Families of quadratic sets on the Klein quadric

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Abstract

Consider the Klein quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q)$. A set of points of $Q^+(5,q)$ is called a *quadratic set* if it intersects each plane π of $Q^+(5,q)$ in a possibly reducible conic of π, i.e. in a singleton, a line, an irreducible conic, a pencil of two lines or the whole of π. A quadratic set is called *good* if at most two of these possibilities occur as π ranges over all planes of $Q^+(5,q)$. Good quadratic sets can come into 15 possible types and in [3] we already discussed 11 of these types. The present paper is devoted to the remaining types. We will describe several infinite families of good quadratic sets of $Q^+(5,q)$. This will show that there are examples of quadratic sets for each of these four types and for each value of the prime power q.

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1 Introduction

This paper is a sequel to the paper [3] in which we classified (so-called) good quadratic sets of the Klein quadric $Q^+(5,q)$. Such quadratic sets fall into 15 types and in [3] we were able to deal in a geometric and combinatorial way with 11 of these 15 types, obtaining nonexistence results or complete classifications for 10 of the types and indicating how the quadratics sets of one of the types are equivalent with so-called ovoids of $Q^+(5,q)$.

In the present paper, we discuss the four remaining types. We do not obtain complete classification results here, but we are able to show that there are examples for any of these types and this for all possible values of the prime power q. The techniques used here differ from those of [3]. The main aim here is to construct several infinite families of quadratic sets in an entirely algebraic way.

A *quadratic set* of a nonsingular quadric Q of Witt index at least 3 is defined as a set of points meeting each subspace π of Q is a possibly reducible quadric of π. The classical examples of such sets are the intersections of Q with other quadrics of the ambient projective space of Q. This notion has some similarity with the notion of a quadratic set of a projective space defined in [2] (see also [14, Chapter 5]) as a set of points having similar structural properties as quadrics.

A set S of points of the Klein quadric $Q^+(5,q)$ is thus a quadratic set if and only if every plane π of $Q^+(5,q)$ meets S in either a singleton, a line, a conic, a pencil or the whole of π. Here and in the sequel of this paper, we use the words "conic" and "pencil" as abbreviations for respectively "irreducible conic" and "pencil of two lines". We say that the intersection $\pi \cap S$ has *type* (S), (L), (C), (P) or (W) depending on whether $\pi \cap S$ is a singleton, a line, a conic, a pencil or the whole of π. If all plane intersections have the same type (X), then we say that the quadratic set S has *type (X)*. If there are exactly two possible types for the plane intersections, say (X) and (Y), then the quadratic set is said to be of *type* (XY). A quadratic set S of $Q^+(5,q)$ is called *good* if there are at most two possible types for the plane intersections. There are thus 15 possible types for a good quadratic set: (S), (L), (C), (P), (W), (SL), (SC), (SP), (SW), (LC), (LP), (LW), (CP), (CW), (PW).

In [3] we gave a computer assisted classification of all good quadratic sets of the (smallest) Klein quadric $Q^+(5,2)$, showing that there are up to isomorphism 27 of them. In [3], we also obtained a complete classification of all good quadratic sets of $Q^+(5,q)$ for which the type is equal to either (L), (P), (W), (SL), (SP), (SW), (LP), (LW), (CW) or (PW). We also noted there that the good quadratic sets of type (S) are precisely the images under the Klein correspondence of the line spreads of the projective space $\mathrm{PG}(3,q)$. The cases of good quadratic sets of types (C), (SC), (LC) and (CP) were not discussed in [3]. These form the content of the present paper. In fact, one of these cases was already discussed in the literature.

In [7], Glynn described for every prime power q a line set $\mathcal{L}$ in $\mathrm{PG}(3,q)$ that satisfies the following two properties:

- for every plane π of $\mathrm{PG}(3,q)$, the set of lines of $\mathcal{L}$ contained in π is a conic envelope or a dual conic of π (i.e., a conic in the dual plane of π);
- for every point p, the set of lines of $\mathcal{L}$ containing p is a conic in the quotient projective space $\mathrm{PG}(3,q)_x \cong \mathrm{PG}(2,q)$ of $\mathrm{PG}(3,q)$ defined by the point x.

This set $\mathcal{L}$ was obtained as orbit of a line under a Singer group of $\mathrm{PG}(3,q)$ and its image under the Klein correspondence gives rise to a quadratic set of type (C).

In Section 5 of the present paper, we describe six families of quadratic sets of type (LC) of $Q^+(5,q)$, three for q even and three for q odd. All the quadratic sets of type (LC) are examples of so-called $(q+1)$-ovoids, where an *m-ovoid* of $Q^+(5,q)$ is defined as a set of points meeting each plane of $Q^+(5,q)$ in exactly m points. m-ovoids of polar spaces have been widely investigated, see e.g. [1, 13]. We are not aware that the $(q+1)$-ovoids that arise from our quadratic sets of type (LC) have already occurred in the literature.

In Section 4, we also describe two families of quadratic sets of type (SC) of $Q^+(5,q)$ (one for q even and one for q odd) and in Section 6 we describe three families of good quadratic sets of type (CP) of $Q^+(5,q)$ (two for q even and one for q odd).

The proofs that we give for each family provide information on which planes of the Klein quadric have a particular type. This information can be useful to establish non-isomorphism between two quadratic sets of the same type (such as two quadratic sets occurring in the present paper). For each of the families, we will collect some of this information in a proposition at the end of the discussion.

The classical examples of quadratic sets of $Q^+(5,q)$ are obtained by intersecting $Q^+(5,q)$ with quadrics $\mathcal{Q}$ of $\mathrm{PG}(5,q)$. It is therefore no surprise that all families of good quadratic sets we describe later are obtained in this way. Good quadratic sets of the types considered here are quite rare objects and probably hard to find. The only way we were able to find these examples was via prior computer computations for the smallest values of the prime power q which suggested some candidates for the quadratic forms that describe the quadrics $\mathcal{Q}$. The number of such quadratic forms is huge and only a tiny fraction of them seem to work. Several of the described families were ultimately found via a trial-and-error method.

To verify that the described families of point sets are indeed good quadratic sets of the given types, a lot of computations are necessary. We have verified each of these computations with the aid of the Computer Algebra System SageMath [11]. The computer code we used can be found in [4]. Although the proofs follow a unified approach, also some ad hoc arguments are necessary for several families to obtain the desired conclusions.

Good quadratic sets are not only rare but also very special in the sense that for all plane intersections (and there are usually many of these), we always have the same type or the same two types that occur. It might therefore not be surprising that such point sets will turn out to be useful in the future for other interesting geometrical problems. This is in fact already the case for two of such problems.

The quadratic sets we construct in Section 5.6 are used in [6] to answer an open problem from the paper [10]. The authors of [10] studied line sets in $\mathrm{PG}(3,q)$, q odd, that satisfy a list of axioms. Their main theorem states that for $q \geq 7$ each such line set is either the set of secant lines with respect to a hyperbolic quadric or belongs to a hypothetical family of line sets. The family of quadratic sets we describe in Section 5.6 are explicitly used in [6] to provide examples of line sets in the hypothetical family, hereby showing that this family is nonempty for every odd prime q. In fact, the research on quadratic sets in [3] and the current paper was motivated by an observation (to be found in [6]) that the line sets in the hypothetical family are related to quadratic sets of $Q^+(5,q)$. We would also like to mention that the quadratic sets constructed in Section 5.1 of the present paper also play some role in [6].

The family of quadratic sets constructed in Section 4.2 can be used to construct an infinite family of hyperovals of $Q^+(5,q)$ (see [5]), the first infinite family of hyperovals in polar spaces of rank at least three with more than three points per line. The smallest example in this family provides a computer-free description of a hyperoval of $Q^+(5,4)$ that was already discovered in [9] by means of a backtrack search. One of the open problems of [9] precisely asked for a computer-free description of this hyperoval. A further investigation of these hyperovals is currently under way by the author.

Additional applications might be possible. For instance, the various examples of quadratic sets of type (LC) constructed in Section 5 might be helpful to construct examples of $(q+1)$-ovoids in the elliptic quadric $Q^-(7,q)$. In this context, it is worthwhile to mention Theorem 13 in [1] which implies that the smallest value of $m > 0$ for which

m-ovoids of $Q^-(7,q)$ can exist is $m = q+1$.

2 The Klein correspondence

Let V be a 4-dimensional vector space over the finite field $\mathbb{F}_q$ of order q. Associated with V, there is the 3-dimensional projective space $\mathrm{PG}(3,q) = \mathrm{PG}(V)$. The second exterior power $\bigwedge^2 V$ of V is a 6-dimensional vector space over $\mathbb{F}_q$ whose associated projective space $\mathrm{PG}(\bigwedge^2 V)$ will also be denoted by $\mathrm{PG}(5,q)$. In this section, we will describe a certain connection between the lines of $\mathrm{PG}(3,q)$ and certain points of $\mathrm{PG}(5,q)$. For more background information on this correspondence, we refer to [8] and [12].

Let L be a line of $\mathrm{PG}(V)$. If $\langle \bar{v}_1 \rangle$ and $\langle \bar{v}_2 \rangle$ are two distinct points of L, then we denote by $\kappa(L)$ the point $\langle \bar{v}_1 \wedge \bar{v}_2 \rangle$ of $\mathrm{PG}(\bigwedge^2 V) = \mathrm{PG}(5,q)$. We note here that the point $\langle \bar{v}_1 \wedge \bar{v}_2 \rangle$ does not depend on the chosen points $\langle \bar{v}_1 \rangle$ and $\langle \bar{v}_2 \rangle$ on the line L. The map κ is thus well-defined. In fact, κ defines a bijection between the set of lines of $\mathrm{PG}(3,q)$ and a certain nonsingular hyperbolic quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q) = \mathrm{PG}(\bigwedge^2 V)$ which is called the *Klein quadric*. The bijective correspondence between the lines of $\mathrm{PG}(3,q)$ and the points of $Q^+(5,q)$ is called the *Klein correspondence*.

