The homogeneous pseudo-embeddings and hyperovals of the generalized quadrangle $H(3, 4)$

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Abstract

In this paper, we determine all homogeneous pseudo-embeddings of the generalized quadrangle $H(3, 4)$ and give a description of all its even sets. Using this description, we subsequently compute all hyperovals of $H(3, 4)$, up to isomorphism, and give computer free descriptions of them. Several of these hyperovals, but not all of them, have already been described before in the literature.

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1 Introduction

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with the property that the number of points on each line is finite and at least three. A *pseudo-embedding* of $\mathcal{S}$ is a mapping ϵ from the point set $\mathcal{P}$ of $\mathcal{S}$ to the set of points of a projective space $\mathrm{PG}(V)$, with V an $\mathbb{F}_2$-vector space, such that the following two properties are satisfied:

- the image $\epsilon(\mathcal{P})$ of ϵ generates $\mathrm{PG}(V)$;
- if L is a line with points $x_1, x_2, \ldots, x_k$, then $\epsilon(x_1), \epsilon(x_2), \ldots, \epsilon(x_k)$ is a frame of the subspace $\langle \epsilon(L) \rangle$ of $\mathrm{PG}(V)$, i.e. there exist linearly independent vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k-1}$ of V such that $\epsilon(x_i) = \langle \bar{v}_i \rangle$, $\forall i \in \{1, 2, \ldots, k-1\}$, and $\epsilon(x_k) = \langle \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{k-1} \rangle$.

We often denote such a pseudo-embedding by $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$. Two pseudo-embeddings $\epsilon_1 : \mathcal{S} \to \mathrm{PG}(V_1)$ and $\epsilon_2 : \mathcal{S} \to \mathrm{PG}(V_2)$ of $\mathcal{S}$ are called *isomorphic* if there exists a linear isomorphism θ from V_1 to V_2 such that $\epsilon_2 = \theta \circ \epsilon_1$.

A pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is called *G-homogeneous* for some group G of automorphisms of $\mathcal{S}$ if for every $\theta \in G$, there exists a (necessarily unique) $\widetilde{\theta} \in GL(V)$ such that $\widetilde{\theta} \circ \epsilon = \epsilon \circ \theta$. The map $G \to GL(V) : \theta \mapsto \widetilde{\theta}$ then defines a modular representation of G, and V becomes a G-module. The pseudo-embedding ϵ is called *homogeneous* if it is $\mathrm{Aut}(\mathcal{S})$-homogeneous with $\mathrm{Aut}(S)$ the full group of automorphisms of $\mathcal{S}$.

Suppose $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding of $\mathcal{S}$ and α is a subspace of $\mathrm{PG}(V)$ disjoint from $\epsilon(\mathcal{P})$ and all subspaces $\langle \epsilon(L) \rangle$, where $L \in \mathcal{L}$. Then the map $x \mapsto \langle \alpha, \epsilon(x) \rangle$ defines a *pseudo-embedding* ϵ/α of $\mathcal{S}$ into the quotient projective space $\mathrm{PG}(V)/\alpha$ (whose points are the subspaces of $\mathrm{PG}(V)$ that contain α as a hyperplane). We call ϵ/α a *quotient* of ϵ. If ϵ_1, ϵ_2 are two pseudo-embeddings of $\mathcal{S}$, then we write $\epsilon_1 \geq \epsilon_2$ if ϵ_2 is isomorphic to a quotient of ϵ_1.

If $\widetilde{\epsilon}$ is a pseudo-embedding of $\mathcal{S}$ such that $\widetilde{\epsilon} \geq \epsilon$ for any other pseudo-embedding ϵ of $\mathcal{S}$, then $\widetilde{\epsilon}$ is called a *universal pseudo-embedding*. If $\mathcal{S}$ has pseudo-embeddings, then it also has a universal pseudo-embedding which is moreover unique, up to isomorphism. The universal pseudo-embedding is always homogeneous. The vector dimension of the universal pseudo-embedding is called the *pseudo-embedding rank*. In case $|\mathcal{P}| < \infty$, the pseudo-embedding rank is equal to $|\mathcal{P}| - \dim(C)$, where C is the binary code of length $|\mathcal{P}|$ generated by the characteristic vectors of the lines of $\mathcal{S}$.

Pseudo-embeddings were introduced in [9] and further investigated in [8, 10]. We refer to these papers for proofs of the above-mentioned facts. As mentioned above, there exist connections between pseudo-embeddings and certain binary codes. The various results obtained in the papers [8, 9, 10] offer extra tools by which these binary codes can be studied. There also exist connections between G-homogeneous pseudo-embeddings and $\mathbb{F}_2$-representations for G. In practice, many of these $\mathbb{F}_2$-representations turn out to be reducible. Also, given an $\mathbb{F}_2$-representation of G, there is not necessarily an associated G-homogeneous pseudo-embedding, and if there is one, there might not be a natural way to decide which orbit(s) of the G-module correspond(s) to the points of the geometry. It is possible that G occurs as automorphism group of two non-isomorphic geometries and that a certain G-module hosts a G-homogeneous pseudo-embedding for only one of them. For studying G-homogeneous pseudo-embeddings, the group G does therefore not tell the whole story, and the underlying geometric structure of the geometry should still be taken into account. In this paper, we classify all homogeneous pseudo-embeddings of a particular generalized quadrangle, and give some applications of this classification.

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *generalized quadrangle* [17] if every point is incident with at least two lines and if for every non-incident point-line pair (x, L), there exists a unique point on L collinear with x. A generalized quadrangle is said to have *order* (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with exactly $t + 1$ lines. The point-line dual of any generalized quadrangle is again a generalized quadrangle. A standard counting yields that a generalized quadrangle of order (s, t) contains $(s + 1)(st + 1)$ points and $(t + 1)(st + 1)$ lines.

Consider now in $\mathrm{PG}(3, 4)$ the Hermitian surface $\mathcal{H}$ with equation $X_1X_2^2 + X_2X_1^2 + X_3X_4^2 + X_4X_3^2 = 0$. The points and lines of $\mathrm{PG}(3, 4)$ contained in $\mathcal{H}$ then define a generalized quadrangle of order $(4, 2)$ on 45 points which we will denote by $H(3, 4)$. In this paper, we determine the pseudo-embedding rank and all homogeneous pseudo-embeddings of $H(3, 4)$. Our results are as follows.

Theorem 1.1. *The pseudo-embedding rank of $H(3,4)$ is equal to 24. As a consequence, the dimension of the binary code generated by the characteristic vectors of the lines of $H(3,4)$ has dimension $45-24=21$.*

We also show that there are up to isomorphism four homogeneous pseudo-embeddings of $H(3,4)$, with respective vector dimensions 14, 15, 23 and 24. In order to describe these, we need to introduce a number of notations. We put $\mathbb{F}_2=\{0,1\}\subseteq\mathbb{F}_4=\{0,1,\omega,\omega^2\}$, and consider a 24-dimensional $\mathbb{F}_2$-vector space V_{24} with a basis B consisting of the following vectors:

- $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$, $\bar{h}_{34}$, $\bar{k}$,
- $\bar{g}_{12}$, $\bar{h}_{12}$, $\bar{g}_{13}$, $\bar{h}_{13}$, $\bar{g}_{14}$, $\bar{h}_{14}$, $\bar{g}_{23}$, $\bar{h}_{23}$, $\bar{g}_{24}$, $\bar{h}_{24}$,
- $\bar{g}_{123}$, $\bar{h}_{123}$, $\bar{g}_{124}$, $\bar{h}_{124}$, $\bar{g}_{134}$, $\bar{h}_{134}$, $\bar{g}_{234}$, $\bar{h}_{234}$.

Consider in V_{24} the following subspaces:

- V_{23} is generated by the vectors of $B\setminus\{\bar{k}\}$;
- V_{15} is generated by the vectors of $B\setminus\{\bar{k},\bar{g}_{123},\bar{h}_{123},\ldots,\bar{g}_{234},\bar{h}_{234}\}$;
- V_{14} is generated by the vectors of $B\setminus\{\bar{k},\bar{h}_{34},\bar{g}_{123},\bar{h}_{123},\ldots,\bar{g}_{234},\bar{h}_{234}\}$.

Consider also the following summations:

- Σ_1: summation over all $i\in\{1,2,3,4\}$;
- Σ_2: summation over all $i,j\in\{1,2,3,4\}$ with $i<j$;
- $\Sigma_{2'}$: summation over all $i,j\in\{1,2,3,4\}$ with $i<j$ and $(i,j)\neq(3,4)$;
- Σ_3: summation over all $i,j,k\in\{1,2,3,4\}$ with $i<j<k$.

We consider now four maps from $\mathcal{H}$ to the point sets of certain projective spaces. The map ϵ_{24} maps the point $(X_1,X_2,X_3,X_4)\in\mathcal{H}$ to the point of $\mathrm{PG}(V_{24})$ that is generated by the vector

$$(\omega X_3X_4^2+\omega^2X_4X_3^2)\bar{h}_{34}+\Big((X_1^3+X_2^3+X_1^3X_2^3)(X_3^3+X_4^3+X_3^3X_4^3)+1\Big)\bar{k}$$
$$+\sum_1 X_i^3\bar{g}_i+\sum_{2'}\Big((X_iX_j^2+X_jX_i^2)\bar{g}_{ij}+(\omega X_iX_j^2+\omega^2X_jX_i^2)\bar{h}_{ij}\Big)$$
$$+\sum_3\Big((X_iX_jX_k+X_i^2X_j^2X_k^2)\bar{g}_{ijk}+(\omega X_iX_jX_k+\omega^2X_i^2X_j^2X_k^2)\bar{h}_{ijk}\Big).$$

Note that the latter is indeed a vector of V_{24} as $a^3, b+b^2\in\mathbb{F}_2$ for all $a,b\in\mathbb{F}_4$. The map ϵ_{23} maps the point $(X_1,X_2,X_3,X_4)\in\mathcal{H}$ to the point of $\mathrm{PG}(V_{23})$ that is generated by the vector

$$(\omega X_3X_4^2+\omega^2X_4X_3^2)\bar{h}_{34}+\sum_1 X_i^3\bar{g}_i+\sum_{2'}\Big((X_iX_j^2+X_jX_i^2)\bar{g}_{ij}+(\omega X_iX_j^2+\omega^2X_jX_i^2)\bar{h}_{ij}\Big)$$
$$+\sum_3\Big((X_iX_jX_k+X_i^2X_j^2X_k^2)\bar{g}_{ijk}+(\omega X_iX_jX_k+\omega^2X_i^2X_j^2X_k^2)\bar{h}_{ijk}\Big).$$

The map ϵ_{15} maps the point $(X_1,X_2,X_3,X_4)\in\mathcal{H}$ to the point of $\mathrm{PG}(V_{15})$ that is generated by the vector

$$(\omega X_3X_4^2+\omega^2X_4X_3^2)\bar{h}_{34}+\sum_1 X_i^3\bar{g}_i+\sum_{2'}\Big((X_iX_j^2+X_jX_i^2)\bar{g}_{ij}+(\omega X_iX_j^2+\omega^2X_jX_i^2)\bar{h}_{ij}\Big).$$

The map ϵ_{14} maps the point $(X_1, X_2, X_3, X_4) \in \mathcal{H}$ to the point of $\mathrm{PG}(V_{14})$ that is generated by the vector

$$(\omega X_3 X_4^2 + \omega^2 X_4 X_3^2)\bar{h}_{12} + \sum_1 X_i^3 \bar{g}_i + \sum_{2'} \Big((X_i X_j^2 + X_j X_i^2)\bar{g}_{ij} + (\omega X_i X_j^2 + \omega^2 X_j X_i^2)\bar{h}_{ij} \Big).$$

We will prove the following.

