Quadratic sets on the Klein quadric

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Abstract

Consider the Klein quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q)$. A set of points of $Q^+(5,q)$ is called a *quadratic set* if it intersects each plane π of $Q^+(5,q)$ in a possibly reducible conic of π, i.e. in a singleton, a line, an irreducible conic, a pencil of two lines or the whole of π. A quadratic set is called *good* if at most two of these possibilities occur as π ranges over all planes of $Q^+(5,q)$. We obtain several classification results for good quadratic sets. We also provide a complete classification of all good quadratic sets of $Q^+(5,2)$ and give an explicit construction for each of them.

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1 Introduction

One of the most fundamental objects in finite geometry is the Klein quadric. This is a nonsingular hyperbolic quadric $Q^+(5,q)$ in the projective space $\mathrm{PG}(5,q)$. Via the so-called Klein correspondence several relationships between objects of the projective space $\mathrm{PG}(3,q)$ and objects of $Q^+(5,q)$ can be described and this is often very useful. For instance, via the Klein correspondence line spreads of $\mathrm{PG}(3,q)$ correspond to ovoids of $Q^+(5,q)$ [12] and Cameron-Liebler line classes of $\mathrm{PG}(3,q)$ [5] correspond to tight sets of $Q^+(5,q)$ [11]. In the present paper, we study a family of point sets of the Klein quadric which we call here quadratic sets.

We refer to the monograph [13] by Hirschfeld and Thas as a general reference book for the basic properties of (singular and nonsingular) quadrics in finite projective spaces that we will use throughout this paper. Let $\mathcal{Q}$ be a given quadric in $\mathrm{PG}(n,q)$. If $\mathcal{Q}'$ is another quadric in $\mathrm{PG}(n,q)$, then the intersection $S := \mathcal{Q} \cap \mathcal{Q}'$ satisfies the following property:

($*'$) Every nonempty subspace π of $\mathcal{Q}$ intersects S in a quadric of π.

By a subspace of $\mathcal{Q}$, we mean here a subspace of $\mathrm{PG}(n,q)$ that is contained in $\mathcal{Q}$. We call a set S of points of $\mathcal{Q}$ a *quadratic set* if it satisfies property ($*'$). Note that a set of points of $\mathcal{Q}$ is a quadratic set of the following property is satisfied:

($*$) Every maximal subspace π of $\mathcal{Q}$ intersects S in a quadric of π.

The notion of a quadratic set in a projective space was defined and studied by Buekenhout in [3], see also [13, §1.10] and [21, Chapter 5]. Basically, these are sets of points in projective spaces that satisfy similar structural properties as quadrics. In this paper we have defined quadratic sets of a quadric as sets of points of this quadric that satisfy similar structural properties as the intersections of this quadric with the quadrics of its ambient space.

Let us now look at the case of nonsingular quadrics $\mathcal{Q}$ whose Witt index is 3, that is, whose maximal subspaces are planes. By [13], $\mathcal{Q}$ is then either a hyperbolic quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q)$ (the Klein quadric), a parabolic quadric $Q(6,q)$ in $\mathrm{PG}(6,q)$ or an elliptic quadric $Q^-(7,q)$ in $\mathrm{PG}(7,q)$. As a plane and a quadric intersect in a quadric of that plane, we then know by [13] that the following must hold.

Proposition 1.1. *Let $\mathcal{Q}$ be one of the quadrics $Q^+(5,q)$, $Q(6,q)$, $Q^-(7,q)$. Then S is a quadratic set if and only if every plane π of $\mathcal{Q}$ meets S in either a singleton, a line, a conic, a pencil or the whole of π.*

Here and in the sequel of this paper, the words "conic" and "pencil" are abbreviations for respectively "irreducible conic" and "pencil of two lines". If S is a quadratic set of $\mathcal{Q} \in \{Q^+(5,q), Q(6,q), Q^-(7,q)\}$ and π is a plane of $\mathcal{Q}$, then we say that the intersection $\pi \cap S$ has *type* (S), (L), (C), (P) or (W) depending on whether $\pi \cap S$ is a singleton, a line, a conic, a pencil or the whole point set of π. If all plane intersections have the same type (X), then we say that the quadratic set S has *type (X)*. If there are exactly two possible types for the plane intersections, say (X) and (Y), then the quadratic set is said to be of *type* (XY). A quadratic set S of $\mathcal{Q}$ is called *good* if there are at most two possible types for the plane intersections. There are thus 15 possible types for a good quadratic set: (S), (L), (C), (P), (W), (SL), (SC), (SP), (SW), (LC), (LP), (LW), (CP), (CW), (PW).

In this paper, we initiate the study of good quadratic sets of the Klein quadric $Q^+(5,q)$. This study was motivated by an open problem in the paper [15] regarding the existence of certain line sets in $\mathrm{PG}(3,q)$. Subsequent investigations showed that these line sets are related to quadratic sets of the Klein quadric. Some of our investigations on quadratic sets will allow the authors of [10] to prove that the formerly elusive line sets do actually exist.

In this paper, we obtain a complete classification of all good quadratic sets of $Q^+(5,q)$ for which the type is equal to either (L), (P), (W), (SL), (SP), (SW), (LP), (LW), (CW) or (PW). We also observe that the good quadratic sets of type (S) are precisely the images under the Klein correspondence of the line spreads of the projective space $\mathrm{PG}(3,q)$. We keep the study of the good quadratic sets of types (C), (SC), (LC) and (CP) for another paper where we will describe in an algebraic way several infinite families. One of these families will play a crucial role in [10] to prove that certain line sets of $\mathrm{PG}(3,q)$ exist.

The standard examples of quadratic sets are those that are obtained by intersecting the quadric with another quadric of the ambient projective space. One can therefore wonder whether all quadratic sets of $Q^+(5,q)$ arise in this way. We will see in Proposition 3.4 that the answer to this question is false for $q \geq 3$. For $q = 2$ however, the following can be proved.

Proposition 1.2 ([9, Corollary 1.7]). *Let $\mathcal{Q}$ be one of the quadrics $Q^+(5,2)$, $Q(6,2)$, $Q^-(7,2)$. Then every quadratic set of $\mathcal{Q}$ arises by intersecting $\mathcal{Q}$ with a quadric of the ambient projective space of $\mathcal{Q}$.*

Quadratic sets of $\mathcal{Q} \in \{Q^+(5,2), Q(6,2), Q^-(7,2)\}$ are examples of so-called *pseudo-hyperplanes* of the geometry of the points and planes of $\mathcal{Q}$. The proof of Proposition 1.2 given in [9] used this observation along with the various connections between pseudo-hyperplanes, pseudo-embeddings and pseudo-generating ranks described in [7]. In the present paper, the connection between pseudo-hyperplanes and pseudo-embeddings is used to give a computer-assisted classification of all (good) quadratic sets of $Q^+(5,2)$. Using a certain model for the quadric $Q^+(5,2)$, we will also provide computer free constructions for all good quadratic sets. There are up to isomorphism 27 of them.

2 The Klein quadric

Let V be a 4-dimensional vector space over the finite field $\mathbb{F}_q$ of order q. Associated with V, there is the 3-dimensional projective space $\mathrm{PG}(3,q) = \mathrm{PG}(V)$. The second exterior power $\bigwedge^2 V$ of V is a 6-dimensional vector space over $\mathbb{F}_q$ whose associated projective space $\mathrm{PG}(\bigwedge^2 V)$ will also be denoted by $\mathrm{PG}(5,q)$. In this section, we will describe a certain connection between the lines of $\mathrm{PG}(3,q)$ and certain points of $\mathrm{PG}(5,q)$. For more background information on this correspondence, we refer to [12] and [17].

Let L be a line of $\mathrm{PG}(V)$. If $\langle \bar{v}_1 \rangle$ and $\langle \bar{v}_2 \rangle$ denote two distinct points of L, then we denote by $\kappa(L)$ the point $\langle \bar{v}_1 \wedge \bar{v}_2 \rangle$ of $\mathrm{PG}(\bigwedge^2 V) = \mathrm{PG}(5,q)$. We note here that the point $\langle \bar{v}_1 \wedge \bar{v}_2 \rangle$ does not depend on the chosen points $\langle \bar{v}_1 \rangle$ and $\langle \bar{v}_2 \rangle$ on the line L. The map κ is thus well-defined. In fact, κ defines a bijection between the set of lines of $\mathrm{PG}(3,q)$ and a certain hyperbolic quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q) = \mathrm{PG}(\bigwedge^2 V)$ which is called the *Klein quadric*. The bijective correspondence between the lines of $\mathrm{PG}(3,q)$ and the points of $Q^+(5,q)$ is called the *Klein correspondence*. With respect to a certain reference system in $\mathrm{PG}(5,q) = \mathrm{PG}(\bigwedge^2 V)$, the Klein quadric $Q^+(5,q)$ has equation $X_1X_2 + X_3X_4 + X_5X_6 = 0$. The Klein quadric satisfies the following properties on which we will often rely.

Lemma 2.1. (1) *The subspaces of maximal possible dimension contained in $Q^+(5,q)$ are the planes of $Q^+(5,q)$. These planes can be partitioned in two families such that two planes belong to the same family if and only if they are equal or intersect in a point.*

(2) *Every point of $Q^+(5,q)$ is contained in $2(q+1)$ planes of $Q^+(5,q)$. Every line of $Q^+(5,q)$ is contained in two planes of $Q^+(5,q)$, one of each family of planes.*

(3) *For every non-incident point-line pair (p,L) of $Q^+(5,q)$, either one or all points of L are collinear with p on $Q^+(5,q)$.*

(4) *For every non-incident point-plane pair (p,π) of $Q^+(5,q)$, there exists a unique plane of $Q^+(5,q)$ through p intersecting π in a line.*

(5) *For every line-plane pair (L, π) of $Q^+(5,q)$ such that $L \cap \pi$ is a singleton, there exists a unique plane of $Q^+(5,q)$ through L intersecting π in a line.*

Hereby, two points of $Q^+(5,q)$ are said to be *collinear on* $Q^+(5,q)$ if there is some line of $Q^+(5,q)$ containing them. If x is a point of $Q^+(5,q)$, then the set $x^\perp$ of all points of $Q^+(5,q)$ that are collinear with x on $Q^+(5,q)$ is obtained by intersecting $Q^+(5,q)$ with a hyperplane Π_x of $\mathrm{PG}(5,q)$. Such a hyperplane Π_x of $\mathrm{PG}(5,q)$ is called a *tangent hyperplane* of $\mathrm{PG}(5,q)$, more specifically the hyperplane of $\mathrm{PG}(5,q)$ that is tangent to $Q^+(5,q)$ in the point x. A tangent hyperplane Π_x of $\mathrm{PG}(5,q)$ intersects $Q^+(5,q)$ in a quadric of type $xQ^+(3,q)$ which is a cone whose kernel is the point x and whose base is a hyperbolic quadric $Q^+(3,q)$ in a 3-dimensional subspace not containing x. The point-line geometry $\mathcal{S}_x$ whose points and lines are the lines and planes of $Q^+(5,q)$ through x, with incidence being containment, is thus isomorphic to the geometry of the points and lines contained in the hyperbolic quadric $Q^+(3,q)$, i.e. to a $(q+1)\times(q+1)$-grid. Such a grid is an example of a generalized quadrangle, i.e. a partial linear space having two disjoint lines such that for every non-incident point-line pair (x, L) there exists a unique point on L collinear with x.

If Π is a non-tangent hyperplane of $\mathrm{PG}(5,q)$, then $\Pi \cap Q^+(5,q)$ is a parabolic quadric of type $Q(4,q)$ [13]. We call such an intersection a $Q(4,q)$*-quadric*. The points and lines contained in a $Q(4,q)$-quadric define a generalized quadrangle which we will also denote by $Q(4,q)$. This generalized quadrangle has $q+1$ points on each line and $q+1$ lines through each point.

With $Q^+(5,q)$ there is associated a polarity [13] which maps each point $x \in Q^+(5,q)$ to the tangent hyperplane Π_x and each point $y \notin Q^+(5,q)$ to a nontangent hyperplane. This polarity is orthogonal if q is odd and symplectic if q is even.

An *ovoid* of a generalized quadrangle is a set of points intersecting each line in a singleton. Every ovoid is an example of a hyperplane, where a *hyperplane* of a point-line geometry with point set P is defined as a proper subset of P meeting each line in a singleton or the whole line. We will later need the properties that the generalized quadrangle $Q(4,2)$ has six ovoids and that through each point x of $Q(4,2)$, there are two ovoids which partition the set of points noncollinear with x.

We will also need some information about the intersection of $Q^+(5,q)$ with three-dimensional subspaces. If β is a 3-dimensional subspace of $\mathrm{PG}(5,q)$, then $\beta \cap Q^+(5,q)$ is either the union of two distinct planes through a line, a quadric of type $xQ(2,q)$, a hyperbolic quadric of type $Q^+(3,q)$ or an elliptic quadric of type $Q^-(3,q)$. In the latter two cases, these intersections are also called $Q^+(3,q)$*-quadrics* and $Q^-(3,q)$*-quadrics*, respectively. A $Q^+(3,q)$-quadric contains lines, while a $Q^-(3,q)$-quadric does not. An intersection of type $xQ(2,q)$ is the union of $q+1$ lines of $Q^+(5,q)$ through the point x no two of which are contained in the same plane of $Q^+(5,q)$, i.e. these $q+1$ lines form an ovoid of $\mathcal{S}_x$. In fact, such an intersection is a cone with kernel a point $x \in Q^+(5,q)$ whose base is a conic in a plane not containing x.

One of the following two cases occurs for two disjoint lines L_1 and L_2 of $Q^+(5,q)$:

(1) For every $i \in \{1,2\}$, there exists a unique point $x_i \in L_i$ which is collinear on $Q^+(5,q)$

with all points of L_{3-i}. Then the 3-dimensional subspace $\langle L_1, L_2\rangle$ intersects $Q^+(5,q)$ in the union of the two planes $\langle x_1x_2, L_1\rangle \cup \langle x_1x_2, L_2\rangle$.

(2) For every $i \in \{1,2\}$ and every $x \in L_i$, there exists a unique point on L_{3-i} collinear on $Q^+(5,q)$ with x. Then the 3-dimensional subspace $\langle L_1, L_2\rangle$ intersects $Q^+(5,q)$ in a $Q^+(3,q)$-quadric.

If case (2) occurs, then the lines L_1 and L_2 are called *opposite*.

Lemma 2.2. *Let p_1 and p_2 be two points of $Q^+(5,q)$ which are noncollinear on $Q^+(5,q)$. Let K_1 and L_1 be two lines of $Q^+(5,q)$ through p_1 which are not contained in a plane of $Q^+(5,q)$. Then $\langle K_1, L_1, p_2\rangle$ is a 3-space intersecting $Q^+(5,q)$ in a $Q^+(3,q)$-quadric.*

Proof. Obviously, $\langle K_1, L_1\rangle$ is a plane contained in the tangent hyperplane Π_{p_1}. As $p_2 \notin \Pi_{p_1}$, $\alpha := \langle K_1, L_1, p_2\rangle$ is a 3-dimensional subspace. There are four possibilities for the intersection $\alpha \cap Q^+(5,q)$:
(1) quadric of type $Q^+(3,q)$;
(2) a quadric of type $Q^-(3,q)$;
(3) a quadric of type $pQ(2,q)$;
(4) the union of two distinct planes.
As there exists two distinct intersecting lines K_1 and L_1 through a point p_1 and an additional point p_2 such that there are no planes of $Q^+(5,q)$ containing $\langle K_1, L_1\rangle$ and no line of $Q^+(5,q)$ containing p_1 and p_2, we see that case (1) must occur. □

In Lemma 2.2, we thus see that the two lines through p_2 meeting K_1 or L_1 are also not contained in a plane of $Q^+(5,q)$. So, if $\mathcal{L}$ is a set of lines of $Q^+(5,q)$ through p_1 forming an ovoid of $\mathcal{S}_{p_1}$, then the set of all lines of $Q^+(5,q)$ through p_2 meeting a line of $\mathcal{L}$ is an ovoid of $\mathcal{S}_{p_2}$.

For a set S of points of $Q^+(5,q)$, an S-*line* is defined as a line of $Q^+(5,q)$ having all its points in S.

3 Good quadratic sets of type (S)

Quadratic sets of type (S) of $Q^+(5,q)$ are also known as *ovoids* of $Q^+(5,q)$, and it is well known that these are related to line spreads of PG$(3,q)$. A *line spread* of PG$(3,q)$ is a set of q^2+1 lines of PG$(3,q)$ partitioning its point set. The connection between line spreads of PG$(3,q)$ and ovoids of $Q^+(5,q)$ is described in the following proposition.

Proposition 3.1 ([12])**.** *The ovoids of $Q^+(5,q)$ are the images under the Klein correspondence of the line spreads of* PG$(3,q)$.

Classifying good quadratic sets of type (S) of $Q^+(5,q)$ is thus equivalent with classifying line spreads of PG$(3,q)$. Several isomorphism classes of line spreads of PG$(3,q)$ are known to exist. The standard examples are the *regular spreads* which correspond via the Klein correspondence to the $Q^-(3,q)$-quadrics of $Q^+(5,q)$, see [12]. These $Q^-(3,q)$-quadrics of $Q^+(5,q)$ are also known as the *classical ovoids* of $Q^+(5,q)$. By [12, page 55], we know that every spread of $\mathrm{PG}(3,2)$ is regular, or equivalently, that every ovoid of $Q^+(5,2)$ is classical. We will need that result later.

A *regulus* $\mathcal{R}$ of $\mathrm{PG}(3,q)$ is a set of $q+1$ mutually disjoint lines contained in a hyperbolic quadric $Q^+(3,q) \subseteq \mathrm{PG}(3,q)$. The set $\mathcal{R}'$ of the $q+1$ remaining (mutually disjoint) lines of $Q^+(3,q)$ is then called the *opposite regulus* of $\mathcal{R}$. A regular spread of $\mathrm{PG}(3,q)$ contains many reguli. In fact, any two distinct lines of a regular spread S of $\mathrm{PG}(3,q)$ are contained in a unique regulus $\mathcal{R} \subseteq S$.

By the following lemma, we know that the classical ovoids of $Q^+(5,q)$ can be obtained by intersecting $Q^+(5,q)$ with a suitable quadric of $\mathrm{PG}(5,q)$.

Lemma 3.2. *Let Π and Π' be two (not necessarily distinct) hyperplanes of $\mathrm{PG}(5,q)$. Then each of the intersections $S_1 := (\Pi \cap \Pi') \cap Q^+(5,q)$ and $S_2 := (\Pi \cup \Pi') \cap Q^+(5,q)$ are of the form $Q^+(5,q) \cap \mathcal{Q}$ for some suitable quadric $\mathcal{Q}$ of $\mathrm{PG}(5,q)$.*

Proof. Suppose Π and Π' are described by the respective equations $a_1X_1 + a_2X_2 + \cdots + a_6X_6 = 0$ and $a'_1X_1 + a'_2X_2 + \cdots + a'_6X_6 = 0$. If $f(X,Y) \in \mathbb{F}_q[X,Y]$ is an irreducible homogeneous polynomial of degree 2 in the variables X and Y, then S_1 is also obtained by intersecting $Q^+(5,q)$ with the quadric $\mathcal{Q}_1$ whose equation is given by

$$Q_1(X_1, X_2, \ldots, X_6) = f(a_1X_1 + a_2X_2 + \cdots + a_6X_6, a'_1X_1 + a'_2X_2 + \cdots + a'_6X_6) = 0.$$

On the other hand, S_2 is obtained by intersecting $Q^+(5,q)$ with the quadric $\mathcal{Q}_2$ whose equation is

$$Q_2(X_1, X_2, \ldots, X_6) = (a_1X_1 + a_2X_2 + \cdots + a_6X_6)(a'_1X_1 + a'_2X_2 + \cdots + a'_6X_6) = 0.$$

□

Suppose $\mathcal{Q}$ is a quadric of the projective space $\mathrm{PG}(5,q)$. For every point x of $\mathcal{Q}$, the union of all lines L through x that meet $\mathcal{Q}$ in either $\{x\}$ or the whole line L is a subspace of $\mathrm{PG}(5,q)$ which is either a hyperplane or the whole of $\mathrm{PG}(5,q)$. We call this subspace the *subspace at x tangent to $\mathcal{Q}$*. In case $\mathcal{Q}$ is a nonsingular quadric, all tangent subspaces are hyperplanes.

Lemma 3.3. *Suppose S is an ovoid of $Q^+(5,q)$ obtained by intersecting $Q^+(5,q)$ with a quadric $\mathcal{Q}$. Then for every point $x \in S$, the tangent hyperplane T_x at x to the quadric $Q^+(5,q)$ is contained in the tangent subspace T'_x at x to the quadric $\mathcal{Q}$.*

Proof. The tangent hyperplane T_x is generated by all lines through x contained in $Q^+(5,q)$. As $S = Q^+(5,q) \cap \mathcal{Q}$, all these lines contain a unique point of $\mathcal{Q}$ and so are contained in T'_x. We must therefore have that $T_x \subseteq T'_x$. □

Proposition 3.4. *Let α be a 3-dimensional subspace of $\mathrm{PG}(5,q)$ intersecting $Q^+(5,q)$ in a $Q^-(3,q)$-quadric $\mathcal{Q}$. Let π be a plane of α intersecting $Q^+(5,q)$ in a conic C. Put $\pi' := \pi^\zeta$, where ζ is the polarity of $\mathrm{PG}(5,q)$ associated with $Q^+(5,q)$. Put $C' := \pi' \cap Q^+(5,q)$. Then the following hold:*

(1) $(\mathcal{Q} \cup C') \setminus C$ *is a good quadratic set of type (S);*

(2) *if* $q \geq 3$, *then this good quadratic set cannot be obtained by intersecting* $Q^+(5,q)$ *with a quadric.*

Proof. First note that $C \cap C' = \emptyset$.

