The completion of the classification of the finite dense near $2d$-gons, $d \in \{3, 4\}$, with four points on each line

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Abstract

We classify finite near hexagons that satisfy the following three properties for certain $s, t_2 \in \mathbb{N} \setminus \{0, 1\}$: every line contains precisely $s+1$ points; every two points at distance 2 have either 2 or $t_2 + 1$ common neighbours; if Q is a quad of order (s, t_2), then $\Gamma_2(Q)$ does not contain lines. As a consequence of our treatment, we are able to complete the classification of all finite dense near hexagons and octagons with four points on each line.

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1 Introduction

This paper is concerned with the classification of certain near polygons. As we will see in Section 2, for classification purposes we may often assume that the near polygons under consideration have a constant number of points on each line. We will also make this assumption here. Specifically, we classify finite near hexagons that satisfy the following three properties for certain numbers $s, t_2 \in \mathbb{N} \setminus \{0, 1\}$:

(P1) Every line contains precisely $s + 1$ points.

(P2) Every two points at distance 2 have either 2 or $t_2 + 1$ common neighbours.

(P3) If Q is a quad of order (s, t_2), then $\Gamma_2(Q)$ does not contain lines.

For the definitions of the basic notions and notations that occur in this introductory section, the reader is referred to Section 2. We already note here that Properties (P1) and (P2) imply that $\mathcal{S}$ has an order (s, t) and that every two points at distance 2 are contained in a unique quad, which is either an $(s + 1) \times (s + 1)$-grid or a quad of order (s, t_2). If one of the quads of order (s, t_2) is big, then necessarily all quads of order (s, t_2)

are big. In the following theorem, which is our first main result, we make use of the following sets of tuples:

$$\begin{aligned}
\mathcal{T}_1 &:= \{(4,77),(5,10),(5,146),(8,569),(11,1442),(12,171),(14,11),(14,56),(18,6139),\\
&\quad (21,365),(23,12674),(32,33761),(53,151634),(56,92),(65,65),(99,66),\\
&\quad (129,1861),(158,3969119),(204,1683)\};\\
\mathcal{T}_2 &:= \{(8,4,68),(8,13,221),(9,3,57),(12,6,195),(14,7,287),(17,7,623),\\
&\quad (20,40,14705),(22,6,930),(26,52,2756),(35,7,1085),(41,7,623),(41,7,659),\\
&\quad (41,7,1289)\}.
\end{aligned}$$

Theorem 1.1 *Suppose $\mathcal{S}$ is a finite near hexagon satisfying Properties (P1), (P2), (P3) for certain $s, t_2 \in \mathbb{N} \setminus \{0,1\}$, and let (s,t) denote the order of $\mathcal{S}$. Then the following hold.*

(1) *If $s \leq 500$ and all quads of $\mathcal{S}$ are grids, then either $\mathcal{S} \cong \mathbb{E}_1$, $\mathcal{S}$ is a Hamming near hexagon $H(3,s+1)$ or $(s,t) \in \mathcal{T}_1$.*

(2) *If $s \leq 50$ and $\mathcal{S}$ contains non-big quads of order (s,t_2), then either $\mathcal{S} \cong \mathbb{E}_2$ or $(s,t_2,t) \in \mathcal{T}_2$.*

(3) *If $\mathcal{S}$ contains big quads of order (s,t_2), then $\mathcal{S}$ is one of the following:*

   (a) *a dual polar space $DW(5,q)$, $DQ(6,q)$, $DQ^-(7,q)$, $DH(5,q^2)$, $DH(6,q^2)$ for some prime power q;*

   (b) *a product near hexagon of the form $\mathbb{L}_{s+1} \times Q$, where $\mathbb{L}_{s+1}$ is a line of size $s+1$ and Q is a generalized quadrangle of order (s,t_2);*

   (c) *a glued near hexagon of type $Q_1 \otimes Q_2$, where Q_1 and Q_2 are two generalized quadrangles of order (s,t_2);*

   (d) *the near hexagon $\mathbb{H}_3$.*

No example of a near hexagon is known for each of the possibilities for (s,t) and (s,t_2,t) mentioned in Theorem 1.1. In fact, among the thirteen mentioned possibilities for (s,t_2,t) there are only two for which there exists a (currently) known generalized quadrangle of order (s,t_2), namely $(8,4,68)$ and $(9,3,57)$. For each of the mentioned possibilities for (s,t_2,t), there also exists a constant b such that every point is contained in b quads of order (s,t_2), see Section 10.

As we will see in Section 2, Property (P3) is satisfied if Q does not admit a partition in (induced) ovoids, an *induced ovoid* being a set of points of Q of the form $\Gamma_2(x) \cap Q$ where $x \in \Gamma_2(Q)$. This is for instance the case when Q is a generalized quadrangle isomorphic to $W(q)$, q odd, or $Q(5,q)$. It is also the case if the generalized quadrangle Q is isomorphic to $H(3,q^2)$ or $Q(4,q)$ with all induced ovoids being classical. We refer to [21, Chapter 3] for the definitions of the mentioned generalized quadrangles and for proofs of these facts about ovoids.

Classification results in the direction of Theorem 1.1 have already been obtained in the papers [5, 10]. In Sections 3 till 9, we derive all kinds of restrictions (equalities, inequalities, divisibility conditions) involving the parameters of near hexagons satisfying (P1), (P2), (P3) and having non-big quads of order (s, t_2). In fact, several of these restrictions are generalizations of arguments already given in [5, 10], but we will also derive several new restrictions. In generalizing some of the arguments in [5, 10], we try to optimize the obtained bounds with the aim of excluding as many cases for (s, t_2, t) as possible. And indeed, as we can see in Theorem 1.1, there are not many possibilities for (s, t_2, t) that will ultimately survive all these restrictions.

In [10], we classified finite dense near hexagons with four points per line. This classification was "almost complete" in the sense that only four open cases remained with specific information about s, t and some other parameters. In recent work, we excluded three of these four cases (one in [17] and two in [14]), resulting in one case still being open. The elusive near hexagons corresponding to this final case are in fact near hexagons of order $(3, 27)$ satisfying (P1), (P2), (P3) with $s = t_2 = 3$ and all quads of order $(3, 3)$ being non-big and isomorphic to $Q(4, 3)$. The nice thing is that one of the new restrictions we derive here will allow us to kill this final case (see Section 10), resulting in a complete classification of all finite dense near hexagons with four points per line, almost twenty years after the original incomplete classification was obtained in [10]. The following is thus our second main result.

Theorem 1.2 *Suppose $\mathcal{S}$ is a finite dense near hexagon with four points per line. Then $\mathcal{S}$ is one of the* 10 *near hexagons described in Section* 1 *of [10].*

In Proposition 1.1 and Theorem 1.2 of [12], we obtained an incomplete classification of the finite dense near octagons with four points per line. In that paper, we described an explicit list of 28 near octagons, and showed that any other example must contain an "exceptional near hexagon" as subgeometry. Such an exceptional near hexagon corresponds to one of the four open cases in the classification of the finite dense near hexagons with four points per line as obtained in [10]. In view of Theorem 1.2, we thus have:

Theorem 1.3 *Suppose $\mathcal{S}$ is a finite dense near octagon with four points per line. Then $\mathcal{S}$ is isomorphic to one of the* 28 *near octagons mentioned in Proposition 1 of [12].*

Theorems 1.2 and 1.3 thus complete existing results from the literature. In fact, with these results available, we now have a complete classification of all finite dense near $2d$-gons with three or four points per line if the diameter d is at most 4. Indeed, disregarding the trivial cases $d = 0$ and $d = 1$ (where the near polygons are points and lines), we may assume here that $d \geq 2$. The dense near $2d$-gons with three points per line have been classified in [5] for $d = 3$, in [16] for $d = 4$ and in [21, Section 6.1] for $d = 2$ (although the latter classification was already folklore long before [21] was published). The classification of the finite generalized quadrangles with four points per line was obtained in [18], see also [21, Section 6.2].

2 Preliminaries

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with non-empty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near polygon* if every two distinct points are incident with at most one line, and if for every point-line pair (x, L), there exists a unique point $\pi_L(x)$ on L that is nearest to x with respect to the distance in the collinearity graph Γ of $\mathcal{S}$. This is the graph whose vertices are the points of $\mathcal{S}$, with two distinct points being adjacent whenever they are incident with the same line. If $d \in \mathbb{N}$ is the diameter of Γ, then the near polygon is called a *near $2d$-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles having two disjoint lines are also known as *generalized quadrangles* [21].

If x_1 and x_2 are two points of a near polygon, then $\mathrm{d}(x_1, x_2)$ denotes the distance between x_1 and x_2 (in the collinearity graph). If x is a point and Y a non-empty set of points, then $\mathrm{d}(x, Y)$ denotes the smallest distance between x and a point of Y. If Y_1 and Y_2 are two non-empty sets of points, then $\mathrm{d}(Y_1, Y_2)$ denotes the smallest distance between a point of Y_1 and a point of Y_2. If $*$ is a point or a non-empty set of points, then $\Gamma_i(*)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from $*$. For every point x, we denote $\Gamma_0(x) \cup \Gamma_1(x)$ also by $x^\perp$.

A set X of points of a point-line geometry $\mathcal{S}$ is called a *subspace* if every line having two of its points in X has all its points in X. For every non-empty subspace X, we denote by $\widetilde{X}$ the subgeometry of $\mathcal{S}$ defined on the point set X by all those lines of $\mathcal{S}$ that have all their points in X. A set X of points of $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of X is also contained in X. If X is a non-empty convex subspace of a near polygon $\mathcal{S}$, then $\widetilde{X}$ is also a near polygon. If $\widetilde{X}$ is moreover a generalized quadrangle, then X is called a *quad* of $\mathcal{S}$. Two distinct quads of a near polygon intersect in either a singleton, a line or the empty set. If $*_1, *_2, \ldots, *_k$ is a collection of objects of a near polygon (like points, lines and non-empty sets of points), then $\langle *_1, *_2, \ldots, *_k \rangle$ denotes the smallest convex subspace containing $*_1, *_2, \ldots, *_k$. This new object is well-defined as it equals the intersection of all convex subspaces containing the mentioned objects.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. If $\mathcal{S}$ is a dense near polygon, then every two points at distance 2 are contained in a unique quad, as well as any two distinct intersecting lines.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two near polygons with respective collinearity graphs Γ_1 and Γ_2, then a new near polygon $\mathcal{S}_1 \times \mathcal{S}_2$ can be constructed whose collinearity graph is isomorphic to the cartesian product $\Gamma_1 \times \Gamma_2$ of Γ_1 and Γ_2, see [7, p. 146] or [13, Section 6.6]. The near polygon $\mathcal{S}_1 \times \mathcal{S}_2$ is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and its diameter is the sum of the diameters of $\mathcal{S}_1$ and $\mathcal{S}_2$. If $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are three near polygons, then $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $\mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3) \cong (\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3$ and so the direct product $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_k$ of $k \geq 2$ near polygons $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ is well-defined. The direct product of a number of lines is called a *Hamming near polygon*. If there are d lines of size $s+1$ involved in this construction, then the Hamming near polygon is denoted by $H(d, s+1)$.

If $\mathcal{S}$ is a near polygon having the property that every two points at distance 2 have at least two common neighbours (as it is the case for dense near polygons), then by [7, Theorem 1] (see also [13, Corollary 6.17]) we know that $\mathcal{S}$ is isomorphic to some product near polygon $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_k$, where each $\mathcal{S}_i$ is a near polygon having a constant number of points on each line. For this reason, when classifying near polygons, we may often assume that there is a constant number $s+1$ of points on each line. A near polygon with two points per line is nothing else than a bipartite graph ([13, Theorem 6.3]). Such near polygons are usually excluded from classification purposes as there are too many such graphs.

A *line spread* of a point-line geometry $\mathcal{S}$ is a set of lines partitioning the point set. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two generalized quadrangles having line spreads S_1 and S_2 satisfying certain nice properties, then by [9] (a) new near hexagon(s) $\mathcal{S}$ can be constructed from $(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2)$. Any such near hexagon is said to be *glued of type* $\mathcal{Q}_1 \otimes \mathcal{Q}_2$. By [9], glued near hexagons of type $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ are characterized by the following properties:

- for every $i \in \{1, 2\}$, $\mathcal{S}$ has a partition $\mathcal{R}_i$ in quads isomorphic to $\mathcal{Q}_i$;
- every quad of $\mathcal{R}_1$ intersects every quad of $\mathcal{R}_2$ in a line;
- every line is contained in a quad of $\mathcal{R}_1$ or a quad of $\mathcal{R}_2$.

Suppose x is a point and Q a quad of a near polygon such that $\mathrm{d}(x, Q) = i$. The point x is called *classical with respect to* Q if there exists a unique point $\pi_Q(x)$ in Q at distance i from x, in which case it holds that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_Q(x)) + \mathrm{d}(\pi_Q(x), y)$ for every $y \in Q$. The point x is called *ovoidal with respect to* Q if $\Gamma_i(x) \cap Q$ is an *ovoid* of $\widetilde{Q}$ (or shortly of Q). This is a set of points of $\widetilde{Q}$ meeting each line of $\widetilde{Q}$ in a singleton. If the latter case occurs, then the set $\Gamma_i(x) \cap Q$ is called a *subtended ovoid*. For every point-quad pair (x, Q) in a dense near polygon, the point x is classical or ovoidal with respect to Q, see [23, Proposition 2.6] (or [13, Theorem 6.24]).

Suppose Q is a quad of a dense near hexagon $\mathcal{S}$. Then the maximal distance from a point of $\mathcal{S}$ to Q is equal to 2. Points of $Q \cup \Gamma_1(Q)$ are classical with respect to Q, and points of $\Gamma_2(Q)$ are ovoidal with respect to Q. If $\Gamma_2(Q) = \emptyset$, then Q is called *big*. By [7, (b)] (see also [13, Theorem 6.25]), one of the following cases occurs for a line L of $\mathcal{S}$:

(i) $L \subseteq Q$.

(ii) L intersects Q in a unique point.

(iii) L is contained in $\Gamma_1(Q)$. In this case, $\pi_Q(L) := \{\pi_Q(x) \,|\, x \in L\}$ is a line of Q.

(iv) L contains a unique point $y_L \in \Gamma_1(Q)$ and all remaining points of L lie in $\Gamma_2(Q)$. If z_L denotes the unique point of Q collinear with y_L, then the subtended ovoids determined by the points of $L \setminus \{y_L\}$ all contain z_L and partition the set of points of $Q \cap \Gamma_2(z_L)$.

