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A Study of Incidence Structures and Codes related to Regular Two-Graphs

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Preface

In mathematics you don't understand things.
You just get used to them.
— Johann von Neumann

That is what I, a PhD student in mathematics, have been doing for the past three and a half years: get used to certain things. Get used to, or acquainted with, various mathematical structures, in the hope of finding inspiration for new results. Get used to presenting these results in a clear, precise way, at conferences or in papers. But also get used to having to start all over again after finding a mistake in a calculation or proof. What you are now holding in your hands is the reflection of this process.

This thesis is somewhat uncommon in the sense that it does not focus on one class of objects, but rather explores several kinds of structures and their connections. In a fairly long introductory chapter we collect the notions and results that form the background for our research.

The first problem we worked on is situated between graphs and partial geometries. In 1998, Mathon constructed in an algebraic way a new class of partial geometries living on a Hermitian curve in the projective plane. Their point graphs are the Hermitian graphs $\mathcal{H}'(q)$, $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$. Being related to classical geometric structures, these partial geometries were thought to admit a geometric description as well. While translating Mathon's description into a more geometric language and trying to see a pattern, the turning point came when we started using a different model for the Hermitian two-graphs, involving the split Cayley hexagon $H(q)$. The geometric construction that we found, and that is given in Chapter 3, is also valid if $q = 3^{2h+1}$, $h \in \mathbb{N}$, yielding a new class of partial geometries.

The two-graph geometries treated in Chapter 2 have a connection with two-graphs that resembles the relation between a partial geometry and its point graph. The main result here is a proof of the non-existence of a two-graph geometry on the symplectic two-graphs $\Sigma(2m, 2)$ for $m \geq 3$. We also discuss a generalisation to semipartial geometries, and construct a class of examples.

In Chapter 4, the two-graphs disappear from the picture for a while. We introduce axioms for incidence structures called distance-regular geometries, which have a distance-regular point graph. As such, they generalise the (semi)partial geometries, the point graphs of which can be seen as distance-regular graphs having diameter two. We discuss some examples, both sporadic ones and infinite classes.

Chapter 5 is devoted to two-graphs again. The results in it arose from cooperation with Dr. Ir. Willem Haemers, which explains the use of matrix techniques and codes. We investigate regular graphs which belong to the switching class corresponding to a regular two-graph. Using an argument relying on the dimensions of the associated codes, we succeed in excluding the existence of such graphs for some of the doubly transitive two-graphs. In other cases we can give explicit constructions.

In Chapter 6, which is based on joint work with Dr. Ir. Willem Haemers as well, we concentrate on the second smallest Hermitian two-graph $\mathcal{H}(5)$. It is related to sporadic objects like the Hoffman–Singleton graph, which provides us with a complementary point of view. We discuss the strongly regular graph $\mathrm{srg}\,(126, 50, 13, 24)$ in $\mathcal{H}(5)$ and prove its uniqueness. Our knowledge of $\mathcal{H}(5)$ allows us to construct in a combinatorial way the words of certain weights in the associated code.

Generalised hexagons are at the centre of Chapter 7. Bader and Lunardon found a model for the split Cayley hexagon $H(q)$, q odd and not divisible by three. Prof. Dr. Tim Penttila suggested that we try to extend it to the case where q is divisible by three. A slight modification turns out to suffice. In addition, our model is more compatible with Lunardon's model for the twisted triality hexagon $H(q^3, q)$, q odd.

Let me mention some people and institutions I received various kinds of help from. I acknowledge the financial support of the Fund for Scientific Research – Flanders (Belgium) and of the European Community, which awarded me a Marie Curie Fellowship from January to March 2001.

I am most grateful to my supervisor, Prof. Dr. Frank De Clerck, without whom this work would simply not exist. He proposed some of the subjects for my research and showed me the way by asking the right questions. I appreciated a lot his optimism and encouraging words, as well as his advice in practical matters. Besides, he allowed me a taste of the international aspects of mathematical life. It was a privilege to be able to visit international conferences and, most of all, to go on research visits abroad.

At this point I would like to express my gratitude to Dr. Ir. Willem Haemers (Tilburg University, The Netherlands) and to Prof. Dr. Tim Penttila (University of Western Australia) for inviting me. In a limited period of time, they could guide me towards challenging problems. Discussing our ideas together led to the solution of some of them, apart from being a very pleasant way of doing mathematics.

I consider myself lucky for conducting my research in the flourishing Department of Pure Mathematics and Computer Algebra at Ghent University. Not only it is a stimulating environment from a mathematical point of view, it is also good to have some colleagues around to share the ups and downs of daily work with. I appreciate the efforts of the Department of Econometrics and Operations Research at Tilburg University and of the Department of Mathematics and Statistics at the University of Western Australia to make my stay there a fruitful and enjoyable one.

Finally, I thank my family and friends for being present in my life outside mathematics; my parents, brother and sister for their sympathy; and Thomas for the moral support and the delicious cakes.

Elisabeth Kuijken
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Contents

Chapter 1

Introduction

1.1 Incidence structures

An *incidence structure* is a triple $(\mathcal{P}, \mathcal{B}, \mathrm{I})$ of a set $\mathcal{P}$ of *points*, a set $\mathcal{B}$ of *blocks* and a symmetric *incidence relation* $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$. The sets $\mathcal{P}$ and $\mathcal{B}$ will always be finite here. The *dual* of an incidence structure $(\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the incidence structure $(\mathcal{B}, \mathcal{P}, \mathrm{I})$. An incidence structure can be represented by its *incidence matrix*. This is a matrix N of which the rows (columns) are indexed by the points (blocks); the entry n_{pL} is 1 if the point p is incident with the line L, and 0 otherwise. Clearly dualising corresponds to transposing incidence matrices. An *isomorphism* between incidence structures is a bijection between the point sets together with a bijection between the line sets such that incidence is preserved. An *automorphism* of an incidence structure is an isomorphism from the incidence structure to itself. An incidence structure is called *self-dual* if it is isomorphic to its dual; the isomorphism is a *duality* of the incidence structure. A *polarity* of an incidence structure is a duality θ such that $\theta^2 = \mathbb{1}$. An element of the incidence structure is *absolute* with respect to the polarity θ if it is incident with its image under θ.

If any two blocks of an incidence structure are incident with at most one common point, it is a *partial linear space*[1], and the blocks are usually called *lines*. Points (lines) of a partial linear space which are incident with a common line (point) are *collinear* (*concurrent*). An incident point-line pair is a *flag*, while a non-incident point-line pair is an *antiflag*. The *incidence number* of an antiflag $\{p, L\}$ is the number of points incident with the line L and collinear with the point p. If each line is incident with a constant

[1] A more correct, but non-standard name would be *partially linear space*. We will use the Dutch version of the latter in Appendix A.

number $s + 1$ of points and each point is incident with a constant number $t + 1$ of lines, the partial linear space is said to have *order* (s, t), or order s if $s = t$.

We will sometimes identify a block of an incidence structure with the set of points incident with it. Incidence will also be referred to in expressions like "a point *on* the line L" or "a line *through* the point p", which are not likely to be misunderstood.

1.2 Graphs

1.2.1 Terminology

A *graph* is a pair (V, E) of a *vertex* set V and a set E of *edges* which are unordered pairs of vertices. Only graphs on a finite vertex set will be considered here. Two distinct vertices x and y are said to be *adjacent* if the pair $\{x, y\}$ is an edge, and we write $x \sim y$. A graph with an empty edge set is *void*, and a graph in which all pairs of vertices are edges is *complete*; both are called *trivial* graphs. A vertex of a graph is *isolated* if it is adjacent to no other vertex. The *complement* $\overline{\Gamma}$ of a graph $\Gamma = (V, E)$ is the graph with vertex set V in which the edges are the unordered pairs of vertices which are not in E. The *subgraph* of a graph (V, E) *induced on* a subset W of V is the graph (W, F), where F consists of all edges of (V, E) which are contained in W. A set of mutually adjacent vertices is called a *clique*, while a set of mutually non-adjacent vertices is a *coclique*. If the vertex set of a graph can be partitioned into cocliques, it is *multipartite*; in particular the vertex set of a *bipartite* graph is the disjoint union of two cocliques. The *line graph* $L(\Gamma)$ of a graph $\Gamma = (V, E)$ has E as a vertex set, and two edges of Γ are adjacent if and only if they have a vertex of Γ in common. An *isomorphism* between graphs is a bijection between the vertex sets such that edges are mapped to edges and non-edges are mapped to non-edges. An *automorphism* of a graph is an isomorphism from the graph to itself.

Let x and y be (not necessarily distinct) vertices of a graph. A *path* of *length* $n \geq 1$ from x to y is an $(n+1)$-tuple $(x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ of vertices such that $x_i \sim x_{i+1}$ for all $i \in \{0, \ldots, n-1\}$ and $x_i \neq x_{i+2}$ for all $i \in \{0, \ldots, n-2\}$ (if $n \geq 2$). A *circuit* is a path of length at least one (hence at least three) from a vertex to itself. For completeness, a path of length zero from a vertex to itself is defined as just that vertex. The *girth* of a graph is the length of the shortest circuit. A graph is *connected* if there exists a path between any two distinct vertices, otherwise it is called *disconnected*. The *distance* between two distinct vertices is the length of the shortest path between them. The *diameter* of a connected graph is the maximal distance between two of its vertices. In a graph with diameter d, the set of vertices at distance i from a vertex x is written as $\Gamma_i(x)$, for $i \in \{0, \ldots, d\}$. Clearly $\Gamma_1(x)$ is nothing else than the set of vertices adjacent to x. A graph is *regular* of *degree* or *valency* $k > 0$, or k*-regular*, if each vertex is adjacent to k vertices.

1.2.2 Matrix and eigenvalue techniques

A convenient way to represent graphs is by their $(0,1)$ *adjacency matrix*. This is a symmetric matrix A of which the rows and corresponding columns are indexed by the vertices of the graph; the entry a_{ij} is 1 if $i \sim j$ and 0 otherwise. In the context of two-graphs and switching (see Section 1.9) the $(0,1,-1)$ *adjacency matrix* B sometimes occurs. Again the rows and corresponding columns are indexed by the vertices of the graph, but now $b_{ij} = 1$ if $i \not\sim j$, $b_{ij} = -1$ if $i \sim j$ and $b_{ij} = 0$ if $i = j$. The relation between both adjacency matrices is $B = J - I - 2A$, where J and I are the all-one matrix and the identity matrix, respectively, both of the appropriate size. Unless otherwise stated, we will stick to the $(0,1)$ adjacency matrix, however. The *eigenvalues* of a graph are the eigenvalues of its $(0,1)$ adjacency matrix. One easily sees that the $(0,1)$ adjacency matrix of a k-regular graph has the all-one vector $\underline{1}$ as an eigenvector, with eigenvalue k; it can be proved that each other eigenvalue ρ satisfies $|\rho| \leq k$ (see [5]). An eigenvalue is called *restricted* if it has an eigenvector perpendicular to $\underline{1}$.

Let $\lambda_1 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \ldots \geq \mu_m$ be two sequences of real numbers, with $n > m$. The second sequence *interlaces* the first one if $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for all $i \in \{1, \ldots, m\}$. The interlacing is said to be *tight* if there exists a $k \in \{1, \ldots, m\}$ such that $\lambda_i = \mu_i$ for $1 \leq i \leq k$ and $\lambda_{n-m+i} = \mu_i$ for $k+1 \leq i \leq m$ (if $k < m$). Interlacing can be a powerful tool when applied to eigenvalues of graphs. An overview can be found in [45]. A corollary of a more general result of Courant and Hilbert is the following.

Theorem 1.2.1 ([26]) *If C is a principal submatrix of a symmetric real matrix A, then the eigenvalues of C interlace those of A.*

This yields upper bounds for the size of cliques and cocliques in graphs.

Theorem 1.2.2 ([27]) *Let Γ denote a graph on v vertices having eigenvalues $\lambda_1 \geq \ldots \geq \lambda_v$. If C is a coclique in Γ, then $|C| \leq |\{i \mid \lambda_i \geq 0\}|$ and $|C| \leq |\{i \mid \lambda_i \leq 0\}|$. If C is a clique in Γ, then $|C| \leq |\{i \mid \lambda_i \geq -1\}|$ and $|C| \leq 1 + |\{i \mid \lambda_i \leq -1\}|$.*

Proof. Let C be a coclique. Then after an appropriate permutation of the rows and corresponding columns, the adjacency matrix A of Γ has the all-zero matrix O of size $|C|$ as a principal submatrix. The eigenvalues of O are $\mu_1 = \ldots \mu_{|C|} = 0$. By Theorem 1.2.1, $\lambda_{|C|} \geq \mu_{|C|} = 0$ and $0 = \mu_1 \geq \lambda_{v-|C|+1}$, whence the coclique bounds. Now let C be a clique; this implies that after an appropriate permutation of the rows and columns, A has the matrix $J - I$ of size $|C|$ as a principal submatrix. The eigenvalues of $J - I$ are $\mu_1 = |C| - 1$ and $\mu_2 = \ldots = \mu_{|C|} = -1$. By Theorem 1.2.1, $\lambda_{|C|} \geq \mu_{|C|} = -1$ and $-1 = \mu_2 \geq \lambda_{v-|C|+2}$. The clique bounds now easily follow. □

Let A be a real $n \times n$ matrix and let $\{X_1, \ldots, X_d\}$ be a partition of the index set $\{1, \ldots, n\}$ of the rows and columns of A. After an appropriate permutation of the rows and corresponding columns, A can be written as

$$\begin{bmatrix} A_{11} & \cdots & A_{1d} \\ \vdots & & \vdots \\ A_{d1} & \cdots & A_{dd} \end{bmatrix},$$

where A_{ij} is the submatrix of A obtained by restricting the index set of the rows to X_i and the index set of the columns to X_j. Define b_{ij} to be the average of the row sums in A_{ij}, $i, j \in \{1, \ldots, d\}$. The matrix $B = (b_{ij})_{i,j \in \{1,\ldots,d\}}$ is called the *quotient matrix* of A with respect to the partition. If all row sums in A_{ij} are equal to b_{ij} for all $i, j \in \{1, \ldots, d\}$, the partition is said to be *regular* or *equitable*. Here is another corollary of Courant and Hilbert's result.

Theorem 1.2.3 ([26]) *Let B be the quotient matrix of a real symmetric $n \times n$ matrix A with respect to a partition $\{X_1, \ldots, X_d\}$ of the index set $\{1, \ldots, n\}$ of the rows and columns. Then the eigenvalues of B interlace those of A, and the interlacing is tight if and only if the partition is regular.*

A corollary is the Hoffman bound for the size of cocliques in regular graphs.

Theorem 1.2.4 ([57]) *If Γ is a k-regular graph on v vertices with eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_v$, then the size of a coclique is at most $-v\lambda_v/(k - \lambda_v)$. If a coclique C meets this bound, then any vertex not in C is adjacent to precisely $-\lambda_v$ vertices of C.*

Proof. Consider the partition of the vertex set $\{1, \ldots, v\}$ of Γ into a coclique C and the remaining vertices. The quotient matrix of the adjacency matrix A of Γ with respect to this partition is

$$B = \begin{bmatrix} 0 & k \\ \frac{k|C|}{v-|C|} & k - \frac{k|C|}{v-|C|} \end{bmatrix}.$$

The eigenvalues of B are $\mu_1 = k$ and $\mu_2 = -k|C|/(v - |C|)$. Theorem 1.2.3 states that they interlace those of A; the inequality $\mu_2 \geq \lambda_v$ leads to the bound. If the coclique C meets this bound, then the interlacing is tight and by Theorem 1.2.3 the partition is regular. From the entry b_{21} of B one deduces that every vertex in C is adjacent to $-\lambda_v$ vertices of C. $\square$

1.2.3 Strongly regular graphs

A *strongly regular graph* $\mathrm{srg}\,(v, k, \lambda, \mu)$ is a k-regular graph on v vertices such that for any two adjacent (respectively non-adjacent) vertices there are exactly λ (respectively μ) vertices adjacent to both. The complement of an $\mathrm{srg}\,(v, k, \lambda, \mu)$ is an $\mathrm{srg}\,(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$. The strongly regular graph and its complement are connected if and only if $0 < \mu < k < v - 1$. The defining properties of a strongly regular graph $\mathrm{srg}\,(v, k, \lambda, \mu)$ can be rephrased in terms of its adjacency matrix A, yielding the matrix equation

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

or equivalently

$$A^2 - (\lambda - \mu)A + (\mu - k)I = \mu J. \tag{1.1}$$

Clearly the all-one vector $\underline{1}$ is an eigenvector of A, with eigenvalue k. If the strongly regular graph satisfies $0 < \mu < k < v-1$, then the multiplicity of the eigenvalue k is 1. As any symmetric $n \times n$ matrix gives rise to an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors, the eigenvectors corresponding to the remaining eigenvalues of A are all perpendicular to $\underline{1}$. Equation (1.1) implies that A has precisely two restricted eigenvalues which will be called r and l here[2], with $r > l$. Their multiplicities are called f and g, respectively. One calculates r and l from Equation (1.1):

$$\begin{cases} r + l = \lambda - \mu \\ rl = \mu - k, \end{cases} \tag{1.2}$$

and f and g can be deduced using the fact that all diagonal entries of A are zero:

$$\begin{cases} f + g + 1 = v \\ k + rf + lg = 0. \end{cases} \tag{1.3}$$

The following theorem gives a converse.

[2] The author is aware that the use of l, instead of s, as one of the restricted eigenvalues may seem non-standard to some readers. The idea is to avoid confusion with the parameter s of a (semi)partial geometry (Section 1.3) or a generalised polygon (Section 1.5).

Theorem 1.2.5 *A regular graph* Γ *with exactly three eigenvalues, of which two restricted, is strongly regular.*

Proof. Let v be the number of vertices of Γ, k its valency, r and l its restricted eigenvalues, and A its adjacency matrix. Any vector x of length v can be written as $x = \alpha\underline{1} + \beta u + \gamma w$, where $\alpha, \beta, \gamma \in \mathbb{R}$ and u (respectively w) is an eigenvector of A corresponding to the eigenvalue r (respectively l). Hence $(A^2 - (r+l)A + rlI)x = \alpha(k^2 - (r+l)k + rl)\underline{1}$. On the other hand $(1/v)Jx = \alpha\underline{1}$. Consequently $(A^2-(r+l)A+rlI)x = ((k^2-(r+l)k+rl)/v)Jx$ for all vectors x of length v, which implies

$$A^2 - (r+l)A + rlI = ((k^2 - (r+l)k + rl)/v)J.$$

Since this is an equality of the form of (1.1), Γ is strongly regular. $\square$

Several restrictions on the parameters of strongly regular graphs are known, see for instance [9] or [11]. If one counts in two ways the number of triples of vertices containing two edges in an srg (v, k, λ, μ), one learns

$$(v - k - 1)\mu = k(k - 1 - \lambda). \tag{1.4}$$

Let us also mention the following well-known result, which can for instance be found in [9].

Theorem 1.2.6 *The restricted eigenvalues* r *and* l *of a strongly regular graph* srg (v, k, λ, μ) *are either integers, or* $r = -l - 1 = (\sqrt{v} - 1)/2$, $f = g = k = (v-1)/2$, $\mu = \lambda + 1 = (v-1)/4$.

In the latter case the graph is said to be a *conference graph* because of its relation to conference matrices. A *conference matrix* is a $v \times v$ matrix C in which the diagonal entries are zero and the off-diagonal entries are 1 or -1, and which satisfies $CC^T = (v-1)I$. For more on conference matrices we refer to [102]. A symmetric conference matrix can be seen as the $(0, 1, -1)$ adjacency matrix of a graph which appears to be switching equivalent (see Section 1.9) to a graph consisting of a conference graph and an isolated vertex. Conversely the $(0, 1, -1)$ adjacency matrix of a graph consisting of a conference graph and an isolated vertex is a symmetric conference matrix. If a $v \times v$ conference matrix exists, then $v \equiv 2 \pmod 4$ and $v - 1$ is the sum of two squares (see [3]).

A clique bound for strongly regular graphs follows from Theorem 1.2.3.

Theorem 1.2.7 ([57]) *Let* Γ *be a strongly regular graph* srg (v, k, λ, μ) *with restricted eigenvalues* r *and* l*. Then a clique has at most* $v(r+1)/(v-k+r)$ *vertices. If a clique* C *meets this bound, then every vertex not in* C *is adjacent to exactly* $(r+1)(k-r)/(v-k+r)$ *vertices in* C*.*

Proof. If A is the adjacency matrix of Γ, then the complement $\overline{\Gamma}$ of Γ has adjacency matrix $\overline{A} = J - I - A$ and hence has eigenvalues $\overline{k} = v - k - 1$ (the valency), $\overline{r} = -l - 1$ and $\overline{l} = -r - 1$. As a clique in Γ is a coclique in $\overline{\Gamma}$, Theorem 1.2.3 yields the clique bound for Γ stated above. Now let C be a clique in Γ (so a coclique in $\overline{\Gamma}$) which meets this bound. In $\overline{\Gamma}$, every vertex not in C is adjacent to precisely $r + 1$ vertices of C, and therefore non-adjacent to $v(r + 1)/(v - k + r) - (r + 1) = (r + 1)(k - r)/(v - k + r)$ vertices of C. The claim follows by taking complements again. □

Lots of examples of strongly regular graphs are known, infinite classes as well as sporadic ones. They are often associated with interesting incidence structures, as in Chapter 3. Some will turn up as descendants of the doubly transitive two-graphs in Subsection 1.9.3. In Chapter 5 we will use some sporadic strongly regular graphs.

1.2.4 Graphs associated with partial linear spaces

Let $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. The *point graph* (respectively *block graph*) of $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ has $\mathcal{P}$ (respectively $\mathcal{L}$) as a vertex set, and adjacency is collinearity (respectively concurrency). We use "block graph" instead of "line graph" to avoid confusion with the notion of the line graph of a graph (see Subsection 1.2.1). The *incidence graph* of $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ has $\mathcal{P} \cup \mathcal{L}$ as a vertex set, adjacency being incidence; it is obviously bipartite.

Let N be the incidence matrix of a partial linear space having order (s, t). Then the adjacency matrix of its point graph is $A = NN^T - (t + 1)I$ and the adjacency matrix of its block graph is $C = N^T N - (s + 1)I$. As NN^T and $N^T N$ have the same non-zero eigenvalues, it is possible to determine the eigenvalues of A from those of C or conversely.

1.3 (Semi)partial geometries

Partial geometries were introduced by Bose in the paper [7]. A partial geometry $\mathrm{pg}(s, t, \alpha)$ is an incidence structure $\mathcal{S}$ satisfying the following axioms.

pg1 $\mathcal{S}$ is a partial linear space of order (s, t).

pg2 Each antiflag of $\mathcal{S}$ has incidence number α.

Here $s, t \geq 1$ and $1 \leq \alpha \leq \min\{s + 1, t + 1\}$. The dual of a partial geometry $\mathrm{pg}(s, t, \alpha)$ is a partial geometry $\mathrm{pg}(t, s, \alpha)$. Partial geometries with $\alpha = 1$ are known as *generalised quadrangles*; we will discuss them in Section 1.5.

Partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$ are 2-$(v, s+1, 1)$ *designs* (see Section 1.8) and their duals. If a partial geometry has $\alpha = t$, it is called a (*Bruck*) *net* of *order* $s+1$ and *degree* $t+1$ (see [13]). Its dual is called a *dual* (*Bruck*) *net*. Finally partial geometries with $1 < \alpha < \min\{s,t\}$ are said to be *proper*.

The point graph of a partial geometry $\mathrm{pg}\,(s,t,\alpha)$ with $\alpha \leq s$ is a strongly regular graph

$$\mathrm{srg}\,((s+1)(st/\alpha+1), (t+1)s, s-1+t(\alpha-1), (t+1)\alpha)$$

with restricted eigenvalues $r = s-\alpha$ and $l = -t-1$. The requirement $\alpha \leq s$ is needed to exclude partial geometries in which any two points are collinear and which therefore have a complete point graph. Note that a line of a partial geometry, seen as the set of points incident with it, is (by Theorem 1.2.7) a maximal clique in the point graph. A strongly regular graph which has the above parameters, for some $s, t, \alpha \in \mathbb{N}$, $s, t \geq 1$, $1 \leq \alpha \leq \min\{s, t+1\}$, is called *pseudo-geometric*; if it is really the point graph of a partial geometry it is *geometric*. One can show that the parameters of a geometric graph uniquely determine the parameters of the partial geometry it supports.

Semipartial geometries were introduced by Debroey and Thas [29]. A *semipartial geometry* $\mathrm{spg}\,(s,t,\alpha,\mu)$ is an incidence structure $\mathcal{S}$ for which the following axioms hold.

spg1 $\mathcal{S}$ is a partial linear space of order (s,t).

spg2 Each antiflag of $\mathcal{S}$ has incidence number 0 or α.

spg3 For any two non-collinear points of $\mathcal{S}$ there are μ points collinear with both.

Here $s,t \geq 1$, $1 \leq \alpha \leq \min\{s+1, t+1\}$ and $1 \leq \mu \leq (t+1)\alpha$. We have $\mu = (t+1)\alpha$ if and only if the semipartial geometry is a partial geometry; otherwise it is called *proper*. A semipartial geometry with $\alpha = s+1$ is a 2-$(v, s+1, 1)$ design. Semipartial geometries with $\alpha = 1$ are known as *partial quadrangles*; they were introduced by Cameron [19]. We see that semipartial geometries generalise both the partial geometries and the partial quadrangles. The dual of a semipartial geometry is again a semipartial geometry if and only if $s = t$ or it is a partial geometry [29].

The point graph of a semipartial geometry $\mathrm{spg}\,(s,t,\alpha,\mu)$ with $\alpha \leq s$ is a strongly regular graph

$$\mathrm{srg}\,(1+(t+1)s(\mu+t(s+1-\alpha))/\mu, (t+1)s, s-1+t(\alpha-1), \mu)\,.$$

A strongly regular graph having such parameters, for some $s, t, \alpha, \mu \in \mathbb{N}$ with $s, t \geq 1$, $1 \leq \alpha \leq \min\{s, t+1\}$, $1 \leq \mu < (t+1)\alpha$, is called *pseudo-semigeometric*; if it is really the point graph of a semipartial geometry it is *semigeometric*. A line of a proper semipartial geometry, seen as the set of points incident with it, does not meet the clique bound in Theorem 1.2.7 for the point graph, since some points not on this line are collinear with none of the points on it. Also, the parameters of a semigeometric graph do not determine the parameters of the semipartial geometry in general.

For more on (semi)partial geometries, including a list of known examples, we refer to [31] and [33].

1.4 Polar spaces of rank at least 3

Polar spaces were introduced by Veldkamp [101] and have been thoroughly studied, see for instance [17], [21] or [55]. They are at the base of many interesting incidence structures which are often the canonical examples for a more general class.

A *finite polar space of rank* $n \geq 3$ is a finite set $\mathcal{P}$ of *points* together with a set of *subspaces* which are subsets of $\mathcal{P}$, such that the following axioms are satisfied.

PS1 A subspace together with the subspaces contained in it is isomorphic to a projective space $\mathrm{PG}(d, q)$, $-1 \leq d \leq n-1$, q a prime power. Such a subspace is said to have *dimension* d.

PS2 The intersection of two subspaces is a subspace.

PS3 If π is a subspace of dimension $n-1$ and $p \in \mathcal{P} \setminus \pi$, then there is a unique subspace π' of dimension $n-1$ which contains p and intersects π in an $(n-2)$-dimensional subspace. The points of $\pi \cap \pi'$ are exactly the points r of π such that there exists a one-dimensional subspace containing p and r.

PS4 There exist disjoint $(n-1)$-dimensional subspaces.

This definition is cited from [55]. As we will not consider infinite polar spaces, we will omit the adjective “finite” from now on. Distinct points x and y are said to be *collinear* if there exists a one-dimensional subspace containing x and y. The subspaces of maximal dimension $n-1$ are also called *generators*. An *isomorphism* between polar spaces of rank at least 3 is a bijection between the point sets which maps subspaces to subspaces. An *automorphism* of a polar space of rank at least 3 is an isomorphism from the polar space to itself.

Examples of polar spaces of rank at least 3 arise from quadrics, Hermitian varieties and symplectic polarities; for an extensive treatment of these objects we refer to [55].

- For $n \geq 3$ and q a prime power, consider a non-degenerate quadric $Q^+(2n-1, q)$, $Q(2n, q)$ or $Q^-(2n+1, q)$. The set of points and projective subspaces on the quadric is a polar space of rank n.
- For $n \geq 3$ and q a prime power, consider a non-degenerate Hermitian variety $H(2n-1, q^2)$ or $H(2n, q^2)$. The set of points and projective subspaces on the Hermitian variety is a polar space of rank n.
- Let π be a symplectic polarity of $\mathrm{PG}(2n-1, q)$, $n \geq 3$, q a prime power. Then the set of points of $\mathrm{PG}(2n-1, q)$ together with the set of totally isotropic subspaces of $\mathrm{PG}(2n-1, q)$ with respect to π is a polar space of rank n, usually written as $W(2n-1, q)$.

By a theorem of Tits [97] these are (up to isomorphism) the only polar spaces of rank at least 3. If q is even and $n \geq 3$, the polar spaces $Q(2n, q)$ and $W(2n-1, q)$ are isomorphic. The number of points on each of the polar spaces is

$$\begin{aligned}
|Q^+(2n-1, q)| &= (q^{n-1}+1)(q^n-1)/(q-1),\\
|Q(2n, q)| &= (q^{2n}-1)/(q-1),\\
|Q^-(2n+1, q)| &= (q^n-1)(q^{n+1}+1)/(q-1),\\
|H(2n-1, q^2)| &= (q^{2n}-1)(q^{2n-1}+1)/(q^2-1),\\
|H(2n, q^2)| &= (q^{2n+1}+1)(q^{2n}-1)/(q^2-1),\\
|W(2n-1, q)| &= (q^{2n}-1)/(q-1).
\end{aligned}$$

1.5 Generalised polygons

1.5.1 Definitions and basic results

As no infinite incidence structures appear in this thesis, the word "finite" will mostly be omitted in the following definitions, although generalised polygons are also defined in the infinite case. The standard reference is [100].

An *ordinary k-gon*, $k \geq 3$, is the incidence structure of vertices and edges of a graph consisting of one circuit of length k. An *ordinary 2-gon* is an incidence structure with two points and two blocks in which each point is incident with each block. A *generalised n-gon*, $n \geq 3$, is an incidence structure which

GP1 contains no ordinary k-gons for $2 \le k < n$;

GP2 contains an ordinary n-gon or *apartment* through any two of its elements (points or lines).

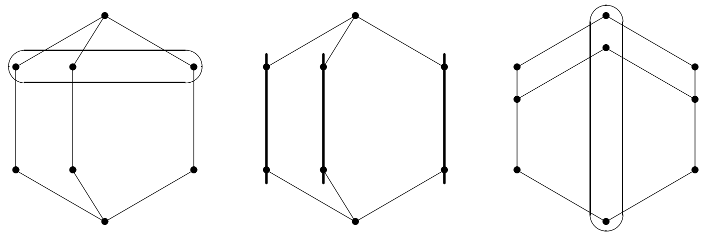
A generalised n-gon is *thick*, i.e. every point (line) is incident with at least three lines (points), if and only if it contains an ordinary $(n+1)$-gon [100]. A generalised n-gon can also be seen as a partial linear space of which the incidence graph has diameter n and girth $2n$. Clearly the dual of a generalised n-gon is again a generalised n-gon. Feit and Higman determined all possible n for which thick examples can be found.

Theorem 1.5.1 ([38]) *Finite thick generalised n-gons, with $n \ge 3$, exist only for $n \in \{3, 4, 6, 8\}$.*

A finite thick generalised n-gon always has an *order* (s, t) for certain $s, t > 1$; if $s = t$ it is also said to have order s. The number of points and lines are given in the following table.

n	number of points	number of lines
3	s^2+s+1	s^2+s+1
4	$(1+s)(1+st)$	$(1+t)(1+st)$
6	$(1+s)(1+st+s^2t^2)$	$(1+t)(1+st+s^2t^2)$
8	$(1+s)(1+st)(1+s^2t^2)$	$(1+t)(1+st)(1+s^2t^2)$

Generalised 3-gons are projective planes; we refer to [59] for more details. The only known restriction on the order s, apart from the fact that no projective plane of order 10 exists (see [65]), comes from the Bruck–Ryser theorem in [14] which states that if s is congruent to 1 or 2 modulo 4, then s is the sum of two perfect squares. Generalised 4-gons, or *generalised quadrangles*, are extensively treated in [75]; they are partial geometries $\mathrm{pg}(s, t, 1)$ (see Section 1.3). They are sometimes referred to as polar spaces of rank 2. Quite some examples are known, a few of which will be discussed in Subsection 1.5.2. In [51] it is proved that for a generalised quadrangle of order (s, t) the inequalities $s \le t^2$ and dually $t \le s^2$ hold. A different proof of this result, using a method called the variance trick, can be found in [19]. For generalised 6-gons, or *generalised hexagons*, of order (s, t) there is a similar result in [49]: $s \le t^3$ and dually $t \le s^3$. The known finite generalised hexagons, two infinite classes and their duals, will be treated in Subsection 1.5.3. Finally a generalised 8-gon, or *generalised octagon*, of order (s, t) satisfies $s \le t^2$ and dually $t \le s^2$ (see [51]). Up to duality only one class of finite generalised octagons is known.



distance-2-trace line regulus point regulus

Figure 1.1: Some configurations in a generalised hexagon.

Some more concepts and results on generalised polygons are needed. Terminology referring to the incidence graph will often be used, especially the notions of path and distance and the notation $\Gamma_i(x)$, which are explained in Subsection 1.2.1. Let Γ be a generalised n-gon, $n \geq 3$. Two elements (points or lines) are said to be *opposite* if they are at maximal distance n. Let x and y be non-opposite elements; then there is a unique shortest path $(x = x_0, x_1, \ldots, x_{k-1}, x_k = y)$ of a certain length $k < n$ between them. The *projection* $\mathrm{proj}_x(y)$ of y onto x is defined to be the element x_1 of the path closest to x. For two opposite elements x and y, and a natural number i with $2 \leq i \leq n/2$, the set $\Gamma_i(x) \cap \Gamma_{n-i}(y)$ is called a *distance-i-trace with respect to x*. In Figure 1.1 a distance-2-trace in a generalised hexagon is drawn. An element x is *distance-i-regular* if distinct distance-i-traces with respect to x intersect in at most one element. If an element is distance-i-regular for all i such that $2 \leq i \leq n/2$, it is called *regular*. A generalised n-gon in which all points, respectively lines, are (distance-i-)regular is said to be point-, respectively line-(distance-i-)regular. Now consider a generalised hexagon with a distance-3-regular point p. Any pair $\{L, M\}$ of opposite lines which are both at distance 3 from p uniquely defines a distance-3-trace with respect to p which is called the *line regulus* $\mathcal{R}(L, M)$ determined by L and M. The set of points at distance 3 from all lines of $\mathcal{R}(L, M)$ is known as a *point regulus*; it clearly contains p and we will write $\mathcal{P}(p, q)$, where q is one of its elements different from p. Figure 1.1 clarifies the construction of line and point reguli. The reguli $\mathcal{R}(L, M)$ and $\mathcal{P}(p, q)$ are said to be *complementary*, and they determine each other completely. Also note that a regulus is fully determined by any two of its elements. One can verify that point-distance-3-regularity and line-distance-3-regularity in generalised hexagons are equivalent. By a result of Ronan [77], a point-distance-2-regular generalised hexagon is also point- and line-distance-3-regular.

1.5.2 Some generalised quadrangles

The *classical* generalised quadrangles arise from quadrics, Hermitian varieties and symplectic polarities in a similar way as the polar spaces of rank at least 3, which were described in Section 1.4. The points and lines on a quadric $Q^+(3,q)$, $Q(4,q)$ or $Q^-(5,q)$, q a prime power, form a generalised quadrangle of order $(q,1)$, q and (q,q^2), respectively. The points and lines on a Hermitian variety $H(3,q^2)$ or $H(4,q^2)$, q a prime power, form a generalised quadrangle of order (q^2,q), respectively (q^2,q^3). Finally the points of $\mathrm{PG}(3,q)$, q a prime power, and the lines which are totally isotropic with respect to a symplectic polarity of $\mathrm{PG}(3,q)$ form a generalised quadrangle of order q which is written as $W(3,q)$ or $W(q)$. There exist isomorphisms between certain (dual) classical generalised quadrangles: $Q(4,q)$ is isomorphic to the dual of $W(q)$, $Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$, and $W(q)$ is self-dual if and only if q is even. For the proofs of these well-known facts we refer to [75].

For two points x and y at distance four in the incidence graph of a generalised quadrangle of order (s,t), the distance-2-trace $\Gamma_2(x) \cap \Gamma_2(y)$ is simply called *trace*. It is usually written as $\{x,y\}^\perp$ and has $t+1$ elements. The *span* or *hyperbolic line* $\{x,y\}^{\perp\perp}$ determined by x and y is the set of points at distance two from all elements of the trace $\{x,y\}^\perp$. The point x is said to be *regular* if the hyperbolic line $\{x,y\}^{\perp\perp}$ has $t+1$ elements for all points y at distance four from x. Similarly one defines regular lines. It is not difficult to show that this definition of regularity is equivalent with the more general definition of (distance-2-)regularity given in Subsection 1.5.1.

Payne [74] found a construction for a non-classical generalised quadrangle $\mathcal{P}(\mathcal{S},p)$ of order $(s-1,s+1)$ from a generalised quadrangle $\mathcal{S}$ of order $s>1$ with a regular point p. The points of $\mathcal{P}(\mathcal{S},p)$ are the points of $\mathcal{S}$ different from and not collinear with p; the lines of $\mathcal{P}(\mathcal{S},p)$ are the lines of $\mathcal{S}$ not incident with p and the hyperbolic lines $\{p,x\}^{\perp\perp}$, x a point of $\mathcal{S}$ at distance four from p. Incidence is as in $\mathcal{S}$, or (reverse) containment. The generalised quadrangles arising from this construction are the only thick examples where s and t are not powers of the same prime. Regular elements do occur in the classical generalised quadrangles of order s: all points of $W(q)$ are regular, and so are all lines if q is even.

Generalised quadrangles of order 2, $(2,4)$, 4, $(3,5)$ or $(3,9)$ are determined up to isomorphism by their order, and hence are isomorphic to $W(2)$, $Q^-(5,2)$, $W(4)$, $\mathcal{P}(W(4),p)$ (p may be any point of $W(4)$) and $Q^-(5,3)$, respectively. Of course their duals are also determined up to isomorphism by their order. Any generalised quadrangle of order 3 is isomorphic to either $W(3)$ or its dual $Q(4,3)$.

1.5.3 The known generalised hexagons

The *classical* generalised hexagons live on quadrics. Let q be a prime power and consider the hyperbolic quadric $Q^+(7,q)$ in $\mathrm{PG}(7,q)$. The generators of $Q^+(7,q)$ are three-dimensional subspaces. The set of generators can be partitioned into two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that generators belong to the same set if and only if they intersect in a subspace of odd dimension (the empty set, a line or a three-dimensional subspace). Let $\mathcal{P}_0$ be the set of points of $Q^+(7,q)$, and call the elements of $\mathcal{P}_i$ the *i-points*, $i \in \{0,1,2\}$; let $\mathcal{L}$ denote the set of lines on $Q^+(7,q)$. A counting argument yields $|\mathcal{P}_0| = |\mathcal{P}_1| = |\mathcal{P}_2|$. Incidence between a 0-point and an i-point, $i \in \{1,2\}$, or between a line and an i-point, $i \in \{0,1,2\}$, is defined as (reverse) containment, while a 1-point and a 2-point are said to be incident if and only if they intersect in a plane. A *triality* of $Q^+(7,q)$ is a bijection θ which maps $\mathcal{P}_0$ to $\mathcal{P}_1$, $\mathcal{P}_1$ to $\mathcal{P}_2$, $\mathcal{P}_2$ to $\mathcal{P}_0$ and $\mathcal{L}$ to $\mathcal{L}$, preserves incidence, and satisfies $\theta^3 = \mathbb{1}$. An *absolute i-point* is an i-point p which is incident with its image p^θ under the triality θ, $i \in \{0,1,2\}$; an *absolute line* is a line which is fixed by θ. For certain trialities the incidence structure formed by the absolute i-points, for an $i \in \{0,1,2\}$, and the absolute lines, with incidence as explained above, is a generalised hexagon [95]. The choice of i does not matter; the same generalised hexagon (up to isomorphism) is obtained. In [100, Section 2.4], the explicit description of such a triality is given; it involves an automorphism σ of $\mathrm{GF}(q)$ with $\sigma^3 = \mathbb{1}$. The order of the generalised hexagon arising from the triality is (q,q'), where q' is the order of the fixed field $\{x \in \mathrm{GF}(q) \mid x^\sigma = x\}$ of σ. Replacing σ by its inverse σ^{-1} yields an isomorphic generalised hexagon.

By choosing $\sigma = \mathbb{1}$, one finds that a generalised hexagon of order q exists for any prime power q; it is called the *split Cayley hexagon* $H(q)$. Tits [95] proved that all points and lines of $H(q)$ lie in a hyperplane of $\mathrm{PG}(7,q)$ which intersects $Q^+(7,q)$ in a non-degenerate quadric $Q(6,q)$, and gave the following description. Let $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$ be the equation of $Q(6,q)$. The points of $H(q)$ are all points of $Q(6,q)$, and the lines of $H(q)$ are the lines on $Q(6,q)$ of which the Grassmann coordinates (see [55]) satisfy the six linear equations

$$p_{12} = p_{34},\ p_{20} = p_{35},\ p_{01} = p_{36},\ p_{03} = p_{56},\ p_{13} = p_{64},\ p_{23} = p_{45}.$$

Now let σ be the automorphism $x \mapsto x^q$ of $\mathrm{GF}(q^3)$, where q is any prime power. The corresponding triality of $Q^+(7,q^3)$ yields a generalised hexagon $H(q^3,q)$ of order (q^3,q) called the *twisted triality hexagon*. By restricting the coordinates to $\mathrm{GF}(q)$, which is precisely the fixed field of σ, one shows that $H(q)$ is a subhexagon of $H(q^3,q)$. Up to duality, $H(q)$ and $H(q^3,q)$ are the only known finite thick generalised hexagons. In [23] it is proved that any generalised hexagon of order 2 is isomorphic to $H(2)$ or to its dual, and that $H(8,2)$ is, up to isomorphism, the only generalised hexagon of order $(8,2)$. The split Cayley hexagon $H(q)$ is self-dual if and only if $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$, and admits a polarity if and only if $q = 3^{2h+1}$, $h \in \mathbb{N}$. In $H(q)$ as well as in $H(q^3, q)$, points are opposite if and only if they are not collinear on the underlying quadric, and the points at distance 2 from a point p are all contained in a plane through p on the quadric. Both $H(q)$ and $H(q^3, q)$ are point-distance-2-regular and hence point- and line-distance-3-regular [77]. The lines of $H(q)$ are distance-2-regular if and only if $H(q)$ is self-dual; no line of $H(q^3, q)$ is distance-2-regular. A line regulus in $H(q)$ is actually a *regulus*, i.e. one of the two sets of $q+1$ mutually disjoint lines on a $Q^+(3,q)$ which is the intersection of the underlying $Q(6,q)$ with a three-dimensional subspace. A point regulus in $H(q)$ is a non-degenerate conic $Q(2,q)$ which is the intersection of $Q(6,q)$ with a plane.

1.5.4 Coordinatisation

Coordinatisation is a process of labelling the elements of a generalised polygon. The very explicit viewpoint obtained by this approach can be a welcome supplement to more abstract descriptions as the ones in Subsection 1.5.3. On the other hand, working solely with coordinates one may get lost in enormous calculations and needs geometric interpretations to clarify the situation. To us, coordinatisation will prove useful in Chapter 7. The first generalised polygons to be coordinatised were the projective planes, see for instance [59]. General coordinatisation theory for finite or infinite generalised polygons is explained in [100]. We will only treat coordinatisation of finite generalised hexagons, as this is the only case we need. Our description is based on [35].

Consider a finite generalised hexagon of order (s,t), and let R_1 and R_2 be sets of cardinality s and t, respectively, both containing two distinct elements 0 and 1. The set R_1 will be used to label the points on a line except for one, and similarly the lines through a point except for one will be labelled by the set R_2. The starting point of the coordinatisation is an apartment $\mathcal{A}$ containing a flag $\{(\infty), [\infty]\}$ which will play a special role. Points different from (∞) are labelled by i-tuples between round brackets, and lines different from $[\infty]$ are labelled by i-tuples between square brackets. Here the number $i \in \{1,2,3,4,5\}$ is determined by the distance to the special flag $\{(\infty), [\infty]\}$: if x is an element different from (∞) and $[\infty]$, and i is the minimum of the distances from x to (∞) and $[\infty]$, then x is represented by an i-tuple. In Figure 1.2 one sees how the elements of the apartment $\mathcal{A}$ are labelled. Now the remaining elements will be labelled in such a way that the incidences are as clearly visible as possible. Choose a bijection between the points

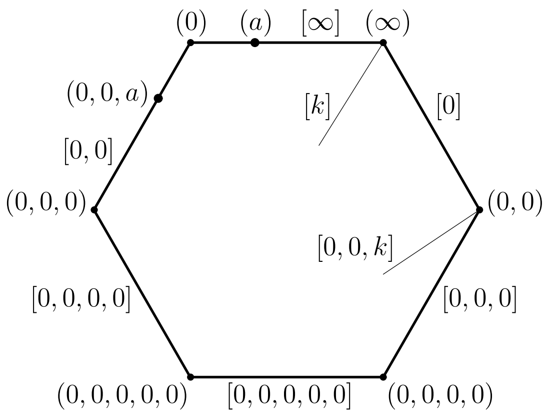


Figure 1.2: Coordinatisation.

on $[\infty]$ different from (∞) and the elements of R_1, assuring that the point (0) of $\mathcal{A}$ corresponds to the element 0 of R_1, and accordingly write these points as (a), $a \in R_1$. Similarly label the lines through (∞) different from $[\infty]$ as $[k]$, $k \in R_2$, such that the line $[0]$ of $\mathcal{A}$ indeed corresponds to the element 0 of R_2. These labellings are drawn in Figure 1.2. The unique point on the line $[0,0,0,0,0]$ closest to the point (a), $a \in R_1$, is labelled $(a,0,0,0,0)$, and dually the unique line through $(0,0,0,0,0)$ closest to the line $[k]$, $k \in R_2$, is labelled $[k,0,0,0,0]$. Let p_a denote the point on $[1,0,0,0,0]$ closest to (a), $a \in R_1$, define $(0,a)$ to be the point on $[0]$ closest to p_a, and let $(0,0,0,0,a)$ be the point on $[0,0,0,0]$ closest to $(0,a)$. Dually define $[0,k]$ and $[0,0,0,0,k]$, $k \in R_2$. Now label the points on $[0,0]$ different from (0) as $(0,0,a)$, $a \in R_1$, using any bijection which maps the element $0 \in R_1$ to the point $(0,0,0)$. Let $(0,0,0,a)$ be the point on $[0,0,0]$ closest to $(0,0,a)$. Dually define $[0,0,k]$ and $[0,0,0,k]$, $k \in R_2$. The point $(0,0,a)$ and the line $[0,0,k]$ can be found in Figure 1.2. Thus a label has been given to all elements incident with an element of $\mathcal{A}$; this allows us to label all other elements as well. The point on $[k]$ closest to $(0,0,0,0,b)$ is (k,b), $k \in R_2$, $b \in R_1$; dually $[a,l]$ is defined, for $a \in R_1$ and $l \in R_2$. The point on $[a,l]$ closest to $(0,0,0,a')$ is (a,l,a'), $a,a' \in R_1$, $l \in R_2$, and dually we find the lines $[k,b,k']$, $k,k' \in R_2$, $b \in R_1$. Let (k,b,k',b') denote the point on $[k,b,k']$ closest to $(0,0,b')$, $k,k' \in R_2$, $b,b' \in R_1$, and dually define $[a,l,a',l']$, $a,a' \in R_1$, $l,l' \in R_2$. Finally the point on $[a,l,a',l']$ closest to $(0,a'')$ is labelled as (a,l,a',l',a''), $a,a',a'' \in R_1$, $l,l' \in R_2$; dually the lines $[k,b,k',b',k'']$ are obtained, $k,k',k'' \in R_2$, $b,b' \in R_1$. By definition (∞) is incident with $[\infty]$ and all lines $[k]$, $k \in R_2$, and dually $[\infty]$ is incident with (∞) and all points

(a), $a \in R_1$. Many of the other incidences are directly visible: two elements which are different from (∞) and $[\infty]$ and are not both 5-tuples are incident if and only if deleting the last coordinate of one of them and changing the kind of brackets yields the other one. The remaining incidences are somewhat more difficult to express. Define

$$\begin{aligned}
&S_1(k,a,l,a',l',a'') = b \\
&\quad \Leftrightarrow d((k,b),(a,l,a',l',a'')) = 4, \\
&S_2'(k,a,l,a',l',a'') = k' \\
&\quad \Leftrightarrow d([k,S_1(k,a,l,a',l',a''),k'],(a,l,a',l',a'')) = 3, \\
&S_2(a,k,b,k',b',k'') = l \\
&\quad \Leftrightarrow d([a,l],[k,b,k',b',k'']) = 4, \\
&S_1'(a,k,b,k',b',k'') = a' \\
&\quad \Leftrightarrow d((a,S_2(a,k,b,k',b',k''),a'),[k,b,k',b',k'']) = 3,
\end{aligned}$$

where $d(x,y)$ denotes the distance between x and y, and $a,a',a'',b,b' \in R_1$ and $k,k',k'',l,l' \in R_2$. One verifies that a necessary and sufficient condition for (a,l,a',l',a'') and $[k,b,k',b',k'']$ to be incident is

$$\begin{cases}
S_1(k,a,l,a',l',a'') = b \\
S_2'(k,a,l,a',l',a'') = k' \\
S_2(a,k,b,k',b',k'') = l \\
S_1'(a,k,b,k',b',k'') = a',
\end{cases}$$

for any $a,a',a'',b,b' \in R_1$ and $k,k',k'',l,l' \in R_2$. This completely determines the generalised hexagon.

