On hyperovals of $Q^+(5,q)$, q even

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1 Constructions of hyperovals of $Q^+(5,q)$, q even, from quadratic sets of type (SC)

Let X be a quadratic set of type (SC) of $Q^+(5,q)$, q even. We then define the following sets of points of $Q^+(5,q)$:

- A_1 is the set of all points $x \in X$ that are contained in a plane of type (S) with respect to X;
- A_2 is the set of all kernels of all conic intersections $\pi \cap X$, where π is a plane of type (C) with respect to X.

Note that $A_1 \subseteq X$ and $A_2 \cap X = \emptyset$. Suppose the following hold:

(1) Every plane of $Q^+(5,q)$ through a point of A_1 has type (S).

(2) Every plane π of $Q^+(5,q)$ through a point $x \in A_2$ has type (C) and x is the kernel of the irreducible conic $\pi \cap X$ of π.

Note that $A_1 \subseteq X$ and $A_2 \cap X = \emptyset$.

Lemma 1.1. *No two points of $A_1 \cup A_2$ are collinear in $Q^+(5,q)$.*

Proof. It suffices to prove that no plane of $Q^+(5,q)$ contains two points of $A_1 \cup A_2$. Suppose to the contrary that π is a plane of $Q^+(5,q)$ containing two distinct points of $A_1 \cup A_2$.

If π has type (S) with respect to X, then by properties (1) and (2), we know that $x_1, x_2 \in A_1$. As $x_1, x_2 \in X \cap \pi$, we then know that $|X \cap \pi| \geq 2$, an obvious contradiction.

If π has type (C) with respect to X, then by properties (1) and (2), we know that $x_1, x_2 \in A_2$. Property (2) also implies that both x_1 and x_2 must then be equal to the kernel of the irreducible conic $\pi \cap X$ of π, again a contradiction. □

Lemma 1.2. *$A_1 \cup A_2$ is an ovoid of $Q^+(5,q)$.*

Proof. Let π be an an arbitrary plane of $Q^+(5,q)$. By Lemma 1.1, we need to prove that $(A_1 \cup A_2) \cap \pi \neq \emptyset$. If π has type (S) with respect to X, then $\pi \cap X$ is a singleton contained in A_1. If $\pi \cap X$ is an irreducible conic of π, then the kernel of this conic belongs to A_2. In any case, we have $(A_1 \cup A_2) \cap \pi \neq \emptyset$. □

Lemma 1.3. *$(X \setminus A_1) \cup A_2$ is a hyperoval of $Q^+(5,q)$.*

Proof. As there are planes of type (C), we have $A_2 \neq \emptyset$ and hence also $(X \setminus A_1) \cup A_2 \neq \emptyset$.

Let π be an arbitrary plane of $Q^+(5,q)$. If π is a plane of type (S) containing a unique point of A_1, then $(\pi \cap X) \setminus A_1 = \pi \cap A_2 = \emptyset$ by Lemma 1.1, and so π is disjoint from $(X \setminus A_1) \cup A_2$. If π is a plane of type (C), then $X \cap \pi$ is an irreducible conic of π and Lemma 1.1 implies that $\pi \cap A_1 = \emptyset$ and $\pi \cap A_2$ is a singleton consisting of the kernel k of the irreducible conic $X \cap \pi$ of π. We then have that $(X \setminus A_1) \cup A_2$ intersects π in the hyperoval $(X \cap \pi) \cup \{k\}$ of π. □

Lemma 1.4. *We have $|A_2| = q^2+1-|A_1|$, $|X| = (q^2+1)(q+1)-q|A_1|$ and $|(X\setminus A_1)\cup A_2| = (q^2+1-|A_1|)(q+2)$.*

Proof. Note that through each point of $Q^+(5,q)$ there are exactly $2(q+1)$ planes of $Q^+(5,q)$. Property (1) thus implies that the total number of planes of type (S) is equal to $2(q+1)|A_1|$. Hence, the total number of planes of type (C) is equal to $2(q+1)(q^2+1) - 2(q+1)|A_1| = 2(q+1)(q^2+1-|A_1|)$. By property (2), we then know that

$$|A_2| = \frac{1}{2(q+1)} \cdot 2(q+1)(q^2+1-|A_1|) = q^2+1-|A_1|.$$

We then also find that

$$|X| = \frac{1}{2(q+1)}\Big(2(q+1)|A_1|\cdot 1 + 2(q+1)(q^2+1-|A_1|)\cdot(q+1)\Big) = (q^2+1)(q+1)-q|A_1|$$

and

$$|(X \setminus A_1) \cup A_2| = |X| - |A_1| + |A_2| = (q^2+1-|A_1|)(q+2).$$

□

2 A family of hyperovals of size $q^2(q+2)$ of $Q^+(5,q)$, q even

Let V be a 6-dimensional vector space over the finite field $\mathbb{F}_q = \mathrm{GF}(q)$, q even, and Q a quadratic form on V such that the set of all points $\langle \bar{v} \rangle$ of $\mathrm{PG}(V)$ for which $Q(\bar{v}) = 0$ is a hyperbolic quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q) := \mathrm{PG}(V)$. Let $B : V \times V \to \mathbb{F}_q$ denote the bilinear form associated with Q, i.e. $B(\bar{v}_1, \bar{v}_2) = Q(\bar{v}_1 + \bar{v}_2) - Q(\bar{v}_1) - Q(\bar{v}_2)$ for all $\bar{v}_1, \bar{v}_2 \in V$. With B, there is associated a symplectic polarity ζ of $\mathrm{PG}(5,q)$. For every point $x \in Q^+(5,q)$, x^ζ is the tangent hyperplane T_x in the point $x \in Q^+(5,q)$. This tangent hyperplane T_x intersects $Q^+(5,q)$ in a cone of type $xQ^+(3,q)$. For every line $L \subseteq Q^+(5,q)$, L^ζ intersects $Q^+(5,q)$ in the union of two planes through L. So, there cannot be 3-dimensional subspaces of $\mathrm{PG}(5,q)$ that meet $Q^+(5,q)$ in a single line K as this 3-dimensional subspace would otherwise need to coincide with K^ζ, but as said above $K^\zeta \cap Q^+(5,q)$ is the union of two planes.

Now, let $Q^-(3,q)$ be an elliptic quadric obtained by intersecting $Q^+(5,q)$ with a 3-dimensional subspace α. Then α^ζ is a line. This line is disjoint from $Q^+(5,q)$ as for every point $y \in \alpha^\zeta \cap Q^+(5,q)$, we would have $Q^-(3,q) \subseteq \alpha \subseteq y^\zeta = T_y$, which is impossible as said above.

Let $p^* = \langle \bar{v}^* \rangle$ be an arbitrary point of $Q^-(3,q)$ and for every point $\langle \bar{v} \rangle$ of $\mathrm{PG}(V) \setminus T_{p^*}$, we define

$$A(p) := B(\bar{v}^*, \bar{v})^{q-3} Q(\bar{v}) \in \mathbb{F}_q.$$

Note that this is well-defined as

$$B(\bar{v}^*, \lambda\bar{v})^{q-3} Q(\lambda\bar{v}) = \lambda^{q-1} B(\bar{v}^*, \bar{v}) Q(\bar{v}) = B(\bar{v}^*, \bar{v}) Q(\bar{v})$$

for all $(\lambda, \bar{v}) \in \mathbb{F}_q^* \times V$.

Now, consider a point $p \in Q^+(5,q) \setminus Q^-(3,q)$ not collinear with p^* on the quadric $Q^+(5,q)$, i.e. not contained in the tangent hyperplane T_{p^*} at $p^* \in Q^+(5,q)$. As the line α^ζ is disjoint from $Q^+(5,q)$, we have $p \notin \alpha^\zeta$ and so α is not contained in $p^\zeta = T_p$. So, T_p intersects α in a plane β_p not containing p^*. If $\beta_p \subseteq \alpha$ is a tangent plane to the elliptic quadric $Q^-(3,q)$ with tangency point u, then the 3-dimensional subspace $\langle p, \beta_p \rangle$ would intersect $Q^+(5,q)$ in the line pu, an impossibility. So, β intersects $Q^-(3,q)$ in an irreducible conic $\mathcal{C}_p$ of β_p with kernel k_p. The tangent lines through k_p contained in α are precisely the lines through k_p contained in β_p. As $p^* \notin \beta_p$, $k_p p^*$ is not a tangent line and so $k_p \notin T_{p^*}$. We then define $B(p) := A(k_p)$.