Claim (1) is a known property. Every plane of $Q^+(5,q)$ containing a (unique) point of C also contains a unique point of C'. Therefore $(\mathcal{Q} \cup C') \setminus C$ is a good quadratic set of type (C). Alternatively, one can argue as follows. Put $S := \kappa^{-1}(\mathcal{Q})$, $\mathcal{R} := \kappa^{-1}(C)$ and $\mathcal{R}' := \kappa^{-1}(C')$. Then $\mathcal{R}$ is a regulus of the regular spread S and $\mathcal{R}'$ is its opposite regulus [12]. Therefore, $(S \cup \mathcal{R}') \setminus \mathcal{R}$ is a line spread of $\mathrm{PG}(3,q)$ and so $\kappa((S \cup \mathcal{R}') \setminus \mathcal{R}) = (Q \cup C') \setminus C$ is a quadratic set of type (S).

As to Claim (2), suppose to the contrary that there exists a quadric $\mathcal{Q}'$ in $\mathrm{PG}(5,q)$ for which $(\mathcal{Q} \cup C') \setminus C = \mathcal{Q}' \cap Q^+(5,q)$.

We first prove that this is impossible in case $q \geq 3$ is odd. The intersection $\pi \cap \mathcal{Q}'$ must contain a point x, and this point does not belong to C. Let L be a line of π through x which is external to C. In α there are two planes through L that are tangent to $\mathcal{Q}$ and $q-1$ planes through L that meet $\mathcal{Q}$ in a conic. Let π_1 be one of the $q-2 \geq 1$ planes of α through L distinct from π for which $\pi \cap \mathcal{Q}$ is a conic. As $\mathcal{Q} \setminus C \subseteq \mathcal{Q}'$, the $q+2$ points in $\{x\} \cup (\pi_1 \cap \mathcal{Q})$ all belong to $\mathcal{Q}'$. We now show that π_1 is contained in $\mathcal{Q}'$.

If this were not the case, then the fact that $\pi_1 \cap \mathcal{Q}$ is a conic of π_1 contained in $\mathcal{Q}'$ implies that $q = 3$ and $\pi_1 \cap \mathcal{Q}'$ is the union of two lines L_1 and L_2. Moreover, each of L_1, L_2 must contain two points of $\pi_1 \cap \mathcal{Q}$ and $L_1 \cap L_2$ is not contained in $\pi_1 \cap \mathcal{Q}$. Now, take a point $u \in \pi_1 \cap \mathcal{Q}$ belonging to $L_1 \setminus L_2$. As L_1 is a line through u contained in $\mathcal{Q}'$ and intersecting $\mathcal{Q}$ in exactly two points, we know from Lemma 3.3 that the tangent subspace at u to the quadric $\mathcal{Q}'$ is the whole space $\mathrm{PG}(5,q)$. Now, the lines of π_1 through u all contain an additional point of $L_2 \subseteq \mathcal{Q}'$ and so are completely contained in $\mathcal{Q}'$. Hence, $\pi_1 \subseteq \mathcal{Q}'$. But this is in contradiction with the fact that $\pi_1 \cap \mathcal{Q}'$ is the union of two lines.

So, we must have that $\pi_1 \subseteq \mathcal{Q}'$. In particular, the line L is contained in $\mathcal{Q}'$. As any line L' of π that is external with respect to C contains a point of $L \subseteq \mathcal{Q}'$, we can repeat the above argument for the line L' to conclude that $L' \subseteq \mathcal{Q}'$. As any point of $\pi \setminus C$ is contained in a line of π that is external to C, we can then conclude that $\pi \setminus C \subseteq \mathcal{Q}'$. So, π contains at least q^2 points of $\mathcal{Q}'$. It has therefore all its points in $\mathcal{Q}'$, in contradiction with the fact that $C \cap \mathcal{Q}' = \emptyset$.

We now also derive a contradiction in the case that $q \geq 4$ is even. As before, the intersection $\pi \cap \mathcal{Q}'$ must contain a point x that does not belong to C. Let L be a line of π through x which is tangent to C and denote by x_L the tangency point. In α there is a unique plane through L that is tangent to $\mathcal{Q}$ (necessarily in the point x_L). In α, there is also a unique plane β through L such that $\beta \cap \mathcal{Q}$ is a conic with nucleus x [12]. As $q \neq 2$, there exists a plane π_1 of α through L distinct from π which intersects $\mathcal{Q}$ in a conic for which x is not the nucleus of the conic $\pi_1 \cap \mathcal{Q}$. As $\mathcal{Q} \setminus C \subseteq \mathcal{Q}'$, the $q+1$ points in $\{x\} \cup ((\pi_1 \cap \mathcal{Q}) \setminus \{x_L\})$ all belong to $\mathcal{Q}'$.

Suppose $\pi_1 \cap \mathcal{Q}'$ is a conic. Note that two distinct conics of π_1 have at most four points in common. As $x \in (\pi_1 \cap \mathcal{Q}') \setminus (\pi_1 \cap \mathcal{Q})$, the conics $\pi_1 \cap \mathcal{Q}'$ and $\pi_1 \cap \mathcal{Q}$ of π_1 are distinct. As they also have at least q points in common, namely the points in $A := (\pi_1 \cap \mathcal{Q}) \setminus \{x_L\}$, we must have $q = 4$. But then x_L must be the nucleus of the conic $\pi_1 \cap \mathcal{Q}' = A \cup \{x\}$ and x must be the nucleus of the conic $\pi_1 \cap \mathcal{Q} = A \cup \{x_L\}$, an obvious contradiction.

So, $\pi_1 \cap \mathcal{Q}'$ cannot be a conic. We show that π_1 is contained in $\mathcal{Q}'$.

If this were not the case, then the fact that $\pi_1 \cap \mathcal{Q}$ is a conic of π_1 for which $(\pi_1 \cap \mathcal{Q}) \setminus \{x_L\} \subseteq \mathcal{Q}'$ implies that $q = 4$ and that $\pi_1 \cap \mathcal{Q}'$ is the union of two lines L_1 and L_2. Moreover, each of L_1, L_2 must contain two points of $(\pi_1 \cap \mathcal{Q}) \setminus \{x_L\}$ and $L_1 \cap L_2$ is not contained in $(\pi_1 \cap \mathcal{Q}) \setminus \{x_L\}$. Now, take a point $u \in (\pi_1 \cap \mathcal{Q}) \setminus \{x_L\}$ belonging to $L_1 \setminus L_2$. As L_1 is a line through u contained in $\mathcal{Q}'$ and intersecting $\mathcal{Q}$ in exactly two points, we know from Lemma 3.3 that the tangent subspace at u to the quadric $\mathcal{Q}'$ is the whole space $\mathrm{PG}(5, q)$. Now, the lines of π_1 through u all contain an additional point of $L_2 \subseteq \mathcal{Q}'$ and so are completely contained in $\mathcal{Q}'$. Hence, $\pi_1 \subseteq \mathcal{Q}'$. But this is in contradiction with the fact that $\pi_1 \cap \mathcal{Q}'$ is the union of two lines.

So, we must have that $\pi_1 \subseteq \mathcal{Q}'$. In particular, the line L is contained in $\mathcal{Q}'$. But then the point x_L would also be contained in $\mathcal{Q}'$, in contradiction with $\mathcal{Q}' \cap C = \emptyset$. $\square$

4 Good quadratic sets of types (L), (W), (SP), (SW), (LW) and (CW)

Classifying good quadratic sets of type (W) of $Q^+(5, q)$ is a trivial problem, as the following clearly holds.

Proposition 4.1. *Let S be a set of points of $Q^+(5, q)$ having the property that each plane of $Q^+(5, q)$ is contained in S. Then $S = Q^+(5, q)$.*

In the following proposition, we classify all good quadratic sets of types (L) and (LW).

Proposition 4.2. (1) *Let S be a set of points of $Q^+(5, q)$ having the property that each plane of $Q^+(5, q)$ intersects S in a line. Then S is obtained by intersecting $Q^+(5, q)$ with a nontangent hyperplane, i.e. S is a $Q(4, q)$-quadric.*

(2) *Let S be a set of points of $Q^+(5, q)$ having the property that each plane π of $Q^+(5, q)$ intersects S in either a line or the whole of π, and suppose that both possibilities occur. Then S is a set of points that arises by intersecting $Q^+(5, q)$ with a tangent hyperplane.*

Proof. Let S be a set of points as in (1) or (2). Then any line L of $Q^+(5, q)$ is contained in a plane π of $Q^+(5, q)$ which meets S in either a line or the whole of π, showing that $S \cap L$ is either a singleton or the whole of L. It follows that S is a hyperplane of the point-line geometry induced on $Q^+(5, q)$. By Cohen and Shult [6, Theorem 5.12], we know that such a hyperplane arises by intersecting $Q^+(5, q)$ with a hyperplane Π of $\mathrm{PG}(5, q)$. If Π is a tangent hyperplane, then situation (2) occurs. If Π is a nontangent hyperplane, then situation (1) occurs. $\square$

We thus see that all good quadratic sets of types (W), (L) and (LW) are obtained by intersecting $Q^+(5,q)$ with a suitable quadric $\mathcal{Q}$ of $\mathrm{PG}(5,q)$. For the unique good quadratic set of type (W), we can take $\mathcal{Q} = Q^+(5,q)$. For the good quadratic sets of types (L) and (LW), we know that this holds by Lemma 3.2 and Proposition 4.2.

In the following two propositions, we prove the nonexistence of good quadratic sets of types (SW), (CW) and (SP).

Proposition 4.3. (1) *There are no sets S of points of $Q^+(5,q)$ having the property that each plane π of $Q^+(5,q)$ intersects S in either a singleton or the whole of π, with both possibilities occurring.*

(2) *There are no sets S of points of $Q^+(5,q)$ having the property that each plane π of $Q^+(5,q)$ intersects S in either a conic or the whole of π, with both possibilities occurring.*

Proof. Let S be a set of points as in (1) or (2). The graph defined on the planes of $Q^+(5,q)$ by calling two planes adjacent whenever they meet in a line is connected. So, there exist two planes π_1 and π_2 intersecting in a line such that $\pi_1 \cap S = \pi_1$ and $\pi_2 \cap S$ is a conic or a singleton. This is clearly impossible. □

Proposition 4.4. *There are no sets S of points of $Q^+(5,q)$ having the property that each plane of $Q^+(5,q)$ intersects S in either a singleton or a pencil, with both possibilities occurring.*

Proof. We determine an upper bound for $|S|$. Let π_1 be a plane of $Q^+(5,q)$ meeting S in a singleton $\{x\}$. By Lemma 2.1(4), the planes of $Q^+(5,q)$ meeting π_1 in a line partition the set $S \setminus \pi_1$. There are now q^2 planes of $Q^+(5,q)$ meeting π_1 in a line not containing x and each of these planes must contain a unique point of $S \setminus \pi_1$. There are also $q+1$ planes of $Q^+(5,q)$ meeting π_1 in a line containing x and these planes contain at most $2q$ points of $S \setminus \pi_1$. Hence, $|S \setminus \pi_1| \leq q^2 + 2q(q+1) = 3q^2 + 2q$ and $|S| \leq 3q^2 + 2q + 1$.

We also determine a lower bound for $|S|$. Let π_2 be a plane of $Q^+(5,q)$ meeting S in the union of two lines L_1 and L_2 through a point u. The planes of $Q^+(5,q)$ meeting π_2 in a line partition the set $S \setminus \pi_2$. There are now q^2 planes of $Q^+(5,q)$ meeting π_2 in a line not containing u and each of these planes contains exactly $2q-1$ points of $S \setminus \pi_2$. There are also two planes of $Q^+(5,q)$ meeting π_2 in either L_1 or L_2, and each of these planes contains exactly q points of $S \setminus \pi_2$. It follows that $|S \setminus \pi_2| \geq q^2(2q-1) + 2q = 2q^3 - q^2 + 2q$ and hence $|S| \geq 2q^3 - q^2 + 4q + 1$.

So, we have $2q^3 - q^2 + 4q + 1 \leq |S| \leq 3q^2 + 2q + 1$. It follows that $2q^3 - 4q^2 + 2q = 2q^2(q-2) + 2q \leq 0$, a contradiction. □

5 Good quadratic sets of type (P)

5.1 Examples and basic properties

In the following proposition, we describe the standard examples of good quadratic sets of type (P).

Proposition 5.1. *The union of two $Q(4,q)$-quadrics intersecting in a $Q^-(3,q)$-quadric is a good quadratic set of type (P).*

Proof. Let Π_1 and Π_2 be two distinct hyperplanes of $\mathrm{PG}(5,q)$ such that the 3-dimensional subspace $\Pi_1 \cap \Pi_2$ intersects $Q^+(5,q)$ in an elliptic quadric $Q^-(3,q)$. Put $\mathcal{Q}_1 = \Pi_1 \cap Q^+(5,q)$, $\mathcal{Q}_2 = \Pi_2 \cap Q^+(5,q)$ and $S := \mathcal{Q}_1 \cup \mathcal{Q}_2$. Then $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two $Q(4,q)$-quadrics intersecting in the elliptic quadric $Q^-(3,q)$. We prove that S is a good quadratic set of type (P).

Let π be an arbitrary plane of $Q^+(5,q)$. Then π cannot be contained in Π_1 nor in Π_2 as neither of $\mathcal{Q}_1$, $\mathcal{Q}_2$ contains planes. So, $\pi \cap \Pi_i$ with $i \in \{1,2\}$ is a line L_i. Now, $\pi \cap S = \pi \cap (Q^+(5,q) \cap (\Pi_1 \cup \Pi_2)) = \pi \cap (\Pi_1 \cup \Pi_2) = L_1 \cup L_2$. Note also that L_1 and L_2 are mutually distinct as otherwise the line $L_1 = L_2$ must be contained in $Q^-(3,q)$ which cannot be true. $\square$

Note that if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two $Q(4,q)$-quadrics intersecting in a $Q^-(3,q)$-quadric and $S := \mathcal{Q}_1 \cup \mathcal{Q}_2$, then each point of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is contained in $2(q+1)$ S-lines and every point of $\mathcal{Q}_1 \Delta \mathcal{Q}_2$ is contained in exactly $q+1$ S-lines.

For $q = 2$, we can also construct the following family of examples.

Proposition 5.2. *Let O be a (necessarily classical) ovoid of $Q^+(5,2)$, let $x \in O$ and let $\{L_1, L_2, L_3\}$ be a set of three lines of $Q^+(5,2)$ through x forming an ovoid of $\mathcal{S}_x$. Put $\overline{S} := (L_1 \cup L_2 \cup L_3 \cup O) \setminus \{x\}$ and $S := Q^+(5,2) \setminus \overline{S}$. Then S is a good quadratic set of type (P).*

Proof. Let π be a plane of $Q^+(5,2)$ through x and let L_i with $i \in \{1,2,3\}$ be the unique line of $\{L_1, L_2, L_3\}$ contained in π. Then $\pi \cap S$ is the union of the two lines of π through x distinct from L_i.

Let π be a plane of $Q^+(5,2)$ not containing x, let π' be the unique plane through x meeting π in a line and let L_i with $i \in \{1,2,3\}$ be the unique line of $\{L_1, L_2, L_3\}$ contained in π' and let o be the unique point of O contained in π. Then $\pi \cap S$ consists of all points of π, except for the point o and the unique point in $L_i \cap \pi$. It follows that $\pi \cap S$ is the union of two distinct lines of π. $\square$

By Proposition 1.2 and Lemma 3.2, the quadratic sets described in Propositions 5.1 and 5.2 are obtained by intersecting $Q^+(5,q)$ with quadrics of $\mathrm{PG}(5,q)$.

We can divide the good quadratic sets of type (P) of $Q^+(5,2)$ constructed in Proposition 5.2 into two subfamilies. Suppose as in Proposition 5.2 that O is an ovoid of $Q^+(5,2)$, that $x \in O$ and that $\{L_1, L_2, L_3\}$ is an ovoid of $\mathcal{S}_x$. The ovoid O is classical and thus is obtained by intersecting $Q^+(5,2)$ with a 3-dimensional subspace α of $\mathrm{PG}(5,2)$. Let Π_1, Π_2 and Π_3 denote the three hyperplanes of $\mathrm{PG}(5,2)$ through α and put $\mathcal{Q}_i := \Pi_i \cap Q^+(5,2)$. Then $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are three $Q(4,2)$-quadrics which mutually intersect in O. In fact, O is an ovoid of each of $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ if we regard it as a generalized quadrangle. For every $i \in \{1,2,3\}$, denote by $\mathcal{L}_i$ the set of three lines of $Q^+(5,2)$ through x contained in Π_i (or equivalently, in $\mathcal{Q}_i$). Then $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ is a partition of the point set of $\mathcal{S}_x$ in three ovoids. There are thus two possibilities:

(a) $\{L_1, L_2, L_3\} = \mathcal{L}_i$ for some $i \in \{1, 2, 3\}$;

(b) $\{L_1, L_2, L_3\}$ has a unique line in common with each of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

We prove that case (a) corresponds to the standard examples of good quadratics sets of type (P), namely those that are discussed in Proposition 5.1, and that case (b) gives rise to other (nonstandard) examples.

As in Proposition 5.2, we put $\overline{S} := (L_1 \cup L_2 \cup L_3 \cup O) \setminus \{x\}$ and $S = Q^+(5, 2) \setminus \overline{S}$. Let O' denote the unique ovoid of $\mathcal{Q}_1 \cong Q(4, 2)$ through x distinct from O. Then O' is also an ovoid of $Q^+(5, 2)$ (as every plane of $Q^+(5, 2)$ intersects $\mathcal{Q}_1$ in a line and thus O' in a singleton) and O' is contained in S. Obviously, the point $x \in O'$ is contained in six S-lines, namely the lines of $Q^+(5, 2)$ through x distinct from L_1, L_2 and L_3.

Suppose that case (b) occurs. Then without loss of generality, we may suppose that $L_i \in \mathcal{L}_i$ for every $i \in \{1, 2, 3\}$. We claim that every point $y \in O' \setminus \{x\} \subseteq S$ is contained in precisely four S-lines. Note first that each of the three lines of $\mathcal{Q}_1$ through y contains a point of O and can therefore not be an S-line. Now, a line of $Q^+(5, 2)$ through y not contained in $\mathcal{Q}_1$ is an S-line if and only if it is disjoint from $L_2 \cup L_3$, i.e. if and only if it is distinct from the two lines of $Q^+(5, 2)$ through y meeting L_2 and L_3. So, we indeed see that y is contained in precisely four S-lines. The good quadratic sets of type (P) corresponding to case (b) can therefore not be included in the standard examples constructed in Proposition 5.1, as otherwise y would be contained in either three or six S-lines.

Suppose that case (a) occurs. Then without loss of generality, we assume that $\{L_1, L_2, L_3\} = \mathcal{L}_1$. The ovoid O' of $Q^+(5, 2)$ is obtained by intersecting $Q^+(5, 2)$ with a 3-dimensional subspace α' of $\mathrm{PG}(5, 2)$. Besides Π_1, there are two other hyperplanes of $\mathrm{PG}(5, 2)$ through α which we will denote by Π'_2 and Π'_3. Put $\mathcal{Q}'_i := \Pi'_i \cap Q^+(5, 2)$ for every $i \in \{2, 3\}$. Then $\mathcal{Q}'_2$ and $\mathcal{Q}'_3$ are two $Q(4, 2)$-quadrics meeting in O'. Now, as $\overline{S} = (L_1 \cup L_2 \cup L_3 \cup O) \setminus \{x\} = \mathcal{Q}_1 \setminus O'$, we have $S = \mathcal{Q}'_2 \cup \mathcal{Q}'_3$. The good quadratic sets of type (P) corresponding to case (a) are therefore included in the standard examples constructed in Proposition 5.1.

The following will be the main result of this section.

Proposition 5.3. *If $q \geq 3$, then every good quadratic set of type (P) of $Q^+(5, q)$ is as obtained in Proposition 5.1. If $q = 2$, then every good quadratic set of type (P) of $Q^+(5, q)$ is obtained as in Proposition 5.2.*

As the complements of the 2-ovoids of $Q^+(5, 2)$ are precisely the good quadratic sets of type (P) of $Q^+(5, 2)$, we thus have:

Corollary 5.4. *The 2-ovoids of $Q^+(5, 2)$ are precisely the complements of the sets described in Proposition 5.2.*

Corollary 5.4 in combination with the discussion preceding Proposition 5.3 allows us to see that there are two families of 2-ovoids of $Q^+(5, 2)$ (see also Section 9). m-ovoids of general polar spaces have been studied at various places in the literature, see e.g. [1, 18]. We did not find a reference in the literature for the complete classification of 2-ovoids of $Q^+(5,2)$.

