A group theoretic approach to $(0, 2)$-geometries

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Abstract. A $(0, 2)$-geometry is a geometry in which for any non-incident point-line pair (x, L) there are exactly 0 or 2 lines incident with x and concurrent with L. In this paper we use the special properties of a $(0, 2)$-geometry to define groups of projectivities and Moufang-like conditions, in a similar way as is done for generalized polygons. These definitions are explored and some partial classification results obtained.

2000 Mathematics Subject Classification: 51E14, 51E26,

1. Introduction

There are two ways in which one can define a group related to a generalized polygon. Either one looks at the *automorphism group* (sometimes called the *collineation group*), or one considers the group of *projectivities*. The latter is the group of permutations of all points on a line arising from the bijections between the point sets of two opposite lines given by a pair of points being not opposite. The existence and special properties of the relation "being opposite" is essential in this context. Consequently, the notion of "group of projectivities" has only been considered for generalized polygons (as a generalization of this notion for projective planes; more generally, one can consider spherical or twin buildings, but there has been very little, to our knowledge, done in this direction in the literature). In the present paper, we observe that $(0, 2)$-geometries also have a special geometric property that enables one to define a group of projectivities in a very natural way. It also allows us to characterize some classical nets by means of that group.

Moreover, the geometry of $(0, 2)$-geometries permits us to define Moufang-like conditions. We introduce these conditions, develop some theory, and prove some characterization theorems.

[*]The first author acknowledges the support of the Australian Research Council.

[‡]Research Fellow supported by the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), grant no. IWT/SB/13367/Tonesi

[**]Partly supported by a Research Grant of the Fund for Scientific Research - Flanders (FWO - Vlaanderen)

It is a valuable exercise to compare the notions introduced in the present paper with the existing notions for generalized polygons; we therefore refer to [12, 13]. Note that generalized polygons were introduced by Jacques Tits [11] and are arguably the most important rank 2 incidence geometries, see [6]. Almost all other rank 2 geometries are modelled after the generalized polygons by weakening some axioms. The generalized polygons earn their status partly because of the properties of their automorphism groups and groups of projectivities. In this respect, the present paper shows that $(0,2)$-geometries are also fundamental. Unfortunately, no complete classification theorem has yet been proved, however many partial results are available.

2. Definitions and Preliminary Results

$(0,2)$-geometries.

A $(0,2)$-geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consists of a *point set* $\mathcal{P}$, a *line set* $\mathcal{L}$ and a symmetric incidence relation $\mathtt{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$, satisfying the axioms (ZT1), (ZT2), (ZT3) and (ZT4) below. Before stating these, we give some standard terminology of incidence geometries.

If $x\mathtt{I}L$, with $x \in \mathcal{P}$ and $L \in \mathcal{L}$, then we say that L *contains* x, or L *goes through* x, or x *is contained in* L, or x *is on* L. When $x\mathtt{I}L\mathtt{I}y\mathtt{I}M$, with $x, y \in \mathcal{P}$, $L, M \in \mathcal{L}$, $x \neq y$ and $L \neq M$, then we say that L *and* M *intersect in* y, that L *and* M *are concurrent*, that x *and* y *are joined by* L, that L *joins* x *and* y, or that x *and* y *are collinear*, and we denote this by $x \sim y$ and $L \sim M$. We will sometimes write L as xy.

The *incidence graph* of Γ is the graph $(\mathcal{P} \cup \mathcal{L}, \mathtt{I})$, while the *point graph* of Γ is the graph $(\mathcal{P}, \sim)$.

(ZT1) Every line contains at least two points and every point is contained in at least two lines.

(ZT2) Two lines intersect in at most one point.

(ZT3) For every line L and every point x not incident with L, there are either exactly two lines incident with x and concurrent with L, or no line is incident with x and concurrent with L.

(ZT4) The incidence graph of Γ is a connected graph.

The *dual* of Γ is the geometry $\Gamma^{\mathrm{dual}} = (\mathcal{L}, \mathcal{P}, \mathtt{I})$, obtained from Γ by interchanging the point set with the line set.

It is easy to see that the dual of a $(0,2)$-geometry is again a $(0,2)$-geometry. Hence everything we say or prove about $(0,2)$-geometries has a dual meaning, which we often do not state explicitly.

Also note that Axiom (ZT4) is equivalent to the point graph of Γ being connected. For a point or line α, and a natural number i, we denote by $\Gamma_i(\alpha)$ the set of vertices of the graph $(\mathcal{P} \cup \mathcal{L}, \mathtt{I})$ at distance i from α.

A *semipartial* $(0, 2)$-geometry (often a semipartial geometry with $\alpha = 2$ in the literature) is a $(0, 2)$-geometry the collinearity graph of which is a strongly regular graph. In other words, this is a $(0, 2)$-geometry where two noncollinear points are collinear to a constant number μ of points. A semipartial $(0, 2)$-geometry in which for every non-incident, point-line pair (x, L) there are exactly 2 lines incident with x and concurrent with L is called a *partial* $(0, 2)$-geometry (often a partial geometry with $\alpha = 2$ in the literature). A semipartial geometry that is not a partial geometry is called *proper*. Finally, a partial $(0, 2)$-geometry with $t = 2$ is known as a *(Bruck) net* with order $s + 1$ and degree 3. (See [6] for more details on (semi)partial geometries, including nets.) Such a Bruck net gives rise to Latin squares and loops (see [2], for instance). In order to avoid confusion between the order $s + 1$ of a Bruck net and the order (s, t) of it as a $(0, 2)$-geometry, we will usually use the more systematic terminology of $(0, 2)$-geometries.

