Maximal Arcs in Finite Projective Planes and Associated Structures in Projective Spaces

Nicholas Hamilton

This thesis is presented for the degree of
Doctor of Philosophy
of The University of Western Australia
Department of Mathematics
June 1995

Maximal Arcs in Finite Projective Planes and
Associated Structures in Projective Spaces

SUMMARY

The aim of this thesis has been to answer various questions concerning construction, embedding and characterisation of maximal arcs and associated structures in finite projective planes.

The work of this thesis is divided into five chapters and one appendix. As well as original work the main known results on maximal arcs are given. The core results of the thesis are the constructions of Chapters 3 and 5, as well as the characterisation of Thas 1974 maximal arcs in Chapter 2. In the following, apart from where clearly stated, the results are my own and original.

Chapter 1: Introduction to Maximal Arcs in Projective Planes and Associated Structures

This chapter is an introduction to the study of maximal arcs. Maximal arcs are defined and various well known combinatorial properties are proved. Existence and non-existence results for parameter sets of maximal arcs are listed. The constructions of partial geometries, balanced incomplete block designs and some of their properties are outlined. The bulk of this chapter consists of well known results, though some new (minor) theorems on embedding of certain associated partial geometries and the order of the collineation stabiliser of a maximal arc in a Desarguesian plane are also proved.

Chapter 2: Known Constructions of Maximal Arcs

This chapter is an investigation of the constructions and characterisations of maximal arcs known prior to this thesis (apart from hyperovals which have a literature of their own.) The full collineation stabiliser of Denniston maximal arcs is calculated in a Theorem by myself and T. Penttila, and we also characterise the maximal arcs constructed by J.A. Thas in 1974. In this chapter it is also proved that a construction due to Thas in 1980, when applied to Desarguesian planes, gives rise to Denniston maximal arcs. Maximal arcs in Desarguesian planes whose collineation stabiliser is transitive on the points of the maximal arc are investigated and shown not to exist for a large number of parameters. Finally, the concept of a *linear* maximal arc is introduced with a view to providing a framework to study known examples, and perhaps give new constructions in the future.

Chapter 3: Inherited Maximal Arcs

In this chapter a number of new constructions of maximal arcs in projective planes are given. Firstly a class of maximal arcs is constructed in certain derived dual translation planes that are inherited from the duals of the Thas 1974 maximal

arcs. In particular this gives maximal arcs in certain derived duals of the Lüneburg planes, as well as the Hall planes of even order. An alternative proof of the maximal arcs in the Hall planes of even order is then given. A class of maximal arcs in certain derived duals of the Hall planes of even order is also constructed via inheriting.

Chapter 4: More Maximal Arcs and Associated Structures

New structures in projective planes are generated using Denniston maximal arcs. Disjoint degree 4 Denniston maximal arcs are found and classified in $PG(2, 16)$, and an example of disjoint degree 8 maximal arcs in $PG(2, 64)$ is given. These results complete the spectral problem for existence of sets of type (m, n) in $PG(2, 16)$. A conjecture is also made about the existence of inherited degree 4 maximal arcs in Hall planes of even order. The conjecture has been shown to be true (by computer) for Hall planes of order 16, 64 and 256. A number of new degree 4 maximal arcs in certain projective planes of order 16 are also exhibited and pairs of disjoint degree 4 maximal arcs are also shown to exist in certain of these planes.

Chapter 5: Hyperovals and Unitals in Figueroa Planes

The main result (due to myself and M.J. de Resmini) is to construct hyperovals in the finite Figueroa planes of even order. I also show that all finite Figueroa planes (of square order) admit a unitary polarity and hence contain unitals. The final result of the chapter is to classify polarities of the finite Figueroa planes.

Appendix: Inherited Unitals in Derived Dual Translation Planes

The work of this appendix arose from applying the techniques developed for inheriting maximal arcs to unitals. The main result (by T. Penttila and myself) is to show that if a Buekenhout-Metz unital exists in a translation plane, then there exist unitals in certain derived duals of the translation plane. Unfortunately, at the time the work for the appendix was submitted for publication to a journal it was discovered that the results had been previously published, and so the paper was withdrawn. But since the work is a natural extension of the ideas developed for maximal arcs it seemed worth including it here. The appendix is in its original form as a paper and is intended to be independent of the rest of the thesis.

TABLE OF CONTENTS

Preface

In 1955, A. Barlotti introduced the idea of a *maximal arc* in a projective plane. Originally maximal arcs were studied as a combinatorial extremal problem, and were also of interest in the study of algebraic curves. But later with the papers of W.D. Wallis and J.A. Thas (see Chapter 1), maximal arcs were shown to give rise to many other interesting structures such as designs and partial geometries. Constructions by J.A. Thas also show connections with ovoids, quadrics and polar spaces.

Apart from hyperovals and their duals, which have an extensive literature of their own, prior to this thesis only 3 constructions of maximal arcs were known; one due to R.H.F. Denniston published in 1969, and two due to J.A. Thas published in 1974 and 1980, respectively (see Chapter 2). All their constructions occur in projective planes of even order. In 1974, J.A. Thas proved that degree 3 maximal arcs do not exist in Desarguesian projective planes of odd order, and has also conjectured that (non-trivial) maximal arcs do not exist in Desarguesian planes of odd order. Prior to this thesis, these results were essentially the only ones known concerning construction and existence of maximal arcs. And the only characterisation was one of Denniston maximal arcs by V. Abatangelo and B. Larato.

This thesis aims to provide a comprehensive study of maximal arcs in projective planes. A number of new constructions and characterisations are given, and all the main known theorems are presented and most of them are proved in the text.

Fundamental to my approach to finding new constructions are the ideas of derivation, inheriting and embedding. Many techniques of deriving projective planes to get new projective planes are known and have been studied extensively. For example, the Hall planes can be derived from Desarguesian planes by replacing certain parallel classes of lines with Baer subplanes. An interesting question is: when does the derivation process preserve a particular substructure? For example, can the set of points of a maximal arc in one plane derive to the set of points of a maximal arc in another? This process is often known as inheriting.

This approach has turned out to be quite fruitful, and forms the bulk of the thesis. By starting with the Thas 1974 construction of maximal arcs, inheriting techniques are used to construct maximal arcs in the Hall planes and derived dual Lüneburg planes of even order (Chapter 3.) The maximal arcs in the Hall planes so constructed are also shown to inherit to maximal arcs in the derived dual Hall planes (Chapter 3.) Apart from hyperovals and their duals, prior to this thesis no examples of maximal arcs were known in planes that were not translation planes or dual to translation planes. The results of Chapter 3 give such examples. Similar techniques are used to construct hyperovals in the Figueroa planes of even order that are inherited from regular hyperovals in Desarguesian planes (Chapter 5.) Unitals are also constructed in the Figueroa planes by showing that a unitary polarity of a Desarguesian projective plane can inherit to a unitary polarity of the Figueroa plane (in fact polarities of Figueroa planes are classified in the thesis and shown to be essentially certain inherited polarities from the Desarguesian planes.)

Though it is not very apparent in the thesis itself, computers were used quite extensively during its preparation. The approach has been to use computers as an experimental tool. For instance, virtually all of the constructions mentioned in the previous paragraph were found by first making an educated guess as to what might be true, then checking by computer the first few cases, and finally guessing a generalisation and finding a computer free proof of the generalisation. The advantage in this is that it cuts down the time between making a conjecture and finding the generalisation by weeding out the bad guesses swiftly. With the exception of Chapter 4 all of the new results proved are computer free. Chapter 4 presents a large number of new examples of maximal arcs (and disjoint maximal arcs) constructed by computer in planes of order 16, as well as some of maximal arcs of degree 4 in certain Hall planes. The hope is that in the future it will be possible to generalise certain of these, and that by presenting a large number of examples conjectures that might be made about the structure of maximal arcs can be tested.

The other main part of the thesis has been understanding the constructions known prior to the thesis (Chapter 2). The Thas 1980 maximal arcs, when they occur in Desarguesian projective planes, are shown to be of Denniston type. The Thas 1974 maximal arcs are characterised by T. Penttila and myself. This, together with Abatangelo and Larato's result on Denniston maximal arcs, means that there now exist characterisations of all the classes of known maximal arcs in Desarguesian projective planes (apart from hyperovals and their duals.) The full collineation stabilisers are also calculated for the Denniston and Thas constructions when they occur in Desarguesian projective planes. The idea of a *linear* maximal arc in a translation plane is also introduced. Linear maximal arcs unify the study of the Denniston and both Thas constructions by showing that they are all associated with cones subtended by subspaces of quadrics in projective spaces. Though this idea is not fully developed in the thesis it does raise some interesting questions and possible approaches to new constructions in the future.

In summary, the thesis aims to provide a comprehensive study of construction and characteristion of maximal arcs in projective planes.

In the text, apart from where otherwise attributed, the results are my own.

The thesis is divided into chapters, with sections and subsections. The n^{th} Theorem stated in Section s of Chapter c is referred to as Theorem $s.n$ within Chapter c, and as Theorem $c.s.n$ in other chapters. Similarly for Lemmas.

Acknowledgments

I would like to thank Tim Penttila who has been a good friend for many years and has been the best supervisor a PhD. student could have.

Gordon Royle and Ivano Pinneri deserve extra special thanks for too many things to list here. They along with Tim and myself were the founding members of the Combinatorial Computing Research Group in 1992. Being a part of such a strong and enthusiastic group made the whole process of becoming a researcher lively and exciting. It's a shame that Ivano and I have leave the group now to move onto other places.

During my canditature I was fortunate enough to meet many geometers from around the world. It amazes me that one area of mathematics can contain so many nice people. In particular, Marialuisa de Resmini's enormous enthusiasm, encouragement and occasional "pestering" have been an inspiration.

Jillian Duffield also deserves a lot of thanks, firstly because she's the best person I know, and secondly for helping keep me sane in some of the darker moments of this thesis.

And, of course, I would like to thank my parents, Kerttu and Ian, who knew I could do it before I did.

Chapter 1
INTRODUCTION TO MAXIMAL ARCS IN PROJECTIVE PLANES AND ASSOCIATED STRUCTURES

1. INTRODUCTION

This chapter is divided into two sections.

In Section 2, maximal arcs and other geometric sets in projective planes are introduced. In subsection 2.1, maximal arcs are defined and various combinatorial properties are proved. In subsection 2.2, known existence and non-existence results for parameter sets of maximal arcs are listed. In subsection 2.3, maximal arcs are seen in the context of the study of k-sets of type (m, n) in projective planes. Subsequently in the thesis Lemma 2.6 will prove to be surprisingly useful.

In Section 3, some constructions of geometric structures that can be obtained from maximal arcs are given. In subsection 3.1, partial geometries and strongly regular graphs are introduced, and it is shown that maximal arcs can be used to construct both. Some (new) embedding and automorphism results are proved for one of the partial geometry constructions. In subsection 3.2, balanced incomplete block designs (BIBDs) are defined. Maximal arcs are shown to lead to BIBDs, and some interesting work by D.R. Stinson on the properties of these designs is outlined.

In the large part this chapter is a review of known results.

2. MAXIMAL ARCS IN PROJECTIVE PLANES

2.1 Combinatorial properties of maximal arcs in projective planes

In a finite projective plane of order q, a $\{k; n\}$**-arc** $\mathcal{K}$ is a non-empty proper subset of k points of the plane such that some line of the plane meets $\mathcal{K}$ in n points, but no line meets $\mathcal{K}$ in more than n points, where $2 \le n \le q$ [11] [78]. A line that meets a $\{k; n\}$-arc in m points is known as an m**-secant**.

LEMMA 2.1 (Tallini Scafati [124]) *If $\mathcal{K}$ is a $\{k; n\}$-arc in a projective plane of order q, then $k \le (q + 1)(n - 1) + 1$.*

Proof: Let P be any point of $\mathcal{K}$. Then there are $q + 1$ lines on P, each meeting $\mathcal{K}$ in at most $n - 1$ points of $\mathcal{K} - \{P\}$. ∎

If equality occurs the set is called a **maximal arc** [11].

LEMMA 2.2 (Cossu [42]) *If $\mathcal{K}$ is a maximal $\{k; n\}$-arc in a projective plane of order q, then every line meets $\mathcal{K}$ in either 0 or n points.*

Proof: Let $\mathcal{K}$ be a maximal $\{k; n\}$-arc in a projective plane of order q. Suppose some line l meets $\mathcal{K}$ in m points, where $1 \le m < n$. Let $P \in \mathcal{K}$ be a point incident with l. Then, other than l, there are q lines incident with P, each meeting $\mathcal{K}$ in at most $n-1$ points. Hence $k \leq m + q(n-1) < n + q(n-1)$, and $\mathcal{K}$ is not a maximal arc. ∎

So equivalently, a maximal $\{k;n\}$-arc is a subset of points such that every line meets the set in 0 or n points, for some n with $2 \leq n \leq q$ (it necessarily has size $k = q(n-1)+n$.) If a line meets $\mathcal{K}$ it is said to be **secant** to $\mathcal{K}$, otherwise it is **external** to $\mathcal{K}$. We shall call n the **degree** of a maximal $\{k;n\}$-arc.

Given a projective plane π the **dual** of π is the incidence structure whose points are the lines of π, whose lines are the the points of π, and whose incidence is as in π. Clearly the dual of a projective plane is a projective plane.

LEMMA 2.3 (Cossu [42]) *If $\mathcal{K}$ is a maximal $\{q(n-1)+n;n\}-$arc in a projective plane π of order q and $n < q$, the set of lines external to $\mathcal{K}$ is a maximal $\{q(q-n+1)/n; q/n\}-$arc in the dual plane.*

Proof: Let $\mathcal{K}^D$ be set of points of the dual of π corresponding to the external lines of $\mathcal{K}$. We show that every line of the dual plane meets $\mathcal{K}^D$ in 0 or q/n points.

Let P be a point of $\mathcal{K}$. Then every line on P is secant to $\mathcal{K}$. It follows that the line of the dual plane corresponding to P does not meet $\mathcal{K}^D$.

Let P be a point not in $\mathcal{K}$. Since every line on P is either secant or external to $\mathcal{K}$, it follows that there are $|\mathcal{K}|/n = \frac{q}{n}(n-1)+1$ lines on P secant to $\mathcal{K}$, and thus $q+1-\frac{q}{n}(n-1)-1 = q/n$ lines external to $\mathcal{K}$ on P. It follows that the line of the dual plane corresponding to P meet $\mathcal{K}^D$ in q/n points. ∎

We call the set of external lines to a maximal arc the **dual** of $\mathcal{K}$. It follows that a necessary condition for the existence of a maximal $\{q(n-1)+n;n\}-$arc in a projective plane of order q is that n divides q. But it is not sufficient (see subsection 2.2).

For example, in a projective plane of order q the complement of any line is a maximal $\{q^2;q\}-$arc. In this thesis this type of maximal arc will be referred to as **trivial**.

Other (non-trivial) examples of maximal arcs are given by *hyperovals* in projective planes of even order, which arose in the study of k-arcs. In a finite projective plane of order q, a **k-arc** is a set of k points no three collinear [78]. Lines that meet a k-arc in 1 point are known as **tangent** lines. A k-arc is **complete** if it is not contained in a $(k+1)$-arc. The maximum number of points a k-arc can have is $q+2$ if q is even, and $q+1$ if q is odd. A $(q+1)$-arc is known as an **oval**. A $(q+2)$-arc is known as a **hyperoval**, in which case every line meets the set in 0 or 2 points. Hence hyperovals are exactly the maximal $\{q+2;2\}$-arcs in projective planes of order q.

For instance, in $PG(2,q)$ the points of a non-degenerate conic form an oval. If q is odd, then the oval is complete, and Segre's celebrated result shows that these are the only ovals in $PG(2,q)$, q odd [119]. If q is even, there is a unique tangent line on each point of the conic, and these tangents are all concurrent on a point known as the **nucleus** of the conic [78]. The points of a non-degenerate conic together with its nucleus, then form a hyperoval which is called **regular** [96].

Hyperovals and ovals have been extensively studied, and are connected to many interesting geometrical structures such as *generalised quadrangles* [112] and *q-clans* [6], though few examples are known. See Chapter 5 for more on hyperovals, and also Chapter 4, Section 3. For a recent survey of known hyperovals see [113][117].

The following Theorem and Lemma give information on completing $\{k, n\}$-arcs to maximal arcs.

THEOREM 2.1 (Barlotti [11]) *If $\mathcal{K}$ is a $\{q(n-1)+n-1;n\}$-arc in a projective plane π of order q, it is incomplete and can be uniquely completed to a maximal $\{q(n-1)+n;n\}$-arc*

Proof: Let P be any point of $\mathcal{K}$. Then the $q+1$ lines incident with P meet $\mathcal{K}$ in at most $n-1$ points other than P. Since $|\mathcal{K}| = 1 + q(n-1) + (n-2)$ it follows that q lines on P meet $\mathcal{K}$ in n points, and there exists a unique line on P meeting $\mathcal{K}$ in $n-1$ points. Hence on any point of $\mathcal{K}$ there exists a unique $(n-1)$-secant.

The total number of such $(n-1)$-secants is then $\frac{|\mathcal{K}|}{n-1} = q+1$. We show that these $(n-1)$-secants are concurrent and hence that $\mathcal{K}$ can be completed to a maximal $\{q(n-1)+n;n\}$-arc by adding the point of concurrency.

First note that if a set l of $q+1$ points has the property that every line of the plane meets l, then l is the set of points on a line. For suppose l is not the set of points on a line. Then there exists some line m that meets l in at least 2 points, but such that there is a point $Q \notin l$ on m. Since there are at most $q-1$ points of l not on m, but there are q lines not equal to m on Q, there must be some line on Q that does not meet l. This contradicts the hypothesis that every line meets the set l, and so it must be that l is the set of points of a line. Dually it follows that a set of $q+1$ lines, such that every point of the plane in on some line, is a set of concurrent lines.

So suppose the set of $(n-1)$-secants are not concurrent. Then by the comments of the last paragraph there must be some point $Q \in \pi - \mathcal{K}$ such that Q is on no $(n-1)$-secant to $\mathcal{K}$. Hence every line incident with Q meets $\mathcal{K}$ in either 0 or n points. But $q(n-1)+n-1 \equiv -1 \bmod n$, giving a contradiction. Hence the $(n-1)$-secants must be concurrent. ∎

LEMMA 2.4 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be degree n maximal arcs in a projective plane of order q. If $|\mathcal{K}_1 \cap \mathcal{K}_2| > k - (q - q/n + 1)$ then $\mathcal{K}_1 = \mathcal{K}_2$, where $k = q(n-1) + n$ is the number of points in each maximal arc.*

Proof: Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are distinct. Then there exists a point P contained in $\mathcal{K}_1 - \mathcal{K}_1 \cap \mathcal{K}_2$. Since the external lines to a maximal arc form a maximal arc in the dual plane, there are q/n lines on P external to $\mathcal{K}_2$ (all of which are secant to $\mathcal{K}_1$.) On these lines there are $\frac{q}{n}(n-1)+1$ points of $\mathcal{K}_1$. Hence if $\mathcal{K}_1$ and $\mathcal{K}_2$ are distinct $|\mathcal{K}_1 \cap \mathcal{K}_2| \leq k - (q - q/n + 1)$. ∎

As an example, Lemma 2.4 shows that two distinct degree 4 maximal arcs in a plane of order 16 can meet in at most 39 points. For hyperovals (degree 2 maximal arcs), the bound becomes that two distinct hyperovals can meet in at most $q/2+1$ points. Examples of hyperovals that meet in exactly half their points are known in $PG(2,4)$ and $PG(2,8)$.

This subsection is concluded by examining what restrictions there are on the order of the collineation stabiliser of a maximal arc in a Desarguesian projective plane. Following lemma and proof is analogous to one given for k-arcs in [96].

LEMMA 2.5 *Let $\mathcal{K}$ be any set in $PG(2,q)$ such that $\mathcal{K}$ contains $m \neq 0$ ordered quadrangles. Then the order of $PGL(3,q)_{\mathcal{K}}$, the stabiliser of $\mathcal{K}$ in the projective general linear group, divides m.*

Proof: First note that $PGL(3,q)$ acts regularly on ordered quadrangles in $PG(2,q)$, and so the number of ordered quadrangles in $PG(2,q)$ is the same as the order of $PGL(3,q)$. The order of $PGL(3,q)$ is $(q^2+q+1)(q^2+q)q^2(q-1)^2$.

We count in two ways the size of the set

$$\{(Q, \mathcal{K}') \colon Q \text{ an ordered quadrangle}, \mathcal{K}' = \mathcal{K}^g \text{ for some } g \in PGL(3,q), Q \subseteq \mathcal{K}'\}.$$

Let $H = PGL(3,q)_{\mathcal{K}}$. Let N denote the number of images of $\mathcal{K}$ under $PGL(3,q)$ of a fixed ordered quadrangle in $\mathcal{K}$. Note that N does not depend on the choice of quadrangle since $PGL(3,q)$ is transitive on ordered quadrangles in $PG(2,q)$. Then the count gives:

$$(q^2+q+1)(q^2+q)q^2(q-1)^2.N = \frac{(q^2+q+1)(q^2+q)q^2(q-1)^2}{|H|}.m$$

Hence $|H|$ divides m. ∎

COROLLARY Let $\mathcal{K}$ be a maximal $\{q(n-1)+n; n\}$-arc in $PG(2,q)$, with $PGL(3,q)_{\mathcal{K}}$ the stabiliser of $\mathcal{K}$ in the projective general linear group. Then the order of $PGL(3,q)_{\mathcal{K}}$ divides $k(k-1)(k-n)(k-3n+3)$, where $k = q(n-1)+n$.

Proof: Count the number of ordered quadrangles in $\mathcal{K}$. There are k choices for the first point, and $k-1$ choices for the second. The third point may be any point of $\mathcal{K}$ not on the line joining the first two, and so there are $k - n$ choices for this point. The fourth point may be any point of $\mathcal{K}$ not on the lines of the triangle formed by the first three points, and so there are $k-(3n-3)$ choices for this point. ∎

Lemma 2.5 and its Corollary can often provide useful information about what sorts of collineations can stabilise a maximal arc in a Desarguesian plane. For instance, we might wonder what the orders of collineations that stabilise a maximal $\{52;4\}$-arc in $PG(2,16)$ can be. The order of $PGL(3,16)$ is $2^{12}.3^3.5^2.7.13.17.$, but a maximal $\{52;4\}$-arc contains $52.51.48.43 = 2^6.3^2.13.17.43$ ordered quadrangles. From the Lemma it follows that no elements of order 5 or 7 in $PGL(3,16)$ can stabilise a degree 4 maximal arc in $PG(2,16)$. So the only prime order elements that can stabilise the maximal arc are of order 2, 3, 13 or 17. The primes 2, 3 and 17 do in fact occur in orders of the stabilisers of the known degree 4 maximal arcs in $PG(2,16)$ (see next chapter.) By Sylow's Theorem all subgroups of order 13 are conjugate in $PGL(3,16)$. In fact such a subgroup of order 13 is a subgroup of a Singer cycle. Easy (but tedious) hand calculations show that it is not possible to take 4 orbits of such a subgroup to construct a degree 4 maximal arc in $PG(2,16)$.

2.2 The spectral problem for existence of maximal arcs.

In $PG(2,q)$, q even, R. H. F. Denniston has given a construction of maximal $\{k;n\}-$arcs for all n dividing q [59](see next chapter). But, in general, the necessary condition that n divides q is not sufficient.

In [115], Penttila, Royle and Simpson classify hyperovals and hence degree 8 maximal arcs in known projective planes of order 16. In particular they show that certain of the planes do not admit hyperovals, and so the dual planes do not admit degree 8 maximal arcs (see Chapter 4 for more details.)

For projective planes of odd order no non-trivial constructions of maximal arcs are known. In 1974, J.A. Thas proved the following Theorem.

THEOREM 2.2 (Thas [129]) *In the Desarguesian projective plane $PG(2,q)$ of order q, $q > 3$, there are no maximal $\{2q+3;3\}-$arcs, and no maximal $\{q(q-2)/3;q/3\}-$arcs.* ∎

The proof is highly dependent on the hypothesis that the plane is Desarguesian. It involves showing that if such a maximal arc existed in a Desarguesian plane then points of the maximal arc together with secant lines would form an affine space $AG(m,3)$, $m > 2$. Noting that the number of points in such a space cannot be the number of points of a degree 3 maximal arc gives a contradiction. Thas also conjectured the following.

CONJECTURE (Thas [129]) *No non-trivial maximal arcs exist in $PG(2,q)$ for q odd.*

No progress has been made on this conjecture in the 20 years since it was posed.†

2.3 Sets of type (m,n) in projective planes

THEOREM 2.3 (Tallini Scafati [123]) *Let $\mathcal{K}$ be a set of k points of a projective plane π of order q. For each i, with $0 \le i \le q+1$, let t_i be the number of lines of π that meet $\mathcal{K}$ in i points. Then*

$$\sum_{0\le i\le q+1} t_i = q^2+q+1 \tag{1}$$

$$\sum_{1\le i\le q+1} it_i = k(q+1) \tag{2}$$

$$\sum_{2\le i\le q+1} i(i-1)t_i = k(k-1) \tag{3}$$

Proof: (1) The total number of lines in the plane is q^2+q+1.
(2) Count in two ways the size of the set
$\{(P,l):\ P\in\mathcal{K},\ P$ incident with the line $l\}$.
(3) Count in two ways the size of the set
$\{(P,Q,l):\ P,Q\in\mathcal{K},\ P\ne Q,\ P$ and Q incident with the line $l\}$. ∎

A **k-set of type** (m,n) in a projective plane is a set of k points such that every line of the plane meets the set in either m or n points, and such that m-secants and n-secants to the set both exist.

THEOREM 2.4 (Tallini Scafati [124]) *Let $\mathcal{K}$ be a k-set of type (m,n) for some m and n, in a projective plane of order q. Then*
(i) $(n-m)$ divides q,
(ii) $k^2-k(q(n+m-1)+n+m)+mn(q^2+q+1)=0$.

Proof: (1) Let $P\in\mathcal{K}$, and μ be the number of n-secants to $\mathcal{K}$ on P. Then counting the points of $\mathcal{K}$ on the lines on P gives $k-1=\mu(n-1)+(q+1-\mu)(m-1)$. Hence $\mu=\frac{k-1-(q+1)(m-1)}{n-m}$.

Let $Q\notin\mathcal{K}$, and λ be the number of n-secants to $\mathcal{K}$ on Q. Then counting the points of $\mathcal{K}$ on the lines on Q gives $k=\lambda n+(q+1-\lambda)m$. Hence $\lambda=\frac{k-(q+1)m}{n-m}$. Then

$$\mu-\lambda=\frac{q}{n-m}.$$

† As we go to press it appears that Blokhuis and Mazzocca have a proof that degree 5 maximal arcs do not exist in $PG(2,25)$.

(2) For a k-set of type (m,n) the equations of Theorem 2.3 become $t_n + t_m = q^2+q+1$, $mt_m + nt_n = k(q+1)$, $m(m-1)t_m + n(n-1)t_n = k(k-1)$. Solving to eliminate terms in t_n and t_m gives the required result. ∎

COROLLARY *Let $\mathcal{K}$ be a k-set of type (m,n) for some m and n, in a projective plane of order q. Then the set of n-secants to $\mathcal{K}$ form a $(n(q^2+q+1)-k(q+1))/(n-m)$-set of type $([k-m(q+1)]/(n-m), [k+q-m(q+1)]/(n-m))$ in the dual plane.*

Proof: Follows from the calculation of μ and λ in part (i) of the Theorem. ∎

It is clear that if $m=0$ then the sets of type $(0,n)$ in projective planes are the maximal arcs in those planes. In the same paper that Theorem 2.4 was proved, Tallini Scafati considered the case $m=1$.

THEOREM 2.5 (Tallini Scafati [124]) *Let $\mathcal{K}$ be a k-set of type $(1,n)$ in a projective plane of order q, q a prime power, $n \neq q+1$. Then q is a square, $n=\sqrt{q}+1$, and either $k = q^{3/2}+1$ or $k = q+\sqrt{q}+1$.* ∎

In a projective plane of order q^2, a (q^3+1)-set of type $(1,q+1)$ is called a **unital**. A line is called **tangent** if it meets the unital in one point, otherwise **secant**. The set of tangent lines to a unital form a unital in the dual plane called the **dual unital**. Unitals have been much studied in the last 20 years. See, for instance, [2][8][9][30][36][53][69][97][101][106][118], or the Appendix for more on unitals.

In a projective plane π of order q^2, a (q^2+q+1)-set $\mathcal{B}$ of type $(1,q+1)$ is called a **Baer subplane**. If a line of π meets $\mathcal{B}$ in $q+1$ points, then the set of points of intersection is known as a **Baer subline**, and the line is called a **secant** line. It is not difficult to show that the set of points of $\mathcal{B}$ together with the secant lines form a projective plane of order q. Baer subplanes have been much studied, and can be used to construct all sorts of useful objects such as new projective planes (see Chapter 3), and other sets of type (m,n) [50].

Other than maximal arcs (including hyperovals), unitals and Baer subplanes, k-sets of type (m,n) do not have individual names. For examples of such sets with $m>1$ see [51][52][114][25][50] and also Chapter 4, Section 3.

This subsection is concluded with the most useful Lemma in the whole thesis. It will be referred to again and again throughout the thesis. It deals with how sets of type (m,n) in projective planes may meet.

LEMMA 2.6 (Grüning [69]) *Let π be a projective plane of order q. Let $\mathcal{M}$ be a set of type (m,n), and $\mathcal{K}$ a set of type (k,l), both in π. Let t_l be the total number of l-secants to $\mathcal{K}$, λ be the number of l-secants to $\mathcal{K}$ on a point not in $\mathcal{K}$, and μ be the number of l-secants to $\mathcal{K}$ on a point in $\mathcal{K}$. Let b be the number of l-secants to $\mathcal{K}$ that are m-secants to $\mathcal{M}$, then*

$$|\mathcal{M}\cap\mathcal{K}|\mu + (|\mathcal{M}| - |\mathcal{M}\cap\mathcal{K}|)\lambda = bm + n(t_l - b).$$

Proof: Count in two ways the size of the set

$$\{(P,h)| \quad P\in\mathcal{M}, \quad h \text{ a } l\text{-secant to } \mathcal{K}, \quad P \text{ incident with } h\}.$$

$\blacksquare$

COROLLARY 1 *Let $\mathcal{M}$ be a degree n maximal arc, and $\mathcal{B}$ a Baer subplane in a plane π of order q^2. Let b be the number of Baer sublines which when extended to π are external to $\mathcal{M}$, then*

$$|\mathcal{M}\cap\mathcal{B}|q + nb = q^2 + nq.$$

Proof: Let the maximal arc be the set $\mathcal{M}$ in the Lemma, so $m=0$ and $n=n$. Let the Baer subplane be the set $\mathcal{K}$ in the Lemma, so $k=1$ and $l=q+1$. The total number of secants to a Baer subplane is just the number of lines of a projective plane of order q and so $t_l = q^2+q+1$. The number of secants on a point of a Baer subplane is $q+1$ and so $\mu = q+1$.

For any point $P\in\pi-\mathcal{B}$ there are q^2+1 lines on P, each of which is either a tangent or secant to $\mathcal{B}$. There are q^2+q+1 points in $\mathcal{B}$, so at least one line on P is a secant to $\mathcal{B}$. But if there were two secants to $\mathcal{B}$ on P, then these two lines would meet in both P and a point of $\mathcal{B}$ (since any two lines of the subplane must meet). Hence on any point of $\pi-\mathcal{B}$ there is a unique secant line to $\mathcal{B}$, and so $\lambda=1$. $\blacksquare$

COROLLARY 2 *Let $\mathcal{M}$ be a degree $n>2$ maximal arc in a projective plane π of order q, π embedded as a Baer subplane of a plane π' of order q^2. Then the set of points of $\mathcal{M}$ cannot be extended to the set of points of a degree n maximal arc $\mathcal{M}'$ of π'.*

Proof: From Corollary 1 it follows that a degree n maximal arc $\mathcal{M}'$ of π' meets a Baer subplane in at most $q+n$ points. If this intersection contains the set of points of a degree n maximal arc M of π, then we require that $q(n-1)+n\le q+n$, and hence that $n=2$. $\blacksquare$

Corollary 2 was originally proved by O'Keefe in [92, Cor. 4.2.11].

3. ASSOCIATED STRUCTURES

3.1 Partial Geometries and Strongly Regular Graphs

A (finite) **partial geometry** $(\mathcal{P}, \mathcal{L})$ with parameters (s, t, α) is a finite non-empty set $\mathcal{P}$ of points together with a family $\mathcal{L}$ of subsets of $\mathcal{P}$ called lines, incidence being containment such that
(i) Any two distinct points are incident with at most one line;
(ii) Each line is incident with exactly $s+1$ points;
(iii) Each point is incident with exactly $t+1$ lines;
(iv) For any point $P \in \mathcal{P}$ and any line $l \in \mathcal{L}$ not containing P there are exactly α points on l collinear with P.

Simple counts then show that the number of points is $v = (s+1)(st+\alpha)/t$, and the number of lines is $b = (t+1)(st+\alpha)/t$ [20]. Note that the dual of a partial geometry with parameters s, t and α is a partial geometry with parameters $s' = t$, $t' = s$ and $\alpha' = \alpha$.

Partial geometries were first introduced by R. C. Bose in [20] to study strongly regular graphs (see below), though now they have developed an independent life of their own. Examples of partial geometries are the generalised quadrangles (partial geometries with $\alpha = 1$ [112]), the $2-(v, s+1, 1)$ designs and their duals [82], as well as nets and their duals [82]. These partial geometries are known as **improper** (a **proper** partial geometry satisfies $1 < \alpha < min(s,t)$) [44]. Few examples of proper partial geometries are known: there are two constructions ${T_2}^*(\mathcal{K})$ and $S(\mathcal{K})$ arising from maximal arcs in projective planes (given below); a construction by J.A. Thas involving a spread of a hyperbolic quadric in $PG(4h+3, 3)$ [135] (the only known such spread has $h = 1$, so at present the construction gives only one partial geometry); a construction by F. De Clerck et al involving a hyperbolic quadric in $PG(4n-1, 2)$ [48] (one of the partial geometries in this class was originally due to Cohen in [41], this was later shown in [136] to be dual to a construction by Haemers and van Lint [71]); various sporadic constructions: one due to Haemers [70], one due to van Lint and Schrijver [102] (Cameron and van Lint give another construction for this second sporadic partial geometry in [33]), and one due to Mathon [104](with the same parameters, but not isomorphic to the example of Haemers and van Lint). Few characterisations of proper partial geometries are known. For examples see [43][44][131][133].

A graph $\mathcal{G}$ with v vertices is said to be **strongly regular** if there exist integers k, μ and λ such that
(i) each vertex is adjacent to k other vertices,
(ii) for any pair of adjacent vertices there are λ vertices adjacent to both,
(iii) for any pair of non-adjacent vertices there are μ vertices adjacent to both.

Strongly regular graphs have been studied extensively. See for instance [27] [34][121][32]. The following theorem shows how partial geometries give rise to strongly regular graphs (and is why partial geometries were originally defined).

THEOREM 3.1 (Bose [20]) *Given a partial geometry with parameters s, t and α form a graph $\mathcal{G}$ with vertices the points of the partial geometry, and two vertices are adjacent if and only if the corresponding points are on a line of the partial geometry. Then the graph $\mathcal{G}$ is strongly regular with parameters $v = (s+1)(st+\alpha)/t$, $k = s(t+1)$, $\lambda = (\alpha-1)t + s - 1$ and $\mu = \alpha(t+1)$.* ∎

The following Theorems 3.2 and 3.3 show how partial geometries and hence strongly regular graphs may be obtained from maximal arcs in projective planes.

THEOREM 3.2 (Thas [127], Wallis [140]) *Let $\mathcal{K}$ be a maximal $\{q(n-1)+n; n\}$-arc in a projective plane π of order q. Define an incidence structure $S(\mathcal{K})$ with the points of $S(\mathcal{K})$ being the points of $\pi - \mathcal{K}$, the lines of $S(\mathcal{K})$ being the lines secant to $\mathcal{K}$, and incidence induced from π. Then $S(\mathcal{K})$ is a partial geometry with parameters $s = q - n$, $t = q - q/n$ and $\alpha = q - q/n - n + 1$.*

Proof: Any line secant to $\mathcal{K}$ is incident with $q + 1 - n$ points of $\pi - \mathcal{K}$, hence $s = q - n$. Any point of $\pi - \mathcal{K}$ is incident with $(q(n-1)+n)/n$ lines secant to $\mathcal{K}$, hence $t = q - q/n$.

Suppose P is a point of $\pi - \mathcal{K}$, and l a line secant to $\mathcal{K}$ but not incident with P. The line l meets $\mathcal{K}$ in n points. There are $q - q/n + 1$ secant lines to $\mathcal{K}$ incident with P. Of these secant lines exactly n of them meet l in a point of $\mathcal{K}$. Hence the number of lines on P that are secant to $\mathcal{K}$, but meet l in a point of $\pi - \mathcal{K}$, is $q - q/n + 1 - n$, and so $\alpha = q - q/n - n + 1$. ∎

Given a maximal arc $\mathcal{K}$, with dual maximal arc $\mathcal{K}'$ in the dual plane, it is not difficult to see that the dual of the $S(\mathcal{K})$ partial geometry is $S(\mathcal{K}')$, i.e. the dual of an $S(\mathcal{K})$ partial geometry is the partial geometry formed from the dual maximal arc.

Certain of the $S(\mathcal{K})$ partial geometries have a further interesting property. The point-line incidence graph of an $S(\mathcal{K})$ partial geometry with $\mathcal{K}$ a degree q maximal arc in a plane of order q^2, q even, is a bipartite distance regular graph of diameter 4 [26, 12.6.1] .

