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Universiteit Gent
Vakgroep Zuivere Wiskunde
en Computeralgebra

Projective Spaces and Linear Codes

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Proefschrift voorgelegd aan de Faculteit Wetenschappen
tot het behalen van de graad van
Doctor in de Wetenschappen: Wiskunde

Promotor: Prof. Dr. L. Storme

Preface

Suppose there are many important messages to be sent through a noisy communication channel. In order to give these messages some protection against errors arising from transmission through the channel, they are encoded into codewords which then are sent through the channel. The set consisting of these codewords is called a code. We will always restrict ourselves to linear codes, which are most useful for practical applications.

The best linear codes are codes with the shortest possible length n, the largest possible dimension k and the largest possible minimum distance d; because (a) the smaller the length, the faster the transmission, (b) the larger the dimension, the greater the variety of codewords that can be sent through, and (c) the larger the minimum distance, the greater the number of errors that can be corrected.

An important way of studying linear codes is by using projective geometry. Assume that we have an $[n, k, d; q]$-code C, then we can view the columns of a generator matrix as being a multi set K of points of $PG(k-1, q)$, the $(k-1)$-dimensional projective space over the field $GF(q)$. Further more, stating that there is no hyperplane intersecting K in $n-i$ points, counted with multiplicities, is equivalent with stating that there are no codewords of weight i. The multiplicity of a point in a multi set will be called the weight of the point. Hence studying $[n, k, d; q]$-codes is equivalent with studying multi sets K of size n of points in $PG(k-1, q)$ with the property that a hyperplane intersects these multi sets K in at most $n-d$ points, counted with weights.

In this thesis, it is our aim to obtain new results concerning linear codes, using techniques from projective geometry.

The two main subjects of the thesis are so-called minihypers and caps.

Results on minihypers

In a first chapter, we state general notions on linear codes and projective geometry. We describe the link between linear codes and projective geometry. And we give an overview of the results on blocking sets which are needed in later chapters.

In Chapter 2, we extend results of Hamada, Helleseth and Maekawa [40, 44]. They characterized projective minihypers in which the blocking sets which occur are lines. Our extensions of these results consist of characterizing also minihypers in which Baer subplanes occur as blocking sets.

In Chapter 3, we continue to characterize projective minihypers. This time, we limit ourselves to a particular class of minihypers in three dimensions; namely a class of minihypers which is related to maximal partial spreads. In that class of minihypers, an other type of blocking sets appears, namely projected $PG(3,p)$ onto a plane.

Next, we describe in Chapter 4 the minihypers with the same parameters as the ones in Chapter 3; but we do not longer assume the connection with maximal partial spreads. This has as a consequence that also non-projective minihypers may occur. The way to describe such minihypers is by using a weight function.

The results of Chapter 4 are extended to general dimensions in Chapter 5; where we also allow a more general form for the parameters.

Results on caps

In Chapter 6, we determine the largest size of a cap in $AG(5,3)$, and we prove that such a cap is projectively unique. Namely, the largest size of a cap in $AG(5,3)$ is 45, and such a cap is obtained by deleting an 11-hyperplane from the 56-cap in $PG(5,3)$.

In Chapter 7, we improve (if $h > 3$) the bound on the size of the largest cap in $PG(3,2^h)$ not contained in an ovoid. Furthermore, we improve (if $h > 9$) a result stating that there is no complete k-cap in $PG(3,2^h)$ for certain intervals for the size of k.

Finally, we mention in the appendix a construction of a new class of non-projective two-weight codes meeting the Griesmer bound. We give the link between strongly regular graphs and projective two-weight codes and give a partial result on a projective two-weight code meeting the Griesmer bound.

Acknowledgement

I got highly interested in incidence geometry by following the Intensive course *Galois geometry and generalized polygons* (1998) (Ghent University, April 14-25, 1998) organized by Prof. Dr. F. De Clerck. As a consequence, I did my master's thesis under the supervision of Prof. Dr. F. De Clerck.

By that time, I also had followed an interesting course on coding theory by Prof. Dr. J. A. Thas. By obtaining a grant, I was ready to prepare a PhD thesis on Projective Spaces and Linear Codes, under the supervision of Prof. Dr. L. Storme.

First of all, I wish to thank my supervisor Leo Storme. Throughout all four years, he was a continuous source of mathematical ideas and suggestions. I also would like to thank him for the careful proof reading of my manuscripts, as well as the preprints of the articles as this thesis.

I also would like to thank the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), grant No. IWT/SB/991011/ Ferret, for giving me the grant which enabled me to prepare this PhD thesis

Furthermore, I want to thank all people who supported me on a mathematical or a non-mathematical way; including

the research group Incidence Geometry at Ghent University,

my co-authors Dr. Y. Edel, Prof. Dr. I. Landjev, Prof. Dr. L. Storme,

Prof. Dr. R. Hill, Prof. Dr. T. Maruta whom I visited at Salford University (United Kingdom),

Prof. Dr. T Helleseth, whom I visited at Bergen University (Norway)

family and friends.

Ghent, January 2003
Sandy Ferret

Contents

Chapter 1

Preliminaries

Let $GF(q)$ denote the Galois field of order q.

1.1 Introduction to linear codes

Consider the field $GF(q)$ and the vector space $V(n,q)$ of dimension n over $GF(q)$.

A *code* C is a subset of $V(n,q)$. The elements of C are called *codewords*. The integer n is called the *length* of the code C.

The *Hamming distance* $d(\bar{x},\bar{y})$ between two codewords $\bar{x} = (x_1,\ldots,x_n)$ and $\bar{y} = (y_1,\ldots,y_n)$ of C is the number of positions j for which $x_j \neq y_j, 1 \leq j \leq n$.

The *minimum distance* $d(C)$ of a code C, is the minimum of all distances between two different codewords of C.

A *message* of C is a vector of $V(k,q)$. The *weight* of a vector $\overline{x}$ of $V(n,q)$ is the number of non-zero positions of $\overline{x}$.

A code C over $GF(q)$ with length n is *linear* if C is a subspace of $V(n,q)$.

A linear $[n,k,d;q]$, code C is a k-dimensional subspace of $V(n,q)$ (resp. with minimum distance d).

Write a basis of C as the rows of a $(k \times n)$-matrix G. Such a matrix is called a *generator matrix* of the code C.

Encoding of linear codes

Error-correcting codes are used to correct errors when messages are transmitted through a noisy communication channel.

The object of an error-correcting code is to encode the data by adding a certain amount of *redundancy* to the messages, so that the original messages can be recovered if (not too many) errors have occurred. The encoding of a message goes as follows.

Let C be a linear $[n, k, d; q]$-code with generator matrix G. Instead of sending the message $\bar{u}$ through the communication channel, the *codeword* $\bar{v} = \bar{u}G$ is transmitted.

Decoding of linear codes

Suppose a codeword $\bar{x}$ of C, unknown to the receiver, has been transmitted through a communication channel and assume that the vector $\bar{y}$ of $V(n,q)$, which may have been distorted by noise, is received. Assume $\bar{y} = \bar{x} + \bar{e}$. The vector $\bar{e}$ is called the *error vector* of $\bar{x}$.

The following theorem tells how many errors can be corrected in the codeword $\overline{x}$ of C.

Theorem 1.1.1 *If C is a linear $[n, k, d; q]$ code, then all errors $\bar{e}$ in $V(n,q)$ with weight at most $\frac{d-1}{2}$ can be corrected.*

Dual codes

Definition 1.1.2 Let C be a linear $[n, k, d; q]$ code. The *dual code* of C is defined as $C^{\perp} = \{\bar{x} \mid \bar{v} \cdot \bar{x} = 0, \forall \bar{v} \in C\}$ where $\bar{v} \cdot \bar{x} = \sum_{i=1}^{n} v_i x_i, \bar{x} = (x_1, \ldots, x_n), \bar{v} = (v_1, \ldots, v_n)$. The dual code $C^{\perp}$ of C is a $[n, n-k, d'; q]$-code.

Definition 1.1.3 A generator matrix of the dual code $C^{\perp}$ of a linear code C is called a *parity check matrix* of C.

Theorem 1.1.4 *Let H be a parity check matrix of a linear code C. Then d is the minimum distance of C if and only if every $d-1$ columns of H are linearly independent, and there exist d columns in H which are linearly dependent.*

Let A_i denote the number of codewords of C of weight i.

We call $A(x) = \sum_{i=0}^{n} A_i x^i$ the *weight enumerator* of C.

Let B_i denote the number of codewords of $C^{\perp}$ of weight i.

The following three problems are important problems in coding theory.

Problem 1.1.5

1. *For given values of k, d and q, find an $[n, k, d; q]$-code C whose length n is minimal.*
2. *For given values of n, d and q, find an $[n, k, d; q]$-code C whose dimension k is maximal.*

3 For given values of n, k and q, find an $[n, k, d; q]$-code C whose minimum distance d is maximal.

Codes satisfying one of the above conditions, are called *optimal codes.*

Notation 1.1.6 Let $\lceil x \rceil$ denote the smallest integer greater than or equal x, and analogously $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x.

The following bound is well-known and will be studied in great detail in this thesis (Chapters 2, 3, 4, and 5).

Theorem 1.1.7 (The Griesmer bound [36, 76]) *A linear $[n, k, d; q]$-code satisfies*

$$n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil = g_q(k, d). \tag{1.1}$$

Lemma 1.1.8 ([59]) *Let C be an $[n, k, d; q]$-code. Then $B_i = 0$ for every $1 \leq i \leq k$, for which there does not exist an $[n - i, k - i + 1, d; q]$-code.*

Lemma 1.1.9 ([59]) *Let C be an $[n, k, d; q]$-code attaining the Griesmer bound. Assume j is a positive integer such that $d \leq q^{k-j+1}$. Then $B_j = 0$.*

Remark 1.1.10 If $B_2 = 0$, then we have a *projective code.*
If $B_3 = 0$, then we have a *cap-code.*

1.2 Linear codes and projective geometry

Let $V(n+1, q)$ be the $(n+1)$-dimensional vector space over $GF(q)$. Denote the set of subspaces of $V(n+1, q)$ by $D(V)$. Let I be the incidence relation on $D(V)$ defined as $UIW \Leftrightarrow U \subseteq W$. Then the n-dimensional projective space over $GF(q)$ is the incidence structure $(D(V), I)$. If $U \in D(V)$ is an $(m+1)$-dimensional vector space, then the *(projective) dimension* of U is by definition m. For instance, with vector lines correspond *projective points*, having dimension zero. A projective space of dimension $1, 2, 3, n-1$, is called respectively *(projective line, plane, solid, hyperplane.* Note that there are $v_{i+1} = (q^{i+1} - 1)/(q - 1)$ (projective) points in an i-dimensional projective space.

Let W be a hyperplane of $PG(n, q)$, and let $D(V)^W$ be the set of subspaces not contained in W, then the incidence structure $(D(V)^W, I \cap (D(V)^W \times D(V)^W))$ is called the *n-dimensional affine space* over $GF(q)$. This is denoted by $AG(n, q)$.

An *automorphism* of $PG(n, q)$ is a bijective relation on $D(V)$ preserving incidence. The group of all automorphisms of $PG(n, q)$ is denoted by $\mathrm{P\Gamma L}(n+1, q)$.

Remark 1.2.1 An $[n, k, d; q]$-code C is *equivalent* to an other code C' if a generator matrix of C' can be obtained from the generator matrix of C by using the following operations.

1. A permutation of the rows.
2. Multiplication of a row by an element of $GF(q) \setminus \{0\}$.
3. Adding a multiple of a row to an other row.
4. Permutating the columns.
5. Multiplication of a column by an element of $GF(q) \setminus \{0\}$.

Assume the generator matrix of C does not have zero columns. If we consider the columns of the generator matrix G of C as points of $PG(k-1, q)$, then we see that the projective equivalence class of this multi set of n points corresponds exactly to the equivalence class of codes of C. Stating htat the minimum distance of the code is d, is exactly the same as stating that a hyperplane of $PG(k-1, q)$ intersects the corresponding multi set in at most $n-d$ points (counted with multiplicities).

Remark 1.2.2 We can reverse Remark 1.2.1 and start from projective geometry.
Then we can define an $[n, k, d; q]$-code C as being a multi set $\mathcal{C}$ of n points in $PG(k-1, q)$ such that each hyperplane of $PG(k-1, q)$ intersects $\mathcal{C}$ in at most $n-d$ points and there is a hyperplane which intersects $\mathcal{C}$ in exactly $n-d$ points.

Two linear codes are equivalent if there exists a collineation $\pi \in \mathrm{P\Gamma L}(k-1, q)$ which maps the corresponding multi sets on each other.

1.3 Blocking sets

Throughout the thesis, we will heavily rely on results on planar blocking sets. As will become clear later, planar blocking sets can indeed be seen as the *building blocks* of linear codes meeting the Griesmer bound.

In this section we will overview some results on blocking sets, which will be used in later chapters.

Definition 1.3.1 A *t-fold blocking set* B of $PG(n, q)$ is a set of points of $PG(n, q)$ intersecting every hyperplane of $PG(n, q)$ in at least t points.

If $t = 1$, we will speak of a *blocking set*.

A blocking set is called *minimal* when no proper subset of it is still a blocking set; and we call a blocking set *non-trivial* when it contains no line.

A planar blocking set is called *small* when it has less than $3(q+1)/2$ points.

If $q = p^h$, p prime, we call the *exponent* of the minimal planar blocking set B the maximal integer e such that every line intersects B in 1 modulo p^e points.

From a result of Szőnyi [80], it follows that $e \geq 1$, for every small non-trivial minimal blocking set in $PG(2,q)$.

Remark 1.3.2 In [80], it is proven that, if e is the exponent of a small non-trivial minimal blocking set in $PG(2,q)$, $q=p^h$, p prime, then $1 \leq e \leq h/2$, and the size of the blocking set must lie in certain intervals depending on p^e. We note that the bounds given in [80] are improved in [68] and in [70]. For t-fold blocking sets, there exists a t modulo p result; see [57, 31]

The results of [80] have been used to classify all small minimal blocking sets of $PG(2,q)$, $q=p^h$, of exponent $e \geq h/3$.

Theorem 1.3.3 (Polverino, Polverino and Storme [69, 70, 71]) *The smallest minimal blocking sets in $PG(2,p^3)$, $p=p_0^h$, p_0 prime, $p_0 \geq 7$, with exponent $e \geq h$, are:*

(1) *a line,*
(2) *a Baer subplane of cardinality $p^3+p^{3/2}+1$, when p is a square,*
(3) *a set of cardinality p^3+p^2+1, equivalent to*

$$\{(x,T(x),1) \mid x \in GF(p^3)\} \cup \{(x,T(x),0) \mid x \in GF(p^3) \setminus \{0\}\},$$

with T the trace function from $GF(p^3)$ to $GF(p)$: $T(x)=x+x^p+x^{p^2}$, see Figure 1.1,
(4) *a set of cardinality p^3+p^2+p+1, equivalent to*

$$\{(x,x^p,1) \mid x \in GF(p^3)\} \cup \{(x,x^p,0) \mid x \in GF(p^3) \setminus \{0\}\},$$

see Figure 1.2*,*

This result is also the complete classification of all small minimal blocking sets in $PG(2,p^3)$, p prime, $p \geq 7$. Regarding small minimal blocking sets in $PG(2,p^3)$, $p=p_0^h$, p_0 prime, $p_0 \geq 7$, $h>1$, of exponent smaller than h, there exists a minimal blocking set

$$\{(x,T_0(x),1) \mid x \in GF(p^3)\} \cup \{(x,T_0(x),0) \mid x \in GF(p^3) \setminus \{0\}\},$$

with T_0 the trace function from $GF(p^3)$ to $GF(p_0^l)$, with l the largest divisor of $3h$ smaller than h. This minimal blocking set has size p^3+p^3/p_0^l+1 and has exponent l. In particular, this blocking set has size at most p^3+p^3/p_0+1.

We describe more in detail the two latter blocking sets of Theorem 1.3.3.

Remark 1.3.4 (1) The minimal blocking set of size p^3+p^2+1 has a unique point, called the *vertex*, lying on exactly $p+1$ lines containing p^2+1 points of the blocking set. These $p+1$ lines form a dual $PG(1,p)$. All other lines intersect the blocking set in 1 or in $p+1$ points.
Furthermore, these (p^2+1)-sets which are the intersection of the blocking set

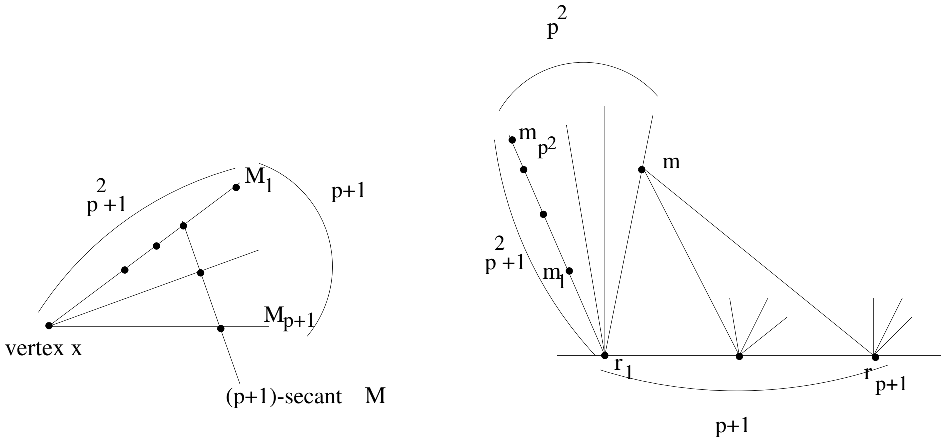

Figure 1.1: A blocking set of size $p^3 + p^2 + 1$, with its dual

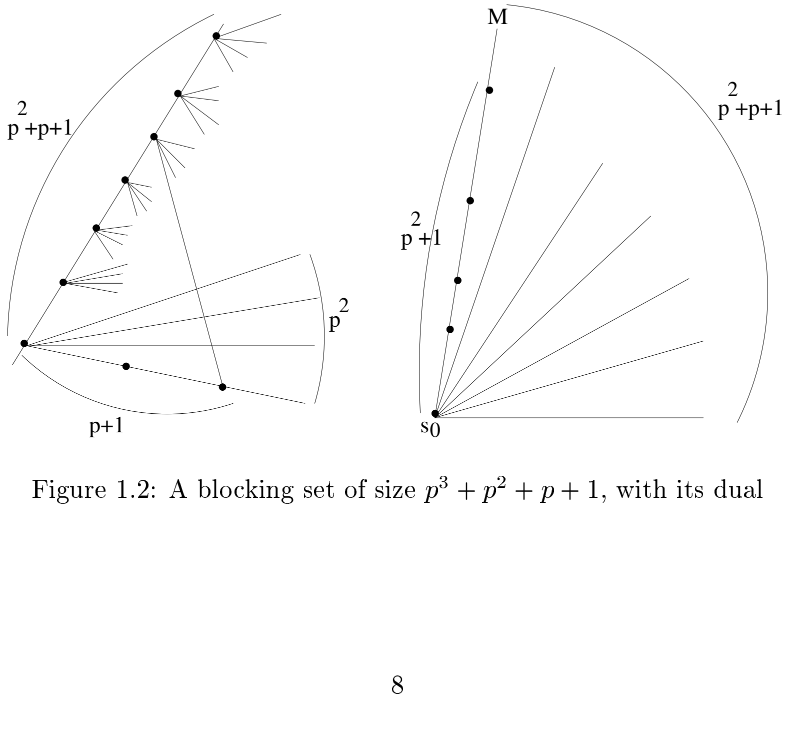

Figure 1.2: A blocking set of size $p^3 + p^2 + p + 1$, with its dual

with these (p^2+1)-secants are equivalent to the set $\{\infty\} \cup \{x \in GF(p^3) \mid x + x^p + x^{p^2} = 0\}$, with ∞ corresponding to the vertex of the blocking set.

Later on, we will refer to the point corresponding to ∞ as being the *special point* of this (p^2+1)-set.

The lines sharing $p+1$ points with this blocking set intersect the blocking set in a subline $PG(1,p)$.

(2) The minimal blocking set of size p^3+p^2+p+1 has p^2+p+1 points in common with exactly one line; all other lines intersect the blocking set in 1 or in $p+1$ points.
This (p^2+p+1)-set which is the intersection of the blocking set with the (p^2+p+1)-secant is equivalent to $\{x \in GF(p^3) \mid x^{p^2+p+1} = 1\}$. The $(p+1)$-secants intersect the blocking set in a subline $PG(1,p)$.

(3) These two blocking sets are also characterized [58] as being a projected $PG(3,p)$ in the plane $PG(2,p^3)$. Namely, embed the plane $PG(2,p^3)$ in a 3-dimensional space $PG(3,p^3)$. Consider a subgeometry $PG(3,p)$ of $PG(3,p^3)$ and a point r not belonging to this subgeometry $PG(3,p)$ and not belonging to the plane $PG(2,p^3)$.

Project $PG(3,p)$ from r onto $PG(2,p^3)$.

If the point r belongs to a line of the subgeometry $PG(3,p)$, then this $PG(3,p)$ is projected onto the blocking set of size p^3+p^2+1; else we obtain the blocking set of size p^3+p^2+p+1.

To simplify notations in this thesis, when we talk of a (p^2+1)-set or a (p^2+p+1)-set, we respectively mean a set of points on a line $PG(1,p^3)$ projectively equivalent to the set $\{\infty\} \cup \{x \mid x + x^p + x^{p^2} = 0\}$ or $\{x \mid x^{p^2+p+1} = 1\}$.

Theorem 1.3.5 (Storme and Weiner [78]) *A minimal blocking sets in $PG(N,p^3)$ of size at most p^3+p^2+p+1 ($p = p_0^h$, p_0 prime, $p_0 \geq 7$, $N \geq 3$), with respect to hyperplanes is one of the following:*

(1) *a line,*
(2) *a Baer subplane, when p is a square,*
(3) *a minimal planar blocking set of cardinality p^3+p^2+1,*
(4) *a minimal planar blocking set of cardinality p^3+p^2+p+1,*
(5) *a subgeometry $PG(3,p)$.*

Theorem 1.3.6 (Blokhuis, Storme and Szőnyi [9]) *Let B be an s-fold blocking set in $PG(2,q)$, $q = p^f$, p prime, of size $s(q+1)+c$. Let $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$.*
(1) *If $q = p^{2d+1}$ and $s < q/2 - c_p q^{2/3}/2$, then $c \geq c_p q^{2/3}$, unless $s = 1$ in which case B, with $|B| < q+1+c_p q^{2/3}$, contains a line.*
(2) *If $4 < q$ is a square, $s < q^{1/4}/2$ and $c < c_p q^{2/3}$, then $c \geq s\sqrt{q}$ and B contains the union of s disjoint Baer subplanes, except for $s = 1$ in which case*

B contains a line or a Baer subplane. If $s \geq 2$, necessarily $s < c_p q^{1/6}$.
(3) *If $q = p^2$, p prime, and $s < q^{1/4}/2$ and $c < p\lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \rceil$, then $c \geq s\sqrt{q}$ and B contains the union of s disjoint Baer subplanes, except for $s = 1$ in which case B contains a line or a Baer subplane.*

Theorem 1.3.7 (Barát and Storme [4]) *Let B be an s-fold blocking set in $PG(n, q)$, $q = p^f$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| < sq + c_p q^{2/3} - (s-1)(s-2)/2$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $s < \min(c_p q^{1/6}, q^{1/4}/2)$. Then B contains a union of s disjoint lines and/or Baer subplanes.*

Theorem 1.3.8 (Ball [2]) *A t-fold blocking set in $PG(2, q)$ which does not contain a line has at least $tq + \sqrt{tq} + 1$ points.*

If B is a t-fold blocking set in $PG(2, p)$, where $p > 3$ is prime, and if $t < p/2$ then $|B| \geq (t + 1/2)(p + 1)$, while if $t > p/2$, then $|B| \geq (t + 1)p$.

Theorem 1.3.9 (Bruen [13]) *A t-fold blocking set B in $AG(2, q)$ has size at least $(t + 1)q - t$.*

Chapter 2

Characterizations of minihypers

In view of Problem 1.1.5, it is important to determine, for given k, d and q, if there exist $[g_q(k,d),k,d;q]$-codes, and if they exist to characterize them.

This coding-theoretical problem, which has been studied in great detail can be translated into a geometrical problem on *minihypers in projective spaces*.

2.1 Introduction

We will now introduce the definition of a minihyper.

Definition 2.1.1 (Hamada and Tamari [45]) An $\{f,m;N,q\}$-*minihyper* is a pair (F,w), where F is a subset of the point set of $PG(N,q)$ and w is a weight function $w: PG(N,q) \to \mathbb{N} : x \mapsto w(x)$, satisfying
(1) $w(x) > 0 \Leftrightarrow x \in F$,
(2) $\sum_{x \in F} w(x) = f$, and
(3) $min(|(F,w) \cap H| = \sum_{x \in H} w(x) | H \in \mathcal{H}) = m$; where $\mathcal{H}$ denotes the set of hyperplanes.

In the case that w is a mapping onto $\{0,1\}$, the minihyper (F,w) can be identified with the set F and is simply denoted by F.

Suppose there exists an $[n,k,d;q]$-code meeting the Griesmer bound ($d \geq 1, k \geq 3$), then we can write d in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^{\lambda_i}$ such that $\theta \geq 1$ and $0 \leq \epsilon_i < q$.

Using this expression for d, the Griesmer bound for an $[n,k,d;q]$ code can be expressed as: $n \geq \theta v_k - \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}$ where $v_l = (q^l-1)/(q-1)$, for any integer $l \geq 0$.

Let $E(t,q)$ denote the set of all ordered tuples $(\zeta_0,\ldots,\zeta_{t-1})$ of integers ζ_i such that $(\zeta_0,\ldots,\zeta_{t-1}) \neq (0,\ldots,0)$ and $0 \leq \zeta_0 \leq q-1, 0 \leq \zeta_1 \leq q-1, \ldots, 0 \leq$

$\zeta_{t-1} \leq q-1$.

Let $\bar{E}(t,q)$ denote the set of all ordered tuples $(\zeta_0,\ldots,\zeta_{t-1})$ of integers ζ_i such that $(\zeta_0,\ldots,\zeta_{t-1}) \neq (0,\ldots,0)$ and either: (a) $(\zeta_0,\ldots,\zeta_{t-1}) \in E(t,q)$, or (b) $\zeta_0 = q, 0 \leq \zeta_1 \leq q-1,\ldots,0 \leq \zeta_{t-1} \leq q-1$, or (c) $\zeta_0 = \ldots = \zeta_{\lambda-1} = 0$, $\zeta_\lambda = q$, $0 \leq \zeta_{\lambda+1} \leq q-1,\ldots,0 \leq \zeta_{t-1} \leq q-1$ for some integer $\lambda \in \{1,\ldots,t-1\}$.

From now on, we suppose that $(\epsilon_0,\ldots,\epsilon_{k-2})$ belongs to $E(k-1,q)$

Hamada and Helleseth [42, 43] showed that there is a one-to-one correspondence between the set of all non-equivalent $[n,k,d;q]$ codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=0}^{k-2}\epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2}\epsilon_i v_{\lambda_i}; k-1,q\}$-minihypers (F,w), such that $1 \leq w(p) \leq \theta$ for every point $p \in F$.

More precisely, the link is described in the following way. Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n,k,d;q]$ code, meeting the Griesmer bound. We look at a column of G as being the coordinates of a point in $PG(k-1,q)$. Let the point set of $PG(k-1,q)$ be $\{s_1,\ldots,s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $m(G)$ be the maximum value in $\{m_i(G) \mid i = 1,2,\ldots,v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code C and we call it the *maximum multiplicity* of the code. Define the weight function $w : PG(k-1,q) \to \mathbb{N}$ as $w(s_i) = \theta - m_i(G)$, $i = 1,2,\ldots,v_k$. Let $F = \{s_i \in PG(k-1,q) \mid w(s_i) > 0\}$, then (F,w) is a $\{\sum_{i=0}^{k-2}\epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2}\epsilon_i v_{\lambda_i}; k-1,q\}$-minihyper with weight w.

Remark 2.1.2 If we restrict ourselves to the case $d < q^{k-1}$, we have projective codes (see Remark 1.1.10). This means that no different columns $g_i, i = 1,\ldots,n$, of the generator matrix of the code can correspond to the same projective point; i.e. there are no two columns which are scalar multiples of each other.
In this case, the minihyper is just $PG(k-1,q) \setminus \{g_1,\ldots,g_n\}$. In later chapters we will consider also non-projective codes meeting the Griesmer bound.

We now prove that if $d < q^{k-1}$, we always have projective codes.
Proof. Assume the generator matrix G of an $[n,k,d;q]$-code with $d < q^{k-1}$ has two linearly dependent columns. We may assume that $G = (g_1, \lambda g_1, g_3,\ldots,g_n)$. Consider the dual code $C^\perp$. Then $\bar{x} = (x_1,x_2,\ldots,x_n) \in C^\perp \Leftrightarrow \bar{x}.G^T = \bar{0} \Leftrightarrow \sum_i x_i g_i = \bar{0}$. Hence $\bar{y} = (\lambda,-1,0,\ldots,0) \in C^\perp$. Hence $B_2 > 0$; contradicting Lemma 1.1.9. $\square$

An important class of minihypers, so of linear codes meeting the Griesmer bound, is obtained by taking in $PG(k-1,q)$ a disjoint union of ϵ_0 points, ϵ_1 lines, ..., ϵ_{k-2} $(k-2)$-dimensional subspaces. Then such a set defines a $\{\sum_{i=0}^{k-2}\epsilon_i v_{i+1}, \sum_{i=0}^{k-2}\epsilon_i v_i;\ k-1,q\}$-minihyper. The linear codes associated to these minihypers are the linear codes meeting the Griesmer bound, discovered by Belov, Logachev and Sandimirov [6].

Strong results on minihypers for general values of n,k,d and q were obtained in [40, 41, 44] by Hamada, Helleseth and Maekawa, who found the following characterizations for minihypers in finite projective spaces. Here, $\mathcal{F}(\lambda_1,\ldots,\lambda_h;t,q)$ denotes the set of all unions $\cup_{i=1}^h V_i$ of λ_i-dimensional subspaces V_i of $PG(t,q)$ which are mutually disjoint, where $1 \leq h \leq t(q-1)$,

$0 \leq \lambda_1 \leq \cdots \leq \lambda_h < t$ and where at most $q-1$ of the integers λ_i take the same value [40, Definition 1.2].

Theorem 2.1.3 (Hamada, Helleseth [40], Hamada, Maekawa [44]) *Let t, q, h and λ_i, $i = 1, \dots, h$, be any integers such that $t \geq 2, h \geq 1$, $q > (h-1)^2$ and $0 \leq \lambda_1 \leq \cdots \leq \lambda_h < t$.*
(1) If $t < \lambda_{h-1} + \lambda_h + 1$, there is no $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q\}$-minihyper.
(2) If $t \geq \lambda_{h-1} + \lambda_h + 1$, then F is a $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q\}$-minihyper if and only if $F \in \mathcal{F}(\lambda_1, \lambda_2, \dots, \lambda_h; t, q)$.

Remark 2.1.4 The result of Theorem 2.1.3 is a sharp result, in the sense that for $h = \sqrt{q}+1$ there are examples of minihypers $F \notin \mathcal{F}(\lambda_1, \lambda_2, \dots, \lambda_h; t, q)$. For instance, consider a Baer subplane, which is a $\{v_2 + \sqrt{q}, 1; 2, q\}$-minihyper.

We will improve on these results by showing that for q non-square, the upper bound $h < \sqrt{q}+1$ can be replaced by $h \leq q^{6/9}/(1+q^{1/9})$ when $q = p^f$, p prime, $p > 3$, $q \geq 661$, and to $h \leq c_p q^{5/9}$ when $q = p^f$, $p = 2, 3$, $q > 7687$, $c_p = 2^{-1/3}$. We also improve the above mentioned results for q square, $q \geq 2^{12}$ if $p > 3$ and $q \geq 2^{14}$ if $p = 2, 3$, to respectively $h \leq \min(2\sqrt{q}-1, q^{6/9}/(1+q^{1/9}))$ and $h \leq \min(2\sqrt{q}-1, c_p q^{5/9})$. We show that a minihyper with the above described parameters is the disjoint union of subspaces of $PG(t, q)$, up to maybe one subgeometry defined over the subfield $GF(\sqrt{q})$ when q is square. These results appeared in Ferret and Storme, Minihypers and linear codes meeting the Griesmer bound: Improvements to results of Hamada, Helleseth and Maekawa, [30].

The following results of Hamada [37] and Hamada and Helleseth [40] play a key role in the induction arguments of the theorems and lemmas which follow.

Theorem 2.1.5 (Hamada [37]) *Let F be a $\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=1}^{t-1} \epsilon_i v_i; t, q\}$-minihyper where $t \geq 2$, $0 \leq \epsilon_i \leq q-1$, $i = 0, \dots, t-1$.*
Then there exists a $(t-2)$-dimensional subspace Δ intersecting F in $\sum_{i=1}^{t-1} \epsilon_i v_{i-1}$ points, and the hyperplanes H_j, $j = 1, \dots, q+1$, through Δ intersect F in a $\{\sum_{i=1}^{t-1} \epsilon_i v_i + \delta_j, \sum_{i=1}^{t-1} \epsilon_i v_{i-1}; t-1, q\}$-minihyper, where $\sum_{j=1}^{q+1} \delta_j = \epsilon_0$.

Theorem 2.1.6 (Hamada and Helleseth [40]) *Let F be a $\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=1}^{t-1} \epsilon_i v_i; t, q\}$-minihyper where $t \geq 2$, $h \geq 2$, $q \geq h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{t-1} \epsilon_i = h$.*
(1) If there exists a hyperplane H of $PG(t, q)$ such that $|F \cap H| = \sum_{i=1}^{t} m_i v_i$ for some ordered set $(m_1, \dots, m_t) \in \bar{E}(t, q)$, then $F \cap H$ is a $\{\sum_{i=1}^{t} m_i v_i, \sum_{i=1}^{t} m_i v_{i-1}; t-1, q\}$-minihyper in H.
(2) There does not exist a hyperplane H in $PG(t, q)$ such that $|F \cap H| = \sum_{i=1}^{t} m_i v_i$ for any ordered set $(m_1, \dots, m_t) \in \bar{E}(t, q)$ such that $\sum_{i=1}^{t} m_i > h$.
(3) In the case $\epsilon_0 = 0$ and $q \geq 2h-1$, there is no hyperplane H in $PG(t, q)$ such that $|F \cap H| = \sum_{i=1}^{t} m_i v_i$ for any ordered set $(m_1, \dots, m_t) \in \bar{E}(t, q)$ such that $\sum_{i=1}^{t} m_i < h$.

Proof. (cfr. Hamada and Helleseth [40])

1. Suppose there exists a $(t-2)$-dimensional space Δ in H such that $|F\cap\Delta| \leq \sum_{i=1}^{t} m_i v_{i-1} - 1$. Let H_l, $l = 1, 2, \ldots, q$, be the hyperplanes, different from H, through Δ. Since $|F| = \sum_{i=0}^{t-1} \epsilon_i v_{i+1}$, $|F \cap H| = \sum_{i=1}^{t} m_i v_i$ and $|F \cap H_l| \geq \sum_{i=1}^{t-1} \epsilon_i v_i$ $(l = 1, 2, \ldots, q)$, it follows from $qv_i = v_{i+1} - 1$, $q \geq h$ and $(m_1, \ldots, m_t) \neq (0, \ldots, 0)$ that

$$\begin{aligned}
|F| &= |F\cap H| + \sum_{l=1}^{q} |F \cap H_l| - q|F \cap \Delta| \\
&\geq \sum_{i=1}^{t} m_i v_i + \sum_{i=1}^{t-1} \epsilon_i (v_{i+1} - 1) + q \\
&\geq \sum_{i=1}^{t-1} \epsilon_i v_{i+1} - (h - \epsilon_0) + \sum_{i=1}^{t} m_i v_i + q \\
&> \sum_{i=0}^{t-1} \epsilon_i v_{i+1};
\end{aligned}$$

since $h \leq q$ and since $(m_1, \ldots, m_t) \neq (0, \ldots, 0)$. This is a contradiction. Hence $|F \cap \Delta| \geq \sum_{i=1}^{t} m_i v_{i-1}$ for any $(t-2)$-dimensional space Δ in H, and the result follows from [40, Theorem A.2].

2. Suppose H exists, then there exists a $(t-2)$-dimensional space Δ in H with $|F \cap \Delta| = \sum_{i=2}^{t} m_i v_{i-1}$. Using the notations of above, we find $|F| \geq \sum_{i=0}^{t-1} \epsilon_i v_{i+1} - h + \sum_{i=1}^{t} m_i > |F|$; a contradiction.

3. Let Δ be a $(t-2)$-dimensional space such that $|F \cap \Delta| = \sum_{i=2}^{t} m_i v_{i-1}$. Let H_l be as above. Then, $|F \cap H_l| = \sum_{i=1}^{t-1} \epsilon_i v_i + \delta_l$, where $\delta_l \geq 0$ $(l = 1, 2, \ldots, q)$. Hence

$$\begin{aligned}
\sum_{i=1}^{q} |F \cap (H_l \setminus \Delta)| &= |F| - |F \cap H| \\
&= \sum_{i=1}^{t-1} \epsilon_i v_{i+1} - \sum_{i=1}^{t} m_i v_i \\
&= q(\sum_{i=1}^{t-1} \epsilon_i v_i - \sum_{i=1}^{t} m_i v_{i-1}) + h - \sum_{i=1}^{t} m_i.
\end{aligned}$$

Also $|F \cap (H_l \setminus \Delta)| = |F \cap H_l| - |F \cap \Delta| = \sum_{i=1}^{t-1} \epsilon_i v_i - \sum_{i=1}^{t} m_i v_{i-1} + \delta_l$. Hence, $\sum_{l=1}^{q} \delta_l = h - \sum_{i=1}^{t} m_i$. Without loss of generality, we may assume $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_q \geq 0$. Since $1 \leq \sum_{i=1}^{t} m_i \leq h - 1$, we have $1 \leq \delta_1 \leq h - 1$. Let $\tilde{m_1} = \epsilon_1 + \delta_1$ and $\tilde{m_i} = \epsilon_i$, for $i > 1$ and let $\tilde{m_t} = 0$. Then $|F \cap H_1| = \sum_{i=1}^{t-1} \epsilon_i v_i + \delta_1 = \sum_{i=1}^{t} \tilde{m_i} v_i$. Since $1 \leq \tilde{m_1} \leq \epsilon_1 + h - 1$ and

$0 \leq \tilde{m_i} \leq h$, for $i = 2, 3, \ldots, t$; it follows from $q \geq 2h - 1 \geq \epsilon_1 + h - 1$ that $(\tilde{m_1}, \tilde{m_2}, \ldots, \tilde{m_t}) \in \bar{E}(t, q)$ and $\sum_{i=1}^{t} \tilde{m_i} = h + \delta_1 > h$; a contradiction with part (2).

$\square$

Remark 2.1.7 (1) The statement of the preceding theorem is slightly different from the one of [40, Lemma 2.2] since the latter lemma is proved for $|F \cap H| = \sum_{i=1}^{t-1} m_i v_i$. Here we allow $|F \cap H| = \sum_{i=1}^{t} m_i v_i$.
(2) If $h < q$, then Theorem 2.1.6 (2) proves that any hyperplane intersects F in a $\{\sum_{i=1}^{t} m_i v_i, \sum_{i=1}^{t} m_i v_{i-1}; t-1, q\}$-minihyper where $\sum_{i=1}^{t} m_i \leq h$; so all m_i satisfy $0 \leq m_i \leq q - 1$.

Hence, this implies that we can apply the preceding theorem inductively. This leads to the following corollary.

Corollary 2.1.8 *Let F be a $\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=1}^{t-1} \epsilon_i v_i; t, q\}$-minihyper where $t \geq 2$, $h \geq 2$, $q > h$, $0 \leq \epsilon_i \leq q - 1$, $\sum_{i=0}^{t-1} \epsilon_i = h$.*

Then a plane of $PG(t, q)$ is either contained in F or intersects F in an $\{m_0 + m_1(q + 1), m_1; 2, q\}$-minihyper with $m_0 + m_1 \leq h$.

From now on until the end of Section 2.4, let q be a square, with $q \geq 2^{12}$ if $q = p^f$, p prime, $p > 3$, and with $q \geq 2^{14}$ when $q = p^f$, $p = 2, 3$. For simplicity of notations, we will write $q \geq q_0$ where q_0 is equal to the appropriate value 2^{12} or 2^{14}.

Remark 2.1.9 We now will proceed in proving Theorems 2.4.9 and 2.5.6.

To obtain these characterization results, we will proceed by induction on the dimension of the largest dimensional subspaces and subgeometries contained in the minihypers. We first characterize in the next section minihypers in $PG(3, q)$ consisting of lines and/or a subgeometry $PG(3, \sqrt{q})$. This result is then extended to the similar result in arbitrary dimensional spaces $PG(t, q)$.

Then we investigate minihypers in $PG(t, q)$ containing the disjoint union of planes, lines, points, and at most one subgeometry $PG(l, \sqrt{q})$, with $l \leq 5$ (Section 2.3).

Then we proceed in Section 2.4 by induction on the dimension of the largest dimensional subspaces and subgeometries contained in the minihypers. The general idea of the induction principle can be described in the following way: intersecting lines contained in the minihyper eventually give us planes contained in the minihyper, intersecting planes contained in the minihyper give us 3-dimensional spaces contained in the minihyper, ..., and finally, we obtain the largest dimensional subspaces contained in the minihyper. A similar argument is used for the subgeometry over $GF(\sqrt{q})$ contained in the minihyper.

As indicated in the beginning of this remark, we first obtain a classification result on minihypers of $PG(3, q)$ which contain a disjoint union of lines and, at most one, subgeometry $PG(3, \sqrt{q})$.

2.2 First improvements to results on minihypers

Assume F to be a $\{\sum_{i=0}^{t-1}\epsilon_i v_{i+1}, \sum_{i=1}^{t-1}\epsilon_i v_i; t, q\}$-minihyper, q square, with $\sum_{i=0}^{t-1}\epsilon_i = h \leq \min(2\sqrt{q}-1, 2^{-1/3}q^{5/9})$ when $p = 2, 3$, and with $\sum_{i=0}^{t-1}\epsilon_i = h \leq \min(2\sqrt{q}-1, q^{6/9}/(1+q^{1/9}))$ when $p > 3$, and with $q \geq q_0$. For simplicity of notations, we write $\sum_{i=0}^{t-1}\epsilon_i = h \leq h_0$.

We first present a result on plane intersections with F that will already give important results for the inductive construction principle stated in Remark 2.1.9.

Remark 2.2.1 Suppose we are working in PG(t, q). It is well-known that a hyperplane PG$(t-1, q)$ intersects a subgeometry PG$(k, \sqrt{q})$ $(t \geq k \geq 2)$ in at least a subgeometry PG$(k-2, \sqrt{q})$. Namely, consider a subgeometry PG$(t, \sqrt{q})$ containing this PG$(k, \sqrt{q})$. A hyperplane intersects this PG$(t, \sqrt{q})$ in at least a PG$(t-2, \sqrt{q})$, being the intersection of the hyperplane and its conjugate hyperplane with respect to $PG(t, \sqrt{q})$. Finally in PG$(t, \sqrt{q})$, a PG$(k, \sqrt{q})$ and a PG$(t-2, \sqrt{q})$ intersect in at least a PG$(k-2, \sqrt{q})$.

Lemma 2.2.2 *A plane of $PG(t, q)$, $q \geq q_0$, not contained in F, intersects F in less than $2\sqrt{q}$ points, or in an $\{m_0 + (q+1), 1; 2, q\}$-minihyper, with $m_0 < 2\sqrt{q} - 1$, containing a line or a Baer subplane.*

Proof. By Corollary 2.1.8, a plane Π not contained in F intersects F in an $\{m_0 + m_1(q+1), m_1; 2, q\}$-minihyper, with $m_0 + m_1 \leq h < 2\sqrt{q}$.

If $m_0 = 0$, then the lemma is proved. If $m_1 = 1$, then $F \cap \pi$ contains a line or Baer subplane (Theorem 1.3.6). Assume $m_1 \geq 2$.

If $m_1 < c_p q^{1/6}$, then it follows by (2) in Theorem 1.3.6 that $m_0 \geq m_1\sqrt{q}$, i.e. $m_0 + m_1 \geq m_1(\sqrt{q}+1) \geq 2(\sqrt{q}+1)$, but this is false.

If $m_1 \geq c_p q^{1/6}$, then we have two cases.

If $F \cap \Pi$ does not contain a line, then Theorem 1.3.8 implies that $|F \cap \Pi| \geq m_1 q + \sqrt{m_1 q} + 1 \geq m_1 q + \sqrt{c_p} q^{7/12} + 1$. But this contradicts $|F \cap \Pi| = m_1(q+1) + m_0 < m_1 q + 2\sqrt{q}$ since $(m_1 q + \sqrt{c_p} q^{7/12} + 1) - (m_1 q + 2\sqrt{q}) = q^{1/2}((c_p q^{1/6})^{1/2} - 2) + 1 > 0$.

If $F \cap \Pi$ contains a line, then delete this line to obtain an $(m_1 - 1)$-fold blocking set in an affine plane. By Theorem 1.3.9, this implies $|F \cap \Pi| \geq m_1(q-1) + 1 + (q+1) = m_1(q+1) + q + 2 - 2m_1$, contradicting $m_0 < 2\sqrt{q}$. □

Remark 2.2.3 The importance of this lemma for the induction principle stated in Remark 2.1.9 is as follows.

(1) The preceding result shows that two intersecting lines contained in F define a plane completely contained in F.

(2) A second result implied by the preceding lemma is as follows. Let Π be a plane, $\Pi \not\subseteq F$, sharing two distinct Baer sublines $\tilde{L}$ and $\tilde{M}$ with F. Assume that $\tilde{L}$ and $\tilde{M}$ do not lie on a line completely contained in F. Such a plane necessarily shares a Baer subplane with F containing $\tilde{L}$ and $\tilde{M}$.

For, $|\tilde{L} \cup \tilde{M}| \geq 2\sqrt{q}$ since two distinct Baer sublines share at most 2 points. So, from Lemma 2.2.2, Π shares a line or a Baer subplane with F. If it would share a line with F, then it would share at least $q + 2\sqrt{q}$ points with F which contradicts the preceding lemma. So, it shares a Baer subplane with F. And this Baer subplane must contain $\tilde{L}$ and $\tilde{M}$, or Π would share at least $q + 2\sqrt{q}$ points with F.

Corollary 2.2.4 (1) *There does not exist an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_1; 2, q\}$-minihyper, $q \geq q_0$, where $\epsilon_0 + \epsilon_1 \leq h_0$ and $\epsilon_1 \geq 2$.*
(2) *There does not exist an $\{\epsilon_0 v_1 + \epsilon_1 v_2 + \epsilon_2 v_3, \epsilon_1 v_1 + \epsilon_2 v_2; 3, q\}$-minihyper, $q \geq q_0$, where $\epsilon_0 + \epsilon_1 + \epsilon_2 \leq h_0$ and $\epsilon_2 \geq 2$.*

Proof. Part (1) follows from the proof of the preceding lemma, and the existence of a minihyper with the parameters described in (2) would imply the existence of a minihyper with the parameters in (1) (Theorem 2.1.6 (1)). $\square$

Theorem 2.2.5 *Let F be an $\{\epsilon_0 v_1 + v_2, \epsilon_0 v_0 + v_1; t, q\}$-minihyper, $t \geq 2$, with $\epsilon_0 + 1 \leq h_0$. Then F contains a line and ϵ_0 points, or a Baer subplane and $\epsilon_0 - \sqrt{q}$ points.*

Proof. This follows from Theorem 1.3.5. $\square$

From now on, we study $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; 3, q\}$-minihypers F, $\epsilon_0 + \epsilon_1 = h$, where h satisfies the conditions mentioned at the beginning of this section. From Lemma 2.2.2, a plane which intersects F in more than $\epsilon_0 + \epsilon_1$ points intersects F in a 1-fold blocking set which contains a line or a Baer subplane. This observation will enable us to apply techniques from Hamada and Helleseth [40].

Lemma 2.2.6 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; 3, q\}$-minihyper, with $\epsilon_0 + \epsilon_1 = h \leq h_0, \epsilon_1 \geq 2$, $q \geq q_0$.*

If F does not contain a Baer subplane, then F is the disjoint union of ϵ_1 lines and ϵ_0 points.

Proof. If $\epsilon_1 < c_p q^{1/6}$, then the result follows from Theorem 1.3.7. So assume $\epsilon_1 \geq c_p q^{1/6}$. Let θ be the number of lines contained in F. Denote these lines by $L_1, \ldots, L_\theta$.

Since F does not contain a Baer subplane, every plane containing more than h points of F contains a line of F.

Corollaries 2.1.8 and 2.2.4 imply that a plane H intersects in $\epsilon_1 \leq |F \cap H| \leq h$ points or in $q + 1 \leq |F \cap H| \leq q + h$ points. Let x_i and y_k hyperplanes of $PG(3, q)$ contain respectively $i = \epsilon_1, \ldots, h$ and $q + k$, $k = 1, \ldots, h$ points of F. Let

$$a = \sum_{k=l}^{h} y_k, \quad b = \sum_{k=1}^{h} (q + k) y_k, \quad c = \sum_{k=1}^{h} \binom{q+k}{2} y_k.$$

Using standard counting arguments, we obtain

$$\sum_{i=\epsilon_1}^{h} x_i + a = v_4, \tag{2.1}$$

$$\sum_{i=\epsilon_1}^{h} ix_i + b = |F|v_3, \tag{2.2}$$

$$\sum_{i=\epsilon_1}^{h} \binom{i}{2} x_i + c = \binom{|F|}{2} v_2. \tag{2.3}$$

Calculating (2.1)$\times h\epsilon_1$ - (2.2)$\times(h+\epsilon_1-1)$+(2.3)$\times 2$, and using $\sum i = {\epsilon_1}^h (i-\epsilon_1)(i-h)x_i \leq 0$, we obtain

$$\begin{array}{c} h\epsilon_1 a - (h+\epsilon_1-1)b + 2c \geq \\ \frac{\epsilon_1 q^4 - (h-1)(h-\epsilon_1)q^3 + (h^2-h-{\epsilon_1}^2)q^2 - \epsilon_1(h-\epsilon_1 q)}{q-1}. \end{array} \tag{2.4}$$

Since $k \leq h$, in the formula for c, we have $2c \leq (q+h-1)b$. Together with $b \geq qa$ we have

$$h\epsilon_1 a - (h-1+\epsilon_1)b + 2c \leq \epsilon_1(h-q)a + qb. \tag{2.5}$$

Since there are $q+1$ planes, denoted by H_{ij} $(j = 1, \ldots, q+1)$, which contain L_i; it follows that $a = \theta(q+1)$ and

$$\begin{aligned} b &= \sum_{i=1}^{\theta}\sum_{j=1}^{q+1} |F \cap H_{ij}| = \sum_{i=1}^{\theta}((|F| - |L_i|) + |L_i|(q+1)) \\ &= \frac{\theta(q^3 + \epsilon_1 q^2 + (h-1-\epsilon_1)q - h)}{q-1} \end{aligned}$$

Together with (2.4) and (2.5), we obtain

$$\theta \geq \epsilon_1 - \frac{(h-1)(h-\epsilon_1)q^3 - (h^2 - (\epsilon_1^2 + \epsilon_1 + 1)h + \epsilon_1)q^2 - \epsilon_1^2 h}{q^4 + (\epsilon_1 h + h - 1 - \epsilon_1)q^2 - (h-\epsilon_1)q - \epsilon_1 h}.$$

The conditions on h and q imposed at the beginning of this section imply $\theta > \epsilon_1 - c_p^2 q^{1/9} - 1$.

Consider $\epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor - 1$ lines contained in F, and replace every such line by $\lfloor c_p^2 q^{1/9} \rfloor + 1$ points lying on the considered line; to obtain a set G. A plane containing one of the lines L_i contains at least $\lfloor q^{1/9} c_p^2 \rfloor + 1$ points of G; and a plane not containing one of the lines L_i, contains at least $\epsilon_1 - (\epsilon_1 - \lfloor q^{1/9} c_p^2 \rfloor - 1)$ points of G. Hence G is a $\{s(q+1) + \epsilon_0 + (\epsilon_1 - s)s, s; 3, q\}$-minihyper, where $s = \lfloor q^{1/9} c_p^2 \rfloor + 1$.

Since $|G| < sq + c_p q^{2/3} - (s-1)(s-2)/2$, Theorem 1.3.7 learns us that G is the disjoint union of lines and points.

