

Pseudo-embeddings and quadratic sets of quadrics

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Abstract

A quadratic set of a nonsingular quadric Q of Witt index at least three is defined as a set of points intersecting each subspace of Q in a possibly reducible quadric of that subspace. By using the theory of pseudo-embeddings and pseudo-hyperplanes, we show that if Q is one of the quadrics $Q^+(5, 2)$, $Q(6, 2)$, $Q^-(7, 2)$, then the quadratic sets of Q are precisely the intersections of Q with the quadrics of the ambient projective space of Q . In order to achieve this goal, we will determine the universal pseudo-embedding of the geometry of the points and planes of Q .

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1 Introduction

In [4], Buekenhout defined the notion of a quadratic set in a projective space, see also [16, Chapter 5]. These are sets of points in projective spaces that satisfy similar structural properties as quadrics. In [8], the notion of a quadratic set of a nonsingular quadric Q is defined as a set of points that satisfies similar structural properties as the intersections of Q with the quadrics of the ambient projective space of Q . Specifically, a quadratic set of a nonsingular quadric Q of Witt index at least three is defined as a set of points intersecting each subspace of Q in a possibly reducible quadric of that subspace.

The authors of [13] studied line sets in $\text{PG}(3, q)$ that satisfy a list of axioms. Their main theorem states that for $q \geq 7$ each such line set is either the set of secant lines with respect to a hyperbolic quadric or belongs to a hypothetical family of line sets. The question whether this hypothetical family of line sets is nonempty was left open in [13]. Further investigations showed that these line sets are related to quadratic sets of the Klein quadric, see [10]. This fact urged the first author of the present paper to initiate the study of quadratic sets of the Klein quadric in [8, 9]. In fact, one of the several families of quadratic sets described in [9] will explicitly be used in [10] to provide examples of line sets in the hypothetical family.

Also the results of the present paper arose as a by-product of these investigations. We obtain here a classification of all quadratic sets of the nonsingular quadrics $Q^+(5, 2)$, $Q(6, 2)$ and $Q^-(7, 2)$. Using the theory of pseudo-embeddings and pseudo-hyperplanes developed in [6], we show that each such quadratic set is “standard”, i.e. obtained by intersecting the considered quadric with another quadric of the ambient projective space. In the theory of quadratic sets, it is often the case that a certain argument fails to work for the smallest value(s) of the prime power q . In this regard, it can be important to have a separate classification available for small values of q . In fact, the results of the current paper will explicitly be used in [8] to show that a certain result remains true for $q = 2$.

As already alluded above, our classification of the quadratic sets of the quadrics $Q^+(5, 2)$, $Q(6, 2)$ and $Q^-(7, 2)$ requires that we study the pseudo-embeddings of some geometries related to these quadrics. In particular, we will determine the universal pseudo-embeddings of the geometries of the points and planes of these quadrics. We therefore start by defining the notion of pseudo-embedding of a general point-line geometry.

Consider a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ having the property that the number of points on each line is finite and at least 3. For such a geometry, we can define the notion of a *pseudo-embedding*. This is a map ϵ from \mathcal{P} to the point set of a projective space $\text{PG}(V)$ defined by a vector space V over $\mathbb{F}_2 = \{0, 1\}$ for which the following properties are satisfied:

- (PS1) the image $\epsilon(\mathcal{P})$ of ϵ generates the whole of $\text{PG}(V)$;
- (PS2) ϵ maps in a bijective way the point set of each line L of \mathcal{S} to a frame of a subspace Σ_L of $\text{PG}(V)$.

With a *frame* of a projective space of dimension n , we mean a set of $n + 2$ points no $n + 1$ of which are contained in a hyperplane. A pseudo-embedding ϵ as above will shortly be denoted by $\epsilon : \mathcal{S} \rightarrow \text{PG}(V)$. Two pseudo-embeddings $\epsilon_1 : \mathcal{S} \rightarrow \text{PG}(V_1)$ and $\epsilon_2 : \mathcal{S} \rightarrow \text{PG}(V_2)$ of the same point-line geometry \mathcal{S} are called *isomorphic* if there exists an isomorphism θ from $\text{PG}(V_1)$ to $\text{PG}(V_2)$ such that $\epsilon_2 = \theta \circ \epsilon_1$.

Suppose $\epsilon : \mathcal{S} \rightarrow \text{PG}(V)$ is a pseudo-embedding of $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ and α is a subspace of $\text{PG}(V)$ disjoint from $\epsilon(\mathcal{P})$ and all subspaces Σ_L with $L \in \mathcal{L}$. Then the map $x \mapsto \langle \alpha, \epsilon(x) \rangle$ defines a pseudo-embedding of \mathcal{S} into the quotient projective space $\text{PG}(V)/\alpha$ whose points are those subspaces of $\text{PG}(V)$ that contain α as a hyperplane. We then call ϵ/α a *quotient* of ϵ . We write that $\epsilon_1 \geq \epsilon_2$ for two pseudo-embeddings ϵ_1 and ϵ_2 of \mathcal{S} if ϵ_2 is isomorphic to a quotient of ϵ_1 .

If $\tilde{\epsilon}$ is a pseudo-embedding of \mathcal{S} with the property that $\tilde{\epsilon} \geq \epsilon$ for any other pseudo-embedding ϵ of \mathcal{S} , then $\tilde{\epsilon}$ is called *universal*. If \mathcal{S} has pseudo-embeddings, then it also has a universal pseudo-embedding, which is moreover unique, up to isomorphisms. The vector dimension $\text{er}(\mathcal{S})$ of the universal pseudo-embedding is called the *pseudo-embedding rank*. In case $|\mathcal{P}| < \infty$, we have $\text{er}(\mathcal{S}) = |\mathcal{P}| - \dim(C)$ where C is the binary code of length $|\mathcal{P}|$ generated by the characteristic vectors of the lines of \mathcal{S} . This gives a link between coding theory and the theory of pseudo-embeddings. Pseudo-embeddings were introduced in [6],

and we refer to this paper for more background information, in particular for proofs of the above facts.

The geometries under consideration in this paper are related to nonsingular quadrics of Witt index 3 in finite projective spaces over \mathbb{F}_2 . There are three types of such quadrics, namely $Q^+(5, 2)$ in $\text{PG}(5, 2)$, $Q(6, 2)$ in $\text{PG}(6, 2)$ and $Q^-(7, 2)$ in $\text{PG}(7, 2)$. The maximal subspaces of the projective space contained in each of these quadrics are planes which are also called *generators*. The generators of $Q^+(5, 2)$ can be partitioned into two isomorphic families, with two generators belonging to the same family if they intersect in a subspace of even dimension. If Q is one of the above quadrics, then \mathcal{G}_Q denotes the set of generators of Q . For every $\mathcal{G} \subseteq \mathcal{G}_Q$, we define the point-line geometry $\mathcal{S}_{Q, \mathcal{G}}$ as $(Q, \mathcal{G}, \text{I})$, where $\text{I} \subseteq Q \times \mathcal{G}$ is the natural incidence relation defined by containment. We will prove here the following.

Theorem 1.1. *Let Q be one of the quadrics $Q^+(5, 2)$, $Q(6, 2)$, $Q^-(7, 2)$, and let \mathcal{G} be the set of all generators of Q or one family of generators if $Q = Q^+(5, 2)$. Then the geometry $\mathcal{S} = \mathcal{S}_{Q, \mathcal{G}}$ has pseudo-embeddings. Moreover, the pseudo-embedding rank of \mathcal{S} is equal to 20 if $Q = Q^+(5, 2)$ and \mathcal{G} is the set \mathcal{G}_Q of all generators, equal to 24 if $Q = Q^+(5, 2)$ and \mathcal{G} is one of the two families of generators, equal to 27 if $Q = Q(6, 2)$ and equal to 35 if $Q = Q^-(7, 2)$.*

If \mathcal{S} is one of the geometries as in Theorem 1.1, then we know that the dimension of the binary code generated by the characteristic vectors of the lines of \mathcal{S} is equal to $|\mathcal{P}| - \text{er}(\mathcal{S})$ with \mathcal{P} the point set of \mathcal{S} . These dimensions are thus respectively equal $35 - 20 = 15$, $35 - 24 = 11$, $63 - 27 = 36$ and $119 - 35 = 84$. This fact is certainly known if $Q = Q^+(5, 2)$ and \mathcal{G} is one of the families of generators of Q . Indeed, every pseudo-embedding of $\text{PG}(3, 2)$ is an ordinary embedding and so $\text{er}(\text{PG}(3, 2)) = 4$, implying that the incidence matrix of $\text{PG}(3, 2)$ has \mathbb{F}_2 -rank $15 - 4 = 11$. Using the Klein correspondence [2, 14], we see that this is also the \mathbb{F}_2 -rank of the incidence matrix of $\mathcal{S}_{Q, \mathcal{G}}$.