THEOREM 3.3 (Thas [127]) *Let $\mathcal{K}$ be a maximal $\{q(n-1)+n; n\}$-arc in $PG(2,q)$. Embed $PG(2,q)$ as a hyperplane π of $PG(3,q)$. Define an incidence structure ${T_2}^*(\mathcal{K})$ as follows. The points of ${T_2}^*(\mathcal{K})$ are the points of $PG(3,q) - \pi$. The lines of ${T_2}^*(\mathcal{K})$ are the lines of $PG(3,q)$ that meet π in a unique point of $\mathcal{K}$. Incidence is the incidence induced from the incidence of $PG(3,q)$. Then ${T_2}^*(\mathcal{K})$ is a partial geometry with parameters $s = q - 1$, $t = (q+1)(n-1)$ and $\alpha = n - 1$.*

Proof: Note that any line of $PG(3,q)$ that meets π in a unique point of $\mathcal{K}$ is necessarily not contained in π. Hence any such line is incident with q points of $PG(3,q)-\pi$, and so $s=q-1$. On any point P of $PG(3,q)-\pi$ there are $q(n-1)+n$ lines incident with $\mathcal{K}$ (one for each point of $\mathcal{K}$), hence $t=q(n-1)+n-1$.

Let P be any point of $PG(3,q)-\pi$, and l any line of $PG(3,q)$ not incident with P and meeting π in a unique point of $\mathcal{K}$. Any line of $PG(3,q)$ incident with P and a point of l is contained in the plane π' given by the span of P and l. In $PG(3,q)$ two distinct planes meet in a line, so let l' be the line of intersection of π and π'. Note that l' contains the point $l \cap \pi \in \mathcal{K}$ and so is a secant line to $\mathcal{K}$ in π. Hence there are n lines in π' incident with P and a point of $\mathcal{K}$ (and necessarily meeting the line l.) Of these n lines only the line joining the points P and $l \cap \pi$ meets l in a point of π. Hence on P there are exactly $n-1$ lines that meet l in a point of $PG(3,q)-\pi$ and also meet π in a point of $\mathcal{K}$. Hence $\alpha=n-1$. ∎

A similar construction (due to Mr. Folklore) to that of Theorem 3.3 can be used to obtain strongly regular graphs from k-sets of type (m,n). Suppose $\mathcal{K}$ is a k-sets of type (m,n) in $PG(2,q)$. As before, embed $PG(2,q)$ as a hyperplane π of $PG(3,q)$. Define a graph $\mathcal{G}$ as follows. The vertices of $\mathcal{G}$ are the points of $PG(3,q)-\pi$. Two vertices of $\mathcal{G}$ are adjacent if and only if the line joining them in $PG(3,q)$ meets π in a point of $\mathcal{K}$. Then $\mathcal{G}$ is a strongly regular graph (see for instance [32, Thm. 3.2].) Also, in the *Handbook of Incidence Geometry* [31], Frank DeClerck (Chapter 10) notes that the construction of Theorem 3.3 gives rise to a *semi-partial geometry* if $\mathcal{K}$ is either a unital or a Baer subplane.

In Chapter 3 several new constructions of maximal $\{q^3-q^2+q;q\}$-arcs will be given in certain (non-Desarguesian) projective planes of order q^2. A reasonable question to ask is whether the $S(\mathcal{K})$ partial geometries associated with these maximal arcs are isomorphic to previously known $S(\mathcal{K})$ partial geometries. The following answers this question.

An **ovoid** of a partial geometry is a collection $\mathcal{O}$ of points such that every line of the partial geometry is incident with a unique point of the ovoid [49]. A **spread** of a partial geometry is a collection $\mathcal{S}$ of lines such that every point of the partial geometry lies on a unique line of the spread. Both ovoids and spreads (if they exist) of a partial geometry with parameters s, t and α, necessarily have size $(st+\alpha)/\alpha$ [49]. Ovoids and spreads of partial geometries are dual concepts: an ovoid of a partial geometry is a spread of the dual partial geometry, and a spread of a partial geometry is a ovoid of the dual partial geometry.

Interesting examples of ovoids and spreads of $S(\mathcal{K})$ partial geometries are given by the following. For an $S(\mathcal{K})$ partial geometry in a plane of order q, take any external line to the maximal arc $\mathcal{K}$, then the collection of $q+1$ points on that line form an ovoid of the partial geometry. Such ovoids of the $S(\mathcal{K})$ partial geometries are known as **linear**. Dually, if we take the collection of $q+1$ lines of the plane on a point of $\mathcal{K}$, then these lines form a spread of the partial geometry which we shall also call **linear** [49]. It is clear that linear ovoids are dual to linear spreads. A number of results on ovoids and spreads of partial geometries are known, see for instance [49][134].

LEMMA 3.1 *Let $\mathcal{K}$ be a degree n maximal arc in a projective plane π of order q, and $S(\mathcal{K})$ the associated partial geometry as above. Then if all the spreads and ovoids of $S(\mathcal{K})$ are linear then*

(i) Given another partial geometry $S(\mathcal{K}')$, $S(\mathcal{K}')$ is isomorphic to $S(\mathcal{K})$ if and only if the maximal arcs $\mathcal{K}$ and $\mathcal{K}'$ occur in isomorphic projective planes, and $\mathcal{K}$ is isomorphic to $\mathcal{K}'$.

(ii) The automorphism group of the $S(\mathcal{K})$ partial geometry is exactly the stabiliser of the maximal arc in the collineation group of the plane.

Proof: Let $\mathcal{K}$ be a maximal arc and $S(\mathcal{K})$ the associated partial geometry as in the statement of the Lemma. We show that given such a $S(\mathcal{K})$ (and its dual) the projective plane can be uniquely reconstructed from the linear spreads.

Under the assumption that all spreads of the partial geometry are linear we may identify each point of $\mathcal{K}$ with a unique spread of $S(\mathcal{K})$. Similarly, if all ovoids of $S(\mathcal{K})$ are linear we may identify the lines external to $\mathcal{K}$ with the ovoids of $S(\mathcal{K})$.

Define an incidence structure $\mathcal{I}$ by the following. The points of $\mathcal{I}$ are the points of $S(\mathcal{K})$ together with the spreads of $S(\mathcal{K})$. The lines of $\mathcal{I}$ are the lines of $S(\mathcal{K})$ together with the ovoids of $S(\mathcal{K})$. Incidence in $\mathcal{I}$ is symmetrised containment. Then it is clear that $\mathcal{I}$ is isomorphic to the projective plane π.

The automorphism group G of $S(\mathcal{K})$ certainly admits a group isomorphic to the stabiliser of $\mathcal{K}$ in the collineation group of the plane. The group G must also map ovoids to ovoids and spreads to spreads, and so must preserve the incidence structure $\mathcal{I}$, and so is isomorphic to a subgroup of the collineation group of the plane. Hence G is exactly the stabiliser of $\mathcal{K}$ in the collineation group of π. Hence (ii) follows.

Embedding the $S(\mathcal{K})$ partial geometry as a partial geometry $S(\mathcal{K}')$ with $\mathcal{K}'$ a degree n maximal arc not isomorphic to $\mathcal{K}'$ cannot occur. This is because the identification of spreads of $S(\mathcal{K})$ with points of $\mathcal{K}$, and ovoids of $S(\mathcal{K})$ with lines external to $\mathcal{K}$, is unique (i.e. the plane cannot be reconstructed in any other way.) Hence (i) follows. ∎

This subsection is concluded by showing some situations in which the ovoids and spreads of certain $S(\mathcal{K})$ partial geometries can be proved to be linear.

THEOREM 3.4 (Blokhuis & Wilbrink [18]) *Let A and B be disjoint sets in the projective plane $PG(2,q)$, $|A| > q-1$, $|B| = q+1$, such that each line through a point of A meets B. Then B is a line.* ∎

The following Theorem was noted for ovoids in [49].

THEOREM 3.5 *If $\mathcal{K}$ is a maximal arc in a $PG(2,q)$ then every ovoid and spread of the associated $S(\mathcal{K})$ partial geometry is linear.*

Proof: Put $A = \mathcal{K}$, let B be an ovoid (necessarily of size $q+1$) of $S(\mathcal{K})$. Then by definition every line that meets A meets B, and A certainly contains more than $q-1$ points. Applying Theorem 3.4 shows that the ovoid B is linear.

Similarly every ovoid of the dual partial geometry is linear. Since a linear ovoid in the dual partial geometry is a linear spread of the partial geometry, it follows that every spread of the $S(\mathcal{K})$ partial geometry is linear. ∎

COROLLARY *If $\mathcal{K}$ is a maximal arc in $PG(2,q)$ then the associated $S(\mathcal{K})$ is not isomorphic to any other $S(\mathcal{K}')$ partial geometry where $\mathcal{K}'$ is a maximal arc in a non-Desarguesian plane of order q. Further, the automorphism group of $S(\mathcal{K})$ is isomorphic to the stabiliser of $\mathcal{K}$ in the collineation group of the plane.*

Proof: Follows immediately from Lemma 3.1 and Theorem 3.5. ∎

LEMMA 3.2 *Let $\mathcal{K}$ be a degree n maximal arc in a projective plane π of order q. If $n^2 - n < q$ then every spread of $S(\mathcal{K})$ is linear.*

Proof: Let $\mathcal{K}$ be a degree n maximal arc in a plane π of order q. Note that a spread $\mathcal{S}$ of $S(\mathcal{K})$ is a collection of $q+1$ lines, which when considered as lines of π pairwise meet in points of $\mathcal{K}$. Suppose that the spread is not linear, i.e. that the lines of $\mathcal{S}$ are not all concurrent on a point of $\mathcal{K}$. Let $l \in \mathcal{S}$, then considered as a line of π, l meets $\mathcal{K}$ in n points. Every line of $\mathcal{S} - \{l\}$ meets l in one of these n points, and so some point $P \in l \cap \mathcal{K}$ must be incident with at least q/n lines of $\mathcal{S} - \{l\}$. Since $\mathcal{S}$ is not linear there is some line $k \in \mathcal{S}$ such that k is not incident with P. The line k meets $\mathcal{K}$ in $n-1$ points of $\mathcal{K}$ not on l. But k meets each of the lines of $\mathcal{S} - \{l\}$ on P in a unique point of $\mathcal{K}$. Hence $n - 1 \geq q/n$.

So if $n - 1 < q/n$ then the lines of $\mathcal{S}$ must be concurrent on a point of $\mathcal{K}$. ∎

COROLLARY *Let $\mathcal{K}$ be a maximal $\{q^3 - q^2 + q; q\}$-arc in a projective plane of order q^2. Then all spreads and ovoids of $S(\mathcal{K})$ are linear. Further, the associated $S(\mathcal{K})$ partial geometry is not isomorphic to any other $S(\mathcal{K}')$ partial geometry where $\mathcal{K}'$ is a maximal $\{q^3-q^2+q; q\}$-arc not isomorphic to $\mathcal{K}$. The automorphism group of $S(\mathcal{K})$ is isomorphic to the stabiliser of $\mathcal{K}$ in the collineation group of the plane.*

Proof: Let $\mathcal{K}$ be a maximal $\{q^3 - q^2 + q; q\}$-arc in a plane of order q^2. It follows immediately from the Lemma that every spread of the $S(\mathcal{K})$ partial geometry is linear. The dual maximal arc $\mathcal{K}^D$ to $\mathcal{K}$ in the dual plane is also a maximal $\{q^3 - q^2 + q; q\}$-arc, and so it follows that every spread of the $S(\mathcal{K}')$ maximal arc is linear. But the spreads of $S(\mathcal{K}')$ are the ovoids of $S(\mathcal{K})$ and so every ovoid of $S(\mathcal{K})$ is linear. Apply Lemma 3.1. ∎

In Chapter 3 a number of new constructions of classes of degree q maximal arcs in projective planes of order q^2 are given. The Corollary shows that the associated $S(\mathcal{K})$ partial geometries are also new. In Chapter 4 a number of new degree 4 maximal arcs in planes of order 16 are found. These together with the previously known (Denniston) maximal arcs in $PG(2,16)$ give rise to 16 non-isomorphic partial geometries with parameters $s = t = 12$ and $\alpha = 9$ (some of these partial geometries are dual to one another; counting a partial geometry and its dual as one partial geometry the results of Chapter 4 give 11 known partial geometries with these parameters).

The proof of Lemma 3.2 is a dumb one. It seems likely that a more cunning method of proof would give a similar result for less restricted values of the degree of the maximal arc. Blokhuis and Wilbrink's result (Theorem 3.4) shows that this is the case in Desarguesian planes. Is a similar result true for non-Desarguesian planes (at least in the case where the set A is a maximal arc)?

3.2 Designs

A **balanced incomplete block design** (or BIBD) with parameters (v, b, r, k, λ) is an incidence structure $(\mathcal{P}, \mathcal{B}, I)$, with the following properties:
(i) $\mathcal{P}$ is a set of v elements (called *points*),
(ii) $\mathcal{B}$ is a set of b elements (called *blocks*), and each block is incident with exactly k points,
(iii) Each point is incident with exactly r blocks,
(iv) Every pair of distinct points is incident with exactly λ blocks.

Simple counts establish the following two relations [60]:

$$vr = bk \quad \text{and} \quad \lambda(v-1) = r(k-1).$$

Since given any three of the parameters the other two can be deduced, a BIBD is often denoted as a (v, k, λ)-BIBD.

The study of designs is the subject of many books and hundreds of papers. For an introduction to the theory of designs see, for instance, [14][82][60] or [55].

In a BIBD, a partition of the pointset into blocks is called a **parallel class**. A parallel class consists of v/k blocks.

A BIBD is said to be **resolvable** if the set of blocks can be partitioned into parallel classes.

THEOREM 3.6 (Wallis [140]) *If there exists a degree n maximal arc $\mathcal{K}$ in a projective plane π of order q then there exist resolvable BIBD's with parameters*
(i) $v = q(n-1) + n$, $k = n$, $\lambda = 1$
(ii) $v = q(\frac{q}{n} - 1) + q/n$, $k = q/n$, $\lambda = 1$.

Proof: *(i)* Form an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$ where $\mathcal{P}$ is the set of points of $\mathcal{K}$, $\mathcal{B}$ is the set of lines secant to $\mathcal{K}$, and incidence is induced from the plane. Then $v = |\mathcal{K}| = q(n-1) + n$. Every secant line to $\mathcal{K}$ meets $\mathcal{K}$ in n points, and so there are n points on any block of $\mathcal{I}$. Also, any pair of points of $\mathcal{K}$ is on a unique secant line to $\mathcal{K}$. It follows that $\mathcal{I}$ is a BIBD with parameters as in *(i)*. We now show that $\mathcal{I}$ is resolvable.

Let l be any line of the plane that is external to $\mathcal{K}$. Let P be any point on l. Then there are $q+1-q/n$ lines on P that are secant to $\mathcal{K}$, and they form a parallel class of lines in $\mathcal{I}$ (since the lines meet only at $P \notin \mathcal{K}$, and they partition the points of $\mathcal{K}$). Hence corresponding to the $q+1$ points on l there are $q+1$ parallel classes of lines of $\mathcal{I}$. Since every line of π meets l it follows that this collection of parallel classes is a resolution of $\mathcal{I}$.

Part *(ii)* is the same as *(i)*, but applied to the dual maximal arc in the dual plane. ∎

Note that the designs of Theorem 3.6 and their duals are also (improper) partial geometries.

In [122], D.R. Stinson introduces the idea of a hyperresolution of a BIBD. A **hyperresolution** of a $(v, k, 1)$-BIBD is a set of parallel classes $\mathcal{P}$ such that for every pair of disjoint blocks A and B in the BIBD there is a unique parallel class $P \in \mathcal{P}$ such that $\{A, B\} \subset P$. The reason for introducing the idea is the following Theorem.

THEOREM 3.7 (Stinson [122]) *If there is a hyperresolution of a $(v, k, 1)$-BIBD then there is a partial geometry with parameters*

$$s = \frac{v}{k} - 1, \quad t = \frac{v - k^2}{k - 1}, \quad \alpha = \frac{v}{k} - k.$$

Proof: Let $\mathcal{P}$ be a hyperresolution of the BIBD. Denote the set of blocks of the BIBD by $\mathcal{B}$. It is shown that the incidence structure given by $\mathcal{I} = (\mathcal{B}, \mathcal{P}, I)$, where incidence I is symmetrised containment, is a partial geometry. Call $\mathcal{B}$ the *points*, and $\mathcal{P}$ the *lines* of $\mathcal{I}$.

For any two lines of the BIBD that meet there is no parallel class of $\mathcal{P}$ containing both lines. For any two lines of the BIBD that are parallel, by the definition of hyperresolution, there is a unique parallel class in $\mathcal{P}$ containing both lines. Hence for any two (distinct) *points* of $\mathcal{I}$ there is at most one line of $\mathcal{I}$ incident with both points.

Every parallel class of the BIBD contains v/k lines and so $s = v/k - 1$.

Since on any point of a block B of the $(v, k, 1)$-BIBD there are $r = (v-1)/(k-1)$ blocks incident with that point (including B), and there are k points on a block, it follows that there are $\frac{k(v-1)}{k-1}$ blocks incident with any given block B. There are $b = \frac{v(v-1)}{k(k-1)}$ blocks in total, so given any block B there are

$$\frac{v(v-1)}{k(k-1)} - 1 - \frac{k(v-1)}{k-1}$$

blocks disjoint from it. The block B together with any block disjoint from B is contained in a unique parallel class of $\mathcal{P}$. Each parallel class contains $\frac{v}{k} - 1$ blocks other than B. Hence the are

$$t + 1 = \frac{\frac{v(v-1)}{k(k-1)} - 1 - \frac{k(v-1)}{k-1}}{\frac{v}{k} - 1} = \frac{v - k^2 + k - 1}{k - 1}$$

parallel classes of $\mathcal{P}$ containing B.

Let B be a block of the $(v, k, 1)$-BIBD and $P \in \mathcal{P}$ a parallel class of blocks not containing B. Then B meets k of the v/k blocks of P in a single point, and so is disjoint from $v/k - k$ blocks of P. For each of the blocks $B' \in P$ disjoint from B there is a unique element of the hyperresolution containing both B and B'. Hence $\alpha = v/k - k$. ∎

THEOREM 3.8 (Stinson [122]) *Let $\mathcal{K}$ be a degree n maximal arc in a projective plane π of order q. Let $\mathcal{D}$ be the BIBD with points being the points of $\mathcal{K}$ and blocks being the secant lines to $\mathcal{K}$. Then $\mathcal{D}$ admits a hyperresolution.*

Proof: For any point $P \in \pi - \mathcal{K}$ there is an associated parallel class of blocks of $\mathcal{D}$ given by the lines on P that are secant to $\mathcal{K}$. Take the collection $\mathcal{P}$ of all such parallel classes associated with points of $\pi - \mathcal{K}$. We show that $\mathcal{P}$ is a hyperresolution.

Let B and B' be any two disjoint blocks of $\mathcal{D}$. Then considered as lines of π, B and B' meet in some point $P \in \pi - \mathcal{K}$. The unique element of $\mathcal{P}$ containing both B and B' is then given by the secant lines to $\mathcal{K}$ on P. ∎

In Theorem 3.8, since we are identifying the elements of the hyperresolution with the points of $\pi - \mathcal{K}$, it is not difficult to see that a partial geometry so constructed is isomorphic to the dual of the $S(\mathcal{K})$ partial geometry. But as D.R. Stinson points out, what is interesting about the last two Theorems is that it breaks the process *maximal arc* $\Rightarrow$ *partial geometry* into *maximal arc* $\Rightarrow$ *hyperresolution* $\Rightarrow$ *partial geometry*.

It may be that there are other hyperresolutions of the maximal arc BIBD's which might give rise to non-isomorphic partial geometries with the same parameters.

For instance, suppose there exists a degree q maximal arc $\mathcal{K}$ in a plane π of order q^2, and a Baer subplane $\mathcal{B}$ disjoint from $\mathcal{K}$. Then by Grünings Lemma (Lemma 2.6) there are exactly $q^2 - q + 1$ lines of $\mathcal{B}$ which when extended to π are secant to $\mathcal{K}$. Each of these lines meets $\mathcal{K}$ in q points, and any pair of these lines meet in $\mathcal{B}$. Since $\frac{|\mathcal{K}|}{q} = q^2 - q + 1$ the collection forms a parallel class of lines of the BIBD associated with $\mathcal{K}$. In the case of Thas 1974 maximal arcs (see Chapters 3 and 4) it is not difficult to construct many disjoint Baer subplanes. It would be interesting to know if these "non-linear" parallel classes could be used to construct new hyperresolutions.

In the same paper as above Stinson also shows how *group divisible designs* with *orthogonal resolutions* can be constructed from maximal arcs in projective planes. But the study of these objects is (once again) a whole area in itself and I don't want to get into it here.

As a final note on maximal arcs and designs we note that the definition of a maximal arc in a projective plane can be generalised to that of a maximal arc in a BIBD. A $\{\mu, n\}$-arc in a (v, k, λ)-BIBD is a set of μ points of the BIBD such that no block meets the set in more that n points. Simple counts, as in Section 1, show that $\mu \leq 1 + r(n-1)/\lambda$. If equality is achieved then the set is known as a *maximal arc*, and as in Section 1 it follows that every block meets a maximal $\{\mu, n\}$-arc in 0 or n points. In [137], T. van Trung gives an interesting construction of maximal arcs in certain designs. It involves constructing a design from multiple copies of an affine plane, and then noting that the set of points of a maximal arc in one of the affine planes is a maximal arc in the design.

Chapter 2
KNOWN CONSTRUCTIONS OF MAXIMAL ARCS

1. INTRODUCTION

This chapter is an investigation of the constructions and characterisations of maximal arcs known prior to this thesis (apart from hyperovals which have a literature of their own).

In Section 2, Denniston's maximal arcs in Desarguesian planes are investigated. A characterisation Theorem due to V. Abatangelo and B. Larato is proved (Theorem 2.2). And the full collineation stabiliser of Denniston maximal arcs is calculated in Theorem 2.3 by myself and T. Penttila. In subsection 2.2, the duals of Denniston maximal arcs (which are also Denniston) are examined, and their parameters calculated in Theorem 2.4.

In Section 3, a construction due to J.A. Thas in 1974 is given. These are characterised in Theorem 3.4, and another characterisation is given for those that occur in Desarguesian planes (Theorem 3.5.) A result due to A. Blokhuis and H. Wilbrink shows that Thas 1974 maximal arcs do not exist in planes of odd order (Theorem 3.3).

In Section 4, a construction due to J.A. Thas in 1980 is given. It is proved that this construction when applied to Desarguesian planes gives rise to Denniston maximal arcs (Theorem 4.2). Exactly which Denniston maximal arcs arise in this way in shown in Lemma 4.1 and its corollary.

In Section 5, maximal arcs in Desarguesian planes whose collineation stabiliser is transitive on the points of the maximal arc are investigated and shown not to exist for a large number of parameters. And finally in Section 6, the concept of a *linear* maximal arc is introduced with a view to providing a framework to study known examples, and perhaps give new constructions in the future.

2. DENNISTON MAXIMAL ARCS IN $PG(2,q)$, q EVEN.

2.1 The construction and characterisation of Denniston maximal arcs

In this chapter $trace_{\frac{GF(2^{ed})}{GF(2^e)}}$ will denote the usual trace function from $GF(2^{ed})$ to $GF(2^e)$, i.e. for $\lambda \in GF(2^{ed})$,

$$trace_{\frac{GF(2^{ed})}{GF(2^e)}}(\lambda) = \lambda + \lambda^{2^e} + \cdots + \lambda^{(2^e)^{d-1}}.$$

In the following the Desarguesian plane $PG(2,2^e)$ is represented via homogeneous coordinates over the Galois field $GF(2^e)$. I.e. represent the points of $PG(2,2^e)$ by $\langle(x,y,z)\rangle$, $x,y,z \in GF(2^e)$ and $(x,y,z) \neq (0,0,0)$, and similarly lines by $\langle[a,b,c]\rangle$ $a,b,c \in GF(2^e)$ and $[a,b,c] \neq [0,0,0]$. Incidence is given by the dot product $\langle(x,y,z)\rangle I \langle[a,b,c]\rangle \Leftrightarrow ax+by+cz=0$. To avoid awkward notation the angle brackets will be dropped in the following.

Let $\xi^2+\alpha\xi+1$ be an irreducible polynomial over $GF(2^e)$, and let $\mathcal{F}_{std}$ be the set of conics given by the pencil

$$F_\lambda : x^2+\alpha xy+y^2+\lambda z^2=0, \quad \lambda \in GF(2^e)\cup\{\infty\}.$$

Then F_0 is the point $(0,0,1)$, F_∞ is the line $z^2=0$ (which we shall call the *line at infinity*). Every other conic in the pencil is non-degenerate and has nucleus F_0. Further, the pencil is a partition of the points of the plane. For convenience, this pencil of conics will be referred to as the **standard pencil**.

In it is well known (see for instance [1]) that the pencil $\mathcal{F}_{std}$ is stabilised by the following cyclic group of order 2^e+1, the orbits being the conics of the pencil.

$$C_{2^e+1} = \left\{ \begin{pmatrix} a+\alpha b & b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} : a^2+\alpha ab+b^2=1 \right\}.$$

Note that all cyclic subgroups of $PGL(3,2^e)$ of order 2^e+1 are conjugate in $PGL(3,2^e)$ (see for instance [1, Lemma 6] or [96, Lemma 3.2].) Hence, up to isomorphism, the choice of pencil of conics is arbitrary (as long as the pencil is stabilised by a cyclic linear group of order 2^e+1).

In 1969, R. H. F. Denniston proved the following Theorem.

THEOREM 2.1 (Denniston [59]) *If A is an additive subgroup of $GF(2^e)$ of order n, then the set of points of all F_λ for $\lambda \in A$ form a maximal $\{2^e(n-1)+n;n\}-$arc $\mathcal{K}$ in $PG(2,2^e)$.*

Proof: We show every line of the plane meets $\mathcal{K}$ in either 0 or n points. The proof given here is essentially that given in [78, 12.2.2] with the slight simplification that since C_{2^e+1} stabilises $\mathcal{K}$, and acts regularly on the points on the line at infinity, we only need show that the lines on a given point on the line at infinity meet $\mathcal{K}$ in 0 or n points.

Let $P=(1,1,0)$. We show every line on P meets $\mathcal{K}$ in 0 or n points. The lines on P are $\{[1,1,d] : d \in GF(2^e)\} \cup \{[0,0,1]\}$. Notice that $[0,0,1]$ is external to $\mathcal{K}$. Also the line $[1,1,0]$ is the line joining P to the nucleus of the pencil. Any line on the nucleus is tangent to each of the conics of the pencil, and so $[1,1,0]$ meets $\mathcal{K}$ in $|A|=n$ points.

Assume $d \neq 0$. Then any line $[1,1,d]$ does not meet the nucleus of the conics and so meets each conic of the pencil in either 0 or 2 points. Since $[1,1,d]$ is incident with 2^e points not on the line at infinity it must meet $2^e/2$ of the conics of the pencil in 2 points, and $2^e/2$ of the conics in 0 points.

The points on a line $[1,1,d]$ are $\{(x, x+d, 1) : x \in GF(2^e)\} \cup \{P\}$. So $[1,1,d]$, $d \neq 0$, does not meet the conic F_λ, $\lambda \in GF(2^e)$ if and only if

$$x^2 + \alpha x(x+d) + (x+d)^2 + \lambda = 0$$

has no solution for any $x \in GF(2^e)$. This is equivalent to

$$trace_{\frac{GF(2^e)}{GF(2)}}(\frac{d^2+\lambda}{\alpha d^2}) = 1.$$

The trace map is additive, and $trace_{\frac{GF(2^e)}{GF(2)}}(1/\alpha) = 1$ (since $\xi^2 + \alpha\xi + 1$ is irreducible), so we get the condition

$$trace_{\frac{GF(2^e)}{GF(2)}}(\frac{\lambda}{\alpha d^2}) = 0.$$

Notice that if $[1,1,d]$ does not meet the conics $F_{\lambda_1}, F_{\lambda_2}$ then this condition shows that it also does not meet $F_{\lambda_1+\lambda_2}$. Hence, if we define the set

$$H_d = \{\lambda \in GF(2^e) : [1,1,d] \cap F_\lambda = \emptyset\},$$

then H_d is an (additive) subgroup of index 2 in $GF(2^e)$. Define $H'_d = GF(2^e) - H_d$, H'_d is the set of $\lambda \in GF(2^e)$ such that $[1,1,d]$ meets F_λ in 2 points. Now if A is a subgroup of H_d, then $A \cap H'_d = \emptyset$, and so $[1,1,d] \cap \mathcal{K} = \emptyset$. If A is not a subgroup of H_d then $GF(2^e) = AH_d$ and so

$$\frac{GF(2^e)}{H_d} = \frac{AH_d}{H_d} \cong \frac{A}{H_d \cap A}.$$

Hence since $A \cap H_d$ has index 2 in H_d, $|A \cap H'_d| = \frac{|A|}{2} = n/2$. Also every $\lambda \in A \cap H'_d$ corresponds to $[1,1,d]$ meeting the conic F_λ in 2 points, so it follows that $[1,1,d]$ meets $\mathcal{K}$ in n points. ∎

Since additive subgroups of $GF(2^e)$ of size n exist for all n dividing 2^e, Denniston's result shows that the necessary condition for a maximal arc of degree n to exist, that n divide 2^e, is also sufficient in $PG(2, 2^e)$.

Now suppose that $\mathcal{K}$ is a maximal arc in $PG(2, 2^e)$ stabilised by a linear cyclic group of order $2^e + 1$. Then, by the remarks before Theorem 2.1, without loss of generality, there is a subset A of $GF(2^e)$ with $0 \in A$, and such that $\mathcal{K}$ is the union of points of F_λ, $\lambda \in A$ (where $F_\lambda : \lambda \in GF(2^e) \cup \{\infty\} = \mathcal{F}_{std}$). Then we can define

$$X = \{d \in GF(2^e)^* : [1,1,d] \cap \mathcal{K} = \emptyset\}$$

Then with H_d defined as in the proof of the theorem, it is clear that $A \subseteq H_d$ for any $d \in X$, and further that defining a set H by

$$H = \bigcap_{d \in X} H_d \qquad \text{then} \qquad A \subseteq H \leq (GF(2^e), +).$$

In fact $\mathcal{K}$ being a maximal arc forces $A = H$ as the following shows. Let $\lambda \notin A$. We show that $\lambda \notin H$. Since $F_\lambda \cap \mathcal{K} = \emptyset$, on any point of F_λ there is some line l such that $l \cap \mathcal{K} = \emptyset$ and $|l \cap F_\lambda| = 2$. Now l meets the line at infinity in a point, and C_{2^e+1} is regular on the points on the line at infinity, and so $\exists h \in C_{2^e+1}$ such that $l^h = [1, 1, d']$ for some $d' \in GF(2^e)^*$. But C_{2^e+1} stabilises each of the conics of the pencil, and so preserves $\mathcal{K}$. It follows that $l^h \cap \mathcal{K} = \emptyset$ and $|l^h \cap F_\lambda| = 2$, and hence that $\lambda \notin H_{d'}$, for some $d' \in X$, and so $\lambda \notin H$. Thus $H = A$. Also H is the intersection of additive subgroups of $GF(2^e)$, and so is an additive subgroup itself. *Hence for the set of points of F_λ, $\lambda \in A$ to define a maximal arc A must be additive.* This result is summarised in the following characterisation theorem.

THEOREM 2.2 (Abatangelo & Larato [1]) *If a maximal arc $\mathcal{K}$ in $PG(2, 2^e)$, is invariant under a linear collineation group of $PG(2, 2^e)$ which is cyclic and has order $2^e + 1$, then $\mathcal{K}$ is a Denniston maximal arc.* ∎

COROLLARY The dual of Denniston maximal arc is a Denniston maximal arc.

∎

In the next Theorem the full collineation stabilisers of the Denniston maximal arcs are calculated. The Theorem does not hold for Denniston maximal arcs that are (regular) hyperovals or duals of (regular) hyperovals, so we first state some well known results for these two cases.

The full collineation group of a regular hyperoval in $PG(2, 2^e)$ (and so the dual of a regular hyperoval) is isomorphic to $P\Gamma L(2, 2^e)$ for $e > 2$ [96], and is isomorphic to S_4 and S_6 as $e = 1$ and 2, respectively [78, 8.4.2 Cor.6]. A maximal arc formed from the dual of a regular hyperoval has the interesting property that the full collineation stabiliser of the maximal arc is transitive on the points of the maximal arc (see Section 5.)

In the proof of the Theorem the following two types of collineations of a projective plane will be of interest.

A point x is a **centre** of a collineation of a projective plane if all the lines on x are fixed by the collineation. A line L of a projective plane is an **axis** of a collineation of the plane if every point of L is fixed by the collineation. A non-trivial collineation has at most one centre and at most one axis, and has a centre if and only if it has an axis [55, 1.4.9]. An **elation** is a collineation that has a centre, and an axis incident with the centre. A **homology** is a collineation that has a centre, and an axis not incident with the centre. In a projective plane of order q, the order of an elation must divide q, and the order of a homology must divide $q-1$ [55, 1.4.12]. The orbits of the group generated by an elation or homology on the non-fixed points all have length the order of the elation or homology [78, p.152, p.153].

THEOREM 2.3 (Hamilton & Penttila) *In $PG(2, 2^e)$, $e > 2$, let $\mathcal{K}$ be a degree n Denniston maximal arc, $2 < n < q/2$, with additive subgroup A. Define the group G acting on $GF(2^e)$ by*

$$G = \{x \mapsto ax^\sigma : \quad a \in GF(2^e)^*, \quad \sigma \in \ Aut\ GF(2^{2e})\}.$$

Then the collineation stabiliser of $\mathcal{K}$ is isomorphic to $C_{2^e+1} \rtimes G_A$, the semidirect product of a cyclic group of order (2^e+1) with the stabiliser of A in G.

Proof: Let $\mathcal{F} = \{F_\lambda : \lambda \in GF(2^e) \cup \{\infty\}\}$, with F_0 the nucleus and F_∞ the line at infinity, be the pencil of conics associated with $\mathcal{K}$, and C_{2^e+1} be a linear cyclic group of order 2^e+1 stabilising $\mathcal{K}$. Notice that the orbits of C_{2^e+1} are the elements of the pencil $\mathcal{F}$. We show that the group $P\Gamma L(3,q)_{\mathcal{K}}$ of collineations that stabilise $\mathcal{K}$ is a subgroup of the normaliser $N_{P\Gamma L(3,q)}(C_{2^e+1})$ of C_{2^e+1} in the collineation group of the plane.

Let $g \in P\Gamma L(3,q)_{\mathcal{K}}$, and let F_μ, $\mu \in A - \{0\}$ be any conic of $\mathcal{F}$ contained in $\mathcal{K}$. Then ${F_\mu}^g$ is also a conic contained in $\mathcal{K}$. There are $n-1$ conics in the set $\{F_\lambda : \lambda \in A - \{0\}\}$. Hence since $n \leq q/4$ and there are at least q points of ${F_\mu}^g$ not on the nucleus of the pencil, it follows that

$$max\{|{F_\mu}^g \cap F_\lambda| : \lambda \in A - \{0\}\} \geq \frac{q}{n-1} \geq \frac{q}{q/4-1} > 4$$

and so ${F_\mu}^g$ meets some (non-degenerate) conic of $\mathcal{F}$ in at least 5 points. But 5 points of a non-degenerate conic determine that conic, and so ${F_\mu}^g = F_\lambda$ for some $\lambda \in A$. Hence g permutes of the conics of $\mathcal{F}$ in $\mathcal{K}$, and g also fixes the nucleus F_0. Dually, g fixes the nucleus of the dual (Denniston) maximal arc, and so fixes the line at infinity F_∞ of $\mathcal{F}$.

Notice that a non-degenerate conic F_μ of $\mathcal{F}$ together with the line at infinity F_∞ uniquely determine a pencil of the form required for the Denniston construction. One way to see this is to notice that the equation of any conic of the pencil can be written as a linear combination of the equation of F_μ and the equation of F_∞, and so any collineation group (i.e. C_{2^e+1}) that preserves F_μ and F_∞ must preserve all the conics of the pencil. Another way to see this is to note that there is a unique homology group H of order 2^e-1 with centre F_0 and axis F_∞. The conics of the pencil $\mathcal{F}$ are then (uniquely) given as $\{{F_\mu}^h : h \in H\}$.

Now g permutes the set of conics of $\mathcal{F}$ in $\mathcal{K}$, and since g fixes the line at infinity, it follows from the previous paragraph that g also permutes the conics of $\mathcal{F}$.

Hence g is an element of the collineation stabiliser of the pencil. We show that the collineation stabiliser of the pencil is the normaliser $N_{P\Gamma L(3,q)}(C_{2^e+1})$, and so $g \in N_{P\Gamma L(3,q)}(C_{2^e+1})$.

First note that any $n \in N_{P\Gamma L(3,q)}(C_{2^e+1})$ permutes the orbits of C_{2^e+1}, and so is an element of the collineation stabiliser of the pencil.

The stabiliser of the pencil is transitive on the $2^e - 1$ non-degenerate conics of the pencil. From the comments in the previous paragraphs, the order of the collineation stabiliser of the pencil is then $2^e - 1$ times the order of the collineation stabiliser of a non-degenerate conic F_μ of the pencil and the line F_∞. But the collineation stabiliser of a non-degenerate conic is isomorphic to $P\Gamma L(2,q)$, which is transitive on lines external to the conic. There are $\frac{1}{2}(2^{2e} - 2^e)$ such lines (the number of points of the dual of a hyperoval), and so the order of the collineation stabiliser of F_μ and F_∞ is $|P\Gamma L(2,2^e)|/(\frac{1}{2}(2^{2e} - 2^e)) = 2e(2^e+1)$. Hence the order of the collineation stabiliser of the pencil is $2e(2^{2e} - 1)$.

The centraliser $C_{P\Gamma L(3,q)}(C_{2^e+1})$ is normal in $N_{P\Gamma L(3,q)}(C_{2^e+1})$, and it follows by [83, Theorem $II.7.3$] that

$$N_{P\Gamma L(3,q)}(C_{2^e+1}) \cong GF(2^{2e})^* \rtimes \text{ Aut } GF(2^{2e}).$$

So the order of $N_{P\Gamma L(3,q)}(C_{2^e+1})$ is also $2e(2^{2e} - 1)$. And hence $N_{P\Gamma L(3,q)}(C_{2^e+1})$ is exactly the collineation stabiliser of the pencil, and so $g \in N_{P\Gamma L(3,q)}(C_{2^e+1})$.

So we have shown that $P\Gamma L(3,q)_{\mathcal{K}} \leq N_{P\Gamma L(3,q)}(C_{2^e+1})$. We now examine how this normaliser acts on the plane.