The remainder of this section is devoted to the proof of Proposition 5.3. So, we assume here that S is a set of points of $Q^+(5,q)$ such that every plane π of $Q^+(5,q)$ intersects S in a pencil. If L_1 and L_2 are the two lines contained in $\pi \cap S$, then the unique point in $L_1 \cap L_2$ is called the *center* of π.

A point $x \in S$ is said to be of *type (1)* if there exists a plane of $Q^+(5,q)$ through x for which x is the center. A point $x \in S$ is said to be of *type (2)* if there exists a plane of $Q^+(5,q)$ through x for which x is not the center. In principle, a point of S can thus have both types (1) and (2).

Lemma 5.5. *We have* $|S| = (2q+1)(q^2+1)$.

Proof. There are $2(q+1)(q^2+1)$ planes contained in $Q^+(5,q)$. Each of these planes contains $2q+1$ points of S, and conversely each point of S is contained in $2(q+1)$ planes of $Q^+(5,q)$. It follows that $|S| = \frac{2(q+1)(q^2+1)\cdot(2q+1)}{2(q+1)} = (2q+1)(q^2+1)$. □

Lemma 5.6. *The number of S-lines is equal to* $2(q+1)(q^2+1)$.

Proof. There are $2(q+1)(q^2+1)$ planes contained in $Q^+(5,q)$ and each of these planes contains exactly two S-lines. As each S-line is contained in precisely two planes of $Q^+(5,q)$, we see that the total number of S-lines is equal to $\frac{2(q+1)(q^2+1)\cdot 2}{2} = 2(q+1)(q^2+1)$. □

Lemma 5.7. *Let x be a point of type* (1) *of S. Then x is contained in either* $2q$, $2q+1$ *or* $2q+2$ *S-lines. Moreover, if x does not have type* (2)*, then x is contained in exactly* $2q+2$ *S-lines.*

Proof. Let π be a plane of $Q^+(5,q)$ for which x is the center and denote by L_1 and L_2 the two S-lines contained in π.

Each S-line through x that does not lie in π is contained in a unique plane of $Q^+(5,q)$ that meets π in a line. There are now $q-1$ planes of $Q^+(5,q)$ through x meeting π in a line distinct from L_1 and L_2. Each of these planes has x as center and contains two S-lines that do not lie in π. There are also two planes of $Q^+(5,q)$ meeting π in a line L_i for some $i \in \{1,2\}$. The center of such a plane lies in L_i. If x is the center (as it is the case if x does not have type (2), then there is a unique S-line in the plane through x not contained in π. If the center is distinct from x, then there is no such S-line. Taking into account that there are also two S-lines in π through x (namely L_1 and L_2), we thus see that the total number of S-lines through x is either N, $N+1$ or $N+2$, with $N := 2(q-1)+2 = 2q$. If x is not a point of type (2), then the number of S-lines through x is equal to $N+2 = 2q+2$. □

Lemma 5.8. *Let x be a point of type* (2) *of S. Then x is contained in either* $q+1$ *or* $q+2$ *S-lines. Moreover, if x does not have type* (1)*, then x is contained in exactly* $q+1$ *S-lines.*

Proof. Let π be a plane of $Q^+(5,q)$ through x for which x is not the center and denote by L_1 and L_2 the two S-lines contained in π such that $x \in L_1$.

Each S-line through x that does not lie in π is contained in a unique plane of $Q^+(5,q)$ that meets π in a line. There are now q planes π' of $Q^+(5,q)$ through x meeting π in a line distinct from L_1. As the line $\pi \cap \pi'$ meets S in two points, the center of π' lies in $\pi' \setminus \pi$ and so there is a unique S-line through x contained in π' (but not in π). There is also a unique plane meeting π in the line L_1, and the center of this plane lies on L_1. If the center is distinct from x (as it is the case if x does not have type (1)), then the plane does not contain an S-line through x distinct from L_1. If the center equals x, then there is a unique such S-line. Taking into account that there is also a unique S-line in π through x (namely L_1), we thus see that the total number of S-lines through x is either N or $N+1$, with $N := q+1$. If x is not a point of type (1), then the number of S-lines through x is equal to $N = q+1$. □

The following is a consequence of Lemmas 5.7, 5.8 and the fact that $q+2 < 2q$ for every $q \geq 3$.

Corollary 5.9. *If $q \geq 3$, then no point of $\mathcal{S}$ has both types* (1) *and* (2)*. Moreover, every point of type* (1) *is contained in exactly* $2q+2$ *S-lines and every point of type* (2) *is contained in exactly* $q+1$ *S-lines.*

We can now distinguish two cases:

- **Case 1:** No point of S has both types (1) and (2).
- **Case 2:** There is at least one point in S that has types (1) and (2).

By Corollary 5.9, we know that q must be equal to 2 if case 2 occurs.

We show in this section that case (1) corresponds to the standard examples described in Proposition 5.1 and that case (2) corresponds to the extra examples described in Proposition 5.2, see Corollary 5.17 and Propositions 5.20, 5.23.

5.2 Treatment of Case 1

We classify here sets S of points of $Q^+(5,q)$ that satisfy the following two properties:

(I) every plane of $Q^+(5,q)$ intersects S in a pencil;

(II) no point of S has types (1) and (2).

By Lemmas 5.7 and 5.8, we know that the following must hold.

Corollary 5.10. *Every S-point of type* (1) *is contained in* $2(q+1)$ *S-lines and every S-point of type* (2) *is contained in* $q+1$ *S-lines.*

Lemma 5.11. *Any line L of $Q^+(5,q)$ containing a point $x \in S$ of type* (2) *either is contained in S or contains exactly two points of S which both have type* (2)*.*

Proof. Let π be a plane of $Q^+(5,q)$ through L. Then $\pi \cap S = L_1 \cup L_2$ for two distinct lines L_1 and L_2 in π. Without loss of generality, we may suppose that $x \in L_1 \setminus L_2$. The line L_1 through x is entirely contained in S. Every other line of π through x contains precisely two points of S and those two points have both type (2). □

Lemma 5.12. *Any line L of $Q^+(5,q)$ containing at least three points of S is completely contained in S.*

Proof. Let π be a plane of $Q^+(5,q)$ through L. As $\pi \cap S$ is a pencil of two lines and L has at least three points in common with $\pi \cap S$, the line L must be one of the two lines of this pencil. □

Lemma 5.13. *Any line L of $Q^+(5,q)$ with $|L \cap S| \geq 2$ containing an S-point of type* (1) *is completely contained in S.*

Proof. Let π be a plane of $Q^+(5,q)$ through L with center c. Recall that $\pi \cap S$ is a pencil of two lines. As $L \subseteq \pi$ has an S-point of type (1), it is a line of π through c. As L has at least two points in common with $\pi \cap S$, it is one of the two lines of the pencil. □

Lemma 5.14. *Every S-line L contains one point of type (1) and q points of type* (2)*.*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$ containing L and let c be the center of π. Then $c \in L$ has type (1) and every point of $L \setminus \{c\}$ has type (2). □

Lemma 5.15. *Let L_1 and L_2 be two distinct S-lines through a given point x of type* (2)*. Then there is no plane of $Q^+(5,q)$ containing L_1 and L_2.*

Proof. If there were a plane π of $Q^+(5,q)$ containing L_1 and L_2, then $\pi \cap S = L_1 \cup L_2$ and x would be the center of π, which is not the case. □

Proposition 5.16. *The set O of points of type* (1) *is an ovoid of $Q^+(5,q)$ and thus contains q^2+1 points of S. If this ovoid is classical, then $S = \mathcal{Q}_1 \cup \mathcal{Q}_2$ for two distinct $Q(4,q)$-quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $Q^+(5,q)$ through O.*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$. Then $\pi \cap S = L_1 \cup L_2$ for two distinct lines L_1 and L_2 of $Q^+(5,q)$. The unique point c in $L_1 \cap L_2$ is then the unique point of type (1) in π, and all points of $(L_1 \cup L_2) \setminus \{c\}$ have type (2). This proves that O is an ovoid of $Q^+(5,q)$.

Suppose now that O is classical. Then $\langle O \rangle$ is 3-dimensional and $O = \langle O \rangle \cap Q^+(5,q)$. Let K_1 be an S-line through a point $x \in O$. The 4-dimensional subspace $\langle O, K_1 \rangle$ intersects $Q^+(5,q)$ then in a $Q(4,q)$-quadric $\mathcal{Q}_1$. We note the following.

(1) If $y \in \mathcal{Q}_1 \setminus O$, then there are $q+1$ lines of $Q^+(5,q)$ through y meeting O, namely the $q+1$ lines of $\mathcal{Q}_1$ through y. All these lines are thus contained in $\mathcal{Q}_1$.

(2) If $y \in S \setminus O$, then by Corollary 5.10 and Lemmas 5.13, 5.14 there are $q+1$ lines of $Q^+(5,q)$ through y meeting O. All these lines are contained in S.

By [2, Theorem 7.3] or [19, Lemma 6.1], we know that the complement of O in $\mathcal{Q}_1$ is connected and since $K_1 \setminus O \subseteq S \cap \mathcal{Q}_1$, we see from (1) and (2) that the whole of $\mathcal{Q}_1$ is contained in S.

Now, there are $2(q+1)$ lines of $Q^+(5,q)$ through x contained in S and $q+1$ lines of $Q^+(5,q)$ through x contained in $\mathcal{Q}_1$. Repeating the above argument for a line K_2 through x not contained in $\mathcal{Q}_1$, we find that there exists another $Q(4,q)$-quadric $\mathcal{Q}_2$ that is contained in S, namely $Q^+(5,q) \cap \langle O, K_2 \rangle$.

We now have $\mathcal{Q}_1 \cup \mathcal{Q}_2 \subseteq S$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2 = O$. As $|\mathcal{Q}_1 \cup \mathcal{Q}_2| = (2q+1)(q^2+1) = |S|$, we have $S = \mathcal{Q}_1 \cup \mathcal{Q}_2$. □

Corollary 5.17. *If $q = 2$, then S is the union of two $Q(4,q)$-quadrics that meet in a $Q^-(3,q)$-quadric.*

Proof. This follows from Proposition 5.16 and the fact that every ovoid of $Q^+(5,2)$ is a classical. □

In view of Corollary 5.17, we may from now on suppose that $q \geq 3$.

Lemma 5.18. *Let x_1, x_2 and x_3 be three distinct points of O and let L be an S-line through x_1 such that the unique points x_2' and x_3' of L collinear on $Q^+(5,q)$ with respectively x_2 and x_3 are distinct. Then there is a unique $Q^+(3,q)$-quadric G containing $\{x_1,x_2,x_3\} \cup L$. Moreover, all points of G lie in S.*

Proof. The three points x_1, x_2 and x_3 cannot lie on the same line of $\mathrm{PG}(5,q)$ as otherwise the unique line of $\mathrm{PG}(5,q)$ containing them contains three points of $Q^+(5,q)$ and so would be contained in $Q^+(5,q)$. So, $\langle x_1,x_2,x_3 \rangle$ is a plane necessarily intersecting $Q^+(5,q)$ in a conic. It follows that $\langle x_1,x_2,x_3,L \rangle$ is a 3-dimensional subspace of $\mathrm{PG}(5,q)$, whose intersection with $Q^+(5,q)$ contains the lines L, x_2x_2' and x_3x_3'. By looking at the possible intersections of $Q^+(5,q)$ with 3-dimensional subspaces (see Section 2), we see that $\langle x_1,x_2,x_3,L \rangle$ intersects $Q^+(5,q)$ in a $Q^+(3,q)$-quadric G. Obviously, G is the unique $Q^+(3,q)$-quadric containing $\{x_1,x_2,x_3\} \cup L$. Recall that the points and lines contained in G define a $(q+1) \times (q+1)$-grid.

Put $K_2 = x_2x_2'$ and $K_3 = x_3x_3'$. By Lemma 5.13, K_2 and K_3 are S-lines. Let L' denote the unique line of G through x_2 meeting K_3, and let K_1 denote the unique line of G through x_1 meeting L'. By applying Lemma 5.13 two consecutive times, once for L' and another time for K_1, we see that all points of L' and all points of K_1 are contained in S. Now, the lines of G meeting the three mutually disjoint S-lines K_1, K_2 and K_3 have all their points in S by Lemma 5.12. These lines cover all points of G. □

Lemma 5.19. *Let L_1 and L_2 be two S-lines intersecting in a point x of type (2). Then there are precisely q $Q^+(3,q)$-quadrics through $L_1 \cup L_2$ that have all their points in S.*

Proof. Through x, there are $q+1$ S-lines and each of these S-lines contains a unique point of type (1) by Lemma 5.14. We denote by A the set of $q+1$ S-points of type (1) that arise in this way. The set consisting of the remaining $(q^2+1)-(q+1) = q^2-q$

S-points of type (1) will be denoted by B. We denote by x_i, $i \in \{1,2\}$, the unique point of type (1) contained in L_i.

By Lemma 5.14, each $Q^+(3,q)$-quadric through $L_1 \cup L_2$ that is entirely contained in S has $q+1$ S-points of type (1) which consist of x_1, x_2 and $q-1$ points of the set B. Conversely, by Lemma 5.18 we know that for each of the q^2-q points y of B there is a unique $Q^+(3,q)$-quadric through $\{x_1,x_2,y\}$ containing L_1 (and thus also the unique line L_2 through x_2 meeting L_1) and entirely consisting of points of S. We can therefore conclude that the number of $Q^+(3,q)$-quadrics through $L_1 \cup L_2$ entirely consisting of points of S is equal to $\frac{|B|}{q-1} = \frac{q^2-q}{q-1} = q$. □

Proposition 5.20. *If $q \geq 3$, then we also have that $S = \mathcal{Q}_1 \cup \mathcal{Q}_2$ for two distinct $Q(4,q)$-quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ intersecting in an elliptic quadric.*

Proof. Let x be an S-point of type (2) and let L_1, L_2 be two distinct S-lines through x. By Lemma 5.19, we know that there exist two $Q^+(3,q)$-quadrics G_1 and G_2 through $L_1 \cup L_2$ that have all their points in S.

We show that the subspace $\langle G_1, G_2 \rangle$ intersects $Q^+(5,q)$ in a $Q(4,q)$-quadric. Suppose that this is not the case. Then $\langle G_1, G_2 \rangle$ is a tangent hyperplane Π_y for some point $y \in Q^+(5,q) \setminus (G_1 \cup G_2)$. By considering a collection of $q+1$ planes of $Q^+(5,q)$ through y intersecting G_1 in a collection of $q+1$ mutually disjoint lines of G_1, we see that $y \notin S$ and $|\Pi_y \cap S| = (q+1)(2q+1)$. Indeed, q of these $q+1$ planes intersect $G_1 \cup G_2 \subseteq S$ in a pencil not containing y. As $|G_1 \cup G_2| = 2q^2+2q+1$, we see that $|(\Pi_y \cap S) \setminus (G_1 \cup G_2)| = q$. In fact, q points of $(\Pi_y \cap S) \setminus (G_1 \cup G_2)$ must be contained in the plane $\langle y, L_1 \rangle$ and q points of $(\Pi_y \cap S) \setminus (G_1 \cup G_2)$ must be contained in $\langle y, L_2 \rangle$. This implies that $(\Pi_y \cap S) \setminus (G_1 \cup G_2) = yx \setminus \{x\}$. But that is impossible as $y \notin S$.

So, the subspace $\langle G_1, G_2 \rangle$ intersects $Q^+(5,q)$ in a $Q(4,q)$-quadric $\mathcal{Q}_1$. We show that $\mathcal{Q}_1$ is completely contained in S.

Suppose that the only lines of $\mathcal{Q}_1$ through x that are contained in S are the lines L_1 and L_2. Since $q \geq 3$, there are at least two additional lines of $\mathcal{Q}_1$ through x, two of which we will denote by L_3 and L_4. By Lemma 5.11, the set $L_i \setminus \{x\}$ with $i \in \{3,4\}$ contains a unique S-point x_i and this S-point has type (2).

For every $i \in \{1,2\}$, the set of points of type (1) contained in G_i is an ovoid O_i of G_i by Lemma 5.14. This ovoid O_i contains points $x_1 \in L_1$ and $x_2 \in L_2$. In G_i, the points x_1 and x_2 have two neighbours, namely x and a certain point u_i. Obviously, also $O_i' := (O_i \setminus \{x_1,x_2\}) \cup \{x,u_i\}$ is an ovoid of G_i. Note that also $O_i'' := x_3^\perp \cap G_i$ and $O_i''' := x_4^\perp \cap G_i$ are ovoids of G_i. We show that every point v of $O_i \setminus \{x_1,x_2\} = O_i' \setminus \{x,u_i\}$ is collinear with x_3 and x_4. It suffices to show that $v \sim x_3$ as the reasoning showing that $v \sim x_4$ is then completely similar. Let $x_3' \neq x$ denote the unique point of L_3 collinear with v. The line vx_3' contains an S-point of type (1), namely v, and an additional point of S, namely the unique point in $vx_3' \cap G_{3-i}$ and so must be contained in S by Lemma 5.13. But then $x_3' \in S \cap L_3$ implies that $x_3' = x_3$.

We thus see that all points of $O_1' \setminus \{x,u_1\}$ and $O_2' \setminus \{x,u_2\}$ are collinear with x_3 and x_4. As x is also collinear with x_3 and x_4, we see that $O_1'' = O_1''' = O_1'$ and $O_2'' = O_2''' = O_2'$, i.e. also u_1 and u_2 are collinear with x_3 and x_4. So, we have located $2q+1$ common neighbours of x_3 and x_4 in $\mathcal{Q}_1$, namely the points of $O_1' \cup O_2'$, but that is impossible as there are only $q+1$ such neighbours.

We thus see that there exists an S-line L_3 of $\mathcal{Q}_1$ through x that is distinct from L_1 and L_2. We now prove that every point y of $\mathcal{Q}_1$ noncollinear with x belongs to S. Let L be the unique line of $\mathcal{Q}_1$ through y meeting L_3. This line contains three points of S, namely the unique points in $L \cap G_1$, $L \cap G_2$ and $L \cap L_3$ and so all points of L belong to S by Lemma 5.12. In particular, the point y belongs to S. We now also show that every point z of $\mathcal{Q}_1 \setminus \{x\}$ collinear with x belongs to S. Let K be a line of $\mathcal{Q}_1$ through z distinct from xz. Then all $q \geq 3$ points of $K \setminus \{z\}$ belong to S and so K itself is also contained in S by Lemma 5.12. In particular, we have $z \in S$.

We thus see that the $Q(4,q)$-quad $\mathcal{Q}_1$ is contained in S. This already amounts for $(q+1)(q^2+1)$ points of the $(2q+1)(q^2+1)$ points of S. The points of $O \cap \mathcal{Q}_1$ form an ovoid of $\mathcal{Q}_1$ having exactly q^2+1 points, showing that $O \cap \mathcal{Q}_1 = O$, i.e. $O \subseteq \mathcal{Q}_1$. Now, let $x' \in S \setminus \mathcal{Q}_1$. Then x' necessarily has type (2). Taking two S-lines L_1' and L_2' through x' and repeating the above argument, we see that there exists a $Q(4,q)$-quad $\mathcal{Q}_2$ through x' and O completely contained in S. As $\mathcal{Q}_1 \neq \mathcal{Q}_2$ and $O \subseteq \mathcal{Q}_1 \cap \mathcal{Q}_2$, we have $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is a $Q^-(3,q)$-quadric, $O = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $|\mathcal{Q}_1 \cap \mathcal{Q}_2| = |\mathcal{Q}_1| + |\mathcal{Q}_2| - |O| = (2q+1)(q^2+1) = |S|$. So, we must have that $S = \mathcal{Q}_1 \cup \mathcal{Q}_2$. □

5.3 Treatment of Case 2

We classify here sets S of points of $Q^+(5,q)$ that satisfy the following two properties:

(I) every plane of $Q^+(5,q)$ intersects S in a pencil of two lines;

(II) there are points in S that have both types (1) and (2).

By Corollary 5.9, we then know that $q = 2$. The number of S-lines through a point $x \in S$ will be denoted by $I(x)$ and called the *index of* x. By Lemmas 5.7 and 5.8, we know the following.

Lemma 5.21. *The index of a point that has both types* (1) *and* (2) *is equal to* 4.

Lemma 5.22. *One of the following two cases occurs for a plane π of $Q^+(5,2)$ intersecting S in the union $L_1 \cup L_2$ of two distinct lines L_1 and L_2.*

(*a*) *The center c of π has index* 6 *and the four points of $(L_1 \cup L_2) \setminus \{c\}$ have index* 3.

(*b*) *The center c of π has index* 4 *and for each $i \in \{1,2\}$, one point of $L_i \setminus \{c\}$ has index* 4 *while the other has index* 3.

Moreover, there exists a plane π of $Q^+(5,2)$ for which case (b) occurs.

Proof. For every $i \in \{1,2\}$, let π_i be the unique plane of $Q^+(5,2)$ through L_i distinct from π and denote by c_i the center of π_i. The indices of the points of $L_1 \cup L_2$ depend on the precise position of the points c_1 and c_2 and can be computed with the information provided in the proofs of Lemmas 5.7 and 5.8. If $c_1 = c_2 = c$, then case (a) occurs. If $c_1 \neq c \neq c_2$, then case (b) occurs, with the points $c_1 \in L_1 \setminus \{c\}$ and $c_2 \in L_2 \setminus \{c\}$ having index 4. If precisely one of c_1, c_2 equals c, then the following third possibility occurs.