Perspectivities, projectivities and their duals.

Now let L and M be concurrent lines of a $(0, 2)$-geometry Γ intersecting in the point x. Axiom (ZT3) implies that for every point $y \neq x$ on L there is a unique point $y^{\pi_{L,M}} \neq x$ on M collinear with y. Hence if we define the mapping $\pi_{L,M} : \Gamma_1(L) \to \Gamma_1(M)$, with $x^{\pi_{L,M}} = x$, then $\pi_{L,M}$ is a bijection with inverse $\pi_{M,L}$. Such a bijection is called a *perspectivity*, and the composition of two or more perspectivities $\pi_{L_1,L_2}\pi_{L_2,L_3} \ldots \pi_{L_{n-1},L_n}$, for lines $L_1, L_2, \ldots, L_n$, with $L_1 \sim L_2 \sim \ldots \sim L_n$, is called a *projectivity*. For $L_1 = L_n$, the projectivity is called a *self-projectivity of* L_1 and the set of all self-projectivities of a line L forms a permutation group under the usual composition, called the *group of projectivities* of Γ, and denoted $\Pi(\Gamma)$, since it is clearly independent of the choice of L by a standard argument using connectivity of the point graph (see [13] for instance). As usual, one can also restrict to the self-projectivities that are composed of an even number of perspectivities. We thus obtain the *special group of projectivities* of the line L and denote this group by $\Pi^+(\Gamma)$. It is a subgroup of index 1 or 2 of $\Pi(\Gamma)$. If we consider perspectivities between lines of Γ containing a fixed point x, then, for $L\mathtt{I}x$, we obtain the *restricted group of projectivities of* L *relative to* x, and denote this by $\Pi_x(\Gamma)$. We can also consider the *special restricted* group $\Pi_x^+(\Gamma)$ of all elements of $\Pi_x(\Gamma)$ that are the composition of an even number of perspectivities between lines containing x (and so not necessarily equal to $\Pi_x(\Gamma) \cap \Pi^+(\Gamma)$). Notice that both $\Pi_x(\Gamma)$ and $\Pi_x^+(\Gamma)$ in general depend on the choice of x.

All previous notions may be dualized; then we speak of *dual perspectivities* and of the *(special) (restricted) group of dual projectivities*.

We remark that, since perspectivities are bijections, and since every $(0, 2)$-geometry is connected by definition, all lines are incident with the same number of points. We denote that constant by $1 + s$. likewise, all points are incident with constant number $1+t$ of lines. We say that the pair (s, t) is the *order* of Γ. Neither

s nor t needs to be finite, but for finite Γ they both are. If $s = 1$, then Γ is a complete graph.

Collineations.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a $(0,2)$-geometry. A permutation $\theta : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ that induces a graph automorphism in the incidence graph $(\mathcal{P} \cup \mathcal{L}, \mathtt{I})$ will be called a *correlation of* Γ. If the correlation θ maps at least one point to a point, than it is a *collineation.* The group of all collineations of Γ will be denoted by $\mathrm{Aut}\Gamma$. Again, this group may be viewed as a permutation group, either on $\mathcal{P}$, or on $\mathcal{L}$, or on $\mathcal{P} \cup \mathcal{L}$.

If θ is a collineation of Γ fixing all points on some line L, then we call L *an axis* of θ. Dually, one defines a *center* of θ.

The following lemma will be responsible for the existence of a rather natural notion of "elation" in $(0,2)$-geometries.

Lemma 2.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a $(0,2)$-geometry, and let θ be a collineation with some axis L and some center x. If $x\mathtt{I}L$ or $x \in \Gamma_3(L)$, then θ is the identity. If $x \in \Gamma_5(L)$, then θ has order at most 2.*

Proof. First suppose that $x\mathtt{I}L$. Consider an arbitrary point y collinear with x. We claim that y is fixed under θ. Indeed, this is trivial if $y\mathtt{I}L$. Otherwise, there is a unique point $z\mathtt{I}L$, $z \neq x$, collinear with y. Since y is the unique point on xy collinear with z and different from x, the claim follows. Now, every line M through y meets a unique fixed line through x different from xy; hence y is a center for θ. By connectivity, every point of Γ is a center, and so θ is trivial.

Now suppose $x \in \Gamma_3(L)$ and let M be a line through x meeting L in, say, the point y. The line M is fixed and since every point on M is collinear with a unique point of L different from y, we deduce that M is an axis. The assertion now follows from the first paragraph.

Finally, suppose $x \in \Gamma_5(L)$, let $y \in \Gamma_2(x) \cap \Gamma_3(L)$, and let $z \in \Gamma_2(y) \cap \Gamma_1(L)$. There are exactly two points y, y' on xy collinear with z. Hence θ^2 fixes both y and y'. As above, this implies that y is a center of θ^2. The second paragraph of our proof now shows that θ^2 is the identity. $\square$

Remark 2.2. *In general, using similar arguments as above, one can prove that, if $x \in \Gamma_{2n+1}(L)$, and if x is a center of θ and L an axis, then the order of θ is a divisor of 2^{n-1} ($n \geq 1$).*

We call a line L of the $(0,2)$-geometry Γ an *axis of transitivity* if, for some point $x\mathtt{I}L$, the group of collineations $G_{[L]}$ with axis L acts transitively on $\Gamma_1(x) \setminus \{L\}$. Dually, one defines a *center of transitivity.* An axis of transitivity is called an *elation line* (dually, *elation point*) if, for some point $x\mathtt{I}L$, there is a group $E_{[L]}$ of collineations with axis L acting regularly on $\Gamma_1(x) \setminus \{L\}$. Finally, an elation line is a *Moufang line* (dually, a *Moufang point*) if, for some point $x\mathtt{I}L$, there is a group U_L of collineations with axis L acting regularly on $\Gamma_1(x)\setminus\{L\}$ and such that U_L is normal in $(\mathrm{Aut}\Gamma)_L$. If every point and line are Moufang, then Γ is called a *Moufang* $(0,2)$-geometry.

