Two-intersection sets with respect to lines on the Klein quadric

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Abstract

We construct new examples of sets of points on the Klein quadric $Q^+(5,q)$, q even, having exactly two intersection sizes 0 and α with lines on $Q^+(5,q)$. By the well-known Plücker correspondence, these examples yield new $(0,\alpha)$-geometries embedded in $\mathrm{PG}(3,q)$, q even.

1 Preliminaries

A $(0,\alpha)$-*geometry* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a connected partial linear space of order (s,t) (i.e., every line is incident with $s+1$ points, while every point is incident with $t+1$ lines) such that for every anti-flag $\{p, L\}$ the number of lines through p and intersecting L is 0 or α. The concept of a $(0,\alpha)$-geometry, introduced by Debroey, De Clerck and Thas [5, 20], generalizes a lot of well-studied classes of geometries such as semipartial geometries [8], partial geometries [2] and generalized quadrangles [16].

A $(0,\alpha)$-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is fully embedded in $\mathrm{PG}(n,q)$ if $\mathcal{L}$ is a set of lines of $\mathrm{PG}(n,q)$ not contained in a proper subspace and $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n,q)$ on the lines of $\mathcal{S}$. In [20] the $(0,\alpha)$-geometries $(\alpha > 1)$ fully embedded in $\mathrm{PG}(n,q)$, $n > 3, q > 2$, are classified. For $\alpha = 1$ as well as for the $(0,\alpha)$-geometries with $q = 2$ a classification of the embeddings is out of reach as explained for instance in [6, 20]. As for $\mathrm{PG}(3,q)$, in [5] it is proven that if $\mathcal{S}$ is a $(0,\alpha)$-geometry $(\alpha > 1)$ fully embedded in $\mathrm{PG}(3,q)$, $q > 2$, then every planar pencil of $\mathrm{PG}(3,q)$ (i.e., the $q+1$ lines through a point in a plane) contains 0 or α lines of $\mathcal{S}$. Conversely one easily verifies

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that a set of lines of $\mathrm{PG}(3,q)$ which shares 0 or α ($\alpha > 1$) lines with every pencil of $\mathrm{PG}(3,q)$ yields a $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(3,q)$.

We can use the well-known Plücker correspondence, in order to see the set of lines of the $(0,\alpha)$-geometry as a set of points on the Klein quadric $Q^+(5,q)$.

For the remainder of the paper we will always assume that $\alpha > 1$ and $q > 2$, and we may conclude that the following objects are equivalent.

- A $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(3,q)$.
- A set of lines of $\mathrm{PG}(3,q)$ sharing 0 or α lines with every pencil of $\mathrm{PG}(3,q)$.
- A set of points on the Klein quadric $Q^+(5,q)$ sharing 0 or α points with every line on $Q^+(5,q)$. We call such a set a $(0,\alpha)$-*set* on $Q^+(5,q)$.

A *maximal arc of degree* α in $\mathrm{PG}(2,q)$ is a set of points such that every line of $\mathrm{PG}(2,q)$ intersects it in 0 or α points. Examples of maximal arcs in $\mathrm{PG}(2,2^h)$ were first constructed by Denniston [10]. Examples of maximal arcs in $\mathrm{PG}(2,2^h)$ which are not of Denniston type were constructed by Thas [18, 19] and by Mathon [15]. Ball, Blokhuis and Mazzocca [1] proved that maximal arcs of degree $1 < \alpha < q$ in $\mathrm{PG}(2,q)$ do not exist if q is odd.

