Université Libre de Bruxelles
Faculté des Sciences
Département de Mathématique

Année Académique 1997-1998

Classification of RWPRI *Geometries for the Suzuki Simple Groups*

Dimitri Leemans

Thèse présentée en vue
de l'obtention du grade de
Docteur en Sciences

Promoteurs de Thèse:
Francis Buekenhout
Michel Dehon

Acknowledgements

I would like to thank my supervisors, Francis Buekenhout and Michel Dehon, for their support, their kindness and their disponibility throughout the elaboration of this work.

During the making of this thesis, I had the chance to visit Professor John Cannon at the University of Sydney. I would like to thank him and all his team for having made my stay there as nice as it was.

I would also like to thank several people with whom I had interesting conversations while working on this thesis: Wieb Bosma, Philippe Cara, Frank De Clerck, Jean Doyen, Harald Gottschalk, Cécile Huybrechts, Antonio Pasini, Leo Storme, Gernot Stroth, Joseph Thas and Hendrik Van Maldeghem.

Thanks must go to all my colleagues for their support, and also for the very nice working atmosphere that reigns in the Mathematics Department of the Université Libre de Bruxelles.

I also thank my parents for the opportunity they always gave me to let me choose what I wanted to do.

And last but certainly not least, I thank my wife Murielle for having pushed me into doing what I really wanted to do: mathematics.

Contents

Introduction

In this thesis, we classify geometries satisfying a set of properties for the Suzuki simple groups.

The classification of the finite simple groups, summarized around 1981, is one of the most spectacular achievement in mathematics of all time. This is due to the efforts of hundreds of researchers, over a period of thirty years, in around five hundred publications, covering more than fiftheen thousand pages. A coherent and revised presentation and proof is presently in the course of publication. The importance of the subject is due, to a large extent to the fact that the groups are the best mathematical expression of symmetry and, as their name may suggest, the simple groups are the basis that constitutes the more general groups. The classification leads to eighteen infinite families of groups and twenty-six sporadic groups, the largest of them being the Monster.
While some people could think that the classification finishes the study of these groups, there are still open questions in this field. One of them is the classification of all the maximal subgroups of the simple groups.
Also, these families of groups are known relatively to other things that are supposed to be known, as for example the prime numbers. As we can see in the work we do here, certain prime numbers lead to results that are completely different from others.

One of the most important open questions on these groups concerns a unified geometric interpretation of them which should help understanding them better. This theory already exists for the majority of them, namely for the groups of Lie type, which means all of them except the alternating groups and the sporadic groups : it is the theory of Buildings, due to Jacques Tits, for which he received the Wolf prize in 1993 (see [5], chapter 11 for a historical survey). This theory has given one of the most used geometrico-combinatorial structures in other fields of mathematics, and in particular, in the classification itself. Its extension to the sporadic groups is far from being easy; many efforts have been made in that sense during the last twenty years (see [5], chapter 22 for a historical survey). One of the reasons of the difficulty is precisely the sporadic character of these groups, and the fact that geometrically speaking, some groups of Lie type might need to be considered as groups isomorphic to further sporadic groups.

In [5] (chapter 22), the approach which is developed is the following: given a diagram Δ, try to find a geometry Γ belonging to Δ and a group G which is an automorphism group of Γ.

Here we work with another approach: we start from a group and we try to determine all geometries, satisfying a set of axioms, that can be constructed from this group and some of its subgroups. This approach is strongly based on experimental work.

We really think that experimental work can play an important role in mathematics, helping to solve theoretical problems. Some mathematical journals are now entirely devoted to experimentation, as for example *Experimental Mathematics*, whose advisory board is composed of great mathematicians as Hendrik W. Lenstra Jr., Jean-Pierre Serre and William P. Thurston.

It seems appropriate to elaborate an exhaustive experimental search of geometries satisfying some properties and corresponding to the smallest simple groups, be them sporadic or not. Such a research concentrates on reasonable conditions to be able to determine the geometries of all simple groups.

Experimentation has already been achieved with two slightly different conditions denoted PRI and RWPRI, with help of the computer algebra packages CAYLEY [10, 11] and MAGMA [2, 12, 13]. For experimental results in the PRI case, we refer to [7, 8, 17]. This work covers the groups G with $PSL(2,q) \leq G \leq Aut(PSL(2,q))$, for $5 \leq q \leq 19$, the Mathieu group M_{11}, and $PSU(4,2)$.
For the RWPRI case, we refer to [9, 18, 19, 20, 21, 22, 23]. These references cover experimental results for the groups $Alt(n)$ and $Sym(n)$ with $5 \leq n \leq 7$, the groups G with $PSL(2,q) \leq G \leq Aut(PSL(2,q))$ for $q = 7, 8$, the Mathieu group M_{11}, $PSU(4,2)$, $PSL(3,4)$ and the Janko group J_1.

If G is one of the studied groups, we would like to determine the class $\Gamma(G)$ of geometries in the Buekenhout-Tits sense, in which G acts transitively on the maximal flags, and which satisfy additional conditions, as for example the maximality, in G, of the stabilizers of some elements. We have, at present, a set of conditions or axioms which apparently permits that the size of $\Gamma(G)$ remains reasonable, as in the building case, and which accepts an efficient programming using the computer algebra packages CAYLEY or MAGMA. Also, it should now be possible to start a purely theoretical approach, in the same way than experimentation.

In its actual state, experimental work showed us that an interactive study of the geometries associated to a group G depends on the knowledge of the geometries corresponding to some subgroups of G. The next step might reasonably begin with a family of groups having a well known subgroup structure. Here the candidates that seem more appropriate are the $PSL(2,q)$ groups and the Suzuki groups $Sz(q)$. The latter appear to us as the easiest to start with for such a general study. Most of the subgroups of these groups are soluble groups. Also, we notice very early that the subgroups that play an important role are most of the time dihedral groups. In that spirit, we first study in general geometries associated to the dihedral groups.

Most of the definitions and notation that are used throughout this thesis are introduced in the first chapter. There are notions about incidence geometries and Suzuki groups, but also some notational shortcuts that are described in distinct sections.

Chapter 2 is devoted to the classification of the RWPRI geometries of dihedral groups. We start by giving an exhaustive list of the maximal subgroups of a dihedral group D_{2n} for n an integer. Then we classify the rank two geometries. And finally we determine the geometries of rank at least three.

In Chapter 3 we start the classification of the RWPRI geometries of the Suzuki groups. Again, we first recall what are the maximal subgroups of a Suzuki group *Sz(q)*. Then we classify the rank two geometries. We also determine how many rank two geometries there are up to isomorphism for any Suzuki group *Sz(q)*. Some very surprising results appear here. Let us give a few examples.

Let $q = 2^{2e+1}$ and $G \cong Sz(q)$.

- If $2e+1$ is a prime number, that is, the corresponding Suzuki group does not have proper subgroups isomorphic to a Suzuki group, then it has either one or a huge number of geometries.

- By looking at these geometries, we see that if we add a condition called $(2T)_1$, that is two transitivity on rank 1 residues, then all geometries except one (or three for the case $q = 8$) are rejected. So this property which we tested the last few years seems to be a good candidate to enter our set of axioms.

In Chapter 4, we classify the RWPRI geometries of rank three for all Suzuki simple groups $Sz(q)$ with q an odd power of 2. One big observation arises again from the results obtained there: if we impose the $(2T)_1$ property, only Suzuki simple groups $Sz(q)$ with $q-1$ prime have geometries of rank greater than 2. All of these are thin geometries of rank 3.