(iv) $L \subseteq \Gamma_2(Q)$. In this case, the subtended ovoids determined by the points of L form a partition of Q.

We thus see that $Q \cup \Gamma_1(Q)$ is a subspace of $\mathcal{S}$.

A near polygon is called *classical* if every two points at distance 2 are contained in a quad and if every point is classical with respect to any quad. It follows from [8] that classical near polygons are so-called dual polar spaces, a class of point-line geometries closely related to the polar spaces of Tits [24]. In the finite case, it follows from Tits' classification of polar spaces (see e.g. [13, Sections 7.7, 7.9, 8.2]) that every classical dense near hexagon is isomorphic to one of the following geometries:

• a product near polygon $\mathcal{L} \times \mathcal{Q}$, where $\mathcal{L}$ is a finite line and $\mathcal{Q}$ is a finite generalized quadrangle, each having at least three points on each line;
• the symplectic dual polar space $DW(5,q)$ for some prime power q;
• the orthogonal dual polar space $DQ(6,q)$ for some prime power q;
• the elliptic dual polar space $DQ^-(7,q)$ for some prime power q;
• the Hermitian dual polar space $DH(5,q^2)$ for some prime power q;
• the Hermitian dual polar space $DH(6,q^2)$ for some prime power q.

A near polygon is said to have *order* (s,t) if every line is incident with precisely $s+1$ points and if every point is incident with exactly $t+1$ lines. A near hexagon is said to be *regular with parameters* (s,t,t_2) if it has order (s,t) and if every two points at distance 2 have precisely t_2+1 common neighbours. If $\mathcal{S}$ is a regular near hexagon with parameters (s,t,t_2), $s \geq 2$, then an inequality of Haemers and Mathon [19] states that $t \leq s^3 + t_2(s^2-s+1)$, see also [7, (i)], [20, p. 207] or [13, Theorem 6.48]. The near hexagons $\mathbb{E}_1$ and $\mathbb{E}_2$ mentioned in Theorem 1.1 are regular near hexagons whose parameters (s,t_2,t) are respectively equal to $(2,1,11)$ and $(2,2,14)$ (and so for each of them, we have equality in the Haemers-Mathon bound). They were constructed in [23] from the extended ternary Golay code and the Witt design $S(5,8,24)$. By [3] and [4], these near hexagons are uniquely determined (up to isomorphism) by their parameters.

Suppose now again that $\mathcal{S}$ is a regular near hexagon with parameters (s,t,t_2). It is then known, see e.g. [22], [7, (i)] or [13, Theorems 3.13, 6.46 and 6.48] that the collinearity graph Γ of $\mathcal{S}$ has exactly four distinct eigenvalues $\lambda_1 = s(t+1)$, λ_2, λ_3, $\lambda_4 = -(t+1)$, where $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ and λ_2, λ_3 are the roots of the quadratic polynomial

$$x^2 - (s-1)(t_2+2)x + (s^2-s+1)t_2 - st + (s-1)^2 \in \mathbb{R}[x].$$

We have

$$\lambda_2 = \frac{(s-1)(t_2+2)+\sqrt{D}}{2}, \qquad \lambda_3 = \frac{(s-1)(t_2+2)-\sqrt{D}}{2},$$

where $D = (s-1)^2 t_2^2 + 4s(t-t_2)$. Putting

$$k_0 = 1, \quad k_1 = s(t+1), \quad k_2 = \frac{s^2(t+1)t}{t_2+1}, \quad k_3 = \frac{s^3 t(t-t_2)}{t_2+1},$$

we have that $|\Gamma_i(x)| = k_i$ for every $i \in \{0,1,2,3\}$ and every point x of $\mathcal{S}$. The total number of points of $\mathcal{S}$ is then equal to

$$v = 1 + s(t+1) + \frac{s^2(t+1)t}{t_2+1} + \frac{s^3 t(t-t_2)}{t_2+1} = (s+1)(1+st+\frac{s^2 t(t-t_2)}{t_2+1}). \tag{1}$$

We denote by m_i, $i \in \{1,2,3,4\}$, the multiplicity of the eigenvalue λ_i of $\mathcal{S}$. Since Γ is a connected graph, its valency $\lambda_1 = s(t+1)$ has multiplicity $m_1 = 1$. The multiplicity m_4 of $-(t+1)$ can be computed from Biggs' formula [1] (see also [6, Theorem 4.1.4] or [13, Theorem 3.19]). We have

$$m_4 = \frac{v}{\sum_{i=0}^3 k_i s^{-2i}} = s^3 \frac{(t_2+1)+s(t_2+1)t+s^2t(t-t_2)}{s^2(t_2+1)+st(t_2+1)+t(t-t_2)}. \tag{2}$$

As $m_1 + m_2 + m_3 + m_4 = v$ and $\lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 + \lambda_4 m_4 = Tr(A) = 0$, where A is the adjacency matrix of Γ, we have

$$\begin{aligned} m_2 + m_3 &= v - m_1 - m_4 = v - 1 - m_4, \\ \lambda_2 m_2 + \lambda_3 m_3 &= -\lambda_1 m_1 - \lambda_4 m_4 = -s(t+1) + (t+1)m_4. \end{aligned}$$

We deduce that

$$m_2 = \frac{\lambda_3(v-1-m_4) - (t+1)(m_4-s)}{\lambda_3 - \lambda_2}, \qquad m_3 = \frac{\lambda_2(v-1-m_4) - (t+1)(m_4-s)}{\lambda_2 - \lambda_3}.$$

It follows that

$$m_3 = \frac{((s-1)(t_2+2) + \sqrt{D})(v-1-m_4) - 2(t+1)(m_4-s)}{2\sqrt{D}}.$$

The multiplicities m_1, m_2, m_3 and m_4 should all be integral. This is the case if and only if m_3 and m_4 are integral. We note that if D is not a square, then m_3 is integral if and only if $v - 1 - m_4$ is even and

$$m_4 = \frac{(s-1)(t_2+2)(v-1) + 2s(t+1)}{2(t+1) + (s-1)(t_2+2)}. \tag{3}$$

Lemma 2.1 *If $s \geq 2$ and $t_2 = 1$, then D must be a square.*

Proof. Suppose to the contrary that D is not a square. Equating the values of m_4 obtained in (2) and (3) (and using (1)), we see that t must satisfy the equation $\frac{s^3-s}{2}t(t+1)(at^2+bt+c) = 0$, i.e. $at^2 + bt + c = 0$, where $a = 3s$, $b = 2s^2 - 9s + 2$ and $c = -(2s^2 - 10s + 2)$. The discriminant $d = b^2 - 4ac = 4s^4 - 12s^3 - 31s^2 - 12s + 4$ must therefore be a square. If $s \geq 21$, then one verifies that

$$(2s^2-3s-11)^2 = 4s^4-12s^3-35s^2+66s+121 < d < 4s^4-12s^3-31s^2+60s+100 = (2s^2-3s-10)^2$$

and so d cannot be a square. For $s \in \{2, 3, \ldots, 20\}$, one verifies individually that d is only a square if $s = 5$. In this case, $t \in \mathbb{N}$ satisfies the quadratic equation $at^2 + bt + c = 15t^2 + 7t - 2 = (5t-1)(3t+2)$, which is impossible. ■

In the following proposition, we prove the first claim of Theorem 1.1.

Proposition 2.2 *If $2 \leq s \leq 500$ and $t_2 = 1$, then either $\mathcal{S} \cong \mathbb{E}_1$, $\mathcal{S} \cong H(3, s+1)$ or $(s,t) \in \mathcal{T}_1$.*

Proof. We have $t \geq t_2 + 1 = 2$. If $t = 2$, then no two (grid-)quads can intersect in a singleton, implying that every (grid-)quad is big. So, the near hexagon is classical and necessarily isomorphic to the Hamming near hexagon $H(3, s+1)$. So, we may assume that $t \geq 3$. By the Haemers–Mathon inequality, we also have $t \leq s^3 + t_2(s^2 - s + 1) = s^3 + s^2 - s + 1$. With the aid of a computer, see [15], we have determined all $(s,t) \in \mathbb{N} \times \mathbb{N}$ for which $2 \leq s \leq 500$, $3 \leq t \leq s^3 + s^2 - s + 1$ such that the numbers $\sqrt{D}$, m_2

and m_4 are integral. This turned out to be the case if (s,t) is equal to $(2,11)$, $(3,9)$, $(3,34)$ or to one of the elements of $\mathcal{T}_1$. As mentioned before, any regular near hexagon with parameters $(s,t,t_2)=(2,11,1)$ must be isomorphic to $\mathbb{E}_1$. By [2], no regular near hexagon with parameters $(s,t,t_2)=(3,9,1)$ exists, and by [17, Proposition 4.7] no regular near hexagon with parameters $(s,t,t_2)=(3,34,1)$ exists. ■

3 Basic inequalities and divisibility conditions

Suppose $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ is a finite near hexagon that satisfies the Properties (P1), (P2), (P3) of Section 1 for certain $s,t_2\in\mathbb{N}\setminus\{0,1\}$. Then $\mathcal{S}$ is a dense near polygon and by [7, Lemma 19] (see also [13, Theorem 6.26(3)]), there exists a $t\in\mathbb{N}\setminus\{0,1\}$ such that $\mathcal{S}$ has order (s,t). Let v denote the total number of points of $\mathcal{S}$. The following lemma is precisely Theorem 1(2) of [14]. It gives an extremely restrictive divisibility condition not available at the time of the writing of [5, 10]. (In fact, the treatments given in [5, 10] can often be simplified if one would rely on this condition.)

Lemma 3.1 ([14]) *The number*

$$\frac{s^5 v}{(s+1)^2(s-1)(s^2+1)+st(s-1)(s+1)^2+v}$$

is integral.

For every point x of $\mathcal{S}$, let $\mathcal{L}_x$ denote the set of lines through x. For every incident point-quad pair (x,Q), we denote by $\mathcal{L}_{x,Q}$ the set of lines through x contained in Q. The linear space $\mathcal{S}_x$ defined on the set $\mathcal{L}_x$ by all sets $\mathcal{L}_{x,Q}$, where Q is a quad through x, is called the *local space at* x. Put

$$\delta_0:=1,\quad \delta_1:=s(t+1),\quad \delta_2:=\frac{v}{s+1}-1+s^2t-st,\quad \delta_3:=\frac{sv}{s+1}-s-s^2t.$$

The following lemma is a special case of Lemma 3 of [11].

Lemma 3.2 ([11]) *For every point x of $\mathcal{S}$ and every $i\in\{0,1,2,3\}$, we have $|\Gamma_i(x)|=\delta_i$.*

Lemma 3.3 *Every point of $\mathcal{S}$ is contained in $a:=\frac{\delta_2(t_2+1)}{s^2(t_2-1)}-\frac{t(t+1)}{t_2-1}$ grid-quads and $b:=\frac{1}{t_2(t_2-1)}\Big(t(t+1)-\frac{2\delta_2}{s^2}\Big)$ quads of order (s,t_2).*

Proof. Let x be a given point of $\mathcal{S}$. We denote by A_x, respectively B_x, the number of grid-quads, respectively quads of order (s,t_2), through x. As two distinct lines through x are contained in a unique quad, we have

$$2A_x+t_2(t_2+1)B_x=t(t+1). \tag{4}$$

Every grid-quad through x contains s^2 points of $\Gamma_2(x)$ and every quad of order (s, t_2) through x contains $s^2 t_2$ points of $\Gamma_2(x)$. As any two points at distance 2 are contained in a unique quad, we have

$$s^2 A_x + s^2 t_2 B_x = |\Gamma_2(x)| = \delta_2 \tag{5}$$

by Lemma 3.2. From (4) and (5), we find $A_x = a$ and $B_x = b$. ∎

In view of Theorem 1.1 (which we need to prove) and Proposition 2.2, we may assume that quads of order (s, t_2) exist, i.e. that $b > 0$. Lemma 3.3 then implies the following.

Corollary 3.4 *We have $v < (s+1)(\frac{1}{2}s^2 t(t-1) + st + 1)$.*

Corollary 3.5 *The total number of grid-quads is equal to $\frac{va}{(s+1)^2}$ and the total number of quads of order (s, t_2) is equal to $\frac{vb}{(s+1)(st_2+1)}$. As a consequence, these numbers are integral.*

Proof. This follows from Lemma 3.3 and the facts that each grid-quad contains $(s+1)^2$ points and that each quad of order (s, t_2) contains $(s+1)(st_2+1)$ points. ∎

Lemma 3.6 *Let Q be a quad of order (s, t_2) and $x \in \Gamma_1(Q)$. Then the number of lines through x contained in $\Gamma_1(Q)$ is bounded above by $t_2(t_2+1)$.*

Proof. Suppose L is a line through x contained in $\Gamma_1(Q)$. The unique quad through the lines L and $x\pi_Q(x)$ then contains the line $\pi_Q(L) \subseteq Q$ through $\pi_Q(x)$. Now, there are $t_2 + 1$ quads through $x\pi_Q(x)$ intersecting Q in a line, and each of these quads contains at most t_2 lines through x contained in $\Gamma_1(Q)$. The required number is thus bounded above by $t_2(t_2+1)$. ∎

Put

$$\Delta_0 := (s+1)(st_2+1), \quad \Delta_1 := s(s+1)(st_2+1)(t-t_2), \quad \Delta_2 := v-(s+1)(st_2+1)(1+s(t-t_2)).$$

Lemma 3.7 *Let Q be a quad of order (s, t_2). Then $\mathrm{d}(x, Q) \leq 2$ for every point x of $\mathcal{S}$. Moreover, $|\Gamma_i(Q)| = \Delta_i$ for every $i \in \{0, 1, 2\}$.*

Proof. Suppose $\mathrm{d}(x, Q) \geq 3$ for some point x of $\mathcal{S}$. Then every line L of Q contains a point at distance at least $\mathrm{d}(x, Q)+1 \geq 4$ from x, an obvious contradiction. So, $\mathrm{d}(x, Q) \leq 2$ for every point x of $\mathcal{S}$.

Obviously, $|\Gamma_0(Q)| = |Q| = (s+1)(st_2+1) = \Delta_0$. As every point of $\Gamma_1(Q)$ is collinear with a unique point of Q, we have $|\Gamma_1(Q)| = |Q| \cdot s(t-t_2) = \Delta_1$. Hence, $|\Gamma_2(Q)| = v - |\Gamma_0(Q)| - |\Gamma_1(Q)| = v - (s+1)(st_2+1) - (s+1)(st_2+1)s(t-t_2) = \Delta_2$. ∎

Corollary 3.8 *If a quad of order (s, t_2) is big, then every quad of order (s, t_2) is big.*

Proof. A quad and thus all quads of order (s, t_2) are big if and only if $\Delta_2 = 0$. ∎

The case where none of the quads of order (s, t_2) is big will be treated in Sections 4 till 10. The case where all quads of order (s, t_2) are big will be treated in Section 11.

