The two smallest minimal blocking sets of $Q(2n, 3)$, $n \geqslant 3$

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Abstract

We describe the two smallest minimal blocking sets of $Q(2n, 3)$, $n \geqslant 3$. To obtain these results, we use the characterization of the smallest minimal blocking sets of $Q(6, 3)$, different from an ovoid. We also present some geometrical properties of ovoids of $Q(6, q)$, q odd.

1 Introduction

Let $Q(2n, q)$, $n \geqslant 2$, be the non-singular parabolic quadric in $PG(2n, q)$. An *ovoid* of the polar space $Q(2n, q)$ is a set of points $\mathcal{O}$ of $Q(2n, q)$, such that every maximal singular subspace (or *generator*) of $Q(2n, q)$ intersects $\mathcal{O}$ in exactly one point. For $Q(2n, q)$, the generators are spaces of dimension $n - 1$. A *blocking set* of the polar space $Q(2n, q)$ is a set of points $\mathcal{K}$ of $Q(2n, q)$ such that every generator intersects $\mathcal{K}$ in at least one point. If $\mathcal{O}$ is an ovoid of $Q(2n, q)$, then $\mathcal{O}$ has size $q^n + 1$. So if $\mathcal{K}$ is a blocking set of $Q(2n, q)$ different from an ovoid, then $\mathcal{K}$ has size $q^n + 1 + r$, with $r > 0$. A blocking set $\mathcal{K}$ is called *minimal* if for every point $p \in \mathcal{K}$, $\mathcal{K} \setminus \{p\}$ is not a blocking set, or equivalently, if for every point $p \in \mathcal{K}$, there is a generator α such that $\alpha \cap \mathcal{K} = \{p\}$.

We suppose in this article that q is odd. We recall known results about ovoids of the parabolic quadric in 4, 6 and 8 dimensions.

Theorem 1 (Ball [1]) *Suppose that $\mathcal{O}$ is an ovoid of $Q(4, q)$, $q = p^h$, p prime, $h \geqslant 1$, then every elliptic quadric $Q^-(3, q)$ of $Q(4, q)$ intersects $\mathcal{O}$ in $1 \bmod p$ points.*

This result has interesting applications. One of them is the classification of all ovoids of $Q(4, q)$, q prime.

Theorem 2 (Ball et al. [2]) *The only ovoids of $Q(4, q)$, q prime, are elliptic quadrics $Q^-(3, q)$.*

When $q = p^h$, p an odd prime, $h > 1$, and $q = 2^{2h+1}$, $h \geq 1$, other classes of ovoids of $Q(4, q)$ are known ([9, 12, 15, 16]).

The classification of the ovoids of $Q(4, q)$, q prime, leads to the following theorem, using a result of [10].

Theorem 3 *When q is an odd prime, $q \geqslant 5$, $Q(6, q)$ does not have ovoids.*

When $q = 3^h$, $h \geqslant 1$, $\mathrm{Q}(6,q)$ always has ovoids ([9, 13, 14]), and when q is even, then $\mathrm{Q}(6,q)$ does not have ovoids ([14]). For all other values of q, the existence or non-existence of ovoids of $\mathrm{Q}(6,q)$ is not known, although it is conjectured in [10] that $\mathrm{Q}(6,q)$ has ovoids if and only if $q = 3^h$, $h \geqslant 1$.

Finally, we recall the following theorem about ovoids of higher dimensional parabolic quadrics.

Theorem 4 (Gunawardena and Moorhouse [8]) *The parabolic quadric* $\mathrm{Q}(8,q)$*, q odd, does not have ovoids. This implies also that* $\mathrm{Q}(2n,q)$*, q odd, $n \geqslant 5$, does not have ovoids.*

We now recall known results about blocking sets different from ovoids. Suppose that $\alpha\mathcal{B}$ is a cone with vertex the k-dimensional subspace α and base some set $\mathcal{B}$ of points, lying in some subspace π, $\pi \cap \alpha = \emptyset$. Then the *truncated cone* $\alpha^*\mathcal{B}$ is defined as $\alpha\mathcal{B} \setminus \alpha$, hence, as the set of points of the cone $\alpha\mathcal{B}$ where the points of the vertex α are removed from. If α is the empty subspace, then $\alpha^*\mathcal{B} = \mathcal{B}$.

The case $q = 3$ of the following theorem was proven in [5]. The theorem for $q > 3$ odd prime was proven in [4]. We denote the polarity associated to the quadric by $\perp$.