In chapter 5, we determine all RWPRI pre-geometries of rank ≥ 4 for any Suzuki simple group $Sz(q)$. To do that, we start from the classification theorems stated in chapter 4 for the rank 3 case. We get, as final result, eleven constructions leading to all RWPRI pre-geometries of rank ≥ 4 for any $Sz(q)$ (see theorem 5.3.1).

This thesis ends with two appendices: the first one gives the classification of all firm, residually connected geometries on which the group $Sz(8)$ acts flag-transitively and residually weakly primitively. It is the smallest Suzuki group, but also the only one that is currently possible to investigate fully with a computer. It helped us very well in guessing the way to follow for the general case. The second one gives all thin residually connected geometries on which $Sz(8)$ acts flag-transitively. We hoped that this work would help us to determine, in the general case, how many thin geometries are appearing up to isomorphism. Unfortunately, we didn't succeed. In this appendix, we also put more structure on the diagrams and we introduce a new property that seems very promising to us. It is the $(SIP)_2$ property. It seems promising because it is a necessary condition to have the intersection property, and also, it reduces quite significantly the huge number of thin geometries obtained.

The classification we obtain in this thesis for the Suzuki groups and for the dihedral groups strenghten our belief that the RWPRI condition is a very efficient one to impose on the subgroups forming the geometries. Indeed, even if this condition gives many geometries for some groups, their number seems to stay under control.

Chapter 1

Definitions and Notation

In a first section, we recall the basics of incidence geometry and fix some notation.
In a second section, we introduce the Suzuki simple groups as done in [33].
In a third section, we fix some notation that is used throughout this thesis.

1.1 Incidence Geometry

We recall the basic definitions about incidence geometry and group-geometry pairs. For a complete introduction on that subject, we refer to [5].

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [41] (see also [5], chapter 3).
Let I be a finite set and let G be a group together with a finite family of subgroups $(G_i)_{i\in I}$. We define the *pre-geometry* $\Gamma = \Gamma(G, (G_i)_{i\in I})$ as follows. The set X of *elements* of Γ consists of all cosets gG_i, $g \in G$, $i \in I$. We define an *incidence relation* $*$ on X by :

$$g_1G_i * g_2G_j \text{ iff } g_1G_i \cap g_2G_j \text{ is non-empty in } G.$$

The *type function* t on Γ is defined by $t(gG_i) = i$. The *type* of a subset Y of X is the set $t(Y)$; its *rank* is the cardinality of $t(Y)$ and we call $\mid t(X) \mid$ the *rank* of Γ. The *Borel subgroup* of the pre-geometry is the subgroup $B = \cap_{i\in I} G_i$. A *flag*

is a set of pairwise incident elements of X and a *chamber* of Γ is a flag of type I. An element of type i is also called an *i-element*.
The group G acts on Γ as an automorphism group, by left translation, preserving the type of each element.
As in [17], we call Γ a *geometry* provided that every flag of Γ is contained in some chamber and we call Γ *flag-transitive* (FT) provided that G acts transitively on all chambers of Γ, hence also on all flags of any type J, where J is a subset of I. It is obvious that any rank two pre-geometry $\Gamma(G; G_0, G_1)$ is a flag-transitive geometry.

Lemma 1.1.1 *Let $\Gamma(G; G_0, G_1, G_2)$ be a rank 3 pre-geometry. Then Γ is a geometry (not necessarily flag-transitive).*

Proof: We have to show that every flag is contained in a chamber. Suppose without loss of generality that we have a flag $F = \{g_0G_0, g_1G_1\}$. Then, there exists an element $h \in G$ such that $h \in g_0G_0 \cap g_1G_1$. Thus the coset hG_2 is incident to both g_0G_0 and g_1G_1 and F is contained in the chamber $\{g_0G_0, g_1G_1, hG_2\}$. $\square$

This lemma permits us to talk about geometries instead of pre-geometries in the rank three case.
Let $\Gamma(G; G_0, \ldots, G_{n-1})$ be a rank n pre-geometry. We call $C = \{G_0, \ldots, G_{n-1}\}$ the *maximal parabolic chamber* of Γ. Assuming that F is a subset of C, the *residue* of F is the pre-geometry

$$\Gamma_F = \Gamma(\cap_{j \in t(F)} G_j, (G_i \cap (\cap_{j \in t(F)} G_j))_{i \in I \setminus t(F)})$$

If $F = \{G_i\}$ for some $i \in I = \{0, \ldots, n-1\}$ then Γ_F is also called the *G_i-residue* of Γ. In that case, we sometimes write Γ_i instead of Γ_F. If Γ is a flag-transitive geometry and if F is any flag of Γ, of type $t(F)$, then the residue Γ_F of Γ is isomorphic to the residue of the flag $\{G_i, i \in t(F)\} \subseteq C$.
Let J be a subset of I. The *J-truncation* of Γ is the pre-geometry consisting of the elements of type $j \in J$, together with the restricted type-function and induced incidence relation. In group-geometry terms, the J-truncation of $\Gamma(G, (G_i)_{i \in I})$ is the pre-geometry $\Gamma(G, (G_j)_{j \in J})$.

Assume Γ is a pre-geometry. We call Γ *firm* (F) (resp. *thick*, *thin*) provided that every flag of rank $\mid I \mid -1$ is contained in at least two (resp. three, exactly two)

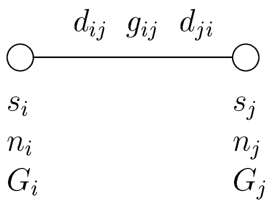
chambers. We call Γ *residually connected* (RC) provided that the incidence graph of each residue of rank ≥ 2 is a connected graph.
Let $\Gamma(G; G_0, \ldots, G_{n-1})$ be a pre-geometry and put $I = \{0, \ldots, n-1\}$. As in [23], for any nonempty subset $J \subset I$, we set $G_J = \bigcap_{j \in J} G_j$, $B = G_I$ and $G_\emptyset = G$. Then we call $\mathcal{L}(\Gamma) = \{G_J : J \subseteq I\}$ the *sublattice* (on the subgroup lattice of G) *spanned* by the collection $(G_i)_{i \in I}$. The elements of the lattice are called the *parabolic subgroups* and the subgroups G_i's are the *maximal parabolic subgroups*. When the context is clear, we write "sublattice" instead of "sublattice spanned by ...".
We call Γ *primitive* (PRI) provided that G acts primitively on the set of i-elements of Γ for each $i \in I$. We call Γ *residually primitive* (RPRI) provided that for any $J \subset I$ and for all $i \in I \backslash J$, the group $G_{J \cap \{i\}}$ is a maximal subgroup of G_J. Notice that this definition of RPRI differs slightly from the one given in [9]. Moreover, if Γ is a flag-transitive geometry, the definition given here is equivalent to the one given in [9]. We call the pre-geometry Γ *residually weakly primitive* (RWPRI) provided that for any $J \subset I$ there exists at least one element $i \in I \backslash J$ such that $G_{J \cup \{i\}}$ is maximal in G_J. Again, this definition of RWPRI differs slighty from the one given in [9]. If the pre-geometry Γ is a flag-transitive geometry, then the definition of RWPRI given here is equivalent to the one given in [9].
If Γ is a RWPRI pre-geometry then, without loss of generality, we may assume that

- $\cap_{j \in \{0, \ldots, i\}} G_j$ is a maximal subgroup of $\cap_{j \in \{0, \ldots, i-1\}} G_j$ for all $i = 1, \ldots, n-1$;
- $\cap_{j \in I} G_j$ is a maximal subgroup of $\cap_{j \in I \backslash \{i\}} G_j$ for all $i \in I$.