Let $\mathcal{K}$ be a $(0,\alpha)$-set on $Q^+(5,q)$. Clearly every plane on $Q^+(5,q)$ is either disjoint from $\mathcal{K}$ or intersects $\mathcal{K}$ in a maximal arc of degree α. Consider the $(0,\alpha)$-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, fully embedded in $\mathrm{PG}(3,q)$, which corresponds to $\mathcal{K}$. Then every plane of $\mathrm{PG}(3,q)$ contains either no line of $\mathcal{S}$ or $q\alpha - q + \alpha$ lines of $\mathcal{S}$ which constitute a dual maximal arc of degree α. Similarly through every point p of $\mathrm{PG}(3,q)$ there are either 0 lines of $\mathcal{S}$ or $q\alpha - q + \alpha$ lines of $\mathcal{S}$ which intersect a plane not containing p in a maximal arc of degree α. Let π be a plane of $\mathrm{PG}(3,q)$ containing $q\alpha - q + \alpha$ lines of $\mathcal{S}$, and let d be such that π contains $q^2+q+1-d$ points of $\mathcal{S}$. Then counting the lines of $\mathcal{S}$ by their intersection with π we get that $|\mathcal{L}| = |\mathcal{K}| = (q\alpha - q + \alpha)(q^2 + 1 - d)$. We call d the *deficiency* of the $(0,\alpha)$-geometry $\mathcal{S}$ and of the $(0,\alpha)$-set $\mathcal{K}$.

In this paper we will give an overview of the known examples so far and we will give new examples, α being any proper divisor of q, q even.

2 The known examples

It is clear that the design of all points and lines of $\mathrm{PG}(3,q)$ is the only $(0,q+1)$-geometry fully embedded in $\mathrm{PG}(3,q)$.

On the other hand let $\mathcal{S}$ be a $(0,q)$-geometry fully embedded in $\mathrm{PG}(3,q)$ and let Π be the set of planes containing at least two lines of $\mathcal{S}$. Then for every plane $\pi \in \Pi$, the incidence structure of points and lines of $\mathcal{S}$ in π is a dual affine plane, while the incidence structure with point set the set of lines of $\mathcal{S}$ through a fixed point p of $\mathcal{S}$ and with line set the set of planes of Π through p is an affine plane. These geometries are classified, there are two non-isomorphic examples, see for instance [9, 13, 14]. Here we summarize the result in the terminology of a $(0,q)$-set on the Klein quadric $Q^+(5,q)$.

Theorem 2.1 *The points of $Q^+(5,q)$ not on a hyperplane U of $\mathrm{PG}(5,q)$, $q>2$, are the only $(0,q)$-sets on $Q^+(5,q)$. If U is a tangent hyperplane, then the deficiency is 1. If U is a secant hyperplane then the deficiency is 0.*

Remark

For the $(0,q)$-set of deficiency 1 the corresponding $(0,q)$-geometry in $\mathrm{PG}(3,q)$ is the well known dual net denoted by H_q^3. For the $(0,q)$-set of deficiency 0 the corresponding $(0,q)$-geometry in $\mathrm{PG}(3,q)$ is the semipartial geometry denoted by $\overline{W(3,q)}$. For a detailed description of both examples as $(0,q)$-geometries embedded in $\mathrm{PG}(3,q)$ we refer for instance to [6].

In [1] it is proved that in desarguesian planes of order q, q odd, maximal arcs of degree α, $1<\alpha<q$, do not exist. Hence we can conclude that if q is odd, no other $(0,\alpha)$-set, $\alpha>1$, on the Klein quadric $Q^+(5,q)$ exists. Hence, for other examples we may restrict ourselves to the case q even, $1<\alpha<q$.

Here is an other example. The points of $Q^+(5,q)$, q even, corresponding to the external lines of a nonsingular hyperbolic quadric in $\mathrm{PG}(3,q)$ form a $(0,\alpha)$-set on $Q^+(5,q)$ with $\alpha=q/2$ and deficiency $q+1$. The corresponding $(0,q/2)$-geometry is denoted by $\mathrm{NQ}^+(3,q)$.

It was conjectured in [5] that H_q^3, $\overline{W(3,q)}$ and $\mathrm{NQ}^+(3,q)$ are the only $(0,\alpha)$-geometries, with $\alpha>1$, fully embedded in $\mathrm{PG}(3,q)$, $q>2$. This conjecture is false as will be clear from the remainder of the paper.