Theorem 1.1 will be proved by means of the notion of pseudo-generating set. Again under the assumption that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ is a point-line geometry with the property that the number of points on each line is finite and at least 3, a *pseudo-subspace* of \mathcal{S} is defined as a set X of points having the property that no line intersects $\mathcal{P} \setminus X$ in a singleton. Note that the whole point set \mathcal{P} is always an example of a pseudo-subspace. Given a set X of points of \mathcal{S} , the intersection $[X]$ of all pseudo-subspaces containing X is again a pseudo-subspace, obviously equal to the smallest pseudo-subspace that contains X . We call $[X]$ the *pseudo-subspace generated by X* . If $[X] = \mathcal{P}$, then X is called a *pseudo-generating set*. The smallest size $\text{gr}(\mathcal{S})$ of a pseudo-generating set of \mathcal{S} is called the *pseudo-generating rank*. By [6], we know that if \mathcal{S} has pseudo-embeddings, then $\text{er}(\mathcal{S}) \leq \text{gr}(\mathcal{S})$. This fact is often an important tool for determining both $\text{er}(\mathcal{S})$ and $\text{gr}(\mathcal{S})$, see e.g. Proposition 2.1. It turns out that for many point-line geometries \mathcal{S} , the numbers $\text{er}(\mathcal{S})$ and $\text{gr}(\mathcal{S})$ are equal. We will prove that this is again the case for the point-line geometries under consideration here.

Theorem 1.2. *Let Q be one of the quadrics $Q^+(5, 2)$, $Q(6, 2)$ and $Q^-(7, 2)$, and let \mathcal{G} be the set of all generators of Q or the generators of one family if $Q = Q^+(5, 2)$. Then the pseudo-embedding and pseudo-generating ranks of $\mathcal{S}_{Q, \mathcal{G}}$ are equal.*

In this paper, we also give explicit descriptions of the universal pseudo-embeddings. Let $V(8, 2)$ be an 8-dimensional vector space over \mathbb{F}_2 for which $\text{PG}(7, 2)$ is the associated projective space. We choose an ordered basis $(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_8)$ in $V(8, 2)$ with respect to which $Q^-(7, 2) \subseteq \text{PG}(7, 2)$ has equation $X_1X_2 + X_3X_4 + X_5X_6 + X_7^2 + X_7X_8 + X_8^2 = 0$. Here, $(X_1, X_2, \dots, X_8) = \langle X_1\bar{e}_1 + X_2\bar{e}_2 + \dots + X_8\bar{e}_8 \rangle$ denotes a generic point of $\text{PG}(7, 2)$. We denote by $\text{PG}(6, 2)$ and $\text{PG}(5, 2)$ the subspaces of $\text{PG}(7, 2)$ with respective equations $X_8 = 0$ and $X_7 = X_8 = 0$. Points (X_1, X_2, \dots, X_8) belonging to these subspaces will also be abbreviated to respectively (X_1, X_2, \dots, X_7) and (X_1, X_2, \dots, X_6) . Obviously, $\text{PG}(6, 2) \cap Q^-(7, 2)$ is the quadric $Q(6, 2)$ of $\text{PG}(6, 2)$ with equation $X_1X_2 + X_3X_4 + X_5X_6 + X_7^2 = 0$ and $\text{PG}(5, 2) \cap Q^-(7, 2)$ is the quadric $Q^+(5, 2)$ of $\text{PG}(5, 2)$ with equation $X_1X_2 + X_3X_4 + X_5X_6 = 0$. We put $Q_5 = Q^+(5, 2)$, $Q_6 = Q(6, 2)$ and $Q_7 = Q^-(7, 2)$. Let \mathcal{G}^* be the family of generators of Q_5 containing the plane with equation $X_1 = X_3 = X_5 = X_7 = X_8 = 0$.

Consider a 35-dimensional vector space V_{35} over \mathbb{F}_2 with a basis consisting of the vectors \bar{g}_i and \bar{h}_{jk} with $i, j, k \in \{1, 2, \dots, 8\}$ such that $j < k$ and $(j, k) \neq (1, 2)$. The base elements \bar{g}_i and \bar{h}_{jk} with $i, j, k \leq 7$ define a 27-dimensional vector subspace V_{27} of V_{35} and those with $i, j, k \leq 6$ a 20-dimensional subspace V_{20} . Let V_{24} be a 24-dimensional vector space generated by V_{20} and four additional vectors which we will denote by \bar{f}_{135} , \bar{f}_{146} , \bar{f}_{236} and \bar{f}_{245} .

In the sequel, we denote by Σ^n with $n \in \{6, 7, 8\}$ the summation ranging over all $j, k \in \{1, 2, \dots, n\}$ with $j < k$ and $(j, k) \neq (1, 2)$. We denote by Σ^* the summation ranging over all $(i, j, k) \in \{(1, 3, 5), (1, 4, 6), (2, 3, 6), (2, 4, 5)\}$.

If $n \in \{6, 7, 8\}$ and $l = \frac{(n+2)(n-1)}{2}$, then $\epsilon_{n,l}$ is the map from the point set of Q_{n-1} to the point set of $\text{PG}(V_l)$ sending the point (X_1, X_2, \dots, X_n) of Q_{n-1} to the point $\langle \sum_{i=1}^n X_i\bar{g}_i + \Sigma^n X_jX_k\bar{h}_{jk} \rangle$ of $\text{PG}(V_l)$. We will prove the following.

Theorem 1.3. *Let $n \in \{6, 7, 8\}$ and $l = \frac{(n+2)(n-1)}{2}$. Then $\epsilon_{n,l}$ is a pseudo-embedding of $\mathcal{S}_{Q_{n-1}, \mathcal{G}_{Q_{n-1}}}$ which is moreover universal.*

Let $\epsilon_{6,24}^*$ be the map from the point set of Q_5 to the point set of $\text{PG}(V_{24})$ sending the point (X_1, X_2, \dots, X_6) of Q_5 to the point $\langle \sum_{i=1}^6 X_i\bar{g}_i + \sum^6 X_jX_k\bar{h}_{jk} + \Sigma^* X_iX_jX_k\bar{f}_{ijk} \rangle$ of $\text{PG}(V_{24})$. We will show the following.

Theorem 1.4. *The map $\epsilon_{6,24}^*$ is a pseudo-embedding of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ which is moreover universal.*

In [7], we considered the point-line geometry \mathcal{S}_n , $n \geq 2$, whose points and lines are the points and planes of the projective plane $\text{PG}(n, 2)$, with incidence being containment, and showed that \mathcal{S}_n has pseudo-embeddings. From the description of the universal pseudo-embedding of \mathcal{S}_n , we will derive the following.

Theorem 1.5. *The sets of points of $\text{PG}(n, 2)$, $n \geq 2$, that intersect each plane of $\text{PG}(n, 2)$ in a possibly reducible conic of that plane are precisely the possibly reducible quadrics of $\text{PG}(n, 2)$.*

Using the description of the universal pseudo-embedding of the geometry $\mathcal{S}_{Q,\mathcal{G}}$ with $Q \in \{Q^+(5, 2), Q(6, 2), Q^-(7, 2)\}$ and $\mathcal{G} = \mathcal{G}_Q$, we derive the following.

Theorem 1.6. *Let Q be one of the quadrics $Q^+(5, 2)$, $Q(6, 2)$ or $Q^-(7, 2)$. Then the sets of points of Q intersecting each plane π of Q in a possibly reducible conic of π are precisely the intersections of Q with the possibly reducible quadrics of its ambient projective space.*

For nonsingular quadrics Q of Witt index 3, the quadratic sets of Q are precisely the sets of points of Q that intersect each plane of Q in a possibly reducible conic. Theorem 1.6 can thus be rephrased as follows.

Corollary 1.7. *Let Q be one of the quadrics $Q^+(5, 2)$, $Q(6, 2)$ or $Q^-(7, 2)$. Then the quadratic sets of Q are precisely the intersections of Q with the quadrics of its ambient projective space.*

It is interesting to note that the conclusion of Corollary 1.7 is not valid for the quadrics $Q^+(5, q)$ with $q \geq 3$. Indeed, we show in [8] that each of these quadrics has quadratic sets that do not arise as intersection of $Q^+(5, q)$ with another quadric.

The conclusion of Theorem 1.6 is no longer valid for $Q = Q^+(5, 2)$ if we restrict to the planes π of one family \mathcal{G}_1 of generators of $Q^+(5, 2)$. Indeed, each $\pi_1 \in \mathcal{G}_1$ meets each $\pi \in \mathcal{G}_1$ in either a point or the whole of π , and so in a reducible conic of π . A plane π_1 of \mathcal{G}_1 however cannot be obtained by intersecting $Q^+(5, 2)$ with a quadric of $\text{PG}(5, 2)$, as otherwise every plane of $Q^+(5, 2)$ would have to contain a point of π_1 , in particular those planes of $Q^+(5, 2)$ that are disjoint from π_1 , an obvious impossibility.

At the very end of the paper, we will see that the conclusion of Theorem 1.5 is also no longer valid for $\text{PG}(5, 2)$ if we restrict to those planes that are totally isotropic with respect to a given symplectic polarity of $\text{PG}(5, 2)$.