Hence, also F consists of the disjoint union of ϵ_1 lines and ϵ_0 points since the points we added to the deleted lines from F to get the minihyper G were selected in an arbitrary way. □

Lemma 2.2.7 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; 3, q\}$-minihyper, with $\epsilon_0 + \epsilon_1 = h \leq h_0, \epsilon_1 \geq 2$, $q \geq q_0$.*

If F contains exactly one Baer subplane, then F is the mutually disjoint union of that unique Baer subplane, of $\epsilon_1 - 1$ lines, and of $\epsilon_0 - \sqrt{q}$ points.

Proof. We modify the proof of Hamada and Helleseth [40] to the case where there is a Baer subplane. Let θ be the number of lines $L_1, \ldots, L_\theta$ contained in F and let B be the Baer subplane contained in F.

We first prove that these lines and the Baer subplane B are disjoint. For suppose that a line L_i intersects B in a point r. Let M be a tangent line to F through r in the plane of B. Then there are at most $|F| - (q + \sqrt{q} + 1) - q < (\epsilon_1 - 1)(q-1)$ points left which must lie in the $q-1$ planes through M distinct from the plane of B and the plane $\langle L_i, M \rangle$. And each of these planes must contain at least $\epsilon_1 - 1$ of these points. This is impossible.

We now use similar arguments as in Lemma 2.2.6. We count the numbers x_i and y_k of planes of $PG(3,q)$ containing respectively $i = \epsilon_1, \ldots, h$ and $q + k$, $k = 1, \ldots, h$ points of F. The number of planes containing a line of F or containing B is $a = \sum_{k \geq 0} y_k = 1 + \theta v_2$. From Lemma 2.2.2, we know that any plane with at least $2\sqrt{q}$ points of F shares a line or a Baer subplane with F.

The second parameter that is used is $b = \sum_{k \geq 0} (q+k) y_k = \sum_{i=1}^{\theta} (|F| - v_2 + |L_i| v_2) +$ (the number of points of F in the plane of B). Hence, $b \leq \theta((\epsilon_1 - 1)q + h - 1 + v_2^2) + q + 2\sqrt{q}$.

Similarly as in the proof of Lemma 2.2.6,

$$\frac{\epsilon_1 q^4 - (h-1)(h-\epsilon_1)q^3 + (h^2-h)q^2 - \epsilon_1^2 q^2 - \epsilon_1(h-\epsilon_1)q}{q-1} \leq \epsilon_1(h-q)a + qb.$$

Hence, filling in the value of a and the upper bound on b, and using $h \leq 2\sqrt{q}$; we obtain after standard calculations

$$\epsilon_1 q^4 - (1 + (h-1)(h-\epsilon_1))q^3 + (h^2 - h - \epsilon_1^2 + \epsilon_1 - 2\sqrt{q} + 1)q^2$$

$$+q(\epsilon_1^2 - \epsilon_1 - 2\epsilon_1 h + 2\sqrt{q}) + \epsilon_1 h$$

$$\leq \theta(q^4 + q^2(\epsilon_1 h + h - \epsilon_1 - 1) + q(\epsilon_1 - h) - \epsilon_1 h).$$

This latter inequality implies that $\theta > \epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor - 1$ lines are contained in F. Hence, F contains a Baer subplane and at least $\epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor - 1$ lines, which all are pairwise disjoint. Replace every such line by $s = \lfloor c_p^2 q^{1/9} \rfloor + 1$ points lying on the considered line. Then a new $\{s(q+1) + \epsilon_0 + (\epsilon_1 - s)s, s; 3, q\}$-minihyper G is obtained.

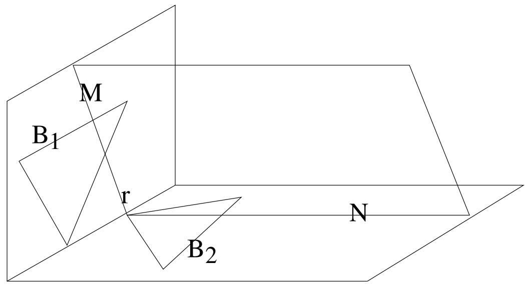

Figure 2.1: A Baer subline through which there are two Baer subplanes

Since $|G| < sq + c_p q^{2/3} - (s-1)(s-2)/2$, Theorem 1.3.7 learns us that G contains the disjoint union of one Baer subplane and ϵ_1 lines.

Hence, also F consists of the disjoint union of one Baer subplane, $\epsilon_1 - 1$ lines and $\epsilon_0 - \sqrt{q}$ points since the points we added to the deleted lines from F to get the minihyper G were selected in an arbitrary way. $\square$

Lemma 2.2.8 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; 3, q\}$-minihyper, with $\epsilon_0 + \epsilon_1 = h \leq h_0, \epsilon_1 \geq 2$, $q \geq q_0$.*

If F contains at least two Baer subplanes, then F contains a unique Baer subgeometry $PG(3, \sqrt{q})$, and there are no other Baer subplanes contained in F than those of this Baer subgeometry $PG(3, \sqrt{q})$.

Proof. Let Π_1 and Π_2 be planes both sharing a Baer subplane, B_1 and B_2 respectively, with F. Let $L = \Pi_1 \cap \Pi_2$. We show that we can find a line which shares the same Baer subline with two Baer subplanes B and C contained in F.

If L shares a Baer subline with B_1 and B_2, then these Baer sublines coincide, or else Π_1 shares at least $q + \sqrt{q} + 1 + (\sqrt{q} - 1)$ points with F, since two distinct Baer sublines share at most two points. This however is too much (Lemma 2.2.2).

Suppose that L does not share a Baer subline with B_2, then it also does not share a Baer subline with B_1, or else again $|F \cap \Pi_2| \geq q + 2\sqrt{q}$. Assume that L intersects B_2 in the point r; see Figure 2.1. Consider a secant M to B_1 passing through r and a secant N to B_2 passing through r. Then the plane $\alpha = \langle N, M \rangle$ intersects F in a minihyper containing a Baer subplane B_3 and containing the Baer subline of B_2 on N, respectively the Baer subline of B_1 on M (Remark 2.2.3 (2)).

Two Baer subplanes B and C not lying in the same plane and sharing a Baer subline define a unique subgeometry $PG(3, \sqrt{q})$.

Consider a point r in this subgeometry, then this point lies in a Baer subplane B_4 sharing distinct Baer sublines with B and with C; hence, reasoning as above, B_4 is contained in F. So the whole subgeometry $PG(3, \sqrt{q})$ is contained in F.

There cannot be any other Baer subplanes contained in F than the ones lying in $PG(3,\sqrt{q})$, since the preceding reasoning also implies that every Baer subplane contained in F lies in a subgeometry $PG(3,\sqrt{q})$ contained in F. Now two subgeometries $PG(3,\sqrt{q})$ share at most $q+\sqrt{q}+2$ points [79]; and the cardinality of F is smaller than the minimal number of points in two distinct subgeometries $PG(3,\sqrt{q})$. □

Lemma 2.2.9 *Let F be an $\{\epsilon_0 v_1+\epsilon_1 v_2, \epsilon_0 v_0+\epsilon_1 v_1; 3, q\}$-minihyper, with $\epsilon_0+\epsilon_1 = h \le h_0, \epsilon_1 \ge 2$, $q \ge q_0$.*

If F contains a subgeometry $PG(3,\sqrt{q})$, then F is the disjoint union of this subgeometry, of $\epsilon_1-\sqrt{q}-1$ lines and of ϵ_0 points.

Proof. Replace the subgeometry $PG(3,\sqrt{q})$ contained in F by a subplane $PG(2,\sqrt{q})$ contained in it to obtain a new $\{(\epsilon_1-\sqrt{q})v_2+\epsilon_0+\sqrt{q}, \epsilon_1-\sqrt{q}; 3, q\}$-minihyper G.

From Lemma 2.2.7, G contains the disjoint union of this unique Baer subplane and of $\epsilon_1-\sqrt{q}-1$ lines.

Since the Baer subplane selected in the subgeometry $PG(3,\sqrt{q}$ was arbitrarily chosen, also these latter lines in G are disjoint to the subgeometry $PG(3,\sqrt{q})$ of F. □

Theorem 2.2.10 *Let F be an $\{\epsilon_0 v_1+\epsilon_1 v_2, \epsilon_0 v_0+\epsilon_1 v_1; 3, q\}$-minihyper, $q=p^f$, p prime, q square, with $\epsilon_0+\epsilon_1 \le \min(2\sqrt{q}-1, 2^{-1/3}q^{5/9})$ when $p=2,3$, and with $\epsilon_0+\epsilon_1 = h \le \min(2\sqrt{q}-1, q^{6/9}/(1+q^{1/9}))$ when $p>3$, $q \ge q_0$.*

Then F consists of the disjoint union of either:

(1) *ϵ_1 lines and ϵ_0 points,*

(2) *ϵ_1-1 lines, one Baer subplane and $\epsilon_0-\sqrt{q}$ points,*

(3) *one subgeometry $PG(3,\sqrt{q})$, $\epsilon_1-\sqrt{q}-1$ lines and ϵ_0 points.*

Proof. This result follows from the preceding lemmas. □

This result improves the results of Hamada and Helleseth (Theorem 2.1.3) since in their results $\epsilon_0+\epsilon_1 \le \sqrt{q}$.

Theorem 2.2.11 *Let F be an $\{\epsilon_0 v_1+\epsilon_1 v_2+v_3, v_2+\epsilon_1 v_1; 3, q\}$-minihyper, with $\epsilon_0+\epsilon_1+1 \le \min(2\sqrt{q}-1, 2^{-1/3}q^{5/9})$ when $p=2,3$, and with $\epsilon_0+\epsilon_1+1 \le \min(2\sqrt{q}-1, q^{6/9}/(1+q^{1/9}))$ when $p>3$, $q \ge q_0$.*

Then $\epsilon_1 = 0$ and F consists of one plane and ϵ_0 points.

Proof. Here we already start using the ideas of the induction principle of Remark 2.1.9. We apply the methods of Hamada and Helleseth [40].

By Theorem 2.1.5, there exists a line Δ sharing exactly one point r with F. The $q+1$ hyperplanes H_i, $i=1,\ldots,q+1$, through Δ intersect F in a $\{q+1+\epsilon_1+\delta_i, 1; 2, q\}$-minihyper, with $\sum_{i=1}^{q+1}\delta_i = \epsilon_0$.

By Lemma 2.2.2, their intersections with F all either contain a line or a Baer subplane.

Assume that there are at least two such intersections $F \cap H_i$ which contain a Baer subplane. Then these Baer subplanes intersect in r only. By Remark 2.2.3 (2) or by using the proof of Case 1 of Lemma 2.3.2, these two Baer subplanes define a $PG(4, \sqrt{q})$ completely contained in F. This is impossible since we are working in $PG(3, q)$.

It is also impossible that they define a projected $PG(4, \sqrt{q})$ in $PG(3, q)$ since this is a cone with vertex a point and base a $PG(2, \sqrt{q})$. But then any two lines of this cone define a plane contained in F (Remark 2.2.3(1)). But this is impossible.

Assume now that at most one intersection $F \cap H_i$ contains a Baer subplane, then the other intersections of F with the hyperplanes through Δ contain lines passing through r. By Remark 2.2.3, this implies that F contains a plane, and the case that only one intersection contains a Baer subplane does not occur.

Replace now in F this plane by an arbitrarily selected line L in this plane. Then a new $\{\epsilon_0 + (\epsilon_1 + 1)v_2, \epsilon_1 + 1; 3, q\}$-minihyper G is obtained.

Hence, G is described by one of the possibilities in Theorem 2.2.10. All possibilities consist of disjoint unions. Apart from the line L, the possible subgeometries $PG(3, \sqrt{q})$, $PG(2, \sqrt{q})$ and lines contained in G must all be disjoint to the plane contained in F since L is an arbitrarily selected line in the plane contained in F. This means that there is no subgeometry $PG(3, \sqrt{q})$, $PG(2, \sqrt{q})$ and no other line contained in F than the lines contained in the plane of F.

Hence, F consists of one plane and ϵ_0 points. □

2.3 Further improvements to results on minihypers

First we prove two more general lemmas. Here, we consider a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=1}^{s} \epsilon_i v_i; t, q\}$-minihyper F, $t \geq 4$, with $\sum_{i=0}^{s} \epsilon_i \leq \min(2\sqrt{q} - 1, c_p q^{5/9})$ when $p = 2, 3$, $c_p = 2^{-1/3}$, and with $\sum_{i=0}^{s} \epsilon_i \leq \min(2\sqrt{q} - 1, q^{6/9}/(1 + q^{1/9}))$ when $p > 3$, $q \geq q_0$. Again, we use the notation $h = \sum_{i=0}^{s} \epsilon_i \leq h_0$.

The second lemma will play a crucial role in the induction principle stated in Remark 2.1.9.

Lemma 2.3.1 *Let F be a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=1}^{s} \epsilon_i v_i; t, q\}$-minihyper, $t \geq 4$, with $\sum_{i=0}^{s} \epsilon_i = h \leq h_0$, $q \geq q_0$.*

Then a line contained in F and a Baer subplane contained in F, with both of them not contained in planes completely contained in F, are always disjoint to each other.

Proof. Suppose a line L contained in F and a Baer subplane B contained in F intersect.

Consider the 3-dimensional space generated by L and the plane of B. This space intersects F in an $\{m_0 + m_1 v_2 + m_2 v_3, m_1 v_1 + m_2 v_2; 3, q\}$-minihyper, with $m_0 + m_1 + m_2 \leq h_0$ (Theorem 2.1.6).

Since there is a Baer subplane contained in this minihyper, it follows from Corollary 2.2.4 and Theorem 2.2.11 that $m_2 = 0$. So we are considering a minihyper described in Theorem 2.2.10. We have a contradiction. □

Lemma 2.3.2 *Let F be a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=1}^{s} \epsilon_i v_i; t, q\}$-minihyper, $t \geq 4$, with $\sum_{i=0}^{s} \epsilon_i = h \leq h_0$, $q \geq q_0$.*

Suppose F contains a subgeometry $PG(k, \sqrt{q})$ and a subgeometry $PG(l, \sqrt{q})$, $k, l \geq 1$, completely disjoint from i-dimensional subspaces, $i = 1, \ldots, s$, contained in F.

Then there is a subgeometry $PG(m, \sqrt{q})$, which contains $PG(k, \sqrt{q})$ and $PG(l, \sqrt{q})$, completely contained in F.

Proof. Case 1. Suppose that the two subgeometries $PG(k, \sqrt{q})$ and $PG(l, \sqrt{q})$ intersect.

Consider all Baer subplanes defined by intersecting Baer sublines $\tilde{L}$ and $\tilde{M}$ in respectively $PG(k, \sqrt{q})$ and $PG(l, \sqrt{q})$. Then this Baer subplane already shares two Baer sublines with F, so by the arguments of Remark 2.2.3 (2), this Baer subplane is completely contained in F. Hence, $PG(k, \sqrt{q})$ and $PG(l, \sqrt{q})$ define a subgeometry $PG(m, \sqrt{q})$, with $PG(k, \sqrt{q}), PG(l, \sqrt{q}) \subset PG(m, \sqrt{q})$, completely contained in F.
Case 2. Now we assume that $PG(k, \sqrt{q}) \cap PG(l, \sqrt{q}) = \emptyset$. Here, consider a line L sharing a Baer subline $\tilde{L}$ with $PG(k, \sqrt{q})$ and a line M sharing a Baer subline $\tilde{M}$ with $PG(l, \sqrt{q})$. Then L and M generate a 3-dimensional space intersecting F in an $\{m_0 + m_1 v_2 + m_2 v_3, m_1 v_1 + m_2 v_2; 3, q\}$-minihyper which is either the disjoint union of one plane and m_0 points, where $m_0 < 2\sqrt{q} - 1$ (Theorem 2.2.11), or of the type described in Theorem 2.2.10. In the cases of Theorem 2.2.11 and of Theorem 2.2.10 (1), (2), it is impossible that there are two disjoint Baer sublines, completely disjoint from i-dimensional subspaces, $i \geq 1$, contained in F. When Case (3) of Theorem 2.2.10 occurs, then the two Baer sublines $\tilde{L}$ and $\tilde{M}$ are contained in the subgeometry $PG(3, \sqrt{q})$ in $F \cap \langle L, M \rangle$; we now apply the result of Case 1 for the subgeometries $\langle PG(k, \sqrt{q}), PG(3, \sqrt{q}) \rangle$ and $\langle PG(l, \sqrt{q}), PG(3, \sqrt{q}) \rangle$ both contained in F. □

Remark 2.3.3 We now discuss $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; t, q\}$-minihypers F, $t \geq 4$, with $\epsilon_0 + \epsilon_1 = h \leq h_0$, $q \geq q_0$.

A hyperplane π intersects F in an $\{m_0 v_1 + m_1 v_2, m_0 v_0 + m_1 v_1; t-1, q\}$-minihyper, $m_0 + m_1 \leq h_0$.

By induction from $t = 3$ on, Theorem 2.2.10 shows that this minihyper in π consists of either:
(1) m_1 mutually disjoint lines and m_0 points,
(2) $m_1 - 1$ lines and one Baer subplane which are pairwise disjoint, together with $m_0 - \sqrt{q}$ points,
(3) a subgeometry $PG(3, \sqrt{q})$ and $m_1 - \sqrt{q} - 1$ lines, which are pairwise disjoint, together with m_0 points.

We now prove that the same result is valid for $PG(t, q)$.

Lemma 2.3.4 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; t, q\}$-minihyper, $t \geq 4$, $h = \epsilon_0 + \epsilon_1 \leq h_0$, $q \geq q_0$.*

If F contains two Baer subplanes, then F contains a Baer subgeometry $PG(3, \sqrt{q})$ and the Baer subplanes of this subgeometry are the only Baer subplanes contained in F.

Proof. This follows from Lemma 2.3.2. The subgeometry contained in F can maximally be a 3-dimensional subgeometry if one considers the cardinality of F. □

Lemma 2.3.5 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; t, q\}$-minihyper, $t \geq 4$, $\epsilon_0 + \epsilon_1 = h \leq h_0$, $q \geq q_0$.*

If F does not contain a subgeometry $PG(3, \sqrt{q})$, then F consists of either:
(1) ϵ_1 mutually disjoint lines and ϵ_0 points,
(2) $\epsilon_1 - 1$ lines and one Baer subplane which are pairwise disjoint, and $\epsilon_0 - \sqrt{q}$ points.

Proof. From the results for $t = 3$ and by induction from $t = 3$ on, we deduce that every hyperplane intersecting F in more than h points, intersects F in an $\{m_0 + m_1(q+1), m_1; t-1, q\}$-minihyper, where $m_1 \geq 1$, $m_0 + m_1 \leq h$, and where F contains either m_1 pairwise disjoint lines, or $m_1 - 1$ lines and one Baer subplane which are pairwise disjoint.

Since $t \geq 4$, a hyperplane H intersects in $\epsilon_1 \leq |F \cap H| \leq h$ points or in $l(q+1) \leq |F \cap H| \leq l(q+1) + h - 1$ points, where $l \in \{1, 2, \ldots, \epsilon_1\}$.

We do a counting of the numbers x_i and y_{lk} of hyperplanes of $PG(t,q)$ containing respectively $i = \epsilon_1, \ldots, h$ and $lq + k$, $l = 1, \ldots, \epsilon_1$, $k = l, \ldots, h$ points of F.

The same argument as in Lemma 2.2.6 shows that

$$h\epsilon_1 a - (h + \epsilon_1 - 1)b + 2c \geq \frac{\epsilon_1 q^{t+1} - (h-1)(h-\epsilon_1)q^t + (h^2-h)q^{t-1} - \epsilon_1{}^2 q^2 - \epsilon_1(h - \epsilon_1 q)}{q-1}, \tag{2.6}$$

with $a = \sum_{l=1}^{\alpha} \sum_{k=l}^{l} y_{lk}$, $b = \sum_{l=1}^{\alpha} \sum_{k=l}^{l} (lq+k) y_{lk}$ and $c = \sum_{l=1}^{\alpha} \sum_{k=l}^{l} \binom{lq+k}{2} y_{lk}$. Let $\tilde{a}$ be the number of incident pairs (line or Baer subplane contained in F, hyperplane containing this line or this Baer subplane). Then $\tilde{a} = \sum_{l,k} l y_{lk}$. More precisely, $\tilde{a} = \theta v_{t-1}$ when F contains exactly θ lines $L_1, \ldots, L_\theta$ and no Baer subplane, while $\tilde{a} = (\theta - 1) v_{t-1} + v_{t-2}$ when F contains exactly $\theta - 1$ lines $L_1, \ldots, L_{\theta-1}$ and one Baer subplane.

Let $\tilde{b} = \sum_{l,k} l(lq+k) y_{lk}$ be the number of triples (point of F, line L or Baer subplane B contained in F, hyperplane through this point of F and this line L or this Baer subplane B).

Since $q\tilde{b} + (h-1)b = \sum_{l=1}^{\alpha} \sum_{k=l}^{h} lq(lq+k) y_{lk} + (h-1) \sum_{l=1}^{\alpha} \sum_{k=l}^{h} (lq+k) y_{lk}$ and $k \leq h$, we have $2c \leq q\tilde{b} + (h-1)b$. Together with $b \geq q\tilde{a}$ and $a \leq \tilde{a}$, we get

$$h\epsilon_1 a - (h - 1 + \epsilon_1)b + 2c \leq \epsilon_1(h-q)\tilde{a} + q\tilde{b}. \tag{2.7}$$

Case 1. Suppose there is a Baer subplane B contained in F. Then the plane of B shares at most $q+2\sqrt{q}$ points (Lemma 2.2.2) with F and so $\tilde{b} \leq (q+2\sqrt{q})v_{t-2} + (|F| - (q+1+\sqrt{q}))v_{t-3} + \sum_{i=1}^{\theta-1}((|F|-|L_i|)v_{t-2} + |L_i|v_{t-1})$.

Then, by (2.6) and (2.7),

$$\begin{aligned}
&\frac{\epsilon_1 q^{t+1} - (h-1)(h-\epsilon_1)q^t + (h^2-h)q^{t-1} - \epsilon_1^2 q^2 - \epsilon_1(h-\epsilon_1)q}{q-1} \\
\leq\ & \epsilon_1(h-q)\left[(\theta-1)\left(\frac{q^{t-1}-1}{q-1}\right) + \frac{q^{t-2}-1}{q-1}\right] + q(q+2\sqrt{q})\left(\frac{q^{t-2}-1}{q-1}\right) \\
& +q((\epsilon_1-1)q + h - \sqrt{q} - 1)\left(\frac{q^{t-3}-1}{q-1}\right) + \\
& (\theta-1)(q^t + \epsilon_1 q^{t-1} + (h-1)q^{t-2} - \epsilon_1 q - h)q\left(\frac{1}{q-1}\right).
\end{aligned}$$

Assuming $h \geq \sqrt{q}+1$, for otherwise we are reduced to the results of Theorem 2.1.3, we obtain $\theta - 1 \geq \epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor - 1$. So F contains a Baer subplane and at least $\epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor - 1$ lines. Proceeding as in Lemma 2.2.7, we obtain (2) of this lemma.

Case 2. There is no Baer subplane contained in F.

Assume there are θ lines contained in F. As in Lemma 2.2.6,

$$\theta \geq \epsilon_1 - \frac{(h-1)(h-\epsilon_1)q^t - h^2 q^{t-1} + ((\epsilon_1{}^2 + \epsilon_1 + 1)h - \epsilon_1)q^{t-1} - \epsilon_1{}^2 h}{q^{t+1} + (\epsilon_1 h + h - 1)q^{t-1} - \epsilon_1 q^2 - (h-\epsilon_1)q - \epsilon_1 h},$$

and this implies $\theta \geq = \epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor$.

We now proceed as in Case 1 to obtain (1) of this lemma. □

Lemma 2.3.6 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; t, q\}$-minihyper, with $\epsilon_1 \geq 2$, $t \geq 4$, $h = \epsilon_0 + \epsilon_1 \leq h_0$, $q \geq q_0$.*

If F contains a subgeometry $PG(3, \sqrt{q})$, then F is the disjoint union of this subgeometry, of $\epsilon_1 - \sqrt{q} - 1$ lines and of ϵ_0 points.

Proof. This is proved in the same way as Lemma 2.2.9 by using the preceding lemmas. □

Theorem 2.3.7 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2, \epsilon_0 v_0 + \epsilon_1 v_1; t, q\}$-minihyper, with $\epsilon_1 \geq 2$, $t \geq 3$, $h = \epsilon_0 + \epsilon_1 \leq h_0$, $q \geq q_0$.*

Then F consists of the disjoint union of either:

(1) *ϵ_1 lines and ϵ_0 points,*

(2) *$\epsilon_1 - 1$ lines, one Baer subplane and $\epsilon_0 - \sqrt{q}$ points,*

(3) *one subgeometry $PG(3, \sqrt{q})$, $\epsilon_1 - \sqrt{q} - 1$ lines and ϵ_0 points.*

Proof. This result follows from the preceding lemmas and of Theorem 2.2.10 for $t = 3$. □

We now have characterized the considered minihypers of cardinality $\epsilon_0 v_1 + \epsilon_1 v_2$. We now concentrate on minihypers of cardinality $\epsilon_0 v_1 + \epsilon_1 v_2 + \epsilon_2 v_3$. Our goal is to prove that they contain a disjoint union of planes and lines, up to maybe one subgeometry defined over $GF(\sqrt{q})$. As indicated in Remark 2.1.9, the basic idea is to use intersecting lines contained in F to define a plane completely contained in F. The techniques we use follow from Hamada and Helleseth [40].

Lemma 2.3.8 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2 + \epsilon_2 v_3, \epsilon_0 v_0 + \epsilon_1 v_1 + \epsilon_2 v_2; t, q\}$-minihyper, $t \geq 4$, $h = \epsilon_0 + \epsilon_1 + \epsilon_2 \leq h_0$, $q \geq q_0$.*

Then F contains the disjoint union of either:
(1) *a $PG(5, \sqrt{q})$ and $\epsilon_2 - \sqrt{q} - 1$ planes,*
(2) *a $PG(4, \sqrt{q})$ and $\epsilon_2 - 1$ planes,*
(3) *ϵ_2 planes.*

Proof. By Theorem 2.1.5, there exists a $(t-2)$-dimensional subspace Δ such that $|F \cap \Delta| = \epsilon_2$. Let H_i, $i = 1, \ldots, q+1$, be the $q+1$ hyperplanes containing Δ. Then $F \cap H_i$ is an $\{\epsilon_2(q+1) + \epsilon_1 + \delta_i, \epsilon_2; t-1, q\}$-minihyper, where $\sum_{i=1}^{q+1} \delta_i = \epsilon_0$.

By Theorem 2.3.7, $F \cap H_i$ contains either: (1) ϵ_2 lines, (2) $\epsilon_2 - 1$ lines and one Baer subplane, or (3) $\epsilon_2 - \sqrt{q} - 1$ lines and one subgeometry $PG(3, \sqrt{q})$.

(1) Assume that there are at least two intersections $F \cap H_1$ and $F \cap H_2$ containing a subgeometry $PG(3, \sqrt{q})$. Call them respectively π_1 and π_2. Then Δ shares a Baer subline $PG(1, \sqrt{q})$ with these subgeometries π_1 and π_2. Since $\epsilon_2 < 2\sqrt{q}$, these Baer sublines of π_1 and π_2 in Δ coincide.

Two subgeometries $PG(3, \sqrt{q})$ which share exactly one Baer subline define a subgeometry $PG(5, \sqrt{q})$ and Lemma 2.3.2 shows that this 5-dimensional subgeometry is contained in F.

Lemma 2.3.2 also shows that there are no other Baer subplanes contained in F than the ones of this subgeometry $PG(5, \sqrt{q})$, for otherwise they would imply that F contains even a subgeometry $PG(m, \sqrt{q})$ of dimension $m > 5$, which is impossible. The $\epsilon_2 - \sqrt{q} - 1$ points of $F \cap \Delta$ not lying in this subgeometry $PG(5, \sqrt{q})$ lie on lines contained in F, if one considers the description of the intersections $F \cap H_1$, $F \cap H_2$ (Theorem 2.3.7 (3)). By Remark 2.2.3 (1), lines of F intersecting in the same point of $F \cap \Delta$ define a plane completely contained in F.

The disjointness of this subgeometry and planes contained in F can be seen by looking at the intersections of F with all hyperplanes H_i, $i = 1, \ldots, q+1$.

Hence, F satisfies (1).

(2) Assume $F \cap H_1$ is the only intersection containing a subgeometry $PG(3, \sqrt{q})$. Assume that an intersection, for instance $F \cap H_2$, contains a Baer subplane B.

Then Δ shares a Baer subline $PG(1, \sqrt{q})$ with this subgeometry $PG(3, \sqrt{q})$ in $F \cap H_1$ and shares at least one point with B.

If B and this subgeometry $PG(3, \sqrt{q})$ intersect in exactly one point, then they generate a 5-dimensional subgeometry $PG(5, \sqrt{q})$ completely contained in

F (Lemma 2.3.2). But then H_2 intersects this $PG(5, \sqrt{q})$ in a 3-dimensional subgeometry and not in a Baer subplane B.

If B and this subgeometry $PG(3, \sqrt{q})$ share a Baer subline in Δ, then a contradiction is obtained. Namely, $F \cap H_2$ contains B and exactly $\epsilon_2 - 1$ lines which all are pairwise disjoint. Then at least one of those $\epsilon_2 - 1$ lines in $F \cap H_2$ intersects the subgeometry $PG(3, \sqrt{q})$ contained in $F \cap H_1$ in a point of $F \cap \Delta$. This contradicts Lemma 2.3.1.

If B and this subgeometry $PG(3, \sqrt{q})$ in $F \cap H_1$ are disjoint, then the point $r \in B \cap \Delta$ lies on a line contained in $F \cap H_1$; this again contradicts Lemma 2.3.1.

(3) Assume two hyperplane sections, for instance $F \cap H_1$ and $F \cap H_2$, contain Baer subplanes B_1 and B_2, but none of the hyperplane sections $F \cap H_i$ contains a subgeometry $PG(3, \sqrt{q})$. Then these two Baer subplanes B_1 and B_2 intersect in the same point r of $F \cap \Delta$, or else there are intersecting lines and Baer subplanes contained in F. Then these two Baer subplanes B_1 and B_2 define a unique Baer subgeometry $PG(4, \sqrt{q})$ completely contained in F. There are no other Baer subplanes contained in F than the ones contained in $PG(4, \sqrt{q})$, or else F would even contain a higher dimensional subgeometry; so the $\epsilon_2 - 1$ points of $(F \cap \Delta) \setminus \{r\}$ lie on lines contained in F if one considers the intersections $F \cap H_i$. These lines then define $\epsilon_2 - 1$ planes completely contained in F since intersecting lines define a plane completely contained in F (Remark 2.2.3 (1)).

(4) Suppose one hyperplane intersection $F \cap H_i$, for instance $F \cap H_1$, contains a Baer subplane B or even a subgeometry $PG(3, \sqrt{q})$, but no other hyperplane intersections $F \cap H_i$, $i > 1$, contain a Baer subplane or a 3-dimensional subgeometry. Let $r \in F \cap \Delta$ be a point belonging to this subgeometry contained in $F \cap H_1$. Then r lies on a line contained in $F \cap H_2$. This contradicts Lemma 2.3.1. So this possibility does not occur.

(5) Assume that no intersections $F \cap H_i$, $i = 1, \ldots, q+1$, contain a Baer subplane $PG(2, \sqrt{q})$ or a subgeometry $PG(3, \sqrt{q})$. Then every point of $F \cap \Delta$ lies on lines contained in F if one considers the intersections $F \cap H_i$. Two lines contained in F and intersecting in the same point of $F \cap \Delta$ define a plane completely contained in F. Hence, F contains ϵ_2 pairwise disjoint planes. $\square$

Now that in the preceding lemma, the largest dimensional subspaces and subgeometries contained in F have been characterized, we can characterize these minihypers completely.

Theorem 2.3.9 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2 + \epsilon_2 v_3, \epsilon_0 v_0 + \epsilon_1 v_1 + \epsilon_2 v_2; t, q\}$-minihyper, q square, $q = p^f$, p prime, $t \geq 4$, $\epsilon_0 + \epsilon_1 + \epsilon_2 \leq \min(2\sqrt{q} - 1, c_p q^{5/9})$ when $p = 2, 3$, $c_p = 2^{-1/3}$, and with $\epsilon_0 + \epsilon_1 + \epsilon_2 \leq \min(2\sqrt{q} - 1, q^{6/9}/(1 + q^{1/9}))$ when $p > 3$, $q \geq q_0$.*

Then F consists of the disjoint union of either:
(1) *a $PG(5, \sqrt{q})$, $\epsilon_2 - \sqrt{q} - 1$ planes, ϵ_1 lines and ϵ_0 points,*
(2) *a $PG(4, \sqrt{q})$, $\epsilon_2 - 1$ planes, $\epsilon_1 - \sqrt{q}$ lines and ϵ_0 points,*
(3) *ϵ_2 planes, a subgeometry $PG(3, \sqrt{q})$, $\epsilon_1 - \sqrt{q} - 1$ lines and ϵ_0 points,*

(4) ϵ_2 *planes, a subplane* $PG(2, \sqrt{q})$, $\epsilon_1 - 1$ *lines and* $\epsilon_0 - \sqrt{q}$ *points,*
(5) ϵ_2 *planes,* ϵ_1 *lines and* ϵ_0 *points.*

For $t = 4$, *case (1) cannot occur and the other cases also do not occur when* $\epsilon_2 > 1$.

Proof. We continue the study of the minihypers made in the preceding lemma.

Suppose that F contains ϵ_2 pairwise disjoint planes. Replace these planes in F each by a line contained in them. Then a new $\{(\epsilon_2+\epsilon_1)v_2+\epsilon_0 v_1, \epsilon_1+\epsilon_2; t, q\}$-minihyper G is obtained. By Theorem 2.3.7, G is the disjoint union of either: (1) $\epsilon_1 + \epsilon_2$ lines and ϵ_0 points, (2) $\epsilon_1 + \epsilon_2 - 1$ lines, one Baer subplane, and $\epsilon_0 - \sqrt{q}$ points, or (3) one $PG(3, \sqrt{q})$, $\epsilon_1 + \epsilon_2 - \sqrt{q} - 1$ lines and ϵ_0 points. Exactly ϵ_2 lines come from the selected lines in the planes of F.

Hence, cases (3), (4) and (5) are obtained. The descriptions of F written in (3), (4) and (5) are disjoint unions since the lines selected in the planes of F to construct the new minihyper G are arbitrarily selected, and these are disjoint to the other lines, Baer subgeometries and points in G.

Suppose that we are in case (2) of the preceding lemma. Replace the subgeometry $PG(4, \sqrt{q})$ by a subgeometry $PG(3, \sqrt{q})$ contained in it, and replace the $\epsilon_2 - 1$ planes of F each by a line contained in them. Then a new $\{(\epsilon_1 + \epsilon_2)(q+1) + \epsilon_0, \epsilon_1 + \epsilon_2; t, q\}$-minihyper G is obtained. By Theorem 2.3.7, G is the disjoint union of this 3-dimensional subgeometry, $\epsilon_2 + \epsilon_1 - \sqrt{q} - 1$ lines, and ϵ_0 points. Exactly $\epsilon_2 - 1$ lines come from the selected lines in the planes of F. Hence, case (2) is obtained.

Suppose now that we are in case (1) of the preceding lemma. Now we replace the subgeometry $PG(5, \sqrt{q})$ by a subgeometry $PG(4, \sqrt{q})$ contained in it, and replace the $\epsilon_2 - \sqrt{q} - 1$ planes contained in F each by a line contained in them. Then a new $\{\epsilon_0 v_1 + (\epsilon_2 + \epsilon_1 - 1)v_2 + v_3, (\epsilon_2 + \epsilon_1 - 1)v_1 + v_2; t, q\}$-minihyper G is obtained. By case (2) proved in the preceding paragraph, G is the disjoint union of this subgeometry $PG(4, \sqrt{q})$, $\epsilon_2 + \epsilon_1 - 1 - \sqrt{q}$ lines and ϵ_0 points. Again $\epsilon_2 - \sqrt{q} - 1$ of these lines are the selected lines in the planes of F. So F is of the type described in case (1) of this theorem. □

2.4 The general results on minihypers

Note that $|\mathrm{PG}(2l+1, \sqrt{q})| = (\sqrt{q}+1)v_{l+1}$ and $|\mathrm{PG}(2l, \sqrt{q})| = v_{l+1} + \sqrt{q}v_l$.

Remark 2.4.1 Let F be a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper, where $\sum_{i=0}^{s} \epsilon_i = h \leq h_0$, $q \geq q_0$.

Now that in the preceding sections, the cases $s = 1$ and $s = 2$ have been described completely, we can characterize F by induction on s. Here we assume $s \geq 3$.

We will prove that F consists of the disjoint union of either:
(1) ϵ_s spaces $PG(s, q), \epsilon_{s-1}$ spaces $PG(s-1, q), \ldots, \epsilon_0$ points,
(2) one subgeometry $PG(2l+1, \sqrt{q})$, for some $l \leq s$, ϵ_s spaces $PG(s, q), \ldots, \epsilon_{l+1}$

spaces $PG(l+1,q), \epsilon_l - \sqrt{q} - 1$ spaces $PG(l,q), \epsilon_{l-1}$ spaces $PG(l-1,q), \ldots, \epsilon_0$ points,
(3) one subgeometry $PG(2l, \sqrt{q})$, for some $l \leq s$, ϵ_s spaces $PG(s,q), \ldots, \epsilon_{l+1}$ spaces $PG(l+1,q), \epsilon_l - 1$ spaces $PG(l,q), \epsilon_{l-1} - \sqrt{q}$ spaces $PG(l-1,q), \epsilon_{l-2}$ spaces $PG(l-2,q), \ldots, \epsilon_0$ points.

Remark 2.4.2 To obtain this characterization result, by Theorem 2.1.5, there exists a $(t-2)$-dimensional subspace Δ intersecting F in $\sum_{i=1}^{s} \epsilon_i v_{i-1}$ points. Let H_i, $i = 1, \ldots, q+1$, be the $q+1$ hyperplanes through Δ. For these hyperplanes, $|F \cap H_i| = \sum_{j=1}^{s} \epsilon_j v_j + \delta_i$, with $\sum_{i=1}^{q+1} \delta_i = \epsilon_0$.

By induction, $F \cap \Delta$ is the disjoint union of, at most one, subgeometry $PG(l, \sqrt{q})$, and of subspaces of dimension $s-2, \ldots, 0$.

Since Δ is a hyperplane in H_i, $F \cap H_i$ contains at most a subgeometry $PG(l+2, \sqrt{q})$, and subspaces of dimension $s-1, \ldots, 0$.

We will prove in the following theorems that F contains either: (1) ϵ_s disjoint subspaces $PG(s,q)$, (2) one subgeometry $PG(2s, \sqrt{q})$ and $\epsilon_s - 1$ subspaces $PG(s,q)$, or (3) one subgeometry $PG(2s+1, \sqrt{q})$ and $\epsilon_s - \sqrt{q} - 1$ subspaces $PG(s,q)$.

By induction, the largest subspaces in $F \cap \Delta$ are either: (1) ϵ_s disjoint $PG(s-2,q)$, (2) one $PG(2s-4, \sqrt{q})$ and $\epsilon_s - 1$ disjoint $PG(s-2,q)$, or (3) one $PG(2s-3, \sqrt{q})$ and $\epsilon_s - \sqrt{q} - 1$ disjoint $PG(s-2,q)$.

By induction, the largest subspaces in an intersection $H_i \cap F$ are either: (1) ϵ_s disjoint $PG(s-1,q)$, (2) one $PG(2s-2, \sqrt{q})$ and $\epsilon_s - 1$ disjoint $PG(s-1,q)$, or (3) one $PG(2s-1, \sqrt{q})$ and $\epsilon_s - \sqrt{q} - 1$ disjoint $PG(s-1,q)$.

Lemma 2.4.3 *If two intersections $F \cap H_i$, for instance when $i = 1, 2$, contain a subgeometry $PG(2s-1, \sqrt{q})$, then F contains the disjoint union of a subgeometry $PG(2s+1, \sqrt{q})$ and of $\epsilon_s - \sqrt{q} - 1$ spaces $PG(s,q)$.*

Proof. These two subgeometries $PG(2s-1, \sqrt{q})$ intersect Δ in the same subgeometry $PG(2s-3, \sqrt{q})$ since a line and a Baer subplane contained in F are always disjoint (Lemma 2.3.1). So the two subgeometries of dimension $2s-1$ define a unique subgeometry of dimension $2s+1$ completely contained in F (Lemma 2.3.2).

For the remainder, $F \cap H_1$ and $F \cap H_2$ contain $\epsilon_s - \sqrt{q} - 1$ subspaces $PG(s-1,q)$ intersecting Δ in the $\epsilon_s - \sqrt{q} - 1$ subspaces $PG(s-2,q)$ contained in $F \cap \Delta$. So two such subspaces $PG(s-1,q)$ in respectively $F \cap H_1$ and $F \cap H_2$ intersecting in the same $(s-2)$-dimensional subspace of $F \cap \Delta$ define an s-dimensional subspace completely contained in F (Remark 2.2.3 (1)). □

Lemma 2.4.4 *It is impossible that there is one intersection $F \cap H_i$, for instance for $i = 1$, containing a subgeometry $PG(2s-1, \sqrt{q})$ and an other intersection $F \cap H_j$, for instance when $j = 2$, containing a subgeometry $PG(2s-2, \sqrt{q})$, but not a subgeometry $PG(2s-1, \sqrt{q})$.*

Proof. From the induction hypotheses which led to the only possible descriptions for $F \cap \Delta$ given in Remark 2.4.2, we deduce the following contradiction.

The hyperplane H_1 intersects $\Delta \cap F$ in a subgeometry $PG(2s-3, \sqrt{q})$, $\epsilon_s - \sqrt{q} - 1$ subspaces $PG(s-2, q)$, ϵ_{s-1} spaces $PG(s-3, q), \ldots, \epsilon_2$ points, while the hyperplane H_2 intersects $\Delta \cap F$ into a subgeometry $PG(2s-4, \sqrt{q})$, $\epsilon_s - 1$ spaces $PG(s-2, q), \epsilon_{s-1} - \sqrt{q}$ spaces $PG(s-3, q), \ldots, \epsilon_2$ points.

These are distinct intersections. □

Lemma 2.4.5 *It is impossible that, for instance, $F \cap H_1$ contains a subgeometry $PG(2s-1, \sqrt{q})$, but all the other intersections $F \cap H_i$, $i > 1$, do not contain a subgeometry $PG(2s-1, \sqrt{q})$ nor a subgeometry $PG(2s-2, \sqrt{q})$.*

Proof. The description of $F \cap H_1$ would imply that $F \cap \Delta$ contains a subgeometry $PG(2s-3, \sqrt{q})$, while the description of $F \cap H_2$ would imply that $F \cap \Delta$ contains ϵ_s spaces $PG(s-2, q)$.

These are different descriptions. □

Lemma 2.4.6 *Assume that no intersections $F \cap H_i$, $i = 1, \ldots, q+1$, contain a subgeometry $PG(2s-1, \sqrt{q})$.*

If two intersections $F \cap H_i$, for instance when $i = 1, 2$, contain a subgeometry $PG(2s-2, \sqrt{q})$, then F contains the disjoint union of a subgeometry $PG(2s, \sqrt{q})$ and $\epsilon_s - 1$ spaces $PG(s, q)$.

Proof. From the induction hypothesis, $F \cap H_1$ and $F \cap H_2$ contain the disjoint union of a $PG(2s-2, \sqrt{q})$ and $\epsilon_s - 1$ spaces $PG(s-1, q)$.

So $F \cap \Delta$ contains a subgeometry $PG(2s-4, \sqrt{q})$ and $\epsilon_s - 1$ spaces $PG(s-2, q)$.

So the two subgeometries of dimension $2s-2$ in $F \cap H_1$ and $F \cap H_2$ share a subgeometry of dimension $2s-4$; so they define a subgeometry $PG(2s, \sqrt{q})$ completely contained in F (Lemma 2.3.2).

The $\epsilon_s - 1$ subspaces of dimension $s-1$ in respectively $F \cap H_1$ and $F \cap H_2$ together define $\epsilon_s - 1$ subspaces of dimension s contained in F. □

Lemma 2.4.7 *Assume that no intersections $F \cap H_i$, $i = 1, \ldots, q+1$, contain a subgeometry $PG(2s-1, \sqrt{q})$.*

It is impossible that, for instance, $F \cap H_1$ contains a subgeometry $PG(2s-2, \sqrt{q})$, and all the other intersections $F \cap H_i$, $i > 1$, contain ϵ_s disjoint subspaces $PG(s-1, q)$.

Proof. The description of $F \cap H_1$ would imply the existence of a subgeometry $PG(2s-4, \sqrt{q})$ in $F \cap \Delta$ while the description of $F \cap H_2$ would imply that such a subgeometry does not belong to $F \cap \Delta$. □

Lemma 2.4.8 *If all intersections $F \cap H_i$, $i = 1, \ldots, q+1$, contain ϵ_s subspaces $PG(s-1, q)$, then F contains ϵ_s pairwise disjoint subspaces $PG(s, q)$.*

Proof. This is proved in a similar way as the preceding lemmas. □

Theorem 2.4.9 *Let F be a $\{\sum_{i=0}^{s}\epsilon_i v_{i+1}, \sum_{i=0}^{s}\epsilon_i v_i; t, q\}$-minihyper, q square, where $\sum_{i=0}^{s}\epsilon_i \leq \min(2\sqrt{q}-1, c_p q^{5/9})$, $c_p = 2^{-1/3}$, when $q = p^f$, p prime, $p = 2, 3$, $q \geq 2^{14}$, and where $\sum_{i=0}^{s}\epsilon_i \leq \min(2\sqrt{q}-1, q^{6/9}/(1+q^{1/9}))$ when $q = p^f$, p prime, $p > 3$, $q \geq 2^{12}$.*

Then F consists of the disjoint union of either:
(1) ϵ_s spaces $PG(s,q), \epsilon_{s-1}$ spaces $PG(s-1,q), \ldots, \epsilon_0$ points,
(2) one subgeometry $PG(2l+1,\sqrt{q})$, for some integer l with $1 \leq l \leq s$, ϵ_s spaces $PG(s,q), \ldots, \epsilon_{l+1}$ spaces $PG(l+1,q), \epsilon_l - \sqrt{q} - 1$ spaces $PG(l,q), \epsilon_{l-1}$ spaces $PG(l-1,q), \ldots, \epsilon_0$ points,
(3) one subgeometry $PG(2l,\sqrt{q})$, for some integer l with $1 \leq l \leq s$, ϵ_s spaces $PG(s,q), \ldots, \epsilon_{l+1}$ spaces $PG(l+1,q), \epsilon_l - 1$ spaces $PG(l,q), \epsilon_{l-1} - \sqrt{q}$ spaces $PG(l-1,q), \epsilon_{l-2}$ spaces $PG(l-2,q), \ldots, \epsilon_0$ points.

Proof. (1) Assume that, following the results of the preceding lemmas, F contains the disjoint union of ϵ_s subspaces $PG(s,q)$. Replace each s-dimensional subspace Π_s of F by an $(s-1)$-dimensional subspace contained in Π_s. Then a new $\{(\epsilon_s+\epsilon_{s-1})v_s + \sum_{i=0}^{s-2}\epsilon_i v_{i+1}, (\epsilon_s+\epsilon_{s-1})v_{s-1} + \sum_{i=0}^{s-2}\epsilon_i v_i; t, q\}$-minihyper G is obtained.

From the induction hypothesis, G consists of the disjoint union of $\epsilon_s + \epsilon_{s-1}$ subspaces $PG(s-1,q)$, and for the remainder the disjoint union of lower dimensional subspaces, up to the occurrence of at most one subgeometry.

Exactly ϵ_s subspaces $PG(s-1,q)$ arise from the selected $PG(s-1,q)$ in the s-dimensional subspaces of F. Hence, F is of the description given in the theorem.

(2) Suppose that F contains a subgeometry $PG(2s,\sqrt{q})$ and $\epsilon_s - 1$ subspaces $PG(s,q)$. Now replace the subgeometry $PG(2s,\sqrt{q})$ by a subgeometry $PG(2s-1,\sqrt{q})$ contained in it and replace each one of the $\epsilon_s - 1$ subspaces $PG(s,q)$ of F by a hyperplane contained in it. Then an $\{(\epsilon_s+\epsilon_{s-1})v_s + \sum_{i=0}^{s-2}\epsilon_i v_{i+1}, (\epsilon_s+\epsilon_{s-1})v_{s-1} + \sum_{i=0}^{s-2}\epsilon_i v_i; t, q\}$-minihyper G is obtained. By induction, G consists of the disjoint union of one subgeometry $PG(2s-1,\sqrt{q})$, $\epsilon_s+\epsilon_{s-1}-\sqrt{q}-1$ subspaces $PG(s-1,q)$, ϵ_{s-2} spaces $PG(s-2,q), \ldots, \epsilon_0$ points. Continuing as in (1) proves the theorem for this case.

(3) If F contains a subgeometry $PG(2s+1,\sqrt{q})$ and $\epsilon_s - \sqrt{q} - 1$ spaces $PG(s,q)$, then replace in F this subgeometry $PG(2s+1,\sqrt{q})$ by a hyperplane $PG(2s,\sqrt{q})$ contained in it, and replace each one of the s-dimensional spaces contained in F by an $(s-1)$-dimensional subspace contained in it. A new $\{v_{s+1} + (\epsilon_s+\epsilon_{s-1}-1)v_s + \sum_{i=0}^{s-2}\epsilon_i v_{i+1}, v_s + (\epsilon_s+\epsilon_{s-1}-1)v_{s-1} + \sum_{i=0}^{s-2}\epsilon_i v_i; t, q\}$-minihyper G is obtained. Such a minihyper G was completely described in the preceding paragraph; hence, similar arguments lead to the description of F stated in this theorem. □

From Theorem 2.4.9 and the correspondence between minihypers and linear codes meeting the Griesmer bound, we have proven the following results on linear codes meeting the Griesmer bound.

Corollary 2.4.10 *Let k, q, h and λ_i ($i = 1, 2, \ldots, h$) be any integers such that*

$k \geq 3$, $h < 2\sqrt{q}$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_h < k-1$. Let

$$d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}, \qquad n = v_k - \sum_{i=1}^{h} v_{\lambda_i+1}.$$

1 In the case $k \geq \lambda_{h-1} + \lambda_h + 2$, we obtained the complete classification of linear $[n,k,d;q]$-codes meeting the Griesmer bound.

2 In the case $k \leq \lambda_{h-1} + \lambda_h + 1$, there is no linear $[n,k,d;q]$-code meeting the Griesmer bound.

2.5 Results for q non-square

Let $q = p^f$, p prime, $f \geq 1$, let $c_p = 1$ for $p > 3$, and let $c_2 = c_3 = 2^{-1/3}$.

We will now prove that a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper, where $h = \sum_{i=0}^{s} \epsilon_i \leq q^{6/9}/(1+q^{1/9})$ for $p > 3$, $q \geq 661$, and $h = \sum_{i=0}^{s} \epsilon_i \leq c_p q^{5/9}$ for $p = 2, 3$, $q > 7687$, is the disjoint union of ϵ_s spaces $PG(s,q)$, ϵ_{s-1} spaces $PG(s-1,q)$, …, ϵ_0 points. We collect the two upper bounds on h within the notation $h \leq h_0$, and we suppose that q satisfies the lower bounds imposed above.

We apply the techniques from the preceding sections. Since we do not need to consider Baer subgeometries, we only focus on the main steps.

Lemma 2.5.1 *Let F be a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper, where $h = \sum_{i=0}^{s} \epsilon_i \leq h_0$, $q \geq q_0$.*

Then every plane, not contained in F, intersects F in at most h points, or in a 1-fold blocking set of size smaller than or equal to $q+h$. In this latter case, this plane shares a line with F.

Proof. Assume that Π is this plane, and assume that $F \cap \Pi$ is an $\{m_0 + m_1(q+1), m_1; 2, q\}$-minihyper, with $m_1 \geq 2$.

Then Theorem 1.3.6 implies that $m_0 \geq c_p q^{2/3}$. This is false since $m_0 \leq q^{5/9}$. Hence $m_1 = 1$ and Π shares a line with F (Theorem 1.3.6). □

Remark 2.5.2 (1) As in Remark 2.2.3 (1), the preceding lemma implies that two intersecting lines L and M, completely contained in a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper F, where $\sum_{i=0}^{s} \epsilon_i \leq h_0$, define a plane completely contained in F.

(2) And as for q square (Corollary 2.2.4 (2)), it also implies that there is no $\{\epsilon_0 v_1 + \epsilon_1 v_2 + \epsilon_2 v_3, \epsilon_1 v_1 + \epsilon_2 v_2; 3, q\}$-minihyper, with $\epsilon_0 + \epsilon_1 + \epsilon_2 \leq h_0$ and with $\epsilon_2 > 1$.