Theorem 1.2. *Up to isomorphism, $H(3,4)$ has four homogeneous pseudo-embeddings. Specifically, every pseudo-embedding of $H(3,4)$ is isomorphic to either ϵ_{24}, ϵ_{23}, ϵ_{15} or ϵ_{14}. The pseudo-embedding ϵ_{24} is universal.*

An *even set* of a point-line geometry is a set of points meeting each line in an even number of points. The empty point set is the trivial example of an even set. A nontrivial even set intersecting each line in either 0 or 2 points is called a *hyperoval.* Hyperovals of generalized quadrangles play a fundamental role in the study of the so-called extended generalized quadrangles, see e.g. [3, 13, 14, 15]. The more general notion of a hyperoval (sometimes also called a *local subspace*) of a polar space has also been studied; these objects first arose in [2] in view of their connection with so-called locally polar spaces.

The complements of the nontrivial even sets are also called *pseudo-hyperplanes.* If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry for which the number of points on each line is finite and at least three such that $\mathcal{S}$ has a pseudo-embedding, then by [9, Theorem 1.3] all pseudo-hyperplanes of $\mathcal{S}$ have the form $H_\Pi := \widetilde{\epsilon}^{-1}(\widetilde{\epsilon}(\mathcal{P}) \cap \Pi)$, where $\widetilde{\epsilon} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ is the universal pseudo-embedding of $\mathcal{S}$ and Π is a hyperplane of $\mathrm{PG}(\widetilde{V})$. The correspondence $\Pi \leftrightarrow H_\Pi$ is moreover bijective. In view of Theorem 1.2, we thus have the following.

Corollary 1.3. *The even sets of $H(3,4)$ are precisely the subsets of $\mathcal{H}$ satisfying an equation of the form*

$$\sum_1 (a_i X_i^3) + a_5(\omega X_3 X_4^2 + \omega^2 X_4 X_3^2) + a_6\Big((X_1^3 + X_2^3 + X_1^3 X_2^3)(X_3^3 + X_4^3 + X_3^3 X_4^3) + 1 \Big)$$

$$+ \sum_{2'} (b_{ij} X_i X_j^2 + b_{ij}^2 X_j X_i^2) + \sum_3 \Big(b_{ijk} X_i X_j X_k + b_{ijk}^2 X_i^2 X_j^2 X_k^2 \Big) = 1,$$

with the a_i's belonging to $\mathbb{F}_2$ and the b_{ij}'s and b_{ijk}'s belonging to $\mathbb{F}_4$.

There are thus 2^{24} even sets in $H(3,4)$. In fact, Corollary 1.3 gives a bijective correspondence between the elements $\bar{a} = (a_1, \ldots, a_6, b_{12}, \ldots, b_{24}, b_{123}, \ldots, b_{234}) \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ and the even sets $E(\bar{a})$ of $H(3,4)$. With the aid of a computer, we will determine for which $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$, $E(\bar{a})$ is a hyperoval. We also investigate when two elements $\bar{a}_1, \bar{a}_2 \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ give rise to isomorphic even sets $E(\bar{a}_1)$ and $E(\bar{a}_2)$. This allows us to conclude the following.

Theorem 1.4. *The generalized quadrangle $H(3,4)$ has 70648 hyperovals which fall into 23 isomorphism classes.*

Although many of these hyperovals have already been described before in the literature, several of them also appear to be new. In fact, for each construction mentioned in the literature we will indicate with which of the 23 hyperovals it corresponds. Based on Corollary 1.3, we also give explicit unified descriptions for representatives of the various isomorphism classes of hyperovals. In order to keep these descriptions as simple as possible, we have chosen representatives for which the corresponding elements $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ have smallest possible weights.

We will also determine some properties of the hyperovals. This will make identification easier each time a hyperoval of $H(3,4)$ emerges somewhere. One of these properties is a description of the 3-regular subgraph of the collinearity graph induced on the hyperoval. In fact, these regular graphs were also classified in [13] (up to isomorphism) for the hyperovals on 6, 8, 10, 12 and 14 vertices. Our classification of the hyperovals shows that some 3-regular graphs on 12 and 14 vertices are missing in the discussion in [13] (and others do not correspond to hyperovals). We will see that it is also possible that non-isomorphic hyperovals can have isomorphic underlying 3-regular subgraphs. Note that a list of the 3-regular subgraphs that can occur as induced subgraphs of the hyperovals gives important structural information about the hyperovals, but not at all a complete classification.

2 Enlarging pseudo-embeddings

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Suppose $\mathcal{S}$ has a pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$. Then for every hyperplane Π of $\mathrm{PG}(V)$, the set $H_\Pi := \epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi)$ is a pseudo-hyperplane of $\mathcal{S}$. We will say that H_Π *arises* from ϵ. We denote by $\mathcal{H}_\epsilon$ the set of all pseudo-hyperplanes that arise from ϵ. If $\mathcal{S}$ has pseudo-embeddings and $\widetilde{\epsilon} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ denotes its universal pseudo-embedding, then as mentioned before $\mathcal{H}_{\widetilde{\epsilon}}$ is the set of all pseudo-hyperplanes.

Under certain circumstances, it is possible to modify an existing pseudo-embedding to obtain a "larger" pseudo-embedding. This procedure is explained in the following proposition.

Proposition 2.1. *Let $\epsilon : \mathcal{S} \to \Sigma$ be a pseudo-embedding of $\mathcal{S}$ and suppose H is a pseudo-hyperplane of $\mathcal{S}$ not arising from ϵ. Embed Σ as a hyperplane in $\overline{\Sigma}$, and let $p \in \overline{\Sigma} \setminus \Sigma$. For every point $x \in H$, we define $\bar{\epsilon}(x) := \epsilon(x)$ and for every point y of $\mathcal{S}$ not contained in H, let $\bar{\epsilon}(y)$ denote the third point on the line $\epsilon(y)p$. Then $\bar{\epsilon} : \mathcal{S} \to \overline{\Sigma}$ is a pseudo-embedding of $\mathcal{S}$.*

Proof. Put $\Sigma = \mathrm{PG}(V)$ and $\overline{\Sigma} = \mathrm{PG}(\overline{V})$, where V is a hyperplane of the $\mathbb{F}_2$-vector space $\overline{V}$. Let $\bar{v}^* \in \overline{V}$ such that $p = \langle \bar{v}^* \rangle$. Let $L = \{x_1, x_2, \ldots, x_k\}$ be an arbitrary line of $\mathcal{S}$. Without loss of generality, we may suppose that there exists an $l \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$ such that $x_1, x_2, \ldots, x_{k-2l} \in H$ and $x_{k-2l+1}, \ldots, x_k \notin H$. Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$ such that $\epsilon(x_i) = \langle \bar{v}_i \rangle$ for every $i \in \{1, 2, \ldots, k\}$. For every $i \in \{1, 2, \ldots, k-2l\}$, we put $\bar{v}_i' := \bar{v}_i$, and for every $i \in \{k-2l+1, \ldots, k\}$, we put $\bar{v}_i' = \bar{v}_i + \bar{v}^*$. Then $\bar{\epsilon}(x_i) = \langle \bar{v}_i' \rangle$ for every

$i \in \{1, 2, \ldots, k\}$. We have $\bar{v}_1' + \bar{v}_2' + \cdots + \bar{v}_k' = \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_k + 2l \cdot \bar{v}^* = \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_k = \bar{o}$. Moreover, since the vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k-1}$ are linearly independent, also $\bar{v}_1', \bar{v}_2', \ldots, \bar{v}_{k-1}'$ are linearly independent. So, $\bar{\epsilon}$ is a pseudo-embedding of $\mathcal{S}$ in a suitable subspace of $\overline{\Sigma}$.

We still need to show that the image of $\bar{\epsilon}$ generates the whole projective space $\overline{\Sigma}$. Suppose the image is contained in a hyperplane Π. Then $\Pi \neq \Sigma$ and so $\Pi' := \Pi \cap \Sigma$ is a hyperplane of Σ. If Π would contain p, then the image of ϵ would be contained in Π', which is impossible. So, Π is the unique hyperplane through Π' distinct from $\langle p, \Pi' \rangle$ and Σ. Now, $\bar{\epsilon}$ maps the points of H to points of Π' and the points not in H to points of $\Pi \setminus \Pi'$. This implies that ϵ maps points of H to points of Π' and points not in H to points of $\Sigma \setminus \Pi'$. But that is impossible as H does not arise from ϵ. □

3 An upper bound for the pseudo-embedding rank of $H(3,4)$

Not all point-line geometries with the property that the number of points on each line is finite and at least three have pseudo-embeddings. In [9] (see Corollary 3.11(1)), it was however shown that every finite generalized quadrangle of order (s,t), $s \geq 2$, admits pseudo-embeddings. In particular, this applies to the generalized quadrangle $H(3,4)$. In this section, we show that the pseudo-embedding rank of $H(3,4)$ is at most 24. This upper bound is essential for the treatment given in Section 4, where we will show among other things that the pseudo-embedding rank is precisely 24.

Again, let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. A *pseudo-subspace* of $\mathcal{S}$ is a set S of points with the property that no line L intersects the complement of S in a singleton, i.e. if there are k points on L and $k-1$ of these points are known to belong to S then also the remaining point must belong to S. The whole point set $\mathcal{P}$ is an example of a pseudo-subspace, and the intersection of any number of pseudo-subspaces is again a pseudo-subspace. So, given a nonempty set of points X, the intersection $[X]$ of all pseudo-subspaces containing X is the smallest pseudo-subspace that contains X. If $[X]$ coincides with the whole point set, then X is called a *pseudo-generating set*. The smallest size of a pseudo-generating set is called the *pseudo-generating rank*. We following proposition is precisely Theorem 1.5 of [9].

Proposition 3.1. *Suppose $\mathcal{S}$ has pseudo-embeddings. Then the following hold.*

(1) *The pseudo-embedding rank of $\mathcal{S}$ is bounded above by the pseudo-generating rank of $\mathcal{S}$.*

(2) *If there exists a pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ and a pseudo-generating set X of $\mathcal{S}$ such that $|X| = \dim(V) < \infty$, then the pseudo-embedding and pseudo-generating ranks of $\mathcal{S}$ are equal to $\dim(V)$ and ϵ is isomorphic to the universal pseudo-embedding of $\mathcal{S}$.*

Lemma 3.2. *$H(3,4)$ has a pseudo-generating set of size 24.*

Proof. We will reason in the dual generalized quadrangle $H^D(3,4)$ of $H(3,4)$. We recall some facts about this dual generalized quadrangle taken from [17]. The generalized quadrangle $H^D(3,4)$ is isomorphic to the generalized quadrangle $Q(5,2)$ of the points and lines of an elliptic quadric in the projective space $\mathrm{PG}(5,2)$. Its order is $(2,4)$ and it contains a sub(generalized)quadrangle $Q(4,2)$ of order $(2,2)$. For every point x of $Q(5,2)$, $x^\perp$ denotes the set of points of $Q(5,2)$ equal to or collinear with x. If x does not belong to $Q(4,2)$, then $O_x = x^\perp \cap Q(4,2)$ is an ovoid of $Q(4,2)$, i.e. a set of five points meeting each line in a singleton. We call O_x the ovoid of $Q(4,2)$ subtended by the point x. The generalized quadrangle $Q(4,2)$ has six ovoids, which two by two intersect in a singleton. Each of these ovoids is subtended by exactly two points of $Q(5,2) \setminus Q(4,2)$. Every point x of $Q(4,2)$ is contained in precisely two ovoids. If $\{x, y_1, y_2\}$ is a line of $Q(5,2)$ through x not contained in $Q(4,2)$, then these ovoids are precisely $y_1^\perp \cap Q(4,2)$ and $y_2^\perp \cap Q(4,2)$.

Now, let O be an ovoid of $Q(4,2)$ and L a line of $Q(4,2)$. Put $\{x^*\} = L \cap O$. Let $\mathcal{L}_1$ be the set of 14 lines of $Q(4,2)$ distinct from L and let $\mathcal{L}_2$ be a set of 10 lines of $Q(5,2)$ not contained in $Q(4,2)$ such that each of the 10 points of $Q(4,2) \setminus O$ is contained in a unique line of $\mathcal{L}_2$. We show that the 24-set $\mathcal{L}_1 \cup \mathcal{L}_2$ is a pseudo-generating set of the point-line dual $Q^D(5,2) \cong H(3,4)$ of $Q(5,2)$. Let $\mathcal{L}$ denote the smallest pseudo-subspace of $Q^D(5,2)$ containing $\mathcal{L}_1 \cup \mathcal{L}_2$.

