On blocking sets of the tangent lines to a nonsingular quadric in PG$(3, q)$, q prime

Bart De Bruyn Francesco Pavese Puspendu Pradhan
Binod Kumar Sahoo

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Abstract

Let $Q^-(3, q)$ be an elliptic quadric and $Q^+(3, q)$ a hyperbolic quadric in PG$(3, q)$. For $\epsilon \in \{-, +\}$, let $\mathcal{T}^\epsilon$ denote the set of all tangent lines of PG$(3, q)$ with respect to $Q^\epsilon(3, q)$. If k is the minimum size of a $\mathcal{T}^\epsilon$-blocking set in PG$(3, q)$, then it is known that $q^2+1 \leq k \leq q^2+q$. For an odd prime q, we prove that there are no $\mathcal{T}^+$-blocking sets of size $q^2 + 1$ and that the quadric $Q^-(3, q)$ is the only $\mathcal{T}^-$-blocking set of size $q^2 + 1$ in PG$(3, q)$. When $q = 3$, we show with the aid of a computer that there are no minimal $\mathcal{T}^-$-blocking sets of size 11 and that, up to isomorphisms, there are eight minimal $\mathcal{T}^-$-blocking sets of size 12 in PG$(3, 3)$. We also provide geometrical constructions for these eight mutually nonisomorphic minimal $\mathcal{T}^-$-blocking sets of size 12.

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1 Introduction

1.1 Nonsingular quadrics in finite projective spaces

In this section, we recall the basic properties of nonsingular quadrics in finite projective spaces. These properties as well as additional background information on quadrics can be found in the standard works on this topic such as [11, 12].

Let PG(n, q), $n \geq 1$, denote the n-dimensional projective space defined over the finite field $\mathbb{F}_q$ of order q. A quadric and a line of PG(n, q) intersect each other in either $0, 1, 2$ or $q + 1$ points. With respect to a quadric $\mathcal{Q}$, a line L of PG(n, q) is called *external* if $|L \cap \mathcal{Q}| = 0$, *secant* if $|L \cap \mathcal{Q}| = 2$, and *tangent* if $|L \cap \mathcal{Q}| \in \{1, q + 1\}$.

There are two types of nonsingular quadrics in PG(n, q) if $n \geq 1$ is odd: (1) *elliptic quadrics* which are of Witt index $(n - 1)/2$, and (2) *hyperbolic quadrics* which are of Witt index $(n+1)/2$. An elliptic quadric in PG(n, q), n odd, will be denoted by $Q^-(n, q)$ while a hyperbolic quadric in this projective space will be denoted by $Q^+(n,q)$. An elliptic quadric in $\mathrm{PG}(1,q)$ is empty while a hyperbolic quadric is a set of two points.

If $n \geq 2$ is even, then every nonsingular quadric of $\mathrm{PG}(n,q)$ has Witt index $n/2$. Such a quadric is called a *parabolic quadric* and denoted by $Q(n,q)$. A parabolic quadric in $\mathrm{PG}(2,q)$ is nothing else than an irreducible conic.

Let $\mathcal{Q}$ be a nonsingular quadric in $\mathrm{PG}(n,q)$, $n \geq 1$, and suppose that q is odd if n is even. With the quadric $\mathcal{Q}$, there is naturally associated a polarity ζ of $\mathrm{PG}(n,q)$ which is symplectic if q is even (and n odd) and orthogonal if q is odd. For every point x of $\mathcal{Q}$, x^ζ is a hyperplane *tangent to* $\mathcal{Q}$ *at the point* x, and the tangent lines through x are precisely the lines through x contained in x^ζ. For every point x of $\mathrm{PG}(n,q) \setminus \mathcal{Q}$, x^ζ is a nontangent hyperplane intersecting $\mathcal{Q}$ in a nonsingular quadric $\mathcal{Q}_x$ of x^ζ. If q is odd, then $x \notin x^\zeta$ and the tangent lines through x are precisely the lines through x containing a point of $\mathcal{Q}_x$. If q is even, then $x \in x^\zeta$ and the tangent lines through x are precisely the lines through x contained in x^ζ.

Consider now a parabolic quadric $Q(n,q)$ in $\mathrm{PG}(n,q)$ where $n \geq 2$ is even and q is odd. If x is a point of $\mathrm{PG}(n,q) \setminus Q(n,q)$, then the intersection $x^\zeta \cap Q(n,q)$ is either an elliptic or hyperbolic quadric of $x^\zeta \cong \mathrm{PG}(n-1,q)$. If $n=2$, so if $Q(2,q)$ is an irreducible conic $\mathcal{C}$ of the projective plane $\mathrm{PG}(2,q)$, then such a point is called an *interior* or *exterior point* (with respect to $\mathcal{C}$) depending on whether the former or the latter case occurs. In $\mathrm{PG}(2,q)$, there are $\frac{q(q+1)}{2}$ exterior points and $\frac{q(q-1)}{2}$ interior points. Every tangent line (with respect to $\mathcal{C}$) contains one point of $\mathcal{C}$ and q exterior points. More information on the basic properties of points and lines of $\mathrm{PG}(2,q)$ with respect to an irreducible conic can be found in [13].

Of special interest in this paper are the elliptic and hyperbolic quadrics in $\mathrm{PG}(3,q)$. The elliptic quadric $Q^-(3,q)$ contains q^2+1 points such that no three of them are on the same line. Every point of $Q^-(3,q)$ is contained in $q+1$ tangent lines, this gives $(q+1)(q^2+1)$ tangent lines to $Q^-(3,q)$. We also have $q^2(q^2+1)/2$ secant lines and then $(q^2+1)(q^2+q+1)-q^2(q^2+1)/2-(q+1)(q^2+1) = q^2(q^2+1)/2$ external lines to $Q^-(3,q)$. Every point of $Q^-(3,q)$ is contained in q^2 secant lines. Every point of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ is contained in $q+1$ tangent lines, $q(q-1)/2$ secant lines and $q(q+1)/2$ external lines.

The hyperbolic quadric $Q^+(3,q)$ contains $(q+1)^2$ points and $2(q+1)$ (tangent) lines. Every point of $Q^+(3,q)$ is contained in $q+1$ tangent lines and q^2 secant lines. Every point of $\mathrm{PG}(3,q) \setminus Q^+(3,q)$ is contained in $q+1$ tangent lines, $q(q+1)/2$ secant lines and $q(q-1)/2$ external lines. There are $(q+1)(q^2+1)$ tangent lines, $q^2(q+1)^2/2$ secant lines and $q^2(q-1)^2/2$ external lines to $Q^+(3,q)$.

Consider now the quadric $Q^\epsilon(3,q)$ in $\mathrm{PG}(3,q)$ where $\epsilon \in \{+,-\}$ and q is even. The polarity ζ is then symplectic and so there an associated symplectic generalized quadrangle $W(q)$, whose points are the points of $\mathrm{PG}(3,q)$ and whose lines are the lines of $\mathrm{PG}(3,q)$ that are totally isotropic with respect to ζ, with incidence being containment. The lines of $W(q)$ are precisely the lines of $\mathrm{PG}(3,q)$ that are tangent to $Q^\epsilon(3,q)$. One can refer to [17] for the basics on finite generalized quadrangles.

1.2 Blocking sets in $\mathrm{PG}(3,q)$

For a given nonempty set $\mathcal{L}$ of lines of $\mathrm{PG}(n,q)$, a set X of points of $\mathrm{PG}(n,q)$ is called an *$\mathcal{L}$-blocking set* if each line of $\mathcal{L}$ meets X. The first step in the study of blocking sets has been to determine the smallest cardinality of a blocking set and to characterize, if possible, all blocking sets of that cardinality. If $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(n,q)$ and X is an $\mathcal{L}$-blocking set in $\mathrm{PG}(n,q)$, then $|X| \geq (q^n-1)/(q-1)$ and equality holds if and only if X is the point set of a hyperplane of $\mathrm{PG}(n,q)$. This classical result was proved by Bose and Burton in [4, Theorem 1].

We denote by $\mathcal{E}^\epsilon$, $\mathcal{S}^\epsilon$ and $\mathcal{T}^\epsilon$ the set of lines of $\mathrm{PG}(3,q)$ that are external, secant and tangent, respectively, with respect to the quadric $Q^\epsilon(3,q)$, $\epsilon \in \{-,+\}$. If $\mathcal{L}^\epsilon$ is one of the line sets $\mathcal{E}^\epsilon$, $\mathcal{S}^\epsilon$, $\mathcal{E}^\epsilon \cup \mathcal{T}^\epsilon$, $\mathcal{E}^\epsilon \cup \mathcal{S}^\epsilon$ and $\mathcal{T}^\epsilon \cup \mathcal{S}^\epsilon$, then the minimum size $\mathcal{L}^\epsilon$-blocking sets in $\mathrm{PG}(3,q)$ are characterized in the papers [3, 6, 7, 21] for $\epsilon = -$ and in the papers [2, 3, 8, 19, 20, 21] for $\epsilon = +$.

In this paper, we study $\mathcal{T}^\epsilon$-blocking sets in $\mathrm{PG}(3,q)$. A $\mathcal{T}^\epsilon$-blocking set X in $\mathrm{PG}(3,q)$ is said to be *minimal* if X has no proper subset which is also a $\mathcal{T}^\epsilon$-blocking set in $\mathrm{PG}(3,q)$. Every $\mathcal{T}^\epsilon$-blocking set in $\mathrm{PG}(3,q)$ of minimum size is a minimal $\mathcal{T}^\epsilon$-blocking set. Two $\mathcal{T}^\epsilon$-blocking sets X_1 and X_2 in $\mathrm{PG}(3,q)$ are said to be *isomorphic* if there exists a collineation of $\mathrm{PG}(3,q)$ stabilizing $Q^\epsilon(3,q)$ and mapping X_1 to X_2.