Lemma 2.5.3 *Let F be an $\{\epsilon_0 + \epsilon_1(q+1), \epsilon_1; 3, q\}$-minihyper, where $h = \epsilon_0 + \epsilon_1 \leq h_0$, $q \geq q_0$.*

Then F consists of ϵ_1 pairwise disjoint lines and of ϵ_0 points.

Proof. Let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x.

We follow the proof of Lemma 2.2.6. The division made in the proof of Lemma 2.2.6 gives $\theta > \epsilon_1 - c_p^2 q^{1/9} - 1$.

Consider $\epsilon_1 - \lfloor c_p^2 q^{1/9} \rfloor - 1$ lines contained in F, and replace every such line by $\lfloor c_p^2 q^{1/9} \rfloor + 1$ points lying on the considered line. Then a new $\{s(q+1) + \epsilon_0 + (\epsilon_1 - s)s, s; 3, q\}$-minihyper G, with $s = \lfloor q^{1/9} c_p^2 \rfloor + 1$, is obtained.

Now Theorem 1.3.7 learns us that G is the disjoint union of lines and points.

Hence, also F consists of the disjoint union of ϵ_1 lines and ϵ_0 points. □

Lemma 2.5.4 *Let F be an $\{\epsilon_0 v_1 + \epsilon_1 v_2 + v_3, \epsilon_1 v_1 + v_2; 3, q\}$-minihyper, with $\epsilon_0 + \epsilon_1 + 1 = h \leq h_0$, $q \geq q_0$. Then $\epsilon_1 = 0$ and F consists of one plane and ϵ_0 points.*

Proof. Let Δ be a line only intersecting F in one point. Then the $q+1$ planes H_i, $i = 1, \ldots, q+1$, through Δ intersect F in $\{q+1+\epsilon_1+\delta_i, 1; 2, q\}$-minihypers, where $\sum_{i=1}^{q+1} \delta_i = \epsilon_0$; so $1 + \epsilon_1 + \delta_i \leq h$, and so all the planes H_i through Δ intersect F in a 1-fold blocking set containing a line L_i.

These lines L_i, $i = 1, \ldots, q+1$, intersect in the unique point of $\Delta \cap F$; so F contains a plane α (Remark 2.5.2 (1)).

Replace the plane α by a line L contained in it. Then a new $\{\epsilon_0 + (\epsilon_1 + 1)(q+1), \epsilon_1 + 1; 3, q\}$-minihyper G is obtained. So, by the preceding lemma, G is the disjoint union of $\epsilon_1 + 1$ lines and ϵ_0 points; so F is the disjoint union of one plane, ϵ_1 lines and ϵ_0 points. Since we are working in $PG(3, q)$, $\epsilon_1 = 0$. □

Lemma 2.5.5 *Let F be an $\{\epsilon_0 + \epsilon_1(q+1), \epsilon_1; t, q\}$-minihyper, where $h = \epsilon_0 + \epsilon_1 \leq h_0$, $t \geq 4$, $q \geq q_0$.*

Then F consists of ϵ_1 pairwise disjoint lines and of ϵ_0 points.

Proof. This is proven by using the ideas of Lemma 2.5.3 and of Lemma 2.3.5, Case 2. □

Theorem 2.5.6 *Let F be a $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper, $t \geq 3$, where*
(1) $\sum_{i=0}^{s} \epsilon_i \leq q^{6/9}/(1 + q^{1/9})$, $q = p^f$, f odd, p prime, $p > 3$, $q \geq 661$,
(2) $\sum_{i=0}^{s} \epsilon_i \leq c_p q^{5/9}$, $q = p^f$, f odd, $p = 2, 3$, $q > 7687$, $c_p = 2^{-1/3}$.

Then F is the disjoint union of ϵ_s spaces $PG(s, q)$, ϵ_{s-1} spaces $PG(s-1, q)$, $\ldots$, ϵ_0 points.

Proof. This is now proved by induction on s by using the techniques of Lemma 2.3.8 and Theorem 2.3.9, and by using the techniques of Section 2.4. □

Remark 2.5.7 It is known (cf. Hamada [39],NH:93) that in the case $(h, q) = (3, 3), (3, 4), (4, 3)$ or $(4, 4)$, Theorems 2.1.3, 2.4.9 and 2.5.6 do not hold.

From Theorem 2.5.6 and the correspondence between minihypers and linear codes meeting the Griesmer bound, we have proven the following results on linear codes meeting the Griesmer bound.

Corollary 2.5.8 *Let k, q, h and λ_i $(i = 1, 2, \ldots, h)$ be any integers such that $k \geq 3$, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_h < k - 1$ and either*
(1) $h \leq q^{6/9}/(1 + q^{1/9})$, $q = p^f$, f odd, p prime, $p > 3$, $q \geq 661$, or
(2) $h \leq c_p q^{5/9}$, $q = p^f$, f odd, $p = 2, 3$, $q \geq 7687$, $c_p = 2^{-1/3}$. Let

$$d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}, \qquad n = v_k - \sum_{i=1}^{h} v_{\lambda_i+1}.$$

1 *In the case $k \geq \lambda_{h-1} + \lambda_h + 2$, C is a linear $[n, k, d; q]$-code meeting the Griesmer bound if and only if C is an $[n, k, d; q]$-code of the type discovered by Belov, Logachev and Sandimirov [6].*

2 *In the case $k \leq \lambda_{h-1} + \lambda_h + 1$, there is no linear $[n, k, d; q]$-code meeting the Griesmer bound.*

Chapter 3

Minihypers and maximal partial spreads

This chapter classifies all $\{\delta(q+1), \delta; 3, q\}$-minihypers, δ small, $q = p_0^h$, $h \geq 1$, for a prime number $p_0 \geq 7$, which arise from a maximal partial spread of deficiency δ. When q is a third power, the minihyper is the disjoint union of projected $PG(5, \sqrt[3]{q})$; when q is a square, also Baer subgeometries $PG(3, \sqrt{q})$ can occur. This leads to a discrete spectrum for the small values of the deficiency δ of the corresponding maximal partial spreads.

3.1 Known results

An *s-spread* of $PG(N, q)$ is a set of s-dimensional subspaces which partitions the point set of $PG(N, q)$. There exist s-spreads in $PG(N, q)$ if and only if $(s+1)|(N+1)$.
If $s = 1$, we have a partition of the point set of $PG(N, q)$ in lines, and we shortly speak of a *spread*.

A *partial s-spread* S of $PG(N, q)$ is a set of mutually skew s-dimensional subspaces of $PG(N, q)$. We call a partial s-spread *maximal* if it cannot be extended to a larger (partial) s-spread.

A point of $PG(N, q)$ not contained in an element of the partial s-spread S is called a *hole* of S. Suppose $(s + 1)|(N + 1)$ and that S is a maximal partial s-spread of $PG(N, q)$ of size $|S| = (q^{N+1} - 1)/(q^{s+1} - 1) - \delta$. We call δ the *deficiency* of S. We will study the set of holes of S by means of so-called *minihypers*.

In [34], it has been proven that the set of holes of a maximal partial s-spread S in $PG(t, q)$, $(s+1)|(t+1)$, of deficiency $\delta < q$ forms a $\{\delta v_{s+1}, \delta v_s; t, q\}$-minihyper F. Our goal is to prove that there is a discrete spectrum of small values of $\delta > 0$ for which there exists a maximal partial spread in $PG(3, q)$, q a cube power. Presently, the best known results are of Metsch and Storme; see Theorems 3.1.2-3.1.5.

A plane of $PG(3,q)$ not containing a line of a partial spread S is called a *poor plane* of S, and a plane containing a line of S is called a *rich plane* of S.

The following properties are fundamental in the study of maximal partial spreads of $PG(3,q)$ having deficiency δ.

Lemma 3.1.1 (1) *The poor planes of S are the planes containing exactly $q+\delta$ holes; the rich planes contain exactly δ holes.*
(2) *The holes in a poor plane form a non-trivial blocking set in the poor plane.*

Proof. For a proof, we refer to [60, Lemma 2.1]. □

In the calculations, we will restrict ourselves to the case $\delta \leq q/8$. For the exact upper bound on δ, we refer to Remark 3.6.12.

Since the definition of maximal partial spread is self-dual, if we consider the poor planes through a hole r, then they form a dual blocking set in the quotient geometry of r. We will always describe this quotient geometry by means of a plane π_r skew to r, and denote the dual blocking set of poor planes in π_r by ${B_r}^D$.
Using the results on non-trivial minimal blocking sets and the self-duality in the definition of a maximal partial spread, the following results on maximal partial spreads in $PG(3,q)$ have been obtained.

Theorem 3.1.2 (Metsch and Storme [60]) *Suppose that δ is a positive integer and q is a square prime power, $q > 4$, such that*
(1) $2\delta \leq q+1$,
(2) *every non-trivial blocking set of $PG(2,q)$, q square, $q > 4$, with at most $q+\delta$ points contains a Baer subplane.*

Let S be a maximal partial spread of $PG(3,q)$ with $q^2+1-\delta$ lines, then
(a) $\delta = s(\sqrt{q}+1)$ *for an integer $s \geq 2$,*
(b) *the set of holes is the disjoint union of s Baer subgeometries $PG(3,\sqrt{q})$.*

Corollary 3.1.3 (Metsch and Storme [60]) (a) *Suppose that δ is a positive integer and q square, $q = p^h$, $h > 2$, $p \geq 5$, p prime, such that $\delta < q^{2/3}+1$.*

If S is a maximal partial spread of $PG(3,q)$ with $q^2+1-\delta$ lines, then $\delta = s(\sqrt{q}+1)$ for an integer $s \geq 2$ and the set of holes is the disjoint union of s Baer subgeometries $PG(3,\sqrt{q})$.

(b) *Suppose that δ is a positive integer and $q = p^2$, p prime, $q > 4$, such that $2\delta \leq q+1$.*

If S is a maximal partial spread of $PG(3,q)$ with $q^2+1-\delta$ lines, then $\delta = s(\sqrt{q}+1)$ for an integer $s \geq 2$ and the set of holes is the disjoint union of s Baer subgeometries $PG(3,\sqrt{q})$.

Theorem 3.1.4 (Metsch and Storme [60]) *Let S be a maximal partial spread of $PG(3,p^3)$, $p = p_0^h$, $h \geq 2$, h even, p_0 prime, $p_0 \geq 7$, of deficiency $0 < \delta \leq p^2+p+1$. Then $\delta \equiv 0 \pmod{p^{3/2}+1}$, $\delta \geq 2(p^{3/2}+1)$, and the set of holes is* *the union of disjoint subgeometries* $PG(3, p^{3/2})$, *or* $\delta = p^2 + p + 1$ *and the set of holes is a projected subgeometry* $PG(5, p)$ *in* $PG(3, p^3)$.

Theorem 3.1.5 (Metsch and Storme [60]) *Let* S *be a maximal partial spread of* $PG(3, p^3)$, $p = p_0^h$, $h \geq 1$, h *odd,* p_0 *prime,* $p_0 \geq 7$, *of deficiency* $0 < \delta \leq p^2 + p + 1$. *Then* $\delta = p^2 + p + 1$ *and the set of holes is a projected subgeometry* $PG(5, p)$ *in* $PG(3, p^3)$.

The goal of this chapter is to improve the two latter results. By investigating in detail the sets of holes in poor planes, we can again obtain a discrete spectrum for the deficiencies δ of maximal partial spreads in $PG(3, p^3)$. The results are taken from Ferret and Storme, Results on maximal partial spreads in $PG(3, p^3)$ and on related minihypers, [29].

3.2 Introductory results

Let $q = p^3$, $p = p_0^h$, $p_0 \geq 7$ a prime.

We recall Theorem 1.3.3 and Remark 1.3.4, since we will have to investigate these minimal blocking sets in poor planes and also their dual minimal blocking sets in the quotient geometries of holes.
As mentioned in Section 3.1, a plane not containing a line of the maximal partial spread is called a *poor* plane. We now present some useful standard results on poor planes and holes. For explicit proofs, we refer to [33].

Lemma 3.2.1 *Let* S *be a maximal partial spread of* $PG(3, q)$ *of deficiency* $\delta < q$. *Let* F *be the* $\{\delta(q + 1), \delta; 3, q\}$*-minihyper of holes of* S.
(1) *A line* L *contains* α *points of* F *if and only if there are exactly* α *poor planes through* L.
(2) *Through a point of* F, *there are exactly* $q + \delta$ *poor planes. Through a point not belonging to* F, *there are exactly* δ *poor planes.*

Our goal is to describe the minihyper associated to the maximal partial spread as a union of pairwise disjoint $PG(3, \sqrt{q})$ and projected subgeometries $PG(5, q^{1/3})$. To find these (projected) subgeometries, we will use the minimal blocking sets, completely consisting of holes, in the poor planes. The dual blocking set of poor planes ${B_r}^D$ through a hole will play an important role here.

As a first objective, we study the sizes of intersection of distinct $(p^2 + p + 1)$-sets and $(p^2 + 1)$-sets, described in Remark 1.3.4.

3.3 Intersections of $(p^2 + p + 1)$-sets and $(p^2 + 1)$-sets

In this section we will show that distinct $(p^2 + p + 1)$-sets and $(p^2 + 1)$-sets of Remark 1.3.4 intersect in only a small number of points. Assume $p = p_0^h$ for a prime $p_0 \geq 7$.

Lemma 3.3.1 *Two different (p^2+p+1)-sets share at most $2p+2$ points.*

Proof. All such sets are projectively equivalent to $\{x \in GF(p^3) \mid x^{p^2+p+1} = 1\}$. Hence we can take as sets $\{x \mid x^{p^2+p+1} = 1\}$ and $\{x \mid ((ax+b)/(cx+d))^{p^2+p+1} = 1\}$, where $ad - bc \neq 0$.

This latter equality can be rewritten as $(ax+b)^{p^2+p+1} = (cx+d)^{p^2+p+1}$, which is equal to

$$\begin{aligned} & a^{p^2+p+1}x^{p^2+p+1} + a^{p^2+p}x^{p^2+p}b + a^{p^2+1}x^{p^2+1}b^p + a^{p^2}x^{p^2}b^{p+1} \\ & \quad + b^{p^2}x^{p+1}a^{p+1} + b^{p^2+1}x^pa^p + b^{p^2+p}ax + b^{p^2+p+1} \\ = & \ c^{p^2+p+1}x^{p^2+p+1} + c^{p^2+p}x^{p^2+p}d + c^{p^2+1}x^{p^2+1}d^p + c^{p^2}x^{p^2}d^{p+1} \\ & \quad + d^{p^2}x^{p+1}c^{p+1} + d^{p^2+1}x^pc^p + d^{p^2+p}cx + d^{p^2+p+1}. \end{aligned}$$

Substituting $x^{p^2+p+1} = 1$ in $x^{p^2+p+1}, x^{p^2+p}, x^{p^2+1}, x^{p^2}$ and multiplying again by x^{p+1} to obtain a polynomial expression in x yields a polynomial equation of degree $2(p+1)$. Since we assume the two (p^2+p+1)-sets to be different, this shows that they share at most $2p+2$ points. □

Lemma 3.3.2 *A (p^2+p+1)-set and a (p^2+1)-set share at most $2p+2$ points.*

Proof. As intersection points different from the special point, we can take solutions of $X^{p^2+p+1} = 1$ and $((aX+b)/(cX+d))^{p^2} + ((aX+b)/(cX+d))^p + (aX+b)/(cX+d) = 0$. Again manipulating these equations to reduce the degree of one of them as much as possible, this reduces to a polynomial equation of degree $2p+1$. Together with the special point, we obtain at most $2p+2$ points in the intersection of the two sets, unless this polynomial equation reduces to the identity.

Suppose the (p^2+1)-set is contained in the (p^2+p+1)-set, then using the cyclic group of the (p^2+p+1)-set, we can assume that the element $X = 0$ of the (p^2+1)-set is mapped onto $X = 1$. For such a mapping, we can assume $b = d = 1$. After substituting $X^{p^2+p+1} = 1$ in $((aX+1)/(cX+1))^{p^2} + ((aX+1)/(cX+1))^p + (aX+1)/(cX+1) = 0$, after multiplying numerator and denominator in the first quotient by X^{p+1}, and after multiplying by $(c^{p^2} + X^{1+p})(c^pX^p + 1)$, the equation

$$\begin{aligned} & (a^{p^2} + X^{1+p})(c^pX^p + 1) + (a^pX^p + 1)(c^{p^2} + X^{1+p}) \\ & \quad + (aX+1)(c^{p^2} + X^{1+p})(cX+1)^{p-1} = 0 \end{aligned}$$

is obtained.

Multiply also by $cX + 1$.

Imposing that all coefficients of the distinct powers of X are zero, we obtain the conditions $(a+2c)^p = 0$ (coefficient of X^{2p+1}) and $c^{p+1} + a^pc + ac^p = 0$ as coefficient of X^{2p+2}. Substituting $a = -2c$ into this latter coefficient, we obtain $3c^{p+1} = 0$. If $c = 0$, then also $a = 0$ and then $ad - bc = 0$, which is false. So $p_0 = 3$, but then $a = -2c = c$, and again $ad - bc = 0$. □

Lemma 3.3.3 *Two different (p^2+1)-sets S_1 and S_2 intersect in at most $2p+2$ points.*

Proof. This is proven using similar polynomial manipulations as in the proofs of the preceding two lemmas. □

Lemma 3.3.4 *The union of $\lfloor p/2 \rfloor$ pairwise distinct (p^2+p+1)-sets and/or (p^2+1)-sets on $PG(1,p^3)$ has at least $(p^3+p^2+3p-5)/4$ points when p is odd, and $(p^3+p^2+4p)/4$ points when p is even.*

Proof. Since two distinct (p^2+p+1)-sets and/or (p^2+1)-sets share at most $2p+2$ points, we get at least

$$\begin{aligned} f(i) &\geq \sum_{j=1}^{i}(p^2+1-(j-1)(2p+2)) \\ &\geq i(p^2+1)-(i^2-i)(p+1) \end{aligned}$$

points in the union of i distinct $(p^2(+p)+1)$-sets. The function $f(i)$ attains its maximum for $i = p/2+1/(p+1)$, hence we get at least $f(\lfloor p/2 \rfloor)$ points. □

Corollary 3.3.5 *For $i \leq (p-1)/2$ different $(p^2(+p)+1)$-sets, the points of a $(p^2(+p)+1)$-set cannot be divided over the other $i-1$ $(p^2(+p)+1)$-sets.*

Proof. A $(p^2(+p)+1)$-set intersects an other $(p^2(+p)+1)$-set in at most $2p+2$ points, so it is impossible to divide such a set over at most $(p-3)/2$ other $(p^2(+p)+1)$-sets. □

3.4 The dual blocking set contains a Baer subplane

We will prove that if the dual blocking set ${B_r}^D$ of poor planes through a hole r contains a Baer subplane, then this hole r is contained in a subgeometry $PG(3,\sqrt{q})$ of holes.

We rely on the results by Metsch and Storme [60].

So suppose ${B_r}^D$ contains a Baer subplane E. Consider the Baer cone $\mathcal{C} = \langle r, E \rangle$.

Lemma 3.4.1 *Every line of $\mathcal{C}$ contains at least $\sqrt{q}+1$ holes.*

Proof. Every line of $\mathcal{C}$ lies in at least $\sqrt{q}+1$ planes of $\mathcal{C}$ which all are poor planes. So, they contain at least $\sqrt{q}+1$ holes (Lemma 3.2.1). □

Consider a poor plane α through r defining a line of E.

Lemma 3.4.2 *The minimal blocking set of holes contained in α is a Baer subplane.*

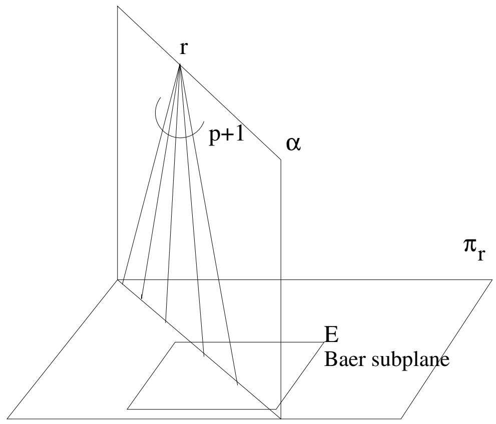

Figure 3.1: Suppose r is the vertex of a minimal blocking set of size p^3+p^2+1.

Proof. Suppose α contains a minimal blocking set B of holes of size p^3+p^2+1. First of all, r belongs to B. For r lies on $\sqrt{q}+1$ lines of $\mathcal{C}$ in α all containing at least $\sqrt{q}+1$ holes; so this gives already at least $q+\sqrt{q}+1$ holes. Since there are exactly $q+\delta$ holes in α, there are at most $\delta-\sqrt{q}-1$ holes on the remaining $q-\sqrt{q}$ lines through r in α; so r must belong to B.

If r is the vertex of B, then r lies on $p+1$ lines containing p^2+1 holes (Remark 1.3.4). There are at least $\sqrt{q}-p$ other points on $(E\setminus B)\cap\pi_r$; see Figure 3.1. By Lemma 3.4.1, this yields at least $(p+1)p^2+(\sqrt{q}-p)\sqrt{q}+1 > q+\delta$ holes in α; a contradiction. If r is not the vertex, then r belongs to p^2+1 secants to B, and so there are at least $p^2-\sqrt{q}$ points in $(B\setminus E)\cap\pi_r$. So using the secants of B and the lines of $\mathcal{C}$ through r, yielding at least $(\sqrt{q}+1)\sqrt{q}+(p^2-\sqrt{q})p+1 > q+\delta$ holes in α; a contradiction.

Now, suppose α contains the minimal blocking set B of holes of size p^3+p^2+p+1, then also here $r \in B$. Here r belongs to p^2+1 or p^2+p+1 secants to B, all containing at least $p+1$ points of B. Since r belongs to exactly $\sqrt{q}+1$ lines of $\mathcal{C}$, using Lemma 3.2.1, there are at least $(\sqrt{q}+1)\sqrt{q}+1+(p^2-\sqrt{q})p$ holes in α. This is more than $q+\delta$.

Lemma 3.1.1 and Theorem 1.3.3 yield the result. □

Theorem 3.4.3 *If ${B_r}^D$ contains a Baer subplane E, then r is contained in a $PG(3,\sqrt{q})$ of holes.*

Proof. The situation is now completely reduced to that of [60, Lemma 2.2].; in which Metsch and Storme proved the results on maximal partial spreads in $PG(3,q)$, q square, of Theorem 3.1.2 and Corollary 3.1.3. □

Remark 3.4.4 Since we have completely finished the discussion when the dual minimal blocking set of poor planes through a hole is a Baer subplane, we can restrict ourselves to the cases that the dual minimal blocking set of poor planes

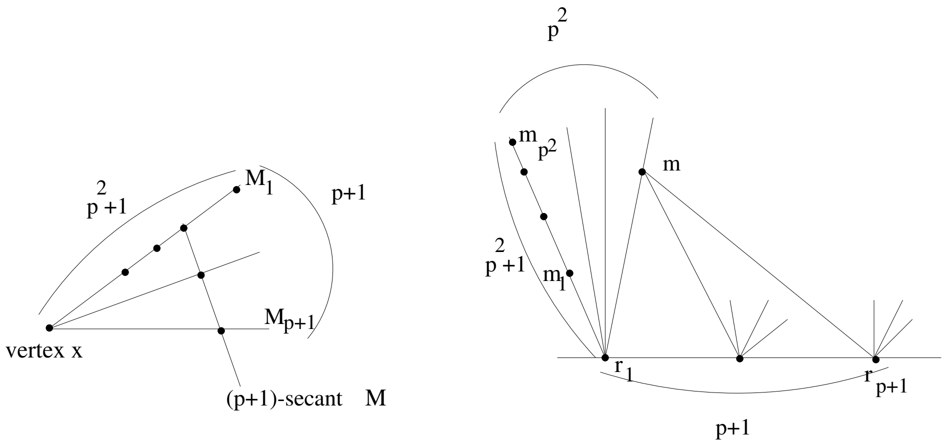

Figure 3.2: A blocking set of size $p^3 + p^2 + 1$, with its dual

through a hole is the dual minimal blocking set of size $p^3 + p^2 + 1$ or of size $p^3 + p^2 + p + 1$.

We will also continue the discussion for $q = p^3$, and $p \geq 7$ a prime; in this case we assume $\delta \leq q/8$. The arguments are completely the same for $q = p^3$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h > 1$.

For the exact upper bound on δ, we refer to Remark 3.6.12.

3.5 The dual minimal blocking set has size $p^3 + p^2 + 1$

Assume ${B_r}^D$ contains the dual minimal blocking set E of size $p^3 + p^2 + 1$. We describe this dual minimal blocking set explicitly in order to construct a cone $\mathcal{C}$, as was done in the preceding calculations. Dualize Remark 1.3.4.

We recall the figure of the dual minimal blocking set of size $p^3 + p^2 + 1$ and its dual; Figure 3.2.

The dual vertex in π_r is a $(p+1)$-secant $L = \langle r_1, r_2, \ldots, r_{p+1} \rangle$ to the dual blocking set. Since in a minimal blocking set of size $p^3 + p^2 + 1$, a point different from the vertex lies on p^2 $(p+1)$-secants, we have that in π_r, on p^2 lines, different from $r_1 r_2$, through one of the points r_i, there are p^2 points m_j lying on $p + 1$ lines of ${B_r}^D$.

We will consider these p^4 points m_i, together with the $p + 1$ points r_i as being the base of a cone $\mathcal{C}$ with vertex r.

We summarize the description of the cone $\mathcal{C}$ in the following remark.

Remark 3.5.1 (1) There is one particular line containing $p+1$ points $r_1, \ldots, r_{p+1}$ of the base of $\mathcal{C}$. These points $r_1, \ldots, r_{p+1}$ form a subline $PG(1,p)$.
(2) There are exactly p^2 lines, different from $r_1 r_2$, through a point r_i containing p^2 extra points m_j of the base of $\mathcal{C}$; such a point m_j lies on exactly $p+1$ lines of ${B_r}^D$; these lines are in fact the lines $\langle m_j, r_k \rangle$; $k = 1, \ldots, p+1$.

The point r_i and these p^2 other points m_j on such a line form a (p^2+1)-set with special point r_i.
(3) The other lines, differen from $r_1 r_2$, through r_i only have r_i in common with the base of $\mathcal{C}$.
(4) A line not through one of the points r_i contains exactly p^2 points of the base of $\mathcal{C}$.

We call the planes through r which intersect π_r in a line of E the *planes of the cone* $\mathcal{C}$. The lines of E are the line $r_1 r_2$ and the lines through the points r_i containing p^2+1 points of the base of $\mathcal{C}$.

Lemma 3.5.2 *The lines rr_i have at least p^2+1 holes and the lines rm_j have at least $p+1$ holes.*

Proof. This follows from the fact that the points r_i and m_j lie on respectively p^2+1 and $p+1$ lines of E (Lemma 3.2.1 and Remark 3.5.1). □

Lemma 3.5.3 *There is no plane α of $\mathcal{C}$ containing a Baer subplane B of holes.*

Proof. If $\alpha \cap \pi_r$ is the line $r_1 r_2$, then there are at least $(p+1)p^2 + (\sqrt{q}-p)\sqrt{q}$ holes in α, since each line rr_i contains at least p^2+1 holes (Lemma 3.2.1) and there are at least $\sqrt{q}-p$ other lines of B through r all containing at least $\sqrt{q}+1$ holes.

If $\alpha \cap \pi_r$ is different from the line $r_1 r_2$, there are at least $\sqrt{q}(\sqrt{q}+1) + (p^2 - \sqrt{q})p$ holes in α.

In each case we have more than $q + \delta$ holes in α; a contradiction. □

Lemma 3.5.4 *Every plane α of $\mathcal{C}$ has a projected $B = PG(3,p)$ of holes contained in the cone $\mathcal{C}$.*

Proof. First we will prove the result for the plane $\langle r, r_1, r_2 \rangle$.

Lemma 3.5.2 yields that on every line rr_i, there are at least p^2+1 holes. Since $\delta \leq p^3/8$, we have at most $p^3/8 - p^2 - 1$ holes on the remaining $p^3 - p$ lines through r in the plane $\langle r, r_1, r_2 \rangle$. So $r \in B$. Suppose B does not lie on the lines $rr_1, rr_2, \ldots, rr_{p+1}$. We will discuss all possibilities for the minimal blocking set of holes in $\langle r, r_1, r_2 \rangle$.
Case 1. Assume r is the vertex of the minimal blocking set of holes of size $p^3 + p^2 + 1$ in α. Two distinct sublines $PG(1,p)$ share at most two points, hence at least $(p-1)p^2$ holes would lie outside the lines $rr_1, rr_2, \ldots, rr_{p+1}$ when $B \not\subset \mathcal{C}$; which is false.
Case 2. Assume $|B| = p^3 + p^2 + 1$, but r is not the vertex. Then r lies on p^2+1 secants; so at least $p^2 - p$ secants lie outside the lines $rr_1, rr_2, \ldots, rr_{p+1}$. Hence,

in total, at least $(p^2-p)p = p^3-p^2$ holes lie outside the lines $rr_1, rr_2, \ldots, rr_{p+1}$; which is false.
Case 3. Assume $|B| = p^3+p^2+p+1$. Then r lies on p^2+p+1 secants when r does not lie on the unique (p^2+p+1)-secant to B or on p^2+1 secants when r lies on the unique (p^2+p+1)-secant to B. So at least p^2-p secants lie outside the lines $rr_1, rr_2, \ldots, rr_{p+1}$; which gives the same contradiction as above.

Now, we will prove the result for an other plane $\alpha = \langle r, L\rangle$, with $L \in E$, through r. We may assume that α contains the point r_1. On rr_1, there are at least p^2+1 holes and on the other p^2 lines of $\mathcal{C}$ through r in α, there are in total at least pp^2 extra holes. Hence, of the $p^3+\delta$ holes in α, there are at most $p^3/8-p^2-1$ holes outside the cone $\mathcal{C}$ on the remaining p^3-p^2 lines through r in α. Necessarily, r belongs to B. Suppose B does not lie in the cone $\mathcal{C}$.
Case 1. Suppose r is the vertex of the minimal blocking set B of size p^3+p^2+1. Then there are at least $(p+1)p^2+1+(p^2-p)p$ holes in the plane $\langle r, L\rangle$ where we first counted the holes on the $p+1$ (p^2+1)-secants through r to B; a contradiction.
Case 2. Assume $|B| = p^3+p^2+1$, but r is not the vertex. By Lemma 3.3.3, there are at least $(p^2+1-(2p+2))p$ holes of $\langle r, L\rangle$ outside the cone $\mathcal{C}$; again a contradiction.
Case 3. Assume $|B| = p^3+p^2+p+1$. Then r lies on p^2+p+1 secants or on p^2+1 secants to this minimal blocking set. By Lemmas 3.3.1 and 3.3.2, at least $(p^2+1-(2p+2))p$ holes of $\langle r, L\rangle$ lie outside the cone $\mathcal{C}$; a contradiction. □

Remark 3.5.5 From the previous discussion, the projected $B = PG(3,p)$ of holes of a plane α of $\mathcal{C}$, different from $\langle r, r_1, r_2\rangle$, lies inside the cone $\mathcal{C}$, and the only possibilities are that rr_i has p^2+1 or p^2+p+1 holes of B; and r lies on p^2 other $(p+1)$-secants of holes of B.

This follows again from the fact that two distinct (p^2+p+1)- and/or (p^2+1)-sets only share a very small number of points. Looking at a line of E passing through r_i, different from r_1r_2, we see that this line contains p^2+1 points of the base of $\mathcal{C}$ which form a (p^2+1)-set with special point r_i. This can only happen when r belongs to the $(p^2(+p)+1)$-set of the blocking set of size $p^3+p^2(+p)+1$, and when r_1 is the special point of the (p^2+1)-set on $\alpha \cap \pi_r$.

Lemma 3.5.6 *It is impossible that, for a hole r, the set E has size p^3+p^2+1.*

Proof. In the plane $\langle r, r_1, r_2\rangle$, we have a minimal blocking set B of size p^3+p^2+1 with vertex r since r lies on $p+1$ lines with at least p^2+1 holes. Suppose on every line rr_i, $i = 1, \ldots, p+1$, we have at least either two (p^2+1)-sets, or a (p^2+1)-set and a (p^2+p+1)-set. Then on each such line we have at least $2(p^2+1)-(2p+2) = 2p^2-2p$ holes. So in total, there would be at least $(2p^2-2p-1)(p+1)+1$ holes in $\langle r, r_1, r_2\rangle$; a contradiction.

Hence we can find a line, for instance rr_1, having only one (p^2+1)-set of holes and no (p^2+p+1)-set of holes. Consider a plane α through r and a line of E through r_1, with $\alpha \neq \langle r, r_1, r_2\rangle$. Let B' be the minimal blocking set of holes

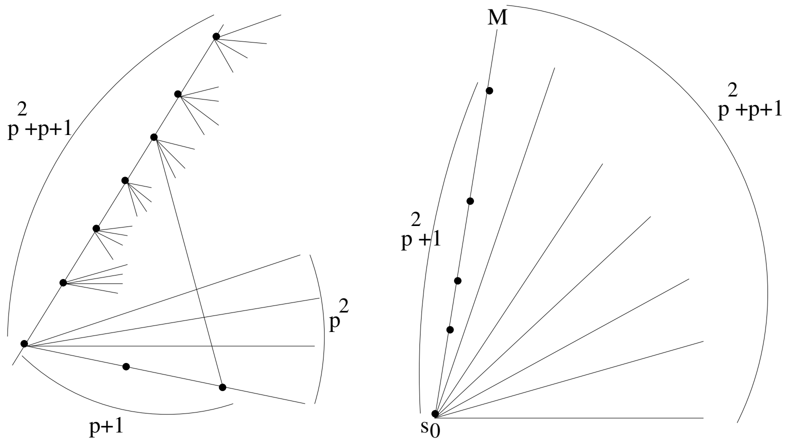

Figure 3.3: A blocking set of size $p^3 + p^2 + p + 1$, with its dual

in α. Since B' lies completely in $\alpha \cap \mathcal{C}$, and since r lies on one $(p^2 + 1)$-secant of holes and on p^2 $(p + 1)$-secants of holes, necessarily, since $B' \subset \mathcal{C}$, B' must be a blocking set of size $p^3 + p^2 + 1$.

Since r is the vertex of the minimal blocking set of holes in $\langle r, r_1, r_2 \rangle$, r is the special point of the $(p^2 + 1)$-secant of holes on rr_1. But this point must also be the vertex of the minimal blocking set of holes in α. This follows from Lemmas 3.4 and 3.5 of [60]. For, the special point of the $(p^2 + 1)$-set is the unique point stabilized under an elementary abelian group of order p^2 fixing the $(p^2 + 1)$-set. In our case, this is the point r, so r is the vertex of B'.

This however implies that r lies on $p + 1$ $(p^2 + 1)$-secants of holes of B' in α. This contradicts the fact above that r lies on one $(p^2 + 1)$-secant of holes and on p^2 $(p + 1)$-secants of holes of B'.

Hence, E cannot be a dual minimal blocking set of size $p^3 + p^2 + 1$. $\square$

3.6 The dual minimal blocking set is of size $p^3 + p^2 + p + 1$

Assume ${B_r}^D$ contains a dual minimal blocking set E of size $p^3 + p^2 + p + 1$; see Remark 1.3.4.

Remark 3.6.1 Again we will construct a cone $\mathcal{C}$ with vertex r and base the points of π_r lying on more than one line of E.

We recall the figure of the minimal blocking set of size $p^3 + p^2 + p + 1$, with its dual; see Figure 3.3.

Dualising the description of a minimal blocking set of size $p^3 + p^2 + p + 1$, we get the following description for the base of $\mathcal{C}$.

(1) There is one special point s_0, the dual of the (p^2+p+1)-secant, lying on p^2+p+1 lines of E which intersect the base of $\mathcal{C}$ in p^2 extra points.

The point s_0, and all these p^2 points on such a line of E form a (p^2+1)-set with s_0 as special point.

(2) Another point of the base of $\mathcal{C}$ lies on one (p^2+1)-secant to the base of $\mathcal{C}$, that is the line through s_0, p lines which are (p^2+p+1)-secants, intersecting the base of $\mathcal{C}$ in (p^2+p+1)-sets, and on p^3-p tangents to the base of $\mathcal{C}$.

Every line of E, not through s_0, intersects the base of $\mathcal{C}$ in a (p^2+p+1)-set.

The *planes of* $\mathcal{C}$ are the planes $\langle r, L\rangle$, with $L \in E$.

Lemma 3.6.2 *The line rs_0 has at least p^2+p+1 holes and the other lines of $\mathcal{C}$ have at least $p+1$ holes.*

Proof. This follows from Lemma 3.2.1. $\square$

Lemma 3.6.3 *There is no plane α of $\mathcal{C}$ containing a Baer subplane B of holes.*

Proof. First of all, it is again possible to prove that $r \in B$.

If $s_0 \in \alpha \cap \pi_r$, then we have at least $q+\sqrt{q}+1+(p^2-\sqrt{q})p$ holes in α, where we first counted the holes in B and then the p extra holes on the, at least $p^2-\sqrt{q}$, other lines of $\mathcal{C}$ through r.

If $s_0 \notin \alpha \cap \pi_r$, then, similarly, we have at least $q+\sqrt{q}+1+(p^2+p-\sqrt{q})p$ holes in α.

In both cases, we have found more than $q+\delta$ holes in α. This is false. $\square$

Lemma 3.6.4 *If L is a line of E, then the poor plane $\langle r, L\rangle$ has its projected $B = PG(3,p)$ of holes inside the cone $\mathcal{C}$.*

Proof. This is proved in the same way as Lemma 3.5.4. Here, we use the fact that a line of E through s_0 intersects the base of $\mathcal{C}$ in a (p^2+1)-set with special point s_0, and a line of E not through s_0 intersects the base of $\mathcal{C}$ in a (p^2+p+1)-set. $\square$

Remark 3.6.5 If the proof of Lemma 3.6.4 would be made explicitly, then also here it would follow that if the minimal blocking set of holes in $\langle r, L\rangle$, with $s_0 \in \langle r, L\rangle$, is of size p^3+p^2+1, it has a (p^2+1)-set on rs_0; if the minimal blocking set is of size p^3+p^2+p+1, it has its (p^2+p+1)-set lying on rs_0.

Construction of a projected $PG(5,p)$ of holes

Consider a hole r for which ${B_r}^D$ contains the dual E of a minimal blocking set of size p^3+p^2+p+1. We will prove that r is contained in a projected $PG(5,p)$ of holes. We still describe the quotient geometry of r by a plane π_r skew to r, and we let s_0 denote the dual of the (p^2+p+1)-secant of E.

Let α and β be two planes of $\mathcal{C}$ through rs_0 which share the same (p^2+1)-set or the same (p^2+p+1)-set of holes on rs_0 contained in the projected $PG(3,p)$

of holes in α and β. Then these two $PG(3,p)$ define a $PG(4,p)$, say S, projected on $PG(3,p^3)$.

The following lemma shows that these planes α and β exist.

Lemma 3.6.6 *There exist two planes α and β of $\mathcal{C}$ through rs_0 sharing the same (p^2+1)- or (p^2+p+1)-set with the projected $PG(3,p)$ of holes in α and β.*

Proof. Consider $p/2$ such planes through rs_0. If they all share distinct (p^2+1)- or (p^2+p+1)-sets with the projected $PG(3,p)$ of holes within them, then Lemma 3.3.4 states that there are at least $(p^3+p^2+3p-5)/4$ holes on rs_0. But there are at least $q=p^3$ other holes in a plane of $\mathcal{C}$ through rs_0 since such a plane has at least p^2 other lines through r each having at least p holes different from r; so such a plane of $\mathcal{C}$ through rs_0 has more than $q+\delta$ holes; this is false. □

Lemma 3.6.7 *The point r lies on p^2 lines which are $(p+1)$-secants to the projected $PG(3,p)$ of holes in α (call them L_i) and on p^2 lines which are $(p+1)$-secants to the projected $PG(3,p)$ of holes in β (call them M_i).*

Proof. This is Remark 3.6.5. □

Lemma 3.6.8 *More than $(7p^2-p-2)/8$ of the $(p+1)$-secants L_i (respectively M_j) in Lemma 3.6.7 contain only one $PG(1,p)$ of holes.*

Or equivalently, there are less than $(p^2+p+2)/8$ lines L_i or M_j through r in α or β which contain at least two $PG(1,p)$'s of holes.

Proof. Since $\delta \leq p^3/8$, there lie at most $p^3/8$ extra holes on these p^2 lines. They contain already one $PG(1,p)$ of holes. A second $PG(1,p)$ through r would give at least $p-1$ extra holes. Hence, at most $p^3/(8(p-1))$ lines through r contain an extra $PG(1,p)$ of holes. □

Remark 3.6.9 Call such a line L_i or M_j having only one $PG(1,p)$ of holes, a *good line*. Suppose a line L_i lies in a plane of $\mathcal{C}$, different from $\langle L_i, rs_0\rangle$, intersecting β in a good line M_j. Then these two lines L_i and M_j define a unique $PG(2,p)$ of holes, already sharing two sublines $PG(1,p)$ with S; hence this $PG(2,p)$ lies completely in S.

Lemma 3.6.10 *There exists a plane η of $\mathcal{C}$ defined by a good line L_i and a good line M_j.*

Proof. Consider a line L_i. Suppose all p planes of $\mathcal{C}$ through L_i, different from $\langle L_i, s_0\rangle$, intersect β in a bad line M_j. Then these p bad lines M_j define a dual $PG(1,p)$ of lines, including the line rs_0; see Figure 3.4.

Since i such dual $PG(1,p)$ contain at least $\sum_{j=1}^{i}(p-(j-1)) = ip - i(i-1)/2$ lines, and there are less than $(p^2+p+2)/8$ bad lines M_j; there are $i<(p+1)/4$ such dual $PG(1,p)$.

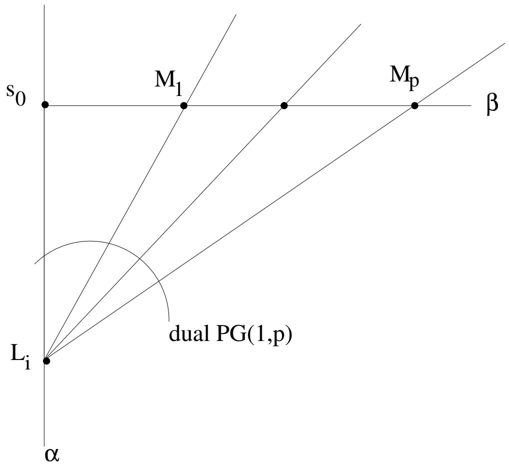

Figure 3.4: A dual $PG(1, p)$ defined by p bad lines M_j (intersection with π_r)

An additional property of these dual $PG(1, p)$ containing p bad lines M_j is that it is impossible to partition another dual $PG(1, p)$ over these, less than $(p + 1)/4$, bad $PG(1, p)$. So a dual $PG(1, p)$ of p bad lines M_j is one of those, less than $(p + 1)/4$, bad sublines $PG(1, p)$ considered above.

Now consider an arbitrary line L_i whose p planes of $\mathcal{C}$ all intersect β in a bad line M_j. Suppose two such lines L_i, L_i' define the same dual $PG(1, p)$ of bad lines M_j. Then the lines rs_0, L_i, L_i' define a dual $PG(1, p)$ of lines of $\mathcal{C}$ in α through r since the planes of $\mathcal{C}$ through M_j intersect α into a dual $PG(1, p)$. So, a correspondence arises between a dual $PG(1, p)$ of lines L_i in α and a dual $PG(1, p)$ of lines M_j in β having the property that we cannot use the lines L_i since they lie on p planes of $\mathcal{C}$ intersecting β in bad lines M_j.

There can be at most $(p + 1)/4$ such bad dual sublines $PG(1, p)$ of lines L_i. These sublines can contain at most $p(p + 1)/4$ lines L_i. There are less than $(p^2 + p + 2)/8$ lines L_i which are bad in the sense that they have at least two sublines $PG(1, p)$ of holes. So, since there are exactly p^2 lines L_i, there is a good line L_i lying in a plane of $\mathcal{C}$ intersecting β in a good line M_j. $\square$

Now, this $PG(2, p)$ of holes in η, containing the subline of holes on the good lines L_i and M_j lies in the union of $p + 1$ planes $\pi_1, \ldots, \pi_{p+1}$ through rs_0 of the dual blocking set E; see Figure 3.5.

This includes the planes $\pi_1 = \alpha$ and $\pi_{p+1} = \beta$. We will show that the projected $PG(3, p)$'s of holes in the planes $\pi_1, \ldots, \pi_{p+1}$ completely lie in S, and so S consists entirely of holes. We do this by using an argument similar to that of the proof of Lemma 3.6.10.

Lemma 3.6.11 *The projected $PG(4, p) \equiv S$ defined by the projected $PG(3, p)$ of holes in α and β consists completely of holes.*

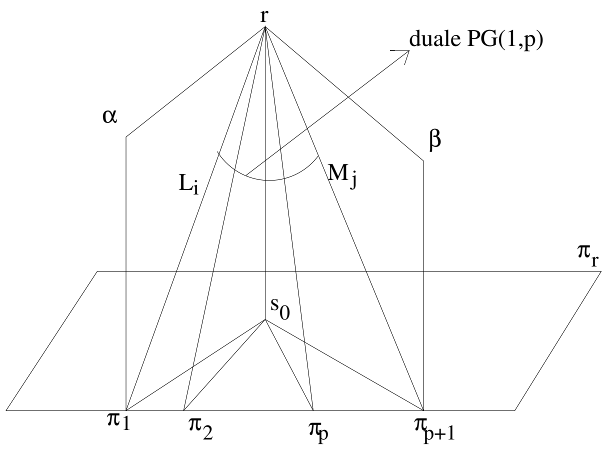

Figure 3.5: The $PG(2, p)$ of holes in η inside the planes $\pi_1, \ldots, \pi_{p+1}$

Proof. Consider a line L of $\mathcal{C}$ through r contained in π_2 sharing one, but not two, $PG(1,p) \equiv \tilde{L}$, with the set of holes. Then this line lies in p poor planes of $\mathcal{C}$ different from π_2. If such a poor plane intersects α and β in good lines L_k and M_l, then $\tilde{L}$ also lies in S; by the reasoning of Lemma 3.6.10.

Suppose all poor planes through L of $\mathcal{C}$, different from π_2, intersect either α or β in a bad line.

Then either α or β has at least $p/2$ bad lines in one dual $PG(1,p)$ of lines through r, which contains rs_0. Suppose this happens j times for α. Then α has at least

$$\sum_{i=1}^{j} (p/2 - (i-1)) = (p/2+1)j - j(j+1)/2$$

bad lines. The maximum of this sum is $(p^2 + 2p + 1)/8$, and this is attained for $j = (p+1)/2$.

From the condition imposed on δ, there are less than $(p^2 + p + 2)/8$ bad lines in α; so there are at most $p/2$ such bad dual $PG(1,p)$ in α containing at least $p/2$ bad lines L_i, and similarly in β. If some dual $PG(1,p)$ in α through r contains at least $p/2$ bad lines in α, then it must coincide with one of these $j \leq p/2$ dual $PG(1,p)$ in α. The same is true for β.

The way in which we proceed goes as follows. We search for a plane of $\mathcal{C}$ through a line good L of $\mathcal{C}$ on $\pi_2 \cap \mathcal{C}$ intersecting α and β in good lines. Such a line L is defined by its intersection point with $\pi_r \cap \pi_2$.

If this is not possible for such a point on $\pi_r \cap \pi_2$, it must lie on p lines of π_r sharing at least $p/2$ points with such a $PG(1,p)$ of, say, α which is the intersection of such a dual $PG(1,p)$ described above, and $\alpha \cap \pi_r$. Note that we have more than $(7p^2 - p - 2)/8$ good lines on π_2.

Consider a line N_i of $\mathcal{C} \cap \pi_2$ lying on p planes of $\mathcal{C}$ of which at least $p/2$ intersect α in bad lines L_i. Consider two such bad lines L_i and L_i', then L_i, L_i', rs_0 define a dual $PG(1,p)$ of sublines through r in α.

Suppose another line N_i' of $\mathcal{C}$ in π_2 cannot be used because it lies on p planes of $\mathcal{C}$ intersecting α or β in bad lines, and suppose these planes also intersect α in the lines of the dual subline $PG(1,p)$ defined by L_i, L_i', rs_0. Then also rs_0, N_i, N_i' define a dual $PG(1,p)$ of lines through r in π_2; and this dual $PG(1,p)$ consists of the intersection lines with π_2 of the planes of $\mathcal{C}$, different from α, through a line of the dual subline L_i, L_i', rs_0. Here we use again the fact that the $p+1$ planes of $\mathcal{C}$ through such a line L_i form a dual $PG(1,p)$. So, also here, a correspondence arises between: (1) a dual subline, defined by L_i, L_i', rs_0, in α containing at least $p/2$ bad sublines L_i, and (2) a dual subline, defined by N_i, N_i', rs_0, of lines of $\mathcal{C}$ in π_2. The lines N_i'' of this latter dual subline defined by N_i, N_i', rs_0 might have the property that they do not lie in a plane of $\mathcal{C}$ intersecting α and β in good lines L_i and M_j.

We determined the bound on δ, such that there are at least $p+2$ lines of $\mathcal{C}$ in π_2 that lie in a plane of $\mathcal{C}$ intersecting α and β in good lines; see Remark 3.6.12. Hence, at least $p+2$ sublines $PG(1,p)$ of the projected $PG(3,p)$ of holes in π_2 are contained in S. So this projected $PG(3,p)$ of holes in π_2 is completely contained in S.

The same is true for $\pi_3, \ldots, \pi_p$, and hence S consists completely of holes. □

Remark 3.6.12 In this remark, we determine a bound on δ, such that in the proof of Lemma 3.6.11 we can indeed find the $p+2$ lines of $\mathcal{C}$ in π_2 that lie in a plane of $\mathcal{C}$ intersecting α and β in good lines.

We do this by looking for an upper bound for the number j of dual sublines L_i, L_i', rs_0 having at least $p/2$ bad sublines L_i. The j corresponding sublines $PG(1,p)$ in α and β each contain at least $p/2$ lines containing at least two $PG(1,p)$'s of holes. This gives at least $(p/2+1)j - j(j+1)/2$ bad lines with at least one extra $PG(1,p)$ of holes in α; see the proof of Lemma 3.6.11.

The j sublines $PG(1,p)$ on α or β which we cannot use each give at most jp lines in π_2 of dual sublines N_i, N_i', rs_0 which we cannot use.

The same reasoning is true for β. Hence, there are at most $2jp$ lines on π_2 which are bad in the sense that they can define planes of $\mathcal{C}$ intersecting α and/or β in a bad line.

Now we take into account the bad lines on π_2 containing at least two $PG(1,p)$'s.

The j bad $PG(1,p)$'s in α contain at least $(p/2+1)j - j(j+1)/2$ bad lines L_i. They all contain, besides the $p+1$ points of the minimal blocking set of holes inside the plane $\langle s_0, L_i\rangle$, $p-1$ extra holes. We will impose further on that

$$((p/2+1)j - j(j+1)/2)(p-1) \leq \delta - p^2 - 1. \tag{3.1}$$

Then we find our upper bound on δ. Repeating the arguments of the proof of Lemma 3.6.11, this condition (3.1) gives us the upper bound on j we want to have to make sure that at least $p+2$ lines of $\mathcal{C}$ in π_2 share a subline $PG(1,p)$ of S with the projected $PG(3,p)$ of holes in π_2.

We impose the condition that

$$jp + jp + (p/2+1)j - j(j+1)/2 \leq p^2 - p - 2. \tag{3.2}$$

Condition (3.2) implies

$$j \leq \frac{5p+1-\sqrt{17p^2+18p+17}}{2}.$$

This is satisfied for

(1) $j \leq 2p/5$ when $p \geq 17$.

This leads to less than $(p/2+1)j - j(j+1)/2$ lines of $\mathcal{C}$ in a plane through rs_0 having a second $PG(1,p)$ of holes. Each such second subline $PG(1,p)$ gives at least $p-1$ extra holes leading to already $((p/2+1)2p/5 - 2p(2p+5)/50)(p-1) = (3p^3+2p^2-5p)/25$ holes.

Since each poor plane with a projected $PG(3,p)$ of holes contains already at least p^3+p^2+1 holes, we obtain the condition $p^3+\delta < p^3+p^2+1+(3p^3+2p^2-5p)/25$; so $\delta < (3p^3+27p^2-5p+25)/25$.

(2) For $p=7$, a detailed calculation gives $\delta \leq 90$; for $p=11$, we obtain $\delta \leq 285$; for $q=13$, $\delta \leq 441$.