If $\mathcal{L}$ is a set of lines of $\mathrm{PG}(3,q)$, then we define $\kappa(\mathcal{L}) := \{\kappa(L) \mid L \in \mathcal{L}\}$. If p is a point of $\mathrm{PG}(3,q)$, then we denote by $\mathcal{L}_p$ the set of lines of $\mathrm{PG}(3,q)$ through p, and if π is a plane of $\mathrm{PG}(3,q)$, then we denote by $\mathcal{L}_\pi$ the set of lines of $\mathrm{PG}(3,q)$ contained in π. The following properties are well-known.

Lemma 2.1.

- *For every point p of $\mathrm{PG}(3,q)$, $\kappa(\mathcal{L}_p)$ is a plane of $Q^+(5,q)$.*
- *For every plane π of $\mathrm{PG}(3,q)$, $\kappa(\mathcal{L}_\pi)$ is a plane of $Q^+(5,q)$.*
- *If α is a plane of $Q^+(5,q)$, then either $\alpha = \kappa(\mathcal{L}_p)$ for some point p of $\mathrm{PG}(3,q)$ or $\alpha = \kappa(\mathcal{L}_\pi)$ for some plane π of $\mathrm{PG}(3,q)$.*

The planes of $Q^+(5,q)$ of the form $\kappa(\mathcal{L}_p)$ for points p of $\mathrm{PG}(3,q)$ are called *Latin planes*, and the planes of $Q^+(5,q)$ of the form $\kappa(\mathcal{L}_\pi)$ for planes π of $\mathrm{PG}(3,q)$ are called *Greek planes*. The following holds.

Lemma 2.2.

(1) *Two distinct Latin planes intersect in a singleton.*

(2) *Two distinct Greek planes intersect in a singleton.*

(3) *A Latin and a Greek plane are either disjoint or meet in a line.*

(4) *Every line of $Q^+(5,q)$ is contained in precisely two planes of $Q^+(5,q)$, a Latin and a Greek plane.*

(5) *Every point of $Q^+(5,q)$ is contained in precisely $2(q+1)$ planes of $Q^+(5,q)$. Among these planes, there are $q+1$ Latin planes and $q+1$ Greek planes.*

Lemma 2.3. (1) *The Latin planes are precisely the planes of* $\mathrm{PG}(\bigwedge^2 V)$ *generated by three points* $\langle \bar v_1 \wedge \bar v_2\rangle$, $\langle \bar v_1 \wedge \bar v_3\rangle$ *and* $\langle \bar v_1 \wedge \bar v_4\rangle$, *where* $(\bar v_1, \bar v_2, \bar v_3, \bar v_4)$ *is an ordered basis of* V. *The Latin plane is then equal to* $\kappa(\mathcal{L}_p)$ *where* $p = \langle \bar v_1\rangle$.

(2) *The Greek planes are precisely the planes of* $\mathrm{PG}(\bigwedge^2 V)$ *generated by the points* $\langle \bar v_1 \wedge \bar v_2\rangle$, $\langle \bar v_1 \wedge \bar v_3\rangle$ *and* $\langle \bar v_2 \wedge \bar v_3\rangle$ *where* $\bar v_1$, $\bar v_2$, $\bar v_3$ *is a linearly independent collection of vectors of* V. *The Greek plane is then equal to* $\kappa(\mathcal{L}_\pi)$ *where* π *is the plane of* $\mathrm{PG}(3,q)$ *generated by* $\langle \bar v_1\rangle$, $\langle \bar v_2\rangle$ *and* $\langle \bar v_3\rangle$.

Suppose now that $(\bar e_1, \bar e_2, \bar e_3, \bar e_4)$ is an ordered basis of V and put $\bar e_{ij} := \bar e_i \wedge \bar e_j \in \bigwedge^2 V$ for all $i, j \in \{1,2,3,4\}$. Then $(\bar e_{12}, \bar e_{34}, \bar e_{13}, \bar e_{42}, \bar e_{14}, \bar e_{23})$ is an ordered basis of $\bigwedge^2 V$. If L is a line of $\mathrm{PG}(3,q)$, then the coordinates of $\kappa(L)$ with respect to this basis are called the *Plücker coordinates* of L. The coordinates of a point of $\mathrm{PG}(5,q)$ with respect $(\bar e_{12}, \bar e_{34}, \bar e_{13}, \bar e_{42}, \bar e_{14}, \bar e_{23})$ will be denoted by $(p_{12}, p_{34}, p_{13}, p_{42}, p_{14}, p_{23})$. The point $(p_{12}, p_{34}, p_{13}, p_{42}, p_{14}, p_{23})$ belongs to $Q^+(5,q)$ if and only if $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$.

If p is a point of $\mathrm{PG}(3,q)$ with coordinates (X_1, X_2, X_3, X_4) with respect to $(\bar e_1, \bar e_2, \bar e_3, \bar e_4)$, then we denote by $L(X_1, X_2, X_3, X_4)$ the Latin plane $\kappa(\mathcal{L}_p)$. If π is a plane of $\mathrm{PG}(3,q)$ with equation $a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = 0$ where $(a_1, a_2, a_3, a_4) \in \mathbb{F}_q^4 \setminus \{(0,0,0,0)\}$, then we denote by $G(a_1, a_2, a_3, a_4)$ the Greek plane $\kappa(\mathcal{L}_\pi)$.

The Latin planes are thus precisely the planes $L(1,x,y,z)$, $L(0,1,x,y)$, $L(0,0,1,x)$ and $L(0,0,0,1)$ with $x, y, z \in \mathbb{F}_q$. The Greek planes are precisely the planes $G(1,x,y,z)$, $G(0,1,x,y)$, $G(0,0,1,x)$ and $G(0,0,0,1)$ with $x, y, z \in \mathbb{F}_q$. If π is one of these Latin or Greek planes, then by Lemma 2.3 the points of π are the points $p_\pi(\alpha,\beta,\gamma) := \langle \bar v_\pi(\alpha,\beta,\gamma)\rangle$ with $(\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$. Here, $\bar v_\pi(\alpha,\beta,\gamma) = \alpha \cdot \bar v_1 \wedge \bar w_1 + \beta \cdot \bar v_2 \wedge \bar w_2 + \gamma \cdot \bar v_3 \wedge \bar w_3$, with $\bar v_1$, $\bar w_1$, $\bar v_2$, $\bar w_2$, $\bar v_3$ and $\bar w_3$ the vectors of V as mentioned in the following table.

	$L(1,x,y,z)$	$L(0,1,x,y)$	$L(0,0,1,x)$	$L(0,0,0,1)$
$\bar v_1$	$\bar e_1 + x\bar e_2 + y\bar e_3 + z\bar e_4$	$\bar e_2 + x\bar e_3 + y\bar e_4$	$\bar e_3 + x\bar e_4$	$\bar e_4$
$\bar w_1$	$\bar e_2$	$\bar e_1$	$\bar e_1$	$\bar e_1$
$\bar v_2$	$\bar e_1 + x\bar e_2 + y\bar e_3 + z\bar e_4$	$\bar e_2 + x\bar e_3 + y\bar e_4$	$\bar e_3 + x\bar e_4$	$\bar e_4$
$\bar w_2$	$\bar e_3$	$\bar e_3$	$\bar e_2$	$\bar e_2$
$\bar v_3$	$\bar e_1 + x\bar e_2 + y\bar e_3 + z\bar e_4$	$\bar e_2 + x\bar e_3 + y\bar e_4$	$\bar e_3 + x\bar e_4$	$\bar e_4$
$\bar w_3$	$\bar e_4$	$\bar e_4$	$\bar e_4$	$\bar e_3$
	$G(1,x,y,z)$	$G(0,1,x,y)$	$G(0,0,1,x)$	$G(0,0,0,1)$
$\bar v_1$	$-x\bar e_1 + \bar e_2$	$\bar e_1$	$\bar e_1$	$\bar e_1$
$\bar w_1$	$-y\bar e_1 + \bar e_3$	$-x\bar e_2 + \bar e_3$	$\bar e_2$	$\bar e_2$
$\bar v_2$	$-x\bar e_1 + \bar e_2$	$\bar e_1$	$\bar e_1$	$\bar e_1$
$\bar w_2$	$-z\bar e_1 + \bar e_4$	$-y\bar e_2 + \bar e_4$	$-x\bar e_3 + \bar e_4$	$\bar e_3$
$\bar v_3$	$-y\bar e_1 + \bar e_3$	$-x\bar e_2 + \bar e_3$	$\bar e_2$	$\bar e_2$
$\bar w_3$	$-z\bar e_1 + \bar e_4$	$-y\bar e_2 + \bar e_4$	$-x\bar e_3 + \bar e_4$	$\bar e_3$

We thus obtain the following points $p_\pi(\alpha,\beta,\gamma)$.

π	$p_\pi(\alpha,\beta,\gamma)$
$L(1,x,y,z)$	$(\alpha,-z\beta+y\gamma,\beta,z\alpha-x\gamma,\gamma,-y\alpha+x\beta)$
$L(0,1,x,y)$	$(-\alpha,-y\beta+x\gamma,-x\alpha,-\gamma,-y\alpha,\beta)$
$L(0,0,1,x)$	$(0,\gamma,-\alpha,x\beta,-x\alpha,-\beta)$
$L(0,0,0,1)$	$(0,-\gamma,0,\beta,-\alpha,0)$
$G(1,x,y,z)$	$(y\alpha+z\beta,\gamma,-x\alpha+z\gamma,-\beta,-x\beta-y\gamma,\alpha)$
$G(0,1,x,y)$	$(-x\alpha-y\beta,\gamma,\alpha,x\gamma,\beta,y\gamma)$
$G(0,0,1,x)$	$(\alpha,0,-x\beta,-\gamma,\beta,-x\gamma)$
$G(0,0,0,1)$	$(\alpha,0,\beta,0,0,\gamma)$

3 On the intersection of $Q^+(5,q)$ with quadrics

Recall that $Q^+(5,q)$ is the Klein quadric of $\mathrm{PG}(5,q)$ having equation $X_1X_2+X_3X_4+X_5X_6=0$ with respect to the ordered basis $(\bar{e}_{12},\bar{e}_{34},\bar{e}_{13},\bar{e}_{42},\bar{e}_{14},\bar{e}_{23})$ of $\bigwedge^2 V$. Let $\mathcal{Q}$ be another quadric of $\mathrm{PG}(5,q)$ described by the quadratic form $Q:\bigwedge^2 V\to\mathbb{F}_q$.