A first counterexample has been given by Ebert, Metsch and Szőnyi [11]. A *k-cap* in $\mathrm{PG}(n,q)$ is a set of k points, no three on a line. It is called *maximal* if it is not contained in a larger cap. Quite some research has been done on caps in $\mathrm{PG}(5,q)$ that are contained in the Klein quadric $Q^+(5,q)$. Since the maximum size of a cap in $\mathrm{PG}(2,q)$ is $q+1$ if q is odd and $q+2$ if q is even, a cap in $Q^+(5,q)$ has size at most $(q+1)(q^2+1)$ if q is odd and at most $(q+2)(q^2+1)$ if q is even. Glynn [12] constructs a cap of size $(q+1)(q^2+1)$ in $Q^+(5,q)$ for any prime power q (see also [17]). Ebert, Metsch and Szőnyi construct caps of size $q^3+2q^2+1=(q+2)(q^2+1)-q-1$ in $Q^+(5,q)$ for q even. They show that a cap in $Q^+(5,q)$, q even, of size q^3+2q^2+1 is either maximal in $Q^+(5,q)$ and is then a $(0,2)$-set of deficiency 1 together with one extra point, or it is contained in a cap of size $(q+2)(q^2+1)$. One easily verifies that caps of size $(q+2)(q^2+1)$ in $Q^+(5,q)$, q even, and $(0,2)$-sets of deficiency 0 are equivalent. A cap of size $(q+2)(q^2+1)$ is only known to exist for $q=2$.

The construction of Ebert, Metsch and Szőnyi is as follows. Let Σ be a 3-space intersecting $Q^+(5,q)$ in a nonsingular elliptic quadric E. Let $L=\Sigma^\beta$ where β is the symplectic polarity associated with $Q^+(5,q)$. Then the line L is external to $Q^+(5,q)$. Consider an ovoid O in Σ which has the same set of tangent lines as E. Let $\mathcal{K}$ be the intersection of $Q^+(5,q)$ with the cone with vertex L and base $E\cup O$. Then $\mathcal{K}$ is a cap of size $(q+1)\,|\,O\setminus E\,|\,+q^2+1$ which is maximal in $Q^+(5,q)$ [11], and $\mathcal{K}\setminus(E\cap O)$ is a $(0,2)$-set in $Q^+(5,q)$ of deficiency $\,|\,E\cap O\,|\,$.

We have the following possibilities for O. The ovoid O can be an elliptic quadric. Then E and O intersect in either one point or $q+1$ points which form a conic in a plane of Σ (Types 1(i) and 3(g)(ii) in Table 2 of [3]). We will denote the corresponding $(0,2)$-set by $\mathcal{E}_1$ if $\,|\,E\cap O\,|\,=1$ and by $\mathcal{E}_{q+1}$ if $\,|\,E\cap O\,|\,=q+1$. On the other hand when q is an odd power of 2 the ovoid O can be a Suzuki-Tits ovoid. Then E and O intersect in $q\pm\sqrt{2q}+1$ points and both intersection sizes do occur [7]. We will denote the corresponding $(0,2)$-set by $\mathcal{T}_{q-\sqrt{2q}+1}$ if $\,|\,E\cap O\,|\,=q-\sqrt{2q}+1$ and by $\mathcal{T}_{q+\sqrt{2q}+1}$ if $\,|\,E\cap O\,|\,=q+\sqrt{2q}+1$.

3 Unions of elliptic quadrics

Consider a $(0,2)$-set $\mathcal{K}\in\{\mathcal{E}_1,\mathcal{E}_{q+1}\}$ in $Q^+(5,q)$, $q=2^h$. Let Π be a hyperplane containing Σ and let $p=\Pi\cap L$, where $L=\Sigma^\beta$. Then Π intersects $Q^+(5,q)$ in a nonsingular parabolic quadric $Q(4,q)$ with nucleus p. Since $\mathcal{K}$ is the intersection of $Q^+(5,q)$ with the cone with vertex L and base the symmetric difference $E\bigtriangleup O$ we find that $\mathcal{K}\cap\Pi$ is the intersection of $Q(4,q)$ with the cone with vertex p and base $E\bigtriangleup O$.

