Maximal Arcs in PG$(2, q)$ and partial flocks of the quadratic cone

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Abstract

In this paper we show that there are several other structures that arise from the functions associated with the maximal arcs of Mathon type. So it is shown that maximal arcs of Mathon type are equivalent to additive partial flocks of the quadratic cone in PG$(3, q)$ and to additive partial q-clans. Further they yield partial ovoids of $Q^+(5, q)$, partial spreads of lines of PG$(3, q)$, translation k-arcs of PG$(2, q)$ and m-ovoids of a certain classical generalized quadrangle.

Keywords: maximal arc, flock, q-clan, k-arc, m-ovoid.

1 Introduction

A *maximal* $\{q(n-1)+n; n\}$*-arc* in a projective plane of order q is a subset of $q(n-1)+n$ points such that every line meets the set in 0 or n points for some $2 \leq n \leq q$. For such a maximal arc n is called the *degree*. In [12], R. Mathon gave a construction method for maximal arcs in Desarguesian projective planes that generalised a previously known construction of R.H.F. Denniston [1]. We begin by describing this construction method.

In the following the order of the fields will always be even. Let Tr be the usual absolute trace map from the finite field GF(q) onto GF(2). We represent the points of the Desarguesian projective plane PG$(2, q)$ via homogeneous coordinates (a, b, c) over GF(q), lines similarly as triples $[u, v, w]$ over GF(q), and incidence by the usual inner product $au + bv + cw = 0$. For $\alpha, \beta \in$ GF(q) such that the absolute trace $Tr(\alpha\beta) = 1$, and $\lambda \in$ GF$(q)^*$, define $F_{\alpha,\beta,\lambda}$ to be the conic

$$F_{\alpha,\beta,\lambda} = \{(x, y, z) : \alpha x^2 + xy + \beta y^2 + \lambda z^2 = 0\}$$

and let $\mathcal{F}$ be the set of all such conics. Note that the non-singular conics in $\mathcal{F}$ have the point $F_0 = (0, 0, 1)$ as their nucleus, and they are all disjoint from the line $z = 0$ which we will denote F_∞.

For given $0 \neq \lambda \neq \lambda' \neq 0$, define a composition

$$F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'} = F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda+\lambda'}$$

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where the operator $\oplus$ is defined on $\mathrm{GF}(q) \times \mathrm{GF}(q)$ by

$$a \oplus b = \frac{\lambda a + \lambda' b}{\lambda + \lambda'}.$$

Given some subset $\mathcal{C}$ of $\mathcal{F}$, with $\mathcal{C} \neq \phi$ and $\mathcal{C} \neq \{F_0\}$, we say $\mathcal{C}$ is *closed* if for every $F_{\alpha,\beta,\lambda}, F_{\alpha',\beta',\lambda'} \in \mathcal{C}$, with $F_{\alpha,\beta,\lambda} \neq F_{\alpha',\beta',\lambda'}$, we have that $F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda+\lambda'}$ is defined and belongs to $\mathcal{C}$. In [12], the following theorem is proved.

Theorem 1
[12, Thm. 2.4] Let $\mathcal{C}$ be a closed set of conics with nucleus F_0 in $\mathrm{PG}(2,q)$, q even. Then the union of the points of the conics of $\mathcal{C}$ together with F_0 form the points of a degree $|\mathcal{C}|+1$ maximal arc in $\mathrm{PG}(2,q)$.

Note that a closed set of conics may be written in the form

$$\mathcal{F} = \bigcup_{\lambda \in A} \{(x,y,z) : p(\lambda)x^2 + xy + r(\lambda)y^2 + \lambda z^2 = 0\}$$

where A is a subset of $\mathrm{GF}(q) - \{0\}$ such that $A \cup \{0\}$ is closed under addition and p and r are functions from A to $\mathrm{GF}(q)$.

In the paper Mathon also gave several classes of examples of new maximal arcs using his method. We shall say that maximal arcs constructed from closed sets of conics are of *Mathon type*. See also [4], [3] and [5] for more examples arising using this construction method, as well as structural results on the maximal arcs obtained. We will mention here just two examples.

In 1969, R.H.F. Denniston constructs a class of maximal arcs of the form

$$\mathcal{F} = \bigcup_{\lambda \in B} \{(x,y,z) : x^2 + xy + \alpha y^2 + \lambda z^2 = 0\},$$

where α is a fixed element of $\mathrm{GF}(q)$ of non-zero trace, and B is an additive subgroup of $\mathrm{GF}(q)$. It is clear that the Denniston maximal arcs are of Mathon type.

Choose $b_0, b_1, b_2 \in \mathrm{GF}(2^h)$ with $b_2 \neq 0$ and $Tr(b_0) = 1$. Define the quadratic form $Q(\lambda) = Tr(b_1\lambda + b_2\lambda^3)$ on $\mathrm{GF}(2^h)^*$ considered as a projective space $\mathrm{PG}(h-1,2)$, and let A be a subspace of the associated quadric. Then the set

$$\{x^2 + xy + (b_0 + b_1\lambda + b_2\lambda^3)y^2 + \lambda z^2 = 0 : \lambda \in A\}$$

is a closed set of conics giving rise to a maximal arc of degree $|A|+1$ (see [5]).