A *Moufang set* $\mathcal{M} = (X, G; U_x : x \in X)$ consists of a set X, a permutation group G acting faithfully on X, and for each $x \in X$ a subgroup U_x of the stabilizer G_x of x in G such that

- each U_x is a normal subgroup of G_x and acts regularly on $X \setminus \{x\}$;
- the family $\mathcal{U} := \{U_x : x \in X\}$ is a conjugacy class of subgroups in G;
- the group G is generated by $\mathcal{U}$.

The following lemma is straightforward.

Lemma 2.3. *Let x and y be two collinear Moufang points of the $(0,2)$-geometry Γ, with corresponding groups U_x and U_y. If $G := \langle U_x, U_y\rangle$, then $\mathcal{M}_{xy} := (\Gamma_1(xy), G; U_x^G)$ is a Moufang set.*

A $(0,2)$ geometry in which all points are centers of transitivity will be called a $(0,2)$-*geometry with central transitivity.* Dually, one has the notion of $(0,2)$-*geometries with axial transitivity.*

Lemma 2.4. *If a $(0,2)$-geometry Γ has two centers of transitivity, x and y and two axes of transitivity, L and M, such that $x\mathtt{I}L\mathtt{I}y\mathtt{I}M$, then Γ is a $(0,2)$-geometry with both axial and central transitivity.*

Proof. If Γ or its dual is a complete graph the result follows, so we suppose that each point and each line is incident with at least three elements.

We first claim that we can map y to any point collinear with y, and M to any line through y. Using $G_{[x]}$ and $G_{[y]}$ we see that all points of L are in the orbit of y. Using $G_{[L]}$ and $G_{[M]}$, the claim follows.

Now let z be an arbitrary point, and let i be such that $z \in \Gamma_i(y)$ (i exists by connectivity). Let $y' \in \Gamma_{i-2}(z) \cap \Gamma_2(y)$. By the previous claim, there is a collineation θ mapping y' to y; hence $z^\theta \in \Gamma_{i-2}(y)$. An inductive argument on i now shows that z is a center of transitivity.

The dual argument completes the proof of the lemma. $\square$

3. Examples

In this section we gather some examples of $(0,2)$-geometries with emphasis on the caes with centers and/or axes of transitivity, elation points and/or elation lines, and Moufang points and/or Moufang lines.

3.1. Some examples with $s = 3$.

Consider a projective space $\mathrm{PG}(n, 2)$, with $n \geq 3$. Let $H = \mathrm{PG}(n-2, 2)$ be a fixed subspace of dimension $n-2$. Then the geometry H_2^{n*} has as point set the set of lines of $\mathrm{PG}(n, 2)$ which do not intersect H, and as line set the set of planes of $\mathrm{PG}(n, 2)$ which intersect H in exactly one point (and with natural incidence relation) is a $(0, 2)$-geometry (see [4] for more details). This construction can be generalized to projective spaces of arbitrary order giving a semipartial geometry H_q^{n*} with $\alpha = q$.

3.2. Some linear representations.

Let $\mathcal{S}$ be a set of points of the projective space $\mathrm{PG}(d, q)$, $d > 0$ and q any prime power, with the property that any line of $\mathrm{PG}(d, q)$ meets $\mathcal{S}$ in either $0, 1$ or 3 points. We also assume that $\mathcal{S}$ spans $\mathrm{PG}(d, q)$ linearly. Now embed $\mathrm{PG}(d, q)$ as a hyperplane π_∞ in $\mathrm{PG}(d+1, q)$ and let the point set of a geometry $T_d^*(\mathcal{S})$ (standard in the literature) be all points of $\mathrm{PG}(d+1, q)$ that do not belong to $\mathrm{PG}(d, q)$. A line of $T_d^*(\mathcal{S})$ is a line of $\mathrm{PG}(d+1, q)$ that intersects π_∞ in a unique point belonging to $\mathcal{S}$. Then $T_d^*(\mathcal{S})$ is usually called the *linear representation geometry* of $\mathcal{S}$. It is easy to see that if $T_d^*(\mathcal{S})$ is connected, then it is indeed a $(0, 2)$-geometry (actually, with diameter at least $2d+2$). All points are Moufang points, as is easily verified. Obviously, each line of $T_d^*(\mathcal{S})$ is an axis of transitivity if the collineations of $\mathrm{PG}(d, q)$ leaving $\mathcal{S}$ invariant acts 2-transitively on $\mathcal{S}$. In the special case where $q = 1$ and hence $|\mathcal{S}| = 3$ for the geometry to be connected, the geometry $T_1^*(\mathcal{S})$ is a net of order q and degree 3. This net has the property that every line is a Moufang line. In this case we will also denote $T_2^*(\mathcal{S})$ by $\Gamma_{1,q}$.