The projection of $Q(4,q)$ from p on Σ yields an isomorphism from the classical generalized quadrangle $Q(4,q)$ to the classical generalized quadrangle $W(q)$ consisting of the points of Σ and the lines of Σ that are tangent to E. This isomorphism induces a bijection from the set of ovoids of $Q(4,q)$ to the set of ovoids of $W(q)$. Since the ovoid O has the same set of tangent lines as E, it is an ovoid of the generalized quadrangle $W(q)$. Hence O is the projection from p on Σ of an ovoid $\overline{O}$ of $Q(4,q)$. So $\mathcal{K}\cap\Pi$ is the symmetric difference $E\bigtriangleup\overline{O}$. Since O is a nonsingular elliptic quadric in Σ, $\overline{O}$ is a nonsingular elliptic quadric in a 3-space $\overline{\Sigma}\subseteq\Pi$. Now Σ and $\overline{\Sigma}$ intersect in a plane $\overline{\pi}$ and we may also write $\mathcal{K}\cap\Pi=Q(4,q)\cap(\Sigma\cup\overline{\Sigma})\setminus\overline{\pi}$.

From the definition of $\mathcal{E}_1$ and $\mathcal{E}_{q+1}$ it follows that there is exactly one plane $\pi \subseteq \Sigma$ such that $\pi \cap Q(4,q) = E \cap O$. Indeed, if $\mathcal{K} = \mathcal{E}_1$ then E and O intersect in exactly one point and π is the unique tangent plane in Σ to E at this point. If $\mathcal{K} = \mathcal{E}_{q+1}$ then E and O intersect in a nondegenerate conic and π is the ambient plane of this conic. We prove that $\overline{\pi} = \pi$. Since O is the projection of $\overline{O}$ from p on Σ, $E \cap \overline{O} = E \cap O$. Since $\overline{O} = \overline{\Sigma} \cap Q(4,q)$, $\overline{\pi} \cap Q(4,q) = \Sigma \cap \overline{\Sigma} \cap Q(4,q) = \Sigma \cap \overline{O} = E \cap \overline{O} = E \cap O$. So $\overline{\pi}$ is a plane in Σ such that $\overline{\pi} \cap Q(4,q) = E \cap O$. This means that $\overline{\pi} = \pi$.

So $\mathcal{K} \cap \Pi$ is the symmetric difference of elliptic quadrics E and $\overline{O}$ on $Q(4,q)$ with ambient 3-spaces Σ and $\overline{\Sigma}$ intersecting in the plane π. Since this holds for all hyperplanes Π containing Σ we conclude that there exist 3-spaces $\Sigma_0 = \Sigma, \Sigma_1, \ldots, \Sigma_{q+1}$ mutually intersecting in the plane π, such that each intersects $Q^+(5,q)$ in an elliptic quadric and such that

$$\mathcal{K} = Q^+(5,q) \cap (\Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_{q+1}) \setminus \pi.$$

What remains to be verified is the position of the 3-spaces Σ_i. Consider a plane π' spanned by L and a point $r \in O \setminus E$. One verifies in the respective cases $\mathcal{K} = \mathcal{E}_1$ and $\mathcal{K} = \mathcal{E}_{q+1}$ that $\pi \cap O = \pi \cap E = E \cap O$, so $r \notin \pi$. Hence π' is skew to π. We determine the points of intersection of Σ_i, $i = 0, \ldots, q+1$, with π'. Clearly $\Sigma_0 \cap \pi' = \Sigma \cap \pi' = r$. Let $i \in \{1, \ldots, q+1\}$ and let $p_i \in L$ be such that $\Sigma_i \subseteq \langle p_i, \Sigma \rangle$. Let r_i be the unique point of $Q^+(5,q)$ on the line $\langle p_i, r \rangle$. Since $r \in O \setminus E$, r_i is a point of $\mathcal{K}$ and hence of Σ_i. But also $r_i \in \pi'$, so $\Sigma_i \cap \pi' = r_i$. Repeating this reasoning for all points p_i on L we see that the 3-spaces Σ_i, $i = 1, \ldots, q+1$, intersect π' in the points of the nondegenerate conic $C' = \pi' \cap Q^+(5,q)$ and that Σ intersects π' in the point r which is the nucleus of the conic C'. We have now proven the following theorem which completely determines the structure of the $(0,2)$-sets $\mathcal{E}_1$ and $\mathcal{E}_{q+1}$.