Theorem 5 *The smallest minimal blocking sets of* $\mathrm{Q}(6,q)$*, q an odd prime, different from an ovoid of* $\mathrm{Q}(6,q)$*, are truncated cones* $p^*\mathrm{Q}^-(3,q)$*,* $p \in \mathrm{Q}(6,q)$*,* $\mathrm{Q}^-(3,q) \subseteq p^\perp \cap \mathrm{Q}(6,q)$*, and have size* $q^3 + q$*.*

When $q > 3$ is an odd prime, this theorem generalizes to the following theorem.

Theorem 6 ([5]) *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q)$*, $q > 3$ prime, $n \geqslant 4$, are truncated cones* $\pi^*_{n-3}\mathrm{Q}^-(3,q)$*,* $\pi_{n-3} \subseteq \mathrm{Q}(2n,q)$*,* $\mathrm{Q}^-(3,q) \subseteq \pi^\perp_{n-3} \cap \mathrm{Q}(2n,q)$*, and have size* $q^n + q^{n-2}$*.*

Ovoids of $\mathrm{Q}(6,q)$ can be used to construct smaller examples in higher dimension. For $q = 3$, the following result is known.

Theorem 7 ([5]) *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q=3)$*, $n \geqslant 4$, are truncated cones* $\pi^*_{n-4}\mathcal{O}$*,* $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6,q=3)$*,* $\mathcal{O} \subset \pi^\perp_{n-4}, \pi_{n-4} \subset \mathrm{Q}(2n,q)$*, and have size* $q^n + q^{n-3}$*.*

Theorems 6 and 7 express the difference between $q > 3$ odd prime and $q = 3$. Furthermore, considering $\mathrm{Q}(2n,q=3)$, $n \geqslant 4$, it is clear that a truncated cone $\pi^*_{n-3}\mathrm{Q}^-(3,q)$, contained in $\mathrm{Q}(2n,q)$, constitutes a minimal blocking set of size $q^n + q^{n-2}$. We show in this article that minimal blocking sets of $\mathrm{Q}(2n,3)$ of size k, $q^n + q^{n-3} < k < q^n + q^{n-2}$ do not exist, and we characterize the minimal blocking sets of $\mathrm{Q}(2n,q=3)$ of size $q^n + q^{n-2}$, as described in the following theorem. Finally, we show that minimal blocking sets of $\mathrm{Q}(2n,q=3)$, $n \geqslant 3$, of size $q^n + q^{n-2} + 1$ do not exist.

Theorem 8 *The minimal blocking sets of* $\mathrm{Q}(2n,q=3)$*, $n \geqslant 3$, of size at most $q^n + q^{n-2}$, are truncated cones* $\pi^*_{n-4}\mathcal{O}$*,* $\pi_{n-4} \subseteq \mathrm{Q}(2n,3)$*,* $\pi^\perp_{n-4} \cap \mathrm{Q}(2n,q=3) = \pi_{n-4}\mathrm{Q}(6,q=3)$*,* $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6,3)$*, and* $\pi^*_{n-3}\mathrm{Q}^-(3,q=3)$*,* $\pi_{n-3} \subseteq \mathrm{Q}(2n,3)$*,* $\pi^\perp_{n-3} \cap \mathrm{Q}(2n,q=3) = \pi_{n-3}\mathrm{Q}(4,q=3)$*,* $\mathrm{Q}^-(3,q=3) \subseteq \mathrm{Q}(4,q=3)$*. Furthermore, a minimal blocking set of size* $q^n + q^{n-2} + 1$ *of* $\mathrm{Q}(2n,q=3)$ *does not exist.*

Before presenting the proof of this theorem, we first mention some geometrical properties of ovoids of $\mathrm{Q}(6,q)$, q odd.

2 Geometrical results on ovoids of $\mathrm{Q}(6,q)$, q odd

For the next three lemmas, we suppose that $\mathrm{Q}(6,q)$ has ovoids. This implies that q is odd, since $\mathrm{Q}(6,q)$, q even, does not have ovoids [14], and this hypothesis is satisfied when $q = 3^h$, $h \geqslant 1$. Denote an ovoid of $\mathrm{Q}(6,q)$ by $\mathcal{O}$.