(c) The center c of π has index 5, there exists a unique $i \in \{1,2\}$ such that the two points of $L_i \setminus \{c\}$ have index 3, one point of $L_{3-i} \setminus \{x\}$ has index 3 and the other point of $L_{3-i} \setminus \{c\}$ has index 4.

We prove however that possibility (c) cannot occur. Indeed, in this case the center c_{3-i} of the plane π_{3-i} would have index 4 and case (b) would occur for this plane π_{3-i}. This means that all points of $\pi_{3-i} \cap S$ have index 3 or 4, but that is not possible as the point $c \in \pi_{3-i} \cap S$ has index 5.

If case (b) never occurs for a plane π of $Q^+(5,2)$, then all points of S have index 3 or 6, in contradiction with Lemma 5.21. □

Proposition 5.23. (1) *There is a unique point x with index 6, all 12 points of $x^\perp \cap S$ distinct from x have index 3 and all 12 points of $S \setminus x^\perp$ have index 4.*

(2) *There exists an ovoid O of $Q^+(5,2)$ containing x and an ovoid $\{L_1, L_2, L_3\}$ of $\mathcal{S}_x$ such that $S = Q^+(5,2) \setminus \overline{S}$, where $\overline{S} = (L_1 \cup L_2 \cup L_3 \cup O) \setminus \{x\}$.*

(3) *There exists no $Q(4,2)$-quadric containing L_1, L_2, L_3 and O.*

Proof. Let π be a plane of $Q^+(5,2)$ for which case (b) of Lemma 5.22 occurs. Put $\pi \cap S = L_1 \cup L_2$ with $L_1 = \{c, x_1, y_1\}$ and $L_2 = \{c, x_2, y_2\}$ where $I(c) = I(x_1) = I(x_2) = 4$ and $I(y_1) = I(y_2) = 3$.

Let π_1 be the unique plane of $Q^+(5,2)$ through y_1y_2 distinct from π. All other six planes meeting π in a line contain a point with index 4 and so correspond to case (b) of Lemma 5.22, implying that each such plane contains three S-points with index 4 and two S-points with index 3. There are now two possibilities.

(1) π_1 is a plane of type (a) containing a unique S-point x with index 6 and four S-points with index 3. As each point of $S \cap \pi$ is contained in three planes of $Q^+(5,2)$ meeting π in a line and each point of $S \setminus \pi$ is contained in a unique plane of $Q^+(5,2)$ meeting π in a line, we see that there exists a unique S-point with index 6 (namely x), $6 \cdot 3 + 1 \cdot 0 - 2 \cdot 3 = 12$ S-points with index 4 and $6 \cdot 2 + 1 \cdot 4 - 2 \cdot 2 = 12$ S-points with index 3. By Lemma 5.22, the 12 S-points of index 3 are necessarily the S-points of $x^\perp \cap S$ distinct from x. There are three lines L_1, L_2 and L_3 through x not contained in S and no plane through x can contain two of these lines, i.e. $\{L_1, L_2, L_3\}$ is an ovoid of $\mathcal{S}_x$. If π' is a plane not containing x, then π' contains a unique point of $O' := Q^+(5,2) \setminus (S \cup x^\perp)$. This also shows that $O := \{x\} \cup O'$ is an ovoid of $Q^+(5,2)$. We can now see that S is as described in the proposition. Note that there cannot exist a $Q(4,2)$-quadric containing L_1, L_2, L_3 and O, as otherwise the good quadratic set would belong to the standard ones constructed in Proposition 5.1 and these only have S-points with indices 3 and 6.

(2) π_1 is a plane of type (b) containing three S-points with index 4 and two S-points with index 3. As each point of $S \cap \pi$ is contained in three planes of $Q^+(5,2)$ meeting π in a line and each point of $S \setminus \pi$ is contained in a unique plane of $Q^+(5,2)$ meeting π in a line, we see that there exist $7 \cdot 3 - 2 \cdot 3 = 15$ S-points with index 4 and $7 \cdot 2 - 2 \cdot 2 = 10$ S-points with index 3. We prove that there are no examples corresponding to this situation. To

that end, we consider the model of $Q^+(5,2)$ where the points are the subsets of size 3 of $\{1,2,3,4,5,6,7\}$ and where two distinct points are adjacent whenever they meet in a singleton (see Section 9). For every $\sigma \in S_7$, we define

$$P[\sigma] := \{\{1^\sigma,2^\sigma,3^\sigma\},\{1^\sigma,4^\sigma,5^\sigma\},\{1^\sigma,6^\sigma,7^\sigma\},\{2^\sigma,4^\sigma,6^\sigma\},\{2^\sigma,5^\sigma,7^\sigma\},\{3^\sigma,4^\sigma,7^\sigma\},\{3^\sigma,5^\sigma,6^\sigma\}\}.$$

Then $\{P[\sigma]\,|\,\sigma \in S_7\}$ is the set of all 30 planes of $Q^+(5,2)$. The lines of $Q^+(5,2)$ have the form $\{\{x,y_1,z_1\},\{x,y_2,z_2\},\{x,y_3,z_3\}\}$, where $\{x,y_1,z_1,x_2,y_2,x_3,y_3\} = \{1,2,3,4,5,6,7\}$.

Consider now an S-point x with index 4 which is the center of a plane α_1. In α_1, there exists a unique line L through x such that none of the two points of $L \setminus \{x\}$ belongs to S. If α_2 is the other plane of $Q^+(5,2)$ through L, then $(\alpha_1 \cup \alpha_2) \setminus L$ is contained in S and the four S-lines through x are the four lines through x contained in $\alpha_1 \cup \alpha_2$ and distinct from L. If K is a line of $Q^+(5,2)$ through x not contained in $\alpha_1 \cup \alpha_2$, then the fact that the unique plane through K meeting π_1 in a line (necessarily contained in S) has type (P) implies that K contains besides x one other point of S.

Without loss of generality, we may suppose that $x = \{1,2,3\}$ and $L = \{\{1,2,3\},\{1,4,5\},\{1,6,7\}\}$. So, we have

(1) $\{1,2,3\} \in S$ and $\{1,4,5\},\{1,6,7\} \in \overline{S} := Q^+(5,2) \setminus S$.

Expressing that $(\alpha_1 \cup \alpha_2) \setminus L \subseteq S$ leads to

(2) $\{2,4,6\},\{2,4,7\},\{2,5,6\},\{2,5,7\},\{3,4,6\},\{3,4,7\},\{3,5,6\},\{3,5,7\} \in S$.

The lines of $Q^+(5,2)$ through x not contained in α_1 nor in α_2 are the lines $\{x,\{1,5,6\},\{1,4,7\}\}$, $\{x,\{1,4,6\},\{1,5,7\}\}$, $\{x,\{2,4,5\},\{2,6,7\}\}$ and $\{x,\{3,4,5\},\{3,6,7\}\}$. Each of these four lines contains a unique point of $\overline{S}$. This leads to 16 possibilities. Specifically, if we also take (1) into account we see that $\overline{S}$ contains either $A_1 := \{\{1,4,5\},\{1,4,6\},\{1,5,6\},\{1,6,7\},\{2,4,5\},\{3,4,5\}\}$, $A_2 := \{\{1,4,5\},\{1,4,6\},\{1,5,6\},\{1,6,7\},\{2,4,5\},\{3,6,7\}\}$, $A_3 := \{\{1,4,5\},\{1,4,6\},\{1,5,6\},\{1,6,7\},\{2,6,7\},\{3,6,7\}\}$, $A_1^{(47)(56)}$, $A_1^{(4756)}$, $A_1^{(67)}$, $A_2^{(23)}$, $A_2^{(23)(47)(56)}$, $A_2^{(47)(56)}$, $A_2^{(23)(4756)}$, $A_2^{(4756)}$, $A_2^{(67)}$, $A_2^{(23)(67)}$, $A_3^{(47)(56)}$, $A_3^{(4756)}$ and $A_3^{(67)}$. Without loss of generality, we thus suppose that either $A_1 \subseteq \overline{S}$, $A_2 \subseteq \overline{S}$ or $A_3 \subseteq \overline{S}$. We now give a unified treatment of these three cases.

If $A_1 \subseteq \overline{S}$, then we define

$$A = A_1, \qquad u = \{2,3,7\}, \qquad v = \{4,5,7\},$$

$$\pi_1 = P[(1236)], \quad \pi_2 = P[(1367)], \quad \pi_3 = P[(136)], \quad \pi_4 = P[(126)], \quad \pi_5 = P[(17)],$$

$$E = \{\{1,2,4\},\{1,2,5\},\{1,2,7\},\{1,3,4\},\{1,3,5\},\{1,3,7\},\{2,3,6\},\{4,6,7\},\{5,6,7\}\},$$

$$\mathcal{L}_u = \{\{u,\{1,3,4\},\{3,5,6\}\},\{u,\{1,5,7\},\{4,6,7\}\},\{u,\{1,4,7\},\{5,6,7\}\},$$
$$\{u,\{1,2,4\},\{2,5,6\}\},\{u,\{1,2,5\},\{2,4,6\}\},\{u,\{1,3,5\},\{3,4,6\}\}\},$$

$$\mathcal{L}_v = \{\{v,\{1,3,7\},\{2,6,7\}\},\{v,\{1,2,7\},\{3,6,7\}\},\{v,\{1,2,4\},\{3,4,6\}\},$$
$$\{v,\{1,3,4\},\{2,4,6\}\},\{v,\{1,2,5\},\{3,5,6\}\},\{v,\{1,3,5\},\{2,5,6\}\}\}.$$

If $A_2 \subseteq \overline{S}$, then we define

$$A = A_2, \qquad u = \{1,2,6\}, \qquad v = \{1,3,7\},$$

$$\pi_1 = P[(1236)], \quad \pi_2 = P[(1246)], \quad \pi_3 = P[(126)], \quad \pi_4 = P[(135)], \quad \pi_5 = P[(14)],$$

$$E = \{\{1,2,7\},\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{4,5,7\},\{4,6,7\},\{5,6,7\}\},$$

$$\mathcal{L}_u = \{\{u,\{3,4,6\},\{5,6,7\}\},\{u,\{3,5,6\},\{4,6,7\}\},\{u,\{1,3,4\},\{1,5,7\}\},$$
$$\{u,\{2,3,4\},\{2,5,7\}\},\{u,\{1,3,5\},\{1,4,7\}\},\{u,\{2,3,5\},\{2,4,7\}\}\},$$

$$\mathcal{L}_v = \{\{v,\{2,3,4\},\{3,5,6\}\},\{v,\{2,4,7\},\{5,6,7\}\},\{v,\{2,6,7\},\{4,5,7\}\},$$
$$\{v,\{2,5,7\},\{4,6,7\}\},\{v,\{2,3,5\},\{3,4,6\}\},\{v,\{2,3,6\},\{3,4,5\}\}\}.$$

If $A_3 \subseteq \overline{S}$, then we define

$$A = A_3, \qquad u = \{2,3,6\}, \qquad v = \{4,5,6\},$$

$$\pi_1 = P[(124)], \quad \pi_2 = P[(135)], \quad \pi_3 = P[(1235)], \quad \pi_4 = P[(1246)], \quad \pi_5 = P[(167)],$$

$$E = \{\{1,2,4\},\{1,2,5\},\{1,2,7\},\{1,3,4\},\{1,3,5\},\{1,3,7\},\{2,3,4\},\{2,3,5\},\{4,5,7\}\},$$

$$\mathcal{L}_u = \{\{u,\{1,2,7\},\{2,4,5\}\},\{u,\{1,2,4\},\{2,5,7\}\},\{u,\{1,3,7\},\{3,4,5\}\},$$
$$\{u,\{1,3,4\},\{3,5,7\}\},\{u,\{1,2,5\},\{2,4,7\}\},\{u,\{1,3,5\},\{3,4,7\}\}\},$$

$$\mathcal{L}_v = \{\{v,\{1,4,7\},\{2,3,4\}\},\{v,\{1,3,4\},\{2,4,7\}\},\{v,\{1,5,7\},\{2,3,5\}\},$$
$$\{v,\{1,3,5\},\{2,5,7\}\},\{v,\{1,2,4\},\{3,4,7\}\},\{v,\{1,2,5\},\{3,5,7\}\}\}.$$

We thus have

(3) $A \subseteq \overline{S}$.

Now, if π is a plane intersecting $A \subseteq \overline{S}$ in two points, then $\pi \setminus A$ necessarily is contained in S. Applying this observation to the planes π_1, π_2, π_3 and π_4, we then see that

(4) $E \subseteq S$.

Consider now the point u. If $u \in S$, then by (1), (2) and (4) the six elements of $\mathcal{L}_u$ are six lines of $Q^+(5,2)$ through u contained in S. That is impossible as there no S-points with index 6. So,

(5) $u \in \overline{S}$.

Now, the plane π_5 intersects $A \cup \{u\} \subseteq \overline{S}$ in two points. This implies that $\pi_5 \setminus (A \cup \{u\}) \subseteq S$. In particular, we have that

(6) $v \in S$.

Now, by (1), (2), (4) and (6) the six elements of $\mathcal{L}_v$ are six lines of $Q^+(5,2)$ through v contained in S. As there are no S-points with index 6, we have reached our final contradiction. $\square$

6 Good quadratic sets of type (SL)

In the following two propositions, we describe the two standard examples of good quadratic sets of type (SL).

Proposition 6.1. *Let x be a point of $Q^+(5,q)$ and $\mathcal{L}$ a set of lines of $Q^+(5,q)$ through x forming an ovoid of $\mathcal{S}_x$. Then $S := \bigcup_{L\in\mathcal{L}} L$ is a good quadratic set of type (SL) containing q^2+q+1 points.*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$.

Suppose $x \in \pi$. The fact that $\mathcal{L}$ is an ovoid of $\mathcal{S}_x$ implies that there is a unique line $L \in \mathcal{L}$ in π. Then $S \cap \pi$ is the line L.

Suppose $x \notin \pi$. Let π' denote the unique plane through x meeting π in a line. Let L' be the unique line of $\mathcal{L}$ contained in π'. Then $L' \cap \pi$ is a singleton $\{x'\}$ and hence $\pi \cap S = \{x'\}$.

Obviously, we have $|S| = 1 + q \cdot |\mathcal{L}| = q^2 + q + 1$. □

Proposition 6.2. *Every $Q^+(3,q)$-quadric $\mathcal{Q}$ is a good quadratic set of type (SL).*

Proof. $\mathcal{Q}$ is obtained by intersecting $Q^+(5,q)$ with a 3-dimensional subspace α. Let π be an arbitrary plane of $Q^+(5,q)$. As $\mathcal{Q}$ does not contain planes and since π and α meet, there are two possibilities: π meets $\mathcal{Q}$ in a point or in a line. We prove that both possibilities occur.

Let x be a point of $\mathcal{Q}$ and L_1, L_2 the two lines of $\mathcal{Q}$ through x. There are four planes of $Q^+(5,q)$ containing (precisely) one of the lines L_1, L_2 and each of these planes intersects $\mathcal{Q}$ in a line. The remaining $2(q+1)-4 = 2(q-1)$ planes of $Q^+(5,q)$ through x intersect $\mathcal{Q}$ in the singleton $\{x\}$. □

By Lemma 3.2, the quadratic set of type (SL) constructed in Proposition 6.2 can be obtained by intersecting $Q^+(5,q)$ with a quadric of $\mathrm{PG}(5,q)$. Assuming that $Q^+(5,q)$ has equation $X_1X_2 + X_3X_4 + X_5X_6 = 0$, a situation as in Proposition 6.2 occurs if we intersect $Q^+(5,q)$ with the subspace with equation $X_1 = X_2 = 0$, or with the quadric with equation $a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 = 0$ where $a_{11}, a_{22} \in \mathbb{F}_q^*$ and $a_{12} \in \mathbb{F}_q$ such that the polynomial $x_{11}X^2 + a_{12}X + a_{22}$ is irreducible in $\mathbb{F}_q[X]$.

The good quadratic sets constructed in Proposition 6.2 contain $(q+1)^2$ points and so cannot be isomorphic to the ones constructed in Proposition 6.1.

We now prove that every good quadratic set of type (SL) of $Q^+(5,q)$ can be obtained as in Proposition 6.1 or as in Proposition 6.2. So, assume now that S is a set of points of $Q^+(5,q)$ meeting each plane of $Q^+(5,q)$ in either a singleton or a line.

Lemma 6.3. *Any line L of $Q^+(5,q)$ containing at least two points of S has all its points in S.*

Proof. Consider a plane π of $Q^+(5,q)$ through L. As $|\pi \cap S| \geq |L \cap S| \geq 2$, we then have that $\pi \cap S$ is a line, necessarily equal to L. □

The following is a rephrasing of Lemma 6.3.

Corollary 6.4. *S is a subspace of the geometry of the points and lines of $Q^+(5,q)$.*

Lemma 6.5. *Any two disjoint S-lines L_1 and L_2 are opposite.*

Proof. Suppose this is not the case. Then there is a line L of $Q^+(5,q)$ meeting L_1 and L_2 such that $\langle L, L_1\rangle$ and $\langle L, L_2\rangle$ are planes of $Q^+(5,q)$. But each of these planes would then contain at least $q+2$ points of S, an obvious contradiction. □

Lemma 6.6. *If L_1 and L_2 are two disjoint S-lines, then $\langle L_1, L_2\rangle \cap Q^+(5,q)$ is a $Q^+(3,q)$-quadric which is entirely contained in S.*

Proof. By Lemma 6.5, we know that $\langle L_1, L_2\rangle \cap Q^+(5,q)$ is a $Q^+(3,q)$-quadric. As this $Q^+(3,q)$-quadric is the smallest subspace containing $L_1 \cup L_2 \subseteq S$, we know from Corollary 6.4 that it is contained in S. □

Proposition 6.7. *If there are two disjoint S-lines, then S is a $Q^+(3,q)$-quadric.*

Proof. By Lemma 6.6, we know that there is some $Q^+(3,q)$-quadric $\mathcal{Q}$ which is entirely contained in S. We show that $S = \mathcal{Q}$.

Suppose to the contrary that $S \neq \mathcal{Q}$ and let $x \in S \setminus \mathcal{Q}$. There is a point $y \in \mathcal{Q}$ collinear on $Q^+(5,q)$ with x. By Lemma 6.3, the line xy is completely contained in S. We show that $\langle y, \mathcal{Q}\rangle \cap Q^+(5,q)$ is a $Q(4,q)$-quadric.

Suppose that this is not the case. Then $\langle L, \mathcal{Q}\rangle$ is a hyperplane and there is a plane of $Q^+(5,q)$ through the tangency point containing $xy \subseteq S$ and another line in $\mathcal{Q} \subseteq S$. This is clearly impossible.

So, $\mathcal{Q}' = \langle xy, \mathcal{Q}\rangle \cap Q^+(5,q)$ is a $Q(4,q)$-quadric. As $\mathcal{Q} \cup xy \subseteq S$ and $\mathcal{Q}$ is a maximal subspace of $\mathcal{Q}'$ (regarded as generalized quadrangle), the fact that $S \cap \mathcal{Q}'$ is a subspace of $\mathcal{Q}'$ then implies that $S \cap \mathcal{Q}' = \mathcal{Q}'$, i.e. $\mathcal{Q}' \subseteq S$. But as every plane of $Q^+(5,q)$ intersects $\mathcal{Q} \subseteq S$ in a line, there can then not exist planes meeting S in singletons, a contradiction. So, we must have $S = \mathcal{Q}$. □

In the sequel, we may therefore assume that any two S-lines meet in a singleton.

Lemma 6.8. *There exists a point x^* which is contained in all S-lines. Moreover, the set $\mathcal{L}_{x^*}$ of all S-lines through x^* is a partial ovoid of $\mathcal{S}_{x^*}$.*

Proof. This follows from the fact that there are no disjoint S-lines and no planes of $Q^+(5,q)$ containing at least two S-lines. □

Lemma 6.9. *No point of $Q^+(5,q)$ that is noncollinear with x^* on $Q^+(5,q)$ can be contained in S.*

Proof. Suppose y is such a point. Let L be an S-line through x^* and let z be the unique point of L collinear with y on $Q^+(5,q)$. As S is a subspace, yz must be an S-line meeting the S-line L in a point distinct from x^*, a contradiction. □

Lemma 6.10. *$\mathcal{L}_{x^*}$ is an ovoid of $\mathcal{S}_{x^*}$.*

Proof. Suppose that this is not the case. Then by Lemma 6.8, we know that there is a plane π through x^* meeting S in the singleton $\{x^*\}$. If L is a line of π not containing x^* and π' is the plane of $Q^+(5,q)$ through L distinct from π, then π' would be disjoint from S by Lemma 6.9, an obvious contradiction. □

We have thus proved the following.

Proposition 6.11. *If any two S-lines meet in a singleton, then S is as in Proposition 6.1.*

7 Good quadratic sets of type (LP)

7.1 Examples and basic properties

In the following proposition, we describe the standard examples of good quadratic sets of type (LP). By Lemma 3.2, these quadratic sets arise as intersections of $Q^+(5,q)$ with quadrics of PG$(5,q)$.

