspaces of maximal dimension $r-1$ are also called *generators*. The set of points of Q together with the subspaces contained in Q define a polar space. The dimension $\dim(\Sigma)$ of the projective space Σ is at least equal to $2r-1$. If $\dim(\Sigma) = 2r-1$, then Q has equation $X_1X_2+X_3X_4+\cdots+X_{2r-1}X_{2r} = 0$ with respect to a suitable reference system. Such a quadric is called a *hyperbolic quadric* and often denoted by $Q^+(2r-1,\mathbb{F})$. If $\dim(\Sigma) = 2r$, then Q has equation $X_0^2 + X_1X_2 + X_3X_4 + \cdots + X_{2r-1}X_{2r} = 0$ with respect to a suitable reference system. Such a quadric is called a *parabolic quadric* and often denoted by $Q(2r,\mathbb{F})$.

The dual polar spaces associated with $Q^+(2r-1,\mathbb{F})$ and $Q(2r,\mathbb{F})$ will respectively be denoted by $DQ^+(2r-1,\mathbb{F})$ and $DQ(2r,\mathbb{F})$. The dual polar space $DQ^+(2r-1,\mathbb{F})$ is an example of a thin near polygon.

Suppose Δ is a dual polar space of quadratic type and F is a convex subspace of diameter at least 1 of Δ. Then the corresponding polar space Π arises from some quadric Q in a projective space Σ, and there exists some subspace α in Q such that F consists of all maximal subspaces of Q through α. Let Σ' be a subspace of Σ complementary to α and let Q' be the nonsingular quadric $\Sigma' \cap Q$ of Σ'. By [5, Theorem 8.6], the dual polar space $\widetilde{F}$ is associated with the quotient polar space Π_α. This quotient polar space is obviously isomorphic to the polar space associated with the quadric Q' of Σ'. So, we have:

Lemma 2.4 *Suppose Δ is a dual polar space of quadratic type and F is a convex subspace of diameter at least 1 of Δ. Then $\widetilde{F}$ is also a dual polar space of quadratic type.*

3 Definition and basic properties of the near polygon $\overline{\mathcal{S}}$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near $2d$-gon. For every two points x and y of $\mathcal{S}$, we define $\epsilon_{xy} = +$ if $\mathrm{d}_{\mathcal{S}}(x, y)$ is even and $\epsilon_{xy} = -$ if $\mathrm{d}_{\mathcal{S}}(x, y)$ is odd.

Let $\overline{\Gamma}$ be the graph whose vertices are the elements of the set $\overline{\mathcal{P}} := \mathcal{P} \times \{+, -\}$, with two vertices (x, ϵ_x) and (y, ϵ_y) being adjacent if $\mathrm{d}_{\mathcal{S}}(x, y) \leq 1$ and $\epsilon_y = -\epsilon_x$. The following is then obvious.

Lemma 3.1 *$\overline{\Gamma}$ is a bipartite graph with bipartite parts $\{(x, \epsilon_x) \in \overline{\mathcal{P}} \,|\, \epsilon_x = +\}$ and $\{(y, \epsilon_y) \in \overline{\mathcal{P}} \,|\, \epsilon_y = -\}$.*

For every point x of $\mathcal{S}$, we denote by L_x the edge $\{(x, +), (x, -)\}$ of $\overline{\Gamma}$. The following is obvious.

Lemma 3.2 (a) *Let x, y be two points of $\mathcal{S}$ and $\epsilon \in \{+, -\}$. Then the distance in $\overline{\Gamma}$ between (x, ϵ) and L_y is equal to $d_{\mathcal{S}}(x, y)$. The vertex $(y, \epsilon \cdot \epsilon_{xy})$ is the unique vertex of L_y nearest to (x, ϵ).*

(b) *Let x and y be two distinct points of $\mathcal{S}$, and let $\epsilon_x, \epsilon_y \in \{+, -\}$. Then the distance in $\overline{\Gamma}$ between the vertices (x, ϵ_x) and (y, ϵ_y) is equal to $d_{\mathcal{S}}(x, y)$ if $\epsilon_y = \epsilon_x \cdot \epsilon_{xy}$ and equal to $d_{\mathcal{S}}(x, y) + 1$ otherwise.*

(c) *The maximal distance between two vertices of $\overline{\Gamma}$ is equal to $d + 1$.*

Let $\overline{\mathcal{S}}$ be the point-line geometry whose points are the vertices of $\overline{\Gamma}$ and whose lines are the edges of $\overline{\Gamma}$, with incidence being containment. The following is an immediate consequence of Lemma 3.2(a)+(c).

Corollary 3.3 *$\overline{\mathcal{S}}$ is a near $2(d + 1)$-gon.*

If we define $S := \{L_x \,|\, x \in \mathcal{P}\}$, then we see that S is a line spread of $\overline{\mathcal{S}}$. By Lemma 3.2(b), we see that the following holds.

Corollary 3.4 *If x and y are two points of $\mathcal{S}$, then the lines L_x and L_y are parallel and lie at distance $d_{\mathcal{S}}(x, y)$ from each other in $\overline{\mathcal{S}}$. As a consequence, S is an admissible spread of $\overline{\mathcal{S}}$.*

Proposition 3.5 *If $\mathcal{S}$ is a point, then $\overline{\mathcal{S}}$ is a (thin) line. If $\mathcal{S}$ is a line of size $s + 1$, then $\overline{\mathcal{S}}$ is the dual of an $(s + 1) \times (s + 1)$-grid.*

Proof. The first claim is trivial. The second claim follows from the fact that the graph $\overline{\Gamma}$ is a complete bipartite graph with bipartite parts of size $s + 1$, see Lemma 3.1. ∎

The following proposition, whose proof is straightforward, implies that $\overline{\mathcal{S}}$ has an automorphism group isomorphic to $Aut(\mathcal{S}) \times C_2$.

Proposition 3.6 (1) *The map $\sigma : \mathcal{P} \to \mathcal{P}; (x,\epsilon) \mapsto (x,-\epsilon)$ is an automorphism of $\overline{\mathcal{S}}$.*

(2) *For every automorphism θ of $\mathcal{S}$, the map $\overline{\theta} : \overline{\mathcal{P}} \to \overline{\mathcal{P}} : (x,\epsilon) \mapsto (x^\theta,\epsilon)$ is an automorphism of $\overline{\mathcal{S}}$.*

(3) *The map $\theta \mapsto \overline{\theta}$ defines an isomorphism between $Aut(\mathcal{S})$ and a certain group of automorphisms of $\overline{\mathcal{S}}$.*

(4) *The groups $\langle\sigma\rangle \cong C_2$ and $\overline{Aut(\mathcal{S})} := \{\bar{\theta} \,|\, \theta \in Aut(\mathcal{S})\} \cong Aut(\mathcal{S})$ trivially intersect and commute.*

This automorphism group can be characterized as follows.

Proposition 3.7 *The automorphisms of $\overline{\mathcal{S}}$ stabilizing the spread S are precisely the automorphisms of $\langle\sigma,\overline{Aut(\mathcal{S})}\rangle \cong Aut(\mathcal{S}) \times C_2$.*

Proof. (a) Every automorphism of $\langle\sigma,\overline{Aut(\mathcal{S})}\rangle$ stabilizes the spread S. Indeed, σ fixes each line of S, and if ϕ is an automorphism of $\mathcal{S}$, then the automorphism $\overline{\phi}$ of $\overline{\mathcal{S}}$ maps each line L_x, $x \in \mathcal{P}$, to the line L_{x^ϕ}.

(b) Conversely, suppose that θ is an automorphism of $\overline{\mathcal{S}}$ stabilizing the spread S. Then there exists a permutation ϕ of $\mathcal{P}$ such that $(L_x)^\theta = L_{x^\phi}$ for every point x of $\mathcal{S}$. Corollary 3.4 implies that two points x_1 and x_2 of $\mathcal{S}$ lie at distance 1 if and only if the lines L_{x_1} and L_{x_2} of $\overline{\mathcal{S}}$ lie at distance 1. Now, the lines L_{x_1} and L_{x_2} lie at distance 1 from each other if and only if the lines $L_{x_1}^\theta = L_{x_1^\phi}$ and $L_{x_2}^\theta = L_{x_2^\phi}$ are at distance 1. So, the points x_1 and x_2 are collinear if and only if x_1^ϕ and x_2^ϕ are collinear, showing that ϕ is an automorphism of $\mathcal{S}$.

In order to prove the proposition, it thus suffices to show that for every automorphism ϕ of $\mathcal{S}$ there are at most two automorphisms θ of $\overline{\mathcal{S}}$ such that $(L_x)^\theta = L_{x^\phi}$, $\forall x \in \mathcal{P}$. By part (a) of the proof we know that there are at least two such automorphisms. In fact, by part (a) it suffices to prove the claim in the case that ϕ is the trivial automorphism of $\mathcal{S}$. This is equivalent with showing that there are at most two automorphisms θ of $\overline{\mathcal{S}}$ fixing each line of S.