A subgroup H of a group G is called *quasi-maximal* if there exists a unique chain of intermediate subgroups from G to H.
We call Γ *quasi primitive* (QPRI) provided that every subgroup G_i forming the geometry is a quasi-maximal subgroup of G.
If Γ is a geometry of rank 2 with $I = \{0, 1\}$ such that every of its 0-elements is incident with each of its 1-elements, we call Γ a *generalized digon*.
We call the pre-geometry Γ *locally 2-transitive* and we write $(2T)_1$ for this, provided that the stabilizer G_F of any flag $F \subset C$ of rank $\mid I \mid -1$ acts 2-transitively on the residue Γ_F.
Again here, if Γ is a flag-transitive geometry, then the $(2T)_1$ property is the same as the one given in [9].

We denote by $\sigma_0(F)$ the 0-shadow of a flag F, that is the set of elements of type 0 that are incident with the flag F. We say that a geometry Γ, over a set of types I, satisfies the *intersection property* (IP) when for every types $0, 1 \in I$, for every 1-element x, and for every flag F, $\mid \sigma_0(x) \cap \sigma_0(F) \mid > 1$ implies that there exists a flag F' such that x and F are incident with F' and $\sigma_0(F') = \sigma_0(x) \cap \sigma_0(F)$.
We say that Γ satisfies the *rank two intersection property* $(IP)_2$ if in every rank 2 residue of Γ other than a generalized digon, any two elements of the same type are incident with at most one element of the other type.

Following [3] and [4], the *diagram* of a firm, RC, FT geometry Γ is a graph together with additional structure, whose vertices are the elements of I, which is further described as follows. To each vertex $i \in I$, we attach the *order* s_i which is $\mid \Gamma_F \mid -1$, where F is any flag of type $I \backslash \{i\}$, the *number* n_i *of varieties of type* i, which is the index of G_i in G, and the subgroup G_i. Elements i, j of I are not joined by an edge of the diagram provided that a residue Γ_F of type $\{i, j\}$ is a generalized digon. Otherwise, i and j are joined by an edge endowed with three positive integers d_{ij}, g_{ij}, d_{ji} where g_{ij} (the *gonality*) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i, j\}$ and d_{ij} (resp. d_{ji}), the *i-diameter* (resp. *j-diameter*) is the greatest distance from some fixed i-element (resp. j-element) to any other element in the incidence graph of Γ_F.
On a picture of the diagram, this structure will often be depicted as follows.



If $g_{ij} = d_{ij} = d_{ji} = n$, then Γ_F is called a *generalized n-gon* and on a picture, we do not write d_{ij} and d_{ji}.
If $(d_{ij}, g_{ij}, d_{ji}) = (5, 5, 6)$ we write P on the corresponding edge instead of the 3 parameters. This is because such a rank 2 residue is a Petersen graph.
If $d_{ij} = g_{ij} = 3$, $d_{ji} = 4$ and $s_i = 1$, we write $\subset$ on the corresponding edge.
We sometimes call a rank 2 residue with parameters (d_{ij}, g_{ij}, d_{ji}) a $d_{ij} - g_{ij} - d_{ji}$-residue.
A *star-diagram* is a diagram with one vertex joined to all other and no other edge.

The ordered pairs (Γ, G) and (Γ', G) are *isomorphic* (resp. *conjugate*) if there exists an automorphism (resp. internal automorphism) of G mapping Γ onto Γ'.

As to notation for groups, we follow the conventions of the Atlas [14] up to slight variations. The symbol ":" stands for split extensions, the "hat" symbol "$\hat{\cdot}$"stands for non split extensions and the symbol $\times$ stands for direct products. Sometimes it is not known whether an extension is split or not. In that case, we write "$\cdot$" or nothing to denote that it is an extension. We denote by Z_n (or just n when the context is not ambiguous) a cyclic group of order n.

Following ideas developped in [39], we say that a group G acts *chirally* on a geometry Γ if the chambers of Γ are divided in two orbits of the same length on which G acts transitively. We also say that the group G acts *regularly* on Γ if G is transitive on the set of chambers of Γ.

1.2 The Suzuki simple groups

We recall the definition of the Suzuki groups as given in [33]. Let $\mathcal{K}$ be a field of characteristic 2 with $\mid \mathcal{K} \mid > 2$. Let σ be an automorphism of $\mathcal{K}$ such that $x^{\sigma^2} = x^2$ for each x in $\mathcal{K}$. Let $\mathcal{B}$ be the 3-dimensional projective space over $\mathcal{K}$ and let (x_0, x_1, x_2, x_3) be the coordinates of a point of $\mathcal{B}$. Let E be the plane defined by the equation $x_0 = 0$ and put $U = (0, 1, 0, 0)\mathcal{K}$. We introduce coordinates in the affine space $\mathcal{B}_E$ by $x = \frac{x_2}{x_0}$, $y = \frac{x_3}{x_0}$ and $z = \frac{x_1}{x_0}$. Finally, let $\mathcal{D}$ be the set of points of $\mathcal{B}$ consisting of U and all those points of $\mathcal{B}_E$ whose coordinates (x, y, z) satisfy the equation

$$z = xy + x^{\sigma+2} + y^{\sigma}$$

We denote by *Sz($\mathcal{K}$, σ)* the group of all projective collineations of $\mathcal{B}$ which leave $\mathcal{D}$ invariant.

Lemma 1.2.1 *[33]*

- *If $\mathcal{K}$ is isomorphic to $GF(2^{2e+1})$ with $e \geq 1$, then $\mathcal{K}$ admits exactly one automorphism σ with $x^{\sigma^2} = x^2$ for all x in $\mathcal{K}$.*
- *If $\mathcal{K}$ is isomorphic to $GF(2^{2e})$, then $\mathcal{K}$ does not possess an automorphism σ with $x^{\sigma^2} = x^2$ for all x in $\mathcal{K}$.*

This theorem yields that we may write *Sz(q)* instead of *Sz($\mathcal{K}$, σ)* if $\mathcal{K}$ is isomorphic to *GF(q)* with $q = 2^{2e+1}$. The groups *Sz(q)* are the *Suzuki groups* named after Michio Suzuki who found them in 1960. The generalizations *Sz($\mathcal{K}$, σ)* are due to Rimhak Ree and Jacques Tits (see for example [42]).
The set $\mathcal{D}$ is an *ovoid*, i.e. a non-empty point-set of a projective 3-space that satisfies the following three conditions.

1. No three points are collinear.
2. If $p \in \mathcal{D}$, there exists a plane E of $\mathcal{B}$ with $\mathcal{D} \cap E = \{p\}$.
3. If $p \in \mathcal{D}$ and if E is a plane of $\mathcal{B}$ with $\mathcal{D} \cap E = \{p\}$, then all lines l through p which are not contained in E carry a point of $\mathcal{D}$ distinct from p.

For $a, b \in \mathcal{K}$, we denote by $\tau(a, b)$ the mapping defined by

$$(x, y, z)^{\tau(a,b)} = (x + a, y + b + a^{\sigma} x, z + ab + a^{\sigma+2} + b^{\sigma} + ay + a^{\sigma+1} x + bx)$$

We have $\tau(a, b)\tau(c, d) = \tau(a + c, ac^{\sigma} + b + d)$.
For $k \in \mathcal{K}^*$, we define the collineation $\eta(k)$ by

$$(x, y, z)^{\eta(k)} = (kx, k^{\sigma+1} y, k^{\sigma+2} z)$$

A straightforward computation shows that $\tau(a, b)\eta(k) = \eta(k)\tau(ka, k^{\sigma+1} b)$.
Let ω be the collineation of $\mathcal{B}$ defined by $(x_0, x_1, x_2, x_3)^{\omega} = (x_1, x_0, x_3, x_2)$.
The following lemma, due to Tits, will be used in a further section to study the structure of the stabilizer of a point of the ovoid.

