A characterization of the family of secant lines to a hyperbolic quadric in PG$(3, q)$, q odd, Part II

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Abstract

In [9], two of us classified line sets in PG$(3, q)$, q odd, that satisfy a certain list of properties. It was shown there that if $q \geq 7$, then each such line set is either the set of secant lines with respect to a hyperbolic quadric of PG$(3, q)$ or belongs to a certain "hypothetical family" of line sets (for which no examples were known in [9]). In the present paper, we achieve two goals. On the one hand, we extend the mentioned classification result to all odd prime powers q. On the other hand, we study the hypothetical family of line sets and show that they are related to quadratic sets of the Klein quadric. This will allow us to show that such line sets exist for every odd prime power q.

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1 Introduction

If $\mathcal{Q}$ is a quadric of a projective space PG(n, q), then a line L of PG(n, q) is called an *external*, a *secant* or a *tangent line* according to as $|L \cap \mathcal{Q}| = 0$, $|L \cap \mathcal{Q}| = 2$ or $|L \cap \mathcal{Q}| \in \{1, q + 1\}$. From Theorem 1.1 and Lemmas 3.1, 4.4 of [9], the following characterization result for the set of secant lines to a hyperbolic quadric in PG$(3, q)$ follows.

Proposition 1.1. *Let $\mathcal{S}$ be a set of lines of* PG$(3, q)$*, $q \geq 7$ odd, for which the following properties hold:*

(P1) There are $\frac{1}{2}q(q + 1)$ or q^2 lines of $\mathcal{S}$ through a given point of PG$(3, q)$*. Further, there exists a point which is contained in $\frac{1}{2}q(q + 1)$ lines of $\mathcal{S}$ and a point which is contained in q^2 lines of $\mathcal{S}$.*

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(P2) Every plane π of $\mathrm{PG}(3,q)$ *contains at least one line of $\mathcal{S}$ and one of the following two cases occurs:*

(P2a) every pencil of lines in π contains 0 *or q lines of $\mathcal{S}$;*

(P2b) every pencil of lines in π contains $\frac{1}{2}(q-1)$, $\frac{1}{2}(q+1)$ or q lines of $\mathcal{S}$.

Then $\mathcal{S}$ is one of the following:

(1) *the set of all secant lines with respect to a hyperbolic quadric in* $\mathrm{PG}(3,q)$*;*

(2) *a hypothetical family of $\frac{1}{2}(q^4+q^3+2q^2)$ lines such that the set of points that are incident with q^2 lines of $\mathcal{S}$ is a line $L \notin \mathcal{S}$ and the planes π for which* (*P2a*) *holds are precisely the planes containing L.*

The question whether line sets as in (2) of Proposition 1.1 can exist remained open in [9]. In this paper, we achieve the following two goals.

(1) We extend Proposition 1.1 to all odd q, i.e. we show that Proposition 1.1 is also valid for $q \in \{3,5\}$.

(2) We study the hypothetical families of $\frac{1}{2}(q^4+q^3+2q^2)$ lines of $\mathrm{PG}(3,q)$ alluded to in Proposition 1.1(2). In particular, we show that such line sets exist for every odd prime power q.

The first goal will be realized in Sections 3 and 4, and the second goal in Sections 5, 6, 7, 8 and 9. In both treatments, we need to make use of an observation about dual conics which we will derive in the following section.

2 An observation on dual conics

If $\mathcal{C}$ is an irreducible conic of $\mathrm{PG}(2,q)$, q odd, then every point x of $\mathrm{PG}(2,q)\setminus\mathcal{C}$ is contained in either two or zero tangent lines. The point x is called *exterior* or *interior* according to as the former or the latter case occurs. A *dual conic* of $\mathrm{PG}(2,q)$ is a set of lines of $\mathrm{PG}(2,q)$ forming an irreducible conic in the dual plane $\mathrm{PG}(2,q)^*$ of $\mathrm{PG}(2,q)$.

Lemma 2.1. (1) *A set of lines of* $\mathrm{PG}(2,q)$*, q odd, is a dual conic if and only if it is the set of tangent lines with respect to an irreducible conic of* $\mathrm{PG}(2,q)$*.*

(2) *A set of lines of* $\mathrm{PG}(2,q)$*, q odd, is the set of exterior points with respect to an irreducible conic of* $\mathrm{PG}(2,q)^*$ *if and only if it is the set of secant lines with respect to an irreducible conic of* $\mathrm{PG}(2,q)$*.*

(3) *A set of lines of* $\mathrm{PG}(2,q)$*, q odd, is the set of interior points with respect to an irreducible conic of* $\mathrm{PG}(2,q)^*$ *if and only if it is the set of external lines with respect to an irreducible conic of* $\mathrm{PG}(2,q)$*.*

Proof. Suppose $\mathcal{C}$ is an irreducible conic in $\mathrm{PG}(2,q)$, q odd. There are $q+1$ tangent lines with respect to $\mathcal{C}$ and no three of these lines are concurrent, i.e. the set $\mathcal{C}^*$ of these tangent lines is a dual conic by Segre [11] (it is also possible to show this directly by using coordinates). As each of the $q+1$ points of $\mathcal{C}$ is contained in a unique tangent line, $\mathcal{C}$ is the set of points of $\mathrm{PG}(2,q)$ corresponding to the tangent lines of the dual conic $\mathcal{C}^*$.

Now, a line L of $\mathrm{PG}(2,q)$ not belonging to $\mathcal{C}^*$ is a secant line with respect to $\mathcal{C}$ if and only if it contains two points of $\mathcal{C}$, i.e. if and only if $L \notin \mathcal{C}^*$ regarded as a point of $\mathrm{PG}(2,q)^*$ is contained in two tangent lines with respect to $\mathcal{C}^*$, in other words if and only if L is an exterior point with respect to $\mathcal{C}^*$.

By the above, Claims (1), (2) and (3) will be valid if any dual conic $\widetilde{\mathcal{C}}^*$ of $\mathrm{PG}(2,q)^*$ consists of the tangent lines with respect to a conic of $\mathcal{C}$. But this follows by the same argument as in the first paragraph of this proof (with $\mathcal{C}$ replaced by $\widetilde{\mathcal{C}}^*$). $\square$

3 Sets of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$ in $\mathrm{PG}(2,q)$, q odd

Before we can achieve the first goal of this paper, we need to derive some results regarding certain sets of points in $\mathrm{PG}(2,3)$ and $\mathrm{PG}(2,5)$.

If $m_1, m_2, \ldots, m_k$ are k mutually distinct integers satisfying $0 \leq m_1 < m_2 < \cdots < m_k \leq q+1$, then a set of points of $\mathrm{PG}(2,q)$ is said to be of *class* $[m_1, m_2, \ldots, m_k]$ if it meets every line in either m_1, m_2, …, or m_k points.

3.1 Sets of class $[1,2,3]$ in $\mathrm{PG}(2,3)$

If $\mathcal{C}$ is an irreducible conic in $\mathrm{PG}(2,3)$ and $\mathcal{E}_{\mathcal{C}}$ denotes the set of exterior points of $\mathrm{PG}(2,3)$ with respect to $\mathcal{C}$, then every tangent line meets $\mathcal{E}_{\mathcal{C}}$ in three points, every secant line meets $\mathcal{E}_{\mathcal{C}}$ in one point and every external line meets $\mathcal{E}_{\mathcal{C}}$ in two points. So, $\mathcal{E}_{\mathcal{C}}$ is a set of class $[1,2,3]$. Note also that the complement of any set of class $[1,2,3]$ in $\mathrm{PG}(2,3)$ is again a set of class $[1,2,3]$.

Lemma 3.1. *Every set of class* $[1,2,3]$ *in* $\mathrm{PG}(2,3)$ *is either the set of exterior points with respect to an irreducible conic or the complement of such a set. Equivalently, a set of class* $[1,2,3]$ *in* $\mathrm{PG}(2,3)$ *is either the set of points on the sides of a triangle except for the vertices, or the complement of such a set.*

Proof. Let X be a set of class $[1,2,3]$ in $\mathrm{PG}(2,3)$. Then X is either a set of class $[1,2,3]$ with $|X| \leq 6$ or the complement of such a set of points.

Suppose therefore that $|X| \leq 6$. If there are no lines intersecting X in three points, then $|X| \leq 4$ and X must be a subset of an irreducible conic, implying that there are lines disjoint from X, a contradiction.

So, there exists a line L meeting X in precisely three points x_1, x_2 and x_3. Let y denote the unique point of L, not belonging to X. Through y, there are three lines L_1, L_2 and L_3 distinct from L.

Each of these three lines contains an extra point of X. As $|X| \leq 6$, we thus see that each of these three lines contains precisely one point of X. Put $L_i \cap X = \{y_i\}$ for every

$i \in \{1,2,3\}$. The points y_1, y_2, y_3 cannot be collinear, otherwise the line containing these points would be entirely contained in X as this line also contains a point of $L \setminus \{y\}$. Each y_i with $i \in \{1,2,3\}$ is now contained in two lines meeting X in exactly three points, namely the lines $y_i y_j$ with $j \in \{1,2,3\} \setminus \{i\}$ and one line meeting X in exactly one point, namely $y_i y$. So, each y_i, $i \in \{1,2,3\}$, is contained in a unique line meeting X in exactly two points. Repeating the above argument with L replaced by either $y_1 y_2$ or $y_1 y_3$, we then also see that every point of $L \cap X$ is contained in a unique line meeting X in exactly two points.

There are thus three lines U_1, U_2 and U_3 that contain exactly two points of X, resulting in the 6 points for X. Moreover, any two of these lines intersect in a point outside X. If U_1, U_2 and U_3 go through the same point, then the fourth line through that point would be disjoint from X, which is impossible. So, the lines U_1, U_2 and U_3 form a triangle and

$$X = (U_1 \cup U_2 \cup U_3) \setminus ((U_1 \cap U_2) \cup (U_1 \cap U_3) \cup (U_2 \cup U_3)). \qquad (*)$$

Conversely, every set of 6 points of $\mathrm{PG}(2,3)$ obtained as in $(*)$ is a set of class $[1,2,3]$. So, there exists up to isomorphism only one set of class $[1,2,3]$ having 6 points. Such a set of class $[1,2,3]$ thus necessarily is the set of exterior points with respect to an irreducible conic. □

3.2 Sets of class $[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3)]$ in $\mathrm{PG}(2,q)$, $q \geq 5$ odd

We take the following result from Theorem 4.6 of [7].

Proposition 3.2 ([7])**.** *Let $\mathcal{K}$ be a set of class $[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3)]$ in $\mathrm{PG}(2,q)$, q odd and $q \geq 5$. Then one of the following cases occurs:*

(1) *$q = 5$ and $\mathcal{K}$ consists of all points on the sides of a triangle, except the vertices;*

(2) *$\mathcal{K}$ consists of an irreducible conic and its interior points.*

3.3 Two consequences

From Lemma 3.1 and Proposition 3.2, the following can be derived.