The centraliser $C_{P\Gamma L(3,q)}(C_{2^e+1}) \cong GF(2^{2e})^*$ acts regularly on the points of the plane not on F_∞ and not equal to F_0, and so we may identify such points with elements of $GF(2^{2e})^*$ by taking an arbitrary point P not on F_∞ and not equal to F_0, and identifying it with $1 \in GF(2^{2e})$. The group $N_{P\Gamma L(3,q)}(C_{2^e+1})$ then acts on $GF(2^{2e})^*$ by

$$x \mapsto ax^\sigma, \quad a \in GF(2^{2e})^*, \quad \sigma \in \text{ Aut } GF(2^{2e}).$$

Let η be primitive in $GF(2^{2e})$. Define $r = \frac{2^{2e}-1}{2^e-1} = 2^e + 1$ and $s = \frac{2^{2e}-1}{2^e+1} = 2^e - 1$, then η^r has order $2^e - 1$ and η^s has order $2^e + 1$ in $GF(2^{2e})^*$. The orbits of $\langle \eta^s \rangle$ on the plane are then the conics of the pencil. The conics can then be described by the sets

$$\begin{aligned}
C_{\eta^r} &= \{\eta^r, \eta^{s+r}, \eta^{2s+r}, \ldots, \eta^{qs+r}\} \\
&\;\vdots \qquad\quad \vdots \\
C_{\eta^{ir}} &= \{\eta^{ir}, \eta^{s+ir}, \eta^{2s+ir}, \ldots, \eta^{qs+ir}\}. \\
&\;\vdots \qquad\quad \vdots \\
C_{\eta^{2^e-1}} &= \{1, \eta^s, \eta^{2s}, \ldots, \eta^{qs}\}
\end{aligned}$$

Notice that η^r is a homology that cyclicly permutes the conics. Also $\langle\eta^r\rangle = GF(2^e)^* < GF(2^{2e})^*$, and as long as we chose the point of the plane corresponding to 1 to be a point of the conic F_1, then $F_{\eta^{ir}} = C_{\eta^{ir}}$ for every $i = 1, \ldots, 2^e - 1$.

We now show that $N_{P\Gamma L(3,q)}(C_{2^e+1}) \cong \langle\eta^s\rangle \rtimes (\langle\eta^r\rangle \rtimes \text{ Aut } GF(2^{2e}))$.

First notice that $\langle\eta^s\rangle$ and $\langle\eta^r\rangle$ are normal subgroups of $N_{P\Gamma L(3,q)}(C_{2^e+1})$. Consider the group given by the product $\langle\eta^s\rangle\text{Aut } GF(2^{2e})$. The group $\langle\eta^s\rangle$ acts semiregularly on $GF(2^{2e})^*$, in particular the stabiliser of 1 is the identity. The stabiliser of 1 in the group Aut $GF(2^{2e})$ is Aut $GF(2^{2e})$, and so Aut $GF(2^{2e})$ is a complement for $\langle\eta^s\rangle$ in $\langle\eta^s\rangle\text{Aut } GF(2^{2e})$. Hence $\langle\eta^s\rangle\text{Aut } GF(2^{2e}) \cong \langle\eta^s\rangle \rtimes \text{ Aut } GF(2^{2e})$.

Since 2^e-1 and 2^e+1 are coprime, it follows that the group given by the product $\langle\eta^r\rangle\text{Aut } GF(2^{2e})$ is a complement for $\langle\eta^s\rangle$ in $N_{P\Gamma L(3,q)}(C_{2^e+1})$. Also, arguing as in the previous paragraph, it follows that Aut $GF(2^{2e})$ is a complement for $\langle\eta^r\rangle$ in $\langle\eta^r\rangle\text{Aut } GF(2^{2e})$. Hence $N_{P\Gamma L(3,q)}(C_{2^e+1}) \cong \langle\eta^s\rangle \rtimes (\langle\eta^r\rangle \rtimes \text{ Aut } GF(2^{2e}))$.

The stabiliser of the maximal arc $\mathcal{K}$ is then given by

$$\langle\eta^s\rangle \rtimes (\langle\eta^r\rangle \rtimes \text{ Aut } GF(2^{2e}))_A.$$

The result follows. ∎

COROLLARY (Hamilton & Penttila) *In $PG(2, 2^e)$ let $\mathcal{F}$ be a pencil of conics stabilised by a linear cyclic group of order $2^e + 1$, with F_0 the nucleus, and F_∞ the line at infinity. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Denniston maximal arcs constructed from $\mathcal{F}$ and additive subgroups A_1 and A_2 respectively. Then $\mathcal{K}_1$ is isomorphic to $\mathcal{K}_2$ if and only if $A_1 = {A_2}^g$ for some $g \in G$, where*

$$G = \{x \mapsto ax^\sigma : \quad a \in GF(2^e)^*, \quad \sigma \in Aut(GF(2^e))\}.$$

Proof: The Corollary follows from the details of the proof of the Theorem and by considering the action of Aut $GF(2^{2e})$ on $GF(2^e)^*$. The automorphism $x \mapsto x^{2^e}$ acts as the identity on $GF(2^e)^*$. Hence Aut $GF(2^{2e})$ acts on $GF(2^e)$ as Aut $GF(2^e)$ with kernel $\langle x \mapsto x^{2^e}\rangle$. ∎

Notice that since $x \mapsto x^{2^e}$ acts as the identity on $GF(2^e)$, it fixes pointwise the line containing the points corresponding to $GF(2^e)$. It follows that this line is the axis of the collineation, and since the collineation has order 2 it must be an elation. Also, since each conic $C_{\eta^{ir}}$ contains exactly one point of this line, each of the conics is stabilised by this collineation. Direct calculation then show that $x \mapsto x^{2^e}$ together with the cyclic group C_{2^e+1} generate a dihedral group which stabilises the set of conics of the pencil pointwise, and so any Denniston maximal arc in $PG(2, 2^e)$ admits a dihedral group of order $2(2^e + 1)$ in its collineation stabiliser.

Using this Corollary it is easy to show by direct calculation in $GF(16)$ that there are exactly 2 degree 4 Denniston maximal arcs in $PG(2, 16)$ up to isomorphism. Let ω be a primitive element in $GF(16)$ such that $\omega^4 + \omega = 1$, then representatives for the additive subgroups may be described by

$$A_1 = GF(4) = \{0, 1, \omega^5, \omega^{10}\} \quad \text{and} \quad A_2 = \{0, 1, \omega^1, \omega^4\}.$$

They give rise to maximal arcs with full collineation stabilisers of order 408 and 68 respectively. Since the collineation stabilisers have different orders these maximal arcs have the interesting property that each is isomorphic to its dual maximal arc.

Similarly, in $PG(2, 32)$ it can be shown that there is up to isomorphism a unique degree 4 Denniston maximal arc (and so a unique degree 8 Denniston maximal arc.)

2.2 The duals of Denniston maximal arcs

As was shown in the Corollary to Theorem 2.2, the dual of a Denniston maximal arc is also a Denniston maximal arc. In this subsection some calculations are done to identify the dual maximal arcs.

THEOREM 2.4 *Let $\mathcal{F}_{std}$ be the standard pencil of conics. Let A be an additive subgroup of $GF(2^e)$, and $\mathcal{K}$ the Denniston maximal arc in $PG(2, 2^e)$ determined by A and $\mathcal{F}_{std}$. Then the dual maximal arc $\mathcal{K}'$ of $\mathcal{K}$ has points determined by the standard pencil and additive subgroup*

$$A' = \{\alpha^2 s : s \in GF(2^e)^* \ \textit{and}\ trace_{\frac{GF(2^e)}{GF(2)}}(\lambda s) = 0 \quad \forall \lambda \in A\} \cup \{0\}.$$

Proof: Firstly, suppose we have a matrix $g \in GL(3, q)$ acting on $PG(2, q)$ by

$$P^g = g \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for any point $P = (a, b, c)$. Then it is easily verified that the action of g on homogenous coordinates for lines is given by

$$l^g = g^{-T} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for any line $[x, y, z]$ (where g^{-T} is the inverse transpose of g).

We consider the pencil of conics $\mathcal{F}'$ stabilised by C_{2^e+1} (defined in the last subsection) in the dual plane. Clearly there is a unique such pencil in the dual plane. Also $g^T = g$ for every $g \in C_{2^e+1}$, so the orbits of C_{2^e+1} on the lines of $PG(2, 2^e)$ are exactly the same as that on points. Hence we may set $\mathcal{F}' = \mathcal{F}_{std}$.

We now calculate the elements of the additive subgroup A' which, together with $\mathcal{F}' = \mathcal{F}_{std}$, give rise to the dual (Denniston) maximal arc. To attempt to avoid confusion we shall use italics to distinguish between points of $PG(2, 2^e)$ and *points* of the dual plane, similarly for lines. Note that (a, b, c) is a point of $PG(2, 2^e)$ and is a *line* of the dual plane.

First notice that as with $\mathcal{K}$ the nucleus of $\mathcal{K}'$ is the *point* $[0, 0, 1]$ and the *line* at infinity is $(0, 0, 1)$.

Let $P = (1, 1, 0)$. Using arguments similar to those of Theorem 2.1 we find the lines on P which are external to $\mathcal{K}$.

The lines on P are $\{[1, 1, d] : d \in GF(2^e)\} \cup \{[0, 0, 1]\}$. Notice that $[0, 0, 1]$ is external to $\mathcal{K}$, and the line $[1, 1, 0]$ is the line joining P to the nucleus of $\mathcal{K}$, and is secant to $\mathcal{K}$, so we assume $d \neq 0$. The points on a line $[1, 1, d]$ are $\{(x, x + d, 1) : x \in GF(2^e)\} \cup \{P\}$. So $[1, 1, d]$, $d \neq 0$, is external to $\mathcal{K}$ if and only if

$$x^2 + \alpha x(x + d) + (x + d)^2 + \lambda = 0$$

has no solution for any $\lambda \in A$. Which is equivalent to

$$trace_{\frac{GF(2^e)}{GF(2)}}(\frac{d^2 + \lambda}{\alpha d^2}) = 1, \quad \forall \lambda \in A.$$

The trace map is additive, and $trace_{\frac{GF(2^e)}{GF(2)}}(1/\alpha) = 1$ (since $\xi^2 + \alpha\xi + 1$ is irreducible), so we get the condition

$$trace_{\frac{GF(2^e)}{GF(2)}}(\frac{\lambda}{\alpha d^2}) = 0, \quad \forall \lambda \in A.$$

So the lines on P that are external to $\mathcal{K}$ are

$$\{[1, 1, d] : d \in GF(2^e)^* \text{ and } trace_{\frac{GF(2^e)}{GF(2)}}(\frac{\lambda}{\alpha d^2}) = 0, \quad \forall \lambda \in A\} \cup \{[0, 0, 1]\}.$$

Substituting each of these *points* into $x^2 + \alpha xy + y^2 + \mu z^2 = 0, \mu \in A'$ and solving for μ then gives that

$$A' = \{\alpha/d^2 : d \in GF(2^e)^* \text{ and } trace_{\frac{GF(2^e)}{GF(2)}}(\frac{\lambda}{\alpha d^2}) = 0 \quad \forall \lambda \in A\} \cup \{0\}.$$

Putting $s = 1/\alpha d^2$ gives the required result. Note that we have implicitly used the fact that $P = (1, 1, 0)$ is a *line* through the nucleus of $\mathcal{K}'$ so that the elements of

A' are in one to one correspondence with the *points* of $\mathcal{K}'$ on the *line* $P = (1, 1, 0)$. ∎

The following Corollary will be of interest when Thas 1980 maximal arcs are considered in Section 4.

COROLLARY *In $PG(2, 2^{ed})$, the dual of a Denniston maximal arc with additive subgroup $A = GF(2^e)$ is isomorphic to a Denniston maximal arc described by the standard pencil and additive subgroup $A' = Kernel(trace_{\frac{GF(2^{ed})}{GF(2^e)}})$.*

Proof: By the Theorem

$$A' = \{\alpha^2 s : s \in GF(2^{ed})^* \text{ and } trace_{\frac{GF(2^{ed})}{GF(2)}}(\lambda s) = 0 \quad \forall \lambda \in GF(2^e)\} \cup \{0\}.$$

First notice that by the Corollary to Theorem 2.3 that $A'' = \alpha^{-2}A'$ determines an equivalent Denniston maximal arc. We now use some elementary properties of trace to get the required result.

Let $\lambda \in GF(2^e)$, then $\lambda^{2^e} = \lambda$, and let $s \in Kernel(trace_{\frac{GF(2^{ed})}{GF(2^e)}})$. Then

$$\begin{aligned}
trace_{\frac{GF(2^{ed})}{GF(2)}}(\lambda s) &= trace_{\frac{GF(2^e)}{GF(2)}}(trace_{\frac{GF(2^{ed})}{GF(2^e)}}(\lambda s)) \\
&= trace_{\frac{GF(2^e)}{GF(2)}}(\lambda s + (\lambda s)^{2^e} + \cdots + (\lambda s)^{(2^e)^{d-1}}) \\
&= trace_{\frac{GF(2^e)}{GF(2)}}(\lambda[s + (s)^{2^e} + \cdots + (s)^{(2^e)^{d-1}}]) \\
&= trace_{\frac{GF(2^e)}{GF(2)}}(\lambda . trace_{\frac{GF(2^{ed})}{GF(2^e)}}(s)) \\
&= trace_{\frac{GF(2^e)}{GF(2)}}(\lambda . 0) \\
&= 0
\end{aligned}.$$

Hence every $s \in Kernel(trace_{\frac{GF(2^{ed})}{GF(2^e)}})$ is an element of A''.
Also $|Kernel(trace_{\frac{GF(2^{ed})}{GF(2^e)}})| = (2^e)^{d-1}$ which is the degree of the maximal arc dual to $\mathcal{K}$. Hence result. ∎

3. THAS 1974 MAXIMAL ARCS IN TRANSLATION PLANES OF ORDER q^2 WITH KERNEL $GF(q)$

In 1974, J. A. Thas constructed a class of maximal arcs in certain translation planes of order q^2 [127]. We shall refer to these as Thas 1974 maximal arcs to distinguish them from another construction due to Thas (see Section 4).

In this section the construction is given and the Thas 1974 maximal arcs are characterised as being exactly those maximal arcs that are stabilised by a homology of order $q-1$ (Theorem 3.4), with an equivalent condition involving Baer sublines being given in the Desarguesian case (Theorem 3.5). The characterisation was joint work between myself and Tim Penttila and appeared in a different form in [72].

3.1 The Thas 1974 *construction and associated structures*

A projective plane π is a **translation plane** if there is a line L in π such that the group of elations of π with axis L is transitive on the points of $\pi\backslash L$. The line L is called a **translation line** of π. In particular, a finite Desarguesian plane is a translation plane, and any line of the plane is a translation line. The order of a finite translation plane is necessarily a power of a prime [103, 1.9].

A **spread** $\mathcal{S}$ of $PG(3,q)$ is a set of lines such that every point of $PG(3,q)$ lies on exactly one line of $\mathcal{S}$. Equivalently, it is a set of q^2+1 pairwise skew lines [79, 17.1].

An **ovoid** of $PG(3,q)$, $q>2$, is a set of q^2+1 points no 3 collinear. (If $q=2$ an ovoid is a set of 5 points no 4 coplanar.) [79, p.52]. A line of $PG(3,q)$ is said to be **external**, **tangent** or **secant** to an ovoid $\mathcal{O}$ if it meets $\mathcal{O}$ in 0, 1 or 2 points, respectively. Ovoids arise in many areas of projective geometry such as in the study of inversive planes, generalised quadrangles and polar spaces (for instance see [64][112]). Ovoids have many interesting properties, but the only one of concern to us here is that every plane of $PG(3,q)$ meets $\mathcal{O}$ in either 1 or $q+1$ points [79, 16.1.6].

Let $PG(3,q)$ be embedded as a hyperplane H in $PG(4,q)$, and let $\mathcal{S}$ be a spread of H. Construct a new incidence structure $\pi=\pi(\mathcal{S})$ as follows:

The *points* of π are the points of $PG(4,q)\backslash H$, together with the elements of the spread.
The *lines* of π are the planes of $PG(4,q)$ which intersect H in a member of $\mathcal{S}$, together with the line (often denoted L_∞) whose points are the elements of the spread.
The *incidence* relation of π is that induced by the incidence in $PG(4,q)$.

Then π is a translation plane of order q^2, with translation line L_∞. This construction is a particular case of a general construction (see Section 4) of translation planes due to R. H. Bruck and R. C. Bose [28], [29]. The general Bruck-Bose construction is the projective version of an affine construction due to André [4].

We shall refer to a spread which gives rise to a Desarguesian projective plane as a **Desarguesian spread**. A Desarguesian spread of $PG(3,q)$ corresponds to a *regular* spread [55, 5.1.3].

The Thas 1974 maximal arcs are constructed in the following manner.

THEOREM 3.1 (Thas [127]) *Let $PG(3,q)$ be embedded as a hyperplane H in $PG(4,q)$. Let $\mathcal{O}$ be an ovoid and $\mathcal{S}$ a spread of H, such that each line of $\mathcal{S}$ has exactly one point in common with $\mathcal{O}$. Let $x \in PG(4,q)\backslash H$, define $\mathcal{K}$ to be the union of the points of $PG(4,q)\backslash H$ that are on the lines of $PG(4,q)$ given by the span of x and each of the points of $\mathcal{O}$ (including the point x itself). Then $\mathcal{K}$ is a maximal $\{q^3-q^2+q;q\}-$arc in the translation plane π defined by the spread $\mathcal{S}$.*

Proof: We show that every line meets $\mathcal{K}$ in 0 or q points.

Clearly L_∞ does not meet $\mathcal{K}$, so consider any other line L of π represented as a plane in $PG(4,q)$.

(a) Suppose x is incident with L. Then the plane representing L meets the 3-space H in a line s of the spread, and s contains a unique point o of the ovoid $\mathcal{O}$. Hence the plane representing L meets $\mathcal{K}$ in the q points in $PG(4,q)\backslash H$ on the line $\bar{xo}$.

(b) Suppose L is not incident with x. The projection through x of the plane representing L onto H is then a plane of H. This plane meets $\mathcal{O}$ in either 1 or $q+1$ points (each case including the point of the ovoid that is contained in the plane representing L). The former case corresponds to L being external to $\mathcal{K}$, the later to meeting $\mathcal{K}$ in q points. ∎

For reasons that will become apparent in the next subsection, we shall call x the **centre** of the Thas 1974 construction.

The known ovoids of $PG(3,q)$, q even, are the elliptic quadrics and the Tits ovoids (Tits ovoids are only defined for $q = 2^{2m+1}, m \geq 1$).

A canonical form for an elliptic quadric is

$$\mathcal{O} = \{(t,s,1,t^2+st+as^2) : s,t \in GF(q)\} \cup \{(0,0,0,1)\}$$

where $\xi^2+\xi+a$ is irreducible over $GF(q)$ [93]. A canonical form for a Tits ovoid is

$$\mathcal{O} = \{(t,s,1,t^\sigma+st+s^{\sigma+2}) : s,t \in GF(q)\} \cup \{(0,0,0,1)\}$$

where $q = 2^{2m+1}, m \geq 1$ and $\sigma = 2^{m+1}$ [79, Theorem 16.4.5].

It is well known that any ovoid of $PG(3,q)$, q even, gives rise to a symplectic polarity of $PG(3,q)$[55]. The totally isotropic lines with respect to this polarity are the tangent lines to the ovoid. In fact, the points of $PG(3,q)$ together with the totally isotropic lines form a generalised quadrangle $W(q)$ [112, Chapter 3][126]. In this setting, the ovoid of $PG(3,q)$ is an ovoid (see Chapter 1, Section 3) of $W(q)$, and a spread of tangent lines is a spread of $W(q)$. If q is even, a spread of $W(q)$ can easily be constructed by taking the spread formed from the image of any ovoid of $W(q)$ under a duality of the associated generalised quadrangle. Other methods of construction exist, such as exploiting certain actions of the Suzuki group on $PG(3,q)$, see for instance [103, Theorem 27.3].

Given either a Tits ovoid or an elliptic quadric, the known spreads of $PG(3,q)$ of the form required for the Thas construction are the Desarguesian (regular) spreads and the Lüneburg spreads (Lüneburg spreads are only defined for $q=2^{2m+1}$, $m \geq 1$, in fact they are the duals of the Tits ovoids under a duality of $W(q)$) [126]. Note that given an elliptic quadric both Desarguesian and Lüneburg spreads of the required form exist; similarly, given a Tits ovoid both Desarguesian and Lüneburg spreads of the required form exist [126]. Hence the Thas construction is known to give maximal arcs in the Desarguesian and Lüneburg planes of even order.

In Section 4 it will be shown that an elliptic quadric-Desarguesian spread pair of the form required for the Thas 1974 construction gives rise to Denniston maximal arcs (an observation that was first made in [127].) It is also shown that up to isomorphism there is one such elliptic quadric-Desarguesian spread pair. T. Penttila has calculated (up to isomorphism) the Tits ovoid-Desarguesian spread pairs. He shows there are two orbits on Tits ovoid-Desarguesian spread pairs in the stabiliser of the Desarguesian spread and proves the following Theorem.

THEOREM 3.2 (Penttila [116]) *There are, up to equivalence under $P\Gamma L(3,q^2)$, two Thas 1974 maximal arcs in $PG(2,q^2)$ arising from Tits ovoids. They have stabilisers in $P\Gamma L(3,q^2)$ given by the semidirect product of a dihedral group of order $4(q \pm (2q)^{1/2}+1)(q-1)$ by a cyclic group of order $2e+1$, where $q=2^{2e+1}$.* ∎

The proof of Theorem 3.2 follows fairly quickly from results of [61, Chapter 2] or [7] on the intersections of ovoids of $W(q)$.

Until recently it was unknown whether Thas 1974 maximal arcs exist for q odd. Ovoids of $PG(3,q)$, q odd, were shown by A. Barlotti in 1955 to be elliptic quadrics ([10], see also [79, 16.1.7].) So the problem is reduced to whether a spread of tangent lines to an elliptic quadric exists for q odd. A. Blokhuis and H. Wilbrink use an elegant colouring argument to show that such a spread does not exist. We reproduce here their proof in full (with thanks).

THEOREM 3.3 (Blokhuis & Wilbrink [19]) *Given an elliptic quadric $\mathcal{O}$ in $PG(3,q)$, q odd, there does not exist a spread of $PG(3,q)$ such that each line of the spread is tangent to $\mathcal{O}$.*

Proof: The proof involves properties of elliptic quadrics and quadratic forms in projective spaces. Since this isn't a thesis about quadrics, the reader is referred to [80] or [125] for more information.

Let the elliptic quadric be given by the quadratic form Q. We color a point x

not on the quadric red if $Q(x)$ is a square in $GF(q)$, and blue if it is a non-square. On a tangent line then all non quadric points have the same color. Given a spread we can extend the coloring to the quadric by painting a point red if it is on a red tangent line (and blue otherwise). After this half of the points (of the quadric) will be red, and half will be blue. Now we consider the conic determined by the intersection of a plane π and the quadric. The internal points of this conic all have the same color, blue say, and there are $(q^2 - q)/2$ of them. The plane π cannot contain a blue tangent, so all blue tangents intersect π in a single point, which is either an internal point, or a point of the conic. It follows that exactly $(q+1)/2$ points of the conic are blue. So all conics have exactly half their points blue, and the other half red. Now consider the pencil of conics on a line through two red points of the quadric. Counting the total number of blue points on these conics we get a contradiction. ∎

COROLLARY *Thas* 1974 *maximal arcs do not exist for q odd.* ∎

3.2 Characterisation of Thas 1974 *maximal arcs*

A collineation g of $PG(t,q)$ is called an **homology** if there exists a hyperplane H of $PG(t,q)$ and a point $x \notin H$, such that every point of H, and every line through x, is fixed by g. The point x is called a **centre**, and H an **axis** of the homology g.

The following result is fundamental to our approach. It is a Corollary to results from [103, p.4, 1.12].

A translation plane π of order q^2 arises from the Bruck-Bose construction in $PG(3,q)$ detailed above if and only if π has a homology of order $q-1$ with axis a translation line of π. Moreover, an homology of $\pi(\mathcal{S})$ of order $q-1$ with axis L_∞ induces an homology of $PG(4,q)$ of order $q-1$ with axis H, and conversely.

THEOREM 3.4 (Hamilton & Penttila [72]) *Let π be a translation plane of order q^2. Then a non-trivial maximal arc $\mathcal{K}$ in π is a Thas* 1974 *maximal arc if and only if it is stabilised by a homology of order $q-1$ with axis a translation line of π.*

Proof: We first remark that there is a unique projective plane of order 4, and up to equivalence there is a unique non-trivial maximal arc in the plane. Since for $q = 2$ an homology of order $q-1$ is the identity, the result follows from the fact that this is a Thas maximal arc. So assume $q > 2$.

By the comments prior to the Theorem, there is a homology g of $\pi(\mathcal{S})$ of order $q-1$ with axis L_∞. Since the automorphism group of a translation plane is transitive on points not on L_∞, we can choose the centre of g to be the centre x. Corresponding to g, there is an homology $\hat{g}$ of $PG(4,q)$ of order $q-1$ that fixes the points of H and the lines of $PG(4,q)$ though x. Hence $\hat{g}$, and thus g stabilises $\mathcal{K}$.

Conversely, suppose that $\mathcal{K}$ is a maximal arc in a translation plane of order q^2 and is stabilised by an homology g of order $q-1$. We note that this translation plane can then be represented as $\pi = \pi(\mathcal{S})$ as above, with the axis of g being L_∞.

To prove that $\mathcal{K}$ is a Thas maximal arc we need the following lemma.

LEMMA 3.1 *Let $\mathcal{K}$ be a maximal $\{q^2(n-1)+n; n\}-$arc in a projective plane of order q^2, $q > 2$ a power of a prime. If $\mathcal{K}$ is stabilised by an homology of order $q-1$ then $\mathcal{K}$ is trivial or $n = q$. In the latter case the centre x is in $\mathcal{K}$, and the axis L is external.*

Proof: Let g be the homology stabilising $\mathcal{K}$. Let M be any secant line to $\mathcal{K}$ that contains x (i.e. M is fixed by g). Then since the orbits of $\langle g \rangle$ on M are all of length $q-1$ or are the fixed points x and $L \cap M$ we have,

$$n = |M \cap \mathcal{K}| \equiv 0, 1 \text{ or } 2 \pmod{q-1}$$

corresponding to the various cases of the centre being in $\mathcal{K}$, and whether or not $L \cap M$ is contained in $\mathcal{K}$. We know that n divides q^2, but $q-1$ does not divide q^2; so $n \not\equiv 0 \pmod{q-1}$.

Suppose that L is secant to $\mathcal{K}$, and $x \in \mathcal{K}$. Then if $L \cap M \notin \mathcal{K}$, we have $n = |M \cap \mathcal{K}| \equiv 1 \pmod{q-1}$. But if $L \cap M \in \mathcal{K}$, $n = |M \cap \mathcal{K}| \equiv 2 \pmod{q-1}$. Contradiction. Similarly, if L is secant to $\mathcal{K}$ and $x \notin \mathcal{K}$ we get a contradiction. So the axis L must be external to $\mathcal{K}$, and since $n \not\equiv 0 \pmod{q-1}$, $x \in \mathcal{K}$.

Put $q = p^h$, for some prime p and integer h, then $n = p^l$ with $l \leq 2h$. Consider the line M again. The set $M \cap \mathcal{K} \backslash \{x\}$ is the union of orbits of $\langle g \rangle$, so $p^h - 1$ divides $p^l - 1$, giving h divides l. Hence $l = 0$, h or $2h$ corresponding to $n = 1$ (trivial), $n = q$, or $n = q^2$ (trivial), respectively. ∎

Let x be the centre of g. Now, given a line L on x, it follows from the details of the proof for the forward implication that the union of x, an orbit of $\langle g \rangle$ on L of length $q-1$, and the point $L \cap L_\infty$ corresponds to a line of $PG(4,q)$ not in H. So it remains to be shown that the set $\mathcal{L}$ of lines of $PG(4,q)$ through x which meet $\mathcal{K}$ in q points intersects H in an ovoid of H (clearly $\mathcal{O} = \{L \cap H | L \in \mathcal{L}\}$ meets each element of the spread in exactly one point). To do this we need the following well known Lemma.

LEMMA 3.2 *Let $\mathcal{O}$ be a set of q^2+1 points in $PG(3,q)$, $q > 2$. If every plane of $PG(3,q)$ meets $\mathcal{O}$ in 1 or $q+1$ points, then $\mathcal{O}$ is a ovoid.*

Proof: Let N be a line of $PG(3,q)$ such that $|N \cap \mathcal{O}| = i \geq 2$. Then there are

$q+1$ planes through N, each intersecting $\mathcal{O}$ in $q+1$ points, thus

$$(q+1)(q-i+1)+i = q^2+1$$

giving $i=2$. So no 3 points of $\mathcal{O}$ are collinear and $\mathcal{O}$ is a ovoid. ∎

Let Σ be any plane in H. Then Σ contains a unique element $s \in \mathcal{S}$, and hence $|\Sigma \cap \mathcal{O}| \geq 1$. Choose $L \neq L_\infty$ to be any line of $\pi(\mathcal{S})$ not on x and such that the plane representing L satisfies $\langle x, L\rangle \cap H = \Sigma$ (in other words, choose L to be a plane on s contained in the projective 3-space $\langle x, \Sigma\rangle$).

Suppose $|\langle x, L\rangle \cap \mathcal{O}| \geq 2$, i.e. Σ contains more than 2 points of $\mathcal{O}$.

Let $y = L \cap \mathcal{O}$ and $z \in \Sigma \cap \{\mathcal{O}\backslash\{y\}\}$. Since a line and a plane contained in a projective 3-space must be incident, it follows that the line xz (of $PG(4,q)$) intersects L in a point of $PG(4,q)\backslash H$ (they are both contained in $\langle x, L\rangle$). Therefore $|L \cap \mathcal{K}| \neq 0$, and $|L \cap \mathcal{K}| = q$ since every line of $\pi(\mathcal{S})$ meets $\mathcal{K}$ in 0 or q points.

Thus there exist exactly q lines of $PG(4,q)$, w_1x, w_2x, ..., w_qx, with

$$w_i \in \mathcal{O} \text{ and } |L \cap w_i x| = 1 \quad \forall i = 1, 2, .., q.$$

Hence $|\langle x, L\rangle \cap \mathcal{O}| = q+1$. And so every plane Σ in H meets $\mathcal{O}$ in 1 or $q+1$ points, and hence $\mathcal{O}$ is an ovoid of H. ∎

This completes the proof of Theorem 3.4. ∎

THEOREM 3.5 (Hamilton & Penttila [72]) *Let $\mathcal{K}$ be a non-trivial maximal arc in $PG(2,q^2)$. Then $\mathcal{K}$ is* Thas 1974 *if and only if there exists a point $x \in \mathcal{K}$ and a line L external to $\mathcal{K}$ such that for every line M on x, $\{M \cap \mathcal{K}\} \cup \{M \cap L\}$ is a Baer subline.*

Proof: As before, the result is trivially true for $q = 2$ (noting that any 3 collinear points are a Baer subline).

We require the following well known Lemma.

LEMMA 3.3 *In the Bruck-Bose representation of the Desarguesian plane $PG(2,q^2)$ given above, the Baer sublines which meet L_∞ in a point are the lines of $PG(4,q)$ which meet but are not contained in the hyperplane H.*

Proof: A plane in $PG(4,q)$ that meets H in a line L not contained in the spread $\mathcal{S}$ corresponds to a Baer subplane of $PG(2,q^2)$ [139, 1.2], [138, Prop. 1]. Hence the lines of $PG(4,q)$ that meet but are not contained in H correspond to Baer sublines in $PG(2,q^2)$. It remains to be shown that these are the only Baer sublines that meet L_∞.

Let x be any point not on L_∞, and y any point on L_∞. Now, any two distinct Baer subplanes of $PG(2,q^2)$ containing 3 points on a line L of $PG(2,q^2)$ meet in exactly $q+1$ points of L, i.e. are the same Baer subline [40]. Since each Baer subline containing x and y contains $q+1$ points it follows that there can be at most

$$\frac{(q^2+1)-2}{(q+1)-2} = q+1$$

Baer sublines containing x and y. But there are $q+1$ distinct lines of $PG(4,q)$ joining any point $x \in PG(4,q)\backslash H$ to an element of the spread, each of which corresponds to a Baer subline of $PG(2,q^2)$. Hence these lines of $PG(4,q)$ correspond exactly to the Baer sublines which meet L_∞ in $PG(2,q^2)$. ∎

The forward implication of Theorem 3.5 then follows from Lemma 3.3 by choosing x to be the centre in the Thas construction, and L to be L_∞.

Conversely, assume there exists a point $x \in \mathcal{K}$ and a line L external to $\mathcal{K}$ such that for every line M incident with x, $\{M \cap \mathcal{K}\} \cup \{M \cap L\}$ is a Baer subline.

Put $L = L_\infty$ in the Bruck-Bose representation of $PG(2,q^2)$. Then the Baer sublines through x that meet L_∞ correspond to the lines of $PG(4,q)$ though x that meet H. Hence by the details of the proof of Theorem 3.4, $\mathcal{K}$ is stabilised by a homology of order $q-1$, and so is a Thas maximal arc. ∎

Note that there exist counterexamples to Lemma 3.3 for non-Desarguesian translation planes [15], [138], [139], [66], but if we replace "Baer subline" by "set of points corresponding to a line of $PG(4,q)$ that meets but is not contained in H", Lemma 3.3 and Theorem 3.5 hold for the non-Desarguesian cases.

4. THAS 1980 MAXIMAL ARCS IN SYMPLECTIC TRANSLATION PLANES

In 1980, J. A. Thas gave a construction of maximal $\{q^{2d-1}-q^d+q^{d-1}; q^{d-1}\}$–arcs in certain translation planes of order q^d [130]. The construction relied upon the existence of a pair of spreads of a particular form, one of a quadric, and one of a projective space. In this section the Thas 1980 construction is given (Theorem 4.1), and some existence and isomorphism problems are considered. In particular, it is shown that Thas 1980 maximal arcs in Desarguesian planes of even order are of Denniston type (Theorem 4.2), and the Denniston maximal arcs that they give rise to are identified.

Much of this section relies on knowledge of polar spaces and quadrics in projective spaces. But this thesis is not intended to be an exposition of these so the reader is referred to [80] and [125] for the relevant definitions and background information.

4.1 The Thas 1980 maximal arcs in translation planes

In Section 3 a particular case of a general construction of translation planes was given (those in Section 3 were the translation planes of order q^2 with kernel containing $GF(q)$.) We give the more general construction here. Any translation plane may be represented via the following construction due to R. H. Bruck and R. C. Bose [28],[29].

A $(t-1)$-**spread** $\mathcal{S}$ of $PG(2t-1, q)$ is a set of projective spaces of dimension $t-1$ such that every point of $PG(2t-1, q)$ lies in exactly one element of $\mathcal{S}$. Equivalently, it is a set of $q^t + 1$ pairwise disjoint $t-1$ dimensional projective spaces.

Let $PG(2t-1, q)$ be embedded as a hyperplane H in $PG(2t, q) = P$, and let $\mathcal{S}$ be a $(t-1)$-spread of H. Construct a new incidence structure $\pi(\mathcal{S})$ as follows:

The *points* of $\pi(\mathcal{S})$ are the points of $P \backslash H$, together with the elements of the spread. The *lines* of $\pi(\mathcal{S})$ are the t dimensional subspaces of P which intersect H in a member of $\mathcal{S}$, together with the line (often denoted L_∞) whose points are the elements of the spread.

The *incidence* relation of $\pi(\mathcal{S})$ is that induced by the incidence in P.

Then $\pi(\mathcal{S})$ is a translation plane of order q^t, with translation line L_∞ [29]. The collineation group of the plane, if the plane is non-Desarguesian, is well known to be the semi-direct product of the group of translations with the stabiliser of the spread $\mathcal{S}$ in the group of all invertible semi-linear transformations $\Gamma L(2t, q)$ of the underlying vector space [103].

We shall refer to a spread which gives rise to a Desarguesian projective plane as a **Desarguesian spread**. If $q > 2$, a Desarguesian spread corresponds to a *regular* spread [55, 5.1.3].

Let x be an affine point of a translation plane, then the group of all homologies of the plane with axis L_∞ and center x is well known to be isomorphic to the multiplicative group of a field [87, p.73]. Notice that the different choices of x give rise to conjugate homology groups (since the translation group is transitive on the affine points). This field is known to be isomorphic to a structure known as the **kernel** of the translation plane. The formal definition of the kernel involves co-ordinatising structures called *quasifields* or any number of other things depending on who you read [87, p.73] [103, p.3], and is not required for this thesis. We only need note that the multiplicative group of the kernel is isomorphic to the group of all homologies with center a fixed point x and axis the translation line. It is easy to see that the above construction of translation planes of order q^t gives rise to translation planes with kernel *containing* $GF(q)$.

In 1980 J.A. Thas gave the following construction of maximal arcs in certain translation planes of order q^d whose kernel contains $GF(q)$.

THEOREM 4.1 (Thas [130]) *Let $Q^- = Q^-(2t-1, q)$ be a non-singular elliptic quadric in $PG(2t-1, q)$, $t > 1$, and let $\mathcal{S}^-$ be a $(t-2)-$spread of Q^-. Suppose there exists a $(t-1)-$spread $\mathcal{S} = \{s_1, s_2, \ldots, s_{q^t+1}\}$ of $PG(2t-1, q)$ such that $\mathcal{S}^- = \{Q^- \cap s_1, Q^- \cap s_2, \ldots, Q^- \cap s_{q^t+1}\}$. Embed $PG(2t-1, q)$ as a hyperplane H in $PG(2t, q)$ and choose any point $x \in PG(2t, q)\backslash H$. Let $\mathcal{K}$ be the union of the points of $PG(2t, q)\backslash H$ that are on the lines of $PG(2t, q)$ given by the span of x and each of the points of Q^- (including the point x itself), then $\mathcal{K}$ is the set of points of a maximal $\{q^{2t-1} - q^t + q^{t-1}; q^{t-1}\}-$arc in the translation plane $\pi(\mathcal{S})$ of order q^t determined by the spread $\mathcal{S}$.*

Proof: The proof is essentially a projection argument as in the proof of the Thas 1974 maximal arcs, but using some elementary properties of elliptic quadrics and their associated polar spaces to get the intersection properties required.