4 Basic restrictions in case no quad of order (s, t_2) is big

In this section, as well as Sections 5 till 10, we assume that no quad of order (s, t_2) is big, or equivalently, that $\Delta_2 > 0$. Recall that if Q is a quad of order (s, t_2) and $x \in \Gamma_2(Q)$, then $\Gamma_2(x) \cap Q$ is an ovoid of Q.

Lemma 4.1 *If Q is a quad of order (s, t_2) and $x \in \Gamma_2(Q)$, then every line through x contains a unique point of $\Gamma_1(Q)$ (besides s points of $\Gamma_2(Q)$).*

Proof. If this were not the case, then there exists a line L through x contained in $\Gamma_2(Q)$, in contradiction with Property (P3). ■

Lemma 4.2 *Let Q be a quad of order (s, t_2) and $x \in \Gamma_2(Q)$. Then x is contained in $\gamma_1 := \frac{st_2^2+t_2+st_2-t}{t_2-1}$ grid-quads and $\gamma_2 := \frac{t-2st_2-1}{t_2-1}$ quads of order (s, t_2) meeting Q in singletons.*

Proof. We show that every line L through x is contained in a unique quad $\langle x, y \rangle$, where $y \in O_x := \Gamma_2(x) \cap Q$. Indeed, if y_L denotes the unique point on L contained in $\Gamma_1(Q)$, then y necessarily is the unique point of Q collinear with y_L. Now, suppose there are M_1, respectively M_2, grid-quads, respectively quads of order (s, t_2), through x meeting Q in a singleton (necessarily contained in O_x). Then

$$M_1 + M_2 = |O_x| = st_2 + 1. \tag{6}$$

As each grid-quad through x meeting Q contributes 2 lines to $\mathcal{L}_x$ and each quad of order (s, t_2) through x meeting Q contributes $t_2 + 1$ lines to $\mathcal{L}_x$, we have

$$2M_1 + (t_2 + 1)M_2 = t + 1. \tag{7}$$

From (6) and (7), it follows that $M_1 = \frac{st_2^2+t_2+st_2-t}{t_2-1}$ and $M_2 = \frac{t-2st_2-1}{t_2-1}$. ■

Corollary 4.3 *We have $st_2^2 + t_2 + st_2 - t \leq (t_2 - 1)a$ and $t - 2st_2 - 1 \leq (t_2 - 1)b$.*

Proof. Lemmas 3.3 and 4.2 imply that $\gamma_1 \leq a$ and $\gamma_2 \leq b$. ■

The following is another consequence of Lemma 4.2.

Corollary 4.4 *We have*

(1) $2(st_2 + 1) \leq t + 1 \leq (t_2 + 1)(st_2 + 1)$,

(2) *$t_2 - 1$ is a divisor of $t - 2s - 1$.*

The following lemma says something about the structure of $\mathcal{S}$ if t attains the upper bound in Corollary 4.4(1).

Lemma 4.5 *If $t = (t_2 + 1)st_2 + t_2$, then $\mathcal{S}$ is a regular near hexagon with parameters (s, t, t_2), i.e. $\mathcal{S}$ has no grid-quads.*

Proof. If $t = (t_2+1)st_2+t_2$, then we know from Lemma 4.2 that there are no grid-quads that meet a quad of order (s, t_2) in a singleton.

Now, let x be an arbitrary point of $\mathcal{S}$, let Q be a quad of order (s, t_2) through x, L a line through x not contained in Q and y a point of $L \setminus \{x\}$. By Lemma 3.6, there are at most $(t_2 + 1)t_2 < t$ lines through y contained in $\Gamma_1(Q)$. So, there is some line M through y meeting $\Gamma_2(Q)$. As the quad $Q' := \langle L, M\rangle$ intersects Q in a singleton, it has order (s, t_2). Now, consider another line L' through x not contained in $Q \cup Q'$ and let $y' \in L' \setminus \{x\}$. Through y', there are at most $2t_2(t_2 + 1) < t$ lines contained in $\Gamma_1(Q)$ or $\Gamma_1(Q')$ and so there exists a line M' through y' containing points of $\Gamma_2(Q)$ and $\Gamma_2(Q')$. The quad $Q'' := \langle L', M'\rangle$ meets each of Q, Q' in the singleton $\{x\}$. Now, there cannot exists a grid-quad through x as such a grid-quad would intersect Q, Q' and Q'' in lines, but there are only two lines through a point in a grid-quad. ■

Based on the above divisibility conditions and inequalities, we now define a set $\mathcal{F}(s, t_2)$ consisting of all possible values for (t, b). For given s and t_2, we know that $2st_2 + 1 \leq t \leq (t_2 + 1)st_2 + t_2$ by Corollary 4.4(1). As $b \geq 1$ and $2a + t_2(t_2 + 1)b = t(t + 1)$, we have $1 \leq b \leq \lfloor \frac{t(t+1)}{t_2(t_2+1)} \rfloor$ and $a = \frac{t(t+1)}{2} - \frac{t_2(t_2+1)}{2}b$. Also the number v can be expressed in terms of s, t_2, t and b. We have

$$\delta_2 = s^2t_2b + s^2a = s^2t_2b + \frac{1}{2}s^2t(t+1) - \frac{1}{2}s^2t_2(t_2+1)b = \frac{1}{2}s^2t(t+1) - \frac{1}{2}s^2t_2(t_2-1)b$$

and hence

$$v = (s+1)(\delta_2 + 1 - s^2t + st) = (s+1)\Big(\frac{1}{2}s^2t(t-1) - \frac{1}{2}s^2t_2(t_2-1)b + st + 1\Big).$$

Note that the condition $\Delta_2 > 0$ is equivalent with $t(t-1) - t_2(t_2-1)b > 2t_2(t - t_2)$.

Let $\mathcal{F}(s, t_2)$ denote the set of all (t, b) satisfying

$$2st_2 + 1 \leq t \leq (t_2+1)st_2 + t_2, \qquad 1 \leq b \leq \lfloor \frac{t(t+1)}{t_2(t_2+1)} \rfloor,$$

for which

- $t(t-1) - t_2(t_2-1)b > 2t_2(t - t_2)$,
- $st_2^2 + t_2 + st_2 - t \leq (t_2 - 1)a$,
- $t - 2st_2 - 1 \leq (t_2 - 1)b$,
- $\frac{t-2s-1}{t_2-1} \in \mathbb{N}$,
- $\frac{s^5v}{(s+1)^2(s-1)(s^2+1)+st(s-1)(s+1)^2+v} \in \mathbb{N}$,
- $\frac{va}{(s+1)^2} \in \mathbb{N}$,
- $\frac{vb}{(s+1)(st_2+1)} \in \mathbb{N}$,

where

$$a = \frac{t(t+1)}{2} - \frac{t_2(t_2+1)}{2}b,$$

and

$$v = (s+1)\Big(\frac{1}{2}s^2t(t-1) - \frac{1}{2}s^2t_2(t_2-1)b + st + 1\Big).$$

For a given $s \in \mathbb{N} \setminus \{0,1\}$, we know that $\sqrt{s} \leq t_2 \leq s^2$ and $s+t_2 \mid st_2(s+1)(t_2+1)$ by [21, 1.2.2 & 1.2.3]. As no quad of order (s,t_2) is big, every quad of order (s,t_2) has induced ovoids, implying that $t_2 \leq s^2-s$ by [21, 1.8.3]. We also know that $\{s,t_2\} \neq \{3,6\}$ ([18], [21, 6.2.2]). For a given $s \in \mathbb{N} \setminus \{0,1\}$, we denote by $\mathcal{F}(s)$ the set of all triples (t_2,t,b), where t_2 satisfies the just-mentioned restrictions and $(t,b) \in \mathcal{F}(s,t_2)$. We computed $\mathcal{F}(s)$ for $2 \leq s \leq 50$, see [15]. We found that

$$\mathcal{F}(2) = \{(2,9,12),(2,10,9),(2,13,15),(2,14,35)\},$$

$$\mathcal{F}(3) = \{(3,21,22),(3,23,5),(3,25,8),(3,27,43),(3,35,83),(3,39,35)\},$$

$$\mathcal{F}(4) = \{(2,17,16),(2,20,46),(2,22,20),(2,25,56),(2,25,65),$$
$$(2,26,115),(4,51,65),(4,66,170),(6,99,225),(8,233,582)\}.$$

The sizes of the sets $\mathcal{F}(s)$ for $s \leq 50$ have been mentioned in the following table.

s	$\|\mathcal{F}(s)\|$	s	$\|\mathcal{F}(s)\|$	s	$\|\mathcal{F}(s)\|$	s	$\|\mathcal{F}(s)\|$	s	$\|\mathcal{F}(s)\|$	s	$\|\mathcal{F}(s)\|$	s	$\|\mathcal{F}(s)\|$
2	4	9	104	16	32	23	274	30	96	37	7	44	311
3	6	10	54	17	273	24	84	31	4	38	74	45	657
4	10	11	206	18	90	25	69	32	271	39	75	46	37
5	177	12	110	19	17	26	232	33	188	40	196	47	112
6	62	13	34	20	302	27	78	34	86	41	938	48	40
7	13	14	739	21	187	28	272	35	1670	42	88	49	84
8	196	15	159	22	51	29	182	36	282	43	17	50	1009

For $2 \leq s \leq 50$, there are thus 10259 possibilities for (s,t_2,t,b). Sections 5 till 10 have as goal to show for as many of these quadruples as possible that they cannot occur as parameters of a near hexagon satisfying (P1), (P2), (P3) and having non-big quads of order (s,t_2). And indeed, as we can see from Theorem 1.1, from the 10259 original possibilities for (s,t_2,t,b) there are only $|\mathcal{T}_2|+1 = 14$ that will survive the various restrictions that we will derive in these sections. We also note that an example is known only for the case $(s,t_2,t,b) = (2,2,14,35)$, in which case it is even unique and isomorphic to $\mathbb{E}_2$.

Remark. Lemma 3.1 seems to be the most restrictive among all the above restrictions. E.g., without this divisibility condition $|\mathcal{F}(7)|$ would be equal to 149694 instead of 13.

5 Restrictions arising from the mutual position of two quads

Lemma 5.1 *Let Q be a quad of order (s,t_2). Then the number of grid-quads meeting Q in a singleton is equal to $\Phi_1' := \frac{\Delta_2\gamma_1}{s^2}$ and the number of quads of order (s,t_2) meeting Q in a singleton is equal to $\Phi_1 := \frac{\Delta_2\gamma_2}{s^2 t_2}$.*

Proof. If x is one of the Δ_2 points of $\Gamma_2(Q)$, then by Lemma 4.2, x is contained in γ_1 grid-quads and γ_2 quads of order (s,t_2) meeting Q in a singleton.

Conversely, for every grid-quad R meeting Q in a singleton $\{u\}$, we have $|\Gamma_2(Q)\cap R| = |\Gamma_2(u)\cap R| = s^2$ and for every quad S of order (s,t_2) meeting Q in a singleton $\{v\}$, we have $|\Gamma_2(Q)\cap S| = |\Gamma_2(v)\cap S| = s^2 t_2$. The numbers stated in the lemma now easily follow from straightforward counting. ∎

Lemma 5.2 *Let Q be a quad of order (s,t_2). Then the number of grid-quads meeting Q in a line is equal to $\Phi_2' := (st_2+1)a - \frac{\Delta_2\gamma_1}{s^2(s+1)}$ and the number of quads of order (s,t_2) meeting Q in a line is equal to $\Phi_2 := (st_2+1)(b-1) - \frac{\Delta_2\gamma_2}{s^2t_2(s+1)}$. As a consequence, $a \geq \frac{\Delta_2\gamma_1}{s^2(s+1)(st_2+1)}$, $b-1 \geq \frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}$ and the numbers $\frac{\Delta_2\gamma_1}{s^2(s+1)}$ and $\frac{\Delta_2\gamma_2}{s^2t_2(s+1)}$ are integral.*

Proof. Through each of the $(s+1)(st_2+1)$ points of Q, there are a grid-quads. The grid-quads that meet Q in a singleton are counted only once in this way, but the grid-quads that meet Q in a line are counted $s+1$ times. Invoking Lemma 5.1, we thus see that the total number of grid-quads meeting Q in a line is given by

$$\frac{1}{s+1}\Big((s+1)(st_2+1)a - \Phi_1'\Big) = (st_2+1)a - \frac{\Delta_2\gamma_1}{s^2(s+1)}.$$

A similar argument applies to the quads of order (s,t_2) if we take into account that through each of the $(s+1)(st_2+1)$ points of Q, there are $b-1$ quads of order (s,t_2) distinct from Q. ∎

Lemma 5.3 *Let Q be a quad of order (s,t_2). Then the number of grid-quads disjoint from Q is equal to $\Phi_3' := \frac{va}{(s+1)^2} - \frac{\Delta_2\gamma_1}{s(s+1)} - (st_2+1)a$ and the number of quads of order (s,t_2) disjoint from Q is equal to $\Phi_3 := \frac{vb}{(s+1)(st_2+1)} - \frac{\Delta_2\gamma_2}{st_2(s+1)} - (st_2+1)b + st_2$.*

Proof. As the total number of grid-quads is equal to $\frac{va}{(s+1)^2}$ (Corollary 3.5), it follows from Lemmas 5.1 and 5.2 that the number of grid-quads disjoint from Q is equal to $\frac{va}{(s+1)^2} - \Phi_1' - \Phi_2' = \Phi_3'$.