Corollary 3.3. *Let $\mathcal{K}$ be a set of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$ in $\mathrm{PG}(2,q)$ where $q \in \{3,5\}$. Then $\mathcal{K}$ is one of the following:*

(1) *the set of exterior points with respect to an irreducible conic;*

(2) *a set of the form $(\mathrm{PG}(2,q) \setminus (L_1 \cup L_2 \cup L_3)) \cup (L_1 \cap L_2) \cup (L_1 \cap L_3) \cup (L_2 \cap L_3)$, where L_1, L_2 and L_3 are three nonconcurrent lines.*

Proof. For $q = 3$, this is a consequence of Lemma 3.1. For $q = 5$, this is a consequence of Proposition 3.2 if we take into account that the complement of a set of class $[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3)]$ in $\mathrm{PG}(2,q)$, q odd, is a set of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$. □

From Proposition 3.2 and Corollary 3.3, the following can be derived.

Corollary 3.4. *Let X be a set of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$ in $\mathrm{PG}(2,q)$, q odd, having exactly $\frac{1}{2}q(q+1)$ points. Then there exists an irreducible conic $\mathcal{C}$ in $\mathrm{PG}(2,q)$ such that X coincides with the set of points of $\mathrm{PG}(2,q)$ that are exterior with respect to $\mathcal{C}$.*

Proof. If $q \in \{3,5\}$, then one of the cases (1) or (2) of Corollary 3.3 must occur. As $|X| = \frac{1}{2}q(q+1)$, the former case must occur.

If $q \geq 7$, then as the complement of X is a set of class $[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3)]$, we know from Proposition 3.2 that X is the set of exterior points with respect to an irreducible conic. $\square$

4 Proof of Proposition 1.1 for all odd q

Let $\mathcal{S}$ be a family of lines of $\mathrm{PG}(3,q)$ with q odd satisfying the properties (P1) and (P2) of Proposition 1.1. A plane of $\mathrm{PG}(3,q)$ is said to be *tangent* or *secant* according as it satisfies the property (P2a) or (P2b). For a given plane π of $\mathrm{PG}(3,q)$, we denote by $\mathcal{S}_\pi$ the set of lines of $\mathcal{S}$ contained in π.

4.1 Recollection of some results from [9]

The following three results are respectively Lemma 2.1, Lemma 2.2 and Corollary 2.3 of [9]. The proofs of these results in [9] work for all odd q.

Lemma 4.1. *If π is a tangent plane, then $|\mathcal{S}_\pi| = q^2$.*

Lemma 4.2. *Let π be a tangent plane. Then there are exactly q^2+q points in π each of which is contained in q lines of $\mathcal{S}_\pi$. Equivalently, there is only one point of π that is not contained in any line of $\mathcal{S}_\pi$.*

For a tangent plane π, we denote by p_π the unique point of π which is contained in no lines of $\mathcal{S}_\pi$ (Lemma 4.2) and call it the *pole* of π.

Corollary 4.3. *Let π be a tangent plane. Then the $q+1$ lines of π not contained in $\mathcal{S}_\pi$ are precisely the lines of π through the pole p_π.*

The following is an immediate consequence of property (P2b).

Lemma 4.4. *If π is a secant plane, then $\mathcal{S}_\pi$ is a set of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$ in the dual plane of π.*

The following is precisely Lemma 2.5 of [9]. It is a consequence of Proposition 3.2 and Lemmas 2.1, 4.4, taking into account that the complement of a set of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$ is a set of class $[1, \frac{1}{2}(q+1), \frac{1}{2}(q+3)]$.

Corollary 4.5. *If $q \geq 7$, then for every secant plane π, the set $\mathcal{S}_\pi$ consists of all secant lines with respect to an irreducible conic of π.*

Using Corollary 4.5, the authors of [9] were able to complete the proof of Proposition 1.1 without making further use of the restriction that $q \geq 7$. Although they never stated this explicitly in their paper, the authors of [9] have thus shown the following.

Proposition 4.6. *Suppose that for every secant plane π, the set $\mathcal{S}_\pi$ consists of all secant lines to an irreducible conic of π. Then $\mathcal{S}$ is one of the following:*

(1) *the set of all secant lines with respect to a hyperbolic quadric in* $\mathrm{PG}(3,q)$*;*

(2) *a set of $\frac{1}{2}(q^4+q^3+2q^2)$ lines such that the set of points that are incident with q^2 lines of $\mathcal{S}$ is a line $L \notin \mathcal{S}$ and the planes π for which (P2a) holds are precisely the planes containing L.*

4.2 Completion of the proof

The following is a consequence of Lemmas 2.1, 4.4 and Corollaries 3.3, 4.5.

Corollary 4.7. *One of the following cases occurs for a secant plane π:*

(1) *$\mathcal{S}_\pi$ consists of all secant lines with respect to an irreducible conic;*

(2) *there exist three noncollinear points x_1, x_2 and x_3 such that $\mathcal{S}_\pi$ consists of the lines x_1x_2, x_1x_3, x_2x_3 and all lines disjoint from $\{x_1, x_2, x_3\}$.*

Case (2) *cannot occur if $q > 5$.*

If case (1) occurs, then we call π a *secant plane of type (1)*. If case (2) occurs (and so $q \in \{3, 5\}$), then we call π a *secant plane of type (2)*. Every point of $\mathrm{PG}(3,q)$ is contained in q^2 or $\frac{1}{2}q(q+1)$ lines of $\mathcal{S}$ by property (P1). We call a point of $\mathrm{PG}(3,q)$ *black* (respectively *white*) if it is contained in q^2 (respectively $\frac{1}{2}q(q+1)$) lines of $\mathcal{S}$.

Lemma 4.8. *Suppose x is a black point. Then there exists a unique plane π_x through x such that the lines of $\mathcal{S}$ through x are precisely the lines through x not contained in π_x. This plane π_x is a tangent plane.*

Proof. We will reason in the quotient projective space $\mathrm{PG}(3,q)_x$. There are $q+1$ lines through x not belonging to $\mathcal{S}$. Each line of $\mathrm{PG}(3,q)_x$ contains at least one element of this collection of $q+1$ lines. By Theorem 1 of [2], we then know that the $q+1$ lines through x not belonging to $\mathcal{S}$ are the $q+1$ lines through x contained in a certain plane π_x. As no line of π_x through x belongs to $\mathcal{S}$, π_x is a tangent plane. □

Consider the map θ between the set of black points and the set of tangent planes mapping x to π_x.

Lemma 4.9. *The map θ is injective.*

Proof. Let π be a tangent plane. If $\pi = \theta(x)$ for some black point x, then we know that x necessarily coincides with the pole of π. So, θ is injective. □

Lemma 4.10. (1) *The map θ is bijective.*

(2) *There are no secant planes of type* (2).

Proof. Consider now the following numbers:
- B is the number of black points;
- W is the number of white points;
- N_1 is the number of tangent planes;
- N_2 is the number of secant planes of type (1);
- N_3 is the number of secant planes of type (2).

By Lemma 4.9, we have $B \leq N_1$, with equality if and only if θ is bijective.

We count in two different ways the triples (x, L, π), where x is a point of $\mathrm{PG}(3, q)$, L a line of $\mathcal{S}$ through x and π is a plane of $\mathrm{PG}(3, q)$ through L.

Counting according to the sequence x, L, π yields that the number A of such triples equals

$$A = \Big(Bq^2 + W\frac{q(q+1)}{2}\Big)(q+1). \tag{1}$$

Counting according to the sequence π, L, x yields that the number A of triples also equals

$$A = \Big(N_1q^2 + N_2\frac{q(q+1)}{2} + N_3(q^2 - 2q + 4)\Big)(q+1). \tag{2}$$

By equations (1) and (2), we find

$$Bq^2 + W\frac{q(q+1)}{2} = N_1q^2 + N_2\frac{q(q+1)}{2} + N_3(q^2 - 2q + 4).$$

Replacing W by $q^3 + q^2 + q + 1 - B$ and N_2 by $q^3 + q^2 + q + 1 - N_1 - N_3$, we find

$$\begin{aligned} B\frac{q(q-1)}{2} + (q^3 + q^2 + q + 1)\frac{q(q+1)}{2} = N_1\frac{q(q-1)}{2} &+ (q^3 + q^2 + q + 1)\frac{q(q+1)}{2} \\ &+ N_3\Big(q^2 - 2q + 4 - \frac{q(q+1)}{2}\Big), \end{aligned}$$

i.e.

$$B\frac{q(q-1)}{2} = N_1\frac{q(q-1)}{2} + N_3\frac{(q-3)(q-2)+2}{2}.$$

We thus find that $B \geq N_1$. Hence, $B = N_1$ and $N_3 = 0$. As $B = N_1$, the map θ must be bijective. $\square$

Theorem 4.11. *The conclusion of Proposition 1.1 is valid for all odd prime powers q.*

Proof. For $q > 5$, this was shown in [9], see also Proposition 4.6 and the remark prior to that proposition. For $q \in \{3, 5\}$, this now follows from Proposition 4.6, Corollary 4.7 and Lemma 4.10. $\square$

For each prime power q, we denote by $\mathcal{H}(q)$ the set of all line sets in $\mathrm{PG}(3, q)$ that satisfy the properties (P1), (P2) of Proposition 1.1 and which correspond to case (2) of that proposition. The question whether $\mathcal{H}(q)$ is non-empty was left open in [9]. In the following sections, we show that $\mathcal{H}(q)$ is non-empty for every odd prime power. We also obtain some classification results for the line sets in $\mathcal{H}(q)$, some of which have been obtained by means of computer computations.

5 Alternative descriptions for the line sets in $\mathcal{H}(q)$

Consider the following properties for a set $\mathcal{S}$ of lines of $\mathrm{PG}(3,q)$, q odd:

(P3) $\mathcal{S}$ is not the set of secant lines with respect to a hyperbolic quadric of $\mathrm{PG}(3,q)$.

(P1′) One of the following two cases occurs for a point p of $\mathrm{PG}(3,q)$:

- (P1a′) there exists a plane π through p such that the lines of $\mathcal{S}$ through p are precisely the lines through p not contained in π;
- (P1b′) there exists an irreducible conic $\mathcal{C}_p$ in the quotient projective space $\mathrm{PG}(3,q)_p \cong \mathrm{PG}(2,q)$ such that the lines of $\mathcal{S}$ through p are precisely the points of $\mathrm{PG}(3,q)_p$ that are exterior with respect to $\mathcal{C}_p$.

(P2′) One of the following two cases occurs for a plane π of $\mathrm{PG}(3,q)$:

- (P2a′) there exists a point p in π such that the lines of $\mathcal{S}$ in π are precisely the lines in π not containing p;
- (P2b′) there exists an irreducible conic $\mathcal{C}_\pi$ in π such that the lines of $\mathcal{S}$ in π are precisely the secant lines with respect to $\mathcal{C}_\pi$.

(P3′) There exists a line $L^* \notin \mathcal{S}$ in $\mathrm{PG}(3,q)$ such that the points p of $\mathrm{PG}(3,q)$ for which case (P1a′) occurs are precisely the points on L^* and the planes π for which case (P2a′) occurs are precisely the planes through L^*.

We hereby note that property (P3′) is only defined for line sets that also satisfy properties (P1′) and (P2′).

Recall that the members of $\mathcal{H}(q)$ are precisely the line sets satisfying properties (P1), (P2) and (P3). We now show that the members of $\mathcal{H}(q)$ can also be described by means of the properties (P1′), (P2′) and (P3′).