Let π be a plane of $Q^+(5,q)$. Then we define

$$Q_\pi(\alpha,\beta,\gamma):=Q(\bar{v}_\pi(\alpha,\beta,\gamma)),$$

where $\bar{v}_\pi(\alpha,\beta,\gamma)$ is as in Section 2. Put

$$Q_\pi(\alpha,\beta,\gamma)=a_{11}\alpha^2+a_{22}\beta^2+a_{33}\gamma^2+a_{12}\alpha\beta+a_{13}\alpha\gamma+a_{23}\beta\gamma,$$

where $a_{11},a_{22},a_{33},a_{12},a_{13},a_{23}\in\mathbb{F}_q$.

If q is odd, then we define

$$\Delta_\pi:=\begin{vmatrix} a_{11} & \frac{a_{12}}{2} & \frac{a_{13}}{2} \\ \frac{a_{12}}{2} & a_{22} & \frac{a_{23}}{2} \\ \frac{a_{13}}{2} & \frac{a_{23}}{2} & a_{33} \end{vmatrix}.$$

If $\Delta_\pi\neq 0$, then $\pi\cap\mathcal{Q}$ is a conic of π. If $\Delta_\pi=0$, then $\pi\cap\mathcal{Q}$ is either a singleton, a line, a pencil or the whole of π.

If q is even, then we define $k_\pi=(a_{23},a_{13},a_{12})$ and $D_\pi:=Q_\pi(a_{23},a_{13},a_{12})$. If $D_\pi\neq 0$, then $\pi\cap\mathcal{Q}$ is a conic of π with kernel equal to k_π. If $D_\pi=0$, then $\pi\cap\mathcal{Q}$ is either a singleton, a line, a pencil or the whole of π. The following can also be said in case $D_\pi=0$.

Lemma 3.1. (1) *If $(a_{23},a_{13},a_{12})=(0,0,0)$, then $D_\pi=0$ and $\pi\cap\mathcal{Q}$ is a line.*

(2) *If $D_\pi=0$ and $a_{ij}\neq 0$ for a certain $(i,j)\in\{(1,2),(1,3),(2,3)\}$, then $Q_\pi(\alpha,\beta,\gamma)=a_{ii}(\alpha')^2+a_{ij}\alpha'\beta'+a_{jj}(\beta')^2$, where $[\alpha'\,\beta'\,\gamma']^T=A\cdot[\alpha\,\beta\,\gamma]^T$ for a certain nonsingular (3×3)-matrix A over $\mathbb{F}_q$.*

Proof. (1) If $(a_{23},a_{13},a_{12})=(0,0,0)$, then $D_\pi=Q_\pi(0,0,0)=0$, Moreover, $Q_\pi(\alpha,\beta,\gamma)=a_{11}\alpha^2+a_{22}\beta^2+a_{33}\gamma^2=(\sqrt{a_{11}}\alpha+\sqrt{a_{22}}\beta+\sqrt{a_{33}}\gamma)^2$ and so $\pi\cap\mathcal{Q}$ is the line with equation $\sqrt{a_{11}}\alpha+\sqrt{a_{22}}\beta+\sqrt{a_{33}}\gamma=0$.

(2) Suppose $D_\pi = a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 + a_{12}a_{13}a_{23} = 0$ and $a_{ij} \neq 0$ for a certain $(i,j) \in \{(1,2),(1,3),(2,3)\}$. Without loss of generality, we may suppose that $a_{12} \neq 0$. If we put $\alpha = \alpha' + a_{23}\gamma'$, $\beta = \beta' + a_{13}\gamma'$ and $\gamma = a_{12}\gamma'$, then we compute that $Q_\pi(\alpha,\beta,\gamma) = a_{11}(\alpha')^2 + a_{12}\alpha'\beta' + a_{22}(\beta')^2$. So, the claim of the lemma holds with A the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & a_{23} \\ 0 & 1 & a_{13} \\ 0 & 0 & a_{12} \end{bmatrix}.$$

$\square$

Lemma 3.1 can be useful to determine which type of quadric a certain intersection $\pi \cap \mathcal{Q}$ is. Also information about the number of roots of quadratic equations can be useful. Still under the assumption that q is even, we put $q = 2^h$ with $h \in \mathbb{N}^*$ and we define $\mathrm{Tr}(k) := k + k^2 + k^4 + \cdots + k^{2^{h-1}}$ for every $k \in \mathbb{F}_q$. We then have that $\mathrm{Tr}(k^2) = \mathrm{Tr}(k)$ and $\mathrm{Tr}(k_1 + k_2) = \mathrm{Tr}(k_1) + \mathrm{Tr}(k_2)$ for all $k, k_1, k_2 \in \mathbb{F}_q$. We also have $\mathrm{Tr}(k) \in \{0,1\}$ for every $k \in \mathbb{F}_q$ with $\mathrm{Tr}(k) = 0$ if and only if the equation $X^2 + X + k = 0$ has a root r in $\mathbb{F}_q$ (the other root is then $r+1$). Note that if $k_1, k_2, k_3 \in \mathbb{F}_q$ with $k_1 \neq 0 \neq k_2$, then the quadratic equation $k_1X^2 + k_2X + k_3 = 0$ is equivalent with $(\frac{k_1X}{k_2})^2 + (\frac{k_1X}{k_2}) + \frac{k_1k_3}{k_2^2} = 0$ and so there is a solution in $\mathbb{F}_q$ if and only if $\mathrm{Tr}(\frac{k_1k_3}{k_2^2}) = 0$.

4 Quadratic sets of type (SC)

4.1 A first family of quadratic sets of type (SC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q odd, defined by the quadratic form

$$X_1^2 + a_{33}a_{44}X_2^2 + a_{33}X_3^2 + a_{44}X_4^2,$$

where $a_{33}, a_{44} \in \mathbb{F}_q^*$ such that $-a_{33}$ and $-a_{44}$ are non-squares in $\mathbb{F}_q$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (1 + a_{44}z^2)\alpha^2 + (a_{33} + a_{33}a_{44}z^2)\beta^2 + (a_{33}a_{44}y^2 + a_{44}x^2)\gamma^2 - (2a_{44}xz)\alpha\gamma - (2a_{33}a_{44}yz)\beta\gamma$ and $\Delta_\pi = a_{33}a_{44}(1 + a_{44}z^2)(x^2 + a_{33}y^2)$. If $(x,y) \neq (0,0)$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = (1 + a_{44}z^2)(\alpha^2 + a_{33}\beta^2)$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (0,0,1)$.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (1 + a_{33}x^2)\alpha^2 + (a_{33}a_{44}y^2)\beta^2 + (a_{33}a_{44}x^2 + a_{44})\gamma^2 - (2a_{33}a_{44}xy)\beta\gamma$ and $\Delta_\pi = a_{33}a_{44}^2y^2(1 + a_{33}x^2)$. If $y \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $y = 0$, then $Q_\pi(\alpha,\beta,\gamma) = (1 + a_{33}x^2)(\alpha^2 + a_{44}\gamma^2)$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (0,1,0)$.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + (a_{44}x^2)\beta^2 + (a_{33}a_{44})\gamma^2$ and $\Delta_\pi = a_{33}^2a_{44}^2x^2$. If $x \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}(\alpha^2 + a_{44}\gamma^2)$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (0,1,0)$.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{44}(\beta^2 + a_{33}\gamma^2)$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (1,0,0)$.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (y^2+a_{33}x^2)\alpha^2+(z^2+a_{44})\beta^2+a_{33}(a_{44}+z^2)\gamma^2+(2yz)\alpha\beta-(2a_{33}xz)\alpha\gamma$ and $\Delta_\pi = a_{33}a_{44}(y^2+a_{33}x^2)(z^2+a_{44})$. If $(x,y) \neq (0,0)$, then $\Delta_\pi \neq 0$ and so $\pi\cap\mathcal{Q}$ is a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = (z^2+a_{44})(\beta^2+a_{33}\gamma^2)$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (1,0,0)$.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (x^2+a_{33})\alpha^2 + y^2\beta^2 + a_{44}(a_{33}+x^2)\gamma^2 + (2xy)\alpha\beta$ and $\Delta_\pi = a_{33}a_{44}y^2(a_{33}+x^2)$. If $y \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $y = 0$, then $Q_\pi(\alpha,\beta,\gamma) = (x^2+a_{33})(\alpha^2+a_{44}\gamma^2)$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (0,1,0)$.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2 + (a_{33}x^2)\beta^2 + a_{44}\gamma^2$ and $\Delta_\pi = a_{33}a_{44}x^2$. If $x \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi\cap\mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2 + a_{44}\gamma^2$ and so $\pi\cap\mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (0,1,0)$.

If $\pi = G(0,0,0,1)$, then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2 + a_{33}\beta^2$ and so $\pi \cap \mathcal{Q}$ is the singleton consisting of the point of π for which $(\alpha,\beta,\gamma) = (0,0,1)$.