Each of the eight points of $Q(4,2) \setminus (L \cup O)$ is contained in three lines of $\mathcal{L}_1$ and one line of $\mathcal{L}_2$ and so all five lines through that point belong to $\mathcal{L}$.

Now, let O' be the unique ovoid of $Q(4,2)$ through x^* distinct from O. This ovoid O' is subtended by two points u_1 and u_2. Then x^*u_1 and x^*u_2 are the two lines of $Q(5,2)$ through x^* not contained in $Q(4,2)$. The four lines through u_i, $i \in \{1,2\}$, meeting $(O' \setminus L) \subseteq (Q(4,2) \setminus (L \cup O))$ are contained in $\mathcal{L}$ and so also the fifth line through u_i also belongs to $\mathcal{L}$. This implies that all two lines of $Q(5,2)$ through x^* not contained in $Q(4,2)$ belong to $\mathcal{L}$. As there are also two lines of $\mathcal{L}_1 \subseteq \mathcal{L}$ through x^*, we see that the fifth line L through x^* also belongs to $\mathcal{L}$. Every point of $L \setminus \{x^*\}$ is thus contained in three lines of $\mathcal{L}$ that are in $Q(4,2)$ and one line of $\mathcal{L}_2 \subseteq \mathcal{L}$, implying that also the fifth line through that point belongs to $\mathcal{L}$.

The above allows to conclude that all lines of $Q(4,2)$ as well as all lines intersecting $Q(4,2)$ in a point not contained in O belong to $\mathcal{L}$.

We now show that also every line K meeting $Q(4,2)$ in a point $x \in O$ belongs to $\mathcal{L}$. Take to that end the point $w \in K \setminus \{x\}$ such that $w^\perp \cap Q(4,2) \neq O$. Then the four lines through w distinct from K belong to $\mathcal{L}$, implying that also K must belong to $\mathcal{L}$. □

The following is a consequence of Proposition 3.1(1) and Lemma 3.2.

Corollary 3.3. *The pseudo-embedding and pseudo-generating ranks of* $H(3,4)$ *are at most* 24.

4 Three homogeneous pseudo-embeddings of $H(3,4)$

The generalized quadrangle $H(3,4)$ is naturally embedded in the projective space $\mathrm{PG}(3,4)$, such that every automorphism of $H(3,4)$ is induced by an automorphism of $\mathrm{PG}(3,4)$. This implies that every homogeneous pseudo-embedding of $\mathrm{PG}(3,4)$ induces a homogeneous pseudo-embedding of $H(3,4)$. By [8, Theorem 1.4], there are up to isomorphism two homogeneous pseudo-embeddings of $\mathrm{PG}(3,4)$, the universal pseudo-embedding (which has vector dimension 24) and the Hermitean Veronese embedding (which has vector dimension 16).

Let V'_{24} be an $\mathbb{F}_2$-vector space of dimension 24 that meets V_{24} in the subspace V_{23}. We denote by $\bar{g}'_{34}$ an arbitrary vector of $V'_{24} \setminus V_{23}$, and put $V'_{16} = \langle V_{15}, \bar{g}'_{34} \rangle$. We define $\bar{g}'_{12} := \bar{g}_{12} + \bar{g}'_{34}$ and $\bar{g}'_{ij} := \bar{g}_{ij}$ for all $i,j \in \{1,2,3,4\}$ with $i<j$ and $(1,2) \neq (i,j) \neq (3,4)$.

By [8, Theorem 1.1], the map $\epsilon_u : \mathrm{PG}(3,4) \to \mathrm{PG}(V'_{24})$ that maps the point (X_1, X_2, X_3, X_4) to the point of $\mathrm{PG}(V'_{24})$ generated by the vector

$$\sum_1 X_i^3 \bar{g}_i + \sum_2 \Big((X_i X_j^2 + X_j X_i^2) \bar{g}'_{ij} + (\omega X_i X_j^2 + \omega^2 X_j X_i^2) \bar{h}_{ij} \Big)$$
$$+ \sum_3 \Big((X_i X_j X_k + X_i^2 X_j^2 X_k^2) \bar{g}_{ijk} + (\omega X_i X_j X_k + \omega^2 X_i^2 X_j^2 X_k^2) \bar{h}_{ijk} \Big)$$

is isomorphic to the universal pseudo-embedding of $\mathrm{PG}(3,4)$. For points (X_1, X_2, X_3, X_4) of $\mathcal{H}$ (satisfying $X_1 X_2^2 + X_2 X_1^2 + X_3 X_4^2 + X_4 X_3^2 = 0$), the images of ϵ_u and ϵ_{23} (as defined in Section 1) coincide. So, ϵ_{23} determines a homogeneous pseudo-embedding of $H(3,4)$ in a suitable subspace Σ_{23} of $\mathrm{PG}(V_{23})$.

By [8, Section 3.2], the map $\epsilon_h : \mathrm{PG}(3,4) \to \mathrm{PG}(V'_{16})$ that maps the point (X_1, X_2, X_3, X_4) to the point of $\mathrm{PG}(V'_{16})$ generated by the vector

$$\sum_1 X_i^3 \bar{g}_i + \sum_2 \Big((X_i X_j^2 + X_j X_i^2) \bar{g}'_{ij} + (\omega X_i X_j^2 + \omega^2 X_j X_i^2) \bar{h}_{ij} \Big)$$

is a homogeneous pseudo-embedding (the so-called Hermitean Veronese embedding of $\mathrm{PG}(3,4)$). For points (X_1, X_2, X_3, X_4) of $\mathcal{H}$, the images of ϵ_h and ϵ_{15} (as defined in Section 1) coincide. So, ϵ_{15} determines a homogeneous pseudo-embedding of $H(3,4)$ in a suitable subspace Σ_{15} of $\mathrm{PG}(V_{15})$.

Suppose now that $\mathrm{PG}(3,4) = \mathrm{PG}(V)$ for some vector space V of dimension 4 over $\mathbb{F}_4$ and that $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ is a basis of V such that the point (X_1, X_2, X_3, X_4) of $\mathrm{PG}(3,4)$ coincides with the point $\langle \sum_1 X_i \bar{b}_i \rangle$.

Lemma 4.1. *We have* $\Sigma_{15} = \mathrm{PG}(V_{15})$.

Proof. Considering the points $\langle \bar{b}_i \rangle$ ($i \in \{1,2,3,4\}$) and $\langle \bar{b}_i + \bar{b}_j \rangle$ ($i,j \in \{1,2,3,4\}$ with $i<j$) of $\mathcal{H}$, we see that the points $\langle \bar{g}_i \rangle$ ($i \in \{1,2,3,4\}$), $\langle \bar{h}_{ij} \rangle$ ($i,j \in \{1,2,3,4\}$ with $i<j$) belong to Σ_{15}. Let Σ'' denote the subspace of $\mathrm{PG}(V_{15})$ generated by all these points, and let Σ' denote the subspace of $\mathrm{PG}(V_{15})$ generated by the points $\langle \bar{g}_{ij} \rangle$, $i,j \in \{1,2,3,4\}$ with $i<j$ and $(i,j) \neq (3,4)$. Then $\Sigma'' \subseteq \Sigma_{15}$. As $\mathrm{PG}(V_{15}) = \langle \Sigma', \Sigma'' \rangle$, it thus suffices to show that also $\Sigma' \subseteq \Sigma_{15}$. The fact that $\Sigma'' \subseteq \Sigma_{15}$ implies the following.

For every point x for which $\epsilon_{15}(x) \notin \Sigma''$, we have $\epsilon(x) \in \Sigma_{15}$, where $\epsilon(x)$ is the unique point of Σ' in the subspace $\langle \Sigma'', \epsilon_{15}(x) \rangle$.

We will now apply this fact to points of $\mathcal{H}$. Note that the above point $\epsilon(x)$ is equal to $\langle \sum_{2'} (X_i X_j^2 + X_j X_i^2) \bar{g}_{ij} \rangle$ which is symmetric in the subindices 1 and 2, and also in the subindices 3 and 4.

Considering the points $(1, \omega, \omega^2, 1)$ and $(1, \omega, \omega, 1)$ of $\mathcal{H}$, we find

$$\langle \bar{g}_{12} + \bar{g}_{13} + \bar{g}_{23} + \bar{g}_{24} \rangle \in \Sigma_{15}, \tag{1}$$

$$\langle \bar{g}_{12} + \bar{g}_{13} + \bar{g}_{24} \rangle \in \Sigma_{15}. \tag{2}$$

From equations (1) and (2), we find $\langle \bar{g}_{23} \rangle \in \Sigma_{15}$. By applying the symmetries $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, we then find

$$\langle \bar{g}_{13} \rangle,\ \langle \bar{g}_{14} \rangle,\ \langle \bar{g}_{23} \rangle,\ \langle \bar{g}_{24} \rangle \in \Sigma_{15}. \tag{3}$$

From equation (2) and (3), we then find

$$\langle \bar{g}_{12} \rangle \in \Sigma_{15}. \tag{4}$$

By (3) and (4), we then know that $\Sigma' \subseteq \Sigma_{15}$. □

Lemma 4.2. *We have* $\Sigma_{23} = \mathrm{PG}(V_{23})$.

Proof. Considering the points $\langle \bar{b}_i \rangle$ $(i \in \{1,2,3,4\})$, $\langle \bar{b}_i + \bar{b}_j \rangle$ $(i, j \in \{1,2,3,4\}$ with $i < j)$ and $\langle \bar{b}_i + \bar{b}_j + \bar{b}_k \rangle$ $(i, j, k \in \{1,2,3,4\}$ with $i < j < k)$ of $\mathcal{H}$, we see that the points $\langle \bar{g}_i \rangle$ $(i \in \{1,2,3,4\})$, $\langle \bar{h}_{ij} \rangle$ $(i, j \in \{1,2,3,4\}$ with $i < j)$, $\langle \bar{h}_{ijk} \rangle$ $(i, j, k \in \{1,2,3,4\}$ with $i < j < k)$ belong to Σ_{23}. Let Σ'' denote the subspace of $\mathrm{PG}(V_{23})$ generated by all these points, and let Σ' denote the subspace of $\mathrm{PG}(V_{23})$ generated by the points

$\langle \bar{g}_{ij} \rangle$ $(i, j \in \{1,2,3,4\}$ with $i < j$ and $(i, j) \neq (3,4))$, $\langle \bar{g}_{ijk} \rangle$ $(i, j, k \in \{1,2,3,4\}$ with $i < j < k)$.

Then $\Sigma'' \subseteq \Sigma_{23}$. As $\mathrm{PG}(V_{23}) = \langle \Sigma', \Sigma'' \rangle$, it thus suffices to show that also $\Sigma' \subseteq \Sigma_{23}$. The fact that $\Sigma'' \subseteq \Sigma_{23}$ implies the following.

For every point x for which $\epsilon_{23}(x) \notin \Sigma''$, we have $\epsilon(x) \in \Sigma_{23}$, where $\epsilon(x)$ is the unique point of Σ' in the subspace $\langle \Sigma'', \epsilon_{23}(x) \rangle$.

We will now apply this fact to several points of $\mathcal{H}$. Note that the above point $\epsilon(x)$ is equal to

$$\langle \sum_{2'} (X_i X_j^2 + X_j X_i^2) \bar{g}_{ij} + \sum_{3} (X_i X_j X_k + X_i^2 X_j^2 X_k^2) \bar{g}_{ijk} \rangle$$

which is symmetric in the subindices 1 and 2, and also in the subindices 3 and 4.

