The CRC Handbook of Combinatorial Designs

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1 Partial Geometries

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1.1 Definitions and Examples

1.1 **Definition** A (finite) *partial geometry* is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ in which P and B are disjoint (nonempty) sets of objects called *points* and *lines*, and for which I is a symmetric point-line incidence relation, i.e., $\mathrm{I} \subseteq (P \times B) \cup (B \times P)$, satisfying the following axioms:

1. each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
2. each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
3. if x is a point and L is a line not incident with x, then there are exactly α ($\alpha \geq 1$) points $y_1, y_2, \cdots, y_\alpha$ and α lines $M_1, M_2, \cdots, M_\alpha$ such that $x\mathrm{I}M_i\mathrm{I}y_i\mathrm{I}L$, with $i = 1, 2, \cdots, \alpha$.

The integers s, t and α are the *parameters* of the partial geometry. Put $|P| = v$ and $|B| = b$.

1.2 **Remarks**

1. There is a point-line duality for partial geometries for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged.
2. R.C. Bose used r, k, t for the parameters $t+1, s+1, \alpha$ given above.

1.3 **Examples**

1. A *generalized quadrangle* is a partial geometry with $\alpha = 1$. See §IV.**??** for many constructions and examples.
2. Let P be the set of all points of the n-dimensional projective space $\mathrm{PG}(n, q)$ over the Galois field $\mathbb{F}_q$, which are not contained in a fixed subspace $\mathrm{PG}(n-2, q)$, with $n \geq 2$. Let B be the set of all lines of $\mathrm{PG}(n, q)$ having no point in common with $\mathrm{PG}(n-2, q)$. Finally, let I be the natural incidence. Then (P, B, I) is a partial geometry with $s = q, t = q^{n-1} - 1, \alpha = q$. This geometry is denoted by H_q^n [16].

1.2 Existence and Pseudo-Geometric Graphs

1.4 **Theorem** If $\mathcal{S} = (P, B, \mathrm{I})$ is a partial geometry with parameters s, t, α, then $v = (s+1)(st+\alpha)/\alpha$ and $b = (t+1)(st+\alpha)/\alpha$ [9].

1.5 **Theorem** The point graph of a partial geometry with parameters s, t, α is a strongly regular graph with parameters $v, k = s(t+1), \lambda = s - 1 + t(\alpha - 1), \mu = \alpha(t+1)$.

1.6 **Definition** Each strongly regular graph Γ having the parameters of Theorem 1.5 with $t \geq 1, s \geq 1, 1 \leq \alpha \leq s+1$, and $1 \leq \alpha \leq t+1$ is a *pseudo-geometric* (t, s, α)-graph. If the pseudo-geometric graph Γ is the point graph of at least one partial geometry, then it is *geometric*.

1.7 **Theorem** Any pseudo-geometric $(s^2, s, 1)$-graph is geometric [3].

1.8 **Theorem** Let $\mathcal{S} = (P, B, \mathrm{I})$ be a partial geometry with parameters s, t, α. Then s, t and α satisfy:

1. the integer $\alpha(s + t + 1 - \alpha)$ divides $st(s + 1)(t + 1)$ [9];
2. the Krein inequalities [3], namely $(s + 1 - 2\alpha)t \leq (s - 1)(s + 1 - \alpha)^2$ and $(t + 1 - 2\alpha)s \leq (t - 1)(t + 1 - \alpha)^2$.

1.9 **Definition** Partial geometries $\mathcal{S}$ can be divided into four (nondisjoint) classes.

1. $\mathcal{S}$ has $\alpha = s + 1$ or, dually, $\alpha = t + 1$; when $\alpha = s + 1$, then $\mathcal{S}$ is a 2-$(v, s + 1, 1)$ design.
2. $\mathcal{S}$ has $\alpha = s$ or, dually, $\alpha = t$; when $\alpha = t$, then $\mathcal{S}$ is a *net* of *order* $s + 1$ and *degree* $t + 1$.
3. When $\alpha = 1$, $\mathcal{S}$ is a generalized quadrangle.
4. When $1 < \alpha < \min(s, t)$, then $\mathcal{S}$ is *proper*.

1.10 **Remark** Example 1.3(2) is a dual net.

1.11 **Remark** The dual of a net of order $s + 1$ and degree $t + 1$ is a transversal design $TD(t + 1, s + 1)$.