Lemma 1.2.2 *[42]*
Let $\mathcal{K}$ be a commutative field of characteristic 2 with $\mid \mathcal{K} \mid > 2$ and assume that $\mathcal{K}$ admits an automorphism σ such that $x^{\sigma^2} = x^2$ for all $x \in \mathcal{K}$. If $\mathcal{D}$ is the point set defined above in the projective space of dimension 3 over $\mathcal{K}$, then $Sz(\mathcal{K}, \sigma)$ acts doubly transitively on $\mathcal{D}$.
Moreover, if $\gamma \in Sz(\mathcal{K}, \sigma)_U$, then there exists exactly one triple $(k, a, b) \in \mathcal{K}^* \times \mathcal{K} \times \mathcal{K}$ with $\gamma = \eta(k)\tau(a, b)$, and if $\gamma \in Sz(\mathcal{K}, \sigma) \backslash Sz(\mathcal{K}, \sigma)_U$, then there exists exactly one 5-tuple $(k, a, b, c, d) \in \mathcal{K}^* \times \mathcal{K} \times \mathcal{K} \times \mathcal{K} \times \mathcal{K}$ with $\gamma = \eta(k)\tau(a, b)\omega\tau(c, d)$.

Let $PG(3, q)$ be the projective space over the field $GF(q)$ and let $\mathcal{D}$ be an ovoid of $PG(3, q)$. If Π is a plane of $PG(3, q)$ such that $\mid \Pi \cap \mathcal{D} \mid > 1$, then we call $\Pi \cap \mathcal{D}$ a *circle*. The following lemma ensures that every circle has the same number of points, and moreover that these circles are ovals.

Lemma 1.2.3 *[33] If $\mathcal{D}$ is an ovoid in $PG(3,q)$, then $\mid \mathcal{D} \mid = q^2+1$. If E is a plane of $PG(3,q)$, then E is either a tangent plane of $\mathcal{D}$ or $E \cap \mathcal{D}$ consists of the $q+1$ points of an oval of E.*

1.3 Some useful notation

The symbol n_p denotes the number $\frac{n}{p}$ when p is a divisor of n. The function ϕ is Euler's function that associates to every positive integer n the number of positive integers $i < n$ such that $(i,n) = 1$.

For some properties of this function, we refer to [37]. Since it is a well known function, almost every book on number theory is a good reference for this.

There are four numbers that play an important role in the subgroup structure of *Sz(q)*. They are respectively q, $q-1$, $q+r+1$ and $q-r+1$ where $r = \sqrt{2q}$. We note $q+r+1 = \alpha_q$ and $q-r+1 = \beta_q$.

When n is a proper divisor of m, we write $n \mid_P m$.
When n is a maximal divisor of m, we write $n \mid_M m$.

Chapter 2

The Dihedral Groups

This chapter is taken from [25].

2.1 Introduction

The main part of this thesis is devoted to the classification of all firm residually connected geometries on which a group $Sz(q)$, with q an odd power of 2, acts residually weakly primitively. In order to classify the geometries of rank greater than 2, we need to know the geometries of (some of) their maximal subgroups because of the RWPRI condition. Actually, every Suzuki simple group has maximal subgroups isomorphic to dihedral groups and other families of finite simple groups have some. So, this chapter may be seen as a necessary prelude to a classification process for some infinite families of finite simple groups.

The chapter is organised as follows. In section 2, we compute the automorphism group of a dihedral group. In section 3, we list the maximal subgroups of a dihedral group, as given in [36]. In section 4, we classify all the rank 2 geometries. Finally in section 5, we classify all the geometries of rank greater than 2.

2.2 The Automorphism group of a dihedral group

Let us recall that a dihedral group $G = D_{2n}$, with $n \geq 1$ an integer, is a group generated by two elements a and b characterized by the relations

$$a^2 = b^2 = 1 = (ab)^n$$

as defined, for example, in [16].
The elements a and b are called *generating reflections*. The case $n = 1$ gives the cyclic group of order 2, and the case $n = 2$ gives the elementary abelian group 2^2.
We find it convenient to represent G as the automorphism group of a regular n-gon denoted by P.

Observe that $\mid Aut(G) \mid= 1$ (resp. 6) for $n = 1$ (resp. 2).

Lemma 2.2.1 *If $n \geq 3$, then $\mid Aut(G) \mid= \phi(n) \cdot n$*

Proof: The n-gon P has n symmetry axes. The elements a and b can be seen as two generating reflections, each of them stabilizing an axis of P. Remember that a rotation of P is the product of two reflections. The automorphism group of G is then the group that sends two generating reflections onto two other generating reflections. We can apply a onto one of the n reflections. Since the images of a and b must generate the cyclic group of rotations Z_n of D_{2n}, we may still apply b onto $\phi(n)$ different reflections. Thus $\mid Aut(G) \mid= n \cdot \phi(n)$. □

2.3 On the subgroups of a dihedral group

The maximal subgroups of a dihedral group are given in [36]. Let $G = D_{2n}$ be a dihedral group. Its maximal subgroups have the following structure.

- Z_n, which is the group of rotations of P.
- D_{2m} with m a maximal divisor of n. These subgroups are the stabilizers of the maximal subpolygons of P.

Remark that if n is even, there are two non-conjugate subgroups D_n in D_{2n}. One of them may be seen as the stabilizer of a $\frac{n}{2}$-subpolygon of the n-gon, and the other is the group generated by the rotations of the n-gon and the involution that exchanges the two $\frac{n}{2}$-subpolygons of the n-gon.

The following lemma is very useful in the next section while trying to construct RWPRI geometries.

Lemma 2.3.1 *If $\frac{n}{m} \neq 2$ and $m \neq 1$, then D_{2m} is self-normalizing in D_{2n}. If $\frac{n}{m} = 2$ then D_{2m} is a normal subgroup of D_{2n}.*

Proof: The second part of the lemma is obvious. Let us prove the first. Suppose $\frac{n}{m} \neq 2$. We know that D_{2m} is a maximal subgroup of D_{2n} since its index is a prime number. The normalizer of D_{2m} is then either itself or D_{2n}. We know that there are $\frac{n}{m}$ subgroups isomorphic to D_{2m} in D_{2n}. They are the stabilizers of the $\frac{n}{m}$ different m-gons that we can construct in a n-gon by starting from the same edge. All of them are conjugate. □

2.4 Geometries of rank 2

To construct a RWPRI rank two geometry, we first choose a G_0 which is maximal in G, then a G_{01} maximal in G_0. And finally, we look at all the G_1 that contain G_{01} as a maximal subgroup.

2.4.1 $G_0 = Z_n$

All maximal subgroups of G_0 are cyclic subgroups of prime index. Thus $G_{01} = Z_m$ with m a maximal divisor of n. The subgroups of G that contain G_{01} as a maximal subgroup are Z_n and D_{2m}. The first possibility leads to a contradiction since G_0 must be distinct from G_1. Hence $G_1 = D_{2m}$. Because $[D_{2n} : Z_n] = [D_{2m} : Z_m]$, we can conclude that this geometry is a generalized digon.
For given integers n and m, we now count how much geometries appear up to isomorphism.
If $n \neq 2$ and $\frac{n}{m} \neq 2$, there is only one conjugacy class of D_{2m} in G. Then we may assume G_1 is given and since there is only one Z_n in D_{2n}, we can construct only

one such geometry up to isomorphism.
If $n \neq 2$ and $\frac{n}{m} = 2$, there are two classes of D_{2m} that are not fused under the action of $Aut(D_{2n})$. Thus is this case, we get two geometries that are non-isomorphic.
If $n = 2$, there are three conjugacy classes of $D_2 = Z_2$ in $D_4 = 2^2$. The automorphism group acts doubly transitively on the involutions of G. Thus there is only one such geometry up to isomorphism.
Let $n = p_1^{e_1} \ldots p_i^{e_i}$ be the factorization of n. Then, there are i possible choices for G_{01} and thus i (resp. $i+1$) for G_1 if n is odd (resp. even).