Let L^* be a fixed line of S. If L is a line of S, then L and L^* are parallel by Corollary 3.4. So, for every point x of L, the image of x under θ is equal to $\pi_L(y)$, where y is the image of the point $\pi_{L^*}(x)$ under θ. As there are at most two possibilities for the restriction $\theta_{|L^*}$ of θ to L^*, there are at most two possibilities for θ itself. ∎

Proposition 3.8 *Let $\mathcal{S}$ be a thin near polygon. Then $\overline{\mathcal{S}}$ is the direct product of $\mathcal{S}$ with a line of size* 2.

Proof. The collinearity graph of any thin near polygon is a bipartite graph. So, we can write $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$ such that there no adjacencies between points belonging to the same $\mathcal{P}_\epsilon$, $\epsilon \in \{+,-\}$. For every point (x,ϵ) of $\overline{\mathcal{S}}$, we define $(x,\epsilon)^\theta = (x,\epsilon)$ if $x \in \mathcal{P}_+$ and $(x,\epsilon)^\theta = (x,-\epsilon)$ if $x \in \mathcal{P}_-$.

Suppose (x_1, ϵ_1) and (x_2, ϵ_2) are two distinct collinear points of $\overline{\mathcal{S}}$. Then $\epsilon_2 = -\epsilon_1$. If x_1 and x_2 are equal, say to x, then θ maps the line $\{(x, \epsilon_1), (x, \epsilon_2)\}$ to the set $\{(x, \epsilon_1), (x, \epsilon_2)\}$. If $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = 1$, then θ maps the line $\{(x_1, \epsilon_1), (x_2, \epsilon_2)\}$ to the set $\{(x_1, \epsilon), (x_2, \epsilon)\}$, where $\epsilon = \epsilon_1$ if $x_1 \in \mathcal{P}_+$ and $\epsilon = -\epsilon_1$ if $x_1 \in \mathcal{P}_-$. Indeed, $(x_1, \epsilon_1)^\theta = (x_1, \epsilon)$. Moreover, if $x_1 \in \mathcal{P}_+$, then $x_2 \in \mathcal{P}_-$ and $(x_2, \epsilon_2)^\theta = (x_2, -\epsilon_1)^\theta = (x_2, \epsilon_1) = (x_2, \epsilon)$. Also, if $x_1 \in \mathcal{P}^-$, then $x_2 \in \mathcal{P}^+$ and $(x_2, \epsilon_2)^\theta = (x_2, -\epsilon_1)^\theta = (x_2, -\epsilon_1) = (x_2, \epsilon)$.

So, the map θ defines an isomorphism between $\overline{\mathcal{S}}$ and the product near polygon $\mathcal{S} \times \mathbb{L}_2$ defined on the point set $\mathcal{P} \times \{+, -\}$. ∎

4 Convex subspaces of Type I of $\overline{\mathcal{S}}$

We now initiate the study of the convex subspaces of $\overline{\mathcal{S}}$. We shall need to rely on the following lemma.

Lemma 4.1 *Suppose X is a convex subspace of $\mathcal{S}$, and x and y are two points of X at distance k from each other. Then every path of length at most $k+1$ connecting x and y is contained in X.*

Proof. Since X is convex, it suffices to prove that every path of length $k+1$ connecting x and y is contained in X. We prove this by induction on k. The case $k = 0$ is void. Suppose $k = 1$, and x, z, y is a path of length 2 connecting x and y. As $x \sim z \sim y \sim x$, the points x, y and z are contained in a line, and we conclude that $z \in xy \subseteq X$.

Suppose therefore that $k \geq 2$ and that $x = z_0, z_1, \ldots, z_{k+1} = y$ is a path of length $k+1$ connecting x and y. We have $\mathrm{d}(x, z_k) \in \{k-1, k\}$. If $\mathrm{d}(x, z_k) = k-1$, then $z_k \in X$ since X is convex and the induction hypothesis then implies that every point of the path $x = z_0, z_1, \ldots, z_k$ is contained in X. Suppose therefore that $\mathrm{d}(x, z_k) = \mathrm{d}(x, z_{k+1}) = k$. Then the line $z_k z_{k+1}$ contains a point u at distance $k-1$ from x. Since X is convex, we know that $u \in X$ and hence $u z_{k+1} \subseteq X$, in particular $z_k \in X$. Since X is convex, we then know that every point of the path $x = z_0, z_1, \ldots, z_k$ is contained in X. ∎

Let X be a nonempty subspace of $\mathcal{S}$ such that $\widetilde{X}$ is an isometrically embedded full subgeometry of $\mathcal{S}$. Then $\mathcal{S}_X := \widetilde{X}$ is a near polygon and so there is a thin near polygon $\overline{\mathcal{S}_X}$ associated with $\mathcal{S}_X$ with collinearity graph $\overline{\Gamma_X}$. The verification of the following lemma is straightforward.

Lemma 4.2 *The subgraph of $\overline{\Gamma}$ induced on the set $\overline{X} := X \times \{+, -\}$ is precisely $\overline{\Gamma_X}$.*

We shall use Lemma 4.2 to prove the following.

Lemma 4.3 *Two vertices of $\overline{X}$ have the same distance in $\overline{\Gamma}$ as in $\overline{\Gamma_X}$. As a consequence, $\overline{\mathcal{S}_X}$ is an isometrically embedded subgeometry of $\overline{\mathcal{S}}$.*

Proof. Suppose (x_1, ϵ_1) and (x_2, ϵ_2) are two distinct points of $\overline{X}$ and denote by (x_2, ϵ_2') the point $(x_2, \epsilon_1 \cdot \epsilon_{x_1 x_2})$ of $\overline{\mathcal{S}}$. The points (x_1, ϵ_1) and (x_2, ϵ_2') have distance $\mathrm{d}_{\mathcal{S}}(x_1, x_2)$ in $\overline{\Gamma}$, and so their distance δ' in $\overline{\Gamma_X}$ is at least $\mathrm{d}_{\mathcal{S}}(x_1, x_2)$. Now, any path $x_1 = y_0, y_1, \dots, y_k = x_2$ of length $k = \mathrm{d}_{\mathcal{S}}(x_1, x_2)$ in $\overline{\Gamma}$ with all y_i's belonging to X induces a path $(x_1, \epsilon_1) = (y_0, \tilde{\epsilon}_0), (y_1, \tilde{\epsilon}_1), \dots, (y_k, \tilde{\epsilon}_k) = (x_2, \epsilon_2')$ in $\overline{\Gamma}$ all whose vertices belong to $\overline{X}$ showing that $\delta' = \mathrm{d}_{\mathcal{S}}(x_1, x_2)$.

Suppose now that $\epsilon_2' \neq \epsilon_2$. Then the points (x_1, ϵ_1) and (x_2, ϵ_2) have distance $\mathrm{d}_{\mathcal{S}}(x_1, x_2) + 1$ in $\overline{\Gamma}$ and so their distance δ in $\overline{\Gamma_X}$ is at least $\mathrm{d}_{\mathcal{S}}(x_1, x_2) + 1$. Since $(x_1, \epsilon_1), (x_2, \epsilon_2')$ have distance $\mathrm{d}_{\mathcal{S}}(x_1, x_2)$ in $\overline{\Gamma_X}$ and (x_2, ϵ_2'), (x_2, ϵ_2) have distance 1 in $\overline{\Gamma_X}$, we see that $\delta = \mathrm{d}_{\mathcal{S}}(x_1, x_2) + 1$. ■

Proposition 4.4 *If X is a convex subspace of $\mathcal{S}$, then the set $\overline{X}$ is a convex set of vertices of $\overline{\Gamma}$, i.e. $\overline{X}$ is a convex subspace of $\overline{\mathcal{S}}$.*

Proof. Let (x, ϵ) and (y, ϵ') be two arbitrary vertices of $\overline{X}$. Suppose they lie at distance k from each other in $\overline{\mathcal{S}}$. Then $\mathrm{d}_{\mathcal{S}}(x, y) \in \{k, k-1\}$ by Lemma 3.2(b). Let $(z_0, \epsilon_0), (z_1, \epsilon_1), \dots, (z_k, \epsilon_k)$ be a path of length k in $\overline{\Gamma}$ connecting $(z_0, \epsilon_0) = (x, \epsilon)$ with $(z_k, \epsilon_k) = (y, \epsilon')$. Then $\mathrm{d}(z_{i-1}, z_i) \in \{0, 1\}$ for every $i \in \{1, 2, \dots, k\}$, and so $x = z_0, z_1, \dots, z_k = y$ defines a path of length at most k connecting x and y. By Lemma 4.1, we know that all points of $z_0, z_1, \dots, z_k$ are contained in X. Hence, all (z_i, ϵ_i) are contained in $\overline{X} = X \times \{+, -\}$. ■

Every convex subspace of $\overline{\mathcal{S}}$ of the form $X \times \{+, -\}$ with X a convex subspace of $\mathcal{S}$ is called a *convex subspace of Type I* of $\overline{\mathcal{S}}$. By Propositions 3.5, 4.4 and Lemma 4.2, we know that the following holds.