For every $\lambda \in \mathbb{F}_q^*$, let H_λ be the set $(Q^-(3,q) \setminus \{p^*\}) \cup G_\lambda$, where G_λ is the set of all points $p \in Q^+(5,q) \setminus (Q^-(3,q) \cup T_{p^*})$ for which $B(p) = \lambda$. We prove the following.

Theorem 2.1. *For every $\lambda \in \mathbb{F}_q^*$, H_λ is a hyperoval of size $q^2(q+2)$ of $Q^+(5,q)$. In fact, if γ is a plane of $Q^+(5,q)$ then $\gamma \cap H_\lambda = \emptyset$ if $p^* \in \gamma$ and $\gamma \cap H_\lambda$ is a hyperoval of γ if $p^* \notin \gamma$.*

Proof. Let γ be a plane of $Q^+(5,q)$ through p^*. Then γ is disjoint from both $Q^-(3,q) \setminus \{p^*\}$ and G_λ and so is disjoint from H_λ.

Let γ be a plane of $Q^+(5,q)$ not containing p^*. Then γ intersects $Q^-(3,q) \setminus \{p^*\}$ in a point x. For every $p \in \gamma \setminus \{x\}$, the irreducible conic $\mathcal{C}_p = T_p \cap \alpha \cap Q^-(3,q)$ of $\beta_p = T_p \cap \alpha$ contains x and so the kernel k_p of this irreducible conic is contained in the tangent plane π_x through x to the elliptic quadric $Q^-(3,q)$. We show that the map

$$p \mapsto k_p \text{ if } p \in \gamma \setminus \{x\}, \qquad x \mapsto x,$$

defines an isomorphism between the planes γ and π_x. This follows from the following observations:

(i) For every $y \in \gamma$, T_y contains γ. The map $y \mapsto T_y$ defines an isomorphism between the projective plane γ and the dual projective plane of the quotient projective space $\mathrm{PG}(5,q)_\gamma$ (whose points and lines are the 3-dimensional and 4-dimensional subspaces of $\mathrm{PG}(5,q)$ through γ).

(ii) Because of (i), the map $y \mapsto T_y \cap \alpha$ defines an isomorphism between the projective plane γ and the dual projective plane of the quotient space α_x (whose points and lines are the lines and planes of α through x).

(iii) The map which associates with each tangent plane $\omega \subseteq \alpha$ with respect to $Q^-(3,q)$ its tangency point and with each secant plane $\omega' \subseteq \alpha$ with respect to $Q^-(3,q)$ the kernel of the irreducible conic $\omega' \cap Q^-(3,q)$ is induced by a duality of α (which is even a symplectic polarity of α). This duality maps π_x to x.

Now, let G'_λ denote the set of all points $p \in \pi_x \setminus \{x\}$ for which $A(p) = \lambda$. In view of the above isomorphism between γ and π_x, we need to prove that $\{x\} \cup G'_\lambda$ is a hyperoval of π_x, or equivalently $|L \cap G'_\lambda| = 1$ for every line L of π_x through x and $|K \cap G'_\lambda| \in \{0,2\}$ for every line K of π_x not containing x.

The line L intersects T_{p^*} in a point $\langle \bar{w}_2 \rangle$. If we put $x = \langle \bar{w}_1 \rangle$, then $L \setminus (\{x\} \cup T_{p^*})$ consists of all points of the form $\langle \bar{w}_2 + \mu \bar{w}_1 \rangle$ with $\mu \in \mathbb{F}_q^*$. Note that

$$B(\bar{v}^*, \bar{w}_2 + \mu\bar{w}_1)^{q-3} Q(\bar{w}_2 + \mu\bar{w}_1) = B(\bar{v}^*, \bar{w}_1)^{q-3} \mu^{q-3} Q(\bar{w}_2) = \frac{Q(\bar{w}_2) B(\bar{v}^*, \bar{w}_1)^{q-3}}{\mu^2}.$$

As every element of $\mathbb{F}_q$ is a square and $Q(\bar{w}_2)B(\bar{v}^*, \bar{w}_1)^{q-3} \neq 0$, there is a unique $\mu \in \mathbb{F}_q^*$ for which $\frac{Q(\bar{w}_2)B(\bar{v}^*,\bar{w}_1)^{q-3}}{\mu^2} = \lambda$.

Again the line K contains a point $\langle \bar{w}_2 \rangle$ of T_{p^*}, and we denote by $\langle \bar{w}_1 \rangle$ any other point of K. As $\pi_x \cap Q^+(5,q) = \{x\}$, $B(\bar{w}_1, \bar{w}_2) \neq 0$. The points of $K \setminus T_{p^*}$ are then the points $\langle \bar{w}_1 + \mu \bar{w}_2 \rangle$ with $\mu \in \mathbb{F}_q$. Note then that

$$B(\bar{v}^*, \bar{w}_1 + \mu\bar{w}_2)^{q-3} Q(\bar{w}_1 + \mu\bar{w}_2) = B(\bar{v}^*, \bar{w}_1)^{q-3} (Q(\bar{w}_1) + \mu B(\bar{w}_1, \bar{w}_2) + \mu^2 Q(\bar{w}_2)).$$

This value is equal to λ if and only if

$$Q(\bar{w}_2)\mu^2 + B(\bar{w}_1, \bar{w}_2)\mu + Q(\bar{w}_1) - \frac{\lambda}{B(\bar{v}^*, \bar{w}_1)^{q-3}} = 0.$$

As $B(\bar{w}_1, \bar{w}_2) \neq 0$ and $Q(\bar{w}_2) \neq 0$, this equation in $\mu \in \mathbb{F}_q$ has 0 or 2 solutions.

Since every plane of $Q^+(5,q)$ intersects H_λ in either the empty set or a hyperoval of that plane, H_λ must be a hyperoval of $Q^+(5,q)$.

As there are $2(q+1)$ planes of $Q^+(5,q)$ disjoint from H_λ and $2q^2(q+1)$ planes of $Q^+(5,q)$ meeting H_λ in exactly $q+2$ points, the fact that each point of $Q^+(5,q)$ is contained in $2(q+1)$ planes of $Q^+(5,q)$ then implies that

$$|H_\lambda| = \frac{2(q+1) \cdot 0 + 2q^2(q+1) \cdot (q+2)}{2(q+1)} = q^2(q+2).$$

□

Some special cases

(1) The case $q = 2$. Then $\mathbb{F}_q = \mathbb{F}_2 = \{0, 1\}$ and $\lambda = 1$. In this case, $H_1 = (Q^-(3, q) \setminus \{p^*\}) \cup G_1$ is precisely the complement of $T_{p^*} \cap Q^+(5, 2)$. This is obviously a hyperoval of $Q^+(5, 2)$. In fact, the hyperovals of $Q^+(5, q)$ are precisely the complements of the geometric hyperplanes of $Q^+(5, 2)$, and there are two such geometric hyperplanes, the intersections of $Q^+(5, 2)$ with the tangent hyperplanes and the intersections of $Q^+(5, 2)$ with the nontangent hyperplanes.

(2) The case $q = 4$. Then we obtain a hyperoval of size 96 of $Q^+(5, 4)$. This hyperoval was found in [7] by means of a backtrack search. A computer free construction was left as an open problem in [7].