Lemma 1 *The ovoid* $\mathcal{O}$ *spans the* 6*-dimensional space* $\mathrm{PG}(6,q)$*.*

Proof. Let $\Omega = \langle \mathcal{O} \rangle$.

It is impossible that $\Omega \cap \mathrm{Q}(6,q)$ is a singular quadric. For, assume that $\langle \mathcal{O} \rangle \cap \mathrm{Q}(6,q) = \pi_s Q$, a cone with vertex π_s, an s-dimensional subspace, $s \geqslant 0$, and with base Q, a non-singular quadric of dimension at most 4. Then π_s projects $\mathcal{O}$ onto an ovoid of Q. However, no non-singular quadric of dimension at most four has ovoids of size q^3+1.

If $\Omega \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$, then $\mathcal{O}$ must necessarily be an ovoid of $\mathrm{Q}(4,q)$; impossible since $|\mathcal{O}| > q^2+1$. If $\langle \mathcal{O} \rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$, then $\mathcal{O}$ must be an ovoid of $\mathrm{Q}^+(5,q)$; impossible since $|\mathcal{O}| > q^2+1$. Finally, $\langle \mathcal{O} \rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$ is impossible, since $\mathrm{Q}^-(5,q)$ does not have ovoids [11].

Lemma 2 *No elliptic quadric* $\mathrm{Q}^-(3,q)$ *is contained in* $\mathcal{O}$*.*

Proof. Suppose that some $\mathrm{Q}^-(3,q) \subseteq \mathcal{O}$. Since $\mathcal{O}$ spans the 6-dimensional space, there is a point $p \in \mathcal{O} \setminus \mathrm{Q}^-(3,q)$. The space $\langle p, \mathrm{Q}^-(3,q) \rangle$ intersects $\mathrm{Q}(6,q)$ in a parabolic quadric $\mathrm{Q}(4,q)$, containing at least q^2+2 points of $\mathcal{O}$, a contradiction, since any $\mathrm{Q}(4,q)$ can intersect $\mathcal{O}$ in at most q^2+1 points, the number of points of an ovoid of $\mathrm{Q}(4,q)$. $\square$

The following lemma is an application of Theorem 1.

Lemma 3 *The ovoid* $\mathcal{O}$ *does not contain an ovoid* $\mathcal{O}'$ *of* $\mathrm{Q}(4,q)$*, with* $\mathrm{Q}(4,q)$ *contained in* $\mathrm{Q}(6,q)$*.*

Proof. Suppose the contrary, i.e., suppose that there is some ovoid $\mathcal{O}'$ of $\mathrm{Q}(4,q) \subseteq \mathrm{Q}(6,q)$, with $\mathcal{O}' \subseteq \mathcal{O}$. By the previous lemma, we may suppose that $\mathcal{O}'$ is not an elliptic quadric and hence, $\langle \mathcal{O}' \rangle$ is a 4-dimensional projective space α, such that $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$. Since $\mathcal{O}$ spans the 6-dimensional space, we can choose a point $p \in \mathcal{O} \setminus \alpha$. Since α contains an ovoid of $\mathrm{Q}(4,q)$, $p \notin \alpha^\perp$, hence $p^\perp \cap \mathrm{Q}(4,q) = \mathrm{Q}^\pm(3,q)$, or $p^\perp \cap \mathrm{Q}(4,q) = r\mathrm{Q}(2,q)$ which is a tangent cone to $\mathrm{Q}(4,q)$. All these 3-dimensional quadrics intersect $\mathcal{O}'$ in 1 mod p points, hence, at least one point $r \in \mathcal{O}'$ belongs to $p^\perp$, a contradiction. $\square$

We call a hyperplane α of $\mathrm{PG}(6,q)$ *hyperbolic, elliptic* respectively, if $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$, $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$ respectively.

Corollary 1 *Any hyperbolic hyperplane* α *has the property that* $\langle \alpha \cap \mathcal{O} \rangle = \alpha$*.*

Proof. Suppose that α is a 5-dimensional subspace such that $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$. Then necessarily α intersects $\mathcal{O}$ in an ovoid $\mathcal{O}'$ of $\mathrm{Q}^+(5,q)$. Since any ovoid of $\mathrm{Q}(4,q)$ is not contained in $\mathcal{O}$, the ovoid $\mathcal{O}'$ spans the 5-dimensional space α. $\square$

With the aid of the software package `pg` [3], we also found the following result for $q = 3$. The software package `pg` is a package written in the language of the computer algebra system `GAP` [7]. Checking the mentioned property can be done with a few commands of the package `pg`.