There are a lot of sets $\mathcal{S}$ known, but we mention two special immediate cases. One case is when $\mathcal{S}$ is the point set of a projective subspace $\mathrm{PG}(d, 2)$ arising from $\mathrm{PG}(d, q)$, q even, by restricting coordinates from $\mathrm{GF}(q)$ down to $\mathrm{GF}(2)$. Another special case arises for $q = 3^e$, $e > 1$, with $\mathcal{S}$ the point set of an affine subspace of $\mathrm{PG}(d, q)$ isomorphic to $\mathrm{AG}(d, 3)$ by first deleting a hyperplane of $\mathrm{PG}(d, q)$ and then restricting the coordinates from $\mathrm{GF}(q)$ down to $\mathrm{GF}(3)$.

3.3. Some generalizations of linear representations with symmetry.

3.3.1. The case $d = 1$ The case $d = 1$ above generalizes to the class of partial $(0, 2)$-geometries with $t = 2$. Such a geometry is a net of degree 3 and is also equivalent to a Latin square. Here we give a particular generalization that has a large collineation group.

Let G be an arbitrary group containing at least three elements. Then the points of the geometry Γ_G, are all pairs of elements in G. The lines consist of the sets $H_a := \{(g, a) : g \in G\}$, $V_a := \{(a, g) : g \in G\}$ and $D_a := \{(g, ga) : g \in G\}$, for all $a \in G$ (where H, V, D stand for horizontal, vertical and diagonal, respectively).

The direct product $G \times G$ acts (at the right) in a natural way on Γ_G as a regular permutation group on the point set. Moreover, for each $a \in G$, the involutive mapping $(x,y) \mapsto (ay^{-1}x, ay^{-1}a)$ maps H_b to $H_{ab^{-1}a}$, maps V_b onto $D_{ab^{-1}}$ and maps D_b onto $V_{ab^{-1}}$. Hence it fixes the line H_a pointwise and interchanges V_e, where e is the identity element of G, with D_a. So the line H_a is a Moufang line, and by symmetry, every line V_a is also a Moufang line. It follows that all lines are Moufang lines.

Suppose now that Γ_G admits a nontrivial collineation θ fixing a point. By the sharply transitive action of $G \times G$ on the point set of Γ_G we may assume that θ fixes the point (e,e). By the above observation that each line is a Moufang line, we may assume that θ fixes all lines through (e,e). Hence θ induces a permutation σ on G via the action $(x,e)^\theta = (x^\sigma, e)$ of θ on the points of the line H_e. Note that $e^\sigma = e$.

Now the vertical line V_a, $a \in G \setminus \{e\}$, is mapped under θ to a vertical line (as every other line meets V_e, which is fixed by θ); hence V_a is mapped onto V_{a^σ}. Similarly the diagonal line $D_{a^{-1}b}$ is mapped onto another diagonal line which must then be $D_{((b^{-1}a)^\sigma)^{-1}}$. Hence the point (a,b), which is the intersection of V_a with $D_{a^{-1}b}$ is mapped onto the point $(a^\sigma, a^\sigma((b^{-1}a)^\sigma)^{-1})$. In view of the fact that horizontal lines must be mapped onto (horizontal) lines, the second coordinate, namely $a^\sigma((b^{-1}a)^\sigma)^{-1}$, is independent of a. Putting $a = b$, we see that it must be equal to b^σ, and we obtain the identity $a^\sigma = b^\sigma(b^{-1}a)^\sigma$, from which it follows that θ is an automorphism of G.

Hence Γ_G has centers of transitivity if and only if G admits a transitive automorphism group. In the finite case this is equivalent to G being elementary abelian. In this case we say that Γ_G is a classical net. It arises from $\mathrm{AG}(2,q)$ by deleting $q-2$ parallel classes of lines.

More generally, if in the above construction we allow G to be a quasigroup, then Γ_G is a net of degree 3. Further, any net of degree 3 may be constructed in such a manner and in particular with G a loop (see [2] or [1], for instance). For any such net Γ a *translation* is a collineation of Γ which fixes each of the three parallel classes of Γ and each line of one of the parallel classes. The parallel class fixed elementwise is called the *axis* of the translation. If the group of translations with a fixed axis acts transitively on the set of points incident with one of the lines of the axis, then the axis is called *transitive* (not to be confused with an axis of transitivity defined earlier). A collineation β of Γ which fixes each of the parallel classes is called a *homology* if all elements of $\langle\beta\rangle$, different from the identity, have exactly one fixed point x which is called the *centre* of β. If the group of homologies with centre x acts transitively on the points, different from x, on a line incident with x, then x is called a *transitive centre* (not to be confused with an centre of transitivity defined earlier). Any elation point or Moufang point of Γ can be shown to be a transitive centre of Γ. In [1] a Lenz classification for loops and nets is given in terms of transitive axes and transitive centres. The paper [1] also contains many other results connecting the collineation group of a net to the algebraic structure of a loop giving rise to the net.

3.3.2. The affine case Let G be a group of exponent 3, that is, a group in which every non-identity element has order 3. Let n be a positive integer. We define a geometry $\Gamma_{n,G}$ as follows. The point set of $\Gamma_{n,G}$ is the Cartesian product $G \times G \times \cdots \times G$ ($n+1$ factors). For each pair of nonnegative integers (k, ℓ), with $k + \ell \leq n$, and each n-tuple $(a_1, a_2, \ldots, a_n)$ of elements of G, the set $\{(g, ga_1, ga_2, \ldots, ga_k, g^{-1}a_{k+1}, \ldots, g^{-1}a_{k+\ell}, a_{k+\ell+1}, \ldots, a_n) : g \in G\}$ and every set obtained from this one by permuting the coordinates, but leaving the first coordinate fixed, is a line of $\Gamma_{n,G}$.