Under these conditions, we have at most $2jp + (\delta - p^2 - 1)/(p-1)$ bad lines in π_2; which is at most p^2-p-2.

Note that the previous bound on δ was $\delta \leq p^2+p+1$, by Metsch and Storme; see Theorems 3.1.4 and 3.1.2.

Theorem 3.6.13 *If the dual blocking set ${B_r}^D$ in the quotient geometry of a hole r contains a dual minimal blocking set of size p^3+p^2+p+1, then r is contained in a projected $PG(5,p)$ of holes.*

Proof. We continue the discussion of Lemma 3.6.11.

So now we have already found a projected subgeometry $PG(4,p)$ of holes defined by $p+1$ planes of $\mathcal{C}$ through rs_0. We now prove that there is even a projected $PG(5,p)$ of holes defined by all p^2+p+1 planes of $\mathcal{C}$ through rs_0.

With the same notation as above, consider the p^2+p+1 lines T_i of E through s_0. The preceding results show that if the planes $\langle r, T_1\rangle$ and $\langle r, T_2\rangle$ share the same (p^2+1)-set or (p^2+p+1)-set of holes on rs_0, then we can construct a projected $PG(4,p)$ of holes S.

Now suppose there are lines T_3 and T_4 of E through s_0 whose planes $\langle r, T_3\rangle$ and $\langle r, T_4\rangle$ share the same (p^2+p+1)- or (p^2+1)-set of holes on rs_0, but where this $(p^2 + p + 1)$- or $(p^2 + 1)$-set of holes on rs_0 is different from the one of $\langle r, T_1 \rangle$ and $\langle r, T_2 \rangle$ on rs_0. Then the projected $PG(3, p)$ of holes in the planes of $\mathcal{C}$ through T_3 and T_4 again define a projected $PG(4, p)$ of holes, but distinct from the one defined by T_1 and T_2. Since these two $PG(4, p)$'s of holes define distinct $(p^2 + 1)$- or $(p^2 + p + 1)$-sets of holes on rs_0, they also define disjoint dual $PG(1, p)$'s in π_r of poor planes of $\mathcal{C}$ through s_0. It is impossible to partition the $p^2 + p + 1$ lines of E through s_0 into sets of size $p + 1$; so there must be at least $p + 2$ lines $T_1, \ldots, T_{p+2}$ of E through s_0 in π_r whose planes $\langle r, T_i \rangle$ share the same $(p^2 + p + 1)$- or $(p^2 + 1)$-set of holes on rs_0.

Consider the projected $PG(4, p) \equiv S$ of holes defined by T_1 and T_2. Let $\langle r, T_k \rangle$ be another plane of $\mathcal{C}$ through s_0 sharing the same $(p^2 + p + 1)$- or $(p^2 + 1)$-set of holes on rs_0 as $\langle r, T_1 \rangle$, but not lying in S. Then S and the projected $PG(3, p)$ of holes in $\langle r, T_k \rangle$ define a projected $PG(5, p)$ in $PG(3, p^3)$.

Fix the line T_k and combine it with a line T_j of E through s_0 lying in S. Then the projected $PG(3, p)$'s of holes in $\langle r, T_j \rangle$ and $\langle r, T_k \rangle$ define a projected $PG(4, p)$ of holes. Hence, letting vary T_j, we find a pencil of projected $PG(4, p)$'s through the projected $PG(3, p)$ of holes in $\langle r, T_k \rangle$. These projected $PG(4, p)$'s consist completely of holes. Hence, this projected $PG(5, p)$ consists completely of holes. $\square$

We now prove the uniqueness of this projected $PG(5, p)$ of holes passing through r.

Theorem 3.6.14 *If the dual blocking set ${B_r}^D$ in the quotient geometry of a hole r contains a dual minimal blocking set of size $p^3 + p^2 + p + 1$, then r is contained in a unique projected $PG(5, p)$ of holes, having $p^5 + p^4 + p^3 + p^2 + p + 1$ holes of weight one.*

Proof. Embed the space $\Pi = PG(3, p^3)$, containing the maximal partial spread, in $PG(5, p^3)$. Then the projected $PG(5, p)$ is the projection from a line L of a subgeometry $\Lambda = PG(5, p)$ of $PG(5, p^3)$ onto Π. This line L is skew to the subgeometry $PG(5, p)$. It has two conjugate lines L^p and L^{p^2} with respect to Λ. We discuss the different possibilities for $\dim\langle L, L^p, L^{p^2} \rangle$. Let $\pi_0 = \langle L, L^p, L^{p^2} \rangle$.

Case 1. $\dim \pi_0 = 2$. Then $\dim(\pi_0 \cap \Lambda) = 2$ since π_0 is invariant under conjugation with respect to Λ. Consider a 3-space over $GF(p^3)$ through π_0 intersecting Λ in a subgeometry $\pi_3 = PG(3, p)$. The points x of this subgeometry are projected from L onto the points x' of a line M of Π.

Suppose two points x_1 and x_2 of $\pi_3 \setminus \pi_0$ are projected onto the same point of M. Then the subline of π_3 through x_1 and x_2 intersects L in a point; since all points x_1, x_2 and their projection lie in a plane through L. But this intersection point also lies in π_0. This would imply that L is not skew to π_3. So, no two points of $\pi_3 \setminus \pi_0$ are projected onto the same point of M. This however implies that M consists completely of holes, namely the p^3 projections of points in $\pi_3 \setminus \pi_0$ and the projection of π_0. This contradicts the maximality of the partial spread.

Case 2. $\dim \pi_0 = 3$. Then $\pi_0 \cap \Lambda$ is a 3-dimensional space π_3 of Λ. The points of π_3 are projected onto the points of a line N of Π. Consider an arbitrary plane α of $\pi_0 \equiv PG(3, p^3)$ through L. Then α and its conjugate planes intersect in at least one point belonging to π_3. All $q^3 + 1$ possiblities of α yield different points of π_3, and as in Case 1, they are all projected on different points of N. But this again implies that N consists completely of projected points of π_3; so the partial spread would not be maximal.

Case 3. $\dim \pi_0 = 4$. Then $\pi_0 \cap \Lambda$ is a 4-dimensional space π_4 of Λ. The points of π_4 are projected onto the points of a plane M of Π. This plane M has $p^3 + \delta$ holes, but π_4 has $p^4 + p^3 + p^2 + p + 1$ points. So several points of π_4 are projected onto the same points of M. It is impossible that the points of a subplane $\pi_2 = PG(2, p)$ are projected onto the same point of M; or else L and its conjugate lines L^p, L^{p^2} would lie in the plane of π_2, which is invariant under conjugation; so they would not generate a 4-dimensional space. So at most the points of a subline π_1 of π_4 are projected onto the same point of M. Consider two such sublines π_1 and π_1' of π_4 whose points are projected from L onto the same points. Then π_1 and π_1' do not intersect in a point, or else L, L^p, L^{p^2} lie in the plane $\langle \pi_1, \pi_1' \rangle$. So π_1 and π_1' generate a 3-space. Both the lines of $PG(5, p^3)$ containing π_1, respectively π_1' intersect L, but then L, L^p, L^{p^2} lie in this 3-space generated by π_1 and π_1'. So at most the points of one subline π_1 of π_4 are projected onto the same point of M; but then M has more than $q + \delta$ holes. This is again a contradiction.

Case 4. $\dim \pi_0 = 5$. Then L, L^p and L^{p^2} define a regular 2-spread in Λ. There exists a unique regular 2-spread, and the planes of this regular 2-spread are defined by the intersections $\langle x, x^p, x^{p^2} \rangle \cap \Lambda$, where $x \in L$. No subline $PG(1, p)$ of points of Λ is projected onto the same point. Otherwise, this subline, extended to the field $GF(p^3)$, intersects L; then L, L^p, L^{p^2} only generate a 4-dimensional space. This already shows that the projected $PG(5, p)$ has $p^5 + \cdots + p + 1$ points.

Consider an arbitrary point $s \in \Lambda$ and consider the unique plane α of the regular 2-spread of Λ through s. Then the points of $\alpha \cap \Lambda$ are projected onto $p^2 + p + 1$ distinct points of the line $\langle L, \alpha \rangle \cap \Pi$. Consider another plane $\pi_2 = \langle r, r^p, r^{p^2} \rangle$ ($r \in L$) of the regular 2-spread, but with $s \notin \pi_2$. Then the points of the 3-space $\langle s, \pi_2 \rangle$ which lie in Λ are projected onto the points of a plane M of Π; being generated by the projection of s and the line containing the $(p^2 + p + 1)$-set which is the projection of π_2.

It also shows that s', the projection of s, lies in p^3 such planes M containing $p^3 + p^2 + p + 1$ holes; they are the projections of all solids $\langle s, \pi_i \rangle$, where $\pi_i \neq \alpha$ is a plane of the regular 2-spread. We also need to count the $p^2 + p + 1$ planes of Π through the line containing the projections of the points of $\alpha \cap \Lambda$, containing $p^3 + p^2 + p + 1$ projected points of Λ. So s' lies in $p^3 + p^2 + p + 1$ planes of Π with $p^3 + p^2 + p + 1$ projected holes of Λ. $\square$

Theorem 3.6.15 *There is at most one projected $PG(5, p)$ of holes through a* *hole r.*

Proof. Suppose that r lies in two projected $PG(5,p)'s$ of holes. Call them Λ' and Λ''. Since r lies in $p^3 + \delta < 9p^3/8$ poor planes, Λ' and Λ'' share at least $7p^3/8 + 2p^2 + 2p + 2$ poor planes. In these poor planes, they share the same projected $PG(3,p)$ of holes.

Consider two of those common projected $PG(3,p)$'s of holes. If they share only a common subline $PG(1,p)$, then $\Lambda' = \Lambda''$. If they intersect in a projected subplane $PG(2,p)$, then this projected subplane is a (p^2+p+1)-set; then Λ' and Λ'' already share a projected subgeometry $PG(4,p)$. Using the notations of the preceding lemmas and theorems, let rs_0 be the line containing the (p^2+p+1)-set. Then Λ' and Λ'' share already $p+1$ projected $PG(3,p)$'s in $p+1$ planes through rs_0. Consider another plane through r, but not through rs_0, containing a common projected $PG(3,p)$ of holes with Λ' and Λ''. Then this latter common projected $PG(3,p)$ of holes implies that $\Lambda' = \Lambda''$. □

Remark 3.6.16 We now present the main theorems of this chapter. Here we also use the explicit upper bound δ_0 on the deficiency δ.

For $q = p^3$, p prime, $p \geq 17$, δ_0 is the largest integer smaller than $(3p^3 + 27p^2 - 5p + 25)/25$ (Remark 3.6.12). For $p = 7, 11, 13$, $\delta_0 = 90, \delta_0 = 285$ and $\delta_0 = 441$ respectively.

For $q = p^3$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h > 1$, δ_0 is defined as the largest integer smaller than $(3p^3 + 27p^2 - 5p + 25)/25$ and smaller than the value δ' for which $p^3 + \delta'$ is the cardinality of the smallest non-trivial minimal blocking set in $PG(2,p^3)$ of cardinality larger than $p^3 + p^2 + p + 1$. Presently, this value is still unknown, but we know $\delta' \leq p^3/p_0 + 1$ (See Section 1.3).

Theorem 3.6.17 *Let $p = p_0^h$, $p_0 \geq 7$ a prime, $h \geq 1$ odd. Then the minihyper corresponding to a maximal partial spread in $PG(3,p^3)$ of deficiency $\delta \leq \delta_0$, is the disjoint union of projected $PG(5,p)$ of cardinality $p^5 + p^4 + p^3 + p^2 + p + 1$.*

Theorem 3.6.18 *Let $p = p_0^h$, $p_0 \geq 7$ a prime, $h > 1$ even. Then the minihyper corresponding to a maximal partial spread in $PG(3,p^3)$ of deficiency $\delta \leq \delta_0$, is the disjoint union of $PG(3,p^{3/2})$ and of projected $PG(5,p)$.*

Proof. Arguments similar to those used in the proof of Theorem 3.6.15 can be used to show that the union of $PG(3,p^{3/2})$'s and of projected $PG(5,p)$'s of holes is a disjoint union. □

Corollary 3.6.19 *Under the conditions of Theorems* 3.6.17 *and* 3.6.18*, the deficiency δ of a maximal partial spread in $PG(3,p^3)$ can be written as $\delta = r(p^{3/2}+1) + s(p^2+p+1)$ for some non-negative integers r and s.*

Remark 3.6.20 By [8], it is not possible to cover all points of $PG(3,q)$ except for the points of a Baer subspace, by lines. This implies that in Corollary 3.6.19, $(r,s) = (1,0)$ is not possible.

Chapter 4

Weighted $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers

The *excess* of a point r of a minihyper (F, w) is the weight $w(r)$ of the point r minus 1.

The *excess* e of a minihyper (F, w) is the number $\sum_{x \in F}(w(x) - 1)$.

We classify all $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, $\delta \leq 2p^2 - 4p$, $p = p_0^h \geq 9$, $h \geq 1$, for a prime number $p_0 \geq 7$, with excess $e \leq p^3$. Such a minihyper is a sum of lines and of possibly one projected subgeometry $PG(5, p)$, or a sum of lines and a minihyper which is a projected subgeometry $PG(5, p)$ minus one line. When p is a square, also (possibly projected) Baer subgeometries $PG(3, p^{3/2})$ can occur. The results are taken from Ferret and Storme, A classification result on weighted $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, [26].

4.1 A sum of geometric structures

The easiest way to construct weighted minihypers is to construct a *sum* of certain geometrical objects.

Consider a number of geometrical objects, such as subspaces $PG(d, q = p^h)$ of $PG(N, q = p^h)$, subgeometries $PG(d, p^t)$ of $PG(N, q = p^h)$, where $t|h$, and even projected subgeometries $PG(d, p^t)$ in $PG(N, q = p^h)$, where $t|h$. In the first two cases, a point of respectively $PG(d, q)$ or $PG(d, p^t)$ has weight one, while all the other points not belonging to respectively $PG(d, q)$ or $PG(d, p^t)$ have weight zero. In the latter case, let Π be a projected $PG(d, p^t)$. The *weight* of a point $s \in \Pi$ is the number of points s' of $PG(d, p^t)$ that are projected onto s; all other points s not belonging to Π have weight zero.

Then the *sum* of these subspaces and (projected) subgeometries is the weighted set (F, w), where the weight $w(s)$ of a point s of (F, w) is the sum of all the weights of s in the subspaces and (weighted) subgeometries of (F, w).

We will characterize the $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, $\delta \leq 2p^2 - 4p$,

with excess $e \leq p^3$, as being either:
(1) a sum of lines, (projected) subgeometries $PG(3, p^{3/2})$ when p is a square, and of at most one projected $PG(5, p)$, or
(2) a sum of lines, (projected) subgeometries $PG(3, p^{3/2})$ when p is a square, and one $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper which is a projected $PG(5, p)$ minus one line.

Remark 4.1.1 Sometimes, we will intersect the minihyper (F, w) with a set of points (for example, the point set of a plane) α, and shortly write $(F, w) \cap \alpha$. With this, we mean the point set $F \cap \alpha$ with as weight function the restriction of w to the points of $F \cap \alpha$. If we take an element or a point of the minihyper (F, w), then we mean a point of F.

Crucial in our classification results on $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers are the recent classification results on non-trivial minimal blocking sets in $PG(2, p^3)$ of Theorem 1.3.3 and Remark 1.3.4.

For a plane intersecting a $\{\delta(q+1), \delta; 3, q\}$-minihyper (F, w) in an $\{m_0 + m_1(q+1), m_1; 2, q\}$-minihyper, we will call m_1 the *multiplicity* of that plane with respect to (F, w). If $m_1 \geq 1$, then we call the plane a *blocking plane* of (F, w). Note, that in Chapter 3 such a plane was called a *poor plane*; since in that case it did not contain a line of the corresponding maximal partial spread.

4.2 Projected $PG(5, p)$ in $PG(3, p^3)$

The main problem in the classification results on $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers (F, w) is that such minihypers might contain projected subgeometries $PG(5, p) \equiv \Omega$. We will always be able to prove that a large subset of a projected subgeometry $PG(5, p) \equiv \Omega$ is contained in (F, w), but the next step is then to prove that Ω is completely contained in (F, w). This is obtained by checking every point s of Ω. We check the planes of $PG(3, p^3)$ through s which contain a projected $PG(3, p)$ of Ω. We show that these planes are blocking planes with respect to (F, w). If we have more than δ such blocking planes through s, then it follows immediately that $s \in F$ (Lemma 3.2.1). If we have less than δ such blocking planes, we can show that at least one such plane π intersects (F, w) in a 1-fold blocking set B. This latter 1-fold blocking set B contains a minimal blocking set B_0 which is a projected subgeometry $PG(3, p)$ (Remark 1.3.4). We show that this latter projected $PG(3, p)$ coincides with the one arising from the intersection $\pi \cap \Omega$; again implying that $s \in F$.

We now repeat the detailed description of the different types of points s in a projected subgeometry $PG(5, p) \equiv \Omega$, and of the planes of $PG(3, p^3)$ passing through s which share a projected subgeometry $PG(3, p)$ with Ω.

Consider a subgeometry $\Lambda = PG(5, p)$ naturally embedded in $PG(5, p^3)$. Let L be a line of $PG(5, p^3)$ skew to Λ. Then the line L has two conjugate lines with respect to Λ, denoted by L^p and L^{p^2}.

Case 1. Suppose Ω is the projection of Λ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Then every projected point s in Ω has weight one. Every point $s \in \Omega$ lies on exactly one (p^2+p+1)-set of Ω, on $p^4+p^3+p^2$ $(p+1)$-secants to Ω, and lies in p^3+p^2+p+1 planes of $PG(3,p^3)$ sharing a minimal 1-fold blocking set of size p^3+p^2+p+1 with Ω.
We refer to the proof of Theorem 3.6.14 for the proofs.

In general, a plane of $PG(3,p^3)$ intersects Ω in either a $PG(2,p)$, a (p^2+p+1)-set, or in a minimal blocking set of size p^3+p^2+p+1.

Case 2. Suppose Ω is the projection of Λ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Then the 4-dimensional space $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ is called the *special* 4-space of Λ, and similarly, its projection is called the *special* projected 4-space of Ω. We will denote this special 4-space $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ by $\mathcal{P}$.

As explained in the proof of Theorem 3.6.14; for exactly one point r of L, $\dim\langle r, r^p, r^{p^2}\rangle = 1$. This line $M = \langle L, L^p\rangle \cap \langle L^p, L^{p^2}\rangle \cap \langle L, L^{p^2}\rangle = \langle r, r^p, r^{p^2}\rangle$ is projected from L onto a point m of Ω of weight $p+1$. The other p^3 points r of L satisfy $\dim\langle r, r^p, r^{p^2}\rangle = 2$. These latter planes are projected onto (p^2+p+1)-sets of Ω.

Every plane π of Λ passing through M and not lying in $\mathcal{P}$ is projected from L onto a (p^2+1)-set with special point m [60]. Each such plane π lies in p^2+p+1 solids of Λ which are projected onto planar minimal blocking sets of size p^3+p^2+1; thus implying that m lies in $p^4+p^3+p^2$ planes of $PG(3,p^3)$ sharing a 1-fold blocking set of size p^3+p^2+1 with Ω.

Let s be a point of Ω different from m and not lying in the special 4-space of Ω. Assume s is the projection of $s' \in \Lambda$. Then each solid $\langle r, r^p, r^{p^2}, s'\rangle \cap \Lambda$, with $r \in L \setminus M$, is projected onto a planar minimal blocking set of size p^3+p^2+p+1; hence, s lies in p^3 such planes. And every solid of Λ passing through M and s' is projected onto a planar minimal blocking set of size p^3+p^2+1 passing through s; thus giving p^2+p+1 extra planes through s intersecting Ω in a projected $PG(3,p)$.

Let s be a point of weight one of Ω which is the projection of a point s' of $\mathcal{P}$. Then the plane $\langle M, s'\rangle$ lies in p^2 distinct 3-spaces of Λ not contained in $\mathcal{P}$ which are projected onto planar blocking sets of size p^3+p^2+1 through s.

Case 3. Suppose Ω is the projection of a $PG(5,p) \equiv \Lambda$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

Let $\mathcal{P} = \langle L, L^p, L^{p^2}\rangle \cap \Lambda$.

Every plane α through L in $\langle L, L^p, L^{p^2}\rangle$ has two conjugate planes α^p, α^{p^2} with respect to Λ, and these three planes intersect in at least one point of $\mathcal{P}$; see Figure 4.1. Hence every plane through L in $\langle L, L^p, L^{p^2}\rangle$ contains at least one point of $\mathcal{P}$ and the projection of $\mathcal{P}$ is a line N of $PG(3,p^3)$. There are $p+1$ skew lines $L_1, \ldots, L_{p+1}$ in $\mathcal{P}$ which are projected onto points of weight $p+1$,

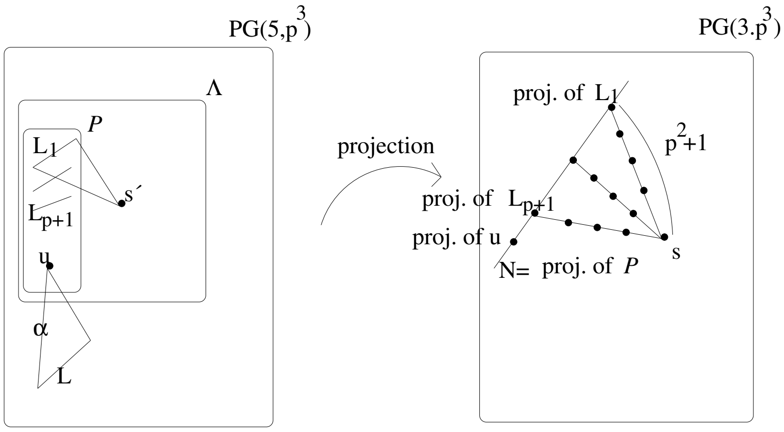

Figure 4.1: Case 3

and the remaining p^3-p points of $\mathcal{P}$ are projected onto points of weight one of the line N.

Then we call the 3-dimensional space $\mathcal{P}$ the *special* 3-space of Λ, and its projection will always be denoted by the line N.

A point s' of $\Lambda \setminus \mathcal{P}$ is projected onto a point s lying on $p+1$ different (p^2+1)-secants to Ω, which are the projections of planes $\langle s', L_i\rangle \cap \Lambda$, $i = 1,\ldots,p+1$ (see Figure 4.1). Each such (p^2+1)-secant through s lies in p^2 planes of $PG(3,p^3)$ containing a projected $PG(3,p)$ of Λ, which is a minimal blocking set of size p^3+p^2+1; hence, s' lies in p^3+p^2 such planes. Considering these $PG(3,p)$'s in Λ; these are the $PG(3,p)$'s through a plane $\langle s', L_i\rangle$ only intersecting $\mathcal{P}$ in L_i.

Furthermore, $\mathcal{P}$ is projected on the line N through which there are $p+1$ planes of $PG(3,p^3)$ containing $p^4+p^3+p^2+p+1$ projected points of Λ. The other planes through N contain p^3+p^2+p+1 projected weighted points; the points with weights larger dan one, all lie on N.

Hence, this projection forms a $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper containing the line N. Reducing the weight of every point on N by one yields a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$.

Case 4. Suppose Ω is the projection of Λ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

As in the proof of Theorem 3.6.14, all p^2+p+1 three-dimensional spaces of Λ through $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ are projected onto a line. Then this projection is a cone of p^2+p+1 lines; the vertex of the cone is a point having weight p^2+p+1 arising from the projection of the points of the plane $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$, and the base of the cone is a subplane $PG(2,p)$.

Remark 4.2.1 (1) The symbols Ω, Λ and N will always have the following meaning. The symbol Ω will always denote the projection of a $PG(5,p) \equiv \Lambda$ from a line L, and if $\dim\langle L, L^p, L^{p^2}\rangle = 3$, then N will always denote the line contained in Ω.

(2) In the latter case, when Ω contains a line N, then $(\Omega \setminus N, w')$ denotes the $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper obtained by reducing the weight of every point of N by one.

4.3 $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers for p non-square

Let (F,w) be a $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper.

We first state a theorem on $\{\delta v_{\mu+1}, \delta v_\mu; N, q\}$-minihypers.

Theorem 4.3.1 ([33]) *Let (F,w) be a $\{\delta v_{\mu+1}, \delta v_\mu; N, q\}$-minihyper satisfying $1 \leq \delta \leq (q+1)/2$, $0 \leq \mu \leq N-1$ and containing a μ-dimensional subspace π_μ. Then the minihyper (F',w') defined by the weight function w', where*

- *$w'(p) = w(p) - 1$, for $p \in \pi_\mu$, and*
- *$w'(p) = w(p)$ for $p \in PG(N,q) \setminus \pi_\mu$,*

is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; N, q\}$-minihyper.

With abuse of notation, the minihyper (F',w') will also be denoted by $(F \setminus \pi_\mu, w')$ (even if F' contains points in π_μ.

$(F \setminus \pi_\mu, w')$

Hence, from now on, we can assume that (F,w) contains no lines.

Remark 4.3.2 Consider a point $r \in F$ with weight one. As in Chapter 3, if we consider the $p^3+\delta$ blocking planes of (F,w) through r according to their multiplicities, then they form a dual blocking set in the quotient geometry of r. We will always describe this quotient geometry by means of a plane π_r skew to r, and denote the dual blocking set in π_r by ${B_r}^D$.

This dual blocking set contains a minimal blocking set E.

We will distinguish the different possibilities for E (Theorem 1.3.3).

4.3.1 Assume E has size p^3+p^2+p+1.

Assume ${B_r}^D$ contains a dual minimal blocking set E of size p^3+p^2+p+1. Then there are at most $p^3+\delta-(p^3+p^2+p+1) \leq p^2-5p-1$ planes through r intersecting (F,w) in a multiple blocking set.

Remark 4.3.3 We construct the cone $\mathcal{C}$ with vertex r and base the points of π_r lying on more than one line of E (see Remark 3.6.1).

Lemma 4.3.4 *Two different $(p^2(+p)+1)$-sets intersect in at most $2p+2$ points.*

Proof. See Lemmas 3.3.1, 3.3.2 and 3.3.3. □

Lemma 4.3.5 *There are at most two different $(p^2(+p)+1)$-sets of elements of (F,w) on rs_0.*

Proof. At least $p^2+p+1-(p^2-5p-1)=6p+2$ planes of $\mathcal{C}$ through rs_0 intersect (F,w) in 1-fold blocking sets.

If there would be three different $(p^2(+p)+1)$-sets on rs_0, then by Lemma 4.3.4, there are at least $p^2+1+(p^2+1-(2p+2))+(p^2+1-2(2p+2))$ points of $\pi\cap(F,w)$ on rs_0. This contradicts Corollary 2.1.8; a non-trivial blocking set $\pi\cap(F,w)$ of size at most p^3+2p^2-4p shares at most $2p^2-4p$ points with a line.

The line rs_0 also cannot be contained in (F,w) since we assume that no lines are contained in (F,w). $\square$

Remark 4.3.6 Now it is possible to explain why the upper bound $\delta\leq 2p^2-4p$ has been selected.

If $\delta\leq 2p^2-4p$, then it is possible to find two planes α and β of E through rs_0 sharing the same $(p^2(+p)+1)$-set of (F,w) with rs_0 and sharing a 1-fold blocking set with (F,w). These two 1-fold blocking sets in α and β each contain a minimal blocking set, which is a projected $PG(3,p)$. These latter projected $PG(3,p)$'s define a projected $PG(4,p)=\Omega_4$ in $PG(3,p^3)$ since they share the same projected $PG(2,p)$ with rs_0.

It is even possible to find a third plane γ of E through rs_0 sharing the same $(p^2(+p)+1)$-set of (F,w) with rs_0 as α and β, and sharing a 1-fold blocking set with (F,w) containing a projected $PG(3,p)$, which together with Ω_4 defines a projected $PG(5,p)$, which we denote by Ω. We will then show that $\Omega\subseteq F$ or $\Omega\setminus N\subseteq F$, where N is the line contained in Ω.

Again, the *planes of* $\mathcal{C}$ are the planes $\langle r,L\rangle$, with $L\in E$.

Lemma 3.6.2 states that the line rs_0 has at least p^2+p+1 elements of (F,w) and the other lines of $\mathcal{C}$ have at least $p+1$ elements of (F,w).

Remark 4.3.7 Consider two planes α and β of E through rs_0 intersecting (F,w) in a 1-fold blocking set and sharing the same $(p^2+(p+)1)$-set with (F,w) on rs_0. These planes α and β have a minimal blocking set, contained in $\alpha\cap(F,w)$ and $\beta\cap(F,w)$, lying inside the cone $\mathcal{C}$; see Lemma 3.6.4.

Now, r lies on p^2 lines L_i and M_j of $\mathcal{C}$ in respectively α and β, different from rs_0. We determine a lower bound for the number of such lines L_i and M_j containing only one subline $PG(1,p)$ of points of (F,w). Since two distinct sublines $PG(1,p)$ share at most two points, a line with two sublines $PG(1,p)$ contained in (F,w) would necessarily contain the $p+1$ points of one $PG(1,p)$, and at least $p-1$ other elements of (F,w). There are at most $p^3+\delta-(p^3+p^2+1)<p^2$ points of $(F,w)\cap\alpha$ and $(F,w)\cap\beta$ not lying in the minimal blocking set contained in $(F,w)\cap\alpha$ and $(F,w)\cap\beta$. Hence, at least $p^2-(p^2-1)/(p-1)\geq p^2-p-1$ lines L_i and M_j contain exactly one $PG(1,p)$ of elements of (F,w). As in Remark 3.6.9, we call these lines *good lines.* Otherwise, we call these lines *bad* lines.

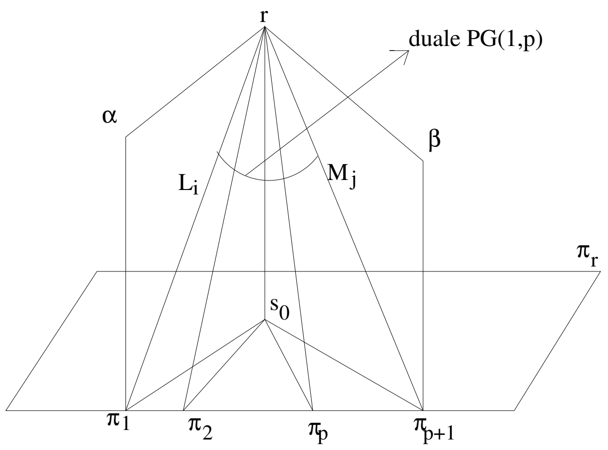

Figure 4.2: The planes $\pi_1, \ldots, \pi_{p+1}$ through rs_0

The projected $PG(3,p)$'s contained in $(F,w) \cap \alpha$ and in $(F,w) \cap \beta$ define a projected $PG(4,p) \equiv \Omega_4$ since both projected $PG(3,p)$'s share the same $(p^2(+p)+1)$-set with rs_0.

Suppose there is a plane η of $\mathcal{C}$ not passing through rs_0 which intersects (F,w) in a 1-fold blocking set and which intersects α and β in good lines L_i and M_j, then $(F,w) \cap \eta$ contains a minimal 1-fold blocking set which is a projected $PG(3,p)$ completely contained in $\mathcal{C}$ (see Lemma 3.6.4). This latter projected $PG(3,p)$ contains the sublines $PG(1,p)$ of (F,w) on the good lines L_i and M_j; so $\langle L_i, M_j\rangle$ defines a subplane $PG(2,p)$ completely contained in (F,w) and in Ω_4.

Consider the subplane $PG(2,p) \subset \Omega_4$ defined by this good line L_i in α and this good line M_j in β. This $PG(2,p)$ defines the $p+1$ planes $\pi_1, \pi_2, \ldots, \pi_{p+1}$, including α and β, through rs_0 intersecting Ω_4 in (projected) $PG(3,p)$'s; see Figure 4.2.

We now show that such a plane η exists.

Lemma 4.3.8 *There is a plane η of $\mathcal{C}$, not passing through rs_0, which intersects (F,w) in a 1-fold blocking set, and which intersects α and β in good lines L_i and M_j.*

Proof. There are p^3 planes of $\mathcal{C}$ not through rs_0. There are at most $p+1$ bad lines M_j in β, lying in at most $(p+1)p$ planes of $\mathcal{C}$ not through rs_0 (Remark 4.3.3). A similar argument is valid for the bad lines L_i in α. Hence there are at most $2(p+1)p$ planes of $\mathcal{C}$ intersecting α or β in a bad line.

Taking into account that there are at most p^2-5p-1 planes through r intersecting (F,w) in multiple blocking sets, we obtain at least p^3-3p^2+3p+1 planes of $\mathcal{C}$ intersecting α and β in good lines and intersecting (F,w) in 1-fold blocking sets. □

Remark 4.3.9 We have at least $3p+1$ planes of $\mathcal{C}$ through rs_0 intersecting (F,w) in 1-fold blocking sets and intersecting rs_0 in the same $(p^2(+p)+1)$-set (Lemma 4.3.5). Consider three such planes α, β and γ such that the projected $PG(3,p)$ of elements of (F,w) in α, β and γ define a projected $PG(5,p) \equiv \Omega$.

Consider another plane η of $\mathcal{C}$ through r, but not through s_0, and sharing a good line L_i, M_j, N_k with α, β, γ respectively. Assume also that η intersects (F,w) in a 1-fold blocking set; then $\eta \cap (F,w)$ contains a projected $PG(3,p)$ contained in $\mathcal{C}$ (Lemma 3.6.4).

Then the subplanes $PG(2,p)$ of elements of (F,w), defined by any two of the lines L_i, M_j, N_k lie in Ω; so the projected $PG(3,p)$ of elements of (F,w) in η is completely contained in Ω.

There are at least $p^3-(p^2-5p-1)$ planes η of $\mathcal{C}$ through r, but not through s_0, intersecting (F,w) in a 1-fold blocking set.

There are at most $p+1$ bad lines L_i, M_j, N_k in α, β, γ (Remark 4.3.7; each one of them lying in p planes of $\mathcal{C}$ not through rs_0. Hence, at least $p^3-p^2+5p+1-3(p+1)p = p^3-4p^2+2p+1$ planes η satisfy the conditions above. They all give us a $PG(3,p)$ of elements of (F,w) contained in Ω.

We call these latter p^3-4p^2+2p+1 planes of $\mathcal{C}$ the *good planes* of $\mathcal{C}$ through r.

Lemma 4.3.10 *The projected subgeometry Ω cannot be the projection of a subgeometry $PG(5,p) \equiv \Lambda$ from a line L, for which* $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

Proof. Such a projected subgeometry is a cone over a subplane $PG(2,p)$. The only planes of $PG(3,p^3)$ intersecting such a cone in at least p^3 points are planes containing exactly one or exactly $p+1$ lines of this cone. But we know that there are planes through r sharing a non-trivial minimal blocking set with Ω. So we have a contradiction. □

Theorem 4.3.11 *The projected subgeometry Ω either consists entirely of elements of (F,w), or it is the projection of a subgeometry $PG(5,p) \equiv \Lambda$ from a line L for which* $\dim\langle L, L^p, L^{p^2}\rangle = 3$ *and then $\Omega \setminus N$ is contained in F, where N is the line contained in Ω.*

Proof. Suppose some point $s \in \Omega$ is not an element of (F,w), then it lies on $\delta \leq 2p^2$ blocking planes (Lemma 3.2.1). Then $s \notin rs_0$ since the projected subgeometry Ω was constructed starting from the points of projected $PG(3,p)$ of (F,w) in three planes α, β and γ through rs_0.
Case 1. Assume Ω was the projection of a subgeometry Λ from a line L, with $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Here Ω consists of $p^5+p^4+p^3+p^2+p+1$ distinct points; and every point of Ω lies in p^3+p^2+p+1 planes of $PG(3,p^3)$ containing p^3+p^2+p+1 points of Ω (Section 4.2).

Let r', s' be the points of Λ projected from L onto r, s.

Select a plane π of $PG(3,p^3)$ passing through s, but not through r, intersecting Ω in a 1-fold blocking set of size p^3+p^2+p+1. First we show that there are at least p^3+p^2 such planes.

Since rs_0 is a (p^2+p+1)-set to Ω, and $s \notin rs_0$, rs is a $(p+1)$-secant to Ω and similarly, $r's'$ is a line of Λ not lying in a plane of the regular 2-spread defined by L on Λ. Hence, the line $r's'$ lies in $p+1$ distinct 3-spaces of Λ containing a plane of this 2-spread. Consequently, rs lies in $p+1$ planes of $PG(3,p^3)$ intersecting Ω in a 1-fold blocking set.

So at least $p^3+p^2+p+1-(p+1)$ planes π of $PG(3,p^3)$ pass through s, but not through r, and share a 1-fold blocking set with Ω.

In π we have at least $p+1$ intersection points with every plane of the at least p^3-4p^2+2p+1 good planes η, through r, sharing a projected $PG(3,p)$ of elements of (F,w) with Ω (Remark 4.3.9). In that way, we obtain at least $(p^3-4p^2+2p+1)(p+1)$ intersection points. Each such point is counted at most p times since we are only considering planes η of $\mathcal{C}$ through r not containing rs_0. Hence we obtain at least $(p^2-4p+2)(p+1) > \delta$ intersection points; so π is a blocking plane.

We conclude that s lies in more than δ blocking planes; so $s \in F$.

Case 2. Assume Ω is the projection of a subgeometry Λ from a line L, with $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

There is one $PG(1,p) = M$ of Λ which is projected onto one point m. There are $p^5+p^4+p^3+p^2$ points in Ω of weight 1 and one point m of weight $p+1$. There is one plane of $PG(3,p^3)$ containing $p^4+p^3+p^2+p+1$ weighted projected points. The point m lies in $p^4+p^3+p^2$ planes of $PG(3,p^3)$ sharing a minimal projected $PG(3,p)$ with Ω. There are p^3 planes of the special 4-space $\mathcal{P}$, defined by $\langle x, x^p, x^{p^2}\rangle \cap \Lambda$, where $x \in L$, $M \neq \langle x, x^p, x^{p^2}\rangle \cap \Lambda$, projected onto (p^2+p+1)-sets. A point s' of $\Lambda \setminus \mathcal{P}$ is therefore projected onto a point s of Ω lying in p^3 planes of $PG(3,p^3)$ containing a projected $PG(3,p)$ of size p^3+p^2+p+1 of Ω.

So the good planes of $\mathcal{C}$ through r do not contain m since m lies in planes of $PG(3,p^3)$ sharing a projected $PG(3,p)$ of size p^3+p^2+1 with Ω, and every good plane π through r shares a 1-fold blocking set of size p^3+p^2+p+1 with Ω. This follows from the fact that r lies on p^2+p+1 different $(p+1)$-secants to $\pi \cap \Omega$. These latter p^2+p+1 different $(p+1)$-secants are the lines of $\mathcal{C}$ in π. This also shows that $r \neq m$.

Consider a point $s \in \Omega$ which is the projection of a point $s' \in \Lambda \setminus \mathcal{P}$. Then there are p^2+p+1 $PG(3,p)$'s $\langle M, s'', s'\rangle$, where $s'' \in \mathcal{P} \setminus M$, which are all projected onto planar minimal blocking sets of size p^3+p^2+1 passing through s.

If $s \notin F$, then s lies in $\delta \leq 2p^2 - 4p$ blocking planes. Let r' be the point of Λ which is projected onto r. We distinguish three cases:

(1) $r' \notin \mathcal{P}$ and the line $r's'$ intersects M.
(2) $r' \notin \mathcal{P}$ and the line $r's'$ is skew to M.
(3) $r' \in \mathcal{P} \setminus M$.

In the latter two cases, $\langle M, r' \rangle$ lies in one such $PG(3,p) = \langle M, s'', s' \rangle$ of Λ. So, we still have at least $p^2 + p$ such $PG(3,p)$'s $\langle M, s'', s' \rangle$ not passing through r. After projecting, as in Case 1, we again have a plane of $PG(3,p^3)$ containing at least $(p^2-4p+2)(p+1)$ points of (F,w). Not all these planes can intersect (F,w) in a multiple blocking set since $2(p^2+p) > \delta$; so at least one of them, call it π_2, intersects (F,w) in a 1-fold blocking set. This blocking set contains a projected $PG(3,p)$, and this projected $PG(3,p)$ must coincide with the projected $PG(3,p) = \pi_2 \cap \Omega$. For, otherwise through a point of $(\pi_2 \cap \Omega) \setminus F$, there are at least $p^3 - p^2 - p$ lines which contain a point of the 1-fold blocking set of $(F,w) \cap \pi_2$ not lying in $\pi_2 \cap \Omega$; so we have at least $p^3 - p^2 - p + (p^2 - 4p + 2)(p+1)$ points in the 1-fold blocking set $(F,w) \cap \pi_2$; a contradiction to Corollary 2.1.8. Hence $s \in F$.

In the first case, consider all p^3 $PG(3,p)$'s of Λ passing through s' and through a plane $\langle x, x^p, x^{p^2} \rangle \cap \Lambda$, $x \in L \setminus M$. Since $r's'$ intersects M, these $PG(3,p)$'s do not contain r'. So after projecting, they do not contain r. As before, they contain at least $(p^2 - 4p + 2)(p+1)$ points of (F,w), so $s \in F$.

We now prove that the projection of the special 4-space $\mathcal{P}$ is contained in (F,w). Consider a point $s'' \in \mathcal{P} \setminus M$ projected onto the point s. The plane $\langle M, s'' \rangle$ is projected onto a (p^2+1)-set lying in p^2 projected $PG(3,p)$'s of Ω not contained in the projection of $\mathcal{P}$. All these $PG(3,p)$'s are projected onto planar blocking sets through this (p^2+1)-set. These planar blocking sets contain already p^3 elements of (F,w), namely the p^3 points lying in Ω, but not in the projection of $\mathcal{P}$. Hence these planes are blocking planes. Again at least one of these planes, denoted by π_2, intersects (F,w) in a 1-fold blocking set if $s \notin F$, and this 1-fold blocking set contains the projected $PG(3,p) = \pi_2 \cap \Omega$. Otherwise, there are at least $p^3 - p^2 - p$ lines through s which contain a point of the 1-fold blocking set of (F,w) not lying in the projected $PG(3,p)$ of Ω in π_2. And there would be at least $p^3 - p^2 - p + (p^2 - 4p + 2)(p+1) > p^3 + \delta$ points of (F,w) in π_2; a contradiction. Hence $s \in F$.

Also m is contained in (F,w), since it lies in $p^4 + p^3 + p^2$ planes of $PG(3,p^3)$, sharing already $p^3 + p^2$ points with (F,w).

We conclude that Ω is contained in (F,w).

Case 3. Assume Ω was the projection of a subgeometry Λ from a line L, with $\dim\langle L, L^p, L^{p^2} \rangle = 3$.

Let $\mathcal{P} = \langle L, L^p, L^{p^2} \rangle \cap \Lambda$ be the special $PG(3,p)$, and let the line N be the projection of $\mathcal{P}$. Then, as indicated in Section 4.2, there are $p+1$ skew lines L_i of $\mathcal{P}$ which are projected onto points of weight $p+1$ of Ω. And, also following from Section 4.2, a plane of $PG(3,p^3)$ sharing a projected $PG(3,p)$ with Ω must pass through one of those points of weight $p+1$ of Ω.

The point r lies in at least p^3-4p^2+2p+1 good planes of $\mathcal{C}$ sharing a 1-fold blocking set B with (F,w) and with Ω. These 1-fold blocking sets B intersect N in exactly one point. But N is the projection of $\mathcal{P}$ and two $PG(3,p)$'s in Λ intersect in at least a $PG(1,p)$. So, after projecting, the intersection is exactly one point, hence N and B intersect in one of the projections of the lines $L_1,\ldots,L_{p+1}$. As before, r cannot be the projection of a line $L_1,\ldots,L_{p+1}$. Also, $r\notin N$ since it cannot be the intersection of a good plane of $\mathcal{C}$ and the line N. So, r is the projection of a point r' of $\Lambda\setminus\mathcal{P}$.

We show that every point s of $\Omega\setminus N$ lies in (F,w). Let s' be the point of Λ projected onto s.

The only planes of $PG(3,p^3)$ intersecting Ω in a projected $PG(3,p)$ are planes through one of the projections of the lines L_i, and which only intersect N in one point. In this case, these planes contain the projections of $PG(3,p)$ of Λ passing through a line L_i and intersecting $\mathcal{P}$ only in this latter line L_i.

If $r's'$ intersects $\mathcal{P}$ in a point not lying on one of the lines L_i, $i=1,\ldots,p+1$, then each plane $\langle L_i,s'\rangle$ lies in p^2 3-spaces of Λ only intersecting $\mathcal{P}$ in L_i. Thus implying that s lies in p^3+p^2 planes π_2 of $PG(3,p^3)$ intersecting Ω in a 1-fold blocking set of size p^3+p^2+1 not passing through r.

If $r's'$ intersects $\mathcal{P}$ in a point on a line L_i, a similar argument as in the preceding paragraph gives p^3 such planes π_2 of $PG(3,p^3)$ passing through s and not containing r.

If $r's'$ is skew to $\mathcal{P}$, this line defines with each of the $p+1$ lines L_i a unique 3-space of Λ only intersecting $\mathcal{P}$ in this line L_i, and so each plane $\langle L_i,s'\rangle$ lies in p^2-1 distinct 3-spaces of Λ only intersecting $\mathcal{P}$ in L_i, and not passing through r'. So, in this latter case, if one projects these $(p+1)(p^2-1)$ 3-spaces from L, s lies in p^3+p^2-p-1 planes π_2 of $PG(3,p^3)$ intersecting Ω in a 1-fold blocking set, and not containing r.

Consider such a projected $PG(3,p)$ of (F,w) in a good plane through r, and consider the projected $PG(3,p)\equiv\pi_2\cap\Omega$. The corresponding $PG(3,p)$ in Λ intersect in at least $p+1$ different points since we already know that every good plane through r shares a minimal 1-fold blocking set B of size p^3+p^2+p+1 with (F,w). So, we obtain at least $(p^3-4p^2+2p+1)(p+1)$ intersection points.

These intersection points are counted at most p times.

Hence, we have at least $(p^3-4p^2+2p+1)(p+1)/p$ intersection points. Hence π_2 is a blocking plane and s lies in at least p^3 blocking planes; yielding $s\in F$ (Lemma 3.2.1).

We now prove that the projections of the lines L_i, $i=1,\ldots,p+1$, are in (F,w).

A line L_i lies in p^4 $PG(3,p)$'s of Λ which only intersect $\mathcal{P}$ in L_i, and which then are projected onto planes of $PG(3,p^3)$ sharing a projected $PG(3,p)$ with Ω. All these planes have already p^3+p^2 points in common with (F,w); yielding that the projections of L_i, $i=1,\ldots,p+1$, lie in more than δ blocking planes and hence the projection of a line L_i is contained in (F,w).

So we already have that $\Omega \setminus N$ and the projections of the lines L_i, $i = 1, \ldots, p+1$, are contained in F $\square$

4.3.2 Assume E has size $p^3 + p^2 + 1$

Assume ${B_r}^D$ contains a dual minimal blocking set E of size $p^3 + p^2 + 1$. We describe this dual minimal blocking set explicitly in order to construct a cone $\mathcal{C}$ as was done in the preceding subsection.

The base of the cone $\mathcal{C}$ consists of the points of π_r lying on more than one line of E.

We refer to Remark 3.5.1 for the explicit description of this cone.

Remark 4.3.12 The point r lies on $p^3 + \delta$ blocking planes; of which $p^3 + p^2 + 1$ contain a line of E. Hence there are at most $\delta - p^2 - 1 < p^2 - 4p$ other blocking planes through r. So we can find a point r_i through which there are at most $(p^2 - 4p)/(p+1) < p - 1$ planes different from $\langle r, r_0, r_1 \rangle$ intersecting (F, w) in a multiple blocking set.

So, suppose r_0 lies on at least $p^2 - p + 1$ planes (different from $\langle r, r_0, r_1 \rangle$) of $\mathcal{C}$ intersecting (F, w) in a 1-fold blocking set. They all must intersect rr_0 in one of at most two $(p^2(+p) + 1)$-sets since there are at most $2p^2$ elements of (F, w) on rr_0 (similar proof as for Lemma 4.3.5). Consider two planes α and β of $\mathcal{C}$ having the same $(p^2(+p)+1)$-set of points of (F, w) on rr_0, and intersecting (F, w) in 1-fold blocking sets. The minimal 1-fold blocking sets in $\alpha \cap (F, w)$ and $\beta \cap (F, w)$ then define a projected $PG(4, p) \equiv \Omega_4$.

As in Lemma 4.3.8, we find a plane π of $\mathcal{C}$ not through rr_0 intersecting (F, w) in a 1-fold blocking set and intersecting α and β in good lines L_i and M_j; see Figure 4.3. Similarly, we obtain a subplane $PG(2, p)$ of elements of (F, w), which is defined by L_i and M_j, and which lies in Ω_4. This defines $p+1$ planes $\pi_1, \ldots, \pi_{p+1}$ through rr_0 which intersect Ω_4 in projected $PG(3, p)$'s. Consider a further plane γ of $\mathcal{C}$, different from $\langle r, r_0, r_1 \rangle$ and different from $\pi_1, \ldots, \pi_{p+1}$, through rr_0 intersecting (F, w) in a 1-fold blocking set and sharing the same $(p^2 + (p+)1)$-set with $rr_0 \cap (F, w)$ as α and β, then the projected $PG(3, p)$'s of elements of (F, w) in α, β and γ define a projected $PG(5, p) \equiv \Omega$.

There are p^3 planes of $\mathcal{C}$ through one of the lines $rr_1, \ldots, rr_p$ which are different from $\langle r, r_0, r_1 \rangle$, and at least $p^3 - p^2 + 5p + 1 > p^3 - p^2$ of them intersect (F, w) in a 1-fold blocking set. There are at most $p+1$ lines L_i, M_j, N_k of the cone $\mathcal{C}$ in α, β, γ on which there is more than one $PG(1, p)$ of elements of (F, w). Each such line lies on p planes of $\mathcal{C}$ different from the plane through rr_0. Hence, at least $p^3 - p^2 - (3p(p+1)) = p^3 - 4p^2 - 3p$ planes π of $\mathcal{C}$ intersect α, β and γ in good lines and intersect (F, w) in a 1-fold blocking set. Lemma 3.5.4 shows that the projected $PG(3, p)$ inside the intersection $\pi \cap (F, w)$ lies inside the cone $\mathcal{C}$. This implies that this latter projected $PG(3, p)$ lies inside Ω.

We again call these $p^3 - 4p^2 - 3p$ planes π of $\mathcal{C}$ the *good* planes through r.

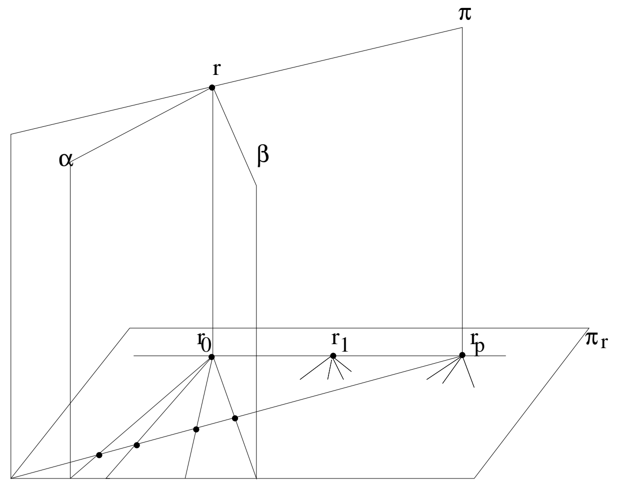

Figure 4.3: A plane of $\mathcal{C}$ intersecting α and β in good lines

Theorem 4.3.13 *The projected subgeometry Ω of Remark 4.3.12 either consists entirely of elements of (F,w), or it is the projection of a subgeometry Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$ and then $\Omega \setminus N$ is contained in F, where N is the line contained in Ω.*

Proof. Consider a good plane η of $\mathcal{C}$. This plane η shares a 1-fold blocking set with (F,w) containing a minimal projected $PG(3,p)$ which has size p^3+p^2+1 or p^3+p^2+p+1. In the former case, there is one point, the vertex of the projected $PG(3,p)$, lying on $p+1$ (p^2+1)-sets; all the other points of this projected $PG(3,p)$ lie on one (p^2+1)-set and on p^2 $(p+1)$-sets.

Suppose the 1-fold blocking set $\eta\cap(F,w)$ contains a minimal blocking set B of size p^3+p^2+1. Since r lies on p^2+1 lines of $\mathcal{C}$ in η, r is not the vertex of B, and so r cannot be the projection of a $(p+1)$-secant of Λ. This plane η contains one of the points $r_1,\ldots,r_p$. Assume this plane contains r_i, then the possible projection of a $(p+1)$-secant of Λ must lie on rr_i. For, the (p^2+1)-secant through r and the p^2 $(p+1)$-secants through r to B form a (p^2+1)-set in the quotient geometry of r in η. This latter (p^2+1)-set consists of r_i and of the p^2 points m_j (see the description for the base of the cone in Remark 3.5.1) of η in $\mathcal{C}$. Since r_i is the special point of this latter (p^2+1)-set, necessarily, rr_i is a (p^2+1)-secant to B; so the vertex of B lies on rr_i.