In this paper we show that there are several other structures that arise from the functions associated with the maximal arcs of Mathon type. In Section 2, the existence of maximal arcs of Mathon type is shown to be equivalent to the existence of additive partial flocks of the quadratic cone in $\mathrm{PG}(3,q)$. In Section 3 they are shown to give partial ovoids of $Q^+(5,q)$ and partial spreads of lines of $\mathrm{PG}(3,q)$. In Section 4 they are shown to be equivalent to the existence of additive partial q-clans. In Section 5 they are shown to give rise to translation k-arcs in $\mathrm{PG}(2,q)$. In Section 6 they are shown to give m-ovoids of a certain classical generalised quadrangle. The paper is concluded by giving another construction of a partial flock from a maximal arc of Mathon type.

2 Partial flocks from maximal arcs

Let K be a quadratic cone in $\mathrm{PG}(3,q)$ with vertex x. A partial flock $\mathcal{F}$ of K is a set of disjoint (non-singular) conics on K. A partial flock is *complete* if it is not contained in a larger partial flock. A *flock* $\mathcal{F}$ of K is a partial flock of size q. If all the planes containing the elements of the partial flock contain a common line, then $\mathcal{F}$ is called *linear*. Partial flocks and their associated structures were studied in [16].

Suppose the cone K has equation $X_1X_3 = X_2^2$. The vertex $x = (1,0,0,0)$ does not belong to any plane of the partial flock $\mathcal{F}$. There are then k planes Π_t, $t \in \{1,\dots,k\}$, defining the conics of $\mathcal{F}$ and we may write their equations in the form

$$X_0 + f(t)X_1 + tX_2 + g(t)X_3 = 0, \qquad t \in B,$$

where B is some subset of $\mathrm{GF}(q)$, and f and g are functions from B to $\mathrm{GF}(q)$. The k conics $\Pi_t \cap K$ form a partial flock of K if and only if

$$Tr\left(\frac{(f(s)+f(t))(g(s)+g(t))}{(s+t)^2}\right) = 1, \text{ for every } s \neq t.$$

We now use the functions associated with a closed set of conics to give a partial flock. Suppose A is a subset of $\mathrm{GF}(q)$, $0 \notin A$, and p and q are functions from A to $\mathrm{GF}(q)$ that define a closed set of conics. Put $B = A \cup \{0\}$, define functions f and g on B by $f(0) = g(0) = 0$, and $f(t) = tp(t)$, $g(t) = tr(t)$ for $t \in A$. Now A, p and r define a closed set of conics and the closure property gives us that

$$\frac{sp(s)+tp(t)}{s+t} = p(s+t) \text{ and } \frac{sr(s)+tr(t)}{s+t} = r(s+t), \text{ for } s,t \in A, \text{ with } s \neq t. \quad (1)$$

Since $s+t \in A$ then the trace condition for the closed set of conics gives that

$$\begin{aligned} Tr(p(s+t)r(s+t)) &= Tr\left(\left(\frac{sp(s)+tp(t)}{s+t}\right)\left(\frac{sr(s)+tr(t)}{s+t}\right)\right) \\ &= Tr\left(\frac{(f(s)+f(t))(g(s)+g(t))}{(s+t)^2}\right) = 1. \end{aligned}$$

It follows immediately that f, g and B define a partial flock.

The equations (1) show that the functions f and g arising from a closed set of conics have the interesting property that they are additive on B, and also that B must be closed under addition. A partial flock with these properties will be called *additive*.

Conversely, suppose there are given an additive partial flock with functions f and g on an additive subgroup B of $\mathrm{GF}(q)$. We may define $A = B - \{0\}$, and functions $p(t) = f(t)/t$ and $r(t) = g(t)/t, t \in A$, and it is readily checked that these functions have the required trace and closure conditions on A to give a closed set of conics, and hence a maximal arc in $\mathrm{PG}(2,q)$. We have the following theorem.

Theorem 2
A maximal arc of degree n of Mathon type gives rise to an additive partial flock of size n of the quadratic cone in $\mathrm{PG}(3,q)$, and conversely.

Assume that $\mathcal{F}$ is a maximal arc of degree n of Mathon type and that the corresponding partial flock is linear. Calculating the line of intersection of pairs of planes of the partial flock and noting that it must be in the plane with equation $X_0 = 0$, it follows that $\mathcal{F}$ is a Denniston maximal arc. Conversely, if $\mathcal{F}$ is the Denniston maximal arc described in Section 1, then clearly the corresponding partial flock is linear.

When $B = \mathrm{GF}(q)$, additive flocks are known as *semifield flocks*. In [8], N.L. Johnson showed that when q is even every semifield flock is linear; clearly this is not the case when $B \neq \mathrm{GF}(q)$.

The additive partial flocks are particularly nice in that they admit a regular abelian group acting on them. Define a set of matrices $G_{\mathcal{F}}$ as follows:

$$G_{\mathcal{F}} = \{ \begin{pmatrix} 1 & f(t) & t & g(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in B\}.$$

Then it is easily verified that $G_{\mathcal{F}}$ is an abelian group that acts regularly on the planes of the flock and also fixes the quadratic cone K. This is in sharp contrast to the collineation stabilisers of the maximal arcs arising from closed sets of conics. In many cases these stabilisers are trivial.