Considering the points $(1, 1, \omega, 0)$,$(1, 1, 0, \omega)$,$(1, 1, \omega, \omega)$ of $\mathcal{H}$, we see that

$$\langle \bar{g}_{13} + \bar{g}_{23} + \bar{g}_{123} \rangle \in \Sigma_{23}, \tag{5}$$

$$\langle \bar{g}_{14} + \bar{g}_{24} + \bar{g}_{124} \rangle \in \Sigma_{23}, \tag{6}$$

$$\langle \bar{g}_{13} + \bar{g}_{14} + \bar{g}_{23} + \bar{g}_{24} + \bar{g}_{123} + \bar{g}_{124} + \bar{g}_{134} + \bar{g}_{234} \rangle \in \Sigma_{23}. \tag{7}$$

From (5), (6) and (7), we deduce

$$\langle \bar{g}_{134} + \bar{g}_{234} \rangle \in \Sigma_{23}. \tag{8}$$

Similarly, by considering the points $(0, \omega, 1, 1)$, $(\omega, 0, 1, 1)$ and $(\omega, \omega, 1, 1)$, we find

$$\langle \bar{g}_{123} + \bar{g}_{124} \rangle \in \Sigma_{23}. \tag{9}$$

Considering the points $(1, \omega, \omega^2, 1)$ and $(1, \omega, \omega, 1)$ of $\mathcal{H}$, we find

$$\langle \bar{g}_{12} + \bar{g}_{13} + \bar{g}_{23} + \bar{g}_{24} + \bar{g}_{124} + \bar{g}_{134} \rangle \in \Sigma_{23}, \tag{10}$$

$$\langle \bar{g}_{12} + \bar{g}_{13} + \bar{g}_{24} + \bar{g}_{123} + \bar{g}_{124} + \bar{g}_{134} + \bar{g}_{234} \rangle \in \Sigma_{23}. \tag{11}$$

From equations (10) and (11), we find

$$\langle \bar{g}_{23} + \bar{g}_{123} + \bar{g}_{234} \rangle \in \Sigma_{23}. \tag{12}$$

By applying the symmetry $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ to (12), we find

$$\langle \bar{g}_{14} + \bar{g}_{124} + \bar{g}_{134} \rangle \in \Sigma_{23}. \tag{13}$$

By (8), (9), (12) and (13), we find

$$\langle \bar{g}_{14} + \bar{g}_{23} \rangle \in \Sigma_{23}. \tag{14}$$

By applying the symmetry $3 \leftrightarrow 4$ to (14), we also find

$$\langle \bar{g}_{13} + \bar{g}_{24} \rangle \in \Sigma_{23}. \tag{15}$$

By (8), (9), (11) and (15), we find

$$\langle \bar{g}_{12} \rangle \in \Sigma_{23}. \tag{16}$$

By considering the point $(\omega, 0, 1, 1)$ of $\mathcal{H}$, we see that

$$\langle \bar{g}_{13} + \bar{g}_{14} + \bar{g}_{134} \rangle \in \Sigma_{23}. \tag{17}$$

By (6), (13), (15) and (17), we find $\langle \bar{g}_{14} \rangle \in \Sigma_{23}$. Using the symmetries $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, we then have

$$\langle \bar{g}_{13} \rangle,\ \langle \bar{g}_{14} \rangle,\ \langle \bar{g}_{23} \rangle,\ \langle \bar{g}_{24} \rangle \in \Sigma_{23}. \tag{18}$$

By (5), (6) and (18), we then have $\langle \bar{g}_{123} \rangle, \langle \bar{g}_{124} \rangle \in \Sigma_{23}$. Combining this with (12), (13) and (18), we then find that

$$\langle \bar{g}_{123} \rangle,\ \langle \bar{g}_{124} \rangle,\ \langle \bar{g}_{134} \rangle,\ \langle \bar{g}_{234} \rangle \in \Sigma_{23}. \tag{19}$$

By (16), (18) and (19), we then know that $\Sigma' \subseteq \Sigma_{23}$. □

If X is a nonempty set of points of a generalized quadrangle $\mathcal{Q}$, then $X^\perp$ denotes the set of all points of $\mathcal{Q}$ collinear with all points of X. If x, y are two noncollinear points of $\mathcal{Q}$, then $\{x,y\}^{\perp\perp} := (\{x,y\}^\perp)^\perp$ is called a *hyperbolic line* of $\mathcal{Q}$. If x and y are two points of $H(3,4)$, then $L = \{x,y\}^\perp$ and $K = \{x,y\}^{\perp\perp}$ are two hyperbolic lines of $H(3,4)$. Not only do we have $K = L^\perp$, but also $L = K^\perp$. The hyperbolic lines of $H(3,4)$ are precisely the intersections of size 3 that arise by intersecting $\mathcal{H}$ with the lines of $\mathrm{PG}(3,4)$.

Now, let L_{12} be the hyperbolic line $\{(1,0,0,0),(0,1,0,0),(1,1,0,0)\}$ of $H(3,4)$. Then $L_{34} := L_{12}^\perp = \{(0,0,1,0),(0,0,0,1),(0,0,1,1)\}$ is also a hyperbolic line. A line of $H(3,4)$ contains a (necessarily unique) point of L_{12} if and only if it contains a (necessarily unique) point of L_{34}. So, $L_{12} \cup L_{34}$ is an even set, and $H^* := \mathcal{H} \setminus (L_{12} \cup L_{34})$ is a pseudo-hyperplane.

Lemma 4.3. *The pseudo-hyperplane H^* does not arise from the pseudo-embedding ϵ_{23}.*

Proof. If H^* arises from ϵ_{23}, then H^* has an equation of the form

$$\sum_1 a_i X_i^3 + \sum_{2'} a_{ij}(X_i X_j^2 + X_j X_i^2) + \sum_2 b_{ij}(\omega X_i X_j^2 + \omega^2 X_j X_i^2)$$
$$+ \sum_3 a_{ijk}(X_i X_j X_k + X_i^2 X_j^2 X_k^2) + b_{ijk}(\omega X_i X_j X_k + \omega^2 X_i^2 X_j^2 X_k^2) = 0, \qquad (*)$$

where

- $a_i \in \mathbb{F}_2$ for every $i \in \{1,2,3,4\}$;
- $a_{ij} \in \mathbb{F}_2$ for all $i,j \in \{1,2,3,4\}$ with $i < j$ and $(i,j) \neq (3,4)$;
- $b_{ij} \in \mathbb{F}_2$ for all $i,j \in \{1,2,3,4\}$ with $i < j$;
- $a_{ijk}, b_{ijk} \in \mathbb{F}_2$ for all $i,j,k \in \{1,2,3,4\}$ with $i < j < k$.

If $i \in \{1,2\}$ and $j \in \{3,4\}$, then the fact that the points $\langle \bar{b}_i + \bar{b}_j \rangle$, $\langle \bar{b}_i + \omega \bar{b}_j \rangle$ and $\langle \bar{b}_i + \omega^2 \bar{b}_j \rangle$ of H^* satisfy equation $(*)$ imply that

$$a_i + a_j = a_{ij} = b_{ij} = 0. \qquad (20)$$

The latter also implies that

$$a_1 = a_2 = a_3 = a_4. \qquad (21)$$

By (20), (21) and the fact that the points $(1,1,1,0)$, $(1,1,\omega,0)$, $(1,1,\omega^2,0)$, $(1,1,0,1)$, $(1,1,0,\omega)$, $(1,1,0,\omega^2)$, $(0,1,1,1)$, $(0,\omega,1,1)$, $(0,\omega^2,1,1)$, $(1,0,1,1)$, $(\omega,0,1,1)$, $(\omega^2,0,1,1)$ of H^* satisfy equation $(*)$, we find

$$a_1 = a_2 = a_3 = a_4 = b_{12} = b_{34}, \qquad (22)$$

and

$$a_{ijk} = b_{ijk} = 0 \qquad (23)$$

for all $i,j,k \in \{1,2,3,4\}$ with $i < j < k$. By (20), (21), (22), (23) and the fact that the points $(1,\omega,1,\omega)$ and $(1,\omega,1,\omega^2)$ of H^* satisfy equation $(*)$, we then also find that

$$a_1 = a_2 = a_3 = a_4 = a_{12} = b_{12} = b_{34} = 0. \qquad (24)$$

So, all coefficients should be zero, which is impossible. ∎

By Proposition 2.1 and Lemma 4.3, we can enlarge the embedding ϵ_{23} to a pseudo-embedding whose vector dimension is 24. As the pseudo-hyperplane $H^* \subseteq \mathcal{H}$ is described by the equation $(X_1^3 + X_2^3 + X_1^3 X_2^3)(X_3^3 + X_4^3 + X_3^3 X_4^3) = 1$, the construction given in Proposition 2.1 tells us that this pseudo-embedding is isomorphic to ϵ_{24}. Proposition 3.1 and Lemma 3.2 then imply the following.

Corollary 4.4. (1) *The pseudo-embedding and pseudo-generating ranks of $H(3,4)$ are equal to 24.*

(2) *The universal pseudo-embedding of $H(3,4)$ is isomorphic to ϵ_{24}.*

So far, we have constructed three homogeneous pseudo-embeddings of $H(3,4)$, namely ϵ_{15}, ϵ_{23} and ϵ_{24}. In Section 6, we classify all homogenous pseudo-embeddings of $H(3,4)$. From that treatment, it follows that there is an additional homogeneous pseudo-embedding of $H(3,4)$. Before we can achieve these goals, we describe in Section 5 a set of generators for the automorphism group of $H(3,4)$ and explain how these act on the even sets of $H(3,4)$.

5 Generators for the automorphism group of $H(3,4)$

As before, we put $\mathrm{PG}(3,4) = \mathrm{PG}(V)$, where V is some 4-dimensional $\mathbb{F}_4$-vector space, and we suppose the coordinates (X_1, X_2, X_3, X_4) of the points of $\mathrm{PG}(3,4)$ are those with respect to a fixed basis $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ of V. We denote by $h : V \times V \to \mathbb{F}_4$ the Hermitean form described by

$$h(\sum_{i=1}^{4} X_i \bar{b}_i, \sum_{j=1}^{4} Y_j \bar{b}_j) = X_1 Y_2^2 + X_2 Y_1^2 + X_3 Y_4^2 + X_4 Y_3^2.$$

Note that $\mathcal{H}$ consists of all points $\langle \bar{x} \rangle$ of $\mathrm{PG}(V)$ for which $h(\bar{x}, \bar{x}) = 0$.

A basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ of V is called a *hyperbolic basis* of (V, h) if $h(\bar{e}_i, \bar{e}_j) = h(\bar{f}_i, \bar{f}_j) = 0$ and $h(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, 2\}$. The fixed basis $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ is thus a hyperbolic basis of (V, h).

Every automorphism ϕ of $H(3,4)$ is induced by an automorphism of $\mathrm{PG}(3,4)$ stabilizing $\mathcal{H}$ and so also by a semi-linear transformation $\theta = \theta_\phi$ of V. For every such θ, there exists an $e_\theta \in \{1, 2\} \subseteq \mathbb{N}$ such that $h(\bar{x}^\theta, \bar{y}^\theta) = h(\bar{x}, \bar{y})^{e_\theta}$. We denote by G the subgroup of $\mathrm{Aut}(H(3,4))$ consisting of all $\phi \in \mathrm{Aut}(H(3,4))$ for which $e_{\theta_\phi} = 1$, i.e. θ_ϕ leaves h invariant, or equivalently, θ_ϕ maps hyperbolic bases of (V, h) to hyperbolic bases of (V, h).

If $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2)$ is a hyperbolic basis of (V, h), then obviously

(1) $(\bar{e}_2, \bar{f}_2, \bar{e}_1, \bar{f}_1)$ is also a hyperbolic basis of (V, h);

(2) $(\frac{\bar{e}_1}{\omega}, \omega^2 \bar{f}_1, \bar{e}_2, \bar{f}_2)$ is also a hyperbolic basis of (V, h);

(3) $(\bar{e}_1 + \bar{e}_2, \bar{f}_1, \bar{e}_2, \bar{f}_1 + \bar{f}_2)$ is also a hyperbolic basis of (V, h);

(4) $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2 + \bar{e}_2)$ is also a hyperbolic basis of (V, h);

(5) $(\bar{e}_1, \bar{f}_1, \bar{e}_2 + \bar{f}_2, \bar{e}_2)$ is also a hyperbolic basis of (V, h).

For every $i \in \{1, 2, 3, 4, 5\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of hyperbolic bases of (V, h) such that B_2 can be obtained from B_1 as described in (i) above.