If X is a minimum size $\mathcal{T}^\epsilon$-blocking set in $\mathrm{PG}(3,q)$, then $q^2+1 \leq |X| \leq q^2+q$, and $|X| = q^2+1$ if and only if every tangent line contains a unique point of X. This was proved in [9, Lemmas 2.1, 2.2] for $\epsilon = +$ and the same arguments work for $\epsilon = -$ as well.

If q is even, then the $\mathcal{T}^\epsilon$-blocking sets in $\mathrm{PG}(3,q)$ of size q^2+1 are precisely the ovoids[1] of the generalized quadrangle $W(q)$ associated with $Q^\epsilon(3,q)$. There are two known types of ovoids of $W(q)$: ovoids which are elliptic quadrics of the ambient projective space $\mathrm{PG}(3,q)$ and the so-called *Suzuki-Tits ovoids*. The former ovoids exist for each power of 2 and the latter only for the prime powers of the form 2^{2m+1} for positive integers m. One can refer to [11, Section 16.4] for more on Suzuki-Tits ovoids. By [10, 14, 15, 16], every ovoid of $W(q)$ is an elliptic quadric if $q \in \{2,4,16\}$, and either an elliptic quadric or a Suzuki-Tits ovoid if $q \in \{8,32\}$. However, classifying all ovoids of $W(q)$ for other even q is still an open problem.

In the q odd case, not much is known for the minimum size $\mathcal{T}^\epsilon$-blocking sets in $\mathrm{PG}(3,q)$. In $\mathrm{PG}(3,3)$, by [9, Theorem 1.1], there is no $\mathcal{T}^+$-blocking set of size 10 and there are exactly two $\mathcal{T}^+$-blocking sets of size 11 up to isomorphisms. By means of the computer algebra systems GAP [22] and Sage [18], it was proved that there exist no $\mathcal{T}^+$-blocking sets in $\mathrm{PG}(3,q)$ of size q^2+1 for each odd prime power $q \leq 13$, see [9, Theorem 1.2]. We generalize this result to all odd primes and prove the following.

Theorem 1.1. *If q is an odd prime, then there are no $\mathcal{T}^+$-blocking sets in $\mathrm{PG}(3,q)$ of size q^2+1.*

For every odd q, the quadric $Q^-(3,q)$ itself is an example of a $\mathcal{T}^-$-blocking set in $\mathrm{PG}(3,q)$ of size q^2+1. We prove the following result for all odd primes.

[1] An ovoid of a point-line geometry is a set of points meeting each line in a unique point.

Theorem 1.2. *Suppose that q is an odd prime. If B is a $\mathcal{T}^-$-blocking set in* $\mathrm{PG}(3,q)$ *of size q^2+1, then $B = Q^-(3,q)$.*

When $q = 3$, Theorem 1.2 implies that $Q^-(3,3)$ is the only minimal $\mathcal{T}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10. By means of the computer algebra systems GAP [22] and Sage [18], we are able to show the following.

Theorem 1.3. *Suppose that $q = 3$. Then there are no minimal $\mathcal{T}^-$-blocking sets in* $\mathrm{PG}(3,3)$ *of size* 11*. Up to isomorphisms, there are* 8 *minimal $\mathcal{T}^-$-blocking sets in* $\mathrm{PG}(3,3)$ *of size* 12.

We also provide geometrical constructions of the eight mutually nonisomorphic minimal $\mathcal{T}^-$-blocking sets of size 12 in $\mathrm{PG}(3,3)$.

2 Some useful facts

Let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$, where q is odd and $n \geq 2$. Consider in the associated projective space $\mathrm{PG}(n,q) = \mathrm{PG}(V)$ a nonsingular quadric $\mathcal{Q} = \mathcal{Q}(n)$ described by a quadratic form Q on V. Let B denote the bilinear form associated with Q, i.e. $B(\bar{x},\bar{y}) = Q(\bar{x}+\bar{y}) - Q(\bar{x}) - Q(\bar{y})$ for all $\bar{x},\bar{y} \in V$. Note that $\mathcal{Q}$ can be of *parabolic*, *hyperbolic* or *elliptic* type. We denote by ζ the orthogonal polarity of $\mathrm{PG}(n,q)$ naturally associated with $\mathcal{Q}$. The following lemma is well-known, see e.g. [12].

Lemma 2.1. *One of the following cases occurs for a plane π of* $\mathrm{PG}(n,q)$ *not contained in $\mathcal{Q}$:*

(I) *$\pi \cap \mathcal{Q}$ is a singleton;*

(II) *$\pi \cap \mathcal{Q}$ is a line;*

(III) *$\pi \cap \mathcal{Q}$ is the union of two distinct (intersecting) lines;*

(IV) *$\pi \cap \mathcal{Q}$ is an irreducible conic in π.*

Let S (respectively, N) denote the set of all points $\langle \bar{x} \rangle$ of $\mathrm{PG}(n,q) \setminus \mathcal{Q}$ for which $Q(\bar{x})$ is a nonzero square (respectively, a nonsquare) in $\mathbb{F}_q$. The sets S and N are well-defined. If $\lambda \in \mathbb{F}_q^*$ and $\bar{x} \in V \setminus \{\bar{0}\}$, then $Q(\bar{x})$ is a (non)square if and only if $Q(\lambda\bar{x}) = \lambda^2 \cdot Q(\bar{x})$ is a (non)square.

Lemma 2.2. *If $\langle \bar{x} \rangle$ and $\langle \bar{y} \rangle$ are two distinct points of* $\mathrm{PG}(n,q) \setminus \mathcal{Q}$ *on a tangent line to $\mathcal{Q}$, then $Q(\bar{x}) \cdot Q(\bar{y})$ is a nonzero square.*

Proof. Let $\langle \bar{z} \rangle$ denote the tangency point. If we take $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$ such that $\bar{y} = \lambda_1\bar{z} + \lambda_2\bar{x}$, then $Q(\bar{y}) = \lambda_1^2 \cdot Q(\bar{z}) + \lambda_2^2 \cdot Q(\bar{x}) + \lambda_1\lambda_2 \cdot B(\bar{z},\bar{x}) = \lambda_2^2 \cdot Q(\bar{x})$ and so the claim of the lemma holds. □

Lemma 2.3. *If L is a secant line to $\mathcal{Q}$, then $\frac{q-1}{2}$ points of $L \setminus \mathcal{Q}$ belong to S and the remaining $\frac{q-1}{2}$ points of $L \setminus \mathcal{Q}$ belong to N.*

Proof. Put $L \cap \mathcal{Q} = \{\langle \bar{x} \rangle, \langle \bar{y} \rangle\}$. As L is a secant line, we have $B(\bar{x}, \bar{y}) \neq 0$. Then the $q-1$ points of $L \setminus \mathcal{Q}$ are the $q-1$ points $\langle \bar{x} + \lambda \bar{y} \rangle$, $\lambda \in \mathbb{F}_q^*$. Clearly,

$$Q(\bar{x} + \lambda \bar{y}) = Q(\bar{x}) + \lambda^2 Q(\bar{y}) + \lambda \cdot B(\bar{x}, \bar{y}) = \lambda \cdot B(\bar{x}, \bar{y})$$

takes all values of $\mathbb{F}_q^*$ as λ ranges over all elements of $\mathbb{F}_q^*$. Since $\frac{q-1}{2}$ of these values are nonzero squares and the other $\frac{q-1}{2}$ values are nonsquares, the claim of the lemma holds. □

Lemma 2.4. *Let μ be a nonsquare in $\mathbb{F}_q$. Then there are $a, b \in \mathbb{F}_q$ such that $a^2 - \mu b^2 = \mu$.*

Proof. There are $\frac{q+1}{2}$ squares in $\mathbb{F}_q$ and so there are $\frac{q+1}{2}$ elements of the form $\mu + \mu b^2$, $b \in \mathbb{F}_q$. One of these elements must therefore also be a square. The claim follows. □

Lemma 2.5. *If L is an external line to $\mathcal{Q}$, then $\frac{q+1}{2}$ points of L belong to S and the remaining $\frac{q+1}{2}$ points belong to N.*

Proof. We choose linearly independent vectors $\bar{e}_1, \bar{e}_2 \in V$ such that $\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle \in L$ and $B(\bar{e}_1, \bar{e}_2) = 0$. Then there exists a $\lambda \in \mathbb{F}_q^*$ and a nonsquare $\mu \in \mathbb{F}_q^*$ such that $f(X_1, X_2) := Q(X_1 \bar{e}_1 + X_2 \bar{e}_2) = \lambda(X_1^2 - \mu X_2^2)$, where $(X_1, X_2) \in \mathbb{F}_q^2$. For every $\eta \in \mathbb{F}_q^*$, let N_η denote the number of $(X_1, X_2) \in \mathbb{F}_q^2 \setminus \{(0,0)\}$ for which $f(X_1, X_2) = \eta$.

As $f(kX_1, kX_2) = k^2 f(X_1, X_2)$ for every $k \in \mathbb{F}_q^*$ and every $(X_1, X_2) \in \mathbb{F}_q^2$, we have $N_{\eta_1} = N_{\eta_2}$ if $\eta_1, \eta_2 \in \mathbb{F}_q^*$ with $\frac{\eta_1}{\eta_2}$ a square. Thus if η_1, η_2 are both nonzero squares or both nonsquares, then $N_{\eta_1} = N_{\eta_2}$.

By Lemma 2.4, we can take $a, b \in \mathbb{F}_q$ such that $a^2 - \mu b^2 = \mu$. We then have $f(aX_1 + \mu b X_2, bX_1 + aX_2) = \mu f(X_1, X_2)$. This implies that $f(X_1, X_2) = 1$ if and only if $f(aX_1 + \mu b X_2, bX_1 + aX_2) = \mu$. As the map $\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q \times \mathbb{F}_q; (X_1, X_2) \mapsto (aX_1 + \mu b X_2, bX_1 + aX_2)$ is bijective, we thus have $N_1 = N_\mu$. Combining this with the previous paragraph, we see that N_η is constant for all $\eta \in \mathbb{F}_q^*$, necessarily equal to $m = \frac{q^2-1}{q-1} = q+1$.