As the total number of quads of order (s,t_2) is equal to $\frac{vb}{(s+1)(st_2+1)}$ (Corollary 3.5), it follows from Lemmas 5.1 and 5.2 that the number of quads of order (s,t_2) disjoint from Q is equal to $\frac{vb}{(s+1)(st_2+1)} - \Phi_1 - \Phi_2 - 1 = \Phi_3$. ∎

Lemma 5.4 *We have $\Phi_2 \leq (t_2+1)(st_2+1)\lfloor\frac{t-t_2}{t_2}\rfloor$, with equality if and only if every line is contained in either 0 or $\lfloor\frac{t}{t_2}\rfloor$ quads of order (s,t_2). If this is the case, then the number $\lfloor\frac{t}{t_2}\rfloor$ is a divisor of $b(t_2+1)$.*

Proof. Let Q be a given quad of order (s,t_2). By Lemma 5.2, there are Φ_2 quads of order (s,t_2) meeting Q in a line. On the other hand, each of the $(t_2+1)(st_2+1)$ lines of Q is contained in at most $\lfloor\frac{t-t_2}{t_2}\rfloor$ quads of order (s,t_2) distinct from Q. So, $\Phi_2 \leq (t_2+1)(st_2+1)\lfloor\frac{t-t_2}{t_2}\rfloor$, with equality if and only if every line of Q is contained in precisely $\lfloor\frac{t}{t_2}\rfloor$ quads of order (s,t_2). The first claim of the lemma now follows from the fact that Q was an arbitrary quad of order (s,t_2).

As to the second claim, consider a point x. A standard double counting then gives that the number of lines through x that are contained in some quad of order (s, t_2) is equal to $b(t_2 + 1)\lfloor \frac{t}{t_2} \rfloor^{-1}$. ■

Remark. A similar reasoning as in the proof of Lemma 5.4 would show that $\Phi_2' \leq (t_2 + 1)(st_2 + 1)(t - t_2)$. However, this inequality already follows from earlier results. Indeed, it can be shown from earlier derived formulas (see e.g. the remark at the end of Section 7) that $\Phi_2' + t_2\Phi_2 = (t_2 + 1)(st_2 + 1)(t - t_2)$.

Lemma 5.5 *Let Q be a quad of order (s, t_2) and R a quad disjoint from Q. Then $\Gamma_1(Q) \cap R$ is either R or a hyperplane of R (i.e. a proper subspace of R meeting each line of R).*

Proof. As the set $\Gamma_0(Q) \cup \Gamma_1(Q)$ is a subspace of $\mathcal{S}$, the intersection $(\Gamma_0(Q) \cup \Gamma_1(Q)) \cap R = \Gamma_1(Q) \cap R$ is a subspace of R. As every line L of R meets $\Gamma_1(Q)$ (Lemma 4.1), this subspace is either R or a hyperplane of R. ■

The following are two consequences of Lemma 5.5 and [21, 2.3.1].

Corollary 5.6 *Let Q be a quad of order (s, t_2) and R a grid-quad disjoint from Q. Then precisely one of the following cases occurs:*

(1) $R \subseteq \Gamma_1(Q)$;

(2) $R \cap \Gamma_1(Q)$ *is the union of two intersecting lines;*

(3) $R \cap \Gamma_1(Q)$ *is an ovoid of R.*

As a consequence, $|R \cap \Gamma_2(Q)| \leq s(s + 1)$.

Corollary 5.7 *Let Q be a quad of order (s, t_2) and R a quad of order (s, t_2) disjoint from Q. Then precisely one of the following cases occurs:*

(1) $R \subseteq \Gamma_1(Q)$;

(2) $R \cap \Gamma_1(Q) = x^\perp \cap R$ *for some point $x \in R$;*

(3) $R \cap \Gamma_1(Q)$ *is a sub(generalized)quadrangle of order $(s, \frac{t_2}{s})$ of R;*

(4) $R \cap \Gamma_1(Q)$ *is an ovoid of R.*

As a consequence, $|R \cap \Gamma_2(Q)| \leq s(st_2 + 1)$.

Lemma 5.8 *Let Q be a quad of order (s, t_2). Then the number of grid-quads disjoint from Q is at least $\frac{\Delta_2(a-\gamma_1)}{s(s+1)}$ and the number of quads of order (s, t_2) disjoint from Q is at least $\frac{\Delta_2(b-\gamma_2)}{s(st_2+1)}$.*

Proof. By Lemmas 3.3, 3.7 and 4.2 each of the Δ_2 points $x \in \Gamma_2(Q)$ is contained in $a-\gamma_1$ grid-quads disjoint from Q. Conversely, every grid-quad R disjoint from Q contains at most $s(s+1)$ points of $\Gamma_2(Q)$ by Corollary 5.6. It follows that the number of grid-quads disjoint from Q is at least $\frac{\Delta_2(a-\gamma_1)}{s(s+1)}$.

A similar reasoning can be applied to obtain a lower bound for the number of quads of order (s,t_2) disjoint from Q. This time one needs to rely on Corollary 5.7. ■

The following is an immediate consequence of Lemmas 5.3 and 5.8 (and the latter's proof).

Corollary 5.9 *We have*

$$\frac{va}{(s+1)^2} - \frac{\Delta_2\gamma_1}{s(s+1)} - (st_2+1)a \geq \frac{\Delta_2(a-\gamma_1)}{s(s+1)},$$

$$\frac{vb}{(s+1)(st_2+1)} - \frac{\Delta_2\gamma_2}{st_2(s+1)} - (st_2+1)b + st_2 \geq \frac{\Delta_2(b-\gamma_2)}{s(st_2+1)}.$$

Equality holds in the first (respectively, second) inequality if and only if every grid-quad (respectively, quad of order (s,t_2)) disjoint from a quad Q of order (s,t_2) meets $\Gamma_1(Q)$ is an ovoid of that quad.

Taking into account that $\Delta_2 = v-(s+1)(st_2+1)(1+s(t-t_2))$, the first inequality of Corollary 5.9 reduces to the following.

Corollary 5.10 *We have $v \leq (s+1)^2(st_2+1)(1+s(t-t_2-1))$, with equality if and only if every grid-quad disjoint from a quad Q of order (s,t_2) meets $\Gamma_1(Q)$ in an ovoid of that quad.*

In certain cases, the second inequality of Corollary 5.9 can be improved as follows.

Lemma 5.11 *Let Q be a quad of order (s,t_2). Suppose no ovoid of Q intersects a subtended ovoid of Q in precisely t_2+1 points. Then*

$$\frac{vb}{(s+1)(st_2+1)} - \frac{\Delta_2\gamma_2}{st_2(s+1)} - (st_2+1)b + st_2 \geq \frac{\Delta_2(b-\gamma_2)}{s^2t_2},$$

with equality if and only if for every quad R of order (s,t_2) disjoint from Q, we have $R \cap Q = x^\perp \cap R$ for some point $x \in R$.

Proof. Let R be an arbitrary quad of order (s,t_2) disjoint from Q. We show that $R \cap \Gamma_1(Q)$ cannot be an ovoid of R. By Corollary 5.7, we then know that $|R \cap \Gamma_2(Q)| \leq s^2t_2$, and the inequality would then follow from a similar reasoning as in Corollary 5.9.

So, suppose $R \cap \Gamma_1(Q)$ is an ovoid O_R of R. Then $O_q := Q \cap \Gamma_1(R)$ is an ovoid of Q, and every point of O_Q is collinear with a unique point of O_R (implying that every such point is classical with respect to R). Now, take a point $x \in R \setminus O_R$. As x is collinear with precisely t_2+1 points of O_R, the subtended ovoid $\Gamma_2(x) \cap Q$ of Q would intersect the ovoid O_Q of Q in precisely t_2+1 points, a contradiction. ■

Remark. The conditions on the quad Q mentioned in Lemma 5.11 are valid if $(s, t_2) = (2, 2)$, in which case $\widetilde{Q} \cong W(2)$.

Lemma 5.12 *The case $b = 1$ cannot occur.*

Proof. Suppose $b = 1$. Then no two distinct quads of order (s, t_2) can intersect, and so by Lemma 4.2 we have $\gamma_2 = 0$, or equivalently, $t = 2st_2 + 1$.

Let Q be a quad of order (s, t_2). If R is a grid-quad of $\mathcal{S}$, then $|\Gamma_2(Q) \cap R| \in \{0, s^2, s(s+1)\}$ by Corollary 5.6. Let M_i, $i \in \{0, s^2, s(s+1)\}$, be the number of grid-quads R for which $|R \cap \Gamma_2(Q)| = i$.

The grid-quads R for which $|\Gamma_2(Q) \cap R| = s(s+1)$ are precisely the grid-quads disjoint from Q for which $\Gamma_1(Q) \cap R$ is an ovoid of R. Now, through each point x of $\Gamma_1(Q)$, there is one line meeting Q and $t_2 + 1$ lines contained in $\Gamma_1(Q)$. Indeed, as each quad through $x\pi_Q(x)$ is a grid (as $b = 1$), there are $t_2 + 1$ such grids that meet Q in a line, and each of these $t_2 + 1$ grids determines a line through x contained in $\Gamma_1(Q)$. Conversely, every line through x contained in $\Gamma_1(Q)$ is contained together with the line $x\pi_Q(x)$ in a grid-quad that intersects Q in a line. So, through x, there are $t - t_2 - 1$ lines meeting $\Gamma_2(Q)$ in s points. Hence, $M_{s(s+1)} \leq \frac{1}{s+1} \cdot |\Gamma_1(Q)| \cdot \frac{1}{2}(t-t_2-1)(t-t_2-2) = \frac{1}{2}s(st_2+1)(t-t_2)(t-t_2-1)(t-t_2-2)$.

As $b = 1$, there are $t - t_2$ grid-quads through each line of Q and so there are $(t_2+1)(st_2+1)(t - t_2)$ grid-quads that meet Q in a line. The grid-quads R for which $R \cap \Gamma_2(Q) = \emptyset$ are precisely the grid-quads that meet Q in a line or are contained in $\Gamma_1(Q)$.

As $b = 1$, there are $\frac{1}{2}(t-t_2)(t-t_2-1)$ grid-quads through each point of Q that meet Q in a singleton, and so there are $\frac{1}{2}(s+1)(st_2+1)(t-t_2)(t-t_2-1)$ grid-quads that meet Q in a singleton. The grid-quads R for which $|\Gamma_2(Q) \cap R| = s^2$ are precisely the grid-quads meeting Q in a singleton and the grid-quads disjoint from Q for which the intersection with $\Gamma_1(Q)$ is the union of two intersecting lines.

Now, counting triples (R, L_1, L_2), where $L_1, L_2 \subseteq \Gamma_1(Q)$ are two distinct intersecting lines and R is a grid-quad through L_1 and L_2 (necessarily disjoint from Q), we find:

$$(s+1)^2(M_0 - (t_2+1)(st_2+1)(t-t_2)) + (M_{s^2} - \frac{1}{2}(s+1)(st_2+1)(t-t_2)(t-t_2-1))$$

$$\leq |\Gamma_1(Q)| \cdot \frac{(t_2+1)t_2}{2} = s(s+1)(st_2+1)(t-t_2)\frac{(t_2+1)t_2}{2}.$$

It follows that

$$M_0 - (t_2+1)(st_2+1)(t-t_2) + M_{s^2} - \frac{1}{2}(s+1)(st_2+1)(t-t_2)(t-t_2-1)$$

$$\leq s(s+1)(st_2+1)(t-t_2)\frac{(t_2+1)t_2}{2}.$$

Invoking Corollary 3.5, we thus see that

$$\frac{va}{(s+1)^2} = M_0 + M_{s^2} + M_{s^2+s} \leq (t_2+1)(st_2+1)(t-t_2) + \frac{1}{2}(s+1)(st_2+1)(t-t_2)(t-t_2-1)$$

$$+\frac{1}{2}s(s+1)(st_2+1)(t-t_2)(t_2+1)t_2+\frac{1}{2}s(st_2+1)(t-t_2)(t-t_2-1)(t-t_2-2).$$

We denote by z_1, z_2, z_3, z_4 the four terms occurring at the right hand side of the last inequality. Then

$$(s+1)(z_1+z_2+z_3+z_4)-\frac{v}{s+1}a \geq 0. \tag{8}$$

Now,

$$\begin{aligned}
b &= 1,\\
t &= 2st_2+1,\\
a &= \frac{t(t+1)}{2}-\frac{t_2(t_2+1)}{2}b,\\
\frac{v}{s+1} &= \frac{1}{2}s^2t(t-1)-\frac{1}{2}s^2t_2(t_2-1)b+st+1.
\end{aligned}$$

So, we see that each of the numbers $z_1, z_2, z_3, z_4, \frac{v}{s+1}$ and a can be expressed as polynomials in s and t_2. Equation (8) is then equivalent with

$$-\frac{1}{4}(4s^5-2s^4-10s^3+5s^2)t_2^4-\frac{1}{2}(2s^5+2s^4-4s^3-7s^2+3s)t_2^3$$

$$-\frac{1}{4}(6s^4+4s^3-17s^2)t_2^2-\frac{1}{2}(s^3+s^2-3s)t_2 \geq 0.$$

The latter inequality is never satisfied. ■

6 Further restrictions in the case $t = 2st_2 + 1$

In Corollary 4.4, we derived a lower and upper bound for t. In case t attains the upper bound, we provided structural information about $\mathcal{S}$ in Lemma 4.5. In this section, we derive further information in the case t attains the lower bound $2st_2+1$. For every line L of $\mathcal{S}$, we denote by α_L the number of quads of order (s, t_2) through L.

We now define a certain number k. This number is equal to $u+1$, where $u = \frac{t_2}{s}$, unless one of the following conditions is satisfied, in which case k is equal to 1:

- $u \notin \mathbb{N}$;
- $\{s, u\} = \{3, 6\}$;
- $u \in \mathbb{N} \setminus \{0, 1\}$ and $u^2 < s$;
- $u \in \mathbb{N} \setminus \{0, 1\}$ and $s^2 < u$;
- $u \in \mathbb{N} \setminus \{0, 1\}$ and $\frac{su(s+1)(u+1)}{s+u} \notin \mathbb{N}$.

Lemma 6.1 *If $t = 2st_2+1$, then the following hold:*

(1) *No two quads of order (s, t_2) intersect in a singleton.*

(2) $b \leq \max(2s, t_2^2+t_2+1)$.

(3) *If* $\max(2s, 2t_2) < b$*, then for every line* L *of* $\mathcal{S}$*, we have* $\alpha_L \leq k$ *or* $\alpha_L = t_2 + 1$.

(4) *If* $\max(2s, t_2) < b$*, then for every line* L *of* $\mathcal{S}$*, we have* $\alpha_L \leq k$ *or* $\alpha_L \in \{t_2, t_2 + 1\}$.

(5) *If* $\max(2s, 2t_2) < b \neq t_2^2 + t_2 + 1$*, then* $\lceil \frac{b+t_2-(t_2+1)k}{t_2+1-k} \rceil < k + 1$.