Case 1. Assume Ω was the projection of a subgeometry Λ from a line L, with $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

This possibility is excluded in the following way. In such a projection, a point of Ω lies on exactly one (p^2+p+1)-secant.

But in a good plane π through r, the lines of $\mathcal{C}$ form a (p^2+1)-set on one of the lines of E through a point r_i. This shows that the line rr_i contains the (p^2+p+1)-secant to the projected $PG(3,p)$ in $\pi \cap (F,w)$. This latter (p^2+p+1)-set of course contains r. But there are at least two good planes through r, passing through two lines rr_i and rr_j $(i \neq j)$ respectively. In both planes, r lies on a (p^2+p+1)-secant lying on respectively rr_i and rr_j; a contradiction.

Case 2. Assume Ω was the projection of a subgeometry Λ from a line L, with $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Suppose the point m of Ω which is the projection of the line M of Λ lies in one of the good planes through r, then the observation made before Case 1 learns us that m must lie on the line rr_i contained in this good plane.

If it lies on, for instance, rr_1, we do not consider the good planes of Remark 4.3.12 through rr_1, and we keep having at least p^3-5p^2-3p good planes through r having a 1-fold blocking set in common with Ω, and sharing the same projected $PG(3,p) \equiv B$ with Ω and with (F,w); this latter projected $PG(3,p)$ has size p^3+p^2+p+1.

Let $\mathcal{P}$ be the special $PG(4,p)$ of Λ. Suppose $\mathcal{P}$ is projected onto the set π_4 of Ω. We first show that $\Omega \setminus \pi_4$ is contained in (F,w). Let $s \in (\Omega \setminus \pi_4) \setminus F$, and assume that s is the projection of the point $s' \in \Lambda$.

Let r' be the point of Λ which is projected on r. We will again distinguish three cases:

(1) $r' \notin \mathcal{P}$ and the line $r's'$ intersects M.
(2) $r' \notin \mathcal{P}$ and the line $r's'$ is skew to M.
(3) $r' \in \mathcal{P} \setminus M$.

In the latter two cases, $\langle M, s', r'\rangle$ lies in one $PG(3,p)$ of Λ. So we still have p^2+p such $PG(3,p)$'s $\langle M, s'', s'\rangle$ of Λ, $s'' \in \Lambda \setminus \langle M, s'\rangle$, not passing through r'. After projecting, they are planar projected $PG(3,p)$'s of Ω. They share at least a $PG(1,p)$ of $p+1$ distinct points with p^3-5p^2-3p good planes through r since m does not lie in one of these good planes. Hence, there are at least $(p^3-5p^2-3p)(p+1)$ intersection points. If one of these intersection points lies on a line rr_i, $i=1,\ldots,p+1$, it is counted at most p^2 times; the other intersection points are counted at most p times. So we have at least $[(p^3-5p^2-3p)(p+1)-p^2(p+1)]/p \geq p^3-5p^2-9p-3 > 2p^2-4p$ distinct points of (F,w) if $p \geq 9$. Hence we have a blocking plane. Not all p^2+p planes can be intersecting (F,w) in a multiple blocking set since we assume $s \notin F$; so at least one of them, call it π_2, intersects (F,w) in a 1-fold blocking set containing a projected $PG(3,p)$. As in Case 2 of the proof of Theorem 4.3.11, this implies $s \in F$.

In the first case, consider all p^3 $PG(3,p)$'s $\langle r, r^p, r^{p^2}, s'\rangle$, $r \in L$, of Λ skew to M. Since $r's'$ intersects M, these p^3 $PG(3,p)$'s do not contain r'; hence, after projecting they do not contain the point r. These projections are 1-fold blocking sets of size p^3+p^2+p+1. As in the preceding paragraph, these p^3 planes have more than $2p^2-4p$ points of (F,w). So $s \in F$.

We now prove that the projection of $\mathcal{P}$ is completely contained in (F,w).

Let $s' \in \mathcal{P} \setminus M$. The plane $\langle M, s'\rangle$ is projected onto a (p^2+1)-set lying in p^2 projected $PG(3,p)$'s of $\Omega \setminus \pi_4$. All these projected $PG(3,p)$'s contain already p^3 points of (F,w), hence the planes containing them are blocking planes. Again, at least one of these planes intersects (F,w) in a 1-fold blocking set and this again implies $s \in F$.

Finally, $m \in F$ since it lies in $p^4+p^3+p^2$ planes of $PG(3,p^3)$ already sharing p^3+p^2 points with (F,w).

Case 3. Assume Ω was the projection of a subgeometry Λ from a line L, with $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

We again use the observations made in the first paragraphs of Case 3 of the proof of Theorem 3.12.

They show in particular that $s \in \Omega \setminus N$ lies in at least p^3 planes π_2, not through r, of $PG(3,p^3)$ sharing a 1-fold blocking set with Ω. Each one of those planes π_2 has at least $(p^3-5p^2-3p)(p+1)$ intersection points of (F,w) with the good planes through r.

The intersection points of $\pi_2 \cap \Omega$ with the lines $rr_i, i=1,\ldots,p+1$, are counted at most p^2 times, the other points are counted at most p times.

Hence, we again have at least p^3-5p^2-9p-3 intersection points with (F,w). Hence π_2 is a blocking plane and s lies in at least p^3 blocking planes; yielding $s \in F$ (Lemma 3.2.1).

As in Case 3 of the proof of Theorem 4.3.11, the projections of the lines L_i, $i=1,\ldots,p+1$, belong to (F,w). □

4.3.3 The classification result

Lemma 4.3.14 *There is at most one projected $PG(5,p)$ or projected $PG(5,p)\setminus N$ contained in the minihyper (F,w).*

Proof. We have at most $(2p^2-4p)(p^3+1) = 2p^5-4p^4+2p^2-4p$ points in (F,w). Suppose we have two projected $PG(5,p)$'s Ω_1 and Ω_2 contained in (F,w). Let Ω_i be the projection of the subgeometry Λ_i from a line L_i. Consider in Ω_1, respectively Ω_2, a projected hyperplane $\mathcal{P}_1$, respectively $\mathcal{P}_2$. If Ω_i, $i=1,2$, is the projection from a line L_i with $\dim\langle L_i, L_i^p, L_i^{p^2}\rangle \le 4$, select $\mathcal{P}_i$ in such a way that the projection of $\langle L_i, L_i^p, L_i^{p^2}\rangle \cap \Lambda_i$ is contained in $\mathcal{P}_i$.

The two sets $\Omega_i \setminus \mathcal{P}_i$ of size p^5 intersect in at least $4p^4-2p^2+4p$ points.

Using the descriptions of the projected $PG(5,p)$'s of Section 4.2, an arbitrary point s of $\Omega_1 \setminus \mathcal{P}_1$ lies on at least $p^4+p^3+p^2$ or p^4+p^3 $(p+1)$-secants to Ω_1. If $s \notin \Omega_2$, then:

(1) s lies on at most one (p^2+p+1)-set to Ω_2 when $\dim\langle L_2, L_2^p, L_2^{p^2}\rangle = 5$,

(2) s lies on at most one (p^2+1)-secant to Ω_2 when $\dim\langle L_2, L_2^p, L_2^{p^2}\rangle = 4$, and

(3) on at most $p+1$ (p^2+1)-secants to Ω_2 when $\dim\langle L_2, L_2^p, L_2^{p^2}\rangle = 3$.

So in each one of these cases, at least one $PG(1,p)$ of Ω_1 through s contains 3 points of $(\Omega_1 \setminus \mathcal{P}_1) \cap (\Omega_2 \setminus \mathcal{P}_2)$, where these latter three intersection points themselves define a $PG(1,p)$ contained in Ω_2.

Hence, this $PG(1,p)$ is completely contained in $\Omega_1 \cap \Omega_2$, and $s \in \Omega_2$.

So $\Omega_1 \setminus \mathcal{P}_1 = \Omega_2 \setminus \mathcal{P}_2$, and then $\Omega_1 = \Omega_2$. □

We now present our classification result for p non-square.

Theorem 4.3.15 *A $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper (F, w), p non-square, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $p \geq 9$, $\delta \leq 2p^2 - 4p$, with excess $e \leq p^3$, is a sum of either*
(1) δ lines, or
(2) $\delta - p^2 - p - 1$ lines and a projected $PG(5,p) \equiv \Omega$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \in \{3,4,5\}$, or
(3) $\delta - p^2 - p$ lines and a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Proof. As indicated in Theorem 4.3.1, if there is a line T contained in (F, w), then $(F \setminus T, w')$ is a $\{(\delta-1)(p^3+1), \delta-1; 3, p^3\}$-minihyper. So again assume there is no line contained in (F, w).

Since F has at least p^3+1 distinct points and since the excess e satisfies $e \leq p^3$, it is possible to find a point r of weight one in (F, w). Then r belongs to a unique projected $PG(5,p)$ contained in (F, w), or belongs to $\Omega \setminus N$, where Ω again is a projected $PG(5,p)$ and where N is a line contained in Ω (Subsections 4.3.1 and 4.3.2).

The only thing we still have to prove is that the points having weights in the projected $PG(5,p) \equiv \Omega$ must have at least the same weights as in the minihyper.

Case 1. If Ω is the projection of a $PG(5,p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 5$, then all points of Ω have weight one and the projection is a $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper.

If $\delta > p^2+p+1$, then every point of $F \setminus \Omega$ has multiple weight; there are at most p^3 distinct points in $F \setminus \Omega$. Adding the excess $e \leq p^3$, the total weight of (F, w) is at most $(p^2+p+1)(p^3+1) + 2p^3$, so $\delta = p^2+p+2$. This means that there is a point set of weight p^3+1 not yet exactly determined within the minihyper (F, w), and this also implies that there are at most $(p^3+1)/2$ distinct points in $F \setminus \Omega$.

Consider a plane π_2 containing a multiple point s of $F \setminus \Omega$. This plane intersects Ω in p^2+p+1 or p^3+p^2+p+1 distinct points. But since this plane also contains the multiple point s, it has more than $\delta = p^2+p+2$ points in common with (F, w), so it intersects (F, w) in either a 1-fold or a t-fold blocking set ($t > 1$) (Corollary 2.1.8). The first possibility is impossible since this plane then contains at most $p^2+p+1+(p^3+1)/2$ distinct points of F. The second possibility can be eliminated in the following way. There is at least one point of weight one in the intersection of π_2 with Ω. This point lies on at least p^3-p^2-p tangents to $\pi_2 \cap \Omega$. On each of these tangents, there must lie a point of $F \setminus \Omega$ since π_2 intersects (F, w) in a t-fold blocking set ($t > 1$). There are however only at most $(p^3+1)/2$ such points.

We have obtained a contradiction. There are no multiple points in $F \setminus \Omega$. So $F \subseteq \Omega$. Then $|(F,w)| \leq (p^2+p+1)(p^3+1)+p^3$ since the total excess $e \leq p^3$, so $|(F,w)| = (p^2+p+1)(p^3+1)$ and $F = \Omega$ is a $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper.

Case 2. If Ω is the projection of a $PG(5,p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 4$, then Ω has $p^5+p^4+p^3+p^2+1$ distinct points, and one point m of Ω is the projection of a line of Λ; so (F,w) is a weighted $\{\delta'(p^3+1), \delta'; 3, p^3\}$-minihyper with $\delta' \geq p^2+p+1$, as $(p^2+p)(p^3+1) < p^5+p^4+p^3+p^2+1$. Again, $\delta = p^2+p+1$ or $\delta = p^2+p+2$ and there are at most $(p^3+p+1)/2$ distinct multiple points in $F \setminus \Omega$.

Assume $\delta = p^2+p+2$. Then $F \setminus \Omega \neq 0$, for otherwise, the total excess e would be larger than p^3. Consider a (p^2+p+1)-set of Ω on a line T. It is impossible that T contains one of the multiple points in $F \setminus \Omega$, since then T shares more than p^2+p+2 weighted points with (F,w). Then every plane through T intersects (F,w) in at least a 1-fold blocking set (Corollary 2.1.8), but then $|(F,w)| \geq (p^3+1)\cdot p^3$ (false). Then T lies in one plane of $PG(3,p^3)$ containing the projection of the special 4-space $\mathcal{P}$ of Λ, in p^2 planes containing p^3+p^2+p+1 projected points of Ω forming a 1-fold blocking set, and in p^3-p^2 planes only sharing the points of $T \cap \Omega$ with Ω.

It is possible to find a plane through T only sharing the points of $\Omega \cap T$ with (F,w) since there are at most $(p^3+p+1)/2$ distinct points in $F \setminus \Omega$. This then shows that T shares p^2+p+2 weighted points with (F,w).

Then, every plane π_2 through T only sharing the p^2+p+1 points of $T \cap \Omega$ with Ω, only shares the points of $T \cap \Omega$ with (F,w), since otherwise π_2 intersects (F,w) in a 1-fold blocking set, but it contains at most $p^2+p+1+(p^3+p+1)/2$ distinct points of F.

And every plane π_2 through T sharing p^3+p^2+p+1 points with Ω only shares these points with (F,w). Otherwise, it intersects (F,w) in at least a 2-fold blocking set (Corollary 2.1.8). But a similar argument as in the preceding paragraphs shows that this cannot occur when there are at most $(p^3+p+1)/2$ distinct multiple points in $F \setminus \Omega$. This argument also shows that no point of $\pi_2 \setminus T$ can be a multiple point of (F,w).

So all multiple points of $F \setminus \Omega$ lie in the plane Π containing the projection of $\mathcal{P}$.

Now consider a (p^2+1)-secant T through m to Ω, not lying in Π. Select a plane π_2 through T not containing any multiple points, different from m, of (F,w) in Π. This plane exists since $e \leq p^3$.

Then $\pi_2 \cap F \subseteq \pi_2 \cap \Omega$; so π_2 shares p^2+1 or p^3+p^2+1 distinct points with Ω. Since this plane shares p^2+p+2 or p^3+p^2+p+2 weighted points with (F,w), necessarily m has weight $p+2$ since all other points of $\pi_2 \setminus F$ have weight one.

Now select a plane of $PG(3,p^3)$ containing one multiple point of $F \setminus \Omega$. Then this plane shares already p^2+p+2 or p^3+p^2+p+2 weighted points with (F,w) if it contains m, and it shares already p^2+p+1 or p^3+p^2+p+1 points with (F,w) if it does not contain m. In either case, the multiple point of $F \setminus \Omega$

included, this plane must intersect (F, w) in at least a 1-fold or 2-fold blocking set. This again is impossible.

So $\delta = p^2 + p + 1$.

Hence, every plane of $PG(3, p^3)$ must contain at least $p^2 + p + 1$ points of (F, w). Consider a $(p^2 + p + 1)$-set of Ω on a line T. Again, this line T lies in $p^3 + 1$ planes of $PG(3, p^3)$ of which: (a) one contains the projection of the special $PG(4, p) = \mathcal{P}$ of Λ, (b) p^2 planes contain $p^3 + p^2 + p + 1$ projected points of Ω forming a 1-fold blocking set, and (c) the remaining planes through T only share the $p^2 + p + 1$ distinct projected points of the $(p^2 + p + 1)$-set on T with Ω. If a plane as in (c) would contain, besides the $p^2 + p + 1$ points of $\Omega \cap T$, an other point of (F, w), then it would contain at least p^3 extra points of (F, w) (Corollary 2.1.8). Hence the points of $F \cap T$ must be counted with weight one. Again using a plane of $PG(3, p^3)$ through T sharing $p^3 + p^2 + p + 1$ points with Ω, it follows that the points of Ω not lying in the projection of $\mathcal{P}$ also have weight one.

We show that the point m has weight $p + 1$ in (F, w). Select a line of $PG(3, p^3)$ through m containing a $(p^2 + 1)$-set of Ω, but no other points of (F, w). Assume this $(p^2 + 1)$-set is not contained in the projection of $\mathcal{P}$. This line exists since (F, w) has at most excess p. We already know that the points of Ω on this line, different from m, have weight one. Select a plane through this line containing no other points of (F, w). Since this plane must share $p^2 + p + 1$ weighted points with (F, w) (Corollary 2.1.8), m necessarily has weight $p + 1$ in (F, w).

Case 3. If Ω is the projection of a $PG(5, p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, then either Ω completely lies in (F, w), or $\Omega \setminus N$ lies in (F, w) where N is the line contained in Ω. Note that the first possibility $\Omega \subset F$ is not possible since we assume that there are no lines in (F, w).

In the second case, there are already $p^5 + p^4 + p + 1$ distinct projected points in (F, w) since in the proofs of Theorems 4.3.11 and 4.3.13, we also proved that the lines L_i, $i = 1, \ldots, p + 1$ lie in F.

Since $p^5 + p^4 + p + 1 > (p^2 + p - 1)(p^3 + 1)$, and using the fact that the excess $e \leq p^3$, (F, w) is a $\{\delta'(p^3 + 1), \delta'; 3, p^3\}$-minihyper with $\delta' = p^2 + p$, $\delta' = p^2 + p + 1$ or $\delta' = p^2 + p + 2$. The special $PG(3, p) \equiv \mathcal{P}$ is projected onto a line N. Denote by $L_1, \ldots, L_{p+1}$ the lines in $\mathcal{P}$ which are projected from L onto one point. The line N lies in $p + 1$ planes of $PG(3, p^3)$ containing a projected $PG(4, p)$ of Λ. The other planes of $PG(3, p^3)$ through N only contain the projected points of Λ on N. Consider a plane π of $PG(3, p^3)$ intersecting N in one point which is not the projection of one of the lines $L_1, \ldots, L_{p+1}$. This plane can have at most a $PG(2, p)$ in common with Ω.

Hence π has $p^2 + p$ or $p^2 + p + 1$ distinct projected points of Λ lying in (F, w).

Case 3.1. Assume $\delta = p^2 + p$.

Then (F, w) has excess at most $p^2 - 1$.

The preceding observations show that all points of $\Omega \setminus N$ have weight one. A plane of Λ through one of the lines $L_1, \ldots, L_{p+1}$, but not completely contained in $\mathcal{P}$, is projected onto a line with p^2+1 distinct projected points of Λ.

Select such a line T so that it shares these p^2+1 points with (F,w), where the p^2 points different from the projection of L_i have weight one. This line T lies in planes of $PG(3,p^3)$ which contain p^3+p^2+1 or p^2+1 distinct projected points of Λ. Consider a plane through T with p^2+1 distinct points of (F,w). Counting with weights, we have p^2+p points of (F,w) in such a plane (Corollary 2.1.8). Hence, the projection of a line L_i has weight p. So, N has weight p^2+p and $(\Omega \setminus N, w')$ is a $\{(p^2+p)(p^3+1), p^2+p, 3, p^3\}$-minihyper.

Case 3.2. Assume $\delta = p^2+p+1$.

Since at least p^5+p^4+p+1 distinct points of Ω lie in (F,w), only a total weight of size p^3+p^2 has to be determined exactly. We know that the excess is at most p^3. We also know that if there is a point of (F,w) not belonging to Ω, then it is a multiple point since all points of (F,w) of weight one lie in a projected subgeometry $PG(5,p) \equiv \Omega'$ or $\Omega' \setminus N'$, with N' a line contained in Ω', contained in (F,w). However (F,w) only contains one such projected subgeometry (Lemma 4.3.14). This implies that there are at most $(p^3+p^2)/2$ distinct points in $F \setminus \Omega$.

We have at least p^5+p^4+p+1 distinct projected points of Ω in (F,w). Consider a plane π in $PG(3,p^3)$ intersecting N in a point which is not the projection of a line $L_1, \ldots, L_{p+1}$. This plane contains at most $p^2+p+1+(p^3+p^2)/2$ distinct points of (F,w); so this plane cannot intersect (F,w) in a 1-fold blocking set; so this plane shares p^2+p+1 weighted points with (F,w) (Corollary 2.1.8). If a point of Ω, not belonging to N, has multiple weight, then, by letting vary π through this point, no point of N different from the projection points of $L_1, \ldots, L_{p+1}$ can belong to (F,w).

The line N lies in $p+1$ planes of $PG(3,p^3)$, each having at least p^4+p+1 distinct points of (F,w). So, by Lemma 2.1.8, these planes have at least $p(p^3+1)+p^2+1$ points of (F,w). Hence the sum of their multiplicities m_1 is at least p^2+p (Corollary 3.2.1). Since N is not completely contained in (F,w), the sum of the multiplicities of the points of $F \cap N$ is at most $\delta = p^2+p+1$. Otherwise, every plane of $PG(3,p^3)$ through N intersects (F,w) in at least a 1-fold blocking set (Corollary 2.1.8), so this would imply that $|(F,w)| \geq (p^3+1)p^3$ which is false.

Since there is at least one plane through N sharing no other points with (F,w) than those on N, the sum of the multiplicities of the points of $N \cap F$ has to be p^2+p+1.

So, one of the projections of $L_1, \ldots, L_{p+1}$ has weight $x \geq p+1$. Assume the projection of L_1 has weight $x \geq p+1$. Let π' be a plane of Λ through L_1 not contained in $\mathcal{P}$. The projection of π' is a (p^2+1)-set on a line having weight greater than or equal to p^2+x. Again this line is not completely contained in (F,w), so it has at most p^2+p+1 points in common with (F,w); so $x = p+1$; and this implies that no point of Ω, not belonging to N, can have multiple weight.

Now consider the fact that all points of $\Omega \setminus N$ have weight one and that $\delta = p^2+p+1$. Let s be a point of N different from the projections of the lines $L_i, i = 1,\ldots,p+1$. There are exactly p^4+p^3 sublines $PG(1,p)$ through s containing p points of Ω not belonging to N. Select one of these sublines $PG(1,p)$ in such a way that the line T of $PG(3,p^3)$ through this subline only shares the points of Ω with (F,w). Since there are at most $(p^3+p^2)/2$ points of (F,w) not belonging to Ω, it is possible to find a plane π of $PG(3,p^3)$ through T sharing a subplane $PG(2,p)$ with Ω and only sharing the points of Ω with (F,w). Since $\delta = p^2+p+1$, and since all p^2+p points of $\Omega\setminus N$ in π have weight one, necessarily the intersection point of π with N has to have weight one. So this implies that all points of N are in (F,w).

We however assumed that there are no lines contained in (F,w); so $\delta = p^2+p+1$ does not occur in this situation.

Case 3.3. Assume $\delta = p^2+p+2$.

We know that already p^5+p^4+p+1 distinct points of Ω lie in (F,w), so we need to determine exactly a remaining total weight of size $2p^3+p^2+1$.

Assume the excess of these p^5+p^4+p+1 distinct points of Ω in (F,w) is equal to x, then there are at most p^3-x distinct points in $F\setminus\Omega$ since they can contribute at most p^3-x to the total excess. So, there is still a weight of at least p^2+1+x coming from points of Ω. This must come from the points of N different from the projections of the lines $L_1,\ldots,L_{p+1}$. So, adding at least $p+1$ for the weights of the projections of $L_1,\ldots,L_{p+1}$; N shares at least $p^2+p+2+x$ weighted points with (F,w).

This then implies that $|(F,w)\cap N| = p^2+p+2$, that $x=0$, that all the projections of the lines $L_1,\ldots,L_{p+1}$ have weight one in (F,w), and that all points of Ω not belonging to N have weight one.

Consider the point l_1 which is the projection of the line L_1. Consider a (p^2+1)-secant T through l_1 to Ω containing no points of $F\setminus\Omega$. Consider a plane through T different from $\langle N,T\rangle$. This plane either intersects Ω in p^2+1 or p^3+p^2+1 distinct points. But it must share at least p^2+p+2 or p^3+p^2+p+2 weighted points with (F,w), so considering all these p^3 planes, at least a weight of size $(p+1)\cdot p^3$ is still necessary. This is impossible. □

4.4 $\{\delta(p^3+1),\delta;3,p^3\}$-minihypers for p square

Let (F,w) be a $\{\delta(p^3+1),\delta;3,p^3\}$-minihyper, with p square, $p=p_0^h$, p_0 prime, $p_0\geq 7$, $h\geq 2$ even, $\delta\leq 2p^2-4p$, with excess $e\leq p^3$.

Suppose there is a point r of (F,w) of weight one, for which the dual blocking set ${B_r}^D$ contains a Baer subplane E.

Construct the cone $\mathcal{C}$ with vertex r and with base E. A plane through r containing a line of E is called a *plane* of $\mathcal{C}$.

Through r, there are at most $(p^3+\delta)-(p^3+p^{3/2}+1) = \delta - p^{3/2}-1$ planes intersecting (F,w) in a t-fold blocking set with $t>1$.

The next results follow the arguments of Govaerts and Storme [33]; therefore we only repeat the main ideas. To simplify notations, let $q=p^3$.

Lemma 4.4.1 *A plane π of $\mathcal{C}$ intersecting (F,w) in a 1-fold blocking set contains a Baer subplane B_0 and this Baer subplane B_0 is completely contained in $\mathcal{C}$.*

Proof. The arguments of Lemma 3.4.2 show that $\pi\cap(F,w)$ must contain a Baer subplane B_0. If B_0 is not completely contained in $\mathcal{C}$, then, by [33], there are at least $q-\sqrt{q}$ points of B_0 lying outside $\mathcal{C}$. Hence this plane contains at least $2q+1$ points of (F,w); a contradiction. $\square$

Lemma 4.4.2 *The point r is contained in a unique $PG(3,\sqrt{q})\equiv D$ completely contained in (F,w).*

Proof. (cfr. [33]) Let $E_0,\dots,E_{s-1}$ denote the planes of $\mathcal{C}$ through r intersecting (F,w) in a 1-fold blocking set. Then $s\geq q-\delta+2\sqrt{q}+2$.

Let $\pi\in\{E_0,\dots,E_{s-1}\}$ and let $\pi\cap\mathcal{C}=\{L_0,\dots,L_{\sqrt{q}}\}$. Suppose α of these lines L_i contain more than one Baer subline of points of (F,w). From $|\pi\cap(F,w)|=q+\delta\geq q+\sqrt{q}+1+\alpha(\sqrt{q}-1)$, it follows $\alpha<2\sqrt{p}-1$. Call the lines of $\mathcal{C}$ through r containing one Baer subline of (F,w) *good* lines.

Let $\pi,\pi'\in\{E_0,\dots,E_{s-1}\}$ intersect in a good line. Denote the Baer subplanes of points of (F,w), contained in π and π', respectively by B_0 and B_0'. They define a $PG(3,\sqrt{q})\equiv D=\langle B_0,B_0'\rangle$. The good lines of B_0 and B_0' define more than $(\sqrt{q}+1-(2\sqrt{p}-1)-1)^2=(p^{3/2}-2\sqrt{p}+1)^2$ planes of $\mathcal{C}$ intersecting π and π' in good lines. More than $(p^{3/2}-2\sqrt{p})^2-(\delta-\sqrt{q}-1)=p^3-6p^2+8p+p^{3/2}+1$ of these planes intersect (F,w) in a 1-fold blocking set; which necessarily contains a Baer subplane contained in $\mathcal{C}$ and in D (Lemma 4.4.1). Let π^* be one of these planes. Some line of π^* is contained in more than $(p^3-6p^2+8p+p^{3/2}+1)/(p^{3/2}+1)>p^{3/2}-6\sqrt{p}$ such planes. Hence, at least $(p^{3/2}-6\sqrt{p})p^{3/2}+1=p^3-6p^2+1$ lines of $\mathcal{C}$ have a Baer subline of points of (F,w) contained in D.

Suppose some point r' of D does not belong to (F,w). Then it lies on δ blocking planes.

But the q planes of D through r', but not through r, contain more than p^3-6p^2+1 points of (F,w); so they all are rich; a contradiction since we assume $r'\notin F$.

The uniqueness of D follows from [33]. $\square$

We state the final theorem of this chapter. We wish to remark that the description of the minihypers can be done in different ways.

In the statement of the theorem, also the possibility of projected subgeometries $PG(3,\sqrt{q})$ is included. If one projects a subgeometry $PG(3,\sqrt{q})\equiv D$ from a point $s\notin D$, then a cone with base a Baer subline $PG(1,\sqrt{q})$ is obtained. This cone is a $\{(\sqrt{q}+1)(q+1),\sqrt{q}+1;3,q\}$-minihyper if the vertex is giving the weight $\sqrt{q}+1$ and all other points are given weight one.

This cone is also a sum of lines, so it is also possible not to state explicitly these projected Baer subgeometries $PG(3,p^{3/2})$, and simply consider these lines as lines of the sum of lines inside the minihyper.

We however have written them in the formulation of the theorem since in larger dimensions projected subgeometries $PG(2\mu+1,\sqrt{q})$ can occur, and these projections are not equal to sums of spaces $PG(\mu,p^3)$ when $\mu \geq 2$ (see Theorems 5.5.6 and 5.5.8).

Theorem 4.4.3 *A $\{\delta(p^3+1),\delta;3,p^3\}$-minihyper, p square, $p=p_0^h$, p_0 prime, $h \geq 2$ even, $p_0 \geq 7$, $\delta \leq 2p^2-4p$, and with excess $e \leq p^3$, is either:*
(1) a sum of lines, (projected) $PG(3,p^{3/2})$'s, and at most one projected $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines, (projected) $PG(3,p^{3/2})$'s, and a $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihyper $(\Omega \setminus N, w')$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Proof. This is proven in the same way as Theorem 4.3.15. We discuss the different possibilities for the dual minimal blocking set E contained in ${B_r}^D$. If E is a Baer subplane, then the preceding results show that there is a subgeometry $PG(3,p^{3/2}) \equiv D$ through r contained in (F,w). Similarly as in the proof of Theorem 3.6.4, then reducing the weight of every point of D by one, $(F \setminus D, w')$ is a $\{(\delta-p^{3/2}-1)(p^3+1),\delta-p^{3/2}-1;3,p^3\}$-minihyper.

So, from now on, we can assume that E is not a Baer subplane.

If E is a dual minimal blocking set of size p^3+p^2+p+1 or of size p^3+p^2+1, then every good plane through r intersects (F,w) in a 1-fold blocking set containing a projected subgeometry $PG(3,p)$ completely contained in $\mathcal{C}$. The situation is now reduced to that of the preceding subsections. $\square$

Chapter 5

Weighted $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers

This chapter extends the results of Chapter 4 to arbitrary dimensions and where the size of the minihyper has the more general form $f = \delta v_{\mu+1}$.

We classify all $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers, $\delta \leq 2p^2 - 4p$, $N \geq 4$, $p = p_0^h \geq 9$, $h \geq 1$, for a prime number $p_0 \geq 7$, with excess $e \leq p^3 - 4p$ when $\mu = 1$ and with excess $e \leq p^2 + p$ when $\mu > 1$. For $N \geq 4$, p non-square, such a minihyper is a sum of μ-dimensional spaces $PG(\mu, p^3)$ and of at most one (possibly projected) subgeometry $PG(3\mu + 2, p)$; except for one special case when $\mu = 1$. When p is a square, also (possibly projected) Baer subgeometries $PG(2\mu + 1, p^{3/2})$ can occur.
The results are taken from Ferret and Storme, A classification result on weighted $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers [27].

This chapter first of all presents the results for $\{\delta v_2, \delta v_1; N, p^3\}$-minihypers, $N > 3$, and then the results for $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers, $\mu > 1$. In Chapter 4, the $\{\delta v_2, \delta v_1; 3, p^3\}$-minihypers have been discussed. Since this is the basis for the results we present in this chapter, we repeat the characterization results of Chapter 4.

5.1 The result in three dimensions

The following characterization results on $\{\delta v_2, \delta v_1; 3, p^3\}$-minihypers have been obtained in Theorems 4.3.15 and 4.4.3.

Theorem 5.1.1 *A $\{\delta(p^3 + 1), \delta; 3, p^3\}$-minihyper (F, w), p non-square, $p = p_0^h$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, and with excess $e \leq p^3$, is either:*
(1) a sum of lines and at most one projected $PG(5, p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines and a $\{(p^2 + p)(p^3 + 1), p^2 + p; 3, p^3\}$-minihyper $\Omega \setminus N$, where Ω

is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Theorem 5.1.2 *A $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper (F,w), p square, $p = p_0^h$, p_0 prime, $h \geq 2$ even, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, and with excess $e \leq p^3$, is either:*
(1) a sum of lines, (projected) $PG(3, p^{3/2})$, and at most one projected $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) a sum of lines, (projected) $PG(3, p^{3/2})$, and a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $\Omega \setminus N$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

We refer to Section 4.2 for the different types of projected $PG(5,p)$ and for the notations used.

5.2 $\{\delta(p^3+1), \delta; N, p^3\}$-minihypers for p non-square

Let (F,w) be a $\{\delta(p^3+1), \delta; 4, p^3\}$-minihyper.

We suppose the total excess $\sum_{x \in F}(w(x) - 1)$ is at most $p^3 - 4p$ and $\delta \leq 2p^2 - 4p$. Let $r \notin F$ be a point lying on at most $|F|(|F|-1)(p^3-1)/(2(|PG(4,p^3)| - |F|)) < 2p$ secants to (F,w); these latter secants contain at most $4p$ points of (F,w). So we can project (F,w) from r onto a solid Π to obtain a weighted minihyper (F',w'), and at most $2p$ points of (F',w') are the projection of at least two distinct points in (F,w). Moreover, the total excess of (F',w') is $e' \leq p^3 - 4p + 4p = p^3$. So (F',w') satisfies the conditions of Theorem 5.1.1.

Lemma 5.2.1 *There is a bijective relation between the set of the lines contained in (F,w) and the set of lines contained in (F',w').*

Proof. A line of (F,w) is projected onto a line of (F',w'). No two lines are projected onto the same line. Assume there is a line M contained in (F',w'). A point on M which only is the projection of one point in (F,w), defines one point of (F,w) in $\langle M, r\rangle$. The points on M which are the projections of at least two points of (F,w) define at most $4p$ points in $F \cap \langle M, r\rangle$. So, in $\langle M, r\rangle$ lie at most $p^3 + 1 + 4p$ distinct points of (F,w) and they define a 1-fold blocking set in $\langle M, r\rangle$; hence there is a line of (F,w) in $\langle M, r\rangle$ (Theorems 1.3.3 and 2.1.6). □

Theorem 5.2.2 ([33]) *If there is a line L contained in a $\{\delta(q+1), \delta; 4, q\}$-minihyper (F,w), with $\delta \leq (q+1)/2$, then the minihyper obtained by reducing the weight of every point of L by one, is a $\{(\delta-1)(p^3+1), \delta-1; 4, p^3\}$-minihyper (F',w').*

The preceding lemma and theorem imply that from now on, we can assume that (F,w), and its projection (F',w') from r, do not contain any lines. Since by projecting from r, the excess increases by at most $4p$, (F',w') has at most excess p^3. Using Theorem 5.1.1, the only projections (F', w') we have to consider are the following $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers. Since these projected subgeometries $PG(5,p)$ lie in a 3-dimensional space, they are easily described as the projection of a subgeometry $PG(5,p) \equiv \Lambda$ from a line L.

1. $\delta = p^2+p+1$, and (F', w') is the projection of a $PG(5,p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 5$. Then (F', w') only contains points of weight one.

2. $\delta = p^2+p+1$ and (F', w') is the projection of a $PG(5,p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 4$. Then (F', w') contains one point of weight $p+1$ and $p^5+p^4+p^3+p^2$ points of weight one.

3. $\delta = p^2+p$ and $(F', w') = (\Omega \setminus N, w')$ is the projection of a $PG(5,p) \equiv \Lambda$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, minus the line N contained in this projection. Then (F', w') contains exactly $p+1$ points of weight p, lying on N, and all other points of (F', w') have weight one.

The goal now is to prove that (F, w) itself equals a projected subgeometry $PG(5,p)$ or a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$.

To achieve this goal, we discuss the preceding three possibilities one by one. The main difficulty is that we have to use certain properties of solids in projected subgeometries $PG(5,p)$. To make the ideas as clear as possible in Cases 2 and 3, we first describe these, rather obvious, properties originally in the non-projected subgeometry Λ. These are then described in the projected subgeometry (F', w') in Π, and by then describing them in the 4-space $PG(4,p^3)$, the characterization of the original minihyper (F, w) is obtained.

Case 1. Suppose (F', w') is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Every plane of Π intersects (F', w') in p^2+p+1 or p^3+p^2+p+1 points.

Then every solid of $PG(4,p^3)$ through r has p^3+p^2+p+1 or p^2+p+1 points of (F, w). If a solid π_3 through r has p^3+p^2+p+1 points of (F, w), then $F \cap \pi_3$ is a 1-fold blocking set with respect to the planes of π_3 (Theorem 2.1.6). By the results of Theorem 1.3.5, $F \cap \pi_3$ contains a non-projected $PG(3,p)$ or a minimal blocking set of size p^3+p^2+p+1. For if it would share a minimal blocking set of size p^3+p^2+1 with (F, w), we would have $p+1$ different (p^2+1)-sets in the projection (F', w').

Let T be a (p^2+p+1)-secant to (F', w'). Consider two planes π_i, $i = 1, 2$, in Π through T containing p^3+p^2+p+1 points of (F', w'); see Figure 5.1. These planes π_i and r define solids containing p^3+p^2+p+1 points of (F, w). The (projected) $PG(3,p)$ $\langle \pi_i, r\rangle \cap (F, w)$, $i = 1, 2$, sharing p^3+p^2+p+1 points with (F, w), define a (projected) $PG(4,p) = \Omega_4$. Now, select a third plane π_3 of Π through T containing p^3+p^2+p+1 points of (F', w') such that $(F, w) \cap \langle r, \pi_3\rangle$ does not lie in Ω_4. Remark that there are p^2 choices for such a plane π_3. Then $\langle \pi_i, r\rangle \cap (F, w)$, $i = 1, 2, 3$, define a (projected) $PG(5,p) \equiv \Omega$.

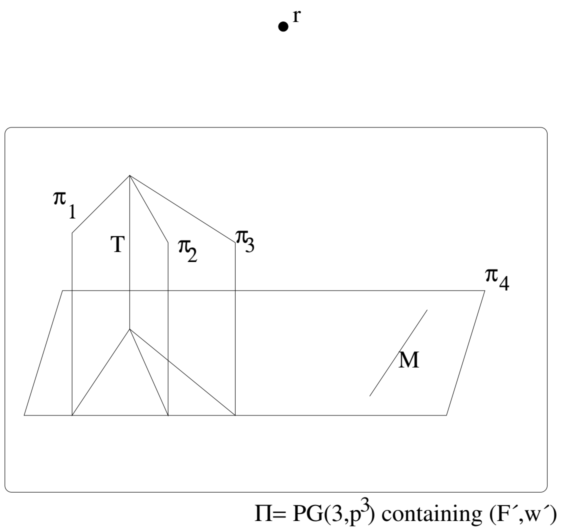

Figure 5.1: The minihyper (F', w')

We show that Ω is completely contained in (F, w). Consider a second (p^2+p+1)-secant M to (F', w'), skew to T, and consider all planes π_4 of Π through M containing p^3+p^2+p+1 points of (F', w') (Figure 5.1). Then $\pi_i \cap \pi_4 \cap (F', w')$, $i = 1, 2, 3$, is a subline $PG(1,p)_i$ since both planes contain a projected $PG(3,p)$ of the projected $PG(5,p) \equiv (F', w')$. Then $\langle \pi_i, r\rangle \cap \langle \pi_4, r\rangle \cap (F, w) \equiv PG(1,p)_i$, $i = 1, 2, 3$, share one point with (F, w), projected from r onto T. Hence the (projected) $PG(3,p) \equiv (F, w) \cap \langle \pi_4, r\rangle$ shares three sublines $PG(1,p)_i$ with Ω. Hence, it also shares the plane $\langle PG(1,p)_1, PG(1,p)_2\rangle$ with Ω. Finally it is contained in Ω since it also shares $PG(1,p)_3$ with Ω and $PG(1,p)_3 \not\subset \langle PG(1,p)_1, PG(1,p)_2\rangle$. Letting vary π_4, we obtain that all $p^5+p^4+p^3+p^2+p+1$ points of Ω lie in (F, w). So, (F, w) is a (projected) $PG(5,p)$ of size $p^5+p^4+p^3+p^2+p+1$ only having points of weight one.

Case 2. Suppose (F', w') is the projection of Λ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Then (F', w') consists of one point of weight $p+1$ and of $p^5+p^4+p^3+p^2$ points of weight one. Let M denote the line of Λ which is projected onto one point m of (F', w').

We first describe some properties of 3-spaces of Λ. These properties allow us to reconstruct Λ from carefully selected solids. We do this since these ideas will then be used to construct the projected $PG(5,p) \equiv \Omega$ contained in (F, w).

Let $\pi = \langle s, s^p, s^{p^2}\rangle \cap \Lambda$, with $s \in L$, be a plane of Λ which is projected onto a (p^2+p+1)-set of (F', w'). Again we select $PG(3,p)_i$, $i = 1, 2, 3$, of Λ

through π defining Λ. This time we select these $PG(3,p)_i$ outside the special $PG(4,p) \equiv \mathcal{P}$ of Λ. Consider a solid $PG(3,p)_4$ through M not lying in $\mathcal{P}$. This solid shares a unique point with π since this plane is skew to M. It shares a subline $PG(1,p)_i$ with $PG(3,p)_i$, $i = 1,2,3$, since they are two solids in Λ and since M is skew to the intersection.

This implies that $PG(3,p)_4$ is contained in Λ. By letting vary the solid $PG(3,p)_4$, the subgeometry Λ is reconstructed.

We now describe the preceding observations in the subgeometry Λ in the projected subgeometry (F', w') to find the projected subgeometry Ω contained in (F, w).

Let T be a (p^2+p+1)-secant to (F', w'). Let π_1, π_2, π_3 be three distinct planes of Π through T intersecting (F', w') in 1-fold blocking sets $PG(3,p)_i$, $i = 1,2,3$, of size p^3+p^2+p+1 such that the 1-fold blocking sets of (F', w') in π_1, π_2, π_3 generate the projected subgeometry (F', w').

Then, as in Case 1, the corresponding solids $\langle \pi_i, r\rangle$ again intersect (F, w) in a non-projected subgeometry $PG(3,p)$ or in a minimal blocking set of size p^3+p^2+p+1. Let Ω be the projected subgeometry $PG(5,p)$ of $PG(4,p^3)$ defined by these 1-fold blocking sets of (F, w) in the solids $\langle \pi_i, r\rangle$, $i = 1,2,3$.

Let π be a plane of Π through m sharing a projected $PG(3,p)$ with (F', w'). Returning to $PG(4,p^3)$, $\langle \pi, r\rangle \cap (F, w)$ is either a (projected) $PG(3,p)$ of size p^3+p^2+p+1; or contains a minimal blocking set of size p^3+p^2+1 and maybe some extra points. In this latter case, since the projection m of M has weight $p+1$, the possible extra points lie on the line $\langle m, r\rangle$. In $PG(4,p^3)$, $(F, w) \cap \langle \pi, r\rangle$ shares a $PG(1,p)_i$ with $(F, w) \cap \langle \pi_i, r\rangle$, $i = 1,2,3$.

So the (projected) $PG(3,p)$ contained in $(F, w) \cap \langle \pi, r\rangle$ lies in Ω.

Letting vary π, this implies that already $p^5+p^4+p^3+p^2+1$ points of (F, w) lie in Ω. If Ω has $p^5+p^4+p^3+p^2+p+1$ distinct points, we will show further on that (F, w) coincides with this set of $p^5+p^4+p^3+p^2+p+1$ points. If Ω has one point of weight $p+1$, we show that (F, w) coincides with the $p^5+p^4+p^3+p^2$ points of weight one and one point of weight $p+1$. The only doubt which remains concerns the points on the line $\langle m, r\rangle =$(say) T'.

We project (F, w) from another point $r' \notin T'$, lying on at most $2p$ secants to F, onto a minihyper (F'', w''). Then (F'', w'') is the projection of a subgeometry $PG(5,p) \equiv \Lambda'$. We obtain at most $4p$ extra multiple points. Hence, the total excess is at most $5p$; and we are again in Case 1 or Case 2. If we are in Case 1, there is nothing left to prove. If we are in Case 2, let $m' \in F''$ be the point of weight $p+1$ of (F'', w''). All points of (F, w) on $\langle m, r\rangle \setminus \langle m', r'\rangle$ must lie in Ω. Remark that there can only be one projected $PG(5,p)$ in $PG(4,p^3)$ containing at least $p^5+p^4+p^3+p^2$ points of (F, w). Since the points of $(\langle m', r'\rangle \setminus \langle m, r\rangle) \cap (F, w)$ have weight one, but the projection m' has weight $p+1$, $\langle m, r\rangle$ has no points of (F, w) outside Ω. Hence, either $\langle m, r\rangle \cap \Omega$ is one point of weight $p+1$ or $p+1$ points of weight one.

Case 3. Suppose (F', w') is the projection of Λ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

We will again describe properties of 3-spaces in Λ. The ideas following from these properties will then be translated into properties of projected 3-spaces in (F', w'). These latter properties will then be interpreted with respect to (F, w) in $PG(4, p^3)$ to construct a projected $PG(5, p) \equiv \Omega$ containing at least $p^5 + p^4$ points of (F, w).

Let L_i, $i = 1, \ldots, p+1$, be the lines of $\mathcal{P} \subset \Lambda$ which contain p points projected onto one point of (F', w'). Select a plane π of Λ through L_1 not contained in $\mathcal{P}$. Select three distinct 3-spaces $PG(3, p)_i$, $i = 1, 2, 3$, through π only sharing L_1 with $\mathcal{P}$ and generating Λ.

Every solid $PG(3, p)$ of Λ through L_2, only intersecting $\mathcal{P}$ in L_2, intersects the solids $PG(3, p)_i$, $i = 1, 2, 3$, in sublines $PG(1, p)_i$. These latter sublines $PG(1, p)_i$, $i = 1, 2, 3$, generate this solid $PG(3, p)$ of Λ through L_2; hence by letting vary $PG(3, p)$ through L_2, we can prove that all points of $\Lambda \setminus \mathcal{P}$ lie in the 5-space generated by the solids $PG(3, p)_i$, $i = 1, 2, 3$.

We now interpret this in the projection (F', w'). Let $l_1, \ldots, l_{p+1}$ be the projections of the lines $L_1, \ldots, L_{p+1}$.

A plane π of Λ through L_1, with $\pi \not\subset \mathcal{P}$, is projected onto a $(p^2 + 1)$-set T through l_1. The solids $PG(3, p)_i$ selected above are projected onto 1-fold blocking sets of size $p^3 + p^2 + 1$ in planes π_1, π_2, π_3 through T.

This implies that in $\langle \pi_i, r \rangle$, $i = 1, 2, 3$, (F, w) shares a 1-fold blocking set of size $p^3 + p^2 + p$ with $\langle \pi_i, r \rangle$, hence it contains a minimal blocking set B_i of size $p^3 + p^2 + 1$ (Theorems 1.3.5 and 2.1.6). The point l_1 had weight p in (F', w'); so possible points of $(F, w) \cap \langle \pi_i, r \rangle$ not in B_i lie on the line $\langle l_1, r \rangle$.

Interpreting everything with respect to (F, w) in $PG(4, p^3)$, the minimal blocking sets B_i, $i = 1, 2, 3$, generate a (projected) subgeometry $PG(5, p) \equiv \Omega$. Remark that B_i, $i = 1, 2, 3$, have the same vertex, since they share the $(p^2 + 1)$-set which is projected onto T.

Every projected $PG(3, p)$ of (F', w'), in a plane π_4 through l_j, $j = 2, \ldots, p+1$, only intersecting the projection of $\mathcal{P}$ in l_j intersects T in exactly one point, and intersects $\pi_i \cap (F', w')$ in a $(p+1)$-secant. By the choice of the planes π_i, $i = 1, 2, 3$, these $(p + 1)$-secants are not coplanar in Ω; hence this projected $PG(3, p)$ is completely contained in (F', w'), and interpreting this in $\langle \pi_4, r \rangle$, the corresponding minimal blocking set of size $p^3 + p^2 + 1$ in $(F, w) \cap \langle \pi_4, r \rangle$ is completely contained in Ω.

Hence we have found a (projected) $PG(5, p) \equiv \Omega$ in $PG(4, p^3)$ having $p^5 + p^4 + p + 1$ points lying in (F, w).

Now Ω has $p + 1$ points through which there pass $(p^2 + 1)$-secants. This is impossible for a $PG(5, p)$ projected from one point onto $PG(4, p^3)$. So Ω lies in a 3-dimensional subspace Π_3 of $PG(4, p^3)$, and Ω is a subgeometry $PG(5, p)$ projected from a line L' for which $\dim\langle L', L'^p, L'^{p^2} \rangle = 3$. Let N be the line contained in Ω.

Since the total weight of the points of (F, w) is equal to $(p^2 + p)(p^3 + 1)$, we still need to determine the exact description of a total weight of order $p^2 - 1$. If there are points of (F, w) not belonging to Ω, then these points must lie on the lines $\langle l_i, r\rangle$, $i = 1,\dots,p+1$.

At most p^2-1 points of (F,w) do not belong to Π_3. Select new points $r' \notin F$, $r' \notin \Pi_3$. For all these points, the projection of (F,w) from r' onto Π_3 must be a projected $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega' \setminus N', w'')$, where N' is the line contained in Ω'.

Furthermore, the points of (F,w) not belonging to Π_3 must always be projected onto the points of weight p in $(\Omega \setminus N, w')$. This is not possible when (F,w) has points not belonging to Π_3. The only possibility is that (F,w) is this $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$. □

We now state the first concluding theorem of this chapter.

Theorem 5.2.3 *A $\{\delta(p^3+1),\delta;N,p^3\}$-minihyper (F,w), $N \geq 4$, p non-square, $p = p_0^h$, $p \geq 9$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2-4p$, and with excess $e \leq p^3-4p$, is either:*
(1) a sum of lines and of at most one (projected) $PG(5,p)$,
(2) a sum of lines and of a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Proof. For $N = 4$, this follows from the preceding lemmas.

The result for $N > 4$ follows from analogous arguments, using an inductive proof with $N = 4$ as induction hypothesis.

Since for $N = 4$, the projection of (F,w) from r can increase the excess by at most $4p$, we only allow here $e \leq p^3-4p$ so that the projection (F',w') has excess $e' \leq p^3$. □

5.3 $\{\delta(p^6+p^3+1),\delta(p^3+1);N,p^3\}$-minihypers for p non-square

In this section, we classify $\{\delta(p^6+p^3+1),\delta(p^3+1);N,p^3\}$-minihypers (F,w), $\delta \leq 2p^2-4p$, $N \geq 3$, p non-square, $p \geq 9$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $h \geq 1$, with excess $e \leq p^2+p$.

Consider a $PG(N-3,p^3)$ skew to (F,w). The $(N-2)$-dimensional subspaces through it intersect (F,w) in δ points; see Theorem 2.1.5. Since the total excess of the points is at most p^2+p, there certainly is a $PG(N-2,p^3) \equiv \Delta$ intersecting (F,w) in δ points of weight one.

The hyperplanes through Δ will be denoted by $H_0,\dots,H_{p^3}$. By Theorem 2.1.6, they intersect (F,w) in weighted $\{\delta(p^3+1),\delta;N-1,p^3\}$-minihypers satisfying the conditions of Theorem 5.2.3. So they intersect (F,w) in a weighted sum of lines and of at most one (projected) $PG(5,p)$, or at most one $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$.

We want to classify the above mentioned $\{\delta(p^6+p^3+1),\delta(p^3+1);N,p^3\}$-minihypers (F,w), $\delta \leq 2p^2-4p$, as sums of planes and of at most one (projected) $PG(8,p)$.

We first show that if there are many lines in the intersections of the hyperplanes H_i with (F, w), then there are planes contained in (F, w).

Lemma 5.3.1 *If $r \in (F, w) \cap \Delta$ lies on two lines L_1, L_2 contained in (F, w), then $\langle L_1, L_2 \rangle$ is contained in (F, w).*

Proof. Assume $\langle L_1, L_2 \rangle$ is not contained in (F, w).

The plane $\langle L_1, L_2 \rangle$ intersects (F, w) in at least a 2-fold blocking set (Theorem 2.1.6). If it intersects (F, w) in a t-fold blocking set, with necessarily $t \leq 2p^2$, by Theorem 2.1.6, it contains at most $tp^3 + 2p^2$ points. On the other hand, it contains the two lines L_1 and L_2, and at least $t - 1$ other points on every other line through r in $\langle L_1, L_2 \rangle$ since r has weight one; yielding $tp^3 + 2p^2 \geq |(F, w) \cap \langle L_1, L_2 \rangle| \geq (t+1)p^3 - t + 2$; a contradiction. Hence $\langle L_1, L_2 \rangle$ is completely contained in (F, w). $\square$

Reducing the weights of all points of $\langle L_1, L_2 \rangle$ by 1, yields a $\{(\delta - 1)(p^6 + p^3 + 1), (\delta - 1)(p^3 + 1); N, p^3\}$-minihyper $(F \setminus \langle L_1, L_2 \rangle, w')$, see Theorem 4.3.1.