To obtain a coordinatisation for the split Cayley hexagon $H(q)$, q a prime power, we take $R_1 = R_2 := \mathrm{GF}(q)$. Assume that the underlying quadric $Q(6,q)$ is described by $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$, and choose the points of the apartment $\mathcal{A}$ in the following way:

$$\begin{aligned}
(\infty) &:= (1,0,0,0,0,0,0), \\
(0) &:= (0,0,0,0,0,0,1), \\
(0,0) &:= (0,0,0,0,0,1,0), \\
(0,0,0) &:= (0,1,0,0,0,0,0), \\
(0,0,0,0) &:= (0,0,1,0,0,0,0), \\
(0,0,0,0,0) &:= (0,0,0,0,1,0,0);
\end{aligned}$$

the lines of $\mathcal{A}$ readily follow from Figure 1.2. The remaining choices that

POINTS	
label	PG$(6,q)$ coordinates
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$
LINES	
label	PG$(6,q)$ coordinates
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 1.1: Coordinatisation of $H(q)$.

have to be made are

$$\begin{aligned}(a) &:= (a,0,0,0,0,0,1),\ a\in \mathrm{GF}(q),\\ [k] &:= \langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle,\ k\in\mathrm{GF}(q),\\ (0,0,a') &:= (0,1,0,0,0,0,-a'),\ a'\in\mathrm{GF}(q),\\ [0,0,k'] &:= \langle(0,0,0,0,0,1,0),(k',0,1,0,0,0,0)\rangle, k'\in\mathrm{GF}(q).\end{aligned}$$

Now all points and lines of $H(q)$ can be assigned labels; the result is found in Table 1.1. The conditions for (a,l,a',l',a'') to be incident with $[k,b,k',b',k'']$ are

$$\begin{cases} a''=ak+b\\ a'=a^2k+b'+2ab\\ k''=ka^3+l-3a^2a''+3aa'\\ k'=k^2a^3+l'-kl-3a^2a''k-3a'a''+3aa''^2,\end{cases}$$

for $a,l,a',l',a'',k,b,k',b',k''\in\mathrm{GF}(q)$. Note that these conditions have more symmetry if q is a power of 3, i.e. in the case where $H(q)$ is self-dual.

1.6 m-Systems of polar spaces

Let $\mathcal{P}$ be a polar space of rank $n \geq 3$. A *partial m-system* of $\mathcal{P}$ is a set $\mathcal{M} = \{\pi_1, \ldots, \pi_r\}$, $r > 1$, of m-dimensional subspaces of $\mathcal{P}$ such that any generator containing an element π_i of $\mathcal{M}$ has an empty intersection with all other elements π_j of $\mathcal{M}$. Clearly $0 \leq m \leq n-1$ holds. Partial m-systems first appeared in [85]; there one also finds upper bounds for their size:

$$\begin{aligned}
\mathcal{P} = Q^+(2n-1,q) &\;\Rightarrow\; |\mathcal{M}| \leq q^{n-1}+1,\\
\mathcal{P} = Q(2n,q) &\;\Rightarrow\; |\mathcal{M}| \leq q^{n}+1,\\
\mathcal{P} = Q^-(2n+1,q) &\;\Rightarrow\; |\mathcal{M}| \leq q^{n+1}+1,\\
\mathcal{P} = H(2n-1,q^2) &\;\Rightarrow\; |\mathcal{M}| \leq q^{2n-1}+1,\\
\mathcal{P} = H(2n,q^2) &\;\Rightarrow\; |\mathcal{M}| \leq q^{2n+1}+1,\\
\mathcal{P} = W(2n-1,q) &\;\Rightarrow\; |\mathcal{M}| \leq q^{n}+1.
\end{aligned}$$

If a partial m-system meets this upper bound, it is called an *m-system*. Note that for a fixed polar space the size of an m-system does not depend on m. A (partial) 0-system is also called a (*partial*) *ovoid*; it is a set of mutually non-collinear points. A (partial) $(n-1)$-system or (*partial*) *spread* is a set of mutually disjoint generators.

1.7 Ovoids and spreads of generalised polygons

1.7.1 Definitions and general results

An *ovoid* of a generalised n-gon Γ, n even and $n \geq 4$, is a set $\mathcal{O}$ of mutually opposite points such that every element (point or line) of Γ is at distance at most $n/2$ from an element of $\mathcal{O}$. Dually a *spread* is a set $\mathcal{S}$ of mutually opposite lines such that every element of Γ is at distance at most $n/2$ from an element of $\mathcal{S}$. Note that n must be even in order for Γ to have opposite points or lines. The following theorem can be proved by counting arguments.

Theorem 1.7.1 *Let Γ be a finite generalised n-gon, n even and $n \geq 4$, of order (s,t). A set of mutually opposite points of Γ is an ovoid if and only if it has cardinality*

- $1 + st$ *for* $n = 4$,
- $\frac{(1+s)(1+st+s^2t^2)}{1+s+st}$ *for* $n = 6$,

- $1 + s^2t^2$ *for* $n = 8$.

The dual statement holds for spreads.

By a result which can for instance be found in [73], a polarity of a generalised polygon yields an ovoid and a spread.

Theorem 1.7.2 *The set of absolute points (lines) with respect to a polarity of a generalised* n*-gon,* n *even and* $n \geq 4$*, is an ovoid (a spread).*

1.7.2 An ovoid of $Q(4, q)$

The generalised quadrangle $Q(4, q)$, q any prime power, has an ovoid which is very easy to construct. Let π be a hyperplane of $\mathrm{PG}(4, q)$ which intersects the quadric $Q(4, q)$ in an elliptic quadric $Q^-(3, q)$. The points of $Q^-(3, q)$ are clearly mutually opposite in $Q(4, q)$, and $Q^-(3, q)$ counts exactly the number of points an ovoid of a generalised quadrangle of order q should have. By Theorem 1.7.1, $Q^-(3, q)$ is indeed an ovoid of $Q(4, q)$. Remark that for most values of q, $Q(4, q)$ also has ovoids which are not hyperplane sections; for an overview we refer to [76].

1.7.3 Some ovoids and spreads of $H(q)$

Recall that the split Cayley hexagon $H(q)$, q a prime power, lives on the quadric $Q(6, q)$, and that points are opposite if and only if they are not collinear on $Q(6, q)$. An ovoid of $H(q)$ is also an ovoid of the polar space $Q(6, q)$ and conversely. A spread of $H(q)$ is a 1-system of $Q(6, q)$, but the converse is not necessarily true, as not every line of $Q(6, q)$ is a line of $H(q)$. Spreads of $H(q)$ exist for any prime power q; an example will be constructed below. If q is a power of 3, a duality can be applied and an ovoid is obtained. It has been proved that $H(q)$ has no ovoids if q is even [92] or if $q \in \{5, 7\}$ [71]. Settling the existence question for all odd q which are not divisible by 3 remains an important open problem.

For any prime power q the generalised hexagon $H(q)$ contains a particular spread which is constructed as follows in [91]. Let Π be a hyperplane of $\mathrm{PG}(6, q)$ which intersects the quadric $Q(6, q)$ underlying $H(q)$ in a $Q^-(5, q)$, and let $\mathcal{S}_H$ be the set of lines of $H(q)$ which are contained in Π. No two elements of $\mathcal{S}_H$ can be at distance two or four in $H(q)$, otherwise $Q^-(5, q)$ would contain planes. A counting argument yields $|\mathcal{S}_H| = q^3 + 1$, so by Theorem 1.7.1 $\mathcal{S}_H$ is a spread of $H(q)$. The line regulus determined by two elements L and M of $\mathcal{S}_H$ consists of $q + 1$ lines lying in the three-dimensional subspace $\langle L, M \rangle \subseteq \Pi$, and hence is completely contained in

$\mathcal{S}_H$. For reasons which will become clear in Subsection 1.8.2, $\mathcal{S}_H$ is called a *Hermitian spread* of $H(q)$. Another geometric construction of $\mathcal{S}_H$ can be found in [100]; in [22] it is described group-theoretically. The *Hermitian ovoid* $\mathcal{O}_H$ arises by dualising from the Hermitian spread $\mathcal{S}_H$ of $H(q)$, q a power of 3. For future use we will calculate an explicit form for $\mathcal{S}_H$ and $\mathcal{O}_H$. Assume that the quadric $Q(6,q)$ underlying $H(q)$, q any prime power, has equation $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$, and that the lines of $H(q)$ are as described by coordinatisation in Subsection 1.5.4. Pick a non-square γ of $\mathrm{GF}(q)$; then the five-dimensional subspace Π with equation $X_5 = \gamma X_1$ intersects $Q(6,q)$ in an elliptic quadric $Q^-(5,q)$ with equation

$$\begin{cases} X_5 = \gamma X_1 \\ X_0X_4 + X_2X_6 + \gamma X_1^2 - X_3^2 = 0. \end{cases}$$

One checks that the Hermitian spread consisting of the lines of $H(q)$ which are contained in Π is

$$\mathcal{S}_H = \{[\infty]\} \cup \{[k, -\gamma^{-1}k'', k', \gamma k, k''] \mid k, k', k'' \in \mathrm{GF}(q)\}.$$

Now assume that $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$. An explicit duality θ of $H(q)$ is given in [35]:

$$\begin{array}{rcl} (\infty) & \mapsto & [\infty], \\ [\infty] & \mapsto & (\infty), \\ (a, l, a', l', a'') & \mapsto & [a^3, l, a'^3, l', a''^3], a, l, a', l', a'' \in \mathrm{GF}(q), \\ [k, b, k', b', k''] & \mapsto & (k, b^3, k', b'^3, k''), k, b, k', b', k'' \in \mathrm{GF}(q). \end{array}$$

The action of θ on the remaining elements of $H(q)$ are deduced by projecting. It follows that a Hermitian ovoid of $H(q)$ can be described by coordinatisation as

$$\mathcal{O}_H = \{(\infty)\} \cup \{(a, -\gamma^{-3}a''^3, a', \gamma^3 a^3, a'') \mid a, a', a'' \in \mathrm{GF}(q)\}.$$

Translated into $\mathrm{PG}(6,q)$ coordinates according to Table 1.1, this becomes

$$\begin{array}{rcl} \mathcal{O}_H & = & \{(1,0,0,0,0,0,0)\} \cup \\ & & \{(-\gamma^{-3}a''^4 + a'^2 - \gamma^3 a^4 + aa'a'', -a'', -a, -a' + aa'', 1, \\ & & \quad -\gamma^{-3}a''^3 - aa' - a^2a'', -\gamma^3a^3 + a'a'') \mid a, a', a'' \in \mathrm{GF}(q)\}. \end{array}$$

The split Cayley hexagon $H(q)$ admits a polarity if and only if $q = 3^{2h+1}$, $h \in \mathbb{N}$; this polarity is unique up to conjugacy with respect to the automorphism group of $H(q)$. By Theorem 1.7.2 it yields an ovoid $\mathcal{O}_R$ and a spread $\mathcal{S}_R$ of $H(q)$ which are called the *Ree–Tits ovoid* and the *Ree–Tits spread*. It is not possible to map a Ree–Tits ovoid to a Hermitian ovoid by an automorphism of $H(q)$. However, the Hermitian ovoid and the Ree–Tits ovoid of $H(3)$ can be mapped to each other by an automorphism of the polar space $Q(6,3)$. We cite explicit descriptions of the Ree–Tits ovoids and spreads from [35]:

$$\begin{aligned}\mathcal{O}_R &= \{(\infty)\} \cup \{(a, a''^s - a^{3+s}, a', a^{3+2s} + a'^s + a^s a''^s, a'') \mid \\ &\qquad a, a', a'' \in \mathrm{GF}(q)\},\\ \mathcal{S}_R &= \{[\infty]\} \cup \{[k, k''^{s/3} - k^{1+s/3}, k', k^{1+2s/3} + k'^{s/3} + k^{s/3}k''^{s/3}, k''] \mid \\ &\qquad k, k', k'' \in \mathrm{GF}(q)\},\end{aligned}$$

where $s = 3^{h+1}$.

1.8 Designs

1.8.1 Definitions

A t-(v,k,λ) *design*, for natural numbers t, v, k and λ with $v > k > 1$ and $k \geq t \geq 1$, is an incidence structure $\mathcal{S}$ satisfying the following axioms.

D1 $\mathcal{S}$ has v points.

D2 Each block of $\mathcal{S}$ is incident with k points.

D3 For any set T of t points there are precisely λ blocks which are incident with all points in T.

We refer to [4] or [60] for more information on designs. Many interesting examples are known; for instance, a projective plane of order n can be seen as a 2-$(n^2+n+1, n+1, 1)$ design. A 1-design is just an incidence structure in which the number of points incident with a block and the number of blocks incident with a point are constants. The 2-designs are also called *block designs*. For a 2-(v,k,λ) design the number r of blocks incident with a point is found by an easy counting argument: $r = \lambda(v-1)/(k-1)$. This implies that the number of blocks in such a design is $b := \lambda v(v-1)/(k(k-1))$. A 2-$(v,k,\lambda)$ design is said to be *square* or *symmetric* if $b = v$, or equivalently $r = k$; in this case any two blocks are incident with λ common points.

Let p be a point of a t-(v,k,λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ with $t > 1$ and $k > 2$. Define $\mathcal{P}_p := \mathcal{P} \setminus \{p\}$ and $\mathcal{B}_p := \{B \in \mathcal{B} \mid (p,B) \in \mathrm{I}\}$, and let I_p denote the restriction of I to $(\mathcal{P}_p \times \mathcal{B}_p) \cup (\mathcal{B}_p \times \mathcal{P}_p)$. Then the design $\mathcal{D}_p := (\mathcal{P}_p, \mathcal{B}_p, \mathrm{I}_p)$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design. It is called the *derived design* of $\mathcal{D}$ with respect to the point p, and $\mathcal{D}$ is said to be a 1-*point extension* of $\mathcal{D}_p$. If $\mathcal{D}$

is a 2-design, then the derived design $\mathcal{D}_p$ of $\mathcal{D}$ with respect to a point p is a partial linear space if and only if no two blocks of $\mathcal{D}$ incident with p are incident with more than two common points.

1.8.2 Unitals

A *unital* is a 2-$(n^3+1, n+1, 1)$ design, for some natural number n. The standard example, the *classical* or *Hermitian unital*, consists of the points and $(q+1)$-secants of a Hermitian curve $H(2, q^2)$ in $\mathrm{PG}(2, q^2)$, q a prime power. As promised in Section 1.7.3 we will now explain why the spread $\mathcal{S}_H$ of the generalised hexagon $H(q)$ is called Hermitian. Recall that $\mathcal{S}_H$ contains the line regulus determined by any two of its elements; therefore the lines and line reguli in $\mathcal{S}_H$ form a unital, which turns out to be isomorphic to a classical unital [91]. Of course the unital formed by the points and point reguli of the Hermitian ovoid $\mathcal{O}_H$ of $H(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$, is also isomorphic to a classical unital. An explicit isomorphism will be established in Section 3.3, as we need it in a proof.

A non-classical unital is constructed from the Ree–Tits ovoid $\mathcal{O}_R$ of $H(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$. The points are the points of $\mathcal{O}_R$; a geometric description of the blocks, due to De Smet and Van Maldeghem [37], reads as follows. Let ρ be the polarity of $H(q)$ having $\mathcal{O}_R$ as a set of absolute points, and let p and t be distinct points of $\mathcal{O}_R$. Define $L := \mathrm{proj}_p(t^\rho)$ and $M := \mathrm{proj}_t(p^\rho)$; then $L^\rho = \mathrm{proj}_{p^\rho}(t)$ and $M^\rho = \mathrm{proj}_{t^\rho}(p)$. One checks that the lines L and M and the points L^ρ and M^ρ are opposite, and that the distances between L and L^ρ, between L and M^ρ, between M and M^ρ, and between M and L^ρ are all equal to 3. It follows that the line regulus $\mathcal{R}(L, M)$ determined by L and M is complementary to the point regulus $\mathcal{P}(L^\rho, M^\rho)$ determined by L^ρ and M^ρ. If N is a line belonging to $\mathcal{R}(L, M)$, then the point N^ρ belongs to $\mathcal{P}(L^\rho, M^\rho)$, implying that the distance between N and N^ρ is 3. Therefore the point $x := \mathrm{proj}_N(N^\rho)$ is incident with its image $x^\rho = \mathrm{proj}_{N^\rho}(N)$ under ρ and hence belongs to $\mathcal{O}_R$. As a consequence the set $\mathcal{B}(p, t)$ of points of $\mathcal{O}_R$ incident with a line of $\mathcal{R}(L, M)$ has $q+1$ elements. One can show that such a set $\mathcal{B}(p, t)$ is uniquely determined by any two of its elements. As a result, the Ree–Tits ovoid $\mathcal{O}_R$ together with the set of blocks $\mathcal{B}(p, t)$, $p, t \in \mathcal{O}_R$, is a unital. It is called the *Ree–Tits unital*, and it is not isomorphic to a Hermitian unital.

1.8.3 Designs containing a unital

It is possible to add blocks to Hermitian or Ree–Tits unitals in such a way that a 2-$(q^3+1, q+1, q+2)$ design is obtained.

Consider a Hermitian unital, seen as the design of points and $(q+1)$-secants of a Hermitian curve $H(2,q^2)$ in $\mathrm{PG}(2,q^2)$, q odd. Hölz [58] described a special class of Baer subplanes of $\mathrm{PG}(2,q^2)$ in the following way. A Baer subplane D satisfies condition (H) if for each point p of $D \cap H(2,q^2)$ the tangent line L_p to $H(2,q^2)$ at p contains $q+1$ points of D. A Baer subplane D satisfying condition (H) and containing at least three points of $H(2,q^2)$ intersects $H(2,q^2)$ in a set of $q+1$ points which are either collinear or form a non-degenerate conic in D. Moreover, if D and D' are Baer subplanes satisfying condition (H) which intersect $H(2,q^2)$ in a conic and for which $|D \cap D' \cap H(2,q^2)| \geq 3$ holds, then $D = D'$. It can be proved that for any two points p and t of $H(2,q^2)$ there are exactly $q+1$ Baer subplanes satisfying condition (H) which contain p and t and intersect $H(2,q^2)$ in a conic. Hence the union of the set of intersections $D \cap H(2,q^2)$, where D is a Baer subplane of $\mathrm{PG}(2,q^2)$ satisfying condition (H) and intersecting $H(2,q^2)$ in a non-degenerate conic of D, and the set of blocks of the Hermitian unital on $H(2,q^2)$ is the block set of a 2-$(q^3+1, q+1, q+2)$ design which is called the *Hölz design*. Thas [93] proved that it is a 1-point extension of the generalised quadrangle $\mathcal{P}(W(q), p)$ of order $(q-1, q+1)$, which is described in Subsection 1.5.2.

Consider a Ree–Tits unital, defined as above on the Ree–Tits ovoid $\mathcal{O}_R$ of $H(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$. For any line L of $H(q)$ incident with no point of $\mathcal{O}_R$, there are exactly $q+1$ points of $\mathcal{O}_R$ which are at distance 3 from L. These sets of $q+1$ points, when added as blocks to the Ree–Tits unital, yield a 2-$(q^3+1, q+1, q+2)$ design, which for $q = 3$ is isomorphic to the Hölz design with the same parameters. Assmus and Key [1] first described these designs in an algebraic way; the geometric construction above comes from [35]. There it is also proved that the blocks intersect in at most two points, so the derived design with respect to any point is a partial linear space. For $q = 3$ it is the unique generalised quadrangle $Q^-(5,2)$ of order $(2,4)$; in [100] one reads that for $q = 3^{2h+1}$, $h \in \mathbb{N}\setminus\{0\}$, it is not a generalised quadrangle.

1.9 Two-graphs and switching of graphs

1.9.1 Switching of graphs

Switching, sometimes also called Seidel switching after the person who introduced it in [99], is an operation which changes some edges of a graph. Let $\{V_1, V_2\}$ be a partition of the vertex set V of a graph Γ, and construct a graph Γ' from Γ by interchanging edges and non-edges between a vertex in V_1 and a vertex in V_2, while leaving (non-)edges inside V_1 or inside V_2 unchanged. This process is called *switching* with respect to the partition $\{V_1, V_2\}$ of V. The $(0, 1, -1)$ adjacency matrix B' of Γ' arises from the $(0, 1, -1)$ adjacency matrix B of Γ by multiplying some rows and the corresponding columns by -1. After an appropriate labelling of the vertices we can write

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \text{ and } B' = \begin{bmatrix} B_{11} & -B_{12} \\ -B_{21} & B_{22} \end{bmatrix},$$

where B_{ij} is the submatrix of B obtained by restricting the index set of the rows to V_i and the index set of the columns to V_j, for $i, j \in \{1, 2\}$. Graphs which can be constructed from each other by switching are called *switching equivalent*; this is indeed an equivalence relation, and the equivalence classes are known as *switching classes*. The following lemma gives another interpretation of switching equivalence.

Lemma 1.9.1 ([82]) *Two graphs Γ and Γ' which have the same vertex set are switching equivalent if and only if for any unordered triple T of vertices the parity of the number of edges of Γ in T equals the parity of the number of edges of Γ' in T.*

Proof. One easily checks that switching does not affect the parity of the number of edges in an unordered triple of vertices. On the other hand, suppose that $\Gamma = (V, E)$ and $\Gamma' = (V, E')$ are two graphs such that for any unordered triple T of vertices the parity of the number of edges of Γ in T equals the parity of the number of edges of Γ' in T. Choose a vertex ω, let V_1 be the set of vertices x such that $\{\omega, x\} \in E$ and $\{\omega, x\} \notin E'$, or $\{\omega, x\} \in E'$ and $\{\omega, x\} \notin E$, and put $V_2 := V \setminus V_1$. Any pair of vertices in V_2, whether it contains ω or not, is either an edge in Γ and in Γ', or a non-edge in both; similarly for any pair of vertices in V_1. A pair of a vertex in V_1 and a vertex in V_2, on the contrary, is either an edge in Γ and a non-edge in Γ', or conversely. Now switch Γ with respect to the partition $\{V_1, V_2\}$ of V; the graph thus constructed is nothing else than Γ'. □

The following observation opens the door for eigenvalue techniques.

Theorem 1.9.2 ([82]) *If two graphs are switching equivalent, then their $(0, 1, -1)$ adjacency matrices have the same eigenvalues with the same multiplicities.*

1.9.2 Two-graphs

A *two-graph* is a pair (V, Δ) of a finite *vertex* set and a set Δ of *coherent* (unordered) *triples* of vertices such that each set of four vertices contains an even number of coherent triples. Two-graphs were introduced by Higman; [82] and [89] are excellent surveys. A two-graph (V, Δ) without coherent triples is called *void*, and a two-graph in which every triple of vertices is coherent is *complete*; both are *trivial* two-graphs. The *complement* of a two-graph (V, Δ) is the two-graph on V in which the coherent triples are the *incoherent* triples of (V, Δ). The *sub-two-graph* of a two-graph (V, Δ) *induced on* a subset W of V is the two-graph (W, Θ), where Θ consists of all coherent triples of (V, Δ) which are contained in W. A set C of vertices in which each triple is (in)coherent is called a(n) *(in)coherent set* of the two-graph. An *isomorphism* between two-graphs is a bijection between the vertex sets mapping coherent triples to coherent triples and incoherent triples to incoherent triples. An *automorphism* of a two-graph is an isomorphism from the two-graph to itself.

The following theorem explains the connection between two-graphs and switching classes of graphs.

Theorem 1.9.3 ([88]) *A switching class of graphs uniquely determines a two-graph on the same vertex set, and conversely a two-graph uniquely leads to a unique switching class of graphs on the same vertex set.*

Proof. Consider a switching class of graphs with vertex set V, let Γ be one of its elements, and let Δ be the set of triples of vertices of Γ containing an odd number of edges. By drawing some pictures, one convinces oneself that any graph on four vertices contains an even number of induced subgraphs on three vertices with an odd number of edges; as a consequence (V, Δ) is a two-graph. Lemma 1.9.1 assures that for any other graph Γ' in the switching class the same set Δ is obtained, while graphs which are not switching equivalent give rise to non-isomorphic two-graphs.
On the other hand, let (V, Δ) be a two-graph, choose $\omega \in V$, and let $\{V_1, V_2\}$ be any partition of $V \setminus \{\omega\}$. Define a set E consisting of the following pairs of vertices: the pairs $\{\omega, x\}$, $x \in V_1$; the pairs $\{x, x'\}$ with $x, x' \in V_1$ and $\{\omega, x, x'\} \in \Delta$; the pairs $\{y, y'\}$ with $y, y' \in V_2$ and $\{\omega, y, y'\} \in \Delta$; and the pairs $\{x, y\}$ with $x \in V_1$, $y \in V_2$ and $\{\omega, x, y\} \notin \Delta$. The set of triples of vertices carrying an odd number of edges in the graph (V, E) is precisely Δ, for any choice of ω and of the partition $\{V_1, V_2\}$ of $V \setminus \{\omega\}$. By Lemma 1.9.1 all graphs thus obtained are switching equivalent. $\square$

A two-graph is *regular* if every pair of vertices is contained in the same number a of coherent triples. This number a and the number n of vertices are called the *parameters* of the regular two-graph. The complement of a regular two-graph with parameters n and a is regular with parameters n and $n-2-a$. It needs no proof that a two-graph admitting a doubly transitive group of automorphisms is regular.

Let (V, Δ) be a regular two-graph with parameters n and a, and choose $\omega \in V$. Define a graph $(V \setminus \{\omega\}, E)$ as follows: a pair $\{x, y\}$ of vertices different from ω is in the edge set E if and only if the triple $\{\omega, x, y\}$ is coherent in (V, Δ). The graph $(V \setminus \{\omega\}, E)$ is called the *descendant* of (V, Δ) with respect to the vertex ω; it is an $\mathrm{srg}\,(n-1, a, (3a-n)/2, a/2)$. On the other hand, let (V, E) be a strongly regular graph $\mathrm{srg}\,(v, k, \lambda, k/2)$, add an isolated vertex ω, and let $(V \cup \{\omega\}, \Delta)$ be the two-graph corresponding to the switching class determined by the graph $(V \cup \{\omega\}, E)$. Then $(V \cup \{\omega\}, \Delta)$ is a regular two-graph with parameters $n = v + 1$ and $a = k$. Let r and l be the restricted eigenvalues of (V, E), with multiplicities f and g, respectively. Using Equations (1.2) and (1.4), one calculates that $k = -2rl$, $\lambda = r + l - rl$ and $v = -(2r+1)(2l+1)$. Upper bounds for the size of an (in)coherent set in the regular two-graph $(V \cup \{\omega\}, \Delta)$ now follow from Theorem 1.2.4 and Theorem 1.2.7.

Theorem 1.9.4 ([89]) *Let (V, Δ) be a regular two-graph such that the descendants have restricted eigenvalues r and l. Then the size of a coherent set in (V, Δ) is at most $2r + 2$, and the size of an incoherent set in (V, Δ) is at most $-2l$.*

The following result of Gunawardena and Moorhouse gives a construction for regular two-graphs from ovoids of $Q(2m, q)$, q odd, $m \geq 2$.

Theorem 1.9.5 ([42]) *Let $\mathcal{O}$ be an ovoid of $Q(2m, q)$, q odd, $m \geq 2$, and let π be the orthogonal polarity associated with $Q(2m, q)$. Define a triple $\{x, y, z\}$ of points of $\mathcal{O}$ to be coherent if and only if the $(2m-3)$-dimensional subspace $\langle x, y, z\rangle^{\pi}$ intersects $Q(2m, q)$ in a $Q^{+}(2m-3, q)$. Then the set $\mathcal{O}$, together with this set of coherent triples, forms a regular two-graph with parameters $n = q^m + 1$ and $a = (q-1)(q^{m-1}+1)/2$.*

It turns out that for $m \geq 4$ such parameters are impossible: they would lead to non-integer multiplicities for the eigenvalues of the descendants of the two-graph. This yields the non-existence of ovoids of $Q(2m, q)$, q odd, $m \geq 4$, which is the main result in [42]. For $m \in \{2, 3\}$ the parameters are feasible, and ovoids of $Q(4, q)$, q odd, and $Q(6, q)$, q a power of 3, do exist, as seen in Subsections 1.7.2 and 1.7.3, respectively.

1.9.3 The doubly transitive two-graphs

As already mentioned above, a two-graph with a doubly transitive automorphism group is regular. However, a much stronger result was proved by Taylor [90]: only six infinite classes of non-trivial regular two-graphs and two sporadic ones (and their complements) admit a doubly transitive automorphism group. We will call these two-graphs *doubly transitive*, although strictly speaking this is a property not of the two-graphs themselves, but of their automorphism groups. The following descriptions are based on [82].

- The *Paley two-graph* $\mathcal{P}(q)$ lives on the projective line $\mathrm{PG}(1,q)$, q an odd prime power such that $q \equiv 1 \pmod 4$. For any two distinct points $x = (x_0, x_1)$ and $y = (y_0, y_1)$ of $\mathrm{PG}(1,q)$ put

  $$D(x,y) := \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}.$$

  The coherent triples of $\mathcal{P}(q)$ are the triples $\{x,y,z\}$ for which the expression $D(x,y)D(y,z)D(z,x)$ is a non-square in $\mathrm{GF}(q)$. The two-graph $\mathcal{P}(q)$ is regular with parameters $n = q+1$ and $a = (q-1)/2$, it is isomorphic to its complement, and its full automorphism group is $\mathrm{P\Sigma L}_2(q)$. For $q \in \{5, 9, 13, 17\}$, $\mathcal{P}(q)$ is uniquely determined by its parameters [78, 81]. These two-graphs carry Paley's name because he constructed (in [72]) the corresponding symmetric conference matrices, see also Subsection 1.2.3. The Paley two-graph $\mathcal{P}(q^2)$, q an odd prime power, can also be obtained by applying Theorem 1.9.5 to the ovoid $Q^-(3,q)$ of $Q(4,q)$, since the stabiliser of $Q^-(3,q)$ in the group leaving $Q(4,q)$ invariant acts doubly transitively on $Q^-(3,q)$.
  The descendant of $\mathcal{P}(q)$ is the *Paley graph* $\mathcal{P}'(q)$; it has restricted eigenvalues $r = (\sqrt{q}-1)/2$ and $l = (-\sqrt{q}-1)/2$. The Paley graph $\mathcal{P}(9)$ is isomorphic to the lattice graph $L_2(3)$, which is the point graph of the generalised quadrangle of order $(2,1)$.

- The *Hermitian two-graphs* were discovered by Taylor [88]; his construction reads as follows. Let H be a non-degenerate Hermitian form of the projective plane $\mathrm{PG}(2,q^2)$, q an odd prime power, and let $\mathcal{U}$ be the Hermitian curve arising from H. The vertex set of the Hermitian two-graph $\mathcal{H}(q)$ is $\mathcal{U}$; a triple $\{x,y,z\}$ of vertices is coherent if and only if the expression $H(x,y)H(y,z)H(z,x)$ is a square or a non-square in $\mathrm{GF}(q^2)$, according as $q \equiv 3 \pmod 4$ or $q \equiv 1 \pmod 4$. One proves that $\mathcal{H}(q)$ is a regular two-graph with parameters $n = q^3+1$ and $a = (q-1)(q^2+1)/2$; its full automorphism group is $\mathrm{P\Gamma U}_3(q^2)$. For

$q = 3$ it is uniquely determined by its parameters [78]. The Hermitian two-graph $\mathcal{H}(3^h)$, $h \in \mathbb{N} \setminus \{0\}$, can also be obtained by applying Theorem 1.9.5 to the Hermitian ovoid $\mathcal{O}_H$ of $H(3^h)$ seen as an ovoid of $Q(6, 3^h)$, as the group leaving $Q(6, 3^h)$ invariant contains a group isomorphic to $\mathrm{P\Gamma U}_3(3^{2h})$ which stabilises $\mathcal{O}_H$ and acts doubly transitively on it. This fact is mentioned in [22].
The descendant of $\mathcal{H}(q)$ is the *Hermitian graph* $\mathcal{H}'(q)$; it has restricted eigenvalues $r = (q-1)/2$ and $l = -(q^2+1)/2$. The smallest Hermitian graph $\mathcal{H}'(3)$ is the point graph of the unique generalised quadrangle $Q^-(5, 2)$ of order $(2, 4)$.

- Let $q = 3^{2h+1}$, $h \in \mathbb{N}$, and apply Theorem 1.9.5 to the Ree–Tits ovoid $\mathcal{O}_R$ of $H(q)$ seen as an ovoid of $Q(6, q)$. As the group leaving $Q(6, q)$ invariant contains a subgroup which stabilises $\mathcal{O}_R$ and acts doubly transitive on it (see [96]), a regular two-graph $\mathcal{R}(q)$ with a doubly transitive automorphism group is obtained. It is called the *Ree two-graph*, has parameters $n = q^3 + 1$ and $a = (q-1)(q^2+1)/2$, and its full automorphism group is the Ree group $R(q)$ extended by $\mathrm{Aut}(\mathrm{GF}(q))$. The smallest Ree two-graph $\mathcal{R}(3)$ is uniquely determined by its parameters [78] and hence is isomorphic to the smallest Hermitian two-graph $\mathcal{H}(3)$.
The descendant $\mathcal{R}'(q)$ of $\mathcal{R}(q)$ is called the *Ree graph*; its restricted eigenvalues are $r = (q-1)/2$ and $l = -(q^2+1)/2$.

- Let $V(2m, 2)$, with $m \geq 2$, be the $2m$-dimensional vector space over $\mathrm{GF}(2)$, equipped with a non-degenerate alternating form f. The vertex set of the *symplectic two-graph* $\Sigma(2m, 2)$ is $V(2m, 2)$, and a triple $\{x, y, z\}$ is coherent if and only if $f(x, y)+f(y, z)+f(z, x) \equiv 0 \pmod 2$. The two-graph $\Sigma(2m, 2)$ is regular with $n = 2^{2m}$ and $a = 2^{2m-1} - 2$; its full automorphism group is the semidirect product of the symplectic group $\mathrm{Sp}(2m, 2)$ with the additive group of $V(2m, 2)$. From [78] it follows that $\Sigma(4, 2)$ is uniquely determined by its parameters.
The descendant $S(2m, 2)$ of $\Sigma(2m, 2)$ is called the *symplectic graph*; it has restricted eigenvalues $r = 2^{m-1} - 1$ and $l = -2^{m-1} - 1$. For $m \geq 3$ its edges are precisely the pairs of collinear points in the symplectic polar space $W(2m-1, 2)$ (which is isomorphic to $Q(2m, 2)$). The smallest symplectic graph $S(4, 2)$ is the point graph of the unique generalised quadrangle $W(2)$ of order 2.

- The *hyperbolic orthogonal two-graph* $\Omega^+(2m, 2)$ and the *elliptic orthogonal two-graph* $\Omega^-(2m, 2)$ are the sub-two-graphs of the symplectic two-graph $\Sigma(2m, 2)$ induced on the set of zeroes of a hyperbolic, respectively

elliptic, quadratic form with associated alternating form f. The parameters of $\Omega^+(2m,2)$ are $n = 2^{2m-1}+2^{m-1}$ and $a = 2^{2m-2}+2^{m-1}-2$, and those of $\Omega^-(2m,2)$ are $n = 2^{2m-1}-2^{m-1}$ and $a = 2^{2m-2}-2^{m-1}-2$. In both cases the full automorphism group is the symplectic group $\mathrm{Sp}(2m,2)$. The smallest elliptic orthogonal two-graph $\Omega^-(4,2)$ is void; the two-graphs $\Omega^+(4,2)$ and $\Omega^-(6,2)$ are uniquely determined by their parameters [78] and hence are isomorphic to the Paley two-graph $\mathcal{P}(9)$ and the Hermitian two-graph $\mathcal{H}(3)$, respectively.
The descendant $O^+(2m,2)$ of $\Omega^+(2m,2)$ is called the *hyperbolic orthogonal graph*, and its restricted eigenvalues are $r = 2^{m-1}-1$ and $l = -2^{m-2}-1$. For $m \geq 3$ its edges are the pairs of collinear points in the polar space $Q^+(2m-1,2)$. The descendant $O^-(2m,2)$ of $\Omega^-(2m,2)$ is called the *elliptic orthogonal graph*, and its restricted eigenvalues are $r = 2^{m-2}-1$ and $l = -2^{m-1}-1$. For $m \geq 4$ its edges are the pairs of collinear points in the polar space $Q^-(2m-1,2)$.

- There is a regular two-graph with parameters $n = 276$ and $a = 112$ which is uniquely determined by its parameters [41] and admits the sporadic Conway group Co_3 (see [24]) as a full automorphism group. A construction can be found in [82]. Its descendant is the McLaughlin graph, an srg$(275,112,30,56)$ which has the sporadic McLaughlin group [24] as a full automorphism group and has restricted eigenvalues $r = 2$ and $l = -28$.

- There is a regular two-graph with parameters $n = 176$ and $a = 72$ which has the sporadic Higman–Sims group HS (see [24]) as a full automorphism group. It is a sub-two-graph of the sporadic regular two-graph on 276 vertices (see [41]). We refer to [82] for a construction. Its descendant is an srg$(175,72,20,36)$ with restricted eigenvalues $r = 2$ and $l = -18$.

A beautiful combinatorial characterisation of the symplectic and orthogonal graphs was proved by Seidel. A graph is said to satisfy the *triangle property* if for each edge $\{x,y\}$ there exists a vertex z adjacent to both x and y such that every vertex different from x, y and z is adjacent to one or three vertices in $\{x,y,z\}$.

Theorem 1.9.6 ([80]) *A non-void graph which contains no vertex adjacent to all other vertices and satisfies the triangle property is isomorphic to a symplectic graph or to an orthogonal graph.*

Shult [84] obtained the same result for non-trivial regular graphs satisfying the triangle property.

Chapter 2

Two-graph geometries

2.1 Definition and small examples

Two-graph geometries were introduced by Haemers in [44]. They are incidence structures $\mathcal{S}$ for which the following axioms hold.

TGG1 $\mathcal{S}$ is a 2-(v, k, λ) design with $v = 1 + (k-1)(2\lambda - 1)$.

TGG1 Two blocks of $\mathcal{S}$ are incident with at most 2 points. Therefore blocks are also called *circles*, and a triple of points lying in a block is *cocircular.*

TGG3 Any set of four points of $\mathcal{S}$ contains an even number of cocircular triples.

It is straightforward that the points and cocircular triples of a two-graph geometry form a two-graph, which is said to be *geometric*. The fact that we started with a design implies that this two-graph is regular with parameters $n = 1 + (k-1)(2\lambda - 1)$ and $a = \lambda(k-2)$. Using Equations (1.2) one sees that the restricted eigenvalues of its descendants are $r = k/2 - 1$ and $l = -\lambda$. By Theorem 1.9.4 a block of a two-graph geometry, seen as the set of points incident with it, is a maximal coherent set in the corresponding two-graph. The derived design of a two-graph geometry with respect to a point p is a partial linear space of order $(k-2, \lambda-1)$ having the descendant of the two-graph with respect to p as a point graph. By Theorem 1.2.7 its lines are maximal cliques in the point graph, and hence we have a partial geometry $\mathrm{pg}\,(k-2, \lambda-1, k/2-1)$. Note that the converse does not hold: a 1-point extension of a partial geometry $\mathrm{pg}\,(s, t, s/2)$ is not necessarily a two-graph geometry. A counterexample is mentioned in [44]. Extensions of generalised quadrangles [20] and partial geometries [56] had been studied before, however without the extra requirement of having an associated two-graph.

Suppose that (V, Δ) is a regular two-graph having descendants with restricted eigenvalues r and l $(r > l)$. For a putative two-graph geometry on (V, Δ) to have an integral number of blocks, the divisibility condition

$$r + 1 \mid l(l + 1)(2l + 1) \tag{2.1}$$

has to be satisfied. If $r = 0$, then (V, Δ) is void. For $r = 1$, condition (2.1) is always satisfied. As the descendants have $\lambda = 1$ in this case, every coherent triple is contained in a unique coherent 4-set. If the coherent 4-sets are taken as the blocks of a design, a unique two-graph geometry on (V, Δ) is found. By a result of Seidel [78], any regular two-graph having descendants with $r = 1$ is isomorphic to the Paley two-graph $\mathcal{P}(9)$ (or equivalently the hyperbolic orthogonal two-graph $\Omega^+(4, 2)$), the symplectic two-graph $\Sigma(4, 2)$, or the Hermitian two-graph $\mathcal{H}(3)$ (or equivalently the Ree two-graph $\mathcal{R}(3)$ or the elliptic orthogonal two-graph $\Omega^-(6, 2)$). The derived design of such a two-graph geometry is a generalised quadrangle of order $(2, t)$ and hence is isomorphic to $Q^+(3, 2)$, $Q(4, 2)$ or $Q^-(5, 2)$, respectively [75]. On the other hand, a generalised quadrangle of order $(2, t)$ has a unique 1-point extension [15] which is necessarily a two-graph geometry [44].

2.2 (Non-)existence of two-graph geometries for the doubly transitive two-graphs

We will discuss what is known to us concerning this question. For the Paley two-graphs only $\mathcal{P}(q^2)$, q and odd prime power, can be a candidate, as the descendants need to have integer eigenvalues. In [39] a two-graph geometry is constructed for which the associated two-graph has the parameters of $\mathcal{P}(q^2)$; in [103] it is proved that the two-graph is indeed $\mathcal{P}(q^2)$. All coherent sets in $\mathcal{P}(q^2)$ meeting the upper bound in Theorem 1.9.4 are blocks of this two-graph geometry, which therefore is unique (see [6]). The derived design with respect to any point is a net $\mathrm{pg}\,(q - 1, (q - 1)/2, (q - 1)/2)$ which can be constructed as follows. Its points are the points of the affine plane $\mathrm{AG}(2, q)$, and its lines are the lines of $\mathrm{AG}(2, q)$ with a slope m such that $1 - \gamma m^2$ is a non-square in $\mathrm{GF}(q)$, where γ is a fixed non-square in $\mathrm{GF}(q)$.

The smallest Hermitian two-graph $\mathcal{H}(3)$ supports a unique two-graph geometry, which is also the smallest Hölz design, see Subsection 1.8.3. In [87] it is proved that $\mathcal{H}'(5)$ and $\mathcal{H}'(7)$ are not geometric, implying that $\mathcal{H}(5)$ and $\mathcal{H}(7)$ are not geometric either. We refer to Section 3.7 for some comments concerning $\mathcal{H}(3^h)$, $h \in \mathbb{N} \setminus \{0\}$. For the remaining cases nothing is known.

The smallest Ree two-graph $\mathcal{R}(3)$ is isomorphic to the Hermitian two-graph $\mathcal{H}(3)$; it is not known whether $\mathcal{R}(3^{2h+1})$, $h \in \mathbb{N} \setminus \{0\}$, is geometric.

The smallest symplectic graph $\Sigma(4,2)$ is geometric in a unique way. From a more general result in [32] it follows that $\Sigma(6,2)$ and $\Sigma(8,2)$ do not support a two-graph geometry. In [34] it was proved that neither $\Sigma(4m,2)$, $m \geq 2$, nor $\Sigma(10,2)$ is geometric. For the latter a computer result by Mathon [69] was used. The following theorem completes the proof of the non-existence of two-graph geometries on the symplectic two-graphs $\Sigma(2m,2)$, $m \geq 3$.

Theorem 2.2.1 *The symplectic two-graph* $\Sigma(4m+2,2)$*,* $m \geq 1$*, is not geometric.*

Proof. We will show that the descendant $S(4m+2,2)$ of $\Sigma(4m+2,2)$ is not geometric; this clearly implies the assertion. So suppose that there exists a partial geometry pg $(2^{2m+1}-2, 2^{2m}, 2^{2m}-1)$ with the symplectic graph $S(4m+2,2)$, $m \geq 1$, as a point graph. In [30] it is proved that the block graph of such a partial geometry is isomorphic to the complement of the triangular graph $T(2^{2m+1}+2)$ (see Section 4.5 for a definition of the latter). It is not difficult to see that $T(2^{2m+1}+2)$ contains exactly $2^{2m+1}+2$ (maximal) cliques of size $2^{2m+1}+1$ and that every vertex is contained in two such cliques. Moreover two adjacent vertices x and y define a unique $(2^{2m+1}+1)$-clique $C(x,y)$, and the unique vertex $z \notin C(x,y)$ adjacent to x and y is adjacent to no other vertex in $C(x,y)$. As these cliques are cocliques in the block graph of the partial geometry, they correspond to sets of $2^{2m+1}+1$ mutually non-concurrent lines; a counting argument learns that the lines of such a set partition the point set of the partial geometry. On the other hand, the symplectic graph $S(4m+2,2)$ is isomorphic to the graph on the points of the polar space $Q(4m+2,2)$ where vertices are adjacent if and only if they are collinear. A line of the partial geometry corresponds to a set of $2^{2m+1}-1$ mutually collinear points of $Q(4m+2,2)$; this can only be the set of points in a generator of $Q(4m+2,2)$. Consequently a set of $2^{2m+1}+1$ mutually non-concurrent lines of the partial geometry is a spread of the polar space $Q(4m+2,2)$. Consider two $(2^{2m+1}+1)$-cliques of $T(2^{2m+1}+2)$, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the corresponding spreads of $Q(4m+2,2)$. Since the two cliques intersect in exactly one vertex, the spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ have precisely one generator L in common. Embed $Q(4m+2,2)$ in a hyperbolic quadric $Q^+(4m+3,2)$, and let Π denote the $(4m+2)$-dimensional subspace of $\mathrm{PG}(4m+3,2)$ spanned by $Q(4m+2,2)$. Let $\mathcal{G}$ be one of the families of generators of $Q^+(4m+3,2)$, and let $\mathcal{S}'_1$, respectively $\mathcal{S}'_2$, be the set of elements of $\mathcal{G}$ containing an element of $\mathcal{S}_1$, respectively $\mathcal{S}_2$. Note that any two elements of $\mathcal{G}$ intersect in a subspace of odd dimension. Suppose that two elements X' and Y' of $\mathcal{S}'_i$, $i \in \{1,2\}$, would intersect in a subspace of dimension at least one. Then this subspace would contain at least one point of Π, which would have to lie in the intersection of the elements $X' \cap \Pi$ and $Y' \cap \Pi$ of $\mathcal{S}_i$, a contradiction. This shows that $\mathcal{S}_1'$ and $\mathcal{S}_2'$ are spreads of $Q^+(4m+3,2)$. Let M be an element of $\mathcal{S}_1 \setminus \{L\}$ and let N be the unique element of $\mathcal{S}_2$ disjoint from M; let L', M' and N' denote the elements of $\mathcal{G}$ containing L, M and N, respectively. Since all elements of $\mathcal{S}_2 \setminus \{L, N\}$ intersect M, all elements of $\mathcal{S}_2' \setminus \{L', N'\}$ intersect M' in at least a line. The elements of $\mathcal{S}_2' \setminus \{L', N'\}$ are mutually disjoint; therefore the union of their intersections with M' contains at least $|\mathcal{S}_2' \setminus \{L', N'\}||\mathrm{PG}(1,2)| = (2^{2m+1}-1) \cdot 3$ points, a contradiction because M' is a $(2m+1)$-dimensional subspace and hence has $2^{2m+2}-1$ points. We conclude that $S(4m+2,2)$, $m \geq 1$, cannot be the point graph of a partial geometry. □

Both the result in [34] and Theorem 2.2.1 also follow from a more general unpublished theorem proved independently by Thas.