Lemma 4 *Any elliptic hyperplane α of $\mathrm{PG}(6,3)$ has the property that $\langle \alpha \cap \mathcal{O} \rangle = \alpha$.*

We end this section with the following result. It was proven in [2], using Theorem 1.

Theorem 9 (Ball, Govaerts and Storme [2]) *Suppose that $\mathrm{Q}(6,q)$, $q = p^h$, $h \geqslant 1$, p an odd prime, has an ovoid $\mathcal{O}$. Then any elliptic hyperplane intersects $\mathcal{O}$ in $1 \bmod p$ points.*

3 Small minimal blocking sets of $\mathrm{Q}(6,3)$

We now consider minimal blocking sets, different from ovoids, of $\mathrm{Q}(6,q)$. Theorem 5 characterizes the smallest minimal blocking sets of $\mathrm{Q}(6,q=3)$ different from ovoids. We will extend this theorem by excluding the existence of minimal blocking sets of size q^3+q+1, with $q = 3$.

We now suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(6,q=3)$ of size at most q^3+q+1. The next two lemmas can be proven by techniques of [6].

Lemma 5 *For every point $r \in \mathcal{K}$, $|r^\perp \cap \mathcal{K}| \leqslant q+1$.*

Lemma 6 *Consider a point $r \in \mathrm{Q}(6,q) \setminus \mathcal{K}$, then the points of $r^\perp \cap \mathcal{K}$ are projected from r onto a minimal blocking set $\mathcal{K}_r$ of $\mathrm{Q}(4,q)$, with $\mathrm{Q}(4,q)$ a base of the cone $r^\perp \cap \mathrm{Q}(6,q)$.*

We call a line of $\mathrm{Q}(2n,q)$ meeting $\mathcal{K}$ in i points an *i-secant* to $\mathcal{K}$. For the next lemma, we use the fact that a minimal blocking set of $\mathrm{Q}(4,3)$, different from an ovoid, contains at least $12 = q^2+q$ points, with $q = 3$. This is proven in e.g. [5].

Lemma 7 *There are no lines of $\mathrm{Q}(6,3)$ meeting $\mathcal{K}$ in exactly 2 points.*

Proof. Suppose that L is a 2-secant to $\mathcal{K}$. Consider a generator π of $\mathrm{Q}(6,3)$ on L such that $\pi \cap \mathcal{K} = L \cap \mathcal{K}$. Count the number of pairs (u,v), $u \in \pi \setminus L$, $v \in \mathcal{K} \setminus L$, $u \in v^\perp$. Since the projection of the set of points $u^\perp \cap \mathcal{K}$ from u is a minimal blocking set of $\mathrm{Q}(4,3)$, and since it cannot be an ovoid of $\mathrm{Q}(4,3)$, it must contain at least q^2+q points of $\mathrm{Q}(4,3)$. We obtain $q^2(q^2+1)$ as lower bound for this number. Using the size of $\mathcal{K}$, we find $(q^3+q-1)q = q^4+q^2-q$ as upper bound, hence, $q^2(q^2+1) \leqslant q^4+q^2-q$, a contradiction. $\square$

Corollary 2 *Every generator π of $\mathrm{Q}(6,q=3)$ intersects $\mathcal{K}$ in 1 point, or in 3 or 4 collinear points.*

Proof. Since there are no 2-secants to $\mathcal{K}$, 2 points of $\mathcal{K}$ in π give rise to 3 or 4 collinear points of $\mathcal{K}$ in π. If there would be 3 points of $\mathcal{K}$ spanning π, then π would contain at least 7 points of $\mathcal{K}$, a contradiction with Lemma 5. $\square$

Lemma 8 *Suppose that L is a line of* $\mathrm{Q}(6,3)$ *meeting $\mathcal{K}$ in 3 or 4 points. Suppose that π is a generator of* $\mathrm{Q}(6,3)$ *on L, then $L \cap \mathcal{K} = \pi \cap \mathcal{K}$, and $|r^{\perp} \cap \mathcal{K}| \leqslant q^2+q+1$ for every $r \in \pi \setminus L$.*