Proposition 5.1. *A set $\mathcal{S}$ of lines of* $\mathrm{PG}(3,q)$*, q odd, satisfies properties (P1), (P2) and (P3) if and only if it satisfies properties (P1′), (P2′) and (P3′).*

Proof. Suppose $\mathcal{S}$ satisfies properties (P1), (P2) and (P3). By Proposition 1.1, Corollaries 4.3, 4.7 and Lemma 4.10, we then know that the following properties hold:

(1) The set $\mathcal{S}$ consists of $\frac{1}{2}(q^4+q^3+2q^2)$ lines.

(2) There exists a necessarily unique line L^* with the property that every point of L^* is incident with exactly q^2 lines of $\mathcal{S}$ and every point not incident with L^* is incident with exactly $\frac{1}{2}q(q+1)$ lines of $\mathcal{S}$.

(3) The line L^* does not belong to $\mathcal{S}$.

(4) Every plane π through L^* contains a unique point $p_\pi \in L^*$ with the property that the lines of $\mathcal{S}$ in π are precisely the lines of π not containing the point p_π.

(5) The map $\pi \mapsto p_\pi$ defines a bijection between the set of planes through L^* and the set of points on L^*.

(6) For every plane π not containing L^*, there exists an irreducible conic $\mathcal{C}_\pi$ in π such that the lines of $\mathcal{S}$ in π are precisely the lines of π that are secant with respect to $\mathcal{C}_\pi$.

For every point x of $\mathrm{PG}(3,q)$, let $\mathcal{S}_x$ denote the set of lines of $\mathcal{S}$ through x. Then $\mathcal{S}_x$ is a set of points of the quotient projective space $\mathrm{PG}(3,q)_x$. By the above properties (4) and (5), we know that the following holds:

(7) If $x \in L^*$, then $\mathcal{S}_x$ is the complement of a line of $\mathrm{PG}(3,q)_x$.

If x is a point of $\mathrm{PG}(3,q) \setminus L^*$ and if π is the plane $\langle x, L^* \rangle$, then with exception of xp_π all lines of π through x belong to $\mathcal{S}$. If π' is another plane through x, then by (P2b) there are either $\frac{1}{2}(q-1)$, $\frac{1}{2}(q+1)$ or q lines of π' through x that belong to $\mathcal{S}$. So, $\mathcal{S}_x$ is a set of class $[\frac{1}{2}(q-1), \frac{1}{2}(q+1), q]$ of $\mathrm{PG}(3,q)_x$. In combination with Corollary 3.4 and property (2), this implies the following:

(8) If x is a point of $\mathrm{PG}(3,q)$ not belonging to L^*, then $\mathcal{S}_x$ is the set of exterior points with respect to an irreducible conic $\mathcal{C}_x$ of $\mathrm{PG}(3,q)_x$.

Note also that if x is a point of $\mathrm{PG}(3,q) \setminus L^*$ and if π is the plane $\langle x, L^* \rangle$, then the fact that with exception of xp_π all lines of π through x belong to $\mathcal{S}$ implies that π is a tangent line of $\mathrm{PG}(3,q)_x$ to the conic $\mathcal{C}_x$ with tangency point xp_π.

Properties (P1′), (P2′) and (P3′) are consequences of the above properties (3), (4), (6), (7) and (8).

Conversely, suppose that $\mathcal{S}$ satisfies the properties (P1′), (P2′) and (P3′). Let L^* be the line of $\mathrm{PG}(3,q)$ such that property (P3′) holds.

If $x \in L^*$, then the fact that property (P1a′) holds for x implies that x is incident with q^2 lines of $\mathcal{S}$. If $x \in \mathrm{PG}(3,q) \setminus L^*$, then the fact that property (P1b′) holds for x implies that x is incident with $\frac{1}{2}q(q+1)$ lines of $\mathcal{S}$. We conclude that $\mathcal{S}$ satisfies (P1).

If π is a plane through L^*, then the fact that property (P2a′) holds for π implies that every pencil of lines in π contains 0 or q lines of $\mathcal{S}$. If π is a plane of $\mathrm{PG}(3,q)$ not containing L^*, then the fact that property (P2b′) holds for π implies that every pencil of lines in π contains $\frac{1}{2}(q-1)$, $\frac{1}{2}(q+1)$ or q lines of $\mathcal{S}$. So, property (P2) holds.

If $\mathcal{S}$ were the set of secant lines with respect to a hyperbolic quadric $Q^+(3,q)$ of $\mathrm{PG}(3,q)$, then the planes of $\mathrm{PG}(3,q)$ for which (P2a′) holds are the tangent planes with respect to $Q^+(3,q)$. As these planes do not intersect in a line (as required by (P3′)), we have a contradiction. So, property (P3) must hold. □

We now give another alternative description of the elements of $\mathcal{H}(q)$ by using the so-called Klein correspondence.

Suppose $\mathrm{PG}(3,q) = \mathrm{PG}(V)$, where V is a 4-dimensional vector space over the finite field $\mathbb{F}_q$, and denote by $\bigwedge^2 V$ the second exterior power of V. The map κ which sends each line L of $\mathrm{PG}(V)$ through two distinct points $\langle \bar{v}_1 \rangle$ and $\langle \bar{v}_2 \rangle$ to the point $\langle \bar{v}_1 \wedge \bar{v}_2 \rangle$ of

$\mathrm{PG}(5,q) = \mathrm{PG}(\bigwedge^2 V)$ is well-defined. It moreover defines a bijection between the set of lines of $\mathrm{PG}(3,q)$ and the set of points of a hyperbolic quadric in $\mathrm{PG}(\bigwedge^2 V)$ which we will denote by $Q^+(5,q)$. The map κ is also called the *Klein correspondence* and the quadric $Q^+(5,q)$ the *Klein quadric*. We recall some known properties of the Klein quadric. For more background information, we refer to [8] and [12].

(1) For every point x of $\mathrm{PG}(3,q)$, we denote by $\mathcal{L}_x$ the set of lines of $\mathrm{PG}(3,q)$ through x. The set $\kappa(\mathcal{L}_x)$ is a plane of $Q^+(5,q)$ which is called a *Latin plane*.

(2) For every plane π of $\mathrm{PG}(3,q)$, we denote by $\mathcal{L}_\pi$ the set of lines of $\mathrm{PG}(3,q)$ contained in π. The set $\kappa(\mathcal{L}_\pi)$ is a plane of $Q^+(5,q)$ which is called a *Greek plane*.

(3) Every plane of $Q^+(5,q)$ is either a Latin or Greek plane.

(4) For every incident point-plane pair (x,π) in $\mathrm{PG}(3,q)$, let $\mathcal{L}_{x,\pi}$ denote the line pencil consisting of all lines through x contained in π. Then $\kappa(\mathcal{L}_{x,\pi}) = \kappa(\mathcal{L}_x) \cap \kappa(\mathcal{L}_\pi)$ is a line of $Q^+(5,q)$. Every line of $Q^+(5,q)$ can be obtained in this way.

(5) Every automorphism or duality θ of $\mathrm{PG}(3,q)$ maps lines to lines and induces an automorphism $\widetilde{\theta}$ of $Q^+(5,q)$ via the relation $\kappa(L^\theta) = \kappa(L)^{\widetilde{\theta}}$ for lines L of $\mathrm{PG}(3,q)$. If θ is an automorphism, then $\widetilde{\theta}$ maps Latin planes to Latin planes and Greek planes to Greek planes. If θ is a duality, then $\widetilde{\theta}$ maps Latin planes to Greek planes and Greek planes to Latin planes. Conversely, every automorphism of $Q^+(5,q)$ is of the form $\widetilde{\theta}$ for some automorphism or duality θ of $\mathrm{PG}(3,q)$.

(6) If (x,L) is a non-incident point-line pair of $Q^+(5,q)$, then either one or all points of L are collinear with x on the quadric $Q^+(5,q)$.

(7) If (x,π) is a non-incident point-plane pair of $Q^+(5,q)$, then there is a unique plane of $Q^+(5,q)$ through x intersecting π in a line.

Consider now the following properties for a set X of points of $Q^+(5,q)$, q odd:

(P1″) For a plane π of $Q^+(5,q)$, the intersection $X \cap \pi$ is either the complement (in π) of a line $L_\pi \subseteq \pi$ or the set of points in π that are exterior with respect to a (necessarily unique) irreducible conic $\mathcal{C}_\pi$ of π.

(P2″) There exists a point $x^* \notin X$ such that the planes π of $Q^+(5,q)$ for which $\pi \cap X$ is the complement of a line are precisely the planes through x^*.

(P3″) If x is a point noncollinear with x^* on $Q^+(5,q)$ and π_1, π_2 are two planes of $Q^+(5,q)$ through x, then $x \in \mathcal{C}_{\pi_1}$ if and only if $x \in \mathcal{C}_{\pi_2}$.

We hereby note that property (P2″) is only defined for point sets that also satisfy property (P1″) and that property (P3″) is only defined for point sets that also satisfy properties (P1″) and (P2″). If X satisfies properties (P1″) and (P2″), then $x^* \in L_\pi$ for every plane π of $Q^+(5,q)$ through x^*.

If X is a set of points of $Q^+(5,q)$ satisfying (P1″), (P2″) and (P3″), then A_X denotes the union of all lines L_π for planes π of $Q^+(5,q)$ through x^* and B_X denotes the set of all points x of $Q^+(5,q)$ noncollinear with x^* on $Q^+(5,q)$ such that $x \in \mathcal{C}_\pi$ for any plane π of $Q^+(5,q)$ through x. Note that $A_X \cap B_X = \emptyset$ and that X is disjoint from $A_X \cup B_X$.

A set X of points of $Q^+(5,q)$ is called a *quadratic set* if it intersects each plane of $Q^+(5,q)$ in a possible reducible conic of that plane. A quadratic set is said to be of *type* (LC) if each plane of $Q^+(5,q)$ intersects it in either a line or an irreducible conic. A quadratic set X is said to be of *type* (LC*) if the planes of $Q^+(5,q)$ that intersect it in a line are precisely the planes of $Q^+(5,q)$ through a given point of X.

Suppose X is a quadratic set of type (LC*) of $Q^+(5,q)$, q odd, and let x^* denote the unique point of X such that the planes of $Q^+(5,q)$ intersecting X in a line are precisely the planes of $Q^+(5,q)$ through x^*. For every plane π of $Q^+(5,q)$ not containing x^*, we denote the irreducible conic $\pi \cap X$ also by $\mathcal{C}_\pi$. We call X a *nice* quadratic set of type (LC*) if the following property is satisfied:

> If x is a point of $Q^+(5,q)$ noncollinear with x^* and π_1, π_2 are two planes of $Q^+(5,q)$ through x, then x is an exterior point with respect to the irreducible conic $\mathcal{C}_{\pi_1}$ if and only if it is an exterior point with respect to $\mathcal{C}_{\pi_2}$.

If this is the case, then C_X denotes the set of points of $Q^+(5,q)$ not contained in X that are collinear on $Q^+(5,q)$ with x^* and D_X denotes the set of all points x of $Q^+(5,q)$ noncollinear with x^* on $Q^+(5,q)$ such that x is an exterior point with respect to $\mathcal{C}_\pi$ for any plane π of $Q^+(5,q)$ containing the point x. Note that $C_X \cap D_X = \emptyset$ and that X is disjoint from $C_X \cup D_X$.