1.12 **Definition** *Axiom of Pasch*, also called the *Axiom of Veblen*: if $L_1 \mathrm{I}\, x \mathrm{I}\, L_2$, $L_1 \neq L_2$, $M_1 \not\mathrm{I}\, x \not\mathrm{I}\, M_2$, and if L_i has a point in common with M_j for all $i, j \in \{1, 2\}$, then M_1 and M_2 have a point in common.

1.13 **Theorem** Let $\mathcal{S}$ be a dual net of order $s + 1$ and degree $t + 1$, with $s < t + 1$ (for such a dual net we always have $s \leq t + 1$ and $s = t + 1$ if and only if $\mathcal{S}$ is a dual affine plane of order s). Then $\mathcal{S}$ satisfies the axiom of Pasch if and only if it is isomorphic to a dual net H_q^n with $n \geq 3$ ($s = q$ and $t = q^{n-1} - 1$) [16].

1.3 Maximal Arcs

1.14 **Definition** In a projective plane $\mathcal{P}$ of order q any nonempty set of k points may be described as a $\{k; d\}$-*arc*, where d ($d \neq 0$) is the greatest number of collinear points in the set. For given q and d ($d \neq 0$), k can never exceed $dq - q + d$, and a $\{dq - q + d; d\}$-arc is a *maximal arc*. Equivalently, a maximal arc may be defined as a nonempty set of points in $\mathcal{P}$ meeting every line in just d points or in none at all.

1.15 **Examples**

1. The point set of $\mathcal{P}$ is a maximal arc with $d = q + 1$.
2. The point set of the affine plane $\mathcal{A}$ obtained by deleting a line from $\mathcal{P}$ is a maximal arc with $d = q$.
3. A single point is a maximal arc with $d = 1$.

4. A *hyperoval* of $\mathcal{P}$, that is, a set of $q+2$ points no three of which are collinear, is a maximal arc with $d=2$.

1.16 Theorem If K is a $\{dq-q+d;d\}$-arc (i.e., a maximal arc) of the projective plane $\mathcal{P}$, where $d \leq q$, then the set K'=$\{$lines L of $\mathcal{P} \mid L \cap K = \phi\}$ is a $\{q(q-d+1)/d; q/d\}$-arc (i.e., a maximal arc) of the dual plane.

1.17 Corollary A necessary condition for the existence of a maximal arc, with $d \leq q$, is that d is a factor of q.

1.18 Remark Theorem 1.19 contains the two most important results on the existence of maximal arcs; Part 1 is due to Ball, Blokhuis and Mazzocca [1], Part 2 to Denniston [8].

1.19 Theorem

1. In PG$(2,q)$, with q odd, there are no $\{qd-q+d;d\}$-arcs for $1<d<q$.
2. For any factor d of q, q even, there exists a $\{dq-q+d;d\}$-arc in the plane PG$(2,q)$.

1.20 Remark Maximal arcs in PG$(2,q), q$ even, were constructed by Denniston [8], Thas [15], Mathon [12], Hamilton and Mathon [7].

1.21 Construction Let K be a maximal $\{qd-q+d;d\}$-arc of a projective plane $\mathcal{P}$ of order q, $2 \leq d < q$. Let P be the set of all points of $\mathcal{P}$ not in K, let B be the set of all lines of $\mathcal{P}$ having a nonempty intersection with K, and let I be the natural incidence. Then $\mathcal{S}(K) = (P, B, \mathrm{I})$ is a partial geometry with parameters

$$s = q-d, \quad t = q(d-1)/d, \quad \alpha = (q-d)(d-1)/d.$$

1.22 Construction Let $\mathcal{P}$ = PG$(2,q)$ and embed $\mathcal{P}$ in the projective space PG$(3,q)$. Further, let P' consist of all points of PG$(3,q)$ not in $\mathcal{P}$, let B' consist of all lines of PG$(3,q)$ having a unique point in common with the maximal $\{qd-q+d;d\}$-arc K of $\mathcal{P}$, and let I' be the natural incidence. Then $\mathcal{T}_2^*(K) = (P', B', \mathrm{I}')$ is a partial geometry with parameters

$$s = q-1, \quad t = (q+1)(d-1), \quad \alpha = d-1.$$

1.23 Remark Construction 1.21 is due independently to Thas and Wallis, Construction 1.22 to Thas [6].

1.24 Corollary By Theorem 1.19 there exist partial geometries with parameters as follows:

Type 1: $s = 2^h - 2^m,\ t = 2^h - 2^{h-m},\ \alpha = (2^{h-m}-1)(2^m-1)$ with $1 \leq m < h$;
Type 2: $s = 2^h - 1,\ t = (2^h+1)(2^m-1),\ \alpha = 2^m - 1$ with $1 \leq m < h$.