We now give an algebraic description of the hyperovals. Let $\omega \in \mathbb{F}_q$ such that the polynomial $X^2 + \omega X + 1 \in \mathbb{F}_q[X]$ is irreducible. We choose a coordinate system in $\mathrm{PG}(5, q)$ such that $Q^+(5, q)$ consists of all points $(X_1, X_2, X_3, X_4, X_5, X_6)$ satisfying $X_1X_2 + X_3X_4 + X_5X_6 = 0$. We suppose that $Q^-(3, q)$ is the elliptic quadric obtained by intersecting $Q^+(5, q)$ with the 3-dimensional subspace α with equations $X_5 = X_6$, $X_4 = X_3 + \omega X_5$. Let p^* be the point $(1, 0, 0, 0, 0, 0)$ of $Q^-(3, q)$. If $p = (y_1, y_2, y_3, y_4, y_5, y_6)$ is a point of $Q^+(5, q) \setminus (Q^-(3, q) \cup T_{p^*})$, then $T_p \cap \alpha$ has equations

$$X_6 = X_5, \qquad X_4 = X_3 + \omega X_5,$$

$$y_2X_1 + y_1X_2 + y_4X_3 + y_3X_4 + y_6X_5 + y_5X_6 = y_2X_1 + y_1X_2 + (y_3 + y_4)X_3 + (y_5 + y_6 + \omega y_3)X_5 = 0.$$

The point $p' = (\omega y_1, \omega y_2, y_5 + y_6 + \omega y_3, y_5 + y_6 + \omega y_4, y_3 + y_4, y_3 + y_4)$ belongs to $T_p \cap \alpha$. Moreover, $(p')^\zeta \cap \alpha$ has equations

$$X_6 = X_5, \qquad X_4 = X_3 + \omega X_5,$$

$$\begin{aligned} \omega y_1X_2 + \omega y_2X_1 + (y_5 + y_6 + \omega y_3)X_4 + (y_5 + y_6 + \omega y_4)X_3 + (y_3 + y_4)X_6 + (y_3 + y_4)X_5 \\ = \omega\Big(y_2X_1 + y_1X_2 + (y_3 + y_4)X_3 + (y_5 + y_6 + \omega y_3)X_5\Big) = 0. \end{aligned}$$

So, $T_p \cap \alpha = T_{p'} \cap \alpha$ and $p' = k_p$.

We thus see that H_λ consists of all points $(X_1, X_2, \ldots, X_6)$ of $Q^+(5, q)$ satisfying

- $X_6 = X_5$ and $X_4 = X_3 + \omega X_5$, with exception of $(1, 0, 0, 0, 0, 0)$,
- $(X_5 + X_6, X_3 + X_4 + \omega X_5) \neq (0, 0)$, $X_2 \neq 0$ and $(\omega X_2)^{q-3}((\omega X_1)(\omega X_2) + (X_5 + X_6 + \omega X_3)(X_5 + X_6 + \omega X_4) + (X_3 + X_4)^2) = \lambda$.

The latter equation is equivalent with

$$\lambda\omega^2X_2^2 + X_3^3 + X_4^2 + X_5^2 + X_6^2 + \omega^2X_5X_6 + \omega(X_3 + X_4)(X_5 + X_6) = 0. \qquad (1)$$

The hyperoval H_λ is thus obtained from a quadratic set of type (SC) by adding an elliptic quadric $Q^-(3, q)$ and removing a point p^*. In fact, if we denote by X the quadratic set of $Q^+(5, q)$ that arises by intersecting $Q^+(5, q)$ with the quadric with equation (1), then by the above, we know that the following hold:

- Every plane of $Q^+(5,q)$ through p^* intersects X in $\{p^*\}$.
- Every plane of $Q^+(5,q)$ not containing p^* intersects X in an irreducible conic. Moreover, the kernels of all the irreducible conics that arise in this way are precisely the points of $Q^-(3,q) \setminus \{p^*\}$.

We thus see that X is a quadratic set of type (SC) satisfying the properties (1) and (2) of the previous section. Using the notation of the previous section, we have

$$\begin{aligned} A_1 &= \{p^*\}, \\ A_2 &= Q^-(3,q) \setminus \{p^*\}. \end{aligned}$$

The hyperoval thus arises as described in the previous section.

3 Constructions of hyperovals of $Q^+(5,q)$, q even, from ovoids of $W(q)$

Let $Q^+(5,q)$ be a hyperbolic quadric in $\mathrm{PG}(5,q)$, q even. Let ζ be the symplectic polarity naturally associated to $Q^+(5,q)$.

Let Π be a 3-dimensional subspace of $\mathrm{PG}(5,q)$ intersecting $Q^+(5,q)$ in an elliptic quadric $Q^-(3,q)$, and let p be a point of $Q^-(3,q)$.

Let $W(q)$ denote the symplectic generalized quadrangle whose points are the points of Π and whose lines are the lines of Π that are tangent to $Q^-(3,q)$. Let O be an ovoid of $W(q)$ distinct from $Q^-(3,q)$.

For every point x of Π, denote by π_x the plane of Π through x containing all lines of $W(q)$ through x. If $x \notin Q^-(3,q)$, then π_x intersects $Q^-(3,q)$ and hence also $Q^+(5,q)$ in an irreducible conic, implying that π_x^ζ also intersects $Q^+(5,q)$ in an irreducible conic of π_x^ζ. We denote this irreducible conic of π_x^ζ by $\mathcal{C}_x$. We also define:

$$H_O := \Big(\bigcup_{x \in O \setminus Q^-(3,q)} \mathcal{C}_x \Big) \cup \Big(Q^-(3,q) \setminus O \Big).$$

Put $L^* := \Pi^\zeta$. Then Π and L^* are disjoint, as well as $Q^+(5,q)$ and L^*. There are two types of planes through L^*: planes intersecting Π in a point of $Q^-(3,q)$ and planes intersecting Π in a point not belonging to $Q^-(3,q)$. The former planes intersect $Q^+(5,q)$ in a singleton and the latter planes intersect $Q^+(5,q)$ in an irreducible conic.

Lemma 3.1. *For every point x of $\Pi \setminus Q^-(3,q)$, we have $\mathcal{C}_x = \langle L^*, x\rangle \cap Q^+(5,q)$.*

Proof. Since $\pi_x \subseteq x^\zeta$, we have $x \in \pi_x^\zeta$. As $\pi_x \subseteq \Pi$, we have $L^* = \Pi^\zeta \subseteq \pi_x^\zeta$. So, $\pi_x^\zeta = \langle L^*, x\rangle$ and $\mathcal{C}_x = \langle L^*, x\rangle \cap Q^+(5,q)$. □

Theorem 3.2. *H_O is a hyperoval of $Q^+(5,q)$ containing $((q^2+1) - |O \cap Q^-(3,q)|)(q+2)$ points. The planes of $Q^+(5,q)$ that are disjoint from H_O are precisely the planes containing a point of $O \cap Q^-(3,q)$.*

Proof. The proof will happen in several steps.

Step 1: *If $x \in O \setminus Q^-(3,q)$ and $y \in \mathcal{C}_x$, then the tangent hyperplane T_y at the point y with respect to $Q^+(5,q)$ intersects Π in the plane π_x.*
PROOF. Since $y \in \pi_x^\zeta$, we have $\pi_x \subseteq y^\zeta = T_y$. As $T_y \cap Q^+(5,q)$ is a cone of type $yQ^+(3,q)$ and $\Pi \cap Q^+(5,q) = Q^-(3,q)$, the hyperplane T_y cannot contain Π and so must intersect Π in the plane π_x.

Step 2: *For every $x \in O \setminus Q^-(3,q)$, $\mathcal{C}_x$ is disjoint from $Q^-(3,q)$.*
PROOF. The irreducible conic $\mathcal{C}_x$ is contained in the plane $\langle L^*, x\rangle$ and $\langle L^*, x\rangle$ intersects Π in the point x which does not belong to $Q^-(3,q)$.

Step 3: *If $x \in O \setminus Q^-(3,q)$ and $y \in \mathcal{C}_x$, then $A_y := T_y \cap \Pi$ is a plane of Π that is secant with respect to $Q^-(3,q)$ and the kernel of the irreducible conic $A_y \cap Q^-(3,q)$ coincides with x.*
PROOF. By Step 1, we know that $A_y = \pi_x$. We already know that $\pi_x \cap Q^-(3,q)$ is an irreducible conic having x as kernel.