If G is elementary abelian, then we obtain exactly the linear representation related to the affine space $\mathrm{AG}(n, 3)$ inside $\mathrm{PG}(n, |G|)$. However there exist nonabelian groups of exponent 3, and they give rise to new $(0, 2)$-geometries with a transitive group of collineations, and with distance regular point graph. The smallest example arises as the multiplicative group of upper diagonal 3×3 matrices over $\mathrm{GF}(3)$ with 1 on each diagonal entry.

3.4. Other representations.

Consider the projective space $\mathrm{PG}(d, q)$ embedded as a hyperplane π_∞ in $\mathrm{PG}(d+1, q)$. Let $\mathcal{S}$ be a set of disjoint n-dimensional subspaces of π_∞, $0 \leq n < d$. We build the geometry $\Gamma_{\mathcal{S}}$ as follows. The points are the points of $\mathrm{PG}(d+1, q)$ not lying in π_∞. The lines are the $n+1$ dimensional subspaces of $\mathrm{PG}(d+1, q)$ intersecting $\mathrm{PG}(d, q)$ in a member of $\mathcal{S}$. It is routine to verify that $\Gamma_{\mathcal{S}}$ is a $(0, 2)$-geometry if and only if $\mathcal{S}$ satisfies the following condition:

(C02) For every pair of members $S, T \in \mathcal{S}$, and for every point $x \in T$, there exists a unique element $U \in \mathcal{S} \in \{X, T\}$ that meets $\langle S, x\rangle$ nontrivially.

If the elements of $\mathcal{S}$ generate $\mathrm{PG}(d, q)$, then we call $\mathcal{S}$ a $(0, 2)$-*representation set* of $\mathrm{PG}(d, q)$.

As an example, consider a spread $\mathcal{S}$ of the generalized quadrangle $\mathrm{Q}(4, 2)$ naturally embedded in $\mathrm{PG}(4, 2)$. Then $\mathcal{S}$ satisfies condition (C02). We will denote the corresponding $(0, 2)$-geometry by $\Gamma_{\mathrm{Q}(4,2)}$.

Note that $\Gamma_{\mathcal{S}}$ is a semipartial geometry if and only if for each point x of $\mathrm{PG}(d, q)$ not in any member of $\mathcal{S}$, there are a constant number of members $S \in \mathcal{S}$ such that $\langle x, S\rangle$ intersects two other members of $\mathcal{S}$ nontrivially. In this case the definition of the $(0, 2)$-representation set (called an *SPG regulus*) and the construction of the semipartial geometry are due to J.A. Thas ([10]).

4. Some classification results

In this section we consider semipartial $(0, 2)$-geometries with the property that every two noncollinear points are collinear with exactly 6 points ($\mu = 6$), in particular with $t = 2$ and s odd. By a theorem of Debroey ([4]) any proper semipartial

$(0,2)$-geometry with $\mu = 6$, satisfying the diagonal axiom is isomorphic to the geometry H_2^{n*}, which has both central and axial transitivity. The diagonal axiom states that, if with four points, no three on a line, one has at least 5 pairs of collinear points, then all six pairs of points are collinear point pairs.

In the case of a proper semipartial $(0,2)$-geometry with $\mu = 6$ and $s < t$ Wilbrink and Brouwer ([14]) showed that the diagonal axiom is satisfied. (Further, in [3] Cuypers observed that a $(0,2)$-geometry with $s < t$ and with the property that every two noncollinear points are collinear with exactly 0 or 6 points, satisfies the diagonal axiom.) If $s = t$, then Wilbrink and Brouwer ([14]) showed that a proper semipartial geometry satisfies the diagonal axiom except possibly in the case $s = t = 28$.

Since for a proper semipartial geometry we must have $s \leq t$ ([5]) it follows that for a semipartial $(0,2)$-geometry with $\mu = 6$ and $t = 2$ the only cases not covered above are the partial geometries, that is, the Bruck nets of degree 3.

Theorem 4.1. *If Γ is a semipartial $(0,2)$-geometry with central and axial transitivity, with the property that every two noncollinear points are collinear with exactly 6 points, and such that the order (s,t) satisfies $t = 2$ and s is odd, then Γ satisfies the diagonal axiom. In other words, if Γ is a net of degree 3 and order $s+1$, s odd, with central and axial transitivity, then Γ satisfies the diagonal axiom.*

Proof. Let p be any point of Γ, and let A, B, C be the three lines incident with p. Let x be any point on A, different from p, and let $\Gamma_2(p)$ denote the set of points of Γ collinear with p, but distinct from p. Define a graph $G = (\Gamma_2(p), E)$ as follows. Two elements of $\Gamma_2(p)$ are adjacent if they are collinear, but lying together on one of the lines A, B, C. Then x is adjacent with exactly two points of $\Gamma_2(p)$, by the fact that $\alpha = 2$. Hence G consists of disjoint polygons, in particular, $3n$-gons, for fixed natural n. Indeed, by the fact that p is a center of transitivity, all these polygons can be mapped onto each other. Now, since A is an axis of transitivity, we can fix exactly n points of such a polygon, preserving it globally. This is impossible if $n > 2$, since an element of a finite dihedral group can have at most two fixed points on the corresponding polygon. Hence $n \in \{1, 2\}$. If $n = 1$, then clearly Γ satisfies the diagonal axiom, and we are done. If $n = 2$, then s is even, contradicting the hypothesis. $\square$

This has some interesting corollaries. The first one follows directly from the previous Theorem and the main result in [9].