As there are $\frac{q-1}{2}$ nonzero squares and $\frac{q-1}{2}$ nonsquares in $\mathbb{F}_q^*$, the fact that $Q(\bar{v}) \cdot Q(\bar{w}) \in S$ for all $\bar{v}, \bar{w} \in V \setminus \{\bar{0}\}$ such that $\langle \bar{v} \rangle = \langle \bar{w} \rangle \in \mathrm{PG}(n,q) \setminus \mathcal{Q}$ implies that $|L \cap S| = |L \cap N| = \frac{m \cdot \frac{q-1}{2}}{q-1} = \frac{q+1}{2}$. □

We define the following equivalence relation R on $\mathrm{PG}(n,q) \setminus \mathcal{Q}$:

> If $u, v \in \mathrm{PG}(n,q) \setminus \mathcal{Q}$, then $(u,v) \in R$ if there exist points $x_0, x_1, \ldots, x_k \in \mathrm{PG}(n,q) \setminus \mathcal{Q}$ for some $k \in \mathbb{N}$ such that $x_0 = u$, $x_k = v$ and $x_{i-1}x_i$ is a tangent line for every $i \in \{1, 2, \ldots, k\}$.

The following is an immediate consequence of Lemma 2.2.

Corollary 2.6. *If C is an equivalence class for R, then either $C \subseteq S$ or $C \subseteq N$.*

Lemma 2.7. *The following hold:*

(1) *If* $n = 2$ *and if* $\mathcal{E}$ *(respectively,* $\mathcal{I}$*) denotes the set of exterior (respectively, interior) points of* $\mathrm{PG}(2, q)$ *with respect to the irreducible conic* $\mathcal{Q}$*, then* $\{S, N\} = \{\mathcal{E}, \mathcal{I}\}$*. The equivalence classes of* R *are then* $\mathcal{E}$ *and the singletons contained in* $\mathcal{I}$*.*

(2) *If* $n \geq 3$*, then* S *and* N *are the only equivalence classes for* R*.*

Proof. Suppose first that $n = 2$. The fact that no interior point is contained in a tangent line then implies that the singletons contained in $\mathcal{I}$ are equivalence classes. If $x, y \in \mathcal{E}$, then by considering a common point of $\mathcal{E}$ on tangent lines through respectively x and y, we see that $(x, y) \in R$, implying that also $\mathcal{E}$ is an equivalence class. The fact that $\mathcal{E}$ is an equivalence class implies by Corollary 2.6 that $\mathcal{E} \subseteq A$, where A is either S or N. In order to complete the proof of the first claim of the lemma, it suffices to prove that no point $u \in \mathcal{I}$ belongs to A. If we take a secant line L through u, then L contains $\frac{q-1}{2}$ exterior points which all belong to A. By Lemma 2.3, we know that $|L \cap A| = \frac{q-1}{2}$, and so the interior point u cannot belong to A.

Suppose next that $n \geq 3$. Let A be either S or N. By Corollary 2.6, it suffices to prove that if x, y are two distinct elements of A, then $(x, y) \in R$. This is clearly the case if the line xy is tangent to $\mathcal{Q}$. So, we may suppose that xy is not a tangent line. Then xy^ζ intersects $\mathcal{Q}$ in a nonsingular quadric $\mathcal{Q}(n-2)$ of xy^ζ ([12]). No plane through xy is of Type II as xy is not a tangent line (see Lemma 2.1). Through xy, there is no Type III plane if xy is an external line and no Type I plane if xy is a secant line. The planes of Type I and III through xy are the planes $\langle u, xy \rangle$, where $u \in xy^\zeta \cap \mathcal{Q} = \mathcal{Q}(n-2)$, and each such plane contains a unique tangent line through x, namely ux. Now, x^ζ intersects $\mathcal{Q}$ in a nonsingular quadric $\mathcal{Q}(n-1)$ of x^ζ and the tangent lines through x are precisely the lines xv, where $v \in \mathcal{Q}(n-1)$. As $n \geq 3$ and $\mathcal{Q}(n-1)$ contains $\mathcal{Q}(n-2)$ as a hyperplane section, we have $|\mathcal{Q}(n-1)| > |\mathcal{Q}(n-2)|$ ([12]) and so there exists a tangent line L through x for which $\pi := \langle xy, L \rangle$ is a plane of type IV, i.e. $\pi \cap \mathcal{Q}$ is an irreducible conic $\mathcal{C}_\pi$ in π. Let $\mathcal{E}$ (respectively, $\mathcal{I}$) denote the set of points of π that are exterior (respectively, interior) with respect to $\mathcal{C}_\pi$. By construction of π, we know that $x \in \mathcal{E}$. By the first claim of the lemma, we know that $x, y \in A \cap \pi \in \{S \cap \pi, N \cap \pi\} = \{\mathcal{E}, \mathcal{I}\}$. So, both x, y belong to $\mathcal{E}$. The first claim of the lemma then implies that $(x, y) \in R$. □

In the case that n is even and that $\mathcal{Q}$ is a parabolic quadric of $\mathrm{PG}(n, q)$, we denote by H (respectively, E) the set of all points y of $\mathrm{PG}(n, q) \setminus \mathcal{Q}$ for which $y^\zeta \cap \mathcal{Q}$ is a hyperbolic (respectively, elliptic) quadric of y^ζ.

Lemma 2.8. *If* n *is even and* L *is a tangent line not entirely contained in* $\mathcal{Q}$*, then* $L \setminus \mathcal{Q}$ *is contained in either* H *or* E*.*

Proof. If $n = 2$, then we have $L \setminus \mathcal{Q} \subseteq H$ since all points of $L \setminus \mathcal{Q}$ are exterior with respect to $\mathcal{Q}$. So, we may assume $n \geq 4$.

Let $x \in L$ denote the tangency point and T_x the tangent hyperplane in the point x. Let α be a hyperplane of T_x not containing x. Then $\alpha \cap \mathcal{Q}$ is a parabolic quadric of α, and $T_x \cap \mathcal{Q}$ is the cone with kernel x and base $\alpha \cap \mathcal{Q}$. Put $L \cap \alpha = \{y\}$. As $y \notin \alpha \cap \mathcal{Q}$, $\beta := y^\zeta$ is a hyperplane through x intersecting $\alpha \cap \mathcal{Q}$ in a hyperbolic or elliptic quadric.

So, $L^\zeta \cap \mathcal{Q} = xy^\zeta \cap \mathcal{Q} = x^\zeta \cap y^\zeta \cap \mathcal{Q}$ is a cone with kernel x and base a hyperbolic or elliptic quadric in $\alpha \cap \beta$. So, all nontangent hyperplanes through L^ζ intersect $\mathcal{Q}$ in quadrics of the same type. These nontangent hyperplanes are precisely the hyperplanes z^ζ, where $z \in L \setminus \{x\}$. □

From Lemmas 2.7 and 2.8, we derive

Corollary 2.9. *If n is even and $\mathcal{Q}$ is a parabolic quadric of* $\mathrm{PG}(n, q)$*, then* $\{H, E\} = \{S, N\}$.

From Lemma 2.7, we also derive the following.

Corollary 2.10. *If θ is a collineation of* $\mathrm{PG}(n, q)$ *stabilizing $\mathcal{Q}$, then either θ stabilizes S and N, or interchanges S and N.*

If n is even, then the fact that $\{S, N\} = \{H, E\}$ implies that every collineation of $\mathrm{PG}(n, q)$ that stabilizes $\mathcal{Q}$ also stabilizes each of S and N. This conclusion is not valid if n is odd.

Lemma 2.11. *If $n \geq 3$ is odd, then there exists a collineation of* $\mathrm{PG}(n, q)$ *stabilizing $\mathcal{Q}$ and mapping S to N.*

Proof. Let μ be a given nonsquare in $\mathbb{F}_q$. If $\mathcal{Q}$ is a hyperbolic quadric with equation $X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0$ with respect to some reference system, then $(X_0, X_1, \ldots, X_n) \mapsto (X_0, \mu X_1, X_2, \mu X_3, \ldots, X_{n-1}, \mu X_n)$ defines the required collineation.

Suppose therefore that $\mathcal{Q}$ is an elliptic quadric with equation $X_0X_1 + \cdots + X_{n-3}X_{n-2} + X_{n-1}^2 - \mu X_n^2 = 0$ with respect to some reference system. By Lemma 2.4, we can take $a, b \in \mathbb{F}_q$ such that $a^2 - \mu b^2 = \mu$. Straightforward calculations show that the map

$$(X_0, X_1, \ldots, X_n) \mapsto (X_0, \mu X_1, \ldots, X_{n-3}, \mu X_{n-2}, aX_{n-1} + \mu bX_n, bX_{n-1} + aX_n)$$

defines the required collineation. □

Lemma 2.12. *Suppose n is odd. Then there exists an $(n + 2)$-dimensional vector space W having V as hyperplane and parabolic quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of* $\mathrm{PG}(W)$ *with associated orthogonal polarities ζ_1 and ζ_2 such that the following hold:*

(1) *$\mathcal{Q}_i \cap \mathrm{PG}(V) = \mathcal{Q}$ for every $i \in \{1, 2\}$;*

(2) *for every $x \in S$, $x^{\zeta_1} \cap \mathcal{Q}_1$ is a hyperbolic quadric of x^{ζ_1} and for every $y \in N$, $y^{\zeta_1} \cap \mathcal{Q}_1$ is an elliptic quadric of y^{ζ_1};*

(3) *for every $x \in S$, $x^{\zeta_2} \cap \mathcal{Q}_2$ is an elliptic quadric of x^{ζ_2} and for every $y \in N$, $y^{\zeta_2} \cap \mathcal{Q}_2$ is a hyperbolic quadric of y^{ζ_2}.*

Proof. Let $W = V \oplus \langle \bar{e} \rangle$ be a vector space containing V as a hyperplane, and let μ be a given nonsquare in $\mathbb{F}_q$.