2.4.2 $G_0 = D_{2m}$ and $G_1 \neq Z_n$

We have to distinguish two cases.

First case: n is prime.
The case $n = 2$ has been analysed in section 2.4.1. So we now assume $n > 2$. Then $G_0 = Z_2$ and $G_1 = Z_2$ give a RWPRI geometry which is a regular n-gon. There are n subgroups of order 2. We may assume G_0 is given. Then, there are $n-1$ subgroups G_1 that give a RWPRI geometry with G_0. These geometries are two by two conjugate, and they all are isomorphic since $\phi(n) = n-1$ when n is a prime number. Thus there is only one such geometry up to isomorphism.

Second case: n is not prime.
The G_{01} candidates are Z_m and D_{2l} with l a maximal divisor of m.
Suppose $G_{01} = Z_m$. The only subgroups containing it maximally have structure Z_n or D_{2m}. The first choice leads to geometries already obtained in section 2.4.1. Thus we may assume $G_1 = D_{2m}$. We know that there are $\frac{n}{m} = p$ subgroups isomorphic to G_0 in G. Each of them contains only one Z_m. And Z_m is a normal subgroup of G. Thus there are p subgroups isomorphic to G_0 which contain the same Z_m. We may assume G_0 and G_{01} are given. The stabilizer of these two subgroups is G_0. There are $p-1$ subgroups that can play the role of G_1. If $p = 2$, then $p - 1 = 1$ which means that there is only one such geometry up to isomorphism. Else, there are $p-1$ candidates for G_1 and $p-1$ is an even number. The stabilizer of G_0 and G_{01} being G_0, we have $\frac{p-1}{2}$ non-conjugate geometries. It is easy to see that $\phi(n) \geq p-1$. Thus the automorphism group $\mid Aut(G) \mid$ fuses all the non-conjugate geometries inside a unique set and we have only one such geometry up to isomorphism.

Suppose $G_{01} = D_{2l}$. If $l = 1$, then m is a prime number and n is the product of two prime numbers. Let's say $n = m{\cdot}p$. Then $G_{01} = Z_2$ and the G_1 candidates are D_{2m} or D_{2p}. The first case gives a geometry which is a p-gon, while the second one gives a generalized digon. These two geometries are unique up to isomorphism. If $l \neq 1$, then G_1 must be a dihedral group. Suppose $n = p_1^{e_1} \dots p_i^{e_i}$. If n is odd, then we may assume $m = n_{p_a}$, $l = m_{p_b}$ and $G_1 = D_{2m'}$ where $m' = n_{p_b}$ and $a \neq b$ thanks to lemma 2.3.1. This lemma implies also that such configurations are unique up to isomorphism. There are i possible choices for G_0 and $i-1$ choices for G_1. Thus there are $\frac{i\cdot(i-1)}{2}$ such geometries. If n is even, we must count twice the configurations with $G_0 = D_n$. Thus we get $\frac{i(i-1)}{2} + i - 1$ geometries.

Now we can state a classification theorem for the rank 2 RWPRI geometries of a dihedral group.

Theorem 2.4.1 *Let $G = 2^2$. Up to isomorphism, G has only one rank 2 geometry satisfying F, RC, FT, and* RWPRI. *It has two maximal parabolic subgroups isomorphic to cyclic groups of order 2 and it is a generalized digon.*
Let $G = D_{2n}$, with $n \geq 3$ an integer, be a dihedral group and suppose $n = p_1^{e_1} \dots p_m^{e_m}$. If n is odd (resp. even), up to isomorphism, the group G has $2m + \frac{m(m-1)}{2}$ (resp. $3m + \frac{m(m-1)}{2}$) geometries of rank 2 satisfying F, RC, FT, and RWPRI. *They are given below.*

- *m (resp. $m+1$) geometries with the following diagram:*

$$\begin{array}{ccccc} \bigcirc & & \bigcirc & & \\ 1 & & p_i - 1 & & B = Z_{n_{p_i}} \\ 2 & & p_i & & \\ Z_n & & D_{2n_{p_i}} & & \end{array}$$

- *m (resp. m) geometries with the following diagram:*

$$\begin{array}{ccccc} \bigcirc & \overset{p_i}{\rule[0.5ex]{6em}{0.4pt}} & \bigcirc & & \\ 1 & & 1 & & B = Z_{n_{p_i}} \\ p_i & & p_i & & \\ D_{2n_{p_i}} & & D_{2n_{p_i}} & & \end{array}$$

- *$\frac{m(m-1)}{2}$ (resp. $\frac{m(m-1)}{2} + m - 1$) geometries with the following diagram:*

$$\begin{array}{ccccc} \bigcirc & & \bigcirc & & \\ p_i - 1 & & p_j - 1 & & B = D_{2n_{p_i p_j}} \\ p_i & & p_j & & \\ D_{2n_{p_i}} & & D_{2n_{p_j}} & & \end{array}$$

2.5 Geometries of rank at least 3

The following result implies that all the RWPRI geometries constructed in this section are residually connected and flag-transitive. As in [1], a *string* is a graph on a set I such that we can order $I = \{1, \dots, n\}$ so that the edges of the graph are $\{i, i+1\}$, $1 \leq i < n$.

Lemma 2.5.1 *[1] Let G be a group, I a finite set, and $\Gamma = (G_i : i \in I)$ a family of subgroups of G. Assume:*
(i) for each subset J of I of corank at least 2, $G_J =< G_{J\cup\{i\}} : i \in I \setminus J >$, and
(ii) the connected components of the graph of the diagram of Γ are strings. Then
(1) G is flag-transitive on Γ.
(2) Γ is residually connected.

We classify directly geometries for all ranks higher than 2. Let $n = p_1^{e_1} \dots p_m^{e_m}$ and $m \geq 1$ (the case $m = 0$ gives the cyclic group of order 2 whose only RWPRI geometry is of rank 1). We construct all RWPRI geometries of rank $\alpha = 3, \dots, m+1$ which is the highest rank.

2.5.1 $G_0 = Z_n$

The residue of G_0 is of the following form.

$$\begin{array}{llll}
\bigcirc & \bigcirc & \cdots & \bigcirc \\
p_{i_1} - 1 & p_{i_2} - 1 & & p_{i_{\alpha-1}} - 1 \\
p_{i_1} & p_{i_2} & & p_{i_{\alpha-1}} \\
Z_{n_{p_{i_1}}} & Z_{n_{p_{i_2}}} & & Z_{n_{p_{i_{\alpha-1}}}}
\end{array}$$

An easy exercise shows that, here, the highest rank of the geometries for G is $m + 1$.
The G_j candidates cannot be cyclic groups. Thus, every G_j is a dihedral group having as subgroup a cyclic group $Z_{n_{p_j}}$. This implies $G_j \cong D_{2n_{p_j}}$. And no two distinct G_j's are isomorphic. Thus we can conclude by saying that the corresponding geometries have the following diagrams.

$$\begin{array}{cccc}
\bigcirc & \bigcirc & \cdots & \bigcirc \\
1 & p_{i_1}-1 & & p_{i_{\alpha-1}}-1 \\
2 & p_{i_1} & & p_{i_{\alpha-1}} \\
Z_n & D_{2n_{p_{i_1}}} & & D_{2n_{p_{i_{\alpha-1}}}}
\end{array}$$

There are $\binom{m}{\alpha-1}$ distinct diagrams. If n is odd, each one corresponds to a unique geometry up to isomorphism, for every rank $\alpha = 3, \ldots, m+1$. If n is even, since there are two subgroups D_n which are non-conjugate under the action of $Aut(G)$, there are $\binom{m-1}{\alpha-1} + 2 \cdot \binom{m-1}{\alpha-2}$ geometries up to isomorphism. Thanks to lemma 2.5.1, we know that all these geometries are residually connected and flag-transitive.