Combining all the above information, we find:

Proposition 4.1. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (SC). There are $4(q+1)$ planes of $Q^+(5,q)$ that meet S in a singleton and $2(q+1)^2(q-1)$ planes of $Q^+(5,q)$ that meet S in a conic. As a consequence, $|S| = \frac{1}{2(q+1)} \cdot \Big(4(q+1)\cdot 1 + 2(q+1)^2(q-1)\cdot(q+1)\Big) = q^3+q^2-q+1$.*

4.2 A second family of quadratic sets of type (SC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q even, defined by the quadratic form

$$X_2^2+a_{33}X_3^2+a_{44}X_4^2+a_{55}X_5^2+a_{66}X_6^2+a_{35}X_3X_5+a_{36}X_3X_6+a_{45}X_4X_5+a_{46}X_4X_6+a_{56}X_5X_6,$$

where $a_{33}, a_{44}, a_{55}, a_{66}, a_{35}, a_{36}, a_{45}, a_{46}, a_{56} \in \mathbb{F}_q^*$ satisfy

$$a_{44} = \frac{a_{33}a_{45}^2}{a_{35}^2},\ a_{46} = \frac{a_{36}a_{45}}{a_{35}},\ a_{55} = \frac{a_{33}a_{45}}{a_{36}},\ a_{56} = \frac{a_{35}a_{36}}{a_{33}},\ a_{66} = \frac{a_{33}a_{36}a_{45}}{a_{35}^2},\ \mathrm{Tr}(\frac{a_{33}^2a_{45}}{a_{35}^2a_{36}}) = 1.$$

We refer to the last condition as the trace condition.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{35}^2}(a_{33}a_{36}a_{45}y^2+a_{35}a_{36}a_{45}yz+a_{33}a_{45}^2z^2)\alpha^2+\frac{1}{a_{35}^2}(a_{33}a_{36}a_{45}x^2+a_{35}^2a_{36}x+a_{35}^2z^2+a_{33}a_{35}^2)\beta^2+\frac{1}{a_{35}^2a_{36}}(a_{33}a_{36}a_{45}^2x^2+a_{35}^2a_{36}a_{45}x+a_{35}^2a_{36}y^2+a_{33}a_{35}^2a_{45})\gamma^2 + \frac{1}{a_{35}}(a_{36}a_{45}xz+a_{35}a_{36}y)\alpha\beta + \frac{1}{a_{33}a_{35}}(a_{33}a_{36}a_{45}xy + a_{35}^2a_{36}y + a_{33}a_{35}a_{45}z)\alpha\gamma + \frac{1}{a_{33}a_{35}}(a_{33}a_{36}a_{45}x^2 + a_{35}^2a_{36}x + a_{33}a_{35}^2)\beta\gamma$ and $D_\pi = \frac{1}{a_{33}^2}(a_{33}a_{36}y^2 + a_{35}a_{36}yz + a_{33}a_{45}z^2)^2$. If $(y,z) \neq (0,0)$, then the trace condition implies that $D_\pi \neq 0$ and $\pi \cap \mathcal{Q}$ is then a conic. If $(y,z) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{33}a_{35}^2a_{36}}(a_{33}a_{36}a_{45}x^2 + a_{35}^2a_{36}x + a_{33}a_{35}^2)(a_{33}a_{36}\beta^2 + a_{35}a_{36}\beta\gamma + a_{33}a_{45}\gamma^2)$ and the trace condition then implies that $\pi \cap \mathcal{Q}$ is the singleton consisting of the unique point of π with $(\alpha,\beta,\gamma) = (1,0,0)$.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{36}}(a_{33}a_{36}x^2 + a_{35}a_{36}xy + a_{33}a_{45}y^2)\alpha^2 + \frac{1}{a_{35}^2}(a_{35}^2y^2 + a_{33}a_{36}a_{45})\beta^2 + \frac{1}{a_{35}^2}(a_{35}^2x^2 + a_{33}a_{45}^2)\gamma^2 + \frac{1}{a_{33}}(a_{33}a_{36}x + a_{35}a_{36}y)\alpha\beta + a_{45}y\alpha\gamma + \frac{1}{a_{35}}a_{36}a_{45}\beta\gamma$ and $D_\pi = \frac{1}{a_{33}^2}(a_{33}a_{36}x^2 + a_{35}a_{36}xy + a_{33}a_{45}y^2)^2$. If $(x,y) \neq (0,0)$, then the trace condition implies that $D_\pi \neq 0$ and $\pi \cap \mathcal{Q}$ is then a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{a_{45}}{a_{35}^2}(a_{33}a_{36}\beta^2 + a_{35}a_{36}\beta\gamma + a_{33}a_{45}\gamma^2)$ and the trace condition then implies that $\pi \cap \mathcal{Q}$ is the singleton consisting of the unique point of π with $(\alpha,\beta,\gamma) = (1,0,0)$.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{36}}(a_{33}a_{45}x^2 + a_{35}a_{36}x + a_{33}a_{36})\alpha^2 + \frac{1}{a_{35}^2}(a_{33}a_{45}^2x^2 + a_{35}a_{36}a_{45}x + a_{33}a_{36}a_{45})\beta^2 + \gamma^2 + \frac{1}{a_{33}}(a_{33}a_{45}x^2 + a_{35}a_{36}x + a_{33}a_{36})\alpha\beta$ and $D_\pi = \frac{1}{a_{33}^2}(a_{33}a_{45}x^2 + a_{35}a_{36}x + a_{33}a_{36})^2 \neq 0$ by the trace condition. So, $\pi \cap \mathcal{Q}$ is then always a conic.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{a_{33}a_{45}}{a_{36}}\alpha^2 + \frac{a_{33}a_{45}^2}{a_{35}^2}\beta^2 + \gamma^2 + a_{45}\alpha\beta$ and $D_\pi = a_{45}^2 \neq 0$. So, $\pi \cap \mathcal{Q}$ is then a conic.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{35}^2}(a_{33}a_{35}^2x^2 + a_{35}^2a_{36}x + a_{33}a_{36}a_{45})\alpha^2 + \frac{1}{a_{35}^2a_{36}}(a_{33}a_{35}^2a_{45}x^2 + a_{35}^2a_{36}a_{45}x + a_{33}a_{36}a_{45}^2)\beta^2 + \frac{1}{a_{36}}(a_{33}a_{45}y^2 + a_{35}a_{36}yz + a_{33}a_{36}z^2 + a_{36})\gamma^2 + \frac{1}{a_{33}a_{35}}(a_{33}a_{35}^2x^2 + a_{35}^2a_{36}x + a_{33}a_{36}a_{45})\alpha\beta + \frac{1}{a_{33}}(a_{33}a_{35}xy + a_{35}a_{36}y + a_{33}a_{36}z)\alpha\gamma + (a_{35}xz + a_{45}y)\beta\gamma$ and $D_\pi = \frac{1}{a_{33}^2a_{35}^2}(a_{33}a_{35}^2x^2 + a_{35}^2a_{36}x + a_{33}a_{36}a_{45})^2 \neq 0$ by the trace condition. So, $\pi \cap \mathcal{Q}$ is then always a conic.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + \frac{a_{33}a_{45}}{a_{36}}\beta^2 + \frac{1}{a_{35}^2}(a_{33}a_{45}^2x^2 + a_{35}a_{36}a_{45}xy + a_{33}a_{36}a_{45}y^2 + a_{35}^2)\gamma^2 + a_{35}\alpha\beta + a_{36}y\alpha\gamma + \frac{1}{a_{33}}(a_{33}a_{45}x + a_{35}a_{36}y)\beta\gamma$ and $D_\pi = a_{35}^2 \neq 0$. So, $\pi \cap \mathcal{Q}$ is then always a conic.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{33}a_{35}^2a_{36}}(a_{33}a_{36}x^2 + a_{35}a_{36}x + a_{33}a_{45})(a_{33}a_{35}^2\beta^2 + a_{35}^2a_{36}\beta\gamma + a_{33}a_{36}a_{45}\gamma^2)$ and by the trace condition $\pi \cap \mathcal{Q}$ is then the singleton consisting of the unique point of π with $(\alpha,\beta,\gamma) = (1,0,0)$.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{35}^2}(a_{33}a_{35}^2\beta^2 + a_{35}^2a_{36}\beta\gamma + a_{33}a_{36}a_{45}\gamma^2)$ and by the trace condition $\pi \cap \mathcal{Q}$ is then the singleton consisting of the unique point of π with $(\alpha,\beta,\gamma) = (1,0,0)$.

Combining all the above information, we find:

Proposition 4.2. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (SC). There are $2(q+1)$ planes of $Q^+(5,q)$ that meet S in a singleton and $2q^2(q+1)$ planes of $Q^+(5,q)$ that meet S in a conic. As a consequence, $|S| = \frac{1}{2(q+1)} \cdot \Big(2(q+1)\cdot 1 + 2q^2(q+1)\cdot(q+1)\Big) = q^3+q^2+1$.*

5 Quadratic sets of type (LC)

In this section, we describe six families of quadratic sets of type (LC) of $Q^+(5,q)$, three for q even and three for q odd. As mentioned before, all the constructed quadratic sets of type (LC) are also examples of $(q+1)$-ovoids.

5.1 A first family of quadratic sets of type (LC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q even, defined by the quadratic form

$$X_2X_5 + a_{26}X_2X_6 + a_{33}X_3^2 + a_{44}X_4^2 + a_{66}X_6^2,$$

where $a_{26}, a_{33}, a_{44}, a_{66} \in \mathbb{F}_q^*$ with $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{66}y^2 + a_{44}z^2)\alpha^2 + (a_{33} + a_{26}xz + a_{66}x^2)\beta^2 + (y + a_{44}x^2)\gamma^2 + (a_{26}yz)\alpha\beta + (a_{26}y^2)\alpha\gamma + (a_{26}xy + z)\beta\gamma$ and $D_\pi = a_{33}a_{26}^2y^4 + a_{66}y^2z^2 + a_{44}z^4$. As $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$, we have $D_\pi \neq 0$ if and only if $(y,z) \neq (0,0)$. So, if $(y,z) \neq (0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(y,z) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = (a_{33} + a_{66}x^2)\beta^2 + (a_{44}x^2)\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{33}x^2)\alpha^2 + (a_{66} + a_{26}y)\beta^2 + a_{44}\gamma^2 + y^2\alpha\beta + xy\alpha\gamma + a_{26}x\beta\gamma$ and $D_\pi = a_{33}a_{26}^2x^4 + a_{66}x^2y^2 + a_{44}y^4$. As $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$, we have $D_\pi \neq 0$ if and only if $(x,y) \neq (0,0)$. So, if $(x,y) \neq (0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{66}\beta^2 + a_{44}\gamma^2$ and so $\pi \cap \mathcal{Q}$ is then a line.