Theorems 1.5 and 1.6 are examples of local characterization results for which many examples in finite geometry do exist. Such characterization theorems basically state that a geometrical object satisfies Property X as soon as certain local substructures satisfy a number of properties that are consistent with this Property X. One of the most known results in this direction is due to Barlotti [1] and Panella [12] and states that an ovoid of $\text{PG}(3, q)$ is a(n elliptic) quadric if and only if all plane intersections are possibly reducible conics. In fact, by Brown [3] we know that such an ovoid is an elliptic quadric as soon as there is one plane intersection that is an irreducible conic.

2 Preliminaries

Throughout this section, $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ is a point-line geometry having the property that the number of points on each line is finite and at least 3. The following proposition is precisely Theorem 1.5 of [6] and is often useful for determining the pseudo-generating rank and universal pseudo-embedding of a given point-line geometry.

Proposition 2.1 ([6]). *If \mathcal{S} has pseudo-embeddings, then $\text{er}(\mathcal{S}) \leq \text{gr}(\mathcal{S})$. Moreover, if there exists a pseudo-embedding $\tilde{\epsilon} : \mathcal{S} \rightarrow \text{PG}(\tilde{V})$ and a pseudo-generating set X of \mathcal{S} such that $|X| = \dim(\tilde{V}) < \infty$, then the pseudo-embedding and pseudo-generating ranks of \mathcal{S} are equal to $\dim(\tilde{V})$ and $\tilde{\epsilon}$ is isomorphic to the universal pseudo-embedding of \mathcal{S} .*

A set X of points of \mathcal{S} distinct from \mathcal{P} is called a *pseudo-hyperplane* if $\mathcal{P} \setminus X$ intersects each line of \mathcal{S} in an even number of points. The complement of the symmetric difference of two distinct pseudo-hyperplanes is again a pseudo-hyperplane.

Suppose $\epsilon : \mathcal{S} \rightarrow \text{PG}(V)$ is a pseudo-embedding of \mathcal{S} . Then $H_\pi := \epsilon^{-1}(\pi \cap \epsilon(\mathcal{P}))$ is a pseudo-hyperplane of \mathcal{S} for every hyperplane π of $\text{PG}(V)$. We will say that H_π *arises from the pseudo-embedding ϵ* . We denote by \mathcal{H}_ϵ the set of all pseudo-hyperplanes of \mathcal{S} arising from ϵ . We also note that the correspondence $\pi \leftrightarrow H_\pi$ between the hyperplanes of $\text{PG}(V)$ and the pseudo-hyperplanes of \mathcal{H}_ϵ is bijective. The case where ϵ is universal is interesting as the following can be shown.

Proposition 2.2 ([6]). *Suppose \mathcal{S} has pseudo-embeddings, and denote by $\tilde{\epsilon} : \mathcal{S} \rightarrow \text{PG}(\tilde{V})$ the universal pseudo-embedding of \mathcal{S} . Then $\mathcal{H}_{\tilde{\epsilon}}$ is the set of all pseudo-hyperplanes of \mathcal{S} . Moreover the correspondence $\pi \leftrightarrow H_\pi = \tilde{\epsilon}^{-1}(\tilde{\epsilon}(\mathcal{P}) \cap \pi)$ between the hyperplanes π of $\text{PG}(\tilde{V})$ and the pseudo-hyperplanes H_π of \mathcal{S} is bijective.*

3 Pseudo-generating sets of the geometries

Proposition 3.1. *Let \mathcal{G} be one of the families of generators of the quadric $Q = Q^+(5, 2)$. Then the geometry $\mathcal{S}_{Q, \mathcal{G}}$ has a pseudo-generating set of size 24.*

Proof. Let x_1 and x_2 be two noncollinear points of $Q^+(5, 2)$. Then $x_1^\perp \cap x_2^\perp$ is a quadric of type $Q^+(3, 2)$. Put $X := Q^+(5, 2) \setminus ((x_1^\perp \cap x_2^\perp) \cup \{x_1, x_2\})$. Then X is a set of size $35 - 9 - 2 = 24$. We show that X is a pseudo-generating set of $\mathcal{S}_{Q, \mathcal{G}}$. Denote by $[X]$ the smallest pseudo-subspace of $\mathcal{S}_{Q, \mathcal{G}}$ containing X .

Let $K_1, K_2, K_3, L_1, L_2, L_3$ be the six lines contained in $x_1^\perp \cap x_2^\perp$ such that K_1, K_2, K_3 are mutually disjoint, as well as L_1, L_2, L_3 . For every $i \in \{1, 2, 3\}$, let α_i denote the unique plane of \mathcal{G} containing K_i , and let β_i denote the unique plane of \mathcal{G} containing L_i . The planes $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ mutually intersect in singletons. Note that for all $i, j \in \{1, 2, 3\}$, the planes α_i and β_j intersect in the singleton $K_i \cap L_j \subseteq x_1^\perp \cap x_2^\perp$. If we denote by k the intersection point of the planes α_1 and α_2 , then as $K_1 \cup K_2 \subseteq k^\perp$, we have $(x_1^\perp \cap x_2^\perp) \subseteq k^\perp$ and so $k \in \{x_1, x_2\}$ and $\alpha_i = \langle k, K_i \rangle$ for every $i \in \{1, 2, 3\}$. In a similar way, one proves that there exists an $l \in \{x_1, x_2\}$ such that $\beta_i = \langle l, L_i \rangle$ for every $i \in \{1, 2, 3\}$. As $\alpha_i \cap \beta_j = K_i \cap L_j \subseteq x_1^\perp \cap x_2^\perp$ for all $i, j \in \{1, 2, 3\}$, we have $k \neq l$ and so $\{k, l\} = \{x_1, x_2\}$.

We first prove that every point p of $x_1^\perp \cap x_2^\perp$ belongs to $[X]$. Suppose this point p is the unique intersection point of the lines K_i and L_j . Denote by γ the unique element of \mathcal{G} through p not containing the line K_i nor the line L_j . The planes α_i, β_j and γ mutually

intersect in the point p , and so the points x_1 and x_2 do not belong to γ . This implies that every point of $\gamma \setminus \{p\}$ belongs to X and thus that $p \in [X]$.

We thus see that $[X]$ contains all points of $Q^+(5, 2)$ with the possible exceptions of x_1 and x_2 .

Now, take a plane δ of \mathcal{G} through x_i , $i \in \{1, 2\}$. This plane δ does not contain x_{3-i} and so $\delta \setminus \{x_i\} \subseteq [X]$. This implies that also $x_i \in [X]$.

We thus see that $[X] = Q^+(5, 2)$, i.e. X is a pseudo-generating set. \square

Proposition 3.2. *If $Q = Q^+(5, 2)$, then the geometry $\mathcal{S}_{Q, \mathcal{G}_Q}$ has a pseudo-generating set of size 20.*

Proof. Let k and l be two noncollinear points of $Q^+(5, 2)$. Then $k^\perp \cap l^\perp$ is a quadric of type $Q^+(3, 2)$. Let K and L be two intersecting lines of $k^\perp \cap l^\perp$. Put $K \cap L = \{u\}$, $K = \{u, v_1, v_2\}$ and $L = \{u, v_3, v_4\}$. Put $\{y_1\} := kv_1 \setminus \{k, v_1\}$, $\{y_2\} := kv_2 \setminus \{k, v_2\}$, $\{y_3\} := lv_3 \setminus \{l, v_3\}$ and $\{y_4\} := lv_4 \setminus \{l, v_4\}$. Also, put $X := Q^+(5, 2) \setminus ((k^\perp \cap l^\perp) \cup \{k, l, y_1, y_2, y_3, y_4\})$. Then X is a set of size 20. We prove that X is a pseudo-generating set of $\mathcal{S}_{Q, \mathcal{G}}$. Denote by $[X]$ the smallest pseudo-subspace of $\mathcal{S}_{Q, \mathcal{G}_Q}$ containing X .

Recall from Proposition 3.1 that the planes $\langle k, K \rangle$ and $\langle l, L \rangle$ belong to the same family \mathcal{G} of generators of $Q^+(5, 2)$. If we take the unique generator π of \mathcal{G} through u distinct from $\langle k, K \rangle$ and $\langle l, L \rangle$, then π has six points in common with X and so the seventh point u belongs to $[X]$. Now, put $k' := ku \setminus \{k, u\}$ and $l' := lu \setminus \{l, u\}$. Then $l'v_1, l'v_2, k'v_3$ and $k'v_4$ are lines of $Q^+(5, 2)$. The unique element of \mathcal{G} containing these lines intersect $\langle k, K \rangle$ and $\langle l, L \rangle$ in singletons and contain six points of X . So, also the remaining points in these planes belong to $[X]$, i.e. $v_1, v_2, v_3, v_4 \in [X]$.

Now, the plane $\langle l, K \rangle$ intersects $\langle k, K \rangle$ in the line $\{u, v_1, v_2\}$, $\langle l, L \rangle$ in the line $\{l, l', u\}$ and $k^\perp \cap l^\perp$ in $\{u, v_1, v_2\}$. As $\{u, v_1, v_2, l'\} \subseteq [X]$, the plane $\langle l, K \rangle$ already contains six points of $[X]$. Hence, also the seventh point l belongs to $[X]$.

In a similar way, one can show that the plane $\langle k, L \rangle$ already contains six points of $[X]$ and that the seventh point k therefore also belongs to $[X]$.