Proposition 4.5 *If L is a line of $\mathcal{S}$, then the set $L \times \{+, -\}$ is a quad of $\overline{\mathcal{S}}$, which is a dual $(s+1) \times (s+1)$-grid if the line L contains $s+1$ points.*

Proposition 4.6 *If A is a convex subspace of $\overline{\mathcal{S}}$ containing a line M of S, then A is a convex subspace of Type I.*

Proof. We show that if A contains a point x, then the unique line U_x of S through x also is contained in A. Let y denote the unique point of M nearest to x. Put $M \setminus \{y\} = \{y'\}$ and $U_x \setminus \{x\} = \{x'\}$. As the lines U_x and M are parallel, we see that x' is on a shortest path between $y' \in A$ and $x \in A$, implying that $x' \in A$ and $U_x \subseteq A$.

Let X denote the set of all points $x \in \mathcal{P}$ for which L_x is contained in A. Then $A = X \times \{+, -\}$.

We show that X is a subspace. Suppose x_1 and x_2 are two distinct collinear points of X for which $L_{x_1} \cup L_{x_2} \subseteq A$. Then L_{x_1} and L_{x_2} are two parallel lines at distance 1 from each other. Let x_3 be a third point on x_1x_2. Every point of L_{x_3} lies at distance 1 from a point of L_{x_1} and at distance 1 from a point of L_{x_2}, implying that L_{x_3} is contained in A (see Lemma 4.1 with $\mathcal{S}$ replaced by $\overline{\mathcal{S}}$ and X replaced by A). This implies that $x_3 \in X$.

We show that X is convex. Let x and y be two points of X, and let $x = z_0, z_1, \dots, z_k = y$ be any shortest path between x and y. Let (z_0, ϵ_0) be an arbitrary point of L_x and let (z_i, ϵ_i) with $i \in \{1, 2, \dots, k\}$ be the unique point of L_{z_i} collinear with $(z_{i-1}, \epsilon_{i-1})$ in $\overline{\mathcal{S}}$. Then $(z_0, \epsilon_0), (z_1, \epsilon_1), \dots, (z_k, \epsilon_k)$ is a shortest path in $\overline{\mathcal{S}}$ connecting the vertices $(z_0, \epsilon_0) \in L_x \subseteq A$

and $(z_k, \epsilon_k) \in L_y \subseteq A$. It follows that $(z_i, \epsilon_i) \in A$ for every $i \in \{0, 1, \ldots, k\}$. Hence, $z_i \in X$ for every $i \in \{0, 1, \ldots, k\}$ and X must be convex. ■

Lemma 4.7 *If X is a classical convex subspace of $\mathcal{S}$, then $\overline{X}$ is a classical convex subspace of $\overline{\mathcal{S}}$.*

Proof. Let (x, ϵ) be an arbitrary point of $\overline{\mathcal{S}}$ and (y, ϵ') be an arbitrary point of $\overline{X}$. If z is the unique point of X nearest to x, then $\mathrm{d}_{\mathcal{S}}(x, y) = \mathrm{d}_{\mathcal{S}}(x, z) + \mathrm{d}_{\mathcal{S}}(z, y)$ and $\epsilon_{xy} = \epsilon_{xz}\epsilon_{zy}$. By Lemma 3.2(b), we have $\delta := \mathrm{d}_{\overline{\mathcal{S}}}[(x, \epsilon), (y, \epsilon')] = \mathrm{d}_{\mathcal{S}}(x, y) + \eta$, where $\eta = 0$ if $\epsilon' = \epsilon_{xy}\epsilon$ and $\eta = 1$ otherwise. Using again Lemma 3.2(b), we find $\delta = \mathrm{d}_{\mathcal{S}}(x, z) + \mathrm{d}_{\mathcal{S}}(z, y) + \eta = \mathrm{d}_{\overline{\mathcal{S}}}[(x, \epsilon), (z, \epsilon_{xz}\epsilon)] + \mathrm{d}_{\overline{\mathcal{S}}}[(z, \epsilon_{xz}\epsilon), (y, \epsilon_{xz}\epsilon_{zy}\epsilon)] + \eta = \mathrm{d}_{\overline{\mathcal{S}}}[(x, \epsilon), (z, \epsilon_{xz}\epsilon)] + \mathrm{d}_{\overline{\mathcal{S}}}[(z, \epsilon_{xz}\epsilon), (y, \epsilon_{xy}\epsilon)] + \eta = \mathrm{d}_{\overline{\mathcal{S}}}[(x, \epsilon), (z, \epsilon_{xz}\epsilon)] + \mathrm{d}_{\overline{\mathcal{S}}}[(z, \epsilon_{xz}\epsilon), (y, \epsilon')]$. We thus see that $\overline{X}$ must be classical in $\overline{\mathcal{S}}$. ■

5 Convex subspaces of Type II of $\overline{\mathcal{S}}$

We now define a second family of convex subspaces of $\overline{\mathcal{S}}$. We start from a nonempty set Y of points of $\mathcal{S}$ satisfying the following properties:

(P1) Y is convex;

(P2) the subgraph of the collinearity graph of $\mathcal{S}$ induced on Y is bipartite.

Lemma 5.1 *A nonempty set Y of points of $\mathcal{S}$ satisfies properties (P1) and (P2) if and only if it satisfies properties (P1) and (P2'), where*

($P2'$) *No three points of Y lie on a common line.*

Proof. Suppose Y satisfies (P1) and (P2). If a line of $\mathcal{S}$ contains three points y_1, y_2 and y_3 of Y, then $\{y_1, y_2, y_3\}$ is a clique in the collinearity graph of $\mathcal{S}$, in contradiction with (P2). So, Y also satisfies (P2′).

Conversely, suppose that Y satisfies (P1) and (P2′). Let y^* be a fixed point of Y. We show that the subgraph of the collinearity graph of $\mathcal{S}$ induced on Y is bipartite with bipartite parts $\{y \in Y \mid \mathrm{d}_{\mathcal{S}}(y^*, y) \text{ is even}\}$ and $\{y \in Y \mid \mathrm{d}_{\mathcal{S}}(y^*, y) \text{ is odd}\}$. To that end, it suffices to prove that if $y_1, y_2, y_3 \in Y$ with $y_2 \sim y_3$, then $\mathrm{d}_{\mathcal{S}}(y_1, y_2) \neq \mathrm{d}_{\mathcal{S}}(y_1, y_3)$. Suppose to the contrary that $\mathrm{d}_{\mathcal{S}}(y_1, y_2) = \mathrm{d}_{\mathcal{S}}(y_1, y_3)$. Let u denote the unique point of y_2y_3 nearest to y_1. Since Y is convex, we have $u \in Y$. But then the three points u, y_2 and y_3 contradict (P2′). So, also property (P2) must be satisfied. ■

In the sequel, we suppose that the nonempty set $Y \subseteq \mathcal{P}$ satisfies (P1) and (P2). From the proof of Lemma 5.1, we can extract the following.

Lemma 5.2 *If $y_1, y_2, y_3 \in Y$ with $y_2 \sim y_3$, then $d_{\mathcal{S}}(y_1, y_2) \neq d_{\mathcal{S}}(y_1, y_3)$.*

For every point $y^* \in \mathcal{P}$ and every $\epsilon^* \in \{+,-\}$, let Φ_{y^*,ϵ^*} denote the map from $\mathcal{P}$ to $\overline{\mathcal{P}}$ that sends each $y \in \mathcal{P}$ to $(y, \epsilon^* \epsilon_{y^* y}) \in \overline{\mathcal{P}}$. Now, let $y^* \in Y$ and $\epsilon^* \in \{+,-\}$ be fixed, and put $\phi := \Phi_{y^*,\epsilon^*}$.

Lemma 5.3 *If y_1 and y_2 are collinear points of Y, then $\phi(y_1)$ and $\phi(y_2)$ are adjacent vertices of Γ.*

Proof. By Lemma 5.2, one of $\{y_1, y_2\}$ is nearer to y^* as the other, implying that $\epsilon_{y^* y_2} = -\epsilon_{y^* y_1}$. So, the vertices $\phi(y_1) = (y_1, \epsilon^* \epsilon_{y^* y_1})$ and $\phi(y_2) = (y_2, \epsilon^* \epsilon_{y^* y_2})$ are indeed adjacent. ■

Lemma 5.4 *For all $y_1, y_2 \in Y$, we have $\epsilon_{y^* y_1} \epsilon_{y_1 y_2} = \epsilon_{y^* y_2}$.*

Proof. We prove this by induction on $k = \mathrm{d}_{\mathcal{S}}(y_1, y_2)$, the case $k = 0$ being trivial (as $\epsilon_{y_1 y_2} = +$ then). So, suppose $k \geq 1$. Let y_3 be an arbitrary vertex of $\Gamma_1(y_2) \cap \Gamma_{k-1}(y_1)$. By the induction hypothesis, we have $\epsilon_{y^* y_1} \epsilon_{y_1 y_3} = \epsilon_{y^* y_3}$. Since $y_3 \sim y_2$, we have $\mathrm{d}_{\mathcal{S}}(y^*, y_3) \neq \mathrm{d}_{\mathcal{S}}(y^*, y_2)$ by Lemma 5.2 and hence $\epsilon_{y^* y_2} = -\epsilon_{y^* y_3}$. We also know $\epsilon_{y_1 y_2} = -\epsilon_{y_1 y_3}$. It follows that $\epsilon_{y^* y_1} \cdot \epsilon_{y_1 y_2} = -\epsilon_{y^* y_1} \cdot \epsilon_{y_1 y_3} = -\epsilon_{y^* y_3} = \epsilon_{y^* y_2}$. ■