Theorem 2.2.2 *The graph on the points of $Q(2m,q)$, $m \geq 3$ and q any prime power, vertices being adjacent if and only if they are collinear, is not geometric.*

Proof. A partial geometry having this point graph is a

$$\mathrm{pg}\left(\frac{q^m - q}{q-1}, q^{m-1}, \frac{q^{m-1}-1}{q-1}\right).$$

Its lines are $(m-1)$-dimensional subspaces on $Q(2m,q)$. Consider a hyperplane Π of $\mathrm{PG}(2m,q)$ which intersects $Q(2m,q)$ in a hyperbolic quadric $Q^+(2m-1,q)$, and let p be a point of $Q^+(2m-1,q)$. Let x denote the number of lines of the partial geometry which are incident with p and are contained in $Q^+(2m-1,q)$. Then the number

$$x\left(\frac{q^m-1}{q-1} - 1\right) + (q^{m-1}+1-x)\left(\frac{q^{m-1}-1}{q-1} - 1\right)$$

is the number of points of $Q^+(2m-1,q)$ which are collinear with p. As this number equals $q(q^{m-1}-1)(q^{m-2}+1)/(q-1)$, we find $x = 2$. First suppose that $m = 2n$, $n \geq 2$. Then the generators of $Q^+(2m-1,q) = Q^+(4n-1,q)$ are $(2n-1)$-dimensional subspaces, and they belong to the same family if and only if they intersect in a subspace of odd dimension. Let L be a line of the partial geometry which is contained in $Q^+(4n-1,q)$. Through any point p of L there is a second line $L(p)$ of the partial geometry which is contained in $Q^+(4n-1,q)$. Since each of the lines $L(p)$ intersects L in just one point (namely p), they are all in the same family. Being lines of the partial geometry, they intersect in at most one point; as a point is a subspace of even dimension, they are two by two disjoint. Thus we constructed $(q^{2n}-1)/(q-1)$ mutually disjoint generators of $Q^+(4n-1,q)$, a contradiction to the fact that a partial spread of $Q^+(4n-1,q)$ has at most $q^{2n-1}+1$ elements (see Section 1.6). Now suppose that $m = 2n+1$, $n \geq 1$. In this case the generators of $Q^+(2m-1,q) = Q^+(4n+1,q)$ are $2n$-dimensional subspaces, and they belong to the same family if and only if their intersection is a subspace of even dimension. A counting argument learns that $2(q^{2n}+1)$ lines of the partial geometry are contained in $Q^+(4n+1,q)$. Let L be one of them; through any point p of L there is a second line $L(p)$ of the partial geometry which is contained in $Q^+(4n+1,q)$. Since all these lines $L(p)$ intersect L in one point (namely p), they all belong to the same family and hence intersect two by two in one point. The set $\{L\} \cup \{L(p) \mid p \in L\}$ consists of $1+(q^{2n+1}-1)/(q-1)$ lines; as this number is strictly smaller than $2(q^{2n}+1)$, there exists a line M of the partial geometry which is contained in $Q^+(4n+1,q)$ and does not intersect L or any line $L(p)$, $p \in L$. Through each point r of M there is a second line $M(r)$ of the partial geometry which is contained in $Q^+(4n+1,q)$. The set $\{L, M\} \cup \{L(p) \mid p \in L\} \cup \{M(r) \mid r \in M\}$ counts $2(1+(q^{2n+1}-1)/(q-1))$ elements. However, this number is strictly greater than $2(q^{2n}+1)$, yielding again a contradiction. This proves the assertion. $\square$

Only the smallest hyperbolic orthogonal two-graph $\Omega^+(4,2)$, which is isomorphic to the Paley two-graph $\mathcal{P}(9)$, satisfies the divisibility condition (2.1). Hence $\Omega^+(2m,2)$, $m \geq 3$, is not geometric. The only elliptic orthogonal two-graph for which the situation is known is $\Omega^-(6,2)$, which is isomorphic to the Hermitian two-graph $\mathcal{H}(3)$.

It is not known whether the sporadic regular two-graph on 276 vertices is geometric. However, Haemers [44] constructed a two-graph geometry associated with the sporadic regular two-graph on 176 vertices. The derived design with respect to any point is a partial geometry pg $(4,17,2)$ which was also found by Haemers [43].

2.3 Generalisation to semipartial geometries

We have briefly looked for 2-(v,k,λ) designs which give rise to a two-graph in the same way as two-graph geometries, but are 1-point extensions of semipartial geometries. Such a semipartial geometry spg (s,t,α,μ) necessarily satisfies $(t+1)s = 2\mu$. However, this condition seems quite restrictive: apart from the rather trivial pentagon, only part of one of the known infinite classes of proper semipartial geometries listed in [31] satisfies it. The semipartial geometries $\overline{W(2n+1,q)}$, $n \geq 1$ and q any prime power, live in the projective space $\mathrm{PG}(2n+1,q)$ equipped with a symplectic polarity π. The points of $\overline{W(2n+1,q)}$ are the points of $\mathrm{PG}(2n+1,q)$, and the lines of $\overline{W(2n+1,q)}$ are the lines of $\mathrm{PG}(2n+1,q)$ which are not totally isotropic with respect to π. With natural incidence, $\overline{W(2n+1,q)}$ is a semipartial geometry spg $(q, q^{2n}-1, q, q^{2n}(q-1))$. The smallest one, $\overline{W(3,2)}$, is isomorphic to the semipartial geometry $U_{2,3}(6)$ (see [33]). One easily checks that $(t+1)s = 2\mu$ is satisfied if and only if $q = 2$. A 1-point extension is found by considering $\mathrm{PG}(2n+1,2)$ as the set of non-zero vectors of the $(2n+2)$-dimensional vector space $V(2n+2,2)$ over GF(2). The symplectic polarity π corresponds to a non-degenerate alternating form of $V(2n+2,2)$, which we will also call π. Let $\mathcal{D}$ be the design on $V(2n+2,2)$ in which the blocks are the two-dimensional subspaces of $V(2n+2,2)$ which are not totally isotropic with respect to π and all their translates. Then $\mathcal{D}$ is a 2-$(2^{2n+2}, 4, 2^{2n})$ design in which two distinct blocks are incident with at most two common points. One checks that the derived design of $\mathcal{D}$ with respect to the zero vector $\underline{0}$ is indeed the semipartial geometry $\overline{W(2n+1,2)}$. Moreover the set of points of $\mathcal{D}$ and the set of unordered triples of points lying in a block of $\mathcal{D}$ form the complement of the symplectic two-graph $\Sigma(2n+2,2)$.

Chapter 3

Hermitian partial geometries

The Hermitian graph $\mathcal{H}'(q)$, q an odd prime power, is an

$$\operatorname{srg}\left(q^3, \frac{(q-1)(q^2+1)}{2}, \frac{(q-1)^3}{4} - 1, \frac{(q-1)(q^2+1)}{4}\right)$$

which is pseudo-geometric for $s = q-1$, $t = (q^2-1)/2$, $\alpha = (q-1)/2$. The smallest Hermitian graph $\mathcal{H}'(3)$ is the point graph of the unique generalised quadrangle $Q^-(5,2)$ of order $(2,4)$ (see [79]). In [87] it is proved that $\mathcal{H}'(5)$ and $\mathcal{H}'(7)$ are not geometric. Mathon [70] showed that $\mathcal{H}'(3^h)$, h even, is geometric; his partial geometries will be described in Section 3.1. Finally the author [63] constructed partial geometries with $\mathcal{H}'(3^h)$ as a point graph for every $h \in \mathbb{N} \setminus \{0\}$. For h even they are isomorphic to Mathon's partial geometries; for h odd, however, they are new, except for the smallest case $q = 3$. The construction will be explained in Section 3.2. For other values of q, i.e. $q \geq 11$ and q not divisible by 3, it is not known whether $\mathcal{H}'(q)$ is geometric.

3.1 The partial geometries $\mathcal{M}_3(h)$, $h \in \mathbb{N} \setminus \{0\}$, and their derived partial geometries

In [70] Mathon gives an algebraic construction of $2^{q(q+1)}$ partial geometries $\operatorname{pg}(q-1, (q^2-1)/2, (q-1)/2)$, $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$, with the Hermitian graph $\mathcal{H}'(q)$ as a point graph. We will slightly modify Mathon's description in order to obtain a nicer isomorphism with the partial geometries which will be constructed in Section 3.2. Let β be a primitive element of $\mathrm{GF}(q^2)$, $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$, put $r := (q+1)/2$ and $\gamma := \beta^{q+1}$, and let β^∞ denote the zero element in $\mathrm{GF}(q^2)$. Define $K := \{\infty\} \cup \{0, \ldots, q-2\}$ and

$I := \{0, \ldots, (q^2-5)/4\}$. For $k, j, l \in K$, $i \in I$ and $\delta \in \{0,1\}$ define

$$\begin{aligned} x_{kl}^{i\delta} &:= \sqrt{\gamma^{-1}}\beta^{2i+\delta}(\beta^{2kr} - \beta^{(2l-1)r}), \\ y_{kjl}^{i\delta} &:= \sqrt{\gamma^{-1}}\left[\beta^{(2(i+\delta)+1)r}\left(\beta^{2kr}(\beta^{2kr} + \beta^{(2i-1)r})^2\right.\right. \\ &\quad \left.\left. - \beta^{(2i-1)r}(\beta^{2kr} - \beta^{(2l-1)r})^2\right) + \beta^{(2j+1)r}\right], \end{aligned}$$

and put

$$\mathcal{P} := \{(x_{kl}^{00}, y_{kjl}^{00}) \mid k, l, j \in K\}.$$

For $k, l \in K$ define

$$L_{kl}^{\infty} := \{(x_{kl}^{00}, y_{kjl}^{00}) \mid j \in K\},$$

and for $j, l \in K$, $i \in I$, $\delta \in \{0,1\}$ define

$$L_{jl}^{i\delta} := \{(x_{kl}^{i\delta}, y_{kjl}^{i\delta}) \mid k \in K\}.$$

Finally define

$$\mathcal{L}_1 := \{L_{jl}^{i\delta} \mid j, l \in K, i \in I, \delta \in \{0,1\}\}$$

and

$$\mathcal{L}_2 := \{L_{kl}^{\infty} \mid k, l \in K\}.$$

The incidence structure $\mathcal{M}_3(h)$ with point set $\mathcal{P}$, line set $\mathcal{L}_1 \cup \mathcal{L}_2$ and natural incidence is a partial geometry $\mathrm{pg}\,(q-1, (q^2-1)/2, (q-1)/2)$ with point graph $\mathcal{H}'(q)$.

From $\mathcal{M}_3(h)$ more partial geometries with the same point graph can be constructed. For $l \in K$, $i \in I$, $\delta \in \{0,1\}$ define

$$R_l^{i\delta} := \{L_{j,l-m}^{i+mr,\delta} \mid j \in K, m \in \{0, \ldots, r-2\}\};$$

it can be calculated that $R_l^{i\delta}$ contains $r-1 = (q-1)/2$ sets of q mutually skew lines (corresponding to the $r-1$ different values of m) covering a set $P_l^{i\delta}$ of q^2 points. Together with the q lines from $\mathcal{L}_2$ which have points in $P_l^{i\delta}$, these lines form a net $\mathrm{pg}\,(q-1, (q-1)/2, (q-1)/2)$ on $P_l^{i\delta}$ which is called a *Baer subnet* of $\mathcal{M}_3(h)$. Now define, for $k, j, l \in K$, $i \in I$ and $\delta \in \{0,1\}$,

$$\begin{aligned} \tilde{x}_{kl}^{i\delta} &:= \sqrt{\gamma}^{-1}\beta^{2i+\delta}(\beta^{2kr} - \beta^{(2l+1)r}), \\ \tilde{y}_{kjl}^{i\delta} &:= \sqrt{\gamma}^{-1}\left[\beta^{(2(i+\delta)-1)r}\left(\beta^{2kr}(\beta^{2kr} + \beta^{(2i+1)r})^2\right.\right. \\ &\quad \left.\left. - \beta^{(2i+1)r}(\beta^{2kr} - \beta^{(2l+1)r})^2\right) + \beta^{(2j-1)r}\right], \end{aligned}$$

and put

$$\tilde{\mathcal{P}} := \{(\tilde{x}_{kl}^{00}, \tilde{y}_{kjl}^{00}) \mid k, l, j \in K\}.$$

For $j, l \in K$, $i \in I$, $\delta \in \{0,1\}$ define

$$\tilde{L}^{i\delta}_{jl} := \{(\tilde{x}^{i\delta}_{kl}, \tilde{y}^{i\delta}_{kjl}) \mid k \in K\},$$

and put

$$\tilde{\mathcal{L}}_1 := \{\tilde{L}^{i\delta}_{jl} \mid j, l \in K, i \in I, \delta \in \{0,1\}\}.$$

For $l \in K$, $i \in I$, $\delta \in \{0,1\}$ define

$$\tilde{R}^{i\delta}_{l} := \{\tilde{L}^{i+mr,\delta}_{j,l-m-1} \mid j \in K, m \in \{0, \ldots, r-2\}\}.$$

The incidence structure $\tilde{\mathcal{M}}_3(h)$ with point set $\tilde{\mathcal{P}}$, line set $\tilde{\mathcal{L}}_1 \cup \mathcal{L}_2$ and natural incidence is a partial geometry pg $(q-1, (q^2-1)/2, (q-1)/2)$ with point graph $\mathcal{H}'(q)$. The set $\tilde{R}^{i\delta}_l$ contains $r-1 = (q-1)/2$ sets of q mutually skew lines (corresponding to the $r-1$ different values of m) covering a set $\tilde{P}^{i\delta}_l$ of q^2 points. Together with the q lines from $\mathcal{L}_2$ which have points in $\tilde{P}^{i\delta}_l$ these lines form a net pg $(q-1, (q-1)/2, (q-1)/2)$ on $\tilde{P}^{i\delta}_l$ which is also called a *Baer subnet* of $\tilde{\mathcal{M}}_3(h)$. Now it can be proved that $\tilde{\mathcal{P}} = \mathcal{P}$ and that $\tilde{P}^{i\delta}_l = P^{i\delta}_l$ for each $l \in K$, $i \in I$, $\delta \in \{0,1\}$. Moreover the sets of edges of $\mathcal{H}'(q)$ covered by the lines of $\tilde{R}^{i\delta}_l$ and $R^{i\delta}_l$, respectively, are the same. Consequently replacing some of the line sets $R^{i\delta}_l$ by the corresponding sets $\tilde{R}^{i\delta}_l$ yields again a partial geometry pg $(q-1, (q^2-1)/2, (q-1)/2)$ with point graph $\mathcal{H}'(q)$. As there are $q(q+1)$ mutually disjoint Baer subnets in $\mathcal{M}_3(h)$, and each of them can be replaced by the corresponding Baer subnet of $\tilde{\mathcal{M}}_3(h)$ independently, $2^{q(q+1)}$ partial geometries with point graph $\mathcal{H}'(q)$ are obtained. They are known as the *derived* partial geometries of $\mathcal{M}_3(h)$.

3.2 A geometric construction of Hermitian partial geometries

We will construct partial geometries using the Hermitian ovoid $\mathcal{O}_H$ of $H(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$, which was described in Subsection 1.7.3. Let $Q(6,q)$ be the parabolic quadric underlying $H(q)$, and let π be the orthogonal polarity associated with $Q(6,q)$. The following property, which follows from a more general lemma in [36], will be useful. As in Subsection 1.5.1, $\mathcal{P}(x,y)$ denotes the point regulus determined by the opposite points x and y.

Lemma 3.2.1 *Under the above assumptions, let* p *be a point of* $\mathcal{O}_H$ *and let* M *be a line not containing* p *in the plane* $\Gamma_2(p) \cup \{p\}$ *on the quadric* $Q(6,q)$*. Then there exists a unique point regulus* $\mathcal{P}(p,x)$*,* $x \in \mathcal{O}_H \setminus \{p\}$*, such that* M *is the distance-2-trace* $\Gamma_2(p) \cap \Gamma_4(x')$ *for all* $x' \in \mathcal{P}(p,x) \setminus \{p\}$*.*

Proof. Recall that $\mathcal{O}_H$ is also an ovoid of $Q(6,q)$. Hence each of the $q+1$ planes through M on $Q(6,q)$ contains exactly one point of $\mathcal{O}_H$, which cannot be on M as all points on M are at distance 2 from p. One of the planes is $\Gamma_2 \cup \{p\}$, and clearly the point of $\mathcal{O}_H$ in it is p. The q remaining points are precisely the points of $\mathcal{O}_H$ which are at distance 4 from all points of M. Let x be such a point; then the construction of point reguli implies that all points $x' \in \mathcal{P}(p,x) \setminus \{p\}$ are also at distance four from all points of M. Consequently these q points necessarily form the set $\mathcal{P}(p,x) \setminus \{p\}$. □

Theorem 1.9.5 applied to $\mathcal{O}_H$, seen as an ovoid of $Q(6,q)$, yields the Hermitian two-graph $\mathcal{H}(q)$. After taking descendants, one obtains a useful model for the Hermitian graph $\mathcal{H}'(q)$.

Theorem 3.2.2 *Let $\mathcal{O}_H$ be a Hermitian ovoid of $H(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$, let π be the orthogonal polarity associated with the quadric $Q(6,q)$ underlying $H(q)$, and choose $p \in \mathcal{O}_H$. Define two points x and y of $\mathcal{O}_H \setminus \{p\}$ to be adjacent if and only if the three-dimensional subspace $\langle p,x,y\rangle^\pi$ intersects $Q(6,q)$ in a $Q^+(3,q)$. The graph on $\mathcal{O}_H \setminus \{p\}$ thus defined is isomorphic to the Hermitian graph $\mathcal{H}'(q)$.*

We will use the explicit description of $\mathcal{O}_H$ given in Subsection 1.7.3, which means that $Q(6,q)$ has equation $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$. Choose a point $p \in \mathcal{O}_H$; without loss of generality we may assume that $p = (1,0,0,0,0,0,0)$. From Table 1.1 it can be deduced that the plane $\Gamma_2(p) \cup \{p\}$ has equations $X_1 = X_2 = X_3 = X_4 = 0$. The point set of the partial geometries is $\mathcal{O}_H \setminus \{p\}$, while lines are of two types. The type 2 lines are the sets $\mathcal{P}(p,t) \setminus \{p\}$, $t \in \mathcal{O}_H \setminus \{p\}$. Clearly they are q-cliques in $\mathcal{H}'(q)$, as $\langle \mathcal{P}(p,t)\rangle^\pi$ intersects $Q(6,q)$ in a $Q^+(3,q)$. For the construction of the type 1 lines we proceed in four steps.

STEP 1. Let $s \in \Gamma_2(p)$, and let M be a line containing s and not containing p in the plane $\Gamma_2(p) \cup \{p\}$. Pick a point $t \in \mathcal{O}_H \setminus \{p\}$ such that $\Gamma_2(p) \cap \Gamma_4(t) = M$ (see Lemma 3.2.1); define $Q(4,q) := p^\pi \cap t^\pi \cap Q(6,q)$ and $Q^+(3,q) := \langle \mathcal{P}(p,t)\rangle^\pi \cap Q(6,q)$. One sees that M is contained in $Q^+(3,q)$, which is in its turn contained in $Q(4,q)$. On $Q(4,q)$ there are $q+1$ generators containing s, two of which (M and another one) lie on $Q^+(3,q)$. Let N_k, $k \in \mathrm{GF}(q) \setminus \{0\}$, denote the $q-1$ remaining generators (see Figure 3.1). No line N_k, $k \in \mathrm{GF}(q) \setminus \{0\}$, contains a point of $\mathcal{O}_H$, since such a point would be at distance at most four from p in $H(q)$. Recall that an ovoid of $H(q)$ is also an ovoid of $Q(6,q)$; consequently each of the $q+1$ planes on $Q(6,q)$ through N_k contains exactly one point of $\mathcal{O}_H$. One of these planes contains p; the q remaining points form the set $C_k := N_k^\pi \cap (\mathcal{O}_H \setminus \{p\})$. For any two distinct points u and v of C_k the quadric $\langle p, u, v\rangle^\pi \cap Q(6, q)$ contains at least one line, namely N_k. As p, u and v are mutually non-collinear on $Q(6, q)$, this quadric is non-degenerate and hence hyperbolic. Theorem 3.2.2 implies that C_k is a q-clique in $\mathcal{H}'(q)$, and thus could be used as a line of a partial geometry with point graph $\mathcal{H}'(q)$. However, not all of these q-cliques C_k, $k \in \mathrm{GF}(q) \setminus \{0\}$, can be used at the same time. To settle the thoughts assume $s = (0, 0, 0, 0, 0, 1, 0)$, $M \equiv X_0 = X_1 = X_2 = X_3 = X_4 = 0$ and $t = (0, 0, 0, 0, 1, 0, 0)$. We may choose N_k to be the line $\langle s, (0, 0, 1, k, 0, 0, k^2)\rangle$ for all $k \in \mathrm{GF}(q) \setminus \{0\}$. Pick two lines N_k and $N_{k'}$, with $k, k' \in \mathrm{GF}(q)$ and $k \neq k'$. Then the intersection of the corresponding cliques C_k and $C_{k'}$ consists of the points of $\mathcal{O}_H \setminus \{p\}$ lying in the three-dimensional subspace $N_k^\pi \cap N_{k'}^\pi$, which is described by the equations

$$\begin{cases} X_1 = 0 \\ k^2 X_2 + kX_3 + X_6 = 0 \\ k'^2 X_2 + k' X_3 + X_6 = 0. \end{cases}$$

Substituting this into a general point of $\mathcal{O}_H \setminus \{p\}$ as described in Subsection 1.7.3, we see that the parameters a, a' and a'' of a point of $C_k \cap C_{k'}$ satisfy

$$\begin{cases} a'' = 0 \\ k^2(-a) + k(-a') + (-\gamma^3 a^3) = 0 \\ k'^2(-a) + k'(-a') + (-\gamma^3 a^3) = 0, \end{cases}$$

or equivalently

$$\begin{cases} a'' = 0 \\ a' = -ka - \gamma^3 k^{-1} a^3 \\ ka + \gamma^3 k^{-1} a^3 = k'a + \gamma^3 k'^{-1} a^3. \end{cases}$$

The latter equation can be rewritten as

$$(k' - k)a = \gamma^3 \left(\frac{k' - k}{kk'}\right) a^3;$$

as $k' \neq k$, we find

$$kk'a = \gamma^3 a^3.$$

It follows that either $a = 0$ or $a^2 = \gamma^{-3} kk'$. The solution $a = 0$ yields the point t, which is by definition contained in both C_k and $C_{k'}$. If kk' is a non-square in $\mathrm{GF}(q)$, then C_k and $C_{k'}$ have two more points in common (the ones corresponding to $a = \pm\sqrt{\gamma^{-3} kk'}$), and hence cannot both be lines of the putative partial geometry. This problem obviously does not occur if kk' is a square in $\mathrm{GF}(q)$. As a consequence, $\{N_k \mid k \in \mathrm{GF}(q) \setminus \{0\}\}$ can

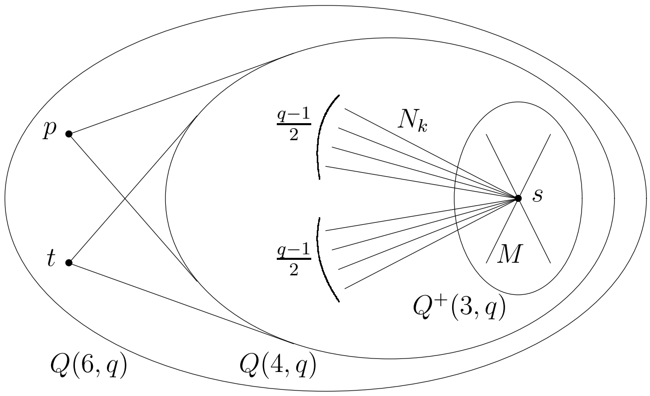


Figure 3.1: Step 1 in the geometric construction.

be partitioned into two sets of size $(q-1)/2$ such that the corresponding q-cliques intersect in exactly one point (namely t) if and only if the lines are in the same set (see Figure 3.1). Choose one of the sets and define the corresponding q-cliques to be the type 1 lines associated with the triple (s, M, t).

STEP 2. By Lemma 3.2.1, $\Gamma_2(p) \cap \Gamma_4(t') = M$ for all $t' \in \mathcal{P}(p,t) \setminus \{p\}$. The type 1 lines associated with the triple (s, M, t') are determined by those associated with (s, M, t) in the following way: the corresponding generators N'_k, $k \in \mathrm{GF}(q) \setminus \{0\}$, of the quadric $p^\pi \cap t'^\pi \cap Q(6,q)$ are the projections from p onto t'^π of the lines N_k, $k \in \mathrm{GF}(q) \setminus \{0\}$, belonging to the set chosen in Step 1. It is easily checked that the type 1 lines associated with (s, M, t') intersect only in t'.

STEP 3. Let $M' \neq M$ be a line through s and not through p in the plane $\Gamma_2(p) \cup \{p\}$, and choose a point $v \in \mathcal{O}_H \setminus \{p\}$ such that $\Gamma_2(p) \cap \Gamma_4(v) = M'$. As $\mathcal{P}(p,t) \setminus \{p\}$ is a maximal clique in $\mathcal{H}'(q)$ not containing v, v is adjacent to exactly $(q-1)/2$ vertices in $\mathcal{P}(p,t) \setminus \{p\}$ (see Theorem 1.2.7). Let t'' be one of these vertices, and determine the "good" generators N''_k of $p^\pi \cap t''^\pi \cap Q(6,q)$ as in Step 2. Using coordinates as in Step 1, one proves that exactly one of these generators lies in v^π. The corresponding type 1 line can also be associated with (s, M', v): it lies in s^π, it contains v, and $\Gamma_2(p) \cap \Gamma_4(v) = M'$. If one repeats this for all $(q-1)/2$ points of $\mathcal{P}(p,t) \setminus \{p\}$ adjacent to v, one obtains $(q-1)/2$ type 1 lines associated with (s, M', v), which intersect only in v.

STEP 4. In Steps 1 to 3, all type 1 lines associated with a triple with s as a first component have been constructed; repeat them for all other points $s' \in \Gamma_2(p)$.

Theorem 3.2.3 *The incidence structure defined above is a partial geometry* $\mathrm{pg}\,(q-1, (q^2-1)/2, (q-1)/2)$ *having* $\mathcal{H}'(q)$ *as a point graph.*

Proof. From Step 3 it follows that type 1 lines are well-defined, although they seem to be constructed from one of their points. In fact a type 1 line L is associated with all triples $(s, M(t), t)$, where $t \in L$, $M(t) := \Gamma_2(p) \cap \Gamma_4(t)$, and s is the unique point in $\Gamma_2(p)$ such that $L \subseteq s^\pi$.
Distinct type 2 lines are not concurrent. Suppose that a type 1 line L would contain two distinct points t and t' of the type 2 line $\mathcal{P}(p,t)\setminus\{p\}$, and let N be the generator of an appropriate quadric $Q(4,q)$ such that $L = N^\pi \cap (\mathcal{O}_H \setminus \{p\})$. Then $p, t, t' \in N^\pi$ and hence $N \subseteq \langle \mathcal{P}(p,t)\rangle^\pi \cap Q(6,q)$, a contradiction to the construction of type 1 lines. Now suppose that distinct type 1 lines L and L' intersect in two distinct points t and t'. Then L (respectively L') is associated with (s, M, t) (respectively (s', M, t)) for some point s (respectively s') of $\Gamma_2(p)$, with $M := \Gamma_2(p) \cap \Gamma_4(t)$. Here s and s' must be distinct because the lines constructed in Step 1 share only one point. This implies that $t' \in s^\pi \cap s'^\pi = M^\pi$, and hence (by Lemma 3.2.1) $t' \in \mathcal{P}(p,t) \setminus \{p\}$, a contradiction. Consequently no two lines intersect in more than one point, implying that the incidence structure is a partial linear space.
Every line is incident with q points, and every point is incident with one type 2 line and $(q^2-1)/2$ type 1 lines. The latter number is obtained as follows: type 1 lines through a point $t \in \mathcal{O}_H \setminus \{p\}$ are associated with $(s, M(t), t)$, where $M(t) := \Gamma_2(p) \cap \Gamma_4(t)$ and $s \in M(t)$, and each of these triples yields $(q-1)/2$ type 1 lines. Thus we find a partial linear space of order $(q-1, (q^2-1)/2)$.
By construction collinear points are adjacent in $\mathcal{H}'(q)$; counting the number of edges learns that $\mathcal{H}'(q)$ is the point graph of the incidence structure. By Theorem 1.2.7, a point not on a line is collinear with $(q-1)/2$ points on that line. This completes the proof. □

We call the partial geometries constructed here the *Hermitian partial geometries*. Note that for each $s \in \Gamma_2(p)$ a choice has to be made between two sets of lines. As these choices are mutually independent, the construction yields $2^{q(q+1)}$ partial geometries $\mathrm{pg}\,(q-1, (q^2-1)/2, (q-1)/2)$. For $q = 3$ the unique generalised quadrangle $Q^-(5,2)$ of order $(2,4)$ is obtained. For $q = 3^{2h+1}$, $h \in \mathbb{N} \setminus \{0\}$, no partial geometries with these parameters were known; therefore the Hermitian ones are new. In Section 3.3 it will be proved that for $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$, the Hermitian partial geometries are isomorphic to $\mathcal{M}_3(h)$ and its derived partial geometries, which were described in Section 3.1.

Not much is known about the number of mutually non-isomorphic Hermitian partial geometries. A remark in [70] implies that for $q = 3^{2h}$, $h \in \mathbb{N}\setminus\{0\}$, each partial geometry is isomorphic to the one where the "complementary" choices are made. Also in [70] there is a list of eleven mutually non-isomorphic partial geometries pg $(8, 40, 4)$ derived from $\mathcal{M}_3(1)$, the size of the automorphism group ranging from 9 to 58320. A general answer to this question seems difficult.

3.3 The isomorphism between Mathon's partial geometries and the Hermitian partial geometries for $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$

Let $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$, and consider the two classes of partial geometries described in Section 3.1 and Section 3.2, respectively. While in the description of the Hermitian ovoid $\mathcal{O}_H$ of $H(q)$ given in Subsection 1.7.3 the symbol γ could denote any non-square in $\mathrm{GF}(q)$, we now choose it to be as in Section 3.1, i.e. $\gamma = \beta^{q+1}$, where β is a primitive element of $\mathrm{GF}(q^2)$.

First we will show that the point set of $\mathcal{M}_3(h)$ (and of its derived partial geometries) is indeed a Hermitian curve minus one point. The set $\{1, \sqrt{\gamma}\}$ is a basis of $\mathrm{GF}(q^2)$ seen as a two-dimensional vector space over $\mathrm{GF}(q)$. Therefore we can write $\beta^{2i+\delta} = b_1(i,\delta) + b_2(i,\delta)\sqrt{\gamma}$, with $b_1(i,\delta), b_2(i,\delta) \in \mathrm{GF}(q)$, for each $i \in I$ and $\delta \in \{0,1\}$. Note that $\beta^r = \sqrt{\gamma}$, and let γ^∞ and $\sqrt{\gamma}^\infty$ denote the zero element in $\mathrm{GF}(q)$. Rewriting the expressions for $x^{i\delta}_{kl}$, $\tilde{x}^{i\delta}_{kl}$, $y^{i\delta}_{kjl}$ and $\tilde{y}^{i\delta}_{kjl}$ ($k, j, l \in K$, $i \in I$, $\delta \in \{0,1\}$) yields

$$\begin{aligned}
x^{i\delta}_{kl} &= \left(-b_1(i,\delta)\gamma^{l-1} + b_2(i,\delta)\gamma^k\right) + \left(-b_2(i,\delta)\gamma^{l-1} + b_2(i,\delta)\gamma^{k-1}\right)\sqrt{\gamma},\\
\tilde{x}^{i\delta}_{kl} &= \left(-b_1(i,\delta)\gamma^{l} + b_2(i,\delta)\gamma^k\right) + \left(-b_2(i,\delta)\gamma^{l} + b_2(i,\delta)\gamma^{k-1}\right)\sqrt{\gamma},\\
y^{i\delta}_{kjl} &= \left(\gamma^{2i+\delta}(\gamma^{-i}\gamma^{3k} + \gamma^i\gamma^{k-1} - \gamma^{k+l-1}) + \gamma^j\right) + \gamma^{2i+\delta}(\gamma^{2k-1} - \gamma^{2l-2})\sqrt{\gamma},\\
\tilde{y}^{i\delta}_{kjl} &= \left(\gamma^{2i+\delta}(\gamma^{-i}\gamma^{3k-1} + \gamma^i\gamma^{k} - \gamma^{k+l}) + \gamma^{j-1}\right) + \gamma^{2i+\delta}(\gamma^{2k-1} - \gamma^{2l})\sqrt{\gamma}.
\end{aligned}$$

Since $b_1(0,0) = 1$ and $b_2(0,0) = 0$, we find

$$\begin{aligned}
x^{00}_{kl} &= -\gamma^{l-1} + \gamma^{k-1}\sqrt{\gamma},\\
y^{00}_{kjl} &= (\gamma^{3k} + \gamma^{k-1} - \gamma^{k+l-1} + \gamma^j) + (\gamma^{2k-1} - \gamma^{2l-2})\sqrt{\gamma},
\end{aligned}$$

for all $k, j, l \in K$. Letting k (respectively l) vary in K is equivalent to letting $a := -\gamma^{k-1}$ (respectively $a'' := -\gamma^{l}$) vary in $\mathrm{GF}(q)$. One also sees that $a' := a(\gamma^3 a^2 + 1 - a'') - \gamma^j$ will take all possible values in $\mathrm{GF}(q)$ as j varies in K. Hence the point set of $\mathcal{M}_3(h)$ becomes

$$\mathcal{P} = \{(\gamma^{-1}a'' - a\sqrt{\gamma}, -a' + aa'' - (\gamma^{-2}a''^2 - \gamma a^2)\sqrt{\gamma}) \mid a, a', a'' \in \mathrm{GF}(q)\}.$$

From $\mathcal{P}$ one easily constructs the set

$$\{(\gamma^{-1}a'' - a\sqrt{\gamma}, 1, -a' + aa'' - (\gamma^{-2}a''^2 - \gamma a^2)\sqrt{\gamma}) \mid a, a', a'' \in \mathrm{GF}(q)\}$$

of q^3 points in $\mathrm{PG}(2, q^2)$; this is the set of points different from $(0, 0, 1)$ on the Hermitian curve $H(2, q^2)$ with equation $\gamma X_0^{q+1} + \sqrt{\gamma} X_1 X_2^q - \sqrt{\gamma} X_2 X_1^q = 0$. Moreover the lines L_{kl}^{∞}, $k, l \in K$, correspond to the $(q+1)$-secants of $H(2, q^2)$ through $(0, 0, 1)$.

Next we need an explicit isomorphism between the design of points and point reguli of the Hermitian ovoid $\mathcal{O}_H$ of $H(q)$ and the design of points and $(q+1)$-secants of $H(2, q^2)$. As in Subsection 1.7.3, intersect the quadric $Q(6, q)$ described by $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$ with the hyperplane Π with equation $X_5 = \gamma X_1$. Then $\mathcal{S}_H$ consists of the lines of $H(q)$ which lie on $Q^-(5, q) := \Pi \cap Q(6, q)$; the description via coordinatisation reads

$$\mathcal{S}_H = \{[\infty]\} \cup \{[k, -\gamma^{-1}k'', k', \gamma k, k''] \mid k, k', k'' \in \mathrm{GF}(q)\}.$$

The quadric $Q^-(5, q)$ seen over $\mathrm{GF}(q^2)$ becomes a $Q^+(5, q^2)$ which contains the planes

$$\eta :\equiv \begin{cases} X_3 = \sqrt{\gamma} X_1 \\ X_0 = \sqrt{\gamma} X_6 \\ X_2 = -\sqrt{\gamma} X_4 \\ X_5 = \gamma X_1 \end{cases} \text{ and } \overline{\eta} :\equiv \begin{cases} X_3 = -\sqrt{\gamma} X_1 \\ X_0 = -\sqrt{\gamma} X_6 \\ X_2 = \sqrt{\gamma} X_4 \\ X_5 = \gamma X_1. \end{cases}$$

These planes are disjoint and conjugate with respect to $\mathrm{GF}(q)$. Moreover the union of the intersection points of the lines (seen over $\mathrm{GF}(q^2)$) of the Hermitian spread $\mathcal{S}_H$ with η, or $\overline{\eta}$, is a Hermitian curve $H(2, q^2)$ (see Figure 3.2). The three-dimensional subspace $\langle L, M\rangle$ spanned by two lines L and M of $\mathcal{S}_H$ intersects $Q^+(5, q^2)$ in a $Q^+(3, q^2)$; clearly the line regulus determined by L and M is a subset of the regulus of $Q^+(3, q^2)$ to which L and M belong. As this line regulus is completely contained in $\mathcal{S}_H$, the quadric $Q^+(3, q^2)$ intersects η in at least $q + 1$ points. If the plane η were contained in $\langle L, M\rangle$, it would lie on the quadric $Q^+(3, q^2)$, a contradiction. Hence η intersects $\langle L, M\rangle$ in a line on $Q^+(3, q^2)$ which does not belong to the regulus containing L and M. This line contains $q + 1$ points of the Hermitian curve $H(2, q^2)$ in η.

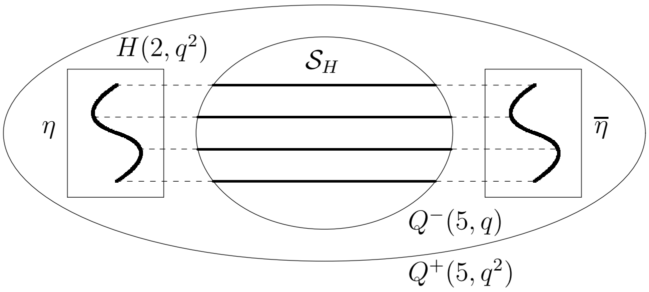


Figure 3.2: The Hermitian spread $\mathcal{S}_H$ of $H(q)$ and the Hermitian curve $H(2,q^2)$.

Consequently line reguli in a Hermitian spread of $H(q)$ indeed correspond to $(q+1)$-secants of a Hermitian curve in $\mathrm{PG}(2,q^2)$, as already mentioned in Subsection 1.7.3. To find an explicit isomorphism, we will determine the coordinates of the intersection points of the lines of $\mathcal{S}_H$ with the plane η. Writing only the "free" variables X_1, X_4 and X_6 in η, the point $[\infty]\cap\eta$ is $(0,0,1)$, while for $k,k',k''\in\mathrm{GF}(q)$ the point $[k,-\gamma^{-1}k'',k',\gamma k,k'']\cap\eta$ is $(\gamma^{-1}k''-k\sqrt{\gamma},1,(-k'+kk'')-(\gamma^{-2}k''^2-\gamma k^2)\sqrt{\gamma})$. This means that the Hermitian curve $H(2,q^2)$ in η has equation

$$\begin{cases} X_3=\sqrt{\gamma}X_1\\ X_0=\sqrt{\gamma}X_6\\ X_2=-\sqrt{\gamma}X_4\\ X_5=\gamma X_1\\ \gamma X_1^{q+1}+\sqrt{\gamma}X_4X_6^q-\sqrt{\gamma}X_6X_4^q,\end{cases}$$

and therefore is precisely the Hermitian curve on which $\mathcal{M}_3(h)$ lives. After applying the duality θ defined in Section 1.7.3 and translating the coordinatisation representation of the points of $\mathcal{O}_H$ into $\mathrm{PG}(6,q)$ coordinates according to Table 1.1, we obtain the following isomorphism φ between the design of points and point reguli in $\mathcal{O}_H$ and the classical unital arising from the Hermitian curve $H(2,q^2)$:

$$\begin{aligned}\varphi\ :\ & (1,0,0,0,0,0,0)\mapsto(0,0,1)\\ & (-\gamma^{-3}a''^4+a'^2-\gamma^3a^4+aa'a'',-a'',-a,-a'+aa'',1,\\ & \qquad -\gamma^{-3}a''^3-aa'-a^2a'',-\gamma^3a^3+a'a'')\\ & \mapsto(\gamma^{-1}a''-a\sqrt{\gamma},1,(-a'+aa'')-(\gamma^{-2}a''^2-\gamma a^2)\sqrt{\gamma}),\\ & \text{for all } a,a',a''\in\mathrm{GF}(q).\end{aligned}$$

Thus we have constructed a bijection between the point sets of the partial geometries which maps the lines L^{∞}_{kl}, $k, l \in K$, of $\mathcal{M}_3(h)$ to the type 2 lines of the Hermitian partial geometries.

Now we will show that φ maps the remaining lines of $\mathcal{M}_3(h)$ and $\tilde{\mathcal{M}}_3(h)$ to the cliques constructed in Step 1 of Section 3.2, and that the choices which have to be made in both constructions have the same meaning.

CASE 1. Suppose that $2i+\delta$ is divisible by $q+1$, or equivalently $b_2(i,\delta) = 0$. As $q+1$ is even, this requires $\delta = 0$, so we can put $2i =: m(i)(q+1)$ for a $m(i) \in \{0, \ldots, (q-3)/2\}$. This means that $\beta^{2i} = b_1(i,0) = \gamma^{m(i)}$ and hence $\gamma^{2i} = \gamma^{2m(i)}$. By definition the line L^{i0}_{jl} consists of all points $(x^{i0}_{kl}, y^{i0}_{kjl})$, $k \in K$. The first component of such a point becomes

$$x^{i0}_{kl} = -\gamma^{m(i)}\gamma^{l-1} + \gamma^{m(i)}\gamma^{k-1}\sqrt{\gamma};$$

putting $a'' := -\gamma^{m(i)}\gamma^{l}$ and $A := -\gamma^{m(i)}\gamma^{k-1}$ we find $x^{i0}_{kl} = \gamma^{-1}a'' - A\sqrt{\gamma}$, where A ranges over $\mathrm{GF}(q)$ as k ranges over K. Substituting this into the expression for y^{i0}_{kjl} yields

$$y^{i0}_{kjl} = (-\gamma^3\gamma^{-2m(i)}A^3 - \gamma^{2m(i)}A - a''A + \gamma^j) - (\gamma^{-2}a''^2 - \gamma A^2)\sqrt{\gamma},$$

where A ranges over $\mathrm{GF}(q)$ as k ranges over K. Hence the line L^{i0}_{jl} corresponds to the set

$$\begin{aligned} L \;\; := \;\; \{(&-\gamma^{-3}a''^4 + A'^2 - \gamma^3 A^4 + AA'a'', -a'', -A, -A' + Aa'', 1, \\ &-\gamma^{-3}a''^3 - AA' - A^2a'', -\gamma^3A^3 + A'a'') \mid A \in \mathrm{GF}(q)\} \end{aligned}$$

of points of $\mathcal{O}_H \setminus \{p\}$, where

$$A' := -\gamma^j + \gamma^{-2m(i)}A(\gamma^3A^2 + \gamma^{4m(i)}) - a''A \tag{3.1}$$

for all $A \in \mathrm{GF}(q)$. Obviously L is contained in s^π with $s := (a'', 0, 0, 0, 0, 1, 0)$ and contains the point $t := (-\gamma^{-3}a''^4 + \gamma^{2j}, -a'', 0, \gamma^j, 1, -\gamma^{-3}a''^3, -\gamma^j a'')$ (choose $A = 0$). We will show that L is actually a clique associated with (s, M, t), $M := \Gamma_2(p) \cap \Gamma_4(t)$, as constructed in Step 1 of the construction in Section 3.2. The quadric $Q(4,q) := p^\pi \cap t^\pi \cap Q(6,q)$ has equations

$$Q(4,q) \equiv \begin{cases} X_4 = 0 \\ X_0 = \gamma^{-3}a''^3X_1 + \gamma^j a''X_2 - \gamma^j X_3 + a''X_5 \\ X_1X_5 + X_2X_6 - X_3^2 = 0; \end{cases}$$

the quadric $Q^+(3,q) := \langle \mathcal{P}(p,t)\rangle^\pi \cap Q(6,q)$ is the intersection of $Q(4,q)$ with the hyperplane with equation $X_3 = a''X_2$. This implies that the generators of $Q(4,q)$ through s which do not lie in $Q^+(3,q)$ are the lines

$$N_{a''-n} := \langle s, (\gamma^j(a''-n), 0, 1, n, 0, 0, n^2)\rangle,$$

with $a'' - n \in \mathrm{GF}(q) \setminus \{0\}$. The four-dimensional space $N^{\pi}_{a''-n}$ is described by

$$N^{\pi}_{a''-n} \equiv \begin{cases} X_1 = -a''X_4 \\ n^2X_2 + nX_3 + \gamma^j(a''-n)X_4 + X_6 = 0. \end{cases}$$

Substituting the $\mathrm{PG}(6,q)$ coordinates of an arbitrary point of $\mathcal{O}_H \setminus \{p\}$ into these equations, one finds that the clique $N^{\pi}_{a''-n} \cap (\mathcal{O}_H \setminus \{p\})$ is the set

$$\{(-\gamma^{-3}a''^4 + A'^2 - \gamma^3A^4 + AA'a'', -a'', -A, -A' + Aa'', 1, \\ -\gamma^{-3}a''^3 - AA' - A^2a'', -\gamma^3A^3 + A'a'') \mid A \in \mathrm{GF}(q)\}$$

with

$$n^2(-A) + n(-A' + Aa'') + \gamma^j(a''-n) - \gamma^3A^3 + A'a'' = 0$$

for all $A \in \mathrm{GF}(q)$. The latter expression can be rewritten as

$$\begin{aligned} (a''-n)(A' + \gamma^j) &= A(\gamma^3A^2 + n^2 - na'') \\ &= A(\gamma^3A^2 + (a''-n)^2) - a''(a''-n)A, \end{aligned}$$

or equivalently

$$A' = -\gamma^j + (a''-n)^{-1}A(\gamma^3A^2 + (a''-n)^2) - a''A,$$

$A \in \mathrm{GF}(q)$. Comparing this to expression (3.1) we find

$$a'' - n = \gamma^{2m(i)}. \tag{3.2}$$

Hence the line L^{i0}_{jl} corresponds to the clique arising from $N_{\gamma^{2m(i)}}$.

In a similar way we find

$$\begin{aligned} \tilde{x}^{i0}_{kl} &= \gamma^{-1}a'' - A\sqrt{\gamma}, \\ \tilde{y}^{i0}_{kjl} &= (-\gamma^2\gamma^{-2m(i)}A^3 - \gamma\gamma^{2m(i)}A - a''A + \gamma^{j-1}) - (\gamma^{-2}a''^2 - \gamma A^2)\sqrt{\gamma} \end{aligned}$$

(here we put $a'' := -\gamma^{m(i)}\gamma^{l+1}$ and $A := -\gamma^{m(i)}\gamma^{k-1}$), where A ranges over $\mathrm{GF}(q)$ as k ranges over K. Hence the line $\tilde{L}^{i0}_{jl}$ corresponds to the set

$$\begin{aligned} \tilde{L} \ := \ \{(&-\gamma^{-3}a''^4 + A'^2 - \gamma^3A^4 + AA'a'', -a'', -A, -A' + Aa'', 1, \\ &-\gamma^{-3}a''^3 - AA' - A^2a'', -\gamma^3A^3 + A'a'') \mid A \in \mathrm{GF}(q)\} \end{aligned}$$

of points of $\mathcal{O}_H \setminus \{p\}$, where

$$A' := -\gamma^{j-1} + \gamma^{-2m(i)+1}A(\gamma A^2 + \gamma^{4m(i)}) - a''A \tag{3.3}$$

for all $A \in \mathrm{GF}(q)$. Again $\tilde{L}$ is contained in s^{π} for $s := (a'', 0, 0, 0, 0, 1, 0)$; it contains the point $\tilde{t} := (-\gamma^{-3}a''^4 + \gamma^{2j-2}, -a'', 0, \gamma^{j-1}, 1, -\gamma^{-3}a''^3, -\gamma^{j-1}a'')$

(choose $A = 0$). The cliques $N^{\pi}_{a''-n} \cap (\mathcal{O}_H \setminus \{p\})$ associated with $(s, M, \tilde{t})$ are the sets

$$\{(-\gamma^{-3}a''^4 + A'^2 - \gamma^3 A^4 + AA'a'', -a'', -A, -A' + Aa'', 1, \\ -\gamma^{-3}a''^3 - AA' - A^2 a'', -\gamma^3 A^3 + A'a'') \mid A \in \mathrm{GF}(q)\}$$

with $a'' - n \in \mathrm{GF}(q) \setminus \{0\}$ and

$$n^2(-A) + n(-A' + Aa'') + \gamma^{j-1}(a'' - n) - \gamma^3 A^3 + A'a'' = 0$$

for all $A \in \mathrm{GF}(q)$. The latter expression is equivalent to

$$A' = -\gamma^{j-1} + \gamma^2 (a'' - n)^{-1} A(\gamma A^2 + \gamma^{-2}(a'' - n)^2) - a''A,$$

$A \in \mathrm{GF}(q)$. Comparing this to expression (3.3) yields

$$a'' - n = \gamma^{2m(i)+1}. \tag{3.4}$$

Hence the line $\tilde{L}^{i0}_{jl}$ corresponds to the clique arising from $N_{\gamma^{2m(i)+1}}$.

Recapitulating, we see that the indices i and l of a line L^{i0}_{jl} or $\tilde{L}^{i0}_{jl}$ uniquely determine a'' and hence s, the index j yields a point t in the clique, and the index i yields a unique clique associated with (s, M, t), $M := \Gamma_2(p) \cap \Gamma_4(t)$. The converse works as well. For a line associated with a triple (s, M, t) with $s := (a'', 0, 0, 0, 0, 1, 0)$, $a'' \in \mathrm{GF}(q)$, the point t can be chosen such that it lies in $X_2 = 0$. The non-zero $\mathrm{GF}(q)$ element $a'' - n$ used in the construction of the line will uniquely determine i, and will also tell whether we will find a line L^{i0}_{jl} of $\mathcal{M}_3(h)$ or a line $\tilde{L}^{i0}_{jl}$ of $\tilde{\mathcal{M}}_3(h)$. If $a'' - n$ is a square in $\mathrm{GF}(q)$, there are two solutions $m(i)$ for Equation 3.2; only one of them corresponds to an i in I. In this case the line will be L^{i0}_{jl}. If $a'' - n$ is a non-square in $\mathrm{GF}(q)$, there are two solutions $m(i)$ for Equation 3.4; again only one of them corresponds to an i in I. In this case the line will be $\tilde{L}^{i0}_{jl}$. Together, a'' and i determine l, via $a'' := -\gamma^{m(i)}\gamma^l$, respectively $a'' := -\gamma^{m(i)}\gamma^{l+1}$. Finally j is deduced from the point t. It follows that φ induces a bijection between the set of lines L^{i0}_{jl} and $\tilde{L}^{i0}_{jl}$ with $j, l \in K$, $i \in I$, $(q+1) \mid 2i$, and the set of cliques associated with a triple with a point $s = (a'', 0, 0, 0, 0, 1, 0)$, $a'' \in \mathrm{GF}(q)$, as a first component.