If $\mathcal{Q}$ is of hyperbolic type, then for every $(\bar{v}, \lambda) \in V \times \mathbb{F}_q$, we define

$$Q_1(\bar{v} + \lambda\bar{e}) := Q(\bar{v}) + \lambda^2, \qquad Q_2(\bar{v} + \lambda\bar{e}) := Q(\bar{v}) + \lambda^2\mu.$$

Then Q_1 and Q_2 are two quadratic forms of W and the associated nonsingular (parabolic) quadrics of $\mathrm{PG}(W)$ will respectively be denoted by $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Obviously, $\mathcal{Q}_i \cap \mathrm{PG}(V) = \mathcal{Q}$ for every $i \in \{1, 2\}$. If $x = \langle \bar{v} \rangle \in S$, then $Q_1(\bar{v}) \cdot Q_1(\bar{e}) = Q(\bar{v})$ is a square, implying by Corollary 2.9 that the quadrics $x^{\zeta_1} \cap \mathcal{Q}_1$ and $\langle \bar{e} \rangle^{\zeta_1} \cap \mathcal{Q}_1 = \mathcal{Q}$ have the same type, i.e. both are hyperbolic. On the other hand, as $Q_2(\bar{v}) \cdot Q_2(\bar{e}) = Q(\bar{v}) \cdot \mu$ is a nonsquare, $x^{\zeta_2} \cap \mathcal{Q}_2$ and $\langle \bar{e} \rangle^{\zeta_2} \cap \mathcal{Q}_2 = \mathcal{Q}$ have different types. i.e. $x^{\zeta_2} \cap \mathcal{Q}_2$ is of elliptic type. The claims regarding the point $y \in N$ then follow from Corollary 2.9.

The proof is similar if $\mathcal{Q}$ is of elliptic type. In this case, we define

$$Q_1(\bar{v} + \lambda\bar{e}) := Q(\bar{v}) + \lambda^2\mu, \qquad Q_2(\bar{v} + \lambda\bar{e}) := Q(\bar{v}) + \lambda^2.$$

for every $(\bar{v}, \lambda) \in V \times \mathbb{F}_q$. □

Lemma 2.13. *Suppose n is even. Then $\mathcal{Q}$ is of parabolic type. Let L be a secant line to $\mathcal{Q}$ and let $x, y \in L \setminus \mathcal{Q}$ such that $y \in x^\zeta$. Then the following hold:*

(1) *If $q \equiv 1 \pmod 4$, then the hyperplanes x^ζ and y^ζ intersect $\mathcal{Q}$ in nonsingular quadrics of the same type (i.e. both hyperbolic or elliptic).*

(2) *If $q \equiv 3 \pmod 4$, then the hyperplanes x^ζ and y^ζ intersect $\mathcal{Q}$ in nonsingular quadrics of different types.*

Proof. Choose vectors $\bar{e}_1, \bar{e}_2 \in V$ such that $Q(\bar{e}_1) = Q(\bar{e}_2) = 0$, $B(\bar{e}_1, \bar{e}_2) = 1$ and $L \cap \mathcal{Q} = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle\}$. Then $x = \langle \bar{e}_1 + \lambda\bar{e}_2 \rangle$ and $y = \langle \bar{e}_1 - \lambda\bar{e}_2 \rangle$ for some $\lambda \in \mathbb{F}_q^*$. By Corollary 2.9, x^ζ and y^ζ intersect $\mathcal{Q}$ in quadrics of the same type if and only if $Q(\bar{e}_1 + \lambda\bar{e}_2) \cdot Q(\bar{e}_1 - \lambda\bar{e}_2) = -\lambda^2$ is a square. This happens if and only if -1 is a square, i.e. if and only if $q \equiv 1 \pmod 4$. □

Lemma 2.14. *Suppose n is even. Then $\mathcal{Q}$ is of parabolic type. Let L be an external line to $\mathcal{Q}$, and let $x, y \in L$ such that $y \in x^\zeta$. Then the following hold:*

(1) *If $q \equiv 1 \pmod 4$, then the hyperplanes x^ζ and y^ζ intersect $\mathcal{Q}$ in nonsingular quadrics of different types.*

(2) *If $q \equiv 3 \pmod 4$, then the hyperplanes x^ζ and y^ζ intersect $\mathcal{Q}$ in nonsingular quadrics of the same type.*

Proof. By Lemma 2.5, we know that there exists an $\bar{e}_1 \in V$ such that $\langle \bar{e}_1 \rangle \in L$ and $Q(\bar{e}_1) = 1$. Choose $\bar{e}_2 \in V$ such that $\langle \bar{e}_2 \rangle \in L$ and $B(\bar{e}_1, \bar{e}_2) = 0$. As $B(\bar{e}_1, \bar{e}_1) = 2Q(\bar{e}_1) \neq 0$, the vectors $\bar{e}_1$ and $\bar{e}_2$ are linearly independent. Put $\mu := -Q(\bar{e}_2)$. The facts that $Q(X\bar{e}_1 + Y\bar{e}_2) = X^2 - \mu Y^2$ and $L \cap \mathcal{Q} = \emptyset$ imply that μ is a nonsquare. As $y \in x^\zeta$, we have $\{x, y\} = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle\}$, or $x = \langle \bar{e}_1 + \lambda\bar{e}_2 \rangle$ and $y = \langle \bar{e}_1 + \frac{1}{\lambda\mu}\bar{e}_2 \rangle$ for some $\lambda \in \mathbb{F}_q^*$. Since

$Q(\bar{e}_1)\cdot Q(\bar{e}_2) = -\mu$ and $Q(\bar{e}_1+\lambda\bar{e}_2)\cdot Q(\bar{e}_1+\frac{1}{\lambda\mu}\bar{e}_2) = \frac{-\mu(1-\mu\lambda^2)^2}{\lambda^2\mu^2}$, Corollary 2.9 implies that the hyperplanes x^ζ and y^ζ intersect $\mathcal{Q}$ in quadrics of the same type if and only if $-\mu$ is a square. This precisely happens if and only if -1 is a nonsquare, i.e. if and only if $q \equiv 3 \pmod 4$. □

3 $\mathcal{T}^\epsilon$-blocking sets in $\mathrm{PG}(3,q)$, $\epsilon \in \{-,+\}$

Let $Q^\epsilon(3,q)$, q odd, be embedded as a hyperplane section of a parabolic quadric $Q(4,q)$ in $\mathrm{PG}(4,q)$, and let ζ be the orthogonal polarity of $\mathrm{PG}(4,q)$ associated with $Q(4,q)$. Let $\Pi = \mathrm{PG}(3,q)$ be the hyperplane of $\mathrm{PG}(4,q)$ such that $\Pi \cap Q(4,q) = Q^\epsilon(3,q)$, and put $x^* := \Pi^\zeta$. Let H, respectively E, denote the set of all points y of $\mathrm{PG}(4,q)\setminus Q(4,q)$ for which $y^\zeta \cap Q(4,q)$ is a hyperbolic, respectively elliptic, quadric in y^ζ.

We denote by $\mathcal{T}^\epsilon$ the set of all lines of $\Pi = \mathrm{PG}(3,q)$ that are tangent to $Q^\epsilon(3,q)$. Recall that if B is a $\mathcal{T}^\epsilon$-blocking set in Π, then $|B| \geq q^2+1$, with equality if and only if every tangent line $L \subseteq \Pi$ contains a unique point of B.

3.1 Proof of Theorem 1.2

Here we take $\epsilon = -$. Put $U := E \cap \Pi$ if $q \equiv 1 \pmod 4$ and $U := H \cap \Pi$ if $q \equiv -1 \pmod 4$. The following three properties follow from Lemmas 2.3, 2.5, 2.8 and Corollary 2.9:

(A) If L is a line of Π external to $Q^-(3,q)$, then $|L\cap H| = |L \cap E| = |L\cap U| = \frac{q+1}{2}$.

(B) If L is a line of Π secant to $Q^-(3,q)$, then $|L\cap H| = |L \cap E| = |L\cap U| = \frac{q-1}{2}$.

(C) If L is a line of Π tangent to $Q^-(3,q)$, then $(|L\cap H|, |L\cap E|)$ is either $(q,0)$ or $(0,q)$. In particular, every tangent to $Q^-(3,q)$ meets U in either 0 or q points.

Note that $(x^*)^\zeta \cap Q(4,q) = \Pi \cap Q(4,q) = Q^-(3,q)$ is an elliptic quadric. The next result comes directly from Lemmas 2.13, 2.14 and the definition of the set U.

Lemma 3.1. *The following hold:*

(1) *If L is a secant line to $Q(4,q)$ through x^*, then the unique point in $L\cap\Pi$ belongs to U.*

(2) *If L is an external line to $Q(4,q)$ through x^*, then the unique point of $L\cap\Pi$ does not belong to U.*

In the sequel, we suppose that B is a $\mathcal{T}^-$-blocking set of size q^2+1 in Π. Then every tangent line of Π intersects B in a unique point.