Suppose $\pi = L(0,0,1,x)$. Then one computes that $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + (a_{44}x^2 + a_{66})\beta^2 + x\alpha\gamma + a_{26}\beta\gamma$ and $D_\pi = a_{44}x^4 + a_{66}x^2 + a_{33}a_{26}^2$. As $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$, we have $D_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{44}\beta^2 + \alpha\gamma$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{66} + a_{33}x^2)\alpha^2 + a_{44}\beta^2 + (y + a_{33}z^2)\gamma^2 + a_{26}\alpha\gamma + x\beta\gamma$ and $D_\pi = a_{33}x^4 + a_{66}x^2 + a_{44}a_{26}^2$. As $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$, we have $D_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + (a_{44}x^2 + a_{66}y^2 + a_{26}y)\gamma^2 + \beta\gamma$ and $D_\pi = a_{33} \neq 0$, showing that $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{33}x^2)\beta^2 + (a_{44} + a_{66}x^2)\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\beta^2 + a_{66}\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Combining all the above information, we find:

Proposition 5.1. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (LC) having $(q+1)(q^2+1)$ points. There are $2(q+1)$ planes of $Q^+(5,q)$ that meet S in a line and $2q^2(q+1)$ planes of $Q^+(5,q)$ that meet S in a conic.*

5.2 A second family of quadratic sets of type (LC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q even, defined by the quadratic form

$$a_{11}X_1^2 + a_{22}X_2^2 + X_3X_5,$$

where $a_{11}, a_{22} \in \mathbb{F}_q^*$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{11}\alpha^2 + a_{22}z^2\beta^2 + a_{22}y^2\gamma^2 + \beta\gamma$ and $D_\pi = a_{11} \neq 0$, showing that $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{11}+xy)\alpha^2+a_{22}y^2\beta^2+a_{22}x^2\gamma^2$, showing that $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = x\alpha^2 + a_{22}\gamma^2$, showing that $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{22}\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{11}y^2\alpha^2 + a_{11}z^2\beta^2 + (a_{22} + yz)\gamma^2 + x^2\alpha\beta + xy\alpha\gamma + xz\beta\gamma$ and $D_\pi = a_{22}x^4$. So, if $x \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = a_{11}y^2\alpha^2 + a_{11}z^2\beta^2 + (a_{22} + yz)\gamma^2$ and so $\pi \cap \mathcal{Q}$ is then a line.

If $\pi = G(0,1,x,y)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{11}x^2\alpha^2+a_{11}y^2\beta^2+a_{22}\gamma^2+\alpha\beta$ and $D_\pi = a_{22} \neq 0$, showing that $\pi \cap \mathcal{Q}$ is a conic.

If $\pi = G(0,0,1,x)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{11}\alpha^2 + x\beta^2$ and so $\pi \cap \mathcal{Q}$ is a line.

If $\pi = G(0,0,0,1)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{11}\alpha^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Combining all the above information, we find:

Proposition 5.2. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (LC) having $(q+1)(q^2+1)$ points. There are $2(q^2+q+1)$ planes of $Q^+(5,q)$ that meet S in a line and $2q^3$ planes of $Q^+(5,q)$ that meet S in a conic.*

5.3 A third family of quadratic sets of type (LC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q even, defined by the quadratic form

$$X_1X_6 + a_{25}X_2X_5 + a_{33}X_3^2 + a_{44}X_4^2 + a_{56}X_5X_6,$$

where $a_{25}, a_{33}, a_{44}, a_{56} \in \mathbb{F}_q^*$ with $\mathrm{Tr}(\frac{a_{25}}{a_{56}^2}) = \mathrm{Tr}(\frac{a_{33}a_{44}a_{56}^2}{a_{25}^2}) = 1$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (y+a_{44}z^2)\alpha^2+a_{33}\beta^2+(a_{44}x^2+a_{25}y)\gamma^2+x\alpha\beta+a_{56}y\alpha\gamma+(a_{25}z+a_{56}x)\beta\gamma$ and $D_\pi = a_{44}(a_{25}z^2+a_{56}xz+x^2)^2+a_{33}a_{56}^2y^2+(a_{25}y)(x^2+a_{56}xz + a_{25}z^2)$. As $\mathrm{Tr}(\frac{a_{25}}{a_{56}^2}) = \mathrm{Tr}(\frac{a_{33}a_{44}a_{56}^2}{a_{25}^2}) = 1$, we have $D_\pi \neq 0$ if and only if $(x,y,z) \neq (0,0,0)$. So, if $(x,y,z) \neq (0,0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(x,y,z) = (0,0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\beta^2$ and $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}x^2\alpha^2+a_{44}\gamma^2+(1+a_{56}y+a_{25}y^2)\alpha\beta+a_{25}xy\alpha\gamma$ and $D_\pi = a_{44}(1 + a_{56}y + a_{25}y^2)^2 \neq 0$. So, $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2+a_{44}x^2\beta^2+(a_{56}x)\alpha\beta+a_{25}x\alpha\gamma = 0$ and $D_\pi = a_{44}a_{25}^2x^4$. If $x \neq 0$, then $D_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{44}\beta^2 + a_{25}\alpha\gamma$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (y+a_{33}x^2)\alpha^2+a_{44}\beta^2+(a_{25}y+a_{33}z^2)\gamma^2+(z+a_{56}x)\alpha\beta+a_{56}y\alpha\gamma+(a_{25}x)\beta\gamma$ and $D_\pi = a_{33}(z^2+a_{56}xz+a_{25}x^2)^2+a_{44}a_{56}^2y^2+a_{25}y(a_{25}x^2+z^2 + a_{56}xz)$. As $\mathrm{Tr}(\frac{a_{25}}{a_{56}^2}) = \mathrm{Tr}(\frac{a_{33}a_{44}a_{56}^2}{a_{25}^2}) = 1$, we have $D_\pi = 0$ if and only if $(x,y,z) = (0,0,0)$. So, if $(x,y,z) \neq (0,0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(x,y,z) = (0,0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{44}\beta^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + a_{44}x^2\gamma^2 + xy\alpha\gamma + (y^2 + a_{56}y + a_{25})\beta\gamma$ and $D_\pi = a_{33}(y^2 + a_{56}y + a_{25})^2 \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{33}x^2)\beta^2 + a_{44}\gamma^2 + x\alpha\gamma + a_{56}x\beta\gamma$ and $D_\pi = a_{33}x^4$. So, if $x \neq 0$, then $D_\pi \neq 0$ and $\pi \cap \mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = a_{44}\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\beta^2 + \alpha\gamma$ and so $\pi \cap \mathcal{Q}$ is a conic.

Combining all the above information, we find:

Proposition 5.3. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (LC) having $(q+1)(q^2+1)$ points. There are 4 planes of $Q^+(5,q)$ that meet S in a line and $2(q^3+q^2+q-1)$ planes of $Q^+(5,q)$ that meet S in a conic.*

5.4 A fourth family of quadratic sets of type (LC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q odd, defined by the quadratic form

$$X_1^2 + \frac{\mu^2}{4}X_2^2 + \mu X_3X_4 - \mu X_5X_6,$$

where $\mu \in \mathbb{F}_q^*$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (\alpha + \frac{\mu z}{2}\beta + \frac{\mu y}{2}\gamma)^2 - (\mu^2yz + 2\mu x)\beta\gamma$. So, if $\mu^2yz + 2\mu x = 0$, then $\pi \cap \mathcal{Q}$ is a line and if $\mu^2yz + 2\mu x \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (\alpha + \frac{\mu y}{2}\beta + \frac{\mu x}{2}\gamma)^2 - \mu^2xy\beta\gamma$. So, if $xy = 0$, then $\pi \cap \mathcal{Q}$ is a line and if $xy \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{\mu^2\gamma^2}{4} - (2\mu x)\alpha\beta$. So, if $x = 0$ then $\pi \cap \mathcal{Q}$ is a line and if $x \neq 0$ then $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \frac{\mu^2\gamma^2}{4}$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (y\alpha - z\beta + \frac{\mu\gamma}{2})^2 + 2(2yz + \mu x)\alpha\beta$. So, if $2yz + \mu x = 0$, then $\pi \cap \mathcal{Q}$ is a line and if $2yz + \mu x \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (x\alpha - y\beta + \frac{\mu\gamma}{2})^2 + 4xy\alpha\beta$. So, if $xy = 0$ then $\pi \cap \mathcal{Q}$ is a line and if $xy \neq 0$ then $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2 + 2\mu x\beta\gamma$. So, if $x = 0$ then $\pi \cap \mathcal{Q}$ is a line and if $x \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2$ and $\pi \cap \mathcal{Q}$ is a line.