We have thus given five elementary constructions for constructing new hyperbolic bases from given ones. Applying these constructions consecutively, we can obtain other hyperbolic bases. E.g., applying (2) a number of times, we conclude that:

(2') $(\frac{\bar{e}_1}{\lambda}, \lambda^2 \bar{f}_1, \bar{e}_2, \bar{f}_2)$ is a hyperbolic basis of (V, h) for every $\lambda \in \mathbb{F}_4^*$.

By applying (2) and (3) a number of times, we can show that also

(3') $(\bar{e}_1 + \lambda \bar{e}_2, \bar{f}_1, \bar{e}_2, \lambda^2 \bar{f}_1 + \bar{f}_2)$ is a hyperbolic basis of (V, h) for every $\lambda \in \mathbb{F}_4^*$.

Indeed, by (2') we know that $(\frac{\bar{e}_1}{\lambda}, \lambda^2 \bar{f}_1, \bar{e}_2, \bar{f}_2)$ is a hyperbolic basis of (V, h). By (3), we then know that also $(\frac{\bar{e}_1}{\lambda} + \bar{e}_2, \lambda^2 \bar{f}_1, \bar{e}_2, \lambda^2 \bar{f}_1 + \bar{f}_2)$ is a hyperbolic basis of (V, h). Claim (3') then follows by applying (2') once more.

Combining Lemma 2.1 of [7] with (2') and (3'), we then have:

Lemma 5.1. *If B and B' are two hyperbolic bases of (V, h), then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ of (V, h) for some $k \in \mathbb{N}$ such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_5$ for every $i \in \{1, 2, \ldots, k\}$.*

Now, consider the linear transformations $\theta_1, \theta_2, \ldots, \theta_5$ of $GL(V)$ defined by

- $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)^{\theta_1} = (\bar{b}_3, \bar{b}_4, \bar{b}_1, \bar{b}_2)$;
- $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)^{\theta_2} = (\frac{\bar{b}_1}{\omega}, \omega^2 \bar{b}_2, \bar{b}_3, \bar{b}_4)$;
- $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)^{\theta_3} = (\bar{b}_1 + \bar{b}_3, \bar{b}_2, \bar{b}_3, \bar{b}_2 + \bar{b}_4)$;
- $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)^{\theta_4} = (\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4 + \bar{b}_3)$;
- $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)^{\theta_5} = (\bar{b}_1, \bar{b}_2, \bar{b}_3 + \bar{b}_4, \bar{b}_4)$.

As θ_i, $i \in \{1, 2, \ldots, 5\}$, maps every hyperbolic basis of (V, h) to another hyperbolic basis of (V, h), it leaves the Hermitian form h invariant and so defines an automorphism ϕ_i of $H(3, 4)$. Put

$$\Theta = \langle \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \rangle \leq GL(V).$$

Then every element of Θ maps hyperbolic bases of (V, h) to hyperbolic bases of (V, h). The following property is obvious.

Lemma 5.2. *Let $\theta \in \Theta$ and $i \in \{1, 2, 3, 4, 5\}$. If $(\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4)$ is the hyperbolic basis of (V, h) such that $((\bar{b}_1^\theta, \bar{b}_2^\theta, \bar{b}_3^\theta, \bar{b}_4^\theta), (\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4)) \in \Omega_i$, then $\theta_i \theta = \theta \circ \theta_i \in \Theta$ maps $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ to $(\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4)$.*

By Lemmas 5.1 and 5.2, we then have

Corollary 5.3. Θ *consists of those elements of $GL(V)$ that map hyperbolic bases of (V, h) to hyperbolic bases of (V, h), or equivalently, that leave the form h invariant.*

We thus also have

Corollary 5.4. *We have* $G = \langle \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \rangle$.

If we denote by ϕ_6 the automorphism $(X_1, X_2, X_3, X_4) \mapsto (X_1^2, X_2^2, X_3^2, X_4^2)$ of $H(3,4)$, then we have

Corollary 5.5. *We have* $\mathrm{Aut}(H(3,4)) = \langle \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6 \rangle$.

We now determine how each of the six generators of $\mathrm{Aut}(H(3,4))$ acts on the even sets of $H(3,4)$, or equivalently, on the corresponding 15-tuples of the cartesian product $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ (recall Corollary 1.3). To that end, we will need to rely on the following three lemmas.

Lemma 5.6. *Suppose E is an even set described by an equation $F(X_1, X_2, X_3, X_4) = 0$ and let $\phi = \phi_i$ with $i \in \{1, 2, 3, 4, 5, 6\}$. Then the following hold.*

- *If $i = 1$, then E^ϕ is described by the equation $F(X_3, X_4, X_1, X_2) = 0$.*
- *If $i = 2$, then E^ϕ is described by the equation $F(\omega X_1, \omega X_2, X_3, X_4) = 0$.*
- *If $i = 3$, then E^ϕ is described by the equation $F(X_1, X_2 + X_4, X_1 + X_3, X_4) = 0$.*
- *If $i = 4$, then E^ϕ is described by the equation $F(X_1, X_2, X_3 + X_4, X_4) = 0$.*
- *If $i = 5$, then E^ϕ is described by the equation $F(X_1, X_2, X_3, X_3 + X_4) = 0$.*
- *If $i = 6$, then E^ϕ is described by the equation $F(X_1^2, X_2^2, X_3^2, X_4^2) = 0$.*

Proof. We give a proof for $i = 3$. The treatment for the remaining cases is similar. Suppose ϕ_3 maps to the point (X_1, X_2, X_3, X_4) to the point (Y_1, Y_2, Y_3, Y_4). Then $X_1(\bar{b}_1 + \bar{b}_3) + X_2\bar{b}_2 + X_3\bar{b}_3 + X_4(\bar{b}_2 + \bar{b}_4) = Y_1\bar{b}_1 + Y_2\bar{b}_2 + Y_3\bar{b}_3 + Y_4\bar{b}_4$. Hence, $(Y_1, Y_2, Y_3, Y_4) = (X_1, X_2 + X_4, X_1 + X_3, X_4)$ and $(X_1, X_2, X_3, X_4) = (Y_1, Y_2 + Y_4, Y_1 + Y_3, Y_4)$. The latter equality shows that (Y_1, Y_2, Y_3, Y_4) satisfies the equation $F(Y_1, Y_2 + Y_4, Y_1 + Y_3, Y_4) = 0$. □

Lemma 5.7. *If (X_1, X_2, X_3, X_4) is a point of $H(3,4)$ and $b \in \mathbb{F}_4$, then*

$$bX_3X_4^2 + b^2X_4X_3^2 = (b + b^2)(\omega X_3X_4^2 + \omega^2 X_4X_3^2) + (b\omega^2 + b^2\omega)(X_1X_2^2 + X_2X_1^2).$$

Proof. We have

$$\begin{aligned} bX_3X_4^2 + b^2X_4X_3^2 &= (b + b^2)(\omega X_3X_4^2 + \omega^2 X_4X_3^2) + (b\omega^2 + b^2\omega)(X_3X_4^2 + X_4X_3^2) \\ &= (b + b^2)(\omega X_3X_4^2 + \omega^2 X_4X_3^2) + (b\omega^2 + b^2\omega)(X_1X_2^2 + X_2X_1^2). \end{aligned}$$
□

Lemma 5.8. *If (X_1, X_2, X_3, X_4) is a point of $H(3,4)$, then*

$$X_1X_3^2X_4^3 + X_1^2X_3X_4^3 + X_1^3X_2X_4^2 + X_1^3X_2^2X_4 = X_1X_2X_4 + (X_1X_2X_4)^2 + (X_1X_3X_4) + (X_1X_3X_4)^2.$$

Proof. As $X_1X_2^2 + X_2X_1^2 + X_3X_4^2 + X_4X_3^2 = 0$, the left hand side is equal to

$$X_1X_4^2(X_1^2X_2 + X_2^2X_1 + X_3X_4^2) + X_1^2X_4(X_3^2X_4 + X_1^2X_2 + X_2^2X_1) + X_1^3X_2X_4^2 + X_1^3X_2^2X_4$$

$$= X_1^2X_2^2X_4^2 + X_1X_3X_4^4 + X_1^2X_3^2X_4^2 + X_1^4X_2X_4 = X_1X_2X_4 + (X_1X_2X_4)^2 + (X_1X_3X_4) + (X_1X_3X_4)^2.$$
□

Lemma 5.9. *Let $\phi = \phi_i \in \{\phi_1, \phi_2, \ldots, \phi_6\}$. Suppose $\bar{a}, \bar{a}' \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ such that $E(\bar{a})^\phi = E(\bar{a}')$. Then $\bar{a}' = (a_1', \ldots, a_6', b_{12}', \ldots, b_{234}')$ can be obtained from $\bar{a} = (a_1, \ldots, a_6, b_{12}, \ldots, b_{234})$ as described in Tables* 1 *and* 2.

Proof. We put the left hand side of the expression in Corollary 1.3 equal to $F(X_1, X_2, X_3, X_4)$. By Lemma 5.6, we should then compute $F(X_3, X_4, X_1, X_2)$, $F(\omega X_1, \omega X_2, X_3, X_4)$, $F(X_1, X_2 + X_4, X_1 + X_3, X_4)$, $F(X_1, X_2, X_3 + X_4, X_4)$, $F(X_1, X_2, X_3, X_3 + X_4)$ and $F(X_1^2, X_2^2, X_3^2, X_4^2)$. Expanding each of the latter polynomials, we obtain a sum of terms of the form:

- X_i^3 with $i \in \{1, 2, 3, 4\}$;
- $\lambda X_i X_j^2 + \lambda^2 X_j X_i^2$ with $\lambda \in \{1, \omega\}$ and $i, j \in \{1, 2, 3, 4\}$ with $i < j$;
- $\lambda X_i X_j X_k + \lambda^2 X_i^2 X_j^2 X_k^2$ with $\lambda \in \{1, \omega\}$ and $i, j, k \in \{1, 2, 3, 4\}$ with $i < j < k$;
- $(X_1^3 + X_2^3 + X_1^3 X_2^3)(X_3^3 + X_4^3 + X_3^3 X_4^3) + 1$;
- $X_1 X_3^2 X_4^3 + X_1^2 X_3 X_4^3 + X_1^3 X_2 X_4^2 + X_1^3 X_2^2 X_4$.

Collecting terms in the expansion, we get a linear combination of the required shape (as in Corollary 1.3), with exception of the terms of the form $X_1 X_3^2 X_4^3 + X_1^2 X_3 X_4^3 + X_1^3 X_2 X_4^2 + X_1^3 X_2^2 X_4$ and $b X_3 X_4^2 + b^2 X_4 X_3^2$ for some $b \in \mathbb{F}_4$. By Lemmas 5.7 and 5.8, also these terms can be written in the right shape.

E.g., for $\phi = \phi_3$, the factor for a_6 in the expression $F(X_1, X_2 + X_4, X_1 + X_3, X_4)$ is equal to

$$((X_1^3 + X_2^3 + X_1^3 X_2^3)(X_3^3 + X_4^3 + X_3^3 X_4^3) + 1) + X_1^3 + X_4^3 + X_2 X_4^2 + X_4 X_2^2$$
$$+ X_1 X_3^2 + X_3 X_1^2 + X_1 X_3^2 X_4^3 + X_1^2 X_3 X_4^3 + X_1^3 X_2 X_4^2 + X_1^3 X_2^2 X_4. \qquad \square$$

6 The homogeneous pseudo-embeddings of $H(3,4)$

In this section, we classify all homogeneous pseudo-embeddings of the generalized quadrangle $H(3,4)$. Our classification relies on the following results from [8, 10].