Lemma 3.2. *If $\gamma \subseteq \Pi$ is a tangent plane for which the tangency point is not contained in B, then $|B\cap\gamma\cap U| = \frac{q+1}{2}$.*

Proof. This follows from properties (A), (C) and the fact that each tangent line of γ through $\gamma \cap Q^-(3,q)$ contains a unique point of B. $\square$

Lemma 3.3. *If $x \in B \setminus Q^-(3,q)$ and if γ is a tangent plane through x, then $\gamma \cap Q^-(3,q)$ is not contained in B.*

Proof. If $\{y\} = \gamma \cap Q^-(3,q)$, then yx is a tangent line and so cannot contain besides x extra points of B. $\square$

Lemma 3.4. *We have $|B \cap U| = \frac{q^2+1-|B\cap Q^-(3,q)|}{2}$.*

Proof. We count in two different ways the number of pairs (x, γ) with $x \in B \cap U$ and γ a tangent plane through x. By Lemma 3.3, we know that γ cannot be a tangent plane through a point of $B \cap Q^-(3,q)$ and so there are $|Q^-(3,q)| - |B \cap Q^-(3,q)| = q^2 + 1 - |B \cap Q^-(3,q)|$ possibilities for γ. By Lemma 3.2, for each such possibility for γ, there are $\frac{q+1}{2}$ possibilities for x . On the other hand, there are $|B \cap U|$ possibilities for x, and for given x there are $q + 1$ possibilities for γ. So,

$$((q^2 + 1) - |B \cap Q^-(3,q)|) \cdot \frac{q+1}{2} = |B \cap U| \cdot (q + 1),$$

from which the claim of the lemma follows. $\square$

Proposition 3.5. *For every point $x \in B \cap U$, let O_x be the set of two points of $Q(4,q)$ on the secant line x^*x (see Lemma 3.1). Then $O := (B \cap Q^-(3,q)) \cup \left(\bigcup_{x\in B\cap U} O_x\right)$ is an ovoid of $Q(4,q)$.*

Proof. By Lemma 3.4, we know that

$$|O| = |B \cap Q^-(3,q)| + 2 \cdot |B \cap U| = q^2 + 1.$$

So, it suffices to prove that no two distinct points u_1, u_2 of O are collinear on $Q(4,q)$. This is clearly the case if $u_1, u_2 \in B \cap Q^-(3,q)$. So, it remains to consider one of the following three cases:

(1) $u_1 \in B \cap Q^-(3,q)$ and $u_2 \in O_x$ for some $x \in B \cap U$;

(2) $u_1, u_2 \in O_x$ for some $x \in B \cap U$;

(3) $u_1 \in O_{x_1}$ and $u_2 \in O_{x_2}$ for some $x_1, x_2 \in B \cap U$ with $x_1 \neq x_2$.

Suppose $u_1 \in B \cap Q^-(3,q)$ and $u_2 \in O_x$ with $x \in B \cap U$ are collinear. The tangent hyperplane T_{u_1} at the point u_1 then contains x^*, u_2 and hence also the unique point x in $x^*u_2 \cap \Pi$. But then the tangent line u_1x would contain two points of B, namely u_1 and x, which is impossible.

If $u_1, u_2 \in O_x$ for some $x \in B \cap U$, then u_1 and u_2 cannot be collinear on $Q(4,q)$ as they are on the secant line x^*x.

Suppose $u_1 \in O_{x_1}$ and $u_2 \in O_{x_2}$ are collinear for some $x_1, x_2 \in B \cap U$ with $x_1 \neq x_2$. In the plane $\langle x^*, x_1x_2 \rangle$, the lines $u_1u_2 \subseteq Q(4,q)$ and x_1x_2 meet in a point $u \in Q^-(3,q)$. The tangent hyperplane T_u at the point u contains x^*, u_1, u_2 and hence also the points x_1 and x_2. But then the tangent line x_1x_2 through u contains two points of B, namely x_1 and x_2, a contradiction. □

Lemma 3.6. *If $\gamma = \langle (B \cap Q^-(3,q)) \cup (B \cap U) \rangle \neq \Pi$, then γ is a nontangent plane of Π, $B \cap Q^-(3,q) = \gamma \cap Q^-(3,q)$ and $B \cap U$ is the set of interior points of γ with respect to the irreducible conic $\mathcal{C}_\gamma = \gamma \cap Q^-(3,q)$.*

Proof. By Lemma 3.4, we have

$$|(B \cap Q^-(3,q)) \cup (B \cap U)| = \frac{q^2 + 1 + |B \cap Q^-(3,q)|}{2} > q + 1, \qquad (1)$$

and so γ is a plane. Equation (1) implies that $|B \cap \gamma| > q + 1$. As each tangent line contains a unique point of B, γ cannot be a tangent plane. So, γ is a nontangent plane.

For every point $x \in (\gamma \cap Q^-(3,q)) \setminus B$, the tangent plane $T_x \subseteq \Pi$ through x to $Q^-(3,q)$ contains $\frac{q+1}{2} \geq 2$ points of $B \cap U$ (by Lemma 3.2) of which at most 1 is contained in $T_x \cap \gamma$. As $B \cap U \subseteq \gamma$, this is impossible. So, $\gamma \cap Q^-(3,q) \subseteq B$ and $B \cap Q^-(3,q) = \gamma \cap Q^-(3,q)$. We conclude by Lemma 3.4 that $|B \cap U| = \frac{q^2-q}{2}$. As every tangent line contains a unique point of B, we see that $B \cap U \subseteq \gamma$ coincides with the set of $\frac{q^2-q}{2}$ points of γ that are interior with respect to the irreducible conic $\gamma \cap Q^-(3,q)$. □

Lemma 3.7. *If $B \neq Q^-(3,q)$, then the ovoid O defined in Proposition 3.5 cannot be contained in a hyperplane of $\mathrm{PG}(4,q)$.*

Proof. Let π be a hyperplane containing O. As $B \neq Q^-(3,q)$, there exists a point $x \in B \cap U$ (see Lemma 3.4). As π contains O_x, we see that $x^* \in \pi$. As $O = (B \cap Q^-(3,q)) \cup \left(\bigcup_{y \in B \cap U} O_y \right) \subseteq \pi$, we see that $(B \cap Q^-(3,q)) \cup (B \cap U)$ is contained in the plane $\Pi \cap \pi$ of Π. By Lemma 3.6, $\langle B \cap Q^-(3,q) \rangle$ is a plane γ and $B \cap U$ is the set of interior points of γ with respect to the irreducible conic $\mathcal{C}_\gamma = \gamma \cap Q^-(3,q)$ of γ.

Now, put $\Pi = \mathrm{PG}(3,q) = \mathrm{PG}(V)$ for some 4-dimensional vector space V over $\mathbb{F}_q$ and let Q be the quadratic form on V describing the quadric $Q^-(3,q) \subseteq \mathrm{PG}(3,q)$. Let S (respectively, N) denote the set of all points $\langle \bar{v} \rangle$ of $\mathrm{PG}(3,q)$ such that $Q(\bar{v})$ is a nonzero square (respectively, a nonsquare). By Corollary 2.9 and Lemma 2.12, the parabolic quadric $Q(4,q)$ could have chosen such that $U = S$ but also such that $U = N$. So, we see that both $B \cap S$ and $B \cap N$ should be equal to the set of interior points of γ with respect to $\mathcal{C}_\gamma$. This is clearly impossible. □

The following proposition proves Theorem 1.2.

Proposition 3.8. *If q is a prime, then $B = Q^-(3,q)$.*

Proof. If q is a prime, then every ovoid of $Q(4,q)$ is an elliptic quadric by [1, Corollary 1] and thus contained in a hyperplane of $\mathrm{PG}(4,q)$. The claim then follows from Lemma 3.7. □

3.2 Proof of Theorem 1.1

Here we take $\epsilon = +$. The lines of $\mathcal{T}^+$ intersect $Q^+(3,q)$ in either 1 or $q+1$ points. Let B be a $\mathcal{T}^+$-blocking set in Π of size q^2+1. Then every line of $\mathcal{T}^+$ intersects B in precisely one point. In order to prove Theorem 1.1, we can almost literally take the above proof given in Section 3.1 for the elliptic quadric $Q^-(3,q)$. The only differences are as follows.

(1) The quadric $Q^-(3,q)$ must now be replaced by $Q^+(3,q)$ ($\epsilon = +$).

(2) The set U must be defined as follows: $U := H \cap \Pi$ if $q \equiv 1 \pmod 4$ and $U := E \cap \Pi$ if $q \equiv -1 \pmod 4$.

(3) In Lemma 3.2, we have the condition $|B \cap \gamma \cap U| = \frac{q-1}{2}$ and the proof makes use of properties (B) and (C).

(4) Lemma 3.3 must become: If $x \in B \setminus Q^+(3,q)$ and γ is a tangent plane through x, then its tangency point is not contained in B.

(5) In Lemma 3.4, we have the equality $|B \cap U| = \frac{q^2-q}{2}$. The proof of Lemma 3.4 must be modified as follows. Considering a set of $q+1$ lines partitioning $Q^+(3,q)$, we see that $|B \cap Q^+(3,q)| = q+1$. The double counting in the proof of Lemma 3.4 then becomes $((q+1)^2 - (q+1)) \cdot \frac{q-1}{2} = |B \cap U| \cdot (q+1)$, from which it indeed follows that $|B \cap U| = \frac{q^2-q}{2}$.

(6) The equation (1) should become $|(B \cap Q^+(3,q)) \cup (B \cap U)| = (q+1) + \frac{q^2-q}{2} > q+1$. To show that $B \cap Q^+(3,q) = \gamma \cap Q^+(3,q)$ as in the proof of Lemma 3.6, the following alternative argument is necessary. For every point $x \in (\gamma \cap Q^+(3,q)) \setminus B$, the tangent plane $T_x \subseteq \Pi$ through x to $Q^+(3,q)$ contains two points of $B \cap Q^+(3,q)$ none of which can be contained in γ. As this is impossible, we must have $(\gamma \cap Q^+(3,q)) \setminus B = \emptyset$.

(7) Lemma 3.7 must become: The ovoid O cannot be contained in a hyperplane of $\mathrm{PG}(4,q)$. Note that as $|B \cap U| = \frac{q^2-q}{2}$, there always exists a point $x \in B \cap U$ and so the proof of Lemma 3.7 keeps working.

(8) Since every ovoid of $Q(4,q)$ is an elliptic quadric if q is a prime [1, Corollary 1], we see that the $\mathcal{T}^+$-blocking set B of size q^2+1 cannot exist if q is a prime.