Lemma 5.5 *The map $y \mapsto \phi(y)$ is a distance-preserving map between Y and $\overline{\mathcal{S}}$.*

Proof. Let $y_1, y_2 \in Y$. By Lemmas 3.2(b) and 5.4, the distance between $\phi(y_1) = (y_1, \epsilon^* \epsilon_{y^* y_1})$ and $\phi(y_2) = (y_2, \epsilon^* \epsilon_{y^* y_2})$ is equal to $\mathrm{d}_{\mathcal{S}}(y_1, y_2)$ since $\epsilon^* \epsilon_{y^* y_1} \cdot \epsilon_{y_1 y_2} = \epsilon^* \epsilon_{y^* y_2}$. ■

Proposition 5.6 *$\phi(Y)$ is a convex subspace of $\overline{\mathcal{S}}$ not containing any line of S.*

Proof. As any line of $\overline{\mathcal{S}}$ contains two points, $\phi(Y)$ is a subspace. By definition, this subspace cannot contain any line of S. It remains to show that $\phi(Y)$ is convex.

Let $y_1, y_2 \in Y$ and suppose $(z_0, \epsilon_0), (z_1, \epsilon_1), \ldots, (z_k, \epsilon_k)$ is a shortest path between $(z_0, \epsilon_0) = \phi(y_1) = (y_1, \epsilon^* \epsilon_{y^* y_1})$ and $(z_k, \epsilon_k) = \phi(y_2) = (y_2, \epsilon^* \epsilon_{y^* y_2})$. As the distance in $\overline{\mathcal{S}}$ between $\phi(y_1)$ and $\phi(y_2)$ is equal to $\mathrm{d}_{\mathcal{S}}(y_1, y_2)$, we thus see that $y_1 = z_0, z_1, \ldots, z_k = y_2$ is a shortest path between y_1 and y_2. As Y is convex, each z_i with $i \in \{0, 1, \ldots, k\}$ is also contained in Y. As $\phi(z_0), \phi(z_1), \ldots, \phi(z_k)$ is a shortest path between $\phi(y_1)$ and $\phi(y_2)$ by Lemma 5.5, we see that $\phi(z_i) = (z_i, \epsilon_i)$ for every $i \in \{0, 1, \ldots, k\}$. So, the path $(z_0, \epsilon_0), (z_1, \epsilon_1), \ldots, (z_k, \epsilon_k)$ is completely contained in $\phi(Y)$. ■

A convex subspace that is of the form $\Phi_{y^*,\epsilon^*}(Y)$, where $\epsilon^* \in \{+,-\}$ and Y is a set of points of $\mathcal{S}$ containing $y^* \in \mathcal{P}$ and satisfying (P1) and (P2), is called a *convex subspace of Type II.* The following proposition, in combination with Proposition 4.6, offers a complete classification of all convex subspaces of $\overline{\mathcal{S}}$.

Proposition 5.7 *Every nonempty convex subspace of $\overline{\mathcal{S}}$ not containing any line of S is of Type II.*

Proof. Suppose A is a convex subspace of $\overline{\mathcal{S}}$ not containing any line of S. Put $Y = \{y \in \mathcal{P} \,|\, (y, \epsilon) \in A$ for some $\epsilon \in \{+,-\}\} \neq \emptyset$. Then for every $y \in Y$, there exists a unique

$\epsilon_y \in \{+,-\}$ such that $\phi(y) := (y, \epsilon_y) \in A$. Then $A = \phi(Y)$. Let y^* be a distinguished point of Y, and put $\epsilon^* = \epsilon_{y^*}$.

Let $y \in Y$. As L_y is not contained in the convex subspace A, the unique point of L_y contained in A is the unique point of L_y nearest to (y^*, ϵ^*), which is the point $(y, \epsilon^* \epsilon_{y^* y})$ by Lemma 3.2(a). This shows that $\phi(y) = (y, \epsilon^* \epsilon_{y^* y})$.

Let $y_1, y_2 \in Y$. As the line L_{y_2} is not contained in A, the unique point of L_{y_2} nearest to $\phi(y_1) = (y_1, \epsilon^* \epsilon_{y^* y_1})$ is the point $\phi(y_2) = (y_2, \epsilon^* \epsilon_{y^* y_2})$. By Corollary 3.4, we thus see that the map ϕ is isometric. Since the collinearity graph of $\overline{\mathcal{S}}$ is bipartite, this shows that the subgraph of the collinearity graph of $\mathcal{S}$ induced on Y is bipartite.

Finally, we show that Y is convex. Let y_1, y_2 be two points of Y, and $y_1 = z_0, z_1, \ldots, z_k = y_2$ be a shortest path between y_1 and y_2. Put $\epsilon_0 = \epsilon_{y_1}$. Then $(z_0, \epsilon_0) \in A$. In $\overline{\mathcal{S}}$, there exists a path of the form $(z_0, \epsilon_0), (z_1, \epsilon_1), \ldots, (z_k, \epsilon_k)$, where (z_k, ϵ_k) is the unique point of L_{y_2} nearest to $(z_0, \epsilon_0) = (y_1, \epsilon_{y_1})$. This point coincides with (y_2, ϵ_{y_2}). As (z_0, ϵ_0) and (z_k, ϵ_k) belong to A, the convexity of A implies that all (z_i, ϵ_i) with $i \in \{0, 1, \ldots, k\}$ belong to A. This implies that all z_i with $i \in \{0, 1, \ldots, k\}$ belong to Y. So, Y is indeed convex.

We conclude that $A = \phi(Y)$ is a convex subspace of Type II. ■

6 Characterizations of $\overline{\mathcal{S}}$ in terms of polygonal triples

We continue with the notation introduced in Section 3. Put $\mathcal{Q} := \{L \times \{+,-\} \mid L \in \mathcal{L}\}$. By Proposition 4.5, $\mathcal{Q}$ is a set of quads of $\overline{\mathcal{S}}$.

Proposition 6.1 *$(\overline{\mathcal{S}}, S, \mathcal{Q})$ is a polygonal triple whose associated near hexagon is isomorphic to $\mathcal{S}$.*

Proof. Let (x, ϵ_x) be an arbitrary point of $\overline{\mathcal{S}}$. The unique line of S containing (x, ϵ_x) is the line $L_x = \{(x,+), (x,-)\}$. Let $\{(x, \epsilon_x), (y, \epsilon_y)\}$ be another line of $\overline{\mathcal{S}}$ through (x, ϵ_x). Then $\mathrm{d}_{\mathcal{S}}(x, y) = 1$ and $\epsilon_y = -\epsilon_x$. Suppose Q is a quad of $\mathcal{Q}$ containing (x, ϵ_x) and (y, ϵ_y). Then $Q = L \times \{+,-\}$ for a certain line L of $\mathcal{S}$. Obviously, $L = xy$ and so there exists a unique quad of $\mathcal{Q}$ containing (x, ϵ_x) and (y, ϵ_y).

Let $\overline{\mathcal{S}}'$ be the point-line geometry whose points and lines are the elements of S and $\mathcal{Q}$, with incidence being containment. The points of $\overline{\mathcal{S}}'$ have the form $\{x\} \times \{+,-\}$ with $x \in \mathcal{P}$ and the lines have the form $L \times \{+,-\}$ with $L \in \mathcal{L}$. It thus follows that $\overline{\mathcal{S}}'$ is isomorphic to $\mathcal{S}$. ■

Theorem 6.2 *Suppose $(\mathcal{S}', S', \mathcal{Q}')$ is a polygonal triple whose associated near polygon is isomorphic to $\mathcal{S}$ such that every line of $\mathcal{S}'$ is thin. Then there exists an isomorphism from $\mathcal{S}'$ to $\overline{\mathcal{S}}$, mapping S' to S and $\mathcal{Q}'$ to $\mathcal{Q}$.*

Proof. Let x^* be a given point of $\mathcal{S}$. Since $\mathcal{S}$ is isomorphic to the associated near polygon, there corresponds with each point x of $\mathcal{S}$ a line U_x of S' and with each line L of $\mathcal{S}$ a quad Q_L of $\mathcal{Q}'$ such that for every point-line pair (y, M) of $\mathcal{S}$, we have $y \in M$ if and only if $U_y \subseteq Q_M$. For any two points x and y in $\mathcal{S}$, we have $\mathrm{d}_{\mathcal{S}'}(U_x, U_y) = \mathrm{d}_{\mathcal{S}}(x, y)$.

We label the two points of $L^* := U_{x^*}$ by $(x^*, +)$ and $(x^*, -)$. Put $\epsilon_x = +$ if $\mathrm{d}(x^*, x)$ is even and $\epsilon_x = -$ if $\mathrm{d}(x^*, x)$ is odd. For every line $U_x \in S'$, we label the unique point of U_x nearest to $(x^*, +)$ by (x, ϵ_x) and the other point by $(x, -\epsilon_x)$. For every point u of $\mathcal{S}'$, we denote the label of u by $L(u)$. We show that the map $u \mapsto L(u)$ defines an isomorphism between $\mathcal{S}'$ and $\overline{\mathcal{S}}$.