Firstly note that L_∞ is an external line to $\mathcal{K}$.

Suppose that L is a line of $\pi(\mathcal{S})$ incident with x. Then in $PG(2t, q)$ L is represented as a t dimensional subspace given by the span of x and some $s \in \mathcal{S}$. Now s meets Q^- in a unique element of the $(t-2)-$spread of Q^-. It follows immediately that L meets $\mathcal{K}$ in q^{t-1} points.

Suppose $L \neq L_\infty$ is a line of $\pi(\mathcal{S})$ not incident with x. Then in $PG(2t, q)$ L is represented as a t dimensional subspace D meeting H in some $s \in \mathcal{S}$. Let D' be the projection of D though x onto H. Then D' is a t dimensional subspace of H containing the element s of the spread. We show that D' meets $Q^-\backslash s$ in 0 or q^{t-1} points. It then follows immediately that D and hence L meets $\mathcal{K}$ in 0 or q^{t-1} points.

Denote the polarity induced by Q^- in H by θ. Let s^- be the (unique) element of $\mathcal{S}^-$ contained in s. The set of points of $D' \cap Q^-$ is a degenerate quadric in D' with singular radical R given by ${D'}^\theta \cap D' \cap Q^-$. We show that R is either s^- or a hyperplane of s^-.

Now $s^- < s < D'$ and so ${D'}^\theta < s^\theta < (s^-)^\theta$. Hence the $(t-1)-$dimensional subspace s^θ contains the $(t-2)-$dimensional subspaces s^- and ${D'}^\theta$. Noting that $(s^-)^\theta \cap Q^- = s^-$ it follows that ${D'}^\theta$ meets Q^- in either s^- or a $(t-3)-$dimensional subspace R of s^-. In the former case $D' = (s^-)^\theta$ and so D meets $Q^-\backslash s^-$ in no points. In the later case there is a non-degenerate quadric in the factor space D'/R corresponding to $D' \cap Q^-$. Since D'/R is isomorphic to $PG(2, q)$, and every (non-empty) non-degenerate quadric of $PG(2, q)$ is a conic [78], it follows that the set of points of $D' \cap Q^-$ is the set of points of $q+1$ $(t-2)-$dimensional subspaces pairwise meeting in R (one such subspace is s^-). Counting the number of points in these subspaces gives that D' meets $Q^-\backslash s^-$ in q^{t-1} points. ∎

In the same paper Thas gave a construction of dual hyperovals in translation planes.

In the above, if $t = 2$ the Thas 1980 construction is just the Thas 1974 construction with the restriction that the ovoid be an elliptic quadric.

If q is odd and $t = 2$, Blokhuis and Wilbrink's result (Theorem 3.3) shows that such maximal arcs do not exist. Currently I am working on a generalisation of Blokhuis and Wilbrink's result to show that Thas 1980 maximal arcs do not exist for q odd.

When q is even we first note that the spread $\mathcal{S}$ of H must be symplectic. The quadric $Q^-(2t-1, q)$ induces a symplectic polarity θ of H [80, Lemma 22.3.3]. Given a $(t-2)$-spread $\mathcal{S}^-$ of $Q^-(2t-1, q)$, any spread $\mathcal{S}$ of H that 'fits' $\mathcal{S}^-$ has elements which are (maximal) totally isotropic with respect to θ ($s^\theta = s$ for every $s \in \mathcal{S}$), and so is a symplectic spread. This follows since any $s \in \mathcal{S}$ can be written as a span $s =< p, s^- >$ for some $s^- \in \mathcal{S}^-$ and $p \in (s^-)^\theta - Q^-(2t-1, q)$. Then $s^\theta =< p, s^- >^\theta = p^\theta \cap (s^-)^\theta$, which contains both p and s^- (θ is symplectic so $p < p^\theta$), and so $s^\theta = s$.

In fact, for q even, as is noted in the original paper, the Thas 1980 construction gives maximal arcs in every translation plane whose spread is symplectic. It is well known that any symplectic spread of $PG(2t-1, q)$ gives rise to a spread of a non-singular quadric $Q(2t, q)$ in $PG(2t, q)$ [62, section 3.2](and conversely). Further, such a spread of $Q(2t, q)$ induces a spread of a quadric $Q^-(2t-1, q)$ in the $PG(2t-1, q)$ which is of the right form with respect to the symplectic spread for the Thas 1980 construction [62, section 3.2]. Another way to say this is that a symplectic spread of $PG(2t-1, q)$ will induce (by intersection) a spread of any quadric $Q^-(2t-1, q)$ with the same polarity.

When q is even, many examples of symplectic spreads are known for all $t > 1$. See, for instance, [62][89][90][91].

4.2 The Thas 1980 *construction in Desarguesian planes*

THEOREM 4.2 *The Thas* 1980 *maximal arcs that occur in Desarguesian planes of even order are of Denniston type.*

Proof: First note that the collineation group of a translation plane is transitive on the affine points and so the different choices of the affine point x in the Thas construction give rise to isomorphic maximal arcs (for a given spreads of $PG(2t-1, q)$ and $Q^-(2t-1, q)$).

Let $\mathcal{S} = \{s_1, \ldots, s_{q^t+1}\}$ be a Desarguesian spread of $PG(2t-1, q)$, $t > 1$, q even. We show that any two non-singular elliptic quadrics $Q^-(2t-1, q)$ and $Q^-(2t-1, q)'$ of the form required for the Thas construction with respect to $\mathcal{S}$

can be mapped to one another by elements of the stabiliser $\Gamma L(2t,q)_{\mathcal{S}}$, and hence give rise to isomorphic maximal arcs. An example is then given of a Desarguesian spread - quadric spread pair that is stabilised by a linear cyclic group of order q^t+1. It then follows that any Desarguesian spread - quadric spread pair of the form required for the Thas construction gives rise to a maximal arc stabilised by a linear cyclic group of order q^t+1, and so the maximal arc is of Denniston type (by Theorem 2.2).

Let $Q^- = Q^-(2t-1,q)$ and $Q'^- = Q^-(2t-1,q)'$ be non-singular elliptic quadrics, with $(t-2)$-spreads $\mathcal{S}^-$ and $\mathcal{S}'^-$ respectively, and such that

$$\mathcal{S}^- = \{Q^- \cap s_1, \ldots, Q^- \cap s_{q^t+1}\} \text{ and } \mathcal{S}'^- = \{Q'^- \cap s_1, \ldots, Q'^- \cap s_{q^t+1}\}.$$

Since q is even, Q^- and Q'^- induce symplectic polarities θ and θ', respectively, of $PG(2t-1,q)$. As was shown in the last section note that $s^\theta = s = s^{\theta'}$ for every $s \in \mathcal{S}$.

The following is most easily understood in the (equivalent) André construction of translation planes [4]. In the André construction our spread $\mathcal{S}$ is considered as a collection of t dimensional subspaces of the vector space $V(2t,q)$ which partition the points of $V(2t,q) - \{0\}$. The points of the (affine) translation plane are then the points of $V(2t,q)$, and the lines are the cosets of the spread $\mathcal{S}$ (i.e. lines are of the form $v+s$ for $v \in V(2t,q)$, $s \in \mathcal{S}$).

We employ an argument used in [89, Theorem 3.5] to show θ and θ' are conjugate in the stabiliser of $\mathcal{S}$ in $\Gamma L(2t,q)$. The product $\theta\theta'$ is a collineation of $PG(2t-1,q)$ and hence is induced by a (invertible) semi-linear transformation on the underlying vector space $V(2t,q)$. By the previous paragraphs, $\theta\theta'$ acts as the identity on $\mathcal{S}$, and so is a homology of the affine translation plane determined by $\mathcal{S}$. Since q is even the order of $\theta\theta'$ must be odd.

Consider the group $D_{2|\theta\theta'|} = <\theta, \theta\theta'>$ acting on $V(2t,q)$. It is easily shown to be dihedral of (odd) order ($\theta^2 = 1 = \theta'^2, \theta(\theta\theta') = (\theta\theta')^{-1}\theta$ Then θ and θ' are elements of order 2 in $D_{2|\theta\theta'|}$ and so are conjugate under some power of $\theta\theta'$ (since $\theta\theta'$ has odd order). The group generated by $\theta\theta'$ stabilises $\mathcal{S}$, and so θ and θ' are conjugate in the $\Gamma L(2t,q)$ stabiliser of $\mathcal{S}$.

It follows that the symplectic spaces arising from θ and θ' can be mapped to one another by some element of the stabiliser of $\mathcal{S}$ in $\Gamma L(2t,q)$. So up to equivalence in $\Gamma L(2t,q)$ we may assume $\theta = \theta'$.

In [62], R.H. Dye constructs a spread denoted $\mathcal{P}_t$ of the symplectic space $Sp(2t,q)$, q even, which is a Desarguesian spread of $PG(2t-1,q)$. Though not explicitly said to be Desarguesian, $\mathcal{P}_t$ admits a group triply transitive on the elements of the spread [62, Theorem 7], and so is Desarguesian (this follows from [103, 39.3]). Dye proves by some lengthy calculations:

THEOREM 4.3 (Dye [62, Theorem 14]) *The stabiliser in the group $Sp(2t, q)$, q even, of $\mathcal{P}_t$ acts transitively on the elliptic quadrics with the same polarity as the space $Sp(2t, q)$.* ∎

Hence, since we have shown up to equivalence in $\Gamma L(2t, q)_{\mathcal{S}}$ that $\theta = \theta'$, it follows from Dye's result that the quadrics Q^- and Q'^- can be mapped to one another by elements of the stabiliser $\Gamma L(2t, q)_{\mathcal{S}}$. It follows that the Thas 1980 maximal arcs of a given degree in a Desarguesian plane are all isomorphic.

In only remains to give an example (for each t) of a Thas 1980 maximal arc in a Desarguesian plane that admits a linear cyclic group of order $q^t + 1$ to complete the proof. We construct a quadric - Desarguesian spread pair of the form required for the Thas construction that admits a linear cyclic group of order $q^t + 1$. There are a number of ways the existence of such a quadric-spread pair could be proved, but here is a fairly concrete one.

Let $\xi^2 + \alpha\xi + 1$ be an irreducible polynomial over $GF(q^t)$, q even. Define a quadratic form on $GF(q^t) \oplus GF(q^t)$ by $f(x, y) = x^2 + \alpha xy + y^2$.

The field $GF(q^t)$ is a vector space over $GF(q)$, so for $a \in GF(q^t)$ denote the corresponding vector in $GF(q)^t$ by $\underline{a}$. Define a quadratic form on the $2t$-dimensional vector space $V(2t, q)$ by $Q(\underline{x}, \underline{y}) = trace_{\frac{GF(q^t)}{GF(q)}}(f(x, y))$. Then it is well known (see for instance [141][89][91]) and not too difficult to verify that Q is a non-degenerate quadratic form on $V(2t, q)$ and is of elliptic type.

Define

$$C_{q^t+1} = \{ \begin{pmatrix} a + \alpha b & b \\ b & a \end{pmatrix} : a^2 + \alpha ab + b^2 = 1 \}.$$

Then C_{q^t+1} is cyclic of order $q^t + 1$ and $f(g(x, y)) = f(x, y)$ for every $g \in C_{q^t+1}$. So C_{q^t+1} preserves the quadratic form f, and so preserves Q when made to act on $V(2t, q)$. It also preserves the (vector space) spread

$$\{\{(\underline{a}, \underline{ab}) : a \in GF(q^t)\} : b \in GF(q^t)\} \cup \{(\underline{0}, \underline{b}) : b \in GF(q^t)\}.$$

Further the points of the quadric determined by Q meet any element of the spread in a subspace, and the spread is maximal totally isotropic with respect to the polar form of the quadratic form. ∎

In the proof of the Theorem it is hardly a coincidence that the group C_{q^t+1} is isomorphic to the group given by Abatangelo and Larato in their characterisation of Denniston maximal arcs.

We can identify which Denniston maximal arcs the Thas 1980 construction gives rise to using the following Lemma.

LEMMA 4.1 *In $PG(2, 2^{et})$, $t > e > 1$, there is up to isomorphism a unique degree $2^{e(t-1)}$ Denniston maximal arc $\mathcal{K}$ stabilised by a homology of order $2^e - 1$.*

Proof: Since in $PG(2,2^{et})$ a degree $2^{e(t-1)}$ maximal arc is dual to a degree 2^e maximal arc in $PG(2,2^{et})$ it is sufficient to show that there is a up to isomorphism a unique degree 2^e Denniston maximal arc $\mathcal{K}'$ stabilised by a homology of order 2^e-1.

First note $\mathcal{K}'$ is isomorphic to a maximal arc $\mathcal{K}''$ described by the following standard pencil of conics, and some additive subgroup A'' of $GF(2^{et})$, $|A''| = 2^e$.

Let $\xi^2+\alpha\xi+1$ be an irreducible polynomial over $GF(2^e)$, and chose an additive subgroup A'' of $GF(2^{et})$ such that the maximal arc $\mathcal{K}''$ described by the set of points of $F_\lambda : x^2+\alpha xy+y^2+\lambda z^2 = 0, \lambda \in A''$, is isomorphic to $\mathcal{K}'$.

The nucleus of $\mathcal{K}''$ is the point $(0,0,1)$, and the line at infinity is $[0,0,1]$. Let h be the homology of order 2^e-1 stabilising $\mathcal{K}''$. Then arguing as in Theorem 2.3 h fixes the nucleus and the line at infinity, and cyclically permutes the conics of $\mathcal{K}''$.

The group of all homologies with centre $(0,0,1)$ and axis $[0,0,1]$ can be described by the following set of matrices.

$$H_{2^{et}-1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix} : b \in GF(2^{et})^* \right\}.$$

Note that the order of $H_{2^{et}-1}$ is $2^{et}-1$, and since the kernel of $PG(2,2^{et})$ is $GF(2^{et})^*$ [103], these are all the homologies with this centre and axis. So $h \in H_{2^{et}-1}$.

Since $H_{2^{et}-1}$ acts regularly on the non-fixed points of any line through the centre, it transitively permutes the orbits of h of length 2^e-1 on any line through the center of h. Hence all degree 2^e Denniston maximal arcs that are a subset of the pencil F_λ, $\lambda \in GF(2^{et})$, and that are stabilised by h are isomorphic. ∎

In the Lemma if $e=1$ then $\mathcal{K}$ is the dual of a conic, and so uniqueness follows (though $2^e-1=1$ and so there is no homology of that order).

COROLLARY *In $PG(2,2^{et})$ there is up to isomorphism a unique degree $2^{e(t-1)}$ Thas 1980 maximal arc. It is isomorphic to a Denniston maximal arc with the standard pencil and additive subgroup $Kernel(trace_{\frac{GF(2^{et})}{GF(2^e)}})$.*

Proof: Uniqueness was shown in the proof of Theorem 4.2. Note that such a Thas 1980 maximal arc is stabilised by a homology of order 2^e-1. Then note that a Denniston maximal arc with standard pencil and additive subgroup $GF(2^e)$ is clearly stabilised by a homology of order 2^e-1. The Corollary to Theorem 2.4 gave the calculation of the dual parameters of such a Denniston maximal arc. ∎

5. TRANSITIVE MAXIMAL ARCS IN DESARGUESIAN PLANES

In this short section maximal arcs in Desarguesian planes whose collineation stabiliser is transitive on the points of the maximal arc are examined. Using simple numerical arguments, it is shown (Corollary to Lemma 5.2) that given $n = 2^i$, there exists an integer e_n such that there are no such transitive maximal arcs in $PG(2, 2^e)$ for any $e > e_n$. This result is then used to classify the degree 2 and degree 4 maximal arcs in Desarguesian planes.

In 1990, Delandtsheer and Doyen classified the maximal arcs in finite projective planes whose collineation stabiliser is transitive on the set of lines secant to the maximal arc. They show the following.

THEOREM 5.1 (Delandtsheer & Doyen [54]) *Let π be a projective plane of order q, q not necessarily a prime power. If $G \leq Aut(\pi)$ stabilises a maximal arc $\mathcal{K}$ of degree n, and acts transitively on the set of lines secant to $\mathcal{K}$, then one of the following holds:*
(i) $q = p^e$, $e \geq 1$, p prime, $n = q$, $\mathcal{K}$ is a line transitive affine translation plane A, and G contains the translation group of A.
(ii) $q = 2^e$, $e \geq 3$, $n = q/2$, $\mathcal{K}$ is the dual of a regular hyperoval in $PG(2, 2^e)$, and $PSL(2, 2^e) \lhd G \leq P\Gamma L(2, 2^e)$.
(iii) $q = 4$, $n = 2$, $\mathcal{K}$ is a (regular) hyperoval in $PG(2, 4)$ and $G = Alt(6)$, $Sym(6)$, $PSL(2, 5)$ or $PGL(2, 5)$.
Further, in each case except (iii), G stabilises a line of π disjoint from $\mathcal{K}$. ∎

LEMMA 5.1 (Camina & Gagen [35]) *Let $\mathcal{S}$ be a $(v, k, 1)$-BIBD with $k < v$. If $k|v$ and $G \leq Aut(\mathcal{S})$ is transitive on the lines of $\mathcal{S}$, then G is transitive on point-line incident pairs.* ∎

Considering a degree n maximal arc in a plane of order q as a $(q(n-1) + n, n, 1)$-BIBD, it follows from the Lemma that transitivity on secant lines to $\mathcal{K}$ implies transitivity on points of $\mathcal{K}$ (but not conversely). So the general problem of classifying maximal arcs whose collineation stabiliser is transitive on points remains open. We shall call such maximal arcs **point transitive**.

In [100], Korchmaros classified the point transitive hyperovals in Desarguesian projective planes (see below). Partial results have also been obtained by various authors for hyperovals in non-Desarguesian projective planes [16].

The following is a simple approach to attempting to classifying point transitive maximal arcs in Desarguesian planes. A necessary condition for a maximal arc in a projective plane to be point transitive is that the size of the maximal arc divides the order of the collineation group of the plane. Using elementary number theory this condition can be used to show the non-existence of transitive maximal arcs of a given degree for a large number of cases. The remaining cases can then often be dealt with using knowledge of the types of group elements that can stabilise a maximal arc and some elementary group theory.

Denote the greatest common factor of two integers x and y by (x, y).

LEMMA 5.2 *Let $\mathcal{K}$ be a maximal arc of degree n in $PG(2,q)$, $n < q = 2^e$. Then a necessary condition for $\mathcal{K}$ to be point transitive is that $q(n-1)+n$ divides $(q/n-n, n(n-1)+1).n.(q-1, 2n-1)^2.e$. In particular, $q(n-1)+n$ must divide $(n(n-1)+1).n.(2n-1)^2e$.*

Proof: For $\mathcal{K}$ to be point transitive, it is necessary that the size of $\mathcal{K}$ divides $|P\Gamma L(3,q)| = (q^2+q+1)(q+1)q^3(q-1)^2e$. We use elementary properties of greatest common divisors, and that n and q are powers of 2, to estimate $(\mathcal{K}, |P\Gamma L(3,q)|)$.

$$
\begin{aligned}
&(q^2+q+1, q(n-1)+n) = (q^2+q+1, q-q/n+1)\\
&= (q(q-q/n+1)+q^2/n+1, q-q/n+1) = (q^2/n+1, q-q/n+1)\\
&= (q^2/n-q+q/n, q-q/n+1) = (q/n(q-n+1), q-q/n+1)\\
&= (q-n+1, q-q/n+1) = (q/n-n, q-q/n+1)\\
&= (q/n-n, (n-1)(q/n-n)+n(n-1)+1) = (q/n-n, n(n-1)+1).
\end{aligned}
$$

$$(q+1, q(n-1)+n) = (q+1, (q+1)(n-1)+1) = 1.$$

$$(q^3, q(n-1)+n) = n(q^3/n, q/n(n-1)+1) = n.$$

$$(q-1, q(n-1)+n) = (q-1, (n-1)(q-1)+2n-1) = (q-1, 2n-1).$$

The result follows. ■

COROLLARY *Let n be a fixed power of 2. Then there exists a fixed integer e_n, such that for any integer $e > e_n$ there are no point transitive degree n maximal arcs in $PG(2, 2^e)$.*

Proof: By the Lemma a necessary condition for the existence of a transitive degree n maximal arc in $PG(2, 2^e)$ is that $2^e(n-1)+n$ divides (and so is less than or equal to) $(n(n-1)+1).n.(2n-1)^2.e$. A suitably large choice of e guarantees this does not occur. ■

The following two Theorems show instances of this method.

THEOREM 5.2 (Korchmaros [100]) *Let $\mathcal{K}$ be a hyperoval in $PG(2, 2^e)$, $e \geq 1$. If $\mathcal{K}$ is point transitive then $\mathcal{K}$ is either a regular hyperoval in $PG(2,2)$ or $PG(2,4)$, or is a Lunelli-Sce hyperoval in $PG(2,16)$.*

Proof: Assume that $e > 1$ (if $e = 1$, then $\mathcal{K}$ is a quadrangle in $PG(2,2)$, and $\mathcal{K}$ is point transitive since $PGL(3,2)$ is transitive on ordered quadrangles).

Applying Lemma 5.2 with $n = 2$ then gives $|\mathcal{K}| = 2^e + 2$ must divide $3.3^2.2.e = 54e$. It follows that $e < 9$, and checking each of the cases $1 < e \leq 8$ shows that $e = 2, 4$.

Hyperovals in $PG(2,2^2)$ are classified [78], and are all regular and point transitive. Hyperovals in $PG(2,2^4)$ are were classified theoretically in [94](Hall originally classified them by computer); they are the regular hyperoval which is not point transitive, and the Lunelli-Sce hyperoval which is point transitive. ∎

THEOREM 5.3 *Let $\mathcal{K}$ be a degree 4 maximal arc in $PG(2,2^e)$, $e \geq 3$. If $\mathcal{K}$ is point transitive then $\mathcal{K}$ is is the dual of the regular hyperoval in $PG(2,8)$.*

Proof: We proceed as in the previous Theorem.

For $\mathcal{K}$ to be point transitive we require that $|\mathcal{K}| = 3.2^e + 4$ divides $13.4.7^2.e = 2584e$. This implies that $e \leq 13$, and checking each of the cases $1 < e \leq 13$ shows that $e = 3, 4$ or 6.

Degree 4 maximal arcs in $PG(2,2^3)$ are classified. They are the duals of regular hyperovals, and are point transitive.

The size of a degree 4 maximal arc in $PG(2,2^4)$ is $52 = 4.13$. So for such a maximal arc to be point transitive it would need to have an element of order 13 in its stabiliser. Since 13^2 does not divide the order of $P\Gamma L(3,2^4)$, it follows by Sylow's Theorem that all subgroups of order 13 of $P\Gamma L(3,2^4)$ are conjugate. In fact such a cyclic group of order 13 is a subgroup of a Singer cycle. As was noted in Chapter 2, hand calculations show that no 4 orbits of such a subgroup form a degree 4 maximal arc. The calculations can be reduced slightly be noticing that the normaliser in $P\Gamma L(3,2^4)$ of a subgroup of order 13 is transitive on the orbits (necessarily of length 13). And so in attempting to construct a degree 4 maximal arc from these orbits, only unions of orbits containing a 'favourite' orbit need be considered.

A point transitive degree 4 maximal arc in $PG(2,2^6)$ can be ruled out by the following. The size of such a maximal arc is $3.64 + 4 = 2^2.7^2$. And the order of a Sylow 7-subgroup of $P\Gamma L(3,2^6)$ is also 7^2. So for a point transitive degree 4 maximal arc to exist in $PG(2,2^6)$ it must be stabilised by a Sylow 7-subgroup of order 7^2.

The group $P\Gamma L(3,2^6)$ contains a group of homologies of order 7 (since $GF(8)^* \leq GF(2^6)^*$). And so any Sylow 7-subgroup contains a group of homologies of order 7. But a homology group of order 7 cannot stabilise a degree 4 maximal arc, giving a contradiction. Hence there are no point transitive degree 4 maximal arcs in $PG(2,2^6)$. ∎

It seems likely that transitive maximal arcs can be classified in Desarguesian planes using something along the lines of the above. It should be possible to prove a stronger version of Lemma 5.2 using a little number theory, and then to exclude the remaining cases (when $|\mathcal{K}|$ does divide the order of $P\Gamma L(3,q)$) with some group theoretic arguments. In all probability though, the answer would be that the only non-trivial point transitive maximal arcs are certain hyperovals and duals of hyperovals.

6. LINEAR MAXIMAL ARCS IN TRANSLATION PLANES

The work of this section is motivated by the observation that all known constructions of maximal arcs in Desarguesian planes (apart from the duals of certain hyperovals) have a property which we call *linearity*. Loosely speaking, a maximal arc in a translation plane is *linear* if its set of points can be described as the span of an affine point with a collection of subspaces (all of the same dimension) of $PG(2t-1,q)$, each subspace contained in a unique element of the spread defining the translation plane.

In this section it is shown that the Denniston, Thas 1974 and Thas 1980 maximal arcs in translation planes are all linear. Certain linear maximal arcs are then characterised using Theorems proved earlier in the Thesis. Some properties required for a collection of subspaces to give rise to linear maximal arcs are then noted, and it is shown that certain linear maximal arcs give rise to designs. Finally, some open problems relating to linear maximal arcs are listed.

The theme of this section is to provide a framework for understanding the known constructions of maximal arcs, and to suggest ways that new constructions of maximal arcs might arise. A theory of linear maximal arcs is not developed in any depth in this section, just some ideas are outlined for future examination.

6.1 Linear maximal arcs

Let $\mathcal{F}_{std}$ be the standard pencil of conics, and A be an additive subgroup of $GF(2^t)$, and $\mathcal{K}$ the Denniston maximal arc in $PG(2,2^t)$ determined by A and $\mathcal{F}_{std}$. Then $F_0=(0,0,1)$ is the nucleus of the conics, and F_∞ is the line $z=0$. We consider what $\mathcal{K}$ 'looks like' in the Bruck-Bose representation of $PG(2,2^t)$ as a translation plane (over $GF(2)$) with translation line $z=0$.

Consider $GF(2^t)$ as a vector space of dimension t over $GF(2)$. For any $a\in GF(2^t)$ denote the corresponding vector of $GF(2)^t$ by $\underline{a}$. The set

$$\{\{(\underline{a},\underline{ab},0) : a\in GF(2^t)^*\} : b\in GF(2^t)\}\cup\{(\underline{0},\underline{b},0) : b\in GF(2^t)^*\}$$

then defines a (Desarguesian) spread of $H=PG(2t-1,2)$, H a hyperplane of $PG(2t,2)$. Considering $PG(2,2^t)$ as a translation plane with translation line $z=0$, and the above spread, the affine points are then of the form $(\underline{a},\underline{b},1)\in PG(2t,2)$, a,$b\in GF(2^t)$.

In the translation plane representation of $PG(2,2^t)$, the points of $\mathcal{K}$ on the line $x=0$ are of the form $(\underline{0},\underline{\sqrt{\lambda}},1)$, $\lambda\in A$. Using the facts that A is additive and that the square root map is a field automorphism, it is easy to verify that these points together with the points $(\underline{0}, \underline{\sqrt{\lambda}}, 0)$ form an projective subspace $\mathcal{P}$. For instance, given (distinct) points $(\underline{0}, \underline{\sqrt{\lambda}}, 1)$ and $(\underline{0}, \underline{\sqrt{\mu}}, 1)$, $\lambda, \mu \in A$, the third point of the line of $PG(2t, 2)$ joining them is given by the sum

$$(\underline{0}, \underline{\sqrt{\lambda}}, 1) + (\underline{0}, \underline{\sqrt{\mu}}, 1) = (\underline{0}, \underline{\sqrt{\lambda}} + \underline{\sqrt{\mu}}, 0) = (\underline{0}, \underline{\sqrt{\lambda + \mu}}, 0).$$

Since A is additive, $\lambda + \mu \in A$, and so $(\underline{0}, \underline{\sqrt{\lambda + \mu}}, 0)$ is a member of $\mathcal{P}$. Similarly it is readily verified that the sum of any two points of $\mathcal{P}$ is a member of $\mathcal{P}$, and so $\mathcal{P}$ is a projective subspace. Note that $\mathcal{P}$ has projective dimension $log_2|A|$.

Note that $\mathcal{P}$ can be considered as the span of the nucleus $(\underline{0}, \underline{0}, 1)$ and the subspace $\{(\underline{0}, \underline{\sqrt{\lambda}}, 0) : \lambda \in A\}$ of H. Now, by Abatangelo and Larato's characterisation of Denniston maximal arcs (Theorem 2.2) there exists a cyclic group C_{2^t+1} of order $2^t + 1$ stabilising $\mathcal{K}$. It fixes both F_0 and F_∞, and acts regularly on the points of the line F_∞ (in this case the line $z = 0$). It follows that the set of points of $\mathcal{K}$ can be considered as the points of the orbit of the set of affine points of $\mathcal{P}$ under C_{2^t+1}. In other words, *in the Bruck-Bose representation over $GF(2)$ of $PG(2, 2^t)$, a Denniston maximal arc can be considered as the set of affine points of the span of an affine point and a collection of $2^t + 1$ subspaces of H, each of these $2^t + 1$ subspaces being of projective dimension $log_2|A| - 1$ and each contained in a unique element of the spread.*

This observation leads naturally to the following definition.

Let π be a translation plane of order q^t whose kernel contains $GF(q)$. Let $\mathcal{K}$ be a degree q^s maximal arc in π such that the translation line L_∞ is external to $\mathcal{K}$. Represent π via Bruck-Bose with a $(t-1)$-spread $\mathcal{S}$ of $H = PG(2t-1, q) \leq PG(2t, q)$. Then $\mathcal{K}$ is **linear with kernel containing** $GF(q)$ if there is a collection of subspaces $\mathcal{K}_\infty$ of H, and a point $x \in \mathcal{K}$ such that:
(i) There are $q^t + 1$ elements of $\mathcal{K}_\infty$, each with projective dimension $s - 1$.
(ii) Each element of the spread $\mathcal{S}$ contains a unique element of $\mathcal{K}_\infty$.
(iii) The points of $\mathcal{K}$ are given by the set of affine points of the span of x with the elements of $\mathcal{K}_\infty$.

If π is Desarguesian every line is a translation line, and so the definition needs to be slightly modified. In which case, if *there exists* a translation line with the required properties, then $\mathcal{K}$ is linear with kernel containing $GF(q)$.

Suppose that $\mathcal{K}$ is linear with kernel containing $GF(q)$ in a translation plane of order q^t with kernel $GF(q^u)$. Then if for every $1 < v|u$, $\mathcal{K}$ is not linear with kernel containing $GF(q^v)$, we say $\mathcal{K}$ is **linear with kernel** $GF(q)$ (i.e. $\mathcal{K}$ is not linear over any larger field).

Note that $\mathcal{K}$ being linear with kernel containing $GF(q)$ implies that the degree of $\mathcal{K}$ is at least q, and that there exists a homology of order $q-1$ stabilising $\mathcal{K}$. The following Theorems follow immediately.

THEOREM 6.1 *Denniston maximal arcs are linear with kernel containing $GF(2)$. Thas* 1974 *maximal arcs of degree q in translation planes of order q^2 are linear with kernel $GF(q)$. Thas* 1980 *maximal arcs of degree q^{d-1} in translation planes of order q^d are linear with kernel $GF(q)$.* ∎

THEOREM 6.2 *Thas* 1974 *maximal arcs in translation planes of order q^2 are exactly the degree q linear maximal arcs with kernel $GF(q)$.*

Proof: Follows immediately from Theorem 3.4. ∎

The following Theorem is a spectral result (for q even) of existence of parameters of linear maximal arcs.

THEOREM 6.3 *Let r, t be positive integers, $t > 1$. Then in $PG(2, 2^{rt})$, for every integer s, such that $1 \leq s < t$, there exists a linear Denniston maximal arc of degree 2^{rs} whose kernel contains $GF(2^r)$.*

Proof: Let $\{a_1, a_2, \ldots, a_t\}$ be a basis for $GF(2^{rt})$ as a vector space over $GF(2^r)$. For any $1 \leq s < t$ define

$$A_s = \{x_1a_1 + x_2a_2 + \cdots + x_sa_s : x_1, x_2, \ldots, x_s \in GF(2^r)\}.$$

Then A_s is an additive subgroup of $GF(2^{rt})$ which is also closed under multiplication by elements of $GF(2^r)$. Proceeding as in the proof that Denniston maximal arcs are linear with kernel containing $GF(2)$, and with the same notation, it is easily verified that A_s gives rise to a subspace in the appropriate Bruck-Bose construction of $PG(2, 2^{rt})$ over $GF(2^r)$. ∎

Other than the Denniston and Thas constructions the only known (non-trivial) constructions of maximal arcs in Desarguesian planes are hyperovals and their duals. By taking any line external to a hyperoval in a Desarguesian plane, and considering that line as a translation line, it is easy to see that hyperovals in Desarguesian planes are linear with kernel $GF(2)$. The duals of such hyperovals may not be linear. Hence, *with the possible exception of the duals of hyperovals, all known examples of maximal arcs in Desarguesian planes are linear with kernel containing $GF(2)$.*

6.2 Subspaces associated with a linear maximal arc

The next thing to consider is what properties a collection of subspaces need so that they give rise to a linear maximal arc.

Let π be a translation plane of order q^t whose kernel contains $GF(q)$. Represent π via Bruck-Bose with a $(t-1)$-spread $\mathcal{S}$ of $H = PG(2t-1, q) \leq PG(2t, q)$. Let

$\mathcal{K}_\infty$ be a collection of q^t+1 subspaces of H, each subspace of dimension $s-1$, and such that each element of $\mathcal{S}$ contains a unique element of $\mathcal{K}_\infty$. Let x be any affine point, and let $\mathcal{K}$ be the set of affine points of the span of x with the subspaces of $\mathcal{K}_\infty$. The question is then what conditions does $\mathcal{K}_\infty$ have to satisfy so that $\mathcal{K}$ is the set of points of a (linear) maximal arc?

Using projection arguments such as those found in the proofs of the Thas 1974 and Thas 1980 maximal arcs, $\mathcal{K}$ is a linear maximal arc of degree q^s with kernel containing $GF(q)$, if and only if, any t-dimensional subspace T of H, such that $\exists S \in \mathcal{S}$, $S < T$, has the property that $T - S$ meets the set of points of elements of $\mathcal{K}_\infty$ in 0 or q^s points.

This simple property of the subspaces associated with a linear maximal arc can be used to show there are designs associated with certain linear maximal arcs.

THEOREM 6.4 *Let $\mathcal{K}$ be a linear maximal arc of degree q with kernel $GF(q)$ in a translation plane π of order q^t. Then there exists a $(q^t+1, q, q-1)$-BIBD and a $(q^t+1, q+1, q+1)$-BIBD.*

Proof: Let $\mathcal{S}$ be the $(t-1)$ spread of $H = PG(2t-1, q) < PG(2t, q)$ which gives rise to the translation plane π, and in which $\mathcal{K}$ is represented as the set of affine points of the span of an affine point and a collection $\mathcal{O}$ of q^t+1 subspaces of H. Then since $\mathcal{K}$ is of degree q, and is linear with kernel $GF(q)$, it follows that $\mathcal{O}$ is a collection of q^t+1 points of H, each element of the spread containing a unique point of $\mathcal{O}$.

Define a new incidence structure $\mathcal{I}$ as follows. The points of $\mathcal{I}$ are the points of $\mathcal{O}$. A subspace $T < H$ of dimension t is a block of $\mathcal{I}$ if
(i) There exists a unique $S \in \mathcal{S}$ such that $S < T$.
(ii) T meets $\mathcal{O}$ in $q+1$ points.

Incidence in $\mathcal{I}$ is inclusion. We show $\mathcal{I}$ is a $(q^t+1, q+1, q+1)$-BIBD.

Firstly, by definition, every block of $\mathcal{I}$ is incident with $q+1$ points. In fact, since $\mathcal{K}$ is a linear maximal arc, it follows that every subspace $T < H$ such that T contains some element of the spread meets $\mathcal{O}$ in either 1 or $q+1$ points.

Let $o \in \mathcal{O}$, and let S be the unique element of $\mathcal{S}$ containing o. Since every subspace of dimension t containing S meets $\mathcal{O} - \{o\}$ in either 0 or q points, there are exactly $(|\mathcal{O}|-1)/q = q^{t-1}$ t-dimensional subspaces containing S and meeting $\mathcal{O}$ in $q+1$ points. There are also q^t blocks of $\mathcal{I}$ incident with o given by the span of o and elements of $\mathcal{S} - \{S\}$. Hence the number of blocks of $\mathcal{I}$ on a point of $\mathcal{I}$ is $q^t + q^{t-1}$.

Let o_1 and o_2 be two distinct points of $\mathcal{O}$. Then any t-dimensional subspace of H containing o_1 and o_2 contains the line $\overline{o_1 o_2}$. Hence the blocks of $\mathcal{I}$ containing

o_1 and o_2 are exactly the subspaces given by the span of $\overline{o_1 o_2}$ and the elements of the spread that $\overline{o_1 o_2}$ meets. Note that any element of the spread meets $\overline{o_1 o_2}$ in at most 1 point, and so there are exactly $q+1$ elements of the spread that $\overline{o_1 o_2}$ meets. It follows that on any pair of points of $\mathcal{I}$ there are exactly $q+1$ blocks containing that pair.

Hence $\mathcal{I}$ is a $(q^t+1, q+1, q+1)$-BIBD.

A $(q^t+1, q, q-1)$-BIBD is given by a very similar construction. Define an incidence structure $\mathcal{I}'$ with points and blocks as for $\mathcal{I}$. Incidence is modified as follows. A point o is incident with a block T if and only if $o < T - S$, where S is the unique element of the spread contained in T.

Essentially the same proof that $\mathcal{I}$ is a BIBD shows that $\mathcal{I}'$ is also a BIBD. ∎

COROLLARY *There exist* $(2^{rt}+1, 2^r, 2^r-1)$*-BIBD and* $(2^{rt}+1, 2^r+1, 2^r+1)$*-BIBD's for all* $r, t \in \mathbf{Z}^+$.

Proof: Follows from Theorems 6.3 and 6.4. ∎

See also [141] for other designs associated with point sets in quadrics (though not associated with linear maximal arcs).