Hence, from now on, we assume that (F, w) does not contain planes.

Remark 5.3.2 *(1)* We can then assume that $\delta = p^2 + p + 1$ or $\delta = p^2 + p$, since if $\delta > p^2 + p + 1$, then every hyperplane through Δ contains at least one line, and hence some point s of $(F, w) \cap \Delta$ lies on at least two lines of (F, w), and so s lies in a plane α contained in (F, w).

We assume the total excess of the points of (F, w) is at most $p^2 + p$. Since Δ intersects (F, w) in points of weight one, it is impossible that all $p^3 + 1$ hyperplanes $H_i, i = 0, \ldots, p^3$, through Δ intersect (F, w) in a $\{(p^2 + p)(p^3 + 1), p^2 + p; N - 1, p^3\}$-minihyper (F_i, w), which is a projected $PG(5, p)$ minus a line, since each such minihyper has $p + 1$ points of weight p.

Hence, there is at least one hyperplane through Δ intersecting (F, w) in a $\{(p^2+p+1)(p^3+1), p^2+p+1; N-1, p^3\}$-minihyper (F_i, w) which is a projected $PG(5, p)$. Hence Δ contains $p^2 + p + 1$ points of weight one and we can assume that $\delta = p^2 + p + 1$.

(2) Also, every $(N-2)$-dimensional subspace Δ' sharing p^2+p+1 points of weight one with (F, w), must intersect (F, w) in either a non-projected $PG(2, p)$, or in a $(p^2 + p + 1)$-set.

For if $\Delta' \cap (F, w)$ would not be of one of these types, then this would imply that no hyperplane through Δ' would intersect (F, w) in a (projected) $PG(5, p)$. So, all hyperplanes H_i' through Δ' would share $p^2 + p + 1$ lines with (F, w), or they would share a minihyper (F_i', w) with (F, w), where (F_i', w) is the sum of a $\{(p^2 + p)(p^3 + 1), p^2 + p; N - 1, p^3\}$-minihyper $\Omega_i \setminus N_i$ and a line N_i', where Ω_i is a projected $PG(5, p)$ and N_i is the line contained in Ω_i. But then there is a point of $(F, w) \cap \Delta'$ lying on two lines of (F, w); so (F, w) would contain at least one plane, and this was excluded.

Lemma 5.3.3 *If there is a hyperplane H_0 through Δ containing $p^2 + p + 1$ lines of (F, w), then these lines form a cone with a (p^2+p+1)-set or a subplane $PG(2, p)$ as base.*

Proof. Case 1. Assume first that the p^2+p+1 points in $\Delta \cap (F,w)$ form a non-projected $PG(2,p)$.

We show that all lines in $(F,w) \cap H_0$ are contained in a unique projected $PG(5,p) \equiv \Omega$ which is projected from a line L, for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

Consider a subline $PG(1,p) \equiv \tilde{M}$ in $\Delta \cap (F,w)$. Construct two $(N-2)$-dimensional subspaces Δ_1 and Δ_2 of H_0 through $\tilde{M}$, only sharing two distinct subplanes $PG(2,p)_1$ and $PG(2,p)_2$ with (F,w). Then there are exactly p^2 lines of (F,w) in H_0 intersecting $PG(2,p)_1$ and $PG(2,p)_2$ in distinct points. The two subplanes $PG(2,p)_i, i=1,2$, define a unique $PG(3,p) \equiv \Omega_3$ and these p^2 lines of $H_0 \cap (F,w)$ share already a subline $PG(1,p)$ with Ω_3.

Select a third $(N-2)$-dimensional subspace Δ_3 in H_0 through $\tilde{M}$ only sharing one subplane $PG(2,p)_3$ with (F,w), such that $PG(2,p)_3 \not\subseteq \Omega_3$. Then $PG(2,p)_3$ and Ω_3 define a (possibly projected) subgeometry $PG(4,p) \equiv \Omega_4$. Then the p^2 lines of (F,w) in H_0 intersecting $PG(2,p)_1$ and $PG(2,p)_2$ in distinct points share an other intersection point with Ω_4, and so they intersect Ω_4 in a (p^2+1)- or (p^2+p+1)-set.

By now selecting a fourth $(N-2)$-dimensional subspace Δ_4 in H_0 through $\tilde{M}$ only sharing one subplane $PG(2,p)_4$ with (F,w), such that $PG(2,p)_4 \not\subseteq \Omega_4$, and by repeating the arguments above, all lines of $H_0 \cap (F,w)$ intersecting $PG(2,p)_1$ and $PG(2,p)_2$ in distinct points now are completely contained in the projected $PG(5,p) \equiv \langle \Omega_4, PG(2,p)_4\rangle \equiv \Omega_5$.

Repeating the arguments for other sublines than $\tilde{M}$, we get that all p^2+p+1 lines of $(F,w) \cap H_0$ are contained in Ω_5. Actually, Ω_5 is contained in a 3-dimensional space over $GF(p^3)$, namely the one generated by the planes over $GF(p^3)$ containing $PG(2,p)_1$ and $PG(2,p)_2$.

The only possibility is that Ω_5 is the projection of a subgeometry Λ from a line L, for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$. The plane $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ is projected onto one point s of weight p^2+p+1 and the p^2+p+1 solids of Λ through this plane are projected onto lines through s.

Case 2. Assume that the p^2+p+1 points in $\Delta \cap (F,w)$ form a (p^2+p+1)-set $\{r_1, \ldots, r_{p^2+p+1}\}$ on a line L_1.

We may assume that every $PG(N-2,p^3)$ of H_0 containing p^2+p+1 points of weight one of $(F,w) \cap H_0$ intersects (F,w) in a (p^2+p+1)-set, since otherwise we are reduced to Case 1.

Consider a point r_1 of this (p^2+p+1)-set $\Delta \cap (F,w)$. It is possible to find an $(N-3)$-dimensional subspace of H_0 only sharing r_1 with (F,w). Then this $(N-3)$-dimensional subspace lies in at least a second $(N-2)$-dimensional subspace of H_0 only intersecting (F,w) in a (p^2+p+1)-set on a line L_2.

Then there are already p^2+p lines of (F,w) lying in the plane $\langle L_1, L_2\rangle$. Since r_1 is an arbitrary point of the (p^2+p+1)-set $(F,w) \cap L_1$, all p^2+p+1 lines $T_1, \ldots, T_{p^2+p+1}$ of $(F,w) \cap H_0$ are contained in $\langle L_1, L_2\rangle$.

The line T_1 has already p^2+p intersection points with the other lines

$T_2, \ldots, T_{p^2+p+1}$, hence we have already the total excess p^2+p of (F,w) on the line T_1. This implies that all other intersections $T_i \cap T_j$, $2 \le i < j$, of the other lines T_i in $(F,w) \cap H_0$ have to coincide with $T_1 \cap T_2$. Hence, we have p^2+p+1 lines through one point. □

We remark that a cone of lines with vertex a point and base a (p^2+p+1)-set also is a projected subgeometry $PG(5,p)$. To obtain such a projected $PG(5,p)$, project first of all a subgeometry $PG(5,p)$ from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 2$. This projection is a cone with base a non-projected subplane $PG(2,p)$ in a plane Π skew to the vertex. Select a point r of Π only lying on tangents to the base of this cone. Project again, but now from r, then the projection is a cone over a (p^2+p+1)-set.

Lemma 5.3.4 *If there is a hyperplane H_0 through Δ intersecting (F,w) in a minihyper which is the sum of a $\{(p^2+p)(p^3+1), p^2+p; N-1, p^3\}$-minihyper $(\Omega \setminus N, w')$ and a line N', where Ω is a projected $PG(5,p)$ containing the line N, then $N = N'$.*

Proof. We know from Remark 5.3.2 (2) that every $(N-2)$-dimensional subspace Δ in H_0 intersecting $(F,w) \cap H_0$ in p^2+p+1 distinct points must intersect $(F,w) \cap H_0$ in a subplane $PG(2,p)$ or (p^2+p+1)-set. At least p^2+p of these points must belong to Ω, so also the latter point must belong to Ω. This latter point lies on N'. So N and N' share at least two points, since there are at least two such $(N-2)$-dimensional subspaces, so $N = N'$. □

Consider again the $(N-2)$-dimensional space Δ containing $\delta = p^2+p+1$ points of weight one of (F,w). The preceding lemmas show that all hyperplanes through Δ intersect (F,w) in a projected $PG(5,p)$. This enables us to prove the following result.

Theorem 5.3.5 *There is a (projected) $PG(8,p)$ completely contained in (F,w).*

Proof. Consider again a $PG(N-2,p^3) \equiv \Delta$ intersecting (F,w) in a $PG(2,p)$. Then the hyperplanes H_i, $i = 0, \ldots, p^3$, through Δ intersect (F,w) in a (projected) $PG(5,p)_i$. Since the total excess e of (F,w) is at most p^2+p, it is possible to find two hyperplanes H_1 and H_2 through Δ intersecting (F,w) in (projected) $PG(5,p)_1$ and $PG(5,p)_2$ without multiple points. These two subgeometries $PG(5,p)_1$ and $PG(5,p)_2$ define a (projected) $PG(8,p)$.

Consider a point r in $PG(5,p)_2 \setminus \Delta$ and consider a line $T \subset H_2$ through r containing a point of $\Delta \cap (F,w)$. Then T intersects (F,w) in $p+1$ or in p^2+p+1 distinct points. If T intersects (F,w) in p^2+p+1 points, consider a $PG(N-3,p^3)$ through T only sharing the points on T with (F,w). There is at least one $PG(N-2,p^3) \equiv \Delta'$ through this $PG(N-3,p^3)$ only sharing these p^2+p+1 points with (F,w); since if a $PG(N-2,p^3)$ shares more than p^2+p+1 points with (F,w), it shares at least p^3+p^2+p+1 points with (F,w) (Theorem 2.1.6). If T intersects (F,w) in $p+1$ points, we similarly find a

$PG(N-2,p^3) \equiv \Delta'$ through T intersecting (F,w) in p^2+p+1 distinct points. The points in $T \cap (F,w)$ generate together with $PG(5,p)_1$ a (projected) $PG(6,p)$ or $PG(7,p)$. We show that this subgeometry consists completely of points of (F,w).

Consider a hyperplane H through Δ'. Then H intersects (F,w) in a (projected) $PG(5,p)$. This hyperplane shares a subgeometry $PG(d,p)$, $d \geq 2$, with $PG(5,p)_1$, and this $PG(d,p)$ defines together with the points in $T \cap (F,w)$ at least a (projected) $PG(3,p)$ in $H \cap (F,w)$ containing r.

Letting H vary, all points of the (projected) subgeometry $\langle PG(5,p)_1, T \cap (F,w)\rangle$ over $GF(p)$ are contained in (F,w).

Letting T vary, we obtain that the, possibly projected, 8-dimensional subgeometry $PG(8,p) = \langle PG(5,p)_1, PG(5,p)_2\rangle$ is contained in (F,w). □

We present a new classification result on minihypers.

Theorem 5.3.6 *Let (F,w) be a $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihyper, $\delta \leq 2p^2-4p$, $N \geq 3$, $p \geq 9$ non-square, $p = p_0^h$, $h \geq 1$, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.*

Then (F,w) is a sum of planes and of at most one (projected) subgeometry $PG(8,p)$.

Proof. This follows from the preceding arguments. We first discussed the possibility of planes in (F,w) and showed that it was possible to remove these planes from (F,w) by reducing in all these planes the weight of their points by one. This either characterized (F,w) completely, or in the other case, we proved that there is a (projected) subgeometry $\Pi_8 = PG(8,p)$ contained in (F,w). Assume that all planes have been removed from (F,w), then $\delta = p^2+p+1$. We now show that in this latter case, (F,w) coincides with Π_8. Let Δ be a $PG(N-2,p^3)$ intersecting (F,w) in δ points of weight one. Every hyperplane H_i, $i = 0,\ldots,p^3$, through Δ intersects (F,w) in a $PG(5,p)$ and intersects Π_8 in at least a $PG(5,p)$. These intersections must coincide, yielding that Π_8 is equal to (F,w).

There is also no problem regarding the weights of the points of (F,w) in this description of (F,w) as a sum of planes and of at most one (projected) $PG(8,p)$.

For the planes, this again follows from the fact that it was possible to remove these planes from (F,w) by reducing in all these planes the weight of their points by one. For the possible remaining (projected) subgeometry $PG(8,p)$ in (F,w), everything is correct regarding the weights of the points by the definition of $(F,w) \cap H_i$, $i = 0,\ldots,p^3$ (Remark 3.6.12). □

5.4 The general result for p non-square

We now prove by induction on μ the following characterization result.

Theorem 5.4.1 *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 3$, $\delta \leq 2p^2 - 4p$, $N \geq 3$, $p \geq 9$ non-square, $p = p_0^h$, $h \geq 1$, $p_0 \geq 7$ prime, with excess $e \leq p^2 + p$.*

Then (F, w) is a sum of μ-dimensional spaces $PG(\mu, p^3)$ and of at most one (projected) subgeometry $PG(3\mu + 2, p)$.

Let Δ be a $PG(N-2, p^3)$ intersecting (F, w) in $\delta v_{\mu-1}$ points of weight one. Then $\Delta \cap (F, w)$ is a $\{\delta v_{\mu-1}, \delta v_{\mu-2}; N-2, p^3\}$-minihyper. Let $H_i, i = 0, \ldots, p^3$, be the hyperplanes through Δ. They intersect (F, w) in $\{\delta v_\mu, \delta v_{\mu-1}; N-1, p^3\}$-minihypers (F_i, w) (Theorem 2.1.6). By the induction hypothesis, these minihypers (F_i, w) are sums of $(\mu-1)$-dimensional spaces $PG(\mu-1, p^3)$ and of at most one (projected) subgeometry $PG(3\mu-1, p)$. This then implies that $\delta \cap (F, w)$ is a disjoint union of $(\mu-2)$-dimensional subspaces $PG(\mu-2, p^3)$ and of at most one (projected) $PG(3\mu-4, p)$.

Lemma 5.4.2 *If $\Delta \cap (F, w)$ contains a $(\mu-2)$-dimensional space $PG(\mu-2, p^3)$ Π, then (F, w) contains a μ-dimensional space $PG(\mu, p^3)$.*

Proof. Consider all hyperplanes $H_i, i = 0, \ldots, p^3$, through Δ. Using the induction hypothesis, they either contain a $PG(\mu-1, p^3)$ of $(F, w) \cap H_i$ or a (projected) $PG(3\mu-1, p) \equiv \Omega$ of $(F, w) \cap H_i$ through Π.

Now assume this latter possibility occurs. Consider a line in Π, this has to be the projection of a subgeometry $PG(l, p)$ in Ω, but then this line contains multiple points in $\Delta \cap (F, w)$; a contradiction.

Hence, Π lies in $p^3 + 1$ subspaces $PG(\mu-1, p^3) \equiv \Pi_i$ of (F, w) in the respective hyperplanes H_i, $i = 0, \ldots, p^3$, through Δ. Every plane $\langle L_1, L_2\rangle$ with $L_1 \subset \Pi_i$ and $L_j \subset \Pi_j$, $i \neq j$, is completely contained in (F, w). Namely, if this plane is not contained in (F, w), then it intersects (F, w) in an $\{m_1(p^3+1) + m_0, m_1; 2, p^3\}$-minihyper, with $m_1 + m_0 \leq 2p^2 - 4p$ (Theorem 2.1.6). But, again using the fact that this plane already contains two lines of (F, w), the arguments of Lemma 5.3.1 imply that this is impossible. We conclude that (F, w) contains a subspace $PG(\mu, p^3)$. □

Removing the subspaces $PG(\mu, p^3)$ from (F, w) (see Theorem 4.3.1) by reducing for each such subspace $PG(\mu, p^3)$ the weight of its points by one shows that (F, w) is either a sum of subspaces $PG(\mu, p^3)$, or there remains a $\{(p^2+p+1)v_{\mu+1}, (p^2+p+1)v_\mu; N, p^3\}$-minihyper (F', w'), and the only case we still need to discuss is that $(F', w') \cap \Delta$ is a projected $PG(3\mu-4, p)$, and all hyperplanes through Δ intersect (F', w') in a (projected) $PG(3\mu-1, p)$. Since there are at most $p^2 + p$ multiple points, we can select two hyperplanes H_1, H_2 through Δ intersecting (F, w) only in points of weight one. Both intersections contain a (projected) $PG(3\mu-1, p)$, call them respectively $PG(3\mu-1, p)_1$ and $PG(3\mu-1, p)_2$. In $PG(3\mu-1, p)_i$, $i = 1, 2$, we do not have lines over $GF(p^3)$ since this would be a projection of at least a subgeometry $PG(t, p)$, $t \geq 3$, of size at least $p^3 + p^2 + p + 1$, and such a projection contains multiple points. So, a line T of H_2, intersecting $PG(3\mu-1, p)_2$ in at least two points, intersects $PG(3\mu-1, p)_2$ in $p+1$ or p^2+p+1 points forming respectively a $PG(1, p)$ or (p^2+p+1)-set.

Lemma 5.4.3 *The* $(3\mu+2)$*-dimensional (projected) subgeometry* $PG(3\mu+2,p) = \langle PG(3\mu-1,p)_1, PG(3\mu-1,p)_2\rangle$ *is completely contained in* (F,w).

Proof. Consider a $(p+1)$- or a (p^2+p+1)-secant T in $H_2 \setminus H_1$ containing a point of $\Delta \cap (F,w)$.

Consider a $PG(N-2,p^3) \equiv \Delta'$ in H_2 through T intersecting (F,w) in a $\{(p^2+p+1)v_{\mu-1},(p^2+p+1)v_{\mu-2}; N-2,p^3\}$-minihyper. Again the hyperplanes $H'_0,\ldots,H'_{p^3}$ through Δ' intersect (F,w) in a (projected) $PG(3\mu-1,p)$, call them respectively $PG(3\mu-1,p)'_i$, $i=0,\ldots,p^3$. Then $H_1 \cap PG(3\mu-1,p)'_i \equiv \mathcal{P}_i$ is at least a (projected) $PG(3\mu-4,p)$. Then $\langle \mathcal{P}_i, T\cap(F,w)\rangle$ is completely contained in $PG(3\mu-1,p)'_i$. Hence $\langle PG(3\mu-1,p)_1, T\cap(F,w)\rangle$ lies completely in (F,w). Letting T vary, we obtain that $\langle PG(3\mu-1,p)_1, PG(3\mu-1,p)_2\rangle = PG(3\mu+2,p) = \Pi_{3\mu+2}$ lies completely in (F,w).

We now show that (F,w) coincides with $\Pi_{3\mu+2}$. Every hyperplane through Δ intersects (F,w) in a $PG(3\mu-1,p)$ and intersects $\Pi_{3\mu+2}$ in at least a $PG(3\mu-1,p)$. These intersections must coincide, yielding that $\Pi_{3\mu+2}$ is equal to (F,w). □

This concludes the proof of Theorem 5.4.1.

5.5 The case where p is a square

5.5.1 $\{\delta(p^3+1),\delta;4,p^3\}$-minihypers

Let $q=p^3$. We will project on a hyperplane Π from a point r lying on at most $2p$ secants to (F,w). This projection (F',w') is a sum of lines, (projected) $PG(3,p^{3/2})$ and of at most one projected $PG(5,p)$ or a $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihyper $(\Omega\setminus N,w')$ (Theorem 4.4.3). As before, we can remove the lines of (F,w); now we will discuss the possible $PG(3,p^{3/2})$ in (F',w').

Lemma 5.5.1 *Every point* s *of* (F,w) *which is projected onto a point* s' *of weight one of* (F',w') *lying in a* $PG(3,\sqrt{q})$ *contained in* (F',w')*, lies in at least two Baer subplanes completely contained in* (F,w).

Proof. We proceed as in [33, Lemma 2.10]. □

Lemma 5.5.2 *Through any point* s *of* (F,w) *which is projected onto* s'*, a point of* (F',w') *of weight one lying in a* $PG(3,\sqrt{q})$ *contained in* (F',w')*, there is a* $PG(3,\sqrt{q})$ *completely consisting of points of* (F,w).

Proof. We proceed as in [33, Lemma 2.11]. □

As in the proof of [33, Theorem 2.1], if (F,w) contains a subgeometry $D \equiv PG(3,\sqrt{q})$, then reducing the weight of every point of D by one, gives a $\{(\delta-\sqrt{q}-1)(p^3+1),\delta-\sqrt{q}-1;4,p^3\}$-minihyper (F',w').

The previous lemma, together with the previous sections implies the following theorem. We however wish to remark that the description of the minihypers can be done in different ways.

In the statement of the theorem, also the possibility of projected subgeometries $PG(3, \sqrt{q})$ is included. In $PG(3,q)$, if one projects a subgeometry $PG(3, \sqrt{q}) \equiv D$ from a point $s \notin D$, then a cone with base a Baer subline $PG(1, \sqrt{q})$ is obtained. This cone is a $\{(\sqrt{q}+1)(q+1), \sqrt{q}+1; 3, q\}$-minihyper if the vertex is giving the weight $\sqrt{q}+1$ and all other points are given weight one.

This cone is also a sum of lines, so it is also possible to simply not state explicitly these projected Baer subgeometries $PG(3, p^{3/2})$, and simply consider these lines as lines of the sum of lines inside the minihyper.

We however have written them in the formulation of the theorem since also in the general case of Theorems 5.5.6 and 5.5.8 projected subgeometries $PG(2\mu+1, \sqrt{q})$ can occur, and these projections are not equal to sums of spaces $PG(\mu, p^3)$ when $\mu \geq 2$.

Theorem 5.5.3 *A $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper (F, w), $N \geq 4$, p square, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, with total excess $e \leq p^3 - 4p$, is a sum of either:*
(1) lines, (projected) $PG(3, p^{3/2})$ (where the projection is from a point), and at most one (projected) $PG(5, p)$,
(2) lines, (projected) $PG(3, p^{3/2})$, and a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.

Proof. The proof for $N = 4$ follows from the preceding lemmas and from the techniques of Section 5.2. To prove the result for $N > 4$, we use induction on N, with $N = 4$ as induction basis. □

5.5.2 $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihypers

Lemma 5.5.4 ([60]) *A Baer subline and a (p^2+p+1)-set in $PG(1, p^3)$ share at most $p + \sqrt{p} + 1$ points.*

Let Δ be a $PG(N-2, p^3)$ intersecting (F, w) in δ points of weight one. As in Lemma 5.3.1, we can assume that a point of $(F, w) \cap \Delta$ lies on at most one line contained in (F, w). Since the total excess e is at most $p^2 + p$, it is also possible to find two distinct hyperplanes H_1 and H_2 through Δ intersecting (F, w) in disjoint unions of $PG(3, \sqrt{q})$ and of at most one $\{(p^2+p+1)(p^3+1), p^2+p+1; N-1, p^3\}$-minihyper which is a (projected) $PG(5,p)_i$, $i = 1, 2$, containing no multiple points. This implies that these latter projected $PG(5,p)_i$ only have $(p+1)$- and (p^2+p+1)-secants.

This also implies that $\Delta \cap (F, w)$ is a disjoint union of Baer sublines $PG(1, \sqrt{q})$ and of at most one subplane $PG(2, p)$ or (p^2+p+1)-set.

There are less than $2p^2/(p^{3/2}+1) < 2\sqrt{p}$ Baer sublines in $\Delta \cap (F, w)$.

Consider a Baer subline in $\Delta \cap (F, w)$, and consider the $PG(3, \sqrt{q})$ of (F, w) in H_1 and H_2 through this Baer subline. Call them $PG(3, \sqrt{q})_1$ and $PG(3, \sqrt{q})_2$ respectively.

Lemma 5.5.5 *The Baer subspace* $PG(5, \sqrt{q}) = \langle PG(3, \sqrt{q})_1, PG(3, \sqrt{q})_2 \rangle$ *is completely contained in* (F, w).

Proof. Consider a Baer subline $\tilde{B}$ of $PG(3, \sqrt{q})_2$ containing exactly one point of $\Delta \cap (F, w)$. The line T containing $\tilde{B}$ satisfies $|T \cap (F, w)| \leq \delta$, and has either a Baer subline, a point, or nothing in common with a $PG(3, \sqrt{q})$ contained in $(F, w) \cap H_2$, and it has either a $PG(1, p)$, a (p^2+p+1)-set, one point or nothing in common with the possible projected $PG(5, p)$ in $(F, w) \cap H_2$. The Baer sublines on T which are the intersection of T with a subgeometry $PG(3, \sqrt{q})$ contained in $H_2 \cap (F, w)$ are the only Baer sublines contained in $(F, w) \cap T$.

We now show that the subgeometry $PG(4, \sqrt{q}) = \Pi_4 = \langle \tilde{B}, PG(3, \sqrt{q})_1 \rangle$ is completely contained in (F, w). Consider a $PG(N-3, p^3)$ in H_2 through T not containing any other points of (F, w), and consider a $PG(N-2, p^3) \equiv \Delta'$ through this $PG(N-3, p^3)$ only containing δ simple points of (F, w).

From the fact above stating that the only Baer sublines in $(F, w) \cap T$ arise from the intersections of T with the subgeometries $PG(3, \sqrt{q})$ contained in $(F, w) \cap H_2$, if some hyperplane H' through Δ' contains lines of (F, w) through a point of $\tilde{B}$, then every point of $\tilde{B}$ lies on a line of $(F, w) \cap H'$.

So, consider all hyperplanes through Δ'; at most one of them contains lines of (F, w) through the points of $\tilde{B}$.

So at least p^3 hyperplanes through Δ' intersect (F, w) in a sum of $PG(3, \sqrt{q})$ and of at most one (projected) $PG(5, p)$. The Baer subline $\tilde{B}$ always lies in such a $PG(3, \sqrt{q})$, and this latter $PG(3, \sqrt{q})$ intersects H_1 in either a subline $PG(1, \sqrt{q})$ or a subplane $PG(2, \sqrt{q})$. Call this intersection B'_1. Then $\langle \tilde{B}, B'_1 \rangle$ is a subgeometry over $GF(\sqrt{q})$ completely contained in $(F, w) \cap H'$.

We conclude that Π_4 is completely contained in (F, w), up to maybe one hyperplane section Π_3.

We now show that Π_4 is completely contained in (F, w).

Consider a point r of $\Pi_4 \setminus F$, then all Baer sublines of Π_4 not lying in Π_3 share $\sqrt{q}$ points with (F, w). Select such a Baer subline N of Π_4 through r intersecting (F, w) in $\sqrt{q}$ points of weight one and such that the line T' containing N only intersects (F, w) in points of weight one. Since T' is not contained in (F, w), it shares at most δ points with (F, w); see [33]. Similarly as for T, the only Baer sublines in $T' \cap (F, w)$ arise from the Baer subline intersections of T' with (F, w). It is impossible to partition the $\sqrt{q}$ points of N in (F, w) over these Baer sublines and at most one $PG(1, p)$ or $(p^2 + p + 1)$-set. This however implies that the $\sqrt{q}$ points of $(F, w) \cap \Pi_4$ on N are contained in a Baer subline contained in (F, w). Hence, $r \in F$. If Π_4 would be a projected subgeometry $PG(4, \sqrt{q})$, then suppose the vertex r of Π_4 does not belong to F. Then r lies on lines containing q points of F; so $r \in F$.

Letting vary $\tilde{B}$ over $PG(3, \sqrt{q})_2$ shows that the 5-dimensional Baer subgeometry $\langle PG(3, \sqrt{q})_1, PG(3, \sqrt{q})_2 \rangle$ is completely contained in (F, w). □

Theorem 5.5.6 *Let (F,w) be a $\{\delta(p^6+p^3+1), \delta(p^3+1); N, p^3\}$-minihyper, $\delta \leq 2p^2-4p$, $N \geq 5$, p square, $p = p_0^h$, $h \geq 2$ even, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.*

Then (F,w) is a sum of planes, (projected) $PG(5,\sqrt{q})$, and of at most one (projected) subgeometry $PG(8,p)$.

Proof. This follows from the preceding lemmas and the techniques of Section 5.3.

Remark 5.5.7 The Baer subgeometry can be at most projected from a point, since otherwise the total excess of the points would be too large.

5.5.3 $\{\delta v_{\mu+1}, \delta v_\mu; N, q\}$-minihypers

We now prove by induction on μ the following characterization result.

Theorem 5.5.8 *Let (F,w) be a $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 3$, $\delta \leq 2p^2-4p$, $N \geq 3$, $p = p_0^h$, $h \geq 2$ even, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.*

Then (F,w) is a sum of μ-dimensional spaces $PG(\mu,p^3)$, (projected) $PG(2\mu+1,\sqrt{q})$, and of at most one (projected) subgeometry $PG(3\mu+2,p)$.

Let Δ be a $PG(N-2,p^3)$ intersecting (F,w) in $\delta v_{\mu-1}$ points of weight one. Then $\Delta \cap (F,w)$ is a $\{\delta v_{\mu-1}, \delta v_{\mu-2}; N-2, p^3\}$-minihyper. Let $H_i, i = 0, \ldots, p^3$, be the hyperplanes through Δ. They intersect (F,w) in $\{\delta v_\mu, \delta v_{\mu-1}; N-1, p^3\}$-minihypers (F_i,w) (Theorem 2.1.6). By the induction hypothesis, these minihypers (F_i,w) are sums of $(\mu-1)$-dimensional spaces $PG(\mu-1,p^3)$, (projected) Baer subgeometries $PG(2\mu-1,\sqrt{q})$, and of at most one (projected) subgeometry $PG(3\mu-1,p)$. This then implies that $\delta \cap (F,w)$ is a disjoint union of $(\mu-2)$-dimensional subspaces $PG(\mu-2,p^3)$ and of at most one (projected) $PG(3\mu-4,p)$.

Lemma 5.5.9 *If $\Delta \cap (F,w)$ contains a $(\mu-2)$-dimensional space $PG(\mu-2,p^3)$ Π, then (F,w) contains a μ-dimensional space $PG(\mu,p^3)$.*

Proof. This proof is similar to that of Lemma 5.4.2. We only need to consider the possibility that Π lies in a (projected) subgeometry $PG(2\mu-1,\sqrt{q})$ contained in some hyperplane intersection $H_i \cap (F,w)$. If this occurs, then there are multiple points of this projected subgeometry in Π since $|PG(2\mu-3,\sqrt{q})| > |PG(\mu-2,q)|$. This contradicts the fact that $(F,w) \cap \Delta$ contains no multiple points. □

Removing the subspaces $PG(\mu,p^3)$ from (F,w) (Theorem 4.3.1) by reducing for every such $PG(\mu,p^3)$ the weight of their points by one, leaves us with a minihyper (F',w') which intersects Δ in a disjoint union of $PG(2\mu-3,\sqrt{q})$, and at most one $\{(p^2+p+1)v_{\mu-1}, (p^2+p+1)v_{\mu-2}; N-2, p^3\}$-minihyper Ω_5, where Ω_5 is a (projected) $PG(3\mu-4,p)$. Since there are at most p^2+p multiple points, we can select two hyperplanes H_1, H_2 through Δ intersecting (F',w') only in points of weight one. Both intersections contain (projected) $PG(2\mu-1,\sqrt{q})$ and at most one (projected) $PG(3\mu-1,p)$.

Suppose that $(F',w')\cap\Delta$ contains a subgeometry $PG(2\mu-3,\sqrt{q})$, then H_1 and H_2 share a subgeometry $PG(2\mu-1,\sqrt{q})_1$ and $PG(2\mu-1,\sqrt{q})_2$ with (F',w') passing through this latter $PG(2\mu-3,\sqrt{q})$.

Lemma 5.5.10 *The (projected) subgeometry* $PG(2\mu+1,\sqrt{q}) = \langle PG(2\mu-1,\sqrt{q})_1, PG(2\mu-1,\sqrt{q})_2\rangle$ *is completely contained in* (F',w').

Proof. Here the arguments of the proof of [33, Theorem 4.1] can be used. □

If $\Delta\cap(F,w)$ contains a (projected) subgeometry $PG(3\mu-4,p)$, proceeding as in Section 5.4, it is possible to consider two hyperplanes H_1 and H_2 through Δ intersecting (F',w') in (projected) $PG(3\mu-1,p)_1$ and $PG(3\mu-1,p)_2$, only having points of weight one. These two subgeometries over $GF(p)$ define a $(3\mu+2)$-dimensional subgeometry over $GF(p)$ completely contained in (F',w').

Lemma 5.5.11 *The* $(3\mu+2)$*-dimensional (projected) subgeometry* $PG(3\mu+2,p) = \langle PG(3\mu-1,p)_1, PG(3\mu-1,p)_2\rangle$ *is completely contained in* (F,w).

The preceding lemmas now finish the proof of Theorem 5.5.8. There is no problem with the weights of the points of (F,w) in this description of (F,w) as a sum of μ-dimensional spaces, (projected) Baer subgeometries $PG(2\mu+1,\sqrt{q})$, and of at most one (projected) subgeometry $PG(3\mu+2,p)$, since by induction, the weights are correct in the hyperplane intersections $(F,w)\cap H_i$, $i=0,\ldots,p^3$, of (F,w) with the hyperplanes through Δ.

Chapter 6

The largest caps in $AG(5,3)$

6.1 Known results on caps

A *k-cap* K in $PG(n,q)$, respectively $AG(n,q)$, is a set of k points, no three of which are collinear. A point r of $PG(n,q)$, respectively $AG(n,q)$, extends a k-cap K to a $(k+1)$-cap if and only if $K \cup \{r\}$ is a $(k+1)$-cap. A k-cap is called *complete* if and only if it is not contained in a $(k+1)$-cap.

The theory of caps is also interesting from a coding-theoretic point of view. A linear code C is a *cap-code* if and only if the columns of a generator matrix of C determine a set of points in the projective space that form a cap. A linear code C is a cap-code if and only if the minimum distance of its dual code $C^{\perp}$ is greater than or equal to four; cfr. Remark 1.1.10.

Let $m_2(n,q)$ denote the maximum value of k for which there exists a (complete) k-cap in $PG(n,q)$. In the binary case, this is a known value. In fact, choosing all nonzero n-tuples as columns we obtain a binary $n \times (2^n - 1)$-matrix. This matrix is the parity check matrix of a binary $[2^n - 1, 2^n - n - 1, 3; 2]$-code. Addition of a parity-check bit yields a $[2^n, 2^n - n - 1, 4; 2]$-code. We conclude $m_2(n,2) = 2^n$. Geometrically, a 2^n-cap in $PG(n,2)$ this corresponds to the complement of a hyperplane. Hence, we will restrict ourselves to the case $q > 2$.

The main problem in the theory of caps is to find the maximal size of a cap in $AG(n,q)$ or $PG(n,q)$.

Presently, only the following exact values are known.

1. In $AG(2,q)$ and $PG(2,q)$, q odd, there are at most $(q+1)$-caps [11].

2. In $AG(2,q)$ and $PG(2,q)$, q even, there are at most $(q+2)$-caps [11].

3. In $AG(3,q)$, $q > 2$, the maximal size of a cap is q^2.

4. In $PG(3,q)$, $q > 2$, the maximal size of a cap is $q^2 + 1$ [11, 72]. A (q^2+1)-cap in $PG(3,q)$ is also called an *ovoid*.

5. And in $AG(n,2)$ and in $PG(n,2)$, the maximal size of a cap is 2^n [11].

In some cases, a complete characterization is known.

1. In $AG(2,q)$ and in $PG(2,q)$, q odd, every $(q+1)$-cap is a conic [73, 74].
2. In $AG(2,q)$ and in $PG(2,q)$, q even, $q \geq 16$, distinct types of $(q+2)$-caps exist; see [52] for a list of the known infinite classes of $(q+2)$-caps.
3. In $PG(3,q)$, q odd, every (q^2+1)-cap is an elliptic quadric [5, 64].
4. In $AG(3,q)$, q odd, every q^2-cap is an elliptic quadric in $PG(3,q)$ minus one point p [5, 64]; where the plane at infinity corresponds to the tangent plane at p.
5. In $PG(3,q)$, $q=2^h$, h odd, $h \geq 3$, next to the elliptic quadric, at least one other type of ovoid exists, called the *Tits ovoid* [84]. The tangent lines to an ovoid $\mathcal{O}$ through a point $p \in \mathcal{O}$ lie in a plane, called the *tangent plane* at p. In $PG(3,16)$, every ovoid is an elliptic quadric [61, 62]. In $PG(3,8)$ and $PG(3,32)$, every ovoid is an elliptic quadric or a Tits ovoid; see [24, 67, 63].
6. In $AG(3,q)$, q even, $q > 2$, every q^2-cap is obtained by deleting one point p from a (q^2+1)-cap in $PG(3,q)$; and the plane at infinity corresponds to the tangent plane at p [75].
7. In $PG(n,2)$, every 2^n-cap is the complement of a hyperplane [75].

Apart from these results which are either valid for arbitrary q or for arbitrary dimension n, only some other sporadic results are known.

1. The maximal size of a cap in $AG(4,3)$ and in $PG(4,3)$ is 20 [66].
2. The maximal size of a cap in $PG(5,3)$ is 56 [46].
3. The maximal size of a cap in $PG(4,4)$ is 41 [20].
4. And the maximal size of a cap in $AG(4,4)$ is 40 [19].

Regarding the characterizations, the following results are known.

1. Exactly one type of 20-cap exists in $AG(4,3)$ and exactly 9 types of 20-caps exist in $PG(4,3)$ [48].
2. The 56-cap in $PG(5,3)$ is projectively unique [46].
3. There are exactly 2 distinct types of 41-caps in $PG(4,4)$ [20, 18].
4. There is a unique type of 40-cap in $AG(4,4)$ [19].

In the other cases, only upper bounds on the sizes of caps in $AG(n,q)$ and $PG(n,q)$ are known. We refer to [52] for a list of the known results.

We wish to explicitly mention the following upper bound on the size of caps in $AG(n,q)$ of Bierbrauer and Edel.

Theorem 6.1.1 (Bierbrauer and Edel [7]) *Let C_n be the largest size of a cap in $AG(n,q)$, q even, $q > 2$, $n \geq 4$, then*

$$C_n \leq \frac{q^{n-1} + q^n C_{n-1}}{q^{n-1} + C_{n-1}}.$$

The second largest value of k for which there exists a complete k-cap in PG(n,q) is denoted by $m_2^{'}(n,q)$. This value $m_2^{'}(n,q)$ is only known in some small projective planes [52, Table 2.4], for $n = 2, q = 2^{2h}$ $(h > 1)$ and for the following cases. Namely

1. $m_2^{'}(2, 2^{2h}) = 2^{2h} - 2^h + 1$ $(h > 1)$ [10, 32, 56],
2. $m_2^{'}(3,3) = 8$ [23],
3. $m_2^{'}(3,4) = 14$ [53, 54],
4. $m_2^{'}(3,5) = 20$ [1],
5. $m_2^{'}(3,7) = 32$ [22],
6. $m_2^{'}(4,3) = 19$ [81],
7. $m_2^{'}(4,4) = 40$ [19],
8. $m_2^{'}(5,3) = 48$ [3] and
9. $m_2^{'}(n,2) = 2^{n-1} + 2^{n-3}$, $n \geq 3$ [17].

The importance of these values is that they can be used to obtain results on the size of the largest cap in higher dimensions.

With respect to the other values of $m_2^{'}(n,q)$, only upper bounds are known. A complete list of these upper bounds for $m_2^{'}(n,q)$ can be found in [52, p. 286].

We focus in this chapter on the maximal size of a cap in $AG(5,3)$ and its relation to the 56-cap in $PG(5,3)$. This latter 56-cap in $PG(5,3)$, called the *Hill cap*, intersects a hyperplane of $PG(5,3)$ in either 20 or 11 points.

Hence, defining $AG(5,3)$ to be $PG(5,3)$ minus an 11-hyperplane of this 56-cap, we obtain that there exists a 45-cap in $AG(5,3)$.

No larger caps are known in $AG(5,3)$.

Presently, the best upper bound on the size of a cap in $AG(5,3)$ is of Bruen, Haddad and Wehlau [14] who proved that the size of a cap in $AG(5,3)$ is at most 48.

We will prove in this chapter that the maximal size of a cap in $AG(5,3)$ is equal to 45, and every 45-cap in $AG(5,3)$ is obtained by deleting an 11-hyperplane from a 56-cap in $PG(5,3)$.

Moreover, this will imply that there is a unique type of 45-caps in $AG(5,3)$. These results appeared in Y. Edel, S. Ferret, I. Landjev and L. Storme, The classification of maximal caps in $AG(5,3)$, [21].

6.2 Preliminary results

The following result was already mentioned in Section 6.1, but we repeat it since it is frequently used.

Lemma 6.2.1 *The largest cap in $AG(3,3)$ is a 9-cap obtained by deleting a 1-hyperplane from an elliptic quadric in $PG(3,3)$.*

Proof. See for instance [50, p. 104]. □

A set K of n points of $PG(k-1,q)$ is called an *$(n,m;k-1,q)$-set*, or *(n,m)-set* for short, if K meets every hyperplane in at most m points.

Given an $(n,n-d)$-set K in $PG(k-1,q)$, we again denote by n_i the number of hyperplanes H in $PG(k-1,q)$ with $|K \cap H| = i$. We call the sequence of integers $(n_i)_{i \geq 0}$ the *spectrum* of K. Simple counting arguments yield the following identities for n-caps in $PG(k-1,q)$:

$$\begin{array}{lcl} \sum_{i \geq 0} n_i & = & \frac{q^k-1}{q-1} \\ \sum_{i \geq 0} i n_i & = & n\frac{q^{k-1}-1}{q-1} \\ \sum_{i \geq 0} i(i-1) n_i & = & n(n-1)\frac{q^{k-2}-1}{q-1} \\ \sum_{i \geq 0} i(i-1)(i-2) n_i & = & n(n-1)(n-2)\frac{q^{k-3}-1}{q-1}. \end{array} \tag{6.1}$$

Let $\mathcal{P}$ be the set of points of $PG(k-1,q)$ and let π and σ be disjoint subspaces of dimensions i and j, respectively, with $i+j = k-2$. Let $\varphi_{\pi,\sigma}$ be the projection from π onto σ; $\varphi_{\pi,\sigma} \colon \mathcal{P} \setminus \pi \to \sigma : Q \mapsto \sigma \cap \langle \pi, Q \rangle$.

Given an (n,m)-set K and a set of points $\mathcal{F} \subset \sigma$, we define $\mu(\mathcal{F}) = |\{P \in K \mid \varphi(P) \in \mathcal{F}\}|$. Let l be a line in σ incident with the points $P_0, P_1, \ldots, P_q$. We call the $(q+1)$-tuple $\big(\mu(P_0), \mu(P_1), \ldots, \mu(P_q)\big)$ the *type* of l, and we call $\mu(P_i)$ the *weight* of the point P_i.

If K is an (n,m)-set, then an *i-line* (with respect to K) is a line l with $|K \cap l| = i$; *i-planes, i-solids*, and *i-hyperplanes* are defined in a similar way.

By [85], there are exactly 7 different $(18,8)$-sets in $PG(4,3)$. Each $(18,8)$-set is uniquely extendable to a $(20,8)$-set. There are exactly 2 types of $(18,8)$-sets, which are also affine (this corresponds to the fact that two of the seven $[18,5,10;3]$-codes have maximum weight 18). Also a $(9,5)$-set in $AG(4,3)$ is uniquely extendable to an $(11,5)$-set in $PG(4,3)$; this corresponds to the dual *Golay code*. We also remark that a solid in $PG(4,3)$ intersects an $(11,5)$-set in 5 or 2 points.

Lemma 6.2.2 *The following sets are caps.*
(1) An $(11,5)$-set in $PG(4,3)$.
(2) A $(20,8)$-set in $PG(4,3)$.
(3) A $(56,20)$-set in $PG(5,3)$.

Proof. The three proofs are similar. Lets for instance prove that an $(18,8)$-set S in $PG(4,3)$ is a cap.

By definition, there are at most 8 points of S in a solid. Hence, considering all solids through a x-plane, we obtain $4(8-x)+x \geq 20$, and hence there are at most 4 points of S contained in a plane. Now, considering all planes through a y-line, we get $13(4-y)+y \geq 20$. Hence we have at most 2 points of S on a line, and S is a cap. □

In $PG(4,3)$, we will project an affine (18,8)-set, respectively an affine (9,5)-set from an empty plane π onto some line l disjoint from π. The next table lists the possible types of the lines which are images of such sets under this projection. It is assumed that π is contained in a 0-solid δ. The column "# of π's" gives the number of choices for the empty plane π in δ, for which we get the particular type for the line l.
Table 1. The types of the images of (18,8)- and (9,5)-sets in $PG(4,3)$ under a projection from an empty plane contained in a 0-solid.

		Type	# of π's
$(18,8)$	(A)	(8,8,2,0)	9
	(B)	(8,5,5,0)	9
	(C)	(7,7,4,0)	18
	(D)	(6,6,6,0)	4
$(9,5)$	(E)	(5,2,2,0)	18
	(F)	(4,4,1,0)	18
	(G)	(3,3,3,0)	4

Remark 6.2.3 Let π be the plane at infinity from which we project. Types (A), (B) (respectively (E)) correspond to the cases that π contains none of the two points which extend the $(18,8)$-set (respectively the $(9,5)$-set). Type (C) (respectively (F)) corresponds to the case that π contains one of the two points which extend the $(18,8)$-set (respectively the $(9,5)$-set). Type (D) (respectively (G)) corresponds to the case that π contains the two points which extend the $(18,8)$-set (respectively the $(9,5)$-set).

6.3 Properties of the Hill-cap

A 4-cap in $PG(2,3)$, a 10-cap in $PG(3,3)$ and a 56-cap in $PG(5,3)$ are known to be unique; the first two being equivalent to respectively conics and elliptic quadrics, while the third, the Hill-cap, is contained in an elliptic quadric $Q^-(5,3)$.

Definition 6.3.1 A *hemisystem* of a geometry is a subset of the point set of the geometry such that a line of the geometry has half its points inside the hemisystem and half its points outside the hemisystem.

The Hill-cap is a hemisystem of $Q^-(5,3)$.

In $PG(4,3)$, there are nine inequivalent types of 20-caps. However, if a 20-cap occurs as an intersection of a 56-cap in $PG(5,3)$ with a hyperplane, only two types are possible for the 20-cap [48].

Let the Hill-cap K be contained in the elliptic quadric $Q^-(5,3)$ in $PG(5,3)$ and let π be a hyperplane of $PG(5,3)$.
(1) If π is the tangent hyperplane of $Q^-(5,3)$ at a point p of the Hill-cap, the intersection of π with K will contain 11 points of K: the vertex p of the cone $\pi \cap Q^-(5,3)$ and one extra point on each line of the cone.
(2) If π is the tangent hyperplane of $Q^-(5,3)$ at a point p not on the Hill-cap, π will contain 20 points of K: two on each line of the cone $\pi \cap Q^-(5,3)$, different from the vertex.
(3) If π is not a tangent hyperplane of $Q^-(5,3)$, the intersection $\pi \cap Q^-(5,3)$ is a quadric containing 20 points of K, forming a hemisystem of $\pi \cap Q^-(5,3)$.

6.4 The size of a largest cap in $AG(5,3)$

Lemma 6.4.1 *Let K be a 45-cap in $AG(5,3)$. Let $P(i) = (i-r_1)(i-r_2)(i-r_3)$, for some constants r_1, r_2, r_3. Then we have the following equality:*

$$\begin{aligned} \sum_i P(i)n_i &= 1106820 + (3 - r_1 - r_2 - r_3)79200 \\ &\quad +(r_1r_2 + r_2r_3 + r_1r_3 + 1 - r_1 - r_2 - r_3)5445 \\ &\quad -363r_1r_2r_3. \end{aligned} \tag{6.2}$$

Proof. We have the following equalities:

$$\begin{aligned} \sum_i n_i &= 363, \\ \sum_i in_i &= 45 \times 121, \\ \sum_i \binom{i}{2} n_i &= \binom{45}{2} 40, \\ \sum_i \binom{i}{3} n_i &= \binom{45}{3} 13. \end{aligned}$$

Equation (6.2) follows from

$$P(i) = 6\binom{i}{3} + (6 - 2r_1 - 2r_2 - 2r_3)\binom{i}{2} + (r_1r_2 + r_2r_3 + r_1r_3 + 1 - r_1 - r_2 - r_3)i - r_1r_2r_3.$$

□

Theorem 6.4.2 *The largest size of a cap in $AG(5,3)$, with at most 18 points in every hyperplane, is 45.*

Moreover, every 45*-cap in* $AG(5,3)$ *contains at least one* 18*-,* 19*-, or* 20*-hyperplane.*

Proof. The first part follows from Theorem 6.1.1. More precisely, the size of a cap in $AG(k,q)$, having at most a c-hyperplane, is at most $q^k(1+cq)/(q^k+cq)$.

We now show that we have at least an 18-hyperplane. If we would have at most 17-hyperplanes, then the size of the cap is at most $3^5(1+17\times 3)/(3^5+17\times 3)<43$. □

Lemma 6.4.3 *Assume there exists a* 45*-cap* K *in* $AG(5,3)$*, for which there exists a hyperplane which intersects in more than* 18 *points. Then we can always find either a* 5*-,* 6*-, or* 7*-hyperplane parallel to a* 20*-hyperplane, or a* 7*- or* 8*-hyperplane parallel to a* 19*-hyperplane.*

Proof. Let $P(i)=(i-11)(i-15)(i-16)$, then by Lemma 6.4.1, $\sum_i P(i)n_i = 0$. Assume that there are no 20-hyperplanes, but there is a 19-hyperplane. Suppose there are no 7-hyperplanes. An 8-hyperplane and its parallel 18- and 19-hyperplane contribute -30 to (6.2) (using $(r_1,r_2,r_3)=(11,15,16)$), while a 9-hyperplane and two parallel 18-hyperplanes, and three parallel 15-hyperplanes contribute zero to (6.2). All other triples of parallel hyperplanes contribute a positive number to (6.2). Hence, if there is no 8-hyperplane, there are only 9-, 15- or 18-hyperplanes; but this contradicts the assumption that there is a 19-hyperplane. So, parallel to some 19-hyperplane, there is a 7- or a 8-hyperplane.

Suppose there is a 20-hyperplane. A 5-hyperplane or a 6-hyperplane is always parallel to a 20-hyperplane. A 7-hyperplane is parallel to a 19- or a 20-hyperplane. So assume $n_5=n_6=n_7=0$. As a 20-hyperplane and its two parallel hyperplanes always induce a positive contribution to (6.2) for $(r_1,r_2,r_3)=(11,15,16)$, there must be a negative contribution. As above, this is only possible for a parallel 8-, 18-, 19-hyperplane triple. □

Lemma 6.4.4 *There is no* 45*-cap in* $AG(5,3)$ *for which there exists a hyperplane intersecting in more than* 18 *points.*

Proof. From [48], we know that there is a unique 20-cap in $AG(4,3)$ and a computer search for all 19-caps in $AG(4,3)$ showed that there is a unique 19-cap.

Using a similar computer search as in [20], we eliminated all cases occurring in Lemma 6.4.3. □

6.5 The classification of the 45-caps in $AG(5,3)$

Remark 6.5.1 There exist 45-caps in $AG(5,3)$, since the Hill-cap is a 56-cap in $PG(5,3)$ which contains an 11-hyperplane [46]. Deleting such an 11-hyperplane yields a 45-cap in $AG(5,3)$. We are going to prove that every 45-cap in $AG(5,3)$ is obtained in that way.

From the preceding lemma, we know that there are at most 18-hyperplanes.

Lemma 6.5.2 *Let K be a 45-cap in $AG(5,3)$. Then every hyperplane intersects K in either 9, 15 or in 18 points, and the spectrum of K is $(n_9, n_{15}, n_{18}) = (55, 198, 110)$.*

Proof. Let $P(i) = (i-11)(i-15)(i-16)$, then Equation (6.2) gives $\sum_i P(i)n_i = 0$. We count the contribution of parallel hyperplane triples to this sum. Only a 9-hyperplane parallel to two 18-hyperplanes, and three parallel 15-hyperplanes give a zero contribution. All other contributions are strictly positive. Hence we only have 9-, 15- and 18-hyperplanes, and $n_{18} = 2n_9$.