2.5.2 $G_0 = D_{2n_{p_i}}$ and $G_i \neq Z_n$

Let us first consider the case where all the G_i's are pairwise non-isomorphic. Then the sublattice spanned by the maximal parabolic subgroups is composed of cyclic groups of prime order or dihedral groups. Thus we obtain the following geometries which are all unique up to isomorphism.

$$\begin{array}{cccc}
\bigcirc & \bigcirc & \cdots & \bigcirc \\
p_{i_1}-1 & p_{i_2}-1 & & p_{i_\alpha}-1 \\
p_{i_1} & p_{i_2} & & p_{i_\alpha} \\
D_{2n_{p_{i_1}}} & D_{2n_{p_{i_2}}} & & D_{2n_{p_{i_\alpha}}}
\end{array}$$

If n is odd, there are $\binom{m}{\alpha}$ such geometries and the highest rank is m. If n is even, there are two D_n that are non-conjugate under the action of $Aut(G)$ and so there are $\binom{m-1}{\alpha} + 2 \cdot \binom{m-1}{\alpha-1}$ geometries up to isomorphism. Thanks to lemma 2.5.1, we know that all these geometries are residually connected and flag-transitive.

Now suppose that exactly two of the G_i's are isomorphic.
While descending in the sublattice, we keep two isomorphic dihedral groups that intersect on a cyclic group at the bottom of the sublattice. This gives a rank 2 residue which is a polygon. All other rank 2 residues are generalized digons. Thus we have the following diagram.

$$\begin{array}{ccccc}
\bigcirc \overset{p_{i_1}}{\rule[0.5ex]{4em}{0.4pt}} & \bigcirc & \bigcirc & \cdots & \bigcirc \\
1 & 1 & p_{i_2}-1 & & p_{i_{\alpha-1}}-1 \\
p_{i_1} & p_{i_1} & p_{i_2} & & p_{i_{\alpha-1}} \\
D_{2n_{p_{i_1}}} & D_{2n_{p_{i_1}}} & D_{2n_{p_{i_2}}} & & D_{2n_{p_{i_{\alpha-1}}}}
\end{array}$$

If n is odd, there are $m \cdot \binom{m-1}{\alpha-2}$ such geometries and the highest rank is $m+1$. If n is even, since there are two D_n that are non-conjugate under $Aut(G)$, there are $\binom{m-1}{\alpha-2} + 2 \cdot (m-1) \cdot \binom{m-2}{\alpha-3} + (m-1) \cdot \binom{m-2}{\alpha-2}$ geometries up to isomorphism. Thanks to lemma 2.5.1, we know that all these geometries are residually connected and flag-transitive.

It is easy to show that there cannot be three G_i's that are isomorphic.
Indeed, suppose it is possible. Then the lattice must have two subgroups that are equal, a contradiction.

It is also easy to show that there cannot be two pairs of isomorphic subgroups in the G_i's.
Again, in the lattice, this would imply that two subgroups are the same, a contradiction.

This ends the classification of the F, RC, FT, RWPRI geometries of a dihedral group. We now resume the preceding discussion in a theorem.

Theorem 2.5.1 *Let $G = D_{2n}$ be a dihedral group with $n = p_1^{e_1} \dots p_m^{e_m} > 2$. If n is odd (resp. even), up to isomorphism, the group G has $\binom{m}{\alpha-1} + \binom{m}{\alpha} + m \cdot \binom{m-1}{\alpha-2}$ (resp. $\binom{m-1}{\alpha} + 3 \cdot \binom{m-1}{\alpha-1} + 3 \cdot \binom{m-1}{\alpha-2} + (m-1) \cdot \binom{m-2}{\alpha-2} + 2 \cdot (m-1) \cdot \binom{m-2}{\alpha-3}$) + geometries of rank α, that satisfy F, RC, FT and* RWPRI *($\alpha = 2 \dots m+1$). They are given below.*

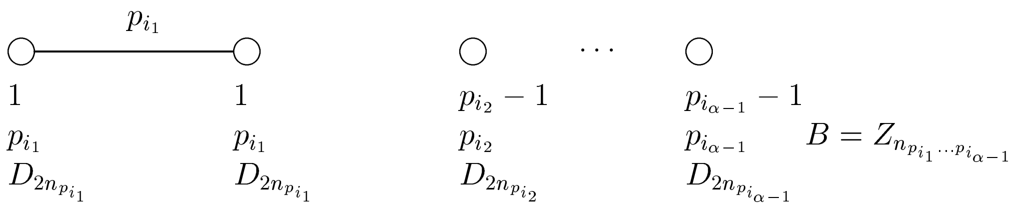
- *$\binom{m}{\alpha-1}$ (resp. $\binom{m-1}{\alpha-1} + 2 \cdot \binom{m-1}{\alpha-2}$) geometries with the following diagram:*

$$\begin{array}{llll} \bigcirc & \bigcirc & \cdots & \bigcirc \\ 1 & p_{i_1}-1 & & p_{i_{\alpha-1}}-1 \\ 2 & p_{i_1} & & p_{i_{\alpha-1}} \\ Z_n & D_{2np_{i_1}} & & D_{2np_{i_{\alpha-1}}} \end{array} \quad B = Z_{np_{i_1}\cdots p_{i_{\alpha-1}}}$$

- *$\binom{m}{\alpha}$ (resp. $\binom{m-1}{\alpha} + 2 \cdot \binom{m-1}{\alpha-1}$) geometries with the following diagram:*

$$\begin{array}{llll} \bigcirc & \bigcirc & \cdots & \bigcirc \\ p_{i_1}-1 & p_{i_2}-1 & & p_{i_\alpha}-1 \\ p_{i_1} & p_{i_2} & & p_{i_\alpha} \\ D_{2np_{i_1}} & D_{2np_{i_2}} & & D_{2np_{i_\alpha}} \end{array} \quad B = D_{2np_{i_1}\cdots p_{i_\alpha}}$$

- *$m \cdot \binom{m-1}{\alpha-2}$ (resp. $\binom{m-1}{\alpha-2} + 2 \cdot (m-1) \cdot \binom{m-2}{\alpha-3} + (m-1) \cdot \binom{m-2}{\alpha-2}$) geometries with the following diagram:*



Remark that this theorem also gives the rank 2 classification as it was stated in theorem 2.4.1, except for $G = 2^2$.
Two easy corollaries can be obtained from the preceding discussion.

Corollary 2.5.1 *All the* RWPRI *geometries of a dihedral group are* RPRI.

Corollary 2.5.2 *All the* RWPRI *geometries of a dihedral group satisfy* $(IP)_2$.