Now, consider the plane of $Q^+(5, 2)$ through $\{k', y_1, v_2\}$ not belonging to \mathcal{G} . This plane intersects $\langle l, L \rangle \in \mathcal{G}$ in the empty set and $k^\perp \cap l^\perp$ in $\{v_2\}$. So, this plane already contains six points of $[X]$, implying that also the seventh point y_1 belongs to $[X]$.

In a similar way, by considering the planes through $\{k', v_1, y_2\}$, $\{l', y_3, v_4\}$, $\{l', y_4, v_3\}$ not belonging to \mathcal{G} , one can show that the points y_2, y_3 and y_4 must belong to $[X]$.

We have thus already shown that all points of $Q^+(5, 2)$ belong to $[X]$, with the possible exception of the four points in $(k^\perp \cap l^\perp) \setminus \{u, v_1, v_2, v_3, v_4\}$. If w is one of these four points, then we can take a plane through w not containing any line of $k^\perp \cap l^\perp$. This plane already contains six points of $[X]$ and so also the seventh point w belongs to $[X]$.

We conclude that $[X] = Q^+(5, 2)$. \square

In order to determine suitable pseudo-generating sets for the geometries $\mathcal{S}_{Q, \mathcal{G}_Q}$, where $Q \in \{Q(6, 2), Q^-(7, 2)\}$, we need some preparatory work.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ is a point-line geometry. A *subspace* of \mathcal{S} is then defined as a set X of points of \mathcal{S} having the property that a line of \mathcal{S} has all its points in X as soon as

it has at least two of its points in X . A subspace X is called *singular* if any two distinct points of X are collinear. A subspace X is called a *hyperplane* if $X \neq \mathcal{P}$ and if every line has at least one of its points in X . The geometry \mathcal{S} is called a *polar space* (in the sense of Buekenhout and Shult [5]) if the following three properties are satisfied:

- (1) for every point x and every line L not incident with x , either one or all points of L are collinear with x ;
- (2) there exists no point x that is collinear with all the remaining points of \mathcal{S} ;
- (3) every strictly ascending chain $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_k$ of singular subspaces of \mathcal{S} has finite length.

The point-line geometry defined by the points and lines of any nonsingular quadric in a projective space is an example of a polar space. The following proposition is well-known. For a proof, see e.g. Shult [15, Lemma 5.2].

Proposition 3.3. *Suppose \mathcal{S} is a polar space having lines and that all these lines contain at least three points. If H is a hyperplane of \mathcal{S} , then the subgraph of the collinearity graph induced on the complement of H is connected.*

Lemma 3.4. *Let $Q \in \{Q(6, 2), Q^-(7, 2)\}$, i.e. $Q = Q_n$ with $n \in \{6, 7\}$. Let Π be the hyperplane of $\text{PG}(n, 2)$ intersecting Q in the quadric $Q_{n-1} \in \{Q^+(5, 2), Q(6, 2)\}$, let $p \in Q_{n-1}$, let Π_p denote the hyperplane of $\text{PG}(n, 2)$ that is tangent to Q at the point p , and let x be a point of $Q \setminus Q_{n-1}$ not collinear with p on Q . Then $Q_{n-1} \cup (\Pi_p \cap Q_n) \cup \{x\}$ is a pseudo-generating set of the geometry $\mathcal{S} = \mathcal{S}_{Q, \mathcal{G}_Q}$.*

Proof. Denote by S the pseudo-subspace generated by $Q_{n-1} \cup (\Pi_p \cap Q_n) \cup \{x\}$.

The subspace $\Pi \cap \Pi_p$, which has co-dimension 2 in $\text{PG}(n, 2)$, is contained in three hyperplanes, namely Π , Π_p and a third one which we will denote by Π' .

The intersection $\Pi \cap \Pi_p \cap Q$ is a cone of type $pQ^+(3, 2)$ if $n = 6$ and a cone of type $pQ(4, 2)$ if $n = 7$. In any case, p is the only point in $\Pi \cap \Pi_p \cap Q$ that is collinear on Q with all points of $\Pi \cap \Pi_p \cap Q$.

We show that Π' is a nontangent hyperplane. Suppose to the contrary that Π' is tangent to Q in the point p' . As $\Pi' \neq \Pi_p$, we have $p' \neq p$ and so by the previous paragraph we have $p' \notin \Pi \cap \Pi_p$. The tangent hyperplane Π' in the point p' would then coincide with $\langle p', \Pi \cap \Pi_p \rangle$, implying that pp' is a line of Q and that $p' \in \Pi_p$. So, $p' \in \Pi' \cap \Pi_p = \Pi \cap \Pi_p$, a contradiction.

So, Π' is a nontangent hyperplane and $\Pi \cap \Pi_p$ is a hyperplane of Π . In case $n = 6$, $\Pi \cap \Pi_p \cap Q$ is a cone of type $pQ^+(3, 2)$ and so $\Pi' \cap Q$ is a quadric of type $Q^+(5, 2)$. In the case $n = 7$, $\Pi \cap \Pi_p \cap Q$ is a cone of type $pQ(4, 2)$ and $\Pi' \cap Q$ is a quadric of type $Q(6, 2)$.

Now, let Γ be the subgraph of the collinearity graph of $\Pi' \cap Q$ induced on the complement of its hyperplane $\Pi \cap \Pi_p \cap Q$. By Proposition 3.3, we then know that Γ is connected. Recall that $\Pi \cap Q = Q_{n-1}$ and $\Pi_p \cap Q$ are contained in S . In order to show that S coincides with the whole point set, we thus still need to show that every vertex of Γ belongs to S .

As the vertex x of Γ belongs to S , it suffices to prove that if u, v are two adjacent vertices of Γ such that $u \in S$, then also $v \in S$.

As $u, v \in \Pi' \setminus (\Pi \cap \Pi_p)$, the line uv of Q meets $\Pi \cap \Pi_p$ in a point w . Let π be a plane of Q through $\{u, v, w\}$ not contained in $\Pi' \cap Q$. This plane intersects $\Pi \cap Q \subseteq S$ in a line L and $\Pi_p \cap Q \subseteq S$ in another line L' . The two points of π not in $L \cup L' \subseteq S$ are the points u and v . As S is a pseudo-subspace, the fact that $u \in S$ thus implies that $v \in S$, as we needed to prove. \square

Lemma 3.5. *Let $Q \in \{Q(6, 2), Q^-(7, 2)\}$, i.e. $Q = Q_n$ with $n \in \{6, 7\}$. Let Π be the hyperplane of $\text{PG}(n, 2)$ intersecting Q in the quadric $Q_{n-1} \in \{Q^+(5, 2), Q(6, 2)\}$, let $p \in Q_{n-1}$ and let x be a point of $Q \setminus Q_{n-1}$ not collinear with p on Q . Denote by L_1, L_2, \dots, L_k the lines of Q through p not contained in Q_{n-1} , and put $L_i = \{p, x_i, y_i\}$ for every $i \in \{1, 2, \dots, k\}$. Then $k = 6$ if $n = 6$ and $k = 12$ if $n = 7$. Moreover, the set $Q_{n-1} \cup \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ is a pseudo-generating set of the geometry $\mathcal{S} = \mathcal{S}_{Q, \mathcal{G}_Q}$.*

Proof. Let Π_p denote the hyperplane that is tangent to Q in the point p . Then $\Pi_p \cap Q$ is a cone of type $pQ(4, 2)$ if $n = 6$ and a cone of type $pQ^-(5, 2)$ if $n = 7$. Also, $\Pi_p \cap Q_{n-1}$ is a cone of type $pQ^+(3, 2)$ if $n = 6$ and a cone of type $pQ(4, 2)$ if $n = 7$. As $|Q(4, 2)| - |Q^+(3, 2)| = 6$ and $|Q^-(5, 2)| - |Q(4, 2)| = 12$, we thus see that $k = 6$ if $n = 6$ and $k = 12$ if $n = 7$.

In view of Lemma 3.4, it suffices to prove that the pseudo-subspace S generated by $Q_{n-1} \cup \{x_1, x_2, \dots, x_k, y_1\}$ contains all lines L_1, L_2, \dots, L_k . We already know that this is the case for the line $L_1 = \{p, x_1, y_1\}$.

Now, consider the graph Γ on the vertex set $U = \{L_1, L_2, \dots, L_k\}$ where two distinct lines of U are adjacent whenever the plane they generate is also a plane of Q . By looking at the quotient polar space at the point p , we see that Γ is isomorphic to the subgraph of the collinearity graph of Q_1 induced on the complement of $Q_2 \subseteq Q_1$, where $(Q_1, Q_2) = (Q(4, 2), Q^+(3, 2))$ if $n = 6$ and $(Q_1, Q_2) = (Q^-(5, 2), Q(4, 2))$ if $n = 7$. As Γ is connected by Proposition 3.3, it suffices to prove that if L and L' are two adjacent vertices of Γ such that $L \subseteq S$, then also $L' \subseteq S$.