3 Partial ovoids of $Q^+(5, q)$ and partial spreads of $\mathrm{PG}(3, q)$

In this section we show that the functions associated with a maximal arc of Mathon type give rise to (nice) partial ovoids of $\mathcal{Q}^+(5, q)$ and (nice) partial spreads of $\mathrm{PG}(3, q)$. We use the following construction of Walker [21] and Thas (unpublished) which was originally used to construct spreads of $\mathrm{PG}(3, q)$ from flocks, but works equally well to construct partial spreads from partial flocks.

Let $\pi_1, \pi_2, \ldots, \pi_k$ be the planes of a partial flock $\mathcal{F}$ of a quadratic cone K with vertex x in $\mathrm{PG}(3, q)$. Embed the cone K in the Klein quadric $Q^+(5, q)$, denote the polar plane of π_i with respect to $Q^+(5, q)$ by $\pi_i^\perp$, and let C_i be the conic $\pi_i^\perp \cap Q^+(5, q)$, for each i. Notice that each of the planes π_i is contained within $x^\perp$, and so each C_i contains x. Further for $i \neq j$, $C_i \cap C_j = \{x\}$, since the planes $\pi_i^\perp$ all meet in the common tangent line given by the perp of the $\mathrm{PG}(3, q)$ containing K. Now, since $\pi_i \cap \pi_j$ is an exterior line of $Q^+(5, q)$ for $i \neq j$, it follows that the span $\langle \pi_i^\perp, \pi_j^\perp \rangle$ meets $\mathcal{Q}^+(5, q)$ in an elliptic quadric of some three-dimensional space and so contains no lines of $Q^+(5, q)$. Hence the union of the conics $\mathcal{O} = C_1 \cup \cdots \cup C_k$ is a partial ovoid of $Q^+(5, q)$ of size $kq + 1$.

In the Klein correspondence (see [6]) the points of $Q^+(5, q)$ are in one to one correspondence with the lines of $\mathrm{PG}(3, q)$. Further, a partial ovoid of $Q^+(5, q)$ corresponds with a partial spread of lines of $\mathrm{PG}(3, q)$ (i.e. a set of pairwise skew lines in $\mathrm{PG}(3, q)$).

Suppose we have a maximal arc of degree n of Mathon type defined by functions p and r on some subset A of $\mathrm{GF}(q)$. Define $p(0) = r(0) = 0$. By Theorem 2 we get a partial flock, and hence a partial ovoid of $Q^+(5, q)$. If the Klein quadric $Q^+(5, q)$ has quadratic form $Q(x_0, x_1, x_2, x_3, x_4, x_5) = x_0x_5 +$

$x_1x_3 + x_2x_4$, then the partial ovoid is given by (cf. [2])

$$\{(u^2{+}ut{+}t^2p(t)r(t), tr(t), u, tp(t), u{+}t, 1) \,:\, u \in \mathrm{GF}(q), t \in A{\cup}\{0\}\}{\cup}\{(1,0,0,0,0,0)\}.$$

Further, the sets

$$\begin{aligned} L_\infty &= \{(c,d,0,0) : c,d \in \mathrm{GF}(q), (c,d) \neq (0,0)\} \\ L_{t,u} &= \{(au + btr(t), atp(t) + b(u+t), a, b) : a,b \in \mathrm{GF}(q), (a,b) \neq (0,0)\} \end{aligned}$$

for $t \in A \cup \{0\}$, $u \in \mathrm{GF}(q)$, form a partial spread $\mathcal{S}$ of lines of $\mathrm{PG}(3,q)$.

Gevaert and Johnson [2] also show that the partial spread admits the following abelian group in its collineation stabiliser:

$$R = \{ \begin{pmatrix} 1 & 0 & v & 0 \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : v \in \mathrm{GF}(q)\}.$$

The group R fixes L_∞, and each of its orbits together with L_∞ forms a regulus (family of generators of a hyperbolic quadric) in $\mathrm{PG}(3,q)$.

Using the additivity of the partial flock that arises from a maximal arc of Mathon type, it is readily verified that the following set E of matrices forms an abelian group that also stabilises $\mathcal{S}$:

$$E = \{ \begin{pmatrix} 1 & 0 & 0 & tr(t) \\ 0 & 1 & tp(t) & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in A \cup \{0\}\}.$$

It is easy to check that the elements of E and R commute and generate a group $G_{\mathcal{S}} = \langle E, R\rangle$ of order nq. In fact $G_{\mathcal{S}}$ acts regularly on the lines of $\mathcal{S} - \{L_\infty\}$. So the partial spreads arising from maximal arcs of Mathon type admit an abelian group of projectivities that fixes a single line of the partial spread and acts regularly on the rest of the lines. It also follows that the partial ovoids admit an abelian group of projectivities fixing $Q^+(5,q)$ that fixes one point of the ovoid and acts regularly on the remaining points.

Since we are assuming that q is even, if the partial flock is an (additive) flock, the partial ovoid is an ovoid which is a non-singular elliptic quadric in some three dimensional subspace of $\mathrm{PG}(5,q)$, and the partial spread is a regular spread [8].