4 Construction of $\mathcal{T}^-$-blocking sets of size 12 in $\mathrm{PG}(3,3)$

We denote by ζ the orthogonal polarity of $\mathrm{PG}(3,q)$, q odd, associated with $Q^-(3,q)$. We have the following facts for every point x of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$. The tangent lines of $\mathrm{PG}(3,q)$ through x are the $q+1$ lines through x containing a point of the irreducible conic $\mathcal{C}_x := x^\zeta \cap Q^-(3,q)$, the $q(q-1)/2$ secant lines through x are the lines through x intersecting x^ζ in a point that is interior with respect to $\mathcal{C}_x$, and the $q(q+1)/2$ external lines through x are the lines through x intersecting x^ζ in a point that is exterior with respect to $\mathcal{C}_x$. Indeed, if y is one of the $q(q+1)/2$ points of x^ζ that are exterior with respect to $\mathcal{C}_x$ and $z \in \mathcal{C}_x$ such that zy is a tangent line, then $z^\zeta = \langle x, y, z \rangle$ is tangent plane and so xy is indeed an external line through x.

Let Q be a quadratic form on a 4-dimensional vector space V over $\mathbb{F}_q$ defining the elliptic quadric $Q^-(3,q)$ in $\mathrm{PG}(3,q) = \mathrm{PG}(V)$. Let S (respectively, N) denote the set of all points $\langle \bar{x} \rangle$ of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ for which $Q(\bar{x})$ is a nonzero square (respectively, a nonsquare) in $\mathbb{F}_q$.

In this section, we construct eight pairwise nonisomorphic $\mathcal{T}^-$-blocking sets in $\mathrm{PG}(3,3)$ of size 12. We denote each constructed blocking set B by B_{i*}, where $i = |B \cap Q^-(3,3)| \in \{0,2,4,5,6,8\}$ and $*$ is either void or equal to a or b if $i \in \{0,4\}$.

4.1 The blocking sets B_{0a} and B_8

We start by giving two straightforward constructions for $\mathcal{T}^-$-blocking sets of size q^2+q.

Proposition 4.1. *For every point $x \in Q^-(3,q)$, the set $B := x^\zeta \setminus \{x\}$ is a $\mathcal{T}^-$-blocking set in $\mathrm{PG}(3,q)$ of size q^2+q which is disjoint from $Q^-(3,q)$.*

Proof. Each tangent line through x is contained in x^ζ and thus contains q points of B. Each tangent line through a point $y \in Q^-(3,q) \setminus \{x\}$ meets x^ζ is a point distinct from x. □

Proposition 4.2. *For two distinct points $x, y \in Q^-(3,q)$, the set $B := (Q^-(3,q) \setminus \{x,y\}) \cup (xy)^\zeta$ is a $\mathcal{T}^-$-blocking set of size q^2+q in $\mathrm{PG}(3,q)$ meeting $Q^-(3,q)$ in q^2-1 points.*

Proof. Each tangent line through a point of $Q^-(3,q) \setminus \{x,y\}$ meets B. Each tangent line through x (resp. y) intersects y^ζ (resp. x^ζ) in a point belonging to $(xy)^\zeta \subseteq B$. □

For $q = 3$, Propositions 4.1 and 4.2 give rise to $\mathcal{T}^-$-blocking sets of size 12 in $\mathrm{PG}(3,3)$ which we will respectively denote by B_{0a} and B_8.

4.2 The blocking set B_{0b}

Let L be an external line with respect to $Q^-(3,q)$. Then L^ζ is a secant line. If we put $L^\zeta \cap Q^-(3,q) = \{w_1, w_2\}$, then $\pi_1 = \langle w_1, L \rangle$ and $\pi_2 = \langle w_2, L \rangle$ are the two tangent planes through L.

Proposition 4.3. *The set $B := (\pi_1 \cap S) \cup (\pi_2 \cap N)$ is a $\mathcal{T}^-$-blocking set in $\mathrm{PG}(3,q)$ of size q^2+q which is disjoint from $Q^-(3,q)$ and contains the line L.*

Proof. As the planes π_1 and π_2 meet in the line L which is disjoint from $Q^-(3,q)$, we have $L \subseteq B$. By Lemmas 2.2 and 2.5, we also have $|B| = \frac{q^2+q}{2} + \frac{q^2+q}{2} = q^2+q$.

Consider now an arbitrary tangent line K with tangency point u. If $u \in \{w_1, w_2\}$, then K is contained in either π_1 or π_2 and so meets $L \subseteq B$. If $u \notin \{w_1, w_2\}$, then K meets $\pi_1 \cap S$ if $K \setminus \{u\} \subseteq S$ and $\pi_2 \cap N$ if $K \setminus \{u\} \subseteq N$. □

For $q = 3$, Proposition 4.3 gives rise to a $\mathcal{T}^-$-blocking set B_{0b} of size 12 in $\mathrm{PG}(3,3)$ disjoint from $Q^-(3,3)$. As there is no plane of $\mathrm{PG}(3,3)$ containing the points of B_{0b}, we see that B_{0a} and B_{0b} are not isomorphic as $\mathcal{T}^-$-blocking sets.

It is possible to give an entirely geometrical construction for the $\mathcal{T}^-$-blocking set B_{0b} without making any reference to the quadratic form Q that defines $Q^-(3,3)$. Put

$L^\zeta = \{x_1, x_2, w_1, w_2\}$. By Lemma 2.3, precisely one of x_1, x_2 belongs to S while the other belongs to N. Assume that we have chosen x_1 and x_2 in such a way that $x_1 \in S$ and $x_2 \in N$. Then every tangent line through x_1 (respectively, x_2) only contains points of $S \cup Q^-(3,3)$ (respectively, $N \cup Q^-(3,3)$).

Lemma 4.4. *Let $i, j \in \{1,2\}$. Then every tangent line through x_i meets w_j^ζ in a point of $w_j^\zeta \setminus (L \cup \{w_j\})$.*

Proof. Note that the line $x_i w_j = L^\zeta$ is a secant line. If $y \in L$, then yw_1, yw_2, yx_1 and yx_2 are the four lines through y contained in the plane $\langle y, L^\zeta \rangle$. As yw_1 and yw_2 are tangent lines, yx_1 and yx_2 cannot be tangent lines. $\square$

For $j \in \{1,2\}$, let A_j denote the set of four points which are obtained as the intersection of the tangent lines through x_j with w_j^ζ. We then have the following.

Corollary 4.5. *For $q = 3$, the $\mathcal{T}^-$-blocking set constructed in Proposition 4.3 is precisely the set $L \cup A_1 \cup A_2$.*

4.3 The blocking set B_{4a}

Suppose here that $q \equiv 3 \pmod 4$. Let π be a nontangent plane, let $x \in \mathcal{C}_\pi := \pi \cap Q^-(3,q)$ and let $U, \overline{U} \in \{S, N\}$ such that $\{U, \overline{U}\} = \{S, N\}$ and $y = \langle \bar{v} \rangle := \pi^\zeta$ belongs to U.

Proposition 4.6. *The set $B := \mathcal{C}_\pi \cup (\pi \cap \overline{U}) \cup ((x^\zeta \cap U) \setminus \{y\})$ is a $\mathcal{T}^-$-blocking set in $\mathrm{PG}(3,q)$ of size $q^2 + q$ intersecting $Q^-(3,q)$ in a conic.*

Proof. We obviously have $B \cap Q^-(3,q) = \mathcal{C}_\pi$. If $z = \langle \bar{w} \rangle$ is a point of π that is exterior with respect to $\mathcal{C}_\pi$, then the fact that the line yz is disjoint from $Q^-(3,q)$ implies that $-Q(\bar{v}) \cdot Q(\bar{w})$ is a nonsquare, i.e. $Q(\bar{v}) \cdot Q(\bar{w})$ is a square. The set $\pi \cap U$ thus consists of all $q(q+1)/2$ points of π that are exterior with respect to $\mathcal{C}_\pi$ and $\pi \cap \overline{U}$ consists of all $q(q-1)/2$ points of π that are interior with respect to $\mathcal{C}_\pi$. The set B thus contains $q+1+\frac{q(q-1)}{2}+\left(\frac{q(q+1)}{2}-1\right) = q^2+q$ points. We show that B is a $\mathcal{T}^-$-blocking set. To that end, consider an arbitrary tangent line K with tangency point $u \notin \mathcal{C}_\pi$. Then K meets $\pi \cap \overline{U}$ if $K \setminus \{u\} \subseteq \overline{U}$ and $(x^\zeta \cap U) \setminus \{y\}$ if $K \setminus \{u\} \subseteq U$. $\square$

For $q = 3$, Proposition 4.6 gives rise to a $\mathcal{T}^-$-blocking set B_{4a} of size 12 in $\mathrm{PG}(3,3)$ which intersects $Q^-(3,3)$ in a conic. Note also that the plane π then intersects this $\mathcal{T}^-$-blocking set in exactly ten points.

It is again possible to give an entirely geometrical construction for the $\mathcal{T}^-$-blocking set B_{4a} without making any reference to the quadratic form Q that defines $Q^-(3,3)$. Let $\mathcal{I}$ denote the set of three interior points of π with respect to $\mathcal{C}_\pi$, and let T be the line of π that is tangent to $\mathcal{C}_\pi$ in the point x.