Such a partial geometry has $\alpha = 1$ or is proper. A partial geometry of Type 1 is a generalized quadrangle if and only if $h=2$ and $m=1$; then $s=t=2$ and $v=b=15$. A partial geometry of Type 2 is a generalized quadrangle if and only if $m=1$; then K is a hyperoval in $\mathcal{P}$, $s = 2^h-1, t = 2^h+1, v = 2^{3h}, b = 2^{2h}(2^h+2)$ with $h \geq 2$.

1.4 Partial Geometries in Projective and Affine Spaces

1.25 Definition A *projective partial geometry* $\mathcal{S} = (P, B, \mathrm{I})$ is a partial geometry with parameters $s = q, t, \alpha$ for which the point set P is a subset of the point set of some projective space PG(n,q), for which the line set B is a set of lines of PG(n,q), and for which I is the natural incidence. In such a case the partial geometry $\mathcal{S}$ is *embedded* in $\mathrm{PG}(n,q)$. If $\mathrm{PG}(n',q)$ is the subspace of $\mathrm{PG}(n,q)$ generated by all points of P, then $\mathrm{PG}(n',q)$ is the *ambient space* of $\mathcal{S}$.

1.26 Definition Similarly one may define an *affine partial geometry* by replacing the projective space $\mathrm{PG}(n,q)$ by the affine space $\mathrm{AG}(n,q)$.

1.27 Theorem If $\mathcal{S} = (P, B, \mathrm{I})$ is a partial geometry with parameters s, t, α which is projective with ambient space $\mathrm{PG}(n,s), n \geq 2$, then one of the following holds:

1. $\alpha = s + 1$ and $\mathcal{S}$ is the 2-design formed by all points and all lines of $\mathrm{PG}(n,s)$;
2. $\alpha = 1$ and $\mathcal{S}$ is a classical generalized quadrangle;
3. $\alpha = t + 1, n = 2$, P is the complement of a maximal $\{sd - s + d; d\}$-arc K of $\mathrm{PG}(2,s)$ with $d = s/\alpha$ and $2 \leq d < s$, and B consists of all lines of $\mathrm{PG}(2,s)$ having an empty intersection with K;
4. $\alpha = s, n \geq 2$ and $\mathcal{S}$ is the dual net H_s^n [9].

1.28 Remark Part 2 of Theorem 1.27 is due to Buekenhout, Parts 1, 3, and 4 to De Clerck and Thas.

1.29 Remark All partial geometries embedded in the affine space $\mathrm{AG}(n,q), n \geq 2$, were determined by Thas [14]; the three-dimensional case with $\alpha = 1$ was independently settled by Bichara [14]. For $\alpha = 1$ and $n \geq 3$, five interesting sporadic cases occur [14].

1.30 Theorem If $\mathcal{S}$ is a proper partial geometry with parameters s, t, α which is affine with ambient space $\mathrm{AG}(n, s+1), n \geq 2$, then $n = 3$ and $\mathcal{S}$ is a partial geometry $\mathcal{T}_2^*(K)$ with K a maximal arc in the plane at infinity of $\mathrm{AG}(3, s+1)$ [14].

1.5 Enumeration of Partial Geometries

1.31 Theorem Up to duality, the parameters of the known proper partial geometries are the following:

Type 1: $s = 2^h - 2^m, t = 2^h - 2^{h-m}, \alpha = (2^m - 1)(2^{h-m} - 1)$, with $h \neq 2$ and $1 \leq m < h$;
Type 2: $s = 2^h - 1, t = (2^h + 1)(2^m - 1), \alpha = 2^m - 1$, with $1 < m < h$;
Type 3: $s = 2^{2h-1} - 1, t = 2^{2h-1}, \alpha = 2^{2h-2}$, with $h > 1$;
Type 4: $s = 3^{2m} - 1, t = (3^{4m} - 1)/2, \alpha = (3^{2m} - 1)/2, m \geq 1$;
Type 5: $s = 26, t = 27, \alpha = 18$;
Type 6: $s = t = 5, \alpha = 2$;
Type 7: $s = 4, t = 17, \alpha = 2$.
Type 8: $s = 8, t = 20, \alpha = 2$.