4 Partial q-clans

Let $\mathcal{C} = \{A_t : t \in \mathrm{GF}(q)\}$ be a family of q distinct 2×2 matrices with entries in $\mathrm{GF}(q)$. Then $\mathcal{C}$ is called a *q-clan* if $A_u - A_r$ is anisotropic for any $u \neq r$, that is, if

$$(x \quad y)(A_u - A_r)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

has just the solution $x = y = 0$. Suppose that q is even, and let $\mathcal{C} = \{A_t : t \in \mathrm{GF}(q)\}$ have matrices of the form

$$A_t = \begin{pmatrix} a_t & b_t \\ 0 & c_t \end{pmatrix}.$$

Then $\mathcal{C}$ is a q-clan if and only if

$$Tr\left(\frac{(a_s+a_t)(c_s+c_t)}{(b_s+b_t)^2}\right)=1 \text{ and } b_s\neq b_t, \text{ for all } s,t\in \mathrm{GF}(q), \text{ with } s\neq t.$$

The connections between q-clans, flocks, hyperovals and generalised quadrangles have been much studied. See for instance [14].

Here we define a *partial q-clan*, q even, to be a set $\mathcal{C}=\{A_t : t\in B\}$ of distinct 2×2 matrices with entries in $\mathrm{GF}(q)$, $B\subseteq \mathrm{GF}(q)$, such that the difference of any distinct pair of matrices is anisotropic. We may assume that the matrices of $\mathcal{C}$ have the form

$$A_t=\begin{pmatrix} f(t) & t \\ 0 & g(t)\end{pmatrix} \tag{2}$$

where f and g are functions from B to $\mathrm{GF}(q)$. In this case the condition on the matrices is that

$$Tr\left(\frac{(f(s)+f(t))(g(s)+g(t))}{(s+t)^2}\right)=1 \text{ for all } s,t\in B, \text{ with } s\neq t.$$

It is then clear that if we have a maximal arc of Mathon type defined by functions p,r on a subset A of $\mathrm{GF}(q)$, then similarly to the flock case the set of matrices

$$\{\begin{pmatrix} tp(t) & t \\ 0 & tr(t)\end{pmatrix} : t\in A\}\cup\{\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}\}$$

is a partial q-clan. Notice that such a partial q-clan $\mathcal{C}$ has the property that it is closed under addition, that is, the sum of any two matrices in $\mathcal{C}$ is also in $\mathcal{C}$. Define a partial q-clan to be *additive* if its matrices are closed under addition. Then it is clear that given an additive partial q-clan with matrices of the form of (2), we can define $p(t)=f(t)/t$ and $r(t)=g(t)/t$ for $t\in B-\{0\}$ and get the following.

Theorem 3
A maximal arc of degree n of Mathon type gives rise to an additive partial q-clan of size n, and conversely.

An interesting transformation to get new q-clans from old is the following which was used in [14, Thm. 1] to construct herds of hyperovals. We apply it to partial q-clans. Suppose we have a partial q-clan of the form

$$A_t=\begin{pmatrix} af(t) & t \\ 0 & g(t)\end{pmatrix} \text{ for } t\in B \tag{3}$$

where a is some fixed element of $\mathrm{GF}(q)$, q even. Then for any fixed $s\in \mathrm{GF}(q)^*$ such that $1+s+as^2\neq 0$, and putting $b=a+1/s+1/s^2$, the set of matrices

$$A'_t=\begin{pmatrix} bf(t) & t \\ 0 & \frac{f(t)+as^2g(t)+st}{as^2+s+1}\end{pmatrix} \text{ for } t\in B \tag{4}$$

is also a partial q-clan.

Notice that for an additive partial q-clan the new partial q-clan is still additive, so we might wonder if new maximal arcs might be constructed from old using this transformation. If the maximal arc corresponding to the matrices in (3) has functions $p(t) = af(t)/t$, $r(t) = g(t)/t, t \neq 0$, then the functions corresponding to (4) are $p'(t) = (b/a)p(t)$, $r'(t) = (p(t)/a + as^2 r(t) + s)/(1+s+as^2)$. However, the maximal arcs defined by p, r and p', r' are isomorphic. Applying the transformation

$$x \mapsto x\left(\frac{as^2+s+1}{as^2}\right)^{\frac{1}{2}} + y\left(\frac{1}{a(1+s+as^2)}\right)^{\frac{1}{2}}, y \mapsto y\left(\frac{as^2}{1+s+as^2}\right)^{\frac{1}{2}}$$

and $z \mapsto z$ to the *equations* of the conics of the first maximal arc gives the equations of the conics of the second maximal arc.

5 Translation k-arcs in $\mathrm{PG}(2, q)$

In a projective plane of order q a *k-arc* is a set of k points, no three of which are collinear [7]. A k-arc is *complete* if it is not contained in a $(k+1)$-arc.

When q is even k is at most $q+2$ in which case the arc is known as an *hyperoval*. A *translation hyperoval* in $\mathrm{PG}(2, q)$, q even, is a hyperoval whose points (up to isomorphism) may be written in the form

$$\{(1, F(t), t) : t \in \mathrm{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

where F is an additive function on $\mathrm{GF}(q)$, i.e. $F(s+t) = F(s) + F(t)$ for $s, t \in \mathrm{GF}(q)$ (see [7]).