Proof. Let r_0 be one of the points of $\mathcal{K} \cap \pi$. Suppose that $r \in \pi \setminus L$. Then there exists a generator π' of $\mathrm{Q}(6,3)$ through r meeting $\mathcal{K}$ only in r_0. The q^2-q lines of π' not through r_0 or r lie in q generators of $\mathrm{Q}(6,3)$ different from π'. Hence, at least q^3-q^2 points of $\mathcal{K}$ lie outside $r^{\perp}$, and so, $|r^{\perp} \cap \mathcal{K}| \leqslant q^2+q+1$. $\square$

Lemma 9 *Suppose that L is a 3-secant to $\mathcal{K}$, then the point $r \in L \setminus \mathcal{K}$ only lies on 3-secants to $\mathcal{K}$ and $\mathcal{K} = r^*\mathcal{O}$, $\mathcal{O}$ an ovoid of* $\mathrm{Q}(4,3)$*, with* $\mathrm{Q}(4,3)$ *the base of the cone $r^{\perp} \cap \mathrm{Q}(6,3)$.*

Proof. Put $\mathcal{K} \cap L = \{r_1, r_2, r_3\}$ and $r \in L \setminus \mathcal{K}$. Since $|(r_1^{\perp} \cup r_2^{\perp} \cup r_3^{\perp}) \cap \mathcal{K}| \leqslant 3+1+1+1$, necessarily $|r^{\perp} \cap \mathcal{K}| \geqslant q^3+q+1-6 = q^3-2 > q^2+q+1$, so, using the proof of Lemma 8, r does not lie in a generator with 1 point of $\mathcal{K}$, so r only lies in generators containing at least 3 points of $\mathcal{K}$. Moreover, these 3 or 4 points are collinear with r by Corollary 2 and Lemma 8. If r projects the points of $r^{\perp} \cap \mathcal{K}$ onto an ovoid of $\mathrm{Q}(4,3)$, then $|\mathcal{K}| = q(q^2+1)$; else $|\mathcal{K}| \geqslant q(q^2+2)$. Since $|\mathcal{K}| \leqslant q^3+q+1$, necessarily $\mathcal{K} = r^*\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(4,3)$, with $\mathrm{Q}(4,3)$ the base of the cone $r^{\perp} \cap \mathrm{Q}(6,3)$. $\square$

Theorem 10 *A minimal blocking set $\mathcal{K}$ of size $|\mathcal{K}| \leqslant q^3+q+1$, $q=3$, of* $\mathrm{Q}(6,3)$ *is an ovoid $\mathcal{O}$ or a truncated cone $r^*\mathcal{O}$, $\mathcal{O}$ an elliptic quadric $\mathrm{Q}^-(3,3) \subseteq \mathrm{Q}(4,3)$, with* $\mathrm{Q}(4,3)$ *the base of the cone $r^{\perp} \cap \mathrm{Q}(6,3)$. In particular, there does not exist a minimal blocking set of size q^3+q+1 on* $\mathrm{Q}(6,3)$*.*

Proof. Assume that $\mathcal{K}$ is not an ovoid of $\mathrm{Q}(6,3)$, then a line of $\mathrm{Q}(6,3)$ is either a 1-, 3-, or 4-secant to $\mathcal{K}$. By Lemma 9, we can assume that there is no 3-secant to $\mathcal{K}$. So a line of $\mathrm{Q}(6,3)$ containing at least 2 points of $\mathcal{K}$ contains 4 points of $\mathcal{K}$. Suppose that L is a 4-secant to $\mathcal{K}$. By Lemma 5, we find that $|\mathcal{K}| \leqslant 4$, since a point of $\mathrm{Q}(6,3) \setminus L$ is perpendicular to at least one point of L. But $|\mathcal{K}| > q^3+1$, a contradiction. $\square$

4 Small minimal blocking sets of $\mathrm{Q}(2n,3)$

Consider the parabolic quadric $\mathrm{Q}(2n, q=3)$, $n \geqslant 4$. For this section, we assume that the following hypothesis is true for $\mathrm{Q}(2k,3)$, $k=3,\ldots,n-1$.

The minimal blocking sets of size at most $q^k+q^{k-2}+1$ in $\mathrm{Q}(2k,q=3)$ are truncated cones $\pi_{k-4}^*\mathcal{O}$, $\pi_{k-4}^{\perp} \cap \mathrm{Q}(2k,q=3) = \pi_{k-4}\mathrm{Q}(6,q=3)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q=3)$; and truncated cones $\pi_{k-3}^*\mathrm{Q}^-(3,q=3)$, $\pi_{k-3}^{\perp} \cap \mathrm{Q}(2k,q=3) = \pi_{k-3}\mathrm{Q}^-(3,q=3)$, π_i an i-dimensional subspace contained in $\mathrm{Q}(2k,q=3)$. These examples have respectively size q^k+q^{k-3} and q^k+q^{k-2}. This hypothesis is true for $n=4$.