Proof. (1) Suppose Q_1 and Q_2 are two quads of order (s, t_2) that intersect in the singleton $\{x\}$. Let $y \in Q_2 \cap \Gamma_2(x)$. Then $y \in \Gamma_2(Q_1)$ and so by Lemma 4.2, there exists no quad of order (s, t_2) through y intersecting Q_1 in a singleton. This is a contradiction as Q_2 is such a quad.

(2) Let x be a point of $\mathcal{S}$. In the local space $\mathcal{S}_x$, there are only lines of size $t_2 + 1$ and 2. There are b lines of size $t_2 + 1$ and by part (1), we know that any two such lines of size $t_2 + 1$ meet.

Suppose now that $b > 2s$ and let L_1, L_2 be two distinct lines of size $t_2 + 1$ in $\mathcal{S}_x$. Put $\{p\} = L_1 \cap L_2$. Through p, there are at most $\lfloor \frac{t}{t_2} \rfloor = 2s$ lines of size t_2+1. As $b > 2s$, there exists a line M of size $t_2 + 1$ in $\mathcal{S}_x$ not containing $\{p\}$. As any line of size $t_2 + 1$ through p meets M, there are at most $t_2 + 1$ such lines. Any line of size $t_2 + 1$ not containing p meets $L_1 \setminus \{p\}$ and $L_2 \setminus \{p\}$ and so there are at most t_2^2 such lines. We conclude that if $b > 2s$ then there are at most $(t_2 + 1) + t_2^2$ lines of size $t_2 + 1$ in $\mathcal{S}_x$.

(3) Let x and y be two distinct points of L. The line L is contained in at most $\lfloor \frac{t}{t_2} \rfloor = 2s$ quads of order (s, t_2). As $b > 2s$, there exists a quad Q of order (s, t_2) through x not containing L. Any quad of order (s, t_2) through L meets Q in a line and so we have $\alpha_L \leq t_2 + 1$. Suppose $l := \alpha_L \in [k + 1, t_2]$. Denote by $Q_1, Q_2, \ldots, Q_l$ the l quads of order (s, t_2) through L (necessarily intersecting Q in a line), and by $R_1, R_2, \ldots, R_{t_2+1-l}$ the grid-quads through L intersecting Q in a line through x. For every $i \in \{1, 2, \ldots, t_2 + 1 - l\}$, let K_i denote the unique line of R through y distinct from L. Any quad S of order (s, t_2) through y not containing L intersects $\Gamma_1(Q)$ in a set that contains the $l > k$ lines $Q_1 \cap S, Q_2 \cap S, \ldots, Q_l \cap S$. By Corollary 5.7 and [21, 1.2.2, 1.2.3, 6.2.2], every line of S through y is contained in $\Gamma_1(Q)$, implying that the line K_1 is contained in S. As any quad of order (s, t_2) through K_1 intersects Q_1 in a line through y distinct from L, there are at most t_2 such quads. So, the number of quads of order (s, t_2) through y is bounded above by $l + t_2 \leq 2t_2$. As $b > 2t_2$, this is a contradiction. It follows that $l \leq k$ or $l = t_2 + 1$.

(4) The proof is similar to the proof of part (3). Here, we assume that $l := \alpha_L \in [k + 1, t_2 - 1]$. Following the same notational conventions as in (3), we must have that K_1 and K_2 are contained in S. Every quad of order (s, t_2) through y not containing L should therefore coincide with $\langle K_1, K_2 \rangle$, implying that the number of quads of order (s, t_2) through y is bounded above by $l + 1 \leq t_2$. As $b > t_2$, this is a contradiction. It follows that $l \leq k$ or $l \in \{t_2, t_2 + 1\}$.

(5) If $\max(2s, 2t_2) < b \neq t_2^2 + t_2 + 1$, then we know from (2) that $b < t_2^2 + t_2 + 1$. Suppose to the contrary that $\lceil \frac{b+t_2-(t_2+1)k}{t_2+1-k} \rceil \geq k + 1$. Let x be a point of $\mathcal{S}$ and consider a line U of size $t_2 + 1$ in the local space $\mathcal{S}_x$. By (3) we know that $\alpha_u \leq k$ or $\alpha_u = t_2 + 1$ for every point u of U. Let N_1 denote the number of points $u \in U$ for which $\alpha_u = t_2 + 1$. Then there are $t_2 + 1 - N_1$ points u of U for which $\alpha_u \leq k$. Since any two lines of size

t_2+1 of $\mathcal{S}_x$ meet, we have $\sum_{u\in\mathcal{U}} \alpha_u = b+t_2$ and hence

$$N_1(t_2+1)+(t_2+1-N_1)k \geq b+t_2,$$

i.e.

$$N_1 \geq \frac{b+t_2-(t_2+1)k}{t_2+1-k}.$$

This implies that

$$N_1 \geq \lceil \frac{b+t_2-(t_2+1)k}{t_2+1-k} \rceil \geq k+1.$$

Since $\sum_{u\in U} \alpha_u = b+t_2 < t_2^2+2t_2+1=(t_2+1)^2$, there exists a point $u_2\in U$ for which $\alpha_{u_2}\leq k$. As $N_1\geq k+1$, there also exists a point $u_1\in U$ for which $\alpha_{u_1}=t_2+1$. Let U' be a line of size t_2+1 through u_1 distinct from U. As $\alpha_{u_2}\leq k$, there exists a point $v\in U'\setminus\{u_1\}$ such that vu_2 is a line of size 2. As $N_1\geq k+1$, there are at least $k+1$ lines of size t_2+1 through v, i.e. $\alpha_v\geq k+1$. But this would imply that $\alpha_v=t_2+1$ which is impossible as vu_2 is a line of size 2. ■

7 Derivation of some other constants

The following lemma is an improvement of one of the claims of Lemma 5.1.

Lemma 7.1 *Let Q be a quad of order (s,t_2). If $\gamma_2\neq 1$, then through every point x of Q, there are $\phi_1 := \frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}$ quads of order (s,t_2) that meet Q in the singleton $\{x\}$.*

Proof. For every point u of Q, let N_u denote the number of quads of order (s,t_2) that meet Q in the singleton $\{u\}$. If u_1 and u_2 are two noncollinear points of Q, then counting in two different ways the number of pairs (R_1,R_2), where R_1 and R_2 are two quads of order (s,t_2) satisfying $R_1\cap Q=\{u_1\}$, $R_2\cap Q=\{u_2\}$ and $|R_1\cap R_2|=1$, we find by Lemma 4.2 that $N_{u_1}\cdot(\gamma_2-1)=N_{u_2}\cdot(\gamma_2-1)$, i.e. $N_{u_1}=N_{u_2}$. Now, since the collinearity relation defined on the point set of Q gives rise to a connected graph (as $s\geq 2$), we see that all N_u's are equal, necessarily to $\frac{\Phi_1}{|Q|}=\phi_1$. ■

For every line L, we denote by α_L, respectively β_L, the number of quads of order (s,t_2), respectively grid-quads, through L. As any two intersecting lines are contained in a unique quad, we have $t_2\alpha_L+\beta_L=t$. Hence, $\alpha_L\leq\lfloor\frac{t}{t_2}\rfloor$.

Lemma 7.2 (1) *We have*

$$\Big(\lceil b-1-\frac{\Phi_1}{(s+1)(st_2+1)}\rceil\Big)\Big(\lceil b-t_2^2-\lfloor\frac{t}{t_2}\rfloor-\frac{\Phi_1}{(s+1)(st_2+1)}\rceil\Big)\leq\frac{\Phi_1 b}{(s+1)(st_2+1)}.$$

(2) *If $\frac{\Phi_1}{(s+1)(st_2+1)}\in\mathbb{N}$ and*

$$\Big(b-\frac{\Phi_1}{(s+1)(st_2+1)}\Big)\Big(b+1-t_2^2-\lfloor\frac{t}{t_2}\rfloor-\frac{\Phi_1}{(s+1)(st_2+1)}\Big)>\frac{\Phi_1 b}{(s+1)(st_2+1)},$$

then through every point x of a quad Q of order (s,t_2), there are $\phi_1 = \frac{\Phi_1}{(s+1)(st_2+1)}$ quads of order (s,t_2) meeting Q in the singleton $\{x\}$.

Proof. For a quad Q of order (s,t_2) and a point $x \in Q$, let $N_{x,Q}$ denote the number of quads of order (s,t_2) that intersect Q in the singleton $\{x\}$. By Corollary 3.5 and Lemma 5.1, we have

$$\sum_x \sum_Q N_{x,Q} = \sum_Q \sum_x N_{x,Q} = \sum_Q \Phi_1 = \frac{\Phi_1 vb}{(s+1)(st_2+1)}, \tag{9}$$

where the summation ranges over all points x and all quads Q of order (s,t_2) such that $x \in Q$. Now, let x be a fixed point and consider the local space $\mathcal{S}_x$. For a line L of size t_2+1 of $\mathcal{S}_x$, we define $A_L := \sum_{p\in L}\alpha_p$. Let M be one of the $A_L-(t_2+1)$ lines of size t_2+1 meeting L in a singleton, say $\{p_1\}$. There are at least $\sum_{p\in L\setminus\{p_1\}}(\alpha_p-(t_2+1)) = A_L-\alpha_{p_1}-t_2(t_2+1) \geq A_L - \lfloor\frac{t}{t_2}\rfloor - t_2(t_2+1)$ lines of size t_2+1 disjoint from M. Hence,

$$\sum_Q N_{x,Q} \geq \Big(A_L-(t_2+1)\Big)\Big(A_L - \lfloor\frac{t}{t_2}\rfloor - t_2(t_2+1)\Big) \tag{10}$$

for every line L of size t_2+1 of $\mathcal{S}_x$.

For a line L of size t_2+1 of $\mathcal{S}_x$, there are $b-(A_L-(t_2+1))-1 = b+t_2-A_L$ lines of size t_2+1 disjoint from L. Hence,

$$\sum_Q N_{x,Q} = \sum_L (b+t_2-A_L), \tag{11}$$

where the latter summation ranges over all lines L of size t_2+1 of $\mathcal{S}_x$.

Suppose now that

$$\Big(\lceil b-1-\frac{\Phi_1}{(s+1)(st_2+1)}\rceil\Big)\Big(\lceil b-t_2^2-\lfloor\frac{t}{t_2}\rfloor-\frac{\Phi_1}{(s+1)(st_2+1)}\rceil\Big) > \frac{\Phi_1 b}{(s+1)(st_2+1)}.$$

Then $b-1-\frac{\Phi_1}{(s+1)(st_2+1)} > 0$ (by Lemma 5.2) and $b-t_2^2-\lfloor\frac{t}{t_2}\rfloor-\frac{\Phi_1}{(s+1)(st_2+1)} > 0$. The displayed inequality in combination with (10) implies that if x is a point such that $\mathcal{S}_x$ has a line L of size t_2+1 for which $A_L \geq b+t_2-\frac{\Phi_1}{(s+1)(st_2+1)}$ (i.e. $A_L \geq \lceil b+t_2-\frac{\Phi_1}{(s+1)(st_2+1)}\rceil$), then $\sum_Q N_{x,Q} > \frac{\Phi_1 b}{(s+1)(st_2+1)}$. On the other hand, if x is a point such that $A_L < b+t_2-\frac{\Phi_1}{(s+1)(st_2+1)}$ for every line L of size t_2+1 of $\mathcal{S}_x$, then (11) implies that $\sum_Q N_{x,Q} > \frac{\Phi_1 b}{(s+1)(st_2+1)}$. This allows us to conclude that

$$\sum_x\sum_Q N_{x,Q} > \frac{\Phi_1 bv}{(s+1)(st_2+1)},$$

in contradiction with (9). So, Part (1) of the theorem is certainly valid.

Suppose now that $\frac{\Phi_1}{(s+1)(st_2+1)} \in \mathbb{N}$ and

$$\Big(b - \frac{\Phi_1}{(s+1)(st_2+1)}\Big)\Big(b + 1 - t_2^2 - \lfloor \frac{t}{t_2} \rfloor - \frac{\Phi_1}{(s+1)(st_2+1)}\Big) > \frac{\Phi_1 b}{(s+1)(st_2+1)}.$$

Then $b - \frac{\Phi_1}{(s+1)(st_2+1)} > 0$ and $b+1-t_2^2 - \lfloor \frac{t}{t_2} \rfloor - \frac{\Phi_1}{(s+1)(st_2+1)} > 0$. Also (10) implies that if x is a point such that $\mathcal{S}_x$ has a line L of size t_2+1 for which $A_L \geq b+t_2+1-\frac{\Phi_1}{(s+1)(st_2+1)}$, then $\sum_Q N_{x,Q} > \frac{\Phi_1 b}{(s+1)(st_2+1)}$. On the other hand, if x is a point such that $A_L \leq b+t_2-\frac{\Phi_1}{(s+1)(st_2+1)}$ for every line L of size t_2+1 of $\mathcal{S}_x$, then (11) implies that $\sum_Q N_{x,Q} \geq \frac{\Phi_1 b}{(s+1)(st_2+1)}$. As $\sum_x \sum_Q N_{x,Q} = \frac{\Phi_1 v b}{(s+1)(st_2+1)}$, we thus see that for every point x and every line L of size t_2+1 of $\mathcal{S}_x$, we have $A_L = b + t_2 - \frac{\Phi_1}{(s+1)(st_2+1)}$.

Now, let Q be a quad of order (s, t_2) and $x \in Q$. Let L be the line of size t_2+1 of $\mathcal{S}_x$ corresponding to Q. Then the fact that $A_L = b + t_2 - \frac{\Phi_1}{(s+1)(st_2+1)}$ implies that there are $b - 1 - \frac{\Phi_1}{(s+1)(st_2+1)}$ quads of order (s, t_2) that meet Q in a line through x. So, there are $\frac{\Phi_1}{(s+1)(st_2+1)}$ quads of order (s, t_2) that meet Q in the singleton $\{x\}$. ■

Remark: In view of Lemma 7.1, Lemma 7.2(2) only offers new information in the case $\gamma_2 = 1$.