Now it remains to show that the choice between R^{i0}_l and $\tilde{R}^{i0}_l$, $l \in K$, $i \in I$, $(q+1) \mid 2i$, corresponds to the choice between the two sets of $(q-1)/2$ generators through $s = (a'', 0, 0, 0, 0, 1, 0)$, $a'' \in \mathrm{GF}(q)$, on a quadric $Q(4, q)$. From the definition of R^{i0}_l (respectively $\tilde{R}^{i0}_l$) it is clear that $R^{i0}_l = R^{00}_l$ (respectively $\tilde{R}^{i0}_l = \tilde{R}^{00}_l$) for all $l \in K$, $i \in I$, $(q+1) \mid 2i$. Each line $L^{mr,0}_{j,l-m}$ of R^{00}_l gives rise to the same $a'' = -\gamma^m \gamma^{l-m} = -\gamma^l$ and hence to the same point $s = (a'', 0, 0, 0, 0, 1, 0)$, and different $l \in K$ define different $a'' \in \mathrm{GF}(q)$. Similarly each line $\tilde{L}^{mr,0}_{j,l-m-1}$ of $\tilde{R}^{00}_l$ yields the same $a'' = -\gamma^m \gamma^{l-m-1+1} = -\gamma^l$ and hence the same point $s = (a'', 0, 0, 0, 0, 1, 0)$ which was also obtained from R^{00}_l; again different $l \in K$ define different $a'' \in \mathrm{GF}(q)$. Thus we find a one-to-one correspondence between the Baer subnets and the points in $\Gamma_2(p)$ which are relevant in this case. Now fix elements l and j of K, and define $a'' := -\gamma^l$, $s := (a'', 0, 0, 0, 0, 1, 0)$ and $t := (-\gamma^{-3}a''^4 + \gamma^{2j}, -a'', 0, \gamma^j, 1, -\gamma^{-3}a''^3, -\gamma^j a'')$. One checks that $\{L^{mr,0}_{j,l-m} \mid m \in \{0, \ldots, r-2\}\}$ and $\{\tilde{L}^{mr,0}_{j+1,l-m-1} \mid m \in \{0, \ldots, r-2\}\}$ are precisely the sets of cliques corresponding to the two sets of generators through s of the quadric $Q(4, q) := p^\pi \cap t^\pi \cap Q(6, q)$ between which we have to choose in Step 1 of Section 3.2. Letting j vary in K corresponds to projecting generators as in Step 2 of Section 3.2, which proves that indeed the choice for $s = (-\gamma^l, 0, 0, 0, 0, 1, 0)$ corresponds to the choice between R^{00}_l and $\tilde{R}^{00}_l$ for each $l \in K$.

CASE 2. Suppose that $2i + \delta$ is not divisible by $q + 1$, or equivalently $b_2(i, \delta) \neq 0$. The same methods as in Case 1 can be used, but the calculations get longer. As $b_2(i, \delta) \neq 0$, we can write $b_1(i, \delta) = \mu b_2(i, \delta)$ for a certain $\mu \in \mathrm{GF}(q)$, which implies $\beta^{2i+\delta} = b_2(i, \delta)(\mu + \sqrt{\gamma})$ and hence $\gamma^{2i+\delta} = b_2(i, \delta)^2(\mu^2 - \gamma)$. The first component of a point $(x^{i\delta}_{kl}, y^{i\delta}_{kjl})$ of the line $L^{i\delta}_{jl}$ becomes

$$\begin{aligned} x^{i\delta}_{kl} &= b_2(i, \delta)\left((-\mu\gamma^{l-1} + \gamma^k) + (\mu\gamma^{k-1} - \gamma^{l-1})\sqrt{\gamma}\right) \\ &= \gamma^{-1}A'' - A\sqrt{\gamma}, \end{aligned}$$

with $A'' := b_2(i, \delta)(-\mu\gamma^l + \gamma^{k+1})$ and $A := b_2(i, \delta)(-\mu\gamma^{k-1} + \gamma^{l-1})$. Putting $a := b_2(i, \delta)\gamma^{l-1}$, we find the following expression of A in function of A'':

$$A = a - \mu\gamma^{-2}(A'' + \mu\gamma a), \tag{3.5}$$

$A'' \in \mathrm{GF}(q)$. Moreover we can write $\gamma^k = \gamma^{-1}b_2(i, \delta)^{-1}(A'' + \mu\gamma a)$. Substituting this into $y^{i\delta}_{kjl}$ yields

$$\begin{aligned} y^{i\delta}_{kjl} = {} & \left[(\mu^2 - \gamma)\left(\gamma^{-3}\gamma^{-i}b_2(i, \delta)^{-1}(A'' + \mu\gamma a)^3 + \gamma^{-2}\gamma^i b_2(i, \delta)(A'' + \mu\gamma a)\right.\right. \\ & \left.\left. - a\gamma^{-1}(A'' + \mu\gamma a)\right) + \gamma^j\right] \\ & - \left[\gamma^{-2}A''^2 - \gamma(a - \mu\gamma^{-2}(A'' + \mu\gamma a))^2\right]\sqrt{\gamma}. \end{aligned}$$

As a consequence the line $L^{i\delta}_{jl}$ corresponds to the set

$$\begin{aligned} L := {} & \{(-\gamma^{-3}A''^4 + A'^2 - \gamma^3 A^4 + AA'A'', -A'', -A, -A' + AA'', 1, \\ & -\gamma^{-3}A''^3 - AA' - A^2A'', -\gamma^3A^3 + A'A'') \mid A'' \in \mathrm{GF}(q)\} \end{aligned}$$

of points of $\mathcal{O}_H \setminus \{p\}$, with A as in expression (3.5) and

$$\begin{aligned} A' &= -\gamma^j - \mu\gamma a^2 - \gamma^{-3}(\mu^2-\gamma)\gamma^{-i}b_2(i,\delta)^{-1}(A''+\mu\gamma a)^3 \\ &\quad - \mu\gamma^{-2}(A''+\mu\gamma a)^2 - \gamma^{-2}(\mu^2-\gamma)\gamma^i b_2(i,\delta)(A''+\mu\gamma a) \\ &\quad - \gamma^{-1}\mu^2 a(A''+\mu\gamma a) \end{aligned} \tag{3.6}$$

for all $A'' \in \mathrm{GF}(q)$. Expression (3.5) implies that L is contained in s^π with $s := (a - \mu^2\gamma^{-1}a, 0, 0, 0, 0, \mu\gamma^{-2}, 1)$. Putting $a' := -\gamma^j - \mu\gamma a^2$, it is not difficult to see that the point

$$\begin{aligned} t := (&-\gamma^{-3}(-\mu\gamma a)^4 + a'^2 - \gamma^3 a^4 + aa'(-\mu\gamma a), \mu\gamma a, -a, -a' - \mu\gamma a^2, \\ &1, -\gamma^{-3}(-\mu\gamma a)^3 - aa' - a^2(-\mu\gamma a), -\gamma^3 a^3 + a'(-\mu\gamma a)) \end{aligned}$$

is contained in L (choose $A'' = -\mu\gamma a$). After a few calculations one finds that the cliques $N^\pi_{n+a-\mu^2\gamma^{-1}a} \cap (\mathcal{O}_H \setminus \{p\})$ associated with (s, M, t), where $M := \Gamma_2(p) \cap \Gamma_4(t)$, are the sets

$$\begin{aligned} \{(&-\gamma^{-3}A''^4 + A'^2 - \gamma^3 A^4 + AA'A'', -A'', -A, -A' + AA'', 1, \\ &-\gamma^{-3}A''^3 - AA' - A^2A'', -\gamma^3A^3 + A'A'') \mid A'' \in \mathrm{GF}(q)\}, \end{aligned}$$

with $n + a - \mu^2\gamma^{-1}a \in \mathrm{GF}(q) \setminus \{0\}$, A as in expression (3.5) and

$$\begin{aligned} &n^2(-A'') + n(-A' + AA'') + (-\gamma^{-3}A''^3 - AA' - A^2A'') \\ &\quad - \mu\gamma^{-2}(-\gamma^3A^3 + A'A'') + (-\mu a^3 - \gamma^{-1}aa')(\mu^2-\gamma) \\ &\quad + n(a' + \mu\gamma a^2) - n^2\mu\gamma a = 0 \end{aligned}$$

for all $A'' \in \mathrm{GF}(q)$. Rewriting the latter expression we find, after a bit of work,

$$\begin{aligned} A' &= -\gamma^j - \mu\gamma a^2 - \gamma^{-5}(\mu^2-\gamma)(n + a - \mu^2\gamma^{-1}a)^{-1}(A''+\mu\gamma a)^3 \\ &\quad - \mu\gamma^{-2}(A''+\mu\gamma a)^2 - (n+a)(A''+\mu\gamma a) \end{aligned}$$

for all $A'' \in \mathrm{GF}(q)$. This is the same function of A'' as expression (3.6) if and only if

$$n + a - \mu^2\gamma^{-1}a = \gamma^{-2}(\mu^2-\gamma)\gamma^i b_2(i,\delta). \tag{3.7}$$

Hence the line $L^{i\delta}_{jl}$ corresponds to the clique arising from $N_{\gamma^{-2}(\mu^2-\gamma)\gamma^i b_2(i,\delta)}$.

Similarly $\tilde{x}^{i\delta}_{kl}$ can be written as

$$\begin{aligned} \tilde{x}^{i\delta}_{kl} &= b_2(i,\delta)\left((-\mu\gamma^l + \gamma^k) + (\mu\gamma^{k-1} - \gamma^l)\sqrt{\gamma}\right) \\ &= \gamma^{-1}A'' - A\sqrt{\gamma}, \end{aligned}$$

with $A'' := b_2(i,\delta)(-\mu\gamma^{l+1}+\gamma^{k+1})$ and $A := b_2(i,\delta)(-\mu\gamma^{k-1}+\gamma^l)$. Putting $a := b_2(i,\delta)\gamma^l$, we find the following expression of A in function of A'':

$$A = a - \mu\gamma^{-2}(A'' + \mu\gamma a), \tag{3.8}$$

$A'' \in \mathrm{GF}(q)$. We can also write $\gamma^k = \gamma^{-1}b_2(i,\delta)^{-1}(A''+\mu\gamma a)$. Substituting this into $\tilde{y}^{i\delta}_{kjl}$ yields

$$\begin{aligned}\tilde{y}^{i\delta}_{kjl} \;=\; & \big[(\mu^2-\gamma)\big(\gamma^{-4}\gamma^{-i}b_2(i,\delta)^{-1}(A''+\mu\gamma a)^3 + \gamma^{-1}\gamma^i b_2(i,\delta)(A''+\mu\gamma a)\\ & - a\gamma^{-1}(A''+\mu\gamma a)\big) + \gamma^{j-1}\big]\\ & - \left[\gamma^{-2}A''^2 - \gamma(a-\mu\gamma^{-2}(A''+\mu\gamma a))^2\right]\sqrt{\gamma}.\end{aligned}$$

This implies that $\tilde{L}^{i\delta}_{jl}$ corresponds to the set

$$\begin{aligned}\tilde{L} \;:=\; & \{(-\gamma^{-3}A''^4 + A'^2 - \gamma^3A^4 + AA'A'', -A'', -A, -A' + AA'', 1,\\ & -\gamma^{-3}A''^3 - AA' - A^2A'', -\gamma^3A^3 + A'A'') \mid A'' \in \mathrm{GF}(q)\}\end{aligned}$$

of points of $\mathcal{O}_H \setminus \{p\}$, with A as in expression (3.8) and

$$\begin{aligned}A' \;=\; & -\gamma^{j-1} - \mu\gamma a^2 - \gamma^{-4}(\mu^2-\gamma)\gamma^{-i}b_2(i,\delta)^{-1}(A''+\mu\gamma a)^3\\ & - \mu\gamma^{-2}(A''+\mu\gamma a)^2 - \gamma^{-1}(\mu^2-\gamma)\gamma^i b_2(i,\delta)(A''+\mu\gamma a)\\ & - \gamma^{-1}\mu^2 a(A''+\mu\gamma a)\end{aligned} \tag{3.9}$$

for all $A'' \in \mathrm{GF}(q)$. Expression (3.8) implies that $\tilde{L}$ is contained in s^π with $s := (a-\mu^2\gamma^{-1}a, 0, 0, 0, 0, \mu\gamma^{-2}, 1)$. If we put $a' := -\gamma^{j-1} - \mu\gamma a^2$, we see that the point

$$\begin{aligned}\tilde{t} := {} & (-\gamma^{-3}(-\mu\gamma a)^4 + a'^2 - \gamma^3a^4 + aa'(-\mu\gamma a), \mu\gamma a, -a, -a' - \mu\gamma a^2,\\ & 1, -\gamma^{-3}(-\mu\gamma a)^3 - aa' - a^2(-\mu\gamma a), -\gamma^3a^3 + a'(-\mu\gamma a))\end{aligned}$$

is contained in $\tilde{L}$ (choose $A'' = -\mu\gamma a$). After a few calculations one finds that the cliques $N^\pi_{n+a-\mu^2\gamma^{-1}a} \cap (\mathcal{O}_H \setminus \{p\})$ associated with $(s, \tilde{M}, \tilde{t})$, with $\tilde{M} := \Gamma_2(p) \cap \Gamma_4(\tilde{t})$, are the sets

$$\begin{aligned}& \{(-\gamma^{-3}A''^4 + A'^2 - \gamma^3A^4 + AA'A'', -A'', -A, -A' + AA'', 1,\\ & -\gamma^{-3}A''^3 - AA' - A^2A'', -\gamma^3A^3 + A'A'') \mid A'' \in \mathrm{GF}(q)\},\end{aligned}$$

with $n + a - \mu^2\gamma^{-1}a \in \mathrm{GF}(q) \setminus \{0\}$, A as in expression (3.8) and

$$\begin{aligned}& n^2(-A'') + n(-A' + AA'') + (-\gamma^{-3}A''^3 - AA' - A^2A'')\\ & - \mu\gamma^{-2}(-\gamma^3A^3 + A'A'') + (-\mu a^3 - \gamma^{-1}aa')(\mu^2-\gamma)\\ & + n(a' + \mu\gamma a^2) - n^2\mu\gamma a = 0\end{aligned}$$

for all $A'' \in \mathrm{GF}(q)$. The latter expression turns out to be equivalent to

$$\begin{aligned} A' &= -\gamma^{j-1} - \mu\gamma a^2 - \gamma^{-5}(\mu^2-\gamma)(n+a-\mu^2\gamma^{-1}a)^{-1}(A''+\mu\gamma a)^3 \\ &\quad - \mu\gamma^{-2}(A''+\mu\gamma a)^2 - (n+a)(A''+\mu\gamma a), \end{aligned}$$

for all $A'' \in \mathrm{GF}(q)$. This is the same function of A'' as expression (3.9) if and only if

$$n + a - \mu^2\gamma^{-1}a = \gamma^{-1}(\mu^2-\gamma)\gamma^i b_2(i,\delta). \tag{3.10}$$

Hence the line $\tilde{L}^{i\delta}_{jl}$ corresponds to the clique arising from $N_{\gamma^{-1}(\mu^2-\gamma)\gamma^i b_2(i,\delta)}$.

Note that the indices i, δ and l of a line $L^{i\delta}_{jl}$ or $\tilde{L}^{i\delta}_{jl}$ uniquely determine μ, a and hence s; the index j yields a point t in the clique, and the indices i and δ define a unique clique associated with (s, M, t), $M := \Gamma_p \cap \Gamma_4(t)$. The converse works as well. If L is a line associated with a triple (s, M, t), then s uniquely determines μ and a in $\mathrm{GF}(q)$ such that $s = (a - \mu^2\gamma^{-1}a, 0, 0, 0, 0, \mu\gamma^{-2}, 1)$. The point t can be chosen such that it lies in $X_1 = -\mu\gamma X_2$. The non-zero $\mathrm{GF}(q)$ element $n + a - \mu^2\gamma^{-1}a$ will determine i and δ and will tell us whether we have a line $L^{i\delta}_{jl}$ of $\mathcal{M}_3(h)$ or a line $\tilde{L}^{i\delta}_{jl}$ of $\tilde{\mathcal{M}}_3(h)$ in the following way. Put $\beta^{2i+\delta} =: \beta^{g+u(i,\delta)(q+1)}$ with $g \in \{1, \ldots, q\}$; note that $u(i,\delta)$ must be in $\{0, \ldots, (q-3)/2\}$ since $i \in I$. Recall that $\beta^{2i+\delta} = b_2(i,\delta)(\mu + \sqrt{\gamma})$; consequently g is the unique element of $\{1, \ldots, q\}$ such that $\beta^g(\mu + \sqrt{\gamma})$ is an element of $\mathrm{GF}(q)$. Knowing g, it is now possible to determine δ: as $\beta^{2i+\delta} = \beta^{g+u(i,\delta)(q+1)}$, δ and g must be equivalent modulo 2. It follows that $i = u(i,\delta)(q+1)/2 + (g-\delta)/2$; this is indeed an element of I. Now we can write $\gamma^i = \gamma^{u(i,\delta)(q+1)/2+(g-\delta)/2} = (-\gamma)^{u(i,\delta)}\gamma^{(g-\delta)/2}$. If this, together with $b_2(i,\delta) = \gamma^{u(i,\delta)}\beta^g(\mu+\sqrt{\gamma})^{-1}$, is substituted in Equations 3.7 and 3.10, we obtain either

$$(-\gamma^2)^{u(i,\delta)} = \gamma^2(\mu^2-\gamma)^{-1}\gamma^{-(g-\delta)/2}\beta^{-g}(\mu+\sqrt{\gamma})(n+a-\mu^2\gamma^{-1}a)$$

or

$$(-\gamma^2)^{u(i,\delta)} = \gamma(\mu^2-\gamma)^{-1}\gamma^{-(g-\delta)/2}\beta^{-g}(\mu+\sqrt{\gamma})(n+a-\mu^2\gamma^{-1}a).$$

Obviously only one of the two can be true, as the right-hand sides differ by a non-square factor γ. If the first equality is true, we obtain a line $L^{i\delta}_{jl}$, while if the second one is true we have a line $\tilde{L}^{i\delta}_{jl}$. Two possible solutions for $u(i,\delta)$ are found, of which only one is in $\{0, \ldots, (q-3)/2\}$. This $u(i,\delta)$ together with g determines i. From i, δ and a the index l can be deduced, and j follows from the point t. Thus it is proved that the bijection φ induces a bijection from the set of lines $L^{i\delta}_{jl}$ of $\mathcal{M}_3(h)$ and $\tilde{L}^{i\delta}_{jl}$ of $\tilde{\mathcal{M}}_3(h)$ with $j, l \in K$, $i \in I$, $\delta \in \{0,1\}$, $(q+1) \nmid 2i+\delta$, to the set of cliques associated with a triple with a point $s = (a - \mu^2\gamma^{-1}a, 0, 0, 0, 0, \mu\gamma^{-2}, 1)$, $a, \mu \in \mathrm{GF}(q)$, as a first component.

Now it has to be proved that the choice between $R_l^{i\delta}$ and $\tilde{R}_l^{i\delta}$, $l \in K$, $i \in I$, $\delta \in \{0,1\}$, $(q+1) \nmid 2i + \delta$, corresponds to the choice between two sets of $(q-1)/2$ generators through $s = (a - \mu^2\gamma^{-1}a, 0, 0, 0, 0, \mu\gamma^{-2}, 1)$, with $a, \mu \in \mathrm{GF}(q)$, on a quadric $Q(4,q)$. From the definition of $R_l^{i\delta}$, respectively $\tilde{R}_l^{i\delta}$, it is obvious that we may choose i and δ such that $2i + \delta \in \{1, \ldots, q\}$. One calculates that $b_1(i + mr, \delta) = \gamma^m b_1(i, \delta)$ and $b_2(i + mr, \delta) = \gamma^m b_2(i, \delta)$, so all lines of $R_l^{i\delta}$ and $\tilde{R}_l^{i\delta}$ define the same μ and the same a, and hence the same point $s = (a - \mu^2\gamma^{-1}a, 0, 0, 0, 0, \mu\gamma^{-2}, 1)$, while different Baer subnets lead to different points in $\Gamma_2(p)$. Thus we have a one-to-one correspondence between the Baer subnets and the points in $\Gamma_2(p)$ which are relevant in this case. Now fix elements $i \in I$, $\delta \in \{0,1\}$ and $l, j \in K$, and put $a := b_2(i, \delta)$, $\mu := b_1(i, \delta)/b_2(i, \delta)$, $a' := -\gamma^j - \mu\gamma a^2$ and

$$\begin{aligned} t := (&-\gamma^{-3}(-\mu\gamma a)^4 + a'^2 - \gamma^3 a^4 + aa'(-\mu\gamma a), \mu\gamma a, -a, -a' - \mu\gamma a^2, \\ &1, -\gamma^{-3}(-\mu\gamma a)^3 - aa' - a^2(-\mu\gamma a), -\gamma^3 a^3 + a'(-\mu\gamma a)) \end{aligned}$$

Then one verifies that the two sets $\{L_{j,l-m}^{i+mr,\delta} \mid m \in \{0, \ldots, (q-3)/2\}\}$ and $\{\tilde{L}_{j+1,l-m-1}^{i+mr,\delta} \mid m \in \{0, \ldots, (q-3)/2\}\}$ are precisely the sets of cliques corresponding to the two sets of generators through s contained in the quadric $Q(4,q) := p^\pi \cap t^\pi \cap Q(6,q)$ between which we had to choose in Step 1 of Section 3.2. Letting j vary in K corresponds to projecting generators as in Step 2 of Section 3.2, while all possibilities for $i \in I$, $\delta \in \{0,1\}$ and $l \in K$ account for all points $s \in \Gamma_2(p)$ which were not yet treated in Case 1.

Thus we have proved the following theorem.

Theorem 3.3.1 *For $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$, the Hermitian partial geometries are isomorphic to $\mathcal{M}_3(h)$ and its derived partial geometries.*

3.4 Slices and Baer subnets

Let $\mathcal{O}$ be any ovoid of $Q(6,q)$, q an odd prime power, and let π denote the orthogonal polarity associated with $Q(6,q)$. Choose a point x which does not belong to $\mathcal{O}$. An easy counting argument learns that precisely $q^2 + 1$ points of $\mathcal{O}$ are collinear on $Q(6,q)$ with x. Equivalently, these points lie in the cone $x^\pi \cap Q(6,q)$. Project them from x onto a four-dimensional subspace Π of x^π which does not contain x. Then q^2+1 points of the quadric $Q(4,q) := \Pi \cap Q(6,q)$ are obtained. If two of them were collinear, their inverse images under the projection would also be collinear, a contradiction. As a consequence, the q^2+1 points form an ovoid $\mathcal{O}'$ of $Q(4,q)$ which is called a *slice* of $\mathcal{O}$ (see [94]). Not surprisingly, the two-graph obtained from $\mathcal{O}'$ using Theorem 1.9.5 is precisely the sub-two-graph induced on $\mathcal{O} \cap x^\pi$ of the two-graph associated with $\mathcal{O}$.

The *Kantor ovoid* of $Q(4,q)$, q an odd prime power, has the following standard form if the quadric is described by $X_0X_3 + X_1X_4 - X_2^2 = 0$:

$$\mathcal{O}_K(\sigma) = \{(1,0,0,0,0)\} \cup \{(-\nu a^{\sigma+1} + l^2, a, l, 1, \nu a^\sigma) \mid a, l \in \mathrm{GF}(q)\},$$

where ν is any non-square in $\mathrm{GF}(q)$ and $\sigma \neq \mathbb{1}$ is an automorphism of $\mathrm{GF}(q)$. One verifies that $\mathcal{O}_K(\sigma)$ is the union of q conics which two by two intersect in the point $(1,0,0,0,0)$. Any slice of a Hermitian ovoid of $H(3^h)$ with $h \in \mathbb{N} \setminus \{0,1\}$, seen as an ovoid of $Q(6,3^h)$, is a Kantor ovoid $\mathcal{O}_K(\sigma)$ where the field automorphism is $\sigma : x \mapsto x^3$. A slice of a Hermitian ovoid of $H(3)$, seen as an ovoid of $Q(6,3)$, is always an elliptic quadric $Q^-(3,3)$ (see [76]).

Now we will determine the effect of the slice construction on the Hermitian partial geometries constructed in Section 3.2. Let p be the point of the Hermitian ovoid $\mathcal{O}_H$ of $H(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0,1\}$, which does not belong to the point set of the partial geometries. Let π be the orthogonal polarity associated with the quadric $Q(6,q)$ underlying $H(q)$. For any point $s \in \Gamma_2(p)$ there are precisely q^2 points of $\mathcal{O}_H \setminus \{p\}$ which are contained in s^π. If t is such a point, then the line $M := \Gamma_2(p) \cap \Gamma_4(t)$ contains s. By Lemma 3.2.1, the point regulus $\mathcal{P}(p,t)$ is completely contained in M^π and hence in s^π. It follows that the set $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$ is the union of q lines of type 2 of the Hermitian partial geometries. All other type 2 lines are disjoint from $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$. A line of type 1 of a Hermitian partial geometry is contained in s'^π for a unique point $s' \in \Gamma_2(p)$; this s' is the first component of each of the q triples to which it is associated. Consequently, there are $q(q-1)/2$ lines of type 1 which are contained in $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$. All other type 1 lines intersect $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$ in one point or are disjoint from it. Now project from s onto a four-dimensional subspace Π, with $s \notin \Pi \subseteq s^\pi$, all points of $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$ and all type 1 and type 2 lines which are contained in it. This yields a partial linear space of order $(q-1,(q-1)/2)$ on the Kantor ovoid $\mathcal{O}_K(\sigma)$ where σ is the automorphism of $\mathrm{GF}(q)$ which maps every element to its third power. Since the point graph of this partial linear space has the same parameters as the Paley graph $\mathcal{P}(q^2)$, Theorem 1.2.7 implies that we have a $\mathrm{pg}\,(q-1,(q-1)/2,(q-1)/2)$. If $q = 3^{2h}$, $h \in \mathbb{N} \setminus \{0\}$, then it is one of the Baer subnets of $\mathcal{M}_3(h)$ or $\tilde{\mathcal{M}}_3(h)$.

An explicit description of this net is found by choosing for s a point with easy coordinates, for instance $s = (0,0,0,0,0,1,0)$. Then s^π is the hyperplane described by $X_1 = 0$, and $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$ is the set

$$\{(-\gamma^3 a^4 + a'^2, 0, -a, -a', 1, -aa', -\gamma^3 a^3) \mid a, a' \in \mathrm{GF}(q)\}.$$

The type 2 lines contained in $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$ are the sets

$$\{(-\gamma^3 a^4 + A'^2, 0, -a, -A', 1, -aA', -\gamma^3 a^3) \mid A' \in \mathrm{GF}(q)\},$$

with $a \in \mathrm{GF}(q)$. With the help of Section 3.3, one calculates that the lines of type 1 which are contained in $s^\pi \cap (\mathcal{O}_H \setminus \{p\})$ are of the form

$$\{(-\gamma^3 A^4 + A'^2, 0, -A, -A', 1, -AA', -\gamma^3 A^3) \mid A \in \mathrm{GF}(q)\},$$

where A' depends on A as $A' = a' - k^{-1}\gamma^3 A^3 - kA$. Here a' ranges over $\mathrm{GF}(q)$, and the set of all possible k is either the set of non-squares in $\mathrm{GF}(q)$ or the set of non-zero squares in $\mathrm{GF}(q)$. After projecting from s onto the four-dimensional subspace $\Pi \equiv X_1 = X_5 = 0$ and putting $x := -a$, $y := -a'$ and $\nu := \gamma^3$ we obtain the Kantor ovoid

$$\{(1, 0, 0, 0, 0)\} \cup \{(-\nu x^4 + y^2, x, y, 1, \nu x^3) \mid x, y \in \mathrm{GF}(q)\}$$

on the quadric $Q(4, q)$ in Π which is described by $X_0 X_3 + X_1 X_4 - X_2^2 = 0$. The type 1 lines are projected onto the sets

$$\{(-\nu x^4 + y^2, x, y, 1, \nu x^3) \mid y \in \mathrm{GF}(q)\},$$

for $x \in \mathrm{GF}(q)$, while the lines of type 2 become sets of the form

$$\{(-\nu x^4 + y^2, x, y, 1, \nu x^3) \mid x \in \mathrm{GF}(q)\},$$

where $y := -a' - k^{-1}\nu x^3 - kx$ and a' and k as before.

By modifying the above description, we can construct a net on *any* Kantor ovoid of $Q(4, q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0, 1\}$. Let $\sigma \neq \mathbb{1}$ be an automorphism of $\mathrm{GF}(q)$, and let

$$\mathcal{O}_K(\sigma) = \{(1, 0, 0, 0, 0)\} \cup \{(-\nu x^{\sigma+1} + y^2, x, y, 1, \nu x^\sigma) \mid x, y \in \mathrm{GF}(q)\},$$

with ν a non-square in $\mathrm{GF}(q)$, be the corresponding Kantor ovoid. Put $p := (1, 0, 0, 0, 0)$ and let π be the orthogonal polarity associated with $Q(4, q)$. Let $\mathcal{S}$ be the incidence structure with point set $\mathcal{O}_K(\sigma) \setminus \{(1, 0, 0, 0, 0)\}$ in which the lines are the following sets:

$$\{(-\nu x^{\sigma+1} + y^2, x, y, 1, \nu x^\sigma) \mid y \in \mathrm{GF}(q)\},$$

for $x \in \mathrm{GF}(q)$, and

$$\{\{(-\nu x^{\sigma+1} + y^2, x, y, 1, \nu x^\sigma) \mid x \in \mathrm{GF}(q)\},$$

with $y := -a' - k^{-1}\nu x^\sigma - kx$. Here a' ranges over $\mathrm{GF}(q)$ and the set of all possible k is either the set of non-squares in $\mathrm{GF}(q)$ or the set of non-zero squares in $\mathrm{GF}(q)$. Lines of the first type are never concurrent, while any line of the first type intersects any line of the second type in exactly one point. An easy calculation learns that no two lines of the second type have more than one common point either. Hence $\mathcal{S}$ is a partial linear space of order $(q-1,(q-1)/2)$. Moreover, one can verify that for any two points t and t' which are collinear in $\mathcal{S}$ the line $p^\pi \cap t^\pi \cap t'^\pi$ intersects $Q(4,q)$ in two points. Consequently the lines of $\mathcal{S}$ are cliques in the descendant with respect to p of the two-graph associated with $\mathcal{O}_K(\sigma)$. By Theorem 1.2.7, we have indeed constructed a $\mathrm{pg}\,(q-1,(q-1)/2,(q-1)/2)$ on this graph. Thus we obtain the following result.

Theorem 3.4.1 *Let $\mathcal{O}_K(\sigma)$ be a Kantor ovoid of $Q(4,q)$, where $q = 3^h$, $h \in \mathbb{N} \setminus \{0,1\}$, and $\sigma \in \mathrm{Aut}(\mathrm{GF}(q)) \setminus \{\mathbb{1}\}$. Let p denote the point which is the intersection of the q conics in $\mathcal{O}_K(\sigma)$. Then the descendant with respect to p of the regular two-graph associated with $\mathcal{O}_K(\sigma)$ is geometric.*

The slice construction can also be applied in the case $q = 3$; then one finds the net $Q^+(3,2)$ as a substructure of the generalised quadrangle $Q^-(5,2)$.

3.5 Spreads

A *spread* of a partial geometry $\mathrm{pg}\,(s,t,\alpha)$ is a set $\mathcal{S}$ of $st/\alpha + 1$ lines such that every point is on exactly one line of $\mathcal{S}$. In [70] one reads that $\mathcal{M}_3(h)$, $h \in \mathbb{N} \setminus \{0\}$, contains a lot of spreads. There is even a set of $(q^2+1)/2$ spreads which partitions the line set of $\mathcal{M}_3(h)$ and is invariant under its full automorphism group. These spreads are $\mathcal{S}^\infty := \{L^\infty_{kl} \mid k,l \in K\}$ and $\mathcal{S}^{i\delta} := \{L^{i\delta}_{jl} \mid j,l \in K\}$, for $i \in I$ and $\delta \in \{0,1\}$. In fact, however, all the Hermitian partial geometries contain a great number of spreads. We will give a general construction and show how, in the case of $\mathcal{M}_3(h)$, the spreads $\mathcal{S}^{i\delta}$ are obtained in a geometric way.

Let L be a line of $H(q)$ through p; then L lies in the plane $\Gamma_2(p) \cup \{p\}$. For each point $s \neq p$ of L pick one generator $N(s)$ belonging to the set of $(q-1)/2$ generators through s chosen in Step 1 of the construction in Section 3.2. Then the set of q^2 type 1 lines $N'(s)^\pi \cap (\mathcal{O}_H \setminus \{p\})$, where $N'(s)$ is a line different from L through s in $\langle p, N(s)\rangle$ and s is a point different from p on L, is a spread. Suppose that two of these lines would have a point x in common. As p is contained in the span of the corresponding generators through a point of L, x lies in p^π, a contradiction because x is also a point of $\mathcal{O}_H$. Since L can be chosen in $q+1$ ways, and there are $((q-1)/2)^q$ choices for the generators $N(s)$, this construction yields $(q+1)((q-1)/2)^q$ spreads. Note that the construction can be repeated with other sets of generators $N(s)$ disjoint from the chosen one, or with different lines L. As a consequence the line set of any Hermitian partial geometry admits a partition into $(q^2+1)/2$ spreads in many different ways.

To construct the spreads $\mathcal{S}^{i\delta}$, we need the concept of a *normal rational cubic scroll*. This is the union of $q+2$ lines on a degenerate quadric in $\mathrm{PG}(4,q)$ with vertex a line L and base a non-degenerate conic $\mathcal{C}$. One of these lines is L; it is called the *directrix line* of the normal rational cubic scroll. The remaining $q+1$ lines are spanned by a point of L and its image under a *linear projectivity*, i.e. a bijection preserving cross-ratios, from L to $\mathcal{C}$. The directrix line of a normal rational cubic scroll on the degenerate quadric described by the equation $X_0X_2-X_1^2=0$ is the line $\langle(0,0,0,1,0),(0,0,0,0,1)\rangle$; in the standard form the remaining lines on the normal rational cubic scroll are $\langle(0,0,0,0,1),(0,0,1,0,0)\rangle$ and $\langle(0,0,0,1,t),(1,t,t^2,0,0)\rangle$, $t\in\mathrm{GF}(q)$. A normal rational cubic scroll is uniquely determined by the conic $\mathcal{C}$, the directrix line, and three of the other lines on it. For a more complete treatment we refer to [83].

Let again L be a line of $H(q)$ through p, let $s\neq p$ be a point on L, and let $N(s)$ be one of the generators through s chosen in Step 1 of Section 3.2. Consider the quadric $L^\pi\cap Q(6,q)$, which is a cone with vertex the line L and base a non-degenerate conic. Intersection of this quadric with a plane η in L^π disjoint from L yields a non-degenerate conic $\mathcal{C}$. The plane $\Gamma_2(p)\cup\{p\}$ intersects η in a point $c(p)$ of $\mathcal{C}$, and the plane $\langle p,N(s)\rangle$ intersects η in a point $c(s)$ of $\mathcal{C}$. The lines $\langle p,c(p)\rangle$ and $\langle s,c(s)\rangle$ lie on $q-1$ normal rational cubic scrolls with directrix line L on $L^\pi\cap Q(6,q)$. One calculates that just one of them contains no lines lying in the forbidden $Q^+(3,q)$s from Step 1 in Section 3.2. Assume that the partial geometry is $\mathcal{M}_3(h)$, $h\in\mathbb{N}\setminus\{0\}$, and apply the above construction to the set of q lines not through p on this normal rational cubic scroll. The result is a spread $\mathcal{S}^{i\delta}$ for certain $i\in I$ and $\delta\in\{0,1\}$. To prove this, we need the isomorphism between Mathon's partial geometries and the Hermitian ones (see Section 3.3). The line L determines the ratio of $b_1(i,\delta)$ and $b_2(i,\delta)$; we will show the claim in case $L=\langle p,(0,0,0,0,0,1,0)\rangle$, or equivalently $b_2(i,\delta)=0$, but the reasoning is similar in the other cases. Let s be the point $(0,0,0,0,0,1,0)$; then the generator $N(s)$ is of the form $\langle s,(0,0,1,n,0,0,n^2)\rangle$ for an $n\in\mathrm{GF}(q)\setminus\{0\}$ (as we consider the plane $\langle p,N(s)\rangle$ later, we may choose the second point defining $N(s)$ in $X_0 = 0$). The quadric $L^\pi \cap Q(6,q)$ has equations

$$\begin{cases} X_1 = 0 \\ X_4 = 0 \\ X_2X_6 - X_3^2 = 0; \end{cases}$$

intersection with the plane η with equations $X_0 = X_1 = X_4 = X_5 = 0$ yields the conic $\mathcal{C}$ in η with equation $X_2X_6 - X_3^2 = 0$. One checks that the point $c(p) := (\Gamma_2(p) \cup \{p\}) \cap \eta$ is $(0,0,0,0,0,0,1)$, and that the point $c(s) := \langle p, N(s)\rangle \cap \eta$ is $(0,0,1,n,0,0,n^2)$. The remaining lines on a normal rational cubic scroll with directrix line L and containing $\langle p, c(p)\rangle$ and $\langle s, c(s)\rangle$ are of the form $\langle (t,0,0,0,0,1,0),(0,0,1,n+bt,0,0,(n+bt)^2)\rangle$, $t \in \mathrm{GF}(q)\setminus\{0\}$, for a certain $b \in \mathrm{GF}(q) \setminus \{0\}$. Note that $n+bt$ determines the generator through $(t,0,0,0,0,1,0) \in L$. To avoid the forbidden $Q^+(3,q)$s, $n+bt$ must be different from t for all $t \in \mathrm{GF}(q) \setminus \{0\}$. If $b \neq 1$, then there is always a $t \in \mathrm{GF}(q) \setminus \{0\}$ such that $n+bt = t$, namely $t = n/(1-b)$. Hence $b = 1$, and the normal rational cubic scroll is uniquely determined. Now it has to be checked that the lines of the corresponding spread indeed form $\mathcal{S}^{i\delta}$. The fact that the partial geometry is $\mathcal{M}_3(h)$ implies that n is a non-zero square in $\mathrm{GF}(q)$. Let $m(i)$ be the unique element of $\{0,\ldots,(q-3)/2\}$ such that $-n = \gamma^{2m(i)}$; then the generator $N(s)$ yields a line $L_{jl}^{i\delta}$ of $\mathcal{M}_3(h)$ with $i = m(i)(q+1)$ and $\delta = 0$. The lines different from L through s in $\langle p, N(s)\rangle$ are of the form $\langle s, (-n\gamma^j, 0, 1, n, 0, 0, n^2)\rangle$, $j \in K$; as the same n occurs here, they also correspond to lines $L_{jl}^{i\delta}$ with $i = m(i)(q+1)$ and $\delta = 0$. Now consider a point $s' := (a'',0,0,0,0,1,0)$ on L different from p and s, with $a'' \in \mathrm{GF}(q) \setminus \{0\}$. The generators through s' are of the form $\langle s', ((a''-n')\gamma^j, 0, 1, n', 0, 0, n'^2)\rangle$ with $a''-n' \in \mathrm{GF}(q) \setminus \{0\}$ and $j \in K$; the ones lying in the plane spanned by p and a line on the normal rational cubic scroll have $n' = n+a''$ and hence $a''-n' = -n = \gamma^{2m(i)}$, so these are also lines $L_{jl}^{i\delta}$ with $i = m(i)(q+1)$ and $\delta = 0$, for all $a'' \in \mathrm{GF}(q) \setminus \{0\}$. The elements $j \in K$ correspond to the q lines different from L in the plane spanned by p and a line on the normal rational cubic scroll not containing p, while the $l \in K$ correspond to the q points on L different from p. Thus we have indeed found the spread $\mathcal{S}^{i\delta}$.

3.6 Block graphs

The block graphs of the Hermitian partial geometries are

$$\mathrm{srg}\left(\frac{q^2(q^2+1)}{2}, \frac{q(q^2-1)}{2}, q(q-2), \frac{q(q-1)}{2}\right);$$

no such graphs seem to have been known before, except for $q = 3$. For $q = 9$ and $q = 27$ they are not in Brouwer's database of distance-regular graphs [8]. These graphs contain a great number of cocliques of size q^2 arising from the spreads described in Section 3.5 ; they can even be seen as $(q^2 + 1)/2$-partite graphs in many different ways.

3.7 Do Hermitian two-graphs support two-graph geometries?

As already mentioned in Subsection 2.2, $\mathcal{H}(3)$ is geometric (in a unique way), while $\mathcal{H}(5)$ and $\mathcal{H}(7)$ are not; it is not known whether $\mathcal{H}(q)$, $q \geq 9$, supports a two-graph geometry. For $q = 3^h$, $h \in \mathbb{N} \setminus \{0, 1\}$, one could attempt to construct a two-graph geometry by trying to glue together the right number of Hermitian partial geometries and hoping that the blocks of the structure thus obtained are well-defined. Unluckily this method will turn out not to work. Suppose that $\mathcal{H}(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0, 1\}$, supports a two-graph geometry $\mathcal{T}$ such that the derived structures $\mathcal{T}_p$ and $\mathcal{T}_t$ with respect to two distinct points p and t, respectively, are Hermitian partial geometries. To settle the thoughts, assume $p = (1, 0, 0, 0, 0, 0, 0)$ and $t = (0, 0, 0, 0, 1, 0, 0)$. For the blocks of $\mathcal{T}$ to be well-defined, the lines of $\mathcal{T}_p$ through t and the lines of $\mathcal{T}_t$ through p must coincide, after adding the point p, respectively t. This is true for the lines of type 2. The type 1 lines through t in $\mathcal{T}_p$ are cliques of the form $N^\pi \cap (\mathcal{O}_H \setminus \{p\})$ and hence become $N^\pi \cap \mathcal{O}_H$, for a generator N of $Q(4, q) := p^\pi \cap t^\pi \cap Q(6, q)$ which is not contained in $Q^+(3, q) := \langle \mathcal{P}(p, t)\rangle^\pi \cap Q(6, q)$ and contains a point of $M := \Gamma_2(p) \cap \Gamma_4(t)$. As $\mathcal{T}_t$ is also a Hermitian partial geometry, these type 1 lines lie in K^π for a generator K of $Q(4, q)$ containing a point of $\Gamma_2(t)$. One checks that $h \geq 2$ implies that the set of points of a type 1 line corresponding to a generator N spans the four-dimensional space N^π. Therefore the only possibility is $K = N$, so N must contain a point of $\Gamma_2(t)$. It is not difficult to calculate that either $N = \langle (0, 0, 0, 0, 0, 1, 0), (0, 0, 1, k, 0, 0, k^2)\rangle$, $k \in \mathrm{GF}(q) \setminus \{0\}$, or $N = \langle (0, 0, 0, 0, 0, \mu, 1), (0, 1, -\mu, k, 0, k^2, 0)\rangle$, $\mu, k \in \mathrm{GF}(q)$, $k \neq 0$. However, from the description of $H(q)$ via coordinatisation (see Subsection 1.5.4) it follows that the plane $\Gamma_2(t) \cup \{t\}$ has equations $X_0 = X_3 = X_5 = X_6 = 0$ and hence does not intersect any of these lines N.

This does not necessarily mean that $\mathcal{H}(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0, 1\}$, is not geometric. It is still possible that there exists a Hermitian two-graph geometry such that the derived structures with respect to at least two distinct points are Hermitian partial geometries. The reasoning above, nevertheless, suggests that such a two-graph geometry would probably not have a nice model on $\mathcal{O}_H$. Another possibility is that a Hermitian two-graph geometry exists, but has at most one point such that the derived structure with respect to that point is a Hermitian partial geometry. This would imply the existence of (probably many) presently unknown partial geometries with a Hermitian point graph, of which the description could hardly be simpler than the description of the Hermitian partial geometries (which is itself not really simple!). In fact, however, the author thinks that the Hermitian two-graph $\mathcal{H}(3^h)$ is not geometric for $h \geq 2$.

Chapter 4

Interlude: distance-regular geometries

4.1 Distance-regular graphs

A connected graph Γ with diameter $d \geq 2$ is *distance-regular* if there exist integers b_i, $i \in \{0, \dots, d-1\}$, and c_i, $i \in \{1, \dots, d\}$, such that

drg1 Γ is regular with valency b_0;

drg2 for any two vertices x and y at distance $i \in \{1, \dots, d-1\}$ in Γ the vertex y is adjacent to precisely c_i vertices in $\Gamma_{i-1}(x)$ and to precisely b_i vertices in $\Gamma_{i+1}(x)$;

drg3 for any two vertices x and y at distance d the vertex y is adjacent to precisely c_d vertices in $\Gamma_{d-1}(x)$.

The sequence $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$ is known as the *intersection array* of Γ, and its elements are called the *intersection numbers*. Clearly $c_1 = 1$ holds. The case $d = 2$ corresponds to the strongly regular graphs: an srg (v, k, λ, μ) is distance-regular with intersection array $\{k, k-1-\lambda; 1, \mu\}$. Distance-regular graphs are extensively treated in the standard work [9].

4.2 Distance-regular geometries

As explained above, distance-regular graphs generalise the strongly regular graphs. It is a natural question whether one can find axioms for partial linear spaces with a distance-regular point graph which generalise the (semi)partial geometries. An important difference between partial and semipartial geometries is that in the incidence graph of a partial geometry the distance between a point and a line is either 1 or 3, while any proper semipartial geometry contains a point and a line at distance 5. A similar difference will occur in the partial linear spaces with a distance-regular point graph. Therefore we need the concepts of *point diameter*, *line diameter* and *point-line diameter* of the incidence graph of a partial linear space (see also [16]). They indicate the maximal possible distance between two vertices corresponding to two points, two lines, and a point and a line, respectively. It is obvious that point and line diameters are even, while point-line diameters are odd. The difference between the point or line diameter and the point-line diameter of the incidence graph of a partial linear space is 1; the point and line diameter differ by two or are equal.

Let $\mathcal{S}$ be an incidence structure, let Φ be its incidence graph, and suppose that the following axioms hold.

DRG1 $\mathcal{S}$ is a partial linear space of order (s,t).

DRG2 The point diameter of Φ is $2d \geq 4$.

DRG3 There exist integers α_{2i-1}, $1 \leq i \leq d$, such that for any point p and line L of $\mathcal{S}$ which are at distance $2i-1$ in Φ there are precisely α_{2i-1} points incident with L and at distance $2i-2$ from p.

DRG4 There exist integers t_{2i}, $1 \leq i \leq d$, such that for any two points p and q of $\mathcal{S}$ which are at distance $2i$ in Φ there are precisely $t_{2i}+1$ lines incident with q and at distance $2i-1$ from p.

Note that $\alpha_1 = 1$ and $t_2 = 0$. It is clear that the point graph Γ of $\mathcal{S}$ is regular with valency $(t+1)s$ and has diameter d. Vertices x and y of Γ at distance $i \in \{1,\dots,d\}$ correspond to points of $\mathcal{S}$ which are at distance $2i$ in the incidence graph Φ; the axioms learn that there are $(t_{2i}+1)\alpha_{2i-1}$ vertices of Γ which are adjacent to y and at distance $i-1$ from x. Similarly, if $i \leq d-1$ there are $(t-t_{2i})(s+1-\alpha_{2i+1})$ vertices adjacent to y and at distance $i+1$ from x. We conclude that Γ is distance-regular with diameter d and intersection numbers

$$\begin{aligned} b_0 &= (t+1)s, \\ b_i &= (t-t_{2i})(s+1-\alpha_{2i+1}),\ 1 \leq i \leq d-1, \\ c_i &= (t_{2i}+1)\alpha_{2i-1},\ 1 \leq i \leq d. \end{aligned} \tag{4.1}$$

We will call such partial linear spaces *distance-regular geometries*. If $d = 2$ and $t_4 = t$, the distance-regular geometry is a partial geometry $\mathrm{pg}\,(s,t,\alpha_3)$; if $d = 2$ and $t_4 < t$, we have a semipartial geometry $\mathrm{spg}\,(s,t,\alpha_3,(t_4+1)\alpha_3)$. Distance-regular geometries with $\alpha_{2i-1} = 1$ for all $i \in \{1,\dots,d\}$ are *regular* *near polygons*. Near polygons were introduced by Shult and Yanushka in [86]. We will call a distance-regular geometry *proper* if it is neither a (semi)partial geometry nor a regular near polygon.

In the most general case, a distance-regular geometry is described by $2d$ relevant parameters: s, t, α_{2i-1} for $i \in \{2, \ldots, d\}$, and t_{2i} for $i \in \{2, \ldots, d\}$. In Equations (4.1), the one for c_1 is always satisfied, leaving us with only $2d-1$ conditions. Hence the intersection numbers of a distance-regular graph Γ do not necessarily determine the parameters of a putative distance-regular geometry with Γ as a point graph.