Suppose for this section that $\mathcal{K}$ is a minimal blocking set of size at most $q^n+q^{n-2}+1$ of $\mathrm{Q}(2n,q=3)$, $n \geqslant 4$. Since the smallest minimal blocking sets of $\mathrm{Q}(2n, q = 3)$, $n \geqslant 4$, of size $q^n + q^{n-3}$, are already classified [5], we also assume that $|\mathcal{K}| \geqslant q^n + q^{n-3} + 1$.

The next two lemmas are generalizations of Lemma 5 and Lemma 6. They can be proven by using techniques of proofs of [5].

Lemma 10 *For every point $r \in \mathcal{K}$, $|r^\perp \cap \mathcal{K}| \leqslant q^{n-2} + 1$.*

Lemma 11 *Consider a point $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$, then the points of $r^\perp \cap \mathcal{K}$ are projected from r onto a minimal blocking set $\mathcal{K}_r$ of $\mathrm{Q}(2n-2, q)$, with $\mathrm{Q}(2n-2, q)$ the base of the cone $r^\perp \cap \mathrm{Q}(2n, q)$.*

Lemma 12 *No generator π_{n-1} of $\mathrm{Q}(2n, q = 3)$ intersects $\mathcal{K}$ in exactly 2 points.*

Proof. Suppose that for some generator π_{n-1} of $\mathrm{Q}(2n, q)$, $|\pi_{n-1} \cap \mathcal{K}| = 2$, where the two points of $\pi_{n-1} \cap \mathcal{K}$ lie on the line L. Count the number of pairs (u, v), $u \in \pi_{n-1} \setminus L$, $u \in v^\perp$, $v \in \mathcal{K} \setminus \pi_{n-1}$. Since no minimal blocking set of size at most $q^{n-1} + q^{n-3} + 1$ of $\mathrm{Q}(2n-2, q)$ has a 2-secant, we find $|u^\perp \cap \mathcal{K}| \geqslant q^{n-1} + q^{n-3} + 2$. Hence, the lower bound on the number of pairs is $(q^{n-1} + \ldots + q^2)(q^{n-1} + q^{n-3})$. As upper bound, we find $(q^n + q^{n-2} - 1)(q^{n-2} + \ldots + q)$. Necessarily $(q^{n-1} + \ldots + q^2)(q^{n-1} + q^{n-3}) \leqslant (q^n + q^{n-2} - 1)(q^{n-2} + \ldots + q) \leqslant (q^{n-1} + \ldots + q^2)(q^{n-1} + q^{n-3}) - (q^{n-2} + \ldots + q)$, a contradiction. □

Corollary 3 *No line L of $\mathrm{Q}(2n, 3)$ intersects $\mathcal{K}$ in exactly 2 points.*

Proof. Suppose that L is a 2-secant to $\mathcal{K}$. By the minimality of $\mathcal{K}$ and Lemma 10, there exists a generator π_{n-1} on L such that $L \cap \mathcal{K} = \pi_{n-1} \cap \mathcal{K}$, a contradiction. □ .

Lemma 13 *Suppose that π_{n-1} is a generator of $\mathrm{Q}(2n, q)$ such that $|\pi_{n-1} \cap \mathcal{K}| = 1$. For every $r \in \pi_{n-1} \setminus \mathcal{K}$, we have that $|r^\perp \cap \mathcal{K}| \leqslant q^{n-1} + q^{n-2} + 1$.*

Proof. Denote the unique point in $\pi_{n-1} \cap \mathcal{K}$ by s. The $q^{n-1} - q^{n-2}$ hyperplanes of π_{n-1}, not through r or s, all lie in q generators, different from π_{n-1}, all containing at least one point of $\mathcal{K}$. So at least $(q^{n-1} - q^{n-2})q$ points lie in $\mathcal{K} \setminus r^\perp$; so $|r^\perp \cap \mathcal{K}| \leqslant q^{n-1} + q^{n-2} + 1$. □

Lemma 14 *Suppose that $r \notin \mathcal{K}$, and suppose that L is a line of $\mathrm{Q}(2n, 3)$ through r such that $|L \cap \mathcal{K}| = 1$. Then $|r^\perp \cap \mathcal{K}| \leqslant q^{n-1} + q^{n-2} + 1$.*

Proof. Consider a generator through the line $\langle r, s \rangle$, $s \in L \cap \mathcal{K}$, only containing the point $s \in \mathcal{K}$. Such a generator exists; or else $|s^\perp \cap \mathcal{K}| \geqslant q^{n-2} + 2$. The preceding lemma proves the assertion. □ .