Theorem 3.1 *Let $\mathcal{K} \in \{\mathcal{E}_1, \mathcal{E}_{q+1}\}$ and let π be the unique plane in Σ such that $\pi \cap Q^+(5,q) = E \cap O$. Then*

$$\mathcal{K} = (E \cup O_1 \cup \ldots \cup O_{q+1}) \setminus \pi,$$

where O_i, $1 \leq i \leq q+1$, is a nonsingular 3-dimensional elliptic quadric on $Q^+(5,q)$ such that its ambient space Σ_i intersects Σ in the plane π. In particular the 3-spaces $\Sigma_1, \ldots, \Sigma_{q+1}$ intersect each plane $\pi' = \langle r, L \rangle$ with $L = \Sigma^\beta$ and $r \in O \setminus E$ in the points of the nondegenerate conic $C' = \pi' \cap Q^+(5,q)$, while Σ intersects π' in the nucleus r of the conic C'.

Remark

We can apply the same reasoning to the $(0,2)$-sets $\mathcal{T}_{q\pm\sqrt{2q}+1}$. We find then that $\mathcal{T}_{q\pm\sqrt{2q}+1}$ can be written as

$$(E \cup O_1 \cup \ldots \cup O_{q+1}) \setminus (E \cap O),$$

where $O_1, \ldots, O_{q+1}$ are Suzuki-Tits ovoids in the hyperplanes containing Σ, such that for every $p_i \in L = \Sigma^\beta$, there is exactly one $O_i \subseteq \langle p_i, \Sigma \rangle$, and then O is the projection of O_i from p_i on Σ. However this was already known [4].

4 A new construction

The following construction is inspired by Theorem 3.1. Let π be a plane of $\mathrm{PG}(5, q)$, $q = 2^h$, which does not contain any line of $Q^+(5, q)$ and let π' be a plane skew to π. Let $\mathcal{D}$ denote the set of points $p \in \pi'$ such that $\langle p, \pi \rangle$ intersects $Q^+(5, q)$ in a nonsingular elliptic quadric, and suppose that A is a maximal arc of degree α in π' such that $A \subseteq \mathcal{D}$. Then we define the set $\mathcal{M}^\alpha(A)$ to be the intersection of $Q^+(5, q)$ with the cone with vertex π and base A, minus the points of $Q^+(5, q)$ in π.

Theorem 4.1 *The set $\mathcal{M}^\alpha(A)$ is a $(0, \alpha)$-set on $Q^+(5, q)$.*

Proof. Let L be a line on $Q^+(5, q)$ which intersects the plane π. Then the subspace $\Sigma = \langle L, \pi \rangle$ has dimension 3 and it contains a line of $Q^+(5, q)$. Hence $\Sigma \cap \pi' \notin A$. So there are no points of $\mathcal{M}^\alpha(A)$ in Σ and hence also none on L.

Let L be a line on $Q^+(5, q)$ which is skew to π. A point p on L is in $\mathcal{M}^\alpha(A)$ if and only if $\langle p, \pi \rangle \cap \pi' \in A$ if and only if the projection of p from π on π' is a point of A. So if L' is the projection of L from π on π' then $|\, L \cap \mathcal{M}^\alpha(A)\,| = |\, L' \cap A\,| \in \{0, \alpha\}$. So every line on $Q^+(5, q)$ intersects $\mathcal{M}^\alpha(A)$ in 0 or α points. $\square$

Since the plane π does not contain any line of $Q^+(5, q)$, there are two possibilities: either $\pi \cap Q^+(5, q)$ is a single point or it is a nondegenerate conic. In the former case the $(0, \alpha)$-set has deficiency 1 and it is denoted by $\mathcal{M}_1^\alpha(A)$. In the latter case the $(0, \alpha)$-set has deficiency $q+1$ and it is denoted by $\mathcal{M}_{q+1}^\alpha(A)$.