Lemma 5.2. *Let p be a point and π a plane of* $\mathrm{PG}(3,q)$*, q odd. Let $\mathcal{C}_p$ and $\mathcal{C}_\pi$ be irreducible conics in respectively* $\mathrm{PG}(3,q)_p$ *and π. If θ is a duality of* $\mathrm{PG}(3,q)$ *mapping p to π and $\mathcal{C}_p$ to the dual conic in π consisting of all tangent lines to $\mathcal{C}_\pi$ (recall Lemma 2.1), then the set of exterior points of* $\mathrm{PG}(3,q)_p$ *with respect to $\mathcal{C}_p$ is mapped by θ to the set of secant lines with respect to $\mathcal{C}_\pi$.*

Proof. Since there are as many exterior points with respect to $\mathcal{C}_p$ as there are secant lines with respect to $\mathcal{C}_\pi$, namely $\frac{1}{2}q(q+1)$, it suffices to prove that θ maps every exterior point L with respect to $\mathcal{C}_p$ to a secant line with respect to $\mathcal{C}_\pi$. Note that L is a line through p. Let α denote a line in $\mathrm{PG}(3,q)_p$ that contains L and is tangent to $\mathcal{C}_p$. Then α is a plane through L containing exactly one line that belongs $\mathcal{C}_p$. We denote this line by K. In the line pencil $\mathcal{L}_{p,\alpha}$, there is thus one line belonging to $\mathcal{C}_p$, namely K. That implies that the line pencil $(\mathcal{L}_{p,\alpha})^\theta = \mathcal{L}_{\alpha^\theta,p^\theta} = \mathcal{L}_{\alpha^\theta,\pi}$ only contains one tangent line to $\mathcal{C}_\pi$, implying that $\alpha^\theta \in \mathcal{C}_\pi$ and $\mathcal{L}_{\alpha^\theta,p^\theta} \setminus \{K^\theta\}$ contains q secant lines with respect to $\mathcal{C}_\pi$. In particular, L^θ is a secant line with respect to $\mathcal{C}_\pi$. $\square$

The following is an immediate consequence of Lemma 5.2.

Corollary 5.3. *Let θ be a duality of* $\mathrm{PG}(3,q)$ *mapping a point p to a plane π. Then a set $\mathcal{S}$ of lines of* $\mathrm{PG}(3,q)$ *satisfies property* (P1′) *with respect to p if and only if $\mathcal{S}^\theta$ satisfies property* (P2′) *with respect to π. In fact, if property* (P1a′) *(respectively,* (P1b′)*) holds for the point p, then property* (P2a′) *(respectively,* (P2b′)*) holds for the plane π.*

Proposition 5.4. *Let $\mathcal{S}$ be a set of lines of* $\mathrm{PG}(3,q)$*, q odd, and X a set of points of $Q^+(5,q)$ such that $X = \kappa(\mathcal{S})$. Then $\mathcal{S}$ satisfies properties* (P1′)*,* (P2′) *and* (P3′) *if and only if X satisfies properties* (P1″) *and* (P2″)*.*

Proof. We show the following:

(A) Property (P1′) holds for a point p of $\mathrm{PG}(3,q)$ if and only if property (P1″) holds for the Latin plane $\kappa(\mathcal{L}_p)$.

(B) Property (P2′) holds for a plane π of $\mathrm{PG}(3,q)$ if and only if property (P1″) holds for the Greek plane $\kappa(\mathcal{L}_\pi)$.

(C) Property (P2″) is equivalent with (P3′). The line L^* of $\mathrm{PG}(3,q)$ and the point x^* of $Q^+(5,q)$ occurring in these properties are related to each other via the Klein correspondence, i.e. $x^* = \kappa(L^*)$.

We start by proving (A). Let p be a point of $\mathrm{PG}(3,q)$ and π a Latin plane of $Q^+(5,q)$ such that $\pi = \kappa(\mathcal{L}_p)$. If α is a plane of $\mathrm{PG}(3,q)$ through p, then κ maps $\mathcal{L}_p \setminus \mathcal{L}_{p,\alpha}$ to $\kappa(\mathcal{L}_p)\setminus\kappa(\mathcal{L}_{p,\alpha})$, i.e. to the complement of the line $\kappa(\mathcal{L}_{p,\alpha})$ in $\pi = \kappa(\mathcal{L}_p)$. So, it suffices to prove that κ maps the sets of lines of $\mathrm{PG}(3,q)$ through p that arise as sets of exterior points with respect to irreducible conics of $\mathrm{PG}(3,q)_p$ to sets of points of π that arise as sets of exterior points with respect to irreducible conics of π.

Let $(\bar v_1, \bar v_2, \bar v_3, \bar v_4)$ be an ordered basis of V such that $p = \langle \bar v_1\rangle$. Then $\pi = \langle \bar v_1 \wedge \bar v_2, \bar v_1 \wedge \bar v_3, \bar v_1 \wedge \bar v_4\rangle$. There is now a natural bijective correspondence between quadratic forms Q on $\langle \bar v_2, \bar v_3, \bar v_4\rangle$ and quadratic forms $\widetilde{Q}$ on $\langle \bar v_1 \wedge \bar v_2, \bar v_1 \wedge \bar v_3, \bar v_1 \wedge \bar v_4\rangle$ given by

$$\widetilde{Q}(\lambda_2 \cdot \bar v_1 \wedge \bar v_2 + \lambda_3 \cdot \bar v_1 \wedge \bar v_3 + \lambda_4 \cdot \bar v_1 \wedge \bar v_4) := Q(\lambda_2 \bar v_2 + \lambda_3 \bar v_3 + \lambda_4 \bar v_4)$$

for $\lambda_2, \lambda_3, \lambda_4 \in \mathbb{F}_q$. Suppose now that Q and $\widetilde{Q}$ give rise to irreducible conics which we will respectively denote by $\mathcal{C}_Q$ and $\mathcal{C}_{\widetilde{Q}}$. If we denote by S and N the set of nonzero squares and nonsquares in $\mathbb{F}_q$, then we denote by T the unique set in $\{S, N\}$ such that there are $\frac{1}{2}q(q+1)$ points $\langle \bar v\rangle$ in $\langle \bar v_2, \bar v_3, \bar v_4\rangle$ for which $Q(\bar v) \in T$. These are then precisely the exterior points of $\langle \bar v_2, \bar v_3, \bar v_4\rangle$ with respect to $\mathcal{C}_Q$. As $\widetilde{Q}(\bar v_1 \wedge \bar v) = Q(\bar v)$ for all $\bar v \in \langle \bar v_2, \bar v_3, \bar v_4\rangle$, we then also see that the set of all points $\langle \bar v_1 \wedge \bar v\rangle$ of π for which $\widetilde{Q}(\bar v_1 \wedge \bar v) \in T$ is the set of exterior points with respect to $\mathcal{C}_{\widetilde{Q}}$. Now, if we denote by $\mathcal{C}_p$ the irreducible conic of $\mathrm{PG}(3,q)_p$ obtained by connecting p with all points of $\mathcal{C}_Q$, then the map $\langle \bar v_1, \bar v\rangle \mapsto \langle \bar v_1 \wedge \bar v\rangle = \kappa(\langle \bar v_1, \bar v\rangle)$ maps the set of points of $\mathrm{PG}(3,q)_p$ that are exterior with respect to $\mathcal{C}_p$ to the set of points of π that are exterior with respect to $\mathcal{C}_{\widetilde{Q}}$.

We now prove (B). Let θ be an arbitrary duality of $\mathrm{PG}(3,q)$. Then there exists a unique automorphism $\widetilde{\theta}$ of $Q^+(5,q)$ interchanging the Latin and Greek planes such that $\kappa(L^\theta) = \kappa(L)^{\widetilde{\theta}}$ for lines L of $\mathrm{PG}(3,q)$.

By Corollary 5.3, $\mathcal{S}$ satisfies property (P2′) with respect to π if and only if $\mathcal{S}^\theta$ satisfies property (P1′) with respect to the point $p := \pi^\theta$. By (A), we know that this precisely happens if property (P1″) holds for the Latin plane $\kappa(\mathcal{L}_p) = \kappa(\mathcal{L}_{\pi^\theta}) = \kappa(\mathcal{L}_\pi^\theta) = \kappa(\mathcal{L}_\pi)^{\widetilde{\theta}}$ with respect to the set $\kappa(\mathcal{S}^\theta) = \kappa(\mathcal{S})^{\widetilde{\theta}}$, i.e. if and only if property (P1″) holds for the Greek plane $\kappa(\mathcal{L}_\pi)$ with respect to the set $X = \kappa(\mathcal{S})$.

Property (P1″) which we defined above holds for a plane π if precisely one of the following two properties holds:

(P1a$''$) the intersection $\pi \cap X$ is the complement (in π) of a line $L_\pi \subseteq \pi$;

(P1b$''$) the intersection $\pi \cap X$ consists of the points in π that are exterior with respect to a given irreducible conic $\mathcal{C}_\pi$ of π.

From the discussion above, we know that property (P1a$'$) holds for a point p of $\mathrm{PG}(3,q)$ if and only if (P1a$''$) holds for the Latin plane $\kappa(\mathcal{L}_p)$ and that property (P2a$'$) holds for a plane π if and only if property (P1a$''$) holds for the Greek plane $\kappa(\mathcal{L}_\pi)$. Claim (C) follows from this and the fact that for every point-line-plane triple (p, L^*, π) of $\mathrm{PG}(3,q)$, we have $p \in L^* \Leftrightarrow \kappa(L^*) \in \kappa(\mathcal{L}_p)$ and $L^* \subseteq \pi \Leftrightarrow \kappa(L^*) \in \kappa(\mathcal{L}_\pi)$. □

By Propositions 1.1, 5.1 and 5.4, we thus know that every set X of points of $Q^+(5,q)$, q odd, satisfying (P1$''$) and (P2$''$) has size $\frac{1}{2}(q^4+q^3+2q^2)$. This can also be found directly by double counting the pairs (x,π) with π a plane of $Q^+(5,q)$ and $x \in X \cap \pi$.

In the following proposition, we show the equivalence between nice quadratic sets of type (LC*) of $Q^+(5,q)$, q odd, and sets of points of $Q^+(5,q)$ satisfying properties (P1$''$), (P2$''$) and (P3$''$).

Proposition 5.5. *Suppose X is a set of points of $Q^+(5,q)$, q odd, satisfying properties* (P1$''$), (P2$''$), (P3$''$) *and Y is a nice quadratic set of type (LC*) of $Q^+(5,q)$. Then the following hold:*

- *$A_X \cup B_X$ is a nice quadratic set of type (LC*).*
- *$C_Y \cup D_Y$ satisfies properties* (P1$''$), (P2$''$) *and* (P3$''$).
- *We have $Y = A_X \cup B_X$ if and only if $X = C_Y \cup D_Y$.*

Proof. Suppose the point set of $Q^+(5,q)$ can be partitioned in three sets X_1, X_2 and X_3 such that the following three properties are satisfied with respect to a certain point x^* of $Q^+(5,q)$:

- Every plane π of $Q^+(5,q)$ through x^* is contained in $X_1 \cup X_2$. Moreover, $\pi \cap X_1$ is a line of π through x^*.
- If π is a plane of $Q^+(5,q)$ not containing x^*, then $\pi \cap X_1$ is an irreducible conic of π whose set of exterior points coincides with $\pi \cap X_2$ and whose set of interior points coincides with $\pi \cap X_3$.