Now, the plane $\pi = \langle L, L' \rangle$ is a plane of Q intersecting Q_{n-1} in a line $L'' \subseteq S$. Note that L, L' and L'' are the three lines of π through p . As $L, L'' \subseteq S$ and $(L' \setminus \{p\}) \cap S \neq \emptyset$, the fact that S is a pseudo-subspace implies that also $L' \subseteq S$ as we needed to prove. \square

Proposition 3.6. *If $Q = Q(6, 2)$, then the geometry $\mathcal{S}_{Q, \mathcal{G}_Q}$ has a pseudo-generating set of size 27.*

Proof. Let Π denote the hyperplane of $\text{PG}(6, 2)$ for which $\Pi \cap Q(6, 2) = Q^+(5, 2)$, let $p = (1, 0, 0, 0, 0, 0, 0) \in Q^+(5, 2)$ and let Π_p denote the hyperplane of $\text{PG}(6, 2)$ that is tangent to $Q(6, 2)$ in the point p . By Lemma 3.5, there are six lines L_1, L_2, \dots, L_6 of $Q(6, 2)$ through p not contained in $Q^+(5, 2)$. For every $\{1, 2, \dots, 6\}$, we may put $L_i = \{p, x_i, y_i\}$, where $x_1 = (0, 0, 0, 1, 1, 1, 1)$, $y_1 = (1, 0, 0, 1, 1, 1, 1)$, $x_2 = (0, 0, 0, 0, 1, 1, 1)$, $y_2 = (1, 0, 0, 0, 1, 1, 1)$, $x_3 = (0, 0, 1, 0, 1, 1, 1)$, $y_3 = (1, 0, 1, 0, 1, 1, 1)$, $x_4 = (0, 0, 1, 1, 0, 0, 1)$, $y_4 = (1, 0, 1, 1, 0, 0, 1)$, $x_5 = (0, 0, 1, 1, 0, 1, 1)$, $y_5 = (1, 0, 1, 1, 0, 1, 1)$, $x_6 = (1, 0, 1, 1, 1, 0, 1)$ and $y_6 = (0, 0, 1, 1, 1, 0, 1)$. Note that $x = (0, 1, 0, 0, 1, 1, 1)$ is a point of $Q(6, 2) \setminus Q^+(5, 2)$

not collinear with p on the quadric $Q(6, 2)$. For every point z of $Q(6, 2)$, denote by z^\perp the set of points of $Q(6, 2)$ collinear with z on $Q(6, 2)$ (including z itself).

By Proposition 3.2, we know that there exists a pseudo-generating set Z of size 20 of the geometry $\mathcal{S}_{Q', \mathcal{G}_{Q'}}$ with $Q' = Q^+(5, 2)$. We now show that $S := Z \cup \{x_1, x_2, x_3, x_4, x_5, x_6, x\}$ is a pseudo-generating set (of size 27) of $\mathcal{S}_{Q, \mathcal{G}_Q}$. Denote by $[S]$ the pseudo-subspace of $\mathcal{S}_{Q, \mathcal{G}_Q}$ generated by S .

Since $Z \subseteq S$ is a pseudo-generating set of $\mathcal{S}_{Q', \mathcal{G}_{Q'}}$, we have $Q^+(5, 2) \subseteq [S]$. The fact that $[S]$ is a pseudo-subspace then implies the following.

Suppose u, v and w are points of $Q(6, 2)$ such that $\langle u, v, w \rangle$ is a generator intersecting $Q^+(5, 2)$ in the line $vw = \langle v, w \rangle$ and $\langle u, v, w \rangle \setminus (\{u\} \cup vw) \subseteq [S]$, then also $u \in [S]$.

This fact allows us to prove that the three points u_1, u_2 and y_1 of $Q(6, 2)$ are contained in $[S]$, see the following table where also the points u_1 and u_2 are defined.

u	$v, w \in u^\perp \cap Q^+(5, 2)$ with $w \in v^\perp$	$\langle u, v, w \rangle \setminus (\{u\} \cup vw)$
$u_1 := (0, 1, 1, 1, 0, 0, 1)$	$(0, 0, 1, 1, 1, 1, 0), (0, 1, 0, 0, 0, 0, 0)$	x_2, x_4, x
$u_2 := (1, 1, 0, 1, 1, 0, 1)$	$(0, 1, 1, 0, 0, 0, 0), (1, 0, 0, 1, 0, 1, 0)$	x_3, x_6, x
$y_1 = (1, 0, 0, 1, 1, 1, 1)$	$(0, 1, 0, 0, 0, 1, 0), (1, 0, 1, 0, 1, 0, 0)$	x_5, u_1, u_2

Since $[S]$ contains $Q^+(5, 2) \cup \{x_1, x_2, x_3, x_4, x_5, x_6, y_1, x\}$, we know from Lemma 3.5 that the pseudo-subspace $[S]$ coincides with $Q(6, 2)$, i.e. S is a pseudo-generating set of size 27 of $\mathcal{S}_{Q, \mathcal{G}_Q}$. \square

Proposition 3.7. *If $Q = Q^-(7, 2)$, then the geometry $\mathcal{S}_{Q, \mathcal{G}_Q}$ has a pseudo-generating set of size 35.*

Proof. Let Π denote the hyperplane of $\text{PG}(7, 2)$ for which $\Pi \cap Q^-(7, 2) = Q(6, 2)$, let $p = (1, 0, 0, 0, 0, 0, 0, 0) \in Q(6, 2)$ and let Π_p denote the hyperplane of $\text{PG}(7, 2)$ that is tangent to $Q^-(7, 2)$ in the point p . By Lemma 3.5, there are twelve lines L_1, L_2, \dots, L_{12} of $Q^-(7, 2)$ through p not contained in $Q(6, 2)$. For every $\{1, 2, \dots, 12\}$, we may put $L_i = \{p, x_i, y_i\}$, where $x_1 = (0, 0, 0, 1, 1, 1, 0, 1)$, $y_1 = (1, 0, 0, 1, 1, 1, 0, 1)$, $x_2 = (0, 0, 0, 0, 1, 1, 0, 1)$, $y_2 = (1, 0, 0, 0, 1, 1, 0, 1)$, $x_3 = (0, 0, 1, 0, 1, 1, 0, 1)$, $y_3 = (1, 0, 1, 0, 1, 1, 0, 1)$, $x_4 = (0, 0, 1, 1, 0, 0, 0, 1)$, $y_4 = (1, 0, 1, 1, 0, 0, 0, 1)$, $x_5 = (0, 0, 1, 1, 0, 1, 0, 1)$, $y_5 = (1, 0, 1, 1, 0, 1, 0, 1)$, $x_6 = (1, 0, 1, 1, 1, 0, 0, 1)$, $y_6 = (0, 0, 1, 1, 1, 0, 0, 1)$, $x_7 = (0, 0, 0, 0, 1, 1, 1, 1)$, $y_7 = (1, 0, 0, 0, 1, 1, 1, 1)$, $x_8 = (0, 0, 0, 1, 1, 1, 1, 1)$, $y_8 = (1, 0, 0, 1, 1, 1, 1, 1)$, $x_9 = (0, 0, 1, 0, 1, 1, 1, 1)$, $y_9 = (1, 0, 1, 0, 1, 1, 1, 1)$, $x_{10} = (0, 0, 1, 1, 0, 0, 1, 1)$, $y_{10} = (1, 0, 1, 1, 0, 0, 1, 1)$, $x_{11} = (1, 0, 1, 1, 1, 0, 1, 1)$, $y_{11} = (0, 0, 1, 1, 1, 0, 1, 1)$, $x_{12} = (0, 0, 1, 1, 0, 1, 1, 1)$ and $y_{12} = (1, 0, 1, 1, 0, 1, 1, 1)$. Note that $x = (0, 1, 0, 0, 1, 1, 0, 1)$ is a point of $Q^-(7, 2) \setminus Q(6, 2)$ not collinear with p on the quadric $Q^-(7, 2)$.

By Proposition 3.6, we know that there exists a pseudo-generating set Z of size 27 of the geometry $\mathcal{S}_{Q', \mathcal{G}_{Q'}}$ with $Q' = Q(6, 2)$. We now show that $S := Z \cup \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x\}$ is a pseudo-generating set (of size 35) of $\mathcal{S}_{Q, \mathcal{G}_Q}$. Denote by $[S]$ the pseudo-subspace of $\mathcal{S}_{Q, \mathcal{G}_Q}$ generated by S .

Since $Z \subseteq S$ is a pseudo-generating set of $\mathcal{S}_{Q', \mathcal{G}_{Q'}}$, we have $Q(6, 2) \subseteq [S]$. The fact that $[S]$ is a pseudo-subspace then implies the following.

Suppose u, v and w are points of $Q^-(7, 2)$ such that $\langle u, v, w \rangle$ is a generator intersecting $Q(6, 2)$ in the line $vw = \langle v, w \rangle$ and $\langle u, v, w \rangle \setminus (\{u\} \cup vw) \subseteq [S]$, then also $u \in [S]$.