Combining all the above information, we find:

Proposition 5.4. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (LC) having $(q+1)(q^2+1)$ points. There are $2(q+1)^2$ planes of $Q^+(5,q)$ that meet S in a line and $2q(q^2-1)$ planes of $Q^+(5,q)$ that meet S in a conic.*

5.5 A fifth family of quadratic sets of type (LC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q odd, defined by the quadratic form

$$X_1^2 + a_{22}X_2^2 + a_{35}X_3X_5 + a_{46}X_4X_6,$$

where $a_{22}, a_{35}, a_{46} \in \mathbb{F}_q^*$ with $a_{22} = \frac{a_{35}a_{46}}{4}$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (1-a_{46}yz)\alpha^2 + a_{22}z^2\beta^2 + a_{22}y^2\gamma^2 + a_{46}xz\alpha\beta + a_{46}xy\alpha\gamma + (a_{35} - 2a_{22}yz - a_{46}x^2)\beta\gamma$ and $\Delta_\pi = -\frac{1}{4}(a_{35}a_{46}yz + a_{46}x^2 - a_{35})^2$. So, if $a_{35}a_{46}yz + a_{46}x^2 - a_{35} \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic and if $a_{35}a_{46}yz + a_{46}x^2 - a_{35} = 0$, then one computes that $Q_\pi(\alpha,\beta,\gamma) = a_{22}\Big(\frac{2x}{a_{35}}\alpha + z\beta + y\gamma\Big)^2$, showing that $\pi \cap \mathcal{Q}$ is then a line.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (1 + a_{35}xy)\alpha^2 + a_{22}y^2\beta^2 + a_{22}x^2\gamma^2 - (2a_{22}xy + a_{46})\beta\gamma$ and $\Delta_\pi = -\frac{a_{46}^2}{4}(1 + a_{35}xy)^2$. If $1 + a_{35}xy \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $1 + a_{35}xy = 0$, then one computes that $Q_\pi(\alpha,\beta,\gamma) = a_{22}(y\beta + x\gamma)^2$ and so $\pi \cap \mathcal{Q}$ is then a line.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{35}x\alpha^2 - a_{46}x\beta^2 + a_{22}\gamma^2$ and $\Delta_\pi = -a_{22}a_{35}a_{46}x^2$. If $x \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = a_{22}\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{22}\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = y^2\alpha^2 + z^2\beta^2 + (a_{22} - a_{35}yz)\gamma^2 + (2yz - a_{46} + a_{35}x^2)\alpha\beta + a_{35}xy\alpha\gamma - a_{35}xz\beta\gamma$ and $\Delta_\pi = -\frac{1}{4}a_{22}(a_{35}x^2 + 4yz - a_{46})^2$. If $a_{35}x^2 + 4yz - a_{46} \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $a_{35}x^2 + 4yz - a_{46} = 0$, then one computes that $Q_\pi(\alpha,\beta,\gamma) = (y\alpha - z\beta + \frac{a_{35}x}{2}\gamma)^2$ and so $\pi \cap \mathcal{Q}$ is then a line.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = x^2\alpha^2 + y^2\beta^2 + (a_{22} + a_{46}xy)\gamma^2 + (2xy + a_{35})\alpha\beta$ and $\Delta_\pi = -\frac{1}{4}a_{22}(4xy + a_{35})^2$. If $4xy + a_{35} \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $4xy + a_{35} = 0$, then one computes that $Q_\pi(\alpha,\beta,\gamma) = (x\alpha - y\beta)^2$ and so $\pi \cap \mathcal{Q}$ is then a line.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2 - a_{35}x\beta^2 + a_{46}x\gamma^2$ and $\Delta_\pi = -a_{35}a_{46}x^2$. If $x \neq 0$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $x = 0$, then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2$ and so $\pi \cap \mathcal{Q}$ is then a line.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \alpha^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Combining all the above information, we find:

Proposition 5.5. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (LC) having $(q+1)(q^2+1)$ points. If $a_{35}a_{46}$ is a square in $\mathbb{F}_q$, then there are $2(q+1)^2$ planes of $Q^+(5,q)$ that meet S in a line and $2q(q^2-1)$ planes of $Q^+(5,q)$ that meet S in a conic. If $a_{35}a_{46}$ is not a square in $\mathbb{F}_q$, then there are $2(q^2+1)$ planes of $Q^+(5,q)$ that meet S in a line and $2q(q^2+1)$ planes of $Q^+(5,q)$ that meet S in a conic.*

5.6 A sixth family of quadratic sets of type (LC)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q odd, defined by the quadratic form

$$X_2X_5 + d_1X_2X_6 + a_{33}X_3^2 + 2a_{33}d_2X_3X_4 + a_{33}d_2^2X_4^2,$$

where $a_{33}, d_1, d_2 \in \mathbb{F}_q^*$ with $-d_1d_2$ a non-square in $\mathbb{F}_q$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{33}d_2^2z^2)\alpha^2 + (a_{33} - d_1xz)\beta^2 + (y + a_{33}d_2^2x^2)\gamma^2 + (d_1yz + 2a_{33}d_2z)\alpha\beta - (d_1y^2 + 2a_{33}d_2^2xz)\alpha\gamma + (d_1xy - z - 2a_{33}d_2x)\beta\gamma$ and $\Delta_\pi = -\frac{a_{33}}{4}(d_1y^2 + d_2z^2)^2$. If $(y,z) \neq (0,0)$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $(y,z) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}(\beta - d_2x\gamma)^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{33}x^2)\alpha^2 - d_1 y\beta^2 + a_{33}d_2^2\gamma^2 + y^2\alpha\beta + (2a_{33}d_2x - xy)\alpha\gamma + d_1x\beta\gamma$ and $\Delta_\pi = -\frac{a_{33}}{4}(d_1x^2 + d_2y^2)^2$. If $(x,y) \neq (0,0)$, then $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}d_2^2\gamma^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + (a_{33}d_2^2x^2)\beta^2 - (2a_{33}d_2x)\alpha\beta - x\alpha\gamma - d_1\beta\gamma$ and $\Delta_\pi = -\frac{a_{33}}{4}(d_2x^2 + d_1)^2 \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

If $\pi = L(0,0,0,1)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}d_2^2\beta^2 + \alpha\gamma$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{33}x^2)\alpha^2 + (a_{33}d_2^2)\beta^2 + (a_{33}z^2 - y)\gamma^2 + (2a_{33}d_2x)\alpha\beta + (d_1 - 2a_{33}xz)\alpha\gamma - (x + 2a_{33}d_2z)\beta\gamma$ and $\Delta_\pi = -\frac{a_{33}}{4}(x^2 + d_1d_2)^2 \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\alpha^2 + (d_1y + a_{33}d_2^2x^2)\gamma^2 + (2a_{33}d_2x)\alpha\gamma + \beta\gamma$ and $\Delta_\pi = -\frac{1}{4}a_{33} \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

If $\pi = G(0,0,1,x)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}(x\beta + d_2\gamma)^2$ and so $\pi \cap \mathcal{Q}$ is a line.

If $\pi = G(0,0,0,1)$, then $Q_\pi(\alpha,\beta,\gamma) = a_{33}\beta^2$ and so $\pi \cap \mathcal{Q}$ is a line.

Combining all the above information, we find:

Proposition 5.6. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (LC) having $(q+1)(q^2+1)$ points. There are $2(q+1)$ planes of $Q^+(5,q)$ that meet S in a line and $2q^2(q+1)$ planes of $Q^+(5,q)$ that meet S in a conic.*

6 Quadratic sets of type (CP)

6.1 A first family of quadratic sets of type (CP)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q even, defined by the quadratic form

$$X_2^2 + a_{35}X_3X_5 + a_{36}X_3X_6 + a_{45}X_4X_5 + a_{46}X_4X_6 + a_{56}X_5X_6 + a_{66}X_6^2,$$

where $a_{35}, a_{36}, a_{45}, a_{46}, a_{56}, a_{66} \in \mathbb{F}_q^*$ with

$$a_{46} = \frac{a_{36}a_{45}}{a_{35}}, \qquad a_{66} = \frac{a_{36}a_{56}}{a_{35}}, \qquad \mathrm{Tr}(\frac{a_{36}a_{45}}{a_{56}^2}) = 1.$$

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{66}y^2 + a_{46}yz)\alpha^2 + (a_{66}x^2 + a_{36}x + z^2)\beta^2 + (a_{45}x + y^2)\gamma^2 + (a_{46}xz + a_{36}y)\alpha\beta + (a_{46}xy + a_{56}y + a_{45}z)\alpha\gamma + (a_{46}x^2 + a_{56}x + a_{35})\beta\gamma$ and $D_\pi = (a_{36}y^2 + a_{56}yz + a_{45}z^2)^2$. As $\mathrm{Tr}(\frac{a_{36}a_{45}}{a_{56}^2}) = 1$, we have $D_\pi = 0$ if and only if $(y,z) = (0,0)$. So, if $(y,z) \neq (0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(y,z) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{35}}((a_{56}x + a_{35})\beta + (a_{45}x)\gamma)((a_{36}x)\beta + a_{35}\gamma)$. As $(a_{56}x + a_{35})a_{35} + (a_{45}x)(a_{36}x) = (a_{36}a_{45})x^2 + (a_{35}a_{56})x + a_{35}^2$ and $\mathrm{Tr}(\frac{a_{36}a_{45}a_{35}^2}{(a_{35}a_{56})^2}) = \mathrm{Tr}(\frac{a_{36}a_{45}}{a_{56}^2}) = 1$, $\pi \cap \mathcal{Q}$ is then a pencil.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{35}xy)\alpha^2 + (y^2 + a_{66})\beta^2 + x^2\gamma^2 + (a_{36}x + a_{56}y)\alpha\beta + (a_{45}y)\alpha\gamma + a_{46}\beta\gamma$ and $D_\pi = (a_{36}x^2 + a_{56}xy + a_{45}y^2)^2$. As $\mathrm{Tr}(\frac{a_{36}a_{45}}{a_{56}^2}) = 1$, we have $D_\pi = 0$ if and only if $(x,y) = (0,0)$. So, if $(x,y) \neq (0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{a_{36}}{a_{35}}\beta(a_{56}\beta + a_{45}\gamma)$ and $\pi \cap \mathcal{Q}$ is a pencil.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{35}x)\alpha^2 + (a_{46}x + a_{66})\beta^2 + \gamma^2 + (a_{45}x^2 + a_{56}x + a_{36})\alpha\beta$ and $D_\pi = (a_{45}x^2 + a_{56}x + a_{36})^2$. As $\mathrm{Tr}(\frac{a_{45}a_{36}}{a_{56}^2}) = 1$, $D_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \gamma^2 + a_{45}\alpha\beta$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{36}x + a_{66})\alpha^2 + (a_{45}x)\beta^2 + (a_{35}yz + 1)\gamma^2 + (a_{35}x^2 + a_{56}x + a_{46})\alpha\beta + (a_{35}xy + a_{56}y + a_{36}z)\alpha\gamma + (a_{35}xz + a_{45}y)\beta\gamma$ and $D_\pi = (a_{35}x^2 + a_{56}x + a_{46})^2$. As $\mathrm{Tr}(\frac{a_{35}a_{46}}{a_{56}^2}) = \mathrm{Tr}(\frac{a_{36}a_{45}}{a_{56}^2}) = 1$, $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{46}xy + a_{66}y^2 + 1)\gamma^2 + a_{35}\alpha\beta + (a_{36}y)\alpha\gamma + (a_{45}x + a_{56}y)\beta\gamma$ and $D_\pi = a_{35}^2 \neq 0$. So, $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{35}x)\beta^2 + (a_{66}x^2 + a_{46}x)\gamma^2 + (a_{36}x^2 + a_{56}x + a_{45})\beta\gamma = \frac{1}{a_{35}}(a_{35}\beta + a_{36}x\gamma) \cdot (a_{35}x\beta + (a_{56}x + a_{45})\gamma)$. As $a_{35}(a_{56}x + a_{45}) + (a_{36}x)(a_{35}x) = (a_{35}a_{36})x^2 + (a_{35}a_{56})x + (a_{35}a_{45})$ and $\mathrm{Tr}(\frac{(a_{35}a_{36})\cdot(a_{35}a_{45})}{(a_{35}a_{56})^2}) = \mathrm{Tr}(\frac{a_{36}a_{45}}{a_{56}^2}) = 1$, $\pi \cap \mathcal{Q}$ is a pencil.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{66}\gamma^2 + a_{36}\beta\gamma = \gamma(a_{66}\gamma + a_{36}\beta)$ and so $\pi \cap \mathcal{Q}$ is a pencil.