We may similarly define a *translation k-arc* in $\mathrm{PG}(2, q)$, q even, as one whose points (up to isomorphism) may be written in the form

$$\{(1, F(t), t) : t \in B \subseteq \mathrm{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

where F is an additive function on some additive subgroup B of $\mathrm{GF}(q)$. The significance of being a translation $(n+2)$-arc is that there is a group of translations of order n that acts regularly on the points of $\mathcal{K} - \{(0, 1, 0), (0, 0, 1)\}$.

In [16], L. Storme and J.A. Thas show that partial flocks give rise to k-arcs in the Desarguesian projective plane $\mathrm{PG}(2, q)$ of order q, q even. In particular the following was proved.

Theorem 4
[16, Thm. 2.1] The k planes $X_0 + f(t)X_1 + tX_2 + g(t)X_3 = 0, t \in B \subseteq \mathrm{GF}(q)$ with q even, define a partial flock $\mathcal{F}$ of the cone $K : X_1X_3 = X_2^2$ of $\mathrm{PG}(3, q)$ if and only if for all $(a_1, a_2) \in \mathrm{GF}(q)^2 - \{(0, 0)\}$, the set

$$K_{(a_1,a_2)} = \{(1, (a_1^2 f(t) + a_1a_2t + a_2^2 g(t))^{\frac{1}{2}}, t) : t \in B\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

is a $(k+2)$-arc of $\mathrm{PG}(2, q)$.

It then follows from Theorem 2 that a degree n maximal arc in $\mathrm{PG}(2, q)$ that is of Mathon type gives rise to $(n+2)$-arcs in $\mathrm{PG}(2, q)$. If $p, r : A \to \mathrm{GF}(q)$ are the functions defining the maximal arc, define $F_{(a_1,a_2)}(t) = (a_1^2 tp(t) + a_1a_2t + a_2^2 tr(t))^{\frac{1}{2}}$ for $t \in A$, and $F_{(a_1,a_2)}(0) = 0$. It is trivial to check that $F_{(a_1,a_2)}$ is additive on $A \cup \{0\}$ since the functions $f(t) = tp(t)$ and $g(t) = tr(t)$ are, and so the $(n+2)$-arcs are translation $(n+2)$-arcs.

Theorem 5
The conics with equations $p(\lambda)x^2 + xy + r(\lambda)y^2 + \lambda z^2 = 0, t \in A \subset \mathrm{GF}(q)$ *together with the point* F_0 *define a maximal arc of degree* n *of Mathon type if and only if for all* $(a_1, a_2) \in \mathrm{GF}(q)^2 - \{(0,0)\}$*, the set*

$$K_{(a_1,a_2)} = \{(1, F_{(a_1,a_2)}(t), t) : t \in A \cup \{0\}\} \cup \{(0,1,0),(0,0,1)\},$$

with $F_{(a_1,a_2)}(t)$ *as above, is a translation* $(n+2)$*-arc of* $\mathrm{PG}(2,q)$*.*

Proof. From the comments preceding the theorem the forward implication is clear. Conversely, if we choose $a_2 = 0$, $a_1 = 1$, then $F_{(a_1,a_2)}(t) = (a_1^2 tp(t) + a_1 a_2 t + a_2^2 tr(t))^{\frac{1}{2}} = (tp(t))^{\frac{1}{2}}$ being additive implies that the function $f(t) = tp(t)$ on $A \cup \{0\}$, with $f(0) = 0$, is additive on $A \cup \{0\}$; similarly for $g(t) = tr(t)$. Hence the partial flock defined by these two functions on the additive subgroup $A \cup \{0\}$ of $\mathrm{GF}(q)$ is additive, and so gives a maximal arc of Mathon type by Theorem 2. □

It is easily verified that the set of matrices

$$G_{\mathcal{K}} = \{ \begin{pmatrix} 1 & 0 & 0 \\ F_{(a_1,a_2)}(t) & 1 & 0 \\ t & 0 & 1 \end{pmatrix} : t \in B\}$$

forms an abelian group of translations stabilising the $(n+2)$-arc $K_{(a_1,a_2)}$.

We conclude this section by considering the completeness of the translation arcs arising from the maximal arcs. First consider the Denniston maximal arc given at the end of the introduction. This has $p(t) = 1$, $r(t) = \alpha$, where α is a fixed element of $\mathrm{GF}(q)$ of non-zero trace, for every $t \in A \cup \{0\}$. The k-arcs are then

$$K_{(a_1,a_2)} = \{(1, t^{\frac{1}{2}}(a_1^2 + a_1 a_2 + a_2^2 \alpha)^{\frac{1}{2}}, t) : t \in A \cup \{0\}\} \cup \{(0,1,0),(0,0,1)\}.$$

These point sets are each a subset of some regular hyperoval [7], and so are not complete except in the case when $A \cup \{0\} = \mathrm{GF}(q)$ in which case each $K_{(a_1,a_2)}$ is a regular hyperoval.