Take $P(i) = (i-11)(i-16)(i-16)$, then Equation (6.2) gives $-98n_9 + 4n_{15} + 28n_{18} = -1518$. Using $n_9 + n_{15} + n_{18} = 363$, we get the spectrum of K. □

Definition 6.5.3 ([14]) We define for a k-cap K in $AG(5,3)$, an *intersection square* in the following way. Take a hyperplane K_1 and its parallel hyperplanes K_2 and K_3. Take another hyperplane H_1 together with its parallel hyperplanes H_2 and H_3. An intersection square determined by H_1 and K_1 is the 3×3 matrix $[l_{ij}]$, where $l_{ij} = |L_{ij} \cap K|$, with $L_{ij} = H_i \cap K_j$.

$$\begin{matrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{21} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{matrix} \tag{6.3}$$

Remark 6.5.4 We remark that a cap has in general several intersection squares. The hyperplanes $L_{12} \cup L_{21} \cup L_{33}$, $L_{13} \cup L_{22} \cup L_{31}$ and $L_{23} \cup L_{32} \cup L_{11}$ form a parallel hyperplane triple, and also $L_{11} \cup L_{22} \cup L_{33}$, $L_{21} \cup L_{32} \cup L_{13}$ and $L_{31} \cup L_{12} \cup L_{23}$ form a parallel hyperplane triple. Actually, these four parallel hyperplane triples correspond to the parallel hyperplane triples going through the four solids containing the plane at infinity, contained in $H_1 \cap K_1$.
Hence an intersection square is determined by a solid in $AG(5,3)$, since this solid determines the plane at infinity.

Lemma 6.5.5 *If K is a 45-cap in $AG(5,3)$ containing a 9-solid, then K has an intersection square of the form*

$$\begin{matrix} 9 & 0 & 9 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{matrix} \tag{6.4}$$

Proof. Put $l_{11} = 9$. By Lemma 6.5.2, a 9-solid is contained in four 18-hyperplanes. Hence, $l_{11}+l_{12}+l_{13}= l_{11}+l_{22}+l_{33}= l_{11}+l_{21}+l_{31}= l_{11}+l_{23}+l_{32} = 18$. Lemma 6.5.2 implies that an 18-hyperplane is parallel to a 9-hyperplane and an 18-hyperplane. Using a straightforward computer program, we looked for all possibilities to complete our intersection square. Up to equivalence, the only possibility is the intersection square (6.4). □

Lemma 6.5.6 *If K is a 45-cap in $AG(5,3)$, then, up to equivalence, the possible intersection squares are*

$$\begin{array}{ccc} 9 & 0 & 9 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{array}, \quad \begin{array}{ccc} 2 & 8 & 8 \\ 8 & 5 & 5 \\ 5 & 2 & 2 \end{array}, \quad \begin{array}{ccc} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{array}, \quad \begin{array}{ccc} 1 & 7 & 7 \\ 7 & 4 & 4 \\ 7 & 4 & 4 \end{array}, \quad \begin{array}{ccc} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{array}.$$

Proof. In this argument, we heavily rely on Lemma 6.5.2, stating that there are only 9-, 15-, and 18-hyperplanes. Let S be an a-solid and consider the intersection square determined by S; see Remark 6.5.4. Let n'_i denote the number of i-hyperplanes in the intersection square which contain S. Then clearly $n'_9 + n'_{15} + n'_{18} = 4$. Summing the other eight entries in the intersection square corresponds to counting the number of points of $K \setminus S$ in all hyperplanes through S. Hence we find $(9-a)n'_9 + (15-a)n'_{15} + (18-a)n'_{18} = 45 - a$. Eliminating n'_9 from these two equations, we find that $2n'_{15} + 3n'_{18} = 3 + a$.

A 0-solid has to be contained in an 18-hyperplane and for the remainder in three 9-hyperplanes. The solids in the 18-hyperplane, parallel to the 0-solid, have to be 9-solids. Hence we are in the case of Lemma 6.5.5 and, up to equivalence, the only possible intersection square containing a 0-solid is (6.4).

Assume $l_{11} = 1$, we try to complete this to a valid intersection square. By the reasoning above, we can assume that there are no 0-solids in the intersection square. Also, we may assume that we have no 9-solids in the intersection square (Lemma 6.5.5). A 1-solid has to be contained in two 15-hyperplanes and in two 9-hyperplanes. Hence, we may assume that $l_{11} + l_{12} + l_{13} = l_{11} + l_{21} + l_{31} = 15$. If we put $(l_{12}, l_{13}) = (6, 8)$, then we cannot complete this to a valid intersection square, taking into consideration Lemma 6.5.2. So assume $l_{12} = l_{13} = l_{21} = l_{31} = 7$. A 7-solid has to be contained in two 15-hyperplanes and in two 18-hyperplanes. Using $l_{12}+l_{21}+l_{33} \in \{15, 18\}$ and $l_{11}+l_{22}+l_{33} = 9$, we are reduced to two possibilities, namely $\begin{array}{ccc} 1 & 7 & 7 \\ 7 & 7 & \\ 7 & & 1 \end{array}$ or $\begin{array}{ccc} 1 & 7 & 7 \\ 7 & 4 & \\ 7 & & 4 \end{array}$. In the former case, the 7-solid corresponding to L_{12} lies already in two 15-hyperplanes $L_{12} \cup L_{21} \cup L_{33}$ and $L_{11} \cup L_{12} \cup L_{13}$; hence the other hyperplanes containing this solid have to be 18-hyperplanes. So $l_{32} = l_{23} = 4$. But then the hyperplane $L_{31} \cup L_{32} \cup L_{33}$ is a 12-hyperplane; contradicting Lemma 6.5.2. In the latter case, $l_{12} + l_{22} + l_{32}$ has to be 15 or 18. So, l_{32} is 4 or 7. If $l_{32} = 7$, then $l_{23} = 1$ and $l_{13} + l_{23} + l_{33} = 12$. This contradicts Lemma 6.5.2. Hence $l_{32} = 4$ and $l_{23} = 4$.

By a similar reasoning, we determined, up to equivalence, the possible intersection squares containing a 2-solid or a 3-solid: $\begin{array}{ccc} 2 & 8 & 8 \\ 8 & 5 & 5 \\ 5 & 2 & 2 \end{array}$, $\begin{array}{ccc} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{array}$, under the assumption that there are no 0- or 1-solids, and, in the latter case, 2-solids.

Assume $l_{11} = 4$. Assume that we have no solids intersecting in less than 4 points. A 4-solid is contained in a 9-hyperplane, two 15-hyperplanes and an 18-hyperplane. But, since every entry in our intersection square is at least 4, we cannot obtain a 9-hyperplane.

If we assume that there are no solids sharing less than 5 points with the 45-cap, the only possible intersection square containing a 5-solid is $\begin{matrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{matrix}$.

A solid which intersects the cap in more than 5 points, has to be parallel with a solid intersecting in at most 5 points. □

6.5.1 Suppose there is no solid intersecting in 9 points

If there are no 9-solids, Lemma 6.5.6 yields that the 9 points of K lying in a 9-hyperplane form a $(9,5)$-set. Clearly, a solid in H_0, the empty hyperplane at infinity, is contained in either one 9- and two 18-hyperplanes or in three 15-hyperplanes. Since $(n_9, n_{15}, n_{18}) = (55, 198, 110)$ (Lemma 6.5.2), the first possibility occurs for 55 solids; the second possibility occurs for the remaining 66 solids in H_0.

Let H_1 be a 9-hyperplane and let $\delta = H_0 \cap H_1$. Denote by P_δ and Q_δ the two points in δ which extend $K \cap H_1$ to an (11,5)-set in H_1 (Section 6.2). Now define L to be the union of all $\{P_\delta, Q_\delta\}$, where δ runs over all solids in H_0 contained in 9-hyperplanes. We are going to prove that $K \cup L$ is a (56,20)-set, and by Lemma 6.2.2, such a set is always a cap. Let H_2 and H_3 be the other hyperplanes through $\delta = H_0 \cap H_1$. Then $K \cap H_2$ and $K \cap H_3$ are 18-caps; since we excluded 9-solids, these are also $(18,8)$-sets and hence they are uniquely extendable to $(20,8)$-sets (Section 6.2).

We will consider the hyperplanes in $PG(5,3)$, hence in the type of a hyperplane, we will have a fourth entry corresponding to H_0.

Lemma 6.5.7 *The sets* $(K \cap H_2) \cup \{P_\delta, Q_\delta\}$ *and* $(K \cap H_3) \cup \{P_\delta, Q_\delta\}$ *are (equivalent to)* $(20,8)$*-sets in* $PG(4,3)$.

Proof. Let P_1 and Q_1 be the points which extend $K \cap H_2$ to a (20,8)-set. Consider a plane π in δ which contains P_δ and Q_δ. From Remark 6.2.3 and Table 1 (G), we know that $\varphi(H_1)$ is of type $(3,3,3,0)$. From the third intersection square of Lemma 6.5.6, we obtain that for the 18-hyperplane H_2 parallel to H_1, we have that $\varphi(H_2)$ is of type $(6,6,6,0)$. Now it follows from Remark 6.2.3 and Table 1 (D) that π also contains the points P_1 and Q_1. Letting π vary in δ, we have that P_δ, Q_δ, P_1 and Q_1 are collinear.

Assume $\{P_\delta, Q_\delta\} \neq \{P_1, Q_1\}$. Let further $P_\delta \notin \{P_1, Q_1\}$ and consider a plane π in H_0 containing P_δ and none of the remaining three points. A similar reasoning as above shows that, if φ is the projection from π, the line $\varphi(H_1)$ is of type (4,4,1,0) while $\varphi(H_2)$ is of type (8,8,2,0) or (8,5,5,0), by Table 1 and Remark 6.2.3. This contradicts Lemma 6.5.6 since $(4,4,1)$ appears in the fourth intersection square while $(8,8,2)$ and $(8,5,5)$ appear in the second intersection square. □

Now let H_1' and H_1'' be 9-hyperplanes. Let $K \cap H_1'$ and $K \cap H_1''$ be extended to (11,5)-sets by the points P', Q' and P'', Q'', respectively. Set $\pi = H_0 \cap H_1' \cap H_1''$. Consider a projection φ from the plane π. Assume $|\pi \cap \{P', Q'\}| = 2$, then

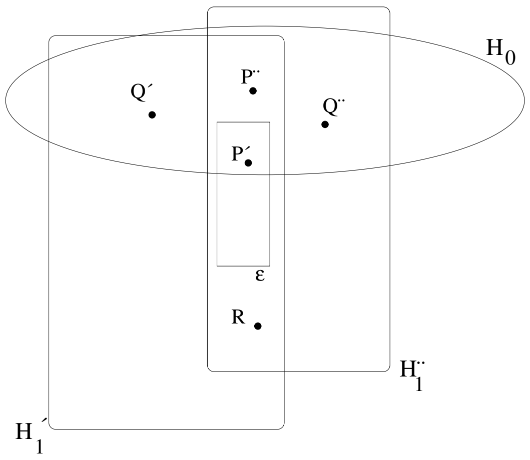

Figure 6.1: $P' \neq P''$

Table 1 and Remark 6.2.3 give that the type of $\varphi(H_1')$ is $(3,3,3,0)$. Hence π determines the intersection square $\begin{matrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{matrix}$, and the only possibility for H_1'' is a 15- or 18-hyperplane; a contradiction.

Hence $|\pi \cap \{P', Q'\}| = |\pi \cap \{P'', Q''\}| = 0$ or 1. For, if $|\pi \cap \{P', Q'\}| = 1$, then there is a $1-4-4$-parallel solid triple in H_1' (Table 1 (F) and Remark 6.2.3). Then the fourth intersection square of Lemma 6.5.6 shows that also in H_1'', there must be a $1-4-4$-parallel solid triple. So also here, using Table 1 (F) and Remark 6.2.3, $|\pi \cap \{P'', Q''\}| = 1$. Let us assume that π contains the points P' and P'' and does not contain the points Q' and Q''. Our next goal is to prove that $P' = P''$.

The types of $\varphi(H_1')$ and $\varphi(H_1'')$ are (4,4,1,0) and $H_1' \cap H_1''$ correspond to the 1-entry or the 4-entry. Since a 4-solid does not lie in two 9-hyperplanes (Lemma 6.5.6), $|K \cap H_1' \cap H_1''| = 1$. Set $K \cap H_1' \cap H_1'' = \{R\}$. Moreover the other two hyperplanes through $H_1' \cap H_1''$ are 15-hyperplanes (Lemma 6.5.6).

Assume that $P' \neq P''$, see Figure 6.1, and consider another projection φ_ε from a plane ε in $H_1' \cap H_1''$ which contains P' and does not contain P'' nor R.

We show that the type of $L_1 = \varphi_\varepsilon(H_1')$ is (4,3,1,1). Consider the $(11,5)$-set $(K \cap H_1') \cup \{P', Q'\}$ in H_1' which is the extension of the $(9,5)$-set $K \cap H_1'$. Every solid in H_1' through ϵ intersects this $(11,5)$-set in 5 or 2 points (Section 6.2). Since ϵ contains one point of the $(11,5)$-set, the solids through ϵ intersect in $5,5,2,2$ points respectively. Going from the $(11,5)$-set in H_1' to the $(9,5)$-set $K \cap H_1'$, we cancel the point P' which lies in ϵ. Hence, we obtain $4,4,1,1$ points in the solids through ϵ. Now, we still need to remove one point P'' in one of the solids. It is impossible that we have a 0-entry in the type of L_1, since a (9,5)-set

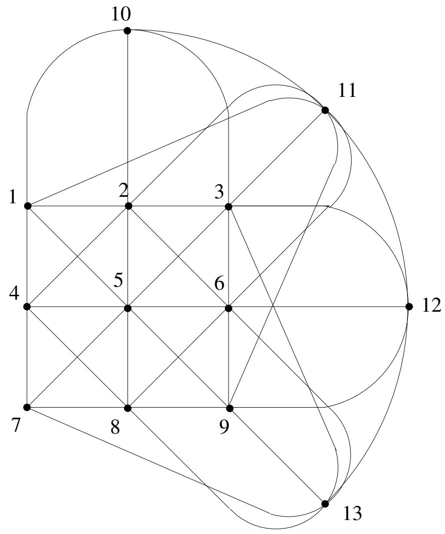

Figure 6.2: The picture of $PG(2,3)$

in $PG(4,3)$ has exactly one 0-solid $H_0 \cap H_1'$ and $\epsilon \not\subseteq H_0 \cap H_1'$. Hence, the only possibility for the type of L_1 is $(4,3,1,1)$.

We now show that the type of $L_2 = \varphi_\varepsilon(H_1'')$ is (5,2,1,1) or (4,2,2,1). Consider the $(11,5)$-set $(K \cap H_1'') \cup \{P'', Q''\}$ in H_1'' which is the extension of the $(9,5)$-set $K \cap H_1''$. Now ϵ does not contain a point of the $(11,5)$-set in H_1''. Note that $\langle \epsilon, P'' \rangle$, which is the solid $H_1' \cap H_1''$, does not contain Q'', since $Q'' \notin \pi$. Hence the two solids $\langle \epsilon, P'' \rangle$ and $\langle \epsilon, Q'' \rangle$ are different, and when we project $H_1'' \cap K$ from ϵ onto L_2, two entries of the type of L_2 differ a unit from the number of points of the (11,5)-set in H_1'' in the corresponding solids through ϵ in H_1''. As in the preceding paragraph, there is no 0-solid through ϵ in H_1'', so we need to decrease two different entries of the $(5,2,2,2)$-type corresponding to the $(11,5)$-set by one, giving $(5,2,1,1)$ or $(4,2,2,1)$.

Case 1. Construct $PG(2,3)$ which represents the quotient geometry of ϵ. First suppose we have a (4,3,1,1)- and a (4,2,2,1)-line L_1 and L_2, respectively. We can fix the entries of the type of L_1 and L_2 without losing generality.

Namely, for the points on L_1, this is certainly true. Then we can use an elation with center $L_1 \cap L_2$ and axis L_1 to choose the weight of a point y on $L_2 \setminus L_1$. Next use the involutory perspectivity with axis L_1 and center y to choose the weights of the other points on L_2.

Since we projected from a plane ϵ which is skew to the 45-cap, all lines must sum to 0 (mod 3), because hyperplanes intersect K in 9, 15 or 18 points.

Consider the picture of $PG(2,3)$ where we number the points from 1 to 13; see Figure 6.2. Here $PG(2,3)$ is considered to be the union of the affine plane of the points represented by the 3×3-grid of points $1, \ldots, 9$ and of the

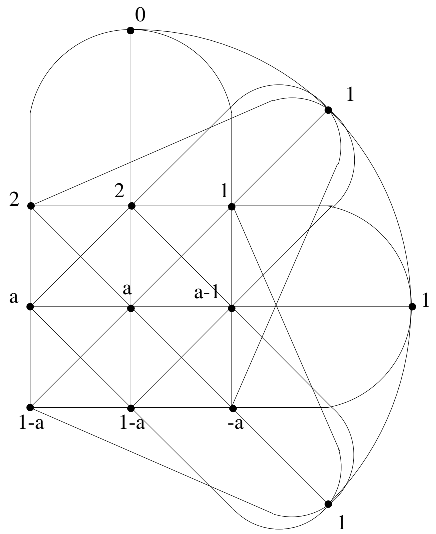

Figure 6.3: The types of the lines

line at infinity $10,\ldots,13$.

Let the $(4,3,1,1)$-line L_1 be the line at infinity and where the points $12,1,2,3$ of the description above form the line L_2. We now fill in the explicit possibilities for the weights of the points of $PG(2,3)$ modulo 3, we obtain Figure 6.3 for the types of the lines.

If we now fill in the explicit possibilities for the weights of the points of $PG(2,3)$, taking into account that every line must have a total weight of 9, 15 or 18; only a limited number of possibilities occur. If one considers such a possibility, one finds that there is a (3,2,3,1)-line L_3.

This line defines a 9-hyperplane intersecting the 45-cap in a $(9,5)$-set. This is always uniquely extendable to an (11,5)-set intersecting every solid in 2 or 5 points. Since the line is a (3,3,2,1)-line, necessarily, the plane ϵ must contain the two points which extend the (9,5)-set to the (11,5)-set; but then the projection from ϵ would imply that the line L_3 is a (3,3,3,0)-line since we lose two points in a 5-solid and in a 2-solid to the $(11,5)$-set.

So we get a contradiction.

Case 2. Now, suppose we have a (4,3,1,1)- and a (5,2,1,1)-line. Using the same arguments, we obtain a contradiction. □

Hence, the following lemma is valid.

Lemma 6.5.8 *For every 9-hyperplane H, we have $|L \cap H| = 2$.*

Denote by δ_i, $i = 1,\ldots,55$, the 55 solids in H_0 that are contained in 9-hyperplanes. We have $|L \cap \delta_i| = 2$ by Lemma 6.5.8 and $|L \cap \delta_i \cap \delta_j| = 0$

or 1 when $i \neq j$; see the discussion following the proof of Lemma 6.5.7. Let $L \cap \delta_1 = \{P, Q\}$. There exist nine planes π_i, $i = 1, \ldots, 9$, in δ_1 that contain P and do not contain Q. If we project from π_i; a 9-hyperplane through δ_1 is projected onto a $(4,4,1,0)$-line (Table 1 (F) and Remark 6.2.3). From the fourth intersection square of Lemma 6.5.6, π_i lies in two 9-hyperplanes, one of them being the 9-hyperplane through δ_1; so π_i lies in one solid $\delta_i \neq \delta_1$. Note that δ_1 cannot lie in two 9-hyperplanes since it lies in H_0, one 9-hyperplane and two 18-hyperplanes. Consequently, the point P of L lies on one solid δ_1 containing P and Q, and lies on nine solids δ_j corresponding to the nine planes in δ_1 through P, but not through Q. Counting in two ways the number of flags (P, δ), where $P \in L$ and $P \in \delta$ with $\delta \in \{\delta_1, \ldots, \delta_{55}\}$, we get $10 \cdot |L| = 2 \cdot 55$. Therefore $|L| = 11$.

Lemma 6.5.9 *The set L is an $(11,5)$-set in H_0.*

Proof. All multiplicities in this proof are meant with respect to the 11-set L defined on the points of H_0.

Consider an empty plane π, with respect to L, and assume it lies in a 9-hyperplane of K. There is a one-to-one correspondence between the pairs of L and the fifty-five 2-solids to L which are the solids at infinity of the 9-hyperplanes of K. It follows from Lemma 6.5.6 that such an empty plane π is contained in two further 2-solids. For, the type of the projection from π of the 9-hyperplane is $(5,2,2,0)$ (Table 1 (E) and Remark 6.2.3), so it determines the 3×3 intersection square only containing the numbers 2, 5 and 8, and this intersection square has three parallel classes containing a 9-hyperplane.

Assume that δ is a w-solid with $2 < w \leq 9$; so there are at least two points of L not in δ. This w-solid is not contained in a 9-hyperplane with respect to K (Lemma 6.5.8). Fifty-five 2-solids are in one-to-one correspondence with the pairs of L. Hence such a 2-solid containing two points from $L \setminus \delta$ intersects δ in a 0-plane π. By the preceding paragraph, π is contained in three 2-solids and one w-solid which is forced to be a 5-solid.

To complete the proof, it remains to be checked that there cannot be 10- or 11-solids with respect to L. Assume there exists a 10- or an 11-solid S. No three of the points of $L \cap S$ can be collinear, since there is a bijection between the fifty-five 2-solids and the pairs of L. Because there are at most 10 points on a cap in a solid, this shows that we cannot have 11-solids. Hence S is a 10-solid, and $S \cap L$ is an elliptic quadric Q. Every pair of the 10-solid S is contained in a 2-solid S', which necessarily intersects S in a plane. This plane shares already 2 points with Q, so shares at least 4 points with Q. But this plane is contained in a 2-solid; a contradiction. □

Theorem 6.5.10 *The set $K \cup L$ is a $(56,20)$-set.*

Proof. Each solid in H_0 contained in two 18- and one 9-hyperplane contains 2 points from L (Lemma 6.5.8) and each solid in H_0 contained in three 15-hyperplanes contains 5 points from L (Lemma 6.5.9). □

The 56-cap of Hill is the only $(56,20)$-set in $PG(5,3)$ [49]. The 11-hyperplanes of the 56-cap are the tangent hyperplanes to the elliptic quadric containing this 56-cap, with the tangent point belonging to the 56-cap. Since the group stabilizing the 56-cap acts transitively on the points of the 56-cap [46]; all these 11-hyperplanes are projectively equivalent; hence, the corresponding 45-caps are unique.

This finishes the discussion of this case.

6.5.2 Suppose there are 9-solids

Embed $AG(5,3)$ in $PG(5,3)$ by adding the hyperplane H_0 at infinity. Then H_0 is a hyperplane skew to this 45-cap in $PG(5,3)$. We identify the affine points with the corresponding projective points.

By Lemma 6.5.5, we have two parallel 9-solids S_1 and S_2, lying in a hyperplane $H \equiv PG(4,3)$. By Lemma 6.2.1, a 9-cap in $AG(3,3)$ is always obtained by deleting a 1-plane of an elliptic quadric in $PG(3,3)$. Hence, working in the projective space, $S_i \cap K$ is an elliptic quadric Q_i minus a point p_i, $i=1,2$. And p_1 and p_2 have the same tangent plane, lying in H_0, to respectively Q_1 and Q_2.

Suppose there is another 9-solid contained in H. Then, this solid contains at least 5 points of one of the two elliptic quadrics, so contains the elliptic quadric completely.

Denote by n_i the number of i-solids contained in H. Then we just showed that $n_9 = 2$.

We now will use parallel classes of solids in $H \setminus H_0$. A parallel class of solids in $H \setminus H_0$ consists of three solids of $H \setminus H_0$ intersecting in a fixed plane of $H \cap H_0$. Every parallel class of solids of $H \setminus H_0$ comes from an intersection square of Lemma 6.5.6. We count how many intersection squares of every type there are. The intersection squares of Lemma 6.5.6 differ from each other in the number of parallel classes of 15-hyperplanes they contain. Note that the latter intersection square of Lemma 6.5.6 only containing the number 5 cannot determine a parallel class in H since the three parallel solids in H would only contain 15 points in total, instead of the 18 points of $K \cap H$. Letting the plane π which determines the intersection square (see Remark 6.5.4) vary in the solid at infinity $H \cap H_0$; we denote by a_i the number of intersection squares with i parallel classes of 15-solids ($i = 0, \ldots, 3$); hence a_0, a_1, a_2, respectively a_3, denote the number of intersection squares of the first, second, fourth, respectively third, type as in Lemma 6.5.6.

We have

$$a_0 + a_1 + a_2 + a_3 = 40 \tag{6.5}$$

$$a_1 + 2a_2 + 3a_3 = 66 \tag{6.6}$$

where the first number equals the number of planes in the solid at infinity of H_0, and where the second number is equal to 66; the total number of parallel classes of 15-solids (Lemma 6.5.2). Let b_1 be the number of parallel $2-8-8$-solid triples in H and let b_2 be the number of parallel $5-5-8$-solid triples in H. Then

$$b_1 + b_2 = a_1 \tag{6.7}$$

since these two types of solid triples only occur in intersection squares of the second type in Lemma 6.5.6.

We now express the spectrum of the 18-cap in H in terms of a_i and b_i: $n_0 = 2$ since we have one 0-solid at infinity and one 0-solid corresponding to the type $(9,0,9)$. Also $n_1 = 0$ since only the fourth intersection square of Lemma 6.5.6 contains a 1-solid. And in this intersection square, a 1-solid only lies in 9- and 15-hyperplanes, but this contradicts the fact that H contains 18 points of the 45-cap. Similarly, $n_3 = 0$ since a 3-solid only lies in the first and third type of intersection squares of Lemma 6.5.6. Only in the first type of intersection square, a 3-solid lies in a 18-hyperplane, but then the parallel class determined by the 3-solid would give rise to a 9-solid different from S_1 and S_2. This was excluded in the beginning of this section. And $n_2 = b_1$, since the only way of having a 2-entry in the type of an 18-hyperplane is $(2,8,8)$, which occurs b_1 times; $n_4 = a_2$ since a 4-solid only lies in the fourth square of Lemma 6.5.6 and this determines a $(7,7,4)$-type in H; $n_5 = 2b_2$ since a 5-solid, contained in H, lies only in the second intersection square of Lemma 6.5.6 and such a square intersects H in a (5,5,8)-triple containing two 5-solids; $n_6 = 3a_3 + 3(a_0 - 1)$, since the third intersection square yields three 6-solids in H and there is one intersection square of the first type, which determines the $(9,0,9)$-type, the other intersection squares of the first type yield three 6-solids in H; $n_7 = 2a_2$ since a 7-solid lies only in the fourth intersection square of Lemma 6.5.6, and such a square determines the $(7,7,4)$-type in H; $n_8 = 2b_1 + b_2$ since there are b_1 $(2,8,8)$-triples and b_2 $(5,5,8)$-triples giving respectively two and one 8-solids; $n_9 = 2$.

Applying (6.1) to $H \cap K$, we have

$$\sum i(i-1)n_i = 18 \times 17 \times 13 \tag{6.8}$$

$$\sum i(i-1)(i-2)n_i = 18 \times 17 \times 16 \times 4. \tag{6.9}$$

Now (6.8) $-$ (6.9)/12 $- 57 \times$ (6.5) $-$ (6.6) $- 58 \times$ (6.7) shows that $a_0 = 0$; while it should be at least one.

We have shown the following lemma.

Lemma 6.5.11 *There is no* 45*-cap in* $AG(5,3)$ *having a* 9*-solid.*

We have discussed all possible configurations that can occur in a 45-cap. Only the 45-cap arising from deleting an 11-hyperplane from a 56-cap in $PG(5,3)$ remains. This proves the result. □

Remark 6.5.12 We remark that the results mentioned in this chapter have been used by B. Lent Davis and D. MacLagan for a paper submitted to *Mathematical Intelligencer* [16] to describe the card game SET. This game was invented in 1974 by the geneticist M. J. Falco in order to study epilepsy in German Shephards. Finally, we remark that H. T. Hall has devised a deck of cards for a SET game of dimension 6, starring the Pokemon characters.

Chapter 7

Caps in $PG(3, q)$, q even

In this chapter, the known upper bound on $m_2'(3, q)$, q even, is improved. We also improve a number of intervals, for k, for which there does not exist a complete k-cap in $PG(3, q)$, q even.
The results are taken from Ferret and Storme, On the size of complete caps in $PG(3, 2^h)$, [28].

7.1 The 3-dimensional case

By a result of Chao [15], the size of the second largest complete cap in $PG(3, q)$, q even, is bounded by $m_2'(3, q = 2^h) \leq q^2 - q + 5$, for $q \geq 8$ [15].

In this chapter, we improve this upper bound on $m_2'(3, q = 2^h)$ to $m_2'(3, q = 2^h) \leq q^2 - q + 2$, for $q \geq 16$.

For smaller values of k, there exists an interval theorem for k, in which there does not exist a complete k-cap in $PG(3, q)$, q even.

Theorem 7.1.1 (Storme and Szőnyi [77]) *There is no complete k-cap K in $PG(3, q)$, q even, $q \geq 64$, with*

$$k \in \left[q^2 - (a-1)q + a\sqrt{q} + 2 - a + \binom{a}{2}, q^2 - (a-2)q - a^2\sqrt{q}\right]$$

and with a an integer satisfying

$$2 \leq a \leq \frac{-2\sqrt{q} + 3 + \sqrt{16q\sqrt{q} + 12q - 44\sqrt{q} - 7}}{4\sqrt{q} + 2}.$$

We will improve the preceding theorem to the following one.

Theorem 7.1.2 *There is no complete k-kap K in $PG(3, q)$, q even, $q \geq 1024$, with*

$$k \in \left[q^2 - (c-1)q + (2c^3 + c^2 - 5c + 6)/2, q^2 - (c-2)q - 2c^2 + 3c\right]$$

and with c an integer satisfying $2 \leq c \leq \sqrt[4]{q}$.

Hence, the size of the gaps between the intervals in now independent from q.

7.2 Preliminaries

Theorem 7.2.1 *Let K be a k-cap in $PG(2,q)$, q even, $q > 2$, for which $q+1 \geq k > q - \sqrt{q} + 1$. Then K can be uniquely extended to a $(q+2)$-cap of $PG(2,q)$.*

Proof. See [51, p. 233] and [83]. □

Definition 7.2.2 A tangent L to a k-cap K in $PG(n,q)$ is a line which has exactly one point in common with K. We call planes, respectively lines, intersecting K in i points, i-planes, respectively i-lines.

By Theorem 7.2.1, a $(q+1)$-cap in $PG(2,q)$, q even, has a unique point n which extends the $(q+1)$-cap to a $(q+2)$-cap. This point n is called the *nucleus* of the $(q+1)$-cap. It is the point through which all tangents to the $(q+1)$-cap pass.

Remark 7.2.3 Let K be a k-cap in $PG(n,q)$. Then each point p of K belongs to exactly $t = q^{n-1} + q^{n-2} + \cdots + q + 2 - k$ tangents.

Theorem 7.2.4 *Let K be a complete k-cap in $PG(n,q)$, q even, and let $t = q^{n-1} + q^{n-2} + \cdots + q + 2 - k$.*

Then each point p of $PG(n,q) \backslash K$ belongs to $\sigma_1(p) \leq t$ tangents to K.

Proof. See [53, Lemma 2.3] and [54, Lemma 27.4.2]. □

7.3 Non-existence intervals for complete k-caps in $PG(3, 2^h)$

Suppose there exists a complete k-kap K in PG(3, q), q even, $q \geq 1024$, with

$$k \in \left[q^2 - (c-1)q + (2c^3 + c^2 - 5c + 6)/2, q^2 - (c-2)q - 2c^2 + 3c\right]$$

and with c an integer satisfying $2 \leq c \leq \sqrt[4]{q}$.

We first present a lemma arising from ideas from Chao [15].

Lemma 7.3.1 (cfr. [15]) *A plane intersects K in at most c^2 points or in at least $q + 2 - c^2$ points; we will call these planes respectively* small *planes and* big *planes.*

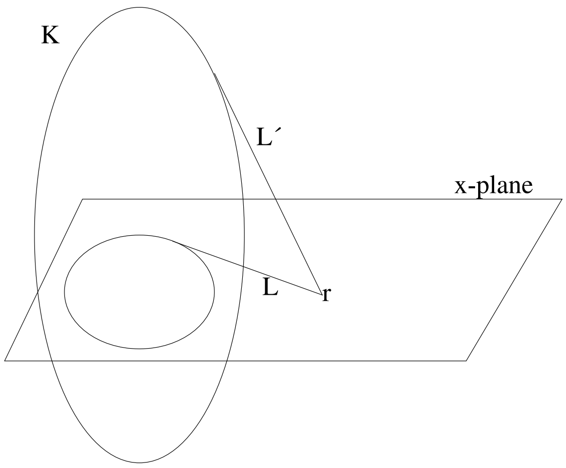

Figure 7.1: An x-plane

Proof. (cfr. [15]) Let π be an x-plane; see Figure 7.1. Let S be the set of pairs (r, L), where $r \in \pi \setminus K$ and L is a tangent through r. By Theorem 7.2.4, we have $|S| \leq t(q^2 + q + 1 - x) = t(t + k - 1 - x)$. We have $t(k - x)$ tangents to K intersecting π in exactly one point of $\pi \setminus K$. In π we have $x(q + 2 - x)$ tangents to K. Hence $|S| = t(k - x) + x(q + 2 - x)q$. Solving this inequality yields the result. $\square$

Remark 7.3.2 Since we impose the condition $c^4 \leq q$ on c, large plane intersections are always contained in $(q + 2)$-caps of these planes (Theorem 7.2.1).

We will use the method of Storme and Szőnyi [77], with the additional information of Lemma 7.3.1.

We will count the cardinality of U, being the set of triples $(L, \alpha, r_{L,\alpha})$, where L is a tangent, α is a big plane through L and $r_{L,\alpha}$ is the unique point on L which extends $K \cap \alpha$ to a larger cap.

Lemma 7.3.3 $|U| \geq (q - c)(q^2 - (c - 2)q - 2c^2 + 3c)(q^2 + q + 2 - (q^2 - (c - 2)q - 2c^2 + 3c))$.

Proof. The k-cap K has $k(q^2 + q + 2 - k) \geq t_1 = (q^2 - (c - 2)q - 2c^2 + 3c)(q^2 + q + 2 - (q^2 - (c - 2)q - 2c^2 + 3c))$ tangents.

Let r be the number of small planes through a tangent, then $k - 1 \leq r(c^2 - 1) + (q + 1 - r)q$. So, there are at most $c + 1$ small planes through a tangent. Hence there are at least $q - c$ big planes through a tangent.

For a fixed tangent L and big plane α, the point $r_{L,\alpha}$ is unique. So $|U| \geq (q - c)t_1$. $\square$

Lemma 7.3.4 $|U| \leq (c - 1)(q + 1)(q^3 + cq + 1 - (2c^3 + c^2 - 5c + 6)/2)$.

Proof. There are at most $|PG(3,q) \setminus K| \leq q^3 + cq + 1 - (2c^3 + c^2 - 5c + 6)/2$ choices for a point r outside the cap.

If r extends a big plane intersection $\alpha \cap K$, then r lies on at least $q - c^2$ tangents in α. Hence, if r extends x big plane intersections through r, then r lies on at least $x(q - c^2) - \binom{x}{2}$ tangents. Since this number is at most t, we have $x \leq c - 1$. For fixed r and α, there are trivially at most $q + 1$ choices for L. So $|U| \leq (c-1)(q+1)(q^3 + cq + 1 - (2c^3 + c^2 - 5c + 6)/2)$. □

Comparing the bounds of Lemmas 7.3.3 and 7.3.4 gives a contradiction; hence there does not exist a complete k-cap whose size lies in the intervals of Theorem 7.1.2.

Remark 7.3.5 Doing the same argument for $c = 3$, we can refine the result to eliminate complete k-caps, with $k \in \{q^2 - q - 8, \ldots, q^2 - q - 5\}$. Hence, for $k > q^2 - 2q + 27$, only the values in $\{q^2 - q - 4, \ldots, q^2 - q + 5\}$ remain open.

7.4 A closer look at $m_2'(3,q)$

We will prove that $m_2'(3, q = 2^h) \leq q^2 - q + 2$, for $q \geq 16$.

In this section, we will eliminate the existence of complete k-caps, where k is $q^2 - q + 3, q^2 - q + 4$ or $q^2 - q + 5$, in $PG(3, 2^h)$, $h \geq 4$. Assume there does exist such a cap K.

Lemma 7.4.1 *The only small planes are* 0*-planes or* 1*-planes.*

Proof. The reasoning of Lemma 7.3.1 gives that for $k = q^2 - q + 3$, we have at most four points in a small plane, and for $k \in \{q^2 - q + 4, q^2 - q + 5\}$, we have at most three points in a small plane.

First, we will eliminate 4-planes for a complete (q^2-q+3)-cap K. Assume there exists a 4-plane α. Take a tangent L in α; see Figure 7.2. If L is contained in a $(q+1)$-plane π, then the nucleus n of this plane π lies on at least one tangent $T \neq L$ contained in α.

Since $\sigma_1(n) \leq t = 2q - 1$, T cannot be contained in a $(q+1)$-plane; or else n is the nucleus of the $(q+1)$-cap of this $(q+1)$-plane since it already lies on two tangents to this $(q+1)$-cap; but then $\sigma_1(n) \geq 2q + 1$. Hence, α always contains a tangent T which is not contained in a $(q+1)$-plane. We have the following planes through T: one 4-plane α, $q - 1$ different q-planes π_i and one $(q-1)$-plane β. Through the unique point n_i on T which extends $\pi_i \cap K$, we have q tangents contained in π_i, at least one tangent different from T in every other q-plane, and at least one tangent different from T in α. Hence, we have found already t tangents through n_i. Since $\sigma_1(n_i) \leq t$, the point n_i lies on the intersection of T and a bisecant in α. There are only three possibilities for such a point, hence we can find indices $i \neq j$ for which $n_i = n_j$, but then we have more than t tangents through n_i; a contradiction.

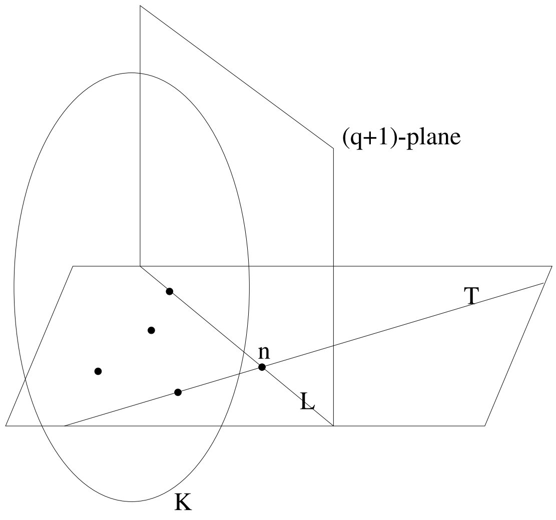

Figure 7.2: There is a tangent T not contained in a $(q + 1)$-plane

Now, we will eliminate 2-planes and 3-planes for $k \in \{q^2-q+4, q^2-q+5\}$. Assume we have a 2-plane α. Take a tangent L in α and let $\{r\} = L \cap K$. Through r go at least $q-1$ tangents in α, hence we have at most $q-1$ tangents left over to divide over the other q planes through L. So we have at least one $(q + 1)$-plane π through L. Take the nucleus n of π and let T be the other tangent through n in α. If T is contained in a $(q + 1)$-plane, then necessarily n is the nucleus of this plane, and we have more than t tangents through n; a contradiction. So, T lies in at most q-planes, and cannot lie in a 2-plane.

Let α be a 3-plane.

Take a tangent T in α, then T is contained in a $(q + 1)$-plane π. By the reasoning above, the nucleus of π is on T and on a bisecant in α. Because a point on T cannot be the nucleus of more than one $(q + 1)$-plane, this implies that through a tangent T in α there is exactly one $(q + 1)$-plane. Hence we obtain at most $k = q^2 - q + 4$ points and the planes through T are one 3-plane α, one $(q+1)$-plane π and $q-1$ different q-planes π_i. Let n be the nucleus of π. In every plane π_i, we must have an even number of tangents through the point n, so besides T, for every q-plane, we have at least one extra tangent through n. But this yields $\sigma_1(n) > t$; a contradiction.

Finally, we will eliminate 2-planes and 3-planes for $k = q^2-q+3$. Assume we have a 2-plane α, and take a tangent L in α. Through L, we have at least one $(q + 1)$-plane π. Take the tangent $T \neq L$, contained in α, passing through the nucleus n of π. This tangent T cannot be contained in a $(q + 1)$-plane, since n would also be the nucleus of this plane and we would have too many tangents through n. Counting incidences with planes through T, we obtain $k - 1 \leq 1 + q(q - 1)$; a contradiction.

Now assume we have a 3-plane α, and take a tangent L in α. Assume there is no $(q+1)$-plane through L, then all planes $\pi_i \neq \alpha$ through L must be q-planes. Now consider the point n_i on L which extends $K \cap \pi_i$. Through n_i, there are q tangents in π_i, and there is at least one extra tangent for every other q-plane $\pi_j \neq \pi_i$. Hence, we have already found t tangents through n_i, and n_i must be the unique point on L, which lies on the unique bisecant to $(K \cap \alpha) \setminus L$. But we can do this reasoning for every q-plane through L, and we have too many tangents through the point $n_1 = \cdots = n_q$.

Hence, there is a $(q+1)$-plane π through L. We can assume that the nucleus n of $\pi \cap K$ must be the point on L, which lies on a bisecant B to $K \cap \alpha$, or else we are reduced to the preceding situation. Also, π is the only $(q+1)$-plane through L. This yields, that there must be at least $q-2$ different q-planes through n, all having a tangent different from L through n. Hence $\sigma_1(n) \geq q-2+q+1 = t$. Now, consider the planes β_i through B different from the 3-plane. Since such a plane β_i intersects π in a tangent, β_i is not a $(q+2)$-plane, and then β_i has to be a $(q+1)$-plane.

Since n lies on a bisecant B, n lies on exactly one tangent in every plane β_i. This gives $\sigma_1(n) = q+1$; a contradiction with the preceding paragraph where we showed that $\sigma_1(n) = t$. $\square$

Lemma 7.4.2 *Every tangent L is contained in a $(q+1)$-plane.*

Proof. Let $\{r\} = L \cap K$. If L is contained in a small plane, the result is trivial, since otherwise $|K| \leq 1 + q(q-1)$ (false). Now suppose that L is only contained in big planes. Hence, there exists a point s on L which extends at least two plane intersections through L.

If $k = q^2 - q + 5$, then the reasoning of Lemma 7.3.1 shows that the big planes contain at least $q-1$ points of K. Then s extends two $(q-1)$-caps lying in planes π_1 and π_2 through L, and there are no tangents through s outside of π_1 or π_2; since $\sigma_1(s) \leq t = 2q-3$. Since the number of tangents through s to a planar q-cap is even, there cannot be a q-plane through L. Hence the planes through L are 3 different $(q+1)$-planes and $q-2$ different $(q-1)$-planes.

If $k = q^2 - q + 4$, then also here the big planes contain at least $q-1$ points of K. Then a first possibility is that s extends two $(q-1)$-caps lying in planes π_1 and π_2 through L, and there is at most one tangent through s not lying in π_1 or π_2. Then we have at most one q-plane through L and the planes through L must be one q-plane, two $(q+1)$-planes and $q-2$ different $(q-1)$-planes. The other possibility is that s extends a $(q-1)$-cap and a q-cap lying in planes through L. This case is similar to the previous one.

If $k = q^2 - q + 3$, then big planes share at least $q-2$ points with K. Then we have at most $t - (q-2+q-2-1) = 4$ different q-planes through L, and equality might only occur when s extends two $(q-2)$-caps lying in planes through L. Assume that there are no $(q+1)$-planes through L, then L lies in at most four q-planes. Once the number of q-planes through L is fixed, the number of $(q-1)$- and $(q-2)$-planes also is fixed. Checking all cases, only the possibility remains that the planes through L are four q-planes and $q-3$ different $(q-1)$-

planes. But then s lies on $2q-3$ tangents to the two $(q-1)$-planes and to four extra tangents to the four q-planes; but then $\sigma_1(s) = 2q+1 > t$. $\square$

Lemma 7.4.3 *All tangents through a nucleus n of a $(q+1)$-plane π lie in this $(q+1)$-plane.*

Proof. Assume there is a tangent M through n not contained in π. Then M lies in a $(q+1)$-plane β. Since n lies on at least two tangents M and $\pi \cap \beta$ in β, n is the nucleus of $\beta \cap K$. But then $\sigma_1(n) > t$; a contradiction. $\square$

The final contradiction

If $k = q^2 - q + 4$, then take the nucleus n of a $(q+1)$-plane. By Lemma 7.4.3, we have $\sigma_1(n) = q+1$; while this number of tangents through n should be even.

If $k \in \{q^2 - q + 3, q^2 - q + 5\}$, then we will first show that the only even plane intersections have size 0 or $q+2$. Suppose we have a plane π which intersects K in q or in $q-2$ points. Then take a tangent L in π, this tangent is contained in a $(q+1)$-plane γ. The number of tangents through the nucleus n of $\gamma \cap K$ to $\pi \cap K$ is even; hence there is a tangent through n not contained in γ; a contradiction (Lemma 7.4.3).

Now, using the standard counting arguments on the numbers $\sum_i n_i, \sum_i i n_i$, $\sum_i i(i-1)n_i, \sum_i i(i-1)(i-2)n_i$, we can compute

$$\sum_i n_i(i-1)(i-(q-1))(i-(q+1)),$$

where n_i denotes the number of i-planes.

For $k = q^2 - q + 3$, we obtain

$$-n_0(q^2-1) + 3n_{q+2}(q+1) = -5q^3 + 17q^2 - 22q + 16.$$

Computing this equation modulo $q+1$, we obtain $60 \equiv 0 \bmod q+1$; a contradiction.

For $k = q^2 - q + 5$, we obtain

$$-n_0(q^2-1) + 3n_{q+2}(q+1) = -7q^3 + 39q^2 - 68q + 96.$$

Computing this equation modulo $q+1$, we obtain $210 \equiv 0 \bmod q+1$; a contradiction. $\square$

7.5 Appendix: larger dimensions

The crucial element in the improvement of Theorem 7.1.1 to Theorem 7.1.2 was the fact that the technique of Chao [15] (cfr. proof of Lemma 7.3.1) showed that planes of $PG(3,q)$ intersect large complete caps of $PG(3,q)$ in either a small number of points or in a large number of points.

A check on the known upper bounds for the sizes of caps in $PG(n,q)$, q even, shows that a similar result is valid for large caps in $PG(n,q)$, q even. We present these results in Theorem 7.5.2.

To find an upper bound on the sizes of caps in $PG(n,q)$, q even, we rely on the upper bound on the size of caps in affine spaces $AG(n,q)$ of Bierbrauer and Edel.

We repeat their result.

Theorem 7.5.1 (Bierbrauer and Edel [7]) *Let C_n be the largest size of a cap in $AG(n,q)$, q even, $q > 2$, $n \geq 4$, then*

$$C_n \leq \frac{q^{n-1} + q^n C_{n-1}}{q^{n-1} + C_{n-1}}.$$

As indicated in [7], starting from $C_3 = q^2$, $C_4 = q^3 - q^2 + q$ follows.

In general, for $4 \leq n \leq 2q/3$, $q \geq 8$, the upper bound

$$C_n \leq q^{n-1} - (n-3)q^{n-2} + (n-3)^2 q^{n-3}$$

follows. Adding the upper bound q^{n-2} for the size of a cap in $PG(n-1,q)$, it follows that

$$m_2(n,q) \leq q^{n-1} - (n-4)q^{n-2} + (n-3)^2 q^{n-3}$$

for even $q \geq 8$ and $4 \leq n \leq 2q/3$.

We now present an interval theorem on the sizes of the hyperplane sections of hyperplanes of $PG(n,q)$, q even, with complete large k-caps.

Theorem 7.5.2 *In $PG(n,q)$, q even, $n \geq 4$, a complete k-cap K, with*

$$k > q^{n-1} - (c-1)q^{n-2} + q^{n-3} + q^{n-4} + \cdots + q + 2 + c^3 q^{n-3}/2,$$

with $c^4 \leq 4q$, intersects a hyperplane in less than $c^2 q^{n-3}$ points or in larger than $q^{n-2} + \cdots + q + 2 - c^2 q^{n-3}$ points.

Proof. We repeat the arguments of the proof of Lemma 7.3.1; see also the case $n = 3$ in [15].

Let $|K| = q^{n-1} - (c-1)q^{n-2} + q^{n-3} + q^{n-4} + \cdots + q + 2 + \epsilon$, then the number of tangents through a point of K is $t = cq^{n-2} - \epsilon$.

Let π be a hyperplane intersecting K in x points.

We count the number of ordered pairs (r, L), where $r \in \pi \setminus K$, and where L is a tangent to K through r.

Then this number equals $t(k-x) + x(q^{n-2} + \cdots + q + 2 - x)q$, and $t(q^{n-1} + \cdots + q + 1 - x)$ is an upper bound for this number (Theorem 7.2.4).

This inequality is equivalent to

$$x^2(q - q^2) + x(q^n + q^2 - 2q) - c^2 q^{2n-3} + c^2 q^{2n-4}$$
$$+(c + 2\epsilon c)q^{n-1} - (c + 2\epsilon c)q^{n-2} - (\epsilon + \epsilon^2)q + \epsilon + \epsilon^2 \leq 0.$$

For $\epsilon > c^3 q^{n-3}/2$, this implies $x < c^2 q^{n-3}$ or $x > q^{n-2} + \cdots + q + 2 - c^2 q^{n-3}$. □

Remark 7.5.3 Presently, no caps in $PG(n, q)$, q even, $n \geq 4$ small, of the size of the upper bounds of Theorem 7.5.1 and Theorem 7.5.2 are known.

Nevertheless, the result of Theorem 7.5.2 might be useful in eliminating the existence of caps of these sizes.

Appendix A

Two weight codes meeting the Griesmer bound

For q odd, we construct $[(3q^2+1)/2, 3, (3q^2-3q)/2; q]$-codes meeting the Griesmer bound. The weights of these codes are $(3q^2-3q)/2$ and $(3q^2-q)/2$; these codes are new if $q > 5$.

We also show that if there exist $[(3q^2+3)/2, 4, (3q^2-3q)/2; q]$-codes, with weights $(3q^2-3q)/2$ and $(3q^2-q)/2$; they must have a *shortened code* obtained from the construction.

This work was done while visiting Prof. Dr. T. Helleseth at Bergen University, Norway [25].

A.1 $[(3q^2+1)/2, 3, (3q^2-3q)/2; q]$-codes

Let C denote a $[(3q^2+1)/2, 3, (3q^2-3q)/2; q]$-code. This code C attains the Griesmer bound since $(3q^2+1)/2 = (3q^2-3q)/2 + (3q-3)/2 + 2$.

Consider a generator matrix G of C. Consider the columns of G as being a multi set C' of points in a projective plane π. Clearly, the code is not projective, since G has more than q^2+q+1 columns.

In this way, the $[(3q^2+1)/2, 3, (3q^2-3q)/2; q]$-code C is equivalent with a multi set C' in π of cardinality $(3q^2+1)/2$ and with the property that a line intersects C' in $(q+1)/2$ or $(3q+1)/2$ points (in this appendix, countings are always with multiplicities).

We will construct such a code C, by constructing a corresponding multi set C'.

Construction A.1.1 Consider a dual $(q+3)/2$-cap in the plane π, together with the set of points on two lines of the dual $(q+3)/2$-cap; this forms the following incidence structure K^D in π:

a set S of $(q+3)/2$ lines,

a set T of $(q^2+4q+3)/8$ points,

there are two lines of S through a point of T,

there are $(q+1)/2$ points of T on a line of S.

Denote the points of π lying on two lines of S by w_i, $i = 1, \ldots, (q^2+4q+3)/8$; these are exactly the points of T.

There are $q+1-(q+1)/2$ points of $\pi \setminus T$ on a line of S, hence there are $(q+3)(q+1)/4$ points v_i of π lying on exactly one line of S.

The other $q^2+q+1-(q^2+4q+3)/8-(q+3)(q+1)/4 = (5q^2-4q-1)/8$ points u_i lie on zero lines of S.

Let C' be the multi set consisting of all points v_i with multiplicity one and all points u_i with multiplicity two.

Hence C' is a multi set of cardinality $(3q^2+1)/2$ and C' generates π.

Theorem A.1.2 *A line L of π intersects the multi set C' in $(q+1)/2$ or $(3q+1)/2$ points.*

Proof. If $L \in S$, then L intersects C' in $(q+1)/2$ points. Suppose $L \notin S$ contains a points u_i, b points v_i, and c points w_i.

Then $a+b+c = q+1$ and counting the lines of S, by considering their intersections with L, yields $b+2c = (q+3)/2$.