4.3 Parameter restrictions

Suppose that $\mathcal{S}$ is a distance-regular geometry of which the incidence graph Φ has point diameter $2d$. Let p and L denote a point and a line at distance $2i-1$ in Φ, $i \in \{2, \ldots, d\}$, and let q be any point which is at distance 2 from p and at distance $2i-3$ from L. Then the α_{2i-3} points incident with L and at distance $2i-4$ from q are all at distance $2i-2$ from p, implying $\alpha_{2i-3} \leq \alpha_{2i-1}$ for all $i \in \{2, \ldots, d\}$. As the point diameter of Φ is $2d$, we also have $\alpha_{2d-1} \leq s$. A similar argument yields $t_{2i-2} \leq t_{2i}$ for all $i \in \{2, \ldots, d\}$. If the point-line diameter of Φ is $2d-1$, then $t_{2d} = t$, otherwise $t_{2d} < t$.

Other restrictions follow from constraints on the parameters of distance-regular graphs, see [9]. Unfortunately, many of them involve rather complicated functions of the intersection numbers. One of the easier restrictions requires that $c_{i-1} \leq c_i$ ($i \in \{2, \ldots, d\}$) and $b_{i-1} \geq b_i$ ($i \in \{1, \ldots, d-1\}$); however, this easily follows from Equations (4.1) and the inequalities for the parameters α_{2i-1} and t_{2i}, $1 \leq i \leq d$. For any $i \in \{1, \ldots, d-1\}$ the number $a_i := b_0 - b_i - c_i$ denotes the number of vertices adjacent to y and at distance i from x, where x and y are at distance i. The condition that a_i be non-negative turns out to be automatically satisfied as well: one calculates that $a_i = (s - \alpha_{2i-1})(t_{2i} + 1) + (t - t_{2i})(\alpha_{2i+1} - 1)$. Finally we will give some divisibility conditions which do not seem very practical in general, but can be of use when checking whether a specific distance-regular graph could support a distance-regular geometry. Let k_i, for $i \in \{1, \ldots, d\}$, denote the number of vertices of a distance-regular graph at distance i from a fixed vertex. Clearly $k_1 = b_0$; the remaining k_i, $i \in \{2, \ldots, d\}$, can be calculated recursively from the formula

$$k_i c_i = k_{i-1} b_{i-1},$$

which follows from an easy counting argument. This can be rewritten as

$$k_i = \frac{\prod_{j=0}^{i-1} b_j}{\prod_{j=1}^{i} c_j}$$

for all $i \in \{1, \ldots, d\}$. Consequently the parameters of a distance-regular geometry satisfy

$$\prod_{j=1}^{i}(t_{2j}+1)\alpha_{2j-1} \mid (t+1)s\prod_{j=1}^{i-1}(t-t_{2j})(s+1-\alpha_{2j+1})$$

for all $i \in \{2, \ldots, d\}$. The number of points of the distance-regular geometry equals

$$v := 1 + \sum_{i=1}^{d} k_i;$$

the fact that the number of lines is an integer yields

$$(s+1) \mid v(t+1).$$

4.4 Two sporadic examples

The *Petersen graph* Pe is the unique strongly regular graph srg$(10, 3, 0, 1)$. Its line graph $L(\text{Pe})$ is distance-regular with diameter 3 and intersection array $\{4, 2, 1; 1, 1, 4\}$. We will show that it supports a distance-regular geometry. First we will deduce the parameters s, t, α_3, t_4, α_5 and t_6 with the help of Equations (4.1):

$$\begin{aligned}
&(t+1)s = 4\\
&t(s+1-\alpha_3) = 2\\
&(t-t_4)(s+1-\alpha_5) = 1\\
&(t_4+1)\alpha_3 = 1\\
&(t_6+1)\alpha_5 = 4.
\end{aligned}$$

As the parameters of a distance-regular geometry must be integers, the fourth equation yields $t_4 = 0$ and $\alpha_3 = 1$, and the third equation yields $t = 1$ and $\alpha_5 = s$. Using the remaining equations we find $s = 2$ and $t_6 = 1$. Hence the lines of a distance-regular geometry on $L(\text{Pe})$ should be 3-cliques. The parameters of Pe imply that it does not contain triangles; consequently any 3-clique in $L(\text{Pe})$ corresponds to a set of 3 edges of Pe containing a common vertex. No two such 3-cliques intersect in more than one vertex, and any vertex is contained in two such 3-cliques. It follows that the edges and vertices of the Petersen graph Pe can be seen (with natural incidence) as the points and lines, respectively, of a partial linear space $\mathcal{S}$ of order $(2, 1)$ having $L(\text{Pe})$ as a point graph. Using the facts that Pe contains no triangles or quadrangles and that its diameter equals 2, one easily proves that $\mathcal{S}$ is a distance-regular geometry with parameters $s = 2$, $t = 1$, $\alpha_3 = 1$, $t_4 = 0$, $\alpha_5 = 2$ and $t_6 = 1$. One verifies that the incidence graph of $\mathcal{S}$ has point diameter 6, line diameter 4 and point-line diameter 5.

A similar construction can be applied to the Hoffman–Singleton graph HoSi, which is the unique $\operatorname{srg}(50, 7, 0, 1)$ (see also Subsection 6.1.2). Its line graph $L(\text{HoSi})$ is distance-regular with diameter 3 and intersection array $\{12, 6, 5; 1, 1, 4\}$. The partial linear space $\mathcal{S}$ in which the points and lines are the edges and vertices, respectively, of HoSi is a distance-regular geometry on $L(\text{HoSi})$ with parameters $s = 6$, $t = 1$, $\alpha_3 = 1$, $t_4 = 0$, $\alpha_5 = 2$ and $t_6 = 1$. Again, the lines of $\mathcal{S}$ are the only possible 7-cliques in $L(\text{HoSi})$, and the incidence graph of $\mathcal{S}$ has point diameter 6, line diameter 4 and point-line diameter 5.

Both examples constructed here are neither (semi)partial geometries (as their point graphs have diameter 3), nor regular near polygons (as $\alpha_5 = 2$). They are both dual partial quadrangles: the vertices and edges of Pe form an $\operatorname{spg}(1, 2, 1, 1)$, while the vertices and edges of HoSi form an $\operatorname{spg}(1, 6, 1, 1)$.

One may wonder whether the above construction can be generalised to other graphs having a distance-regular line graph. In [9, Theorem 4.2.16] it is proved that a connected graph having a distance-regular line graph with diameter at least 3 is one of the following.

1. a circuit of length at least 6
2. the incidence graph of a generalised d-gon ($d \in \{3, 4, 6\}$) of order $s > 1$
3. the Petersen graph Pe
4. the Hoffman–Singleton graph HoSi
5. an $\operatorname{srg}(3250, 57, 0, 1)$ which is not known to exist

It can be proved (see [100]) that in Case (2) the line graph is the point graph of a generalised $2d$-gon of order $(s, 1)$. In Case (1) the line graph is isomorphic to the original graph, and it is the point graph of an ordinary polygon. As a consequence the only proper distance-regular geometries obtained in this way are the ones described above, unless an $\operatorname{srg}(3250, 57, 0, 1)$ turns out to exist.

4.5 Johnson geometries

Let n and e be integers with $n \geq 4$ and $2 \leq e \leq n-2$, and let X be a set with n elements. The vertices of the *Johnson graph* $J(n, e)$ (see [9, Section 9.1]) are the e-subsets of X, and two vertices are adjacent if and only if they intersect in an $(e-1)$-set. This means that two vertices of $J(n, e)$ are at distance i if and only if they intersect in an $(e-i)$-set, implying that the diameter of $J(n, e)$ is $d := \min\{e, n-e\}$. One can prove that $J(n, e)$ is distance-regular with intersection numbers $b_i = (e-i)(n-e-i)$ (for $0 \leq i \leq d-1$) and $c_i = i^2$ (for $1 \leq i \leq d$). Moreover the graphs $J(n, e)$ and $J(n, n-e)$ are isomorphic, which can be seen by considering the complementary subsets of X. The Johnson graphs having diameter 2 are also called the *triangular graphs* $T(n)$, $n \geq 4$.

We will now show that there are only two types of maximal cliques in $J(n, e)$. Note that maximal cliques are cliques which cannot be extended to larger cliques; they do not necessarily meet an upper bound like the one in Theorem 1.2.7. Let x and y be two adjacent vertices in $J(n, e)$, $n \geq 4$, $2 \leq e \leq n-2$, i.e. two e-subsets of a set X of cardinality n which intersect in an $(e-1)$-set. A vertex z which is adjacent to both x and y either contains the $(e-1)$-set $x \cap y$ or is contained in the $(e+1)$-set $x \cup y$. In the first case, any vertex adjacent to x, y and z necessarily contains $x \cap y$, and the only maximal clique to which the clique $\{x, y, z\}$ can be extended is the clique consisting of all e-subsets of X containing the $(e-1)$-set $x \cap y$. In the second case, any vertex adjacent to x, y and z must be contained in $x \cup y$, and the only maximal clique to which the clique $\{x, y, z\}$ can be extended is the clique consisting of all e-subsets of X which are contained in the $(e+1)$-subset $x \cup y$. We conclude that the maximal cliques of $J(n, e)$ can be identified with the $(e-1)$-subsets and the $(e+1)$-subsets of X, and that they have size $n-e+1$ or e, respectively. If one takes complements in X, a maximal clique of the second type corresponds to the set of all $(n-e)$-sets containing a fixed $(n-e-1)$-subset of X, and hence is a maximal clique of the first type in $J(n, n-e)$.

Using our knowledge about the maximal cliques, we can construct a distance-regular geometry on $J(n, e)$ in a straightforward way. Since the parameter α_3 of a distance-regular geometry is at most s, the cliques used as lines should be maximal. If maximal cliques of both types occurred as concurrent lines, the $(e+1)$-set would have to contain the $(e-1)$-set, and they would be incident with two common points, a contradiction. Non-concurrent lines of different type would be connected by a "path" of lines in which at least one pair of concurrent lines of different type would appear. It follows that only one type of maximal cliques can be used at the same time. On the other hand, since two maximal cliques of the same type share at most one vertex, they can all be used together. Let $\mathcal{J}(n,e)$ be the incidence structure in which the points and lines are the e-subsets and the $(e-1)$-subsets of X, respectively, and define incidence as (reverse) containment. As shown above, $\mathcal{J}(n,e)$ is a partial linear space with $J(n,e)$ as a point graph; one easily calculates that its order is $(n-e, e-1)$. Two points of $\mathcal{J}(n,e)$ are at distance $2i$ in the incidence graph Φ if and only if they intersect in an $(e-i)$-set. Suppose that a point p and a line L are at distance $2i-1$ in Φ. Then there is a point on L which is at distance $2i-2$ from p, but no point on L lies at a distance smaller than $2i-2$ from p. Hence p and L intersect in an $(e-i)$-set. A similar reasoning learns that lines of $\mathcal{J}(n,e)$ are at distance $2i$ in Φ if and only if they intersect in an $(e-1-i)$-set. Now it is easily proved that $\mathcal{J}(n,e)$ is indeed a distance-regular geometry with parameters

$$\begin{aligned} s &= n-e, \\ t &= e-1, \\ \alpha_{2i-1} &= t_{2i}+1 = i \ (1 \le i \le d), \end{aligned}$$

with $d := \min\{e, n-e\}$. We call these distance-regular geometries the *Johnson geometries*. If $n \ge 2e$, then the point diameter of the incidence graph Φ of $\mathcal{J}(n,e)$ is $2e$, the line diameter is $2(e-1)$, and hence the point-line diameter is $2e-1$. If $n \le 2(e-1)$, then the point and line diameter of Φ are $2(n-e)$ and $2(n-e+1)$, respectively, implying that the point-line diameter is $2(n-e)+1$. Finally, if $n = 2e-1$, then the point and line diameter of Φ are $2(e-1)$ and the point-line diameter equals $2e-1$. The Johnson geometries having a point graph with diameter 2 are $\mathcal{J}(n,2)$ and $\mathcal{J}(n,n-2)$, for $n \ge 4$. Obviously $\mathcal{J}(n,2)$ is the dual of the (rather trivial) 2-$(n,2,1)$ design formed by the elements and pairs of a set with n elements. By taking complements in X, one shows that $\mathcal{J}(n,n-2)$ is isomorphic to $U_{2,3}(n)$, which is an spg $(2, n-3, 2, 4)$ (see [29]). All other Johnson geometries have a point graph with diameter at least 3 and hence are not (semi)partial geometries. As the parameter α_3 equals 2, they are not near polygons either.

Note that we used the maximal cliques of $J(n,e)$ of the first type for the definition of the lines of $\mathcal{J}(n,e)$. If the other type is chosen, one can take complements in X and obtains nothing else than $\mathcal{J}(n,n-e)$. The dual of $\mathcal{J}(n,e)$ can be seen, after taking complements in X, as the incidence structure consistng of the $(n-e+1)$-subsets and the $(n-e)$-subsets of X. This incidence structure is precisely $\mathcal{J}(n,n-e+1)$. Therefore the dual of a Johnson geometry is again a Johnson geometry, and $\mathcal{J}(n,e)$ is self-dual if and only if $e = (n+1)/2$.

4.6 Grassmann geometries

Let q be any prime power, and let $V(n,q)$ denote the n-dimensional vector space over GF(q), with $n \geq 4$. Choose an integer e such that $2 \leq e \leq n-2$. The vertices of the *Grassmann graph* $G(n,e,q)$ (see [9, Section 9.3]) are the e-dimensional subspaces of $V(n,q)$, and two vertices are adjacent if and only if they intersect in an $(e-1)$-dimensional subspace. Intuitively, one could say that the Grassmann graphs are a "thicker" version of the Johnson graphs; they indeed show very similar behaviour. Two vertices of $G(n,e,q)$ are at distance i if and only if they correspond to e-dimensional subspaces of $V(n,q)$ intersecting in an $(e-i)$-dimensional subspace. It follows that the diameter of $G(n,e,q)$ is $d := \min\{e, n-e\}$. At this point it is practical to introduce the *q-ary Gaussian binomial coefficients*. The number

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{\prod_{i=0}^{m-1}(q^{n-i}-1)}{\prod_{i=1}^{m}(q^i-1)}$$

is precisely the number of m-dimensional subspaces of an n-dimensional vector space over GF(q). It can be proved that $G(n,e,q)$ is a distance-regular graph with intersection numbers

$$b_i = q^{2i+1}\begin{bmatrix} e-i \\ 1 \end{bmatrix}_q \begin{bmatrix} n-e-i \\ 1 \end{bmatrix}_q \quad (0 \leq i \leq d-1),$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q^2 \quad (1 \leq i \leq d).$$

By taking orthogonal complements in $V(n,q)$, one shows that $G(n,e,q)$ and $G(n,n-e,q)$ are isomorphic.

Like the Johnson graphs, the Grassmann graphs contain only two types of maximal cliques. Let x and y be two adjacent vertices of $G(n,e,q)$, $n \geq 4$, $2 \leq e \leq n-2$, q a prime power. A vertex adjacent to x and y contains the $(e-1)$-dimensional subspace $x \cap y$ or is contained in the $(e+1)$-dimensional subspace $\langle x,y\rangle$. If z contains $x \cap y$ and is not contained in $\langle x,y\rangle$, then any vertex adjacent to x, y and z necessarily contains $x \cap y$, and the only maximal clique to which the clique $\{x,y,z\}$ can be extended is the clique consisting of all e-dimensional subspaces of $V(n,q)$ containing the $(e-1)$-dimensional subspace $x \cap y$. If z is contained in $\langle x,y\rangle$ and does not contain $x \cap y$, then any vertex adjacent to x, y and z must be contained in $\langle x,y\rangle$, and the only maximal clique to which the clique $\{x,y,z\}$ can be extended is the clique consisting of all e-dimensional subspaces of $V(n,q)$ which are contained in the $(e+1)$-dimensional subspace $\langle x,y\rangle$. Finally, if z contains $x \cap y$ and is contained in $\langle x, y\rangle$, then the clique $\{x, y, z\}$ can be extended to a maximal clique of either of the types just described, but there are no other possibilities. Thus we see that the maximal cliques of $G(n, e, q)$ correspond to the $(e-1)$-dimensional and $(e+1)$-dimensional subspaces of $V(n, q)$ and therefore have size

$$\begin{bmatrix} n-e+1 \\ 1 \end{bmatrix}_q \text{ or } \begin{bmatrix} e+1 \\ 1 \end{bmatrix}_q,$$

respectively. If one takes orthogonal complements in $V(n, q)$, a maximal clique of the second type becomes the set of all $(n-e)$-dimensional subspaces containing a fixed $(n-e-1)$-dimensional subspace. This clearly is a maximal clique of the first type in $G(n, n-e, q)$.

Again there is a natural way to construct a distance-regular geometry on $G(n, e, q)$. The cliques used as lines should be maximal, as the parameter α_3 of a distance-regular geometry is at most s. Suppose that maximal cliques of both types occur as lines. If they are concurrent, then the $(e-1)$-dimensional subspace is contained in the $(e+1)$-dimensional subspace, and they are incident with $q+1$ common points, a contradiction. If they are not concurrent, then somewhere in the "path" of lines connecting them there is a pair of concurrent lines of different type. Consequently only one type of maximal cliques can be used. On the other hand, maximal cliques of the same type never intersect in more than one vertex, so they can all be used at the same time. Define $\mathcal{G}(n, e, q)$ as the incidence structure in which the points and lines are the e-dimensional and $(e-1)$-dimensional subspaces, respectively, of the n-dimensional vector space $V(n, q)$ over $\mathrm{GF}(q)$. Let incidence be given by (reverse) containment. Then $\mathcal{G}(n, e, q)$ is a partial linear space of order

$$\left(\begin{bmatrix} n-e+1 \\ 1 \end{bmatrix}_q - 1, \begin{bmatrix} e \\ 1 \end{bmatrix}_q - 1\right)$$

which has $G(n, e, q)$ as a point graph. Two points of $\mathcal{G}(n, e, q)$ are at distance $2i$ in the incidence graph Φ of $\mathcal{G}(n, e, q)$ if and only if they intersect in an $(e-i)$-dimensional subspace. Let p and L be a point and a line of $\mathcal{G}(n, e, q)$ which are at distance $2i-1$ in Φ. Then L contains a point at distance $2i-2$ from p, but no points at a smaller distance from p. It follows that p and L intersect in an $(e-i)$-dimensional subspace. Finally, lines are at distance $2i$ if and only if they intersect in an $(e-1-i)$-dimensional subspace. From this one proves that $\mathcal{G}(n,e,q)$ is a distance-regular geometry with parameters

$$\begin{aligned} s &= \begin{bmatrix} n-e+1 \\ 1 \end{bmatrix}_q - 1, \\ t &= \begin{bmatrix} e \\ 1 \end{bmatrix}_q - 1, \\ \alpha_{2i-1} &= t_{2i} + 1 = \begin{bmatrix} i \\ 1 \end{bmatrix}_q \quad (1 \le i \le d), \end{aligned}$$

with $d := \min\{e, n-e\}$. We call these distance-regular geometries the *Grassmann geometries*. If $n \ge 2e$, then the point diameter of the incidence graph Φ of $\mathcal{G}(n,e)$ is $2e$, the line diameter is $2(e-1)$, and hence the point-line diameter is $2e-1$. If $n \le 2(e-1)$, then the point and line diameter of Φ are $2(n-e)$ and $2(n-e+1)$, respectively, implying that the point-line diameter is $2(n-e)+1$. Finally, if $n = 2e-1$, then the point and line diameter of Φ are $2(e-1)$ and the point-line diameter equals $2e-1$. The Grassmann geometries having a point graph with diameter 2 are $\mathcal{G}(n,2,q)$ and $\mathcal{G}(n,n-2,q)$. The dual of $\mathcal{G}(n,2,q)$ is the design of one-dimensional and two-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$, or equivalently of points and lines of $\mathrm{PG}(n-1,q)$. By taking orthogonal complements, one can prove that $\mathcal{G}(n,n-2,q)$ is isomorphic to $\mathrm{LP}(n-1,q)$, which is an

$$\mathrm{spg}\left(q(q+1), \frac{q^{n-2}-q}{q-1}, q+1, (q+1)^2\right)$$

(see [29]). All other Grassmann geometries have a point graph with diameter at least 3 and consequently are not (semi)partial geometries. As $\alpha_3 = q+1$, they are not near polygons either.

Note that we defined the lines of $\mathcal{G}(n,e,q)$ as the maximal cliques of $G(n,e,q)$ of the first type. If the other type is chosen, one obtains, after taking orthogonal complements in $V(n,q)$, the distance-regular geometry $\mathcal{G}(n,n-e,q)$. The dual of $\mathcal{G}(n,e,q)$ can be seen, after taking orthogonal complements in $V(n,q)$, as $\mathcal{G}(n,n-e+1,q)$. As a consequence the dual of a Grassmann geometry is again a Grassmann geometry, and $\mathcal{G}(n,e,q)$ is self-dual if and only if $e = (n+1)/2$.

4.7 Hamming graphs and dual polar graphs

Let $V(d,q)$ be the d-dimensional vector space over $\mathrm{GF}(q)$, $q \ge 2$ and $d \ge 2$. The vertices of the *Hamming graph* $H(d,q)$ (see [9, Section 9.2]) are the elements of $V(d,q)$; two vertices are adjacent if and only if they differ in precisely one coordinate. One can easily construct a distance-regular geometry on $H(d,q)$ by defining the lines as the sets of q elements of $V(d,q)$ which have the same values in a certain set of $d-1$ coordinates. As this distance-regular geometry turns out to be a regular near polygon, we will not discuss it in detail.

Let $\mathcal{P}$ be a polar space of rank $d \geq 3$. The vertices of the *dual polar graph* (see [9, Section 9.4]) associated with $\mathcal{P}$ are the generators of $\mathcal{P}$, and two vertices are adjacent if and only if they intersect in a $(d-2)$-dimensional subspace. The $(d-2)$-dimensional subspaces can be used as the lines of a regular near polygon. The block graphs of the classical generalised quadrangles are also called dual polar graphs. In this case, the associated distance-regular geometry is the dual of the generalised quadrangle.

4.8 Halved cubes and half dual polar graphs

Let $V(n,q)$ be the n-dimensional vector space over GF(2), $n \geq 4$. The vertices of the *halved cube* $D_{n,n}(1)$ (see [9, Section 9.2]) are the elements of $V(n,q)$ in which an even number of coordinates are equal to 1. Two vertices are adjacent if and only if they differ in exactly two coordinates. It is not difficult to prove that $D_{n,n}(1)$ is a distance-regular graph. Moreover, any maximal clique of $D_{n,n}(1)$ is either the set consisting of all vertices which have the same values in a certain set of $n-3$ coordinates, or the set consisting of all vertices which differ in precisely one coordinate from a certain element of $V(n,q)$ having an odd number of ones. If $n \geq 5$, these cliques are different in size and consequently cannot occur together as lines. On the other hand, cliques of the second type cannot be used as concurrent lines, as they would be incident with two common points. This means that in this case only cliques of the first type could yield a distance-regular geometry. There are two possible configurations for a point and such a clique at distance three from it, implying that either $\alpha_3 = 1$ or $\alpha_3 = 3$. However, these values both lead to a contradiction when combined with Equations (4.1). The case $n = 4$ is easily dealt with and produces a contradiction as well. Thus we conclude that the halved cube does not support a distance-regular geometry.

The vertices of the *half dual polar graph* $D_{n,n}(q)$ (see [9, Section 9.4]) are the elements of one family of generators of the quadric $Q^+(2n-1,q)$, $n \geq 4$, $q \geq 2$. Two vertices are adjacent if and only if they intersect in a $(n-3)$-dimensional subspace. By similar arguments as above, one shows that there exists no distance-regular geometry with $D_{n,n}(q)$ as a point graph.

Chapter 5

Two-graphs, related graphs and their codes

In this chapter the codes arising from graphs and two-graphs are investigated, as well as the existence of regular graphs in the switching class corresponding to regular two-graphs. For the Hermitian and Ree two-graphs, the codes yield non-existence results, which can also be found in [46]. For other doubly transitive two-graphs explicit constructions of the regular graphs are given.

5.1 Codes

In this section we give some basic definitions concerning codes; for more information see [98]. A *linear binary code* C of *length* n and *dimension* k is a k-dimensional subspace of the n-dimensional vector space $V(n,2)$ over GF(2). Throughout this thesis, no other codes than linear binary ones will occur, so we will usually omit the adjectives. The elements of a code are called (*code*) *words*. The *dual code* $C^\perp$ of a code C is the $(n-k)$-dimensional subspace of $V(n,2)$ consisting of all vectors which are orthogonal (with respect to the standard inner product) to all words of C. The zero vector $(0,\dots,0)$ and the all-one vector $(1,\dots,1)$ in $V(n,2)$ are written as $\underline{0}$ and $\underline{1}$, respectively. The *complement* of a code word $x \in C$ is the vector $x + \underline{1}$.

The *weight* $w(x)$ of a code word x is the number of coordinate positions where x has a one. For two code words x and y let $x \cap y$ denote the number of coordinate positions where both x and y have a one. Then the following formula for the weight of a sum is straightforward:

$$w(x+y) = w(x) + w(y) - 2(x \cap y). \qquad (5.1)$$

The *minimum weight* of a code C is the smallest natural number w such that C contains a non-zero code word of weight w. The *weight enumerator* of a code C of length n is the polynomial

$$A(X,Y) := \sum_{i=0}^{n} A_i X^i Y^{n-i},$$

where A_i denotes the number of words of weight i in C, for all $i \in \{0,\ldots,n\}$. The *weight distribution* of C is a table where the coefficients A_i are given. The theorem of MacWilliams connects the weight enumerators of a code and its dual code.

Theorem 5.1.1 ([68]) *Let C be a linear binary code of length n and dimension k, and let $C^\perp$ be its dual code. If $A(X,Y)$ is the weight enumerator of C, then the weight enumerator of $C^\perp$ is*

$$A^\perp(X,Y) := 2^{-k} A(Y-X, Y+X).$$

5.2 Codes of graphs

The *code* C_A of an $m \times n$ matrix A with integral entries is defined as the subspace of the n-dimensional vector space $V(n,2)$ over GF(2) generated by the rows of A. The dimension of C_A is equal to the 2-*rank* of A, i.e. the rank of A seen as a matrix over the field GF(2). The *code* of a graph is the code of its $(0,1)$ adjacency matrix. In [10] it is proved that the 2-rank of a symmetric integral matrix with zero diagonal, and hence the dimension of the code of a graph, is always even. The following lemmas are rather simple, but useful.

Lemma 5.2.1 ([48]) *If A is a symmetric $n \times n$ matrix with entries from $\{0,1\}$, then the vector formed by the diagonal entries of A belongs to the code C_A of A.*

Proof. Let x be any element of the dual code $C_A^\perp$ of C_A. Then

$$\sum_{i=1}^{n} a_{ii} x_i \equiv \sum_{i,j \in \{1,\ldots,n\}} a_{ij} x_i x_j \equiv x^T A x \equiv 0 \pmod 2,$$

so x is orthogonal to the vector $(a_{ii})_{i \in \{1,\ldots,n\}}$. Hence this vector is a code word of $C_A^{\perp\perp} = C_A$. □

Lemma 5.2.2 ([48]) *Suppose that A is the adjacency matrix of a graph. Then $C_{A+J} = \langle C_A, \underline{1} \rangle$.*

Proof. By Lemma 5.2.1 the all-one vector $\underline{1}$ belongs to C_{A+J}, implying the assertion. □

Quite some information about the code of a strongly regular graph can be deduced from its parameters.

Theorem 5.2.3 ([48]) *Let Γ be a conference graph on v vertices with adjacency matrix A. If $v \equiv 5 \pmod 8$, then $C_A = \underline{1}^\perp$ and $C_{A+I} = V(v, 2)$. If $v \equiv 1 \pmod 8$, then $C_A^\perp = C_{A+I}$ and $\dim C_A = \dim C_{A+I} - 1 = (v-1)/2$.*

Theorem 5.2.4 ([48]) *Let Γ be a strongly regular graph $\mathrm{srg}\,(v, k, \lambda, \mu)$ having integral restricted eigenvalues r and l ($r > l$), with multiplicity f and g, respectively. Let A be the adjacency matrix of Γ, and let $\overline{A} := J - I - A$ denote the adjacency matrix of the complement of Γ.*

1. *If k, r and l are odd, then $C_A = V(v, 2)$, C_{A+I} is contained in $C_{A+I}^\perp$, and $\dim C_{A+I} \leq \min\{f + 1, g + 1\}$.*
2. *If r and l are odd and k is even, then $C_A = \underline{1}^\perp$, C_{A+I} is contained in $C_A^\perp$, and $\dim C_{A+I} \leq \min\{f + 1, g + 1\}$.*
3. *If $r + l$ is odd and k is even, then $C_{A+I} = C_A^\perp$, $\dim C_A = f'$ and $\dim C_{A+I} = v - f'$, where f' is the multiplicity of the odd eigenvalue.*
4. *If $r + l$ is odd and k is odd, then $C_{\overline{A}} = C_A^\perp$, $\dim C_A = f' + 1$ and $\dim C_{A+I} = v - f'$, where f' is the multiplicity of the odd restricted eigenvalue.*
5. *If r and l are even, then k is even, $C_{A+I} = V(v, 2)$, C_A is contained in $C_A^\perp$, and $\dim C_A$ is even and is at most $\min\{f + 1, g + 1\}$.*

5.3 An example: the symplectic graphs and the codes of their complements

This example is included because it illustrates the interplay between combinatorial properties of the graph and properties of related codes. Recall that the symplectic graph $S(2m, 2)$, $m \geq 2$, can be seen as the graph on the symplectic polar space $W(2m-1, 2)$ with $m \geq 3$, respectively the generalised quadrangle $W(2)$, in which vertices are adjacent if and only if they are collinear. Let π denote the associated symplectic polarity. Consider two points of $\mathrm{PG}(2m-1, 2)$ and let L be the line spanned by them. If u is a point of PG$(2m-1,2)$ not on L, then clearly the hyperplane u^π contains one or three points of L. This proves that $S(2m,2)$ not only satisfies the triangle property (see Subsection 1.9.3), but also the *cotriangle property*: for any two non-adjacent vertices x and y there exists a third vertex z which is non-adjacent to both x and y, such that every further vertex is adjacent to one or three elements of $\{x,y,z\}$.

The triangle and cotriangle properties can be stated in terms of the adjacency matrix of the complement of $S(2m,2)$. Let x and y be two vertices, and let z denote the third vertex from the triangle or cotriangle property. Let $\overline{A}$ denote the adjacency matrix of the complement of $S(2m,2)$, and consider the column of $\overline{A}$ corresponding to a vertex $u \notin \{x,y,z\}$. Then zero or two elements of $\{\overline{a}_{xu},\overline{a}_{yu},\overline{a}_{zu}\}$ are equal to 1, or equivalently $\overline{a}_{xu}+\overline{a}_{yu}+\overline{a}_{zu}$ is even. The fact that $\{x,y,z\}$ contains either zero or three edges of $S(2m,2)$ means that $\overline{a}_{xx}+\overline{a}_{yx}+\overline{a}_{zx}$, $\overline{a}_{xy}+\overline{a}_{yy}+\overline{a}_{zy}$ and $\overline{a}_{xz}+\overline{a}_{yz}+\overline{a}_{zz}$ are even as well. Therefore the row corresponding to z is the sum modulo 2 of the rows corresponding to x and y. This can be done for any pair $\{x,y\}$ of vertices of $S(2m,2)$; it follows that the sum of any two rows of $\overline{A}$ is again a row of $\overline{A}$. Consequently the rows of $\overline{A}$ are precisely all non-zero code words of the code $C_{\overline{A}}$ of the complement of $S(2m,2)$, implying that $C_{\overline{A}}$ has dimension $2m$ and that all non-zero code words of $C_{\overline{A}}$ have the same weight 2^{2m-1}.

5.4 Codes of two-graphs

Codes arise from two-graphs as well as from graphs. Not surprisingly, the codes of certain graphs in the corresponding switching class are used in the definition. Let (V,Δ) be a two-graph, let ω be a vertex, and let Γ_ω denote the graph in the switching class corresponding to (V,Δ) in which ω is isolated. Equivalently, Γ_ω is the graph consisting of the descendant of (V,Δ) with respect to ω and an isolated vertex. Now choose another vertex $\omega' \neq \omega$ and let $\Gamma_{\omega'}$ be the graph in (V,Δ) in which ω' is isolated. As they are in the same two-graph (V,Δ), Γ_ω and $\Gamma_{\omega'}$ are switching equivalent. The set X_1 of vertices adjacent to ω' in Γ_ω determines the partition $\{X_1, V\setminus X_1\}$ needed to switch Γ_ω to $\Gamma_{\omega'}$; clearly X_1 is also the set of vertices adjacent to ω in $\Gamma_{\omega'}$. Let A_ω and $A_{\omega'}$ be the adjacency matrices of Γ_ω and $\Gamma_{\omega'}$, respectively. Let χ be the characteristic vector of X_1; then any row x' of $A_{\omega'}$ can be written as either $x+\chi$ or $x+\underline{1}+\chi$ (modulo 2), for a certain row x of A_ω. A similar statement holds if the roles of Γ_ω and $\Gamma_{\omega'}$ are interchanged. Therefore we have $\langle C_{A_\omega},\underline{1},\chi\rangle = \langle C_{A_{\omega'}},\underline{1},\chi\rangle$. Since χ is equal to the row of A_ω corresponding to ω' and also to the row of $A_{\omega'}$ corresponding to ω, it is a code word in both C_{A_ω} and $C_{A_{\omega'}}$. Using Lemma 5.2.2 we can write $C_{A_\omega+J} = C_{A_{\omega'}+J}$. This motivates the definition of the *two-graph code* of (V, Δ) as $C := C_{A_\omega + J}$, for any vertex ω. An isolated vertex of a graph produces a row and column containing only zeroes in the adjacency matrix; it follows that the all-one vector $\underline{1}$ is not a code word of C_{A_ω}, and that two-graph codes have odd dimension.

5.5 Regular graphs in regular two-graphs

We will mostly use the expression in the title instead of "regular graphs in the switching class corresponding to regular two-graphs". Let (V, Δ) be a non-trivial regular two-graph, and let r and l $(r > l)$ be the restricted eigenvalues of its descendants, with multiplicity f and g, respectively. Recall (see Subsection 1.9.2) that such a descendant Γ has $v = -(2r+1)(2l+1)$ vertices and that its valency is $k = -2rl$. Let Γ_ω be the graph consisting of Γ and an isolated vertex, and let B_ω be its $(0, 1, -1)$ adjacency matrix. Then

$$B_\omega = \begin{bmatrix} 0 & \underline{1} \\ \underline{1}^T & J - I - 2A \end{bmatrix},$$

where A is the $(0, 1)$ adjacency matrix of Γ. Let x be an eigenvector of A corresponding to the restricted eigenvalue r. As x is perpendicular to $\underline{1}$, the vector obtained by extending x with a first component equal to zero is an eigenvector of B_ω with eigenvalue $-2r-1$. Similarly eigenvectors of B_ω with eigenvalue $-2l-1$ arise from the eigenvectors of A corresponding to the other restricted eigenvalue l. Finally one checks that the all-one vector $\underline{1}$ of length $v = -(2r+1)(2l+1)$ extended with a first component equal to $2r+1$, respectively $2l+1$, is an eigenvector of B_ω corresponding to the eigenvalue $-2l-1$, respectively $-2r-1$. Hence the eigenvalues of B_ω are $-2r-1$ and $-2l-1$, with multiplicity $f+1$ and $g+1$, respectively. Now suppose that Γ' is a regular graph in (V, Δ); then its $(0, 1, -1)$ adjacency matrix B' has the same eigenvalues as B_ω, with the same multiplicities. As Γ' is regular, the row sums in B' are constant, or equivalently the all-one vector $\underline{1}$ is an eigenvector of B'. This implies that the row sums in B' are either all equal to $-2r-1$ or all equal to $-2l-1$; from this one easily calculates that the valency of Γ' is $k_g := -(2r+1)l$ or $k_f := -(2l+1)r$, respectively. With the help of equations (1.3) one checks that a strongly regular graph with $f = 0$ or $g = 0$ is trivial. As (V, Δ) is a non-trivial two-graph, its descendant Γ cannot be trivial, hence $f+1$ and $g+1$ are at least 2. It follows that both eigenvalues $-2r-1$ and $-2l-1$ of B_ω have an eigenvector perpendicular to $\underline{1}$. Consequently the $(0, 1)$ adjacency matrix $A' = (J - I - B')/2$ of Γ' has restricted eigenvalues r and l; Theorem 1.2.5 assures that Γ' is strongly regular. If the valency of Γ' is $k_g = -(2r+1)l$, then r has multiplicity f

and l has multiplicity $g+1$; if Γ' has valency $k_f = -(2l+1)r$, then r has multiplicity $f+1$ and l has multiplicity g. Clearly k_f and k_g are integers if and only if r and l are integers.

The existence of a regular graph in a regular two-graph is equivalent to the existence of a certain regular partition of the vertex set of a descendant of the two-graph. To prove this, we need a lemma which is a corollary of Theorem 1.2.3.

Lemma 5.5.1 *Let A be the $(0,1)$ adjacency matrix of a strongly regular graph with eigenvalues k (the valency), r and l $(r > l)$. Let $\{X_1, X_2\}$ be a partition of the vertex set and let*

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be the quotient matrix of A with respect to this partition. Then

1. $b_{11} + b_{12} = b_{21} + b_{22} = k,\ b_{12}|X_1| = b_{21}|X_2|$
2. $l \le b_{11} + b_{22} - k \le r$
3. *One of the inequalities in (2) holds as an equality if and only if the partition is regular.*

Theorem 5.5.2 *Let (V, Δ) be a regular two-graph, and let r and l $(r > l)$ be the restricted eigenvalues of its descendants. Then there exists a regular graph with valency $k_f = -(2l+1)r$ (respectively $k_g = -(2r+1)l$) in (V, Δ) if and only if the vertex set of a descendant Γ admits a regular partition $\{X_1, X_2\}$ with $|X_1| = k_f$ (respectively $|X_1| = k_g$) such that the left (respectively right) hand side in (2) of Lemma 5.5.1 is tight. In both cases the regular graph is obtained by adding an isolated vertex ω to Γ and switching with respect to the partition $\{X_1, X_2 \cup \{\omega\}\}$.*

Proof. Suppose that Γ_f is a regular graph with valency $k_f = -(2l+1)r$ in (V, Δ). Let ω be a vertex of Γ_f, let X_1 be the set of vertices adjacent to it, and consider the partition $\{\{\omega\}, X_1, X_2\}$ with $X_2 := V \setminus (X_1 \cup \{\omega\})$. The quotient matrix of the adjacency matrix A_f of Γ_f with respect to this partition is

$$B_f = \begin{bmatrix} 0 & -(2l+1)r & 0 \\ 1 & l(1-r) & -(r+1)(l+1) \\ 0 & -r(l+1) & -rl \end{bmatrix}.$$

The entry $(b_f)_{22}$ (respectively $(b_f)_{32}$) is nothing else than the parameter λ (respectively μ) of the strongly regular graph Γ_f; the remaining entries follow from the fact that the row sums in B_f must be equal to k_f. Now switch Γ_f with respect to the partition $\{X_1, X_2 \cup \{\omega\}\}$; after deleting the isolated vertex ω, a descendant Γ of (V, Δ) is obtained. The partition $\{X_1, X_2\}$ of the vertex set $V \setminus \{\omega\}$ of Γ obviously satisfies $|X_1| = k_f$; the quotient matrix of the adjacency matrix A of Γ with respect to this partition is

$$B = \begin{bmatrix} l(1-r) & -l(r+1) \\ -rl & -rl \end{bmatrix}.$$

One sees that $b_{11} + b_{22} - (-2rl) = l$; by (3) of Lemma 5.5.1 the partition is regular.
On the other hand, suppose that the vertex set of a descendant Γ of (V, Δ) admits a regular partition $\{X_1, X_2\}$ with $|X_1| = k_f$ such that the left hand inequality in (2) of Lemma 5.5.1 is tight. With the help of Lemma 5.5.1 one calculates that the quotient matrix B of the adjacency matrix A of Γ with respect to this partition is exactly the matrix B above. Add an isolated vertex ω and switch with respect to the partition $\{X_1, X_2 \cup \{\omega\}\}$; call the graph thus obtained Γ_f. As $\{X_1, X_2\}$ is a regular partition of the vertex set of Γ, $\{X_1, X_2, \{\omega\}\}$ is a regular partition of the vertex set of Γ_f, and the corresponding quotient matrix is the matrix B_f above. The row sums of B_f are all equal to k_f, therefore Γ_f is regular with valency k_f. □

The following theorem connects the codes of a regular two-graph (V, Δ), a graph in (V, Δ) having an isolated vertex, and a regular graph in (V, Δ).

Theorem 5.5.3 ([48]) *Let (V, Δ) be a regular two-graph with two-graph code C; let r and l ($r > l$) be the restricted eigenvalues of its descendants. Let ω be a vertex of (V, Δ), and let Γ_ω denote the graph in (V, Δ) in which ω is an isolated vertex. Suppose that Γ' is a k'-regular graph in (V, Δ), and define χ as the characteristic vector of the set of vertices which are adjacent to ω in Γ'. Let A_ω and A' denote the adjacency matrices of Γ_ω and Γ', respectively. Then one of the following holds.*

1. $C_{A'+J} = C_{A_\omega+J} = C$, $\underline{1} \notin C_{A'}$, $\chi \in C_{A_\omega}$
   $\dim C_{A'} = \dim C_{A_\omega} = \dim C - 1$

2. $C_{A'} = \langle C_{A_\omega+J}, \chi\rangle = \langle C, \chi\rangle$, $\underline{1} \in C_{A'}$, $\chi \notin C_{A_\omega}$
   $\dim C_{A'} = \dim C_{A_\omega} + 2 = \dim C + 1$

If k' is even and $r + l$ is odd, then (1) holds. If $k' \equiv 2 \pmod 4$ and $r + l$ is even, or k' is odd, then (2) holds.

The combination of Theorem 5.5.3 with Theorem 5.2.4 can help in excluding the existence of certain regular graphs in regular two-graphs.

Theorem 5.5.4 *Let* (V, Δ) *be a regular two-graph, let* Γ *be the descendant of* (V, Δ) *with respect to a vertex* ω*, and suppose that the restricted eigenvalues* r *and* l $(r > l)$ *of* Γ *are both even. Let* f *and* g *be the multiplicities of* r *and* l*, respectively, and let* A *be the adjacency matrix of* Γ*. If* $r \equiv 2 \pmod 4$ *and* $\dim C_A \in \{f+1, g, g+1\}$*, then no regular graph with valency* $k_f = -(2l+1)r$ *exists in* (V, Δ)*. If* $l \equiv 2 \pmod 4$ *and* $\dim C_A \in \{f, f+1, g+1\}$*, then no regular graph with valency* $k_g = -(2r+1)l$ *exists in* (V, Δ)*.*

Proof. Let Γ_ω denote the graph in (V, Δ) in which the vertex ω is isolated. The adjacency matrix A_ω of Γ_ω is obtained from the adjacency matrix A of Γ by adding a row and a column containing only zeroes. This implies that the codes C_{A_ω} and C_A have the same dimension. Suppose that (V, Δ) contains a regular graph Γ_f with valency $k_f = -(2l+1)r$, and let A_f be its adjacency matrix. Its restricted eigenvalues are r with multiplicity $f+1$ and l with multiplicity g. If $r \equiv 2 \pmod 4$, then also $k_f \equiv 2 \pmod 4$; by Theorem 5.5.3 $\dim C_{A_f} = \dim C_A + 2$. If $\dim C_A \in \{f+1, g, g+1\}$, this yields $\dim C_{A_f} \in \{f+3, g+2, g+3\}$. However, Theorem 5.2.4 implies that the dimension of C_{A_f} is at most $\min\{f+2, g+1\}$. We conclude that such a graph Γ_f cannot exist. Similarly one proves that if $l \equiv 2 \pmod 4$ and $\dim C_A \in \{f, f+1, g+1\}$, then (V, Δ) contains no regular graph with valency $k_g = -(2r+1)l$. □

If the restricted eigenvalues r and l of the descendants of a regular two-graph are both odd, Theorem 5.5.4 can be applied to the complements, where the restricted eigenvalues $\overline{r} = -l-1$ and $\overline{l} = -r-1$ are both even. The hypotheses under which a non-existence result is obtained may seem quite restrictive, but they do apply in certain situations, as we will see.

5.6 (Non-)existence of regular graphs in the doubly transitive two-graphs

In this section we will discuss our knowledge concerning this question; it is collected in Table 5.1.

The Paley two-graphs can only contain regular graphs if they have descendants with integral eigenvalues. If this is the case, regular graphs with both possible valencies can be found (see [82]). Consider the Paley two-graph $\mathcal{P}(q^2)$, q an odd prime power, as the two-graph obtained by applying Theorem 1.9.5 to the ovoid $Q^-(3,q)$ of $Q(4,q)$. Let Π be the three-dimensional subspace spanned by $Q^-(3,q)$, let ω be a point of $Q^-(3,q)$, and let η be the tangent plane in Π to $Q^-(3,q)$ at ω. Suppose that $\{\omega, x, y\}$ is a coherent

two-graph	k_f	k_g
Paley: $\mathcal{P}(q^2)$, q odd	yes	yes
Hermitian: $\mathcal{H}(q)$, q odd	$q \equiv 3 \pmod 8$: no; $q = 5$: yes; rest: ?	yes
Ree: $\mathcal{R}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$	no	yes
symplectic: $\Sigma(2m, 2)$, $m \geq 2$	yes	yes
orthogonal hyperbolic: $\Omega^+(2m, 2)$, $m \geq 2$	yes	?
orthogonal elliptic: $\Omega^-(2m, 2)$, $m \geq 3$	?	yes
276	no	yes
176	yes	yes

Table 5.1: Regular graphs in the doubly transitive two-graphs.

triple in $\mathcal{P}(q^2)$, and let L be the intersection of the plane $\alpha_1 := \langle \omega, x, y \rangle$ with η. Let $\alpha_1, \ldots, \alpha_q$ denote the planes different from η through L in Π. Clearly every α_i, $i \in \{1, \ldots, q\}$ intersects $Q^-(3, q)$ in a non-degenerate conic containing ω. Let π be the orthogonal polarity associated with $Q(4, q)$. As $\{\omega, x, y\}$ is coherent, π maps the plane α_1 to a line intersecting $Q(4, q)$ in exactly two points; consequently the plane L^π intersects $Q(4, q)$ in the union of two lines through ω. The images of the planes α_i, $i \in \{1, \ldots, q\}$, under π are lines through the point Π^π in the plane L^π, and η^π is the line $\langle \Pi^\pi, \omega \rangle$. Since Π^π is not a point of $Q(4, q)$, every line α_i^π, $i \in \{1, \ldots, q\}$, intersects $Q(4, q)$ in exactly two points. It follows that all plane sections $\alpha_i \cap Q^-(3, q)$, $i \in \{1, \ldots, q\}$, are coherent sets in $\mathcal{P}(q^2)$. Taking descendants, one sees that an edge of the Paley graph $\mathcal{P}'(q^2)$ determines a partition of the vertex set into q cliques of size q. These q-cliques are lines of the net $\mathrm{pg}\,(q-1, (q-1)/2, (q-1)/2)$ which has $\mathcal{P}'(q^2)$ as a point graph, so we find spreads in this net. Recall that $\mathcal{P}'(q^2)$ is isomorphic to its complement $\overline{\mathcal{P}'(q^2)}$. Let $\mathcal{S}$ be a spread of the net having $\overline{\mathcal{P}'(q^2)}$ as a point graph, choose $(q-1)/2$ of the q lines of $\mathcal{S}$, and let X_1 be the set of $q(q-1)/2$ points covered by these $(q-1)/2$ lines. In $\overline{\mathcal{P}'(q^2)}$ any vertex in X_1 is adjacent to $q - 1 + (q-3)(q-1)/4$ vertices in X_1, and any vertex not in X_1 is adjacent to $(q-1)^2/4$ vertices in X_1. Equivalently, in $\mathcal{P}'(q^2)$ every vertex in X_1 is adjacent to $(q-3)(q+1)/4$ vertices in X_1, and every vertex not in X_1 is adjacent to $(q^2-1)/4$ vertices in X_1. Thus X_1 determines a regular partition of the vertex set of $\mathcal{P}'(q^2)$ with quotient matrix

$$B = \begin{bmatrix} (q-3)(q+1)/4 & (q+1)^2/4 \\ (q^2-1)/4 & (q^2-1)/4 \end{bmatrix}.$$

The restricted eigenvalues of $\mathcal{P}'(q^2)$ are $r = (q-1)/2$ and $l = (-q-1)/2$; one calculates that $|X_1| = k_f = -(2l+1)r$ and that $b_{11} + b_{22} + 2rl = l$. By Theorem 5.5.2 this implies that switching of the graph consisting of $\mathcal{P}'(q^2)$

and an isolated vertex with respect to the partition determined by X_1 yields a regular graph with valency $k_f = q(q-1)/2$ which is contained in $\mathcal{P}(q^2)$. The complement of this graph is regular with valency $k_g = q(q+1)/2$, and it is contained in $\overline{\mathcal{P}(q^2)}$, which is isomorphic to $\mathcal{P}(q^2)$.