Let us examine when two points u_1 and u_2 with respective labels (x_1, ϵ_1) and (x_2, ϵ_2) are collinear in $\mathcal{S}'$. One way for this to happen is that $x_1 = x_2$ and $\{\epsilon_1, \epsilon_2\} = \{+, -\}$. So, suppose $x_1 \neq x_2$. Then $U_{x_1} \neq U_{x_2}$, $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = 1$ and there exists a unique quad Q of $\mathcal{Q}'$ containing the lines U_{x_1} and U_{x_2} (namely the quad $Q_{x_1x_2}$). Note that Q is a dual grid since every line of $\mathcal{S}'$ is thin. The quad Q is also classical in $\mathcal{S}'$. We denote by x_3 the unique point of x_1x_2 nearest to x^* (in $\mathcal{S}$). Then $U_{x_3} = \pi_Q(L^*)$. We distinguish two cases.

(1) $x_3 = x_i$ for a certain $i \in \{1, 2\}$. Then $\epsilon_{x_2} = -\epsilon_{x_1}$. Since Q is classical in $\mathcal{S}'$, there exists a shortest path in $\mathcal{S}'$ connecting $(x^*, +)$ to $(x_{3-i}, \epsilon_{x_{3-i}})$ via $(x_i, \epsilon_{x_i}) = \pi_Q[(x^*, +)]$. We thus see that $(x_1, +) \sim (x_2, -)$ and $(x_1, -) \sim (x_2, +)$.

(2) Suppose $x_3 \notin \{x_1, x_2\}$. Then $\epsilon := \epsilon_{x_1} = \epsilon_{x_2} = -\epsilon_{x_3}$. Since Q is classical in $\mathcal{S}'$, there exists a shortest path in $\mathcal{S}'$ connecting $(x^*, +)$ to (x_1, ϵ) via $(x_3, -\epsilon) = \pi_Q[(x^*, +)]$, and a shortest path in $\mathcal{S}'$ connecting $(x^*, +)$ to (x_2, ϵ) via $(x_3, -\epsilon)$. We see that $(x_1, \epsilon) \sim (x_3, -\epsilon) \sim (x_2, \epsilon)$. Now, Q is a dual grid containing the lines U_{x_1}, U_{x_3} and U_{x_2}. So, $(x_1, \epsilon) \not\sim (x_2, \epsilon)$. This implies that $(x_1, +) \sim (x_2, -)$ and $(x_1, -) \sim (x_2, +)$.

Combining the above, we indeed see that the map $\theta : u \mapsto L(u)$ defines an isomorphism between $\mathcal{S}'$ and $\overline{\mathcal{S}}$. Moreover, θ maps the lines of S' to the lines of S and hence also the quads of $\mathcal{Q}'$ to the quads of $\mathcal{Q}$ (Recall that the set of quads of a polygonal triple is uniquely determined by the line spread of the triple). ∎

In Example 4 of [6], we showed that the hyperbolic dual polar space $DQ^+(2n+1, \mathbb{F})$ with $n \in \mathbb{N} \setminus \{0\}$ and $\mathbb{F}$ a field has a spread S and a set $\mathcal{Q}$ of quads such that $(DQ^+(2n + 1, \mathbb{F}), S, \mathcal{Q})$ is an admissible triple whose associated near polygon is isomorphic to the parabolic dual polar space $DQ(2n, \mathbb{F})$. Since $DQ^+(2n + 1, \mathbb{F})$ is a thin near polygon, Theorem 6.2 implies the following.

Proposition 6.3 *If $\mathcal{S}$ is isomorphic to the dual polar space $DQ(2d, \mathbb{F})$, where $d \in \mathbb{N} \setminus \{0\}$ and $\mathbb{F}$ a field, then $\overline{\mathcal{S}}$ is isomorphic to the dual polar space $DQ^+(2d + 1, \mathbb{F})$.*

Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are two near polygons. Then $\overline{\mathcal{S}_1}$ and $\overline{\mathcal{S}_2}$ are two thin near polygons having admissible spreads S_1 and S_2 (as defined in Section 3). By [4, Theorem 1], it is possible to obtain glued near polygons from $\overline{\mathcal{S}_1}$ and $\overline{\mathcal{S}_2}$ by glueing along the admissible spreads S_1 and S_2. As each $\overline{\mathcal{S}_i}$ has an automorphism interchanging the two points on each line of S_i (Proposition 3.6(1)), there is essentially one glued near polygon that can be constructed in this way. We denote this glued near polygon by $\overline{\mathcal{S}_1} \otimes \overline{\mathcal{S}_2}$. By Example 7 of [6], the near polygon $\overline{\mathcal{S}_1} \otimes \overline{\mathcal{S}_2}$ has a spread S and a set $\mathcal{Q}$ of quads such that $(\overline{\mathcal{S}_1} \otimes \overline{\mathcal{S}_2}, S, \mathcal{Q})$ is a polygonal triple for which $\mathcal{S}_1 \times \mathcal{S}_2$ is the associated near polygon. By Theorem 6.2, we thus have that $\overline{\mathcal{S}_1} \otimes \overline{\mathcal{S}_2} = \overline{\mathcal{S}_1 \times \mathcal{S}_2}$.

7 Examples

7.1 The existence of quads

We first provide an example of a nonempty set Y satisfying the properties (P1) and (P2) of Section 5. We continue with the notation introduced before. Suppose Q is a quad of $\mathcal{S}$ and x and y are two noncollinear points of Q. Let $\{x,y\}^{\perp} \subseteq Q$ denote the set of points collinear with x and y and denote by $\{x,y\}^{\perp\perp}$ the set of points collinear with each point of $\{x,y\}^{\perp}$. The pair $\{x,y\}$ is called *regular* if for any two distinct points $a,b \in \{x,y\}^{\perp}$, we have $\{a,b\}^{\perp} = \{x,y\}^{\perp\perp}$. If this is the case, then for any two distinct points $c,d \in \{x,y\}^{\perp\perp}$, we also have $\{c,d\}^{\perp} = \{x,y\}^{\perp}$. If $\{x,y\}$ is a regular pair, then the set $Y = \{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp} \subseteq Q$ satisfies the conditions (P1) and (P2). If Q has order (s,t), then for every $y \in Y$ and every $\epsilon \in \{+,-\}$, the set $\Phi_{y,\epsilon}(Y)$ is a quad of $\overline{\mathcal{S}}$ which is the dual of a $(t+1) \times (t+1)$-grid.

Lemma 7.1 *If every two points of $\mathcal{S}$ at distance 2 are contained in a (necessarily unique) quad, and if every pair of noncollinear points of a quad is regular, then every two points at distance 2 in $\overline{\mathcal{S}}$ are contained in a unique quad.*

Proof. Let $u_1 = (x_1, \epsilon_1)$ and $u_2 = (x_2, \epsilon_2)$ be two points of $\overline{\mathcal{S}}$ at distance 2 from each other. We distinguish two cases.

Suppose first that $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = 1$ and $\epsilon_1 = \epsilon_2$. If L is the line x_1x_2, then by Proposition 4.5 $L \times \{+,-\}$ is the (necessarily unique) quad of $\overline{\mathcal{S}}$ containing u_1 and u_2.

Suppose next that $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = 2$ and $\epsilon_2 = \epsilon_1$. Let Q denote the unique quad of $\mathcal{S}$ containing x_1 and x_2. Put $Y := \{x_1,x_2\}^{\perp} \cup \{x_1,x_2\}^{\perp\perp}$. Then Y satisfies (P1) and (P2). By Proposition 5.6, $\Phi_{x_1,\epsilon_1}(Y)$ is the (necessarily unique) quad of $\overline{\mathcal{S}}$ containing u_1 and u_2. ■

Proposition 7.2 *Suppose every line of $\mathcal{S}$ is incident with at least three points. Then every two points of $\overline{\mathcal{S}}$ at distance 2 from each other are contained in a unique quad if and only if the following two properties are satisfied:*

- *every two points of $\mathcal{S}$ at distance 2 have at least two common neighbours;*
- *every pair of noncollinear points of a quad of $\mathcal{S}$ is regular.*

Proof. The former of the two properties and the fact that lines contain at least three points imply by Yanushka's lemma that every two points of $\mathcal{S}$ at distance 2 are contained in a unique quad. So, if these two conditions are satisfied, then we know from Lemma 7.1 that every two points of $\overline{\mathcal{S}}$ are contained in a unique quad.

Conversely, suppose that every two points of $\overline{\mathcal{S}}$ at distance 2 from each other are contained in a unique quad. Take two points x_1 and x_2 in $\mathcal{S}$ at distance 2 from each other, and $\epsilon \in \{+,-\}$. Then the points $u_1 = (x_1, \epsilon)$ and $u_2 = (x_2, \epsilon)$ of $\overline{\mathcal{S}}$ lie at distance 2 from each other. As u_1 and u_2 are contained in a quad, these points have two common neighbours $(x_3, -\epsilon)$ and $(x_4, -\epsilon)$. Then x_3 and x_4 are two common neighbours of x_1 and x_2 in $\mathcal{S}$, and so by Yanushka's lemma we know that there is a unique quad Q of $\mathcal{S}$ containing x_1 and x_2.