Step 4: *If x_1 and x_2 are two distinct points of $O \setminus Q^-(3,q)$, then $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ are disjoint.*
PROOF. If $y \in \mathcal{C}_{x_1} \cap \mathcal{C}_{x_2}$, then by Step 3, both x_1 and x_2 need to be equal to the kernel of the irreducible conic $A_y \cap Q^-(3,q)$ of A_y.

Step 5: *We have $|H| = ((q^2+1) - |O \cap Q^-(3,q)|)(q+2)$.*
PROOF. By Steps 2 and 4, we know that $|H| = |O \setminus Q^-(3,q)| \cdot (q+1) + |Q^-(3,q) \setminus O| = ((q^2+1) - |O \cap Q^-(3,q)|)(q+2)$.

Step 6: *Every plane π of $Q^+(5,q)$ containing a point p of $O \cap Q^-(3,q)$ is disjoint from H_O.*
PROOF. As $p \in \pi \cap O \cap Q^-(3,q)$, the plane π is disjoint from $Q^-(3,q) \setminus O$.

Suppose $y \in \pi \cap \mathcal{C}_x$ for some point $x \in O \setminus Q^-(3,q)$. The plane π_x cannot contain the point p as otherwise the line px of $W(q)$ would contain two points of O, namely p and x. Now, $\{p\} \subseteq \pi \subseteq T_y$ and T_y intersects Π in the plane π_x which does not contain p, an obvious contradiction.

Step 7: *No line L of $Q^+(5,q)$ disjoint from $Q^-(3,q)$ contains more than two points of H_O.*
PROOF. If this were not the case, then the line $\langle L^*, L\rangle$ of Π would contain at least three points of O by Lemma 3.1. This is not possible as a line of $W(q)$ contains exactly one point of O and a hyperbolic line of $W(q)$ contains either 0 or 2 points.

Step 8: *Let L be a line of $Q^+(5,q)$ containing a (unique) point u of $Q^-(3,q) \setminus O$. Then $L \setminus \{u\}$ contains a unique point of $\bigcup_{x \in O \setminus Q^-(3,q)} \mathcal{C}_x$.*
PROOF. The 3-dimensional subspace $\langle L^*, L\rangle$ intersects Π in a line K through u. As u^ζ contains L and L^*, it also contains K and so K is a line of $W(q)$ containing a unique point x of $O \setminus Q^-(3,q)$. The unique point in the intersection $\langle L^*, u\rangle \cap L$ is then by Lemma 3.1 the unique point in $L \setminus \{u\}$ contained in $\bigcup_{x \in O \setminus Q^-(3,q)} \mathcal{C}_x$.

The following step completes in combination with Step 6 the proof of the theorem.

Step 9: *Every plane π of $Q^+(5,q)$ containing a point of $Q^-(3,q) \setminus O$ intersects H_O in a hyperoval of π.*
Proof. By Step 8, we know that $|\pi \cap H_O| = q+2$. By Steps 7 and 8, we know that every line of π intersects H_O in at most two points. So, $\pi \cap H_O$ must be a hyperoval of π. $\square$

Theorem 3.3. *The set*

$$X := \Big(\bigcup_{x \in O \setminus Q^-(3,q)} \mathcal{C}_x \Big) \cup \Big(O \cap Q^-(3,q) \Big)$$

is a quadratic set of type (SC) satisfying the properties (1) *and* (2) *of Section 1.*

Proof. Suppose π is a plane of $Q^+(5,q)$ containing a (necessarily unique) point of $O \cap Q^-(3,q)$. By Step 6 in the proof of Theorem 3.2, we know that π is disjoint from $\bigcup_{x \in O \setminus Q^-(3,q)} \mathcal{C}_x$ and intersects $O \cap Q^-(3,q)$ in a singleton.

Suppose π is a plane of $Q^+(5,q)$ containing a (necessarily unique) point of $Q^-(3,q) \setminus O$. By Step 9 in the proof of Theorem 3.2, we know that π is disjoint from $O \cap Q^-(3,q)$ and intersects $\bigcup_{x \in O \setminus Q^-(3,q)} \mathcal{C}_x$ in an irreducible conic. Moreover, the kernels of all these irreducible conics are precisely the points of $Q^-(3,q) \setminus O$.

We thus see that X is a quadratic set of type (SC) satisfying the properties (1) and (2) of Section 1. In fact, the set A_1 defined there is precisely the set $O \cap Q^-(3,q)$ and the set A_2 defined there is exactly the set $Q^-(3,q) \setminus O$. $\square$

Remark. By Section 1, we know that the set $(X \setminus A_1) \cup A_2$ is a hyperoval of $Q^+(5,q)$. This hyperoval coincides with H_O.

Lemma 3.4. *We have* $|O \cap Q^-(3,q)| \leq \frac{q^2-q}{2}$.

Proof. By Lemma 3.1 of [3], any hyperoval of $Q^+(5,q)$ contains at least $\frac{(q+2)(q^2+q+2)}{2}$ points. Applying this here to the hyperoval H_O of $Q^+(5,q)$, we find that $|O \cap Q^-(3,q)| \leq \frac{q^2-q}{2}$ by Theorem 3.2. $\square$

Some properties

Again, let $q = 2^h$ be an even prime power. For every $x \in \mathbb{F}_q$, we define $Tr(x) := x + x^2 + \cdots + x^{2^{h-1}}$. Note that for $\delta \in \mathbb{F}_q$, the polynomial $X^2 + X + \delta$ is reducible over $\mathbb{F}_q$ if and only if $Tr(\delta) = 0$. Note also that as q is even, every element $x \in \mathbb{F}_q$ has a unique square root in $\mathbb{F}_q$, which we will denote by $\sqrt{x}$.

Let Ω denote the set of all quadratic homogeneous polynomials in the variables X_1, X_2, X_3 and X_4. For every matrix $A \in GL(4, \mathbb{F})$, let φ_A be the permutation of Ω defined by

$$f(X_1, X_2, X_3, X_4) \mapsto f(X_1', X_2', X_3', X_4'),$$

where $[X_1' \, X_2' \, X_3' \, X_4'] := A \cdot [X_1 \, X_2 \, X_3 \, X_4]^T$.

Lemma 3.5. *Let $\delta, b_1, b_2 \in \mathbb{F}_q^*$ with $Tr(\delta) = 1$, $Tr(b_1) = Tr(b_2) = 0$ and $b_1 \neq b_2$. Then there exists no $A \in GL(4, \mathbb{F})$ such that φ_A maps $X_1X_2 + X_3^2 + X_3X_4 + \delta X_4^2$ to $X_1X_2 + X_3^2 + X_3X_4 + \delta X_4^2$ and $X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_1)X_4^2$ to $X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_2)X_4^2$.*

Proof. The map φ_A must map $b_1X_4^2$ to $b_2X_4^2$ and thus X_4 to $\sqrt{\frac{b_2}{b_1}}X_4$. It follows that for every $\eta \in \mathbb{F}_q$, φ_A maps $X_1X_2 + X_3^2 + X_3X_4 + (\delta + \eta)X_4^2$ to $X_1X_2 + X_3^2 + X_3X_4 + (\delta + \eta\frac{b_2}{b_1})X_4^2$. Since φ_A fixes the Witt indices of the nondegenerate quadratic forms in Ω, we must have that the polynomials $Tr(\eta)$ and $Tr(\eta\frac{b_2}{b_1})$ in the variable $\eta \in \mathbb{F}_q$ have the same $\frac{1}{2}\log_2(q)$ (mutually distinct) roots. But as $0 \neq b_1 \neq b_2 \neq 0$, these two polynomials of degree $\frac{1}{2}\log_2(q)$ are distinct and so they cannot have the same roots. □