Proposition 4.7. *For $q = 3$, the $\mathcal{T}^-$-blocking set constructed in Proposition 4.6 is precisely the set $\mathcal{C}_\pi \cup \mathcal{I} \cup (T \setminus \{x\}) \cup (xy \setminus \{x, y\})$.*

Proof. In view of what has already been said in the proof of Proposition 4.6, we still must show that $(x^\zeta \cap U) \cup \{x\} = T \cup xy$. By Lemmas 2.2 and 2.5, we know that $(x^\zeta \cap U) \cup \{x\}$ is the union of two lines through x. As $\pi \cap U$ consists of all $q(q+1)/2 = 6$ points of π that are exterior with respect to $\mathcal{C}_\pi$, we know that the tangent line T is one of these lines. As $y \in U$, the other line is xy. □

4.4 The blocking set B_{4b}

We suppose here that $q = 3$. Consider a point $x \in \mathrm{PG}(3,3) \setminus Q^-(3,3)$. Let $\mathcal{I}$ denote the set of three interior points of x^ζ with respect to the conic $\mathcal{C}_x$. Fix a point y of x^ζ that is exterior to $\mathcal{C}_x$. There is a unique point z of x^ζ exterior to $\mathcal{C}_x$ such that the line yz is external to $\mathcal{C}_x$. Let L_1 and L_2 be the two tangent lines through y not contained in x^ζ and w_i be the tangency point of L_i in $Q^-(3,3)$ for $i \in \{1,2\}$.

Proposition 4.8. *The set $B_{4b} := \mathcal{C}_x \cup \mathcal{I} \cup \{z\} \cup ((L_1 \cup L_2) \setminus \{y, w_1, w_2\})$ is a $\mathcal{T}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 12 meeting $Q^-(3,3)$ in four points. Further, B_{4b} is not isomorphic to B_{4a} as $\mathcal{T}^-$-blocking set.*

Proof. We only need to prove that the tangent lines through any point of $x^\zeta \setminus B_{4b}$ that are not contained in x^ζ must meet B_{4b}. Note that $x^\zeta \setminus B_{4b}$ consists of five points of x^ζ exterior to $\mathcal{C}_x$.

Let u be a point of $x^\zeta \setminus B_{4b}$ and T be a tangent line through u not contained in x^ζ. If $u = y$, then $T \in \{L_1, L_2\}$ and hence it meets B_{4b} at two points. So assume that $u \neq y$. Since both u and y are exterior to $\mathcal{C}_x$ and $u \neq z$, the line uy of x^ζ must be tangent to $\mathcal{C}_x$.

Consider the nontangent plane $\pi := \langle T, uy \rangle$. The point y of π is exterior to the conic $\pi \cap Q^-(3,3)$. So there exists one more tangent line (different from uy) through y in π. Since $\pi \cap x^\zeta = uy$, it follows that L_i is a tangent line of π for some $i \in \{1,2\}$. The tangency points of T and L_i are different as π is a nontangent plane. So the lines T and L_i of π intersect in a point of $L_i \setminus \{y, w_i\}$. Thus T meets B_{4b} in a point of $L_i \setminus \{y, w_i\}$.

In order to show that B_{4a} and B_{4b} are nonisomorphic, it suffices to show that there is no plane of $\mathrm{PG}(3,3)$ intersecting B_{4b} in exactly ten points. Suppose to the contrary that σ is such a plane. As there are two points of B_{4b} outside σ, at least six of the eight points of $x^\zeta \cap B_{4b}$ are contained in σ. As any six points of x^ζ generate x^ζ, we then have that $\sigma = x^\zeta$. But that is impossible as x^ζ intersects B_{4b} in only eight points. □

4.5 The blocking set B_5

We suppose here that $q = 3$. Consider a point x of $\mathrm{PG}(3,3) \setminus Q^-(3,3)$ and an external line E of x^ζ with respect to the conic $\mathcal{C}_x$. Let e_1, e_2 be the two points of E exterior to $\mathcal{C}_x$. Since E is contained in precisely two tangent planes, there are two points w_1, w_2 of $Q^-(3,3) \setminus \mathcal{C}_x$ such that $w_1^\zeta \cap w_2^\zeta = E$. Put $L_1 := w_2 e_1$ and $L_2 := w_2 e_2$ (which are tangent lines through w_2) and denote by $\mathcal{I}$ the set of three interior points of x^ζ with respect to $\mathcal{C}_x$.

Proposition 4.9. *The set* $B_5 := \mathcal{C}_x \cup \mathcal{I} \cup \{w_1\} \cup ((L_1 \cup L_2) \setminus \{w_2, e_1, e_2\})$ *is a* $\mathcal{T}^-$*-blocking set in* $\mathrm{PG}(3,3)$ *of size* 12 *meeting* $Q^-(3,3)$ *in five points.*

Proof. Put $E \cap \mathcal{I} = \{a_1, a_2\}$ and $\mathcal{I} = \{a_1, a_2, a_3\}$. We have then $E = \{a_1, a_2, e_1, e_2\}$. Note that each of the tangent lines through w_2 meets B_5. Let $R := \{w_1, w_2\} \cup \mathcal{C}_x$. It is enough to show that the tangent lines through any of the four points of $Q^-(3,3) \setminus R$ meet B_5.

Let y be a point of $Q^-(3,3) \setminus R$. Then $y^\zeta \cap x^\zeta$ is an external line of x^ζ different from E. So $y^\zeta \cap x^\zeta$ contains two points of $\mathcal{I}$, one of them must be a_3 and the other one is a_i for some $i \in \{1,2\}$. We have $a_i \in w_2^\zeta \cap y^\zeta$ and so $M := w_2^\zeta \cap y^\zeta$ is an external line through a_i contained in w_2^ζ with $M \neq E$. In w_2^ζ, the line M meets L_1 and L_2. Put $M = \{a_i, b_1, b_2, z\}$, where $b_1 \in L_1 \setminus \{w_2, e_1\}$ and $b_2 \in L_2 \setminus \{w_2, e_2\}$.

Note that the tangent lines through a_3 meet $Q^-(3,3)$ in points of $Q^-(3,3) \setminus R$. In particular, ya_3 is a tangent line. We thus have tangent lines ya_i, ya_3, yb_1 and yb_2 through y meeting B_5 in points of $\{a_i, a_3, b_1, b_2\}$.

In order to complete the proof, it suffices to prove that the tangent line ya_3 is different from the tangent lines yb_1 and yb_2. If a_3 is a point on the tangent line yb_j for some $j \in \{1,2\}$, then the points e_j and a_3 of the nontangent plane $\pi := \langle L_j, yb_j \rangle$ are exterior with respect to the conic $\pi \cap Q^-(3,3)$. But this is not possible as the line $e_j a_3$, being secant to $\mathcal{C}_x$ in x^ζ, is also a secant line of π with respect to $\pi \cap Q^-(3,3)$. □

4.6 The blocking sets B_2 and B_6

We suppose here that $q = 3$. Let L be a secant line and π_1 a (nontangent) plane through L. The line L^ζ is disjoint from $Q^-(3,3)$ and meets π_1 in a point x_1. Put $\pi_3 := x_1^\zeta$, and let π_2 be a plane through L distinct from π_1 and π_3. The line L^ζ meets π_2 in a point x_2. Put $\pi_4 := x_2^\zeta$, $\{x_3\} := L^\zeta \cap \pi_3$ and $\{x_4\} := L^\zeta \cap \pi_4$. Then $L^\zeta = \{x_1, x_2, x_3, x_4\}$ and $\pi_1 = x_3^\zeta$, $\pi_2 = x_4^\zeta$, $\pi_3 = x_1^\zeta$, $\pi_4 = x_2^\zeta$ are the four (nontangent) planes through L. We denote by $\mathcal{C}_{\pi_i}$, $i \in \{1,2,3,4\}$, the irreducible conic $\pi_i \cap Q^-(3,3)$ in π_i.

Put $L \setminus Q^-(3,3) = \{y_{13}, y_{24}\}$. In the plane π_i, $i \in \{1,2,3,4\}$, there are two tangent lines through x_i (namely the two lines through x_i meeting $L \cap Q^-(3,3)$), one secant line through x_i and one external line through x_i. So, for every $i \in \{1,2,3,4\}$, exactly one of $x_i y_{13}$, $x_i y_{24}$ is a secant line. Without loss of generality, we may suppose that we have named the points in $L \setminus Q^-(3,3)$ in such a way that $x_1 y_{13}$ is a secant line. As L and $y_{13} x_1$ are two secant lines of π_1 through y_{13}, the point y_{13} must be interior with respect to the conic $\mathcal{C}_{\pi_1}$. As $\pi_1 = x_3^\zeta$, the line $x_3 y_{13}$ must therefore also be a secant line. We have now found all three secant lines through $y_{13} \in \mathrm{PG}(3,3) \setminus Q^-(3,3)$, namely L, $x_1 y_{13}$ and $x_3 y_{13}$. This implies that $x_2 y_{24}$ and $x_4 y_{24}$ must be secant lines.