Theorem 2.5.1 shows that the number of non-isomorphic geometries obtained for a dihedral group D_{2n} depends only on the number of distinct prime divisors of n and on the fact that n is either odd or even. Table 2.1 gives, for every number $n = p_1^{e_1} \dots p_m^{e_m} \geq 3$, with $m \leq 10$, the number of non-isomorphic geometries of rank 2, ..., m+1. For example, if we take D_{60}, we have $n = 30 = 2 \cdot 3 \cdot 5$. Thus $m = 3$. Since n is even, we can say thanks to table 2.1 that D_{60} has 12, (resp. 15, 7) geometries of rank 2 (resp. 3, 4) up to isomorphism. This table, which goes up to $m = 10$, covers all the dihedral groups D_{2n} for $n < 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 = 200,560,490,130$ and, if n is odd, it covers all dihedral groups D_{2n} up to $n < 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 = 3,710,369,067,405$. It is given to show that the number of geometries obtained remains reasonable.

m	n odd	n even
1	2	3
2	5, 3	7, 5
3	9, 10, 4	12, 15, 7
4	14, 22, 17, 5	18, 31, 27, 9
5	20, 40, 45, 26, 6	25, 54, 67, 43, 11
6	27, 65, 95, 81, 37, 7	33, 85, 135, 126, 63, 13
7	35, 98, 175, 196, 133, 50, 8	42, 125, 240, 291, 214, 87, 15
8	44, 140, 294, 406, 364, 204, 65,9	52, 175, 392, 581, 560, 337, 115,17
9	54, 192, 462, 756, 840, 624, 297,82,10	63, 236, 602, 1050, 1246, 988, 501,147,19
10	65, 255, 690, 1302, 1722, 1590,1005,415,101,11	75, 309, 882, 1764, 2478, 2430,1629,712,183,21

Table 2.1: The number of non-isomorphic geometries for a dihedral group D_{2n}, with $n \geq 3$ an integer.

Chapter 3

The rank 2 geometries of *Sz(q)*

This chapter is taken from [26].

3.1 Introduction

We begin the classification of all firm residually connected geometries on which a Suzuki simple group $Sz(q)$, with q an odd power of 2, acts residually weakly primitively, by determining all the rank 2 geometries. In section 2, we give a list of useful properties that are used throughout the chapter to achieve this classification. In section 3, we recall the structure of maximal subgroups of a Suzuki simple group. In section 4, we give the structure of the maximal subgroups of each maximal subgroup of a Suzuki group. In section 5, we determine all rank 2 geometries that are obtained. The results of the quite long discussion and case analysis of section 5 leads to a classification theorem that we state in section 6. Finally, we close this chapter with section 7, in which we make some observations on the results obtained in the preceding sections.

In this chapter, we consider geometries of rank 2, i.e. geometries $\Gamma(G; G_0, G_1)$. We call the quadruple (G, G_0, G_1, G_{01}) an RWPRI *configuration (of rank 2)* provided that G_0 is maximal in G and $G_{01} = G_0 \cap G_1$ is maximal in both G_0 and G_1.

Lemma 3.1.1 *If* Γ *is a* RWPRI *configuration, then* Γ *is firm, residually connected and flag-transitive.*

Proof: This is a direct consequence of the results obtained in [17]. □

This lemma implies that all geometries of rank 2 which are RWPRI are also F, RC, and FT. Thus the search of the rank 2 geometries of the Suzuki group *Sz(q)* that are F, RC, FT and RWPRI is reduced to the search of all RWPRI configurations.

3.2 Some useful properties

In this section, we give some properties concerning $q-1$, α_q, and β_q.

Lemma 3.2.1 *Cataldi, 1603*
If $2^{2e+1}-1$ *is a prime number, then* $2e+1$ *is a prime number.*

The following lemma implies that if $q-1$ is a prime number then the Suzuki group *Sz(q)* does not have any subgroup isomorphic to a Suzuki group of lower order. Let us remark that the converse is not true.

Lemma 3.2.2 *If there exist* $s = 2^{2f+1}$ *and an integer* a *such that* $q = s^a$, *then* $q-1$ *is not a prime number.*

Proof: $s = 2^{2f+1}$ and $q = 2^{2e+1} = 2^{a\cdot(2f+1)}$ implies that $2e+1$ is not a prime number. Suppose $q-1$ is a prime number. Then lemma 3.2.1 says that $2e+1$ must be a prime number. Thus we have a contradiction. □

Lemma 3.2.3 *The numbers q - 1,* α_q *and* β_q *are pairwise coprime.*

Proof: $(q$ - 1, $\alpha_q) = p$ implies that p divides α_q- $(q$ - $1)$ and α_q+ $(q$ - $1)$. Thus p divides $2^{e+1}+1$ and $2^{e+1}+2$. This implies $p=1$.
By a similar argument, we see that $(q$ - 1, $\beta_q) = p$ implies that p divides both $2^{e+1}-2$ and $2^{e+1}-1$. Thus $p=1$.
At last, suppose $(\alpha_q, \beta_q) = p \neq 1$. Then p divides α_q- β_q which is a power of 2. Thus p is a even number dividing two odd numbers, a contradiction. □

Lemma 3.2.4 *Let* $s = 2^{2e+1}$ *and* $q = s^{2f+1}$. *Then* s^2+1 *divides* q^2+1.

Proof: $s^2+1 = 2^{4e+2}+1$ and $q^2+1 = 2^{(4e+2)(2f+1)}+1$. Thus we have

$$\frac{q^2+1}{s^2+1} = \sum_{i=1}^{2f+1} (-1)^{i+1} \cdot 2^{(4e+2)(2f+1-i)}$$

$\square$

Lemma 3.2.5 *Let $s = 2^{2e+1}$ and $q = s^{2f+1}$. Then, s^2+1 does not divide α_q nor β_q.*

Proof: The following argument is due to J.A. Thas.
Assume first that $s^2+1 \mid \alpha_q$. Then

$$2^{2(2e+1)}+1 \quad | \quad 2^{(2e+1)(2f+1)}+2^{2ef+e+f+1}+1$$

Suppose first that f is odd. Then

$$\begin{array}{lll} 2^{2(2e+1)}+1 & | & (2^{2(2e+1)f}+1-1)2^{2e+1}+2^{2ef+e+f+1}+1 \\ & | & -2^{2e+1}+2^{2ef+e+f+1}+1 \end{array}$$

Because f is odd, we can write $f = 2r+1$ with r an integer. Then

$$\begin{array}{lll} 2^{2(2e+1)}+1 & | & -2^{2e+1}+2^{4er+2e+2r+1+e+1}+1 \\ & | & -2^{2e+1}+2^{2(2e+1)r+3e+2}+1 \end{array}$$

Suppose that r is odd. Then

$$\begin{array}{lll} 2^{2(2e+1)}+1 & | & -2^{2e+1}+(2^{2(2e+1)r}+1-1)2^{3e+2}+1 \\ & | & -2^{2e+1}-2^{3e+2}+1 \\ & | & 2^{2e+1}+2^{3e+2}-1 \end{array}$$

But $2^{4e+2}+1 > 2^{2e+1}+2^{3e+2}-1$ thus we have a contradiction.
Suppose that r is even. Then $r = t+1$, with t odd and we have the following.

$$\begin{array}{lll} 2^{2(2e+1)}+1 & | & -2^{2e+1}+(2^{2(2e+1)t}+1-1)2^{7e+4}+1 \\ & | & -2^{2e+1}-2^{7e+4}+1 \\ & | & (2^{2(2e+1}+1-1)2^{3e+2}+2^{2e+1}+1 \\ & | & 2^{3e+2}-2^{2e+1}-1 \end{array}$$

But $2^{4e+2}+1 > 2^{3e+2}-2^{2e+1}-1$ thus we have a contradiction.
Hence we have proved that f cannot be odd. The same reasoning can be made for f even. And the same arguments can be used to show that $2^{2(2e+1)}+1$ cannot divide β_q. $\square$

Lemma 3.2.6 *Let $s = 2^{2e+1}$ and $q = s^{2f+1}$, with e and f some positive integers. Then α_q and β_q are not prime.*

Proof: Easy to see thanks to lemmas 3.2.4, 3.2.5. $\square$

Structure	**Order**	**Index**	**Description**
$(q{\cdot}q){:}(q\text{-}1)$	$q^2 \cdot (q-1)$	q^2+1	Normalizer of a 2-Sylow, stabilizer of a point of the ovoid.
$D_{2(q-1)}$	$2 \cdot (q$ - $1)$	$\frac{(q^2+1)\cdot q^2}{2}$	Stabilizer of a pair of points of the ovoid.
α_q:4	$\alpha_q \cdot 4$	$\frac{q^2(q-1)}{4\beta_q}$	Normalizer of a Z_{α_q}.
β_q:4	$\beta_q \cdot 4$	$\frac{q^2(q-1)}{4\alpha_q}$	Normalizer of a Z_{β_q}.
$Sz(2^{2f+1})$ with $2f+1 \mid_M 2e+1$	$(s^2+1)\cdot s^2 \cdot (s-1)$		

Table 3.1: The maximal subgroups of Sz(q)

Lemma 3.2.7 *Fermat, 1640*
If p is a prime number then p divides $2^p - 2$.