Lemma 3.6. *Let $\delta, b_1, b_2 \in \mathbb{F}_q^*$ with $Tr(\delta) = 1$, $Tr(b_1) = Tr(b_2) = 0$ and $b_1 \neq b_2$. Then there exists no $A \in GL(4, \mathbb{F})$ such that φ_A maps $X_1X_2 + X_3^2 + X_3X_4 + \delta X_4^2$ to $\mu_1(X_1X_2 + X_3^2 + X_3X_4 + \delta X_4^2)$ and $X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_1)X_4^2$ to $\mu_2(X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_2)X_4^2)$ for some $\mu_1, \mu_2 \in \mathbb{F}_q^*$.*

Proof. Note that the map $\varphi_{\sqrt{\mu}\cdot I}$ with $\mu \in \mathbb{F}_q^*$ maps each $f \in \Omega$ to μf. So, without loss of generality, we may suppose that $\mu_1 = 1$. Put $\mu := \mu_2$. The map φ_A then maps $b_1X_4^2 = (\sqrt{b_1}X_4)^2$ to $(\mu + 1)(X_1X_2 + X_3^2 + X_3X_4) + \delta X_4^2 + \mu(\delta + b_2)X_4^2$. The latter polynomial must thus be a square of a linear expression in X_1, X_2, X_3 and X_4. This is only possible when $\mu = 1$. We are then again in the same situation as in the previous lemma. □

Lemma 3.7. *Let O_1 and O_2 be two ovoids of $\mathrm{PG}(3, q)$, q even. Let $\mathcal{Q}_i$ with $i \in \{1, 2\}$ denote the symplectic generalized quadrangle associated to O_i, i.e. the points of $\mathcal{Q}_i$ are the points of $\mathrm{PG}(3, q)$ and the lines of $\mathcal{Q}_i$ are the lines of $\mathrm{PG}(3, q)$ intersecting O_i in a singleton, with incidence being containment. The lines of $\mathcal{Q}_i$ are those lines of $\mathrm{PG}(3, q)$ that are totally isotropic with respect to a certain symplectic polarity ζ_i. The following are then equivalent:*

(1) $\zeta_1 = \zeta_2$;

(2) $\mathcal{Q}_1 = \mathcal{Q}_2$;

(3) *O_1 is an ovoid of $\mathcal{Q}_2$;*

(4) *O_2 is an ovoid of $\mathcal{Q}_1$.*

Proof. The lines of $\mathcal{Q}_i$, $i \in \{1, 2\}$, are precisely those lines of $\mathrm{PG}(3, q)$ that are totally isotropic with respect to ζ_i. So, if $\zeta_1 = \zeta_2$, then $\mathcal{Q}_1 = \mathcal{Q}_2$.

If x is a point of $\mathrm{PG}(3, q)$, then the lines of $\mathcal{Q}_i$, $i \in \{1, 2\}$, through x are precisely the lines through x contained in x^{ζ_i}. So, if $\mathcal{Q}_1 = \mathcal{Q}_2$, then $x^{\zeta_1} = x^{\zeta_2}$ for every point x of $\mathrm{PG}(3, q)$, i.e. $\zeta_1 = \zeta_2$.

We thus see that (1) and (2) are equivalent.

If O_1 is an ovoid of $\mathcal{Q}_2$, then every line of $\mathcal{Q}_2$ intersects O_1 in a singleton and so is a line of $\mathcal{Q}_1$. As both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have exactly $(q + 1)(q^2 + 1)$ lines, we then see that

$\mathcal{Q}_1 = \mathcal{Q}_2$. Conversely, if $\mathcal{Q}_1 = \mathcal{Q}_2$, then every line of $\mathcal{Q}_2$ is a line of $\mathcal{Q}_1$ and so meets O_1 in a singleton, implying that O_1 is an ovoid of $\mathcal{Q}_2$.

We thus see that (2) and (3) are equivalent. In a similar way, one can show that (2) and (4) are equivalent. □

Lemma 3.8. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two distinct elliptic quadrics in* $\mathrm{PG}(3,q)$*, q even, such that $\mathcal{Q}_2$ is an ovoid of the symplectic generalized quadrangle associated to $\mathcal{Q}_1$. Then $|\mathcal{Q}_1 \cap \mathcal{Q}_2|$ is either* 1 *or* $q+1$.

Proof. Suppose $\mathrm{PG}(3,q) = \mathrm{PG}(V)$, where V is a 4-dimensional vector space over $\mathbb{F}_q$. Choose an ordered basis $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ in V and denote the coordinates of a generic point of $\mathrm{PG}(3,q)$ with respect to this basis by (X_1, X_2, X_3, X_4). The quadric $\mathcal{Q}_1$ then consists of all points of $\mathrm{PG}(3,q)$ satisfying $\sum_{1 \le i \le j \le 4} a_{ij} X_i X_j = 0$, where the a_{ij}'s are certain elements in $\mathbb{F}_q$. As the symplectic polarities associated to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the same by Lemma 3.7, there exist $b_1, b_2, b_3, b_4 \in \mathbb{F}_q$ such that $\mathcal{Q}_2$ has equation $\sum_{1 \le i \le j \le 4} a_{ij} X_i X_j + b_1^2 X_1^2 + b_2^2 X_2^2 + b_3^2 X_3^2 + b_4^2 X_4^2 = 0$ with respect to the same reference system. As $\mathcal{Q}_1 \neq \mathcal{Q}_2$, we have $(b_1, b_2, b_3, b_4) \neq (0,0,0,0)$. The common points of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are now precisely the points of $\mathcal{Q}_1$ contained in the plane with equation $b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X_4 = 0$. This plane intersects $\mathcal{Q}_1$ in either a singleton or an irreducible conic, implying that $|\mathcal{Q}_1 \cap \mathcal{Q}_2| \in \{1, q+1\}$. □

Lemma 3.9. *Let Q be an elliptic quadric in* $\mathrm{PG}(3,q)$*, q even, and denote by $W(q)$ the symplectic generalized quadrangle associated to Q. Let $\mathcal{Q}_i$ with $i \in \{1, q+1\}$ denote the set of all elliptic quadrics in* $\mathrm{PG}(3,q)$ *that are ovoids of $W(q)$ and intersect Q in exactly i points. Let G denote the stabilizer of Q inside $P\Gamma L(3,q)$. Then the following hold:*

- *G acts transitively on the elements of $\mathcal{Q}_1$;*
- *the number of orbits of G on $\mathcal{Q}_{q+1}$ equals the number of orbits of $Aut(\mathbb{F}_q)$ on the set of elements in $\mathbb{F}_q^*$ with trace equal to 0.*

Proof. Let δ be an element in $\mathbb{F}_q$ whose trace is equal to 1. Let Q_1 and Q_2 be two elements in $\mathcal{Q}_i$. As G acts transitively on the set of tangent planes with respect to Q and the set of secant planes with respect to Q, we may suppose that $Q_1 \cap Q = Q_2 \cap Q = \pi \cap Q$ for a certain plane π of $\mathrm{PG}(3,q)$ which is a tangent plane if $i = 1$ and a secant plane if $i = q+1$.

Suppose first that $i = 1$. Then we can take a reference system with respect to which Q has equation $X_1 X_2 + X_3^2 + X_3 X_4 + \delta X_4^2 = 0$ and π has equation $X_1 = 0$. Since the symplectic polarities associated to Q, Q_1 and Q_2 are the same, there exist $b_1^2, b_2^2 \in \mathbb{F}_q^*$ such that Q_i with $i \in \{1,2\}$ has equation $b_i^2 X_1^2 + X_1 X_2 + X_3^2 + X_3 X_4 + \delta X_4^2 = 0$. Now, the map $(X_1, X_2, X_3, X_4) \mapsto (\frac{b_1}{b_2} X_1, \frac{b_2}{b_1} X_2, X_3, X_4)$ belongs to G and maps Q_1 to Q_2.

Suppose next that $i = q+1$. Then we can take a reference system with respect to which Q has equation $X_1 X_2 + X_3^2 + X_3 X_4 + \delta X_4^2 = 0$ and π has equation $X_4 = 0$. Since the symplectic polarities associated to Q, Q_1 and Q_2 are the same, there exist $b_1, b_2 \in \mathbb{F}_q^*$ whose trace is 0 such that Q_i with $i \in \{1,2\}$ has equation $X_1 X_2 + X_3^2 + X_3 X_4 + (\delta + b_i) X_4^2 = 0$.