Corollary 4.2. *Under the same assumptions of Theorem 4.1, we have that Γ is embeddable in a Desarguesian affine plane of order $q = s + 1 = 2^h$. hence Γ arises from $AG(2, s+1)$ by deleting $s - 1$ parallel classes of lines and is dual to H_2^{h+1}.*

The next corollary is a translation of the above result to the equivalent results on loops.

Corollary 4.3. *Let G be a loop of even order and let Γ_G be the net of degree 3 constructed from G. If Γ_G has central and axial transitivity, then G is an elementary abelian group of order 2^n for some $n \geq 1$.*

Proof. Apply Corollary 17.3 of [1] to Theorem 4.1. □

In case s is even, we need some stronger assumptions in order to be able to classify. This is achieved by invoking the Moufang condition.

Theorem 4.4. *If Γ is a Moufang semipartial $(0,2)$-geometry, with the property that every two noncollinear points are collinear with exactly 6 points, and such that the order (s,t) satisfies $t=2$ and s is even, then Γ is isomorphic to Γ_H, for a group H admitting a sharply transitive group U of automorphisms, such that the permutation group acting on H and generated by U and the right translations in H is a sharply 2-transitive group. Conversely, let G be a group acting sharply 2-transitively on a set Ω, and let H be the Frobenius kernel. Then Γ_H is a Moufang $(0,2)$-geometry.*

Hence, every Moufang net of degree 3 and order $s+1$, s even, is isomorphic to Γ_H, with H an elementary abelian group of odd order.

Proof. First we assume that Γ is a Moufang semipartial $(0,2)$-geometry with the property that every two noncollinear points are collinear with exactly 6 points, and such that the order (s,t) satisfies $t=2$ and s is odd.

We start by applying Lemma 2.3. So let x and y be two collinear Moufang points of Γ, and denote the corresponding groups U_x and U_y. If $G := \langle U_x, U_y\rangle$, then $\mathcal{M}_{xy} := (\Gamma_1(xy), G; U_x^G)$ is a Moufang set. Since Γ is a net, there are three parallel classes of lines, and we may call them *horizontal, vertical* end *diagonal*, respectively. We also may assume that the line xy is horizontal. Since the groups U_x and U_y fix both the vertical and horizontal class of lines, the group G also fixes each type of parallel class of lines.

Suppose that $G_{x,y}$, the stabilizer of both x and y in G, is nontrivial. We claim that $G_{x,y}$ fixes some point of xy different from x and y. Assume, by way of contradiction, that $G_{x,y}$ does not fix any point on xy except for x and y. There are exactly two points of Γ, say z_1 and z_2, collinear with both x and y, and not incident with xy (since Γ is a net of degree 3). If z_1 were not collinear with z_2, then there would be a point u on xz_2 different from both x and z_2, collinear with z_1, and fixed under $G_{x,y}$. But u would be collinear with a point u' on xy different from both x and y (indeed, $u' \neq y$ since otherwise y is collinear with three points on xz_2), and u' would be fixed under $G_{x,y}$, a contradiction to our assumption. Hence z_1 and z_2 are collinear. It now follows easily that Γ satisfies the diagonal axiom. But then s is odd, using [9]. The claim is proved.

The classification of finite Moufang sets (see [8, 7]) now implies easily that G is a sharply 2-transitive group. Let H be the Frobenius kernel of G. Then H preserves both the vertical and the diagonal class of lines. We now claim that H fixes every horizontal line.

Indeed, suppose H has k orbits on the set of s horizontal lines distinct from xy. Remember that H has order $s+1$ and all nontrivial elements of H are conjugate. Hence all nontrivial elements of H fix equally many horizontal lines, say m. By Burnside's result, the average number of horizontal lines distinct from xy fixed by a nontrivial element of H is equal to

$$\frac{k(s+1)-s}{s} = \frac{k}{s}(s+1)-1,$$

and since this has to be equal to the integer m, we conclude that k is a multiple of s, implying $k = s$ and the claim follows.

Let θ be the nontrivial collineation of Γ with axis the unique diagonal line L through x (and swapping the horizontal and vertical lines). Then H^θ fixes all vertical lines and preserves the other two classes of lines. Hence $[H, H^\theta]$ is trivial (because each element of that commutator fixes all vertical and all horizontal lines). Now, for $h \in H$, it is easy to see that hh^θ stabilizes the line xy; hence if we write the group $\langle H, H^\theta\rangle$ as $H \times H^\theta$, and if we identify a point z of Γ with the group element of $H \times H^\theta$ taking x to z, then the set $\{(h, h) : h \in H\}$ is a diagonal line.

Clearly, the point $(x, y) \in H \times H^\theta$ is mapped onto the point (xa, yb) by the collineation $(a, b) \in H \times H^\theta$. It follows that, for all $a \in H$, the sets $\{(h, a^\theta) : h \in H\}$, $\{(a, h^\theta) : h \in H\}$ and $\{(h, h^\theta a^\theta) : h \in H\}$ represent all lines of Γ. We conclude that Γ is isomorphic to Γ_H, and the first part of the theorem is proved.

The second part of the theorem is now obvious. □

We now easily obtain:

Corollary 4.5. *A Moufang net Γ of degree 3 is isomorphic to the classical net obtained from $\mathrm{AG}(2, q)$ by deleting $q-2$ parallel classes of lines.*

For even q, Γ_H admits only one "Moufang structure", but for odd q there are more, as follows from the classification of sharply 2-transitive permutation groups. Perhaps this is why the case s even is more difficult to treat and why we need the stronger assumption of being Moufang.