Proposition 7.1. *The union of two $Q(4,q)$-quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ intersecting in a $Q^+(3,q)$-quadric or in a quadric of type $xQ(2,q)$ is a good quadratic set of type (LP).*

Proof. Let Π_1 and Π_2 be two hyperplanes of PG$(5,q)$ for which $\mathcal{Q}_1 = \Pi_1 \cap Q^+(5,q)$ and $\mathcal{Q}_2 = \Pi_2 \cap Q^+(5,q)$. Put $S := \mathcal{Q}_1 \cup \mathcal{Q}_2$. Let π be an arbitrary plane of $Q^+(5,q)$. Then π cannot be contained in Π_1 nor in Π_2 as neither of $\mathcal{Q}_1$, $\mathcal{Q}_2$ contains planes. So, $\pi \cap \Pi_i$ with $i \in \{1,2\}$ is a line L_i. Now, $\pi \cap S = \pi \cap (Q^+(5,q) \cap (\Pi_1 \cup \Pi_2)) = \pi \cap (\Pi_1 \cup \Pi_2) = L_1 \cup L_2$. We must therefore prove that we can choose the plane π in such a way that $L_1 = L_2$ and also that we can choose another π such that $L_1 \neq L_2$.

If we choose the plane π such that it contains a line L of $\mathcal{Q}_1 \cap \mathcal{Q}_2$, then we obviously have $L_1 = L_2$. On the other hand, if we choose the plane π such that it contains a line L_1 of $\mathcal{Q}_1$ that is not line of $\mathcal{Q}_1 \cap \mathcal{Q}_2$, then $L_1 \neq L_2$. □

We note that if $q = 2$, then there do not exist two $Q(4,q)$-quadrics meeting in a $Q^+(3,q)$-quadric.

In this section, we will show that all good quadratic sets of type (LP) can be obtained as described in Proposition 7.1. In the sequel, we suppose that S is a good quadratic set of type (LP) of $Q^+(5,q)$. If π is a plane of type (P) of $Q^+(5,q)$ and $\pi \cap S = L_1 \cup L_2$ for two lines L_1 and L_2, then the unique point in $L_1 \cap L_2$ is called the *center* of π.

Lemma 7.2. *There exists an $\epsilon \in \{0,1,\ldots,q+1\}$ such that $|S| = 2q^3 + q + 1 + \epsilon q$. Moreover, if π is a plane of type (P), then the number of planes of type (P) meeting π in a line is equal to $\epsilon + q^2$ and ϵ of these planes contain the center of π.*

Proof. Put $\pi \cap S = L_1 \cup L_2$ and $\{x\} = L_1 \cap L_2$. The points of $S \setminus \pi$ are partitioned by the planes of $Q^+(5,q)$ that meet π in a line. There are now q^2 planes of $Q^+(5,q)$ that meet π in a line not containing x and each of these planes has type (P). The remaining $q+1$ planes of $Q^+(5,q)$ that meet π in a line have type (L) or (P). So,

$$|S| = |\pi \cap S| + |S \setminus \pi| = (2q+1) + q^2(2q-1) + (q-1)q + \epsilon q = 2q^3 + q + 1 + \epsilon q,$$

where $\epsilon \in \{0, 1, \ldots, q+1\}$ is the number of planes of type (P) meeting π in a line through x. $\square$

Lemma 7.3. *If L is an S-line contained in a plane of type (L), then both planes of $Q^+(5,q)$ through L have type (L).*

Proof. Suppose to the contrary that π_1 and π_2 are the two planes of $Q^+(5,q)$ through L such that π_1 has type (L) and π_2 has type (P). Put $\pi_2 \cap S = L \cup L'$ and $\{x\} := L \cap L'$.

We prove that every $y \in L \setminus \{x\}$ is contained in q S-lines distinct from L. Note that every S-line through y distinct from L is contained in a unique plane of $Q^+(5,q)$ distinct from π_1 that meets π_2 in a line. Now, each of the q planes of $Q^+(5,q)$ through y distinct from π_1 that meets π_2 in a line intersects $\pi_2 \cap S$ in two points and so is a plane of type (P) with center outside π_2. These q planes thus give rise to q S-lines through y distinct from L.

We prove that every plane π through $y \in L \setminus \{x\}$ distinct from π_2 that meets π_1 in a line is a plane of type (L). Indeed, there are q such planes and each such plane contributes at least one and hence exactly one S-line through y distinct from L. Now, as π contains one S-line through y and the line $\pi \cap \pi_1$ intersects S in the singleton $\{y\}$, we see that π must be a plane of type (L).

We now compute an upper bound for the number of points of S. Each point of $S \setminus \pi_1$ is contained in a unique plane meeting π_1 in a line. There are q^2 such planes that meet π_1 in a line that does not contain x and each such plane has type (L) and thus contributes q points of $S \setminus \pi_1$. The remaining $q+1$ planes intersecting π_1 in a line have type (L) or (P). So, we have

$$|S| = |S \cap \pi_1| + |\pi_1 \setminus S| = (q+1) + q^2 \cdot q + 1 \cdot q + \epsilon' \cdot 2q + (q - \epsilon')q$$

$$= q^3 + q^2 + 2q + 1 + \epsilon' q \leq q^3 + 2q^2 + 2q + 1,$$

with $\epsilon' + 1 \in \{1, 2, \ldots, q+1\}$ the number of planes of type (P) meeting π_1 in a line (necessarily through x). For $q \geq 3$, we know that $q^3 + 2q^2 + 2q + 1 < 2q^3 + q + 1$ and so we have a contradiction by Lemma 7.2. We therefore have $q = 2$. But we also show that this case cannot occur.

If $q = 2$, then by Lemma 7.2, we have that $|S| = 19 + 2\epsilon$ with $\epsilon \in \{0, 1, 2, 3\}$ and by the above we have $|S| = 17 + 2\epsilon'$ with $\epsilon' \in \{0, 1, 2\}$. There are thus two possibilities. Either, $(\epsilon, \epsilon') = (0, 1)$ or $(\epsilon, \epsilon') = (1, 2)$. We now count the number of S-lines through x not contained in π_2. Each such line is contained in a unique plane of $Q^+(5,q)$ that meets π_1 in a line through x distinct from L. Among the two planes that meet π_1 in a line through x distinct from L, there are ϵ' that have type (P) and $2 - \epsilon'$ that have type (L). So, the number of S-lines through x not contained in π_2 equals $2 \cdot \epsilon' + (2 - \epsilon') = 2 + \epsilon' = 3 + \epsilon$. On the other hand, each S-line through x not contained in π_2 is contained in a unique plane of $Q^+(5, q)$ meeting π_2 in a line. As $3 + \epsilon \geq 3$, we see that the unique plane of $Q^+(5, 2)$ intersecting π_2 in L' contains one S-line through x distinct from L' and the unique plane of $Q^+(5, 2)$ intersecting π_2 in a line through x distinct from L and L' has two S-lines through x. The number of planes of type (P) of $Q^+(5, 2)$ meeting π_2 in a line through x is therefore 2, in contradiction with Lemma 7.2 and the fact that $\epsilon \in \{0, 1\}$. □

The following is an immediate consequence of Lemmas 7.2 and 7.3.

Corollary 7.4. *We have $\epsilon \geq 2$.*

We call an S-line *nice* if the two planes of $Q^+(5, q)$ through it have type (L). We obviously have:

Lemma 7.5. *Let L_1 and L_2 be two distinct nice S-lines meeting in a point. Then no plane of $Q^+(5, q)$ through x can contain L_1 and L_2.*

The following is an immediate consequence of Lemma 7.5.

Corollary 7.6. *Let $x \in Q^+(5, q)$. Then the nice S-lines through x form a partial ovoid of $\mathcal{S}_x$. As a consequence, there are at most $q + 1$ nice S-lines through x.*

Lemma 7.7. *If L_1 and L_2 are two disjoint nice S-lines, then they are opposite.*

Proof. If this were not the case, then there is a plane of $Q^+(5, q)$ containing L_1 and a point x of L_2. But then $\pi \cap S$ cannot be a line. □

Lemma 7.8. *If L_1 and L_2 are two disjoint nice S-lines, then any line L of $Q^+(5, q)$ meeting L_1 and L_2 is a nice S-line.*

Proof. Let π be an arbitrary plane through L. Let π_1 and π_2 denote the planes through respectively L_1 and L_2 that meet π in lines. Put $L_i' := \pi_i \cap \pi$ for every $i \in \{1, 2\}$. As $S \cap \pi$ is disjoint from $L_1' \setminus (L \cap L_i')$ and $L_2 \setminus (L \cap L_2')$, we have $S \cap \pi = L$. We thus have that L is a nice S-line. □

The following is an immediate consequence of Lemma 7.8.

Corollary 7.9. *Let L_1 and L_2 be two disjoint nice S-lines and let G be the unique $Q^+(3, q)$-quadric containing L_1 and L_2. Then all $2(q + 1)$ lines of G are nice S-lines.*

Lemma 7.10. *Let N denote the number of nice S-lines. Then $|S| = (q^2+1)(2q+1) - \frac{Nq}{q+1}$. As a consequence, N is a multiple of $q + 1$.*

Proof. Taking into account that every line of $Q^+(5,q)$ is contained in two planes of $Q^+(5,q)$, we see that there are $2N$ planes of type (L) and $2(q+1)(q^2+1)-2N$ planes of type (P). As every point is contained in $2(q+1)$ planes of $Q^+(5,q)$, we thus have that

$$|S| = \frac{2N(q+1) + (2(q+1)(q^2+1) - 2N)(2q+1)}{2(q+1)} = (q^2+1)(2q+1) - \frac{Nq}{q+1}.$$

□

From Lemmas 7.2 and 7.10, we find:

Corollary 7.11. *We have* $q+1 = \epsilon + \frac{N}{q+1}$.

By Corollaries 7.4 and 7.11, we have:

Corollary 7.12. *We have* $N \leq q^2 - 1$.

Let $\mathcal{G}$ be the geometry whose lines are all nice S-lines and whose points are all points of S that are contained on a nice S-line (natural incidence).

Lemma 7.13. *The geometry* $\mathcal{G}$ *is connected.*

Proof. Let x_1 and x_2 be two points of $\mathcal{G}$. Let L_i with $i \in \{1,2\}$ denote a nice S-line containing x_i. If L_1 and L_2 meet, then x_1 and x_2 are connected by a path. If L_1 and L_2 are disjoint, then they are connected by a path by Lemma 7.8. □

We now consider two cases.

- Case 1: There exist two disjoint nice S-lines.
- Case 2: Any two nice S-lines meet.

7.2 Treatment of case 1

In this subsection, we suppose that there exist two disjoint nice S-lines. We call a $Q^+(3,q)$-quadric *nice* if all its $2(q+1)$ lines are nice S-lines.

Lemma 7.14. *Every nice* S*-line* L *is contained in a nice* $Q^+(3,q)$*-quadric.*

Proof. Suppose first that all nice S-lines meet L. If K_1 and K_2 are two disjoint S-lines, then $K_1 \neq L \neq K_2$ and so K_1 and K_2 meet L in distinct points. But then Corollary 7.9 implies that the unique $Q^+(3,q)$-quadric containing K_1 and K_2 must be nice. This $Q^+(3,q)$-quadric contains L.

Suppose therefore that there is a nice S-line disjoint from L. Then also Corollary 7.9 implies that there is a nice $Q^+(3,q)$-quadric containing L. □

Lemma 7.15. *All points of* $\mathcal{G}$ *are incident with the same number of nice* S*-lines.*

Proof. Bt Lemma 7.13, it suffices to prove that any two distinct $\mathcal{G}$-collinear points x_1 and x_2 are incident with the same number of nice S-lines. Let N_i with $i \in \{1, 2\}$ denote the total number of nice S-lines incident with x_i. We thus need to prove that $N_1 = N_2$. Let L denote the unique nice S-line containing x_1 and x_2. Let $x \in L \setminus \{x_1, x_2\}$. By Lemma 7.14, there must exist a nice S-line K through x distinct from L. By Corollary 7.9 the number of nice $Q^+(3, q)$-quadrics containing $L \cup K$ is equal to both $N_1 - 1$ and $N_2 - 1$, indeed showing that $N_1 = N_2$. □

Assume now that every point of $\mathcal{G}$ is incident with exactly $\alpha + 1$ nice S-lines.

Lemma 7.16. *Every nice S-line L is contained in exactly α^2 nice $Q^+(3, q)$-quadrics.*

Proof. Let x_1 and x_2 be two distinct points of L. Then a nice $Q^+(3, q)$-quadric through L contains a nice S-line $L_1 \neq L$ through x_1 and a nice S-line $L_2 \neq L$ through x_2. There are α^2 possibilities for the nice S-lines L_1 and L_2 and for each such choice for L_1 and L_2, there is by Corollary 7.9 a unique $Q^+(3, q)$-quadric containing L, L_1 and L_2. □

By Corollary 7.9 and Lemma 7.16, we find:

Corollary 7.17. *The number of nice S-lines disjoint from a given S-line is equal to $q\alpha^2$. As a consequence, we have $N = 1 + (q + 1)\alpha + q\alpha^2 = (1 + \alpha)(1 + q\alpha)$.*

Lemma 7.18. *We have $\alpha = 1$, $N = 2(q + 1)$, $\epsilon = q - 1$ and $|S| = 2q^3 + q^2 + 1$.*

Proof. As $q+1$ is a divisor of $N = (1+\alpha)(1+q\alpha)$ by Lemma 7.10, we have that $q+1 \mid \alpha^2-1$. If $\alpha \neq 1$, then we would have that $\alpha \geq \sqrt{q + 2}$. But then $N = (1 + \alpha)(1 + q\alpha) > q\alpha^2 \geq q^2 + 2q$ and this is in contradiction with Corollary 7.12.

So, $\alpha = 1$, $N = (1 + \alpha)(1 + q\alpha) = 2(q + 1)$, $\epsilon = q + 1 - \frac{N}{q+1} = q - 1$ and $|S| = (q^2 + 1)(2q + 1) - \frac{Nq}{q+1} = 2q^3 + q^2 + 1$. □

The following is a consequence of Lemmas 7.14 and 7.18.

Corollary 7.19. *The geometry $\mathcal{G}$ is a $(q + 1) \times (q + 1)$-grid.*

Lemma 7.20. *Let π be a plane of type (P) of $Q^+(5, q)$. Then the center of π is a point of $\mathcal{G}$.*

Proof. Put $\pi \cap S = L_1 \cup L_2$ and $\{x\} = L_1 \cap L_2$. As $\epsilon = q - 1$, we know from Lemma 7.2 that there are two lines K_1 and K_2 of π through x such that the unique planes of $Q^+(5, q)$ through K_1 and K_2 distinct from π are planes of type (L). This implies that there are two nice S-lines through x, i.e. that x is a point of $\mathcal{G}$. □

Corollary 7.21. *Every S-line L is either a line of $\mathcal{G}$ or contains a unique point of $\mathcal{G}$.*

Proof. This follows from Corollary 7.19 and Lemma 7.20 by considering a plane of $Q^+(5, q)$ through L. □

Lemma 7.22. *For every point $x \in S$ not belonging to $\mathcal{G}$, the set of S-lines containing x is an ovoid of $\mathcal{S}_x$ necessarily containing $q+1$ elements.*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$ through x. As x is no point of $\mathcal{G}$, π has type (P). By Lemma 7.20, x cannot be the center of π. So, there is a unique S-line through x contained in π. □

Lemma 7.23. *Every point x of $\mathcal{G}$ is incident with $2q-2$ S-lines not contained in $\mathcal{G}$.*

Proof. Let L_1 and L_2 be the two lines of $\mathcal{G}$ through x. Then there are $2(q+1)-4 = 2q-2$ planes of $Q^+(5,q)$ through x not containing L_1 and L_2. Each of these planes must have type (P) and its center equals x by Lemma 7.20. As each line of $Q^+(5,q)$ is contained in two planes of $Q^+(5,q)$, we thus see that the total number of S-lines through x not contained in $\mathcal{G}$ is equal to $\frac{(2q-2)\cdot 2}{2} = 2q-2$. □

Lemma 7.24. *Let x be a point of $\mathcal{G}$ and let L be one of the $2q-2$ S-lines through x not contained in $\mathcal{G}$. Then $\langle \mathcal{G}, L\rangle$ intersects $Q^+(5,q)$ is a $Q(4,q)$-quadric.*

Proof. Suppose that this is not the case. Then $\langle \mathcal{G}, L\rangle$ is a tangent hyperplane and there is a line L' of $\mathcal{G}$ through x such that L and L' are in the same plane of $Q^+(5,q)$. This is obviously not possible as L' is a nice S-line. □

Lemma 7.25. *Let x be a point of $\mathcal{G}$ and let L be one of the $2q-2$ S-lines through x not contained in $\mathcal{G}$. Then the $Q(4,q)$-quadric $\mathcal{Q} := \langle \mathcal{G}, L\rangle \cap Q^+(5,q)$ is completely contained in S.*

Proof. Note that $\mathcal{G}$ is a hyperplane of $\mathcal{Q}$. In view of the connectedness of hyperplane complements in $\mathcal{Q}$ ([2, Theorem 7.3], [19, Lemma 6.1]) and the fact that $L \subseteq S$, it suffices to prove the following.

> If $y \in \mathcal{Q} \cap S$ is not contained in $\mathcal{G}$, then the $q+1$ lines of $\mathcal{Q}$ through y are all contained in S.

But this follows from the fact that there are $q+1$ S-lines through y (Lemma 7.22) and that each of these S-lines meets $\mathcal{G}$ (Corollary 7.21) and are thus contained in $\mathcal{Q}$. □

Proposition 7.26. *S is the union of two $Q(4,q)$-quadrics through $\mathcal{G}$.*

Proof. By Lemma 7.25, there exists a $Q(4,q)$-quadric $\mathcal{Q}_1$ through $\mathcal{G}$ entirely consisting of points of S. If $x \in \mathcal{G}$, then $q+1$ of the $2q$ S-lines through x are contained in $\mathcal{Q}_1$. So, there is an additional S-line through x not contained in $\mathcal{Q}_1$. This gives rise to a second $Q(4,q)$-quadric $\mathcal{Q}_2$ through $\mathcal{G}$ that entirely consists of points of S. From $\mathcal{Q}_1 \cup \mathcal{Q}_2 \subseteq S$ and $|\mathcal{Q}_1 \cup \mathcal{Q}_2| = |\mathcal{Q}_1| + |\mathcal{Q}_2| - |\mathcal{Q}_1 \cap \mathcal{Q}_2| = (q+1)(q^2+1) + (q+1)(q^2+1) - (q+1)^2 = 2q^3 + q^2 + 1 = |S|$, it follows that $S = \mathcal{Q}_1 \cup \mathcal{Q}_2$. □

7.3 Treatment of case 2

In this subsection, we suppose that there do not exist two disjoint nice S-lines.

Lemma 7.27. *All $N = q + 1$ nice S-lines go through the same point x forming an ovoid O_x of $\mathcal{S}_x$.*

Proof. Let $\mathcal{L} \neq \emptyset$ denote the set of all nice S-lines. As $N = |\mathcal{L}| \neq 0$ is a multiple of $q + 1$ (Lemma 7.10), we have $N \geq q+1$. Let L_1 and L_2 be two arbitrary distinct elements of $\mathcal{L}$ and put $\{x\} := L_1 \cap L_2$. As L_1 and L_2 are not contained in a plane of $Q^+(5, q)$ (Corollary 7.6), no S-line can meet L_1 and L_2 in points distinct from x. So, all lines of $\mathcal{L}$ contain x. By Corollary 7.6, $\mathcal{L}$ is a partial ovoid of $\mathcal{S}_x$. As $|\mathcal{L}| \geq q+1$, we then have that $|\mathcal{L}| = q+1$ and that $\mathcal{L}$ is an ovoid of $\mathcal{S}_x$. □

By Lemmas 7.10, 7.27 and Corollary 7.11, we have:

Corollary 7.28. *We have $\epsilon = q$ and $|S| = 2q^3 + q^2 + q + 1$.*

Let X denote the set of points of $Q^+(5, q)$ that are contained in a nice S-line.

Lemma 7.29. *The planes of $Q^+(5, q)$ through x have type (L). The planes of $Q^+(5, q)$ not through x have type (P) and their centers lie in $X \setminus \{x\}$.*

Proof. By Lemma 7.27, a plane through x contains a unique nice S-line and is therefore a plane of type (L). Suppose therefore that π is a plane not containing x. By Lemma 7.27, π cannot contain nice S-lines and is therefore a plane of type (P). Put $\pi \cap S = L_1 \cup L_2$ for two distinct lines L_1 and L_2. Also put $\{y\} := L_1 \cap L_2$. By Lemma 7.2 and Corollary 7.28, we know that there exists a unique line L_3 in π through y having the property that the plane of $Q^+(5, q)$ through L_3 distinct from π has type (L). So, there is a nice S-line through y, implying that $y \in X \setminus \{x\}$. □

Lemma 7.30. *Every S-line L is either contained in X or contains a unique point of $X \setminus \{x\}$.*

Proof. Let π be an arbitrary plane of $Q^+(5, q)$ through L. If $x \in \pi$, then by Lemma 7.29 L is a nice S-line through x contained in X. If $x \notin \pi$ and y is the center of the plane π (of type (P)), then $y \in L \cap X$ by Lemma 7.29. The unique plane through x meeting π in a line intersects S in the line xy, showing that $L \cap X = \{y\}$. □

Lemma 7.31. *Every $y \in S \setminus X$ is contained in $q + 1$ S-lines forming an ovoid O_y of $\mathcal{S}_y$.*

Proof. Let π be an arbitrary plane of $Q^+(5, q)$ through y. Then π has type (P) and its center belongs to X and is thus distinct from y. This implies that there is a unique S-line in π through y. □

The following is an immediate consequence of Lemmas 7.30 and 7.31.