If these properties hold, then we call (X_1, X_2, X_3) a *nice triple with respect to x^**. If this is the case, then $x^* \in X_1$. The following properties obviously hold:

(A) If (X_1, X_2, X_3) is a nice triple with respect to x^*, then X_2 satisfies properties (P1$''$), (P2$''$) and (P3$''$) (with respect to the same point x^*). If this is the case, then $A_{X_2} \cup B_{X_2} = X_1$ and $X_3 = Q^+(5,q) \setminus (X_1 \cup X_2)$.

(B) If X_2 is a set of points of $Q^+(5,q)$ satisfying properties (P1$''$), (P2$''$) and (P3$''$) (with respect to the same point x^*), then the triple (X_1, X_2, X_3) with $X_1 := A_{X_2} \cup B_{X_2}$ and $X_3 := Q^+(5,q) \setminus (X_1 \cup X_2)$ is a nice triple with respect to x^*.

(C) If (X_1, X_2, X_3) is a nice triple with respect to x^*, then X_1 is a nice quadratic set of type (LC*) (with respect to the same point x^*). If this is the case, then $C_{X_1} \cup D_{X_1} = X_2$ and $X_3 = Q^+(5,q) \setminus (X_1 \cup X_2)$.

(D) If X_1 is a nice quadratic set of type (LC*) with respect to x^*, then the triple (X_1, X_2, X_3) with $X_2 := C_{X_1} \cup D_{X_1}$ and $X_3 := Q^+(5,q) \setminus (X_1 \cup X_2)$ is a nice triple with respect to x^*.

The claims then follow from the above properties (A), (B), (C) and (D). $\square$

6 Examples of quadratic sets of type (LC*)

With respect to a reference system in PG$(5,q)$, the quadric $Q^+(5,q)$ has equation $X_1X_2 + X_3X_4 + X_5X_6 = 0$. For all $x, y, z \in \mathbb{F}_q$, we define the following sets of points of $Q^+(5,q)$:

$$\begin{aligned}
L(1,x,y,z) &:= \{(\alpha, -z\beta + y\gamma, \beta, z\alpha - x\gamma, \gamma, -y\alpha + x\beta) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
L(0,1,x,y) &:= \{(-\alpha, -y\beta + x\gamma, -x\alpha, -\gamma, -y\alpha, \beta) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
L(0,0,1,x) &:= \{(0, \gamma, -\alpha, x\beta, -x\alpha, -\beta) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
L(0,0,0,1) &:= \{(0, -\gamma, 0, \beta, -\alpha, 0) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
G(1,x,y,z) &:= \{(y\alpha + z\beta, \gamma, -x\alpha + z\gamma, -\beta, -x\beta - y\gamma, \alpha) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
G(0,1,x,y) &:= \{(-x\alpha - y\beta, \gamma, \alpha, x\gamma, \beta, y\gamma) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
G(0,0,1,x) &:= \{(\alpha, 0, -x\beta, -\gamma, \beta, -x\gamma) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\},\\
G(0,0,0,1) &:= \{(\alpha, 0, \beta, 0, 0, \gamma) \,|\, (\alpha,\beta,\gamma) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}\}.
\end{aligned}$$

Also, put

$$\begin{aligned}
\mathcal{L}^* &:= \{L(1,x,y,z), L(0,1,x,y), L(0,0,1,x), L(0,0,0,1) \,|\, x,y,z \in \mathbb{F}_q\},\\
\mathcal{G}^* &:= \{G(1,x,y,z), G(0,1,x,y), G(0,0,1,x), G(0,0,0,1) \,|\, x,y,z \in \mathbb{F}_q\}.
\end{aligned}$$

The following was proved in Section 2 of [4].

Proposition 6.1. *The planes of $Q^+(5,q)$ are precisely the elements of $\mathcal{L}^* \cup \mathcal{G}^*$. In fact, the reference system in* PG$(5,q)$ *can be chosen in such a way that $\mathcal{L}^*$ is the set of Latin planes and $\mathcal{G}^*$ is the set of Greek planes.*

The verification of the following lemma is straightforward.

Lemma 6.2. *Let $x, y, z \in \mathbb{F}_q$ and let p be the point $(1,0,0,0,0,0)$ of $Q^+(5,q)$. Then the following hold:*

- *The point p belongs to $L(1,x,y,z)$ if and only if $(y,z)=(0,0)$.*
- *The point p belongs to $L(0,1,x,y)$ if and only if $(x,y)=(0,0)$.*
- *The point p never belongs to $L(0,0,1,x)$, $L(0,0,0,1)$, $G(1,x,y,z)$ and $G(0,1,x,y)$.*
- *The point p always belongs to $G(0,0,1,x)$ and $G(0,0,0,1)$.*

If q is even, say $q=2^h$ with $h\in\mathbb{N}^*$, then we define $\mathrm{Tr}(x) := x+x^2+x^4+\cdots+x^{2^{h-1}}$ for every $x\in\mathbb{F}_q$. The following was proved in Section 6.1 of [4].

Proposition 6.3 ([4])**.** *Let $\mathcal{Q}$ be a quadric of $\mathrm{PG}(5,q)$, q even, with an equation of the form*

$$X_2X_5 + a_{26}X_2X_6 + a_{33}X_3^2 + a_{44}X_4^2 + a_{66}X_6^2 = 0,$$

where $a_{26}, a_{33}, a_{44}, a_{66} \in \mathbb{F}_q^*$ with $\mathrm{Tr}(\frac{a_{33}a_{44}a_{26}^2}{a_{66}^2}) = 1$. Then $S := \mathcal{Q}\cap Q^+(5,q)$ is a quadratic set of type (LC). Moreover, the planes of $Q^+(5,q)$ intersecting S in a line are precisely the planes $L(1,x,0,0)$, $L(0,1,0,0)$, $G(0,0,1,x)$ and $G(0,0,0,1)$ with $x\in\mathbb{F}_q$.

The following was proved in Section 6.6 of [4].

Proposition 6.4 ([4])**.** *Let $\mathcal{Q}$ be a quadric of $\mathrm{PG}(5,q)$, q odd, defined by an equation of the form*

$$X_2X_5 + d_1X_2X_6 + a_{33}X_3^2 + 2a_{33}d_2X_3X_4 + a_{33}d_2^2X_4^2 = 0,$$

where $a_{33}, d_1, d_2 \in \mathbb{F}_q^*$ with $-d_1d_2$ a non-square in $\mathbb{F}_q$. Then $S := \mathcal{Q}\cap Q^+(5,q)$ is a quadratic set of type (LC). Moreover, the planes of $Q^+(5,q)$ intersecting S in a line are precisely the planes $L(1,x,0,0)$, $L(0,1,0,0)$, $G(0,0,1,x)$ and $G(0,0,0,1)$ with $x\in\mathbb{F}_q$.

The following is a consequence of Lemma 6.2 and Propositions 6.1, 6.3, 6.4.

Corollary 6.5. *The quadratic sets described in Propositions 6.3 and 6.4 are quadratic sets of type (LC*) with respect to the point $(1,0,0,0,0,0)$.*

7 The set $\mathcal{H}(q)$ is non-empty

By Propositions 5.1, 5.4 and 5.5, we know that nice quadratic sets of type (LC*) in $Q^+(5,q)$, q odd, give rise to line sets in $\mathcal{H}(q)$. The following theorem provides a method for constructing nice quadratic sets of type (LC*).

Theorem 7.1. *Let q be odd. Then every quadratic set X of type (LC*) that arises by intersecting $Q^+(5,q)$ with a quadric $\mathcal{Q}$ of the ambient space of $\mathrm{PG}(5,q)$ is nice.*

Proof. Let V be a 6-dimensional vector space over $\mathbb{F}_q$ for which $\mathrm{PG}(5,q)=\mathrm{PG}(V)$. Suppose $X = Q^+(5,q)\cap\mathcal{Q}$ where $\mathcal{Q}$ is a quadric of $\mathrm{PG}(5,q)$ defined by a quadratic form $Q: V\to\mathbb{F}_q$, i.e.

$$\mathcal{Q} = \{\langle\bar{v}\rangle \,|\, \bar{v}\in V\setminus\{\bar{0}\} \text{ and } Q(\bar{v})=0\}.$$

Let x^* be the unique point of $Q^+(5,q)\cap\mathcal{Q}$ for which the following hold:

(I) every plane π of $Q^+(5,q)$ through x^* intersects $\mathcal{Q}$ in a line L_π through x^*;

(II) every plane π of $Q^+(5,q)$ not containing x^* intersects $\mathcal{Q}$ in an irreducible conic $\mathcal{C}_\pi$ of π.

The set $\mathrm{PG}(5,q) \setminus \mathcal{Q}$ can be partitioned in two subsets S and N such that for a point $\langle \bar{v} \rangle$ of $\mathrm{PG}(5,q)$, we have $\langle \bar{v} \rangle \in S$ if and only if $Q(\bar{v})$ is a nonzero square and $\langle \bar{v} \rangle \in N$ if and only if $Q(\bar{v})$ is a nonsquare. The following fact holds, see e.g. [5, Section 2].

> If L is a line containing a unique point of $\mathcal{Q}$, then $L \setminus \mathcal{Q}$ is contained in either S or N.

Let U denote the set of points of $Q^+(5,q)$ collinear on $Q^+(5,q)$ with x^* but not contained in $\mathcal{Q}$. We can prove the following.

Property 1. *The set U is contained in either S or N.*
proof. Let $\mathcal{G}$ be the point-line geometry whose points and lines are the lines and planes of $Q^+(5,q)$ through x^*, with incidence being containment. Then $\mathcal{G}$ is a $(q+1) \times (q+1)$-grid and the set of lines though x^* contained in $\mathcal{Q}$ is an *ovoid* $\mathcal{O}$ of $\mathcal{G}$, i.e. a set of points of $\mathcal{G}$ intersecting each line of $\mathcal{G}$ in a singleton.

By the above fact, we know that if $x \in U$, then $xx^* \setminus \{x^*\}$ is contained in either S or N. So, it suffices to show that if L_1 and L_2 are two distinct lines through x^* not contained in $\mathcal{Q}$, then $(L_1 \cup L_2) \setminus \{x^*\}$ is contained in either S or N. Since the complement of $\mathcal{O}$ inside $\mathcal{G}$ is connected, it suffices to prove this in the case that L_1 and L_2 are contained in a plane π of $Q^+(5,q)$. As π intersects $\mathcal{Q}$ in a line through x^*, every line joining a point $x_1 \in L_1 \setminus \{x^*\}$ with a point $x_2 \in L_2 \setminus \{x^*\}$ is a tangent line to $\mathcal{Q}$, implying that $\{x_1, x_2\}$ is contained in either S or N. Hence, also $(L_1 \cup L_2) \setminus \{x^*\}$ is contained in either S or N. (qed)

Let $E \in \{S, N\}$ such that $U \subseteq E$ and let I denote the other set in $\{S, N\}$. The following property implies that the quadratic set $X = Q^+(5,q) \cap \mathcal{Q}$ is nice.