5. Moufang $(0, 2)$-geometries arising from $(0, 2)$-representation sets

For the moment it is not feasible to classify all Moufang $(0, 2)$-geometries arising from a linear representation, or arising from a $(0, 2)$-representation set. However, there is one subclass that we can handle. We begin with a lemma.

Lemma 5.1. *Let $\mathcal{S}$ be a $(0, 2)$-representation set of $\mathrm{PG}(5n-1, q)$, $n \geq 1$, consisting of $(2n-1)$-dimensional subspaces. Then q is even.*

Proof. Consider two distinct elements S, T of $\mathcal{S}$. For x a point of T by definition there exists a unique $U \in \mathcal{S} \setminus \{S, T\}$ meeting $\langle \mathcal{S}, x\rangle$ non-trivially and hence a unique elements $U \in \mathcal{S}\setminus\{S, T\}$ such that $x \in \langle S, U\rangle$. It follows that the sets $\langle S, U\rangle \cap T$, $U \in \mathcal{S} \setminus \{S, T\}$, form a partition of T. Since the dimension of $\langle S, U\rangle \cap T$

is at least $n-1$ for $U \in \mathcal{S}\setminus\{S,T\}$, it follows that either the dimension of $\langle S,U\rangle\cap T$ is exactly $n-1$ for all $U \in \mathcal{S}\setminus\{S,T\}$ and $|\mathcal{S}| = (q^n+1)+2 = q^n+3$; or that $|\mathcal{S}| = 3$ and the elements of $\mathcal{S}$ are contained in $\langle S,T\rangle$. In the latter case the elements of $\mathcal{S}$ do not generate $\mathrm{PG}(5n-1,q)$ and so do not form a $(0,2)$-representation set of $\mathrm{PG}(5n-1,q)$.

Now, if we project $\mathcal{S}$ from S onto a $(3n-1)$-dimensional subspace $\mathrm{PG}(3n-1,q)$ skew to S, then we obtain a set $\mathcal{S}'$ of q^n+2 subspaces of $\mathrm{PG}(3n-1,q)$, each of dimension $2n-1$ with the properties

(DA1) two distinct elements of $\mathcal{S}'$ intersect in an $(n-1)$-dimensional subspace, and

(DA2) three distinct elements of $\mathcal{S}'$ meet in the empty set.

Consider distinct $S',T' \in \mathcal{S}'$ and put $R = S'\cap T'$. Let x be any point of $\mathrm{PG}(3n-1,q)$ not contained in $S'\cup T'$, and put $R^* = \langle x,R\rangle$. From (DA1) and (DA2) above it follows that each point of R^* is contained in either 0 or 2 elements of $\mathcal{S}'$. Also, every member of $\mathcal{S}'$ distinct from both S' and T' intersects the n-dimensional space R^* in a point (since, if the intersection contained a line, this line would meet R nontrivially, contradicting (DA2)). We now see that the number of elements of $\mathcal{S}'$ distinct from S' and T' is even, hence the lemma. □

A representation set as in the previous lemma will be called *tight*.

We now have the following classification.

Theorem 5.2. *Let $\mathcal{S}$ be a tight $(0,2)$-representation set of lines in $\mathrm{PG}(4,q)$. Suppose that all lines of the corresponding $(0,2)$-geometry Γ are Moufang and that the corresponding groups are induced by collineations of $\mathrm{PG}(4,q)$. Then Γ is isomorphic to $\Gamma_{\mathrm{Q}(4,2)}$.*

Proof. By Lemma 2.3, the action of the groups related to two intersecting Moufang lines induced on $\mathcal{S}$ defines a Moufang set on $\mathcal{S}$, with $|\mathcal{S}| = q+3$ odd by Lemma 5.1. By the classification of finite Moufang sets in [8, 7], either $q+2$ is a prime power, implying $q = 2$, or $q+3$ is a prime power (and there is a sharply 2-transitive action on $\mathcal{S}$). In any case, the number $(q+3)(q+2)$ must divide the order of the collineation group of $\mathrm{PG}(4,q)$, which is $q^{10}(q^4+q^3+q^2+q+1)(q^3+q^2+q+1)(q^2+q+1)(q+1)(q-1)^4$. Since q is even, $q+3$ does not have any nontrivial divisor in common with q, $q-1$ or $q+1$. Furthermore, the only possible nontrivial common divisors of $q+3$ with q^2+q+1, q^3+q^2+q+1 and $q^4+q^3+q^2+q+1$ are 7, 5 and 61, respectively. Hence either $q = 2$ or $q = 4$. If $q = 2$, then the result follows readily from the fact that $\mathrm{PGL}(5,2)$ admits only one conjugacy class of elements of order 5.