Finally, $|L \cap C'| = 2a+b = (3q+1)/2$. □

A.2 $[(3q^2+3)/2, 4, (3q^2-3q)/2; q]$-codes

Definition A.2.1 A *strongly regular graph with parameters* (N, K, λ, μ) is a graph on N vertices all having valency K, such that for two adjacent vertices, there are λ vertices adjacent with both of them; and for two non-adjacent vertices, there are μ vertices adjacent with both of them.

Strongly regular graphs are well studied and have a lot of applications. We refer the reader to for example [55].

Remark A.2.2 Let V be the k-dimensional vector space over $GF(q)$, and let $\Omega \subset V$ such that $\Omega = -\Omega$ and $(0, \ldots, 0) \notin \Omega$.

We define the graph $G(\Omega)$ as follows. The vertices of $G(\Omega)$ are the vectors of V. Two vertices are adjacent if their difference is in Ω.

Let C denote a projective two weight $[n, k, d; q]$-code with weights d and w.

Assume $G = (g_1 \ldots g_n)$ is a generator matrix of C. Put $\omega = \{\langle g_i \rangle \mid i = 1, \ldots, n\}$, and define Ω as $\Omega = \{v \in V \mid \langle v \rangle \in \omega\}$.

Then $G(\Omega)$ is a strongly regular graph with parameters $N = q^k$, $K = n(q-1)$, $\lambda = K^2+3K-q(K+1)(w+d)+q^2wd$ and $\mu = q^{2-k}wd$.

Definition A.2.3 A *shortened code* C_T of a linear $[n, k, d; q]$-code C is a linear code obtained by selecting the codewords of C which are zero in a certain fixed position, and by deleting this fixed position in these codewords.

We show that if there exist $[(3q^2+3)/2, 4, (3q^2-3q)/2; q]$-codes C, having weights $(3q^2-3q)/2$ and $(3q^2-q)/2$; they must have a shortened code obtained from the construction in Section A.1.

Assume C exists, then it meets the Griesmer bound. Lemma 1.1.9 shows that C is projective.

We consider the columns of a generator matrix of G as a set C' of $(3q^2+3)/2$ points in $PG(3,q)$, having the property that a plane intersects C' in $(3q+3)/2$ or $(q+3)/2$ points.

Definition A.2.4 An *x-line* is a line of $PG(3,q)$ intersecting C' in x points. Similarly, an *x-plane* is defined.

Lemma A.2.5 *Through an x-line, there are $3-x$ different $((q+3)/2)$-planes. In particular, there are at most 3 points of C' on a line.*

Proof. Let L be an x-line and assume there are b different $((q+3)/2)$-planes through L. Counting the number of points of C', not on L, in a plane through L, we obtain $(3q^2+3)/2 - x = b((q+3)/2 - x) + (q+1-b)((3q+3)/2 - x)$; so $b = 3 - x$. □

Lemma A.2.6 *If $p \in C'$, then there are $(q+3)/2$ different $((q+3)/2)$-planes through p, and there are $(5q^2-4q-1)/8$ different 3-lines through p.*

Proof. Assume there are a different $((q+3)/2)$-planes through p. Count the 2-tuples (p', α), where $p' \neq p$ is a point of C' and α is a plane through pp'; we obtain $(q+1)(3q^2+1)/2 = a(q+1)/2 + (q^2+q+1-a)(3q+1)/2$; so $a = (q+3)/2$.

Assume there are z 3-lines through p. By Lemma A.2.5, a 3-line only occurs in a $((3q+3)/2)$-plane. Project from p the $((q+3)/2)$-planes through p on a plane π skew to p. We obtain a set S of $(q+3)/2$ lines. Consider a point s on a line L of S, then, by definition of S, the plane through p and L is a $((q+3)/2)$-plane. Hence, s lies on equally many lines of S, as there are $((q+3)/2)$-planes through ps. Since $p \in C'$, the line ps is a 1-, 2$-$ or 3-line. By Lemma A.2.5, we obtain that a point $s \in \pi$ lies on $0, 1$ or 2 lines of S. Furthermore, the number z of 3-lines through p is equal to the number of points in π lying on 0 lines of S.

Dualizing S yields a $((q+3)/2)$-cap, where z is the number of skew lines to the cap. Since there are $q+2-(q+3)/2$ tangents through a point of the cap, $z = q^2+q+1-(q+2-(q+3)/2) \times (q+3)/2 - (q+3) \times (q+1)/8$. □

Lemma A.2.7 *If $p \in C'$, then there are $x = (q^2+4q+3)/8$ different 1-lines through p, and there are $y = (q^2+4q+3)/4$ different 2-lines through p.*

Proof. We already know the number $z = (5q^2 - 4q - 1)/8$ of 3-lines through p. Obviously, $x + y + z = q^2 + q + 1$. Counting the 2-tuples (p', L), where $p' \neq p$ is a point of C' and L is the line pp', we obtain $y + 2z = (3q^2 + 3)/2 - 1$. □

Theorem A.2.8 *Project the* 1*-lines through* $p \in C'$ *on* π *to obtain a set* T *in* π*. Let* S *again denote the set of* $(q+3)/2$ *lines in* π *obtained by projecting from* p *the* $((q+3)/2)$*-planes through* p *on* π*.*

Then S *is a dual* $(q+3)/2$*-cap.*

Proof. Through a point of T, there are two lines of S, by Lemma A.2.5. On a line $L \in S$, there are $q + 1 - ((q+3)/2 - 1) = (q+1)/2$ points of T, being the number of tangents through p in the plane $\langle p, L\rangle$ intersecting C' in a cap, by Lemma A.2.5. □

Remark A.2.9 We can assume p has coordinates $(0,0,0,1)$ and π has equation $X_3 = 0$. Then if there exists a $[(3q^2+3)/2, 4, (3q^2-3q)/2; q]$-code, it has generator matrix $\begin{pmatrix} 0 & & & \\ 0 & & D & \\ 0 & & & \\ 1 & * & \ldots & * \end{pmatrix}$; where D generates a code obtained from the construction in Section A.1.

As mentioned in the introduction, such codes are only known for $q = 3$ and $q = 5$ [12]. Finding such codes for $q > 5$ yields new strongly regular graphs.

Appendix B

Applications of $\{\delta v_{\mu+1}, \delta v_{\mu};t, q\}$-minihypers

In this appendix, we give a short overview of the many applications of $\{\delta v_{\mu+1}, \delta v_{\mu};t, q\}$-minihypers; the main motivation for studying this particular class of minihypers.

These applications have been collected in the article [35] of Govaerts, Storme and Van Maldeghem. We summarize the new results arising from the results of Chapters 4 and 5. For detailed proofs, we refer to [35].

B.1 Linear codes meeting the Griesmer bound

As described in Chapter 2, the minihypers are equivalent with linear $[n, k, d; q]$-codes meeting the Griesmer bound ($d \geq 1, k \geq 3$), where $n = \theta v_{t+1} - \delta v_{\mu+1}$, $k = t + 1$ and $d = \theta q^t - \delta q^{\mu}$ and where θ denotes the maximum weight of a point of the minihyper.

Hence, classifying these minihypers is equivalent with classifying the above mentioned codes.

B.2 Maximal partial μ-spreads

As mentioned in Chapter 3, the set of holes of a maximal partial μ-spread S in $PG(t, q)$, $(\mu + 1)|(t + 1)$, of deficiency $\delta < q$ forms a $\{\delta v_{\mu+1}, \delta v_{\mu}; t, q\}$-minihyper F. Here, all points of F have weight one.

As demonstrated in Chapter 3, the classification results of the above mentioned minihypers imply that only particular values for δ are allowed. Namely, the minihyper F cannot contain subspaces $PG(\mu.q)$.

Note that the existence of a $\{\delta v_{\mu+1}, \delta v_{\mu}; t, q\}$-minihyper F is however *not* equivalent with the existence of a maximal partial μ-spread S of deficiencey δ

in $PG(t,q)$; see Remark 3.6.20.

For the results on line spreads in $PG(3,q)$, we refer to Corollary 3.6.19. For the results on maximal partial μ-spreads S in $PG(t,q)$, $(\mu+1)|(t+1)$; we can use Theorems 5.2.3 and 5.5.8 with the additional information that all points have weight one. Hence we obtain the following result.

Corollary B.2.1 *Let S be a maximal partial μ-spread in $PG(t,p^3)$, $(\mu+1)|(t+1)$, with deficiency $\delta \leq 2p^2-4p$; $p=p_0^h$, p_0 prime, $p_0 \geq 7$, $p \geq 9$, $h \geq 1$.*

Then $\delta = r(p^{3/2}+1)+s(p^2+p+1)$, where $r \in \mathbb{N}$ and $s \in \{0,1\}$.

Moreover, the case $(r,s)=(1,0)$ cannot occur when $(2\mu+1)(p^{3/2}-1) < q-1$; [8].

The next two sections are based on Govaerts, Storme and Van Maldeghem [35], where the bounds are taken from Theorems 5.2.3 and 5.5.8.

B.3 Partial μ-spreads of polar spaces

The thick finite nondegenerate classical polar spaces are

- $W_{2n+1}(q)$, the polar space arising from a symplectic polarity of $PG(2n+1,q)$, $n \geq 1$.
- $Q^-(2n+1,q)$, the polar space arising from a nonsingular elliptic quadric of $PG(2n+1,q)$, $n \geq 2$.
- $Q(2n,q)$, the polar space arising from a nonsingular quadric of $PG(2n,q)$, $n \geq 2$.
- $Q^+(2n+1,q)$, the polar space arising from a nonsingular hyperbolic quadric of $PG(2n+1,q)$, $n \geq 2$.
- $H(n,q^2)$, the polar space arising from a nonsingular Hermitian variety in $PG(n,q^2)$, $n \geq 3$.

We refer to [54] for more information on these polar spaces. Let $\mathcal{P}$ denote a finite classical polar space. A *μ-spread* of $\mathcal{P}$ is a set of totally isotropic μ-dimensional subspaces of $\mathcal{P}$ partitioning the point set of $\mathcal{P}$. A *partial μ-spread* of $\mathcal{P}$ is a set of pairwise disjoint totally isotropic μ-dimensional subspaces. A partial μ-spread of $\mathcal{P}$ is called *maximal* if it is not contained in a larger partial μ-spread of $\mathcal{P}$.

Necessary conditions [35] for the existence of a μ-spread are:
(1) for $W_{2n+1}(q)$: $(\mu+1)|(n+1)$,
(2) for $Q(2n,q)$: $(\mu+1)|(2n)$,
(3) for $Q^+(2n+1,q)$: $(\mu+1)|(n+1)$,
(4) for $Q^-(2n+1,q)$: $(\mu+1)|n$,
(5) for $H(2n,q^2)$: $(\mu+1)|n$ and,
(6) for $H(2n+1,q^2)$: $(\mu+1)|(n+1)$.

If these conditions are satisfied, we say that *the size of $\mathcal{P}$ admits a μ-spread.*

The *deficiency* of a partial μ-spread S of $\mathcal{P}$ is the size of a hypothetical μ-spread of $\mathcal{P}$ minus $|S|$. A *hole* of a partial μ-spread S of $\mathcal{P}$ is a point of $\mathcal{P}$ not contained in an element of S.

A *generator* is a maximal totally isotropic subspace of $\mathcal{P}$. A *partial ovoid* $\mathcal{O}$ is a set of points of $\mathcal{P}$ such that no generator of $\mathcal{P}$ contains more than one point of $\mathcal{O}$.

Theorem B.3.1 ([35]) *Let $\mathcal{P}$ be a classical polar space in $PG(t,q)$ whose size admits a μ-spread. If S is a partial μ-spread of $\mathcal{P}$ with deficiency $\delta < q$, then the set of holes of S forms a $\{\delta v_{\mu+1}, \delta v_\mu; t, q\}$-minihyper.*

Lemma B.3.2 (1) *A (projected) Baer subspace whose point set is contained in a quadric $\mathcal{P}$ is contained in a generator of $\mathcal{P}$.*
(2) *A (projected) subspace over $GF(\sqrt[3]{q})$ contained in a quadric $\mathcal{P}$ is contained in a generator of $\mathcal{P}$.*

Proof. Assume we have such a (projected) subgeometry Λ contained in $\mathcal{P}$. If a line of Λ is contained in $\mathcal{P}$, then the line over $GF(q)$ generated by it, is contained in $\mathcal{P}$; since a line containing more than two points of a quadric is completely contained in the quadric. This implies the result. $\square$

Remark B.3.3 Let g denote the dimension of a generator of $\mathcal{P}$, then
(1) if there is a (projected) $PG(2\mu+1, \sqrt{q})$ contained in $\mathcal{P}$, we have $((\sqrt{q})^{2\mu+2} - 1)/(\sqrt{q}-1) \leq (q^{g+1}-1)/(q-1)$; implying $\mu < g$.
(2) if there is a (projected) $PG(3\mu+2, \sqrt[3]{q})$ contained in $\mathcal{P}$, we have $((\sqrt[3]{q})^{3\mu+3} - 1)/(\sqrt[3]{q}-1) \leq (q^{g+1}-1)/(q-1)$; implying $\mu < g$.

Theorem B.3.4 ([54]) *For $Q(2n,q)$, respectively $Q^-(2n+1,q)$, $Q^+(2n+1,q)$, the dimension of a generator is $n-1$, respectively $n-1$, n.*

Remark B.3.5 Let S be a partial μ-spread of $\mathcal{P}$ with deficiency δ. We determine an upper bound on δ such that a partial μ-spread of $\mathcal{P}$ with a deficiency δ smaller than or equal to this upper bound, is extendable to a μ-spread of $\mathcal{P}$.

We will use Theorems 5.2.3 and 5.5.8, with the following additional information:
(1) all points of the minihyper have weight one.
(2) by Lemma B.3.2, there are no (projected) $PG(2\mu+1, \sqrt{q})$, nor (projected) subgeometry $PG(3\mu+2, \sqrt[3]{q})$ contained in the minihyper if μ is at least the dimension of a generator of $\mathcal{P}$.

Theorem B.3.6 *Let $q = p^3$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $p \geq 9$, $h \geq 1$. Assume $\mathcal{P}$ is a nonsingular quadric whose size admits a μ-spread, and with μ larger than the dimension of a generator of $\mathcal{P}$.*

If $\delta \leq 2p^2 - 4p$ is the deficiency of a partial μ-spread S of $\mathcal{P}$, then S is extendable to a μ-spread of $\mathcal{P}$.

Proof. By Remark B.3.5 (2), the corresponding minihyper of holes cannot contain (projected) $PG(2\mu+1, \sqrt{q})$, nor (projected $PG(3\mu+2, \sqrt[3]{q})$), so consists completely of μ-dimensional spaces $PG(\mu, q)$, which extend S to a μ-spread of $\mathcal{P}$. □

It is known that $Q^-(5,q)$ has a (line) spread [82]. Since the dual of a partial spread of $Q^-(5,q)$ is a partial ovoid of $H(3,q^2)$ [65], we immediately have the following consequence.

Corollary B.3.7 *If a partial ovoid $\mathcal{O}$ of $H(3,q^2)$ has deficiency $\delta \leq 2q^{2/3} - 4q^{1/3}$, then $\mathcal{O}$ is extendable to an ovoid of $H(3,q^2)$.*

B.4 Partial ovoids of the split Cayley hexagon

Let q be a prime power. Let $x(x_0, \ldots, x_6)$ and $y(y_0, \ldots, y_6)$ be two different points in $PG(6,q)$. The *Grassmann coordinates* of the line xy are $(p_{01}, p_{02}, \ldots, p_{56})$, where

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

The *split Cayley hexagon* is a point-line incidence structure $H(q)$ defined in the following way. The *points* of $H(q)$ are the points of $PG(6,q)$ lying on the quadric $Q(6,q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = 0$. The *lines* of $H(q)$ are the lines of this quadric whose Grassmann coordinates satisfy $p_{12} = p_{34}$, $p_{54} = p_{32}$, $p_{20} = p_{35}$, $p_{65} = p_{30}$, $p_{01} = p_{36}$, $p_{46} = p_{31}$. *Incidence* is containment.

Two points of $H(q)$ are called *opposite* if they are at distance 6 from each other in the incidence graph of $H(q)$ (this is the maximal possible distance). The set of points collinear with a point x of $H(q)$ forms a plane $x^\perp$ contained in $Q(6,q)$. An *ovoid* of $H(q)$ is a set of q^3+1 mutually opposite points. If $\mathcal{O}$ is an ovoid of $H(q)$, then the set of q^3+1 planes $x^\perp$, with $x \in \mathcal{O}$, forms a plane spread of $Q(6,q)$; see [86]. A *partial ovoid* of $H(q)$ is a set of mutually opposite points of $H(q)$, and it is called *maximal* if it cannot be extended to a larger set of mutually opposite points. The *deficiency* of a partial ovoid of $H(q)$ containing N points is $\delta = q^3 + 1 - N$.

The way to obtain results on δ goes as follows [35].
Let $\mathcal{O}$ be a maximal partial ovoid of $H(q)$, then the set of planes $x^\perp$, where $x \in \mathcal{O}$, forms a partial plane spread S of $Q(6,q)$. The set of holes of S forms a $\{\delta v_3, \delta v_2; 6, q\}$-minihyper if $\delta < q$. If the minihyper is a disjoint union S' of planes, then S can be enlarged to a spread $S \cup S'$. By the maximality of $\mathcal{O}$, a plane π of S' cannot be a plane $y^\perp$, with $y \in H(q)$. We will show that S' contains an other plane π' which corresponds in a unique way to π [35]. he point set of π defines a set of q^2+q+1 points of $H(q)$ at mutual distance 4. By [86], there is a unique plane π' of $Q(6,q)$ with the property that the point set of π' is the set of points of $H(q)$ that are collinear with exactly $q+1$ points of π, and such a set of $q+1$ points of π is the point set of a line of π (all lines arise in this way). Note that, since there are $q+1$ lines of $H(q)$ through every

point of $H(q)$, every line of $H(q)$ containing a point of π' contains a point of π and vice versa. Assume by way of contradiction that a point z of π' belongs to a member η of S. Let $\eta = a^\perp$, with $a \in \mathcal{O}$. The line az contains a unique point of π, a contradiction. Hence all points of π' are contained in members of S', implying that $\pi' \in S'$.

Hence, we can partition the planes of S' into pairs, implying that δ is even.

Remark B.4.1 We now determine the upper bound on δ so that the previous reasoning can be done.

We have at most Baer subplanes contained in $H(q)$, and there are no projected $PG(8, \sqrt[3]{q})$ contained in $H(q)$, see Remark B.3.3.

Hence, by Theorem 5.5.8, only a disjoint union of planes occurs in the minihyper.

We have proven the following theorem.

Theorem B.4.2 *Assume $q = p^3$, $p = p_0^h$, $p_0 \geq 7$ prime, $h \geq 1$, $p \geq 9$. If $\mathcal{O}$ is a partial ovoid of $H(p^3)$ of deficiency $\delta \leq 2p^2 - 4p$, then δ is even.*

Remark B.4.3 We remark that $\{\delta v_{\mu+1}, \delta v_\mu; t, q\}$-minihypers also have been used by Govaerts, Storme and Van Maldeghem [35] to study

- partial μ-spreads of $W_{2n+1}(q)$, q even,
- partial μ-spreads of $W_{2\mu+1}(q)$, q odd,
- partial μ-spreads of $H(n, q^2)$,
- *μ-covers* of finite classical polar spaces and projective spaces,
- partial ovoids of finite classical polar spaces,

Appendix C

Nederlandstalige samenvatting

Een lineaire $[n, k, d; q]$ code C kan gedefinieerd worden aan de hand van een $(k \times n)$ *generatormatrix* $G = (g_1 \cdots g_n)$. Dit is een $(k \times n)$ matrix wiens rijen een basis voor C vormen. Bekijken we deze generatormatrix vanuit het standpunt van de kolommen, dan definiëren de kolommen $g_1, \ldots, g_n$ van G vectoren van de k-dimensionale vectorruimte $V(k, q)$ over het eindig veld $GF(q)$. Onderstel dat geen enkele kolom van G gelijk is aan de nulvector.

Als een kolom g_i in de generatormatrix G vervangen wordt door een niet-nul scalair veelvoud, dan ontstaat een *equivalente code*. Daarom is het mogelijk om een kolom van G, tezamen met al zijn niet-nul scalaire veelvouden te interpreteren als zijnde één object. Dit is precies wat gedaan wordt in de *projectieve meetkunde* [50, 51, 54]. De kolommen van G, tezamen met hun niet-nul scalaire veelvouden, worden geïnterpreteerd als *punten van de $(k-1)$-dimensionale projectieve ruimte $PG(k-1, q)$*. Het kan echter zijn dat op deze manier meerdere kolommen corresponderen met eenzelfde projectief punt. In dit geval spreken we van een multiset van punten. In dit geval wordt er gebruik gemaakt van een gewichtsfunctie w. De voorwaarde dat de lineaire code minimale afstand d heeft, wordt in de meetkundige formulering de voorwaarde dat een hypervlak onze multiset van punten in hoogstens $n - d$ punten snijdt.

In deze thesis hebben we onderzoek verricht op: (1) *lineaire codes die de Griesmer grens bereiken*, (2) lineaire codes die in verband staan met *kappen in eindige projectieve ruimten*.

C.1 Minihypers en lineaire codes die de Griesmer grens bereiken

Belangrijke codes, voor praktische toepassingen, zijn deze waarvoor de lengte n zo klein mogelijk is, want dit betekent dat er een klein aantal symbolen doorgestuurd moeten worden om een codewoord te verzenden, waardoor er vlugger informatie doorgestuurd kan worden. Voor vaste k, d, q wordt een theoretische ondergrens op de lengte van een code gegeven door de *Griesmergrens* [36, 76]: $n \geq g_q(k,d) = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$.

De vraag is of er werkelijk $[g_q(k,d), k, d; q]$ codes bestaan, en in het geval van existentie, deze codes te classificeren.

Dit codeertheoretisch probleem kan vertaald worden in een meetkundig probleem over *minihypers in projectieve ruimten.*

Definitie C.1.1 Een $\{f, m; t, q\}$*-minihyper* is een koppel (F, w), waarbij F een deelverzameling is van de puntenverzameling van $PG(t,q)$ en waarbij w een gewichtsfunctie is, $w : PG(t,q) \to \mathbb{N} : w \mapsto w(x)$, die voldoet aan de volgende voorwaarden

1. $w(x) > 0 \Leftrightarrow x \in F$,
2. $\sum_{x \in F} w(x) = f$, en
3. $\min\{|(F,w) \cap H| = \sum_{x \in H} w(x) | H \text{ is een hypervlak}\} = m$.

Indien de minihyper enkel bestaat uit enkelvoudige punten, dan behoort een punt x tot de minihyper dan en slechts dan als $w(x) = 1$, en we noteren de minihyper kortweg als F.

Het verband tussen minihypers en lineaire codes die de Griesmergrens bereiken, wordt als volgt bekomen.

Stel dat $G = (g_1 \cdots g_n)$ een generatormatrix is voor een lineaire $[n, k, d; q]$ code die de Griesmergrens bereikt ($d \geq 1, k \geq 3$). We kunnen d op een unieke manier schrijven als $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^{\lambda_i}$ zodat $\theta \geq 1$ en $0 \leq \epsilon_i < q$.

De Griesmergrens wordt dan $n \geq \theta v_k - \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}$ waarbij $v_l = (q^l - 1)/(q-1)$, voor natuurlijke $l \geq 0$.

We beschouwen de kolommen van G als coördinaten van een punt in $PG(k-1,q)$. Noteer de puntenverzameling van $PG(k-1,q)$ als $\{s_1, \ldots, s_{v_k}\}$. Stel $m_i(G)$ het aantal kolommen in G die het punt s_i bepalen. Stel $m(G) = \max(m_i(G) \mid i \in \{1, \ldots, v_k\})$, dan is $m(G) = \theta$ volledig bepaald door de code. Definieer een gewichtsfunctie $w : PG(k-1,q) \to \mathbb{N} : s_i \mapsto w(s_i) = \theta - m_i(G)$, $i = 1, 2, \ldots, v_k$. Definieer $F = \{s_i \in PG(k-1,q) \mid w(s_i) > 0\}$, dan is (F, w) een $\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k-1, q\}$-minihyper met gewichtsfunctie w.

Opmerking C.1.2 Indien we ons beperken tot $d < q^{k-1}$, dan bepalen alle kolommen van G verschillende projectieve punten; en dan vormt de verzameling $PG(k-1,q) \setminus \{g_1, \ldots, g_n\}$ de minihyper die correspondeert met de code.

In deze thesis classificeren we de minihypers door deze op te bouwen via de doorsneden met deelruimten. De bouwstenen hiervoor zijn zogenaamde *blocking sets.*

Definitie C.1.3 Een *t-voudige blocking set* in $PG(n,q)$ is een verzameling punten in $PG(n,q)$, zodat elk hypervlak van $PG(n,q)$ deze verzameling in minstens t punten snijdt. Indien $t=1$, spreken we kortweg van een *blocking set*. Een *triviale* blocking set in $PG(2,q)$ is een blocking set die een rechte bevat.

Een blocking set wordt *minimaal* genoemd als geen enkele echte deelverzameling ervan nog een blocking set vormt. De exponent e van een blocking set is het grootste natuurlijk getal zodat elke rechte de blocking set snijdt in 1 modulo p^e punten.

Een belangrijk resultaat over blocking sets is de volgende stelling.

Stelling C.1.4 (Polverino, Polverino en Storme [69, 70, 71]) *De kleinste minimale blocking sets in $PG(2,p^3)$, $p=p_0^h$, p_0 priem, $p_0 \geq 7$, met exponent $e \geq h$, zijn:*

(1) *een rechte,*
(2) *een Baer deelvlak,*
(3) *een $PG(3,p)$ geprojecteerd op $PG(2,p^3)$ vanuit een punt dat op een rechte ligt die $PG(3,p)$ snijdt in een deelrechte,*
(4) *een $PG(3,p)$ geprojecteerd op $PG(2,p^3)$ vanuit een punt dat enkel op raaklijnen aan $PG(3,p)$ ligt.*

Opmerking C.1.5 (1) De minimale blocking set van grootte p^3+p^2+1 heeft een uniek punt, de *top*, dewelke op exact $p+1$ rechten ligt die de blocking set snijden in p^2+1 punten; een dergelijke doorsnede wordt ook een (p^2+1)*-set* genoemd. Elke andere rechte snijdt de blocking set in 1 punt of in een $PG(1,p)$. Een (p^2+1)-set is equivalent met $\{\infty\} \cup \{x \in GF(p^3) \mid x + x^p + x^{p^2} = 0\}$. We noemen het punt dat correspondeert met ∞ het *speciale punt* van de (p^2+1)-set.

(2) De minimale blocking set van grootte p^3+p^2+p+1 heeft p^2+p+1 punten gemeen met juist één rechte; een dergelijke doorsnede wordt ook een (p^2+p+1)*-set* genoemd. Alle andere rechten snijden de blocking set in 1 punt of in een $PG(1,p)$. Een (p^2+p+1)-set is equivalent met $\{x \in GF(p^3) \mid x^{p^2+p+1} = 1\}$.

Uitbreiding van de resultaten van Hamada, Helleseth en Maekawa

Sterke resultaten voor algemene waarden van n,k,d en q zijn bekomen in [38, 44] door Hamada, Helleseth en Maekawa, die de volgende karakterisatiestelling over minihypers gevonden hebben.

Stelling C.1.6 (Hamada en Helleseth [38], Hamada en Maekawa [44]) *Stel F een $\{\sum_{i=0}^{s}\epsilon_i v_{i+1}, \sum_{i=0}^{s}\epsilon_i v_i; t, q\}$-minihyper, met $\sum_{i=0}^{s}\epsilon_i \leq \sqrt{q}$, dan is F de disjuncte unie van ϵ_s deelruimten $PG(s,q)$, ϵ_{s-1} deelruimten $PG(s-1,q),\ldots,\epsilon_0$ punten. Indien deze disjuncte unie niet bestaat, bestaat ook de minihyper niet.*

Opmerking C.1.7 De blocking sets die optreden in de bewijzen van de stellingen beschreven in Stelling C.1.6 zijn de rechten.

In Hoofdstuk 2 hebben we deze resultaten verbeterd door de bovengrens $h = \sum_{i=0}^{s} \epsilon_i < \sqrt{q}+1$ te vergroten. Hierbij hielden we ook rekening met Baer deelvlakken als blocking sets.

Stelling C.1.8 (Ferret en Storme [30]) *Stel F een $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper, q een kwadraat, met $\sum_{i=0}^{s} \epsilon_i \leq \min(2\sqrt{q}-1, c_p q^{5/9})$, $c_p = 2^{-1/3}$, en $q = p^f$, $p = 2, 3$, $q \geq 2^{14}$, en waarbij $\sum_{i=0}^{s} \epsilon_i \leq \min(2\sqrt{q}-1, q^{6/9}/(1+q^{1/9}))$ als $q = p^f$, p priem, $p > 3$, $q \geq 2^{12}$.*

Dan bestaat F uit de disjuncte unie van ofwel:
(1) ϵ_s deelruimten $PG(s,q), \epsilon_{s-1}$ deelruimten $PG(s-1,q), \ldots, \epsilon_0$ punten,
(2) één deelmeetkunde $PG(2l+1, \sqrt{q})$, voor een geheel getal l met $1 \leq l \leq s$, ϵ_s deelruimten $PG(s,q), \ldots, \epsilon_{l+1}$ deelruimten $PG(l+1,q), \epsilon_l - \sqrt{q} - 1$ deelruimten $PG(l,q), \epsilon_{l-1}$ deelruimten $PG(l-1,q), \ldots, \epsilon_0$ punten,
(3) één deelmeetkunde $PG(2l, \sqrt{q})$, voor een geheel getal l met $1 \leq l \leq s$, ϵ_s deelruimten $PG(s,q), \ldots, \epsilon_{l+1}$ deelruimten $PG(l+1,q), \epsilon_l - 1$ deelruimten $PG(l,q), \epsilon_{l-1} - \sqrt{q}$ deelruimten $PG(l-1,q), \epsilon_{l-2}$ deelruimten $PG(l-2,q), \ldots, \epsilon_0$ punten.

Voor q een niet-kwadraat kunnen deelmeetkunden over $GF(\sqrt{q})$ niet optreden. Dit impliceert dat met de technieken gebruikt door ons een grotere verbetering bekomen kon worden voor Stelling C.1.6. Dit resultaat wordt vermeld in de volgende stelling.

Stelling C.1.9 (Ferret en Storme [30]) *Stel F een $\{\sum_{i=0}^{s} \epsilon_i v_{i+1}, \sum_{i=0}^{s} \epsilon_i v_i; t, q\}$-minihyper, $t \geq 3$, waarbij*
(1) $\sum_{i=0}^{s} \epsilon_i \leq q^{6/9}/(1+q^{1/9})$, $q = p^f$, f oneven, p priem, $p > 3$, $q \geq 661$,
(2) $\sum_{i=0}^{s} \epsilon_i \leq c_p q^{5/9}$, $q = p^f$, f oneven, $p = 2, 3$, $q > 7687$, $c_p = 2^{-1/3}$.

Dan bestaat F uit de disjuncte unie van ϵ_s deelruimten $PG(s,q)$, ϵ_{s-1} deelruimten $PG(s-1,q)$, $\ldots$, ϵ_0 punten.

In de codeertheoretische formulering classificeren deze stellingen dus de corresponderende $[n = v_k - \sum_{i=0}^{s} \epsilon_i v_{i+1}, k = t+1, d = q^{k-1} - \sum_{i=0}^{s} q^i \epsilon_i; q]$ codes die de Griesmergrens bereiken.

Minihypers en maximale partiële spreads

Minihypers hebben tevens rechtstreekse toepassingen in de studie van maximale partiële spreads in $PG(N,q)$.

Definitie C.1.10 Een *t-spread* in $PG(N,q)$ is een verzameling van t-dimensionale deelruimten die de puntenverzameling van $PG(N,q)$ partitioneert.
Een *partiële t-spread* in $PG(N,q)$ is een verzameling van paarsgewijs disjuncte t-dimensionale deelruimten in $PG(N,q)$.
Een partiële t-spread wordt *maximaal* genoemd indien deze niet kan uitgebreid

worden tot een groter partiële t-spread.
Een *hole* van een partiële t-spread S is een punt van $PG(N,q)$ dat niet bevat is in een element van S.

Als $t=1$, dan spreken we kortweg van een partiële spread.

De volgende resultaten van Metsch en Storme vormden de basis voor een verdere uitbreiding.

Stelling C.1.11 (Metsch en Storme [60]) *Stel δ een natuurlijk getal en q een kwadraat en een priemmacht, $q>4$, zodat*

(1) $2\delta \leq q+1$,

(2) *een blocking set in $PG(2,q)$, met ten hoogste $q+\delta$ punten, een rechte of een Baer deelvlak bevat.*

Stel S een maximale partiële spread van $PG(3,q)$ met $q^2+1-\delta$ rechten, dan

(a) $\delta = s(\sqrt{q}+1)$ *voor een natuurlijk getal $s \geq 2$,*

(b) *de verzameling van holes is de disjuncte unie van s Baer deelmeetkunden $PG(3,\sqrt{q})$.*

Stelling C.1.12 (Metsch en Storme [60]) *Stel S een maximale partiële spread van $PG(3,q)$, $q=p^3$, $p=p_0^h$, $h\geq 2$, h even, p_0 priem, $p_0 \geq 7$, met deficiëntie $0<\delta \leq p^2+p+1$. Dan is $\delta \equiv 0 \pmod{p^{3/2}+1}$, $\delta \geq 2(p^{3/2}+1)$, en de verzameling van holes is de disjuncte unie van deelmeetkunden $PG(3,p^{3/2})$, of $\delta = p^2+p+1$ en de verzameling van holes is een geprojecteerde deelmeetkunde $PG(5,p)$ in $PG(3,p^3)$.*

Stelling C.1.13 (Metsch and Storme [60]) *Stel S een maximale partiële spread van $PG(3,q)$, $q=p^3$, $p=p_0^h$, $h\geq 1$, h oneven, p_0 priem, $p_0\geq 7$, met deficiëntie $0<\delta\leq p^2+p+1$. Dan is $\delta = p^2+p+1$ en de verzameling van holes is een geprojecteerde deelmeetkunde $PG(5,p)$ in $PG(3,p^3)$.*

In Hoofdstuk 3 classificeerden wij $\{\delta(q+1),\delta;3,q\}$-minihypers afkomstig van maximale partiële spreads in $PG(3,q)$, $q=p^3$.

Voor $q=p^3$, p priem, $p\geq 17$, stelt δ_0 het grootste geheel getal kleiner dan $(3p^3+27p^2-5p+25)/25$ voor. Voor $p=7,11,13$, is dit respectievelijk $\delta_0=90, \delta_0=285$ en $\delta_0=441$.

Voor $q=p^3$, $p=p_0^h$, p_0 priem, $p_0\geq 7$, $h>1$, is δ_0 het grootste geheel getal, kleiner dan $(3p^3+27p^2-5p+25)/25$ en kleiner dan de waarde δ' voor dewelke $p^3+\delta'$ de kardinaliteit is van de kleinste niet-triviale minimale blocking set in $PG(2,p^3)$ van kardinaliteit groter dan p^3+p^2+p+1. Het is gekend dat $\delta'\leq p^3/p_0+1$.

Stelling C.1.14 (Ferret en Storme [29]) *Stel $p=p_0^h$, $p_0\geq 7$ priem. Dan is de minihyper corresponderend met een maximale partiële spread in $PG(3,p^3)$ met deficiëntie $0<\delta\leq\delta_0$ de disjuncte unie van $PG(3,p^{3/2})$ en van geprojecteerde $PG(5,p)$ met kardinaliteit $p^5+p^4+p^3+p^2+p+1$.*

Opmerking C.1.15 In de minihypers van Stelling C.1.14 treden alle blocking sets op, beschreven in Stelling C.1.4.

Hierna volgen de verbeteringen voor de resultaten uit Stellingen C.1.11 C.1.12 en C.1.13.

Stelling C.1.16 (Ferret en Storme [29]) *Onder de voorwaarden van Stelling C.1.14, is de deficiëntie δ van een maximale partiële spread in $PG(3,p^3)$ te schrijven als $\delta = r(p^{3/2}+1) + s(p^2+p+1)$ voor bepaalde natuurlijke getallen r en s, met $(r,s) \neq (1,0)$.*

Minihypers met gewichten

In Hoofdstuk 4 classificeren we alle $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, $\delta \leq 2p^2-4p$, $p = p_0^h \geq 9$, $h \geq 1$, voor een priem $p_0 \geq 7$. Een dergelijke minihyper is de som van rechten en hoogstens één geprojecteerde deelmeetkunde $PG(5,p)$, of een som van rechten en een minihyper die een geprojecteerde deelmeetkunde $PG(5,p)$ is, waarbij de gewichten van de punten op een rechte van deze geprojecteerde $PG(5,p)$ met 1 verminderd zijn. Als p een kwadraat is, dan kunnen ook (mogelijks geprojecteerde) Baer deelmeetkunden $PG(3,p^{3/2})$ voorkomen.

We zullen nu dergelijke geprojecteerde deelmeetkunden $PG(5,p)$ beschrijven.

Beschouw een deelmeetkunde $\Lambda = PG(5,p)$ ingebed op natuurlijke wijze in $PG(5,p^3)$. Stel L een rechte van $PG(5,p^3)$ scheef aan Λ. Dan heeft L twee toegevoegde rechten ten opzichte van Λ; we noteren deze met L^p en L^{p^2}.

Geval 1.
Stel dat Ω de projectie is van Λ vanuit een rechte L met $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Dan heeft elk geprojecteerd punt s in Ω gewicht één, een rechte van $PG(3,p^3)$ die Ω in minstens twee punten snijdt, heeft een $PG(1,p)$ of een (p^2+p+1)-set met Ω gemeen (zie Opmerking C.1.5). Verder snijdt een vlak Ω in ofwel een (p^2+p+1)-set, een deelvlak $PG(2,p)$, of in een minimale blocking set van grootte p^3+p^2+p+1.

Geval 2.
Stel dat Ω de projectie is van Λ vanuit een rechte L met $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

De 4-dimensionale ruimte $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ wordt de *speciale 4-ruimte* van Λ genoemd. We noteren deze speciale 4-ruimte $\langle L, L^p, L^{p^2}\rangle \cap \Lambda$ met $\mathcal{P}$.

Voor precies één punt r op L is $\dim\langle r, r^p, r^{p^2}\rangle = 1$. De rechte $M = \langle r, r^p, r^{p^2}\rangle$ wordt geprojecteerd vanuit L op een punt m van Ω met gewicht $p+1$.

Elk vlak π in Λ door M, en niet bevat in $\mathcal{P}$, wordt geprojecteerd vanuit L op een (p^2+1)-set met speciaal punt m. Een dergelijk vlak π ligt in p^2+p+1

solids van Λ dewelke geprojecteerd worden op minimale blocking sets van grootte p^3+p^2+1.

Stel s een punt van Ω niet gelegen in de speciale 4-ruimte van Ω. Stel dat s de projectie is van $s' \in \Lambda$. Elke solid $\langle x, x^p, x^{p^2}, s'\rangle$, met $x \in L \setminus M$, wordt geprojecteerd vanuit L op een planaire minimale blocking set van grootte p^3+p^2+p+1. Elke solid van Λ door M en s' is geprojecteerd op een planaire minimale blocking set van grootte p^3+p^2+1.

Geval 3.
Stel dat Ω de projectie is van Λ vanuit een rechte L met $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

Stel $\mathcal{P} = \langle L, L^p, L^{p^2}\rangle \cap \Delta$ de speciale 3-ruimte van Λ.

De projectie van $\mathcal{P}$ is een rechte N van $PG(3,p^3)$. Er zijn $p+1$ scheve rechten $L_1,\ldots,L_{p+1}$ in $\mathcal{P}$ die geprojecteerd worden op punten met gewicht $p+1$, de overige p^3-p punten van $\mathcal{P}$ worden geprojecteerd op punten met gewicht 1 op de rechte N.

Een punt s' van $\Lambda \setminus \mathcal{P}$ wordt geprojecteerd op een punt s gelegen op $p+1$ (p^2+1)-sets van Ω. Een dergelijke (p^2+1)-set door s ligt in p^2 vlakken van $PG(3,p^3)$ die Ω snijden in een minimale blocking set van grootte p^3+p^2+1.

Door N zijn er $p+1$ vlakken van $PG(3,p^3)$ die $p^4+p^3+p^2+p+1$ geprojecteerde punten van Λ bevatten.

De projectie vormt dus een $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper (Ω, w) die een rechte N bevat. Indien we de gewichten van elk punt op N met 1 verminderen, dan bekomen we een $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$.

Geval 4.
Stel dat Ω de projectie is van Λ vanuit een rechte L met $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

De projectie is een kegel van p^2+p+1 rechten; de top van de kegel is een punt met gewicht p^2+p+1, en de basis van de kegel is een deelvlak $PG(2,p)$.

Definitie C.1.17 Het *exces* van een punt van een minihyper (F, w) is het gewicht van dit punt min 1.

De volgende stellingen vormen de hoofdresultaten van Hoofdstukken 4 en 5.

Stelling C.1.18 (Ferret en Storme [26]) *Een $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper (F, w), p geen kwadraat, $p = p_0^h$, p_0 priem, $h \geq 1$, $p \geq 9$, $p_0 \geq 7$, $\delta \leq 2p^2-4p$, en met totale exces $e \leq p^3$, is ofwel:*
(1) een som van rechten en hoogstens één geprojecteerde $PG(5,p)$, geprojecteerd vanuit een rechte L waarvoor $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) een som van rechten en een $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$, waar Ω een geprojecteerde $PG(5,p)$ is, geprojecteerd vanuit een rechte

L waarvoor $\dim\langle L, L^p, L^{p^2}\rangle = 3$*, en waar N de rechte voorstelt die bevat is in* Ω.

Stelling C.1.19 (Ferret en Storme [26]) *Een $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper (F, w), $p = p_0^h$, p_0 priem, $h \geq 2$ even, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, en met totale exces $e \leq p^3$, is ofwel:*
(1) een som van rechten, (geprojecteerde) $PG(3, p^{3/2})$, en hoogstens één geprojecteerde $PG(5, p)$, geprojecteerd vanuit een rechte L waarvoor $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,
(2) een som van rechten, (geprojecteerde) $PG(3, p^{3/2})$, en een $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$, waar Ω een geprojecteerde $PG(5, p)$ is, geprojecteerd vanuit een rechte L waarvoor $\dim\langle L, L^p, L^{p^2}\rangle = 3$*, en waar N de rechte voorstelt die bevat is in Ω.*

Deze resultaten werden vervolgens uitgebreid in Hoofdstuk 5 naar willekeurige dimensies en waarbij de orde van (F, w) gelijk is aan $\delta v_{\mu+1}$.

Stelling C.1.20 (Ferret en Storme [27]) *Een $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper (F, w), $N \geq 4$, $p = p_0^h$, p_0 priem, $h \geq 1$, $p_0 \geq 7$, $\delta \leq 2p^2 - 4p$, $p \geq 9$, en met totale exces $e \leq p^3 - 4p$, is de som van ofwel:*
(1) rechten, (geprojecteerde) $PG(3, p^{3/2})$ (waarbij de projectie vanuit een punt is), en ten hoogste één (geprojecteerde) $PG(5, p)$,
(2) rechten, (geprojecteerde) $PG(3, p^{3/2})$, en een $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $(\Omega \setminus N, w')$, waarbij Ω een $PG(5, p)$ is, geprojecteerd vanuit een rechte L waarvoor $\dim\langle L, L^p, L^{p^2}\rangle = 3$*, en waarbij N de rechte is die bevat is in Ω.*

Stelling C.1.21 (Ferret en Storme [27]) *Stel (F, w) een $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 2$, $\delta \leq 2p^2 - 4p$, $N \geq 3$, $p = p_0^h$, $h \geq 1$, $p \geq 9$, $p_0 \geq 7$ priem, met totale exces $e \leq p^2 + p$.*

Dan is (F, w) een som van μ-dimensionale deelruimten $PG(\mu, p^3)$, (geprojecteerde) $PG(2\mu+1, \sqrt{q})$, en ten hoogste één (geprojecteerde) deelmeetkunde $PG(3\mu+2, p)$.

Opmerking C.1.22 De resultaten over deze $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihypers hebben vele toepassingen in projectieve ruimten; zie Appendix B. Zo impliceren deze stellingen nieuwe resultaten over
(1) maximale partiële μ-spreads in $PG(N, p^3)$,
(2) maximale partiële μ-spreads in eindige klassieke polaire ruimten, en
(3) maximale partiële ovoiden in de klassieke veralgemeende 6-hoek $H(p^3)$.

C.2 Kappen en kapcodes

Definitie C.2.1 De *pariteit controlematrix* van een lineaire code C is een generatormatrix van de duale code $C^\perp = \{z | z.c = 0, \forall c \in C\}$.

Eén van de belangrijkste stellingen die verbanden beschrijven tussen lineaire codes en eindige projectieve ruimten, is de stelling die *de fundamentele stelling van de codeertheorie* genoemd wordt.

Stelling C.2.2 (MacWilliams en Sloane [59]) *Stel C een lineaire code over $GF(q)$ van lengte n en cimensie k, met pariteit controlematrix $H = (h_1 \cdots h_n)$.*

Dan is de minimale afstand van C gelijk aan d als en slechts als elke $d-1$ kolommen van H lineair onafhankelijk zijn, en er minstens één stel van d lineair afhankelijke kolommen in H zijn.

Bij lineaire codes waarvoor $d \geq 4$ zijn elke drie kolommen van een pariteit controlematrix H lineair onafhankelijk; en omgekeerd. De kolommen van H vormen dus een verzameling van n punten van $PG(n-k-1,q)$, waarvan geen drie van deze punten lineair afhankelijk zijn.

In de theorie van de eindige projectieve ruimten wordt een verzameling van n punten van $PG(N,q)$, waarvan geen drie punten lineair afhankelijk zijn, een n-*kap* genoemd [54].

Op analoge wijze is een k-*kap* K in de affiene ruimte $AG(N,q)$ een verzameling van k punten in $AG(N,q)$ waarvan geen drie op een rechte gelegen zijn.

Een k-kap in $AG(N,q)$, respectievelijk $PG(N,q)$, is *compleet* wanneer ze niet meer kan uitgebreid worden tot een grotere kap van $AG(n,q)$, respectievelijk $PG(n,q)$.

Het hoofdprobleem in de theorie van kappen is het vinden van de grootst mogelijke kappen in $AG(N,q)$ of $PG(N,q)$. De grootte van de grootste kappen in $PG(N,q)$ wordt aangeduid met $m_2(N,q)$.

Als $n > 3$ en $q > 2$, dan zijn slechts de volgende waarden gekend.

(1) De maximale grootte van een kap in $AG(4,3)$ en in $PG(4,3)$ is 20 [66].
(2) De maximale grootte van een kap in $PG(5,3)$ is 56 [46].
(3) De maximale grootte van een kap in $PG(4,4)$ is 41 [20].
(4) De maximale grootte van een kap in $AG(4,4)$ is 40 [19].

Betreffende de karakterisaties van deze kappen hebben we de volgende resultaten.

(1) Er bestaat exact één 20-kap in $AG(4,3)$ en er bestaan precies 9 types van 20-kappen in $PG(4,3)$ [48].
(2) De 56-kap in $PG(5,3)$ is projectief uniek [47].
(3) Er bestaan juist 2 verschillende types 41-kappen in $PG(4,4)$ [18, 20].
(4) Er bestaat exact één 40-kap in $AG(4,4)$ [19].

In Hoofdstuk 6 bewezen wij de volgende stelling.

Theorem C.2.3 (Edel, Ferret, Landjev en Storme [21]) *De maximale orde van een kap in $AG(5,3)$ is 45, en elke 45-kap in $AG(5,3)$ wordt bekomen door een*

11-hypervlak weg te laten uit een 56-kap in $PG(5,3)$. Bovendien is er een uniek type 45-kap in $AG(5,3)$.

In de gevallen waarin de exacte grootte van de grootste kap niet gekend is, wordt gezocht naar bovengrenzen voor deze waarden. De ondergrenzen voor de groottes van kappen worden gegeven door constructies van zo groot mogelijke kappen.

Definitie C.2.4 De waarde $m_2'(N,q)$ is de grootte van de grootste kappen in $PG(N,q)$ die niet bevat zijn in een $m_2(N,q)$-kap van $PG(N,q)$. Dus elke k-kap in $PG(N,q)$, met grootte k groter dan $m_2'(N,q)$, is een deel van een $m_2(N,q)$-kap.

Het belang van deze waarde $m_2'(N,q)$ is dat ze kan gebruikt worden om resultaten te bekomen over de grootste kappen in hogere dimensies.

Door een resultaat van Chao [15], is $m_2'(3,q=2^h) \leq q^2-q+5$, voor $q \geq 8$.

In hoofdstuk 7 vonden we een verbeterde bovengrens op $m_2'(3,q)$, q even.

Stelling C.2.5 (Ferret en Storme [28]) *In $PG(3,2^h)$, $h \geq 4$, is een kap van grootte minstens q^2-q+3 steeds uitbreidbaar tot een (q^2+1)-kap.*

De volgende stelling geeft disjuncte intervallen voor de kardinaliteit k van een k-kap waarvoor geen complete k-kappen kunnen bestaan.

Stelling C.2.6 (Storme en Szőnyi [77]) *Er bestaat geen complete k-kap K in $PG(3,q)$, q even, $q \geq 64$, met*

$$k \in \left[q^2-(a-1)q+a\sqrt{q}+2-a+\binom{a}{2}, q^2-(a-2)q-a^2\sqrt{q}\right]$$

en met a een natuurlijk getal dat voldoet aan

$$2 \leq a \leq \frac{-2\sqrt{q}+3+\sqrt{16q\sqrt{q}+12q-44\sqrt{q}-7}}{4\sqrt{q}+2}.$$

Door op te merken dat vlakken dergelijke kappen snijden in *weinig* of in *veel* punten (cfr. de ideeën van Chao [15]) hebben we bovenstaande stelling kunnen verbeteren tot de volgende stelling.

Stelling C.2.7 (Ferret en Storme [28]) *Er bestaat geen complete k-kap K in $PG(3,q)$, q even, $q \geq 1024$, met*

$$k \in \left[q^2-(c-1)q+(2c^3+c^2-5c+6)/2, q^2-(c-2)q-2c^2+3c\right]$$

en met c een geheel getal dat voldoet aan $2 \leq c \leq \sqrt[4]{q}$.

De grootte van deze intervallen voor k waarvoor het nog niet geweten was of er complete k-kappen bestonden, werd dus vroeger bepaald in functie van $\sqrt{q}$; terwijl de nieuwe intervallen slechts een beperkte vaste grootte hebben; onafhankelijk van q.

Een klasse van twee-gewichtscodes die de Griesmergrens bereiken

In Appendix A beschrijven we een constructie van een nieuwe klasse van twee gewichts $[(3q^2+1)/2, 3, (3q^2-3q)/2; q]$-codes, voor q oneven, met gewichten $(3q^2-3q)/2$ en $(3q^2-q)/2$, die bovendien de Griesmergrens bereiken.

Constructie C.2.8 We construeren generator matrices $G = (g_1 \dots g_n)$ van $[(3q^2+1)/2, 3, (3q^2-3q)/2; q]$-codes, q oneven, met gewichten $(3q^2-3q)/2$ en $(3q^2-q)/2$.
Deze codes bereiken de Griesmergrens en ze zijn nieuw voor $q > 5$.

Vat de kolommen van een generator matrix op als een multiset MS van punten in $PG(2,q)$. We construeren MS.

Beschouw een duale $((q+3)/2)$-kap K in $PG(2,q)$. Definieer MS als volgt. Neem de punten die op geen enkele rechte van K liggen met multipliciteit twee en neem de punten die op één rechte van K liggen met multipliciteit één.

Een *sterk reguliere graaf met parameters* (N, K, λ, μ) is een graaf met N toppen, valentie K, zodat voor twee adjacente toppen, er λ toppen zijn die adjacent zijn met beide toppen; en voor twee niet-adjacente toppen, er μ toppen zijn die adjacent zijn met beide toppen.

Dergelijke grafen zijn veel bestudeerd en hebben vele toepassingen. We verwijzen hiervoor naar [55].

Als er een projectieve twee-gewichts $[n, k, d; q]$-code bestaat met gewichten d en w, dan geeft ze aanleiding tot een sterk reguliere graaf met parameters $N = q^k$, $K = n(q-1)$, $\lambda = K^2 + 3K - q(K+1)(w+d) + q^2wd$ en $\mu = q^{2-k}wd$.

Tot besluit geven we nog een partieel resultaat over $[(3q^2+3)/2, 4, (3q^2-3q)/2; q]$-codes C, met gewichten $(3q^2-3q)/2$ and $(3q^2-q)/2$. Deze codes zijn projectief en bereiken de Griesmergrens, en als ze bestaan, dan hebben ze als verkorte codes, de zonet geconstrueerde 3-dimensionale twee-gewichtscodes.

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