This fact allows us to prove that the 18 points $u_1, u_2, \dots, u_{12}, y_1, x_8, x_9, x_{10}, x_{11}, x_{12}$ of $Q^-(7, 2)$ are contained in $[S]$, see the following table where also the points u_1, u_2, \dots, u_{12} are defined.

u	$v, w \in u^\perp \cap Q(6, 2)$ with $w \in v^\perp$	$\langle u, v, w \rangle \setminus (\{u\} \cup vw)$
$u_1 := (0, 1, 0, 1, 1, 1, 0, 1)$	$(0, 0, 0, 1, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0)$	x_1, x_2, x
$u_2 := (0, 1, 1, 0, 1, 1, 0, 1)$	$(0, 0, 1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0)$	x_2, x_3, x
$u_3 := (0, 1, 1, 1, 0, 0, 0, 1)$	$(0, 0, 1, 1, 1, 1, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0)$	x_2, x_4, x
$u_4 := (1, 1, 0, 0, 0, 1, 0, 1)$	$(0, 1, 1, 1, 1, 1, 0, 0), (1, 0, 0, 0, 1, 0, 0, 0)$	x_4, x_6, x
$u_5 := (1, 1, 0, 1, 1, 0, 0, 1)$	$(0, 1, 1, 0, 0, 0, 0, 0), (1, 0, 0, 1, 0, 1, 0, 0)$	x_3, x_6, x
$u_6 := (1, 1, 1, 0, 1, 0, 0, 1)$	$(0, 1, 0, 1, 0, 0, 0, 0), (1, 0, 1, 0, 0, 1, 0, 0)$	x_1, x_6, x
$u_7 := (0, 1, 1, 1, 0, 0, 1, 1)$	$(0, 1, 1, 1, 1, 1, 0, 0), (1, 0, 1, 1, 0, 1, 1, 0)$	x_6, x_7, u_4
$u_8 := (1, 1, 1, 0, 0, 1, 0, 1)$	$(0, 0, 1, 0, 0, 0, 0, 0), (1, 0, 0, 0, 1, 0, 0, 0)$	x, u_2, u_4
$u_9 := (1, 1, 0, 0, 0, 0, 0, 1)$	$(0, 0, 0, 0, 0, 1, 0, 0), (1, 1, 1, 1, 0, 0, 0, 0)$	x_4, x_5, u_4
$u_{10} := (1, 1, 1, 1, 1, 1, 1, 1)$	$(1, 1, 1, 1, 0, 0, 0, 0), (0, 0, 1, 1, 1, 0, 1, 0)$	x_5, x_7, u_4
$u_{11} := (1, 1, 0, 0, 1, 0, 0, 1)$	$(0, 0, 0, 1, 0, 0, 0, 0), (1, 0, 0, 0, 0, 1, 0, 0)$	x, u_1, u_5
$u_{12} := (0, 1, 1, 1, 0, 1, 1, 1)$	$(0, 0, 0, 0, 0, 1, 0, 0), (1, 0, 1, 1, 0, 0, 1, 0)$	u_4, u_7, u_9
$y_1 = (1, 0, 0, 1, 1, 1, 0, 1)$	$(0, 1, 0, 0, 0, 1, 0, 0), (1, 0, 1, 0, 1, 0, 0, 0)$	x_5, u_3, u_5
$x_8 = (0, 0, 0, 1, 1, 1, 1, 1)$	$(0, 0, 0, 1, 0, 0, 0, 0), (1, 1, 0, 0, 0, 1, 1, 0)$	x_7, u_5, u_{11}
$x_9 = (0, 0, 1, 0, 1, 1, 1, 1)$	$(0, 0, 1, 0, 0, 0, 0, 0), (1, 1, 0, 0, 0, 1, 1, 0)$	x_7, u_6, u_{11}
$x_{10} = (0, 0, 1, 1, 0, 0, 1, 1)$	$(0, 0, 1, 1, 1, 1, 0, 0), (1, 1, 0, 1, 0, 1, 1, 0)$	x_7, u_5, u_8
$x_{11} = (1, 0, 1, 1, 1, 0, 1, 1)$	$(0, 1, 0, 0, 0, 1, 0, 0), (1, 0, 0, 0, 1, 1, 1, 0)$	x_5, u_3, u_{10}
$x_{12} = (0, 0, 1, 1, 0, 1, 1, 1)$	$(0, 0, 0, 0, 0, 1, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0)$	x_{10}, u_7, u_{12}

Since $[S]$ contains $Q(6, 2) \cup \{x_1, x_2, \dots, x_{12}, y_1, x\}$, we know from Lemma 3.5 that the pseudo-subspace $[S]$ coincides with $Q^-(7, 2)$, i.e. S is a pseudo-generating set of size 35 of $\mathcal{S}_{Q, \mathcal{G}_Q}$. \square

4 Proofs of the main results

Let Q be one of the quadrics $Q_5 = Q^+(5, 2)$, $Q_6 = Q(6, 2)$ or $Q_7 = Q^-(7, 2)$. In Section 1, we constructed for every $n \in \{6, 7, 8\}$ a map $\epsilon_{n, l}$ from the point set of Q_{n-1} to the point set of the projective space $\text{PG}(V_l)$ where $l = \frac{(n+2)(n-1)}{2}$. Our first goal is to show that the image of this map generates the whole projective space $\text{PG}(V_l)$. This will be a consequence of the following lemma.

Lemma 4.1. *Let Q be one of the quadrics $Q^+(5, 2)$, $Q(6, 2)$, $Q^-(7, 2)$, and let $\text{PG}(n-1, 2)$ with $n \in \{6, 7, 8\}$ be the ambient projective space of Q . Also, let S be a set of points of $\text{PG}(n-1, 2)$ described by an equation of the form*

$$\sum_{i=1}^n a_i X_i + \sum_{j,k=1}^n b_{jk} X_j X_k = 0,$$

with all coefficients belonging to \mathbb{F}_2 . If $Q \subseteq S$, then $a_i = b_{jk} = 0$ for all $i, j, k \in \{1, 2, \dots, n\}$ with $j < k$ and $(j, k) \neq (1, 2)$.

Proof. For all $j, k \in \{1, 2, \dots, n\}$ with $j < k$ and $(j, k) \neq (1, 2)$, put $b_{kj} := b_{jk}$.

Since every point $\langle \bar{e}_i \rangle$ with $i \in \{1, 2, \dots, 6\}$ belongs to $Q \subseteq S$, we have $a_i = 0$ for every $i \in \{1, 2, \dots, 6\}$.

Since $a_j = a_k = 0$ and $\langle \bar{e}_j + \bar{e}_k \rangle \in Q \subseteq S$ for all $j, k \in \{1, 2, \dots, 6\}$ with $j < k$ and $(j, k) \notin \{(1, 2), (3, 4), (5, 6)\}$ we have that $b_{jk} = 0$ for all such values of j and k .

Since $a_1 = a_2 = \dots = a_6 = b_{13} = b_{14} = b_{23} = b_{24} = b_{15} = b_{16} = b_{25} = b_{26} = 0$, the fact that the points $\langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 \rangle$ and $\langle \bar{e}_1 + \bar{e}_2 + \bar{e}_5 + \bar{e}_6 \rangle$ belong to $Q \subseteq S$ implies that also $b_{34} = b_{56} = 0$. At this stage, we have already proved the lemma in the case $Q = Q^+(5, 2)$. So, suppose $n \geq 7$.

We show that $b_{jk} = a_k = 0$ for all $j, k \in \mathbb{N}$ with $1 \leq j \leq 6$ and $7 \leq k \leq n$. Take $(j', j'') \in \{(3, 4), (5, 6)\}$ such that j, j' and j'' are mutually distinct. As $a_j = a_{j'} = a_{j''} = b_{jj'} = b_{jj''} = b_{j'j''} = 0$, the fact that the points $\langle \bar{e}_{j'} + \bar{e}_{j''} + \bar{e}_k \rangle$ and $\langle \bar{e}_j + \bar{e}_{j'} + \bar{e}_{j''} + \bar{e}_k \rangle$ belong to $Q \subseteq S$ implies that $a_k + b_{j'k} + b_{j''k} = a_k + b_{jk} + b_{j'k} + b_{j''k} = 0$ and hence that $b_{jk} = 0$. As $1 \leq j', j'' \leq 6$, we then also know that $b_{j'k} = b_{j''k} = 0$ and so the above then also implies that $a_k = 0$.

At this stage, we have already proved the lemma in the case that Q is either $Q^+(5, 2)$ or $Q(6, 2)$. In case $Q = Q^-(7, 2)$, we have already showed that all coefficients are zero with the possible exception of b_{78} . However, as the point $\langle \bar{e}_1 + \bar{e}_2 + \bar{e}_7 + \bar{e}_8 \rangle$ belongs to $Q^-(7, 2) \subseteq S$, we can then see that also $b_{78} = 0$. \square

Corollary 4.2. *For every $n \in \{6, 7, 8\}$, the image of the map $\epsilon_{n,l}$ generates the whole projective space $\text{PG}(V_l)$ where $l = \frac{(n+2)(n-1)}{2}$.*

Proof. If this were not the case, then there exists a hyperplane in $\text{PG}(V_l)$ containing all points of the image. But the existence of such a hyperplane would contradict Lemma 4.1. \square

For every $n \in \mathbb{N} \setminus \{0, 1, 2\}$, let \mathcal{S}_{n-1} be the geometry of the points and planes of $\text{PG}(n-1, 2)$, with incidence being containment. In [7], we showed that \mathcal{S}_{n-1} has pseudo-embeddings and determined the universal pseudo-embedding of \mathcal{S}_{n-1} . To describe this universal pseudo-embedding, we need to consider a vector space V' of dimension $\frac{n(n+1)}{2}$ having base elements \bar{g}'_i and h'_{jk} with $i, j, k \in \{1, 2, \dots, n\}$ such that $j < k$.