In order to prove that there do exist $(0, \alpha)$-sets of deficiency 1 and $q+1$ for every $\alpha \in \{2, 2^2, \ldots, 2^{h-1} = q/2\}$ we must show that the set $\mathcal{D}$ in the plane π' contains a maximal arc of degree α for every $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$, and this for both the case where $\pi \cap Q^+(5, q)$ is a single point and the case where $\pi \cap Q^+(5, q)$ is a nondegenerate conic.

If $\pi \cap Q^+(5, q)$ is a single point p then $\mathcal{D}$ is the set of points of π' which are not on the line $\pi' \cap T_p$, where T_p is the tangent hyperplane to $Q^+(5, q)$ at p. Clearly in this case the set $\mathcal{D}$ contains a maximal arc of degree α for every $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$.

If $\pi \cap Q^+(5,q)$ is a nondegenerate conic then the plane π^β also intersects $Q^+(5,q)$ in a nondegenerate conic C. Furthermore β induces an anti-automorphism between the projective plane π^β and the projective plane having as points the 3-spaces through π and as lines the hyperplanes through π. This anti-automorphism is such that a 3-space containing π intersects $Q^+(5,q)$ in a nonsingular elliptic quadric if and only if the corresponding line of π^β is external to the conic C. Hence the set $\mathcal{D}$ in the plane π' is the dual of the set of external lines to a nondegenerate conic. It follows that $\mathcal{D}$ is a Denniston type maximal arc [10] of degree $q/2$, and hence that $\mathcal{D}$ contains a maximal arc of degree α for every $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$. We have proven the following theorem.

Theorem 4.2 *There exist* $(0,\alpha)$*-sets on* $Q^+(5,q)$*,* $q = 2^h$*, of deficiency* 1 *and* $q+1$ *for all* $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$*.*

Corollary 4.3 *There exist* $(0,\alpha)$*-geometries fully embedded in* $\mathrm{PG}(3,q)$*,* $q = 2^h$*, of deficiency* 1 *and* $q+1$ *for all* $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$*.*

By Theorem 3.1 the $(0,2)$-set $\mathcal{E}_d$, $d = 1, q+1$, is of the form $\mathcal{M}_d^2(H)$ with H a regular hyperoval. Let $\mathcal{K}$ be the $(0,q/2)$-set corresponding to the $(0,q/2)$-geometry $\mathrm{NQ}^+(3,q)$, q even. Then $\mathcal{K}$ corresponds to the set of external lines to a nonsingular hyperbolic quadric $Q^+(3,q)$ in $\mathrm{PG}(3,q)$. Let C be the set of points of $Q^+(5,q)$ corresponding to one of the two reguli of lines contained in $Q^+(3,q)$. Then C is a nondegenerate conic in a plane π, and $\mathcal{K}$ is the set of all points of $Q^+(5,q)$ which are not collinear in $Q^+(5,q)$ with any of the points of C. So a point p of $Q^+(5,q)$ is in $\mathcal{K}$ if and only if $p \notin \pi$ and $\langle p, \pi \rangle$ intersects $Q^+(5,q)$ in a nondegenerate elliptic quadric. Hence $\mathrm{NQ}^+(3,q)$ corresponds to the $(0,q/2)$-set $\mathcal{M}_{q+1}^{q/2}(\mathcal{D})$.

We conclude this paper with a list of all the known distinct examples of $(0,\alpha)$-sets $\mathcal{K}$ in $Q^+(5,q)$, $\alpha > 1$, $q > 2$. In this list d is the deficiency of the $(0,\alpha)$-set $\mathcal{K}$.

- $\alpha = q+1$, $d = 0$, and $\mathcal{K}$ is the set of all points of $Q^+(5,q)$.
- $\alpha = q$, $d = 0$, and $\mathcal{K}$ corresponds to $\overline{W(3,q)}$.
- $\alpha = q$, $d = 1$, and $\mathcal{K}$ corresponds to H_q^3.
- $q = 2^h$, $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$, $d \in \{1, q+1\}$ and $\mathcal{K} = \mathcal{M}_d^\alpha(A)$.
- $q = 2^{2h+1}$, $\alpha = 2$, $d = q \pm \sqrt{2q} + 1$, and $\mathcal{K} = \mathcal{T}_{q \pm \sqrt{2q}+1}$.

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