The restricted eigenvalues of the Hermitian graph $\mathcal{H}'(q)$, q an odd prime power, are $r = (q-1)/2$ and $l = -(q^2+1)/2$; if $q \equiv 3 \pmod 4$ they are both odd. In that case the restricted eigenvalues $\overline{r} = (q^2-1)/2$ and $\overline{l} = -(q+1)/2$ of the complement $\overline{\mathcal{H}'(q)}$ are both even, and $q \equiv 3 \pmod 8$ implies $\overline{l} \equiv 2 \pmod 4$. The multiplicities of $\overline{r}$ and $\overline{l}$ are $g := q(q-1)$ and $f := (q-1)(q^2+1)$, respectively. The following theorem assures that the hypotheses of Theorem 5.5.4 are satisfied.

Theorem 5.6.1 ([12]) *The code of the complement $\overline{\mathcal{H}(q)}$ of the Hermitian two-graph $\mathcal{H}(q)$, q an odd prime power such that $q \equiv 3 \pmod 4$, has dimension $q(q-1)+1$.*

Now Theorem 5.5.4 can be applied to $\overline{\mathcal{H}(q)}$, q an odd prime power such that $q \equiv 3 \pmod 8$. After taking complements we obtain the following result.

Theorem 5.6.2 *If q is an odd prime power such that $q \equiv 3 \pmod 8$, then there exists no regular graph with valency $k_f = q^2(q-1)/2$ in the Hermitian two-graph $\mathcal{H}(q)$.*

The second smallest Hermitian two-graph $\mathcal{H}(5)$ contains a regular graph with valency $k_f = 50$, which will be discussed in detail in Section 6.1. It is unclear whether this can be generalised to other $q \equiv 1 \pmod 4$, or what happens if $q \equiv 7 \pmod 8$. On the other hand, all Hermitian two-graphs contain regular graphs with valency $k_g = q(q^2+1)/2$. The construction follows from the observation that the blocks of the Hermitian unital (see Section 1.8.2) are coherent sets in $\mathcal{H}(q)$. Let ω be a vertex of $\mathcal{H}(q)$, choose $(q^2+1)/2$ blocks of the Hermitian unital through ω, and let X_1 denote the set of $q(q^2+1)/2$ points different from ω covered by these blocks. In the descendant $\mathcal{H}'(q)$ of $\mathcal{H}(q)$ with respect to ω, cliques of size q meet the bound in Theorem 1.2.7, which implies that any vertex in X_1 is adjacent to $q-1+(q^2-1)(q-1)/4$ vertices in X_1, and that any vertex not in X_1 is adjacent to $(q^2+1)(q-1)/4$ vertices in X_1. Hence the regular partition of the vertex set of $\mathcal{H}'(q)$ determined by X_1 has quotient matrix

$$B = \begin{bmatrix} (q-1)(q^2+3)/4 & (q-1)(q^2-1)/4 \\ (q-1)(q^2+1)/4 & (q-1)(q^2+1)/4 \end{bmatrix}.$$

The restricted eigenvalues of $\mathcal{H}'(q)$ are $r = (q-1)/2$ and $l = -(q^2+1)/2$; one sees that $|X_1| = k_g = -(2r+1)l$ and that $b_{11} + b_{22} + 2rl = r$. By

Theorem 5.5.2 switching of the graph consisting of $\mathcal{H}'(q)$ and an isolated vertex with respect to the partition determined by X_1 yields a regular graph with valency $k_g = q(q^2+1)/2$ in $\mathcal{H}(q)$. Of course the $(q^2+1)/2$ blocks of the unital can be chosen in many different ways, and a great number of mutually non-isomorphic regular graphs are obtained (see [50]).

The Ree graph $\mathcal{R}'(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$, has the same eigenvalues and multiplicities as the Hermitian graph $\mathcal{H}'(q)$; moreover $q \equiv 3 \pmod 8$ always holds. The proof of Theorem 5.6.1, as given in [12], can be adapted with the help of a result in [66], yielding an analogon for the Ree two-graph.

Theorem 5.6.3 *The code of the complement $\overline{\mathcal{R}(q)}$ of the Ree two-graph $\mathcal{R}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$, has dimension $q(q-1)+1$.*

It follows that $\overline{\mathcal{R}(q)}$ satisfies the hypotheses of Theorem 5.5.4. After taking complements the following result is found.

Theorem 5.6.4 *The Ree two-graph $\mathcal{R}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$, contains no regular graph with valency $k_f = q^2(q-1)/2$.*

Graphs with valency $k_g = q(q^2+1)/2$ in $\mathcal{R}(q)$ can be constructed in a similar way as for the Hermitian two-graphs, since the blocks of the Ree–Tits unital are coherent sets in $\mathcal{R}(q)$.

Let $V(2m, 2)$ be the $2m$-dimensional vector space over GF(2), $m \geq 2$, and let Q^+, respectively Q^-, denote the vertex set of the hyperbolic orthogonal two-graph $\Omega^+(2m, 2)$, respectively the elliptic orthogonal two-graph $\Omega^-(2m, 2)$. Recall that both orthogonal two-graphs are sub-two-graphs of the symplectic two-graph $\Sigma(2m, 2)$. In [80] it is proved that the subset $Q^+ \setminus \{\underline{0}\}$, respectively $Q^- \setminus \{\underline{0}\}$, of the vertex set of the symplectic graph $S(2m, 2)$ satisfies the hypotheses of Theorem 5.5.2, yielding a regular graph with valency $k_g = (2^m - 1)(2^{m-1} + 1)$, respectively $k_f = (2^m + 1)(2^{m-1} - 1)$, in $\Sigma(2m, 2)$. By another result in [80], the subgraph of $S(2m, 2)$ induced on the set $V(2m, 2) \setminus Q^+$ belongs to the two-graph $\Omega^-(2m, 2)$ and is regular with valency $k_g = 2^{2m-2} - 1$, while the subgraph of $S(2m, 2)$ induced on the set $V(2m, 2) \setminus Q^-$ belongs to the two-graph $\Omega^+(2m, 2)$ and is regular with valency $k_f = 2^{2m-2} - 1$. It is not known whether $\Omega^+(2m, 2)$ contains a regular graph with valency $k_g = (2^m - 1)(2^{m-2} + 1)$, or whether $\Omega^-(2m, 2)$ contains a regular graph with valency $k_f = (2^m + 1)(2^{m-2} - 1)$.

Theorem 5.5.4 can also be applied to the sporadic regular two-graph on 276 vertices, of which the descendants have restricted eigenvalues $r = 2$ and $l = -28$, with multiplicities $f = 252$ and $g = 22$, respectively. Clearly $r \equiv 2 \pmod 4$; in [47] it is proved that the code of the descendant has dimension $g = 22$. Consequently there is no srg $(276, 110, 28, 54)$ in this two-graph. Such a strongly regular graph does not even exist at all, as the two-graph is uniquely determined by its parameters. However, this was known before; the parameters of such a strongly regular graph fail to satisfy certain necessary conditions. An srg $(276, 140, 58, 84)$, which is necessarily contained in this two-graph, does exist. The sporadic regular two-graph on 176 vertices contains both an srg $(176, 70, 18, 34)$ and an srg $(176, 90, 38, 54)$. All this can be checked in Brouwer's database of distance-regular graphs [8].

Chapter 6

Graphs and codes related to the Hermitian graph $\mathcal{H}(5)$

The two possibilities for the valency of a regular graph in the second smallest Hermitian two-graph $\mathcal{H}(5)$ are $k_g = 65$ and $k_f = 50$. In Section 5.6 a construction of regular graphs with valency $q(q^2+1)/2$ in $\mathcal{H}(q)$, q any odd prime power, was given, so regular graphs with valency 65 do occur in $\mathcal{H}(5)$. However, $\mathcal{H}(5)$ is the only Hermitian two-graph which is known to contain a regular graph with the other possible valency. We will give two constructions of an $\operatorname{srg}(126, 50, 13, 24)$ in $\mathcal{H}(5)$, and prove its uniqueness. After that, we will describe the code of $\mathcal{H}(5)$ and give constructions of the words of certain weights. The results in this chapter can also be found in [46].

6.1 The $\operatorname{srg}(126, 50, 13, 24)$ in $\mathcal{H}(5)$

6.1.1 Description via switching from $\mathcal{H}'(5)$

Let the Hermitian form defining $\mathcal{H}(5)$ be $H(X,Y) = X_0Y_0^5 + X_1Y_2^5 + X_2Y_1^5$. Any element of GF(25) can be written as $a + bi$, where $a, b \in \mathrm{GF}(5)$ and i is an element of GF(25) such that i^2 is equal to 2 (which is a non-square in GF(5)). Hence the Hermitian curve corresponding to H is

$$\mathcal{U} = \{(0,1,0)\} \cup \{(a+bi, (2a^2+b^2)+ci, 1) \mid a, b, c \in \mathrm{GF}(5)\}.$$

Let $\mathcal{H}'(5)$ be the descendant of $\mathcal{H}(5)$ with respect to the vertex $(0,1,0)$. From the definition of the Hermitian two-graphs in Subsection 1.9.3, and using the fact that an element of GF(25) is a square if and only if its sixth power is a square in GF(5), it can be calculated that two vertices $(a+bi, (2a^2+b^2)+ci, 1)$

and $(a'+b'i,(2a'^2+b'^2)+c'i,1)$ of $\mathcal{H}'(5)$ are adjacent if and only if

$$\left((a-a')^2-2(b-b')^2\right)^2+2\left(a'b-ab'+c-c'\right)^2=\pm 2,$$

for all $a,b,c,a',b',c' \in \mathrm{GF}(5)$.

Spence (personal communication) provided us with all possible sets of 50 vertices in $\mathcal{H}'(5)$ which yield an srg $(126,50,13,24)$ in $\mathcal{H}(5)$ after applying Theorem 5.5.2. One of these 150 sets is the union $S=S_1\cup S_2$, with

$$\begin{array}{rcl} S_1 &=& \{(a,2a^2\pm i,1) \mid a\in \mathrm{GF}(5)\},\\ S_2 &=& \{(a+bi,(2a^2+b^2)+(\pm 2-ab)i,1) \mid a,b\in \mathrm{GF}(5), b\neq 0\}. \end{array}$$

It is useful to have some information about the group stabilising this set S.

Lemma 6.1.1 *The subgroup of the automorphism group of $\mathcal{H}'(5)$ which fixes S setwise has order 40 and acts transitively on S_1 and S_2.*

Proof. An element of the automorphism group $\mathrm{P\Gamma U}_3(25)$ of $\mathcal{H}(5)$ acts on the points of $\mathrm{PG}(2,25)$ as $x\mapsto Mx^\sigma$, where M is a 3×3 unitary matrix and σ is an automorphism of $\mathrm{GF}(25)$. The automorphism group of $\mathcal{H}'(5)$ is the stabiliser $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ of the vertex $(0,1,0)$ in $\mathrm{P\Gamma U}_3(25)$. For a general element of $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$, the matrix M has the following form:

$$M=\begin{bmatrix} e & 0 & -ec^5\\ c & 1 & d\\ 0 & 0 & e^6\end{bmatrix}, \tag{6.1}$$

where $e\in \mathrm{GF}(25)\setminus\{0\}$ and $(c,d,1)\in\mathcal{U}\setminus\{(0,1,0)\}$. Define

$$\begin{array}{rcl}
\varphi &:& \begin{bmatrix} x\\ y\\ 1\end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1\\ 1 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x\\ y\\ 1\end{bmatrix},\\
\theta &:& \begin{bmatrix} x\\ y\\ 1\end{bmatrix} \mapsto \begin{bmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x\\ y\\ 1\end{bmatrix}^5,\\
\psi &:& \begin{bmatrix} x\\ y\\ 1\end{bmatrix} \mapsto \begin{bmatrix} -2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix} x\\ y\\ 1\end{bmatrix}^5.
\end{array}$$

One sees that φ, θ and ψ are elements of $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$; let H be the subgroup generated by them. By routine calculations it is verified that φ, θ and ψ have order 5, 2 and 4, respectively, that $\theta\circ\varphi=\varphi^4\circ\theta$, $\psi\circ\varphi=\varphi^2\circ\psi$ and

$\theta \circ \psi = \psi \circ \theta$, and that H has order 40. The images of a point $(a, 2a^2 \pm i, 1)$ under φ, θ and ψ are

$$\begin{aligned}
(a, 2a^2 \pm i, 1)^{\varphi} &= (a-1, 2(a-1)^2 \pm i, 1),\\
(a, 2a^2 \pm i, 1)^{\theta} &= (-a, 2(-a)^2 \mp i, 1),\\
(a, 2a^2 \pm i, 1)^{\psi} &= (2a, 2(2a)^2 \pm i, 1),
\end{aligned}$$

for $a \in \mathrm{GF}(5)$; consequently H fixes S_1 setwise and acts transitively on it. The images of a point $(a+bi, (2a^2+b^2)+(\pm 2 - ab)i, 1)$ of S_2 under φ, θ and ψ are

$$\begin{aligned}
&(a+bi, (2a^2+b^2)+(\pm 2 - ab)i, 1)^{\varphi}\\
&\quad = ((a-1)+bi, (2(a-1)^2+b^2)+(\pm 2 - (a-1)b)i, 1),\\
&(a+bi, (2a^2+b^2)+(\pm 2 - ab)i, 1)^{\theta}\\
&\quad = (-a+bi, (2(-a)^2+b^2)+(\mp 2 - (-a)b)i, 1),\\
&(a+bi, (2a^2+b^2)+(\pm 2 - ab)i, 1)^{\psi}\\
&\quad = (2a-2bi, (2(2a)^2+(-2b)^2)+(\pm 2 - (2a)(-2b))i, 1),
\end{aligned}$$

for $a, b \in \mathrm{GF}(5)$, $b \neq 0$. This implies that H also fixes S_2 setwise and acts (sharply) transitively on it.

Now we will show that any element of $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ fixing S necessarily fixes S_1. To settle the thoughts, assume that the field automorphism is $\mathbb{1}$; the reasoning is similar if it is not. This means that the element of $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ can be represented by a matrix of the form (6.1). If we put $e =: e_1 + e_2 i$, $c =: c_1 + c_2 i$ and $d =: d_1 + d_2 i$, then $d_1 = 2c_1^2 + c_2^2$ and $e^6 = e_1^2 - 2e_2^2$. The matrix M maps a point $(a, 2a^2 \pm i, 1)$ of S_1 to the point $(x, y, 1)$, with

$$\begin{aligned}
x &= \frac{(e_1(a-c_1) + 2e_2c_2) + (e_1c_2 + e_2(a-c_1))i}{e_1^2 - 2e_2^2},\\
y &= \frac{(2(a-c_1)^2 + c_2^2) + (ac_2 + d_2 \pm 1)i}{e_1^2 - 2e_2^2},
\end{aligned}$$

for $a \in \mathrm{GF}(5)$. Looking at the "imaginary" part of x, we see that either the images of all points of S_1 are in S_1 (if $e_2 = c_2 = 0$), or the images of all points of S_1 are in S_2 (if $e_2 = 0$ and $c_2 \neq 0$), or there is a unique $a \in \mathrm{GF}(5)$ such that the images of the points $(a, 2a^2 \pm i, 1)$ are in S_1 and the images of the remaining points of S_1 are in S_2 (if $e_2 \neq 0$). If one checks the "imaginary" part of y, which has to satisfy certain conditions for the point to belong to S, one can calculate that the second and third possibility lead to a contradiction. Consequently any element of $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ fixing S also fixes S_1, or equivalently the length of the orbit of an element of S_1 under the stabiliser of S in $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ is at most $|S_1| = 10$. On the other hand, one proves (by putting the appropriate restrictions on the matrix) that there are only four elements of $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ which fix the point $(0, i, 1) \in S_1$ and map the points $(0, -i, 1)$, $(i, 1+2i, 1)$ and $(i, 1-2i, 1)$ to points in S. Hence the order of the stabiliser of S in $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ is at most 40. We conclude that this group is precisely the group H constructed above. □

Now we can prove that S indeed yields the desired regular graph in $\mathcal{H}(5)$.

Theorem 6.1.2 *Switching of the graph consisting of $\mathcal{H}'(5)$ and the isolated vertex $(0, 1, 0)$ with respect to the partition determined by the set S described above produces an* srg $(126, 50, 13, 24)$ *in $\mathcal{H}(5)$.*

Proof. Any vertex in S is adjacent to 13 vertices in S; by Lemma 6.1.1 it suffices to check this for one vertex in S_1 and one vertex in S_2. Using Lemma 5.5.1 one sees that the quotient matrix arising from the partition of the vertex set of $\mathcal{H}'(5)$ determined by S is

$$B = \begin{bmatrix} 13 & 39 \\ 26 & 26 \end{bmatrix}.$$

By (3) of Lemma 5.5.1 the partition is regular. The assertion now follows from Theorem 5.5.2. □

As the order of the automorphism group $\mathrm{P\Gamma U}_3(25)_{(0,1,0)}$ of $\mathcal{H}'(5)$ is 6000 (see for instance [54]), Lemma 6.1.1 implies that the orbit of S under this group has length 150. Hence all sets generated by Spence are in the same orbit, which leads to the following result.

Theorem 6.1.3 *Up to isomorphism, there is only one strongly regular graph* srg $(126, 50, 13, 24)$ *in $\mathcal{H}(5)$.*

We tried to generalise the set S to other prime powers $q \equiv 1 \pmod 4$, but did not succeed. The most obvious generalisation, namely replacing ± 1 by "a non-zero square in $\mathrm{GF}(q)$" and ± 2 by "a non-square in $\mathrm{GF}(q)$", yields a set with the right number of vertices, but containing too many edges in the cases $q = 9$ and $q = 13$. Some variations did not work either. We will see that there are reasons to believe that the case $q = 5$ is special.

6.1.2 Description from the Hoffman–Singleton graph

The Hoffman–Singleton graph is a strongly regular graph which does not belong to a known infinite family and is uniquely determined by its parameters. Several constructions can be found in [9]. The construction we will give here relies on the following result.

Theorem 6.1.4 ([25]) *There exists a bijective correspondence between the* 35 *lines of* $\mathrm{PG}(3,2)$ *and the* 35 *unordered triples in a* 7*-set such that lines are concurrent if and only if the corresponding triples have exactly one element in common.*

The vertices of the *Hoffman–Singleton graph* HoSi are the 15 points and 35 lines of $\mathrm{PG}(3,2)$. Points are never adjacent, and lines are adjacent if and only if the corresponding triples are disjoint. A point and a line are adjacent if and only if they are incident. Using the properties of the projective space $\mathrm{PG}(3,2)$ and some counting arguments, one can show that HoSi is a strongly regular graph $\mathrm{srg}\,(50,7,0,1)$.

Goethals discovered that the $\mathrm{srg}\,(126,50,13,24)$ in $\mathcal{H}(5)$ can be constructed from the Hoffman–Singleton graph. Define a graph Φ' as follows: the vertices of Φ' are the edges of HoSi, and two vertices are adjacent if and only if the corresponding edges of HoSi are disjoint edges in a pentagon. By using the properties of HoSi one proves that Φ' is a strongly regular graph $\mathrm{srg}\,(175,72,20,36)$. In [88] it is proved that Φ' is the descendant of the sporadic regular two-graph on 176 vertices described in Subsection 1.9.3. Now fix a vertex u of HoSi and let F be the set of 42 vertices of HoSi which are not adjacent to u. An easy counting argument learns that there are 126 edges of HoSi inside F; the subgraph Γ_f of Φ' induced on the set of 126 vertices which correspond to the edges of HoSi inside F is an $\mathrm{srg}\,(126,50,13,24)$ which belongs to the Hermitian two-graph $\mathcal{H}(5)$ (see [88]). Since Γ_f is a subgraph of Φ', $\mathcal{H}(5)$ is a sub-two-graph of Φ. This "sporadic" behaviour seems to suggest that the existence of the regular graph Γ_f with valency $k_f = 50$ in $\mathcal{H}(5)$ is a coincidence, and not a special case of a property which holds for an infinite class of Hermitian two-graphs.

6.2 The code of $\mathcal{H}(5)$

6.2.1 Two descriptions

We will investigate the code of $\mathcal{H}(5)$ by using both descriptions from Section 6.1. Let A, respectively A_ω, be the adjacency matrix of the Hermitian

weight	C_A	C	weight	C_A	C
0	1	1	64	115290	236250
36	259	525	66	146412	286650
42	1380	2250	68	93240	189000
48	3675	7875	70	90720	189000
50	8568	18900	72	57890	110250
52	4725	7875	74	3150	7875
54	52360	110250	76	10332	18900
56	98280	189000	78	4200	7875
58	95760	189000	84	870	2250
60	140238	286650	90	266	525
62	120960	236250	126	0	1

Table 6.1: Weight distributions of the codes C_A of $\mathcal{H}'(5)$ and C of $\mathcal{H}(5)$.

graph $\mathcal{H}'(5)$, respectively the graph Γ_ω consisting of $\mathcal{H}'(5)$ and an isolated vertex ω; let C_A and C_{A_ω} be the corresponding codes. In [12] it is proved that the code of the Hermitian two-graph $\mathcal{H}(q)$, q an odd prime power such that $q \equiv 1 \pmod 4$, has dimension $q(q-1)+1$; consequently the dimension of the code C of $\mathcal{H}(5)$ is 21, and C_A and C_{A_ω} are 20-dimensional. The definition of two-graph codes implies that the number of words of a certain weight w in C equals the number of words of weight w or $126-w$ in C_{A_ω}, or equivalently in C_A. The weight distributions of C_A and C, generated by the software package GraphToCode described in [48], are given in Table 6.1. In [12] it is proved that the dual code $C^\perp$ of C is the code C_{N^T} generated by the rows of the transpose of the incidence matrix N of the Hermitian unital in PG$(2,25)$. Its weight distribution, which was calculated from the weight enumerator of C using Theorem 5.1.1, is given in Table 6.2.

Another description of the code C of $\mathcal{H}(5)$ follows from Subsection 6.1.2. Recall that the vertices of HoSi are the points and lines of PG$(3,2)$, that a point is adjacent to seven lines, and that a line is adjacent to three points and four lines. Therefore the set of edges of HoSi, or equivalently the set of vertices of Φ', admits a partition into a set of 105 edges between a point and a line and a set K of 70 edges between lines. One verifies that any element of K is adjacent (in Φ') to 18 elements of K. Using Lemma 5.5.1 one can complete the quotient matrix arising from this partition to

$$B = \begin{bmatrix} 18 & 54 \\ 36 & 36 \end{bmatrix}.$$

By (3) of Lemma 5.5.1 the partition is regular. Hence switching of the graph Φ' extended with an isolated vertex with respect to the partition determined

weight	number of code words
0,126	1
6,120	21525
8,118	1228500
10,116	184552200
12,114	18552581250
14,112	1314242167500
16,110	68079082765050
18,108	2667514596045250
20,106	81120550319953200
22,104	1954268046055820250
24,102	37924129974003107625
26,100	601068274770626079480
28,98	7871132213541952114500
30,96	86003428522692699669525
32,94	790676681158633859233875
34,92	6160512966566502512146500
36,90	40933186152747536692851800
38,88	233196885552052675471341375
40,86	1144458561403784200188758850
42,84	4858299700266647804047008000
44,82	17902783039228562237737038750
46,80	57444582099852105080832400050
48,78	160926311556224375790588057000
50,76	394499358043598096283245023710
52,74	847905860048538708672443181000
54,72	1600414904256326262263020974000
56,70	2656273048882773962828716181475
58,68	3880761895373175524889157806000
60,66	4994562484553575572315539465400
62,64	5665434441759763029730883712375

Table 6.2: Weight distribution of the dual code $C^{\perp}$ of C.

weight	$C(\Phi_f)$	$C(\Phi)$
0,176	1	1
50,126	176	0
56,120	1100	1100
64,112	4125	4125
66,110	5600	0
70,106	17600	0
72,104	15400	15400
78, 98	193600	0
80, 96	604450	604450
82, 94	462000	0
86, 90	369600	0
88	847000	847000

Table 6.3: Weight distributions of $C(\Phi_f)$ and $C(\Phi)$.

by K yields, by Theorem 5.5.2, a regular graph Φ_f with valency 70 in Φ. One easily calculates that Φ_f is an srg$\,(176, 70, 18, 34)$. The code $C(\Phi_f)$ of Φ_f is described in [18]. Its dimension is 22, and by Theorem 5.5.3 it contains the 21-dimensional code $C(\Phi)$ of Φ. The weight distributions of $C(\Phi_f)$ and $C(\Phi)$ can be found in Table 6.3. Let D_ω be the adjacency matrix of the graph Φ_ω consisting of Φ' and an isolated vertex. The weight of any non-zero row of D_ω is $72 \equiv 0 \pmod 4$; moreover, the parameters $\lambda = 20$ and $\mu = 36$ of Φ' assure that any two rows of D_ω have an even number of common ones. The weight of the all-one vector $\underline{1}$ of length 176 is divisible by four as well. The formula (5.1) for the weight of a sum of two code words implies that all words in $C(\Phi) = \langle C_{D_\omega}, \underline{1}\rangle$ have a weight divisible by four. Table 6.3 learns that the words of $C(\Phi)$ are precisely the words of $C(\Phi_f)$ of which the weight is divisible by four. Now we show that the vertex set of $\mathcal{H}(5)$ can be found in $C(\Phi_f)$.

Lemma 6.2.1 *There exists a code word χ of weight 126 in $C(\Phi_f)$ which is the characteristic vector of the vertex set of $\mathcal{H}(5)$.*

Proof. Fix a vertex u of HoSi which is a point of PG$(3, 2)$, and let F be the set of 42 vertices of HoSi which are not adjacent to u. Let U be the set of edges of HoSi containing the point u, let L be the set of edges of HoSi which are inside F and connect a point and a line, and let L' be the set of the remaining edges of HoSi between a point and a line. Let K_1, respectively K_2, denote the set of edges of HoSi which connect two lines and are inside F, respectively not inside F. Then $|U| = 7$, $|L'| = 14$, $|L| = 84$, $|K_1| = 42$

and $|K_2| = 28$. By using the properties of HoSi it can be shown that the partition $\{U, L', L, K_1, K_2\}$ of the vertex set of Φ' is regular and that the corresponding quotient matrix is

$$B = \begin{bmatrix} 0 & 12 & 24 & 12 & 24 \\ 6 & 0 & 30 & 30 & 6 \\ 2 & 5 & 29 & 21 & 15 \\ 2 & 10 & 42 & 8 & 10 \\ 6 & 3 & 45 & 15 & 3 \end{bmatrix},$$

where the 1st, 2nd, 3rd, 4th and 5th row and column of B are indexed by U, L', L, K_1 and K_2, respectively. If one adds an isolated vertex ω to Φ' and switches with respect to the partition determined by the set $K = K_1 \cup K_2$ of edges of HoSi between lines, one obtains the graph Φ_f, as explained above. The quotient matrix arising from the partition $\{\{\omega\}, U, L', L, K_1, K_2\}$ of the vertex set of Φ_f is

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 & 42 & 28 \\ 0 & 0 & 12 & 24 & 30 & 4 \\ 0 & 6 & 0 & 30 & 12 & 22 \\ 0 & 2 & 5 & 29 & 21 & 13 \\ 1 & 5 & 4 & 42 & 8 & 10 \\ 1 & 1 & 11 & 39 & 15 & 3 \end{bmatrix},$$

where the 1st row and column of B' are indexed by $\{\omega\}$ and the indexing of the remaining rows and columns follows from the indexing of the rows and columns of B. The transpose of B' is the matrix of column sums of the sub-matrices determined by this partition. The fact that the 1st and 4th column of B' sum up to the vector $(0, 0, 0, 1, 1, 0)$ (modulo 2) means that the rows of the adjacency matrix of Φ_f which correspond to the vertices in $\{\omega\} \cup L$ sum up to the characteristic vector of the set $L \cup K_1$. As this set is nothing else than the set of edges of HoSi inside F, i.e. the vertex set of Γ_f and $\mathcal{H}(5)$, the assertion follows. □

Let ω be a vertex of $\mathcal{H}(5)$. Consider the graph Γ_ω in $\mathcal{H}(5)$ in which ω is an isolated vertex. Recall that $\mathcal{H}(5)$ is a sub-two-graph of Φ; let Φ_ω denote the graph in Φ in which ω is an isolated vertex. Then Γ_ω is a subgraph of Φ_ω. Let A_ω be the adjacency matrix of Γ_ω, and let D_ω be the adjacency matrix of Φ_ω. Let χ denote the characteristic vector (of length 176 and weight 126) of the vertex set of $\mathcal{H}(5)$, as found in Lemma 6.2.1. If the 50 coordinate positions where χ has zeroes are deleted from a row of D_ω corresponding to a vertex of Γ_ω, a row of A_ω is obtained. As $C(\Phi)$ and C both have dimension

21, these rows of D_ω together with the all-one vector $\underline{1}$ generate the whole code $C(\Phi)$. Hence any code word of $C(\Phi)$ becomes, after deleting these 50 coordinate positions, a code word of C, and all words of C are obtained in this way.

6.2.2 Words of weight 52 and 74

Let A_ω be the adjacency matrix of the graph Γ_ω consisting of $\mathcal{H}'(5)$ and an isolated vertex. From the fact that $\mathcal{H}'(5)$ is an srg $(125, 52, 15, 26)$ it readily follows that any non-zero row of A_ω and the sum of any two rows of A_ω corresponding to non-adjacent vertices of $\mathcal{H}'(5)$ have weight 52, while the sum of any two rows of A_ω corresponding to adjacent vertices of $\mathcal{H}'(5)$ has weight 74. The sum of the all-one vector $\underline{1}$ and a word of weight 74 is a word of weight $126 - 74 = 52$. Recall that $C = C_{A_\omega + J} = \langle C_{A_\omega}, \underline{1} \rangle$; as there are 125 vertices and 7750 pairs of vertices in $\mathcal{H}'(5)$, 7875 words of weight 52 and 7875 words of weight 74 in C are obtained. If two such words coincided, one would have four or less rows of A_ω summing up to $\underline{0}$ (modulo 2), a contradiction to the fact that the minimum weight of $C^\perp$ is 6 (see Table 6.2). Therefore the vertices and the pairs of vertices of $\mathcal{H}'(5)$ yield all the words of weight 52 or 74 in C.

6.2.3 Words of weight 50 and 76

We will show that the set S described in Subsection 6.1.1 yields all code words of weight 50 or 76 in C. Let χ be the characteristic vector of S. The parameters of $\mathcal{H}'(5)$ and the srg $(126, 50, 13, 24)$ arising from S imply that we are in Case (1) of Theorem 5.5.3, and hence χ is a word of the code C_{A_ω} of the graph Γ_ω consisting of $\mathcal{H}'(5)$ and the isolated vertex $(0, 1, 0)$. By the construction of two-graph codes, χ is also a code word of the code $C = \langle C_{A_\omega}, \underline{1} \rangle$ of $\mathcal{H}(5)$. In Subsection 6.1.1 we showed that the orbit of S under the stabiliser of Γ_ω has length 150. This holds for each of the 126 descendants of $\mathcal{H}(5)$, and we obtain $150 \cdot 126 = 18900$ words of weight 50 in C. If these words are all distinct, this is the right number (see Table 6.1). Suppose that two of them coincide. Then it is possible to switch the graph consisting of $\mathcal{H}'(5)$ and the isolated vertex $(0, 1, 0)$ to a graph in $\mathcal{H}(5)$ with a different isolated vertex in such a way that in this new graph any vertex in S is still adjacent to precisely 13 vertices in S. Let $\{S', S''\}$ be the partition induced on S by the switching partition; without loss of generality we may assume $|S'| \leq |S''|$. The fact that switching with respect to the partition $\{S', S''\}$ does not affect the valency of the subgraph induced on S means that every vertex in S' is adjacent to precisely $|S''|/2$ vertices in S''. As the row sums of the adjacency matrix of the subgraph induced on S are all equal to 13, the subgraph induced on S' is regular with a certain valency $x \geq 0$ such that $13 = x + |S''|/2$. Now $|S'| \leq |S''|$ implies $|S''| \geq 25$; consequently $|S''| = 26$ and the subgraph induced on S' is void. However, a coclique of size $|S'| = 50 - 26 = 24$ in $\mathcal{H}'(5)$ is forbidden by Theorem 1.2.2, as the multiplicities of the restricted eigenvalues of $\mathcal{H}'(5)$ are $f = 104$ and $g = 20$. It follows that the orbit of the characteristic vector χ of S under the automorphism group $\mathrm{P\Gamma U}_3(25)$ of $\mathcal{H}(5)$ is indeed the set of all code words of weight 50 in C. All code words of weight 76 in C are obtained by adding the all-one vector $\underline{1}$.

6.2.4 Words of weight 36 and 90

Here we will prove that the minimum weight of C is 36, and we will construct all words of this weight. First we need some information about the words of minimal weight in $C(\Phi_f)$.

Lemma 6.2.2 ([52]) *Let $\mathcal{D}$ be the incidence structure in which the points are the 176 coordinate positions of the code words of $C(\Phi_f)$ and the blocks are the 176 words of weight 50 in $C(\Phi_f)$; a code word and a coordinate position are incident if the code word has a one on this coordinate position. Then $\mathcal{D}$ is a symmetric 2-$(176, 50, 14)$ design.*

Theorem 6.2.3 *The code C of $\mathcal{H}(5)$ has minimum weight at least 36.*

Proof. Let χ be the code word of $C(\Phi_f)$ which is the characteristic vector of the vertex set of $\mathcal{H}(5)$. As the restricted eigenvalues of Φ_f are $r = 2$ and $l = -18$ and its valency is $70 \equiv 2 \pmod 4$, the relation between $C(\Phi_f)$ and $C(\Phi)$ is as in (2) of Theorem 5.5.3, and $\underline{1} \in C(\Phi_f)$. Hence the complement $\xi := \underline{1} + \chi$ is also a code word of $C(\Phi_f)$; it clearly has weight 50. Let x be a non-zero code word of $C(\Phi)$ of weight $a + b$, where a is the number of common ones of ξ and x. Obviously $a + b \geq 56$ holds (see Table 6.3). From x arises a word of weight b in C. Note that ξ is not in $C(\Phi)$, as its weight is not divisible by four. Therefore the word $y := \xi + x$ does not belong to $C(\Phi)$ either, or equivalently its weight $50 - a + b$ is not divisible by four and is at least 50. Consequently $b - a$ is non-negative and divisible by four. If the weight of y is 50, then y and ξ are both blocks of the design in Lemma 6.2.2 and hence have precisely 14 common ones, implying $a = b = 36$. If the weight of y is not 50, it is at least 66 (see Table 6.3). In this case, $b \geq 36$ easily follows from $a + b \geq 56$ and $50 - a + b \geq 66$. We conclude that the word of C constructed from x has weight at least 36. □

Words of weight 36 in C do exist. Fix a vertex u of HoSi, let U be the set of 7 vertices adjacent to u, and let F be the set of 42 vertices of HoSi which are not adjacent to u. For a vertex v in U the set F can be partitioned into a set Y of 6 vertices adjacent to v and a set Z of 36 vertices which are not adjacent to v. The set of 126 edges inside F, which is the vertex set of Γ_f, can be partitioned into a set of 36 edges between a vertex of Y and a vertex of Z and a set of 90 edges inside Z (note that Y is a coclique in HoSi). One checks that an element of Y is adjacent (in Γ_f) to 5 vertices of Y; with the help of Lemma 5.5.1 one finds the quotient matrix

$$B = \begin{bmatrix} 5 & 45 \\ 18 & 32 \end{bmatrix},$$

which implies that the partition is regular. Both columns of B are equivalent to the vector $(1, 0)$ (modulo 2). As a consequence the rows of the adjacency matrix A_f of Γ_f corresponding to one set of the partition sum up to a word of weight 36 in C_{A_f}. By Theorem 5.5.3, C_{A_f} is contained in C, so a word of weight 36 in C is obtained. Using GAP [40] and the share package Projective Geometries [28], we verified that the action of the automorphism group $\mathrm{P\Gamma U}_3(25)$ of $\mathcal{H}(5)$ produces all 525 words of weight 36 in C. The words of weight 90 in C arise by adding the all-one vector $\underline{1}$.

6.2.5 Words of weight 42 and 84

Let u be a vertex of HoSi which is a point, and let F be the set of vertices of HoSi which are not adjacent to u. Obviously F consists of 14 points and 28 lines. The set of 126 edges of HoSi inside F, which is the vertex set of Γ_f, can be partitioned into a set L of 84 edges between a point and a line and a set K_1 of 42 edges between lines. It can be verified that an element of K_1 is adjacent (in Γ_f) to 8 elements of K. Lemma 5.5.1 yields the quotient matrix

$$B = \begin{bmatrix} 8 & 42 \\ 21 & 29 \end{bmatrix},$$

which implies that the partition is regular. Both columns of B are equivalent to the vector $(0, 1)$ (modulo 2). Therefore the rows of the adjacency matrix A_f of Γ_f corresponding to one set of the partition sum up to a word of weight 84 in C_{A_f} and hence in C. Again GAP [40] and the share package Projective Geometries [28] can be used to show that the action of the automorphism group $\mathrm{P\Gamma U}_3(25)$ of $\mathcal{H}(5)$ yields all 2250 words of weight 84 in C. The words of weight 42 are the complements of those of weight 84.

6.2.6 Words of weight 6 in the dual code

Let X be a coherent 6-set in $\mathcal{H}(5)$, and consider the descendant $\mathcal{H}'(5)$ of $\mathcal{H}(5)$ with respect to a vertex $\omega \in X$. The clique $X \setminus \{\omega\}$ of $\mathcal{H}'(5)$ meets the bound in Theorem 1.2.7, which implies that every vertex of $\mathcal{H}'(5)$ not in X is adjacent to precisely two vertices in $X \setminus \{\omega\}$. Let A_ω be the adjacency matrix of the graph in $\mathcal{H}(5)$ in which ω is an isolated vertex. Then the sum (modulo 2) of the rows of the matrix $A_\omega + J$ corresponding to the vertices in X is the zero vector $\underline{0}$. Consequently the characteristic vector of X is a code word of $C^\perp$ which clearly has weight 6. In [87] it is proved that there are 21525 coherent 6-sets in $\mathcal{H}(5)$. This number equals the number of code words of weight 6 in $C^\perp$, as can be seen in Table 6.2.

Chapter 7

A model for the split Cayley hexagon $H(q)$, q odd

In this chapter we will discuss an alternative model for the split Cayley hexagon $H(q)$, q odd. Usually the points of $H(q)$ are defined as all points of the parabolic quadric $Q(6,q)$, and the lines are either described vaguely as "certain lines on $Q(6,q)$" or given explicitly by their Grassmann coordinates as in Subsection 1.5.3. An advantage of this description is that no point or line plays a special role, but it can be rather impractical to work with. For instance, to find all lines incident with a certain point, one has to solve a system of equations or use coordinatisation, see Subsection 1.5.4. In [2] Bader and Lunardon construct a model for the dual of $H(q)$, q odd and not divisible by 3, which certainly looks more complicated than the usual one: it has four types of points and three types of lines. However, all incidences are visible without additional calculations. Moreover this model is closely related to the group-theoretical description of $H(q)$, see [2]. Inspired by the combination of some of Kantor's work ([62] and Remark 2 in [61]) with Lunardon's paper [67], Penttila (personal communication) suggested to the author that it could be worth trying to obtain a model for (the dual of) $H(q)$, q a power of 3, by modifying the Bader–Lunardon model. This indeed turned out to work; the result is in Section 7.3 and can also be found in [64]. The modified model allows us to see the dual of $H(q)$ as a subhexagon of the dual of the twisted triality hexagon $H(q^3,q)$, q odd, for which an alternative model was given by Lunardon in [67]. Finally we describe the Hermitian and Ree–Tits spreads of $H(q)$ in this context.

7.1 The twisted cubic

The model of Bader and Lunardon for the dual of $H(q)$, q an odd prime power which is not divisible by 3, relies on properties of the twisted cubic. For any prime power $q > 2$ a *twisted cubic* is a set Σ of $q+1$ points in $\mathrm{PG}(3,q)$ which can be described as

$$\Sigma = \{(0,0,0,1)\} \cup \{(1,u,u^2,u^3) \mid u \in \mathrm{GF}(q)\}$$

after choosing an appropriate coordinate system. The twisted cubic is extensively treated in [53]. No four points of a twisted cubic are coplanar, and consequently no three of its points are collinear. In fact any set of $q+1$ points in $\mathrm{PG}(3,q)$ no four of which are coplanar is necessarily a twisted cubic if q is odd. For each point p of a twisted cubic Σ a special line and a special plane through p can be constructed as follows. Let η be a plane not containing p, and consider the projection of $\Sigma \setminus \{p\}$ from p onto η. This is a set of q points no three of which are collinear; for $q \geq 5$ it can be completed to a conic in a unique way. In that case let p' be the missing point of the conic, and let L' be the tangent line in η to the conic at p'. Then the line $\langle p, p' \rangle$ is called the *tangent line* to Σ at p, and the plane $\langle p, L' \rangle$ is known as the *osculating plane* to Σ at p. Tangent lines and osculating planes to a twisted cubic in $\mathrm{PG}(3,q)$ also have an algebraic description which holds for any prime power $q > 2$. The tangent line at $(0,0,0,1)$ is $\langle (0,0,0,1), (0,0,1,0) \rangle$, and the osculating plane at $(0,0,0,1)$ has equation $X_0 = 0$. For $u \in \mathrm{GF}(q)$ the tangent line at $(1,u,u^2,u^3)$ is $\langle (1,u,u^2,u^3), (0,1,2u,3u^2) \rangle$, and the osculating plane at $(1,u,u^2,u^3)$ is described by $X_3 - 3uX_2 + 3u^2X_1 - u^3X_0 = 0$. It can be checked that the tangent lines to a twisted cubic are two by two disjoint.

Given a twisted cubic Σ in $\mathrm{PG}(3,q)$, $q > 2$, there exists a unique alternating form f of the four-dimensional vector space $V(4,q)$ underlying $\mathrm{PG}(3,q)$ such that for any point p of Σ the plane $p^\perp$ corresponding to the three-dimensional subspace $\{x \in V(4,q) \mid f(p,x) = 0\}$ of $V(4,q)$, where p also denotes any vector in the one-dimensional subspace corresponding to the projective point p, is the osculating plane to Σ at p. If $q \not\equiv 0 \pmod 3$, then f is non-degenerate, so a symplectic polarity of $\mathrm{PG}(3,q)$ is obtained. If $q \equiv 0 \pmod 3$, then f is degenerate; the radical corresponds to a projective line R which is the intersection of all osculating planes to Σ. In particular all tangent lines to Σ contain a point of R. In order to make this text more readable, we will, in the following sections, use expressions like "an alternating form of $\mathrm{PG}(3,q)$" instead of the longer exact statement. Because of the bijective correspondence between subspaces of $V(4,q)$ and subspaces of $\mathrm{PG}(3,q)$, this will probably not confuse the reader.

7.2 The model of Bader and Lunardon for the dual of $H(q)$, q odd and $q \not\equiv 0 \pmod 3$

This model, which was first described in [2], lives in the projective space $\mathrm{PG}(5,q)$, q odd and $q \not\equiv 0 \pmod 3$, equipped with a symplectic polarity $\perp$. Let ∞ be a point of $\mathrm{PG}(5,q)$, and let S be a three-dimensional subspace of the hyperplane $\infty^\perp$ such that S does not contain ∞. Let Σ be a twisted cubic in S such that for each point u of Σ the osculating plane to Σ at u is the plane $u^\perp \cap S$. For each point u of Σ let L_u denote the tangent line to Σ at u. Consider the incidence structure with the following points.

- ∞;
- points at distance[1] 2 from ∞: the points different from ∞ on the lines $\langle \infty, u \rangle$, $u \in \Sigma$;
- points at distance 4 from ∞: the planes which are totally isotropic with respect to $\perp$ and intersect $\infty^\perp$ in a line (not containing ∞) contained in a plane $\langle \infty, L_u \rangle$, $u \in \Sigma$;
- points at distance 6 from ∞: the points of $\mathrm{PG}(5,q)$ which do not lie in $\infty^\perp$.

The lines are defined as follows.

- lines incident with ∞: the lines $\langle \infty, u \rangle$, $u \in \Sigma$;
- lines at distance 3 from ∞: the lines not containing ∞ which are contained in a plane $\langle \infty, L_u \rangle$, $u \in \Sigma$;
- lines at distance 5 from ∞: the lines which are totally isotropic with respect to $\perp$ and intersect $\infty^\perp$ in a point (different from ∞) on a line $\langle \infty, u \rangle$, $u \in \Sigma$.

Incidence is defined by (reverse) containment, respecting distances to ∞. In particular a point $s \neq \infty$ on a line $\langle \infty, u \rangle$, $u \in \Sigma$, and a totally isotropic line intersecting $\infty^\perp$ in s are never incident because they are at distance 2 and 5 from ∞, respectively. In [2] it is proved that this incidence structure is isomorphic to the dual of the split Cayley hexagon $H(q)$.

[1] In fact it does not make sense to use the word "distance" before we have completely defined the incidence structure, but it will turn out that, for instance, the points called "points at distance 4 from ∞" are indeed at distance 4 from ∞. In the author's opinion naming the different types of points and lines in this way is clearer than using terminology like "points of type (3)" or "lines of type (b)".

Since the twisted cubic in $\mathrm{PG}(3,q)$, $q>2$, has different properties if q is divisible by 3, one has to be careful if one wants to extend the Bader–Lunardon model to this case. Let us try to construct it and look where a problem arises. A twisted cubic Σ in a three-dimensional projective space S over $\mathrm{GF}(q)$, $q \equiv 0 \pmod 3$, gives rise to a degenerate alternating form f; the radical of f is a line R which is the intersection of all osculating planes to Σ. The three-dimensional space S can be embedded in a five-dimensional space $\mathrm{PG}(5,q)$ equipped with a degenerate alternating form f' such that the restriction of f' to S is f and the radical of f' is R. For any point x of $\mathrm{PG}(5,q)$ define $x^\perp := \{y \in \mathrm{PG}(5,q) \mid f'(x,y)=0\}$. It is always possible to find a point ∞ of $\mathrm{PG}(5,q)$ such that $\infty \notin S \subseteq \infty^\perp$. Let again L_u denote the tangent line to Σ at u, for any point u of Σ. Every plane $\langle \infty, L_u \rangle$, $u \in \Sigma$, now contains a unique point x_u of the radical R. As any totally isotropic plane intersects the radical, a totally isotropic plane intersecting $\infty^\perp$ in a line lying in the plane $\langle \infty, L_u \rangle$ necessarily contains the point x_u. Hence a line contained in $\langle \infty, L_u \rangle$ and not containing ∞ or x_u is incident with no point at distance 4 from ∞. On the other hand, if L is a line lying in $\langle \infty, L_u \rangle$ which does not contain ∞ but contains x_u, then $L^\perp$ is four-dimensional. One easily sees that there are q^2 planes through L in $L^\perp$ which are not contained in $\infty^\perp$, implying that L is incident with q^2 points at distance 4 from ∞. As a consequence a line at distance 3 from ∞ is incident with either just one point, or q^2+1 points, and never with $q+1$ points, as it should be. We conclude that at least the lines at distance 3 from ∞ need to be redefined if one wants to obtain a model which also holds in characteristic 3.

7.3 The modified model for the dual of $H(q)$, q odd

In this section we propose a slightly modified version of the Bader–Lunardon construction which yields the dual of the split Cayley hexagon $H(q)$ for any odd prime power q. Let $\mathrm{PG}(5,q)$, q odd, be equipped with a symplectic polarity $\perp$ if $q \not\equiv 0 \pmod 3$, and with a degenerate alternating form f' of which the radical is a projective line R if $q \equiv 0 \pmod 3$. In the latter case define $x^\perp := \{y \in \mathrm{PG}(5,q) \mid f'(x,y)=0\}$ for all points x of $\mathrm{PG}(5,q)$. Let ∞ be a point of $\mathrm{PG}(5,q)$, and let S be a three-dimensional subspace of $\infty^\perp$ which does not contain ∞. If $q \equiv 0 \pmod 3$, ∞ and S must be chosen in such a way that $\infty \notin R \subseteq S$. Let Σ be a twisted cubic in S such that $u^\perp \cap S$ is the osculating plane to Σ at u (in S) for each point u of Σ. Let L_u denote the tangent line to Σ at u, for any point u of Σ. Consider the incidence structure $\mathcal{S}$ with the following points.

- ∞;
- points at distance 2 from ∞: the points different from ∞ on the lines $\langle\infty, u\rangle$, $u \in \Sigma$;
- points at distance 4 from ∞: the planes which are totally isotropic with respect to $\perp$ and intersect $\infty^\perp$ in a line (not containing ∞) contained in a plane $\langle\infty, L_u\rangle$, $u \in \Sigma$;
- points at distance 6 from ∞: the points of $\mathrm{PG}(5, q)$ which do not lie in $\infty^\perp$.

Define the lines as follows.

- lines incident with ∞: the lines $\langle\infty, u\rangle$, $u \in \Sigma$;
- lines at distance 3 from ∞: the three-dimensional subspaces intersecting $\infty^\perp$ in a plane $\langle\infty, L_u\rangle$, $u \in \Sigma$;
- lines at distance 5 from ∞: the lines which are totally isotropic with respect to $\perp$ and intersect $\infty^\perp$ in a point (different from ∞) on a line $\langle\infty, u\rangle$, $u \in \Sigma$.

A point s at distance 2 from ∞ and a line Π at distance 3 from ∞ are incident if and only if $\Pi \subseteq s^\perp$. All other incidences are defined by (reverse) containment, respecting distances to ∞ as in the Bader–Lunardon model. In Figure 7.1 all objects are drawn; the numbers indicate the distances to ∞.