Suppose $\{x_1, x_2\}$ is not a regular pair of Q. Then there exist $a, b, c \in \{x_1, x_2\}^\perp$ such that a and b have a common neighbour d that is not a neighbour of c. Since $x_1 \not\sim d \not\sim c$, there exists a point $e \in x_1c \setminus \{x_1, c\}$ collinear with d. Then e is a common neighbour of x_1 and d. Now, any set Y of points satisfying (P1), (P2) and containing x_1, x_2 also contains a, b and c, and hence also d and e. But that is impossible as x_1, e and c are on a line. ■

Examples of generalized quadrangles for which each pair of noncollinear points is regular are the grids, the symplectic generalized quadrangles and quadrangles arising from Hermitian varieties of Witt index 2 in projective spaces of dimension 3, see [8].

7.2 Examples with a regular set of convex subspaces

Consider the family of geometries consisting of all thin near polygons $\overline{\mathcal{S}}$, where $\mathcal{S}$ is a near polygon. In the present section, we take a closer look at those near polygons of this family that have a regular set of convex subspaces. We first derive a criterion to decide whether a near polygon of the family has a regular set of convex subspaces.

Proposition 7.3 *Let $\mathcal{S}$ be a near $2d$-gon and $\delta \in \{0, 1, \ldots, d+1\}$. Then every two points of $\overline{\mathcal{S}}$ at distance δ from each other are contained in a unique convex subspace of diameter δ if and only if the following properties hold.*

(1) *Every two points of $\mathcal{S}$ at distance $\delta - 1$ from each other are contained in a unique convex subspace of diameter $\delta - 1$.*

(2) *Every two points of $\mathcal{S}$ at distance δ from each other are contained in a unique set of points satisfying (P1), (P2) and having diameter δ.*

Proof. Suppose (x_1, ϵ_1) and (x_2, ϵ_2) are two points of $\overline{\mathcal{S}}$ at distance δ. Put $\epsilon_2' = \epsilon_1 \epsilon_{x_1 x_2}$. We distinguish two cases.

(a) $\epsilon_2' \neq \epsilon_2$. Then $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = \delta - 1$ by Lemma 3.2(b). If A is a convex subspace of diameter δ containing (x_1, ϵ_1) and (x_2, ϵ_2), then A also contains (x_2, ϵ_2') as this point is on a shortest path between (x_1, ϵ_1) and (x_2, ϵ_2). So, A contains the line L_{x_2} and is a convex subspace of Type I by Proposition 4.6. So, $A = \overline{B}$, where B is a convex subspace of diameter $\delta - 1$ containing x_1 and x_2. The number of convex subspace of diameter δ of $\overline{\mathcal{S}}$ containing (x_1, ϵ_1) and (x_2, ϵ_2) is thus equal to the number of convex subspaces of diameter $\delta - 1$ of $\mathcal{S}$ containing x_1 and x_2.

(b) $\epsilon_2' = \epsilon_2$. Then $\mathrm{d}_{\mathcal{S}}(x_1, x_2) = \delta$ by Lemma 3.2(b). If A is a convex subspace of diameter δ containing (x_1, ϵ_1) and (x_2, ϵ_2), then A cannot contain the point $(x_2, -\epsilon_2)$ since this point lies at distance $\delta + 1$ from (x_1, ϵ_1) and so A must be a convex subspace of Type II. Hence, $A = \Phi_{x_1, \epsilon_1}(B)$, where B is a set of points of $\mathcal{S}$ having diameter δ, satisfying (P1), (P2) and containing x_1 and x_2. So, the number of convex subspaces of diameter δ containing (x_1, ϵ_1) and (x_2, ϵ_2) is equal to the number of sets of points of $\mathcal{S}$ containing x_1, x_2, satisfying (P1), (P2) and having diameter δ.

■

Note that Properties (1) and (2) of Proposition 7.3 always hold if $\delta \leq 1$. In the near polygon $\overline{\mathcal{S}}$, any two points at distance $\delta \leq 1$ are indeed contained in a unique convex subspace of diameter δ (which is either a singleton or a line).

Our next aim is to show that if $\mathcal{S}$ is a dual polar space of quadratic type, then $\overline{\mathcal{S}}$ has a regular set of convex subspaces. In order to achieve this goal, we need a number of lemmas. Note that every set X of points of a near polygon is contained in a unique smallest convex set, namely the intersection of all convex sets containing X.

Lemma 7.4 *Let x and y be two points of a near $2d$-gon $\mathcal{S}$ at distance d from each other. Let Y be the smallest convex set of points of $\mathcal{S}$ containing x and y. Suppose Y satisfies (P2) (or equivalently (P2')) and also the property that for every point $y_1 \in Y$, there exists a point $y_2 \in Y$ at distance d from x. Then Y is the unique set of points of $\mathcal{S}$ containing x and y and satisfying (P1) and (P2).*

Proof. Suppose Y' is a convex set containing x and y and satisfying properties (P1) and (P2). Then $Y \subseteq Y'$. Suppose $v \in Y' \setminus Y$. Denote by z one of the points of Y nearest to v, and let z' denote a point in Y opposite to z. Let L denote a line through z containing a point u at distance $\mathrm{d}(v,z)-1$ from v. Since Y' is convex and $z, v \in Y'$, we have $u \in Y'$. By the minimality of $\mathrm{d}(v,z)$, we have $u \notin Y$. Now, the line L contains a unique point w at distance $d-1$ from z'. Since Y is convex and $z, z' \in Y$, we have $w \in Y \subseteq Y'$ and hence $w \neq u$. But that is impossible as it would imply that the line L contains three points of Y', namely z, u and w. ■

Lemma 7.5 *Let x and y be two points of a near $2d$-gon $\mathcal{S}$ at distance δ from each other. Let Y be the smallest convex set of points of $\mathcal{S}$ containing x and y. Suppose that F is a convex subspace of diameter δ containing x and y that satisfies the following properties:*

- *F is classical in $\mathcal{S}$;*
- *If u and v are two points of F at distance at most $\delta - 1$ from each other, then there is a line of F through v containing a point at distance $d(u,v)+1$ from u.*

Then:

(*a*) *Y coincides with the smallest convex set of points of $\widetilde{F}$ containing x and y.*

(*b*) *For every point $z_1 \in Y$, there exists a point $z_2 \in Y$ at distance δ from z_1.*

(*c*) *If Y satisfies Property (P2), then Y is the unique set of points of $\mathcal{S}$ containing x and y, satisfying (P1) and (P2), and having diameter δ.*

Proof. The set Y is the intersection of all convex sets of points containing x and y, and hence we should have $Y \subseteq F$. As distances in $\widetilde{F}$ coincide with their corresponding distances in $\mathcal{S}$, the set Y coincides with the smallest convex set of points of $\widetilde{F}$ containing x and y.

We prove that for every point $z_1 \in Y$, there exists a point $z_2 \in Y$ at distance δ from z_1. We show this by induction on the distance $\mathrm{d}(x, z_1)$. The case $\mathrm{d}(x, z_1) = 0$ is trivial, since $z_1 = x$ in this case and we can take z_2 equal to y. Suppose therefore that $\mathrm{d}(x, z_1) > 0$. Let z_1' be a point collinear with z_1 at distance $\mathrm{d}(x, z_1) - 1$ from x. Since Y is convex, we have $z_1' \in Y$. By the induction hypothesis there exists a point $z_2' \in Y$ at distance δ from z_1'. If $\mathrm{d}(z_1, z_2') = \delta$, then we are done. So, we may suppose that $\mathrm{d}(z_1, z_2') = \delta - 1$. Then there is a line L of F through z_2' containing besides z_2' only points at distance δ from z_1. So, the unique point z_2 on L at distance $\delta - 1$ from z_1' must lie at distance δ from z_1. This point belongs to Y as it lies on a shortest path between the points $z_1', z_2' \in Y$.