Property 2. *Let π be a plane of $Q^+(5,q)$ not containing x^*. Then $\pi \cap E$ consists of all points in π that are exterior with respect to $\mathcal{C}_\pi$ and $\pi \cap I$ consists of all points in π that are interior with respect $\mathcal{C}_\pi$.*
Proof. Let π' be the unique plane of $Q^+(5,q)$ through x^* meeting π in a line L. Then $\pi' \cap \mathcal{Q}$ is a line of π' through x^* that meets L in a point x_L. The line L is therefore a tangent line to the conic $\mathcal{C}_\pi$ with tangency point x_L. All points of $L \setminus \{x_L\}$ belong to $U \subseteq E$ and are exterior with respect to $\mathcal{C}_\pi$. The claim then follows from the fact that among the sets $S \cap \pi$, $N \cap \pi$, one corresponds to the set of points of π that are exterior with respect to $\mathcal{C}_\pi$ and the other corresponds to the set of points of π that are interior with respect to $\mathcal{C}_\pi$, see e.g. [5, Section 2]. $\square$

In Proposition 6.4, we constructed for every odd prime power q a family of quadrics of $\mathrm{PG}(5,q)$ which intersect $Q^+(5,q)$ in quadratic sets of type (LC*), see Corollary 6.5. If $\mathcal{Q}$ is such a quadric, then $\kappa^{-1}(C_Y \cup D_Y)$ with $Y := Q^+(5,q) \cap \mathcal{Q}$ belongs to $\mathcal{H}(q)$ by Theorem 7.1 and Propositions 5.1, 5.4, 5.5.

8 The case of nice prime powers

Let V be a 2-dimensional vector space over the finite field $\mathbb{F}_q$, where q is an odd prime power. We denote by $\blacksquare$ the set of nonzero squares in $\mathbb{F}_q$ and by $\square$ the set of nonsquares in $\mathbb{F}_q$. For every basis $(\bar{v}_1, \bar{v}_2)$ of V, we define

$$\mathcal{P}(\bar{v}_1, \bar{v}_2) := \{\langle \bar{v}_1 + \lambda \bar{v}_2 \rangle \mid \lambda \in \blacksquare\}.$$

Then $\mathcal{P}(\bar{v}_1, \bar{v}_2)$ is a set of $\frac{1}{2}(q-1)$ points of the projective line $\mathrm{PG}(1,q) = \mathrm{PG}(V)$. We obviously have that $\mathcal{P}(\bar{v}_1, \bar{v}_2) = \mathcal{P}(\bar{v}_2, \bar{v}_1)$ for every basis $(\bar{v}_1, \bar{v}_2)$ of V. We also see that if $(\bar{v}_1, \bar{v}_2)$ and $(\bar{v}'_1, \bar{v}'_2)$ are two bases of V such that $\{\langle \bar{v}_1 \rangle, \langle \bar{v}_2 \rangle\} = \{\langle \bar{v}'_1 \rangle, \langle \bar{v}'_2 \rangle\}$, then $\mathcal{P}(\bar{v}_1, \bar{v}_2)$ and $\mathcal{P}(\bar{v}'_1, \bar{v}'_2)$ are either equal or disjoint. We say that q is a *nice prime power* if the following property holds:

> For any two bases $(\bar{v}_1, \bar{v}_2)$ and $(\bar{v}'_1, \bar{v}'_2)$ of V such that $\{\langle \bar{v}_1 \rangle, \langle \bar{v}_2 \rangle\} \neq \{\langle \bar{v}'_1 \rangle, \langle \bar{v}'_2 \rangle\}$, the sets $\mathcal{P}(\bar{v}_1, \bar{v}_2)$ and $\mathcal{P}(\bar{v}'_1, \bar{v}'_2)$ are distinct and not disjoint.

We will now classify all nice odd prime powers. In order to achieve this goal, we first need to mention another result from the literature. The following proposition is implied by Theorem 2.2 of [3].

Proposition 8.1 ([3]). *Let q be a prime power and e a positive divisor of $q-1$ distinct from 1. Then the subgroup of the multiplicative group of $\mathbb{F}_q$ consisting of all nonzero eth powers has e cosets in $\mathbb{F}_q^*$ which we will denote by $C_1, C_2, \ldots, C_e$. Let $m \in \mathbb{N}^*$. If $q > \frac{1}{4}(U + \sqrt{U^2 + 4e^{m-1}m})^2$ where $U = \sum_{h=1}^{m} \binom{m}{h}(e-1)^h(h-1)$, then for all mutually distinct $a_1, a_2, \ldots, a_m \in \mathbb{F}_q$ and for all $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, e\}$, there exists a $y \in \mathbb{F}_q$ such that $y - a_j \in C_{i_j}$ for every $j \in \{1, 2, \ldots, m\}$.*

Lemma 8.2. *If $q \geq 47$, then q is a nice prime power.*

Proof. Let $(\bar{v}_1, \bar{v}_2)$ and $(\bar{v}'_1, \bar{v}'_2)$ be two ordered bases of V such that $\{\langle \bar{v}_1 \rangle, \langle \bar{v}_2 \rangle\} \neq \{\langle \bar{v}'_1 \rangle, \langle \bar{v}'_2 \rangle\}$. Then there exist $a, b, c, d \in \mathbb{F}_q$ with $ad - bc \neq 0$ and $(a, d) \neq (0, 0) \neq (b, c)$ such that $\bar{v}'_1 = a\bar{v}_1 + b\bar{v}_2$ and $\bar{v}'_2 = c\bar{v}_1 + d\bar{v}_2$. Then

$$\mathcal{P}(\bar{v}_1, \bar{v}_2) := \{\langle \bar{v}_1 + \lambda \bar{v}_2 \rangle \mid \lambda \in \blacksquare\},$$

$$\mathcal{P}(\bar{v}'_1, \bar{v}'_2) := \{\langle (a + \lambda c)\bar{v}_1 + (b + \lambda d)\bar{v}_2 \rangle \mid \lambda \in \blacksquare\}.$$

Note that $(a, b) \neq (0, 0)$. As $\mathcal{P}(\bar{v}_1, \bar{v}_2) = \mathcal{P}(\bar{v}_2, \bar{v}_1)$, we may without loss of generality suppose that $b \neq 0$.

Suppose first that $c = 0$. Then a, b and d are distinct from 0. As $q \geq 7$, there exists by Proposition 8.1 (with $e = m = 2$) $\lambda_1, \lambda_2 \in \blacksquare$ such that $\lambda_1 + \frac{b}{d} \in \square$ and $\lambda_2 + \frac{b}{d} \in \blacksquare$. It then follows that $\mathcal{P}(\bar{v}_1, \bar{v}_2) \neq \mathcal{P}(\bar{v}'_1, \bar{v}'_2)$ and $\mathcal{P}(\bar{v}_1, \bar{v}_2) \cap \mathcal{P}(\bar{v}'_1, \bar{v}'_2) \neq \emptyset$.

The case $d = 0$ is completely similar. Then a, b and c are distinct from 0. As $q \geq 7$, there exists by Proposition 8.1 (with $e = m = 2$) $\lambda_1, \lambda_2 \in \blacksquare$ such that $\lambda_1 + \frac{a}{c} \in \square$ and $\lambda_2 + \frac{a}{c} \in \blacksquare$. So, we then also have that $\mathcal{P}(\bar{v}_1, \bar{v}_2) \neq \mathcal{P}(\bar{v}'_1, \bar{v}'_2)$ and $\mathcal{P}(\bar{v}_1, \bar{v}_2) \cap \mathcal{P}(\bar{v}'_1, \bar{v}'_2) \neq \emptyset$.

Suppose next that $cd \neq 0$. As $0 \neq \frac{b}{d} \neq \frac{a}{c}$ and $q \geq 47$, we know by Proposition 8.1 (with $e = 2$, $m = 3$ if $a \neq 0$ and $m = 2$ if $a = 0$) that there exist $\lambda_1, \lambda_2 \in \blacksquare$ such that $\lambda_1 + \frac{a}{c} \in \blacksquare$, $\lambda_2 + \frac{a}{c} \in \blacksquare$, $\lambda_1 + \frac{b}{d} \in \blacksquare$ and $\lambda_2 + \frac{b}{d} \in \square$. It then follows that $\mathcal{P}(\bar{v}_1, \bar{v}_2) \neq \mathcal{P}(\bar{v}_1', \bar{v}_2')$ and $\mathcal{P}(\bar{v}_1, \bar{v}_2) \cap \mathcal{P}(\bar{v}_1', \bar{v}_2') \neq \emptyset$. $\square$

Proposition 8.3. *An odd prime power is nice if and only if it is distinct from* 3, 5 *and* 9.

Proof. By Lemma 8.2, we know that each odd prime power $q \geq 47$ is nice. We have verified by computer that every odd prime power $q < 47$ distinct from 3, 5 and 9 is nice. We now show that the prime powers 3, 5 and 9 are not nice. Let $(\bar{e}_1, \bar{e}_2)$ be an ordered basis of V.

If $q = 3$, $\bar{v}_1 = \bar{e}_1$, $\bar{v}_2 = \bar{e}_2$ and $\bar{w} = \bar{e}_1 - \bar{e}_2$, then $\mathcal{P}(\bar{v}_1, \bar{w}) = \{\langle \bar{e}_1 + \bar{e}_2 \rangle\}$ is equal to $\mathcal{P}(\bar{v}_1, \bar{v}_2) = \{\langle \bar{e}_1 + \bar{e}_2 \rangle\}$ and $\mathcal{P}(\bar{v}_2, \bar{w}) = \{\langle \bar{e}_1 \rangle\}$ is disjoint from $\mathcal{P}(\bar{v}_1, \bar{v}_2)$.

If $q = 5$, $\bar{v}_1 = \bar{e}_1$, $\bar{v}_2 = \bar{e}_2$, $\bar{v}_1' = \bar{e}_1 + 2\bar{e}_2$ and $\bar{v}_2' = \bar{e}_1 + 3\bar{e}_2$, then one computes that $\mathcal{P}(\bar{v}_1', \bar{v}_2') = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle\}$ is disjoint from $\mathcal{P}(\bar{v}_1, \bar{v}_2) = \{\langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + 4\bar{e}_2 \rangle\}$.