In general it is known that a complete arc has size greater than $\sqrt{2q} + 1$ in $\mathrm{PG}(2,q)$ [15]. At the time of writing the largest known degree of maximal arcs of Mathon type that are part of classes and are not of Denniston type are those of Hamilton and Mathon in [5]. These are of degree 2^{m+1} in both $\mathrm{PG}(2, 2^{2m})$ and $\mathrm{PG}(2, 2^{2m+1})$, $m \geq 2$. Hence for these maximal arcs it is possible though perhaps unlikely in general that the associated arcs are complete.

In the planes $\mathrm{PG}(2,16)$ and $\mathrm{PG}(2,32)$ the maximal arcs of Mathon type are classified by computer in [12]. We use the notation from that paper for these maximal arcs. In $\mathrm{PG}(2,16)$, the only non-Denniston maximal arc of Mathon type has degree 8 and is the dual of the Lunelli-Sce hyperoval. Choose $\omega \in \mathrm{GF}(16)$ such that $\omega^4 + \omega = 1$; then the functions $p(t) = 1$, $r(t) = \omega^{11} + \omega^{10} t + t^3$, for $t \in \langle 1, \omega, \omega^2 \rangle - \{0\}$ define this maximal arc. It is readily checked by computer that the associated 10-arcs in $\mathrm{PG}(2,16)$ are always complete, the only exception being when $a_2 = 0$ (in which case the arcs are each a subset of some regular hyperoval).

In $\mathrm{PG}(2,32)$, there are 3 non-Denniston maximal arcs of Mathon type of degree 8 and one of degree 16. Computer tests show that none of the degree 8 maximal arcs give complete arcs using the functions given in [12]. The degree 16 maximal arc is the dual of the Cherowitzo hyperoval. Choosing $\omega \in \mathrm{GF}(32)$ such that $\omega^{18} + \omega = 1$, then taking $p(t) = \omega^{25} + \omega^{16}t + \omega^{10}t^3 + \omega^{30}t^7$, $r(t) = \omega^{27} + \omega^5 t + \omega^{11}t^3 + \omega^3 t^7$ with $t \in \langle 1, \omega, \omega^7, \omega^9\rangle - \{0\}$ describes this maximal arc. Testing by computer shows that the associated 18-arcs are always complete.

Note that if any of the k-arcs arising from a partial flock is complete, then the partial flock must also be complete (i.e. is not contained in a larger partial flock). Hence the partial flock of size 8 in $\mathrm{PG}(3, 16)$ associated with the dual of the Lunelli-Sce hyperoval is complete, as is the partial flock of size 16 in $\mathrm{PG}(3, 32)$ associated with the dual of the Cherowitzo hyperoval.

6 m-ovoids of the generalised quadrangle $T_2^*(\mathcal{H})$

J.A. Thas [20] and later G. Lunardon [11] with more details, showed the connection between additive flocks of the quadratic cone and translation ovoids of $Q(4, q)$; see also [9, Section 3.6]. In this section we apply the same techniques to partial flocks to give m-ovoids of the generalised quadrangle $T_2^*(\mathcal{H})$, $\mathcal{H}$ a regular hyperoval. We refer to [13] for definitions and general background on generalised quadrangles. Recall that an *m-ovoid* of a generalised quadrangle is a subset of points $\mathcal{O}$ such that every line of the generalised quadrangle contains exactly m points of $\mathcal{O}$ [18].

Let $\mathcal{H}$ be a regular hyperoval in $\mathrm{PG}(2, q)$, i.e. $\mathcal{H}$ is the set of points of a non-singular conic with its nucleus, q even. Embed $\mathrm{PG}(2, q)$ in $\mathrm{PG}(3, q)$. Define an incidence structure $T_2^*(\mathcal{H})$ as follows (see [17]). The points of $T_2^*(\mathcal{H})$ are the points of $\mathrm{PG}(3, q) - \mathrm{PG}(2, q)$. The lines of $T_2^*(\mathcal{H})$ are the lines of $\mathrm{PG}(3, q)$ that meet $\mathrm{PG}(2, q)$ in a unique point of $\mathcal{H}$. Incidence is the incidence induced from the incidence of $\mathrm{PG}(3, q)$. Then $T_2^*(\mathcal{H})$ is a generalised quadrangle of order $(q - 1, q + 1)$.

To construct m-ovoids of $T_2^*(\mathcal{H})$ we start with an additive partial flock of size n of the quadratic cone K defined by functions f and g on $B \subseteq \mathrm{GF}(q)$ with notation as in Section 2.

We examine the dual space of $\mathrm{PG}(3, q)$ with respect to the standard inner product, i.e. a point (a, b, c, d) gets mapped to the plane with equation $aX_0 + bX_1 + cX_2 + dX_3 = 0$. The vertex x of the cone K then maps to the plane π with equation $X_0 = 0$. Recall that the nuclei of all the conics contained in K are on a line, l say, through the point x. The generators of K, together with l, map to the set of lines of a dual regular hyperoval $\mathcal{H}_d$ in π.