Corollary 7.32. *For every $y \in S \setminus X$, the $q+1$ S-lines through y are precisely the $q+1$ lines through y meeting the lines of O_x.*

The following is a consequence of Corollary 7.32.

Corollary 7.33. *Let $y \in S \setminus X$. Let L_1 and L_2 be two distinct S-lines through y and let K_1, K_2 be the S-lines through x meeting L_1 and L_2. Then the unique $Q^+(3,q)$-quadric containing K_1, K_2, L_1 and L_2 has all its points in S.*

Note that Lemma 2.2 implies that the 4-dimensional subspace generated by K_1, K_2, L_1 and L_2 indeed intersects $Q^+(5,q)$ in a $Q^+(3,q)$-quadric.

Lemma 7.34. *The ovoid O_x of $\mathcal{S}_x$ is classical.*

Proof. Let L_1, L_2 and L_3 be three distinct S-lines through x and let α be the 3-dimensional subspace $\langle L_1, L_2, L_3 \rangle$. Then $\alpha \cap Q^+(5,q)$ is a cone of type $xQ(2,q)$ and Π_x is the only tangent hyperplane through α. So, if $y \in S \setminus X$, then $\langle \alpha, y \rangle$ is a 4-dimensional subspace intersecting $Q^+(5,q)$ in a $Q(4,q)$-quadric $\mathcal{Q}$.

We will show that all lines of $\mathcal{Q}$ through x are S-lines. This then implies that O_x coincides with the set of lines of $\mathcal{Q}$ through x, i.e. O_x is a classical ovoid of $\mathcal{S}_x$.

Suppose to the contrary that there exists a line L_4 of $\mathcal{Q}$ through x that is not an S-line. By considering a plane of $Q^+(5,q)$ through L_4 (which has type (L)), we then see that $L_4 \cap S = \{x\}$. Let y' be the unique point of L_4 that is collinear with y on $Q^+(5,q)$. For all mutually distinct $i,j,k \in \{1,2,3\}$, let G_i denote the unique $Q^+(3,q)$-quadric containing y, L_j and L_k. By Corollary 7.33, we know that G_i has all its points in S. Now, consider a line K through y' distinct from $y'x$ and $y'y$. This line K contains three distinct points of S, namely the points in the singletons $G_1 \cap K$, $G_2 \cap K$ and $G_3 \cap K$. By considering a plane of $Q^+(5,q)$ through K (which has type (L) or (P)), we see that this is only possible when $K \subseteq S$. In particular, we must have $y' \in S$. But that is impossible as $L_4 \cap S = \{x\}$. □

Proposition 7.35. *S is the union of two $Q(4,q)$-quadrics meeting in a quadric of type $xQ(2,q)$.*

Proof. By Lemma 7.34, the union of the lines in O_x is a quadric of type $xQ(2,q)$ obtained by intersecting $Q^+(5,q)$ with a 3-dimensional subspace α. If $y_1 \in S \setminus X$, then $\langle \alpha, y_1 \rangle$ is a 4-dimensional subspace intersecting $Q^+(5,q)$ in a $Q(4,q)$-quadric $\mathcal{Q}_1$. Since the complement of $xQ(2,q)$ in $\mathcal{Q}_1$ is connected ([2, Theorem 7.3], [19, Lemma 6.1]), Corollary 7.32 implies that all points of $\mathcal{Q}_1$ are contained in S. As $(q+1)(q^2+1) = |\mathcal{Q}_1| < |S| = 2q^3+q^2+q+1$, there exists a point $y_2 \in S \setminus (X \cup \mathcal{Q}_1)$. Again $\langle \alpha, y_2 \rangle$ intersects $Q^+(5,q)$ in a $Q(4,q)$-quadric $\mathcal{Q}_2$ that has all its points in S. We have $\mathcal{Q}_1 \cup \mathcal{Q}_2 \subseteq S$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2 = xQ(2,q)$. As $|\mathcal{Q}_1 \cup \mathcal{Q}_2| = |\mathcal{Q}_1| + |\mathcal{Q}_2| - |\mathcal{Q}_1 \cap \mathcal{Q}_2| = (q+1)(q^2+1) + (q+1)(q^2+1) - (q^2+q+1) = 2q^3+q^2+q+1 = |S|$, we have $S = \mathcal{Q}_1 \cup \mathcal{Q}_2$. □

8 Good quadratic sets of type (PW)

8.1 Examples and basic properties

In the following two propositions, we first describe the two standard examples of good quadratic sets of type (PW). By Lemma 3.2, these quadratic sets arise as intersections of $Q^+(5,q)$ with quadrics of $\mathrm{PG}(5,q)$.

Proposition 8.1. *Let x_1 and x_2 be two noncollinear points of $Q^+(5,q)$. Then $S = (\Pi_{x_1} \cup \Pi_{x_2}) \cap Q^+(5,q)$ is a good quadratic set of type (PW).*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$. If either $x_1 \in \pi$ or $x_2 \in \pi$, then $\pi \subseteq S$. Suppose therefore that $\pi \cap \{x_1, x_2\} = \emptyset$. For every $i \in \{1,2\}$, let π_i denote the unique plane through x_i meeting π in a line L_i. Then we have $L_1 \neq L_2$ as otherwise π, π_1 and π_2 would be three distinct planes of $Q^+(5,q)$ through $L_1 = L_2$. So, $S \cap \pi = L_1 \cup L_2$ is a pencil. □

Proposition 8.2. *Let x be a point of $Q^+(5,q)$ and $\mathcal{Q}$ a $Q(4,q)$-quadric containing x. Then $(\Pi_x \cap Q^+(5,q)) \cup \mathcal{Q}$ is a good quadratic set of type (PW).*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$. If $x \in \pi$, then $\pi \subseteq S$. Suppose therefore that $x \notin S$. Then π intersects $\mathcal{Q}$ in a line L_1 not containing x. As there exists a unique point on L_1 collinear with x on $Q^+(5,q)$, there is no plane of $Q^+(5,q)$ containing x and L_1 and so the unique plane of $Q^+(5,q)$ through x intersecting π in a line intersects π in a line L_2 distinct from L_1. Now, $S \cap \pi = L_1 \cup L_2$ is a pencil. □

In this section, we prove that any good quadratic set of type (PW) is obtained as in one of the Propositions 8.1 and 8.2. From now on we suppose that S is a set of points of $Q^+(5,q)$ intersecting each plane of $Q^+(5,q)$ in either a pencil or the whole plane with both possibilities occurring. We call a plane of $Q^+(5,q)$ that has all its points in S an *S-plane.*

Lemma 8.3. *Every line L of $Q^+(5,q)$ containing at least three points of S has all its points in S.*

Proof. Let π be a plane of $Q^+(5,q)$ containing L. Then $\pi \cap S$ is either a pencil or the whole of π. As $|L \cap S| \geq 3$, we then necessarily have that the whole of L is contained in S. □

Lemma 8.4. *Precisely one of the following cases occurs:*

- (a) $|S| = 2q^3 + 2q^2 + q + 1$ *and for every plane π of $Q^+(5,q)$ intersecting $Q^+(5,q)$ in a pencil of two lines L_1 and L_2, precisely one of L_1, L_2 is contained in an S-plane.*
- (b) $|S| = 2q^3 + 3q^2 + 1$. *In this case, any S-line is contained in either 1 or 2 S-planes.*

Proof. Let π be a plane of $Q^+(5,q)$ intersecting S in a pencil of two distinct lines L_1 and L_2. Let $\epsilon \in \{0,1,2\}$ denote the number of lines among L_1 and L_2 that are contained in an S-plane. The points of $S \setminus \pi$ are now partitioned by the planes of $Q^+(5,q)$ intersecting π in a line. There are q^2 planes of $Q^+(5,q)$ intersecting π in a line not containing $L_1 \cap L_2$ and each of these planes contributes $2q-1$ points to $S \setminus \pi$. There are also $q-1$ planes intersecting π in a line through $L_1 \cap L_2$ distinct from L_1 and L_2 themselves and each of these planes contributes $2q$ points to $S \setminus \pi$. There are also two planes intersecting π in L_1 or L_2, and each such plane contributes q^2 or q points to $S \setminus \pi$ depending on whether the plane is an S-plane or not. We thus find that $|S| = |S \cap \pi| + |S \setminus \pi| = 2q+1+q^2(2q-1)+(q-1)2q+\epsilon q^2+(2-\epsilon)q = 2q^3+q^2+2q+1+\epsilon(q^2-q)$. One of the following three situations occur:

- (a) $|S| = 2q^3+2q^2+q+1$ and precisely one of L_1, L_2 is contained in an S-plane;
- (b) $|S| = 2q^3+3q^2+1$ and L_1, L_2 are contained in S-planes;
- (c) $|S| = 2q^3+q^2+2q+1$ and none of L_1, L_2 is contained in an S-plane.

We show that case (c) cannot occur. Suppose to the contrary that case (c) occurs. Then $|S| = 2q^3+q^2+2q+1$ and for every plane π' of $Q^+(5,q)$ intersecting S in a pencil of two lines L_1' and L_2', none of L_1', L_2' is contained in an S-plane. This means that if we take an S-plane π'', then every plane of $Q^+(5,q)$ meeting π'' in a line must be an S-plane. But the planes of $Q^+(5,q)$ meeting π'' in a line cover all points of $Q^+(5,q)$, an obvious contradiction.

So, we always have case (a) or (b). The lemma follows. □

Lemma 8.5. *If $|S| = 2q^3+2q^2+q+1$, then there are precisely $2q+2$ S-planes. If $|S| = 2q^3+3q^2+1$, then there are precisely $4q+4$ S-planes.*

Proof. Let N denote the number of S-planes. Then the number of planes of $Q^+(5,q)$ intersecting S in a pencil is equal to $2(q^3+q^2+q+1)-N$. Counting the number of points of S, we find

$$|S| = \frac{N\cdot(q^2+q+1)+(2(q^3+q^2+q+1)-N)(2q+1)}{2(q+1)}.$$

The lemma follows. □

Lemma 8.6. *Let π be an S-plane. If $|S| = 2q^3+2q^2+q+1$, then there are precisely $q+1$ S-planes meeting π in a line. If $|S| = 2q^3+3q^2+1$, then there are precisely $q+2$ S-planes meeting π in a line.*

Proof. Let N denote the number of S-planes meeting π in a line. Then there are $q^2+q+1-N$ planes of $Q^+(5,q)$ meeting π in a line and S in a pencil. Counting points of S, we find

$$|S| = |S\cap\pi| + |S\setminus\pi| = (q^2+q+1)+Nq^2+(q^2+q+1-N)q,$$

taking into account that the points of $S \setminus \pi$ are partitioned by the planes meeting π in a line. If $|S| = 2q^3+2q^2+q+1$, then we find $N = q+1$. If $|S| = 2q^3+3q^2+1$, then we find $N = q+2$. □

8.2 Treatment of the case $|S| = 2q^3 + 3q^2 + 1$

We call a point x of S *deep* if all lines of $Q^+(5, q)$ through x are contained in S.

Lemma 8.7. *If $|S| = 2q^3 + 3q^2 + 1$, then for every S-plane π, the set $\mathcal{L}_\pi$ of lines of π that are contained in two S-planes is either a dual hyperoval (and then q must be even) or a line pencil in π, plus one extra line. In the latter case, the center of the line pencil of π is a deep point.*

Proof. By Lemma 8.6, we know that $|\mathcal{L}_\pi| = q + 2$. So, it suffices to prove that if L_1, L_2, L_3 are three distinct lines of $\mathcal{L}_\pi$ through a point $x \in \pi$, then all lines of π through x belong to $\mathcal{L}_\pi$. For every $i \in \{1, 2, 3\}$, let π_i denote the unique plane through L_i distinct from π. Let π' be an arbitrary plane of $Q^+(5, q)$ through x belonging to the same family as π. Then for every $i \in \{1, 2, 3\}$, $\pi' \cap \pi_i$ is a line $L'_i \subseteq S$. As $\pi' \cap S$ contains a pencil of three lines, π' is an S-plane. Now, all lines of $Q^+(5, q)$ through x are covered by the planes of $Q^+(5, q)$ through x belonging to the same family as π. It follows that all lines of $Q^+(5, q)$ through x are S-lines and all planes of $Q^+(5, q)$ through x are S-planes. In particular, all lines of π through x belong to $\mathcal{L}_\pi$. □

Lemma 8.8. *If $|S| = 2q^3 + 3q^2 + 1$, then there are no S-planes π for which $\mathcal{L}_\pi$ is a dual hyperoval.*

Proof. Suppose π is an S-plane for which $\mathcal{L}_\pi$ is a dual hyperoval. Let $x \in \pi$ such that x is not contained in a line of $\mathcal{L}_\pi$.

Let π' be an arbitrary plane of $Q^+(5, q)$ through x meeting π in the singleton $\{x\}$. As $x \in S$, the number N of S-lines through x contained in π' is 1, 2 or $q + 1$. If $N \in \{1, 2\}$ and if L is an S-line of π' through x, then by Lemma 8.4(b) the unique plane of $Q^+(5, q)$ through L distinct from π' is an S-plane intersecting π in a line of $\mathcal{L}_\pi$ through x, a contradiction. So, $N = q + 1$ and π' is an S-plane.

So, every plane of $Q^+(5, q)$ through x intersecting π in the singleton $\{x\}$ is an S-plane. As each line of $Q^+(5, q)$ through x not contained in π is contained in precisely one such plane, we see that all lines of $Q^+(5, q)$ through x are S-lines and all planes of $Q^+(5, q)$ through x are S-planes. This is not possible as there are no lines of $\mathcal{L}_\pi$ through the point x. □

The following is a consequence of Lemmas 8.7 and 8.8.

Corollary 8.9. *If $|S| = 2q^3 + 3q^2 + 1$, then for every S-plane π, the set $\mathcal{L}_\pi$ is a line pencil in π, plus one extra line. The center of this line pencil is a deep point.*

Proposition 8.10. *Suppose $|S| = 2q^3 + 3q^2 + 1$. Then there exist two noncollinear points x_1 and x_2 on $Q^+(5, q)$ such that $S = (\Pi_{x_1} \cup \Pi_{x_2}) \cap Q^+(5, q)$.*

Proof. By Corollary 8.9, there exists a deep point x_1. As there are $4(q+1)$ S-planes among which $2(q + 1)$ go through x_1, there exists an S-plane not containing x_1. This S-plane contains a deep point $x_2 \neq x_1$. If x_1 and x_2 are collinear on $Q^+(5, q)$, then every plane of $Q^+(5, q)$ through x_1x_2 would contradict Corollary 8.9. So, x_1 and x_2 are noncollinear. As $(\Pi_{x_1} \cap Q^+(5, q)) \cup (\Pi_{x_2} \cap Q^+(5, q)) \subseteq S$ and both sets have size $2q^3 + 3q^2 + 1$, we have equality. □

8.3 Treatment of the case $|S| = 2q^3 + 2q^2 + q + 1$

From now on, we assume that $|S| = 2q^3 + 2q^2 + q + 1$.

Lemma 8.11. *All $2q + 2$ S-planes go through the same point p^* and p^* is a deep point.*

Proof. Let π be an S-plane. By Lemma 8.6, there are two S-planes π_1 and π_2 meeting π in a line. Put $L_i := \pi \cap \pi_i$ and $\{p^*\} := L_1 \cap L_2$. Let π' be an arbitrary plane of $Q^+(5, q)$ through p^* belonging to the same family as π. Then for every $i \in \{1, 2\}$, $\pi' \cap \pi_i$ is a line $L'_i \subseteq S$. By Lemma 8.4(a), we then know that π' must be an S-plane. Now, all lines of $Q^+(5, q)$ through p^* are covered by the planes of $Q^+(5, q)$ through p^* belonging to the same family as π. It follows that all lines of $Q^+(5, q)$ through p^* are S-lines. Hence, also all $2(q + 1)$ planes of $Q^+(5, q)$ through p^* are S-planes. □

Lemma 8.12. *We have $|S \setminus (p^*)^\perp| = q^3$.*

Proof. As $(p^*)^\perp \subseteq S$, we have $|S\setminus(p^*)^\perp| = |S|-|(p^*)^\perp| = (2q^3+2q^2+q+1)-(1+q(q+1)^2) = q^3$. □

For every point x of $S \setminus (p^*)^\perp$, let $\mathcal{L}_x$ denote the set of S-lines through x.

Lemma 8.13. *For every point x of $S \setminus (p^*)^\perp$, $\mathcal{L}_x$ is an ovoid of $\mathcal{S}_x$ (containing $q + 1$ elements).*

Proof. Let π be an arbitrary plane of $Q^+(5, q)$ through x. As $p^* \notin \pi$, π is a plane of type (P). As $\pi \cap S$ contains the line $(p^*)^\perp \cap \pi$ and $x \notin (p^*)^\perp \cap \pi$, there is a unique S-line through x contained in π. □

Proposition 8.14. *If $q = 2$, then $S = (p^*)^\perp \cup \mathcal{Q}$, where $\mathcal{Q}$ is a $Q(4, 2)$-quadric containing the point p^*.*

Proof. Assuming as before that $Q^+(5, q) = Q^+(5, 2)$ has equations $X_1X_2+X_3X_4+X_5X_6 = 0$, we may without loss of generality assume that $p^* = (1, 0, 0, 0, 0, 0)$. By Proposition 1.2, the set S is obtained by intersecting $Q^+(5, 2)$ with a quadric $\mathcal{Q}'$. We may suppose that $\mathcal{Q}'$ has equation $Q := \sum_{1\leq i\leq j\leq 6} a_{ij}X_iX_j = 0$ with $a_{56} = 0$ (otherwise, replace Q by $Q + X_1X_2 + X_3X_4 + X_5X_6$). The fact that the points $(1, 0, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 0, 1)$ belong to $\mathcal{Q}'$ implies that $a_{11} = a_{33} = a_{44} = a_{55} = a_{66} = 0$. The fact that the points $(1, 0, 1, 0, 0, 0)$, $(1, 0, 0, 1, 0, 0)$, $(1, 0, 0, 0, 1, 0)$ and $(1, 0, 0, 0, 0, 1)$ belong to $\mathcal{Q}'$ then implies that also $a_{13} = a_{14} = a_{15} = a_{16} = 0$. The fact that the points $(0, 0, 1, 0, 1, 0)$, $(0, 0, 1, 0, 0, 1)$, $(0, 0, 0, 1, 1, 0)$ and $(0, 0, 0, 1, 0, 1)$ belong to $\mathcal{Q}'$ implies that $a_{35} = a_{36} = a_{45} = a_{46} = 0$, and finally the fact that the point $(0, 0, 1, 1, 1, -1)$ belongs to $\mathcal{Q}'$ implies that $a_{34} = 0$. So, $\mathcal{Q}'$ has equation

$$X_2(a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + a_{24}X_4 + a_{25}X_5 + a_{26}X_6) = 0$$

and so $S = Q^+(5, 2) \cap \mathcal{Q}'$ is the union of two hyperplane intersections. One of these hyperplane intersections is $(p^*)^\perp$. As there are planes of type (P), the other hyperplane intersection $\mathcal{Q}$ cannot coincide with $(p^*)^\perp$. In fact, as the only S-planes are the planes through p^*, we see that $\mathcal{Q}$ must be a $Q(4,2)$-quadric. If $\mathcal{Q}$ does not contain the point p^*, then we would have $|S| = |(p^*)^\perp \cup \mathcal{Q}| = |(p^*)^\perp| + |\mathcal{Q}| - |(p^*)^\perp \cap \mathcal{Q}| = 19 + 15 - 9 = 25$, in contradiction with $|S| = 2q^3 + 2q^2 + q + 1 = 27$. So, $\mathcal{Q}$ is a $Q(4,2)$-quadric containing p^*. □

Lemma 8.15. *Let Γ be a graph on the ovoids of a (4×4)-grid G, with two distinct ovoids being adjacent whenever they meet in a singleton. Then Γ is a regular graph with valency 8 having two connected components of size 12.*

Proof. Let $\mathcal{C}_1 = \{L_1, L_2, L_3, L_4\}$ and $\mathcal{C}_2 = \{K_1, K_2, K_3, K_4\}$ be two parallel classes of lines of G. For every ovoid O of G, let σ_O be the permutation of $\{1,2,3,4\}$ such that the point in $L_i \cap K_{\sigma(i)}$ belongs to O for every $i \in \{1,2,3,4\}$. The map $O \mapsto \sigma_O$ defines a bijection between the set of 24 ovoids of G and the set S_4 of 24 permutations of the set $\{1,2,3,4\}$.

For every ovoid O of G, there are four points $x \in O$ and for each $x \in O$ there are two ovoids O_1 and O_2 intersecting O in the singleton $\{x\}$. If $\{x\} = L_i \cap K_{i'}$ for some $i, i' \in \{1,2,3,4\}$, then σ_{O_1} and σ_{O_2} are equal to $\sigma_O\sigma_1$ and $\sigma_O\sigma_2$, where σ_1 and σ_2 are the two cycles of length 3 defined on the set $\{1,2,3,4\} \setminus \{i'\}$. The following facts follow from these observations.