In the remainder of the proof, we suppose that Y satisfies Property (P2). If we apply Lemma 7.4 now to the near polygon $\widetilde{F}$, then we see that every set Y' of points of $\mathcal{S}$ containing x and y and satisfying (P1) and (P2) must intersect F in the set Y. So, in order to prove the lemma, it suffices to prove that for every point w not contained in F, there is a point in Y at distance at least $\delta + 1$ from w. Since F is classical in $\mathcal{S}$ and $\mathrm{d}(w, \pi_F(w)) \geq 1$, it thus suffices to show that there is a point in Y at distance at least δ from $\pi_F(w)$. Let w' be a point of Y at maximal distance from $\pi_F(w)$. Then we know that there exists a point $w'' \in Y$ at distance δ from w'. If $\mathrm{d}(\pi_F(w), w') = \delta$, then we are done. So, suppose $\mathrm{d}(\pi_F(w), w') \leq \delta - 1$. Then there exists a line L of F through w' containing besides w' only points at distance $\mathrm{d}(\pi_F(w), w') + 1$ from $\pi_F(w)$. The unique point of L at distance $\delta - 1$ from w'' must belong to Y (since it lies on a shortest path from $w' \in Y$ to $w'' \in Y$) and lies at distance $\mathrm{d}(\pi_F(w), w') + 1$ from $\pi_F(w)$, a contradiction. ∎

Lemma 7.6 *Let x and y be two opposite points of the dual polar space $DQ^+(2n-1, \mathbb{F})$, $n \geq 2$ and $\mathbb{F}$ a field. Then the smallest convex set of points of $DQ^+(2n-1, \mathbb{F})$ containing x and y coincides with the whole point set.*

Proof. In a general dual polar space Δ, the smallest convex subspace of Δ containing two opposite points coincides with the whole point set of Δ, see e.g. [5, Theorem 8.11]. In the case $\Delta = DQ^+(2n-1, \mathbb{F})$, every line of Δ is thin, and so every convex set of points is also a convex subspace. The claim of the lemma is now clear. ∎

Lemma 7.7 *Let Q be a nonsingular quadric of Witt index $n \geq 2$ in a projective space Σ over a field $\mathbb{F}$. Let α and β be two disjoint generators of Q. Let Π be the polar and dual polar spaces corresponding to Q. The subspace $\langle \alpha, \beta \rangle$ of Σ meets Q in a hyperbolic quadric $Q^+(2n-1, \mathbb{F})$. Then the smallest convex set X of points of Δ containing α and β consists of all generators of $Q^+(2n-1, \mathbb{F})$. Moreover, this set of points satisfies Property (P2).*

Proof. By Lemma 2.2, the set X' of generators of $Q^+(2n-1, \mathbb{F}) = Q \cap \langle \alpha, \beta \rangle$ is a convex set of points of Δ. Hence, $X \subseteq X'$, i.e. X is a set of points of the dual polar space $DQ^+(2n-1, \mathbb{F})$ associated with $Q^+(2n-1, \mathbb{F})$. Lemma 2.1 implies that distances between points of $DQ^+(2n-1, \mathbb{F})$ coincide with their distances in Δ. Lemma 7.6 then implies that X consists of all points of $DQ^+(2n-1, \mathbb{F})$, i.e. all generators of $Q^+(2n-1, \mathbb{F})$. The fact that $DQ^+(2n-1, \mathbb{F})$ is a thin near polygon implies that Property (P2$'$) and hence also Property (P2) holds. ∎

By Lemmas 2.3, 2.4, 7.5 and 7.7, we conclude

Proposition 7.8 *Let Δ be a dual polar space of quadratic type having rank $n \geq 2$. Then every two points x and y of Δ at distance δ are contained in a unique set Z of points satisfying (P1), (P2) and having diameter δ. Moreover, for every point $z \in Z$, there exists a point $z' \in Z$ at distance δ from z.*

Recall that any two points of a dual polar space at distance δ from each other are contained in a unique convex subspace of diameter δ. Combining this fact with Propositions 7.3 and 7.8, we find:

Corollary 7.9 *Let Δ be a dual polar space of quadratic type having rank $n \geq 2$. Then $\overline{\Delta}$ has a regular set of convex subspaces. The set of convex subspaces of $\overline{\Delta}$ thus defines a diagram geometry, with diagram as depicted in Section 1.*

Our next aim is to determine the automorphism groups of the thin near polygons $\overline{\Delta}$, where Δ is a quadratic dual polar space of rank at least 2. In order to achieve this goal, we need some preparatory lemmas.

Lemma 7.10 *Let Q be a nonsingular quadric of Witt index $n \geq 2$ in a projective space Σ over a field $\mathbb{F}$. Let α_1 and α_2 be two disjoint generators of Q and suppose the subspace $\langle \alpha_1, \alpha_2 \rangle$ has co-dimension at least 2 in Σ. Then there is a line L of Σ contained in Q and disjoint from $\langle \alpha_1, \alpha_2 \rangle$.*

Proof. The proof of the lemma makes use of the fact that the points of a nonempty nonsingular quadric of a projective space generates the whole projective space.

Since $\langle Q \rangle = \Sigma$, there this exists a point $x \in Q$ not contained in $\langle \alpha_1, \alpha_2 \rangle$. The tangent hyperplane T_x to Q in the point x cannot contain $\langle \alpha_1, \alpha_2 \rangle$, as otherwise $\langle x, \alpha_1 \rangle$ would be a singular subspace properly containing α_1. So, $T_x \cap \langle \alpha_1, \alpha_2 \rangle$ is a subspace of co-dimension at least 2 in T_x. Since the lines of Q through x generate T_x, there exists a line L of Q through x disjoint from $T_x \cap \langle \alpha_1, \alpha_2 \rangle$, i.e. disjoint from $\langle \alpha_1, \alpha_2 \rangle$. ■

Lemma 7.11 *Let Y be a nonempty set of points of $\mathcal{S}$ satisfying (P1) and (P2). Let $y^* \in Y$ and $\epsilon^* \in \{+, -\}$. If there exists a point x of $\mathcal{S}$ such that there are least three points in Y nearest to x, then the convex subspace $\Phi_{y^*,\epsilon^*}(Y)$ of $\overline{\mathcal{S}}$ is not classical.*

Proof. Let y_1, y_2 and y_3 be three points in Y nearest to x. Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}$ such that $(y_i, \epsilon_i) \in \Phi_{y^*,\epsilon^*}(Y)$ for every $i \in \{1, 2, 3\}$. Let $\epsilon_i' \in \{+, -\}$ such that (x, ϵ_i') is the unique point of L_x nearest to (y_i, ϵ_i). Suppose $\epsilon \in \{+, -\}$ occurs at least two times in the sequence $\epsilon_1', \epsilon_2', \epsilon_3'$. Then by Lemma 3.2(b) two of the points (y_i, ϵ_i), $i \in \{1, 2, 3\}$, are among the points of $\Phi_{y^*,\epsilon^*}(Y)$ that are nearest to (x, ϵ). So, $\Phi_{y^*,\epsilon^*}(Y)$ cannot be classical. ■

Lemma 7.12 *Let Q be a nonsingular quadric of Witt index $n \geq 2$ in a projective space Σ over a field $\mathbb{F}$. Let α_1 and α_2 be two disjoint generators of Q such that the subspace*

$\langle \alpha_1, \alpha_2 \rangle$ has co-dimension at least 2 in Σ. Let Y be the set of generators contained in $\langle \alpha_1, \alpha_2 \rangle$. Then in the dual polar space Δ, there exists a point x such that at least three points in Y are nearest to x.

Proof. By Lemma 7.10, we can take a line L of Q disjoint from $\langle \alpha_1, \alpha_2 \rangle$. Let α be a generator through L, and put $\beta = \alpha \cap \langle \alpha_1, \alpha_2 \rangle$. Then β has co-dimension at least 2 in α and hence co-dimension at least 2 in any generator of $Q^+(2n-1, \mathbb{F}) = \langle \alpha_1, \alpha_2 \rangle \cap Q$. So, there must be three generators γ_1, γ_2 and γ_3 of $Q^+(2n-1, \mathbb{F})$ through β. For any generator γ contained in $\langle \alpha_1, \alpha_2 \rangle$, we have $\mathrm{d}_\Delta(\alpha, \gamma) = n-1-\dim(\alpha \cap \gamma) \geq n-1-\dim(\beta) = n-1-\dim(\gamma_i) = \mathrm{d}_\Delta(\alpha, \gamma_i)$ for every $i \in \{1, 2, 3\}$. So, there are at least three points in Y nearest to the point x in Δ corresponding to the maximal singular subspace α. ∎

The following is an immediate consequence of Lemmas 7.11, 7.12 and the classification of the nonempty sets of points satisfying (P1) and (P2).

Corollary 7.13 *Suppose Δ is a quadratic dual polar space of rank $n \geq 2$ associated with a quadric in a projective space of dimension at least $2n+1$. Then in $\overline{\Delta}$, there are no classical convex subspaces of Type II that have diameter at least 2.*

Proposition 7.14 *Let Q be a nonsingular quadric of Witt index $n \geq 2$ in a projective space Σ over a field $\mathbb{F}$. Let Δ be the dual polar space associated with Q. If $\dim(\Sigma) \geq 2n+1$, then $Aut(\overline{\Delta}) \cong Aut(\Delta) \times C_2$.*

Proof. Let x be a point of Δ and y a point of L_x. As the lines of Δ through x intersect in the singleton $\{x\}$, the quads of type I of $\overline{\Delta}$ through y intersect in the line L_x. Invoking Lemma 4.7 and Corollary 7.13, we this see that the classical quads through a given point of $\overline{\Delta}$ intersect in a line of the spread S. So, the spread S is stabilized by any automorphism of $\overline{\Delta}$. Proposition 3.7 then implies that $Aut(\overline{\Delta}) \cong Aut(\Delta) \times C_2$. ∎

In the case that $\dim(\Sigma) = 2n-1 \geq 3$, then $\Delta \cong DQ^+(2n-1, 2)$ and hence $\overline{\Delta} \cong DQ^+(2n-1, 2) \times \mathbb{L}_2$ by Proposition 3.8. Also in this case, we have $Aut(\overline{\Delta}) \cong Aut(\Delta) \times C_2$. In the case that $\dim(\Sigma) = 2n \geq 4$, then $\Delta \cong DQ(2n, 2)$ and $\overline{\Delta} \cong DQ^+(2n+1, 2)$. In this case, $Aut(\overline{\Delta})$ is much bigger than the automorphism group $Aut(\Delta) \times C_2$ mentioned in Proposition 3.7.