Proposition 6.1. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (CP). There are $2q^2(q+1)$ planes of $Q^+(5,q)$ that meet S in a conic and $2(q+1)$ planes of $Q^+(5,q)$ that meet S in a pencil. As a consequence, $|S| = \frac{1}{2(q+1)}\Big(2q^2(q+1)\cdot(q+1) + 2(q+1)\cdot(2q+1)\Big) = q^3 + q^2 + 2q + 1$.*

6.2 A second family of quadratic sets of type (CP)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q odd, defined by the quadratic form

$$X_2^2 + a_{35}X_3X_5 + a_{36}X_3X_6 + a_{45}X_4X_5 + a_{46}X_4X_6 + a_{56}X_5X_6 + a_{66}X_6^2,$$

where $a_{35}, a_{36}, a_{45}, a_{46}, a_{56}, a_{66} \in \mathbb{F}_q^*$ such that $a_{46} = \frac{a_{36}a_{45}}{a_{35}}$, $a_{66} = \frac{a_{36}a_{56}}{a_{35}}$ and $a_{56}^2 - 4a_{36}a_{45}$ is a nonsquare in $\mathbb{F}_q$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{66}y^2 - a_{46}yz)\alpha^2 + (a_{66}x^2 + a_{36}x + z^2)\beta^2 + (y^2 - a_{45}x)\gamma^2 + (a_{46}xz - a_{36}y - 2a_{66}xy)\alpha\beta + (a_{46}xy - a_{56}y + a_{45}z)\alpha\gamma + (a_{56}x - 2yz + a_{35} - a_{46}x^2)\beta\gamma$ and $\Delta_\pi = -\frac{1}{4}\Big(a_{36}y^2 - a_{56}yz + a_{45}z^2\Big)^2$. As $a_{56}^2 - 4a_{36}a_{45}$ is a nonsquare, $\Delta_\pi = 0$ if and only if $(y,z) = (0,0)$. So, if $(y,z) \neq (0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(y,z) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{a_{35}}(a_{36}x\beta + a_{35}\gamma) \cdot ((a_{56}x + a_{35})\beta - (a_{45}x)\gamma)$. As $(a_{36}x)(a_{45}x) + a_{35}(a_{56}x + a_{35}) = (a_{36}a_{45})x^2 + (a_{35}a_{56})x + a_{35}^2$ and $(a_{35}a_{56})^2 - 4(a_{36}a_{45})a_{35}^2 = a_{35}^2(a_{56}^2 - 4a_{36}a_{45})$ is a nonsquare, $\pi \cap \mathcal{Q}$ is then a pencil.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{35}xy)\alpha^2 + (y^2 + a_{66})\beta^2 + x^2\gamma^2 - (a_{36}x + a_{56}y)\alpha\beta + (a_{45}y)\alpha\gamma - (2xy + a_{46})\beta\gamma$ and $\Delta_\pi = -\frac{1}{4}\Big(a_{36}x^2 - a_{56}xy + a_{45}y^2\Big)^2$. As $a_{56}^2 - 4a_{36}a_{45}$ is a nonsquare, $\Delta_\pi = 0$ if and only if $(x,y) = (0,0)$. So, if $(x,y) \neq (0,0)$, then $\pi \cap \mathcal{Q}$ is a conic. If $(x,y) = (0,0)$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{a_{36}}{a_{35}}\beta(a_{56}\beta - a_{45}\gamma)$ and so $\pi \cap \mathcal{Q}$ is a pencil.

Suppose $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{35}x)\alpha^2 + (a_{66} - a_{46}x)\beta^2 + \gamma^2 + (a_{56}x + a_{36} - a_{45}x^2)\alpha\beta$ and $\Delta_\pi = -\frac{1}{4}(a_{45}x^2 - a_{56}x + a_{36})^2$. As $a_{56}^2 - 4a_{45}a_{36}$ is a nonsquare, $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = \gamma^2 - a_{45}\alpha\beta$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{66} - a_{36}x)\alpha^2 + (a_{45}x)\beta^2 + (1 - a_{35}yz)\gamma^2 + (a_{35}x^2 - a_{56}x - a_{46})\alpha\beta + (a_{35}xy - a_{56}y + a_{36}z)\alpha\gamma + (a_{45}y - a_{35}xz)\beta\gamma$ and $\Delta_\pi = -\frac{1}{4a_{35}^2}(a_{35}^2x^2 - a_{35}a_{56}x + a_{36}a_{45})^2$. As $(a_{35}a_{56})^2 - 4a_{35}^2a_{36}a_{45} = a_{35}^2(a_{56}^2 - 4a_{36}a_{45})$ is a nonsquare, $\Delta_\pi \neq 0$ and so $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (a_{46}xy + a_{66}y^2 + 1)\gamma^2 + a_{35}\alpha\beta + (a_{36}y)\alpha\gamma + (a_{45}x + a_{56}y)\beta\gamma$ and $\Delta_\pi = -\frac{1}{4}a_{35}^2 \neq 0$. So, $\pi \cap \mathcal{Q}$ is a conic.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = (-a_{35}x)\beta^2 + (a_{66}x^2 + a_{46}x)\gamma^2 + (a_{36}x^2 - a_{56}x - a_{45})\beta\gamma = \frac{1}{a_{35}}((a_{35}x)\beta + (a_{56}x + a_{45})\gamma) \cdot (-a_{35}\beta + (a_{36}x)\gamma)$. As $(a_{35}a_{56})^2 - 4(a_{35}a_{36})(a_{35}a_{45}) = a_{35}^2(a_{56}^2 - 4a_{36}a_{45})$ is a nonsquare, $(a_{35}x)(a_{36}x) + a_{35}a_{56}x + a_{35}a_{45} \neq 0$ and so $\pi \cap \mathcal{Q}$ is a pencil.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = a_{66}\gamma^2 + a_{36}\beta\gamma = \gamma(a_{66}\gamma + a_{36}\beta)$ is a pencil.

Proposition 6.2. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (CP). There are $2q^2(q+1)$ planes of $Q^+(5,q)$ that meet S in a conic and $2(q+1)$ planes of $Q^+(5,q)$ that meet S in a pencil. As a consequence, $|S| = \frac{1}{2(q+1)}\Big(2q^2(q+1)\cdot(q+1) + 2(q+1)\cdot(2q+1)\Big) = q^3 + q^2 + 2q + 1$.*

6.3 A third family of quadratic sets of type (CP)

Let $\mathcal{Q}$ be the quadric of $\mathrm{PG}(5,q)$, q even, defined by the quadratic form

$$X_3X_5 + a_{36}X_3X_6 + a_{44}X_4^2 + a_{45}X_4X_5 + a_{46}X_4X_6 + a_{56}X_5X_6,$$

where $a_{36}, a_{44}, a_{45}, a_{46}, a_{56} \in \mathbb{F}_q^*$ such that

$$a_{46} = \frac{a_{36}}{a_{45}}, \qquad a_{56} = \frac{a_{44}}{a_{45}}, \qquad \mathrm{Tr}(\frac{a_{36}a_{45}^3}{a_{44}^2}) = 1.$$

As all elements of $\mathbb{F}_q$ are squares, we can put $a_{ij} = b_{ij}^2$ for all $(i,j) \in \{(3,6),(4,4),(4,5),(4,6),(5,6)\}$.