If $q = 9$, we put $\mathbb{F}_9 = \mathbb{F}_3(\alpha)$ where α is a root of the irreducible polynomial $X^2 - X - 1 \in \mathbb{F}_3[X]$. If we put $\bar{v}_1 = \bar{e}_1$, $\bar{v}_2 = \bar{e}_2$, $\bar{v}_1' = \bar{e}_1 + \alpha\bar{e}_2$ and $\bar{v}_2' = \bar{e}_1 + \alpha^5\bar{e}_2$, then one computes that $\mathcal{P}(\bar{v}_1', \bar{v}_2') = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_1 + \alpha^3\bar{e}_2 \rangle, \langle \bar{e}_1 + \alpha^7\bar{e}_2 \rangle\}$ is disjoint from $\mathcal{P}(\bar{v}_1, \bar{v}_2) = \{\langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \alpha^2\bar{e}_2 \rangle, \langle \bar{e}_1 + \alpha^4\bar{e}_2 \rangle, \langle \bar{e}_1 + \alpha^6\bar{e}_2 \rangle\}$. $\square$

Lemma 8.4. *Let $\mathcal{C}$ be an irreducible conic in* $\mathrm{PG}(2, q)$, *q odd, described by a quadratic form Q of the vector space $V(3, q)$ that defines* $\mathrm{PG}(2, q)$. *Let L be a line of* $\mathrm{PG}(2, q)$ *that is secant with respect to $\mathcal{C}$. Put $L \cap \mathcal{C} = \{p_1, p_2\}$. Then there exist vectors $\bar{v}_1, \bar{v}_2 \in V(3, q)$ such that $p_1 = \langle \bar{v}_1 \rangle$, $p_2 = \langle \bar{v}_2 \rangle$ and the set of points of L that are exterior with respect to $\mathcal{C}$ is $\{\langle \bar{v}_1 + \lambda\bar{v}_2 \rangle \,|\, \lambda \in \blacksquare\}$.*

Proof. The quadratic form Q is uniquely determined up to a nonzero factor. So, without loss of generality, we may suppose that Q is such that the set of points of $\mathrm{PG}(2, q)$ that are exterior with respect to $\mathcal{C}$ coincides with $\{\langle \bar{v} \rangle \,|\, \bar{v} \in V(3, q)$ and $Q(\bar{v}) \in \blacksquare\}$. For all $\bar{u}_1, \bar{u}_2 \in V(3, q)$, we define $B(\bar{u}_1, \bar{u}_2) = Q(\bar{u}_1 + \bar{u}_2) - Q(\bar{u}_1) - Q(\bar{u}_2)$, i.e. B is the bilinear form on $V(3, q)$ associated with Q. Now, choose $\bar{v}_1, \bar{v}_2 \in V(3, q)$ such that $p_1 = \langle \bar{v}_1 \rangle$, $p_2 = \langle \bar{v}_2 \rangle$ and $B(\bar{v}_1, \bar{v}_2) \in \blacksquare$. Then a point $\langle \bar{v}_1 + \lambda\bar{v}_2 \rangle$, $\lambda \in \mathbb{F}_q^*$, of $L \setminus \{p_1, p_2\}$ is exterior with respect to $\mathcal{C}$ if and only if

$$Q(\bar{v}_1 + \lambda\bar{v}_2) = Q(\bar{v}_1) + \lambda^2 Q(\bar{v}_2) + \lambda \cdot B(\bar{v}_1, \bar{v}_2) = \lambda \cdot B(\bar{v}_1, \bar{v}_2)$$

is a square, i.e. if and only if λ is a square. $\square$

Proposition 8.5. *If q is a nice prime power, then every set X of points of $Q^+(5, q)$ satisfying properties ($P1''$) and ($P2''$) also satisfies property ($P3''$).*

Proof. Suppose X satisfies properties (P1$''$) and (P2$''$) with respect to the point x^*. Every plane π of $Q^+(5, q)$ not containing x^* then intersects X in the set of exterior points with respect to a unique irreducible conic $\mathcal{C}_\pi$. In order to prove that X also satisfies property (P3$''$), we need to prove that for every point $x \in Q^+(5, q)$ noncollinear with x^* on $Q^+(5, q)$ and any two planes π_1, π_2 of $Q^+(5, q)$ through x, we have $x \in \mathcal{C}_{\pi_1}$ if and only if $x \in \mathcal{C}_{\pi_2}$. Note that if $\pi_1 \cap \pi_2 = \{x\}$, then there is a plane π_3 of $Q^+(5, q)$ through x meeting π_1 and π_2

in lines. So, it suffices to prove the claim in the case that $\pi_1 \cap \pi_2$ is a line L. There are now three possibilities according to whether L is a tangent, external or secant line with respect to $\mathcal{C}_{\pi_1}$:

(1) $|L \cap X| = q$;

(2) $|L \cap X| = \frac{1}{2}(q+1)$;

(3) $|L \cap X| = \frac{1}{2}(q-1)$.

Suppose case (1) occurs. Then L has precisely q points which are exterior with respect to $\mathcal{C}_{\pi_1}$ (namely the points in $L \cap X$) and precisely q points which are exterior with respect to $\mathcal{C}_{\pi_2}$ (namely the points in $L \cap X$). The line L is therefore a tangent line to both $\mathcal{C}_{\pi_1}$ and $\mathcal{C}_{\pi_2}$. In both cases, the tangency points coincide with the unique point in $L \setminus X$. The claim thus obviously holds in this case.

Suppose case (2) occurs. Then L has precisely $\frac{1}{2}(q+1)$ points which are exterior with respect to $\mathcal{C}_{\pi_1}$ (namely the points in $L \cap X$) and precisely $\frac{1}{2}(q+1)$ points which are exterior with respect to $\mathcal{C}_{\pi_2}$ (namely the points in $L \cap X$). The line L is therefore an external line to both $\mathcal{C}_{\pi_1}$ and $\mathcal{C}_{\pi_2}$. The $\frac{1}{2}(q+1)$ points in $L \setminus X$ are interior with respect to both $\mathcal{C}_{\pi_1}$ and $\mathcal{C}_{\pi_2}$. The claim again holds.

Finally, suppose that case (3) occurs. Then L has precisely $\frac{1}{2}(q-1)$ points which are exterior with respect to $\mathcal{C}_{\pi_1}$ (namely the points in $L \cap X$) and precisely $\frac{1}{2}(q-1)$ points which are exterior with respect to $\mathcal{C}_{\pi_2}$ (namely the points in $L \cap X$). The line L is therefore a secant line to both $\mathcal{C}_{\pi_1}$ and $\mathcal{C}_{\pi_2}$. By Lemma 8.4, there exist points $\langle \bar{v}_1 \rangle$, $\langle \bar{w}_1 \rangle$, $\langle \bar{v}_2 \rangle$, $\langle \bar{w}_2 \rangle$ on L such that $\mathcal{C}_{\pi_1} \cap L = \{\langle \bar{v}_1 \rangle, \langle \bar{w}_1 \rangle\}$, $\mathcal{C}_{\pi_2} \cap L = \{\langle \bar{v}_2 \rangle, \langle \bar{w}_2 \rangle\}$ and

$$L \cap X = \{\langle \bar{v}_1 + \lambda \bar{w}_1 \rangle \mid \lambda \in \blacksquare\} = \{\langle \bar{v}_2 + \lambda \bar{w}_2 \rangle \mid \lambda \in \blacksquare\}.$$

Since q is a nice prime power, we necessarily have $\{\langle \bar{v}_1 \rangle, \langle \bar{w}_1 \rangle\} = \{\langle \bar{v}_2 \rangle, \langle \bar{w}_2 \rangle\}$, i.e. $\mathcal{C}_{\pi_1} \cap L = \mathcal{C}_{\pi_2} \cap L$. The claim thus also holds in this case. $\square$

The following is a consequence of Propositions 5.1, 5.4, 5.5, 8.3 and 8.5.

Corollary 8.6. *Suppose q is an odd prime power distinct from 3, 5 and 9. Then the line sets belonging to $\mathcal{H}(q)$ are precisely the sets of the form $\kappa^{-1}(C_Y \cup D_Y)$ where Y is a nice quadratic set of type (LC^*) of $Q^+(5,q)$.*

The computer results described in the following section will show that the conclusion of Corollary 8.6 remains valid for $q = 3$ (see Corollary 9.7).

9 Some computer classification results

Recall that an *ovoid* of a point-line geometry is a set of points having a unique point in common with each line of the geometry. Let x^* be a point of $Q^+(5,q)$ and let $\mathcal{L}$ be a set of lines of $Q^+(5,q)$ through x^* which is an ovoid of the local geometry $\mathcal{G}_{x^*}$ at the point x^*.

This *local geometry* $\mathcal{G}_{x^*}$ is the point-line geometry whose points and lines are the lines and planes of $Q^+(5,q)$ through x^*, with containment as incidence relation. Note that $\mathcal{G}_{x^*}$ is a $(q+1)\times(q+1)$-grid and so has $(q+1)!$ ovoids. If there is some hyperplane Π through x^* such that $\mathcal{L}$ consists of all lines through x^* contained in the intersection $\Pi\cap Q^+(5,q)$, then $\mathcal{L}$ is called a *classical ovoid* of $\mathcal{G}_{x^*}$.

Put $A := \bigcup_{L\in\mathcal{L}} L$. Let $\mathcal{L}_1$ denote the set of lines of $Q^+(5,q)$ not contained in $(x^*)^\perp$ that contain a (necessarily) unique point of A. Let $\mathcal{L}_2$ denote the set of lines of $Q^+(5,q)$ not contained in $(x^*)^\perp$ that are disjoint from A. For every $i\in\{1,2\}$, let $\mathcal{S}_i$ be the point-line geometry with point set $Q^+(5,q)\setminus(x^*)^\perp$, line set $\mathcal{L}_i$ and containment as incidence relation.

Lemma 9.1. *Suppose q is odd or $q\in\{2,4\}$. Then the quadratic sets of type (LC*) containing A are precisely the sets of the form $A\cup O$, where O is an ovoid of $\mathcal{S}_1$ that meets every line of $\mathcal{S}_2$ in at most two points.*

Proof. The following obviously holds:

> The quadratic sets of type (LC*) containing A are precisely the sets of the form $A\cup O$ where O is a set of points of $Q^+(5,q)\setminus(x^*)^\perp$ such that every plane π of $Q^+(5,q)$ not containing x^* intersects $O\cup A$ in an irreducible conic.

Note that an irreducible conic is an example of an *oval*, i.e. a set of $q+1$ points no three of which are collinear. If $q\in\{2,4\}$ or if q is odd, then also the converse is true. For these values of q, every oval of $\mathrm{PG}(2,q)$ is necessarily an irreducible conic by Segre [11]. We thus need to impose that $\pi\cap(O\cup A)$ is set of $q+1$ points intersecting each line of π in at most two points. If we denote the line $(x^*)^\perp\cap\pi$ by L and the unique point in $L\cap A$ by x_L, then L must also be a tangent line to the conic $\pi\cap(O\cup A)$ with tangency point x_L.

The lines of $\mathcal{L}_1$ contained in π are precisely the lines of π through x_L distinct from L and these are secant lines to the conic $\pi\cap(O\cup A)$ and so they must meet O in exactly one point. As every line of $\mathcal{L}_1$ is contained in a plane of $Q^+(5,q)$ not containing x^*, we thus see that every line of $\mathcal{L}_1$ must meet O in exactly one point, i.e. O is an ovoid of $\mathcal{S}_1$. Note that the condition that every line of $\mathcal{L}_1$ meets O in exactly one point implies that $|\pi\cap(O\cup A)|=q+1$.

The lines of $\mathcal{L}_2$ contained in π are precisely the lines in π meeting L in a point distinct from x_L. Each of these lines must meet the conic $\pi\cap(O\cup A)$ and hence O in at most two points. As every line of $\mathcal{L}_2$ is contained in a plane of $Q^+(5,q)$ not containing x^*, we thus see that every line of $\mathcal{L}_2$ must meet O in at most two points.