Another instance where $\overline{\mathcal{S}}$ has a regular set of convex subspaces is the case where $\mathcal{S}$ is a Hamming near polygon. Suppose $\mathcal{S}$ is a Hamming near polygon of diameter d, i.e. the direct product of d lines. Let x and y be two opposite points of $\mathcal{S}$ and let Y be the smallest convex set of points containing x and y. The graph induced on Y by the collinearity relation is then a d-dimensional cube. The set Y satisfies Property (P2). Note that the convex subspaces of Hamming near polygons are classical on which the induced geometries are also Hamming near polygons. Following a completely similar reasoning as in the case of dual polar spaces of quadratic type, we then see that the following should hold.

Proposition 7.15 *Let $\mathcal{S}$ be a Hamming near polygon. Then the following hold.*

- *Every two points x and y of $\mathcal{S}$ at distance δ are contained in a unique set Z of points satisfying (P1), (P2) and having diameter δ. Moreover, for every point $z \in Z$, there exists a point $z' \in Z$ at distance δ from z.*

- *$\overline{\mathcal{S}}$ has a regular set of convex subspaces. The set of convex subspaces of $\overline{\mathcal{S}}$ thus defines a diagram geometry, with diagram as depicted in Section 1.*

In the case that every line of the Hamming near polygon $\mathcal{S}$ is incident with at least three points, we can use Lemma 7.11 (in a similar fashion as above) to show that $Aut(\overline{\mathcal{S}}) \cong Aut(\mathcal{S}) \times C_2$. The restriction that there are at least three points per line cannot be omitted. E.g. if $\mathcal{S}$ is the d-dimensional cube, then by Proposition 3.8, $\overline{\mathcal{S}}$ is the $(d+1)$-dimensional cube, and so we have $|Aut(\mathcal{S})| = 2^d d!$ and $|Aut(\overline{\mathcal{S}})| = 2^{d+1}(d+1)!$.

7.3 Other examples

Let us first look to those finite near polygons $\overline{\mathcal{S}}$ of the family for which every pair of points at distance 2 is contained in a quad of the same size. Suppose all quads are duals of $(s+1) \times (s+1)$-grids. For the Type I quads this means that every line of $\mathcal{S}$ is incident with precisely $s+1$ points. For the Type II quads this implies that every two points of $\mathcal{S}$ at distance 2 have precisely $s+1$ common neighbours. So, every two points of $\overline{\mathcal{S}}$ at distance 2 are contained in a quad of the same size if and only if there exists an $s \in \mathbb{N} \setminus \{0\}$ such that every two points x and y of $\mathcal{S}$ at distance 2 are contained in a quad of order (s,s) in which the pair $\{x,y\}$ is regular. The following is a special case of this situation.

Lemma 7.16 *Suppose $\mathcal{S}$ is a near $2d$-gon, $d \geq 2$, with the property that every two points at distance 2 are contained in a $W(q)$-quad, with q a prime power. Then $\overline{\mathcal{S}}$ is a near $(2d+1)$-gon with the property that every two points at distance 2 are contained in a quad which is the dual of a $(q+1) \times (q+1)$-grid.*

Examples of near $2d$-gons satisfying the property mentioned in Lemma 7.16 are the dual polar spaces $DQ(2d,q)$, $d \geq 2$, and the M_{24} near hexagon $\mathbb{E}_2$ defined in [9, Section 3.6]. If $\mathcal{S} \cong DQ(2d,q)$, then we know that $\overline{\mathcal{S}} \cong DQ^+(2d+1,q)$. In this case, we already know that every two points of $\overline{\mathcal{S}}$ at distance δ are contained in a unique convex subspace of diameter δ and that the automorphism group of $\overline{\mathcal{S}}$ is bigger than what we would expect from Proposition 3.6. Let is now investigate the convex subspaces and automorphism group of the near octagon $\overline{\mathbb{E}_2}$. For every point x of $\mathbb{E}_2$, let L_x be the line $\{(x,+),(x,-)\}$ of $\overline{\mathbb{E}_2}$, and denote by S the set of all lines L_x, where x is a point of $\mathbb{E}_2$.

Lemma 7.17 *The near hexagon $\mathbb{E}_2$ does not have convex sets of diameter 3 that satisfy $(P1)$ and $(P2)$.*

Proof. Suppose Y is such a set of points, and denote by y_1 and y_2 two points of Y at distance 3 from each other. Let Q be a $W(2)$-quad through y_1 not containing a point of $\Gamma_1(y_2)$. Then each line of Q through y_1 contains a unique point of $\Gamma_2(y_2)$, necessarily belonging to Y as this set is convex. The three points z_1, z_2, z_3 that arise in this way belong to the ovoid $\Gamma_2(y_2) \cap Q$ of Q, and so do not form a hyperbolic line. So, if u is a common neighbour of z_1 and z_2 distinct from y_1, then the fact that $z_1 \not\sim z_3 \not\sim u$ implies that $z_3 \sim v$, where v is the third point on the line $z_1 u$. Since Y is convex, we have $u \in Y$ (since u is on a shortest path from $z_1 \in Y$ to $z_2 \in Y$) and $v \in Y$ (since v is on a shortest path from $u \in Y$ and $z_3 \in Y$). This is however impossible as it would imply that the points z_1, v and u are on a common line. ■

The following is a consequence of Lemma 7.17.

Corollary 7.18 *The near octagon $\overline{\mathbb{E}_2}$ has no convex subspaces of Type II that have diameter* 3.

We remarked above that every two points of $\overline{\mathbb{E}_2}$ at distance 2 are contained in a unique quad. Corollary 7.18 implies that not every two points of $\overline{\mathbb{E}_2}$ at distance 3 are contained in a unique convex subspace of diameter 3.

Lemma 7.19 *The convex subspaces of diameter* 3 *through a given point y of $\overline{\mathbb{E}_2}$ intersect in the unique line of S through y.*

Proof. Let x be the point of $\mathbb{E}_2$ such that $y \in L_x$. As the quads of $\mathbb{E}_2$ through x intersect in the singleton $\{x\}$, the convex subspaces of diameter 3 (necessarily of Type I) through y intersect in the line L_x. ■

Lemma 7.19 and Proposition 3.7 immediately imply the following.

Corollary 7.20 *Every automorphism of $\overline{\mathbb{E}_2}$ stabilizes the line spread S, and so the full automorphism group of $\overline{\mathbb{E}_2}$ is isomorphic to $Aut(\mathbb{E}_2) \times C_2 \cong M_{24} \times C_2$.*

The case of the near hexagon $\mathbb{E}_1$ related to the extended ternary Golay code is similar. If x and y are two opposite points of $\mathbb{E}_1$, then we verified with GAP [10] that the smallest convex set of points containing x and y coincides with the whole point set. So, there are no sets of points of diameter 3 satisfying (P1) and (P2). This means that in the near octagon $\overline{\mathbb{E}_1}$ not every pair of points at distance 3 are contained in a convex subspace of diameter 3. In the near octagon $\overline{\mathbb{E}_1}$, there are also two types of quads, the quads of Type I are duals of (3×3)-grids and the quads of Type II are (2×2)-grids. The quads of Type I through a given point intersect in an element of the line spread. So, every automorphism of $\overline{\mathbb{E}_1}$ stabilizes the line spread, and so $\mathrm{Aut}(\overline{\mathbb{E}_1}) \cong \mathrm{Aut}(\mathbb{E}_1) \times C_2 \cong 3^6 : 2 \cdot M_{12} \times 2$.

We can recycle an argument in the proof of Lemma 7.19 to show the following.

Proposition 7.21 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon of diameter at least* 2 *having the property that every two points at distance* 2 *have a unique common neighbour. Then $Aut(\overline{\mathcal{S}}) = \langle \overline{Aut(\mathcal{S})}, \sigma \rangle$ is isomorphic to $Aut(\mathcal{S}) \times C_2$. In particular, this holds if $\mathcal{S}$ is a generalized $2d$-gon with $d \geq 3$.*

Proof. Lemma 5.5 and the fact that every two points of $\mathcal{S}$ at distance 2 have a unique common neighbour imply that $\overline{\mathcal{S}}$ has no quads of Type II. Since the lines through a given point x of $\mathcal{S}$ intersect in the singleton $\{x\}$, the quads (of Type I) through a given point y of $L_x = \{(x,+),(x,-)\}$ intersect in L_x. So, every automorphism of $\overline{\mathcal{S}}$ stabilizes the line spread $S = \{L_x \,|\, x \in \mathcal{P}\}$, and the claim of the proposition follows once more from Proposition 3.7. ∎

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