6.3 Open problems

1. Given a Denniston maximal arc $\mathcal{K}$, let $\mathcal{H}$ be a hyperoval formed by the nucleus of $\mathcal{K}$ and one of the conics contained in $\mathcal{K}$. Then the dual of $\mathcal{H}$ can be described as a Thas 1980 maximal arc. And further the dual of $\mathcal{K}$ is contained in the dual of $\mathcal{H}$. It follows that any Denniston maximal arc can be considered as a subset of a Thas 1980 maximal arc.

The Thas 1980 and Denniston constructions of maximal arcs then both give rise to linear maximal arcs whose subspaces at infinity are subspaces of a spread of an elliptic quadric. In particular, in Desarguesian planes a Denniston maximal arc can be consider as a subset of the points of a Thas 1980 maximal arc. Might Thas 1980 maximal arcs in non-Desarguesian planes also contain subsets of points that are linear maximal arcs? Is there a geometric way of choosing such a collection of subspaces of a spread of a quadric?

2. Are there linear maximal arcs that are *not* associated with a collection of subspaces of a quadric? For those hyperovals in translation planes that can be considered as linear maximal arcs it seems likely that many will give rise to a collection of subspaces at infinity that are not a subset of some elliptic quadric. Also, the Thas 1974 construction with a Tits ovoid is not directly associated with an elliptic quadric. Are there other examples?

3. The Thas 1974 construction gives examples of a collection of subspaces at infinity giving rise to maximal arcs in at least two non-isomorphic planes, i.e. both the Lüneburg and Desarguesian planes may be obtained from a spreads of tangent lines to an elliptic quadric. Are there other examples of collections of subspaces at infinity giving rise to maximal arcs in more than one plane? A spread of tangent lines to an elliptic quadric in $PG(3,q)$ that was not isomorphic to either the Lüneburg or Desarguesian spread would give rise to a new ovoid in $PG(3,q)$ (see comments at the end of Section 3.1). Many people consider the existence of an ovoid of $PG(3,q)$ that is not an elliptic quadric or a Tits ovoid as unlikely, so finding spreads of tangent lines to an elliptic quadric in $PG(3,q)$ is probably not a fruitful approach.

For instance, consider the degree 4 Denniston maximal arc $\mathcal{K}$ with additive subgroup $GF(4)$ in $PG(2,64)$. Then $\mathcal{K}$ is linear with kernel $GF(4)$. Represent $PG(2,64)$ as a translation plane over $GF(4)$, i.e. as a spread of planes of $PG(5,4) < PG(6,4)$, so that $\mathcal{K}$ is linear in this representation. Then the subspaces at infinity of $\mathcal{K}$ are a set $\mathcal{O}$ of 65 points, one in each element of the spread. We can try and construct other planes in which this set of points gives rise to a linear maximal arc by the following. Define a plane π of $PG(5,4)$ to be *good* if: (i) π contains a unique point of $\mathcal{O}$, and (ii) every 3-space containing π in $PG(5,4)$ meets $\mathcal{O}$ in 1 or 5 points. Any translation plane formed from a spread of good planes will then contain a linear maximal arc. Unfortunately, computer searches show that the only good planes with respect to $\mathcal{O}$ in $PG(5,4)$ are the original planes of the Desarguesian spread.

4. It would be interesting to know whether there are other designs or geometric structures such as partial geometries or semi-partial geometries associated with the subspaces at infinity for linear maximal arcs. For instance, W.M. Kantor constructs strongly regular graphs via spreads of certain quadrics [88]. Might similar constructions be possible for the subspaces associated with a linear maximal arc? The only ones I can find are those designs given in the last subsection.

Chapter 3
INHERITED MAXIMAL ARCS

1. INTRODUCTION

In this chapter a number of new constructions of maximal arcs in projective planes are given. In Section 2, a class of maximal arcs is constructed in certain derived dual translation planes that are inherited from the duals of the Thas 1974 maximal arcs (Theorem 2.1). In particular this gives maximal arcs in certain derived duals of the Lüneburg planes, as well as the Hall planes of even order. In Section 3, an alternative proof of the maximal arcs in the Hall planes of even order is given (Theorem 3.1). In Section 4, the duals of the maximal arcs constructed in Section 3 are shown to inherit to maximal arcs in certain derived duals of the Hall planes (Theorem 4.1). In Section 5, the results are summarised and existence and isomorphism problems are examined. Similar techniques to those developed in this chapter were later applied by Hamilton and Penttila to Buekenhout-Metz unitals (see Appendix).

The work of this chapter appeared in a different form in the following series of papers [73][76][74].

2. INHERITED MAXIMAL ARCS IN DERIVED DUAL TRANSLATION PLANES

The main result of this section (Theorem 2.1) is to construct maximal arcs that are inherited from the duals of the Thas 1974 maximal arcs.

2.1 The Bose-Barlotti construction of derived dual translation planes

Let π be a translation plane represented as in Section 3 of the previous chapter. Let s be a fixed element of the spread $\mathcal{S}$, and D a hyperplane of $PG(4,q)$ meeting the projective 3-space H in a plane containing s. Note that every plane in H contains a unique line of the spread, so D contains the line s of the spread and no other. Construct a new incidence structure $\Delta_D(\pi)$ as follows:

The points of $\Delta_D(\pi)$ are of three types, (I) those represented by planes of $PG(4,q)$ not contained in H and passing through lines of $\mathcal{S}$, other than s; (II) those represented by planes of $PG(4,q)$, not contained in H or D and containing s; (III) those represented by points of the line s.

The lines of $\Delta_D(\pi)$ are of three types, (I) those which are represented by points of $PG(4,q)$ not contained in H or D; (II) those which are represented by planes contained in D, and not containing s; (III) a single line represented by the line s.

Incidence is defined to be containment in $PG(4,q)$, except that a point of type (I) or (II) is defined to be incident with a line of type (II) when the planes representing them intersect in a line.

The incidence structure $\Delta_D(\pi)$ is then a projective plane of order q^2 [21]. Note that different choices of D can give rise to non-isomorphic planes. The planes $\Delta_D(\pi)$ are derived planes of the dual plane of π (derived in the sense of Albert [3] and Ostrom [107], [110],[109]), the derivation set being the q lines of π which are represented by planes contained in D on the line s of the spread, together with the line L_∞.

For example, if the plane π is the Desarguesian plane $PG(2, q^2)$ (i.e., the spread is regular [55]) and $q > 2$, the dual of π is isomorphic to $PG(2, q^2)$, and the planes derived from the dual of π are all isomorphic to the Hall plane of order q^2.

2.2 Maximal arcs inherited from dual Thas 1974 *maximal arcs*

In this subsection it is shown that if a Thas maximal $\{q^3 - q^2 + q; q\}-$arc exists in a translation plane of order q^2 then there exist maximal $\{q^3 - q^2 + q; q\}-$arcs in certain derived duals of the translation plane.

Let $\mathcal{K}$ be a Thas maximal $\{q^3 - q^2 + q; q\}-$arc in a translation plane π of order q^2 with notation as in Section 3 of the previous chapter. We first construct a derived dual plane $\Delta_D(\pi)$ of π.

Let o be a point of the ovoid $\mathcal{O}$, and s the unique line of the spread containing o. Then there exists a unique tangent plane Π_o of H to the ovoid on o. Since any plane of H contains a unique line of the spread, it follows that Π_o contains s (if Π_o contained a line of the spread other than s, then Π_o would meet $\mathcal{O}$ in more than one point). Let D be the span of Π_o and the centre x of the Thas maximal arc. The projective 3-space D and the spread $\mathcal{S}$ then define a plane $\Delta_D(\pi)$, as in the previous subsection, that is a derived plane of the dual plane of π.

Define $\mathcal{K}_{\Delta_D}$ to be the set containing the following two types of elements:
(a) planes Π of $PG(4, q)$ not contained in H or D that meet H in a line of the spread not equal to s, and satisfying $\Pi \cap \mathcal{K} = \emptyset$.
(b) points of $PG(4, q)$ on the line s not equal to o.

The set $\mathcal{K}_{\Delta_D}$ is then a set of points of $\Delta_D(\pi)$. The type (a) elements of $\mathcal{K}_{\Delta_D}$ also represent a subset of the lines of the translation plane π that are external to the maximal arc $\mathcal{K}$, so that $\mathcal{K}_{\Delta_D}$ is inherited from the dual of $\mathcal{K}$. We prove that the set $\mathcal{K}_{\Delta_D}$ is the set of points of a maximal $\{q^3 - q^2 + q; q\}-$arc in the plane $\Delta_D(\pi)$ by first showing the set has size $q^3 - q^2 + q$, and then showing that every line of $\Delta_D(\pi)$ meets $\mathcal{K}_{\Delta_D}$ in less than or equal to q points. In summary, we prove the following Theorem.

THEOREM 2.1 ([76]) *Let $\mathcal{K}$ be a Thas maximal $\{q^3-q^2+q;q\}$−arc in a translation plane π of order q^2 with notation as above. Let D be the span of the centre x and a tangent plane in H to the ovoid $\mathcal{O}$. Then $\mathcal{K}_{\Delta_D}$ is the set of points of a maximal $\{q^3-q^2+q;q\}$−arc in the derived dual plane $\Delta_D(\pi)$.*

To prove the Theorem the following result is required.

LEMMA 2.1 *Let π be a translation plane containing a Thas maximal arc $\mathcal{K}$ with notation as above. Let $\Delta_D(\pi)$ be a derived dual translation plane constructed with respect to $\mathcal{K}$ as in this section. Let Π be a line of type (II) of $\Delta_D(\pi)$. Then the plane of $PG(4,q)$ representing Π corresponds to a Baer subplane of π. Further, considering Π as a Baer subplane of π, the external lines to $\mathcal{K}$ on the point corresponding to the line s of the spread are lines of Π (when restricted to Π).*

Proof of Lemma 2.1: A plane Π in $PG(4,q)$ that meets H in a line not contained in the spread $\mathcal{S}$ corresponds to a Baer subplane of the translation plane π [138]. The points of the corresponding Baer subplane are given by the points of $\Pi\backslash H$, together with the points corresponding to the $q+1$ line of the spread that Π meets.

We argue in the projective plane π and its Bruck-Bose representation. The set of external lines to a maximal $\{q^3-q^2+q;q\}$−arc in plane of order q^2 form a maximal $\{q^3-q^2+q;q\}$−arc in the dual plane. Hence every point of π not contained in $\mathcal{K}$ is incident with q lines of π external to $\mathcal{K}$. The lines of the spread $\mathcal{S}$ correspond to points not contained in $\mathcal{K}$, and L_∞ is an external line to $\mathcal{K}$. Hence on any line of the spread there are $q-1$ planes that represent lines of π that are external to $\mathcal{K}$ (the other external line being L_∞).

There are q planes on s in D but not contained in H, exactly one of which meets the maximal arc (the one given by the span of s and the centre x). So there are $q-1$ planes on s in D not contained in H that do not meet $\mathcal{K}$. These, together with L_∞, represent the external lines to $\mathcal{K}$ on s.

Since two planes in a 3-space are either equal or meet in a line, it follows that the plane representing Π meets every plane in D on s in a line, and so meets every external line not equal to L_∞ on s in $q+1$ points. The line L_∞ also meets Π in the $q+1$ points corresponding to the line of the spread that Π meets. The Lemma follows. ∎

As was shown in the proof of Lemma 2.1, on any line of the spread there are $q-1$ planes that represent lines of π that are external to $\mathcal{K}$ (the other external line being L_∞). There are q^2 line of the spread not equal to s, so the number of type (a) elements of $\mathcal{K}_{\Delta_D}$ is $q^2(q-1)$. The number of type (b) elements is q, giving $|\mathcal{K}_{\Delta_D}| = q^3-q^2+q$, the size of a maximal $\{q^3-q^2+q;q\}$−arc in a plane of order q^2. We now show that every line of $\Delta_D(\pi)$ meets $\mathcal{K}_{\Delta_D}$ in at most q points.

Lines of type (I)
Note that in $\Delta_D(\pi)$ a line of type (I) is not incident with a point of type (III), so lines of type (I) are not incident with points of type (b) in $\mathcal{K}_{\Delta_D}$.

A line of type (I) in $\Delta_D(\pi)$ is represented by a point $z \in PG(4,q)\backslash(H \cup D)$. The points of $\Delta_D(\pi)$ on this line are represented as the q^2+1 planes of $PG(4,q)$ given by the span of z and the q^2+1 line of the spread. These planes also represent the lines of π on the point z, and we use this fact to argue that a line of type (I) in $\Delta_D(\pi)$ meets $\mathcal{K}_{\Delta_D}$ in at most q points.

Suppose $z \notin \mathcal{K}$. Then there are exactly q lines of π incident with z and external to $\mathcal{K}$. So there are at most q planes on z that meet H in a line of the spread not equal to s and do not meet $\mathcal{K}$. Hence the line of $\Delta_D(\pi)$ that z represents meets $\mathcal{K}_{\Delta_D}$ in at most q points (in fact exactly q points since the planes on s representing external lines of π to $\mathcal{K}$ are contained in the three space D).

Suppose $z \in \mathcal{K}$. Then every line of π on z meets $\mathcal{K}$, hence every plane of $PG(4,q)$ on z and containing a line of the spread meets $\mathcal{K}$. It follows that no point of $\Delta_D(\pi)$ on the line z is in the set $\mathcal{K}_{\Delta_D}$. So the line of $\Delta_D(\pi)$ that z represents meets $\mathcal{K}_{\Delta_D}$ in 0 points.

Lines of type (II)
Lines of type (II) are represented as planes contained in D, but not containing any line of the spread. Let Π be such a line. The points of $\Delta_D(\pi)$ on Π are of type (I) or (III). The type (III) point on Π is the point $\Pi \cap s$ of $PG(4,q)$. The type (I) points on Π are represented by q^2 planes of $PG(4,q)$ not contained in H, containing a unique line of the spread not equal to s and meeting Π in a line of $PG(4,q)$. Note that the type (I) points on Π represent lines of π.

Suppose that the type (III) point incident with Π is not contained in $\mathcal{K}_{\Delta_D}$ (i.e., Π meets s in the point o of the ovoid). Lemma 2.1 shows that Π represents the set of points of a Baer subplane of the plane π, and that the q external lines to $\mathcal{K}$ in π on the point corresponding to the line s of the spread are lines of Π (when restricted to Π). It follows, by Grünings' Lemma (Lemma 1.2.6), there are at most $2q - q = q$ lines of Π (considered as a Baer subplane of π) that when extended to π are external to $\mathcal{K}$ and not on the point corresponding to s. Hence there are at most q points of type (I) on Π that are contained in $\mathcal{K}_{\Delta_D}$.

Suppose that the type (III) point incident with Π is contained in $\mathcal{K}_{\Delta_D}$. Then the plane representing Π meets the line s in a point other than o. It follows that Π meets the line ox in a point other than o, and so Π meets $\mathcal{K}$ in a single point. Arguing as in the previous paragraph, there are at most $2q-(q+1) = q-1$ lines of Π (considered as a Baer subplane of π) that when extended to π are external to $\mathcal{K}$ and not on the point corresponding to s. Hence there are at most $q-1$ points of type (I) on Π that are contained in $\mathcal{K}_{\Delta_D}$, giving in total at most q points of $\mathcal{K}_{\Delta_D}$ contained in Π.

The line of type (III)
Clearly the line of type (III) meets $\mathcal{K}_{\Delta_D}$ in only the q type (b) points of $\mathcal{K}_{\Delta_D}$.

Thus every line of $\Delta_D(\pi)$ meets $\mathcal{K}_{\Delta_D}$ in at most q points. The Theorem follows.
∎

Using the above arguments and Lemmas it is not difficult (but is notationally awful) to show *directly* that every line of $\Delta_D(\pi)$ meets $\mathcal{K}_{\Delta_D}$ in either 0 or q points.

3. AN ALTERNATE PROOF OF MAXIMAL ARCS IN HALL PLANES

The main result of this section (Theorem 3.1) is to construct maximal arcs in Hall planes that are inherited from the the Thas 1974 maximal arcs. As will be noted in Section 5 this class of maximal arcs is isomorphic to a subclass of those constructed in Section 2. But the method of proof is sufficiently different to make it worth while including.

3.1 A construction of the Hall planes

The Hall plane of order q^2, $q \geq 3$ can constructed from the Desarguesian plane, $PG(2, q^2)$, in the following way. Let L_∞ be a line of $PG(2, q^2)$, and denote the affine plane $PG(2, q^2)\backslash L_\infty$ by $AG(2, q^2)$. For any line L of $AG(2, q^2)$ the **ideal point** of L is the intersection in $PG(2, q^2)$ of L_∞ with L.

Let $\mathcal{D}$ be a Baer subline of L_∞ (see Section 3 of the previous chapter). We define a new incidence structure as follows:

The points are the points of $AG(2, q^2)$.
The lines are of two types:
(i) any line of $AG(2, q^2)$ whose ideal point is not in $\mathcal{D}$.
(ii) The Baer subplanes of $PG(2, q^2)$ which contain $\mathcal{D}$.
Incidence is containment.

This incidence structure is an affine plane of order q^2. It completes to a projective plane of order q^2, called the Hall plane and denoted Hall(q^2) (see [87],[103], [81]). The set $\mathcal{D}$ is often called a **derivation set** for $AG(2, q^2)$. Derivation sets can be defined in an arbitrary affine plane, but we will only be concerned with the Desarguesian planes where the derivation sets are exactly the Baer sublines.

In this section we will be only interested in the Thas 1974 maximal arcs in the Desarguesian planes. Later we examine how certain Baer subplanes of $PG(2, q^2)$ meet a Thas 1974 maximal arc. Representing $PG(2, q^2)$ as previously via a (regular) spread $\mathcal{S}$ in a hyperplane H of $PG(4, q)$, we note the following properties (see [22],[23]):

(i) A plane of $PG(4,q)$ that meets H in a line $L \notin \mathcal{S}$ corresponds to a Baer subplane of $PG(2,q^2)$ that meets L_∞ in $q+1$ points (and conversely).
(ii) Let $\mathcal{B}$ be a Baer subplane that meets L_∞ in a unique point, that point corresponding to a line s of the spread. Then there is a collection $\mathcal{L}$ of $q+1$ skew lines of $PG(4,q)$ with the following properties:
(a) Each line of $\mathcal{L}$ meets s, but is not contained in H.
(b) No three lines of $\mathcal{L}$ are contained in a projective three-space.
(c) Define the set of points

$$\mathcal{P} = \{p \in PG(4,q)\backslash H : p \text{ incident with } L, \text{ for some } L \in \mathcal{L}\}.$$

Then the set comprising of the line s of the spread, together with the points of $\mathcal{P}$, corresponds to the points of the Baer subplane $\mathcal{B}$.

In the Bruck-Bose representation of $PG(2,q^2)$, any line of $PG(4,q)$ not contained in $PG(3,q)$ corresponds to a Baer subline of $PG(2,q^2)$, and hence to a derivation set (Lemma 2.3.3).

3.2 Maximal arcs in Hall planes inherited from Thas 1974 maximal arcs

THEOREM 3.1 ([73]) *Let $\mathcal{K}$ be a Thas 1974 maximal $\{q^3-q^2+q;q\}-$arc in $PG(2,q^2)$, $q>2$, with centre x and axis L_∞. Let L be a line of $PG(2,q^2)$ on the centre x, and let $\mathcal{D} = (L \cap \mathcal{K}) \cup (L \cap L_\infty)$. Then $\mathcal{D}$ is a derivation set and the set of points $\mathcal{K}' = \mathcal{K}\backslash\mathcal{K}\cap\mathcal{D}$ is a set of type $(0, q-1, q)$ in the plane Hall(q^2) associated with $\mathcal{D}$, i.e. every line of Hall(q^2) meets $\mathcal{K}'$ in 0, $q-1$ or q points. Further, $\mathcal{K}'$ is contained in a maximal $\{q^3-q^2+q;q\}-$arc in Hall(q^2).*

Proof: Represent $\mathcal{K}$ as previously in the Bruck-Bose representation of $PG(2,q^2)$, with the axis L_∞ corresponding to the (regular) spread $\mathcal{S}$ in the projective three space H, and an ovoid $\mathcal{O}$ of the appropriate form.

The set of points of $\mathcal{D}$ corresponds to a line $L_{\mathcal{D}}$ of $PG(4,q)$ which is on the centre x of $\mathcal{K}$ and on a point o of the ovoid (o is contained in a unique line s of the spread $\mathcal{S}$). It follows that $\mathcal{D}$ is a Baer subline and hence a derivation set.

We first show that every Baer subplane of $PG(2,q^2)$ containing $\mathcal{D}$ meets $\mathcal{K}'$ in either 0, $q-1$ or q points. Let $\mathcal{B}$ be any Baer subplane of $PG(2,q^2)$ containing the points of $\mathcal{D}$. We examine the two cases:
(i) $|L_\infty \cap \mathcal{B}| = q+1$.
Then there exists a plane Π of $PG(4,q)$ such that the points of $\mathcal{B}$ correspond to the points of $\Pi\backslash(\Pi\cap H)$, together with the lines of the spread that Π meets. Note that $L_{\mathcal{D}} \leq \Pi$.

The line $M = \Pi \cap H$ of $PG(4,q)$ is on the point o of the ovoid. It follows from the definition of the ovoid $\mathcal{O}$, M meets $\mathcal{O}$ in either 1 or 2 points. In the former case $\Pi\cap\mathcal{K} = L_{\mathcal{D}}\backslash\{o\}$ and hence $|\mathcal{B}\cap\mathcal{K}'| = 0$. In the later case there exists $o' \in M\cap\mathcal{O}$,

$o' \neq o$, and $\Pi \cap \mathcal{K} = L_{\mathcal{D}} \cup \bar{xo'} \backslash \{o, o'\}$ ($\bar{xo'}$ is the line of $PG(4, q)$ which contains both x and o'), hence $|\mathcal{B} \cap \mathcal{K}'| = q - 1$.

So every Baer subplane on $\mathcal{D}$ that meets the axis L_∞ in more that one point meets $\mathcal{K}'$ in 0 or $q - 1$ points.

(ii) $|L_\infty \cap \mathcal{B}| = 1$.
$\mathcal{B}$ meets L_∞ in a unique point, that point corresponding to the line s of the spread. Then, as noted previously, there is a collection of $q + 1$ skew lines $\mathcal{L}$ of $PG(4, q)$ that represents $\mathcal{B}$ as described previously. Note that $\mathcal{B}$ contains $\mathcal{D}$, so $L_{\mathcal{D}} \in \mathcal{L}$.

The lines of $PG(2, q^2)$ on the point corresponding to s, that are external to the maximal arc are represented as a collection Π of q planes of $PG(4, q)$ not contained in H. Each plane of Π contains the line s of the spread, and no plane contains a point of $\mathcal{K}$. Since $|\mathcal{D} \cap \mathcal{K}| = q$, there is a unique point of $\mathcal{D}$ not contained in $\mathcal{K}$ (corresponding to $s \in \mathcal{S}$). Hence the external lines to $\mathcal{K}$ that are lines of $\mathcal{B}$ are represented as a subset of the set of planes Π. We examine how the lines $\mathcal{L}$ meet the collection Π of planes.

Consider the projection of the planes of Π from the centre x onto the hyperplane H. Since no plane of Π contains the centre x, the image under the projection of each plane will be a plane of H containing s (and hence o). There is a unique tangent plane Π_T on o in H of the ovoid, and since no plane of Π meets $\mathcal{K}$, the image of each plane under the projection must be Π_T. Hence the planes of Π are all contained in the projective three space $\langle x, \Pi_T \rangle$. No three lines of $\mathcal{L}$ are contained in a three space, so at most two lines of $\mathcal{L}$ are contained in $\langle x, \Pi_T \rangle$ (and at least one since $L_{\mathcal{D}} \leq \langle x, \Pi_T \rangle$). Hence at most one plane of Π contains a line of $\mathcal{L}$. The number b_0 of lines of $\mathcal{B}$ which when extended to $PG(2, q^2)$ are external to $\mathcal{K}$ must then be 0 or 1. It follows from Grüning's Lemma that $|\mathcal{B} \cap \mathcal{K}| = 2q$ or $2q - 1$, i.e. $|\mathcal{B} \cap \mathcal{K}'| = q$ or $q - 1$.

Consider a parallel class of lines of Hall(q^2), the lines of which are of type (ii) (Baer subplanes of $PG(2, q^2)$ containing $\mathcal{D}$). Suppose k of these lines meet $\mathcal{K}'$ in $q - 1$ points, and l of them meet $\mathcal{K}'$ in q points. Then,

$$|\mathcal{K}'| = q^2(q - 1) = k(q - 1) + lq.$$

Noting that the q external lines to $\mathcal{K}'$ of type (ii) are contained in a single parallel class, we have two cases

$$k + l + q = q^2 \quad \text{or} \quad k + l = q^2$$

giving either $k = 0$ and $l = q(q - 1)$ or $l = 0$ and $k = q^2$, respectively. The parallel classes (of type (ii) lines) are then of two types; those whose lines meet $\mathcal{K}'$ in $q - 1$ points, and those whose lines meet $\mathcal{K}'$ in 0 or q points. Let $\mathcal{K}_{Hall}$

be the set of points of $\mathcal{K}'$ together with the q points on the line at infinity of Hall(q^2) of the parallel classes that meet $\mathcal{K}'$ in $q-1$ points. Then it follows that every line of Hall(q^2) meets $\mathcal{K}_{Hall}$ in either 0 or q points, i.e. $\mathcal{K}_{Hall}$ is a maximal $\{q^3-q^2+q;q\}-$arc in Hall(q^2). ∎

4. MAXIMAL ARCS IN DERIVED DUAL HALL PLANES

The main result of this subsection (Theorem 4.1) is that if one of the maximal arcs constructed in the previous section exists in a Hall plane of order q^2 (as they are known to for all $q>2$ even), then there exist maximal $\{q^3-q^2+q;q\}-$arcs in a particular derived dual of the Hall plane.

4.1 Preliminaries

Suppose we have a Thas 1974 maximal $\{q^3-q^2+q;q\}-$arc $\mathcal{K}$ in a Desarguesian plane π or order q^2, represented in the Bruck-Bose construction in $P=PG(4,q)$ via a (regular) spread $\mathcal{S}$ and ovoid $\mathcal{O}$ in a hyperplane H of P, and the centre x of $\mathcal{K}$. Since π is Desarguesian every line is a translation line, and so we can consider how the maximal arc would 'look' if we represented the plane via Bruck-Bose with a different line $L'_\infty \neq L_\infty$ 'at infinity'.

Let L'_∞ be any line of π on the centre x of $\mathcal{K}$, $s\in\mathcal{S}$ be the line of the spread corresponding to the point $L_\infty \cap L'_\infty$, and $o\in\mathcal{O}$ the unique point of the ovoid contained in s. Denote the Bruck-Bose representation of the plane with line at infinity L'_∞ by π'. Similarly, in this model denote the projective four-space by P', the projective three-space at infinity by H', and the (regular) spread in H' by $\mathcal{S}'$.

The line $\bar{xo}$ of P corresponds to a Baer subline of π. It is contained in L'_∞, and so corresponds to a regulus $\mathcal{R}'$ of the spread $\mathcal{S}'$ [23][55]. It follows that in π' the maximal arc $\mathcal{K}$ is represented as a collection of q^3-q^2 points of $P'\backslash H'$, together with q lines of the regulus $\mathcal{R}'$.

If we denote the opposite regulus to $\mathcal{R}'$ by $\mathcal{R}'_{opp}$, a Hall plane π'_{Hall} of order q^2 can be formed from the spread given by $\mathcal{S}''=\{\mathcal{S}'\backslash\mathcal{R}'\}\cup\{\mathcal{R}'_{opp}\}$. Since derivation with respect to a Baer subline of L'_∞ corresponds to switching a regulus in $\mathcal{S}'$, Theorem 3.1 of the previous section shows that the q^3-q^2 points of $\mathcal{K}$ in $P'\backslash H'$, together with a certain collection of q lines of $\mathcal{R}'_{opp}$, is a maximal $\{q^3-q^2+q;q\}-$arc $\mathcal{K}_{Hall}$ in the Hall plane of order q^2 so formed. Exactly which collection of q lines of $\mathcal{R}'_{opp}$ represent points on L'_∞ of $\mathcal{K}_{Hall}$ will become apparent in the proof of Lemma 4.1.

The following Lemma will be used in the next subsection to construct a plane $\Delta_{D'}(\pi'_{Hall})$ that is a derived dual of π'_{Hall}, and such that the dual of $\mathcal{K}_{Hall}$ inherits to a maximal arc in $\Delta_{D'}(\pi'_{Hall})$ in much the same way as those in Theorem 2.1.

LEMMA 4.1 *There is a projective three-space $D' \neq H'$ of P' such that D' contains no points of $\mathcal{K} \backslash H'$, and hence no points of $\mathcal{K}_{Hall} \backslash H'$.*

Proof: Recall that in π, there is a point $o \in \mathcal{O}$ of the ovoid, on the line s of the spread corresponding to the meet of L_∞ with L'_∞ in π. Also recall that there is a regulus $\mathcal{R}'$ in $\mathcal{S}'$ corresponding to the points of the line $\bar{xo}$ in P.

There is a unique tangent plane to o in H, and this tangent plane contains the line s. Consider the collection $\mathcal{B}$ of q planes in P, each plane given by the span of x and a tangent line to o not equal to s. These planes represent Baer subplanes of π that meet both L_∞ and L'_∞ in $q+1$ points. Hence in π' there is a corresponding collection $\mathcal{B}'$ of planes of P' representing the Baer subplanes that the planes of $\mathcal{B}$ represent in π. Also L_∞ is a line of π and hence of π', and so is represented in π' as a plane Π_{L_∞} of P' not contained in H' and meeting H' in a line $r' \in \mathcal{S}'$. In fact r' is a line of the regulus $\mathcal{R}'$ since it corresponds to the intersection of the lines L_∞ and L'_∞.

The planes of $\mathcal{B}$ are parallel in the sense that they all meet pairwise in the same $q+1$ points (corresponding to points of $\bar{xo}$), and no others. It follows, since each Baer subplane represented by a plane in $\mathcal{B}'$ meets L'_∞ in a subline, that there is a unique line $r'_{opp} \in \mathcal{R}'_{opp}$ that is contained in all of the planes of $\mathcal{B}'$.

It is worth noting here that the lines $\mathcal{R}'_{opp} \backslash \{r'_{opp}\}$ correspond to the points of $\mathcal{K}_{Hall}$ on the line at infinity in the associated Hall plane.

Since there are exactly q planes not contained in H' on r'_{opp} that meet Π_{L_∞} in $q+1$ points, and since no plane of $\mathcal{B}'$ contains a point of $\mathcal{K} \backslash H'$, it follows that the projective three-space in P' given by

$$D' = < \Pi_{L_\infty}, r'_{opp} >$$

contains no point of $\mathcal{K} \backslash H'$ and hence no point of $\mathcal{K}_{Hall} \backslash H'$. ∎

At this point it may be useful for the reader to draw a diagram of 'where everything is' in π', as this is where we will be arguing from now on . The main points to note are that:

(a) D' is a projective three-space contained in P' which contains no points of $\mathcal{K}_{Hall} \backslash H'$.

(b) The points of the maximal arc $\mathcal{K}_{Hall}$ in π'_{Hall} are given by those points of $\mathcal{K} \backslash H'$, together with the points represented by the lines of $\mathcal{R}'_{opp} \backslash \{r'_{opp}\}$.

(c) The lines of $\mathcal{R}'_{opp} \backslash \{r'_{opp}\}$ meet D' in the points of $r' \backslash r' \cap r'_{opp}$, where $r' \in \mathcal{R}'$ is the line of the regulus $\mathcal{R}'$ representing the meet of L_∞ with L'_∞ in π'.

(d) The line r'_{opp} represents a point of π'_{Hall} not in $\mathcal{K}_{Hall}$. Since $\mathcal{K}_{Hall}$ is a maximal $\{q^3 - q^2 + q; q\}-$arc arc it follows that there are q lines of π'_{Hall} incident with r'_{opp} that are external to $\mathcal{K}_{Hall}$. Clearly, these external lines are represented by the set $\mathcal{B}'$ of planes which are all contained in the three-space D'.

4.2 Maximal arcs inherited from maximal arcs in dual Hall planes

Let $\mathcal{K}_{Hall}$ be a maximal $\{q^3-q^2+q;q\}-$arc in the Hall plane π'_{Hall} of order q^2 defined by the spread $\mathcal{S}'' = \{\mathcal{S}'\backslash\mathcal{R}'\} \cup \{\mathcal{R}'_{opp}\}$ as in the previous subsection.

The projective three-space D' and the line $r'_{opp} \in \mathcal{S}''$ of the spread define a projective plane $\Delta_{D'}(\pi'_{Hall})$, as in Section 2, that is a derived plane of the dual plane of π'_{Hall}.

Define $\mathcal{K}_{\Delta_{D'}}$ to be the set containing the following two types of elements:
(a) planes Π of P' not contained in H' or D' that meet H' in a line of the spread not equal to r'_{opp}, and satisfying $\Pi \cap \mathcal{K}_{Hall} = \emptyset$.
(b) points of P' on the line r'_{opp} not equal to $r' \cap r'_{opp}$.

The set $\mathcal{K}_{\Delta_{D'}}$ then represents a set of points of $\Delta_{D'}(\pi'_{Hall})$. The type (a) elements of $\mathcal{K}_{\Delta_{D'}}$ also represent a subset of the lines of the translation plane π'_{Hall} that are external to the maximal arc $\mathcal{K}_{Hall}$, so that $\mathcal{K}_{\Delta_{D'}}$ is inherited from the dual of $\mathcal{K}_{Hall}$. We prove that the set $\mathcal{K}_{\Delta_{D'}}$ is the set of points of a maximal $\{q^3-q^2+q;q\}-$arc in the plane $\Delta_{D'}(\pi'_{Hall})$ by first showing the set has size q^3-q^2+q, and then showing that every line of $\Delta_{D'}(\pi'_{Hall})$ meets $\mathcal{K}_{\Delta_{D'}}$ in less than or equal to q points. In summary, we prove the following Theorem.

THEOREM 4.1 ([74]) *The set $\mathcal{K}_{\Delta_{D'}}$ is the set of points of a maximal $\{q^3-q^2+q;q\}-$arc in the derived dual plane $\Delta_{D'}(\pi'_{Hall})$ of π'_{Hall}.*

Proof: In the spread $\mathcal{S}''$ there are q^2+1-q lines which represent points of π'_{Hall} that are not in $\mathcal{K}_{Hall}$, one of which is r'_{opp}. On each of these lines there are q planes that represent lines of π'_{Hall} that are external to $\mathcal{K}_{Hall}$. So the number of type (a) elements of $\mathcal{K}_{\Delta_{D'}}$ is $q(q^2-q)$. The number of type (b) elements is q, giving $|\mathcal{K}_{\Delta_{D'}}| = q^3-q^2+q$, the size of a maximal $\{q^3-q^2+q;q\}-$arc in a plane of order q^2. We now show that every line of $\Delta_{D'}(\pi'_{Hall})$ meets $\mathcal{K}_{\Delta_{D'}}$ in at most q points.

Lines of type (I)
Note that in $\Delta_{D'}(\pi'_{Hall})$ a line of type (I) is not incident with a point of type (III), so lines of type (I) are not incident with points of type (b) in $\mathcal{K}_{\Delta_{D'}}$.

A line of type (I) in $\Delta_{D'}(\pi'_{Hall})$ is represented by a point $z \in P'\backslash(H' \cup D')$. The points of $\Delta_{D'}(\pi'_{Hall})$ on this line are represented as the q^2+1 planes of P' given by the span of z and the q^2+1 lines of the spread $\mathcal{S}''$. These planes also represent the lines of π'_{Hall} on the point z, and we use this fact to argue that a line of type (I) in $\Delta_{D'}(\pi'_{Hall})$ meets $\mathcal{K}_{\Delta_{D'}}$ in at most q points.

Suppose $z \notin \mathcal{K}_{Hall}$. Then there are exactly q lines of π'_{Hall} incident with z and external to $\mathcal{K}_{Hall}$. So there are at most q planes on z that meet H' in an line of the spread $\mathcal{S}''$ not equal to r'_{opp} and do not meet $\mathcal{K}_{Hall}$. Hence the line of $\Delta_{D'}(\pi'_{Hall})$

that z represents meets $\mathcal{K}_{\Delta_{D'}}$ in at most q points (in fact exactly q points since the planes on r'_{opp} representing external lines of π'_{Hall} to $\mathcal{K}_{Hall}$ are contained in the three space D').

Suppose $z \in \mathcal{K}_{Hall}$. Then every line of π'_{Hall} on z meets $\mathcal{K}_{Hall}$, hence every plane of P' on z and containing a line of the spread $\mathcal{S}''$ meets $\mathcal{K}_{Hall}$. It follows that no point of $\Delta_{D'}(\pi'_{Hall})$ on the line z is in the set $\mathcal{K}_{\Delta_{D'}}$. So the line of $\Delta_{D'}(\pi'_{Hall})$ that z represents meets $\mathcal{K}_{\Delta_{D'}}$ in 0 points.

Lines of type (II)
Lines of type (II) are represented as planes contained in D', but not containing r'_{opp}. Let Π be such a line. The points of $\Delta_{D'}(\pi'_{Hall})$ on Π are of type (I) or (III). The type (III) point on Π is the point $\Pi \cap r'_{opp}$ of P'. The type (I) points on Π are represented by q^2 planes of P' not contained in H', containing a unique line of the spread $\mathcal{S}''$ not equal to r'_{opp} and meeting Π in a line of P'. Note that the type (I) points on Π represent lines of π'_{Hall}.

Suppose that the type (III) point incident with Π is not contained in $\mathcal{K}_{\Delta_{D'}}$ (i.e., Π meets r'_{opp} in the point $r'_{opp} \cap r'$). Now Π represents the set of points of a Baer subplane of the plane π'_{Hall}. And the q planes $\mathcal{B}'$ representing external lines to $\mathcal{K}_{Hall}$ in π'_{Hall} on the point corresponding to the line r'_{opp} of the spread are all contained in D' and so are lines of Π (when restricted to Π). It follows, by Grünings' Lemma, there are at most $2q - q = q$ lines of Π (considered as a Baer subplane of π'_{Hall}) that when extended to π'_{Hall} are external to $\mathcal{K}_{Hall}$ and not on the point corresponding to r'_{opp}. Hence there are at most q points of type (I) incident with Π that are contained in $\mathcal{K}_{\Delta_{D'}}$.