Proposition 6.1 ([10, Corollary 2.7])**.** *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three, and let G be a group of automorphisms of $\mathcal{S}$.*

If $\epsilon : \mathcal{S} \to \Sigma$ is a G-homogeneous pseudo-embedding of $\mathcal{S}$, then the set $\mathcal{H} = \mathcal{H}_\epsilon$ satisfies the following properties:

(a) *$\mathcal{H}$ is a union of G-orbits of pseudo-hyperplanes;*

(b) *if H_1 and H_2 are two distinct elements of $\mathcal{H}$, then the complement of the symmetric difference of H_1 and H_2 also belongs to $\mathcal{H}$;*

(c) *if L is a line of $\mathcal{S}$ containing an odd number of points, then for every point x of L there exists a pseudo-hyperplane of $\mathcal{H}$ that only has the point x in common with L;*

(d) *if L is a line of $\mathcal{S}$ containing an even number of points, then for any two distinct points x_1 and x_2 of L, there exists a pseudo-hyperplane of $\mathcal{H}$ having only the points x_1 and x_2 in common with L;*

i	1	2	3
a'_1	a_3	a_1	$a_1 + a_3 + a_6 + b_{13} + b_{13}^2$
a'_2	a_4	a_2	a_2
a'_3	a_1	a_3	a_3
a'_4	a_2	a_4	$a_4 + a_2 + a_6 + b_{24} + b_{24}^2$
a'_5	$b_{12} + b_{12}^2$	a_5	$a_5 + b_{23} + b_{23}^2 + b_{234} + b_{234}^2$
a'_6	a_6	a_6	a_6
b'_{12}	$\omega a_5 + \omega^2 b_{12} + \omega b_{12}^2$	b_{12}	$b_{12} + b_{123}^2 + \omega(b_{23} + b_{23}^2) + \omega^2 b_{234} + \omega b_{234}^2$
b'_{13}	b_{13}^2	ωb_{13}	$b_{13} + a_3 + a_6$
b'_{14}	b_{23}^2	ωb_{14}	$b_{14} + \omega a_5 + b_{12} + b_{23}^2 + b_{123}^2 + b_{124} + b_{134}^2 + b_{234}$
b'_{23}	b_{14}^2	ωb_{23}	b_{23}
b'_{24}	b_{24}^2	ωb_{24}	$b_{24} + a_2 + a_6$
b'_{123}	b_{134}	$\omega^2 b_{123}$	b_{123}
b'_{124}	b_{234}	$\omega^2 b_{124}$	$b_{124} + a_6 + b_{234}$
b'_{134}	b_{123}	ωb_{134}	$b_{134} + a_6 + b_{123}$
b'_{234}	b_{124}	ωb_{234}	b_{234}

Table 1: The action of ϕ_1, ϕ_2 and ϕ_3 on the even sets

i	4	5	6
a'_1	a_1	a_1	a_1
a'_2	a_2	a_2	a_2
a'_3	a_3	$a_3 + a_4 + a_5$	a_3
a'_4	$a_4 + a_3 + a_5$	a_4	a_4
a'_5	a_5	a_5	a_5
a'_6	a_6	a_6	a_6
b'_{12}	$b_{12} + a_3$	$b_{12} + a_4$	$b_{12}^2 + a_5$
b'_{13}	b_{13}	$b_{13} + b_{14} + b_{134}$	b_{13}^2
b'_{14}	$b_{14} + b_{13} + b_{134}$	b_{14}	b_{14}^2
b'_{23}	b_{23}	$b_{23} + b_{24} + b_{234}$	b_{23}^2
b'_{24}	$b_{24} + b_{23} + b_{234}$	b_{24}	b_{24}^2
b'_{123}	b_{123}	$b_{123} + b_{124}$	b_{123}^2
b'_{124}	$b_{124} + b_{123}$	b_{124}	b_{124}^2
b'_{134}	b_{134}	b_{134}	b_{134}^2
b'_{234}	b_{234}	b_{234}	b_{234}^2

Table 2: The action of ϕ_4, ϕ_5 and ϕ_6 on the even sets

(*e*) *for every point x of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{H}$ not containing x.*

Conversely, if $\mathcal{H}$ is a finite set of pseudo-hyperplanes of $\mathcal{S}$ satisfying the conditions (*a*), (*b*), (*c*), (*d*) *and* (*e*) *above, then there exists a pseudo-embedding ϵ of $\mathcal{S}$ such that $\mathcal{H} = \mathcal{H}_\epsilon$. This pseudo-embedding ϵ is moreover unique, up to isomorphism, and G-homogeneous.*

Proposition 6.2 ([8, Theorem 3.1])**.** *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Let V_1 and V_2 be two vector spaces over $\mathbb{F}_2$. For every $i \in \{1,2\}$, let ϵ_i be a map from the point set $\mathcal{P}$ of $\mathcal{S}$ to the point set of $\mathrm{PG}(V_i)$ and let $\mathcal{H}_i$ be the set of all sets of the form $\epsilon_i^{-1}(\epsilon_i(\mathcal{P}) \cap \Pi)$, where Π is some hyperplane of $\mathrm{PG}(V_i)$. If ϵ_1 is a pseudo-embedding of $\mathcal{S}$ and $\mathcal{H}_1 = \mathcal{H}_2$, then also ϵ_2 is a pseudo-embedding of $\mathcal{S}$. Moreover, ϵ_2 is isomorphic to ϵ_1.*

In order to use Proposition 6.1 in the case of $H(3,4)$, we first translate the conditions of Proposition 6.1 to conditions that must be satisfied by the corresponding even sets, and subsequently to conditions that must be satisfied by the corresponding elements of $\mathbb{F}_2^6 \times \mathbb{F}_4^9$.

If $\epsilon : H(3,4) \to \Sigma$ is a pseudo-embedding of $H(3,4)$, then we denote by $\mathcal{E}_\epsilon$ the complements of the elements of $\mathcal{H}_\epsilon$. Every element of $\mathcal{E}_\epsilon$ is an even set. We put $\mathcal{A}_\epsilon := \{\bar{a}_E \,|\, E \in \mathcal{E}_\epsilon\} \cup \{(0,0,\ldots,0)\}$, where $\bar{a}_E$ with $E \in \mathcal{E}_\epsilon$ denotes the unique element $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ for which $E(\bar{a}) = E$.

Lemma 6.3. *If ϵ is a homogeneous pseudo-embedding of $H(3,4)$, then the set $\mathcal{E} := \mathcal{E}_\epsilon$ satisfies the following properties:*

(*a*) *$\mathcal{E}$ can be written as a union of isomorphism classes of even sets;*

(*b*) *the symmetric difference of two distinct even sets of $\mathcal{E}$ is again an even set of $\mathcal{E}$;*

(*c*) *$\mathcal{E} \neq \emptyset$.*

Conversely, if $\mathcal{E}$ is a set of nontrivial even sets of $H(3,4)$ satisfying the properties (*a*), (*b*) *and* (*c*) *above, then there exists a pseudo-embedding ϵ of $\mathcal{S}$ such that $\mathcal{E} = \mathcal{E}_\epsilon$. This pseudo-embedding ϵ is moreover unique, up to isomorphism, and homogeneous.*

Proof. The fact that the conditions (a), (b) and (c) are satisfied by any homogeneous pseudo-embedding of $H(3,4)$ is a consequence of conditions (a), (b) and (c) of Proposition 6.1.

Conversely, suppose that $\mathcal{E}$ is a set of nontrivial even sets of $H(3,4)$ satisfying the conditions (a), (b) and (c). Let $\mathcal{G}$ denote the full group of automorphisms of $H(3,4)$ and let $\mathcal{H}$ denote the set of all complements of the elements of $\mathcal{E}$. Then $\mathcal{H} \neq \emptyset$ by condition (c). The set $\mathcal{H}$ satisfies the conditions (a) and (b) of Proposition 6.1 as the conditions (a) and (b) of the present lemma are satisfied. Also Property (d) of Proposition 6.1 is satisfied as there are no lines containing an even number of points. Property (e) of Proposition 6.1 is satisfied as $\mathcal{H} \neq \emptyset$ and $\mathcal{G}$ is point-transitive. In view of Proposition 6.1, we thus see that the lemma will be valid if we can also show that Property (c) of that proposition is satisfied. For that purpose, we take an arbitrary flag (x, L) of $H(3,4)$.

We show that there exists an element of $\mathcal{E}$ intersecting some line of $H(3,4)$ in four points. Suppose that this is not the case. Then there exists an element E of $\mathcal{E}$ intersecting a certain line M in exactly two points x_1 and x_2. Let x_3 and x_4 be two other points on the line M. As $\mathcal{G}$ acts transitively on the pairs of distinct collinear points, there exists an automorphism ϕ of $H(3,4)$ such that $E^\phi \cap M = \{x_3, x_4\}$. But then $E\Delta E^\phi$ intersects M in $\{x_1, x_2, x_3, x_4\}$. This is a contradiction as $E\Delta E^\phi \in \mathcal{E}$ by properties (a) and (b).

So, there exists an element of $\mathcal{E}$ intersecting some line of $H(3,4)$ in four points. As $\mathcal{G}$ is flag-transitive, we then know that there also exists an element of $\mathcal{E}$ intersecting L in $L \setminus \{x\}$. The complement of this even set belongs to $\mathcal{H}$ and intersects L in $\{x\}$. □

The action of ϕ_i, $i \in \{1, 2, \ldots, 6\}$, on the even sets defines a linear transformation σ_i on the $\mathbb{F}_2$ vector space $\mathbb{F}_2^6 \times \mathbb{F}_4^9$. The description of these linear transformations can be found in Tables 1 and 2.

Lemma 6.4. *If ϵ is a homogeneous pseudo-embedding of $H(3,4)$, then the set $\mathcal{A}_\epsilon$ is a subspace of dimension at least 1 of the $\mathbb{F}_2$-vector space $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ that is closed under the linear maps $\sigma_1, \sigma_2, \ldots, \sigma_6$. Conversely, every subspace $\mathcal{A}$ of dimension at least 1 of the $\mathbb{F}_2$-vector space $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ that is closed under the maps $\sigma_1, \sigma_2, \ldots, \sigma_6$ is of the form $\mathcal{A}_\epsilon$ for some homogeneous pseudo-embedding ϵ of $H(3,4)$. This pseudo-embedding ϵ is moreover unique, up to isomorphism.*

Proof. Suppose $\mathcal{E}$ is a set of nontrivial even sets and $\mathcal{A} := \{\bar{a}_E \,|\, E \in \mathcal{E}\} \cup \{(0, 0, \ldots, 0)\}$. The condition (a) of Lemma 6.3 is equivalent with the claim that $\mathcal{A}$ is closed under the linear maps $\sigma_1, \sigma_2, \ldots, \sigma_6$, the condition (b) of Lemma 6.3 is equivalent with the claim that $\mathcal{A}$ is a subspace of the $\mathbb{F}_2$-vector space $\mathbb{F}_2^6 \times \mathbb{F}_4^9$, and the condition (c) of Lemma 6.3 is equivalent with the claim that the dimension of the subspace $\mathcal{A}$ is at least one. Taking these facts into account, we see that Lemma 6.4 is now a consequence of Lemma 6.3. □

We now use Lemma 6.4 to classify all homogeneous pseudo-embeddings of $H(3,4)$. We define the following subspaces of $\mathbb{F}_2^6 \times \mathbb{F}_4^9$:

- $W_{24} = \mathbb{F}_2^6 \times \mathbb{F}_4^9$;
- W_{23} consists of those $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ for which $a_6 = 0$;
- W_{15} consists of those $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ for which $a_6 = 0$ and $b_{123} = b_{124} = b_{134} = b_{234} = 0$.
- W_{14} consists of those $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ for which $a_6 = 0$, $b_{123} = b_{124} = b_{134} = b_{234} = 0$ and $a_5 = b_{12} + b_{12}^2$. Putting $b_{12} = b'_{12} + b''_{12}\omega$, with $b'_{12}, b''_{12} \in \mathbb{F}_2$, the latter condition is equivalent with $a_5 = b''_{12}$.

Using the explicit actions of $\sigma_1, \sigma_2, \ldots, \sigma_6$, we see that each of these subspaces is stabilized by these maps. By Lemma 6.4, there are associated pseudo-embeddings ϵ'_{24}, ϵ'_{23}, ϵ'_{15} and ϵ'_{14}. By Proposition 6.2, these pseudo-embeddings are respectively isomorphic to ϵ_{24}, ϵ_{23}, ϵ_{15} and ϵ_{14}.

In order to complete the proof of Theorem 1.2, it suffices to prove that W_{24}, W_{23}, W_{15} and W_{14} are the only subspaces of dimension at least one of $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ stabilized by $\sigma_1, \sigma_2, \dots, \sigma_6$.