Proposition 4.3 ([7]). *Let $\tilde{\epsilon}_n$ be the map from the point set of \mathcal{S}_{n-1} to the point set of $\text{PG}(V')$ sending the point (X_1, X_2, \dots, X_n) of \mathcal{S}_{n-1} to the point $\langle \sum_{i=1}^n X_i \bar{g}'_i + \sum_{1 \leq j < k \leq n} X_j X_k \bar{h}'_{jk} \rangle$ of $\text{PG}(V')$. Then $\tilde{\epsilon}_n$ is isomorphic to the universal pseudo-embedding of \mathcal{S}_{n-1} .*

If $Q = Q_{n-1}$ with $n \in \{6, 7, 8\}$ and \mathcal{G} is the set of all generators of Q or one family of generators if $Q = Q_5 = Q^+(5, 2)$, then $\mathcal{S}_{Q, \mathcal{G}}$ is a full subgeometry of \mathcal{S}_{n-1} . The pseudo-embedding $\tilde{\epsilon}_n$ of \mathcal{S}_{n-1} will therefore induce a pseudo-embedding of $\mathcal{S}_{Q, \mathcal{G}}$. The following can be proved about this induced pseudo-embedding.

Proposition 4.4. *If $Q = Q_{n-1}$ with $n \in \{6, 7, 8\}$ and \mathcal{G} is the set of all generators of Q or one family of generators if $Q = Q_5 = Q^+(5, 2)$, then the pseudo-embedding of $\mathcal{S}_{Q, \mathcal{G}}$ induced by the pseudo-embedding $\tilde{\epsilon}_n$ of \mathcal{S}_{n-1} is isomorphic to the map $\epsilon_{n, l}$ defined in Section 1.*

Proof. We define $\bar{g}_i'' = \bar{g}_i'$ for every $i \in \{1, 2, \dots, 6\}$, $\bar{g}_7'' := \bar{g}_7' + \bar{h}'_{12}$ if $n \in \{7, 8\}$ and $\bar{g}_8'' := \bar{g}_8' + \bar{h}'_{12}$ if $n = 8$. We also define $\bar{h}''_{ij} := \bar{h}'_{ij}$ for all $i, j \in \{1, 2, \dots, n\}$ with $i < j$ and $(i, j) \notin \{(1, 2), (3, 4), (5, 6), (7, 8)\}$, $\bar{h}''_{34} := \bar{h}'_{12} + \bar{h}'_{34}$ and $\bar{h}''_{56} := \bar{h}'_{12} + \bar{h}'_{56}$. We also define $\bar{h}''_{78} = \bar{h}'_{12} + \bar{h}'_{78}$ if $n = 8$. Let V'' denote the hyperplane of $V' = \langle V'', \bar{h}'_{12} \rangle$ generated by the vectors \bar{g}_i'' and \bar{h}''_{jk} with $i, j, k \in \{1, 2, \dots, n\}$ such that $j < k$ and $(j, k) \neq (1, 2)$. For every point $p = (X_1, X_2, \dots, X_n)$ of Q_{n-1} , we then have that $\tilde{\epsilon}_n(p)$ is equal to the point $\langle \sum_{i=1}^n X_i \bar{g}_i'' + \sum_{j < k} X_j X_k \bar{h}''_{jk} \rangle$ of $\text{PG}(V'')$. Taking into account Corollary 4.2, we see that $\tilde{\epsilon}_n(Q_{n-1})$ generates the whole of $\text{PG}(V'')$ and that the pseudo-embedding of $\mathcal{S}_{Q, \mathcal{G}}$ induced by $\tilde{\epsilon}_n$ is isomorphic to the map $\epsilon_{n, l}$ defined in Section 1. \square

Denote by V_{28} a 28-dimensional vector space over \mathbb{F}_2 generated by V_{27} and an additional vector \bar{h}_{12} . Let V'_{27} be the hyperplane of V_{28} consisting of all vectors $\sum_{i=1}^7 X_i \bar{g}_i + \sum_{1 \leq j < k \leq 7} X_j X_k \bar{h}_{jk}$ for which $X_7 = 0$. Let V_{21} be the subspace of V_{28} generated by V_{20} and \bar{h}_{12} . The fact that $X_7 + X_1 X_2 + X_3 X_4 + X_5 X_6 = 0$ for points of $Q(6, 2)$ implies that the universal pseudo-embedding $\epsilon_{7, 27}$ of $\mathcal{S}_{Q_6, \mathcal{G}_{Q_6}}$ is isomorphic to the map

$$(X_1, X_2, \dots, X_7) \mapsto \left\langle \sum_{i=1}^6 X_i \bar{g}_i + \sum_{1 \leq j < k \leq 7} X_j X_k \bar{h}_{jk} \right\rangle$$

from the point set of $Q(6, 2)$ to the point set of $\text{PG}(V'_{27})$. We now show that the map $\epsilon_{7, 21}$

$$(X_1, X_2, \dots, X_7) \mapsto \left\langle \sum_{i=1}^6 X_i \bar{g}_i + \sum_{1 \leq j < k \leq 6} X_j X_k \bar{h}_{jk} \right\rangle$$

from the point set of $Q(6, 2)$ to the point set of $\text{PG}(V_{21})$ is also a pseudo-embedding of $\mathcal{S}_{Q_6, \mathcal{G}_{Q_6}}$.

Proposition 4.5. *If $Q = Q(6, 2)$ and \mathcal{G} is the set of all generators of $Q(6, 2)$, then the map $\epsilon_{7, 21}$ is a pseudo-embedding of the geometry $\mathcal{S}_{Q, \mathcal{G}}$.*

Proof. The map $(X_1, X_2, \dots, X_7) \mapsto (X_1, X_2, \dots, X_6)$ defines a bijection between the points of $Q(6, 2)$ and the points of $\text{PG}(5, 2)$ mapping every generator of $Q(6, 2)$ to a plane of $\text{PG}(5, 2)$. This map thus defines a full embedding of $\mathcal{S}_{Q, \mathcal{G}}$ into \mathcal{S}_5 and every pseudo-embedding of \mathcal{S}_5 will induce a pseudo-embedding of $\mathcal{S}_{Q, \mathcal{G}}$. This pseudo-embedding of $\mathcal{S}_{Q, \mathcal{G}}$ is isomorphic to the map $\epsilon_{7, 21}$. \square

Suppose now that $Q = Q^+(5, 2)$. Recall that \mathcal{G}^* is the family of generators of Q to which the plane with equation $X_1 = X_3 = X_5 = X_7 = X_8 = 0$ belongs. Note that if $\mathcal{G} = \mathcal{G}_{Q_5}$, then $\mathcal{S}_{Q_5, \mathcal{G}^*}$ is a full subgeometry of $\mathcal{S}_{Q_5, \mathcal{G}}$ having the same point set. The pseudo-embedding $\epsilon_{6,20}$ of $\mathcal{S}_{Q_5, \mathcal{G}}$ will therefore induce the following pseudo-embedding $\epsilon_{6,20}^*$ of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ in $\text{PG}(V_{20})$:

$$\epsilon_{6,20}^* : (X_1, X_2, \dots, X_6) \mapsto \left\langle \sum_{i=1}^6 X_i \bar{g}_i + \sum_{j,k=1}^6 X_j X_k \bar{h}_{jk} \right\rangle.$$

We will now define an additional pseudo-embedding of $\mathcal{S}_{Q_5, \mathcal{G}^*}$. The construction is based on the following proposition which is a slight improvement of Proposition 2.1 of [11].

Proposition 4.6. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ be a point-line geometry having the property that the number of points on each line is finite and at least 3, let V and \bar{V} be two vector spaces over \mathbb{F}_2 such that V is a hyperplane of \bar{V} , let $\bar{v}^* \in \bar{V} \setminus V$, let $\epsilon : \mathcal{S} \rightarrow \text{PG}(V)$ be a pseudo-embedding of \mathcal{S} and let H be a pseudo-hyperplane of \mathcal{S} . For every point x of H , we define $\bar{\epsilon}(x) := \epsilon(x)$ and for every point y of \mathcal{S} not contained in H , let $\bar{\epsilon}(y)$ denote the third point on the line through $\epsilon(y)$ and $\langle \bar{v}^* \rangle$. If $\text{PG}(W)$ denotes the subspace of $\text{PG}(\bar{V})$ generated by the image of $\bar{\epsilon}$, then the following hold:*

- (1) $\bar{\epsilon} : \mathcal{S} \rightarrow \text{PG}(W)$ is a pseudo-embedding of \mathcal{S} ;
- (2) if $H \notin \mathcal{H}_\epsilon$, then $W = \bar{V}$, $H \in \mathcal{H}_{\bar{\epsilon}}$, and $\epsilon, \bar{\epsilon}$ are nonisomorphic pseudo-embeddings;
- (3) if $H \in \mathcal{H}_\epsilon$, then W is a hyperplane of \bar{V} and $\epsilon, \bar{\epsilon}$ are isomorphic pseudo-embeddings.