(1) Γ is regular with valency 8.

(2) If C is a connected component of Γ, then all permutations σ_O with $O \in C$ have the same parity.

There are now 12 permutations of each parity. The fact that Γ is regular with valency 8 thus implies that there are two connected components of size 12 and that for each connected component C the permutations σ_O with $O \in C$ form one of the two cosets of A_4 in S_4. □

Suppose now that $q = 3$. Then $\mathcal{S}_{p^*}$ is a (4×4)-grid G and we assume that the graph Γ defined in Lemma 8.15 arises from this grid G. By Lemma 8.15, we know that Γ is regular with valency 8. By Lemma 8.13, the subgraph Γ' of the collinearity graph of $Q^+(5,3)$ induced on the set $S \setminus (p^*)^\perp$ is also regular with valency 8. We can prove the following.

Lemma 8.16. *Suppose $q = 3$. For every point $x \in S \setminus (p^*)^\perp$, let O_x denote the set of four lines through p^* meeting the four lines of $\mathcal{L}_x$. Then O_x is an ovoid of $\mathcal{S}_{p^*}$. Moreover, the map $x \mapsto O_x$ between the vertex sets of Γ' and Γ cannot be a cover.*

Proof. We first prove that O_x is an ovoid of $\mathcal{S}_{p^*}$. Suppose two distinct lines L_1 and L_2 of O_x are contained in the same plane π of $Q^+(5,3)$ through p^*. Then the two lines of $\mathcal{L}_x$ through x meeting L_1 and L_2 are contained in the unique plane through x meeting π. This is impossible by Lemma 8.13. Now, as O_x is a partial ovoid of $\mathcal{S}_x$ of size 4, it is also an ovoid.

Suppose that the mentioned map is a cover. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote the two connected components of the graph Γ. Then there exist constants N_1 and N_2 such that that each ovoid O of $\mathcal{O}_i$ with $i \in \{1,2\}$ is the image of precisely N_i vertices of Γ'. The number of vertices of Γ' is then precisely $|\mathcal{O}_1| \cdot N_1 + |\mathcal{O}_2| \cdot N_2 = 12(N_1 + N_2)$, but that is impossible as Γ' has precisely $q^3 = 27$ vertices. $\square$

Proposition 8.17. *If $q = 3$, then $S = (p^*)^\perp \cup \mathcal{Q}$, where $\mathcal{Q}$ is a $Q(4,3)$-quadric containing the point p^*.*

Proof. By Lemma 8.16, we know that there exist two distinct adjacent vertices x and x' in $S \setminus (p^*)^\perp$ such that the ovoids O_x and $O_{x'}$ of $\mathcal{S}_{p^*}$ do not intersect in a singleton. Then $L_1 := xx'$ is a line of $\mathcal{L}_x$. We denote the other lines of $\mathcal{L}_x$ by L_2, L_3 and L_4. For every $i \in \{1,2,3,4\}$, let M_i denote the unique line through p^* meeting L_i, and put $\{u_i\} := L_i \cap M_i$. For every $i \in \{2,3,4\}$, let G_i denote the unique $Q^+(3,3)$-quadric of $Q^+(5,3)$ containing the lines L_1, M_1, L_i and M_i. The unique hyperplane of $\mathrm{PG}(5,3)$ containing $G_2 \cup G_3$ cannot be a tangent hyperplane as otherwise there is a plane of $Q^+(5,3)$ through the tangency point of $\langle G_2, G_3 \rangle$ containing L_2 and L_3, and this is in contradiction with Lemma 8.13. So, there is a unique $Q(4,3)$-quadric $\mathcal{Q}$ containing the $Q^+(3,3)$-quadrics G_2 and G_3. The set of lines of $Q^+(5,3)$ through x contained in $\mathcal{Q}$ is an ovoid of $\mathcal{S}_x$ containing L_1, L_2 and L_3 and so coincides with $\mathcal{L}_x$. This shows that the lines L_1, L_2, L_3, L_4, M_1, M_2, M_3 and M_4 are all contained in $\mathcal{Q}$, as well as the $Q^+(3,3)$-quadric G_4.

The line M_1 now belongs to both ovoids $O_x = \{M_1, M_2, M_3, M_4\}$ and $O_{x'}$. As O_x and $O_{x'}$ do not intersect in the singleton $\{M_1\}$, we may without loss of generality suppose that the line M_2 also belongs to $O_{x'}$, implying that the unique line L_2' through x' meeting M_2 is contained in S. Now, the mutually disjoint lines M_1, L_2' and L_2 of G_2 are all contained in S, implying that the four lines of G_2 meeting M_1, L_2' and L_2 are also completely contained in S as they already contain three points of S. So, we have that $G_2 \subseteq S$.

We prove that among the eight S-lines through a point of $(xu_1 \cup xu_2) \setminus \{x, u_1, u_2\}$ and not contained in G_2, there is a line that is contained in $\mathcal{Q}$. Suppose to the contrary that this is not the case. Then by Lemma 8.16, these eight S-lines meet M_3' and M_4', where $\{M_1, M_2, M_3', M_4'\}$ is the unique ovoid of $\mathcal{S}_{p^*}$ through $\{M_1, M_2\}$ distinct from $\{M_1, M_2, M_3, M_4\}$. Let L_3' denote the unique line of $Q^+(5,3)$ through x meeting M_3' and let G_3' denote the unique $Q^+(3,3)$-quadric containing L_1, L_3', M_1 and M_3'. In G_3', there are four lines meeting L_1 and M_3'. With exception of L_3' all these four lines are S-lines. The four lines in G_3' meeting M_1 and L_3' thus contain at least three points of S and thus are completely contained in S. This implies that $L_3' \subseteq S$. But that is impossible as L_1, L_2, L_3 and L_4 are the only S-lines through z.

We thus see that among the eight S-lines through a point of $(xu_1 \cup xu_2) \setminus \{x, u_1, u_2\}$ not contained in G_2, there is certainly one line that is contained in $\mathcal{Q}$. Without loss of generality, we may suppose that this line meets xu_1 and M_3. If the intersection with xu_1 is equal to u_1', then $O_{u_1'}$ contains the lines M_1, M_2, M_3 and hence also M_4. Similarly as in the previous paragraph, we then know that S contains the $Q^+(3,3)$-quadrics G_3 and G_4.

Now, the S-points in G_2, G_3 and G_4 already cover 34 of the 40 points of $\mathcal{Q}$. The six remaining points of $\mathcal{Q}$ are the points in $(K_1 \cup K_2) \setminus \{u_1\}$ where K_1 and K_2 are the two lines of $\mathcal{Q}$ through u_1 distinct from $u_1 p^* = M_1$ and $u_1 x = L_1$. Each point $y \in (K_1 \cup K_2) \setminus \{u_1\}$

is now contained in a line of $\mathcal{Q}$ distinct from K_1 and K_2. As this line already contains three S-points, it must be completely contained in S, in particular the point y. We thus see that all points of $\mathcal{Q}$ are contained in S.

Now, as $(p^*)^\perp \cup \mathcal{Q} \subseteq S$ and both sets have the same size, we have $S = (p^*)^\perp \cup \mathcal{Q}$. □

For every S-point $p \in (p^*)^\perp \setminus \{p^*\}$, let $\mathcal{L}_p$ denote the set of S-lines through p not entirely contained in $(p^*)^\perp$, plus the line pp^*.

Lemma 8.18. *For every S-point $p \in (p^*)^\perp \setminus \{p^*\}$, we have that $\mathcal{L}_p$ is a partial ovoid of $\mathcal{S}_p$.*

Proof. Let π be an arbitrary plane of $Q^+(5,q)$ through p. If π is one of the two planes of $Q^+(5,q)$ through pp^*, then pp^* is the unique line of $\mathcal{L}_p$ contained in π. Suppose therefore that $p^* \notin \pi$. As π is a plane of type (P) containing the S-line $(p^*)^\perp \cap \pi$, we then see that there is at most one S-line in π through p that is not entirely contained in $(p^*)^\perp$. □

Proposition 8.19. *If $q \geq 4$, then $S = \Pi_{p^*} \cup \mathcal{Q}$, where $\mathcal{Q}$ is a $Q(4,q)$-quadric containing the point p^*.*

Proof. We prove that there exist two $Q^+(3,q)$-quadrics G_1 and G_2 for which the following hold:

(1) $G_1 \cup G_2 \subseteq S$;

(2) $G_1 \cap G_2 = L_1 \cup L_2$ for two distinct lines L_1 and L_2 through a point x;

(3) the point x is not collinear with p^*;

(4) G_1 and G_2 do not contain p^*;

(5) there is a $Q(4,q)$-quadric $\mathcal{Q}$ containing G_1 and G_2.

Let $z \in S \setminus (p^*)^\perp$. There are now $(q^3-1)-(q+1)(q-1) = q^3-q^2$ S-points noncollinear with z and p^* on $Q^+(5,q)$. The number of paths z, u, v of length 2 in the collinearity graph of $Q^+(5,q)$ such that u and v are S-points noncollinear with p^* on $Q^+(5,q)$ and $zu \neq uv$ is equal to $q(q+1)(q-1)^2$. For any such path, the points z and v are not collinear on $Q^+(5,q)$ by Lemma 8.13. As $\frac{q(q+1)(q-1)^2}{q^3-q^2} = q - \frac{1}{q}$, there exists an S-point v noncollinear with z and p^* on $Q^+(5,q)$ and $q \geq 4$ neighbours $u_1, u_2, \ldots, u_q$ of z and v noncollinear on $Q^+(5,q)$ with p^* such that $zu_1, zu_2, \ldots, zu_q, u_1v, u_2v, \ldots, u_qv$ are all S-lines. The partial ovoids $\{zu_1, zu_2, \ldots, zu_q\}$ and $\{vu_1, vu_2, \ldots, vu_q\}$ in respectively $\mathcal{S}_z$ and $\mathcal{S}_v$ can be completed in unique ways to ovoids. Moreover, these ovoids only consist of S-lines by Lemma 8.13. In fact, an S-point u_{q+1} can be chosen in $z^\perp \cap v^\perp$ such that $\{zu_1, zu_2, \ldots, zu_{q+1}\}$ and $\{vu_1, vu_2, \ldots, vu_{q+1}\}$ are these ovoids of respectively $\mathcal{S}_z$ and $\mathcal{S}_v$.

As $\{zu_1, zu_2, \ldots, zu_{q+1}\}$ is an ovoid of $\mathcal{S}_z$, we know from Lemma 2.2 that for every $i \in \{1, 2, \ldots, q\}$, there exists a unique $Q^+(3,q)$-quadric G_i containing the points z, v, u_1 and u_{i+1}. As $p^* \notin u_1z \cup u_1v$, at most one of these $Q^+(3,q)$-quadrics can contain p^*.

So, without loss of generality, we may suppose that G_1 and G_2 do not contain the point p^*. Also, put $x := u_1$, $L_1 := u_1z$ and $L_2 := u_1v$. The smallest subspace of $\mathrm{PG}(5, q)$ containing G_1 and G_2 cannot be a tangent hyperplane otherwise the lines zu_2 and zu_3 would be contained in a plane of $Q^+(5, q)$ through the tangency point. So, this subspace of $\mathrm{PG}(5, q)$ intersects $Q^+(5, q)$ in a $Q(4, q)$-quadric $\mathcal{Q}$. It remains to show that both G_1 and G_2 are contained in S. Let $i \in \{1, 2\}$ be arbitrary.

As $q \geq 4$, there exists a line M in G_i disjoint from u_1z and vu_{i+1} meeting u_1v and zu_{i+1} in points noncollinear with p^* on $Q^+(5, q)$. Such a line M contains at least three points of S, namely the unique points in $M \cap u_1v$, $M \cap zu_{i+1}$ and $M \cap (p^*)^\perp$, and is therefore completely contained in S. Now, the lines of G_i meeting $u_1z \subseteq S$, $vu_{i+1} \subseteq S$ and $M \subseteq S$ are contained in S as they contain at least three points of S. As these lines cover all the points of G_i, we have that $G_i \subseteq S$. This finished the proof of our claims.

Now, let G_1, G_2, L_1, L_2, x and $\mathcal{Q}$ be as in (1), (2), (3), (4) and (5) above. We first prove that $x^\perp \cap \mathcal{Q} \subseteq S$. To that end, it suffices to show that every line L of $\mathcal{Q}$ though x distinct from L_1 and L_2 is contained in S. Put $\{x'\} := L \cap (p^*)^\perp$. In order to show that $L \subseteq S$, it suffices to prove that L contains besides x and x' one other point of S. Note that the fact that $x \notin (p^*)^\perp$ implies that $|(G_i \cap (p^*)^\perp) \setminus (L_1 \cup L_2)| = q - 1$ for every $i \in \{1, 2\}$.

Take $x_1, x_2 \in L \setminus \{x, x'\}$ and consider the set of $2q$ lines of $\mathcal{Q}$ (distinct from L) containing precisely one of the points x_1 and x_2. These $2q$ lines have no overlap in $\mathcal{Q} \setminus L$. As both $|(G_1 \cap (p^*)^\perp) \setminus (L_1 \cup L_2)|$ and $|(G_2 \cap (p^*)^\perp) \setminus (L_1 \cup L_2)|$ have size $q - 1$, one of these lines, say K, must be disjoint from $G_1 \cap (p^*)^\perp$ and $G_2 \cap (p^*)^\perp$. But then K contains at least three points of S, namely the points in $S \cap G_1$, $S \cap G_2$ and $S \cap (p^*)^\perp$, and must therefore be completely contained in S. In particular, at least one of x_1, x_2 is contained in S. It follows that L is contained in S and thus also that $x^\perp \cap \mathcal{Q} \subseteq S$.

We now also prove that every point y of $\mathcal{Q} \setminus x^\perp$ is contained in S. Take a line M through y disjoint from L_1 and L_2. As M contains at least three points of S, namely the unique points in $M \cap G_1$, $M \cap G_2$ and $M \cap x^\perp$, we see that M must be contained in S. In particular, the point y is contained in S.

We thus see that the $Q(4, q)$-quadric $\mathcal{Q}$ is contained in S. We show that $p^* \in \mathcal{Q}$. Suppose to the contrary that $p^* \notin \mathcal{Q}$. Then $|(p^*)^\perp \cup \mathcal{Q}| = |(p^*)^\perp| + |\mathcal{Q}| - |(p^*)^\perp \cap \mathcal{Q}| = 1 + q(q+1)^2 + (q+1)(q^2+1) - (q+1)^2 = 2q^3 + 2q^2 + 1 = |S| - q$. We are still missing q points of S. If L is a line of $\mathcal{Q}$ contained in $(p^*)^\perp$ and π_1, π_2 are the two planes of $Q^+(5, q)$ through L, then the fact that $L = (p^*)^\perp \cap \pi_i = \mathcal{Q} \cap \pi_i$ for every $i \in \{1, 2\}$ implies that q of these missing points must be contained in each of $\pi_1 \setminus L$ and $\pi_2 \setminus L$. This is obviously impossible.

So, $p^* \in \mathcal{Q}$. We then see that $(p^*)^\perp \cup \mathcal{Q} \subseteq S$ and as both sets have the same size, we must have equality. $\square$

9 Good quadratic sets of $Q^+(5,2)$

In this section, we describe all good quadratic sets of $Q^+(5,2)$. The description of all good quadratic sets seems most naturally achieved in a model of $Q^+(5,2)$ that immediately reveals its automorphisms. We therefore start by describing a model of $Q^+(5,2)$ on which $\mathrm{Aut}(Q^+(5,2)) \cong S_8$ has a natural action. The discussion we give here is based on Cameron [4, Section 7.2] and Neumaier [14].

Let Y be a set of size 7. We denote by S_Y and A_Y the symmetric and alternating groups on the set Y. By a *Fano collection* on Y, we mean a set $\mathcal{Y}$ consisting of seven subsets of size 3 of Y such that the point-line geometry defined by Y and $\mathcal{Y}$ is Fano plane. There are 30 Fano collections on Y which are all equivalent under the action of S_Y. Under the action of the subgroup $A_Y \leq S_Y$, the set of 30 Fano collections splits into two orbits of size 15. We call these two orbits the *two systems of Fano collections* on Y.

Let X be a set of size 8. Associated with X, there is the following point-line geometry $\mathcal{S}_X$:

- The points of $\mathcal{S}_X$ are the partitions of X in two subsets of size 4.
- The lines of $\mathcal{S}_X$ are the partitions of X in four subsets of size 2.
- A point of $\mathcal{S}_X$ is incident with a line of $\mathcal{S}_X$ if and only if the line is a refinement of the points (where both are regarded as partitions).

The point-line geometry $\mathcal{S}_X$ is then isomorphic to (the point-line system of the hyperbolic quadric) $Q^+(5,2)$ (of $\mathrm{PG}(5,2)$). Every permutation of X naturally induces a bijection of the point and line sets of $\mathcal{S}_X$ that is an automorphism of $\mathcal{S}_X$. In fact, every automorphism of $\mathcal{S}_X$ can be obtained in this way, i.e. the full automorphism group of $\mathcal{S}_X \cong Q^+(5,2)$ is isomorphic to the symmetric group S_8.

We now give a description of the planes of $\mathcal{S}_X \cong Q^+(5,2)$. Let $x^* \in X$. For every Fano collection F on $X \setminus \{x^*\}$, let $\Pi_{x^*,F}$ consist of all 7 partitions of the form $\{\{x^*\} \cup A, X \setminus (\{x^*\} \cup A)\}$ with $A \in F$. Then $\Pi_{x^*,F}$ is a plane of $\mathcal{S}_X$ and every plane of $\mathcal{S}_X$ can be obtained in this way. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are the two A_Y-orbits of Fano collections on $X \setminus \{x^*\}$, then $\{\Pi_{x^*,F} \,|\, F \in \mathcal{F}_1\}$ and $\{\Pi_{x^*,F} \,|\, F \in \mathcal{F}_2\}$ are the two families of planes of $Q^+(5,2)$.

If we take from a partition of X in two subsets of size 4, this subset containing x^* then after removing x^* we obtain a subset of size 3 of $X \setminus \{x^*\}$. In this way, we see that the collinearity graph of $Q^+(5,2)$ is isomorphic to the graph whose vertices are the subsets of size 3 of $\{1,2,\ldots,7\}$, where two such subsets are adjacent when they meet in a singleton.

Recall that a set P of points of $Q^+(5,2)$ is called a *quadratic set* if every plane π intersects P in a possibly reducible conic to π. Such (possibly reducible) conics of π are precisely the sets of points of π that have odd size. Indeed, those of size 1 are the singletons (type (S)), those of size 3 the lines (type (L)) and irreducible conics (type (C)), those of size 5 the pencils of two lines (type (P)), and the unique one of size 7 consists of all points of π (Type (W)). If we denote by $\mathcal{S}^*$ the geometry of the points and planes of $Q^+(5,2)$, then a quadratic set is nothing else than the complement of an *even set* of $\mathcal{S}^*$. This is a set of points of $\mathcal{S}^*$ meeting each line of $\mathcal{S}^*$ in an even number of points. The even sets of many geometries (including $\mathcal{S}^*$) can easily be found in a computational way based on the fact that a set of points is an even set if and only if its characteristic vector is $\mathbb{F}_2$-orthogonal with the characteristic vectors of all lines. In this way, we were able to computationally classify all quadratic sets of $Q^+(5,2)$ [8]. We found that there are up to isomorphism 131 of them. Among the 131 isomorphism classes, there turned out to be 27 that consisted of good quadratic sets.

We now give a description of all good quadratic sets of $Q^+(5,2)$ using the model of $Q^+(5,2)$ described above, i.e. we will give descriptions of these quadratic sets as sets of points of $\mathcal{S}_X$. If one is using another model of $Q^+(5,2)$ (e.g. based on a quadratic form), then using an explicit isomorphism between both models one can then also obtain a description of all good quadratic sets in this other model.

We start with some definitions. A subset of size 4 of X will shortly be called a *quadruple*. For any set $\mathcal{Q}$ of quadruples satisfying $Q_1 \cup Q_2 \neq X$ for any two $Q_1, Q_2 \in \mathcal{Q}$, we define

$$\Omega(\mathcal{Q}) := \{\{Q, X \setminus Q\} \,|\, Q \in \mathcal{Q}\}.$$

We call $\mathcal{Q}$ an *admissible collection* of quadruples of X if $Q_1 \cup Q_2 \neq X$ for any two $Q_1, Q_2 \in \mathcal{Q}$ and $\Omega(\mathcal{Q})$ is a good quadratic set of $\mathcal{S}_X \cong Q^+(5,2)$. We now give 27 constructions for admissible collections of quadruples of X. We have verified by computer that these are admissible collections of quadruples. This verification could in principle also be done by hand if one would be willing to do the effort. It is possible that a construction can provide two admissible collections Q and Q' of quadruples for which $\Omega(Q) = \Omega(Q')$. We will explicitly mention when and how this is the case. Using this information, we can then easily count the number of good quadratic sets of each type. These numbers can be found in Table 1.

(1) Let $\{A, B\}$ be a partition of X with $|A| = 3$ and $|B| = 5$. Let $\mathcal{Q}_5$ denote the set of all quadruples of the form $A \cup \{x\}$ with $x \in B$.