Lemma 7.3 *Let Q be a quad of order (s, t_2). Suppose that the number of quads of order (s, t_2) meeting Q in a singleton $\{x\} \subseteq Q$ is independent from x. Then through every point $x \in Q$, there are*

- $\phi_1 = \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}$ *quads of order (s, t_2) that meet Q in the singleton $\{x\}$,*
- $\phi_1' := \frac{\Delta_2 \gamma_1}{s^2 (s+1)(st_2+1)}$ *grid-quads that meet Q in the singleton $\{x\}$,*
- $\phi_2 := b - 1 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}$ *quads of order (s, t_2) that meet Q in a line,*
- $\phi_2' := a - \frac{\Delta_2 \gamma_1}{s^2 (s+1)(st_2+1)}$ *grid-quads that meet Q in a line.*

As a consequence, these numbers are integral. In particular, this holds if $\gamma_2 \neq 1$ or $\frac{\Phi_1}{(s+1)(st_2+1)} \in \mathbb{N}$ and

$$\Big(b - \frac{\Phi_1}{(s+1)(st_2+1)}\Big)\Big(b + 1 - t_2^2 - \lfloor \frac{t}{t_2} \rfloor - \frac{\Phi_1}{(s+1)(st_2+1)}\Big) > \frac{\Phi_1 b}{(s+1)(st_2+1)}.$$

Proof. For every point u of Q, we denote by
- N_u the number of quads of order (s, t_2) meeting Q in the singleton $\{u\}$,
- N_u' the number of grid-quads meeting Q in the singleton $\{u\}$,
- M_u the number of quads of order (s, t_2) meeting Q in a line through u,
- M_u' the number of grid-quads meeting Q in a line through u.

Then we have $N_u + M_u = b - 1$, $N_u' + M_u' = a$ and $M_u' + t_2 M_u = (t_2+1)(t - t_2)$ (as any two distinct lines through a point are contained in a unique quad). The fact that all N_u's are constant thus implies that also all M_u's, N'_u's and M'_u's are constant. For every point $u \in Q$, we thus have

$$N_u = \frac{\Phi_1}{(s+1)(st_2+1)} = \phi_1, \qquad N'_u = \frac{\Phi'_1}{(s+1)(st_2+1)} = \phi'_1,$$

$$M_u = b - 1 - N_u = b - 1 - \phi_1 = \phi_2, \qquad M'_u = a - N'_u = a - \phi'_1 = \phi'_2.$$

■

Remark: Making use of Lemma 7.3, we compute that

$$\begin{aligned}
\phi'_2 + t_2\phi_2 &= a + t_2 b - t_2 - \frac{\Delta_2\gamma_1}{s^2(s+1)(st_2+1)} - \frac{\Delta_2\gamma_2}{s^2(s+1)(st_2+1)} \\
&= \frac{\delta_2}{s^2} - t_2 - \frac{\Delta_2(\gamma_1+\gamma_2)}{s^2(s+1)(st_2+1)} \\
&= \frac{1}{s^2}\Big(\frac{v}{s+1} - 1 + s^2 t - st\Big) - t_2 - \frac{\Delta_2}{s^2(s+1)} \\
&= \frac{1}{s^2}\Big(\frac{v}{s+1} - 1 + s^2 t - st\Big) - t_2 - \frac{v - (s+1)(s^2 t_2(t-t_2) + st + 1)}{s^2(s+1)} \\
&= (t_2+1)(t-t_2).
\end{aligned}$$

This inequality also follows from a double counting of the pairs (L_1, L_2), where L_1 and L_2 are two lines through a given point x of a given quad Q of order (s, t_2) such that $L_1 \subseteq Q$ and L_2 is not contained in Q.

As $\phi'_2 + t_2\phi_2 = (t_2+1)(t-t_2)$, the condition that ϕ_1 is integral thus implies that also ϕ'_1, ϕ_2 and ϕ'_2 are integral.

8 Restrictions arising from the α_L's

If Q is a quad of order (s, t_2), then in Lemmas 7.1 and 7.2(2), we gave sufficient conditions for the number of quads of order (s, t_2) intersecting Q in a given singleton $\{y\} \subseteq Q$ to be independent from y. We need the fact that these numbers are constant in the following two lemmas.

Lemma 8.1 *Let Q be a quad of order (s, t_2). Assume that the number of quads of order (s, t_2) intersecting Q in a given singleton $\{y\} \subseteq Q$ is independent from the point y. Then for every $x \in Q$, we have*

$$\sum_{L \in \mathcal{L}_{x,Q}} \alpha_L = b + t_2 - \frac{\Delta_2\gamma_2}{s^2 t_2(s+1)(st_2+1)}.$$

Proof. The number of quads of order (s, t_2) that intersect Q in a line through x is equal to $\sum_{L \in \mathcal{L}_{x,Q}} (\alpha_L - 1) = \sum_{L \in \mathcal{L}_{x,Q}} \alpha_L - (t_2+1)$. By Lemma 7.3, this number is equal to $b - 1 - \frac{\Delta_2\gamma_2}{s^2 t_2(s+1)(st_2+1)}$. ■

Lemma 8.2 *Assume that for any quad Q of order (s,t_2), the number of quads of order (s,t_2) intersecting Q in a given singleton $\{y\} \subseteq Q$ is independent from the point y. Let x be a point of $\mathcal{S}$. Summing over all points p of the local space $\mathcal{S}_x$, we find:*

$$\begin{aligned}\sum_p 1 &= t+1,\\ \sum_p \alpha_p &= (t_2+1)b,\\ \sum_p \alpha_p^2 &= b\Big(b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}\Big).\end{aligned}$$

As a consequence, $(t_2+1)b$ and $b\Big(b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}\Big)$ have the same parity and

$$(t_2+1)^2b \le (t+1)\Big(b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}\Big),$$

with equality of and only if the α_L's are constant, in which case $\alpha_L = \frac{(t_2+1)b}{t+1}$ for every line L of $\mathcal{S}$ (implying that this number is then integral).

Proof. The number $\sum_p 1$ is the number of points of $\mathcal{S}_x$, i.e. $t+1$. The number $\sum_p \alpha_p$ is the number of flags (p,L) of $\mathcal{S}_x$, where L is a line of size t_2+1. As there are b lines of size t_2+1 in $\mathcal{S}_x$, there are $b(t_2+1)$ such flags. The number $\sum_p \alpha_p^2$ is the number of triples (p,L_1,L_2), where p is a point of $\mathcal{S}_x$ and L_1, L_2 are two not necessarily distinct lines of size t_2+1 through p. Now, there are b possibilities for L_1, and for given L_1 there are by Lemma 8.1 $b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}$ possibilities for (x,L_2). Hence,

$$\sum_p \alpha_p^2 = b\Big(b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}\Big).$$

Now, the Cauchy-Schwartz inequality $(\sum_p \alpha_p)^2 \le (\sum_p 1)\cdot(\sum_p \alpha_p^2)$ reduces to the inequality mentioned in the lemma. Obviously, we have equality in this inequality if and only if all α_p's are constant, in which case all α_p's are equal to their average value $\overline{\alpha_p} := (\sum_p \alpha_p)/(\sum_p 1) = \frac{(t_2+1)b}{t+1}$. ■

We now show that the inequality of Lemma 8.2 always holds.

Lemma 8.3 *We always have*

$$(t_2+1)^2b \le (t+1)\Big(b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}\Big).$$

If equality holds, then for a given quad Q of order (s,t_2), the number of quads of order (s,t_2) intersecting Q in a given singleton $\{y\} \subseteq Q$ is independent from the point y (and so the additional conclusions of Lemma 8.2 are also valid.)

Proof. Let x be a point of $\mathcal{S}$. Summing over all points p of the local space $\mathcal{S}_x = (\mathcal{P}_x, \mathcal{L}_x, \mathrm{I}_x)$, we find

$$\sum_{p \in \mathcal{P}_x} 1 = t+1, \qquad \sum_{p \in \mathcal{P}_x} \alpha_p = (t_2+1)b,$$

and hence $\sum_{p \in \mathcal{P}_x} \alpha_p^2 \geq \frac{(t_2+1)^2 b^2}{t+1}$. Invoking Corollary 3.5 and Lemma 5.2, we then have

$$\frac{(t_2+1)^2 b^2}{t+1} v \leq \sum_{x \in \mathcal{P}} \sum_{p \in \mathcal{P}_x} \alpha_p^2 = (s+1) \cdot \sum_{L \in \mathcal{L}} \alpha_L^2$$

$$= (s+1) \frac{bv}{(s+1)(st_2+1)} \Big((t_2+1)(st_2+1) + \Phi_2 \Big) = bv \Big(b + t_2 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)} \Big),$$

from which the stated inequality readily follows. In case of equality, we have that $\sum_{p \in \mathcal{P}_x} \alpha_p^2 = \frac{(t_2+1)^2 b^2}{t+1}$ for every point x of $\mathcal{S}$. Just as in the proof of Lemma 8.2, we then know that α_p is independent from $p \in \mathcal{P}_x$, and so that $\alpha_L = \frac{(t_2+1)b}{t+1}$ for every line L of $\mathcal{S}$. But then for every quad Q of order (s,t_2) and every point $x \in Q$, the number of quads of order (s,t_2) intersecting Q in the singleton $\{x\}$ is equal to

$$b - 1 - (t_2+1) \Big(\frac{(t_2+1)b}{t+1} - 1 \Big)$$

and thus constant. ■

9 Further restrictions arising from the α_L's

Lemma 9.1 *Let L_1 and L_2 be two disjoint lines contained in a grid-quad R. If $\gamma_2 \neq 0$, then $\alpha_{L_1} = \alpha_{L_2}$.*

Proof. Let K be a line meeting L_1 and L_2, say in the points x_1 and x_2. For every $i \in \{1,2\}$, the number α_i of quads of order (s,t_2) through x_i meeting R in a line is equal to $\alpha_K + \alpha_{L_i}$. It thus suffices to prove that $\alpha_1 = \alpha_2$, or equivalently, that the number $b - \alpha_1$ of quads of order (s,t_2) intersecting R in the singleton $\{x_1\}$ equals the number $b - \alpha_2$ of quads of order (s,t_2) intersecting R in the singleton $\{x_2\}$. For every point u of R, let N_u denote the total number of quads of order (s,t_2) meeting R in the singleton $\{u\}$.

Let u_1 and u_2 be two noncollinear points of R. We count in two different ways the pairs (R_1, R_2), where R_1 and R_2 are two quads of order (s,t_2) such that $R \cap R_1 = \{u_1\}$, $R \cap R_2 = \{u_2\}$ and $|R_1 \cap R_2| = 1$. By Lemma 4.2, the number of such pairs is equal to $N_{u_1} \cdot \gamma_2$, but also to $N_{u_2} \cdot \gamma_2$, implying that $N_{u_1} = N_{u_2}$.

Now, since the noncollinearity relation defined on the point set of R gives rise to a connected graph, we see that all N_u's are equal. In particular, we have $N_{x_1} = N_{x_2}$. ■

Now, let Q^* be a given quad of order (s,t_2). We then count the number Ω of pairs (L,R), where R is a quad of order (s,t_2) disjoint from Q^* and L is a line contained in $\Gamma_1(Q^*) \cap R$.

Lemma 9.2 *We have* $\Omega = (t_2 + 1)(st_2 + 1)\Phi_3 - \frac{t_2+1}{s}\Delta_2(b - \gamma_2)$.

Proof. Let $\mathcal{U}$ denote the set of quads of order (s, t_2) disjoint from Q^*. For every $Q \in \mathcal{U}$, let M_Q denote the number of lines of Q contained in $\Gamma_1(Q^*)$. Through each point of $\Gamma_2(Q^*) \cap Q$, there are $t_2 + 1$ lines contained in Q and each of these $t_2 + 1$ lines contains s points of $\Gamma_2(Q^*) \cap Q$ and one point of $\Gamma_1(Q^*) \cap Q$. Hence, there are $\frac{t_2+1}{s} \cdot |\Gamma_2(Q^*) \cap Q|$ lines in Q meeting $\Gamma_2(Q^*)$ and

$$M_Q = (t_2 + 1)(st_2 + 1) - \frac{t_2 + 1}{s} \cdot |\Gamma_2(Q^*) \cap Q|$$

lines in Q contained in $\Gamma_1(Q^*)$. By Lemmas 3.3, 3.7, 4.2 and 5.3, we thus have

$$\begin{aligned}\Omega = \sum_{Q\in\mathcal{U}} M_Q &= (t_2 + 1)(st_2 + 1) \cdot |\mathcal{U}| - \frac{t_2 + 1}{s} \sum_{Q\in\mathcal{U}} |\Gamma_2(Q^*) \cap Q| \\ &= (t_2 + 1)(st_2 + 1)\Phi_3 - \frac{t_2 + 1}{s}\Delta_2(b - \gamma_2).\end{aligned}$$

■

For every line $L \subseteq \Gamma_1(Q^*)$, there exists a unique quad (namely $\langle L, \pi_{Q^*}(L)\rangle$) through L intersecting Q^* in a line. We denote by Ω_1, respectively Ω_2, the number of pairs (L, R) where R is a quad of order (s, t_2) disjoint from Q^* and L is a line contained in $\Gamma_1(Q^*) \cap R$ such that $\langle L, \pi_{Q^*}(L)\rangle$ is a grid-quad, respectively a quad of order (s, t_2). Then $\Omega = \Omega_1+\Omega_2$. ■

Let $\mathcal{L}^*$ denote the set of lines contained in Q^*.

Lemma 9.3 *If* $\gamma_2 \neq 0$, *then* $\Omega_1 = \sum_{L\in\mathcal{L}^*} s\alpha_L(t - t_2\alpha_L)$.

Proof. Let $L \in \mathcal{L}^*$ be fixed. It suffices to prove that the number of pairs (K, R), where R is a quad of order (s, t_2) disjoint from Q^*, K is a line contained in $R\cap\Gamma_1(Q^*)$, $\pi_{Q^*}(K) = L$ and $\langle K, L\rangle$ is a grid-quad is equal to $s\alpha_L(t-t_2\alpha_L)$. Indeed, there are $\beta_L = t-t_2\alpha_L$ choices for the grid-quad $S = \langle K, L\rangle$, and for each such grid-quad S, there are s choices for the line K. As every quad of order (s, t_2) through each such K is disjoint from Q^*, we know from Lemma 9.1 that there are $\alpha_K = \alpha_L$ choices for R for each such K. Hence, the requested number is equal to $(t - t_2\alpha_L) \cdot s\alpha_L$. ■

Lemma 9.4 *We have* $\sum_{L\in\mathcal{L}^*} \alpha_L = (st_2 + 1)(b + t_2) - \frac{\Delta_2\gamma_2}{s^2t_2(s+1)}$.