Theorem 7.3.1 *The incidence structure $\mathcal{S}$ defined above is isomorphic to the dual of the split Cayley hexagon $H(q)$.*

Proof. In $\mathrm{PG}(5, q)$, q odd, the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

defines a symplectic polarity $\perp$ or a degenerate alternating form f' with radical $R := \langle(0,0,1,0,0,0),(0,0,0,1,0,0)\rangle$, according as $q \not\equiv 0 \pmod 3$ or $q \equiv 0 \pmod 3$. In the latter case put $x^\perp := \{y \in \mathrm{PG}(5, q) \mid f'(x, y) = 0\}$

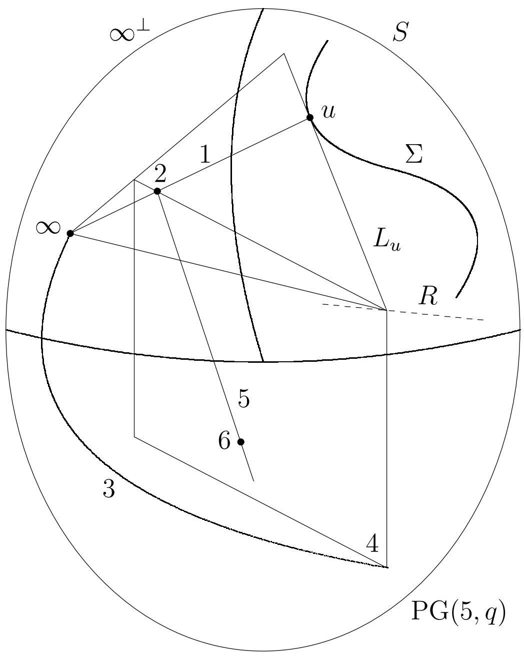


Figure 7.1: The modified model for $H(q)$, q odd.

for each point x of $\mathrm{PG}(5,q)$. If $\infty := (0,0,0,0,0,1)$, then the hyperplane $\infty^\perp$ is described by $X_0 = 0$. Let S be the three-dimensional subspace with equations $X_0 = X_5 = 0$; clearly $\infty \notin S \subseteq \infty^\perp$, and if q is divisible by 3, then S contains the radical R. Let the twisted cubic in S be the set

$$\Sigma := \{(0,0,0,0,1,0)\} \cup \{(0,1,u,u^2,u^3,0) \mid u \in \mathrm{GF}(q)\}.$$

The tangent line to Σ at $(0,0,0,0,1,0)$ is $\langle(0,0,0,0,1,0),(0,0,0,1,0,0)\rangle$; the tangent line to Σ at $(0,1,u,u^2,u^3,0)$ is $\langle(0,1,u,u^2,u^3,0),(0,0,1,2u,3u^2,0)\rangle$, for any $u \in \mathrm{GF}(q)$. The osculating plane to Σ at $(0,0,0,0,1,0)$ is described by $X_0 = X_5 = X_1 = 0$ and hence is precisely the plane $(0,0,0,0,1,0)^\perp \cap S$. For any $u \in \mathrm{GF}(q)$, the osculating plane to Σ at $(0,1,u,u^2,u^3,0)$ is the plane with equations $X_0 = X_5 = X_4 - 3uX_3 + 3u^2X_2 - u^3X_1 = 0$, which is precisely the plane $(0,1,u,u^2,u^3,0)^\perp \cap S$.
Inspired by an idea of Van Maldeghem cited in [2], we define the following bijection between the points and lines of the dual of $H(q)$, q odd (described by coordinatisation as in Subsection 1.5.4) and those of the incidence structure $\mathcal{S}$ described above.

- $[\infty] \leftrightarrow \infty = (0,0,0,0,0,1)$;
- $(a) \leftrightarrow \langle\infty,(0,1,a,a^2,a^3,0)\rangle = \langle(0,0,0,0,0,1),(0,1,a,a^2,a^3,0)\rangle$, for all $a \in \mathrm{GF}(q)$;
- $[a,l] \leftrightarrow (0,1,a,a^2,a^3,-l)$, for all $a,l \in \mathrm{GF}(q)$;
- $(a,l,a') \leftrightarrow$ the three-dimensional subspace with equations

$$\begin{cases} X_4 = 3aX_3 - 3a^2X_2 + a^3X_1 + lX_0 \\ X_3 = 2aX_2 - a^2X_1 + a'X_0, \end{cases}$$

  for all $a,l,a' \in \mathrm{GF}(q)$;
- $[a,l,a',l'] \leftrightarrow$ the plane with equations

$$\begin{cases} X_4 = 3aX_3 - 3a^2X_2 + a^3X_1 + lX_0 \\ X_3 = 2aX_2 - a^2X_1 + a'X_0 \\ X_5 = 3a'X_2 - (l+3aa')X_1 + 2l'X_0, \end{cases}$$

  for all $a,l,a',l' \in \mathrm{GF}(q)$;
- $(a,l,a',l',a'') \leftrightarrow \langle(0,1,a,a^2,a^3,-l),$
  $(1,0,-a'',a'-2aa'',l-3a^2a''+3aa',2l'-3a'a'')\rangle$,
  for all $a,l,a',l',a'' \in \mathrm{GF}(q)$;

- $(\infty) \leftrightarrow \langle\infty,(0,0,0,0,1,0)\rangle = \langle(0,0,0,0,0,1),(0,0,0,0,1,0)\rangle$;
- $[k] \leftrightarrow (0,0,0,0,1,k)$, for all $k \in \mathrm{GF}(q)$;
- $(k,b) \leftrightarrow$ the three-dimensional subspace with equations

$$\begin{cases} X_1 = kX_0 \\ X_2 = -bX_0, \end{cases}$$

  for all $k,b \in \mathrm{GF}(q)$;

- $[k,b,k'] \leftrightarrow$ the plane with equations

$$\begin{cases} X_1 = kX_0 \\ X_2 = -bX_0 \\ X_5 = kX_4 + 3bX_3 + 2k'X_0, \end{cases}$$

  for all $k,b,k' \in \mathrm{GF}(q)$;

- $(k,b,k',b') \leftrightarrow \langle(0,0,0,0,1,k),(1,k,-b,b',0,3bb'+2k')\rangle$, for all $k,b,k',b' \in \mathrm{GF}(q)$;
- $[k,b,k',b',k''] \leftrightarrow (1,k,-b,b',k'',3bb'+2k'+kk'')$, for all $k,b,k',b',k'' \in \mathrm{GF}(q)$.

One verifies that the three-dimensional subspaces corresponding to the elements of the form (a,l,a') or (k,b) intersect $\infty^\perp$ in a plane $\langle\infty,L_u\rangle$, $u \in \Sigma$. The planes corresponding to the elements of the form $[a,l,a',l']$ or $[k,b,k']$ are totally isotropic and intersect $\infty^\perp$ in a line which is contained in a plane $\langle\infty,L_u\rangle$, $u \in \Sigma$. The lines corresponding to the elements of the form (a,l,a',l',a'') or (k,b,k',b') are totally isotropic and intersect $\infty^\perp$ in a point on a line $\langle\infty,u\rangle$, $u \in \Sigma$. Moreover the three-dimensional subspace corresponding to (a,l,a') lies in $(0,1,a,a^2,a^3,-l)^\perp$, and the three-dimensional subspace corresponding to (k,b) lies in $(0,0,0,0,1,k)^\perp$. With the help of these observations, it is not difficult to check that the bijection preserves all flags consisting of elements which do not both correspond to a 5-tuple. The line of $\mathcal{S}$ corresponding to (a,l,a',l',a'') can be described by the equations

$$\begin{cases} X_2 = aX_1 - a''X_0 \\ X_3 = a^2X_1 + (a'-2aa'')X_0 \\ X_4 = a^3X_1 + (l-3a^2a''+3aa')X_0 \\ X_5 = -lX_1 + (2l'-3a'a'')X_0. \end{cases}$$

Therefore the point of $\mathcal{S}$ corresponding to $[k, b, k', b', k'']$ is incident with it if and only if

$$\begin{cases} -b = ak - a'' \\ b' = a^2k + a' - 2aa'' \\ k'' = a^3k + l - 3a^2a'' + 3aa' \\ 3bb' + 2k' + kk'' = -lk + 2l' - 3a'a'', \end{cases}$$

which is equivalent to

$$\begin{cases} a'' = ak + b \\ a' = a^2k + b' + 2ab \\ k'' = ka^3 + l - 3a^2a'' + 3aa' \\ k' = k^2a^3 + l' - kl - 3a^2a''k - 3a'a'' + 3aa''^2. \end{cases}$$

As this is precisely the condition under which (a, l, a', l', a'') and $[k, b, k', b', k'']$ are incident (see Subsection 1.5.4), the incidence structure $\mathcal{S}$ is indeed isomorphic to the dual of $H(q)$. □

Note that for $q \not\equiv 0 \pmod 3$ the modification of the Bader–Lunardon model amounts to replacing the lines at distance 3 from ∞ by their images under the symplectic polarity, and redefining incidence accordingly.

7.4 $H(q)$ as a subhexagon of $H(q^3, q)$, q odd

In [67] Lunardon describes a model for the dual of the twisted triality hexagon $H(q^3, q)$, q odd. It starts from a particular partial ovoid $\mathcal{O}_3$ of the symplectic polar space $W(7, q)$ which has the property that there exists a partial spread $\mathcal{S}_3$ of $W(7, q)$ such that every point p of $\mathcal{O}_3$ is contained in an element T_p of $\mathcal{S}_3$. A construction of $\mathcal{O}_3$ can be found in [67].

The model of Lunardon lives in the projective space $\mathrm{PG}(9, q)$ equipped with a symplectic polarity $\perp$. Let ∞ be a point of $\mathrm{PG}(9, q)$, and let U be a seven-dimensional subspace of the hyperplane $\infty^\perp$ such that U does not contain ∞. Let $W(7, q)$ be the symplectic polar space arising from the restriction of the symplectic polarity $\perp$ to U, and let $\mathcal{O}_3$ denote the partial ovoid mentioned above. Consider the incidence structure with the following points.

- ∞;
- points at distance 2 from ∞: the points different from ∞ on the lines $\langle \infty, u \rangle$, $u \in \mathcal{O}_3$;

- points at distance 4 from ∞: the totally isotropic four-dimensional subspaces intersecting $\infty^\perp$ in a three-dimensional subspace (not containing ∞) contained in a four-dimensional subspace $\langle \infty, T_u \rangle$, $u \in \mathcal{O}_3$;
- points at distance 6 from ∞: the points of $\mathrm{PG}(9, q)$ which do not lie in $\infty^\perp$.

The lines are defined as follows.

- lines incident with ∞: the lines $\langle \infty, u \rangle$, $u \in \mathcal{O}_3$;
- lines at distance 3 from ∞: the three-dimensional subspaces not containing ∞ which are contained in a four-dimensional subspace $\langle \infty, T_u \rangle$, $u \in \mathcal{O}_3$;
- lines at distance 5 from ∞: the totally isotropic lines intersecting $\infty^\perp$ in a point (different from ∞) on a line $\langle \infty, u \rangle$, $u \in \mathcal{O}_3$.

Incidence is defined by (reverse) containment, respecting distances to ∞. In [67] it is proved that this incidence structure is isomorphic to the dual of the twisted triality hexagon $H(q^3, q)$.

Obviously this model is very similar to the Bader–Lunardon model for the dual of $H(q)$, q odd and $q \not\equiv 0 \pmod 3$. Therefore it will probably not surprise the reader that for these values of q the model for the dual of $H(q)$ can be obtained from the model for the dual of $H(q^3, q)$ by intersection with an appropriate five-dimensional subspace. Such a subspace can be found thanks to some nice properties of the partial ovoid $\mathcal{O}_3$ of $W(7, q)$, q odd. Through any three points of $\mathcal{O}_3$ there is a unique three-dimensional subspace S of $\mathrm{PG}(7, q)$ which intersects $\mathcal{O}_3$ in a twisted cubic Σ. Moreover the tangent lines to Σ are the intersections $T_p \cap S$, $p \in \Sigma$, and $T_p \cap S$ is a line if and only if $p \in \Sigma$. If $q \not\equiv 0 \pmod 3$, then the restriction to S of the symplectic polarity determined by $W(7, q)$ is precisely the symplectic polarity of S which maps every point of Σ to its osculating plane. If $q \equiv 0 \pmod 3$, then the restriction to S of the non-degenerate alternating form corresponding to the symplectic polarity determined by $W(7, q)$ is a degenerate alternating form f. For any point $p \in \Sigma$ the plane $p^\perp := \{y \in S \mid f(p, y) = 0\}$ is the osculating plane to Σ at p, and the radical of f is a line R which is the intersection of all osculating planes to Σ.

Consider Lunardon's model for the dual of $H(q^3, q)$, q odd, and let S be a three-dimensional subspace of U which intersects the partial ovoid $\mathcal{O}_3$ in a twisted cubic Σ. Let V be a five-dimensional subspace of $\mathrm{PG}(9, q)$ which contains ∞ and S and is not contained in $\infty^\perp$. Define a substructure of $H(q^3, q)$ by only admitting the points and lines of which the intersection with

V has the maximal possible dimension. For $q \not\equiv 0 \pmod 3$ this substructure is nothing else than the Bader–Lunardon model for the dual of $H(q)$. If $q \equiv 0 \pmod 3$, however, it is the structure which arises if the original Bader–Lunardon model is constructed; as explained in Section 7.3, the lines at distance 3 from ∞ do not behave as they should in this case. This seems to have been overlooked in [67]. One can solve this difficulty by replacing the lines at distance 3 from ∞ in the model for the dual of $H(q^3, q)$, q odd, by their images under the symplectic polarity $\perp$ in $\mathrm{PG}(9, q)$: they become the five-dimensional subspaces intersecting $\infty^\perp$ in a four-dimensional subspace $\langle \infty, T_u \rangle$, $u \in \mathcal{O}_3$. Incidence is redefined accordingly. If one now only takes the points and lines of which the intersection with the five-dimensional subspace V has the maximal possible dimension, one obtains precisely the modified model for the dual of $H(q)$, q odd. We conclude that the modified model is more compatible with Lunardon's model than the original one.

7.5 Spreads of $H(q)$

The Hermitian spread of $H(q)$, q odd, has a simple description in the modified model for the dual of $H(q)$. Recall (see Subsection 1.7.3) that the Hermitian spread can be described by coordinatisation as

$$\mathcal{S}_H = \{[\infty]\} \cup \{[k, -\gamma^{-1}k'', k', \gamma k, k''] \mid k, k', k'' \in \mathrm{GF}(q)\},$$

where γ is a non-square in $\mathrm{GF}(q)$. The elements of $\mathcal{S}_H$ are points of the dual of $H(q)$; using the bijection in Section 7.3 one finds

$$\mathcal{S}_H = \{\infty\} \cup \{(1, k, \gamma^{-1}k'', \gamma k, k'', -k' + kk'') \mid k, k', k'' \in \mathrm{GF}(q)\}.$$

One sees that the elements of $\mathcal{S}_H \setminus \{\infty\}$ are precisely the points not contained in $\infty^\perp$ of the three-dimensional subspace Π_H with equations

$$\begin{cases} X_3 = \gamma X_1 \\ X_4 = \gamma X_2. \end{cases}$$

The intersection of Π_H with $\infty^\perp$ is a plane which intersects every plane $\langle \infty, L_u \rangle$, $u \in \Sigma$, in only the point ∞. For an $x \in \mathcal{S}_H \setminus \{\infty\}$ the line regulus $\mathcal{R}(\infty, x)$ is nothing else than the set of points on the line $\langle \infty, x \rangle$.

The Ree–Tits spread of $H(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$, can also be described in this model. At first sight the description is not easier than the one using coordinatisation. However, if we consider the model over the subfield $\mathrm{GF}(3)$, the Ree–Tits spread turns out to be a union of q subspaces. The four-dimensional projective space $\infty^\perp$ occurring in the model corresponds to a five-dimensional vector space over $\mathrm{GF}(q)$, which can be seen as a vector space of dimension $10h+5$ over $\mathrm{GF}(3)$. From the latter a $(10h+4)$-dimensional projective space Π_∞ over $\mathrm{GF}(3)$ arises. The points of the affine space $\mathrm{AG}(5,q) = \mathrm{PG}(5,q) \setminus \infty^\perp$ are in bijective correspondence with the points of an affine space $\mathrm{AG}(10h+5,3)$, which together with Π_∞ forms a projective space $\mathrm{PG}(10h+5,3)$. The points of $\infty^\perp$ correspond to mutually disjoint $2h$-dimensional subspaces of Π_∞, and similarly the i-dimensional subspaces of $\infty^\perp$, $i \in \{1,2,3\}$, correspond to certain $((i+1)(2h+1)-1)$-dimensional subspaces of Π_∞. The i-dimensional subspaces of $\mathrm{PG}(5,q)$ intersecting $\infty^\perp$ in an $(i-1)$-dimensional subspace, $i \in \{0,1,2,3,4\}$, correspond to the $i(2h+1)$-dimensional subspaces of $\mathrm{PG}(10h+5,3)$ which intersect the hyperplane Π_∞ in the $(i(2h+1)-1)$-dimensional subspace corresponding to the $(i-1)$-dimensional subspace of $\infty^\perp$. Now we will introduce coordinates in $\mathrm{PG}(10h+5,3)$. A basis for $\mathrm{GF}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$, over $\mathrm{GF}(3)$ is given by the set $\{\gamma^{3^i} \mid i \in \{0,\dots,2h\}\}$, where γ is a primitive element of $\mathrm{GF}(q)$. The hyperplane $\infty^\perp$ of $\mathrm{PG}(5,q)$ has equation $X_0 = 0$. For any point $(1,x_1,x_2,x_3,x_4,x_5)$ of $\mathrm{PG}(5,q) \setminus \infty^\perp$ we can write $x_j =: \sum_{i=0}^{2h} x_{j,i}\gamma^{3^i}$, with $x_{j,i} \in \mathrm{GF}(3)$ for all $i \in \{0,\dots,2h\}$, $j \in \{1,2,3,4,5\}$. Hence this point can be associated to a unique point $(1,x_{1,0},\dots,x_{1,2h},\dots,x_{5,0},\dots,x_{5,2h})$ of $\mathrm{PG}(10h+5,3)$ which does not lie in the hyperplane Π_∞ with equation $X_0 = 0$. Recall (see Subsection 1.7.3) that the Ree-Tits spread of $H(q)$ can be described by coordinatisation as

$$\begin{aligned}\mathcal{S}_R &= \{[\infty]\} \cup \{[k, k''^{s/3} - k^{1+s/3}, k', k^{1+2s/3} + k'^{s/3} + k^{s/3}k''^{s/3}, k''] \mid \\ &\qquad k,k',k'' \in \mathrm{GF}(q)\},\end{aligned}$$

where $s = 3^{h+1}$. After applying the bijection in Section 7.3 this becomes

$$\begin{aligned}\mathcal{S}_R &= \{\infty\} \cup \{(1, k, -k''^{s/3} + k^{1+s/3}, k^{1+2s/3} + k'^{s/3} + k^{s/3}k''^{s/3}, \\ &\qquad k'', -k' + kk'') \mid k,k',k'' \in \mathrm{GF}(q)\}.\end{aligned}$$

Let x be any element of $\mathrm{GF}(q)$, and write $x =: \sum_{i=0}^{2h} x_i\gamma^{3^i}$, with $x_i \in \mathrm{GF}(q)$ for all $i \in \{0,\dots,2h\}$. Then

$$x^{s/3} = \left(\sum_{i=0}^{2h} x_i\gamma^{3^i}\right)^{3^h} = \sum_{i=0}^{2h} x_i\gamma^{3^{i+h}} = \sum_{i=0}^{2h} x_{i-h}\gamma^{3^i},$$

where indices are taken modulo $2h+1$. We see that each component of $x^{s/3}$ is a $\mathrm{GF}(3)$-linear function of the components of x. This does not hold for $x^{1+s/3}$ and $x^{1+2s/3}$, as the maps $x \mapsto x^{1+s/3}$ and $x \mapsto x^{1+2s/3}$ are not automorphisms of $\mathrm{GF}(q)$. For any element of $\mathcal{S}_R \setminus \{\infty\}$, the parts which are not GF(3)-linear functions of the components of k, k' or k'' are in the second (the term $k^{1+s/3}$), the third (the terms $k^{1+2s/3}$ and $k^{s/3}k''^{s/3}$) and the fifth (the term kk'') components; in all of them the parameter k is involved. Therefore we will fix a $k \in \mathrm{GF}(q)$ and consider the set

$$\begin{aligned}\mathcal{S}_k \;:=\; \{&(1, k, -k''^{s/3} + k^{1+s/3}, k^{1+2s/3} + k'^{s/3} + k^{s/3}k''^{s/3}, k'', -k' + kk'') \mid \\ &k', k'' \in \mathrm{GF}(q)\}.\end{aligned}$$

Put $k =: \sum_{i=0}^{2h} k_i\gamma^{3^i}$, $k^{1+s/3} =: \sum_{i=0}^{2h} l_i\gamma^{3^i}$ and $k^{1+2s/3} =: \sum_{i=0}^{2h} m_i\gamma^{3^i}$, with $k_i, l_i, m_i \in \mathrm{GF}(3)$ for all $i \in \{0, \ldots, 2h\}$. We can write $k^{s/3} = \sum_{i=0}^{2h} k_{i-h}\gamma^{3^i}$, where indices are taken modulo $2h+1$. Clearly all elements of $\mathcal{S}_k$ satisfy the equations

$$X_{1,i} = k_i X_0,\ i \in \{0, \ldots, 2h\}.$$

Now we will express the relation between the second and the fourth components of an arbitrary element of $\mathcal{S}_k$. Write the fourth component as $\sum_{i=0}^{2h} k''_i\gamma^{3^i}$, where $k''_i \in \mathrm{GF}(3)$ for all $i \in \{0, \ldots, 2h\}$; then the second component equals $\sum_{i=0}^{2h}(-k''_{i-h} + l_i)\gamma^{3^i}$. Consequently the set $\mathcal{S}_k$ is contained in the subspace of $\mathrm{PG}(10h+5, 3)$ with equations

$$X_{2,i} = -X_{4,i-h} + l_i X_0,\ i \in \{0, \ldots, 2h\}.$$

Write $k' =: \sum_{i=0}^{2h} k'_i\gamma^{3^i}$, with $k'_i \in \mathrm{GF}(3)$ for all $i \in \{0, \ldots, 2h\}$, and let $\sum_{i=0}^{2h} y_i\gamma^{3^i}$, with $y_i \in \mathrm{GF}(3)$ for all $i \in \{0, \ldots, 2h\}$, denote the fifth component. We also need to know how products can be expressed in terms of the basis elements; define

$$\gamma^{3^j}\gamma^{3^l} =: \sum_{i=0}^{2h} n_i^{j,l}\gamma^{3^i},$$

with $n_i^{j,l} \in \mathrm{GF}(3)$ for all $i, j, l \in \{0, \ldots, 2h\}$. Then

$$\begin{aligned}\sum_{i=0}^{2h} y_i\gamma^{3^i} &= \sum_{i=0}^{2h}(-k'_i)\gamma^{3^i} + \left(\sum_{j=0}^{2h} k_j\gamma^{3^j}\right)\left(\sum_{l=0}^{2h} k''_l\gamma^{3^l}\right) \\ &= \sum_{i=0}^{2h}\left(-k'_i + \sum_{l=0}^{2h}\left(\sum_{j=0}^{2h} k_j n_i^{j,l}\right)k''_l\right)\gamma^{3^i},\end{aligned}$$

and consequently

$$k'_i = -y_i + \sum_{l=0}^{2h}(\sum_{j=0}^{2h} k_j n_i^{j,l})k''_l$$

for all $i \in \{0, \ldots, 2h\}$. Now the third coordinate becomes

$$\begin{aligned} & \sum_{i=0}^{2h} m_i \gamma^{3^i} + \sum_{i=0}^{2h} k'_{i-h} \gamma^{3^i} + \left(\sum_{j=0}^{2h} k_{j-h} \gamma^{3^j} \right) \left(\sum_{l=0}^{2h} k''_{l-h} \gamma^{3^l} \right) \\ = & \sum_{i=0}^{2h} m_i \gamma^{3^i} + \sum_{i=0}^{2h} \left(-y_{i-h} + \sum_{l=0}^{2h} \left(\sum_{j=0}^{2h} k_j n_{i-h}^{j,l} \right) k''_l \right) \gamma^{3^i} \\ & + \sum_{i=0}^{2h} \left(\sum_{l=0}^{2h} \left(\sum_{j=0}^{2h} k_{j-h} n_i^{j,l} \right) k''_{l-h} \right) \gamma^{3^i} \\ = & \sum_{i=0}^{2h} \left(m_i - y_{i-h} + \sum_{l=0}^{2h} \left(\sum_{j=0}^{2h} k_j (n_{i-h}^{j,l} + n_i^{j+h,l+h}) \right) k''_l \right) \gamma^{3^i}. \end{aligned}$$

One verifies that $n_{i-h}^{j,l} = n_i^{j+h,l+h}$ for all $i, j, l \in \{0, \ldots, 2h\}$. It follows that $\mathcal{S}_k$ is precisely the set of points of $\mathrm{PG}(10h+5, 3) \setminus \Pi_\infty$ which are contained in the $(4h+2)$-dimensional subspace Π_k described by the equations

$$\begin{cases} X_{1,i} = k_i X_0, \ i \in \{0, \ldots, 2h\} \\ X_{2,i} = -X_{4,i-h} + l_i X_0, \ i \in \{0, \ldots, 2h\} \\ X_{3,i} = -X_{5,i-h} - \sum_{l=0}^{2h} (\sum_{j=0}^{2h} k_j n_{i-h}^{j,l}) X_{4,l} + m_i X_0, \ i \in \{0, \ldots, 2h\}. \end{cases}$$

Each two subspaces Π_k and $\Pi_{\overline{k}}$, $k, \overline{k} \in \mathrm{GF}(q)$, $\overline{k} \neq k$, intersect in the $2h$-dimensional subspace Y_R of Π_∞ with equations

$$\begin{cases} X_0 = 0 \\ X_{1,i} = 0, \ i \in \{0, \ldots, 2h\} \\ X_{2,i} = 0, \ i \in \{0, \ldots, 2h\} \\ X_{4,i} = 0, \ i \in \{0, \ldots, 2h\} \\ X_{3,i} = -X_{5,i-h}, \ i \in \{0, \ldots, 2h\}. \end{cases}$$

The subspace Y_R does not correspond to a point of the hyperplane $\infty^\perp$ of $\mathrm{PG}(5, q)$, but is contained in the $(4h+1)$-dimensional subspace of Π_∞ which corresponds to the line with equations $X_0 = X_1 = X_2 = X_4 = 0$ lying in $\infty^\perp$. This is the line spanned by ∞ and the point on the radical R which lies on the tangent line to the twisted cubic at $(0, 0, 0, 0, 1, 0)$. Each intersection $\Pi_k \cap \Pi_\infty$ is contained in the $(6h+2)$-dimensional subspace Z_R of Π_∞ with equations

$$\begin{cases} X_0 = 0 \\ X_{1,i} = 0, \ i \in \{0, \ldots, 2h\} \\ X_{2,i} = -X_{4,i-h}, \ i \in \{0, \ldots, 2h\}. \end{cases}$$

The subspace Z_R does not correspond to a plane contained in $\infty^\perp$, but is contained in the $(8h+3)$-dimensional subspace of Π_∞ which corresponds to the three-dimensional subspace of $\infty^\perp$ with equations $X_0 = X_1 = 0$. This subspace is spanned by ∞ and the osculating plane to the twisted cubic at $(0,0,0,0,1,0)$.

Bijlage A

Nederlandse samenvatting

A.1 Inleiding

In dit proefschrift onderzoeken we eindige *incidentiestructuren*, *grafen*, *tweegrafen* en *codes*, evenals hun onderlinge verbanden. De meeste van de incidentiestructuren die aan bod zullen komen, zijn *partieel lineaire ruimten*[1], d.w.z. twee verschillende punten zijn incident met hoogstens één gemeenschappelijke rechte. Deze verscheidenheid aan objecten wordt in Hoofdstuk 1 ingevoerd. Om de lengte van deze samenvatting te beperken, behandelen we hier enkel de structuren die centraal staan in één van de hoofdstukken. De lezer die niet over voorkennis omtrent bijvoorbeeld sterk reguliere grafen, klassieke polaire ruimten of designs beschikt, kan in Hoofdstuk 1 terecht.

A.1.1 (Semi-)partiële meetkunden

Een *partiële meetkunde* $\mathrm{pg}(s,t,\alpha)$ (zie [7]) is een incidentiestructuur $\mathcal{S}$ die aan de volgende voorwaarden voldoet.

pg1 $\mathcal{S}$ is een partieel lineaire ruimte van de orde (s,t), met $s,t \geq 1$.

pg2 Voor elk punt p en voor elke rechte R niet door p zijn juist α van de punten op R collineair met p, waarbij $1 \leq \alpha \leq \min\{s+1, t+1\}$.

Een partiële meetkunde met $\alpha = 1$ is een *veralgemeende vierhoek*, zie ook Paragraaf A.1.2. Als $\alpha = s+1$, vinden we een *design*. Partiële meetkunden met $\alpha = t$ noemen we (*Bruck*)-*netten* van *orde* $s+1$ en *graad* $t+1$. Het puntgraaf van een $\mathrm{pg}(s,t,\alpha)$ met $\alpha \leq s$ is een sterk regulier graaf

$$\mathrm{srg}\left((s+1)(st/\alpha+1), (t+1)s, s-1+t(\alpha-1), (t+1)\alpha\right).$$

[1] Zie de voetnoot op bladzijde 1.

Een sterk regulier graaf dat dergelijke parameters heeft, voor zekere positieve natuurlijke getallen s, t en α met $\alpha \leq \min\{s, t+1\}$, heet *pseudo-meetkundig*. Als het werkelijk het puntgraaf van een partiële meetkunde is, wordt het *meetkundig* genoemd. De rechten van een partiële meetkunde bereiken, als men ze opvat als cliques in het puntgraaf, de bovengrens in Stelling 1.2.7.

Een *semi-partiële meetkunde* $\mathrm{spg}\,(s, t, \alpha, \mu)$ (zie [29]) is een incidentiestructuur $\mathcal{S}$ waarvoor de volgende axioma's gelden.

spg1 $\mathcal{S}$ is een partieel lineaire ruimte van de orde (s, t), met $s, t \geq 1$.

spg2 Voor elk punt p en voor elke rechte R niet door p zijn hetzij 0, hetzij α van de punten op R collineair met p, waarbij $1 \leq \alpha \leq \min\{s+1, t+1\}$.

spg3 Voor elke twee niet-collineaire punten bestaan er juist μ punten die collineair zijn met beide, met $1 \leq \mu \leq (t + 1)\alpha$.

Als $\mu = (t+1)\alpha$, dan hebben we een partiële meetkunde. Het puntgraaf van een semi-partiële meetkunde $\mathrm{spg}\,(s, t, \alpha, \mu)$ met $\alpha \leq s$ is een

$$\mathrm{srg}\,(1 + (t + 1)s(\mu + t(s + 1 - \alpha))/\mu, (t + 1)s, s - 1 + t(\alpha - 1), \mu)\,.$$

A.1.2 Veralgemeende veelhoeken

We geven hier enkel de meest gebruikte begrippen en verwijzen naar [100] voor meer details. Een *veralgemeende n-hoek*, met $n \geq 3$, is een incidentiestructuur

GP1 die geen gewone k-hoeken bevat voor $2 \leq k < n$;

GP2 waarin door elke twee elementen (punten of rechten) een gewone n-hoek gaat.

Een veralgemeende n-hoek heet *dik* als elk element incident is met minstens drie elementen. Het incidentiegraaf van een veralgemeende n-hoek heeft diameter n en bevat geen circuits (paden van positieve lengte waarvan begin- en eindpunt samenvallen) van lengte kleiner dan $2n$. In de context van veralgemeende veelhoeken zullen we regelmatig over afstanden spreken; we bedoelen dan steeds afstanden in het incidentiegraaf. Punten op maximale afstand noemen we ook *tegenoverliggend*. Door een stelling van Feit en Higman [38] bestaan dikke veralgemeende n-hoeken alleen voor $n \in \{3, 4, 6, 8\}$; bovendien hebben ze altijd een orde (s, t), $s, t > 1$. In dit proefschrift zullen we vooral dikke veralgemeende zeshoeken ontmoeten. Daarvan zijn op dualiteit na slechts twee oneindige klassen gekend. De zeshoek $H(q^3, q)$ bestaat uit de absolute punten en rechten ten opzichte van een bepaalde trialiteit van $Q^+(7, q^3)$ en heeft orde (q^3, q). De zeshoek $H(q)$, die van de orde q is, onstaat door beperking van de coördinaten in $H(q^3, q)$ tot GF(q) en bestaat uit alle punten en bepaalde rechten van een parabolische kwadriek $Q(6, q)$. De expliciete beschrijving van de rechten staat in Paragraaf 1.5.3. Nuttig om te weten is dat punten van $H(q)$ tegenoverliggend zijn als en slechts als ze niet collineair zijn op $Q(6, q)$, en dat elk punt p samen met de verzameling $\Gamma_2(p)$ van punten op afstand 2 ervan een vlak op $Q(6, q)$ vormt. Verder is $H(q)$ zelfduaal als en slechts als $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$.

Stel dat x en y twee tegenoverliggende punten in $H(q)$ zijn. Dan kan men aantonen dat er juist $q + 1$ rechten zijn die op afstand 3 van x en y liggen. Omdat ze een (maximaal) stelsel onderling disjuncte rechten op een hyperbolische kwadriek $Q^+(3, q)$ vormen, spreken we van een *rechtenregulus*. De verzameling $\mathcal{P}(x, y)$ van punten die op afstand 3 van alle rechten van deze rechtenregulus liggen, bevat uiteraard x en y en telt eveneens $q + 1$ elementen. We noemen $\mathcal{P}(x, y)$ de *puntregulus* bepaald door x en y. Een punt- of rechtenregulus wordt eenduidig bepaald door elke twee van zijn elementen.

Bij de *coördinatisatie* van veralgemeende zeshoeken wordt aan elk element een i-tal toegekend, met $i \in \{1, \ldots, 5\}$, op een zodanige manier dat de meeste incidenties onmiddellijk uit de i-tallen af te lezen zijn. In Paragraaf 1.5.4 wordt in het bijzonder de coördinatisatie van $H(q)$ besproken. Tabel 1.1 kan als geheugensteuntje dienen.

A.1.3 Ovoïden en spreads van veralgemeende veelhoeken

Een *ovoïde* van een veralgemeende n-hoek Γ, met n even, is een verzameling $\mathcal{O}$ van twee aan twee tegenoverliggende punten zodanig dat elk element van Γ op afstand hoogstens $n/2$ van een element van $\mathcal{O}$ ligt. Een *spread* is een verzameling $\mathcal{S}$ van twee aan twee tegenoverliggende rechten zodanig dat elk element van Γ op afstand hoogstens $n/2$ van een element van $\mathcal{S}$ ligt.

De zeshoek $H(q)$ bevat spreads voor elke priemmacht q; als q deelbaar is door 3, vindt men door te dualiseren ovoïden. Een expliciete beschrijving van de *Hermitische ovoïden* en *spreads* en van de *Ree–Tits–ovoïden* en *–spreads* vindt men in Paragraaf 1.7.3. Een belangrijke eigenschap van de Hermitische spread is dat hij de rechtenregulus bepaald door elke twee van zijn elementen bevat.

A.1.4 Twee-grafen en switching

Beschouw een partitie $\{V_1, V_2\}$ van de toppenverzameling V van een graaf Γ, en construeer een nieuw graaf Γ' door alle bogen tussen een top in V_1 en een top in V_2 te vervangen door niet-bogen en vice versa, terwijl men bogen en niet-bogen binnen V_1 en binnen V_2 ongemoeid laat. Dit proces is gekend als *switching* ten opzichte van de partitie $\{V_1, V_2\}$ van V. Grafen die in elkaar kunnen worden omgezet door switching, heten *switching-equivalent.* Het gaat hier inderdaad om een equivalentierelatie, waarvan de klassen *switching-klassen* genoemd worden. Men kan bewijzen dat twee grafen Γ en Γ' op dezelfde toppenverzameling switching-equivalent zijn als en slechts als voor elke verzameling T van drie toppen het aantal bogen van Γ in T en het aantal bogen van Γ' in T dezelfde pariteit hebben.

Een *twee-graaf* is een paar (V, Δ) dat bestaat uit een eindige *toppenverzameling* V en een verzameling Δ van (ongeordende) *coherente drietallen*, zodanig dat elke verzameling van vier toppen een even aantal coherente drietallen bevat. Uit een graaf construeert men een twee-graaf door een drietal toppen coherent te noemen als en slechts als het een oneven aantal bogen bevat; men gaat na dat elk graaf op vier toppen inderdaad een even aantal drietallen met een oneven aantal bogen bevat. Switching-equivalente grafen geven aanleiding tot hetzelfde twee-graaf. Anderzijds levert een twee-graaf op eenduidige wijze een switching-klasse van grafen, wat een bijectief verband impliceert.

Een twee-graaf is *regulier* als elk paar toppen bevat is in hetzelfde aantal coherente drietallen. Dit aantal a en het aantal toppen n noemen we de *parameters* van het twee-graaf. Reguliere twee-grafen staan in nauw verband met sterk reguliere grafen, en wel als volgt. Zij (V, Δ) een regulier twee-graaf met parameters n en a, en kies een top $\omega \in V$. Definieer Γ als het graaf op $V \setminus \{\omega\}$ waarin twee toppen x en y adjacent zijn als en slechts als $\{\omega, x, y\}$ coherent is in (V, Δ). Het graaf Γ wordt het *afgeleide graaf* van (V, Δ) ten opzichte van ω genoemd; het is een sterk regulier graaf $\mathrm{srg}\,(n-1, a, (3a-n)/2, a/2)$. Men kan deze constructie ook omkeren: voegt men een geïsoleerde top toe aan een sterk regulier graaf $\mathrm{srg}\,(v, k, \lambda, k/2)$, dan is het twee-graaf dat overeenkomt met de switching-klasse bepaald door dit nieuwe graaf regulier met parameters $n = v + 1$ en $a = k$.

Er zijn verscheidene klassen van interessante reguliere twee-grafen gekend. Sommige ervan kunnen dankzij Stelling 1.9.5 eenvoudig beschreven worden aan de hand van ovoïden van parabolische kwadrieken. We geven hier de volledige lijst van twee-grafen met een tweevoudig transitieve automorfismengroep (zie [90]), die we verder *tweevoudig transitieve* twee-grafen zullen noemen. Voor meer details verwijzen we naar [82] of [89].

- Het *Paley–twee-graaf* $\mathcal{P}(q)$, waarbij q een priemmacht is zodanig dat $q \equiv 1 \pmod 4$, leeft op de projectieve rechte $\mathrm{PG}(1, q)$ en heeft parameters $n = q + 1$ en $a = (q - 1)/2$. Een alternatieve beschrijving van $\mathcal{P}(q^2)$, q een oneven priemmacht, bekomt men door Stelling 1.9.5 toe te passen op de elliptische kwadriek $Q^-(3, q)$ gezien als ovoïde van $Q(4, q)$.

- Het *Hermitische twee-graaf* $\mathcal{H}(q)$, q een oneven priemmacht, zullen we expliciet beschrijven omdat het een belangrijke rol speelt in dit proefschrift. Beschouw een niet-ontaarde Hermitische vorm H van $\mathrm{PG}(2, q^2)$ en noem de overeenkomstige Hermitische kromme $\mathcal{U}$. De toppenverzameling van $\mathcal{H}(q)$ is $\mathcal{U}$; een drietal $\{x, y, z\}$ is coherent als en slechts als $H(x, y)H(y, z)H(z, x)$ een kwadraat (respectievelijk niet-kwadraat) is in $\mathrm{GF}(q^2)$, $q \equiv 3 \pmod 4$ (respectievelijk $q \equiv 1 \pmod 4$). De parameters van $\mathcal{H}(q)$ zijn $n = q^3 + 1$ en $a = (q - 1)(q^2 + 1)/2$. Het Hermitische twee-graaf $\mathcal{H}(3^h)$, $h \in \mathbb{N} \setminus \{0\}$, kan eveneens beschreven worden door Stelling 1.9.5 toe te passen op de Hermitische ovoïde van $H(3^h)$, gezien als ovoïde van $Q(6, 3^h)$.

- Het *Ree–twee-graaf* $\mathcal{R}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N}$, verkrijgt men door Stelling 1.9.5 toe te passen op de Ree–Tits–ovoïde van $H(q)$, gezien als ovoïde van $Q(6, q)$. Het heeft dezelfde parameters als $\mathcal{H}(q)$.

- De toppenverzameling van het *symplectische* twee-graaf $\Sigma(2m, 2)$, met $m \geq 2$, is de $2m$-dimensionale vectorruimte $V(2m, 2)$ over $\mathrm{GF}(2)$. De parameters zijn $n = 2^{2m}$ en $a = 2^{2m-1} - 2$.

- Het *hyperbolische orthogonale twee-graaf* $\Omega^+(2m, 2)$ en het *elliptische orthogonale twee-graaf* $\Omega^-(2m, 2)$, $m \geq 2$, leven op de verzameling van nulpunten van een hyperbolische, respectievelijk elliptische orthogonale vorm in $V(2m, 2)$. De parameters van $\Omega^+(2m, 2)$ zijn $n = 2^{2m-1} + 2^{m-1}$ en $a = 2^{2m-2} + 2^{m-1} - 2$, die van $\Omega^-(2m, 2)$ zijn $n = 2^{2m-1} - 2^{m-1}$ en $a = 2^{2m-2} - 2^{m-1} - 2$.

- Er bestaat een sporadisch regulier twee-graaf met parameters $n = 276$ en $a = 112$ dat de groep Co_3 van Conway als automorfismengroep heeft.

- Er bestaat een sporadisch regulier twee-graaf met parameters $n = 176$ en $a = 72$ dat de groep HS van Higman en Sims als automorfismengroep heeft.

A.2 Twee-graaf-meetkunden

Een *twee-graaf-meetkunde* (zie [44]) is een incidentiestructuur $\mathcal{S}$ die aan de volgende voorwaarden voldoet.

TGG1 $\mathcal{S}$ is een 2-(v, k, λ)-design met $v = 1 + (k-1)(2\lambda - 1)$.

TGG2 Twee blokken van $\mathcal{S}$ snijden elkaar in hoogstens twee punten.

TGG3 Elke verzameling van vier punten van $\mathcal{S}$ bevat een even aantal drietallen die in een blok liggen.

Uit deze definitie blijkt dat de verzameling van punten samen met de verzameling van drietallen die in een blok liggen een regulier twee-graaf met parameters $n = v$ en $a = \lambda(k-2)$ vormt. We noemen een twee-graaf *meetkundig* als het op deze manier met een twee-graaf-meetkunde geassocieerd is. Een twee-graaf-meetkunde is tevens een extensie van een partiële meetkunde pg (s, t, α) met $s = k-2$, $t = \lambda - 1$ en $\alpha = k/2 - 1$.

We geven een overzicht van onze kennis omtrent het bestaan of nietbestaan van twee-graaf-meetkunden verbonden aan de tweevoudig transitieve twee-grafen. De Paley–twee-grafen $\mathcal{P}(q^2)$, q een oneven priemmacht, ondersteunen een unieke twee-graaf-meetkunde [39, 103]. De blokken zijn alle coherente $(q+1)$-verzamelingen van $\mathcal{P}(q^2)$. Het kleinste Hermitische tweegraaf $\mathcal{H}(3)$ is meetkundig [44], terwijl $\mathcal{H}(5)$ en $\mathcal{H}(7)$ dat niet zijn [87]; voor grotere waarden van q is dit probleem nog open. Het kleinste Ree–twee-graaf $\mathcal{R}(3)$ is isomorf met $\mathcal{H}(3)$ en bijgevolg meetkundig; ook hier kennen we de situatie niet voor grotere waarden van q. Het kleinste symplectische twee-graaf $\Sigma(4, 2)$ ondersteunt een twee-graaf-meetkunde [44]. De twee-grafen $\Sigma(2m, 2)$ met $m \geq 3$ zijn echter niet meetkundig. Voor $m \in \{3, 4\}$ wordt dit bewezen in [32], voor $m = 5$ en m even in [34], en voor m oneven door de auteur in Stelling 2.2.1. Het resultaat voor alle $m \geq 3$ volgt eveneens uit een meer algemeen, ongepubliceerd resultaat van Thas (Stelling 2.2.2). Het hyperbolische orthogonale twee-graaf is meetkundig in het kleinste geval [44], en niet meetkundig in alle andere gevallen. Dit volgt uit een delingsvoorwaarde in [44]. Wat het elliptische orthogonale twee-graaf betreft, weten we alleen dat $\Omega^-(6, 2)$ meetkundig is [44] (daar $\Omega^-(4, 2)$ triviaal is, laten we het buiten beschouwing). Het is niet bekend of het sporadische twee-graaf op 276 toppen een twee-graaf-meetkunde ondersteunt, maar bij het sporadische twee-graaf op 176 toppen is dat wel het geval [44].

Men kan het concept twee-graaf-meetkunde wijzigen op een zodanige manier dat de afgeleide structuur een semi-partiële meetkunde is. Na onderzoek van de parameterlijst van gekende semi-partiële meetkunden in [31] blijkt echter slechts één klasse hiervoor in aanmerking te komen, afgezien van wat triviale gevallen. Het gaat hier om $\overline{W(2n+1,2)}$, $n \geq 1$. Deze semi-partiele meetkunde bestaat uit de punten van PG$(2n+1,2)$ en de rechten die niet totaal isotroop zijn ten opzichte van een symplectische polariteit van PG$(2n+1,2)$ (zie [33]). Een 2-$(2^{2n+2}, 4, 2^{2n})$-design dat een extensie van $\overline{W(2n+1,2)}$ is, blijkt eenvoudig te construeren. De puntenverzameling is de $(2n+2)$-dimensionale vectorruimte $V(2n+2,2)$ over GF(2); de blokken zijn de tweedimensionale deelruimten van $V(2n+2,2)$ die niet totaal isotroop zijn ten opzichte van een niet-ontaarde alternerende vorm, en hun beelden onder alle mogelijke translaties. Het geassocieerde twee-graaf is het complement van het symplectische twee-graaf $\Sigma(2n+2,2)$.

A.3 Hermitische partiële meetkunden

A.3.1 Meetkundige beschrijving

Voor elke $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$, construeren we $2^{q(q+1)}$ partiële meetkunden pg$\,(q-1, (q^2-1)/2, (q-1)/2)$ die als puntgraaf het Hermitische graaf $\mathcal{H}'(q)$ hebben. De beschrijving van $\mathcal{H}'(q)$ waarvan we gebruik maken, luidt als volgt. Zij $\mathcal{O}_H$ een Hermitische ovoïde van $H(q)$, kies $p \in \mathcal{O}_H$, en veronderstel dat π de orthogonale polariteit voorstelt die geassocieerd is met de kwadriek $Q(6,q)$ waarop $H(q)$ leeft. Dan is het graaf op $\mathcal{O}_H \setminus \{p\}$ waarbij twee toppen x en y adjacent zijn als en slechts als de driedimensionale ruimte $\langle p,x,y\rangle^\pi$ de kwadriek $Q(6,q)$ snijdt in een hyperbolische kwadriek $Q^+(3,q)$ isomorf met het Hermitische graaf $\mathcal{H}'(q)$. De puntenverzameling van de partiële meetkunden is $\mathcal{O}_H \setminus \{p\}$, en we onderscheiden twee soorten rechten. De rechten van type 2 zijn de verzamelingen van de vorm $\mathcal{P}(p,t) \setminus \{p\}$, met $t \in \mathcal{O}_H$. Vermits $\langle \mathcal{P}(p,t)\rangle^\pi$ de kwadriek $Q(6,q)$ in een $Q^+(3,q)$ snijdt, zijn dit q-cliques in $\mathcal{H}'(q)$. De constructie van de rechten van type 1 is wat ingewikkelder; we gaan in vier stappen te werk.

STAP 1. Neem een punt $s \in \Gamma_2(p)$ en veronderstel dat M een rechte is in het vlak $\Gamma_2(p) \cup \{p\}$ die wel s, maar niet p bevat. Kies een punt $t \in \mathcal{O}_H \setminus \{p\}$ waarvoor $\Gamma_2(p) \cap \Gamma_4(t) = M$, en definieer $Q(4,q) := p^\pi \cap t^\pi \cap Q(6,q)$ en $Q^+(3,q) := \langle \mathcal{P}(p,t)\rangle^\pi \cap Q(6,q)$. Duid met N_k, $k \in \mathrm{GF}(q) \setminus \{0\}$, de $q-1$ generatoren van $Q(4,q)$ door s aan die niet op $Q^+(3,q)$ liggen. We bewijzen dat de verzameling $C_k := N_k^\pi \cap (\mathcal{O}_H \setminus \{p\})$ een q-clique in $\mathcal{H}'(q)$ is en bijgevolg als rechte van een partiële meetkunde zou kunnen dienen. Niet alle q-cliques C_k kunnen echter terzelfdertijd gebruikt worden: het blijkt dat we de verzameling $\{N_k \mid k \in \mathrm{GF}(q) \setminus \{0\}\}$ in twee verzamelingen van $(q-1)/2$

rechten kunnen opsplitsen zodanig dat de q-cliques elkaar in juist één punt (namelijk t) snijden als en slechts als de overeenkomstige generatoren uit dezelfde verzameling komen. Kies één van deze twee verzamelingen en noem de overeenkomstige q-cliques de rechten van type 1 geassocieerd met het drietal (s, M, t).