Suppose $\pi = L(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = (b_{36}^2b_{45}^2yz + b_{44}^2z^2)\alpha^2 + (b_{36}^2x)\beta^2 + (b_{44}^2x^2 + b_{45}^2x)\gamma^2 + (b_{36}^2b_{45}^2xz + b_{36}^2y)\alpha\beta + \frac{1}{b_{45}^2}(b_{36}^2b_{45}^4xy + b_{45}^4z + b_{44}^2y)\alpha\gamma + \frac{1}{b_{45}^2}(b_{36}^2b_{45}^4x^2 + b_{44}^2x + b_{45}^2)\beta\gamma$ and $D_\pi = \frac{b_{44}^2}{b_{45}^4}(b_{36}^2b_{45}^2xy + b_{36}b_{45}^3xz + b_{44}^2xz + b_{36}b_{45}y + b_{45}^2z)^2$. Note also that

$$Q_\pi(\alpha,\beta,\gamma) - \frac{1}{b_{45}^3b_{36}}(b_{36}b_{45}^3z\alpha + b_{36}^2b_{45}^2x\beta + (b_{44}^2x + b_{45}^2)\gamma)((b_{36}^2b_{45}^2y + b_{44}^2z)\alpha + (b_{36}b_{45})\beta + (b_{36}b_{45}^3x)\gamma)$$

$$= \frac{1}{b_{45}^3b_{36}}\alpha(b_{36}^2b_{45}^2\beta + (b_{36}b_{45}^3 + b_{44}^2)\gamma)(b_{36}^2b_{45}^2xy + b_{36}b_{45}^3xz + b_{44}^2xz + b_{36}b_{45}y + b_{45}^2z).$$

So, if $b_{36}^2b_{45}^2xy + b_{36}b_{45}^3xz + b_{44}^2xz + b_{36}b_{45}y + b_{45}^2z \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic. If $b_{36}^2b_{45}^2xy + b_{36}b_{45}^3xz + b_{44}^2xz + b_{36}b_{45}y + b_{45}^2z = 0$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{b_{45}^3b_{36}}(b_{36}b_{45}^3z\alpha + b_{36}^2b_{45}^2x\beta + (b_{44}^2x +$

$b_{45}^2)\gamma)((b_{36}^2 b_{45}^2 y + b_{44}^2 z)\alpha + (b_{36}b_{45})\beta + (b_{36}b_{45}^3 x)\gamma)$. As $\mathrm{Tr}(\frac{b_{36}^2 b_{45}^4 \cdot b_{45}^2}{(b_{44}^2)^2}) = 1$, we then have that

$$\begin{vmatrix} b_{36}^2 b_{45}^2 x & b_{44}^2 x + b_{45}^2 \\ b_{36}b_{45} & b_{36}b_{45}^3 x \end{vmatrix} = b_{36}b_{45}(b_{36}^2 b_{45}^4 x^2 + b_{44}^2 x + b_{45}^2) \neq 0$$

and so $\pi \cap \mathcal{Q}$ is then a pencil.

Suppose $\pi = L(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = xy\alpha^2 + b_{44}^2\gamma^2 + \frac{1}{b_{45}^2}(b_{36}^2 b_{45}^2 x + b_{44}^2 y)\alpha\beta + b_{45}^2 y\alpha\gamma + b_{36}^2 b_{45}^2 \beta\gamma$ and $D_\pi = \frac{b_{44}^2}{b_{45}^4}(b_{36}^2 b_{45}^2 x + b_{36}b_{45}^3 y + b_{44}^2 y)^2$. So, if $b_{36}^2 b_{45}^2 x + b_{36}b_{45}^3 y + b_{44}^2 y \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic. If $b_{36}^2 b_{45}^2 x + b_{36}b_{45}^3 y + b_{44}^2 y = 0$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{b_{45}^2 b_{36}^2}(y\alpha + b_{36}b_{45}\gamma)((b_{36}b_{45}^3 + b_{44}^2)y\alpha + b_{36}^3 b_{45}^3 \beta + b_{36}b_{44}^2 b_{45}\gamma)$ and so $\pi \cap \mathcal{Q}$ is then a pencil.

So, $\pi = L(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = x\alpha^2 + (b_{36}^2 b_{45}^2 x + b_{44}^2 x^2)\beta^2 + \frac{1}{b_{45}^2}(b_{45}^4 x^2 + b_{36}^2 b_{45}^2 + b_{44}^2 x)\alpha\beta = \frac{1}{b_{45}^2}(\alpha + b_{45}^2 x\beta)(b_{45}^2 x\alpha + (b_{36}^2 b_{45}^2 + b_{44}^2 x)\beta)$. As $\mathrm{Tr}(\frac{b_{45}^4 \cdot b_{36}^2 b_{45}^2}{(b_{44}^2)^2}) = 1$, we then have that

$$\begin{vmatrix} 1 & b_{45}^2 x \\ b_{45}^2 x & b_{36}^2 b_{45}^2 + b_{44}^2 x \end{vmatrix} = b_{45}^4 x^2 + b_{44}^2 x + b_{36}^2 b_{45}^2 \neq 0$$

and so $\pi \cap \mathcal{Q}$ is then a pencil.

Suppose $\pi = L(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = b_{44}^2\beta^2 + b_{45}^2\alpha\beta = \beta(b_{45}^2\alpha + b_{44}^2\beta)$ and so $\pi \cap \mathcal{Q}$ is a pencil.

Suppose $\pi = G(1,x,y,z)$. Then $Q_\pi(\alpha,\beta,\gamma) = b_{36}^2 x\alpha^2 + (b_{45}^2 x + b_{44}^2)\beta^2 + yz\gamma^2 + \frac{1}{b_{45}^2}(b_{36}^2 b_{45}^4 + b_{45}^2 x^2 + b_{44}^2 x)\alpha\beta + \frac{1}{b_{45}^2}(b_{36}^2 b_{45}^2 z + b_{45}^2 xy + b_{44}^2 y)\alpha\gamma + (b_{45}^2 y + xz)\beta\gamma$ and $D_\pi = \frac{b_{44}^2}{b_{45}^4}(b_{36}b_{45}^3 y + b_{36}^2 b_{45}^2 z + b_{45}^2 xy + b_{36}b_{45}xz + b_{44}^2 y)^2$. Note also that

$$Q_\pi(\alpha,\beta,\gamma) - \frac{1}{b_{45}^2 b_{36}}(b_{36}x\alpha + b_{36}b_{45}^2\beta + b_{45}y\gamma)((b_{36}^2 b_{45}^2)\alpha + (b_{44}^2 + b_{45}^2 x)\beta + (b_{36}b_{45}z)\gamma)$$

$$= \frac{1}{b_{45}^2 b_{36}}\gamma(b_{36}\alpha + b_{45}\beta)(b_{36}b_{45}^3 y + b_{36}^2 b_{45}^2 z + b_{45}^2 xy + b_{36}b_{45}xz + b_{44}^2 y).$$

So, if $b_{36}b_{45}^3 y + b_{36}^2 b_{45}^2 z + b_{45}^2 xy + b_{36}b_{45}xz + b_{44}^2 y \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic. If $b_{36}b_{45}^3 y + b_{36}^2 b_{45}^2 z + b_{45}^2 xy + b_{36}b_{45}xz + b_{44}^2 y = 0$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{b_{45}^2 b_{36}}(b_{36}x\alpha + b_{36}b_{45}^2\beta + b_{45}y\gamma)((b_{36}^2 b_{45}^2)\alpha + (b_{44}^2 + b_{45}^2 x)\beta + (b_{36}b_{45}z)\gamma)$. As $\mathrm{Tr}(\frac{b_{36}b_{45}^2 \cdot b_{36}^3 b_{45}^4}{(b_{36}b_{44}^2)^2}) = 1$, we then have that

$$\begin{vmatrix} b_{36}x & b_{36}b_{45}^2 \\ b_{36}^2 b_{45}^2 & b_{45}^2 x + b_{44}^2 \end{vmatrix} = b_{36}b_{45}^2 x^2 + b_{36}b_{44}^2 x + b_{36}^3 b_{45}^4 \neq 0$$

and so $\pi \cap \mathcal{Q}$ is then a pencil.

Suppose $\pi = G(0,1,x,y)$. Then $Q_\pi(\alpha,\beta,\gamma) = (b_{36}^2 b_{45}^2 xy + b_{44}^2 x^2)\gamma^2 + \alpha\beta + b_{36}^2 y\alpha\gamma + \frac{1}{b_{45}^2}(b_{45}^4 x + b_{44}^2 y)\beta\gamma$ and $D_\pi = \frac{b_{44}^2}{b_{45}^2}(b_{45}x + b_{36}y)^2$. If $b_{45}x + b_{36}y \neq 0$, then $\pi \cap \mathcal{Q}$ is a conic. If $b_{45}x + b_{36}y = 0$, then $Q_\pi(\alpha,\beta,\gamma) = \frac{1}{b_{45}b_{36}}(\beta + b_{36}b_{45}x\gamma)(b_{36}b_{45}\alpha + (b_{36}b_{45}^3 + b_{44}^2)x\gamma)$ and so $\pi \cap \mathcal{Q}$ is a pencil.

Suppose $\pi = G(0,0,1,x)$. Then $Q_\pi(\alpha,\beta,\gamma) = x\beta^2 + (b_{36}^2 b_{45}^2 x + b_{44}^2)\gamma^2 + \frac{1}{b_{45}^2}(b_{36}^2 b_{45}^2 x^2 + b_{45}^4 + b_{44}^2 x)\beta\gamma = \frac{1}{b_{45}^2}(x\beta + b_{45}^2\gamma)(b_{45}^2\beta + (b_{36}^2 b_{45}^2 x + b_{44}^2)\gamma)$. As $\mathrm{Tr}(\frac{b_{36}^2 b_{45}^2 \cdot b_{45}^4}{(b_{44}^2)^2}) = 1$, we then have that

$$\begin{vmatrix} x & b_{45}^2 \\ b_{45}^2 & b_{36}^2 b_{45}^2 x + b_{44}^2 \end{vmatrix} = b_{36}^2 b_{45}^2 x^2 + b_{44}^2 x + b_{45}^4 \neq 0$$

and so $\pi \cap \mathcal{Q}$ is then a pencil.

Suppose $\pi = G(0,0,0,1)$. Then $Q_\pi(\alpha,\beta,\gamma) = b_{36}^2\beta\gamma$ and so $\pi \cap \mathcal{Q}$ is a pencil.

Proposition 6.3. *The set $S := \mathcal{Q} \cap Q^+(5,q)$ is a quadratic set of type (CP). There are $2q(q^2-1)$ planes of $Q^+(5,q)$ that meet S in a conic and $2(q+1)^2$ planes of $Q^+(5,q)$ that meet S in a pencil. As a consequence, $|S| = \frac{1}{2(q+1)}\Big(2q(q^2-1)\cdot(q+1) + 2(q+1)^2\cdot(2q+1)\Big) = q^3 + 2q^2 + 2q + 1$.*

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