In the sequel, let W be a subspace of dimension at least one of $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ that is stabilized by $\sigma_1, \sigma_2, \dots, \sigma_6$. For every $i \in \{1, 2, \dots, 6\}$, let Ω_i denote the set of all vectors of $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ all whose coordinates are zero, except possibly for the a_i-coordinates. For every $I \in \{12, 13, \dots, 234\}$, let Ω_I denote the set of all vectors of $\mathbb{F}_2^6 \times \mathbb{F}_4^9$ all whose coordinates are zero, except possibly for the b_I-coordinates.

Lemma 6.5. *If $\bar{a} = (a_1, \dots, a_6, b_{12}, \dots, b_{24}, b_{123}, \dots, b_{234})$ belongs to W, then also $(0, 0, 0, 0, 0, 0, b_{13}, b_{14}, b_{23}, b_{24}, b_{123}, b_{124}, b_{134}, b_{234})$ belongs to W.*

Proof. The latter vector is precisely $\bar{a}^{\sigma_2 \sigma_2} + \bar{a}^{\sigma_2}$. □

Lemma 6.6. *If W contains a nonzero vector all whose coordinates are zero with exception of the b_{14}-coordinate, then W contains W_{14}.*

Proof. Since $W^{\sigma_2} = W$, we then see that $\Omega_{14} \subseteq W$. Since $\Omega_{14} \subseteq W$ and $W^{\sigma_1} = W$, we have $\Omega_{23} \subseteq W$. Since $\Omega_{23} \subseteq W$ and $\bar{a} + \bar{a}^{\sigma_4} \in W$ for all $\bar{a} \in W$, we see that $\Omega_{24} \subseteq W$. Since $\Omega_{14} \subseteq W$ and $\bar{a} + \bar{a}^{\sigma_5} \in W$ for all $\bar{a} \in W$, we see that $\Omega_{13} \subseteq W$. Since $\langle \Omega_{13}, \Omega_{24} \rangle \subseteq W$ and $\bar{a} + \bar{a}^{\sigma_3} \in W$ for all $\bar{a} \in W$, we see that $\langle \Omega_1, \Omega_4 \rangle \subseteq W$. Since $\langle \Omega_1, \Omega_4 \rangle \subseteq W$ and $W^{\sigma_1} = W$, we have $\langle \Omega_2, \Omega_3 \rangle \subseteq W$.

Since $\langle \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_{13}, \Omega_{14}, \Omega_{23}, \Omega_{24} \rangle \subseteq W$ and $\bar{a} + \bar{a}^{\sigma_4} \in W$ for all $\bar{a} \in W$, we see that W contains all vectors of $\langle \Omega_5, \Omega_{12} \rangle$ of the form $(0, 0, 0, 0, 0, 0, b_{12}, 0, \dots, 0)$ with $b_{12} \in \mathbb{F}_2$.

Since $\langle \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_{13}, \Omega_{14}, \Omega_{23}, \Omega_{24} \rangle \subseteq W$ and $\bar{a} + \bar{a}^{\sigma_3} \in W$ for all $\bar{a} \in W$, we see that W contains all vectors of $\langle \Omega_5, \Omega_{12} \rangle$ of the form $(0, 0, 0, 0, a_5, 0, a_5\omega, 0, \dots, 0)$ with $a_5 \in \mathbb{F}_2$.

All together, we thus see that W contains the subspace W_{14}. □

Lemma 6.7. *If $b_{123} = 0$ for all vectors of W, then W is equal to either W_{14} or W_{15}.*

Proof. We first prove that $W \subseteq W_{15}$. The fact that certain coordinates are equal to 0 for all vectors of W implies that other coordinates are as well 0 for all vectors of W:

- as $b_{123} = 0$ and $W^{\sigma_1} = W^{\sigma_5} = W$, we have $b_{134} = b_{124} = 0$;
- as $b_{123} = b_{134} = 0$ and $W^{\sigma_3} = W$, we have $a_6 = 0$;
- as $b_{124} = 0$ and $W^{\sigma_1} = W$, we have $b_{234} = 0$.

So, we indeed have $W \subseteq W_{15}$.

We show that it is not possible that $b_{23} = 0$ for all vectors of $W \subseteq W_{15}$. Suppose to the contrary that $b_{23} = 0$ for all vectors of W. The fact that $W^{\sigma_1} = W^{\sigma_5} = W$ then implies that also $b_{14} = b_{24} = 0$ for all vectors of W. Since $W^{\sigma_4} = W$ and $b_{14} = 0$ for all vectors of W, we then know that $b_{13} = 0$ for all vectors of W. As $W^{\sigma_3} = W$ and $b_{13} = b_{24} = 0$ for all vectors of W, we know that $a_2 = a_3 = 0$ for all vectors of W. This in combination with $W^{\sigma_1} = W$ implies that $a_1 = a_4 = 0$ for all vectors of W. Since $W^{\sigma_4} = W$ and $a_3 = a_4 = 0$ for all vectors of W, we also know that $a_5 = 0$ for all vectors of W. Since $W^{\sigma_3} = W$ and

$b_{14} = a_5 = b_{23} = b_{123} = b_{124} = b_{134} = b_{234} = 0$ for all vectors of W, we also know that $b_{12} = 0$ for all vectors of W. So, $W = 0$, an obvious contradiction.

So, we know that W contains a vector whose b_{23}-coordinate is distinct from 0. In combination with Lemma 6.5 this implies that $W \subseteq W_{15}$ contains a vector $\bar{a} = (0,0,0,0,0,0,0, b_{13}, b_{14}, b_{23}, b_{24}, 0,0,0,0)$ with $b_{23} \neq 0$. Lemma 6.5 and the fact that $\bar{a}^{\sigma_3} + \bar{a} \in W$ then implies that $(0,0,0,0,0,0,0,0,b_{23}^2,0,0,0,0,0,0) \in W$. By Lemma 6.6, we know that $W_{14} \subseteq W$. As $W \subseteq W_{15}$ and W_{14} is a hyperplane of W_{15}, we then know that W is either W_{14} or W_{15}. □

Lemma 6.8. *If W contains a vector $\bar{a}$ whose b_{123}-coordinate is distinct from 0, then W is equal to W_{23} or W_{24}.*

Proof. As W_{23} is a hyperplane of W_{24}, it suffices to prove that $W_{23} \subseteq W$.

We first show that W contains a nonzero vector all whose coordinates are 0 with exception of the b_{14}-coordinate. By Lemma 6.5, W contains the vector

$$\bar{a}_1 = (0,0,0,0,0,0,0, b_{13}, b_{14}, b_{23}, b_{24}, b_{123}, b_{124}, b_{134}, b_{234})$$

and hence also the vector

$$\bar{a}_2 = \bar{a}_1 + \bar{a}_1^{\sigma_4} = (0,0,0,0,0,0,0,0, b_{13} + b_{134}, 0, b_{23} + b_{234}, 0, b_{123}, 0, 0).$$

Thus W also contains the vector

$$\bar{a}_2 + \bar{a}_2^{\sigma_3} = (0,0,0, b_{23} + b_{23}^2 + b_{234} + b_{234}^2, 0,0,0,0, b_{123}, 0,0,0,0,0,0)$$

and hence also the vector $(0,0,0,0,0,0,0,0,b_{123},0,0,0,0,0,0)$ by Lemma 6.5. By Lemma 6.6, we thus already know that $W_{14} \subseteq W$. By Lemma 6.5, the fact that $W_{14} \subseteq W$ and the fact that $\bar{a}^{\sigma_4} + \bar{a} \in W$ for all $\bar{a} \in W$, we then know that W contains a nonzero vector of Ω_{124}. As $W^{\sigma_2} = W$, we then know that $\Omega_{124} \subseteq W$. As $\Omega_{124} \subseteq W$ and $W^{\sigma_1} = W$, we then know that $\Omega_{234} \subseteq W$. Since $\Omega_{124} \subseteq W$ and $\bar{a} + \bar{a}^{\sigma_5} \in W$ for all $\bar{a} \in W$, we have $\Omega_{123} \subseteq W$. As $\Omega_{123} \subseteq W$ and $W^{\sigma_1} = W$, we have $\Omega_{134} \subseteq W$. Since $\langle \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_{13}, \Omega_{14}, \Omega_{23}, \Omega_{24}, \Omega_{123}, \Omega_{124}, \Omega_{134}, \Omega_{234} \rangle \subseteq W$ and $W^{\sigma_3} = W$, we have $\langle \Omega_5, \Omega_{12} \rangle \subseteq W$. All together, we thus know that $W_{23} \subseteq W$. □

By Lemmas 6.7 and 6.8, we have:

Corollary 6.9. *W is equal to either W_{14}, W_{15}, W_{23} and W_{24}.*

7 The hyperovals of $H(3,4)$

Every hyperoval of $H(3,4)$ is an example of an *even set*, i.e. a set of the form $E(\bar{a})$, where $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$. With the aid of the Computer Algebra System GAP [19], we have determined for which of the 2^{24} elements $\bar{a} \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$, the even set $E(\bar{a})$ is a hyperoval, see [11]. It turns out that this is the case for 70648 such tuples.

H	v	N	Stabilizer S	O_1	O_2
H_6	6	120	$S_3 \times ((S_3 \times S_3) : C_2)$	1	2
H_8	8	540	$C_2 \times C_2 \times S_4$	1	4
H_{10a}	10	216	$C_2 \times S_5$	1	2
H_{10b}	10	2160	D_{24}	2	5
H_{10c}	10	2592	D_{20}	1	5
H_{12a}	12	120	$((S_3 \times S_3) : C_2) \times S_3$	1	2
H_{12b}	12	2160	$C_2 \times C_2 \times S_3$	1	7
H_{12c}	12	2880	D_{18}	2	5
H_{12d}	12	6480	D_8	3	8
H_{14a}	14	4320	D_{12}	4	6
H_{14b}	14	4320	D_{12}	3	7
H_{14c}	14	4320	D_{12}	3	7
H_{14d}	14	6480	D_8	3	8
H_{16a}	16	540	$GL(2,3) : C_2$	1	3
H_{16b}	16	2160	S_4	2	5
H_{16c}	16	6480	$C_2 \times C_2 \times C_2$	3	9
H_{16d}	16	8640	S_3	4	8
H_{18a}	18	40	$((((C_3 \times ((C_3 \times C_3) : C_2)) : C_2) : C_3) : C_2) : C_2$	1	1
H_{18b}	18	240	$((((C_3 \times C_3) : C_3) : C_2) : C_2) : C_2$	1	2
H_{18c}	18	1440	$S_3 \times S_3$	3	3
H_{18d}	18	1440	$S_3 \times S_3$	3	3
H_{18e}	18	6480	D_8	4	6
H_{18f}	18	6480	D_8	5	6

Table 3: The hyperovals of $H(3,4)$

We call two tuples $\bar{a}_1, \bar{a}_2 \in \mathbb{F}_2^6 \times \mathbb{F}_4^9$ *equivalent* if one of them is obtained from the other by successively applying the maps $\sigma_1, \sigma_2, \ldots, \sigma_6$ (described in Tables 1 and 2). GAP computations [11] show that there are 23 equivalence classes and that their sizes N are as in column 3 of Table 3. For each equivalence class, we have also determined a tuple of minimal possible weight w. Such a tuple (which need not to be unique) can be found in the last column of Table 4, while the value of w itself can be found in the second last column of that table.

We have also determined all even sets and hyperovals of $H(3,4)$ in an alternative way. We have implemented in GAP a computer model of $H(3,4)$ along with a permutation representation of its automorphism group on the set of points. The even sets are precisely the sets of points whose characteristic vectors are orthogonal with the characteristic vectors of the lines. Using this fact, we were able to show that there are up to isomorphism 676 even sets and 23 hyperovals, see [11].