STAP 2. De rechten van type 1 geassocieerd met (s, M, t'), voor elke $t' \in \mathcal{P}(p, t) \setminus \{p, t\}$, worden bepaald door deze geassocieerd met (s, M, t): de overeenkomstige generatoren zijn de projecties vanuit p op t'^{π} van de generatoren die de rechten van type 1 geassocieerd met (s, M, t) leverden.

STAP 3. Zij $M' \neq M$ een rechte door s, maar niet door p in het vlak $\Gamma_2(p) \cup \{p\}$, en kies een punt $v \in \mathcal{O}_H \setminus \{p\}$ waarvoor $\Gamma_2(p) \cap \Gamma_4(v) = M'$. Aangezien v niet in de q-clique $\mathcal{P}(p, t) \setminus \{p\}$ ligt, is v adjacent met precies $(q-1)/2$ elementen van $\mathcal{P}(p, t) \setminus \{p\}$. Veronderstel dat t'' één van die elementen is en construeer de rechten van type 1 geassocieerd met (s, M, t'') zoals beschreven in Stap 2. Juist één van deze rechten bevat v, en kan daardoor ook beschouwd worden als een rechte van type 1 geassocieerd met (s, M', v). Herhaalt men dit voor elk van de $(q-1)/2$ punten in $\mathcal{P}(p, t) \setminus \{p\}$ die adjacent zijn met v, dan bekomt men $(q-1)/2$ rechten van type 1 geassocieerd met (s, M', v), die telkens slechts het punt v gemeen blijken te hebben.

STAP 4. Herhaal Stappen 1 tot en met 3 voor alle andere punten s' van $\Gamma_2(p)$.

In Onderdeel 3.2 tonen we aan dat de aldus gedefinieerde incidentiestructuur, een partieel lineaire ruimte van de orde $(q-1, (q^2-1)/2)$ is. Collineaire punten zijn bovendien steeds adjacent in $\mathcal{H}'(q)$; een eenvoudig telargument leert dat ook het omgekeerde geldt. Het feit dat de rechten maximale cliques zijn in de zin van Stelling 1.2.7, impliceert het volgende resultaat (zie Stelling 3.2.3).

Stelling A.3.1 *De hierboven geconstrueerde incidentiestructuur is een partiële meetkunde* $\mathrm{pg}\,(q-1, (q^2-1)/2, (q-1)/2)$ *met het Hermitische graaf* $\mathcal{H}'(q)$ *als puntgraaf.*

Merk op dat voor elke $s \in \Gamma_2(p)$ een keuze gemaakt dient te worden tussen twee verzamelingen rechten. Vermits al deze keuzes onderling onafhankelijk zijn, levert de constructie $2^{q(q+1)}$ partiële meetkunden, die we de *Hermitische partiële meetkunden* noemen. Voor $q = 3$ vinden we niets anders dan de unieke veralgemeende vierhoek $Q^-(5, 2)$ van de orde $(2, 4)$. Voor $q = 3^{2h}$,

$h \in \mathbb{N} \setminus \{0\}$, zijn deze Hermitische partiële meetkunden isomorf met de partiële meetkunden die door Mathon op een algebraïsche manier beschreven werden in [70] (zie Stelling 3.3.1). Om dit te bewijzen, zoeken we eerst een geschikte bijectie tussen de puntenverzamelingen; na heel wat rekenwerk stellen we vast dat deze ook de rechten op elkaar afbeeldt. Voor $q = 3^{2h+1}$, $h \in \mathbb{N} \setminus \{0\}$, waren tot nog toe geen partiële meetkunden met dergelijke parameters gekend; bijgevolg zijn de Hermitische partiële meetkunden in dit geval nieuw.

A.3.2 Segmenten

Zij $\mathcal{O}$ een ovoïde van $Q(6,q)$, q oneven, en duid met π de orthogonale polariteit aan die geassocieerd is met $Q(6,q)$. Als x een punt van $Q(6,q)$ is dat niet tot $\mathcal{O}$ behoort, dan liggen precies q^2+1 punten van $\mathcal{O}$ in x^π. Na projectie vanuit x op een vierdimensionale deelruimte Π van x^π die x niet bevat, vinden we q^2+1 punten die een ovoïde $\mathcal{O}'$ van de kwadriek $Q(4,q) := \Pi \cap Q(6,q)$ vormen. We noemen $\mathcal{O}'$ een *segment* van $\mathcal{O}$. Op de geassocieerde twee-grafen (zie Stelling 1.9.5) heeft de projectie het verwachte effect: het twee-graaf op $\mathcal{O}'$ is isomorf met het deel-twee-graaf geïnduceerd op $x^\pi \cap \mathcal{O}$ van het twee-graaf op $\mathcal{O}$.

Voor elke oneven priemmacht q heeft de *Kantor–ovoïde* $\mathcal{O}_K(\sigma)$ van de kwadriek $Q(4,q)$ met vergelijking $X_0X_3 + X_1X_4 - X_2^2 = 0$ de volgende standaardgedaante:

$$\mathcal{O}_K(\sigma) = \{(1,0,0,0,0)\} \cup \{(-\nu x^{\sigma+1} + y^2, x, y, 1, \nu x^\sigma) \mid x, y \in \mathrm{GF}(q)\},$$

waarbij ν een niet-kwadraat in $\mathrm{GF}(q)$ is en $\sigma \neq \mathbb{1}$ een veldautomorfisme van $\mathrm{GF}(q)$. Elk segment van een Hermitische ovoïde $\mathcal{O}_H$ van $H(q)$, $q = 3^h$, $h \in \mathbb{N} \setminus \{0,1\}$, gezien als ovoïde van $Q(6,q)$, is een Kantor–ovoïde $\mathcal{O}_K(\sigma)$ met $\sigma : x \mapsto x^3$, terwijl de segmenten van de Hermitische ovoïde van $H(3)$, gezien als ovoïde van $Q(6,3)$, elliptische kwadrieken $Q^-(3,3)$ zijn.

Stel dat $\mathcal{S}$ een Hermitische partiële meetkunde op $\mathcal{O}_H \setminus \{p\}$ is, waarbij $\mathcal{O}_H$ opnieuw een Hermitische ovoïde van $H(q)$ voorstelt, $q = 3^h$, $h \in \mathbb{N} \setminus \{0,1\}$. Beschouw het segment $\mathcal{O}'$ van $\mathcal{O}_H$ (gezien als ovoïde van $Q(6,q)$) dat men verkrijgt door met s^π te snijden, waarbij $s \in \Gamma_2(p)$. Dan vormen de punten van $\mathcal{O}' \setminus \{p\}$ samen met de projecties van de rechten van $\mathcal{S}$ die in s^π liggen een net $\mathrm{pg}\,(q-1, (q-1)/2, (q-1)/2)$ met als puntgraaf het afgeleide graaf ten opzichte van $(1,0,0,0,0)$ van het twee-graaf geassocieerd met de Kantor–ovoïde $\mathcal{O}_K(\sigma)$ met $\sigma : x \mapsto x^3$. Door de expliciete beschrijving van dit net lichtjes aan te passen, vinden we bovendien een net met als puntgraaf het afgeleide graaf ten opzichte van $(1,0,0,0,0)$ van het twee-graaf geassocieerd met een *willekeurige* Kantor–ovoïde van $Q(4,3^h)$, $h \in \mathbb{N} \setminus \{0,1\}$. Dit leidt tot Stelling 3.4.1. Men kan deze constructie ook in het geval $q = 3$ uitvoeren; ze levert dan het net $Q^+(3,2)$ als deelstructuur van de veralgemeende vierhoek $Q^-(5,2)$.

A.3.3 Spreads

Een *spread* van een partiële meetkunde $\mathrm{pg}\,(s,t,\alpha)$ is een verzameling van $st/\alpha + 1$ twee aan twee niet-concurrente rechten. Alle Hermitische partiële meetkunden bevatten spreads; een voor de hand liggend voorbeeld is de verzameling van alle rechten van type 2. Verdere spreads kunnen als volgt geconstrueerd worden. Kies een rechte van $H(q)$ door p en noem ze L. Neem voor elk punt $s \neq p$ van L een generator $N(s)$ uit de in Stap 1 van de constructie gekozen verzameling. Definieer $\mathcal{S}$ als de verzameling van alle rechten van type 1 van de vorm $N'(s)^\pi \cap (\mathcal{O}_H \setminus \{p\})$, waarbij $N'(s)$ een rechte verschillend van L in het vlak $\langle p, N(s)\rangle$ is en $s \neq p$ een punt van L is. Dan kan men aantonen dat $\mathcal{S}$ een spread is. Het variëren van L en van de generatoren $N(s)$ levert duidelijk een zeer groot aantal spreads op. Men kan bovendien deze constructie verscheidene keren met onderling disjuncte verzamelingen generatoren toepassen, zodat men een partitie van de verzameling van rechten van type 1 in spreads bekomt.

In [70] is voor de partiële meetkunde $\mathcal{M}_3(h)$, $h \in \mathbb{N} \setminus \{0\}$, sprake van een partitie van de verzameling van rechten van type 1 in spreads die invariant is onder de volledige automorfismengroep. Meetkundig vindt men deze spreads terug door als generatoren $N(s)$ de rechten van een normaal rationaal kubisch regeloppervlak te kiezen.

A.3.4 Blokgrafen

De blokgrafen van de Hermitische partiële meetkunden zijn

$$\mathrm{srg}\left(\frac{q^2(q^2+1)}{2}, \frac{q(q^2-1)}{2}, q(q-2), \frac{q(q-1)}{2}\right)$$

die volgens onze informatie nieuw zijn, behalve in het kleinste geval $q = 3$.

A.3.5 Hermitische twee-graaf-meetkunden?

De vraag naar het bestaan van een twee-graaf-meetkunde op het Hermitische twee-graaf $\mathcal{H}(q)$ kan voorlopig alleen beantwoord worden in de drie kleinste gevallen: $\mathcal{H}(3)$ is meetkundig [44], $\mathcal{H}(5)$ en $\mathcal{H}(7)$ niet [87]. Aangezien door het bestaan van de Hermitische partiële meetkunden een eerste nodige voorwaarde vervuld is, zou men naar twee-graaf-meetkunden op $\mathcal{H}(3^h)$, $h \in \mathbb{N} \setminus \{0,1\}$, kunnen zoeken. Een naïeve poging wijst echter in de andere richting: het blijkt niet mogelijk om twee Hermitische partiële meetkunden te combineren als afgeleide structuren ten opzichte van verschillende punten van een twee-graaf-meetkunde, althans niet in het hierboven beschreven model. Als er dus een Hermitische twee-graaf-meetkunde bestaat waarvan minstens twee afgeleide structuren Hermitische partiële meetkunden zijn, dan heeft ze vermoedelijk geen elegante beschrijving op de Hermitische ovoïde van $H(q)$. Een andere mogelijkheid is dat er wel een Hermitische twee-graaf-meetkunde bestaat, maar dat hoogstens één van haar afgeleide structuren een Hermitische partiële meetkunde is. Dit zou het bestaan van op dit moment nog niet gekende partiële meetkunden met een Hermitisch puntgraaf met zich meebrengen, waarvan de beschrijving naar alle waarschijnlijkheid niet eenvoudiger zou zijn dan die van de Hermitische partiële meetkunden. Ook in dit geval lijkt er dus weinig hoop op een elegante constructie van een Hermitische twee-graaf-meetkunde. We denken dan ook dat $\mathcal{H}(3^h)$, $h \in \mathbb{N} \setminus \{0,1\}$, geen twee-graaf-meetkunde ondersteunt.

A.4 Afstandsreguliere meetkunden

A.4.1 Afstandsreguliere grafen

Een verbonden graaf Γ met diameter $d \geq 2$ heet *afstandsregulier* als er getallen b_i, $i \in \{0,\ldots,d-1\}$ en c_i, $i \in \{1,\ldots,d\}$ bestaan zodanig dat aan de volgende voorwaarden voldaan is.

drg1 Γ is regulier met valentie b_0;

drg2 voor elke twee toppen x en y van Γ op afstand $i \in \{1,\ldots,d-1\}$ liggen c_i (respectievelijk b_i) van de toppen adjacent met y op afstand $i-1$ (respectievelijk $i+1$) van x;

drg3 voor elke twee toppen x en y van Γ op afstand d liggen c_d van de toppen adjacent met y op afstand $d-1$ van x.

Het is duidelijk dat $c_1 = 1$. De rij $\{b_0,\ldots,b_{d-1}; c_1,\ldots,c_d\}$ heet de *intersectierij*, en de getallen b_i en c_i noemen we de *intersectiegetallen*. Een sterk regulier graaf srg (v,k,λ,μ) is afstandsregulier met intersectierij $\{k, k-1-\lambda; 1, \mu\}$ en vice versa. Afstandsreguliere grafen worden uitgebreid behandeld in [9].

A.4.2 Afstandsreguliere meetkunden

Men kan zich afvragen of er partieel lineaire ruimten bestaan die een afstandsregulier puntgraaf hebben, net zoals sommige sterk reguliere grafen het puntgraaf van een (semi-)partiële meetkunde zijn. Om de axioma's voor een dergelijke incidentiestructuur op te stellen, hebben we de begrippen *punt-diameter*, *rechte-diameter* en *punt-rechte-diameter* nodig. Dit zijn de maximale afstanden in het incidentiegraaf van een partieel lineaire ruimte tussen twee toppen die respectievelijk twee punten, twee rechten, en een punt en een rechte voorstellen. Punt- en rechte-diameters zijn even, terwijl de punt-rechte-diameter altijd oneven is. Het verschil tussen de punt-rechte-diameter en de punt- of rechte-diameter is 1, terwijl punt- en rechte-diameter ofwel gelijk zijn, ofwel 2 verschillen.

Zij $\mathcal{S}$ een incidentiestructuur met incidentiegraaf Φ die aan volgende voorwaarden voldoet.

DRG1 $\mathcal{S}$ is een partieel lineaire ruimte van de orde (s, t).

DRG2 De punt-diameter van Φ is $2d \geq 4$.

DRG3 Er bestaan getallen α_{2i-1}, $1 \leq i \leq d$, zodanig dat voor elk punt p en voor elke rechte L op afstand $2i-1$ van p in Φ precies α_{2i-1} van de punten incident met L op afstand $2i-2$ van p liggen.

DRG4 Er bestaan getallen t_{2i}, $1 \leq i \leq d$, zodanig dat voor elke twee punten p en q op afstand $2i$ in Φ precies $t_{2i}+1$ van de rechten incident met q op afstand $2i-1$ van p liggen.

Het is niet moeilijk aan te tonen dat het puntgraaf van $\mathcal{S}$ afstandsregulier is met diameter d en intersectiegetallen

$$\begin{aligned} b_0 &= (t+1)s, \\ b_i &= (t-t_{2i})(s+1-\alpha_{2i+1}),\ 1 \leq i \leq d-1, \\ c_i &= (t_{2i}+1)\alpha_{2i-1},\ 1 \leq i \leq d. \end{aligned}$$

Een afstandsreguliere meetkunde met $d = 2$ en $t_4 = t$ is een partiële meetkunde pg (s, t, α_3); als $d = 2$ en $t_4 < t$, dan hebben we een semi-partiële meetkunde spg $(s, t, \alpha_3, (t_4+1)\alpha_3)$. Afstandsreguliere meetkunden waarin $\alpha_{2i-1} = 1$ voor elke $i \in \{1, \ldots, d\}$ zijn *reguliere schierveelhoeken* (zie [86]). Een *echte* afstandsreguliere meetkunde is noch een (semi-)partiële meetkunde, noch een reguliere schierveelhoek.

In het algemeen volgen de parameters van een mogelijke afstandsreguliere meetkunde niet eenduidig uit de parameters van het afstandsreguliere graaf. Men kan wel beperkingen op de parameters afleiden. Een telargument leert dat de parameters van een afstandsreguliere meetkunde voldoen aan $\alpha_{2i-3} \leq \alpha_{2i-1} \leq s$ en $t_{2i-2} \leq t_{2i} \leq t$ voor elke $i \in \{2, \ldots, d\}$. Als de punt-rechte-diameter van het incidentiegraaf $2d-1$ is, dan is $t_{2d} = t$, zoniet is $t_{2d} < t$. Andere parameterbeperkingen volgen uit de beperkingen op de intersectiegetallen van afstandsreguliere grafen, zie [9].

A.4.3 Twee sporadische voorbeelden

Het lijngraaf $L(\mathrm{Pe})$ van het Petersen–graaf Pe is afstandsregulier met diameter 3 en intersectierij $\{4, 2, 1; 1, 1, 4\}$. Elke 3-clique in $L(\mathrm{Pe})$ komt overeen met een top van Pe. We bewijzen dat deze cliques kunnen dienen als rechten in een afstandsreguliere meetkunde op $L(\mathrm{Pe})$ met parameters $s = 2$, $t = 1$, $\alpha_3 = 1$, $t_4 = 0$, $\alpha_5 = 2$ en $t_6 = 1$. Een analoge constructie kan toegepast worden op het lijngraaf $L(\mathrm{HoSi})$ van het Hoffman–Singleton–graaf HoSi, dat afstandsregulier is met diameter 3 en intersectierij $\{12, 6, 5; 1, 1, 4\}$. Hier heeft de afstandsreguliere meetkunde parameters $s = 6$, $t = 1$, $\alpha_3 = 1$, $t_4 = 0$, $\alpha_5 = 2$ en $t_6 = 1$. Deze voorbeelden zijn noch (semi-)partiële meetkunden (omdat hun puntgrafen diameter 3 hebben), noch reguliere schierveelhoeken (vermits $\alpha_5 = 2$).

Door gebruik te maken van een stelling in [9] omtrent grafen met een afstandsregulier lijngraaf, tonen we aan dat de bovenstaande voorbeelden de enige echte afstandsreguliere meetkunden zijn die via een dergelijke constructie verkregen kunnen worden, tenzij er een srg $(3250, 57, 0, 1)$ zou bestaan.

A.4.4 Onderzoek van enkele oneindige klassen van afstandsreguliere grafen

Kies natuurlijke getallen n en e waarvoor $n \geq 4$ en $2 \leq e \leq n-2$, en stel dat X een verzameling met n elementen is. De toppen van het *Johnson–graaf* $J(n, e)$ (zie [9, Onderdeel 9.1]) zijn de deelverzamelingen met e elementen van X, en twee toppen zijn adjacent als en slechts als de overeenkomstige deelverzamelingen $e - 1$ elementen gemeen hebben. Men bewijst dat $J(n, e)$ een afstandsregulier graaf met diameter $d := \min\{e, n - e\}$ is. In $J(n, e)$ blijken maar twee soorten maximale cliques te bestaan, die beide (maar niet tegelijk) dienst kunnen doen als rechten van een partieel lineaire ruimte. Dit motiveert de volgende definitie. De punten (respectievelijk rechten) van de *Johnson–meetkunde* $\mathcal{J}(n, e)$ zijn de deelverzamelingen met e (respectievelijk $e - 1$) elementen van X; de incidentie is natuurlijk. Men kan nu aantonen dat $\mathcal{J}(n, e)$ een afstandsreguliere meetkunde is met parameters $s = n - e$,

$t = e - 1$ en $\alpha_{2i-1} = t_{2i} + 1 = i$ voor $i \in \{1, \ldots, d\}$. Door complementen te nemen in X, ziet men in dat de duale van $\mathcal{J}(n, e)$ opnieuw een Johnson–meetkunde is, namelijk $\mathcal{J}(n, n - e + 1)$.

Veronderstel opnieuw dat n en e natuurlijke getallen zijn met $n \geq 4$ en $2 \leq e \leq n - 2$, en beschouw de n-dimensionale vectorruimte $V(n, q)$ over $\mathrm{GF}(q)$, q een priemmacht. De toppen van het *Grassmann–graaf* $G(n, e, q)$ (zie [9, Onderdeel 9.3]) zijn de e-dimensionale deelruimten van $V(n, q)$, en twee toppen zijn adjacent als en slechts als de overeenkomstige deelruimten elkaar in een $(e - 1)$-dimensionale deelruimte snijden. Het graaf $G(n, e, q)$ is afstandsregulier met diameter $d := \min\{e, n-e\}$. Ook hier zijn er slechts twee soorten maximale cliques, die beide (maar niet tegelijk) als rechten van een partieel lineaire ruimte gebruikt kunnen worden. De *Grassmann–meetkunden* $\mathcal{G}(n, e, q)$ lijken erg op de Johnson–meetkunden: de punten (respectievelijk rechten) zijn de e-dimensionale (respectievelijk $(e - 1)$-dimensionale) deelruimten van $V(n, q)$, en de incidentie is natuurlijk. Men bewijst dat $\mathcal{G}(n, e, q)$ een afstandsreguliere meetkunde is met parameters

$$
\begin{aligned}
s &= \begin{bmatrix} n - e + 1 \\ 1 \end{bmatrix}_q - 1, \\
t &= \begin{bmatrix} e \\ 1 \end{bmatrix}_q - 1, \\
\alpha_{2i-1} &= t_{2i} + 1 = \begin{bmatrix} i \\ 1 \end{bmatrix}_q \quad (1 \leq i \leq d),
\end{aligned}
$$

waarbij

$$
\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{\prod_{i=0}^{m-1}(q^{n-i} - 1)}{\prod_{i=1}^{m}(q^i - 1)}.
$$

De duale van $\mathcal{G}(n, e, q)$ is $\mathcal{G}(n, n - e + 1, q)$, wat men kan inzien door orthogonale complementen te nemen in $V(n, q)$.

In de Hamming–grafen [9, Onderdeel 9.2] en in de duale polaire grafen [9, Onderdeel 9.4] vindt men op een voor de hand liggende manier maximale cliques die kunnen dienen als rechten voor een afstandsreguliere meetkunde. Deze meetkunden blijken altijd reguliere schierveelhoeken te zijn.

We bewijzen dat de halve kubussen [9, Onderdeel 9.2] en de halve duale polaire grafen [9, Onderdeel 9.4] geen afstandsreguliere meetkunde ondersteunen door de maximale cliques en hun interacties te bestuderen.

A.5 Twee-grafen, grafen en hun codes

A.5.1 Codes van grafen

Een *lineaire binaire code* C van *lengte* n en *dimensie* k is een k-dimensionale deelruimte van de n-dimensionale vectorruimte $V(n,2)$ over GF(2). Daar we in dit proefschrift enkel met dit soort codes te maken zullen krijgen, laten we in het vervolg de bijvoeglijke naamwoorden achterwege en spreken we kortweg van codes. De *duale code* $C^\perp$ van C is de deelruimte bestaande uit alle vectoren van $V(n,2)$ die loodrecht staan op alle elementen van C. Het *gewicht* $w(x)$ van een codewoord x is het aantal coördinaatposities waar x een 1 heeft. Het *minimumgewicht* van een code C is het kleinste positieve getal w zodanig dat C een codewoord van gewicht w bevat. De *gewichtsverdeling* van C is een tabel waarin weergegeven wordt hoeveel codewoorden van elk gewicht C bezit. Het *complement* van een codewoord x is de vector $x + \underline{1}$.

De code C_A van een $(m \times n)$-matrix A met gehele elementen is de deelruimte van $V(n,2)$ opgespannen door de rijen van A. Het ligt dan ook voor de hand om de code van een graaf te definiëren als de code van zijn adjacentiematrix. Als het graaf sterk regulier is, dan geven de eigenwaarden reeds heel wat informatie over de code (zie Stellingen 5.2.3 en 5.2.4).

A.5.2 Codes van twee-grafen

Beschouw een twee-graaf (V,Δ) en kies een top ω. Stel dat Γ_ω het graaf in (V,Δ) voorstelt waarin ω een geïsoleerde top is, en dat A_ω de adjacentiematrix van Γ_ω is. Dan wordt de code C van (V,Δ) gedefinieerd als $C := \langle C_{A_\omega}, \underline{1}\rangle$. Kiest men een andere top ω', dan vindt men toch dezelfde code.

A.5.3 Reguliere grafen in reguliere twee-grafen

Deze uitdrukking verwijst naar reguliere grafen die tot een switching-klasse behoren die met een regulier twee-graaf overeenkomt. Veronderstel dat de afgeleide grafen van een regulier twee-graaf (V,Δ) beperkte eigenwaarden r en l $(r > l)$ hebben, met respectievelijke multipliciteit f en g. Dan kan men aantonen dat de beperkte eigenwaarden van een regulier graaf in (V,Δ) eveneens r en l zijn. Voor de valentie van een dergelijk regulier graaf zijn er twee mogelijkheden: ze is hetzij $k_f := -(2l+1)r$, hetzij $k_g := -(2r+1)l$. In het eerste geval heeft r multipliciteit $f+1$ en l multipliciteit g, terwijl in het tweede geval r multipliciteit f en l multipliciteit $g+1$ heeft. Stelling 5.5.2 drukt uit dat het bestaan van een regulier graaf in een regulier twee-graaf equivalent is met het bestaan van een bepaalde reguliere partitie van de toppenverzameling van het afgeleide graaf.

Stelling 5.5.3 verduidelijkt de verbanden tussen de codes C en C_{A_ω} en de code van een regulier graaf in (V, Δ). Door ze te combineren met Stelling 5.2.4 en in het bijzonder op de dimensies van de codes te letten, komen we tot het volgende resultaat, dat in de Engelse tekst als Stelling 5.5.4 te vinden is.

Stelling A.5.1 *Zij (V, Δ) een regulier twee-graaf. Veronderstel dat de beperkte eigenwaarden r en l $(r > l)$ van het afgeleide graaf Γ van (V, Δ) ten opzichte van een top ω beide even zijn. Noteer de adjacentiematrix van Γ met A en de multipliciteiten van de eigenwaarden met f, respectievelijk g. Als $r \equiv 2 \pmod 4$ en $\dim C_A \in \{f+1, g, g+1\}$, dan bevat (V, Δ) geen regulier graaf met valentie $k_f = -(2l+1)r$. Als $l \equiv 2 \pmod 4$ en $\dim C_A \in \{f, f+1, g+1\}$, dan bevat (V, Δ) geen regulier graaf met valentie $k_g = -(2r+1)l$.*

A.5.4 (Niet-)bestaan van reguliere grafen in de tweevoudig transitieve reguliere twee-grafen

In het Paley–twee-graaf $\mathcal{P}(q^2)$, q een oneven priemmacht, kan men reguliere grafen met beide mogelijke valenties vinden. Dit werd reeds bewezen in [82]; in Paragraaf 5.6 geven we een constructie.

Op een gelijkaardige manier construeren we in $\mathcal{H}(q)$, q een oneven priemmacht, en in $\mathcal{R}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N} \setminus \{0\}$, reguliere grafen met valentie $k_g = q(q^2+1)/2$. Op het complement van $\mathcal{H}(q)$, $q \equiv 3 \pmod 8$, en op het complement van $\mathcal{R}(q)$ is Stelling A.5.1 van toepassing, wat het volgende impliceert.

Stelling A.5.2 *Als q een oneven priemmacht is waarvoor $q \equiv 3 \pmod 8$, dan bevat het Hermitische twee-graaf $\mathcal{H}(q)$ geen regulier graaf met valentie $k_f = q^2(q-1)/2$.*

Stelling A.5.3 *Het Ree–twee-graaf $\mathcal{R}(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N} \setminus \{0\}$, bevat geen regulier graaf met valentie $k_f = q^2(q-1)/2$.*

Deze resultaten komen overeen met Stellingen 5.6.2 en 5.6.4. Behalve in het geval $q = 5$, dat we in Paragraaf A.6.1 behandelen, weten we niet of in $\mathcal{H}(q)$, met $q \equiv 1 \pmod 4$ of $q \equiv 7 \pmod 8$, een regulier graaf met valentie $k_f = q^2(q-1)/2$ voorkomt.

Tot het symplectische twee-graaf blijken reguliere grafen met beide mogelijke valenties te behoren (zie [80]). Het hyperbolische orthogonale twee-graaf $\Omega^+(2m, 2)$, $m \geq 2$, bevat een regulier graaf met valentie $k_f = 2^{2m-2} - 1$, en het elliptische orthogonale twee-graaf $\Omega^-(2m, 2)$, $m \geq 3$, bevat een regulier graaf met valentie $k_g = 2^{2m-2} - 1$. Dit wordt eveneens in [80] aangetoond. Over het bestaan van reguliere grafen met de andere mogelijke valentie is ons niets bekend.

Tenslotte bestaat er in het sporadische reguliere twee-graaf op 276 toppen geen srg$\,(276, 110, 28, 54)$ (dit was al eerder bekend, maar volgt tevens uit Stelling A.5.1), maar wel een srg$\,(276, 140, 58, 84)$. In het sporadische reguliere twee-graaf op 176 toppen vindt men zowel een srg$\,(176, 70, 18, 34)$ als een srg$\,(176, 90, 38, 54)$. Een samenvatting van al deze resultaten staat in Tabel 5.1.

A.6 Grafen en codes geassocieerd met $\mathcal{H}(5)$

A.6.1 Het srg$\,(126, 50, 13, 24)$ in $\mathcal{H}(5)$

De mogelijke valenties voor een regulier graaf in het Hermitische twee-graaf $\mathcal{H}(q)$ zijn $k_f = q^2(q-1)/2$ en $k_g = q(q^2+1)/2$. Zoals reeds vermeld, bestaan er altijd reguliere grafen met valentie k_g in $\mathcal{H}(q)$; $q = 5$ is echter het enige geval waarvoor we weten dat $\mathcal{H}(q)$ een regulier graaf met valentie k_f bevat. Dit srg$\,(126, 50, 13, 24)$ kan op twee manieren beschreven worden.

Stel dat $H(X, Y) = X_0Y_0^5 + X_1Y_2^5 + X_2Y_1^5$ de Hermitische vorm van $\mathrm{PG}(2, 25)$ is die $\mathcal{H}(5)$ definieert. We kunnen elk element van GF(25) schrijven als $a + bi$ met $a, b \in \mathrm{GF}(5)$, waarbij i een element van GF(25) is zodanig dat $i^2 = 2$. De Hermitische kromme die uit H voortkomt, wordt dan

$$\mathcal{U} = \{(0, 1, 0)\} \cup \{(a + bi, (2a^2 + b^2) + ci, 1) \mid a, b, c \in \mathrm{GF}(5)\}.$$

Zij Γ_ω het graaf in $\mathcal{H}(5)$ waarin $(0, 1, 0)$ een geïsoleerde top is. Spence genereerde met de computer een lijst met alle mogelijke switchings van Γ_ω die een srg$\,(126, 50, 13, 24)$ opleveren. Eén van deze 150 switchings is die ten opzichte van de partitie bepaald door de verzameling $S = S_1 \cup S_2$, met

$$\begin{aligned} S_1 &= \{(a, 2a^2 \pm i, 1) \mid a \in \mathrm{GF}(5)\}, \\ S_2 &= \{(a + bi, (2a^2 + b^2) + (\pm 2 - ab)i, 1) \mid a, b \in \mathrm{GF}(5), b \neq 0\}. \end{aligned}$$

In Lemma 6.1.1 bewijzen we dat de stabilisator van S in de automorfismengroep van Γ_ω orde 40 heeft en transitief werkt op S_1 en S_2. Dankzij dit laatste feit is het niet moeilijk om met de hand aan te tonen dat de partitie bepaald door S inderdaad een srg$\,(126, 50, 13, 24)$ in $\mathcal{H}(5)$ produceert. Dat doen we in Stelling 6.1.2. De automorfismengroep van Γ_ω is de stabilisator van een top in de automorfismengroep $\mathrm{P\Gamma U}_3(25)$ van $\mathcal{H}(5)$. Men gaat na dat hij orde 6000 heeft, waaruit volgt dat S in een baan van lengte 150 ligt. Dit betekent dat de 150 partities die ons door Spence bezorgd werden equivalent zijn. Vandaar het volgende resultaat (zie Stelling 6.1.3).

Stelling A.6.1 *Op een isomorfisme na bevat $\mathcal{H}(5)$ juist één sterk regulier graaf* srg (126, 50, 13, 24)*.*

Dit graaf kan eveneens geconstrueerd worden via het Hoffman–Singleton–graaf HoSi. De volgende beschrijving is van de hand van Goethals. Definieer eerst een graaf Φ' als volgt: de toppen van Φ' zijn de bogen van HoSi, en twee toppen zijn adjacent als de overeenkomstige bogen van HoSi disjuncte zijden in een vijfhoek zijn. Men toont aan dat Φ' een srg $(175, 72, 20, 36)$ is. Het is bovendien het afgeleide graaf van het sporadische reguliere twee-graaf Φ op 176 toppen. Kies nu een top u van HoSi en duid met F de verzameling van 42 toppen van HoSi aan die niet adjacent zijn met u. In F liggen precies 126 bogen van HoSi; het deelgraaf Γ_f van Φ' geïnduceerd op de verzameling van 126 toppen die met deze bogen overeenkomen, is een srg $(126, 50, 13, 24)$ dat tot $\mathcal{H}(5)$ behoort. Dit "sporadische" gedrag van $\mathcal{H}(5)$ verklaart misschien waarom onze pogingen om de verzameling S te veralgemenen naar grotere waarden van q, met $q \equiv 1 \pmod 4$, geen succes kenden.

A.6.2 De code van $\mathcal{H}(5)$

Beide beschrijvingen uit Paragraaf A.6.1 dragen bij tot onze kennis omtrent de code van $\mathcal{H}(5)$. Duid de adjacentiematrix van $\mathcal{H}'(5)$ (respectievelijk van het graaf Γ_ω bestaande uit $\mathcal{H}'(5)$ en een geïsoleerde top ω) aan met A (respectievelijk A_ω). De codes C_A en C_{A_ω} hebben duidelijk dezelfde dimensie; die blijkt 20 te zijn (zie [12]). Uit Stelling 5.5.3 volgt dat de code C van $\mathcal{H}(5)$ dimensie 21 heeft. Aangezien $C = \langle C_{A_\omega}, \underline{1} \rangle$ en $\underline{1} \notin C_{A_\omega}$, is het aantal woorden van gewicht w in C gelijk aan het aantal woorden van gewicht w of $126 - w$ in C_{A_ω} (of in C_A). De gewichtsverdelingen van C_A en C vindt men in Tabel 6.1. De duale code $C^\perp$ van C is de code C_{N^T}, waarbij N de incidentiematrix van het Hermitische unitaal in PG(2, 25) is (zie [12]). In Tabel 6.2 staat de gewichtsverdeling van $C^\perp$.

Beschouw opnieuw het graaf Φ' dat in Paragraaf A.6.1 uit het Hoffman–Singleton–graaf opgebouwd werd. In Paragraaf 6.2.1 beschrijven we een partitie van de toppenverzameling van het graaf bestaande uit Φ' en een geïsoleerde top zodanig dat na switching een regulier graaf Φ_f met valentie 70 in Φ ontstaat. De code $C(\Phi_f)$ van Φ_f is 22-dimensionaal en bevat wegens Stelling 5.5.3 de 21-dimensionale code $C(\Phi)$ van Φ. Een blik op de gewichtsverdelingen (zie Tabel 6.3) leert dat $C(\Phi)$ bestaat uit alle woorden van $C(\Phi_f)$ waarvan het gewicht deelbaar is door 4. Vervolgens construeren we in Lemma 6.2.1 een codewoord ξ van gewicht 126 in $C(\Phi_f)$ dat de karakteristieke vector van de toppenverzameling van $\mathcal{H}(5)$ is. We tonen aan dat men de code C van $\mathcal{H}(5)$ verkrijgt door van alle woorden van $C(\Phi)$ alleen de coördinaatposities over te houden waar in ξ een 1 staat.

Met behulp van de hierboven gegeven beschrijvingen kunnen we de woorden van bepaalde gewichten in C volledig verklaren. Elk woord van gewicht 52 komt overeen met ofwel een niet-nulrij van A_ω, ofwel de som van twee rijen van A_ω die niet-adjacente toppen van $\mathcal{H}'(5)$ voorstellen, ofwel het complement van de som van twee rijen van A_ω die adjacente toppen van $\mathcal{H}'(5)$ voorstellen. De woorden van gewicht 74 zijn de complementen van die van gewicht 52.

Stelling 5.5.3 impliceert dat de karakteristieke vector van de in Paragraaf A.6.1 gedefinieerde verzameling S tot C_{A_ω} behoort, en daardoor ook tot C. We weten dat S in een baan van lengte 150 ligt onder de automorfismengroep van Γ_ω. Dit geldt voor alle 126 grafen met een geïsoleerde top in $\mathcal{H}(5)$; zo bekomt men $126 \cdot 150 = 18900$ woorden van gewicht 50 in C. We kunnen aantonen dat deze twee aan twee verschillend zijn en hebben dus inderdaad alle woorden van gewicht 50 in C teruggevonden. De woorden van gewicht 126 verkrijgt men door de vector $\underline{1}$ op te tellen bij de woorden van gewicht 50.

Met behulp van het feit dat de 176 woorden van gewicht 50 in $C(\Phi_f)$ een symmetrisch 2-$(176, 50, 14)$-design vormen [52], bewijzen we dat 36 het minimumgewicht van C is (zie Stelling 6.2.3). Door gebruik te maken van de eigenschappen van het Hoffman–Singleton–graaf construeren we een partitie van de toppenverzameling van het reguliere graaf Γ_f met valentie 50 in $\mathcal{H}(5)$ zodanig dat de rijen van de adjacentiematrix A_f van Γ_f die met één element van de partitie overeenstemmen als som een woord van gewicht 36 in C_{A_f} hebben. Stelling 5.5.3 verzekert ons dat dit woord ook tot C behoort. De software GAP [40], aangevuld met het pakket Projective Geometries [28], levert het bewijs dat de actie van $\mathrm{P\Gamma U}_3(25)$ op dit woord alle woorden van gewicht 36 in C produceert. De woorden van gewicht 90 zijn de complementen van die van gewicht 36.

Op gelijkaardige wijze vinden we een woord van gewicht 84; opnieuw gebruiken we GAP [40] aangevuld met Projective Geometries [28] om aan te tonen dat de inwerking van $\mathrm{P\Gamma U}_3(25)$ alle woorden van gewicht 84 in C oplevert. De woorden van gewicht 42 volgen door $\underline{1}$ op te tellen bij die van gewicht 84.

Tenslotte bewijzen we dat alle woorden van gewicht 6 in de duale code $C^\perp$ van C geconstrueerd kunnen worden aan de hand van de coherente 6-verzamelingen in $\mathcal{H}(5)$.

A.7 Een model voor $H(q)$, q oneven

A.7.1 De ruimtelijke kubische kromme in $\mathrm{PG}(3,q)$

Een *ruimtelijke kubische kromme* is een verzameling Σ van $q+1$ punten in $\mathrm{PG}(3,q)$, $q>2$, die na de keuze van een geschikte basis als volgt geschreven kan worden:

$$\Sigma = \{(0,0,0,1)\} \cup \{(1,u,u^2,u^3) \mid u \in \mathrm{GF}(q)\}.$$

Voor elk punt $p \in \Sigma$ bestaat de projectie van $\Sigma \setminus \{p\}$ vanuit p op een vlak niet door p uit q punten die op een kegelsnede liggen. Voor $q \geq 5$ is deze kegelsnede uniek bepaald; noem het ontbrekende punt p'. Dan is de rechte $\langle p, p' \rangle$ de *raaklijn* aan Σ in p, en het vlak opgespannen door p en de raaklijn aan de kegelsnede in p' heet het *osculatievlak* aan Σ in p. Voor $q \in \{3,4\}$ worden raaklijnen en osculatievlakken op een algebraïsche manier gedefinieerd.

Een ruimtelijke kubische kromme Σ in $\mathrm{PG}(3,q)$, $q>2$, bepaalt eenduidig een alternerende vorm f van de vierdimensionale vectorruimte $V(4,q)$ die aan $\mathrm{PG}(3,q)$ ten grondslag ligt, zodanig dat voor elk punt p van Σ het hypervlak $p^\perp := \{x \in V(4,q) \mid f(p,x) = 0\}$ precies met het osculatievlak aan Σ in p overeenkomt. Als q niet deelbaar is door 3, dan is f niet-ontaard en kunnen we in feite spreken van een symplectische polariteit van $\mathrm{PG}(3,q)$. Als q deelbaar is door 3, dan is f ontaard; het radicaal is een projectieve rechte R die de doorsnede van alle osculatievlakken aan Σ vormt. In het bijzonder bevatten alle raaklijnen aan Σ een punt van R.

A.7.2 Het model van Bader en Lunardon

In [2] beschrijven Bader en Lunardon een model voor de duale van $H(q)$, q oneven en niet deelbaar door 3, dat vertrekt van een ruimtelijke kubische kromme. Beschouw een projectieve ruimte $\mathrm{PG}(5,q)$, q oneven en niet deelbaar door 3, voorzien van een symplectische polariteit $\perp$. Kies een punt ∞ en een driedimensionale deelruimte S zodanig dat $\infty \notin S \subseteq \infty^\perp$. Stel dat Σ een ruimtelijke kubische kromme is in S zodanig dat de beperking van $\perp$ tot S de punten en osculatievlakken van Σ verwisselt. Noteer verder voor elk punt $u \in \Sigma$ de raaklijn aan Σ in u met L_u. Construeer nu een incidentiestructuur $\mathcal{S}$ met de volgende punten.

- ∞;
- punten op afstand 2 van ∞: de punten verschillend van ∞ op de rechten $\langle \infty, u \rangle$, $u \in \Sigma$;

- punten op afstand 4 van ∞: de totaal isotrope vlakken die $\infty^\perp$ snijden in een rechte die in een vlak $\langle\infty, L_u\rangle$ ligt, met $u \in \Sigma$;
- punten op afstand 6 van ∞: de punten van $\mathrm{PG}(5, q)$ die niet in $\infty^\perp$ liggen.

Definieer de rechten als volgt.

- Rechten incident met ∞: de rechten $\langle\infty, u\rangle$, $u \in \Sigma$;
- rechten op afstand 3 van ∞: de rechten niet door ∞ in de vlakken $\langle\infty, L_u\rangle$, $u \in \Sigma$;
- rechten op afstand 5 van ∞: de totaal isotrope rechten die $\infty^\perp$ snijden in een punt dat op een rechte $\langle\infty, u\rangle$ ligt, met $u \in \Sigma$.

De incidentie is natuurlijk, op voorwaarde dat men de afstanden tot ∞ respecteert. Zo is een punt op afstand 2 van ∞ nooit incident met een rechte op afstand 5 van ∞, ofschoon het er in $\mathrm{PG}(5, q)$ wel op kan liggen. Bader en Lunardon tonen in [2] aan dat $\mathcal{S}$ isomorf is met de duale van $H(q)$.

Dat dit model niet werkt als q deelbaar is door 3, ligt aan het feit dat de ruimtelijke kubische kromme Σ zich in dit geval anders gedraagt. Ze geeft dan immers aanleiding tot een ontaarde alternerende vorm met als radicaal een projectieve rechte R, en niet tot een symplectische polariteit. Men kan niettemin de driedimensionale ruimte S die Σ bevat zonder problemen inbedden in $\mathrm{PG}(5, q)$, waarin men tevens een alternerende vorm vindt die als radicaal R heeft en waarvan de beperking tot S precies de alternerende vorm voortkomend uit Σ is. Ook een punt ∞ waarvoor $\infty \notin S \subseteq \infty^\perp$ bestaat steeds. Als men nu echter probeert de constructie van Bader en Lunardon uit te voeren, stuit men op moeilijkheden bij de rechten op afstand 3 van ∞. Sommige ervan bevatten een punt van R, andere niet, wat met zich meebrengt dat ze in een verschillend aantal totaal isotrope vlakken liggen. Om tot een model te komen dat ook geldig is in karakteristiek 3 zullen we dus in elk geval de rechten op afstand 3 van ∞ moeten herdefiniëren.

A.7.3 Het aangepaste model

Stel dat q oneven is, en beschouw de incidentiestructuur $\mathcal{S}$ die onststaat door in het model van Bader en Lunardon, respectievelijk in het hierboven beschreven equivalent voor q deelbaar door 3, de rechten op afstand 3 van ∞ te vervangen door de volgende objecten.

- Nieuwe rechten op afstand 3 van ∞: de driedimensionale ruimten die $\infty^\perp$ snijden in een vlak $\langle\infty, L_u\rangle$, $u \in \Sigma$.

Zo een nieuwe rechte verklaren we incident met een punt s op afstand 2 van ∞ als en slechts als ze bevat is in het hypervlak $s^\perp$. De overige incidenties zijn natuurlijk, rekening houdend met de afstanden tot ∞. We bewijzen dat $\mathcal{S}$ isomorf is met de duale van $H(q)$ voor alle oneven q (zie Stelling 7.3.1). Dit gebeurt door coördinaten in te voeren in $\mathcal{S}$ en vertrekkende vanuit het punt ∞ een bijectie met de objecten van de duale van $H(q)$ (beschreven door coördinatisatie) op te bouwen. Hierbij kan men er gemakkelijk voor zorgen dat de meeste incidenties kloppen. Gaat men de resterende incidenties na, dan stelt men vast dat ook deze bewaard blijven. Bijgevolg breidt bovenstaande aanpassing het model van Bader en Lunardon uit naar karakteristiek 3.

A.7.4 $H(q)$ als deelzeshoek van $H(q^3, q)$

In [67] beschrijft Lunardon een model voor de duale van $H(q^3, q)$, q oneven, dat sterk op het model van Bader en Lunardon lijkt. De constructie gebeurt in een $\mathrm{PG}(9, q)$ uitgerust met een symplectische polariteit en vertrekt van een bepaalde partiële ovoïde van $W(7, q)$. Als q geen macht van 3 is, dan vindt men door te snijden met een geschikte vijfdimensionale deelruimte juist het model van Bader en Lunardon voor de duale van $H(q)$. Als q deelbaar is door 3, dan levert dit de structuur op waarin de rechten op afstand 3 van ∞ problemen veroorzaken. Vervangt men echter in het model voor de duale van $H(q^3, q)$ de rechten op afstand 3 van het speciale punt ∞ door hun beelden onder de symplectische polariteit, dan bekomt men door te snijden met een geschikte vijfdimensionale deelruimte voor elke oneven q het aangepaste model voor de duale van $H(q)$.

A.7.5 Spreads van $H(q)$

De beschrijving van de Hermitische spread $\mathcal{S}_H$ van $H(q)$, q oneven, kan eenvoudig vertaald worden naar het aangepaste model dat we in Paragraaf A.7.3 geconstrueerd hebben. Men vindt dan dat $\mathcal{S}_H$ bestaat uit het speciale punt ∞ en uit de punten buiten $\infty^\perp$ van een driedimensionale deelruimte Π_H. De doorsnede van Π_H met $\infty^\perp$ is een vlak door ∞ dat geen enkele raaklijn aan Σ snijdt, en dat evenmin het radicaal R snijdt als q deelbaar is door 3.

Voor de Ree–Tits–spread $\mathcal{S}_R$ van $H(q)$, $q = 3^{2h+1}$, $h \in \mathbb{N} \setminus \{0\}$, dan krijgt men een op het eerste gezicht minstens even ingewikkelde beschrijving. Die wordt echter iets eenvoudiger wanneer men de situatie over een deelveld beschouwt. De vierdimensionale deelruimte $\infty^\perp$ over $\mathrm{GF}(q)$ komt overeen met een $(10h+4)$-dimensionale projectieve ruimte Π_∞ over $\mathrm{GF}(3)$. De affiene ruimte $\mathrm{AG}(5, q)$ die ontstaat door $\infty^\perp$ uit $\mathrm{PG}(5, q)$ weg te laten, staat in bijectief verband met een affiene ruimte $\mathrm{AG}(10h + 5, 3)$, die met Π_∞ tot een projectieve ruimte $\mathrm{PG}(10h+5,3)$ aangevuld kan worden. Na herschrijven van de coördinaten blijkt dat $\mathcal{S}_R$ in dit model bestaat uit het speciale punt ∞ en uit de punten buiten Π_∞ van q deelruimten van dimensie $4h+2$.

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Index of notation

GF(q)	the Galois field having q elements, q a prime power
Aut(GF(q))	the group of automorphisms of GF(q)
$V(n, q)$	the n-dimensional vector space over GF(q)
AG(n, q)	the n-dimensional affine space over GF(q)
PG(n, q)	the n-dimensional projective space over GF(q)
$Q^+(2n-1, q)$	the non-degenerate hyperbolic quadric in PG($2n-1, q$)
$Q^-(2n-1, q)$	the non-degenerate elliptic quadric in PG($2n-1, q$)
$Q(2n, q)$	the non-degenerate parabolic quadric in PG($2n, q$)
$H(n, q^2)$	the non-degenerate Hermitian variety in PG(n, q^2)
$W(2n-1, q)$	the polar space ($n \geq 3$) or generalised quadrangle ($n = 2$) arising from a symplectic polarity in PG($2n-1, q$)
$\mathrm{P\Gamma U}_3(q^2)$	the group of collineations stabilising $H(2, q^2)$
$H(q)$	the split Cayley hexagon of order q
$H(q^3, q)$	the twisted triality hexagon of order (q^3, q)
$\mathcal{P}(x, y)$	the point regulus of $H(q)$ determined by x and y
$\Gamma_i(x)$	the set of vertices at distance i from x
$\underline{0}$	the zero vector
$\underline{1}$	the all-one vector
I	the identity matrix
J	the all-one matrix
C_A	the code generated by the rows of the matrix A
$\mathcal{H}(q)$	the Hermitian two-graph
$\mathcal{H}'(q)$	the Hermitian graph

Index