Proof. We have $\sum_{L\in\mathcal{L}^*} \alpha_L = (st_2+1)(t_2+1)+\sum_{L\in\mathcal{L}^*}(\alpha_L-1)$, where $\sum_{L\in\mathcal{L}^*}(\alpha_L-1)$ equals the number of quads of order (s, t_2) meeting Q^* in a line. By Lemma 5.2, $\sum_{L\in\mathcal{L}^*}(\alpha_L-1) = \Phi_2 = (st_2 + 1)(b - 1) - \frac{\Delta_2\gamma_2}{s^2t_2(s+1)}$. ■

Lemma 9.5 *Suppose that the number of quads of order* (s, t_2) *meeting a given quad* Q *of order* (s, t_2) *in a singleton* $\{x\} \subseteq Q$ *is independent from the point* $x \in Q$. *Then the number* Ω_2 *is equal to*

$$\Big((st_2+1)(b-1)-\frac{\Delta_2\gamma_2}{s^2t_2(s + 1)}\Big)\cdot\Big(s(t_2-1)(b-\frac{\Delta_2\gamma_2}{s^2t_2(s + 1)(st_2 + 1)})-st_2\Big)+\sum_{L\in\mathcal{L}^*} s\alpha_L(\alpha_L-1).$$

Proof. Let Q be a quad of order (s, t_2) intersecting Q^* in a line U. Let $\mathcal{L}_1$ denote the set of lines of Q disjoint from U and let $\mathcal{L}_2$ denote the set of lines of Q meeting U in a singleton.

We count the number N_Q of pairs (L, R), where R is a quad of order (s, t_2) disjoint from Q^* and $L = Q \cap R$ is a line.

Since $\sum_{L \in \mathcal{L}_{x,Q}} (\alpha_L - 1) = b - 1 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}$ for every $x \in Q$ (Lemma 7.3), we have $\sum_{L \in \mathcal{L}_2} (\alpha_L - 1) = (s+1)(b - 1 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}) - (s+1)(\alpha_U - 1)$. We also have that $S := \sum_{x \in Q \setminus U} \sum_{L \in \mathcal{L}_{x,Q}} (\alpha_L - 1)$ is equal to

$$\Big(b - 1 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}\Big) \cdot |Q \setminus U| = \Big(b - 1 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}\Big)(s+1) s t_2.$$

On the other hand, S is also equal to

$$\sum_{L \in \mathcal{L}_1} (s+1)(\alpha_L - 1) + \sum_{L \in \mathcal{L}_2} s(\alpha_L - 1) = (s+1) N_Q + s(s+1)\Big(b - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}\Big) - s(s+1)\alpha_U.$$

Combining both expressions for S, we deduce that

$$N_Q = s(t_2 - 1)(b - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}) - s t_2 + s \alpha_U.$$

Let $\mathcal{Q}$ denote the set of all quads that meet Q^* in a line. Summing over all $\Phi_2 = (st_2 + 1)(b-1) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)}$ quads of $\mathcal{Q}$, we find that $\Omega_2 = \sum_{Q \in \mathcal{Q}} N_Q$ is equal to

$$\Big((st_2+1)(b-1) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)}\Big) \cdot \Big(s(t_2-1)(b - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}) - s t_2\Big) + \sum_{L \in \mathcal{L}^*} s \alpha_L (\alpha_L - 1).$$

■

Lemma 9.6 *Suppose $\gamma_2 \neq 0$ and that the number of quads of order (s, t_2) meeting a given quad Q of order (s, t_2) in a singleton $\{x\} \subseteq Q$ is independent from the point $x \in Q$. Then $\Omega = \Omega' - s(t_2 - 1) \sum_{L \in \mathcal{L}^*} \alpha_L^2$, where*

$$\Omega' = \Big((st_2+1)(b-1) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)}\Big) \cdot \Big(s(t_2-1)\Big(b - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}\Big) - s t_2\Big)$$
$$+ s(t-1)\Big((st_2+1)(b+t_2) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)}\Big).$$

Proof. By Lemmas 9.3 and 9.5, we know that $\Omega = \Omega_1 + \Omega_2$ is equal to

$$\Big((st_2+1)(b-1) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)}\Big) \cdot \Big(s(t_2-1)\Big(b - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)(st_2+1)}\Big) - s t_2\Big)$$
$$- s(t_2 - 1) \cdot \sum_{L \in \mathcal{L}^*} \alpha_L^2 + s(t-1) \sum_{L \in \mathcal{L}^*} \alpha_L.$$

By Lemma 9.4, this is equal to $\Omega' - s(t_2 - 1) \sum_{L \in \mathcal{L}^*} \alpha_L^2$. ■

The following is an immediate consequence of Lemmas 9.4 and 9.6.

Corollary 9.7 *Suppose $\gamma_2 \neq 0$ and that the number of quads of order (s, t_2) meeting a given quad Q of order (s, t_2) in a singleton $\{x\} \subseteq Q$ is independent from the point $x \in Q$. Then*

$$\sum_{L \in \mathcal{L}^*} \alpha_L^2 = \frac{\Omega' - \Omega}{s(t_2 - 1)}.$$

As a consequence, the latter number is integral and has the same parity as the number $(st_2 + 1)(b + t_2) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s+1)}$.

Lemma 9.8 *Suppose $\gamma_2 \neq 0$ and that the number of quads of order (s, t_2) meeting a given quad Q of order (s, t_2) in a singleton $\{x\} \subseteq Q$ is independent from the point $x \in Q$. Then*

$$\Big((st_2 + 1)(b + t_2) - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s + 1)}\Big)^2 \leq (t_2 + 1)(st_2 + 1)\frac{\Omega' - \Omega}{s(t_2 - 1)},$$

with equality if and only if every line is contained in either 0 or $\frac{1}{t_2+1}\Big(b+t_2 - \frac{\Delta_2\gamma_2}{s^2 t_2(s+1)(st_2+1)}\Big)$ quads of order (s, t_2). If this is the case, then the numbers

$$\frac{1}{t_2 + 1}\Big(b + t_2 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s + 1)(st_2 + 1)}\Big), \qquad \frac{b(t_2 + 1)^2}{b + t_2 - \frac{\Delta_2\gamma_2}{s^2 t_2 (s+1)(st_2+1)}}$$

are integral.

Proof. The stated inequality is precisely the Cauchy-Schwartz inequality

$$\Big(\sum_{L \in \mathcal{L}^*} \alpha_L\Big)^2 \leq \Big(\sum_{L \in \mathcal{L}^*} 1\Big) \cdot \Big(\sum_{L \in \mathcal{L}^*} \alpha_L^2\Big).$$

We have equality if and only if every line of Q^* is contained in precisely

$$\frac{\sum_{L \in \mathcal{L}^*} \alpha_L}{|\mathcal{L}^*|} = \frac{1}{t_2 + 1}\Big(b + t_2 - \frac{\Delta_2 \gamma_2}{s^2 t_2 (s + 1)(st_2 + 1)}\Big) =: \eta$$

quads of order (s, t_2). Since Q^* was an arbitrary quad of order (s, t_2), we thus see that equality occurs if and only if every line of $\mathcal{S}$ is contained in either 0 or η quads of order (s, t_2). We now also see that the first number should be integral. A standard counting yields that the total number of lines through a point that is contained in a quad of order (s, t_2) is equal to $\frac{b(t_2+1)}{\eta}$. So, also the second number should be integral. ∎

Lemma 9.9 *Suppose $\gamma_2 \neq 0$ and that the number of quads of order (s, t_2) meeting a given quad Q of order (s, t_2) in a singleton $\{x\} \subseteq Q$ is independent from the point $x \in Q$. If $\mathcal{L}$ denotes the set of lines of $\mathcal{S}$, then*

$$\sum_{L \in \mathcal{L}} 1 \;=\; \frac{v(t + 1)}{s + 1},$$

$$\begin{aligned}
\sum_{L\in\mathcal{L}} \alpha_L &= \frac{vb(t_2+1)}{s+1}, \\
\sum_{L\in\mathcal{L}} \alpha_L^2 &= \frac{vb}{s+1}\Big(b+t_2-\frac{\Delta_2\gamma_2}{s^2t_2(s+1)(st_2+1)}\Big), \\
\sum_{L\in\mathcal{L}} \alpha_L^3 &= \frac{\Omega'-\Omega}{s(t_2-1)}\cdot\frac{vb}{(s+1)(st_2+1)}.
\end{aligned}$$

Proof. The first equality follows from the fact that the total number of lines of $\mathcal{S}$ is equal to $\frac{v(t+1)}{s+1}$. The number $\sum_{L\in\mathcal{L}}\alpha_L$ is equal to $(t_2+1)(st_2+1)$ times the number of quads of order (s,t_2), i.e. equal to $(t_2+1)(st_2+1)\frac{vb}{(s+1)(st_2+1)} = \frac{vb(t_2+1)}{s+1}$. The third equality follows from the third equality of Lemma 8.2 (by summing over all points p of $\mathcal{S}$). The fourth equality follows from the equality in Corollary 9.7 (by summing over all quads Q^* of order (s,t_2) of $\mathcal{S}$). ■

Corollary 9.10 *Suppose $\gamma_2 \neq 0$ and that the number of quads of order (s,t_2) meeting a given quad Q of order (s,t_2) in a singleton $\{x\}\subseteq Q$ is independent from the point $x\in Q$. Then the numbers $\frac{vb(t_2+1)}{s+1}$ and $\frac{\Omega'-\Omega}{s(t_2-1)}\cdot\frac{vb}{(s+1)(st_2+1)}$ are congruent modulo 6.*

Proof. For every $L\in\mathcal{L}$, the numbers α_L and α_L^3 are congruent modulo 6. The claim then follows from Lemma 9.9. ■

10 The surviving possibilities

We have used a computer ([15]) to determine which of the original 10259 possibilities for (s,t_2,t,b) survived all conditions mentioned in Sections 5 till 9. In Appendix A, we list the number of cases killed by each of these conditions.

Although we see that Lemma 5.12 and the divisibility conditions of Lemma 8.2 do not kill any of the possibilities, it is still possible that they do for $s\geq 51$. In fact, if we disregard the strong divisibility condition mentioned in Lemma 3.1, these conditions actually kill possibilities. In fact, at the time of the writing of [10] the divisibility condition mentioned in Lemma 3.1 was not yet available and the nonexistence of the case $(s,t_2,t,b)=(3,3,19,1)$ was proved in [10] using the ideas mentioned in the proof of Lemma 5.12. We also note that although Lemma 5.11 "only" looks like a small improvement of the second inequality of Corollary 5.9, it is essential as it gives the only restriction in this paper that would kill the case $(s,t_2,t,b)=(2,2,10,9)$.

From the 10259 possibilities for (s,t,t_2,b) mentioned in Section 4, the following 16 survive all restrictions mentioned in Sections 5 till 9:

$$(2,2,14,35),(3,3,21,22),(8,4,68,33),(8,13,221,25),(9,3,57,30),(12,4,244,2989),$$

$$(12,6,195,397),(14,7,287,750),(17,7,623,5781),(20,40,14705,124413),(22,6,930,20615),$$

$$(26,52,2756,1107),(35,7,1085,15590),(41,7,623,1245),(41,7,659,1851),(41,7,1289,22206).$$

We see that the possibility $(s, t_2, t, b) = (3, 3, 27, 43)$, which is the remaining open case in the classification of the finite dense near hexagons with four points per line, is no longer among these possibilities. In fact, it is only the very last condition (Corollary 9.10) which is responsible for its elimination. At the end of this section we give an alternative proof for the nonexistence of the case $(s, t_2, t, b) = (3, 3, 27, 43)$ still relying on ideas of Section 9 (actually this was the first proof we found). In this section, we also kill two additional possibilities, namely $(12, 4, 244, 2989)$ and $(3, 3, 21, 22)$, and note that the possibility $(2, 2, 14, 35)$ corresponds to the near hexagon $\mathbb{E}_2$. In this way, we have proved the second claim of Theorem 1.1.

The case $(s, t_2, t, b) = (2, 2, 14, 35)$

By Lemma 4.5, we know that $\mathcal{S}$ is a regular near hexagon with parameters $(s, t, t_2) = (2, 14, 2)$. By [4], we know that there exists up to isomorphism a unique such near hexagon, namely the regular near hexagon $\mathbb{E}_2$ related to the Witt design $S(5, 8, 24)$.

The case $(s, t_2, t, b) = (12, 4, 244, 2989)$

By Lemma 4.5, we know that $\mathcal{S}$ is a regular near hexagon with parameters $(s, t, t_2) = (12, 244, 4)$. The eigenvalues and multiplicities can be computed with the aid of the techniques mentioned at the end of Section 2. The near hexagon $\mathcal{S}$ has four eigenvalues, namely $\lambda_1 = s(t+1) = 2940$, $\lambda_4 = -(t+1) = -245$ and the two roots $\lambda_2 = 91$ and $\lambda_3 = -25$ of the quadratic polynomial $x^2 - (s-1)(t_2+2)x + (s^2 - s + 1)t_2 - st + (s-1)^2 = x^2 - 66x - 2275 = (x+25)(x-91)$. If m_i with $i \in \{1, 2, 3, 4\}$ denotes the multiplicity of the eigenvalue λ_i, then

$$\begin{aligned} v &= (s+1)\Big(1 + st + \frac{s^2 t(t-t_2)}{t_2+1}\Big) = 21962941, \\ m_4 &= s^3 \frac{(t_2+1) + s(t_2+1)t + s^2 t(t-t_2)}{s^2(t_2+1) + st(t_2+1) + t(t-t_2)} = 197469, \\ m_3 &= \frac{\lambda_2(v - 1 - m_4) - (t+1)(m_4 - s)}{\lambda_2 - \lambda_3} \notin \mathbb{N}. \end{aligned}$$

The case $(s, t_2, t, b) = (3, 3, 21, 22)$

This case was in fact already treated in [10], but with what we have derived so far, the treatment is very short. For this reason, we have decided to include it. By Lemma 7.2(2), we know that through every point x of a quad Q of order $(s, t_2) = (3, 3)$, there are $\phi_1 = 3$ quads of order $(3, 3)$ meeting Q in the singleton $\{x\}$. So, for each line $L = \{y_1, y_2, y_3, y_4\}$ in the local space $\mathcal{S}_x$, there are three lines of size 4 disjoint from L and 18 lines of size 4 meeting L in a singleton. Suppose M_1 is a line of size 4 meeting L in the singleton $\{y_1\}$. Then there are at least $(\alpha_{y_2} - 4) + (\alpha_{y_3} - 4) + (\alpha_{y_4} - 4) = \alpha_{y_2} + \alpha_{y_3} + \alpha_{y_4} - 12$ lines of size 4 disjoint from M_1, implying that $\alpha_{y_2} + \alpha_{y_3} + \alpha_{y_4} \leq 15$. As $\alpha_{y_1} + \alpha_{y_2} + \alpha_{y_3} + \alpha_{y_4} = 18 + 4 = 22$ and $\alpha_{y_i} \leq \lfloor \frac{t}{t_2} \rfloor = 7$ for every $i \in \{1, 2, 3, 4\}$, we have $\alpha_{y_1} = 7$, and there are at least

(1,7,9,9)	212	(3,7,7,9)	188	(5,5,7,9)	180
(1,8,8,9)	210	(3,7,8,8)	186	(5,5,8,8)	178
(2,6,9,9)	202	(4,4,9,9)	194	(5,6,6,9)	178
(2,7,8,9)	198	(4,5,8,9)	186	(5,6,7,8)	174
(2,8,8,8)	196	(4,6,7,9)	182	(5,7,7,7)	172
(3,5,9,9)	196	(4,6,8,8)	180	(6,6,6,8)	172
(3,6,8,9)	190	(4,7,7,8)	178	(6,6,7,7)	170

Table 1: The possibilities for $(\alpha_{L_1}, \alpha_{L_2}, \alpha_{L_3}, \alpha_{L_4})$

three $i \in \{1,2,3,4\}$ for which $\alpha_{y_i} \geq 2$. Without loss of generality, we may suppose that $\alpha_{y_1}, \alpha_{y_2}, \alpha_{y_3} \geq 2$. Repeating the above argument for lines M_2 and M_3 of size 4 intersecting L in the singletons $\{y_2\}$ and $\{y_3\}$, we see that $\alpha_{y_2} = \alpha_{y_3} = 7$. So, $\alpha_{y_4} = 1$, i.e. every line of size 4 of $\mathcal{S}_x$ contains a unique point y for which $\alpha_y = 1$. As $b = 22 = t+1$, all points y of $\mathcal{S}_x$ satisfy $\alpha_y = 1$, an obvious contradiction.