Suppose that the type (III) point incident with Π is contained in $\mathcal{K}_{\Delta_{D'}}$. Then the plane representing Π meets the line r'_{opp} in a point other than $r'_{opp} \cap r'$. Since D' contains no point of $\mathcal{K}_{Hall} \backslash H'$ it follows that Π meets $\mathcal{K}_{Hall}$ in a single point of π'_{Hall} (corresponding to the line of the spread $\mathcal{S}''$ that contains the point $r'_{opp} \cap \Pi$). Arguing as in the previous paragraph, there are at most $2q - (q+1) = q - 1$ lines of Π (considered as a Baer subplane of π'_{Hall}) that when extended to π'_{Hall} are external to $\mathcal{K}_{Hall}$ and not on the point corresponding to r'_{opp}. Hence there are at most $q-1$ points of type (I) on Π that are contained in $\mathcal{K}_{\Delta_{D'}}$, giving in total at most q points of $\mathcal{K}_{\Delta_{D'}}$ incident with Π.

The line of type (III)
Clearly the line of type (III) meets $\mathcal{K}_{\Delta_{D'}}$ in only the q type (b) points of $\mathcal{K}_{\Delta_{D'}}$.

Thus every line of $\Delta_{D'}(\pi'_{Hall})$ meets $\mathcal{K}_{\Delta_{D'}}$ in at most q points. The Theorem follows. ∎

Using the above arguments and Lemmas it is not difficult (but is notationally awful) to show *directly* that every line of $\Delta_{D'}(\pi'_{Hall})$ meets $\mathcal{K}_{\Delta_{D'}}$ in either 0 or q

points.

In all of the above, supposing we had chosen the Desarguesian spread $\mathcal{S}'$ instead of the Hall spread $\mathcal{S}''$, and chosen $r' \in \mathcal{S}'$ instead of r'_{opp}, then it is clear that the same proof could be used to show that the original Thas maximal arc $\mathcal{K}$ in π' inherited to a maximal arc in a plane that was the derived dual of π'. Since the dual of a Desarguesian plane is Desarguesian, and deriving a Desarguesian plane gives a Hall plane, this would give maximal arcs in Hall planes. However, by examining the derivation sets, it is can be seen that the maximal arcs so constructed in the Hall planes are isomorphic to a subset of those in Section 2 (in fact are isomorphic to those of Section 3, see next section).

5. REMARKS

As it was shown in the previous chapter, the Thas 1974 construction is known not to give maximal arcs in planes of odd order, but is known to give maximal arcs in the Lüneburg and Desarguesian planes of even (square) order. We examine these two cases.

The dual plane of a Desarguesian plane is Desarguesian, and all planes derived from a Desarguesian plane are isomorphic to the Hall plane of the same order. Hence the construction of Section 2 gives maximal $\{q^3 - q^2 + q; q\}-$arcs in all Hall planes of order q^2, $q > 2$ even ($q^2 = 4$ is exceptional in that any plane of order 4 is Desarguesian). Also note that ovoid-Hall spread pairs of the form required for the Thas construction are known not to exist (the Hall spread is not symplectic), so maximal arcs constructed in Section 2 in the Hall planes are not Thas maximal arcs (for $q^2 > 4$).

Using the characterisation Theorem of Thas 1974 maximal arcs (Theorem 2.3.4) it is easy to see that the maximal arcs constructed in Section 2 in the Hall planes are isomorphic to those in Section 3. It follows from this Theorem that the external lines to a Thas maximal arc in a Desarguesian plane form a Thas maximal arc in the dual (also Desarguesian) plane, the centre of the maximal arc in the dual plane being L_∞, and the axis being x. By examining the derivation sets, it is then clear that the maximal arcs of Section 2 that occur in the Hall planes are the same as those in Section 3.

Given a Thas 1974 maximal arc in a Desarguesian plane, different choices for the derivation set may give rise to non-isomorphic maximal arcs in the Hall plane. But in the case of a regular spread - elliptic quadric pair (which gives rise to a Denniston maximal arc), Theorem 2.2.2 shows there is a cyclic group which is regular on the derivation sets of the required form, and so all maximal arcs inherited to Hall planes from such maximal arcs are equivalent.

Given a Lüneburg plane, up to isomorphism there are two projective planes that are its derived duals via the Bose-Barlotti construction. This can be seen from the fact that the stabiliser of the Lüneburg spread in $P\Gamma L(4,q)$ admits the Suzuki group $Sz(q)$ which has two orbits on the planes of $PG(3,q)$ [103, 27.3]. Also the group of translations is transitive on the points not on L_∞. So, up to isomorphism, there are two choices for the three space D, and hence two choices for the derivation set in the construction of Section 2. Neither of the planes derived from the dual of a Lüneburg plane is a translation plane or dual to a translation plane [108]. Apart from the construction of Section 4, the only known constructions of maximal arcs (other than hyperovals or their duals) are in translation planes or their duals, and so the maximal arcs of Section 2 in the derived dual Lüneburg planes are new. Note that the Lüneburg planes are necessarily of order q^2, $q = 2^{2m+1}$, $m \geq 1$, and we have constructed maximal $\{q^3 - q^2 + q; q\}-$arcs in all Bose-Barlotti derived duals of such planes.

Given a Hall plane, up to isomorphism there are two projective planes that are its derived duals via the Bose-Barlotti construction. This is easiest to see by considering a Hall spread in a 3-space H as a regular spread with a single regulus switched. Using a little group theory it is not to difficult to show that the stabiliser in $P\Gamma L(4,q)$ of the Hall spread has two orbits on planes in H, the orbits being those planes that meet the switched regulus in a line of the switched regulus, and those that do not. As in the previous paragraph there are up to isomorphism two choices for the three space D defining the derivation set in the dual Hall plane. The construction of Section 4 requires that D meet the switched regulus in a line of the switched regulus, and so the construction only gives maximal arcs in one of the derived duals of a given Hall plane. If we (arbitrarily) call the Bose-Barlotti derived duals of the Hall planes class 1 or class 2 according to whether or not the 3-space D meets the switched regulus in a line of the switched regulus or not, respectively, then the result of Section 4 can be stated as the following: There exist maximal $\{q^3 - q^2 + q; q\}-$arcs in the class 1 derived duals of the Hall planes of order q^2, $q > 2$ even.

Chapter 4
MORE MAXIMAL ARCS AND ASSOCIATED STRUCTURES

1. INTRODUCTION

The work of this chapter is in two sections.

In Section 2, new structures in projective planes are generated using Denniston maximal arcs. In subsection 2.1, disjoint degree 4 Denniston maximal arcs are found and classified in $PG(2,16)$, and an example of disjoint degree 8 maximal arcs in $PG(2,64)$ is given. The results of this subsection complete the spectral problem for existence of sets of type (m,n) in $PG(2,16)$. The work in subsection 2.1 is my part of a joint paper between Bousfield, Penttila, Royle and myself [25]. In subsection 2.2, a conjecture is made about the existence of inherited degree 4 maximal arcs in Hall planes of even order. The conjecture has been shown to be true (by computer) for Hall planes of order 16, 64 and 256.

In Section 3, a number of new degree 4 maximal arcs in certain projective plane of order 16 are exhibited (subsection 3.2). Pairs of disjoint degree 4 maximal arcs are also shown to exist in certain of these planes (subsection 3.3). Finally, in subsection 3.4, a method for finding inherited structures in translation planes is outlined. The results of subsections 3.2 and 3.3 appeared in a different form as a U.W.A. Department of Mathematics Research Report [75].

2. DENNISTON MAXIMAL ARCS AND ASSOCIATED STRUCTURES

In this section, new structures in projective planes are generated using Denniston maximal arcs. In the first subsection, disjoint degree 4 Denniston maximal arcs are found and classified in $PG(2,16)$, and an example of disjoint degree 8 maximal arcs in $PG(2,64)$ is given. In subsection 2.2, a conjecture is made about the existence of inherited degree 4 maximal arcs in Hall planes of even order.

2.1 Disjoint Denniston maximal arcs

In [51], M. J. de Resmini listed the open cases for sets of type (m,n) in $PG(2,16)$. These were a 104-set of type $(4,8)$, a 156-set of type $(8,12)$, a 130-set of type $(6,10)$ and their duals and complements. It was noted that a 104-set of type $(4,8)$ might be given by a disjoint pair of maximal $\{52;4\}$-arcs, and that a 156-set of type $(8,12)$ might be given by a disjoint triple of maximal $\{52;4\}$-arcs. In this subsection all such disjoint sets of Denniston maximal arcs in $PG(2,16)$ are listed (up to isomorphism). These results, together with a construction of a 130-set of type $(6,10)$ given by Gordon Royle in [25], complete the spectral problem for sets existence of sets of type (m,n) in $PG(2,16)$. A disjoint pair of Denniston maximal $\{456;8\}$-arcs in $PG(2,64)$ is also given, and hence a 912-set of type $(8,16)$ in $PG(2,64)$.

In a plane of order q^2, it is easy to show, using Grüning's Lemma, that disjoint degree q maximal arcs give rise to sets of type (m, n). Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be maximal $\{q^3 - q^2 + q; q\}-$arcs in a plane of order q^2. Let $s = |\mathcal{K}_1 \cap \mathcal{K}_2|$, and b be the number of secant lines to $\mathcal{K}_1$ that are external to $\mathcal{K}_2$. Then the Grüning count gives that

$$s + b = q^3 - q^2 + q.$$

The number of external lines to a maximal $\{q^3 - q^2 + q; q\}-$arc is $q^3 - q^2 + q$. It follows that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are disjoint then every external line to $\mathcal{K}_1$ is secant to $\mathcal{K}_2$. Similarly every external line to $\mathcal{K}_2$ is secant to $\mathcal{K}_1$. Hence every line of the plane meets $\mathcal{K}_1 \cup \mathcal{K}_2$ in either q or $2q$ points. More generally, it follows that given a collection of m pairwise disjoint maximal $\{q^3 - q^2 + q; q\}-$arcs in a plane of order q^2, that the union of the points of such a collection is a set of type $(mq - q, mq)$.

Such sets are also of useful from the point of view of partial geometries. In [46] and [49], F. DeClerck posed the following problem: Given a maximal arc of type $(0, d)$ in a projective plane of order q, does there exist a maximal arc of type $(0, d')$ disjoint from it? Prior to the work of this chapter, no examples of any pair of disjoint maximal arcs were known where both of the maximal arcs were not hyperovals. Such configurations are interesting as the points of the $(0, d')$ maximal arc form a set of type $(0, d')$ in the $S(\mathcal{K})$ partial geometry associated with the $(0, d)$ maximal arc. Sets of this type were studied extensively in [49][134]. The study of disjoint maximal arcs could also be seen in the context of the study of the possibility of disjoint unitals, an area which has generated considerable interest (see for instance [8]). Also disjoint maximal arcs may be of interest in the study of code words for projective planes [98].

For planes of small order it is relatively easy to construct disjoint Denniston maximal arcs by computer. Suppose $\mathcal{K}_1$ is a Denniston maximal arc. A conic with nucleus (f, e, d) has an equation of the form $ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$ [78]. So given any point not on $\mathcal{K}_1$, find all (non-degenerate) conics with this point as nucleus that are disjoint from $\mathcal{K}_1$, and attempt to construct maximal arcs $\mathcal{K}_2$ from them. The number of choices for the nucleus of the disjoint maximal arc can be reduced by only considering representatives of the orbits of the collineation stabiliser of $\mathcal{K}_1$.

Using the corollary to Theorem 2.2.3 it is easy to show that in $PG(2, 16)$ there are up to isomorphism exactly two Denniston maximal $\{52; 4\}$-arcs, one of which is isomorphic to the Thas 1974 maximal arc in $PG(2, 16)$. A computer program as outlined in the previous paragraph was written to find Denniston maximal arcs disjoint from each of these. The relatively small size of the plane meant that a complete classification of disjoint Denniston maximal $\{52; 4\}$-arcs was possible. Brendan McKays' software *nauty* [105] was then used to calculate isomorphism between the sets. Up to isomorphism, 21 sets of type $(4,8)$ were found using these methods.

Let w be a primitive element of $GF(16)$ satisfying $w^4 + w = 1$. Then

$$\begin{aligned} x^2 + yz &= 0 \\ wx^2 + w^{11}y^2 + z^2 + yz &= 0 \\ w^4x^2 + w^8y^2 + w^{12}z^2 + yz &= 0 \end{aligned} \tag{0}$$

describes the set of points of the 3 conics of a Denniston (non-Thas) maximal $\{52;4\}$-arc with nucleus $(1,0,0)$. The following 21 sets describe Denniston maximal $\{52;4\}$-arcs disjoint from (0).

$$\begin{aligned} wx^2 + w^9y^2 + w^4z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \\ w^2x^2 + w^3y^2 + w^{14}z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \\ w^{11}x^2 + w^{11}y^2 + w^2z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \end{aligned} \tag{1}$$

$$\begin{aligned} wx^2 + w^9y^2 + w^4z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \\ w^7x^2 + w^8y^2 + w^{11}z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \\ x^2 + w^{11}y^2 + w^7z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \end{aligned} \tag{2}$$

$$\begin{aligned} wx^2 + w^9y^2 + w^4z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \\ w^{10}x^2 + w^{11}y^2 + w^{13}z^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \\ w^{13}x^2 + w^5y^2 + wz^2 + w^3xy + w^{13}xz + w^{13}yz &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} w^2x^2 + w^2y^2 + w^{10}z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \\ w^9x^2 + w^7y^2 + w^2z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \\ x^2 + w^{11}y^2 + w^8z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \end{aligned} \tag{4}$$

$$\begin{aligned} w^2x^2 + w^2y^2 + w^{10}z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \\ w^9x^2 + w^{13}y^2 + w^{14}z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \\ x^2 + w^9y^2 + wz^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \end{aligned} \tag{5}$$

$$\begin{aligned} w^3x^2 + w^8y^2 + w^9z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \\ w^5x^2 + w^{11}y^2 + w^{11}z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \\ x^2 + wy^2 + w^6z^2 + w^{14}xy + w^3xz + w^{13}yz &= 0 \end{aligned} \tag{6}$$

$$
\begin{aligned}
w^4x^2+w^6y^2+wz^2+w^{14}xy+w^3xz+w^{13}yz&=0\\
w^5x^2+w^{11}y^2+w^{11}z^2+w^{14}xy+w^3xz+w^{13}yz&=0\\
w^9x^2+w^{10}y^2+w^{13}z^2+w^{14}xy+w^3xz+w^{13}yz&=0
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
w^2x^2+w^{14}y^2+wz^2+w^9xy+wxz+w^7yz&=0\\
w^6x^2+w^2y^2+w^{11}z^2+w^9xy+wxz+w^7yz&=0\\
w^{11}x^2+w^9y^2+w^{13}z^2+w^9xy+wxz+w^7yz&=0
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
w^4x^2+w^6y^2+w^{10}z^2+w^9xy+wxz+w^7yz&=0\\
w^7x^2+w^{10}y^2+w^3z^2+w^9xy+wxz+w^7yz&=0\\
w^{11}x^2+w^3y^2+w^6z^2+w^9xy+wxz+w^7yz&=0
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
w^4x^2+w^{14}y^2+w^6z^2+w^9xy+wxz+w^7yz&=0\\
w^6x^2+w^2y^2+w^{11}z^2+w^9xy+wxz+w^7yz&=0\\
w^{14}x^2+w^{11}y^2+z^2+w^9xy+wxz+w^7yz&=0
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
w^6x^2+w^2y^2+w^{11}z^2+w^9xy+wxz+w^7yz&=0\\
w^7x^2+w^4y^2+w^9z^2+w^9xy+wxz+w^7yz&=0\\
x^2+w^{14}y^2+w^3z^2+w^9xy+wxz+w^7yz&=0
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
w^7x^2+w^4y^2+w^9z^2+w^9xy+wxz+w^7yz&=0\\
w^9x^2+y^2+w^{14}z^2+w^9xy+wxz+w^7yz&=0\\
w^{10}x^2+w^3y^2+w^{13}z^2+w^9xy+wxz+w^7yz&=0
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
w^7x^2+w^7y^2+w^{11}z^2+w^9xy+wxz+w^7yz&=0\\
w^9x^2+w^7y^2+w^6z^2+w^9xy+wxz+w^7yz&=0\\
w^{10}x^2+w^7y^2+w^8z^2+w^9xy+wxz+w^7yz&=0
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
w^3x^2+w^6y^2+w^{11}z^2+xy+w^{10}xz+w^{11}yz&=0\\
w^4x^2+w^3y^2+w^{11}z^2+xy+w^{10}xz+w^{11}yz&=0\\
w^{11}x^2+w^{13}y^2+w^{11}z^2+xy+w^{10}xz+w^{11}yz&=0
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
w^7x^2+w^{12}y^2+w^{10}z^2+xy+w^{10}xz+w^{11}yz&=0\\
w^{10}x^2+y^2+w^{14}z^2+xy+w^{10}xz+w^{11}yz&=0\\
w^{14}x^2+w^4y^2+w^7z^2+xy+w^{10}xz+w^{11}yz&=0
\end{aligned} \tag{15}
$$

$$\begin{aligned} w^2x^2 + w^3y^2 + w^9z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^5x^2 + y^2 + w^6z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^9x^2 + wy^2 + w^7z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \end{aligned} \tag{16}$$

$$\begin{aligned} w^2x^2 + w^4y^2 + w^7z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^5x^2 + w^{13}y^2 + w^8z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^9x^2 + w^{12}y^2 + w^{10}z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \end{aligned} \tag{17}$$

$$\begin{aligned} w^7x^2 + y^2 + w^7z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^9x^2 + w^5y^2 + w^6z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^{14}x^2 + w^{12}y^2 + w^8z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \end{aligned} \tag{18}$$

$$\begin{aligned} wx^2 + w^9y^2 + w^{12}z^2 + xy + w^7xz + w^7yz &= 0 \\ w^4x^2 + w^{14}y^2 + w^8z^2 + xy + w^7xz + w^7yz &= 0 \\ w^{14}x^2 + w^2y^2 + w^{13}z^2 + xy + w^7xz + w^7yz &= 0 \end{aligned} \tag{19}$$

$$\begin{aligned} w^8x^2 + y^2 + w^{13}z^2 + xy + w^7xz + w^7yz &= 0 \\ w^9x^2 + w^4y^2 + z^2 + xy + w^7xz + w^7yz &= 0 \\ w^{10}x^2 + w^{10}y^2 + w^9z^2 + xy + w^7xz + w^7yz &= 0 \end{aligned} \tag{20}$$

$$\begin{aligned} w^2x^2 + w^5y^2 + w^2z^2 + xy + w^7xz + wyz &= 0 \\ w^9x^2 + w^5y^2 + w^7z^2 + xy + w^7xz + wyz &= 0 \\ w^{13}x^2 + w^5y^2 + w^8z^2 + xy + w^7xz + wyz &= 0 \end{aligned} \tag{21}$$

The maximal arcs (2), (9), (15), (16) and (17) are Thas 1974 maximal arcs, the others are Denniston (non-Thas) maximal arcs.

The orders of the collineation stabilisers are: $\{(0)(3)\}$ and $\{(0)(5)\}$ have stabilisers of order 8; $\{(0)(4)\}$, $\{(0)(18)\}$ and $\{(0)(21)\}$ have stabilisers of order 4; the rest all have stabiliser of order 2.

Once all the disjoint pairs of maximal arcs were found, a similar program was run to find disjoint triples of Denniston maximal $\{52; 4\}$-arcs. Up to isomorphism only one such set exists, it can be describes as the union of the maximal arcs (0), (6) and the following maximal arc.

$$\begin{aligned} w^7x^2 + y^2 + w^7z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \\ w^9x^2 + w^5y^2 + w^6z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0\,. \\ w^{14}x^2 + w^{12}y^2 + w^8z^2 + w^6xy + w^{10}xz + w^{11}yz &= 0 \end{aligned}$$

It has collineation stabiliser of order 2.

A similar complete search was also done for disjoint quadruples of Denniston maximal $\{52;4\}$-arcs, but they were found not to exist. If such a set existed the complement of the set would have been a unital, and so it would have been possible to partition the points of $PG(2,16)$ into four maximal $\{52;4\}$-arcs and a unital.

Similar programs were run in $PG(2,64)$ to find a disjoint pair of Denniston maximal $\{456;8\}$-arcs, though because of the size of the plane a complete classification was not possible.

Let w be a primitive element of $GF(64)$ satisfying $w^6 + w = 1$.
Then $H = \{0, w^1, w^2, w^5, w^7, w^{17}, w^{25}, w^{34}\}$ is an additive subgroup of $GF(64)$, and the sets

$$w^{10}x^2 + z^2 + xz + \lambda y^2 = 0 \qquad \lambda \in H$$

$$w^{10}x^2 + w^{42}x^2 + w^{43}y^2 + z^2 + w^{21}x^2 + w^{53}xy + xz + \lambda(w^{20}x^2 + y^2) = 0 \qquad \lambda \in H$$

describe a pair of disjoint Denniston maximal $\{456;8\}$-arcs in $PG(2,64)$.

More generally, it may be that disjoint Denniston maximal $\{q^3 - q^2 + q; q\}$-arcs exist in $PG(2,q^2)$, giving rise to $2(q^3 - q^2 + q)$-sets of type $(q, 2q)$, but a proof has so far been elusive.

The above disjoint maximal arcs have interesting configurational properties. Suppose we have a pair of disjoint Denniston maximal arcs $\mathcal{K}_1$ and $\mathcal{K}_2$ with nuclei P_1 and P_2, and lines at infinity l_1 and l_2, respectively. Then, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are any of the maximal arc pairs given in this subsection, it is not difficult to show that the line joining P_1 and P_2, and the lines l_1 and l_2, are all concurrent. The point of concurrency is the centre of an elation of order 2 stabilising the pair of maximal arcs, and the axis of that elation being the line joining the nuclei.

The disjoint triple of maximal arcs in $PG(2,16)$ given above has similar properties. The nuclei of the three maximal arcs are collinear, and all three lines at infinity and the line joining the nuclei are concurrent. Again the point of concurrency is the centre of an elation of order 2 stabilising the triple, the axis of that elation being the line joining the nuclei.

Various computer searchers were attempted in larger order planes (such as $PG(2,64)$ and $PG(2,256)$) to find disjoint maximal arcs under the assumption that the pair would have the properties described in the previous paragraphs. These were unsuccessful in finding new examples, but the searches were far from exhaustive, and so there may well be such sets in these planes.

2.2 Inherited Denniston maximal arcs in Hall planes

As it was noted in the last chapter (Section 3) the Hall planes may be constructed from the Desarguesian planes by derivation with respect to a Baer subline. It was also shown that Thas 1974 maximal arcs in Desarguesian projective planes could inherit to maximal arcs in the Hall planes by a careful choice of the derivation set. This result arose out of "playing" by computer with maximal arcs in the Desarguesian plane of order 16 to see if they might inherit. Once a maximal arc and a derivation set were found such that the maximal arc inherited to a maximal arc in the Hall plane of order 16, I then set about proving the general result found in the previous chapter. The following conjecture is the result of some more "playing" with computers.

Conjecture. Let $\mathcal{K}$ be a Denniston maximal $\{3q^2+4;4\}-$arc in $PG(2,q^2)$. Let L be any line through the nucleus, and $\mathcal{D}$ be the (unique) Baer subline on the three points of $\mathcal{K}$ on L not equal to the nucleus. Suppose that $\mathcal{D}$ does not contain the nucleus. Then the set of points $\mathcal{K}' = \mathcal{K}\backslash L$ are the points of a $\{3q^2;4\}-$arc in the $Hall(q^2)$ determined by $\mathcal{D}$. Further $\mathcal{K}'$ can be completed to the set of points of a maximal $\{3q^2+4;4\}-$arc in $Hall(q^2)$.

For $q^2 = 16$, 64 or 256 the conjecture has been shown to be true by computer, and hence there exist maximal arcs of degree 4 in the Hall planes of these orders. The maximal arcs of the conjecture have the interesting property that the set of (affine) points of each of the conics in the maximal arc form the points of a translation q^2-arc in $Hall(q^2)$ [95]. From this it easily follows that every line of $Hall(q^2)$ meets $\mathcal{K}'$ in at most $3*2=6$ points. But showing in general that every line of $Hall(q^2)$ meets $\mathcal{K}'$ in at most 4 points, and then proving that $\mathcal{K}'$ can be completed to a maximal arc seems to be quite difficult.

3. MAXIMAL ARCS IN PLANES OF ORDER 16

In this section a number of new degree 4 maximal arcs in certain projective planes of order 16 are exhibited (subsection 3.2). Pairs of disjoint degree 4 maximal arcs are also shown to exist in certain of these planes (subsection 3.3). Finally, in subsection 3.4, a method for finding inherited structures in translation planes is outlined.

3.1 Known planes of order 16

The translation planes of order 16 were classified by Dempwolff and Reifart in [58]. They are : the Desarguesian plane (PG(2,16)); the semifield plane with kernel $GF(4)$ (Semifield(4)); the semifield plane with kernel $GF(2)$ (Semifield(2)); the Hall plane (Hall(16)); the Lorimer-Rahilly plane (LMRH); the Johnson-Walker plane (JOWK); the derived semifield plane (DSFP); and the Dempwolff plane. The

Desarguesian plane and the two semifield planes are self dual, while all the others are not.

In addition, the following planes of order 16 (and their duals) are known : two planes that may be obtained by Bose-Barlotti derivation of the Hall plane which we shall denote BB1Hall(16) and BB2Hall(16) (the 1 and 2 corresponding to the classes as described at the end of the previous chapter); a Bose-Barlotti derivation of the semifield with kernel $GF(4)$ plane, which we shall denote BBSemifield(4); the Johnson plane [84]; and the Mathon plane [104]. Of these only BB1Hall(16) is self dual.

In [85], N.L. Johnson shows how DSFP, LMRH and JOWK may be obtained by derivation of Semifield(4). In [56], Dempwolff and Reifart note that deriving the Dempwolff plane gives Semifield(2). Many of the planes of order 16 are obtainable from one another by derivation. Figure 1 (due to T. Penttila) lists all of the planes of order 16 known to the author. A box containing a "D" denotes the dual of the plane (for those planes which are not self dual). A line joining two planes denotes that one plane may be obtained from the other by derivation with respect to a Baer subline (except in the case of dual Johnson-Walker/Mathon in which case a net replacement for six parallel classes of lines is occuring). There may be many other such relations, these are just the ones known to myself.

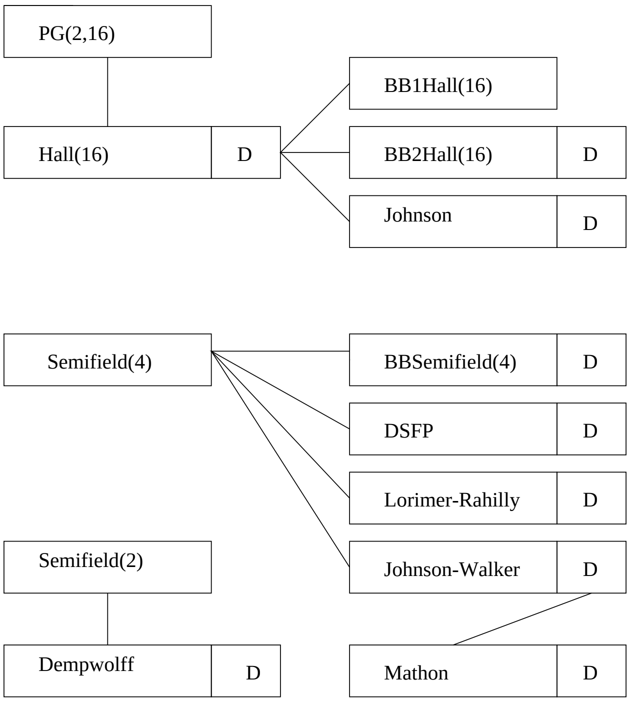

Figure 1: *Known planes of order* 16

Gordon Royle has made these planes available by ftp from ftp.cs.uwa.edu.au in the folder pub/graphs/planes16. They are also accessible as a world wide web page from www.cs.uwa.edu.au/$\sim$gordon.

In [37], W. M. Cherowitzo classified the hyperovals in the translation planes of order 16 (and hence degree 8 maximal arcs in the dual planes). All translation planes of order 16 were found to contain hyperovals. In [115], Penttila, Royle and Simpson classified by computer all hyperovals in all of the known planes of order 16 listed in the previous subsection, and hence classified degree 8 maximal arcs in these planes. In particular Penttila et al showed that hyperovals do not exist in the duals of the Johnson, BB2Hall(16), BBSemifield(4) and DSFP. This disproved a long standing conjecture that every projective plane of even order contains a hyperoval (and also shows that Johnson, BB2Hall(16), BBSemifield(4) and DSFP do not contain degree 8 maximal arcs).

The remaining cases for classification of maximal arcs in these planes of order 16 are then just the degree 4 maximal arcs. The computer searches that I have run suggest that computer classification of such maximal arcs is beyond present techniques (well, my techniques anyway). But I have constructed degree 4 maximal arcs in some of these planes, and these are listed in this subsection.

The following is a list of points of maximal arcs I have found by computer search in planes of order 16. The computer searches were conducted under various symmetry hypothesis, mostly that the maximal arc be stabilised by a homology of order 3, or a Baer collineation of order 3 (a Baer collineation is a collineation whose fixed points and lines form a Baer subplane). Many of the searches were not complete, i.e. not every maximal arc in a given plane stabilised by say, a particular homology of order 3 was found, so I dont list the symmetry hypotheses here. Except in the case of the maximal arcs given in the Hall(16) and BB1Hall(16) the maximal arcs given below were previously unknown. The maximal arcs in Hall(16) and BB1Hall(16) are members of the classes constructed in the previous chapter.

The numbering of the points of the maximal arcs correspond to the point numbering of the planes to be found at the ftp site mentioned in the last subsection.

In the Dempwolff plane :
$Demp.1 = [256, 16, 48, 144, 17, 49, 145, 18, 50, 146, 35, 195, 115, 68, 180, 212, 21, 53, 149, 38, 198, 118, 87, 167, 247, 88, 168, 248, 73, 185, 217, 74, 186, 218, 75, 187, 219, 92, 172, 252, 45, 205, 125, 46, 206, 126, 95, 175, 255, 262, 264, 270]$
$Demp.2 = [256, 16, 48, 144, 33, 193, 113, 18, 50, 146, 67, 179, 211, 20, 52, 148, 37, 197, 117, 70, 182, 214, 71, 183, 215, 40, 200, 120, 89, 169, 249, 26, 58, 154, 91, 171, 251, 76, 188, 220, 93, 173, 253, 94, 174, 254, 47, 207, 127, 262, 264, 270]$
are maximal arcs. Demp.1 has collineation stabiliser of order 24, Demp.2 has collineation stabiliser of order 144.

In the Semifield(4) plane :

$Semi(4).1 = [256, 16, 192, 208, 17, 193, 209, 18, 194, 210, 35, 179, 147, 100, 132, 228, 21, 197, 213, 38, 182, 150, 87, 247, 167, 104, 136, 232, 41, 185, 153, 106, 138, 234, 43, 187, 155, 92, 252, 172, 93, 253, 173, 94, 254, 174, 111, 143, 239, 263, 265, 271]$
is a maximal arc, it has collineation stabiliser of order 96. The maximal arc is self dual.

In the Semifield(2) plane :

$Semi(2).1 = [256, 257, 258, 261, 16, 96, 160, 20, 100, 164, 41, 249, 201, 46, 254, 206, 17, 102, 170, 31, 108, 162, 127, 140, 178, 113, 134, 186, 123, 135, 184, 125, 131, 181, 61, 83, 213, 59, 87, 216, 60, 82, 223, 54, 90, 209, 35, 245, 205, 39, 248, 203]$

$Semi(2).2 = [256, 257, 258, 261, 16, 96, 160, 20, 100, 164, 25, 105, 169, 30, 110, 174, 34, 255, 204, 42, 241, 198, 43, 247, 200, 45, 243, 197, 123, 135, 184, 125, 131, 181, 114, 143, 188, 122, 129, 182, 60, 82, 223, 54, 90, 209, 56, 91, 215, 53, 93, 211]$
are maximal arcs. Semi(2).1 has collineation stabiliser of order 24, Semi(2).2 has collineation stabiliser of order 144. Both maximal arcs are self dual.

In the LMRH plane :

$LMRH.1 = [257, 263, 270, 272, 34, 170, 51, 82, 138, 147, 67, 194, 106, 186, 243, 210, 40, 169, 53, 185, 245, 216, 88, 137, 149, 69, 200, 105, 47, 173, 59, 75, 207, 109, 189, 251, 223, 95, 141, 155, 4, 12, 6, 20, 28, 22, 116, 124, 118, 228, 236, 230]$
is a maximal arc, it has collineation stabiliser of order 96.

In the Mathon plane :

$Math.1 = [257, 264, 265, 269, 16, 240, 144, 3, 99, 227, 21, 117, 245, 6, 230, 134, 98, 152, 185, 92, 135, 125, 17, 139, 42, 110, 207, 244, 122, 193, 187, 1, 123, 186, 95, 228, 158, 36, 94, 159, 45, 140, 183, 18, 232, 201, 13, 44, 247, 82, 200, 105]$
is a maximal arc, it has collineation stabiliser of order 24.

In the Hall(16) plane :

$Hall.1 = [257, 259, 265, 272, 3, 181, 216, 53, 120, 83, 100, 251, 192, 148, 171, 224, 1, 188, 223, 2, 190, 218, 62, 122, 82, 60, 127, 81, 147, 165, 232, 108, 255, 193, 156, 175, 225, 99, 245, 200, 18, 30, 26, 34, 46, 42, 64, 68, 75, 128, 132, 139]$
is a maximal arc, it has collineation stabiliser of order 24.

In the BB1Hall(16) plane :

$BBH1.1 = [261, 26, 47, 250, 50, 86, 198, 21, 33, 240, 104, 149, 173, 139, 189, 231, 57, 89, 192, 129, 191, 236, 28, 34, 248, 108, 159, 174, 136, 186, 226, 142, 176, 233, 107, 154, 166, 53, 85, 204, 109, 144, 161, 60, 88, 203, 18, 38, 241, 262, 263, 264]$
is a maximal arc, it has collineation stabiliser of order 24. It is self dual.

In summary, maximal arcs of degree 4 exist in $PG(2, 16)$, Hall(16), BB1Hall(16), Dempwolff, Semifield(4), Semifield(2), LMRH, Mathon ,and their duals.

As was the case for the maximal arcs constructed in the Hall plane and a derived dual of the Hall plane, it may be that some of the above maximal arcs are members of classes of maximal arcs. Both the semifield planes [99], and the Dempwolff plane [86], are members of infinite classes of planes (in fact the generalisation of the Dempwolff plane may be obtained from derivation of certain semifield planes [86]). In particular, I strongly suspect that the maximal arc given above in Semifield(4) is part of a class of maximal $\{q^3 - q^2 + q; q\}$−arcs in the semifield planes of order q^2 with kernel $GF(q)$.

3.3 Disjoint maximal arcs in planes of order 16

Suppose we have a projective plane π with collineation group G, and a subset $\mathcal{K}$ of π, the collineation stabiliser of $\mathcal{K}$ being a group S. Then, in theory at least, it is possible to contruct all of the isomorphic copies of $\mathcal{K}$ in π. This can be done by taking the images of $\mathcal{K}$ under elements of the transversal of S in G (in its action on π). In particular, such a process can be used to attempt to construct disjoint maximal arcs, i.e. for every element g of the transversal check whether $\mathcal{K}$ and $\mathcal{K}^g$ are disjoint. If a plane contains, up to isomorphism, more than one maximal arc, essentially the same procedure can be used to attempt to find a disjoint pair of maximal arcs where the maximal arcs are not isomorphic to one another.

Using the computer algebra package CAYLEY [24], the collineation groups of the planes of order 16 in the previous subsection have been calculated. With the exception of the Desarguesian plane, the orders of the collineation groups are relatively small. For all of the non-Desarguesian planes of order 16 it is possible to calculate a transversal for S in G, where G is the collineation group of the plane and S is the collineation stabiliser of any of the degree 4 maximal arcs found above in that plane.

Using the above procedure for each of the planes of order 16 that I have constructed maximal arcs in, I have calculated all pairs $(\mathcal{K}_1, \mathcal{K}_2)$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are a pair of disjoint, but not necessarily isomorphic, maximal arcs from the previous subsection. In all but Semifield(4), Semifield(2) and the Mathon plane no disjoint maximal arcs exist (where the maximal arcs are isomomorphic to those degree 4 maximal arcs of the previous subsection). None of the planes admitted disjoint triples of known maximal arcs. Hence 104-sets of type $(4, 8)$ exist in Semifield(4), Semifield(2) and the Mathon plane.

The following is a list of examples of such disjoint sets. See also subsection 2.1 for the Desarguesian case.

In Semifield(4) there is (up to isomorphism) one known degree 4 maximal arc. It has collineation stabiliser of order 96 and is isomorphic to its dual. There are 36 isomorphic copies of it disjoint from itself. Twenty four of these disjoint pairs of maximal arcs have collineation stabiliser of order 16. Up to isomorphism there are two such pairs of disjoint maximal arcs with collineation stabiliser of order 16. Representatives are $(Semi(4).1, Semi(4).2)$ and $(Semi(4).1, Semi(4).3)$, where $Semi(4).1$ is as given in the previous subsection and

$Semi(4).2 = [4, 8, 10, 15, 32, 33, 34, 37, 51, 54, 57, 59, 68, 72, 74, 79, 83, 86, 89, 91, 96, 97, 98, 101, 115, 118, 121, 123, 128, 129, 130, 133, 164, 168, 170, 175, 176, 177, 178, 181, 196, 200, 202, 207, 211, 214, 217, 219, 261, 262, 266, 269].$

$Semi(4).3 = [4, 8, 10, 15, 23, 28, 29, 30, 32, 33, 34, 37, 52, 56, 58, 63, 64, 65, 66, 69, 103, 108, 109, 110, 112, 113, 114, 117, 148, 152, 154, 159, 160, 161, 162, 165, 199, 204, 205, 206, 231, 236, 237, 238, 244, 248, 250, 255, 261, 262, 266, 269].$

Twelve of these disjoint pairs of maximal arcs have collineation stabiliser of order 32. Up to isomorphism there are two such pairs of disjoint maximal arcs with collineation stabiliser of order 32. Representatives are $(Semi(4).1, Semi(4).4)$ and $(Semi(4).1, Semi(4).5)$, where $Semi(4).1$ is as given in the previous subsection and

$Semi(4).4 = [3, 6, 9, 11, 20, 24, 26, 31, 48, 49, 50, 53, 64, 65, 66, 69, 83, 86, 89, 91, 115, 118, 121, 123, 131, 134, 137, 139, 148, 152, 154, 159, 180, 184, 186, 191, 212, 216, 218, 223, 224, 225, 226, 229, 240, 241, 242, 245, 261, 262, 266, 269].$

$Semi(4).5 = [4, 8, 10, 15, 20, 24, 26, 31, 39, 44, 45, 46, 55, 60, 61, 62, 68, 72, 74, 79, 99, 102, 105, 107, 119, 124, 125, 126, 148, 152, 154, 159, 163, 166, 169, 171, 199, 204, 205, 206, 227, 230, 233, 235, 243, 246, 249, 251, 258, 264, 267, 268].$

In Semifield(2) there are (up to isomorphism) two known degree 4 maximal arcs. They have collineation stabilisers of order 24 and 144 and are each isomorphic to their duals. It was found that no pair of disjoint maximal arcs exists where the maximal arcs in the pair were not isomorphic.