Lemma 15 *There does not exist a line of $\mathrm{Q}(2n, 3)$ intersecting $\mathcal{K}$ in 4 points.*

Proof. Suppose that L is a line of $\mathrm{Q}(2n, 3)$ meeting $\mathcal{K}$ in 4 points. By Lemma 10, we find that $|\mathcal{K}| \leqslant 4(q^{n-2} + 1) = (q + 1)(q^{n-2} + 1) = q^{n-1} + q^{n-2} + q + 1 < q^n + 1$, a contradiction. □

Theorem 11 *The minimal blocking sets of $\mathrm{Q}(2n, q = 3)$, $n \geqslant 3$, of size at most $q^n + q^{n-2} + 1$, are truncated cones $\pi^*_{n-4}\mathcal{O}$, $\pi^\perp_{n-4} \cap \mathrm{Q}(2n, q = 3) = \pi_{n-4}\mathrm{Q}(6, q = 3)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, 3)$, and $\pi^*_{n-3}\mathrm{Q}^-(3, q = 3)$, $\pi^\perp_{n-3} \cap \mathrm{Q}(2n, q = 3) = \pi_{n-3}\mathrm{Q}(4, q = 3)$, $\mathrm{Q}^-(3, q = 3) \subseteq \mathrm{Q}(4, q = 3)$. Furthermore, a minimal blocking set of size $q^n + q^{n-2} + 1$ of $\mathrm{Q}(2n, q = 3)$ does not exist.*

Proof. Suppose that L is a line of $\mathrm{Q}(2n, 3)$, which also is a 3-secant to $\mathcal{K}$. Put $L \cap \mathcal{K} = \{r_1, r_2, r_3\}$ and $r \in L \setminus \mathcal{K}$. Then $|(r_1^\perp \cup r_2^\perp \cup r_3^\perp) \cap \mathcal{K}| \leqslant q^{n-2} + 1 + 2(q^{n-2} - 2) \leqslant q^{n-1} - 3$. So $|r^\perp \cap \mathcal{K}| \geqslant q^n + q^{n-3} + 1 - (q^{n-1} - 3) = 2q^{n-1} + q^{n-3} + 4 > q^{n-1} + q^{n-2} + 1$. So every generator through r meets $\mathcal{K}$ in at least 3 points, hence $|r^\perp \cap \mathcal{K}| \geqslant 3(q^{n-1} + 1)$. The projection of $r^\perp \cap \mathcal{K}$ from r contains at least $q^{n-1} + q^{n-4}$ points; so since r lies on 3-secants to the projected points, necessarily $|r^\perp \cap \mathcal{K}| \geqslant 3(q^{n-1} + q^{n-4})$, by the induction hypothesis. The induction hypothesis implies also that $r^\perp \cap \mathcal{K}$ is projected onto a truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, q)$, or a truncated cone $\pi^*_{n-4}\mathrm{Q}^-(3, q)$, since the projection of $\mathcal{K} \cap r^\perp$ must be a minimal blocking set of the base $\mathrm{Q}(2n-2, 3)$ of the cone $r^\perp \cap \mathrm{Q}(2n, 3)$. It follows that $|r^\perp \cap \mathcal{K}| = q^n + q^{n-3}$ or, respectively, $q^n + q^{n-2}$. Hence, $r^\perp \cap \mathcal{K}$ contains a truncated cone $\pi^*_{n-4}\mathcal{O}$, $\pi^\perp_{n-4} \cap \mathrm{Q}(2n, q = 3) = \pi_{n-4}\mathrm{Q}(6, q)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, q)$, or, respectively a truncated cone $\pi^*_{n-3}\mathrm{Q}^-(3, q)$. Since these structures are minimal blocking sets of $\mathrm{Q}(2n, q = 3)$, we conclude that $\mathcal{K}$ is necessarily equal to one of these structures. $\square$

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