(2) Let $\{A, B, C\}$ be a partition of X with $|A| = |B| = 2$ and $|C| = 4$. Let $\mathcal{Q}_7$ denote the set of the quadruples $A \cup B$, and $A \cup U$ with $U \in \binom{C}{2}$. If we denote this set of quadruples by $\mathcal{Q}_7(A, B, C)$, then we have $\Omega(\mathcal{Q}_7) = \Omega(\mathcal{Q}_7')$ where $\mathcal{Q}_7' = \mathcal{Q}_7(B, A, C)$.

(3) Let $\{A, B, C\}$ be a partition of X with $|A| = 2$ and $|B| = |C| = 3$. Let $\mathcal{Q}_{9a}$ denote the set of the quadruples of the form $A \cup \{x, y\}$ with $(x, y) \in B \times C$. If we denote this set of quadruples by $\mathcal{Q}_{9a}(A, B, C)$, then we have $\Omega(\mathcal{Q}_{9a}) = \Omega(\mathcal{Q}_{9a}')$ where $\mathcal{Q}_{9a}' = \mathcal{Q}_{9a}(A, C, B)$.

(4) Let $\{A, B, C, D\}$ be a partition of X with $|A| = 1$, $|B| = |C| = 2$ and $|D| = 3$. Then $\mathcal{Q}_{9b}$ consists of the quadruples $A \cup D$, $A \cup B \cup \{x\}$ with $x \in C$, and $A \cup \{x\} \cup U$ with $x \in B$ and $U \in \binom{D}{2}$.

(5) Let $\{A, B, C, D\}$ be a partition of X with $|A| = 1$, $|B| = |C| = 2$ and $|D| = 3$. Let f be a bijection between B and C. Then $\mathcal{Q}_{11}$ consists of the quadruples $A \cup B \cup \{x\}$ with $x \in C$, $A \cup C \cup \{x\}$ with $x \in D$, and $A \cup \{x, f(x), y\}$ with $(x, y) \in B \times D$.

(6) Let $\{A, B, C\}$ be a partition of X with $|A| = 1$, $|B| = 3$ and $|C| = 4$. Then $\mathcal{Q}_{13a}$ consists of the quadruples $A \cup B$, and $A \cup U \cup \{y\}$ with $U \in \binom{B}{2}$ and $y \in C$.

(7) Let $\{A, B, C, D, E\}$ be a partition of X with $|A| = |B| = |C| = 1$, $|D| = 2$ and $|E| = 3$. Then $\mathcal{Q}_{13b}$ consists of the quadruples $A \cup E$, $A \cup \{x\} \cup U$ with $x \in B \cup C$ and $U \in \binom{E}{2}$, and $A \cup B \cup \{x, y\}$ with $(x, y) \in D \times E$.

(8) Let $\{A, B\}$ be a partition of X with $|A| = 2$ and $|B| = 6$. Let $\mathcal{Q}_{15a}$ denote the set of all quadruples of the form $A \cup U$ with $U \in \binom{B}{2}$.

(9) Let $\{A, B_1, B_2, B_3\}$ be a partition of X with $|A| = |B_1| = |B_2| = |B_3| = 2$ and let $a \in A$. Then $\mathcal{Q}_{15b}$ consists of all quadruples $A \cup B_i$ for some $i \in \{1, 2, 3\}$, and $\{a\} \cup B_i \cup \{y\}$ for some $i \in \{1, 2, 3\}$ and some $y \in (B_1 \cup B_2 \cup B_3) \setminus B_i$. If we denote this set of quadruples by $Q_{15b}(A, B_1, B_2, B_3)$, then we have $\Omega(\mathcal{Q}_{15b}) = \Omega(\mathcal{Q}'_{15b})$, where $\mathcal{Q}'_{15b} = \mathcal{Q}_{15b}(A, B_{\sigma(1)}, B_{\sigma(2)}, B_{\sigma(3)})$ with $\sigma \in S_3$.

(10) Let $\{A, B\}$ be a partition of X with $|A| = 3$ and $|B| = 5$. Let $\mathcal{B}$ be a set of five pairs of B such that the graph $(B, \mathcal{B})$ is a cycle (of length 5). The $\mathcal{Q}_{15c}$ denotes the set of all quadruples of the form $U \cup V$ with $U \in \binom{A}{2}$ and $V \in \mathcal{B}$.

(11) Let $\{A, B, C\}$ be a partition of X with $|A| = |B| = 2$ and $|C| = 4$. Let $(a, b) \in A \times B$. Then $\mathcal{Q}_{15d}$ consists of the quadruples C, $\{x\} \cup U$ with $x \in A$ and $U \in \binom{C}{3}$, and $\{a, b\} \cup U$ with $U \in \binom{C}{2}$. If we denote this set of quadruples by $\mathcal{Q}_{15d}(A, B, C, a, b)$, then we have $\Omega(\mathcal{Q}_{15d}) = \Omega(\mathcal{Q}'_{15d})$ where $\mathcal{Q}'_{15d} = \mathcal{Q}_{15d}(A, B, C, a', b')$ with $A = \{a, a'\}$ and $B = \{b, b'\}$.

(12) Let $\{A, B, C\}$ be a partition of X with $|A| = 2$ and $|B| = |C| = 3$. Let f be a bijection between B and C. Then $\mathcal{Q}_{15e}$ consists of all quadruples $B \cup \{x\}$ with $x \in C$, and $\{x, y\} \cup (C \setminus \{f(x)\})$ with $x \in B$ and $y \in A \cup (B \setminus \{x\})$.

(13) Let $\{A, B, C, D\}$ be a partition of X with $|A| = |B| = 1$, $|C| = 2$ and $|D| = 4$. Then $\mathcal{Q}_{17a}$ consists of the quadruples $A \cup B \cup C$, $A \cup U$ with $U \in \binom{D}{3}$, and $A \cup \{x\} \cup U$ with $x \in C$ and $U \in \binom{D}{2}$.

(14) Let $\{A, B, C\}$ be a partition of X with $|A| = 1$, $|B| = 3$ and $|C| = 4$. Let $\mathcal{C}$ be a set of four pairs of C such that the graph $(C, \mathcal{C})$ is a quadrangle. Then $\mathcal{Q}_{17b}$ consists of the quadruples $A \cup B$, $B \cup \{x\}$ with $x \in C$, and $U \cup V$ with $U \in \binom{B}{2}$ and $V \in \mathcal{C}$.

(15) Let $\{A, B\}$ be a partition of X with $|A| = |B| = 4$. Let $a \in A$. Then $\mathcal{Q}_{19a}$ consists of the quadruples A, and $\{a, x\} \cup U$ with $x \in A \setminus \{a\}$ and $U \in \binom{B}{2}$. If we denote this set of quadruples by $\mathcal{Q}_{19a}(A, B, a)$, then we have $\Omega(\mathcal{Q}_{19a}) = \Omega(\mathcal{Q}'_{19a})$ where $\mathcal{Q}'_{19a} = \mathcal{Q}_{19a}(A', B', a')$ where $\{A', B'\} = \{A, B\}$ and $a' \in A'$.

(16) Let $\{A, B, C, D\}$ be a partition of X with $|A| = 1$, $|B| = |C| = 2$ and $|D| = 3$. Let f be a bijection between B and C. Then $\mathcal{Q}_{19b}$ consists of the quadruples of the form $A \cup U$ with $U \in \binom{B \cup C}{3}$, $A \cup \{x, f(x), y\}$ with $x \in B$ and $y \in D$, $A \cup B \cup \{x\}$ with $x \in D$, and $A \cup \{x\} \cup U$ with $x \in C$ and $U \in \binom{D}{2}$.

(17) Let $\{A, B, C\}$ be a partition of X with $|A| = 2$ and $|B| = |C| = 3$. Then $\mathcal{Q}_{21a}$ consists of the quadruples $A \cup U$ with $U \in \binom{B}{2}$, and $\{x, y\} \cup U$ with $(x, y) \in A \times C$ and $U \in \binom{B}{2}$.

(18) Let $\{A, B, C, D, E\}$ be a partition of X with $|A| = |B| = |C| = 1$, $|D| = 2$ and $|E| = 3$. Then $\mathcal{Q}_{21b}$ consists of the quadruples of the form $A \cup E$, $A \cup \{x\} \cup U$ with $x \in B \cup C \cup D$ and $U \in \binom{E}{2}$, $A \cup B \cup C \cup \{x\}$ with $x \in D$, and $A \cup U \cup \{x\}$ with $x \in E$ and $U \in \binom{C \cup D}{2} \setminus \{D\}$.

(19) Let $\{A, B, C, D\}$ be a partition of X with $|A| = 1$, $|B| = |C| = 2$ and $|D| = 3$. Let f be a bijection between B and C. Then $\mathcal{Q}_{21c}$ consists of the quadruples of the form $A \cup D$, $A \cup \{x\} \cup U$ with $x \in B$ and $U \in \binom{D}{2}$, $A \cup B \cup \{x\}$ with $x \in C$, $A \cup B \cup \{x\}$ with $x \in D$, $A \cup C \cup \{x\}$ with $x \in D$, and $A \cup \{x, f(x), y\}$ with $(x, y) \in B \times D$.

(20) Let $\{A, B, C, D, E\}$ be a partition of X with $|A| = |B| = 1$ and $|C| = |D| = |E| = 2$. Then $\mathcal{Q}_{21d}$ consists of the quadruples of the form $A \cup B \cup U$ with $U \in \binom{D\cup E}{2} \setminus \{D\}$, $A \cup D \cup \{x\}$ with $x \in E$, $A \cup U$ with $U \in \binom{C\cup E}{3}$, and $A \cup \{x\} \cup U$ with $x \in D$ and $U \in \binom{C\cup E}{2} \setminus \{E\}$.

(21) Let $\{A, B\}$ be a partition of X with $|A| = |B| = 4$. Let $a \in A$ and let $\{B_1, B_2\}$ be a partition of B in two pairs. Then $\mathcal{Q}_{23a}$ consists of the quadruples A, $\{a\} \cup U$ with $U \in \binom{B}{3}$, $\{a, x\} \cup U$ with $x \in B$ and $U \in \binom{A\setminus\{a\}}{2}$, and $\{a, x\} \cup B_i$ with $x \in A \setminus \{a\}$ and $i \in \{1, 2\}$. If we denote this set of quadruples by $\mathcal{Q}_{23a}(A, B_1, B_2, a)$, then we have $\Omega(\mathcal{Q}_{23a}) = \Omega(\mathcal{Q}'_{23a})$ where $\mathcal{Q}'_{23a} = \mathcal{Q}_{23a}(A, B_1, B_2, a')$ with $a' \in A$.

(22) Let $\{A, B, C, D\}$ be a partition of X with $|A| = 1$, $|B| = |C| = 2$ and $|D| = 3$. Let f be a bijection between B and C. Then $\mathcal{Q}_{23b}$ consists of the quadruples of the form $A\cup B\cup\{x\}$ with $x \in C\cup D$, $A\cup\{x\}\cup U$ with $x \in B\cup C$ and $U \in \binom{D}{2}$, and $A\cup\{x, f(x), y\}$ with $x \in B$ and $y \in D$.

(23) Let $\{A, B, C\}$ be a partition of X with $|A| = 1$, $|B| = 2$ and $|C| = 5$. Then $\mathcal{Q}_{25a}$ consists of all quadruples of the form $A \cup B \cup \{x\}$ with $x \in C$, and $A \cup \{x\} \cup U$ with $x \in B$ and $U \in \binom{C}{2}$.

(24) Let $\{A, B, C\}$ be a partition of X with $|A| = 1$, $|B| = 3$ and $|C| = 4$. Let $\{C_1, C_2\}$ be a partition of C in two pairs. Then $\mathcal{Q}_{25b}$ consists of all quadruples of the form $A \cup B$, $A \cup \{x\} \cup U$ with $x \in B$ and $U \in \binom{C}{2} \setminus \{C_1, C_2\}$, and $A \cup U \cup \{x\}$ with $U \in \binom{B}{2}$ and $x \in C$.

(25) Let $\{A, B, C\}$ be a partition of X with $|A| = |B| = 2$ and $|C| = 4$. Let $a \in A$. Then $\mathcal{Q}_{27}$ consists of all quadruples of the form $A \cup U$ with $U \in \binom{B\cup C}{2}$, and $\{a, x\} \cup U$ with $x \in B$ and $U \in \binom{C}{2}$. If we denote this set of quadruples by $\mathcal{Q}_{27}(A, B, C, a)$, then we have $\Omega(\mathcal{Q}_{27}) = \Omega(\mathcal{Q}'_{27})$ where $\mathcal{Q}'_{27} = \mathcal{Q}_{27}(A, B, C, a')$ with $\{a, a'\} = A$.

(26) Let $\{A, B, C\}$ be a partition of X with $|A| = 2$ and $|B| = |C| = 3$. Let $a \in A$. Then $\mathcal{Q}_{29}$ consists of all quadruples of the form $\{a\} \cup U$ with $U \in \binom{B\cup C}{3}$, and $A \cup \{x, y\}$ with $(x, y) \in B \times C$. If we denote this set of quadruples by $\mathcal{Q}_{29}(A, B, C, a)$, then we have $\Omega(\mathcal{Q}_{29}) = \Omega(\mathcal{Q}'_{29})$ where $\mathcal{Q}'_{29} = \mathcal{Q}_{29}(A, B', C', a')$ with $\{B', C'\} = \{B, C\}$ and $a' \in A$.

(27) Let $x \in X$ be fixed. $\mathcal{Q}_{35}$ consists of all quadruples containing x. If we denote this set of quadruples by $\mathcal{Q}_{35}(x)$, then we have $\Omega(\mathcal{Q}_{35}) = \Omega(\mathcal{Q}'_{35})$ where $\mathcal{Q}'_{35} = \mathcal{Q}_{35}(x')$ with $x' \in X$.

The complete classification of all good quadratic sets of $\mathcal{S}_X \cong Q^+(5, 2)$ can now be found in the following theorem.

Theorem 9.1. *Up to isomorphism, $\mathcal{S}_X$ has 27 good quadratic sets. These quadratic sets are the sets $\Omega(\mathcal{Q})$, where $\mathcal{Q}$ is one of the 27 above-defined sets of quadruples.*

Properties for these 27 good quadratic sets can be found in Table 1. In this table, N denotes the size of the isomorphism class to which the good quadratic set $\Omega(\mathcal{Q})$ belongs. The stabilizer of $\Omega(\mathcal{Q})$ in the full automorphism group of $\mathcal{S}_X \cong Q^+(5, 2)$ is also mentioned along with the number of orbits this stabilizer has on $\Omega(\mathcal{Q})$ and on the complement of $\Omega(\mathcal{Q})$. If the numbers of these orbits are respectively equal to O_1 and O_2, then we write

$O_1 + O_2$ in the column "Orbits". The number of plane intersections of type (T) $\in \{(\mathrm{S}), (\mathrm{L}), (\mathrm{C}), (\mathrm{P}), (\mathrm{W})\}$ will be denoted by the number T.

References

[1] J. Bamberg, S. Kelly, M. Law and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A* 114 (2007), 1293–1314.

[2] R. J. Blok and A. E. Brouwer. The geometry far from a residue. In: Groups and geometries (Siena, 1996), 29–38, Birkhäuser, 1998.

[3] F. Buekenhout. Ensembles quadratiques des espaces projectifs. *Math. Z.* 110 (1969), 306–318.

[4] P. J. Cameron. *Projective and polar spaces.* QMW Maths Notes 13. Queen Mary and Westfield College, School of Mathematical Sciences, London, 1992.

[5] P. J. Cameron and R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.* 46 (1982), 91–102.

[6] A. M. Cohen and E. E. Shult. Affine polar spaces. *Geom. Dedicata* 35 (1990), 43–76.

[7] B. De Bruyn. Pseudo-embeddings and pseudo-hyperplanes. *Adv. Geom.* 13 (2013), 71–95.

[8] B. De Bruyn. Computer computations for "Quadratic sets on the Klein quadric". Available online from `http://cage.ugent.be/geometry/preprints.php`.

[9] B. De Bruyn and M. Gao. Pseudo-embeddings and quadratic sets of quadrics. Communicated.

[10] B. De Bruyn, P. Pradhan, B. Sahoo and B. Sahu. A characterization of the family of secant lines to a hyperbolic quadric in $PG(3, q)$, q odd, Part II. Preprint.

[11] K. W. Drudge. *Extremal sets in projective and polar spaces.* PhD thesis, The University of Western Ontario (Canada), 1998.

[12] J. W. P. Hirschfeld. *Finite projective spaces of three dimensions.* Oxford Mathematical Monographs, Oxford University Press, 1985.

[13] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries.* Springer Monographs in Mathematics. Springer, London, 2016.

[14] A. Neumaier. Some sporadic geometries related to $PG(3, 2)$. *Arch. Math. (Basel)* 42 (1984), 89–96.

$\mathcal{Q}$	Type of $\Omega(\mathcal{Q})$	N	Stabilizer	Orbits	Plane intersections
$\mathcal{Q}_5$	S	56	$S_3 \times S_5$	1 + 1	$S = 30$
$\mathcal{Q}_7$	SL	210	$D_8 \times S_4$	2 + 2	$(S, L) = (24, 6)$
$\mathcal{Q}_{9a}$	SL	280	$C_2 \times ((S_3 \times S_3) : C_2)$	1 + 3	$(S, L) = (18, 12)$
$\mathcal{Q}_{9b}$	SC	1680	$C_2 \times C_2 \times S_3$	$3 + 5$	$(S, C) = (18, 12)$
$\mathcal{Q}_{11}$	SC	3360	D_{12}	3 + 6	$(S, C) = (12, 18)$
$\mathcal{Q}_{13a}$	SC	280	$S_4 \times S_3$	2 + 2	$(S, C) = (6, 24)$
$\mathcal{Q}_{13b}$	SC	3360	D_{12}	4 + 7	$(S, C) = (6, 24)$
$\mathcal{Q}_{15a}$	L	28	$C_2 \times S_6$	1 + 1	$L = 30$
$\mathcal{Q}_{15b}$	LC	420	$C_2 \times C_2 \times S_4$	2 + 2	$(L, C) = (14, 16)$
$\mathcal{Q}_{15c}$	C	672	$S_3 \times D_{10}$	1 + 2	$C = 30$
$\mathcal{Q}_{15d}$	LC	840	$C_2 \times S_4$	3 + 3	$(L, C) = (6, 24)$
$\mathcal{Q}_{15e}$	LC	3360	D_{12}	3 + 4	$(L, C) = (4, 26)$
$\mathcal{Q}_{17a}$	CP	840	$C_2 \times S_4$	3 + 3	$(C, P) = (24, 6)$
$\mathcal{Q}_{17b}$	CP	840	$D_8 \times S_3$	3 + 2	$(C, P) = (24, 6)$
$\mathcal{Q}_{19a}$	LW	35	$(S_4 \times S_4) : C_2$	2 + 1	$(L, A) = (24, 6)$
$\mathcal{Q}_{19b}$	CP	3360	D_{12}	5 + 4	$(C, P) = (18, 12)$
$\mathcal{Q}_{21a}$	CP	560	$C_2 \times S_3 \times S_3$	2 + 3	$(C, P) = (12, 18)$
$\mathcal{Q}_{21b}$	CP	3360	D_{12}	6 + 5	$(C, P) = (12, 18)$
$\mathcal{Q}_{21c}$	CP	3360	D_{12}	6 + 3	$(C, P) = (12, 18)$
$\mathcal{Q}_{21d}$	CP	5040	$C_2 \times C_2 \times C_2$	7 + 6	$(C, P) = (12, 18)$
$\mathcal{Q}_{23a}$	LP	210	$D_8 \times S_4$	3 + 1	$(L, P) = (6, 24)$
$\mathcal{Q}_{23b}$	CP	3360	D_{12}	5 + 4	$(C, P) = (6, 24)$
$\mathcal{Q}_{25a}$	P	168	$C_2 \times S_5$	2 + 1	$P = 30$
$\mathcal{Q}_{25b}$	P	840	$D_8 \times S_3$	3 + 2	$P = 30$
$\mathcal{Q}_{27}$	PW	420	$C_2 \times C_2 \times S_4$	4 + 1	$(P, A) = (24, 6)$
$\mathcal{Q}_{29}$	PW	280	$C_2 \times ((S_3 \times S_3) : C_2)$	3 + 1	$(P, A) = (18, 12)$
$\mathcal{Q}_{35}$	W	1	S_8	1 + 0	A = 30

Table 1: The good quadratic sets of $Q^+(5, 2)$

[15] P. Pradhan and B. Sahu. A characterization of the family of secant lines to a hyperbolic quadric in $PG(3, q)$, q odd. *Discrete Math.* 343 (2020), 112044, 8 pp.

[16] *Sage Mathematics Software (Version 6.3)*, The Sage Developers, 2014, `http://www.sagemath.org`.

[17] I. R. Shafarevich and A. O. Remizov. *Linear algebra and geometry.* Springer, Heidelberg, 2013.

[18] E. E. Shult and J. A. Thas. m-systems of polar spaces. *J. Combin. Theory Ser. A* 68 (1994), 184–204.

[19] E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[20] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.5, 2014. (http://www.gap-system.org)

[21] J. Ueberberg. *Foundations of incidence geometry. Projective and polar spaces.* Springer Monographs in Mathematics. Springer, Heidelberg, 2011.

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