A plane π_t with equation $X_0 + f(t)X_1 + tX_2 + g(t)X_3 = 0$ of the partial flock maps to the point $(1, f(t), t, g(t))$. The line of intersection of a pair of distinct planes π_s, π_t of the partial flock is disjoint from K, and also from $K \cup \{l\}$, and so the line l_{st} joining the images of π_s and π_t in the dual space is disjoint from the point set covered by the dual regular hyperoval $\mathcal{H}_d$. The point $l_{st} \cap \pi$ is readily calculated to be $(0, f(s + t), s + t, g(s + t))$, using the additivity of the partial flock. Hence if we take the set of points $\mathcal{S} = \{(0, f(t), t, g(t)) : t \in B\}$ we get a set of points in π that is disjoint from the dual regular hyperoval.

Suppose that $q = 2^r$ and $n = 2^v$. Now "blow up" the affine space $\mathrm{AG}(3, q) = \mathrm{PG}(3, q) - \mathrm{PG}(2, q)$ over $\mathrm{GF}(2)$. Then $\mathrm{AG}(3, q)$ becomes an affine space $\mathrm{AG}(3r, 2)$, with π corresponds the hyperplane at infinity $\mathrm{PG}(3r - 1, 2)$ of $\mathrm{AG}(3r, 2)$, with the lines of $\mathcal{H}_d$ correspond $(2r - 1)$-dimensional subspaces of $\mathrm{PG}(3r-1,2)$, with the points $(1, f(t), t, g(t))$ correspond the points of an $\mathrm{AG}(v,2)$. The space at infinity $\mathrm{PG}(v-1,2)$ of $\mathrm{AG}(v,2)$ is disjoint from the $(2r-1)$-dimensional subspaces corresponding to the lines of $\mathcal{H}_d$. Take the dual in $\mathrm{PG}(3r-1,2)$ of these structures. The dual of the set of subspaces corresponding to the lines of $\mathcal{H}_d$ is a set $\bar{\mathcal{H}}$ of $q+2$ subspaces of dimension $r-1$. In fact it is just the set of subspaces that corresponds to a regular hyperoval in $\mathrm{PG}(2,q)$. The dual of $\mathrm{PG}(v-1,2)$ is a subspace $\mathrm{PG}(3r-v-1,2)$ of dimension $3r-v-1$. Since $\mathrm{PG}(v-1,2)$ was disjoint from each of the subspaces corresponding to the lines of $\mathcal{H}_d$, it follows immediately that $\mathrm{PG}(3r-v-1,2)$ meets each element of $\bar{\mathcal{H}}$ in a subspace of dimension $r-v-1$ over $\mathrm{GF}(2)$.

We construct a new incidence structure $T^*(\bar{\mathcal{H}})$ as follows. The *points* of $T^*(\bar{\mathcal{H}})$ will be the points of $\mathrm{PG}(3r,2)-\mathrm{PG}(3r-1,2)$, the *lines* will be r-dimensional subspaces of $\mathrm{PG}(3r,2)$ meeting $\mathrm{PG}(3r-1,2)$ in an element of $\bar{\mathcal{H}}$, incidence is containment. It is clear that $T^*(\bar{\mathcal{H}})$ is isomorphic to the $T_2^*(\mathcal{H})$ constructed above.

Let $\mathrm{PG}(3r-v,2)$ be a subspace of $\mathrm{PG}(3r,2)$ meeting $\mathrm{PG}(3r-1,2)$ in $\mathrm{PG}(3r-v-1,2)$, and let $\mathcal{O}$ be the points of $\mathrm{PG}(3r-v,2)-\mathrm{PG}(3r-v-1,2)$. Then dimensional arguments show that every r-dimensional subspace meeting $\mathrm{PG}(3r-1,2)$ in an element of $\bar{\mathcal{H}}$ must meet $\mathcal{O}$ in an affine space $\mathrm{AG}(r-v,2)$. Hence $\mathcal{O}$ is the set of points of an 2^{r-v}-ovoid of $T^*(\bar{\mathcal{H}})$, and we have the following theorem.

Theorem 6
A maximal arc of degree n of Mathon type in $\mathrm{PG}(2,q)$ gives rise to a q/n-ovoid of the generalised quadrangle $T_2^*(\mathcal{H})$, $\mathcal{H}$ a regular hyperoval in $\mathrm{PG}(2,q)$.

7 Another partial flock from a closed set of conics

Suppose we have a closed set of conics $\mathcal{C}$ of size $n-1$ in a $\mathrm{PG}(2,q)$, q even, with nucleus F_0. Embed $\mathrm{PG}(2,q)$ in $\mathrm{PG}(3,q)$, let x be a point of $\mathrm{PG}(3,q)-\mathrm{PG}(2,q)$, let C be a conic of $\mathcal{C}$, and let K be the quadratic cone with vertex x and base C. Choose a point y on the line joining x to $F_0, y \neq x, y \neq F_0$. Projection from the point y then gives a one to one correspondence between the points of K and $\mathrm{PG}(2,q)$. Recall that the line F_∞, given by the equation $z=0$, is disjoint from the closed set of conics. Hence we may project the set $\{F_\infty\} \cup \mathcal{C}$ onto K to get a partial flock of K of size n. However, this partial flock does not appear to have as many nice properties as that arising from the algebraic approach given in previous sections.