In Table 3, we also mentioned several properties of the hyperovals H (obtained by

H	d	g	B	H	Induced subgraph Γ_H	w	Nonzero coefficients of $\bar{a}_H$
H_6	2	4	Y	Y	$K_{3,3}$	1	$a_6 = 1$
H_8	3	4	Y	Y	Cube graph Q_3	2	$(b_{124}, b_{234}) = (1, 1)$
H_{10a}	2	5	N	N	Petersen graph	6	$(a_4, a_6, b_{12}, b_{14}, b_{24}, b_{124}) = (1, 1, 1, 1, 1, 1)$
H_{10b}	3	4	Y	Y	$\Gamma(\{1,6\}, \{2,9\}, \{3,10\}, \{4,7\}, \{5,8\})$	2	$(b_{123}, b_{124}) = (1, \omega)$
H_{10c}	3	4	Y	Y	$\Gamma(\{1,6\}, \{2,5\}, \{3,8\}, \{4,9\}, \{7,10\})$	7	$(a_5, a_6, b_{12}, b_{123}, b_{124}, b_{134}, b_{234}) = (1, 1, \omega, 1, \omega, \omega^2, 1)$
H_{12a}	∞	4	Y	N	$2K_{3,3}$	1	$b_{12} = 1$
H_{12b}	4	4	Y	Y	$\Gamma(\{1,4\}, \{2,11\}, \{3,6\}, \{5,8\}, \{7,10\}, \{9,12\})$	1	$b_{123} = 1$
H_{12c}	3	5	N	Y	$\Gamma(\{1,7\}, \{2,10\}, \{3,8\}, \{4,12\}, \{5,9\}, \{6,11\})$	7	$(a_1, a_2, a_3, a_6, b_{12}, b_{14}, b_{24}) = (1, 1, 1, 1, \omega^2, \omega, 1)$
H_{12d}	3	4	N	Y	$\Gamma(\{1,6\}, \{2,9\}, \{3,7\}, \{4,10\}, \{5,11\}, \{8,12\})$	3	$(b_{124}, b_{134}, b_{234}) = (\omega, \omega, \omega)$
H_{14a}	4	4	Y	Y	$\Gamma(\{1,8\}, \{2,5\}, \{3,6\}, \{4,9\}, \{7,12\}, \{10,13\}, \{11,14\})$	4	$(a_6, b_{12}, b_{123}, b_{124}) = (1, \omega, 1, \omega)$
H_{14b}	5	4	Y	Y	$\Gamma(\{1,10\}, \{2,5\}, \{3,14\}, \{4,13\}, \{6,9\}, \{7,12\}, \{8,11\})$	4	$(b_{123}, b_{124}, b_{134}, b_{234}) = (1, \omega, \omega, \omega)$
H_{14c}	4	4	N	Y	$\Gamma(\{1,8\}, \{2,13\}, \{3,7\}, \{4,12\}, \{5,11\}, \{6,9\}, \{10,14\})$	3	$(b_{12}, b_{134}, b_{234}) = (1, 1, \omega)$
H_{14d}	3	4	N	Y	$\Gamma(\{1,12\}, \{2,6\}, \{3,9\}, \{4,10\}, \{5,14\}, \{7,11\}, \{8,13\})$	5	$(a_4, a_6, b_{14}, b_{24}, b_{124}) = (1, 1, 1, 1, \omega)$
H_{16a}	4	6	Y	Y	$\Gamma(\{1,12\}, \{2,7\}, \{3,14\}, \{4,9\}, \{5,16\}, \{6,11\}, \{8,13\}, \{10,15\})$	2	$(b_{23}, b_{24}) = (1, \omega)$
H_{16b}	3	4	N	Y	$\Gamma(\{1,14\}, \{2,7\}, \{3,11\}, \{4,12\}, \{5,15\}, \{6,9\}, \{8,13\}, \{10,16\})$	4	$(b_{12}, b_{23}, b_{24}, b_{234}) = (\omega, 1, 1, 1)$
H_{16c}	5	4	Y	Y	$\Gamma(\{1,6\}, \{2,13\}, \{3,16\}, \{4,15\}, \{5,10\}, \{7,12\}, \{8,11\}, \{9,14\})$	4	$(b_{23}, b_{24}, b_{124}, b_{234}) = (1, \omega, 1, 1)$
H_{16d}	4	4	N	Y	$\Gamma(\{1,9\}, \{2,12\}, \{3,11\}, \{4,7\}, \{5,10\}, \{6,15\}, \{8,14\}, \{13,16\})$	6	$(a_4, a_6, b_{12}, b_{14}, b_{24}, b_{123}) = (1, 1, \omega, \omega^2, \omega, 1)$
H_{18a}	∞	4	Y	N	$3K_{3,3}$	2	$(a_6, b_{12}) = (1, 1)$
H_{18b}	4	6	Y	Y	$\Gamma(\{1,8\}, \{2,13\}, \{3,10\}, \{4,15\}, \{5,18\}, \{6,11\}, \{7,14\}, \{9,16\}, \{12,17\})$	6	$(b_{13}, b_{14}, b_{23}, b_{24}, b_{134}, b_{234}) = (1, 1, \omega, \omega, 1, \omega)$
H_{18c}	4	4	Y	Y	$\Gamma(\{1,10\}, \{2,17\}, \{3,18\}, \{4,13\}, \{5,8\}, \{6,9\}, \{7,16\}, \{11,14\}, \{12,15\})$	7	$(a_1, a_2, a_3, a_4, b_{12}, b_{134}, b_{234}) = (1, 1, 1, 1, \omega, 1, \omega)$
H_{18d}	4	6	Y	Y	$\Gamma(\{1,8\}, \{2,13\}, \{3,10\}, \{4,15\}, \{5,18\}, \{6,11\}, \{7,14\}, \{9,16\}, \{12,17\})$	4	$(a_4, a_6, b_{14}, b_{24}) = (1, 1, 1, \omega)$
H_{18e}	4	4	N	Y	$\Gamma(\{1,6\}, \{2,14\}, \{3,13\}, \{4,18\}, \{5,8\}, \{7,10\}, \{9,16\}, \{11,17\}, \{12,15\})$	6	$(a_4, a_6, b_{12}, b_{14}, b_{24}, b_{234}) = (1, 1, 1, 1, \omega, 1)$
H_{18f}	4	4	N	Y	$\Gamma(\{1,6\}, \{2,16\}, \{3,10\}, \{4,18\}, \{5,13\}, \{7,17\}, \{8,11\}, \{9,14\}, \{12,15\})$	7	$(a_4, b_{13}, b_{14}, b_{23}, b_{24}, b_{134}, b_{234}) = (1, 1, 1, \omega, \omega^2, 1, \omega)$

Table 4: The hyperovals of $H(3, 4)$

means of computations). This information includes the total number v of points, the structure of the stabilizer S of the hyperoval, the number O_1 of orbits of S on the hyperoval and the number O_2 of orbits of S on its complement.

Suppose P is a partition of some set $V = \{1, 2, \ldots, 2n\}$ with $n \in \mathbb{N} \setminus \{0, 1\}$ in subsets of size 2 such that $|x - y| \notin \{1, 2n - 1\}$ for every $\{x, y\}$ in P. Then the graph $\Gamma(P)$ with vertex set V and edge set $P \cup \{\{1, 2\}, \{2, 3\}, \ldots, \{2n - 1, 2n\}, \{2n, 1\}\}$ is a Hamiltonian regular graph of valency 3. Conversely, every 3-regular Hamiltonian graph on $2n$ vertices is isomorphic to such a graph.

For each hyperoval H in the classification, we have also implemented (with the aid of Sage [18]) the subgraph Γ_H of the collinearity graph induced on H, see [11]. We verified that with exception of three cases, these graphs are all Hamiltonian (of degree 3) and so they are of the above-described form. A description of these graphs can be found in Table 4. The three non-Hamiltonian graphs are isomorphic to the Petersen graph and the unions $iK_{3,3}$ of $i \in \{2, 3\}$ complete bipartite graphs of type $K_{3,3}$. Certain of the Hamiltonian graphs are known under a certain name: the graph Γ_{H_6} is just $K_{3,3}$, the graph Γ_{H_8} is the cube graph Q_3 of diameter 3, the graph $\Gamma_{H_{10c}}$ is the Möbius ladder on 10 vertices and the graph $\Gamma_{H_{12b}}$ is the prism graph $K_2 \times C_6$ on 12 vertices. All the graphs mentioned in Table 4 are mutually nonisomorphic, with exception of $\Gamma_{H_{18b}}$ and $\Gamma_{H_{18d}}$. So, nonisomorphic hyperovals can have isomorphic induced subgraphs. Besides a construction, Table 4 list some other information about the graphs, like the diameter d, the girth g and the answers B and H (being Y(es) or N(o)) to the questions whether they are bipartite or Hamiltonian.

Several hyperovals of $H(3, 4)$ have already been described in the literature. We give an overview of these constructions and mention which hyperovals in the classification they correspond to. The various properties of the hyperovals mentioned in Tables 3 and 4 have helped in the identification process.

If H is a hyperbolic line of $H(3, 4)$, then $H \cup H^\perp$ is a hyperoval of $H(3, 4)$ by [12, Theorem 2.1]. This hyperoval is isomorphic to H_6.

An *ovoid* of a generalized quadrangle is a set of points meeting each line in a singleton. If O_1 and O_2 are two distinct ovoids of a generalized quadrangle, then the symmetric difference $O_1 \Delta O_2$ of O_1 and O_2 is a hyperoval by [12, Theorem 6.1]. By the main result of [1], we know that the generalized quadrangle $H(3, 4)$ has up to isomorphism two ovoids: the *classical ovoids* arising by intersecting the underlying Hermitian variety with nontangent planes, and the *nonclassical ovoids*. Every nonclassical ovoid is of the form $(O \setminus H) \cup H^\perp$, where O is a classical ovoid and H is a hyperbolic line contained in O.

Computations show that the hyperovals that arise as symmetric differences of two classical ovoids are precisely the hyperovals isomorphic to H_{12a} and H_{16a}. The hyperovals that arise as a symmetric difference of a classical and a nonclassical ovoid are precisely the hyperovals isomorphic to H_6, H_{14a}, H_{18a} and H_{18d}. The hyperovals that arise as symmetric differences of two nonclassical ovoids are precisely the hyperovals isomorphic to H_{10b}, H_{12a}, H_{12b}, H_{16c}, H_{18b} and H_{18c}. By construction, the underlying graphs of these hyperovals are bipartite.

Let π be a nontangent plane to the Hermitean surface $H(3, q^2)$ and let $h_1, h_2, \ldots, h_n$

be a nonempty collection of hyperbolic lines contained in π such that every point of $\pi \cap H(3,q^2)$ is contained in at most two hyperbolic lines of the collection $h_1, h_2, \dots, h_n$. Let B denote the set of all points of $\pi \cap H(3,q^2)$ contained in precisely two hyperbolic lines of this collection. By [6, Theorem 2.2], we then know that the set $H := (h_1 \cup h_2 \cup \cdots \cup h_n \cup h_1^\perp \cup h_2^\perp \cup \cdots \cup h_n^\perp) \setminus B$ is a hyperoval of size $2(n(q+1) - |B|)$ of $H(3,q^2)$. The special cases of this construction where either $B = \emptyset$ or $B = h_1 \cup h_2 \cup \cdots \cup h_n$ were also discussed in Pavese [16] (Examples 1 and 2). Our computations show that the hyperovals of $H(3,4)$ that can be obtained in this way are the hyperovals isomorphic to H_6, H_{10b}, H_{12a}, H_{12c}, H_{14a}, H_{14c}, H_{16a}, H_{18a}, H_{18b}, H_{18c} or H_{18d}.

Let us now mention the remaining constructions from the literature for hyperovals of $H(3,q^2)$. In [4], a hyperoval of size $2(q^3-q)$ of $H(3,q^2)$, q even, was constructed which has its full automorphism group isomorphic to $PSL(2,q) \times C_2 \times C_2$. For $q = 2$, this hyperoval is isomorphic to H_{12b}. In [5], a hyperoval of size $2(q^2+1)$ of $H(3,q^2)$, q even, was constructed whose full automorphism group is a group of type $D_{4(q^2+1)}$, $D_{2(q^2+1)}$ or $C_{q^2+1} \rtimes C_4$. For $q = 2$, this hyperoval is isomorphic to H_{10c}. In [6, Proposition 5.1], a hyperoval of size q^2+2q of $H(3,q^2)$, q even, was constructed. For $q = 2$, this hyperoval is isomorphic to H_8.

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