Another treatment for the case $(s, t_2, t, b) = (3, 3, 27, 43)$

In both Sections 8 and 9, we derived a number of conditions that must be satisfied by the α_L's. It might be possible that contradictions can be obtained by combining these two sets of conditions. We illustrate this principle here by giving a nonexistence proof for the case $(s, t_2, t, b) = (3, 3, 27, 43)$ (which is also excluded by Corollary 9.10). In fact, the proof presented here was our first nonexistence proof for this case.

By the classification of all finite generalized quadrangles with four points per line ([18]; [21, Section 6.2]), we know that there are up to isomorphism two generalized quadrangles of order $(3,3)$, namely $Q(4,3)$ and $W(3)$. As $W(3)$ does not have ovoids ([21, 3.2.1 and 3.4.1]) and no quad of order $(3,3)$ is big in $\mathcal{S}$, we know that all quads of order $(3,3)$ of $\mathcal{S}$ are isomorphic to $Q(4,3)$.

If Q is a $Q(4,3)$-quad and $x \in Q$, then we know from Lemma 8.1 that $\sum_{L \in \mathcal{L}_{x,Q}} \alpha_L = 26$. As $\alpha_L \leq \lfloor \frac{t}{t_2} \rfloor = 9$ for every line L of $\mathcal{S}$, we then know the following.

Lemma 10.1 *Let x be a point of a $Q(4,3)$-quad Q, and put $\mathcal{L}_{x,Q} = \{L_1, L_2, L_3, L_4\}$ such that $\alpha_{L_1} \leq \alpha_{L_2} \leq \alpha_{L_3} \leq \alpha_{L_4}$. Then $(\alpha_{L_1}, \alpha_{L_2}, \alpha_{L_3}, \alpha_{L_4})$ is equal to one of the 4-tuples mentioned in Table 1.*

If x is a point of a $Q(4,3)$-quad Q and $\mathcal{L}_{x,Q} = \{L_1, L_2, L_3, L_4\}$ such that $\alpha_{L_1} \leq \alpha_{L_2} \leq \alpha_{L_3} \leq \alpha_{L_4}$, then we put $\eta(x,Q) := \alpha_{L_1}^2 + \alpha_{L_2}^2 + \alpha_{L_3}^2 + \alpha_{L_4}^2$. For each of the possibilities for $(\alpha_{L_1}, \alpha_{L_2}, \alpha_{L_3}, \alpha_{L_4})$, we have also mentioned the corresponding value of $\eta(x,Q)$ in Table 1. If $\{x_1, x_2, \ldots, x_{10}\}$ is an ovoid of Q, then by Corollary 9.7

$$\sum_{i=1}^{10} \eta(x_i, Q) = \frac{\Omega' - \Omega}{s(t_2 - 1)} = 2090. \tag{12}$$

By Lemma 10.1, we know that

$$\eta(x_i, Q) \in \{170, 172, 174, 178, 180, 182, 186, 188, 190, 194, 196, 198, 202, 210, 212\}$$

for every $i \in \{1, 2, \ldots, 10\}$. Let I_1 denote the set of all $i \in \{1, 2, \ldots, 10\}$ for which $\eta(x_i, Q) > 209$ and put $I_2 = \{1, 2, \ldots, 10\} \setminus I_1$. Equation (12) implies that

$$\sum_{i \in I_1} \Big(\eta(x_i, Q) - 209\Big) = \sum_{i \in I_2} \Big(209 - \eta(x_i, Q)\Big).$$

As $\eta(x_i, Q) - 209 \leq 3$ for every $i \in I_1$ and $209 - \eta(x_i, Q) \geq 7$ for every $i \in I_2$, we have $3 \cdot |I_1| \geq 7 \cdot |I_2| = 7 \cdot \Big(10 - |I_1|\Big)$. This implies that $10 \cdot |I_1| \geq 70$, i.e. $|I_1| \geq 7$.

So, every $Q(4,3)$-quad contains at least seven lines L for which $\alpha_L = 1$. On the other hand, every line L for which $\alpha_L = 1$ is contained in a unique $Q(4,3)$-quad. As there are $\frac{vb}{(s+1)(st_2+1)} = 8944$ $Q(4,3)$-quads, the number of lines L for which $\alpha_L = 1$ is at least $8944 \cdot 7 = 62608$. But this is impossible as there are only $\frac{v(t+1)}{s+1} = 58240$ lines.

11 Proof of Theorem 1.1 in case there are big quads of order (s, t_2)

In this section, we suppose that $\mathcal{S}$ is a finite near hexagon satisfying the properties (P1), (P2) and (P3) for some $s, t_2 \in \mathbb{N} \setminus \{0, 1\}$. We moreover assume that there exist big quads of order (s, t_2). In particular, we have $b > 0$. The following sequence of lemmas proves the last claim of Theorem 1.1. We note that each of the near hexagons mentioned in (a), (b), (c), (d) of Theorem 1.1 satisfies Properties (P1), (P2) and (P3).

Lemma 11.1 *If $\mathcal{S}$ is a classical near hexagon, then $\mathcal{S}$ is one of the following:*

- *a dual polar space $DW(5, q)$, $DQ(6, q)$, $DQ^-(7, q)$, $DH(5, q^2)$, $DH(6, q^2)$ for some prime power q;*
- *a product near hexagon of the form $\mathbb{L}_{s+1} \times Q$, where Q is a generalized quadrangle of order (s, t_2).*

Proof. This follows from Tits' classification of polar spaces, taking into account that $\mathcal{S}$ is a classical dense near hexagon, see Section 2. ∎

Since every quad of order (s, t_2) is big, the following is a special case of Theorem 6.10 of [13].

Lemma 11.2 *If Q is a quad of order (s, t_2), then every other quad intersects Q in the empty set or a line.*

Lemma 11.3 *There exist two quads intersecting in a singleton. Any two such quads are grids.*

Proof. Since $\mathcal{S}$ is not classical, there exists a nonclassical point-quad pair (x, Q). Then $\mathrm{d}(x, Q) = 2$. If $y \in \Gamma_2(x) \cap Q$, then the quads $\langle x, y \rangle$ and Q intersect in the singleton $\{y\}$. The second claim follows from Lemma 11.2. ■

Lemma 11.4 *We have $b \in \{1, 2, 3, 4\}$.*

Proof. By Lemma 11.3, there exist two grid-quads R_1 and R_2 intersecting in a singleton $\{x\}$. By Lemma 11.2, each of the b quads of order (s, t_2) through x intersects R_1 and R_2 in lines and so there are at most four of them. ■

Lemma 11.5 *The case $b = 1$ cannot occur.*

Proof. Let R_1 and R_2 be two grid-quads intersecting in a point x. Denote by Q the unique quad of order (s, t_2) through x. Let L_i with $i \in \{1, 2\}$ denote the unique line of R_i through x not contained in Q. Then $L_1 \neq L_2$ and by Lemma 11.2, the quad $\langle L_1, L_2 \rangle$ must intersect Q in a line through x. As $\langle L_1, L_2 \rangle$ has at least three lines through x, it must be a quad of order (s, t_2), in contradiction with $b = 1$. ■

Lemma 11.6 *If $b = 2$, then $\mathcal{S}$ is a glued near hexagon of type $Q_1 \otimes Q_2$, where Q_1 and Q_2 are two quads of order (s, t_2).*

Proof. Consider a point x and let Q_1, Q_2 denote the two quads of order (s, t_2) through x. Then $L = Q_1 \cap Q_2$ is a line by Lemma 11.2.

We show that $t = 2t_2$. If this were not the case, then there exists a line K through x not contained in $Q_1 \cup Q_2$. Let K' be a line of Q_2 through x distinct from L. As the quad $\langle K, K' \rangle$ meets Q_1 in a line, there are at least three lines through x contained in $\langle K, K' \rangle$ and so $\langle K, K' \rangle$ is a quad of order (s, t_2) distinct from Q_1 and Q_2, in contradiction with $b = 2$. Hence, $t = 2t_2$.

As $t = 2t_2$ and any point is contained in precisely $b = 2$ quads of order (s, t_2), every local space should be isomorphic to the unique linear space on $1 + 2t_2$ points having two lines of size $t_2 + 1$ and t_2^2 lines of size 2. This implies by Theorem 7.2 of [9] that $\mathcal{S}$ is a glued near hexagon, necessarily of type $Q_1 \otimes Q_2$. ■

In view of Lemmas 11.4, 11.5 and 11.6, we may assume that $b \in \{3, 4\}$.

Lemma 11.7 *Let x be a point of $\mathcal{S}$. Then there exists no line L through x such that every quad of order (s, t_2) through x contains L.*

Proof. Suppose to the contrary that this is the case. As $b \in \{3, 4\}$, there then exist three quads Q_1, Q_2 and Q_3 of order (s, t_2) through L. Let R be a quad through x not containing L. Then R must be a grid-quad containing the three distinct lines $R \cap Q_1$, $R \cap Q_2$ and $R \cap Q_3$ through the point x, a contradiction. ■

Lemma 11.8 *There exists a constant α such that every line is contained in α quads of order (s, t_2).*

Proof. Let L be a line and denote by β_L, respectively α_L, the number of grid-quads, respectively quads of order (s, t_2), through the line L. By Lemma 11.7, there exists a quad Q of order (s, t_2) meeting L in a singleton $\{x\}$. As every quad through L intersects Q in a line, we have

$$\alpha_L + \beta_L = t_2 + 1. \tag{13}$$

As the quads through L partition the set of lines through x distinct from L, we have

$$t_2\alpha_L + \beta_L = t. \tag{14}$$

From (13) and (14), we have

$$\alpha_L = \frac{t - t_2 - 1}{t_2 - 1}. \tag{15}$$

So, the α_L's are constant. ∎

Lemma 11.9 *We have $\alpha = 2$.*

Proof. As $b \geq 2$, there exist two quads of order (s, t_2) intersecting in a line and so we have $\alpha \geq 2$.

Now, let R be a grid-quad and L a line meeting R in a singleton. Each of the α quads of order (s, t_2) through L intersects R in a line and so we also have $\alpha \leq 2$. ∎

Lemma 11.10 *We have $t_2 = 2$, $b = 4$ and $t = 5$.*

Proof. Let Q be a quad of order (s, t_2) and $x \in Q$. Each line of Q through x is contained in $\alpha - 1 = 1$ quads of order (s, t_2) distinct from Q. As every quad of order (s, t_2) through x distinct from Q meets Q in a line, we have $b = t_2 + 2$. As $t_2 \geq 2$ and $b \leq 4$, we have $t_2 = 2$ and $b = 4$. Equation (15) then implies that $t = 5$. ∎

Lemma 11.11 *Every local space is isomorphic to the Fano plane in which one point has been removed.*

Proof. Every local space has six points and four lines of size 3. There is only one such linear space, namely the Fano plane in which one point has been removed. ∎

Lemma 11.12 *We have $s \in \{2, 4\}$.*

Proof. As there exists a quad of order $(s, t_2) = (s, 2)$ with $s > 1$, we have $s \in \{2, 4\}$ since $s \leq t_2^2$ and $(s + t_2) \mid st_2(s + 1)(t_2 + 1)$ by [21, 1.2.2 & 1.2.3]. ∎

Lemma 11.13 *If $s = 2$, then $\mathcal{S} \cong \mathbb{H}_3$.*

Proof. This follows from the classification of the finite dense near hexagons with three points per line obtained in [5]. ∎

Lemma 11.14 *The case $s = 4$ cannot occur.*

Proof. Let Q be a quad of order $(s, t_2) = (4, 2)$. Then $|Q| = 45$ and $v = |Q| \cdot (1 + s(t - t_2)) = 585$. As the number $((s + 1)^2(s - 1)(s^2 + 1) + st(s - 1)(s + 1)^2 + v)^{-1}(s^5 v)$ is not integral, a contradiction follows from Lemma 3.1. ∎

A Number of cases killed by the various conditions

Result	Type of condition	# cases killed
Lemma 5.2	Divisibility conditions	6595
Lemma 5.2	Inequalities	9786
Lemma 5.4	Inequality	2466
Lemma 5.4	Divisibility condition	3
Corollary 5.9	Second inequality	7233
Corollary 5.10	Inequality	7245
Lemma 5.11 + Remark	Inequality	2
Lemma 5.12	$b \neq 1$	0
Lemma 6.1(2)+(5)	Inequalities	35
Lemma 7.2(1)	Inequality	5507
Lemma 7.3 + Remark	Divisibility condition	8172
Lemma 8.2	Parity argument	3326
Lemma 8.3	Inequality	7666
Lemmas 8.2 and 8.3	Divisibility conditions	0
Corollary 9.7	Divisibility condition	9098
Corollary 9.7	Parity argument	4685
Lemma 9.8	Inequality	374
Lemma 9.8	Divisibility conditions	42
Corollary 9.10	Divisibility condition	7203

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