Now let $q = 4$. By the transitive action on $\mathcal{S}$, there is an element θ of order 5 cyclically permuting the elements of $\mathcal{S}$. Since the number of points, 341, is equal to 5 modulo 7, there are at least 5 fixed points; dually, there are at least 5 fixed hyperplanes of $\mathrm{PG}(4,4)$. It is easy to see that there are exactly five fixed points, and that they are incident with a common line L (otherwise θ is the identity). Dually, θ fixes a plane π and all five hyperplanes through it. The plane π and the line L are skew. Evidently, no member of $\mathcal{S}$ meets π or L. Since $\langle\theta\rangle$ is the

Frobenius kernel of a sharply 2-transitive group G acting on $\mathcal{S}$, the stabilizer in G of an element of $\mathcal{S}$ in G fixes L and π; hence it stabilizes the projection of $\mathcal{S}$ from L onto π. But in π, there is no group of order 6 permuting transitively six lines and fixing one. $\square$

Theorem 5.3. *Let $\mathcal{S}$ be a tight $(0,2)$-representation set of $(2n-1)$-dimensional subspaces in $\mathrm{PG}(5n-1,q)$, $n \geq 2$. Suppose that all lines of the corresponding $(0,2)$-geometry are Moufang and that all corresponding groups are induced by collineations of $\mathrm{PG}(5n-1,q)$. Then we have the following cases:*
1. $n=2$, $q=2$ (in $PG(9,2)$);
2. $n=2$, $q=4$ (in $PG(9,4)$);
3. $n=3$, $q=2$ (in $PG(14,2)$);
4. $n=4$, $q=2$ (in $PG(19,2)$).

Proof. We have already proved that q is even. As in the previous proof, there is an induced Moufang set on $\mathcal{S}$, and it must arise from a sharply 2-transitive group. Let F be the Frobenius kernel of that group. Then all nontrivial elements of F are mutually conjugate.

Suppose that F is not of prime order. We claim that F fixes a point x. If not, then every element of F acts freely on the point set of $\mathrm{PG}(5n-1,q)$, and hence F is contained in a Singer cycle. But then F is cyclic, a contradiction. The claim follows. Similarly, F fixes a line through x. We can continue this argument until we obtain that F fixes a maximal flag. But then F is contained in the Borel subgroup, which is the normalizer of a Sylow 2-subgroup, and hence the unique prime p that divides $|F|$ also divides $q(q-1)$. Since p is odd, p divides $q-1$ and this contradicts the fact that p also divides q^n+3.

So we have shown that $|F| = q^n+3 = p$ is a prime. Consequently q^n+3 divides some number q^i-1, for some i, with $n+1 \leq i \leq 5n$. We have now to distinguish between $4n < i \leq 5n$, $3n < i \leq 4n$, $2n < i \leq 3n$ and $n < i \leq 2n$.

We give the details of the case $4n < i \leq 5n$, which is the most involved one. The other cases are left to the reader.

Put $i = 5n-k$. Then, modulo q^n+3, the number q^i-1 is equal to $81q^{n-k}-1$, and this must be 0 mod q^n+3. Clearly, this first implies

$$81q^{n-k}-1 \geq q^n+3,$$

hence $k \leq 6$ if $q=2$, or $k \leq 3$, if $q=4$, or $k \leq 2$ if $q=8$, or $k=1$ if $q \geq 16$, and $k=0$ if $q \geq 128$.

In any case, the number q^n+3 divides $81q^n-q^k$, hence it divides $243+q^k$. Since q^k is always a power of 2 and is at most 2^6, we have that q^n+3 divides 244, 245, 247, 251, 259, 275, or 307. Consequently q^n+3 is smaller than 260. the primes of the form 2^j+3 not exceeding 259 are $5, 7, 11, 19, 67$ and 131. Since $k<n$, the only possibilities are $(q,n,k) = (2,2,1)$, $(q,n,k) = (4,2,1)$ and $(q,n,k) = (2,4,2)$. These give rise to cases 1,2 and 4, respectively.

Similarly, the case $3n < i \leq 4n$ gives rise to $(q,n,k) = (2,3,2)$, which is case 3, and $2n < i \leq 3n$ implies $(q,n,k) = (2,2,0)$, which is case 1 again. Finally, $n < i \leq 2n$ yields $(q,n,k) = (2,2,1)$, which is again case 1. $\square$

6. Perspectivities

If a net of degree 3 comes from a Desarguesian affine plane, then it is easy to see that the group of projectivities of a line is a Frobenius group, i.e., it is transitive, but the stabilizer of two points is trivial. Indeed, the group is transitive because the projectivity $A \to B \to C \to A$ for a triangle A, B, C interchanges the two intersection points $A \cap B$ and $A \cap C$.

Conversely, suppose a net of degree 3 and even order has a group of projectivities which is a Frobenius group. Let A, B, C be as above, and let D, E be such that A, D, E form a triangle, with $A \cap B = A \cap D$ and $A \cap C = A \cap E$, with $D \neq B$ and $E \neq C$. Let X be the unique line through $B \cap C$ distinct from both B, C. If we assume that Γ does not satisfy the diagonal axiom, then X, D, E form a triangle. The projectivity $X \to D \to E \to X$ has an involutory pair and a fixed point (namely, $B \cap C$), which is impossible for a Frobenius group acting on an even number of points.

Hence we have proved the following theorem.

Theorem 6.1. *If Γ is a semipartial $(0, 2)$-geometries with the property that every two noncollinear points are collinear with exactly 6 points, and such that the order (s, t) satisfies $t = 2$ and s is odd, and if the group of projectivities of Γ is a frobenius group, then Γ is isomorphic to H_2^{q*}, with $q = (s+1)/2$ an even prime power (hence Γ arises from a Desarguesian projective plane of even order $s+1$ by deleting $s-2$ parallel classes of lines, or, in other words, Γ is a classical net of degree 3 and order $s-1$).*

Note that in the case of a $(0, 2)$-geometry that is a net of degree 3 that our definition of projectivity group is equivalent to that of Barlotti and Strambach in [1]. In [1] there are many interesting results on the groups of projectivities of nets.

Acknowledgements

Thanks to Frank De Clerck for helpful conversations on the material in this paper and for proof reading initial drafts.

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