8 Closed sets of conics and embeddings of projective spaces

In the proof of [4, Thm. 8] it is shown that a closed set of conics of size $n-1$ defines a projective space over $\mathrm{GF}(2)$, the points being the $\mathcal{F}_{\alpha,\beta,\lambda}$ of the set, and the lines being the triples $\{\mathcal{F}_{\alpha,\beta,\lambda}, F_{\alpha',\beta',\lambda'}, \mathcal{F}_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'}\}$. Also note that

$$\frac{\lambda}{\lambda+\lambda'}(\alpha,\beta,\lambda)+\frac{\lambda'}{\lambda+\lambda'}(\alpha',\beta',\lambda')=(\alpha\oplus\alpha',\beta\oplus\beta',\lambda+\lambda').$$

Hence we get an embedding of $\mathrm{PG}((\log_2 n) - 1, 2)$ into $\mathrm{AG}(3, q)$, where lines of $\mathrm{PG}((\log_2 n) - 1, 2)$ are subsets of lines of $\mathrm{AG}(3, q)$. Projective embeddings were classified by M. Limbos in [10, Thm. 1.4] as follows.

Theorem 7
Let S_n be a projective space $\mathrm{PG}(n, q)$, $n \geq 3$, embedded in a space $\mathrm{PG}(j, q^r)$, $j < n$. Then in $\mathrm{PG}(n, q^r) \supset \mathrm{PG}(j, q^r)$ there is a subspace $\mathrm{PG}(n - j - 1, q^r)$ disjoint from $\mathrm{PG}(j, q^r)$ such that S_n may be described as the projection of a natural embedding of $\mathrm{PG}(n, q)$ in $\mathrm{PG}(n, q^r)$ from $\mathrm{PG}(n - j - 1, q^r)$ onto $\mathrm{PG}(j, q^r)$.

So the problem of finding closed sets of conics can be reduced to that of finding embeddings of $\mathrm{PG}((\log_2 n) - 1, 2)$ into $\mathrm{AG}(3, q)$ with certain extra properties.

References

[1] R.H.F. Denniston. Some maximal arcs in finite projective planes. *J. Combin. Theory*, 6:317–319, 1969.

[2] H. Gevaert and N.L. Johnson. Flocks of quadratic cones generalised quadrangles and translation planes. *Geom. Dedicata*, 27:301–317, 1988.

[3] N. Hamilton. Degree 8 maximal arcs in $PG(2, 2^h)$, h odd. *J. Combin. Theory Ser. A*, 100:265–276, 2002.

[4] N. Hamilton and R. Mathon. More maximal arcs in Desarguesian projective planes and their geometric structure. *Adv. Geom., to appear.*

[5] N. Hamilton and R. Mathon. On the spectrum on non-Denniston maximal arcs in $PG(2, 2^h)$. *Submitted to European J. Combin.*.

[6] J.W.P Hirschfeld. *Finite Projective Spaces of Three Dimensions.* Oxford University Press, Oxford, 1985.

[7] J.W.P Hirschfeld. *Projective Geometries over Finite Fields.* Oxford University Press, Oxford, 1998.

[8] N.L. Johnson. Semifield flocks of quadratic cones. *Bull. Belg. Math. Soc. Simon Stevin*, 61:313–326, 1987.

[9] M. Lavrauw. *Scattered subspaces with respect to spreads and eggs in finite projective spaces.* PhD Thesis, Technische Universiteit Eindhoven, 2001.

[10] M. Limbos. *Plongements et Arcs Projectifs.* PhD Thesis, U.L.B, Brussels, 1980-81.

[11] G. Lunardon. Flocks, ovoids of $Q(4, q)$ and designs. *Geom. Dedicata*, 66:163–173, 1997.

[12] R. Mathon. New maximal arcs in Desarguesian planes. *J. Combin. Theory Ser. A*, 97:353–368, 2002.

[13] S.E. Payne and J.A. Thas. *Finite Generalized Quadrangles. Research Notes in Mathematics,* 110. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[14] I. Pinneri, G.F. Royle, W. Cherowitzo, T. Penttila. Flocks and ovals. *Geom. Dedicata*, 60:17–37, 1996.

[15] B. Segre. Le geometrie di Galois. *Ann. Mat. Pura Appl.*, 48:1–97, 1959.

[16] L. Storme and J.A. Thas. k-arcs and partial flocks. *Linear Algebra Appl.*, 226/228:33–45, 1995.

[17] J.A. Thas. Some results concerning $\{(q+1)(n-1); n\}-$arcs and $\{(q+1)(n-1)+1; n\}$-arcs in finite projective planes of order q. *J. Combin. Theory Ser. A*, 19:228–232, 1974.

[18] J.A. Thas. Interesting pointsets in generalized quadrangles and partial geometries. *Linear Algebra Appl.*, 114/115:103–131, 1989.

[19] J.A. Thas. Generalised quadrangles of order (s, s^2), II. *J. Combin. Theory Ser. A*, 79:223–254, 1997.

[20] J. A. Thas. Symplectic spreads in $\mathrm{PG}(3, q)$, inversive planes and projective planes. Proc. *"Combinatorics '94", Discrete Math.*, 174:329–336, 1997.

[21] M. Walker. A class of translation planes. *Geom. Dedicata*, 5:135–146, 1976.

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