Proof. Suppose first that $H \notin \mathcal{H}_\epsilon$, then by Proposition 2.1 of [11] we know that $W = \bar{V}$ and that $\bar{\epsilon} : \mathcal{S} \rightarrow \text{PG}(W)$ is a pseudo-embedding. By construction, it follows that the pseudo-hyperplane H arises from the hyperplane $\text{PG}(V)$ of $\text{PG}(W) = \text{PG}(\bar{V})$. So, $H \in \mathcal{H}_{\bar{\epsilon}}$. As $H \notin \mathcal{H}_\epsilon$, we then see that ϵ and $\bar{\epsilon}$ cannot be isomorphic.

Suppose that $H \in \mathcal{H}_\epsilon$. Then there exists a unique hyperplane π of $\text{PG}(V)$ such that $H = \epsilon^{-1}(\pi \cap \epsilon(\mathcal{P}))$. We denote by $\text{PG}(W')$ the unique hyperplane of $\text{PG}(\bar{V})$ through π distinct from $\text{PG}(V)$ and $\langle \pi, \bar{v}^* \rangle$. The projection from $\text{PG}(V)$ to $\text{PG}(W')$ with center $\langle \bar{v}^* \rangle$ defines an isomorphism θ between $\text{PG}(V)$ and $\text{PG}(W')$ such that $\bar{\epsilon}(x) = \theta(\epsilon(x))$ for every point x of \mathcal{S} . We thus have that $W = W'$ is a hyperplane of \bar{V} and that $\epsilon, \bar{\epsilon}$ are isomorphic pseudo-embeddings. \square

Lemma 4.7. *Let S be a set of points of $\text{PG}(5, 2)$ described by an equation of the form*

$$\sum^* a_{ijk} X_i X_j X_k + \sum_{i=1}^6 b_i X_i + \sum_{j,k=1}^6 c_{jk} X_j X_k = 0,$$

with all coefficients belonging to \mathbb{F}_2 . If $Q^+(5, 2) \subseteq S$, then $a_{135} = a_{146} = a_{236} = a_{245} = b_i = c_{jk} = 0$ for all $i, j, k \in \{1, 2, \dots, 6\}$ with $j < k$ and $(j, k) \neq (1, 2)$.

Proof. Since $\langle \bar{e}_i \rangle \in Q^+(5, 2) \subseteq S$ for every $i \in \{1, 2, \dots, 6\}$, we have $b_i = 0$ for every $i \in \{1, 2, \dots, 6\}$.

Since $\langle \bar{e}_j + \bar{e}_k \rangle \in Q^+(5, 2) \subseteq S$ for all $j, k \in \{1, 2, \dots, 6\}$ with $j < k$ and $(j, k) \notin \{(1, 2), (3, 4), (5, 6)\}$, we have $c_{jk} = 0$ for such values of j and k .

Since $\langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 \rangle$ and $\langle \bar{e}_1 + \bar{e}_2 + \bar{e}_5 + \bar{e}_6 \rangle$ belong to $Q^+(5, 2) \subseteq S$, we then also have $c_{34} = c_{56} = 0$.

Since $\langle \bar{e}_1 + \bar{e}_3 + \bar{e}_5 \rangle$, $\langle \bar{e}_1 + \bar{e}_4 + \bar{e}_6 \rangle$, $\langle \bar{e}_2 + \bar{e}_3 + \bar{e}_6 \rangle$ and $\langle \bar{e}_2 + \bar{e}_4 + \bar{e}_5 \rangle$ are points of $Q^+(5, 2) \subseteq Q$, we finally have that $a_{135} = a_{146} = a_{236} = a_{245} = 0$. \square

Proposition 4.8. *Let \mathcal{G}^* be the family of generators of $Q_5 = Q^+(5, 2)$ containing the plane with equation $X_1 = X_3 = X_5 = X_7 = X_8 = 0$. Then the map $\epsilon_{6,24}^*$ is a pseudo-embedding of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ in the projective space $\text{PG}(V_{24})$.*

Proof. By Lemma 4.7, we know that the image of $\epsilon_{6,24}^*$ generates the whole projective space $\text{PG}(V_{24})$. Combining this with Proposition 4.6, we know that $\epsilon_{6,24}^*$ is a pseudo-embedding of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ if the sets of points of $Q^+(5, 2)$ with equations $X_1X_3X_5 = 0$, $X_1X_4X_6 = 0$, $X_2X_3X_6 = 0$ and $X_2X_4X_5 = 0$ form four pseudo-hyperplanes of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ (we can then apply Proposition 4.6 four consecutive times). We now observe the following for an $(a, b, c) \in \{(1, 3, 5), (1, 4, 6), (2, 3, 6), (2, 4, 5)\}$.

- (1) The set of points of $Q^+(5, 2)$ with equation $X_a = X_b = X_c = 0$, or equivalently $X_aX_bX_c + X_aX_b + X_aX_c + X_bX_c + X_a + X_b + X_c = (X_a + 1)(X_b + 1)(X_c + 1) + 1 = 0$ is a pseudo-hyperplane of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ as this set of points is an element of \mathcal{G}^* and so meets every generator of \mathcal{G}^* in either 1 or 7 points.
- (2) The sets of points of $Q^+(5, 2)$ with equations $X_aX_b = 0$, $X_aX_c = 0$, $X_bX_c = 0$, $X_a = 0$, $X_b = 0$ and $X_c = 0$ are pseudo-hyperplanes of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ arising from $\epsilon_{6,20}^*$.
- (3) If H_1 and H_2 are two distinct pseudo-hyperplanes of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ with respective equations $f_1(X_1, X_2, \dots, X_6) = 0$ and $f_2(X_1, X_2, \dots, X_6) = 0$, then the set of points of $Q^+(5, 2)$ with equation $f_1(X_1, X_2, \dots, X_6) + f_2(X_1, X_2, \dots, X_6) = 0$ is also a pseudo-hyperplane as this set is just the complement of the symmetric difference of H_1 and H_2 .

The properties (1), (2) and (3) imply that the set of points of $Q^+(5, 2)$ with equation $X_aX_bX_c = 0$ is a pseudo-hyperplane of $\mathcal{S}_{Q_5, \mathcal{G}^*}$ for every $(a, b, c) \in \{(1, 3, 5), (1, 4, 6), (2, 3, 6), (2, 4, 5)\}$. \square

Theorems 1.1, 1.2, 1.3 and 1.4 of Section 1 are now consequences of Propositions 2.1, 3.1, 3.2, 3.6, 3.7, 4.4 and 4.8. At this stage, we also have done enough preparatory work to prove Theorems 1.5 and 1.6.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ be one of the geometries \mathcal{S}_n with $n \geq 2$ or $\mathcal{S}_{Q, \mathcal{G}_Q}$ with $Q \in \{Q^+(5, 2), Q(6, 2), Q^-(7, 2)\}$. Note that each $\pi \in \mathcal{L}$ is a plane of the ambient projective space of \mathcal{S} . A set of points of odd size of π is either a point, a line, an irreducible conic, the union of two distinct lines or the whole of π , i.e. a possibly reducible conic of

π . The pseudo-hyperplanes of \mathcal{S} are thus precisely those sets of points of \mathcal{S} distinct from \mathcal{P} that meet each $\pi \in \mathcal{L}$ in a possibly reducible conic of π . Theorems 1.5 and 1.6 are then immediate consequences of Proposition 2.2 if we also take into account the algebraic descriptions of the universal pseudo-embeddings of \mathcal{S} (Theorem 1.3 and Proposition 4.3).

We end this paper with showing that the conclusion of Theorem 1.5 is no longer valid for $\text{PG}(5, 2)$ if we restrict to those planes that are totally isotropic with respect to a given symplectic polarity of $\text{PG}(5, 2)$. The projection of $Q(6, 2)$ from the kernel $(0, 0, 0, 0, 0, 0, 1)$ of $Q(6, 2)$ to $\text{PG}(5, 2)$ defines an isomorphism η between the geometry of points and planes of $Q(6, 2)$ and the geometry of points and planes of a symplectic polar space $W(5, 2)$.

The sets of points of $\text{PG}(5, 2)$ intersecting each plane of $W(5, 2)$ in a possibly reducible conic therefore correspond via η to the sets of points of $Q(6, 2)$ arising from the universal pseudo-embedding $\epsilon_{7,27}$ of $\mathcal{S}_{Q_6, \mathcal{G}_{Q_6}}$.

On the other hand, by Theorem 1.5 the sets of points of $\text{PG}(5, 2)$ intersecting each plane of $\text{PG}(5, 2)$ in a possibly reducible conic are precisely the possibly reducible quadrics of $\text{PG}(5, 2)$. These correspond via η to the sets of points of $Q(6, 2)$ arising from the pseudo-embedding $\epsilon_{7,21}$ (see also the proof of Proposition 4.5).

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