The degree 4 maximal arc with collineation stabiliser of order 24 has 12 isomorphic copies of itself disjoint from itself. These disjoint pairs of maximal arcs all have collineation stabiliser of order 8. Up to isomorphism there are two such pairs of disjoint maximal arcs. Representatives of these pairs are $(Semi(2).1, Semi(2).3)$ and $(Semi(2).1, Semi(2).4)$, where $Semi(2).1$ is as given in the previous subsection and

$Semi(2).3 = [5, 8, 9, 14, 18, 21, 24, 26, 34, 37, 40, 42, 51, 55, 57, 62, 64, 66, 68, 74, 115, 118, 119, 124, 144, 148, 150, 156, 165, 168, 169, 174, 179, 183, 185, 190, 208, 212, 214, 220, 224, 226, 228, 234, 243, 246, 247, 252, 260, 264, 266, 271].$

$Semi(2).4 = [0, 4, 6, 12, 33, 37, 40, 47, 70, 75, 76, 77, 81, 85, 88, 95, 105, 107, 109, 110, 128, 129, 132, 143, 149, 152, 153, 158, 165, 168, 169, 174, 185, 187, 189, 190, 192, 193, 196, 207, 208, 212, 214, 220, 230, 235, 236, 237, 259, 262, 265, 267]$.

The maximal arc with collineation stabiliser of order 144 has 36 isomorphic copies of itself disjoint from itself. These disjoint pairs of maximal arcs all have collineation stabiliser of order 16. Up to isomorphism there are two such pairs of disjoint maximal arcs. Representatives are $(Semi(2).2, Semi(2).5)$ and $(Semi(2).2, Semi(2).6)$, where $Semi(2).2$ is as given in the previous subsection and

$Semi(2).5 = [0, 4, 9, 14, 18, 26, 27, 29, 33, 35, 39, 47, 48, 52, 57, 62, 65, 67, 71, 79, 112, 116, 121, 126, 145, 147, 151, 159, 162, 170, 171, 173, 177, 179, 183, 191, 210, 218, 219, 221, 224, 228, 233, 238, 242, 250, 251, 253, 263, 268, 269, 270]$.

$Semi(2).6 = [5, 6, 8, 12, 21, 22, 24, 28, 33, 35, 39, 47, 49, 51, 55, 63, 66, 74, 75, 77, 113, 115, 119, 127, 146, 154, 155, 157, 162, 170, 171, 173, 177, 179, 183, 191, 210, 218, 219, 221, 229, 230, 232, 236, 245, 246, 248, 252, 263, 268, 269, 270]$.

In Mathon there is (up to isomorphism) one known degree 4 maximal arc. It has collineation stabiliser of order 24. There are 24 isomorphic copies of it disjoint from itself. Twelve of these disjoint pairs of maximal arcs have collineation stabiliser of order 4, and are all isomorphic. A representative of such a pair is $(Math.1, Math.2)$, where $Math.1$ is as given in the previous subsection and

$Math.2 = [2, 9, 15, 22, 25, 28, 32, 33, 37, 48, 49, 50, 65, 70, 73, 85, 86, 90, 102, 103, 111, 114, 116, 127, 130, 131, 138, 145, 151, 154, 163, 164, 172, 176, 179, 191, 192, 196, 197, 212, 213, 215, 231, 234, 236, 243, 249, 252, 256, 258, 263, 271]$.

The other twelve of these disjoint pairs of maximal arcs have collineation stabiliser of order 8. Up to isomorphism there are two such pairs of disjoint maximal arcs. Representatives are $(Math.1, Math.3)$ and $(Math.1, Math.4)$, where $Math.1$ is as given in the previous subsection and

$Math.3 = [4, 8, 14, 22, 25, 26, 33, 35, 40, 60, 62, 63, 67, 75, 78, 86, 90, 93, 104, 106, 108, 113, 116, 124, 137, 141, 143, 145, 148, 153, 166, 171, 174, 179, 184, 191, 196, 198, 205, 217, 218, 219, 225, 235, 237, 243, 252, 255, 256, 258, 262, 272]$.

$Math.4 = [4, 8, 11, 12, 32, 33, 37, 46, 51, 54, 57, 61, 68, 70, 72, 75, 85, 89, 91, 93, 112, 116, 118, 121, 145, 147, 149, 156, 161, 163, 173, 174, 176, 184, 188, 190, 192, 196, 197, 203, 211, 214, 216, 220, 225, 233, 237, 238, 259, 261, 268, 270]$.

In summary, the Desarguesian plane of order 16 [25], both semifield planes of order 16, the Mathon plane, and all of their duals have been shown to admit disjoint degree 4 maximal arcs, and hence contain 104-sets of type $(4, 8)$.

3.4 A technique for inheriting structures in translation planes

In this section a technique, developed from some ideas of Tim Penttila, for finding inherited structures in translation planes is given. The method was successfully used to construct maximal $\{52; 4\}$-arcs in the Dempwolff plane and Semifield(2), both of which are inherited from the Thas 1974 maximal arc in $PG(2,16)$. These maximal arcs were later found using computer searches under symmetry hypotheses, and are contained in the list of subsection 3.2.

Let $\mathcal{K}$ be the Thas 1974 maximal $\{52; 4\}$-arc in $PG(2,16)$ with axis F_∞. Let L_∞ be any line through the centre of $\mathcal{K}$. Represent $PG(2,16)$ via the Bruck-Bose construction, with a 3-spread $\mathcal{S}$ of a hyperplane $H = PG(7,2)$ in $PG(8,2)$, with translation line L_∞. There are then four elements of the spread corresponding to points of $\mathcal{K}$ on L_∞, and 48 points of $P \backslash H$ corresponding to the points of $\mathcal{K}$ not on L_∞. A reasonable question to then ask is are there other 3-spreads $\mathcal{S}'$ of H such that in the translation plane $\pi(\mathcal{S}')$ the points of $\mathcal{K}$ form a maximal arc. It turns out that there are such spreads, and the following is a method of finding them.

Define a projective 3-space Σ of H to be *good* if
(i) Σ is disjoint from the four elements of $\mathcal{S}$ corresponding to points of $\mathcal{K}$ on L_∞.
(ii) Every projective 4-space containing Σ, but not contained in H, meets $\mathcal{K}$ in either 0 or 4 points.

Once all good 3-spaces are found we can then attempt form spreads $\mathcal{S}'$ containing the four elements of $\mathcal{S}$ corresponding to points of $\mathcal{K}$ on L_∞, and good 3-spaces. It follows immediately that $\mathcal{K}$ will correspond to a maximal $\{52; 4\}$-arc in the plane $\pi(\mathcal{S}')$.

It is relatively easy to solve this problem by computer, and it turns out that there is exactly one spread $\mathcal{S}' \neq \mathcal{S}$ such that $\mathcal{K}$ corresponds to a maximal $\{52; 4\}$-arc in the plane $\pi(\mathcal{S}')$. This spread was identified (by the rank of the point-line incidence matrix of the associated translation plane) as a spread giving rise to the Dempwolff plane. The collineation group stabilising the maximal arc can be found using *nauty* and it has order 24. In fact this maximal arc is isomorphic to Demp.1 given in subsection 3.2.

A maximal $\{52; 4\}$-arc in Semifield(2) can be found by a similar method. In the Bruck-Bose representation of $PG(2,16)$ as above let Σ be the projective three space in H corresponding to the point of intersection of the lines L_∞ and F_∞. Then it turns out that Σ is also an element of the Dempwolff spread $\mathcal{S}'$ found as above. Suppose we take the partial spread of size 12 given by the elements of $\mathcal{S}'$ not equal to Σ and not corresponding to points of the maximal arc on L_∞. Then with a slightly modified definition of a *good* 3-space we can try and complete the partial spread to a spread $\mathcal{S}''$ in which the affine points of $\mathcal{K}$ together with four elements of the spread $\mathcal{S}''$ form a maximal arc in the associated plane. By computer, there is a unique such spread. The plane $\pi(\mathcal{S}'')$ has been identified as Semifield(2). The maximal arc has collineation stabiliser of order 24, and is isomorphic to Semi(2).1 given in subsection 3.2.

A modified definition of "good" is required as previously only spread elements corresponding to points not in the maximal arc were replaced. For a spread element to correspond to a point of the maximal arc we require that every 4-space not contained in H on the spread element meet the affine points of the maximal arc in 3 points.

Similar methods can be used to show that LMRH.1 is inherited from Semi(4).1.

The techniques described above can easily be extended to attempt to construct sets of type (m, n) in translation planes that are inherited from set of type (m, n) in other translation planes. It seems likely that these ideas could be used to generate large numbers of such sets. The difficulties are that there is no control over which plane the sets are inherited to, and once such sets are found the method gives little insight as to how they might be generalised.

Chapter 5
HYPEROVALS AND UNITALS IN FIGUEROA PLANES

1. INTRODUCTION

The finite Figueroa planes are a class of projective planes of order q^3, with $q > 2$ a prime power. In the original paper [65], the value of q was restricted, but this restriction was removed in [77]. The description of the planes in [65],[77] was algebraic, but T. Grundhöfer has given a synthetic construction in [68] which is described in Section 2. Figueroa planes are interesting because they are one of the few classes of planes known that are not translation planes, duals of translation planes or obtained from of a chain of derivations of such planes.

Little is known about what types of structures exist in the Figueroa planes. In [38], W. M. Cherowitzo constructed a class of ovals called the *Ovali di Roma* in finite Figueroa planes of odd order. In [111] , Julia M. Nowlin Brown showed that the Ovali di Roma could be described as the absolute points of a particular polarity of the plane, and that the ovals could be generalised to infinite Figueroa planes of characteristic not two.

In this chapter hyperovals are constructed in the finite Figueroa planes of even order (Section 3), and using techniques similar to those in [111] it is shown that all finite Figueroa planes admit a unitary polarity and hence contain unitals (Section 4). A different treatement of these results can be found in [53]. The final result of the chapter is to classify polarities of the finite Figueroa planes. The construction of hyperovals was joint work between myself and M.J. de Resmini.

2. CONSTRUCTION OF FIGUEROA PLANES OF FINITE ORDER

Let $PG(2, q^3)$ be the Desarguesian projective plane of order q^3, over the finite field $GF(q^3)$. Let α be a planar collineation of order 3 of $PG(2, q^3)$. Then the points and lines of $PG(2, q^3)$ fall into three disjoint classes according to the structure of their orbit under α. For a point P,

$$P \text{ is } \begin{cases} \text{type } I, & \text{if } P^\alpha = P, \\ \text{type } II, & \text{if } P, P^\alpha, P^{\alpha^2} \text{ are collinear and distinct,} \\ \text{type } III, & \text{if } P, P^\alpha, P^{\alpha^2} \text{ are not collinear.} \end{cases}$$

Line types are defined dually.

Define an involutory bijection μ between the points of type III and the lines of type III as follows: for a point P and line l, both type III,

$$P^\mu = P^\alpha P^{\alpha^2}, \qquad l^\mu = l^\alpha \cap l^{\alpha^2}.$$

Let $\mathcal{P}$ be the set of points and $\mathcal{L}$ the set of lines of $PG(2,q^3)$. Define a new incidence structure $Fig(q^3) = (\mathcal{P}, \mathcal{L}, I_F)$ in terms of the incidence I of $PG(2,q^3)$:

$$PI_F l \Leftrightarrow \begin{cases} l^\mu I P^\mu, & \text{if } P \text{ and } l \text{ are both of type } III, \\ PIl, & \text{otherwise;} \end{cases}$$

then $Fig(q^3)$ is a projective plane, and is the Figueroa plane of order q^3 [68]. For $q > 2$ such a plane is non-Desarguesian.

Note that the points and lines of type I form a Desarguesian subplane of order q of both $PG(2,q^3)$ and $Fig(q^3)$ referred to as the **fixed subplane**. If the collineation group of the fixed subplane is denoted by A, then the collineation group of $Fig(q^3)$ is given by $\langle\alpha\rangle \cdot A$, the extension of $\langle\alpha\rangle$ by A [57],[77].

3. HYPEROVALS IN FIGUEROA PLANES

THEOREM 3.1 (de Resmini - Hamilton [53]) *Let $\mathcal{H}$ be a regular hyperoval in $PG(2,q^3)$, $q > 2$ even. Let α be a planar collineation of order 3. If $\mathcal{H}$ is stabilised by α then the set of points of $\mathcal{H}$ inherits to the set of points of a hyperoval in the Figueroa plane $Fig(q^3)$ of order q^3 determined by α.*

Proof:
We show that every line of $Fig(q^3)$ meets the set of points of $\mathcal{H}$ in at most two points.

Note that $\mathcal{H}$ can contain no type II points (since the orbit under α of a type II point consists of three collinear points). It is also clear from the following that $\mathcal{H}$ meets the fixed subplane in a hyperoval of the subplane. The stabiliser of the regular hyperoval in the collineation group of $PG(2,q^3)$ fixes the nucleus [78]. Hence for a regular hyperoval to be stabilised by α the hyperoval must meet the fixed subplane in at least the nucleus. The lines of type I (fixed lines) on the nucleus must all meet the hyperoval in a further type I point, and so the fixed subplane meets the hyperoval in a hyperoval of the subplane.

The incidence in $Fig(q^3)$ between points of any type and lines of type I or II is the same as in $PG(2,q^3)$, so every line of type I or II meets the set of points of $\mathcal{H}$ in at most two points. It remains to be shown that every line of type III in $Fig(q^3)$ meets the type III points of $\mathcal{H}$ in at most two points.

Let l be a line of type III with $P = l^\mu$, then either $P \in \mathcal{H}$ or $P \notin \mathcal{H}$. We consider these cases separately.

1. $P \in \mathcal{H}$

Since $\mathcal{H}$ is stabilised by α, the points P^α and P^{α^2} are points of $\mathcal{H}$, and l is the line joining these two points (in $PG(2,q^3)$). It follows that in $Fig(q^3)$ l is incident with the points P^α and P^{α^2}, but not with P. Suppose that l is incident in $Fig(q^3)$ with some point $Q \in \mathcal{H} - \{P, P^\alpha, P^{\alpha^2}\}$. Then PIQ^μ, i.e. in $PG(2,q^3)$ the line Q^μ meets $\mathcal{H}$ in at least the three points $P, Q^\alpha, Q^{\alpha^2}$. This contradicts $\mathcal{H}$ being a hyperoval in $PG(2,q^3)$, and so it must be that l meets $\mathcal{H}$ in $Fig(q^3)$ in only the points P^α, P^{α^2}.

2. $P \notin \mathcal{H}$

We require the following well known results (see, for instance, [17]).

LEMMA 3.1 *(Pascal's Theorem) Let (A, B, C, R, S, T) be an ordered set of six distinct points of a (non-degenerate) conic in a Desarguesian plane. Let AR be the line incident with the points A and R, and similarly for other pairs of points. Let X be the point of intersection of AS and BR, Y the point of intersection of AT and CR, and Z the point of intersection of CS and BT. Then X, Y and Z are collinear.* ∎

We shall call X, Y and Z the *Pascal points* with respect to the ordered hexagon (A, B, C, R, S, T).

LEMMA 3.2 *Let A, B, C and D be four distinct points of a non-degenerate conic in a finite Desarguesian plane of even order. Let X be the point of intersection of AD and BC, and Y the point of intersection of AB and CD. Then the line joining X and Y is incident with the nucleus N of the conic.* ∎

Suppose l is incident in $Fig(q^3)$ with the points Q and R of $\mathcal{H}$. Then PIQ^μ and PIR^μ. Note that the line Q^μ is incident with the points Q^α and Q^{α^2} of $\mathcal{H}$ (in $PG(2,q^3)$), and that the line R^μ is incident with the points R^α and R^{α^2} of $\mathcal{H}$ (in $PG(2,q^3)$).

Consider a Pascal configuration on the hexagon $(Q^\alpha, Q, Q^{\alpha^2}, R^\alpha, R, R^{\alpha^2})$. As in Lemma 3.1, let $X = Q^\alpha R \cap QR^\alpha$, $Y = Q^\alpha R^{\alpha^2} \cap Q^{\alpha^2} R^\alpha$, and $Z = Q^{\alpha^2} R \cap QR^{\alpha^2}$. Then X, Y and Z are collinear, and by examining the images under α of the Pascal configuration it is easy to see that $Y = X^\alpha$ and $Z = X^{\alpha^2}$.

We argue that the nucleus N of the conic together with the points P, P^α, P^{α^2}, X, Y and Z form a subplane of order 2 in $PG(2,q^3)$. It will then be shown that this implies that l can not be incident in $Fig(q^3)$ with any point of $\mathcal{H} - \{Q, R\}$, completing the proof of the Theorem.

By examining the quadrangle Q^α, Q^{α^2}, R^α, R^{α^2}, it follows from Lemma 3.2 that the line of $PG(2,q^3)$ joining Y and P is incident with the nucleus N. Since X, Y and Z form an orbit under $\langle\alpha\rangle$ and the nucleus N is fixed under the action of this group, the lines joining X to P^{α^2} and Z to P^α are also lines of $PG(2,q^3)$ incident with the nucleus.

Consider a Pascal configuration in $\mathcal{H}$ based on the hexagon $(Q^\alpha, Q, R^{\alpha^2}, R^\alpha, R, Q^{\alpha^2})$. Then $X = Q^\alpha R \cap QR^\alpha$, $P = Q^\alpha Q^{\alpha^2} \cap R^\alpha R^{\alpha^2}$ and $P^\alpha = QQ^{\alpha^2} \cap RR^{\alpha^2}$ are collinear by Lemma 3.1. A similar argument (or by considering orbits under $\langle\alpha\rangle$) shows that Y, P^α, P^{α^2} are collinear in $PG(2, q^3)$, and so are Z, P, P^{α^2}. Hence the nucleus N of the conic together with the points P, P^α, P^{α^2}, X, Y and Z form a subplane of order 2 in $PG(2, q^3)$.

In any finite Desarguesian plane of even order a quadrangle determines a unique subplane of order 2, and so given the points N, P, P^α, P^{α^2}, the points X, Y and Z are determined. We use this fact to show that in $Fig(q^3)$ the line l can be incident with no point of $\mathcal{H} - \{Q, R\}$.

Suppose l is also incident in $Fig(q^3)$ with $S \in \mathcal{H} - \{Q, R\}$. Then the Pascal configuration based on the points Q^α, Q, Q^{α^2}, S^α, S and S^{α^2} by the previous paragraph must again have Pascal points X, Y and Z. But for any pair of distinct lines incident with Q and meeting one of the points X, Y or Z, at least one of the lines must meet one of R, R^α or R^{α^2}. Hence there is no Pascal configuration on the points Q^α, Q, Q^{α^2}, S^α, S and S^{α^2} with Pascal points X, Y and Z, for $S \in \mathcal{H} - \{Q, R\}$. Therefore, any line l of $Fig(q^3)$ contains at most two points of $\mathcal{H}$. ∎

It is easy to see that there is exactly one equivalence class of so constructed hyperovals in the Figueroa planes. Indeed the collineation group of a Figueroa plane contains the full collineation group of the fixed (Desarguesian) subplane, and so is transitive on regular hyperovals in the fixed subplane. Since the nucleus and three points of a conic determine a unique regular hyperoval with that nucleus (so the hyperoval in the subplane determines the hyperoval in the larger plane), the result follows. It also follows that the collineation stabiliser of the hyperovals constructed is given by the extension of the group generated by α by the stabiliser in the collineation group of the fixed subplane of the points of the hyperoval in the subplane, i.e. is $\langle\alpha\rangle \cdot P\Gamma O(3, q)$, $q > 4$.

In [113], T. Penttila and I. Pinneri construct two hyperovals in the plane $PG(2, 64)$, one of which has a collineation stabiliser of order 60, the other of order 15. By computer it is possible to show that the hyperoval with collineation stabiliser of order 60 can inherit to the set of points of a hyperoval in $Fig(64)$. Recently [39], the Penttila-Pinneri hyperovals were shown to belong to an infinite class. It would be interesting to know whether these too inherit to hyperovals in Figueroa planes.

A **duality** Δ of a projective plane is a bijection between the set of points and the set of lines of the plane that preserves incidence, i.e. for a point P and a line l, $PIl \Leftrightarrow l^\Delta IP^\Delta$. The group generated by all dualities and collineations of a plane is called the **correlation group**, and the collineation group of a plane has index 1 or 2 in the correlation group.

A duality π of order 2 is called a **polarity**. A point P is **absolute** if PIP^π. A line l is absolute if $l^\pi Il$.

The study of unitals began in 1946 with the paper of R. Baer [5]. Baer showed that a polarity of a projective plane of order q has at least $q+1$ absolute points and that, if equality occurs, then the set of absolute points is a line if q is even and an oval if q is odd. He also showed that if there are more than $q+1$ absolute points, then q is a square, and if every non-absolute line has $q^{1/2}+1$ absolute points, then the set of absolute points is a unital. In 1970, M. Seib sharpened this result by showing that a polarity of a projective plane of order q has at most $q^{3/2}+1$ absolute points and that, if equality occurs, then the set of absolute points is a unital [120]. A polarity is called **unitary** if the number of absolute points is $q^{3/2}+1$, and **orthogonal** if the number of absolute points is $q+1$.

In this section unitals are constructed in all finite Figueroa planes. The method is essentially the same as that used by J. M. Nowlin Brown to show that the Ovali di Roma could be described as the absolute points of a polarity [111]. The final result of the chapter is to classify polarities of finite Figueroa planes.

THEOREM 4.1 ([53]) *The plane $Fig(q^6)$ possesses a unitary polarity, and hence admits unitals.*

To prove this result some notation is required.

Represent the points and lines of $PG(2,q^6)$ via homogeneous coordinates over $GF(q^6)$ as in the first section of Chapter 2. Homogeneous coordinates of the line on two (distinct) points (a,b,c), (a',b',c') are given by the usual cross product

$$(a,b,c)\times(a',b',c')=(bc'-b'c,a'c-ac',ab'-a'b).$$

Define a polarity π by

$$(a,b,c)^\pi=[a,b,c]^{q^3}=[a^{q^3},b^{q^3},c^{q^3}] \quad\text{and}\quad [a,b,c]^\pi=(a,b,c)^{q^3}=(a^{q^3},b^{q^3},c^{q^3}).$$

Then it is well known that the absolute points of π are the points of a unital $\mathcal{U}$ in $PG(2,q^6)$ called the *classical unital*, and the absolute lines are the tangent lines to $\mathcal{U}$ [78, p.148].

Let α be the order 3 automorphism of $PG(2,q^6)$ defined by

$$(a,b,c)^\alpha = (a^{q^2}, b^{q^2}, c^{q^2}) \quad \text{and} \quad [a,b,c]^\alpha = [a^{q^2}, b^{q^2}, c^{q^2}].$$

and $Fig(q^6)$ be the Figueroa plane of order q^6 determined by α, as in Section 2.

LEMMA 4.1

(1) Any point P of $Fig(q^6)$ has the same type as the line P^π.
(2) Any line l of $Fig(q^6)$ has the same type as the point l^π.
(3) For any point P of type III, $P^{\pi\mu} = P^{\mu\pi}$.
(4) For any line l of type III, $l^{\pi\mu} = l^{\mu\pi}$.

Proof:

(1) Let $P = (a,b,c)$, so

$$P^\pi = [a,b,c]^{q^3}, \quad (P^\pi)^\alpha = ([a,b,c]^{q^3})^{q^2} = [a,b,c]^{q^5}, \quad (P^\pi)^{\alpha^2} = [a,b,c]^{q^7}.$$

Then P^π is of type $I \Leftrightarrow (P^\pi)^\alpha = P^\pi \Leftrightarrow [a,b,c]^{q^5} = [a,b,c]^{q^3}$
$\Leftrightarrow [a,b,c]^{q^2} = [a,b,c] \Leftrightarrow P^\alpha = P \Leftrightarrow P$ is a point of type I.

If $P^\alpha \neq P$, then P^π, $P^{\pi\alpha}$ and $P^{\pi\alpha^2}$ are collinear in $PG(2,q^6)$
$\Leftrightarrow (P^\pi \times P^{\pi\alpha}).P^{\pi\alpha^2} = 0$
$\Leftrightarrow [b^{q^3}c^{q^5} - c^{q^3}b^{q^5}, c^{q^3}a^{q^5} - a^{q^3}c^{q^5}, a^{q^3}b^{q^5} - b^{q^3}a^{q^5}].(a,b,c)^{q^7} = 0$
$\Leftrightarrow ([bc^{q^2} - cb^{q^2}, ca^{q^2} - ac^{q^2}, ab^{q^2} - ba^{q^2}].(a,b,c)^{q^4})^{q^3} = 0$
$\Leftrightarrow ((a,b,c) \times (a,b,c)^\alpha).(a,b,c)^{\alpha^2} = 0$
$\Leftrightarrow P$, P^α and P^{α^2} are collinear.
(2) Similarly.
(3) $(a,b,c)^{\pi\mu} = [a^{q^3}, b^{q^3}, c^{q^3}]^{q^2} \times [a^{q^3}, b^{q^3}, c^{q^3}]^{q^4} = ([a,b,c]^{q^2} \times [a,b,c]^{q^4})^{q^3}$
$= [a,b,c]^{\mu\pi}$
(4) Similarly. ∎

Proof (of Theorem 4.1):
We show that π is also a polarity of $Fig(q^6)$. It is sufficient to show that if a point P and a line l are both of type III, then $PI_Fl \Leftrightarrow l^\pi I_F P^\pi$. But for such a point and line

$$\begin{aligned} l^\pi I_F P^\pi &\Leftrightarrow P^{\pi\mu} I l^{\pi\mu} \\ &\Leftrightarrow P^{\mu\pi} I l^{\mu\pi} \text{ (by Lemma 4.1)} \\ &\Leftrightarrow l^\mu I P^\mu \text{ (since } \pi \text{ is a polarity of } PG(2,q^6)) \\ &\Leftrightarrow P I_F l \text{ (definition of } I_F). \end{aligned}$$

Hence π is a polarity of $Fig(q^3)$.

We now determine the absolute points of π in $Fig(q^6)$. Let $\mathcal{U}$ be the set of absolute points of π in $PG(2,q^6)$.

A point P of $Fig(q^6)$ is absolute
$\Leftrightarrow PI_F P^\pi$
$\Leftrightarrow$ (P is of type I or II and PIP^π) or (P is type III and $P^{\pi\mu} IP^\mu$)
$\Leftrightarrow$ (P is a type I or II point of $\mathcal{U}$) or (P is type III and $P^{\mu\pi} IP^\mu$)
$\Leftrightarrow$ (P is a type I or II point of $\mathcal{U}$) or (P is type III and P^μ is a type III tangent line to $\mathcal{U}$).

Hence the absolute points of π in $Fig(q^6)$ are the type I and II points of $\mathcal{U}$ together with the images of the type III tangents to $\mathcal{U}$ under μ. The polarity π is clearly unitary (in $Fig(q^3)$) since the number of absolute points is $q^9 + 1$. ■

The above led the author to wonder whether such a process might be applied to non-classical unitals. I.e. might there be a unital stabilised by α in a Desarguesian plane, such that the type I and II points of the unital together with the images under μ of the type III tangent lines form a unital in the associated Figueroa plane. For instance there is a Buekenhout-Metz unital in $PG(2, 729)$ that is stabilised by an order 3 field automorphism [9]. Unfortunately a short computer program has shown that this unital does not inherit to a unital in $Fig(729)$ by such a process.

We now consider another polarity. Represent $PG(2, q^3)$ as above via homogeneous coordinates. Define a polarity π of $PG(2, q^3)$ by

$$(a, b, c)^\pi = [a, b, c] \quad \text{and} \quad [a, b, c]^\pi = (a, b, c),$$

then it is well known that the absolute points of π are the points of a conic if q is odd, or the points on a line if q is even [78]. If $Fig(q^3)$ is the Figueroa plane described as previously, it is clear that by arguing analogously to the above it can be shown that π induces a polarity of $Fig(q^3)$. The absolute points of π in $Fig(q^3)$ are the points of a type I line if q is even, or the points of an oval if q is odd. This was how J. M. Nowlin Brown showed that the Ovali di Roma in Figueroa planes of odd order could be described as the absolute points of a polarity [111]. (Note that J. M. Nowlin Brown's paper led to Theorem 4.1 being proved, not the other way around.)

In fact it is easy to classify the polarities of finite Figueroa planes.

THEOREM 4.2 *Every polarity π of a finite Figueroa plane induces a polarity of its fixed subplane, and conversely. The absolute points of such a polarity form either a unital, an oval, or a line. Further, all unitary polarities of $Fig(q^3)$ are conjugate, as are all orthogonal polarities of $Fig(q^3)$.*

Proof: Let $Fig(q^3)$ be the Figueroa plane of order q^3 as above, and π any polarity of $Fig(q^3)$. Define a polarity π' of $Fig(q^3)$ by

$$(a, b, c)^{\pi'} = [a, b, c] \quad \text{and} \quad [a, b, c]^{\pi'} = (a, b, c).$$

The automorphism group of $Fig(q^3)$ is $G = \langle\alpha\rangle \cdot P\Gamma L(3,q)$, with $P\Gamma L(3,q)$ acting on the fixed subplane in the usual way [57],[77]. The group G has index 2 in the correlation group of $Fig(q^3)$, so π can be written as a product $\pi = g\pi'$ for some $g \in G$.

Since a polarity has order 2, and α has order 3 and commutes with both π' and elements of G, it follows that g must be an element of $P\Gamma L(3,q)$. Hence a polarity of $Fig(q^3)$ induces a polarity of the fixed subplane, and conversely.

The polarities of $PG(2,q)$ have been classified, and up to conjugation under elements of $P\Gamma L(3,q)$ they are one of the following [78]
(i) $(x,y,z)^\pi = [x,y,z]$
(ii) $(x,y,z)^\pi = [x^{\sqrt{q}}, y^{\sqrt{q}}, z^{\sqrt{q}}]$ (q a square).

In Theorem 4.1 we showed that the absolute points of (ii) are a unital in $Fig(q^3)$. And as was mentioned previously, essentially the same proof as Theorem 4.1 shows that the absolute points of (i) form an oval (q odd) or a type I line (q even) in $Fig(q^3)$. ∎

It is clear that the collineation stabilisers of the absolute points of such polarities in Figueroa planes are given by the extension of the group generated by α by the stabiliser in the collineation group of the fixed subplane of the absolute points in the subplane. I.e. it is $\langle\alpha\rangle \cdot P\Gamma O(3,q)$, $q > 4$, for the ovals, and $\langle\alpha\rangle \cdot P\Gamma U(3,q)$, for the unitals.

As a final note, it is clear that the finite Figueroa planes (of square order) contain Baer subplanes (of Figueroa type). Construct $PG(2,q^6)$ as above via homogeneous coordinates. Take the Baer subplane $\mathcal{B}$ of $PG(2,q^6)$ whose points are given by homogeneous coordinates over $GF(q^3)$. Then the field automorphism α of order 3 in $GF(q^3)$ is also a field automorphism of $GF(q^6)$ of order 3. Acting on the plane as above, α then induces a Figueroa plane of order q^3 (via its action on $\mathcal{B}$) which is contained in a Figueroa plane of order q^6 (via its action on $PG(2,q^6)$). Obviously any image of such a Baer subplane under an element of the collineation group of $Fig(q^6)$ will also be a Baer subplane. In fact, a similar method gives subplanes $Fig(q^3)$ of $Fig(q^{3t})$ provided 3 doesn't divide t (so that α has order 3 in both $GF(q^3)$ and $GF(q^{3t})$).

Index of Definitions

The following is a list of the main definitions and where they occur in the text.

Notation

$GF(q)$	Galois field of order q
$GF(q)^*$	$GF(q) - \{0\}$
$V(n,q)$	n-dimensional vector space over the field $GF(q)$
$AG(n,q)$	n-dimensional affine space over the field $GF(q)$
$PG(n,q)$	n-dimensional projective space over the field $GF(q)$
$PGL(n,q)$	projective general linear invertible transformations of $PG(n-1,q)$
$\Gamma L(n,q)$	semi-linear invertible transformations of $PG(n-1,q)$
$P\Gamma L(n,q)$	projective semi-linear invertible transformations of $PG(n-1,q)$
$\pi(\mathcal{S})$	translation plane determined by a spread $\mathcal{S}$
$\Delta_D(\pi)$	Bose-Barlotti derived dual of a translation plane π
$Fig(q)$	Figueroa plane of order q
$\langle g \rangle$	group generated by a group element g
$N \rtimes G$	semi-direct product of the groups N and G
$\|X\|$	number of elements of a finite set X
$a\|b$	integer a divides integer b
(a,b)	the greatest common divisor of integers a and b

Appendix

INHERITED UNITALS IN DERIVED DUAL TRANSLATION PLANES

The work of this appendix arose from applying the techniques developed for inheriting maximal arcs to unitals. The main result (by T. Penttila and myself) is to show that if a Buekenhout-Metz unital exists in a translation plane, then there exist unitals in certain derived duals of the translation plane. Unfortunately, at the time the work for the appendix was submitted for publication to a journal it was discovered that the results had been previously published [12], and so the paper was withdrawn. But since the work is a natural extension of the ideas developed for maximal arcs it seemed worth including it here. The appendix is in its original form as a paper and is intended to be independent of the rest of the thesis.

1. INTRODUCTION

A **unital** in a projective plane of order q^2 is a set $\mathcal{U}$ of q^3+1 points such that every line meets $\mathcal{U}$ either in 1 point or in $q+1$ points. A line is called **tangent** if it meets the unital in 1 point; otherwise **secant**. The set of tangent lines to a unital form a unital in the dual plane called the **dual unital**.

The study of unitals began in 1946 with the paper of R. Baer [5]. Baer showed that a polarity of a projective plane of order n has at least $n+1$ absolute points and that, if equality occurs, then the set of absolute points is a line if n is even and an oval if n is odd. He also showed that if there are more than $n+1$ absolute points, then n is a square, and that if every non-absolute line has $n^{1/2}+1$ absolute points, then the set of absolute points is a unital. In 1970, M. Seib sharpened this result by showing that a polarity of a projective plane of order n has at most $n^{3/2}+1$ absolute points and that, if equality occurs, then the set of absolute points is a unital [120].

The **classical unitals** in $PG(2,q^2)$ arise in this manner from the unitary polarities of $PG(2,q^2)$.

In [67], M.J. Ganley shows that the (translation) planes coordinatised by the Dickson semifields admit unitary polarities, and so contain unitals. Since a necessary condition for a translation plane to admit a polarity is that the plane be self dual, it follows from [81, Theorem 6.9] that for a translation plane to admit a unitary polarity it must be coordinatisable by a commutative semifield.

Apart from unitals arising from unitary polarities a number of constructions are known.

F. Buekenhout [30] gave two constructions of unitals in translation planes of rank at most two over their kernel : a hyperbolic construction - for which the translation line is secant and a parabolic construction - for which the translation line is tangent. In the parabolic case, an ovoid is used as an ingredient in the construction. This lead to some new unitals in $PG(2,q^2)$, q even, q not a square, from Tits ovoids in $PG(3,q)$. In [106], R. Metz showed that Buekenhouts parabolic construction constructed unitals other than the classical unital in $PG(2,q^2)$ from elliptic quadrics in $PG(3,q)$. K. Grüning [69] studied the hyperbolic Buekenhout unitals in the Hall planes and their duals, from the perspective of the associated design.

A particular class of the unitals constructed by Buekenhout and Metz are known as *Buekenhout-Metz unitals*. Unfortunately, two inequivalent definitions of Buekenhout-Metz unitals occur in the literature, one more general than the other. See Section 3 for the definition used in this paper and comments on the literature.

A class of unitals in the Hughes planes was constructed by L.A. Rosati in [118]. These were also studied in [97]. The Rosati unitals were then shown to inherit to unitals in planes derived from Hughes planes by V. Abatangelo, B. Larato, and L.A. Rosati [2]. Recently, T. Penttila and G.F. Royle classified all unitals (indeed all sets of type (m,n)) in the projective planes of order 9 [114]. For a comprehensive study of unitals see S. Barwick's Ph.D. Thesis [13].

In this paper we prove (Section 4) the following result.

THEOREM A.4.1 *Let $\mathcal{U}$ be a Buekenhout-Metz unital in a non-Desarguesian translation plane π of order q^2 with translation line L_∞. Let s be any point of π on $L_\infty - \mathcal{U}$, then the set of tangent lines to $\mathcal{U}$ on s form a derivation set $\mathcal{D}$ in the dual plane of π. Let $\mathcal{U}'$ be the set of tangent lines to $\mathcal{U}$ not on s. Then $\mathcal{U}'$ corresponds to a set of points in the plane $\Delta_D(\pi)$ that is the derived dual of π determined by $\mathcal{D}$. Further, $\mathcal{U}'$ is contained in a unital in $\Delta_D(\pi)$.*

A similar result is also proved for Buekenhout-Metz unitals in Desarguesian planes, though care has to be taken in the statement as every line of a Desarguesian plane is a translation line.

To prove this we need a number of results and constructions. In Section 2 constructions of translation planes and derived dual translation planes are given. In Section 3 the Buekenhout-Metz construction of unitals is given. Section 5 contains some remarks on the unitals constructed in this paper.