The requirement that every plane π of $Q^+(5,q)$ not containing x^* intersects $O\cup A$ in an irreducible conic is thus equivalent with demanding that O is an ovoid of $\mathcal{S}_1$ that meets every line of $\mathcal{S}_2$ in at most two points. □

Lemma 9.2. *Suppose $q=3$ and let X be a set of points of $Q^+(5,3)$ satisfying properties* (P1″) *and* (P2″) *with respect to x^* such that $A=(x^*)^\perp\setminus X$. Then $X\setminus(x^*)^\perp$ is an ovoid of $\mathcal{S}_1$ that also meets every line of $\mathcal{S}_2$ in at most two points.*

Proof. Put $Y := X\setminus(x^*)^\perp$. We need to prove that $|L_1\cap Y|=1$ and $|L_2\cap Y|\le 2$ for every $L_1\in\mathcal{L}_1$ and every $L_2\in\mathcal{L}_2$. As every line of $\mathcal{L}_1\cup\mathcal{L}_2$ is contained in a plane of $Q^+(5,q)$ (not containing x^*), we may suppose that L_1 and L_2 are contained in a plane π of $Q^+(5,q)$ that has been chosen arbitrarily. As $x^* \notin \pi$, $\pi \cap X$ is the set of exterior points with respect to a conic $\mathcal{C}_\pi$. Put $L := (x^*)^\perp \cap \pi$ and let $L \cap A = \{x_L\}$. As $L \setminus \{x_L\} \subseteq X$ is a set of exterior points with respect to $\mathcal{C}_\pi$, L is tangent to $\mathcal{C}_\pi$ with tangency point $x_L \in \mathcal{C}_\pi$. The points in π that are exterior with respect to $\mathcal{C}_\pi$ are the points in $Y \cap \pi$ and the points in $L \setminus \{x_L\}$.

The lines of $\mathcal{L}_1$ contained in π are precisely the lines of π through x_L distinct from L and as these are secant lines with respect to $\mathcal{C}_\pi$, they contain exactly one exterior point with respect to $\mathcal{C}_\pi$, i.e. one point of Y.

The lines of $\mathcal{L}_2$ contained in π are precisely the lines of π that meet L in a point distinct from x_L. As $L \setminus \{x_L\}$ consists of exterior points with respect to $\mathcal{C}_\pi$, such a line contains $3-1=2$, $2-1=1$ or $1-1=0$ points of Y depending on whether it is a tangent, external or secant line with respect to $\mathcal{C}_\pi$. $\square$

Lemma 9.3. *Let $\mathcal{A}$ be the full automorphism group of $Q^+(5,q)$. Then the following properties hold:*

(1) *$\mathcal{A}$ acts transitively on the points of $Q^+(5,q)$ and $\mathcal{A}_{x^*}$ acts transitively on the ordered triples (L_1, L_2, L_3) of three lines of $Q^+(5,q)$ through x^* no two of which are in the same plane of $Q^+(5,q)$.*

(2) *If L_1, L_2 and L_3 are three lines of $Q^+(5,q)$ through x^* no two of which are in the same plane of $Q^+(5,q)$, then there is a unique classical ovoid of $\mathcal{G}_{x^*}$ containing L_1, L_2 and L_3.*

(3) *$\mathcal{A}_{x^*}$ acts transitively on the classical ovoids of $\mathcal{G}_{x^*}$.*

(4) *If $q \in \{2,3\}$, then every ovoid of $\mathcal{G}_{x^*}$ is classical. If $q \geq 4$, then $\mathcal{G}_{x^*}$ has nonclassical ovoids.*

(5) *If $q \in \{4,5\}$, then $\mathcal{A}_{x^*}$ acts transitively on the nonclassical ovoids of $\mathcal{G}_{x^*}$.*

(6) *Let $q=5$, let $\mathcal{L}$ be a classical ovoid of $\mathcal{G}_{x^*}$ and let π be a plane of $Q^+(5,5)$ not containing the point x^*. Let x be the unique point in $\pi \cap \left(\bigcup_{L \in \mathcal{L}} L\right)$, let T be the line $(x^*)^\perp \cap \pi$ (through x) and let Ω be the set of all irreducible conics of π containing x for which T is a tangent line. Then the stabilizer of x^*, $\mathcal{L}$ and π inside $\mathcal{A}$ acts transitively on Ω.*

Proof. Properties (1) and (2) are well known properties of the Klein quadric $Q^+(5,q)$. Property (3) is a consequence of properties (1) and (2).

As to property (4), we note that every set of three lines of $Q^+(5,q)$ through x^* with the property that no two of them are contained in the same plane of $Q^+(5,q)$ extends in a unique way to a classical ovoid of $\mathcal{G}_{x^*}$, while it extends in a unique way to an ovoid of $\mathcal{G}_{x^*}$ only if $q \in \{2,3\}$ (recall that $\mathcal{G}_{x^*}$ is a $(q+1)\times(q+1)$-grid).

We prove property (5) for $q=4$. Every set of three lines of $Q^+(5,q)$ through x^* with the property that no two of them are contained in the same plane of $Q^+(5,q)$ extends in two ways to an ovoid of $\mathcal{G}_{x^*}$. By (2) we know that precisely one of these two ovoids is nonclassical. The claim then follows from (1).

Property (5) for $q = 5$ and property (6) have been verified by means of computer computations, see [6]. $\square$

Lemma 9.4. *Let S be the stabilizer of A in the full automorphism group of $Q^+(5, q)$.*

(1) *If $q \in \{2, 3\}$, then $\mathcal{L}$ is a classical ovoid of $\mathcal{G}_{x^*}$ and there is only one S-orbit of ovoids of $\mathcal{S}_1$ that meet each line of $\mathcal{S}_2$ in at most two points.*

(2) *If $q \in \{4, 5\}$ and $\mathcal{L}$ is a classical ovoid of $\mathcal{G}_{x^*}$, then there is only one S-orbit of ovoids of $\mathcal{S}_1$ that meets each line of $\mathcal{S}_2$ in at most two points.*

(3) *If $q \in \{4, 5\}$ and $\mathcal{L}$ is a nonclassical ovoid of $\mathcal{G}_{x^*}$, then there are no ovoids in $\mathcal{S}_1$ that meet each line of $\mathcal{S}_2$ in at most two points.*

Proof. Suppose $q \in \{2, 3, 4, 5\}$. Using the Computer Algebra System GAP [13], we have implemented computer models of the geometries $\mathcal{S}_1$ and $\mathcal{S}_2$ for the various cases. By (1), (3), (4) and (5) of Lemma 9.3, we can take for x^* any point of $Q^+(5, q)$ and for $\mathcal{L}$ any classical/nonclassical ovoid of $\mathcal{G}_{x^*}$.

In Section 4 of [1], computer code in SageMath [10] can be found for classifying ovoids of point-line geometries. With the aid of this code, we have classified all ovoids of $\mathcal{S}_1$ for $q \in \{2, 3, 4\}$. Subsequently, we determined which of these ovoids also intersect each line of $\mathcal{S}_2$ in at most two points, and showed that either there were no remaining ovoids (case (3) of the lemma) or that the remaining ovoids were all equivalent under the group S (cases (1) and (2) of the lemma). In fact, we applied the same procedure for $q = 5$ with one difference in the case where $\mathcal{L}$ is a classical ovoid. In order to make the computations feasible we assumed that the ovoid in $\mathcal{S}_1$ already contained a fixed set K of five points in a plane π of $Q^+(5, 5)$ not containing x^* such that $K \cup (A \cap \pi)$ is an irreducible conic of π. By Lemma 9.3(6) we were allowed to make that assumption. All our computations can be found in [6]. $\square$

In Propositions 6.3, 6.4 and Corollary 6.5, we showed that $Q^+(5, q)$ has quadratic sets of type (LC*) for each prime power q. By Lemmas 9.1, 9.3 and 9.4, we now know the following.

Theorem 9.5. *If $q \in \{2, 3, 4, 5\}$, then there is up to isomorphism a unique quadratic set of type (LC*) in $Q^+(5, q)$.*

Theorem 9.6. *The set $\mathcal{H}(3)$ contains up to isomorphism a unique element.*

Proof. Let $\mathcal{A}$ denote the full automorphism group of $Q^+(5, 3)$ and denote by $\widetilde{\mathcal{A}}$ the subgroup of index 2 of $\mathcal{A}$ consisting of all automorphisms in $\mathcal{A}$ that map Latin planes to Latin planes and Greek planes to Greek planes. As before, let κ denote the Klein correspondence between the lines of PG(3, 3) and the points of $Q^+(5, 3)$.

By Propositions 5.1 and 5.4, we know that the elements of $\mathcal{H}(3)$ are precisely the line sets of the form $\kappa^{-1}(X)$, where X is a set of points of $Q^+(5, 3)$ satisfying Properties (P1$''$) and (P2$''$). If X_1 and X_2 are two such sets of points of $Q^+(5, 3)$, then the line sets $\kappa^{-1}(X_1)$ and $\kappa^{-1}(X_2)$ are isomorphic if and only if X_1 and X_2 are equivalent under the group $\widetilde{\mathcal{A}}$.

By Section 7, we know that $Q^+(5,3)$ has a set of points satisfying Properties (P1$''$) and (P2$''$). Indeed, if Y is a quadratic set of type (LC) of $Q^+(5,3)$ as obtained in Proposition 6.4, then by Corollary 6.5, Theorem 7.1 and Proposition 5.5 we know that $C_Y \cup D_Y$ satisfies Properties (P1$''$) and (P2$''$).

By Lemmas 9.2 and 9.4(1), we then know that $Q^+(5,3)$ has up to isomorphism a unique set of points of $Q^+(5,3)$ satisfying (P1$''$) and (P2$''$), i.e. there is a unique $\mathcal{A}$-orbit of sets of points in $Q^+(5,3)$ satisfying Properties (P1$''$) and (P2$''$). In order to prove that all elements in $\mathcal{H}(3)$ are isomorphic, we need to prove that there is only one $\widetilde{\mathcal{A}}$-orbit of such sets of points of $Q^+(5,3)$.

As there exists up to isomorphism a unique set of points in $Q^+(5,3)$ that satisfies (P1$''$) and (P2$''$), we know that each such set has the form $C_Y \cup D_Y$ for some quadratic set Y of type (LC*). In order to prove that there is only one $\widetilde{\mathcal{A}}$-orbit of sets of points of $Q^+(5,3)$ satisfying (P1$''$) and (P2$''$), it thus suffices to prove that there is only one $\widetilde{\mathcal{A}}$-orbit of quadratic sets of type (LC*). As there is only one $\mathcal{A}$-orbit of quadratic sets of type (LC*) by Theorem 9.5, it thus suffices to show that there exists a quadratic set Y of type (LC*) and an automorphism in $\mathcal{A} \setminus \widetilde{\mathcal{A}}$ that stabilizes Y. If Y is the quadratic set of type (LC*) as obtained in Proposition 6.4 with $a_{33} = d_1 = d_2 = 1$, then the map $(X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_1, X_2, X_4, X_3, X_5, X_6)$ belongs to $\mathcal{A} \setminus \widetilde{\mathcal{A}}$ and indeed stabilizes Y. $\square$

Corollary 9.7. *Every line set contained in $\mathcal{H}(3)$ is of the form $\kappa^{-1}(C_Y \cup D_Y)$, where Y is a (necessarily nice) quadratic set of type (LC*) in $Q^+(5,3)$.*

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