Let $\tau \in \mathrm{Aut}(\mathbb{F}_q)$. Note that if (X_1, X_2, X_3, X_4) satisfies $X_1 X_2 + X_3^2 + X_3 X_4 + \delta X_4^2 = 0$, then $(X_1^\tau, X_2^\tau, X_3^\tau, X_4^\tau)$ satisfies $X_1 X_2 + X_3^2 + X_3 X_4 + \delta^\tau X_4^2 = 0$. As $Tr(\delta + \delta^\tau) = 0$, there exists a $b \in \mathbb{F}_q$ such that $b^2+b = \delta+\delta^\tau$. Then $X_1X_2+(X_3+bX_4)^2+(X_3+bX_4)X_4+\delta^\tau X_4^2 = X_1X_2 + X_3^2 + X_3X_4 + \delta X_4^2$ and $X_1X_2 + (X_3 + bX_4)^2 + (X_3 + bX_4)X_4 + (\delta^\tau + b_1^\tau)X_4^2 = X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_1^\tau)X_4^2$. So, if b_1, b_2 are elements of $\mathbb{F}_q$ with trace 0, then by Lemma 3.6 and the above there exists an automorphism of $\mathrm{PG}(3, q)$ with associated field automorphism τ stabilizing Q and mapping $X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_1)X_4^2 = 0$ to $X_1X_2 + X_3^2 + X_3X_4 + (\delta + b_2)X_4^2 = 0$ if and only if $b_2 = b_1^\tau$. The second claim of the lemma now follows. □

Examples:

- if $q = 2$, then $\mathcal{Q}_3$ is empty;
- if $q \in \{4, 8\}$, then G acts transitively on the elements of $\mathcal{Q}_{q+1}$;
- if $q = 16$, then G has three orbits on $\mathcal{Q}_{17}$.

The case where O is a classical ovoid of $W(q)$

For certain values of q even, we know that all ovoids of $W(q)$ are classical, i.e. being an elliptic quadric of the ambient projective space of $W(q)$.

Lemma 3.10. *If $q \in \{2, 4, 16\}$, then every ovoid of $W(q)$ is classical.*

Proof. Proofs of these facts are contained in the papers [1, 4, 5, 6]. □

In this case, we have $|O \cap Q^-(3, q)| \in \{1, q + 1\}$ by Lemma 3.8 and so $|H_O| \in \{q^2(q + 2), (q^2 - q)(q + 2)\}$ by Theorem 3.2.

Suppose first that $q = 2$. Then by the above ($\mathcal{Q}_3 \neq \emptyset$), we know that the case $|O \cap Q^-(3, q)| = q + 1$ cannot occur. So, we then have that $|H_O| = q^2(q + 2) = 16$. For $q = 2$, we also know that every hyperoval of $Q^+(5, 2)$ is the complement of a hyperplane of $Q^+(5, 2)$. The complement of a $Q(4, 2)$-hyperplane of $Q^+(5, 2)$ contains $35 - 15 = 20$ points, while the complement of a $pQ^+(3, 2)$-hyperplane of $Q^+(5, 2)$ contains $35 - 19 = 16$ points. So, in this case, we know that H_O is the complement of a tangent hyperplane intersection of $Q^+(5, 2)$. We can also derive this in another way.

As $|O \cap Q^-(3, q)| = 1$, the intersection $O \cap Q^-(3, q)$ is a singleton $\{p\}$. Every plane of $Q^+(5, q)$ containing p is disjoint from H_O. On the other hand, a plane π of $Q^+(5, q)$ not containing p intersects H_O in a hyperoval of π, necessarily equal to $\pi \setminus p^\perp$. So, H_O must be the complement of the quadric of type $pQ^+(3, 2)$ that arises by intersecting $Q^+(5, 2)$ with the tangent hyperplane at the point p. Combining this with Lemma 3.10, we thus find.

Lemma 3.11.

- *Up to isomorphism, there is a unique hyperoval of $Q^+(5, 2)$ of the form H_O, where O is a classical ovoid of $W(q)$.*
- *Up to isomorphism, there is a unique hyperoval of $Q^+(5, 2)$ of the form H_O, where O is an ovoid of $W(q)$.*

Suppose next that $q = 4$. The both the cases $|O \cap Q^-(3,q)| = 1$ and $|Q \cap Q^-(3,q)| = q+1$ can occur, giving rise to hyperovals of $Q^+(5,q)$ with respective sizes $q^2(q+2) = 96$ and $(q^2-q)(q+2) = 72$. These two hyperplanes were already obtained in the paper of Pasechnik [7] by means of computer backtrack searches. By the above and Lemma 3.10, we then know that the following hold.

Lemma 3.12.
- *Up to isomorphism, there are two hyperovals of $Q^+(5,q)$ of the form H_O, where O is a classical ovoid of $W(q)$.*
- *Up to isomorphism, there are two hyperovals of $Q^+(5,q)$ of the form H_O, where O is an ovoid of $W(q)$.*

A hyperoval of $\mathrm{PG}(2,q)$ with q even is called *regular* if it consists of an irreducible conic union its nucleus. From now on, we suppose that $q \geq 8$.

Lemma 3.13. *If H is a regular hyperoval of $\mathrm{PG}(2,q)$, q even, then there exists a unique point $p \in H$ such that $H \setminus \{p\}$ is an irreducible conic.*

Proof. By the definition of the notion of a regular hyperoval, we know that there exists at least one such point p. Suppose $H \setminus \{p_1\}$ and $H \setminus \{p_2\}$ are irreducible conics of $\mathrm{PG}(2,q)$ for two points $p_1, p_2 \in H$. Note that an irreducible conic of $\mathrm{PG}(2,q)$, q even, is uniquely determined by five of its points. As $|(H \setminus \{p_1\}) \cap (H \setminus \{p_2\})| \geq q \geq 5$, we then have that $H \setminus \{p_1\} = H \setminus \{p_2\}$, i.e. $p_1 = p_2$. □

Lemma 3.14. *Let O be a classical ovoid of $W(q)$ distinct from $Q^-(3,q)$ and let π be a plane of $Q^+(5,q)$ intersecting H_O in a hyperoval of π. Then $\pi \cap H_O$ is a regular hyperoval of π. Moreover, the unique point p of $\pi \cap H_0$ for which $(\pi \cap H_O) \setminus \{p\}$ is an irreducible conic belongs to $Q^-(3,q) \setminus O \subseteq \pi$.*

Proof. As $\pi \cap H_0$ is a hyperoval of π, π intersects $Q^-(3,q) \setminus O$ in a singleton $\{p\}$. By Lemma 3.1 and the definition of H_O, the projection A of $(\pi \cap H_O) \setminus \{p\}$ from L^* on Π is contained in $O \setminus Q^-(3,q)$ and so the plane $\langle L^*, \pi \rangle \cap \Pi$ intersects O in the irreducible conic A. It follows that $(\pi \cap H_O) \setminus \{p\}$ itself must also be an irreducible conic of π. The kernel of this irreducible conic necessarily coincides with $p \in Q^-(3,q) \setminus O \subseteq \Pi$. □

Lemma 3.15. *Let O_1 and O_2 be two classical ovoids of $W(q)$ distinct from $Q^-(3,q)$. Then the following are equivalent:*

(1) *the hyperovals H_{O_1} and H_{O_2} are isomorphic;*

(2) *there exists an automorphism of Π stabilizing $Q^-(3,q)$ mapping O_1 to O_2.*

Proof. Suppose there exists an automorphism θ of Π stabilizing $Q^-(3,q)$ and mapping O_1 to O_2. Then θ extends to an automorphism $\overline{\theta}$ of $Q^+(5,q)$. It is clear that $\overline{\theta}$ maps H_{O_1} and H_{O_2}.