In the following it is assumed that the orders of the projective planes and projective spaces are finite.

2. CONSTRUCTION OF FINITE TRANSLATION PLANES AND DERIVED DUAL TRANSLATION PLANES

A point x is a **centre** of a collineation of a projective plane if all the lines on x are fixed by the collineation. A line L of a projective plane is an **axis** of a collineation of the plane if every point of L is fixed by the collineation. A non-trivial collineation has at most one centre and at most one axis, and has a centre if and only if it has an axis [55, 1.4.9]. An **elation** is a collineation that has a centre, and an axis incident with the centre. A **homology** is a collineation that has a centre, and an axis not incident with the centre. In a projective plane of order q, the order of an elation must divide q, and the order of a homology must divide $q-1$ [55, 1.4.12]. The orbits of the group generated by an elation or homology on the non-fixed points all have length the order of the elation or homology [78, p.152, p.153].

A projective plane π is a **translation plane** if there is a line L in π such that the group of elations of π with axis L is transitive on the points of $\pi - L$. The line L is called a **translation line** of π. In particular, a finite Desarguesian plane is a translation plane, and any line of a Desarguesian plane is a translation line. The order of a finite translation plane is necessarily a power of a prime [103, 1.9].

A **spread** $\mathcal{S}$ of $PG(3,q)$ is a set of lines such that every point of $PG(3,q)$ lies on exactly one line of $\mathcal{S}$. Equivalently, it is a set of q^2+1 pairwise skew lines [79, 17.1].

Let $PG(3,q)$ be embedded as a hyperplane H in $PG(4,q)$, and let $\mathcal{S}$ be a spread of H. Construct a new incidence structure $\pi = \pi(\mathcal{S})$ as follows:

The *points* of π are the points of $PG(4,q) - H$, together with the elements of the spread.
The *lines* of π are the planes of $PG(4,q)$ which intersect H in a member of $\mathcal{S}$, together with the line (often denoted L_∞) whose points are the elements of the spread.
The *incidence* relation of π is that induced by the incidence in $PG(4,q)$.

Then π is a translation plane of order q^2, with translation line L_∞. This construction is a particular case (the translation planes of order q^2 with kernel containing $GF(q)$) of a general construction of translation planes due to R. H. Bruck and R. C. Bose [28], [29]. The general Bruck-Bose construction is the projective version of an affine construction due to André [4].

In [21], R. C. Bose and A. Barlotti constructed a class of projective planes in the following manner.

Let π be a translation plane represented as above. Let s be a fixed element of the spread $\mathcal{S}$, and D a hyperplane of $PG(4,q)$ meeting H in a plane containing s. Note that every plane in H contains a unique spread element, so D contains the spread element s and no other. Construct a new incidence structure $\Delta_D(\pi)$ as follows:

The points of $\Delta_D(\pi)$ are of three types, (I) those represented by planes of $PG(4,q)$ not contained in H and passing through lines of $\mathcal{S}$, other than s; (II) those represented by planes of $PG(4,q)$, not contained in H or D and containing s; (III) those represented by points of the line s.

The lines of $\Delta_D(\pi)$ are of three types, (I) those which are represented by points of $PG(4,q)$ not contained in H or D; (II) those which are represented by planes contained in D, and not containing s; (III) a single line represented by the line s.

Incidence is defined to be containment in $PG(4,q)$, except that a point of type (I) or (II) is defined to be incident with a line of type (II) when the planes representing them intersect in a line.

The incidence structure $\Delta_D(\pi)$ is then a projective plane of order q^2 [21]. Note that different choices of D can give rise to non-isomorphic planes. The planes $\Delta_D(\pi)$ are derived planes of the dual plane of π (derived in the sense of Albert [3] and Ostrom [107] , [110], [109]), the derivation set being the q lines of π which are represented by planes contained in D on the spread element s, together with the line L_∞.

For example, if the plane π is the Desarguesian plane $PG(2,q^2)$ (i.e., the spread is regular [55]) and $q>2$, the dual of π is isomorphic to $PG(2,q^2)$, and the planes derived from the dual of π are all isomorphic to the Hall plane of order q^2.

3. THE BUEKENHOUT-METZ CONSTRUCTION OF UNITALS

An **ovoid** of $PG(3,q)$, $q>2$, is a set of q^2+1 points no 3 collinear. (If $q=2$ an ovoid is a set of 5 points no 4 coplanar.) [79, p.52].

Let H be a hyperplane in $PG(4,q)$ and v be a point of H. Let $\mathcal{O}$ be an ovoid of the quotient space $PG(4,q)/v$ for which the plane H/v is a tangent plane. Then $\mathcal{O}$ is the set of generators of an ovoidal cone $\mathcal{C}$ in $PG(4,q)$ with vertex v, and H contains a unique generator l. Let $\mathcal{S}$ be a spread of H containing l. Then in the Bruck-Bose translation plane π corresponding to the spread $\mathcal{S}$, $\mathcal{U}=\mathcal{C}-l\cup\{l\}$ is a unital. We call such unitals *Buekenhout-Metz unitals.* The line L_∞ is a tangent to $\mathcal{U}$ and so the Buekenhout-Metz unitals are said to be *parabolic.*

An equivalent definition is the following.

Let H, H' be hyperplanes of $PG(4,q)$, intersecting in the plane Π and let v be a point of H not in Π. Let $\mathcal{O}$ be an ovoid of H' to which Π is a tangent plane.

Let $\mathcal{C}$ be the cone subtended by $\mathcal{O}$ with vertex v. Then $\mathcal{C}$ meets H in a line l, namely the line joining v to the point of tangency of Π to $\mathcal{O}$. Let $\mathcal{S}$ be a spread of H containing l. Then in the Bruck-Bose translation plane π corresponding to the spread $\mathcal{S}$, $\mathcal{U} = \mathcal{C} - l \cup \{l\}$ is a unital, to which the line L_∞ is a tangent line.

In either case the the only condition on the spread $\mathcal{S}$ is that it contains the line l, and so Buekenhout-Metz unitals exist in all translation planes of order q^2 with kernel containing $GF(q)$. The known ovoids of $PG(3,q)$ are the elliptic quadrics and the Tits ovoids (Tits ovoids are only defined for $q = 2^{2m+1}, m \geq 1$). Some authors, e.g. [9], define Buekenhout-Metz to be only those unitals constructed as above with the ovoid restricted to being an elliptic quadric. We follow [101] and allow any ovoid.

4. UNITALS INHERITED FROM BUEKENHOUT-METZ UNITALS

THEOREM A.4.1 *Let $\mathcal{U}$ be a Buekenhout-Metz unital in a translation plane π of order q^2 as in the previous section with L_∞ the line at infinity. Let s be any point of π on $L_\infty - \mathcal{U}$, then the set of tangent lines to $\mathcal{U}$ on s form a derivation set $\mathcal{D}$ in the dual plane of π. Let $\mathcal{U}'$ be the set of tangent lines to $\mathcal{U}$ not on s. Then $\mathcal{U}'$ corresponds to a set of points in the plane $\Delta_D(\pi)$ that is the derived dual of π determined by $\mathcal{D}$. Further, $\mathcal{U}'$ is contained in a unital in $\Delta_D(\pi)$.*

Proof: Let $\mathcal{U}$ be a Buekenhout-Metz unital in a translation plane π of order q^2 with notation as above. We first construct a derived dual plane of π.

Let $v \in PG(4,q)$ be the vertex of the ovoidal cone, and l the (unique) line of the spread $\mathcal{S}$ containing v. Let $s \in \mathcal{S} - \{l\}$, then s represents a point of π not contained in $\mathcal{U}$, and so there exist $q+1$ tangent lines to $\mathcal{U}$ on the point corresponding to s. Let Π_t be a plane of $PG(4,q)$ on s representing such a tangent line (necessarily not L_∞). Let D be the three space given by the span of v and Π_t, and x the point of $PG(4,q)$ representing the point of $\mathcal{U}$ on the line that Π_t represents.

It follows that D meets H in the plane given by the span of v and s, and that $\mathcal{U}$ (or rather the representing cone $\mathcal{C}$) meets D in exactly the points on the line vx. For if D contained a point, y say, of the ovoidal cone not on vx, then vy would be a line contained in D, and Π_t would meet the ovoidal cone in the point $vy \cap \Pi_t \neq x$. This contradicts Π_t representing a tangent line.

There are q planes on s not contained in H that meet $\mathcal{C}$ in a single point (on vx). These together with L_∞ represent the tangent lines to $\mathcal{U}$ on the point correponding to s. The projective three space D defines a plane $\Delta_D(\pi)$, as in Section 2, that is a derived plane of the dual plane of π, and with derivation set being the tangent lines to $\mathcal{U}$ on the point corresponding to s.

Define $\mathcal{U}_\Delta$ to be the set containing the following two types of elements:

(a) planes Π of $PG(4,q)$ not contained in H or D that meet H in a spread element not equal to s, and that are tangent to $\mathcal{U}$.
(b) points of $PG(4,q)$ on the line s.

The set $\mathcal{U}_\Delta$ is then a subset of points of $\Delta_D(\pi)$. The type (a) elements of $\mathcal{U}_\Delta$ also represent a subset of the lines of the translation plane π that are tangent to $\mathcal{U}$. We prove that the set $\mathcal{U}_\Delta$ is the set of points of a unital in the plane $\Delta_D(\pi)$ by first showing the set has size q^3+1, and then showing that every line of $\Delta_D(\pi)$ meets $\mathcal{U}_\Delta$ in 1 or $q+1$ points.

We will need the following definition and Lemmas.

A **Baer subplane** of a projective plane of order q^2 is a subplane of order q [55].

LEMMA *A*.4.1 (Grüning [69, Corollary 3.3]) *Let $\mathcal{U}$ be a unital, and $\mathcal{B}$ a Baer subplane of a projective plane π of order q^2. If b_1 is the number of lines of $\mathcal{B}$ which when extended to π are tangent to $\mathcal{U}$, and m is the number of points of $\mathcal{B}\cap\mathcal{U}$, then*

$$b_1 + m = 2(q+1).$$

Proof: We count in two way the set of ordered pairs (p,L) where p is a point of $\mathcal{U}$, L is a line of $\mathcal{B}$ (extended to π) and $p \in L$. The count gives:

$$m(q+1) + (q^3+1-m).1 = b_1.1 + (q^2+q+1-b_1)(q+1).$$

The result follows. ∎

LEMMA *A*.4.2 *Let π be a translation plane containing a Buekenhout-Metz unital $\mathcal{U}$ with notation as above. Let $\Delta_D(\pi)$ be a derived dual translation plane constructed with respect to $\mathcal{U}$ as in this section. Let Π be a line of type (II) of $\Delta_D(\pi)$. Then the plane of $PG(4,q)$ representing Π corresponds to a Baer subplane of π. Further, considering Π as a Baer subplane of π, the tangent lines to $\mathcal{U}$ on the point corresponding to the spread element s are lines of Π (when restricted to Π).*

Proof: A plane Π in $PG(4,q)$ that meets H in a line not contained in the spread $\mathcal{S}$ corresponds to a Baer subplane of the translation plane π [138]. The points of the corresponding Baer subplane are given by the points of $\Pi\backslash H$, together with the points corresponding to the $q+1$ spread elements that Π meets.

As it was noted previously, the planes representing tangent lines to $\mathcal{U}$ on s are contained in D. Since two planes in a three space are either equal or meet in a line, it follows that the plane representing Π meets every plane in D on s in a line, and so meets every tangent line not equal to L_∞ on s in $q+1$ points. The line L_∞ also meets Π in the $q+1$ points corresponding to the spread elements that Π meets. Hence result. ∎

On any line of the spread not equal to l there are q planes that represent lines of π that are tangent to $\mathcal{U}$ (the other tangent line being L_∞). There are $q^2 - 1$ spread elements not equal to s or l, so the number of type (a) elements of $\mathcal{U}_\Delta$ is $(q^2 - 1)q$. The number of type (b) elements is $q + 1$, giving $|\mathcal{U}_\Delta| = q^3 + 1$, the size of a unital in a plane of order q^2. We now show that every line of $\Delta_D(\pi)$ meets $\mathcal{U}_\Delta$ in 1 or $q + 1$ points.

Lines of type (I)

Note that in $\Delta_D(\pi)$ a line of type (I) is not incident with a point of type (III), so lines of type (I) are not incident with points of type (b) in $\mathcal{U}_\Delta$.

A line of type (I) in $\Delta_D(\pi)$ is represented by a point $z \in PG(4, q)\backslash(H \cup D)$. The points of $\Delta_D(\pi)$ on this line are represented as the $q^2 + 1$ planes of $PG(4, q)$ given by the span of z and the $q^2 + 1$ lines of the spread. These planes also represent the lines of π on the point z, and we use this fact to argue that a line of type (I) in $\Delta_D(\pi)$ meets $\mathcal{U}_\Delta$ in 1 or $q + 1$ points.

Suppose $z \notin \mathcal{U}$. Then there are exactly $q + 1$ lines of π incident with z and tangent to $\mathcal{U}$. The (affine) tangent lines to $\mathcal{U}$ on the point corresponding to the spread element s are contained in D, so no tangent line of π to $\mathcal{U}$ on z is incident with s. Hence there are $q + 1$ planes of $PG(4, q)$ not contained in H or D that meet H in a spread element not equal to s, and that are tangent to $\mathcal{U}$. Thus the line of $\Delta_D(\pi)$ that z represents meets $\mathcal{U}_\Delta$ in $q + 1$ points.

Suppose $z \in \mathcal{U}$. Then there is a unique tangent line of π to $\mathcal{U}$ on z. This tangent line is represented as a plane of $PG(4, q)$ given by the span of z and a line of the spread (not equal to s as in the previous paragraph). Hence there is exactly one plane of $PG(4, q)$ not contained in H or D that meets H in a spread element not equal to s, and that is tangent to $\mathcal{U}$. Thus the line of $\Delta_D(\pi)$ that z represents meets $\mathcal{U}_\Delta$ in 1 point.

Lines of type (II)

Lines of type (II) are represented as planes contained in D, but not containing any element of the spread. Let Π be such a line. The points of $\Delta_D(\pi)$ on Π are of type (I) or (III). The type (III) point on Π is the point $\Pi \cap s$ of $PG(4, q)$. The type (I) points on Π are represented by q^2 planes of $PG(4, q)$ not contained in H, containing a unique spread element not equal to s and meeting Π in a line of $PG(4, q)$. Note that the type (I) points on Π represent lines of π.

Now Π contains one type (b) point of $\mathcal{U}_\Delta$, the point given by $\Pi \cap s$.

Suppose Π contains the line vx, then Π meets $\mathcal{U}$ in $q + 1$ points. Lemma $A.4.2$ shows that Π represents the set of points of a Baer subplane of the plane π. Lemma $A.4.1$, then gives that there are $2(q+1) - (q+1) = q+1$ lines of Π that are tangent lines to $\mathcal{U}$ when extended to π. Now, by Lemma $A.4.2$, Π meets the $q+1$ tangents to U on the point corresponding to the spread element s. Hence Π meets no tangent lines of π to $\mathcal{U}$ on spread elements other than s. It follows that Π is not incident with any type (a) points of $\mathcal{U}_\Delta$, and that Π meets $\mathcal{U}_\Delta$ in exactly one (type (b)) point.

Suppose Π does not contain the line vx, then Π meets $\mathcal{U}$ in 1 point (the intersection of Π and vx). Using the same argument as in the previous paragraph it follows that Π contains $q+1$ points of $\mathcal{U}_\Delta$.

The line of type (III)
Clearly the line of type (III) meets $\mathcal{U}_\Delta$ in only the $q+1$ type (b) points of $\mathcal{U}_\Delta$.

Thus every line of $\Delta_D(\pi)$ meets $\mathcal{U}_\Delta$ in 1 or $q+1$ points. Hence result. ∎

5. REMARKS

The dual plane of a Desarguesian plane is Desarguesian, and all planes derived from a Desarguesian plane are isomorphic to the Hall plane of the same order. The dual of a Buekenhout-Metz unital in a Desarguesian plane is Buekenhout-Metz [36], and so it seems likely that the unitals we have constructed in the Hall planes are isomorphic to those of [13]. Also, since every projective plane of order 4 is Desarguesian, and all unitals in $PG(2,4)$ are known to be classical, nothing new has been constructed in the planes of order 4.

An important point to note is that for any given Buekenhout-Metz unital, the present construction may give a number of non-isomorphic unitals in a number of non-isomorphic planes, depending on the choice of the derivation set.

For related constructions for maximal arcs, see [76] and [74] or Chapter 3.

[1] V. Abatangelo & B. Larato, 'A characterisation of Denniston's maximal arcs', *Geom. Dedicata* **30** (1989), 197–203.

[2] V. Abatangelo, B. Larato & L.A. Rosati, 'Unitals in planes derived from Hughes planes', *J. Comb. Info. and System Sciences* **15** (1990), 151–155.

[3] A.A. Albert, 'The finite planes of Ostrom', *Boletin de la Sociedad Matematica Mexicana* **11** (1966), 1–13.

[4] J. André, 'Über nicht-Desaguessche Ebenen mit transitiver Translations - Gruppe', *Math. Zeit.* **60** (1954), 156–186.

[5] R. Baer, 'Polarities in finite projective planes', *Bull. Amer. Math. Soc.* **52** (1946), 77–93.

[6] L. Bader, G. Lunardon & S.E. Payne, 'On q-clan geometry, $q = 2^e$', *Bull. Belg. Math. Soc.* **3** (1994), 301–328.

[7] B. Bagchi & N.S. Narasimha Sastry, 'Intersection pattern of the classical ovoids in symplectic 3-space of even order', *J. Algebra* **126** (1989), 147–160.

[8] R.D. Baker & G.L. Ebert, 'Intersection of unitals in the Desarguesian plane', *Congr. Num.* **70** (1990), 87–94.

[9] R.D. Baker & G.L. Ebert, 'On Buekenhout-Metz unitals of odd order', *J. Combin. Theory Ser. A* **60** (1992), 67–84.

[10] A. Barlotti, 'Un' estensione del teorema di Segre-Kustaanheimo', *Boll. Un. Mat. Ital.* **10** (1955), 96–98.

[11] A. Barlotti, 'Sui $\{k;n\}$-archi di un piano lineare finito', *Boll. Un. Mat. Ital.* **11** (1956), 553–556.

[12] A. Barlotti & G. Lunardon, 'Una classe di unital nei Δ-piani (in Italian)', *Riv. Mat. Univ. Parma* **5** (1979), 781–785.

[13] S. Barwick, '*Substructures of finite geometries*', PhD. Thesis, University of London 1994.

[14] T. Beth, D. Jungnickel & H. Lenz, '*Design Theory*', Cambridge University Press 1986.

[15] M. Biliotti & G. Lunardon, 'Insiemi di derivazione e sottopiani di Baer', *Atti Acc. Lincei, Rendic. Sc. Fis. Mat. e Nat.* **Vol. LXIX, Serie 8a** (1980), 135–141.

[16] M. Biliotti & G. Korchmaros, 'Hyperovals with a transitive collineation group',

Geom. Ded. **24** (1987), 269–281.

[17] J.W. Blattner, '*Projective plane geometry*', Holden-Day, San Fransisco 1968.

[18] A. Blokhuis & H.A. Wilbrink, 'A characterisation of exterior lines of certain points in $PG(2, q)$', *Geom. Ded.* **23** (1987), 253–254.

[19] A. Blokhuis & H. A. Wilbrink, *personal communication.*

[20] R.C. Bose, 'Strongly regular graphs, partial geometries and partially balanced designs', *Pacific J. Math.* **13** (1963), 389–419.

[21] R.C. Bose & A. Barlotti, 'Linear representation of a class of projective planes in a four dimensional representation', *Ann. Mat. Pura Appl.(4)* **88** (1971), 9–31.

[22] R.C. Bose, 'On a representation of the Baer subplanes of the Desarguesian plane $PG(2, q^2)$ in a projective five dimensional space $PG(5, q)$', *Proceedings of the International Colloquium on Combinatorial Theory, held in Rome Sept. 3-15, 1973*, 381–391.

[23] R.C. Bose, J.W. Freeman & D.G. Glynn, 'On the intersection of two Baer subplanes in a finite projective plane', *Utilitas Mathematica* **17** (1980), 65–77.

[24] W. Bosma & J. Cannon, '*A language for computer algebra*', The school of mathematics, University of Sydney.

[25] P. Bousfield, N. Hamilton, T. Penttila & G. Royle, 'A compendium of examples of sets of type (m, n)', *in preparation.*

[26] A.E. Brouwer, A.M. Cohen & A. Neumaier, '*Distance Regular Graphs*', Springer-Verlag 1989.

[27] A.E. Brouwer & J.H. van Lint, 'Strongly regular graphs and partial geometries', *In: Enumeration and Design - Proc. Silver Jubilee conference on Combinatorics, Waterloo* 1982 *(D.M. Jackson & S.A. Vanstone, eds.), Academic Press, Toronto* (1984), 85–122.

[28] R.H. Bruck & R.C. Bose, 'The construction of translation planes from projective spaces', *J. Algebra* **1** (1964), 85–102.

[29] R.H. Bruck & R.C. Bose, 'Linear representations of projective planes in projective spaces', *J. Algebra* **4** (1966), 117–172.

[30] F. Buekenhout, 'Existence of unitals in finite translation planes of order q^2 with a kernel of order q', *Geom. Ded.* **5** (1976), 189–194.

[31] F. Buekenhout, ed., '*Handbook of incidence geometry*', Elsevier Science B.V. 1995.

[32] R. Calderbank & W.M. Kantor, 'The geometry of two weight codes', *Bull. London Math. Soc.* **18** (1986), 97–122.

[33] P.J. Cameron & J.H. van Lint, 'On the partial geometry pg(6,6,2)', *J. Comb. Theory Ser. A* **32** (1982), 252–255.

[34] P.J. Cameron, 'Strongly regular graphs', *In: Selected topics in graph theory (L.W. Beineke & R.J. Wilson, eds.), Acad. Press* (1978), 337–360.

[35] A.R. Camina & T.M. Gagen, 'Block transitive automorphism groups of designs', *J. Algebra* **86** (1984), 549–554.

[36] L.R.A Casse, C.M. O'Keefe, & T. Penttila, 'Characterisations of Buekenhout - Metz unitals', *Geom. Ded., to appear.*

[37] W.M. Cherowitzo, 'Hyperovals in translation planes of order 16', *JCMCC* **9** (1991), 39–55.

[38] W.M. Cherowitzo, 'Ovals in Figueroa planes', *J. Geom.* **37** (1990), 84–86.

[39] W.M. Cherowitzo, T. Penttila, I. Pinneri & G. Royle, 'Flocks and ovals', *Geom. Ded., to appear.*

[40] J. Cofman, 'Baer subplanes in finite projective and affine planes', *Can. J. Math.* **1** (1972), 90–97.

[41] A. M. Cohen, 'A new partial geometry with parameters $(s,t,\alpha) = (7,8,4)$', *J. Geom.* **16** (1981), 181–186.

[42] A. Cossu, 'Su alcune proprietà dei $\{k,n\}-$archi di un piano proiettivo sopra un corpo finito', *Rend. Mat. e Appl.* **20** (1961), 271–277.

[43] F. De Clerck, H. Gevaert & J. A. Thas , 'Translation partial geometries', *Ann. Discrete Math.* **37** (1988), 117–136.

[44] F. De Clerck, M. De Soete & H. Gevaert, 'A characterisation of the partial geometry ${T_2}^*(\mathcal{K})$', *Euro. J. Comb.* **8** (1987), 121–127.

[45] F. De Clerck, H. Gevaert & J. A. Thas , 'Flocks of a quadratic cone in $PG(3,q), q \le 8$.', *Geom. Ded.* **26** (1988), 215–230.

[46] F. De Clerck , 'How computer algebra can help research in finite geometry', *unpublished.*

[47] F. De Clerck & J. A. Thas, 'Partial geometries in finite projective spaces', *Arch. Math.* **30** (1978), 537–540.

[48] F. De Clerck, R. H. Dye & J. A. Thas, 'An infinite class of partial geometries associated with the hyperbolic quadric in $PG(4n-1,2)$', *Europ. J. Comb.* **1** (1980), 323–326.

[49] F. De Clerck, A. Del Fra & D. Ghinelli, 'Pointsets in partial geometries', *In: Advances in finite geometries and designs (eds. J. W. P. Hirschfeld, D.R. Hughes & J.A. Thas)* (Oxford Univ. Press, 1991), 93–110.

[50] M. de Finis, 'On k-sets of type (m, n) in projective planes of square order', *In: Finite geometries and designs (eds. P.J. Cameron, J.W.P. Hirschfeld, D.R. Hughes), London Math. Soc.,Lect. Notes Series* **49** (1981), 98–103.

[51] M.J. de Resmini, 'An infinite family of type (m, n) sets in $PG(2, q^2)$, q a square', *J. Geom.* **20** (1983), 36–43.

[52] M.J. de Resmini, 'On admissible sets with two intersection numbers in a projective plane', *Ann. Discrete Math.* **37** (1988), 137–146.

[53] M.J. de Resmini & N. Hamilton, 'Hyperovals and unitals in Figueroa planes', *Europ. J. Comb., to appear.*

[54] A. Delandtsheer & J. Doyen, 'A classification of line-transitive maximal (v, k)-arcs in finite projective spaces', *Arch. Math.* **55** (1990), 187–192.

[55] P. Dembowski, '*Finite geometries*', Springer, Berlin, Heidelberg, New York 1968.

[56] U. Dempwolff & A. Reifart, 'The translation planes of order 16 admitting a Baer 4-group', *J. Comb. Theory, Ser. A* **32** (1982), 119–124.

[57] U. Dempwolff, 'A note on the Figueroa planes', *Arch. Math. (Basel)* **43** (1984), 285–288.

[58] U. Dempwolff & A. Reifart, 'The classification of translation planes of order 16.', *Arch. Math. (Basel)* **43** (1984), 285–288.

[59] R.H.F. Denniston, 'Some maximal arcs in finite projective planes', *J. Comb. Theory* **6** (1969), 317–319.

[60] J.H. Dinitz & D.R. Stinson, eds., '*Contemporary design theory: A collection of surveys*', John Wiley & Sons 1992.

[61] V. De Smet, '*Substructures of finite classical generalized quadrangles and hexagons*', PhD. Thesis, Universiteit Gent 1994.

[62] R.H. Dye, 'Partitions and their stabilisers for line complexes and quadrics', *Ann. Mat. Pura Appl. Ser IV* **114** (1977), 173–194.

[63] R.H. Dye, 'Maximal subgroups of symplectic groups stabilising spreads', *J. Algebra* **87** (1984), 493–509.

[64] G. Fellegara, 'Gli ovaloidi di uno spazio tridimensionale di Galois di ordine 8', *Atti. Accad. Naz. Lincei Rend.* **32** (1962), 170–176.

[65] R. Figueroa, 'A family of not $(V, l)-$transitive projective planes of order q^3, $q \not\equiv 1 (\text{mod } 3)$ and $q > 2$', *Math. Zeit.* **181** (1982), 471–479.

[66] J.W. Freeman, 'Reguli and pseudo-reguli in $PG(3, s^2)$', *Geom. Dedicata* **9** (1980), 267–280.

[67] M.J. Ganley, 'Polarities in translation planes', *Geom. Ded.* **1** (1972), 103–116.

[68] T. Grundhöfer, 'A synthetic construction of the Figueroa planes', *J. Geom.* **26** (1986), 191–201.

[69] K. Grüning, 'A class of unitals of order q which can be embedded in two different planes of order q^2', *J. Geom.* **29** (1987), 61–77.

[70] W. Haemers, 'A new partial geometry constructed from the Hoffman-Singleton graph', *In: Finite geometries and designs (eds. P. J. Cameron, J. W. P. Hirschfeld, D. R. Hughes), London Math. Soc.,lect. Notes Series* **49** (1981), 119–127.

[71] W. Haemers & J. H. van Lint, 'A partial geometry $pg(9, 8, 4)$', *Ann. Discrete Math.* **15** (1982), 205–212.

[72] N. Hamilton & T. Penttila, 'A characterisation of Thas maximal arcs in translation planes of square order', *J. Geom.* **51** (1994), 60–66.

[73] N. Hamilton, 'Some maximal arcs in Hall planes', *J. Geom.* **52** (1995), 101–107.

[74] N. Hamilton, 'Some maximal arcs in derived dual Hall planes', *Europ. J. Comb.* **15** (1994), 525–532.

[75] N. Hamilton, 'Maximal arcs and disjoint maximal arcs in projective planes of order 16', *Research Report, U.W.A. Mathematics* (March 1995/10).

[76] N. Hamilton, 'Some inherited maximal arcs in derived dual translation planes', *Geom. Ded., to appear.*

[77] Ch. Hering & H.-J. Schaeffer, 'On the new projective planes of R. Figueroa', *In: Combinatorial Theory, Proc. Schloss Rauischholzhausen 1982, eds. D. Jungnickel & K. Vedder, Berlin-Heidelberg-New York,* (1982), 187–190.

[78] J.W.P Hirschfeld, '*Projective geometries over finite fields*', Oxford University Press, Oxford 1979.

[79] J.W.P Hirschfeld, '*Finite projective spaces of three dimensions*', Oxford University Press, Oxford 1985.

[80] J.W.P Hirschfeld & J.A. Thas, '*General Galois geometries*', Oxford University Press, Oxford 1991.

[81] D.R. Hughes & F.C. Piper, '*Projective planes*', Springer-Verlag, New York, Heidelberg, Berlin 1970.

[82] D.R. Hughes & F.C. Piper, '*Design theory*', Cambridge University Press 1985.

[83] B. Huppert, '*Endliche Gruppen I*', Springer-Verlag 1967.

[84] N.L. Johnson, 'A note on semi-translation planes of class $I - 5a$', *Arch. Math.* **21** (1970), 528–532.

[85] N.L. Johnson, 'A note on the derived semifield planes of order 16', *Aequat. Math.* **18** (1978), 103–111.

[86] N.L. Johnson, 'The translation planes of Dempwolff', *Can. J. Math.* **XXXIII, no.5** (1981), 1060–1073.

[87] M.J. Kallaher, '*Affine planes with transitive collineation groups*', Elsevier North Holland, Inc. 1982.

[88] W.M. Kantor, 'Strongly regular graphs defined by spreads', *Israel J. Math* **41** (1982), 298–312.

[89] W.M. Kantor, 'Spreads, translation planes and Kerdock sets. I', *Siam J. Alg. Disc. Meth.* **3** (1982), 151–165.

[90] W.M. Kantor, 'Spreads, translation planes and Kerdock sets. II', *Siam J. Alg. Disc. Meth.* **3** (1982), 308–318.

[91] W.M. Kantor, 'Kerdock sets and related planes', *Discrete Math.* **106/107** (1992), 297–302.

[92] C.M. O'Keefe, '*Concerning t-spreads of $PG((s+1)(t+1)-1, q)$*', PhD. Thesis, Dept. of Pure Mathematics, University of Adelaide 1987.

[93] C.M. O'Keefe & T. Penttila, 'Ovoids of $PG(3, 16)$ are elliptic quadrics', *J. Geom.* **38** (1990), 95–106.

[94] C.M. O'Keefe & T. Penttila, 'Hyperovals in $PG(2, 16)$', *Europ. J. Comb.* **12** (1991), 51–59.

[95] C.M. O'Keefe, A.A. Pascasio & T. Penttila, 'Hyperovals in Hall planes', *Europ. J. Comb.* **13** (1992), 195–199.

[96] C.M. O'Keefe & T. Penttila, 'Symmetries of arcs', *J. Comb. Th., Series A* **66, no. 1** (1994), 53–67.

[97] B. Kestenband, 'A family of unitals in the Hughes plane', *Canad. J. Math.* **42** (1990), 1067–1083.

[98] J.D. Key & M.J. de Resmini, 'Codewords for projective planes from sets of type (s, t)', *Europ. J. Comb.* **15** (1994), 259–268.

[99] D.E. Knuth, 'Finite semifields and projective planes', *J. Algebra* **2** (1965), 182–217.

[100] G. Korchmaros, 'Gruppi di collineazioni transitivi sui punti di una ovale ($(q+2)$-arco) di $S_{2,q}$, q pari', *Atti Sem. Mat. Fis. Univ. Modena* **27** (1978), 89–105.

[101] C. Lefèvre-Percsy, 'Characterisation of Buekenhout - Metz unitals', *Arch. Math.* **36** (1981), 565–568.

[102] J.H. van Lint & A. Schrijver, 'Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields', *Combinatorica* **1** (1981), 63–73.

[103] H. Lüneburg, '*Translation planes*', Springer-Velag, Berlin, Heidelberg, New York 1980.

[104] R. Mathon, *personal communication.*

[105] B.D. McKay, '*Nauty Users Guide (Version 1.5)*', Technical Report TR-CS-90-02, Computer Science Department, Australian National University 1990.

[106] R. Metz, 'On a class of unitals', *Geom. Ded.* **8** (1979), 125–126.

[107] T.G. Ostrom, 'Semi-translation planes', *Trans. Amer. Math. Soc.* **111** (1964), 1–18.

[108] T.G. Ostrom, 'The dual Lüneburg planes', *Math. Z.* **92** (1966), 201–209.

[109] T.G. Ostrom, 'Vector spaces and the construction of finite projective planes', *Arch. Math.* **19** (1968), 1–25.

[110] T.G. Ostrom, 'Translation planes of order p^2', *Combinatorial mathematics and its applications, University of North Carolina Monograph Series in Probability and Statistics, Chapter 26* (1969), 416–425.

[111] J.M. Nowlin Brown, 'Ovals in infinite Figueroa planes', *J. Geom.* **43** (1992), 41–52.

[112] S.E. Payne & J.A. Thas, '*Finite Generalised Quadrangles*', Pitman Pub. Ltd 1984.

[113] T. Penttila & I. Pinneri, 'Irregular hyperovals in $PG(2, 64)$', *J. Geom.* **51** (1994), 89–100.

[114] T. Penttila & G.F. Royle, 'Sets of type (m,n) in affine and projective planes of order 9', *Designs, codes and cryptography, to appear.*

[115] T. Penttila, G.F. Royle & M. Simpson, 'Hyperovals in planes of order 16', *submitted.*

[116] T. Penttila, *personal communication.*

[117] I. Pinneri, '*Hyperovals, generalised quadrangles, flocks and herds*', PhD. Thesis, University of Western Australia, in preparation.

[118] L.A. Rosati, 'Disegni unitari nei piani di Hughes', *Geom. Ded.* **27** (1988), 295–299.

[119] B. Segre, 'Ovals in a finite projective plane', *Canad. J. Math* **7** (1955), 414–416.

[120] M. Seib, 'Unitäre Polaritäten endlicher projectiver Ebenen', *Arch. Math.* **21** (1970), 103–112.

[121] J.J. Seidel, 'Strongly regular graphs', *In: Surveys in Combinatorics, Proc. 7th British Combinatorial Conference, London Math. Soc. Lecture Note Series* 38 *(B. Bollobas ed.), Cambridge University Press, Cambridge* (1979), 48–50.

[122] D.R. Stinson, 'Designs constructed from maximal arcs', *Discrete Math.* **97** (1991), 387–393.

[123] M. Tallini Scafati, 'Sui $\{k,n\}-$archi di un piano grafico finito', *Atti Accad. Naz. Lincei Rend.* **40** (1966), 373–378.

[124] M. Tallini Scafati, '$\{k,n\}-$archi di un piano grafico finito, con particolare riguardo a quelli con due caratteri. (Note I; II)', *Atti Accad. Naz. Lincei Rend.* **40** (1966), 812-818–1020-1025.

[125] D.E. Taylor, '*The geometry of the classical groups*', Heldermann, Berlin 1992.

[126] J.A. Thas, 'Ovoidal translation planes', *Arch. Math.* **23** (1972), 110–112.

[127] J.A. Thas, 'Construction of maximal arcs and partial geometries', *Geom. Dedicata* **3** (1974), 61–64.

[128] J.A. Thas, 'Ovoids and spreads of finite classical polar spaces', *Geom. Dedicata* **10** (1981), 135–144.

[129] J.A. Thas, 'Some results concerning $\{(q+1)(n-1);n\}-$arcs and $\{(q+1)(n-1)+1;n\}-$arcs in finite projective planes of order q', *J. Comb. Theory* **A19** (1974), 228–232.

[130] J.A. Thas, 'Construction of maximal arcs and dual ovals in translation planes', *Europ. J. Comb.* **1** (1980), 189–192.

[131] J.A. Thas, 'Partial geometries in finite affine spaces', *Math Z.* **158** (1978), 1–13.

[132] J.A. Thas, 'Combinatorics of partial geometries and generalised quadrangles', *in Higher combinatorics (ed. M. Aigner), Nato Adv. Sci. Inst. Ser. C: Math. Phys. Sci., Reidel* (1976), 183–199.

[133] J.A. Thas, 'Geometries in finite projective and affine spaces', *In: Surveys in combinatorics (ed. B. Bollobas, Cambridge Univ. Press), London Math. Soc.,Lect. Notes Series* **38** (1979), 157–180.

[134] J.A. Thas, 'Interesting point sets in generalised quadrangles and partial geometries', *Lin. Alg. Appl.* **114/115** (1989), 103–131.

[135] J.A. Thas, 'Some results on quadrics and a new class of finite geometries', *Simon Stevin* **55** (1981), 129–139.

[136] V.D. Tonchev, 'The isomorphism of the Cohen, Haemers-van Lint and De Clerck-Dye-Thas partial geometries', *Discrete Math.* **49** (1984), 213–217.

[137] T. van Trung, 'Maximal arcs and related designs', *J. Comb. Theory, Ser. A* **57** (1991), 294–301.

[138] R. Vincenti, 'A survey of the varieties of $PG(4,q)$ and Baer subplanes of translation planes', *Ann. Discrete Math.* **18** (1983), 775–780.

[139] R. Vincenti, 'Fibrazioni di un $S_{3,q}$ indotte da fibrazioni di un S_{3,q^2} e rappresentazione di sottopiani di Baer di un piano proiettivo', *Atti e Mem. Acc. Sc. Lettere e Arti di Modena* **Serie VI, Vol. XIX** (1977), 1–8.

[140] W.D. Wallis, 'Configurations arising from maximal arcs', *J. Comb. Th. (A)* **14** (1973), 115–119.

[141] M.A. Wertheimer, 'Oval designs in quadrics', *Contemporary Mathematics* **111** (1990), 287–297.