1.32 Remarks

1. Mathon proved by computer that there exist exactly two partial geometries with parameters $s = 6, t = 4, \alpha = 3$ [6]. For $(s, t, \alpha) \neq (6, 4, 3)$ all known partial geometries of Type 1 and 2 are given by the constructions of §1.4.
2. With each partition by $(2h - 1)$-dimensional spaces (i.e., maximal singular subspaces) of the hyperbolic quadric $Q^+(4h - 1, 2)$ of $\mathrm{PG}(4h - 1, 2), h > 1$, there corresponds a partial geometry of Type 3; no other partial geometries with these parameters are known. The geometries of Type 3 are due to De Clerck, Dye, and

Thas; for $h = 2$ the geometry was also constructed by Cohen, and by Haemers and van Lint (the three constructions are completely different) [6].

3. The partial geometries of Type 4 are due to Mathon [11]; their strongly regular graphs arise from the Hermitian two-graph.
4. With each partition by $(2h-1)$-dimensional spaces of the hyperbolic quadric $Q^+(4h-1,3)$ of $\mathrm{PG}(4h-1,3), h > 1$, there corresponds a partial geometry with parameters $s = 3^{2h-1}-1, t = 3^{2h-1}, \alpha = 2.3^{2h-2}$. Up to now only $Q^+(7,3)$ is known to have such a partition; this yields the geometry of Type 4. This construction is due to Thas [6].
5. Spread derivation introduced by De Clerck [4] (and based on Mathon and Street [4]) yields many non-isomorphic partial geometries having the same parameters of the partial geometries of Type 3 and Type 5.
6. The geometry of Type 6 is due to van Lint and Schrijver, the geometry of Type 7 to Haemers, and the geometry of Type 8 to Mathon; one geometry of each type is known [5, 6].

1.33 Table The parameters of possible partial geometries (PG) with $v < 100$, excluding designs and nets and their duals; of a dual pair, the one with $s \leq t$ is given. All conditions in §1.2 are taken into account.

v	s	t	α	b	Comment
15	2	2	1	15	Unique PG exists [13]
27	2	4	1	45	Unique PG exists [13]
28	3	4	2	35	Does not exist [6]
40	3	3	1	40	Exactly two PG exist [13]
45	4	6	3	63	Exactly two PG exist; Remark 1.32 (1)
64	3	5	1	96	Unique PG exists [13]
66	5	8	4	99	Does not exist, Lam et al. [6]
70	6	6	4	70	Unknown
75	4	7	2	120	Unknown
76	3	6	1	133	Does not exist, Dixmier and Zara [13]
81	5	5	2	81	One PG known; Remark 1.32 (6)
85	4	4	1	85	Unique PG exists [13]
91	6	10	5	143	Unknown
95	4	9	2	190	Unknown
96	5	6	2	112	Unknown
96	5	9	3	160	Unknown

1.34 Remark For $v = 70$ and $v = 91$ pseudo-geometric graphs are known; for $v = 75$ and $v = 96$ (both cases) no pseudo-geometric graph is known.

1.6 See Also

§I.??	Partial geometries with $\alpha = s + 1$ are Steiner systems.
§II.??	Partial geometries with $\alpha = s$ are transversal designs; partial geometries with $\alpha = t$ are nets.
§IV.??	Pseudo-geometric graphs are particular association schemes.

§IV.??	Partial geometries with $\alpha = 1$ are generalized quadrangles.
§V.??	Pseudo-geometric graphs are particular strongly regular graphs.
§V. ??	Two-graphs are used to construct partial geometries.
§V.??	Classical geometries and nondesarguesian planes are used to construct partial geometries.
[2]	Excellent survey, without proofs, with extensive bibliography of strongly regular graphs and partial geometries.
[6]	A survey, without proofs, with extensive bibliography. This survey contains most of the information in this section.
[8]	Contains extensive information on maximal arcs.
[9]	Contains a section on projective partial geometries.
[10]	Excellent survey on translation nets.
[13]	Standard reference on generalized quadrangles.
[14]	Determination of all affine partial geometries.
[15]	Contains a survey of maximal arcs.

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