Conversely, suppose that there exists an automorphism $\overline{\theta}$ of $\mathrm{PG}(5,q)$ stabilizing $Q^+(5,q)$ mapping H_{O_1} to H_{O_2}. For every $i \in \{1,2\}$, let Ω_i denote the set of all planes π of

$Q^+(5,q)$ intersecting H_{O_i} in a hyperoval of π. For every $\pi \in \Omega_i$, let k_π denote the unique point of $\pi \cap H_{O_i}$ for which $(\pi \cap H_{O_i}) \setminus \{k_\pi\}$ is an irreducible conic of π. Then $\{k_\pi \,|\, \pi \in \Omega_i\}$ is a set of $|Q^-(3,q) \setminus O_i|$ points of π. By Lemma 3.4, we know that $|Q^-(3,q) \setminus O_i| \geq q^2 + 1 - \frac{q^2-q}{2} = \frac{q^2+q+2}{2} > q+1$. So, the set $\{k_\pi \,|\pi \in \Omega_i\}$ must generate Π. Since $\overline{\theta}$ maps Ω_1 to Ω_2, it maps the set $\{k_\pi \,|\, \pi \in \Omega_1\}$ to the set $\{k_\pi \,|\, \pi \in \Omega_2\}$ and so $\overline{\theta}$ stabilizes Π. Denote by θ the restriction of $\overline{\theta}$ to Π. Then θ stabilizes $Q^-(3,q)$. Also, θ maps $\{k_\pi \,|\, \pi \in \Omega_1\} = Q^-(3,q) \setminus O_1$ to $\{k_\pi \,|\, \pi \in \Omega_2\} = Q^-(3,q) \setminus O_2$ and so $Q^-(3,q) \cap O_1$ to $Q^-(3,q) \cap O_2$. As $\overline{\theta}$ stabilizes $Q^+(5,q)$ and Π, it also stabilizes the line L^*. Note that $O_i \setminus Q^-(3,q)$, $i \in \{1,2\}$, is the projection of $H_{O_i} \setminus \Pi$ from L^* onto Π. Since $\overline{\theta}$ maps $H_{O_1} \setminus \pi$ to $H_{O_2} \setminus \pi$, it must also map $O_1 \setminus Q^-(3,q)$ to $O_2 \setminus Q^-(3,q)$. All together, we thus have that $\overline{\theta}$ and θ map O_1 to O_2. □

The following is a consequence of Lemmas 3.9 and 3.15.

Corollary 3.16. *Let N denote the number of orbits of $\mathrm{Aut}(\mathbb{F}_q)$ on the set of all elements of $\mathbb{F}_q^*$ with trace equal to 0. Then the number of nonisomorphic hyperovals of the form H_O where O is a classical ovoid of $W(q)$ is equal to $N+1$.*

Corollary 3.17. *Suppose $q = 16$. The number of nonisomorphic hyperovals of the form H_O, where O is an ovoid of $W(q)$ is equal to 4.*

The general case

Let $\mathcal{U}$ denote the set of all planes π of $\mathrm{PG}(5,q)$ such that $\pi \cap H_O$ and $\pi \cap Q^+(5,q)$ are coinciding irreducible conics of π. We will prove some results that indicate which planes can belong to $\mathcal{U}$. The following results are useful to that end.

Lemma 3.18. *Suppose $\overline{O}$ is a hyperoval of $\mathrm{PG}(2,q)$, $q \geq 8$ even, and X is a subset of size $q-1$ of $\overline{O}$. Then through every point x of $\mathrm{PG}(2,q) \setminus \overline{O}$, there is a line intersecting X in exactly two points.*

Proof. Through x, there are $\frac{q+2}{2}$ lines intersecting $\overline{O}$ in exactly two points. At most three of these lines contain a point of $\overline{O} \setminus O$. So, at least $\frac{q+2}{2} - 3 = \frac{q-4}{2} > 0$ of these lines contain two points of X. □

Corollary 3.19. *Let X be a set of $q-1$ or q mutually noncollinear points of $\mathrm{PG}(2,q)$, $q \geq 8$ even. Then X is contained in at most one hyperoval of $\mathrm{PG}(2,q)$*

In fact, a better result as the one in Corollary 3.19 is known. By Theorem 3 of [8], we know that the following holds.

Corollary 3.20. *Let X be a set of $q-1$ or q mutually noncollinear points of $\mathrm{PG}(2,q)$, $q \geq 8$ even. Then X is contained in a unique hyperoval of $\mathrm{PG}(2,q)$*

Lemma 3.21. *Suppose $\pi \in \mathcal{U}$. Then no point of $\pi \cap \Pi \cap Q^-(3,q)$ belongs to O.*

Proof. Since $\pi \cap Q^+(5,q)$ and $\pi \cap H_O$ are the same irreducible conic, we have $\pi \cap Q^+(5,q) \cap \Pi = \pi \cap \Pi \cap Q^-(3,q) = \pi \cap H_O \cap \Pi = \Pi \cap (Q^-(3,q) \setminus O)$, proving the validity of the claim. □

Lemma 3.22. *A plane π through L^* belongs to $\mathcal{U}$ if and only if it intersects Π in a point of $O \setminus Q^-(3,q)$.*

Proof. If π intersects Π in a point of $Q^-(3,q)$, then $\pi \cap Q^+(5,q)$ is a singleton and so $\pi \notin \mathcal{U}$.

Suppose $\pi \cap \Pi$ is not contained in $Q^-(3,q)$. By the definition of H_O and Lemma 3.1 we know that $\pi \cap H_O = \emptyset$ if $\pi \cap \Pi$ is not contained in O and $\pi \cap H_O$ is an irreducible conic of π if $\pi \cap \Pi$ is contained in O. Moreover, in the latter case, we have that $\pi \cap H_O = \pi \cap Q^+(5,q)$. □

Lemma 3.23. *If $q \geq 8$, then a plane π intersecting L^* in a singleton can never belong to $\mathcal{U}$.*

Proof. Suppose to the contrary that $\pi \in \mathcal{U}$. Then $\pi \cap Q^+(5,q) = \pi \cap H_O$ is an irreducible conic $\mathcal{C}_\pi$.

The points of $\mathcal{C}_\pi$ contained in Π are precisely the points of $(\pi \cap \Pi) \cap Q^-(3,q)$. As $\pi \cap \Pi$ is a singleton or a line, there are at most two such points.

Each point of $\mathcal{C}_\pi \setminus \Pi$ is by Lemma 3.1 and the definition of H_O contained in a plane of the form $\langle L^*, u\rangle$, where $u \in O \setminus Q^-(3,q)$. Such a point u necessarily is contained in the line $K := \langle L^*, \pi\rangle \cap \Pi$. Now, the line K intersects O and hence also $O \setminus Q^-(3,q)$ in at most two points. If u is a point of K contained in $O \setminus Q^-(3,q)$, then in the three-dimensional subspace $\langle L^*, \pi\rangle$ the intersection of the two planes π and $\langle L^*, u\rangle$ is a line containing at most two points of the irreducible conic $\mathcal{C}_\pi$. We therefore see that there are at most $2 \cdot 2 = 4$ points in $\mathcal{C}_\pi \setminus \Pi$.