Proposition 4.10. *The set* $B_6 := (L \cup x_1 y_{13} \cup x_1 y_{24} \cup x_2 y_{13} \cup x_2 y_{24}) \setminus \{x_1, x_2\}$ *is a* $\mathcal{T}^-$*-blocking set in* $\mathrm{PG}(3,3)$ *of size* 12 *meeting* $Q^-(3,3)$ *in six points.*

Proof. Obviously, the set B_6 has size 12 and meets $Q^-(3,3)$ in exactly six points (namely the points of $L \setminus \{y_{13}, y_{24}\}$, $x_1 y_{13} \setminus \{x_1, y_{13}\}$ and $x_2 y_{24} \setminus \{x_2, y_{24}\}$). We need to prove that a tangent line T with tangency point t meets the set B_6. As every point of $\mathcal{C}_{\pi_1} \cup \mathcal{C}_{\pi_2}$ is contained in B_6, we may suppose that $t \in (\mathcal{C}_{\pi_3} \cup \mathcal{C}_{\pi_4}) \setminus L$. Note that $\pi_3 = x_1^\zeta$ and $\pi_4 = x_2^\zeta$. By symmetry, it thus suffices to consider the case where $t \in \mathcal{C}_{\pi_3} \setminus L = x_3 y_{13} \setminus \{x_3, y_{13}\}$. As $x_3^\zeta = \pi_1$ and y_{13}^ζ contains $x_1 y_{24}$, also t^ζ contains $x_1 y_{24}$ and so $t^\zeta = \langle t, x_1 y_{24} \rangle$. As $x_1 y_{24} \setminus \{x_1\} \subseteq B_6$, all tangent lines through t meet B_6, with the possible exception of the tangent line tx_1. In the plane $\langle y_{13}, L^\zeta \rangle$, as $x_1 y_{13} \cap x_2 y_{13} = \{y_{13}\}$ and $x_1 x_3 \cap x_2 y_{13} = \{x_2\}$, we have $x_1 t \cap x_2 y_{13} \subseteq x_2 y_{13} \setminus \{x_2, y_{13}\} \subseteq B_6$ and so the line tx_1 also meets B_6. □

Proposition 4.11. *The set* $B_2 := (L \cup x_1 y_{24} \cup x_2 y_{13} \cup x_3 y_{24} \cup x_4 y_{13}) \setminus \{x_1, x_2, x_3, x_4\}$ *is a* $\mathcal{T}^-$*-blocking set in* $\mathrm{PG}(3,3)$ *of size* 12 *meeting* $Q^-(3,3)$ *in two points.*

Proof. Obviously, the set B_2 has size 12 and meets $Q^-(3,3)$ in exactly two points (namely the points of $L \cap Q^-(3,3)$). We need to prove that a tangent line T with tangency point t meets the set B_2. By symmetry, it suffices to prove this for $t \in \mathcal{C}_{\pi_3}$. In fact, we may restrict to the case where $t \in \mathcal{C}_{\pi_3} \setminus L = x_3 y_{13} \setminus \{x_3, y_{13}\}$. Similarly as in the proof of Proposition 4.10, we know that $t^\zeta = \langle t, x_1 y_{24} \rangle$. As $x_1 y_{24} \setminus \{x_1\} \subseteq B_2$, all tangent lines through t meet B_2, with the possible exception of the tangent line tx_1. As $x_1 y_{13} \cap x_2 y_{13} = \{y_{13}\}$ and $x_1 x_3 \cap x_2 y_{13} = \{x_2\}$, we have $x_1 t \cap x_2 y_{13} \subseteq x_2 y_{13} \setminus \{x_2, y_{13}\} \subseteq B_2$ and so the line tx_1 also meets B_2. □

5 Computer computations

Let $\mathcal{S}_q$ be the geometry whose points are the points of $\mathrm{PG}(3,q)$ and whose lines are the lines of $\mathrm{PG}(3,q)$ that are tangent to $Q^-(3,q)$, with incidence being containment. The $\mathcal{T}^-$-blocking sets of size q^2+1 in $\mathrm{PG}(3,q)$ are precisely the ovoids of the geometry $\mathcal{S}_q$. If q is even, then $\mathcal{S}_q$ is isomorphic to the symplectic generalized quadrangle $W(q)$ whose ovoids have already been classified for $q \leq 32$, see Section 1 for a discussion. Suppose therefore that q is odd. With the aid of GAP [22] and SageMath [18], we have classified all ovoids of $\mathcal{S}_q$ for $q \in \{3,5,7,9,11,13\}$ [5]. We have used similar computer code as in [9] for computing $\mathcal{T}^+$-blocking sets of size q^2+1. We found that each such ovoid is always the elliptic quadric $Q^-(3,q)$ itself. This corroborates Theorem 1.2 for $q \in \{3,5,7,11,13\}$. For $q = 9$, our computer results give the following additional result.

Theorem 5.1. *If* B *is a* $\mathcal{T}^-$*-blocking set in* $\mathrm{PG}(3,9)$ *of size* $9^2+1 = 82$*, then* $B = Q^-(3,9)$.

We would like to pose the following open problem.

Open problem. What is the minimal size of a minimal $\mathcal{T}^-$-blocking set of size at least q^2+2 in $\mathrm{PG}(3,q)$? How does each such blocking set look like?

For $q = 2$, it is straightforward to give an answer to the above problem.

Theorem 5.2. *Every minimal* $\mathcal{T}^-$*-blocking set of size* $2^2+2 = 6$ *in* $\mathrm{PG}(3,2)$ *is of the form* $\pi \setminus \pi^\zeta$ *with* π *a plane of* $\mathrm{PG}(3,2)$ *and* ζ *the symplectic polarity of* $\mathrm{PG}(3,2)$ *associated with* $Q^-(3,2)$.

Proof. Consider the generalized quadrangle $W(2) \cong \mathcal{S}_2$. If A is a nonempty set of points of $W(2)$, then $A^\perp$ denotes the set of all points of $W(2)$ that are collinear with every point of A. If A is a singleton $\{x\}$, then we denote $A^\perp$ also by $x^\perp$. Obviously, $x^\perp = x^\zeta$.

We thus need to prove that every blocking set of size 6 of $W(2)$ (with respect to its lines) is of the form $x^\perp \setminus \{x\}$ for some point x of $W(2)$. Suppose to the contrary that B is a blocking set of size 6 of $W(2)$ not of this form.

By [17], the generalized quadrangle $W(2)$ has a spread, i.e. a set of five mutually disjoint lines. As each line of such a spread contains at least one point of B, we see that there is some line L containing precisely two points x_1 and x_2 of B. Let y denote the third point of L and let L', L'' be the two lines through y distinct from L. Putting $L' = \{y, x_3, x_3'\}$ and $L'' = \{y, x_4, x_4'\}$, we may without loss of generality assume that $x_3, x_4 \in B$. As $y^\perp \setminus \{y\}$ is not contained in B, we may without loss of generality assume that $x_3' \notin B$. Denote by x_5 and x_6 the two points of $B \setminus \{x_1, x_2, x_3, x_4\}$.

As each of the two lines through x_3' distinct from L' is disjoint from $L \cup L''$ and contains a point of B, these two lines are $x_3'x_5$ and $x_3'x_6$. This implies that $x_4' \notin B$. Repeating the above argument with x_3' replaced by x_4', we then see that also $x_4'x_5$ and $x_4'x_6$ are lines. We thus have that $\{x_3', x_4'\}^\perp = x_3'^\perp \cap x_4'^\perp = \{x_5, x_6, y\}$ and $\{x_5, x_6, y\}^\perp = \{x_3', x_4', x_i\}$ for some $i \in \{1, 2\}$. Note that $\{x_5, x_6, y\}$ and $\{x_3', x_4', x_i\}$ are lines of $\mathrm{PG}(3, 2)$ and that $\{x_5, x_6, y\}^\zeta = \{x_3', x_4', x_i\}$. Each of the three lines through $x_i \in B$ thus contains a point of $B \setminus \{x_i\}$. But then $B \setminus \{x_i\}$ would also be a blocking set, an obvious contradiction. □

With the aid of computer computations [5], we have also solved to above problem for $q = 3$. Our results are as follows.

Theorem 5.3. (1) *There are no minimal $\mathcal{T}^-$-blocking sets of size* 11 *in* $\mathrm{PG}(3, 3)$.

(2) *Up to isomorphism, there are eight minimal $\mathcal{T}^-$-blocking sets of size* 12 *in* $\mathrm{PG}(3, 3)$.

The stabilizer of $Q^-(3, 3)$ in $PGL(4, 3)$ has $14 = |Q^-(3, 3)| + 4$ orbits on the (possibly empty) subsets of $Q^-(3, 3)$, with two orbits corresponding to each of the sizes 4, 5 and 6. If $X_1, X_2, \ldots, X_{14}$ are given representatives for these orbits, then without loss of generality we may suppose that the blocking sets have the form $X_i \cup Y$, where $i \in \{1, 2, \ldots, 14\}$ and $Y \cap Q^-(3, 3) = \emptyset$. In order to significantly reduce the total number of cases, we have made use of this fact in our computer proofs of Theorem 5.3.

Theorem 5.3(1) has the following consequence.

Corollary 5.4. *Let X be a $\mathcal{T}^-$-blocking set in* $\mathrm{PG}(3, 3)$ *of size* 12. *If X does not contain* $Q^-(3, 3)$, *then X is minimal.*

Proof. Suppose that X is not minimal. Then, for some $x \in X$, the set $X_1 = X \setminus \{x\}$ is a $\mathcal{T}^-$-blocking set of size 11. Since there is no minimal $\mathcal{T}^-$-blocking set of size 11 in $\mathrm{PG}(3, 3)$, X_1 cannot be minimal. So X_1 contains a subset X_2 which is a $\mathcal{T}^-$-blocking set of size 10. By Theorem 1.2, $X_2 = Q^-(3, 3)$ and so X contains $Q^-(3, 3)$, a contradiction. □

Although it is very likely that one can also prove this by hand, Corollary 5.4 in fact shows that all eight mutually nonisomorphic $\mathcal{T}^-$-blocking sets of size 12 constructed in Section 4 are minimal. Combining this with Theorem 5.3(2), we thus obtain:

Corollary 5.5. *The minimal $\mathcal{T}^-$-blocking sets of size* 12 *in* PG(3, 3) *are precisely the $\mathcal{T}^-$-blocking sets that are isomorphic to one of B_{0a}, B_{0b}, B_2, B_{4a}, B_{4b}, B_5, B_6, B_8.*

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Addresses:

Bart De Bruyn (Bart.DeBruyn@Ugent.be)
Department of Mathematics: Algebra and Geometry, Ghent University, Krijgslaan 281 (S25), B-9000 Gent, Belgium

Francesco Pavese (francesco.pavese@poliba.it)
Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy

Puspendu Pradhan (puspendu.pradhan@niser.ac.in)
Binod Kumar Sahoo (bksahoo@niser.ac.in)

1) School of Mathematical Sciences, National Institute of Science Education and Research (NISER), Bhubaneswar, P.O.- Jatni, District- Khurda, Odisha-752050, India

2) Homi Bhabha National Institute (HBNI), Training School Complex, Anushakti Nagar, Mumbai-400094, India