Altogether, we have $|\mathcal{C}_\pi| \leq 6$. But that is in contradiction with the fact that $|\mathcal{C}_\pi| = q + 1 \geq 9$. □

Lemma 3.24. *Suppose O is a nonclassical ovoid and $\pi \cap \Pi$ is disjoint from $(Q^-(3,q) \setminus O) \cup L^*$. Then $\pi \notin \mathcal{U}$.*

Proof. Suppose to the contrary that $\pi \cap H_O = \pi \cap Q^+(5,q)$ is an irreducible conic $\mathcal{C}_\pi$ of π. As π is disjoint from $Q^-(3,q) \setminus O$, we see that no point of $\mathcal{C}_\pi$ is contained in Π. Let π' be the projection of π from L^* to Π. By Lemma 3.1 and the definition of H_O, we then see that the projection $\mathcal{C}'_\pi$ of $\mathcal{C}_\pi$ from L^* on Π is an irreducible conic of π' contained in $O \setminus Q^-(3,q)$. So, $O \cap \pi' = \mathcal{C}'_\pi$. But that is impossible. As O is a nonclassical ovoid of Π, we know by the main result of [2] that $O \cap \pi'$ cannot be an irreducible conic. □

Lemma 3.25. *Suppose $q \geq 8$ and suppose $\pi \cap \Pi$ is disjoint from L^* and intersects $Q^-(3,q) \setminus O$ in two points. Then $\pi \notin \mathcal{U}$.*

Proof. Suppose to the contrary that $\pi \in \mathcal{U}$. Then $\pi \cap Q^+(5,q) = \pi \cap H_O$ is an irreducible conic $\mathcal{C}_\pi$ of π. Let x_1 and x_2 be the two points of $\pi \cap \Pi$ contained in $Q^-(3,q) \setminus O$. Then

$\pi \cap \Pi$ is a line x_1x_2. Let π' be the plane of Π that arises as projection of π from L^* on Π, and let $\mathcal{C}'_\pi$ be the irreducible conic of π' that arises as projection of $\mathcal{C}_\pi$ from L^* on Π. By Lemma 3.1 and the definition of H_O, we know that $\mathcal{C}'_\pi \setminus \{x_1, x_2\}$ is a set of $q-1$ points of π' contained in $O \setminus Q^-(3,q)$. As $q \geq 8$, these $q-1$ points extend in a unique way to a hyperoval $\overline{O}$ of π', and $\overline{O}$ coincides with $\mathcal{C}'_\pi$ union its nucleus n. The two points of O not contained in $\mathcal{C}'_\pi \setminus \{x_1, x_2\}$ are then contained in $\{x_1, x_2, n\}$, in contradiction with the fact that none of x_1, x_2 belong to O. □

Lemma 3.26. *Suppose $\pi \in \mathcal{U}$ is disjoint from L^* and intersects $Q^-(3,q) \setminus O$ in a singleton $\{x\}$. Then $\pi \notin \mathcal{U}$.*

Proof. Let π' be the plane of Π that arises as projection of π from L^* on Π, and let $\mathcal{C}'_\pi$ be the irreducible conic of π' that arises as projection of $\mathcal{C}_\pi$ from L^* on Π. By Lemma 3.1 and the definition of H_O, we know that $\mathcal{C}'_\pi \setminus \{x\}$ is a set of q points of π' contained in $O \setminus Q^-(3,q)$. As $q \geq 8$, these q points extend in a unique way to a hyperoval $\overline{O}$ of π', and $\overline{O}$ equals $\mathcal{C}'_x$ union its nucleus $\{n\}$. As $x \in Q^-(3,q) \setminus O$, we have $O \cap \pi' = (\mathcal{C}_x \setminus \{x\}) \cup \{n\}$. So, x is the nucleus of the oval $O \cap \pi'$ of π and all lines of π' through x are tangent to O' and hence also to $Q^-(3,q)$ by Lemma 3.7. We thus have $\pi' \subseteq x^\zeta$. As also $L^* \subseteq x^\zeta$, we have $\pi \subseteq \langle L^*, \pi' \rangle \subseteq x^\zeta$. But then every line of Π through x is either contained in $Q^+(5,q)$ or the singleton $\{x\}$. But that is impossible as $\pi \cap Q^+(5,q)$ is an irreducible conic of π. □

For every $\pi \in \mathcal{U}$, let k_π denote the kernel of the irreducible conic $\pi \cap Q^+(5,q)$ of π. Let K denote the subspace of $\mathrm{PG}(5,q)$ generated by all points k_π, $\pi \in \mathcal{U}$.

Lemma 3.27. *The subspace Π is contained in K.*

Proof. Let π be a plane of the form $\langle L^*, x \rangle$, where $x \in O \setminus Q^-(3,q)$. Then $\Pi = (L^*)^\zeta \supseteq \langle L^*, x \rangle^\zeta$ and so the kernel of the irreducible conic $\mathcal{C}_x = \langle L^*, x \rangle \cap Q^+(5,q)$ is contained in π, i.e. equal to x. So, K contains the subspace $\langle O \setminus Q^-(3,q) \rangle$. Now, $|O \setminus Q^-(3,q)| = q^2 + 1 - |O \cap Q^-(3,q)| \geq q^2 + 1 - \frac{q^2-q}{2} = \frac{q^2+q+2}{2} > q+1$ by Lemma 3.4, implying that $\langle O \setminus Q^-(3,q) \rangle = \Pi$. So, K contains Π. □

Lemma 3.28. *Suppose K is 3-dimensional. Then $K \cap Q^+(5,q)$ is an elliptic quadric of type $Q^-(3,q)$ and so K^ζ is a line disjoint from K. Let X_1 denote the set $K \cap H_O$ and let X_2 denote the projection of $H_O \setminus K$ from K^ζ onto Π. Then $\Pi = K$ and $O = (Q^-(3,q) \setminus X_1) \cup X_2$.*

Proof. By Lemma 3.27, $K = \Pi$ and so $K \cap Q^+(5,q)$ is an elliptic quadric. We have $X_1 = \Pi \cap H_O = Q^-(3,q) \setminus O$ and hence $Q^-(3,q) \setminus X_1 = Q^-(3,q) \cap O$. By the definition of H_O, we also have that $X_2 = O \setminus Q^-(3,q)$. So, $(Q^-(3,q) \setminus X_1) \cup X_2 = O$. □

Lemma 3.29. *Suppose O is a nonclassical ovoid of $W(q)$. Then $K = \Pi$.*

Proof. In view of Lemma 3.27, it suffices to show that each point k_π, $\pi \in \mathcal{U}$, is contained in Π. As O is a nonclassical ovoid, we have $q \geq 8$. The various lemmas above the imply that there are two possible types of planes in $\mathcal{U}$, planes π_1 through L^* and planes π_2 contained in Π. For the former planes, we have already see in Lemma 3.27 that $k_{\pi_1} \in \Pi$. For the latter planes, it is trivial that $k_{\pi_2} \in \Pi$. □

Lemma 3.30. *Let O_1 and O_2 be two ovoids of $W(q)$ distinct from $Q^-(3,q)$. Then the following are equivalent:*

(1) *the hyperovals H_{O_1} and H_{O_2} are isomorphic;*

(2) *there exists an automorphism of Π stabilizing $Q^-(3,q)$ and mapping O_1 to O_2.*

Proof. Suppose there exists an automorphism θ of Π stabilizing $Q^-(3,q)$ and mapping O_1 to O_2. Then θ extends to an automorphism $\overline{\theta}$ of $Q^+(5,q)$. It is clear that $\overline{\theta}$ maps H_{O_1} and H_{O_2}.

Conversely, suppose that there exists an automorphism $\overline{\theta}$ of $\mathrm{PG}(5,q)$ stabilizing $Q^+(5,q)$ mapping H_{O_1} to H_{O_2}. For every $i \in \{1,2\}$, let $\mathcal{U}_i$ denote the set of all planes π of $\mathrm{PG}(5,q)$ satisfying the following property:

$\pi \cap Q^+(5,q)$ is an irreducible conic that is entirely contained in H_{O_i}.

For every $\pi \in \mathcal{U}_i$, let k_π denote the kernel of the irreducible conic $\pi \cap Q^+(5,q)$ of π. Let K_i denote the subspace of $\mathrm{PG}(5,q)$ generated by all points k_π, $\pi \in \mathcal{U}_i$. It is clear that $\overline{\theta}$ maps K_1 to K_2. We distinguish two cases.

(1) Suppose $\dim(K_1) = \dim(K_2) = 3$. By Lemma 3.28, we then know that $K_1 = K_2 = \Pi$ and that $\overline{\theta}$ stabilizes Π and that the restriction θ of $\overline{\theta}$ to Π maps O_1 to O_2.

(2) Suppose $\dim(K_1) = \dim(K_2) > 3$. By Lemma 3.29, we would then know that O_1 and O_2 are classical ovoids of $W(q)$. But then the claim follows from Lemma 3.15. □

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