3.3 The maximal subgroups of *Sz(q)*

Table 3.1 gives a description of the conjugacy classes of maximal subgroups of *Sz(q)*. Those results have been obtained by Suzuki [40]. It is from these subgroups that we start to classify all RWPRI configurations in the subgroup lattice of *Sz(q)*. For a Suzuki simple group *Sz(q)*, with $q = 2^{2e+1}$, if $2e+1$ is a prime number, there are no proper subgroups isomorphic to a Suzuki simple group. This implies that if $2e+1$ is a prime number, the group *Sz(q)* has only four conjugacy classes of maximal subgroups. Some Suzuki groups may have subgroups of Suzuki type which are not isomorphic. For example, *Sz(*32768*)* has maximal subgroups that are respectively isomorphic to *Sz(*8*)* and *Sz(*32*)*. This means we don't have five classes of maximal subgroups there but six. In fact, the number of non-isomorphic classes of *Sz(s)* is equal to the number of prime divisors of $2e+1$. Dividing $2e+1$ by every of the latter primes gives us a number $2f+1$ for which there exists a *Sz(*2^{2f+1}*)* which is a maximal subgroup of *Sz(q)*.

3.4 The maximal subgroups of the maximal subgroups of Sz(q)

Lemma 3.4.1 *The maximal subgroups of (q$\cdot$q) : (q-1) have one of the following structures.*

- *(q$\cdot$q):n where n is a maximal divisor of q - 1.*
- *q:(q - 1)*

Proof: The subgroup G_0 is the stabilizer of a point. Let us take U as the fixed point. Then, a complete description of this group is given in [33]. Lüneburg also shows that the stabilizer of a circle is of order $q(q-1)$. As we defined it in chapter 1, a circle is obtained by intersecting the ovoid with a non-tangent plane. If we take the plane Π defined by a cartesian equation $x = 0$, we see that the $\tau(0, b)$ and the $\eta(k)$ stabilize this plane for $b \in GF(q)$ and $k \in GF(q)^* = GF(q) - \{0\}$. Looking at the subgroup generated by these collineations, we have a subgroup of order $q(q-1)$ which is the stabilizer of a circle. This group has a normal subgroup formed by the $\tau(0, b)$, of order q. And its structure is $q : (q-1)$. We denote it by G_C. We still have to show that it is a maximal subgroup of G_0. Thanks to lemma 1.2.2, every element of G_0 can be written in a unique form as $\eta(k)\tau(a, b)$ with $a, b \in GF(q)$ and $k \in GF(q)^*$. Thus we see that the only elements we could add are of the form $\tau(a, 0)$ with $a \in GF(q)$, $a \neq 0$. If we add any of these elements to G_C, the $\eta(k)$'s act so that every $\tau(a, 0)$ is generated, and thus G_0 is generated. We can then conclude that G_C is maximal in G_0.

Taking all $\tau(a, b)$, we obtain a subgroup of structure $q{\cdot}q$ which is normal in G_0. The subgroup formed by the $\eta(k)$'s, with $k \in GF(q)^*$ is a cyclic group of order $q-1$. Its maximal subgroups are the subgroups of order n, where n is a maximal divisor of $q-1$. If we take one generator of such a subgroup, namely $\eta(k)$ and we look at the subgroup generated by the $\tau(a, b)$ and this $\eta(k)$, we obtain a subgroup of order $q^2 \cdot n$. Again, $q{\cdot}q$ is normal in it and thus this subgroup has structure *(q$\cdot$q):n*. The only elements that are missing use $\eta(l)$ with l not appearing in the multiplicative group generated by k. This multiplicative group is a maximal subgroup of $GF(q)^*$. We can conclude that *(q$\cdot$q):n* is maximal in G_0.

The group generated by all the $\tau(a, 0)$ and the $\eta(k)$, with $a \in GF(q)$ and $k \in GF(q)^*$ is G_0. Thus no other maximal subgroups can be found. □

Lemma 3.4.2 *The maximal subgroups of $n : 4$ have the following structures.*

- *m:4 where m is a maximal divisor of n.*
- *n:2*

Moreover, if n is odd, every maximal divisor m of n gives only one conjugacy class of subgroups isomorphic to $m : 4$.

Proof: Obviously, $n : 4$ has a maximal subgroup whose structure is $n : 2$. In $n : 4$, there exists a normal subgroup isomorphic to m for each divisor m of n. Extending this cyclic group by 4, we obtain a group of order $4m$ which has structure m:4 because $(m, 4) = 1$. If m is a maximal divisor of n, then $[n{:}4 : m{:}4]$ is a prime number, hence m:4 is a maximal subgroup of n:4.
There remains to show that, if n is odd, for a given maximal divisor m of n, all subgroups $m : 4$ are conjugate in $n : 4$. Suppose we have two subgroups A and B isomorphic to $m : 4$. Then, $A =< Z_m, Z_4^A >$ and $B =< Z_m, Z_4^B >$ for some subgroups Z_4^A and Z_4^B of $n : 4$. If n is odd, the subgroups Z_4^A and Z_4^B are 2-Sylow subgroups. Hence they are conjugate in $n : 4$. Thus there exists an element $g \in n : 4$ such that $(Z_4^A)^g = Z_4^B$. Since Z_m is a normal subgroup of $n : 4$, it is then obvious that $A^g = B$. □

Lemma 3.4.3 *The maximal subgroups of $D_{2(q-1)}$ have the following structure.*

- *the cyclic group of order $q - 1$*
- *D_{2n} where n is a maximal divisor of $q - 1$. If $q - 1$ is a prime number then D_2 is the cyclic group of order 2.*

Moreover, if $q-1$ is odd, every maximal divisor n of $q-1$ gives only one conjugacy class of subgroups isomorphic to D_{2n}.

Proof: Straightforward □

As to the maximal subgroups that are isomorphic to some Suzuki simple group, we refer to table 3.1.

3.5 Rank 2 Rwpri geometries

Let G be a group. Let G_0 and G_1 be subgroups of G and let $G_{01} = G_0 \cap G_1$. The Rwpri condition in rank 2 asks that either G_0 or G_1 must be a maximal subgroup of G and that G_{01} must be a maximal subgroup of G_0 and G_1. We use the following method to determine all the Rwpri geometries of the Suzuki group *Sz(q)*. We choose a subgroup G_0 and we impose that it must be a maximal subgroup of *Sz(q)*. Then we determine all the G_{01} possible groups. They are the maximal subgroups of G_0. Finally, we look at all the G_1 candidates which must contain G_{01} as a maximal subgroup and which must be different from G_0.

Lemma 3.5.1 *Let $G \cong$ Sz(q). There are no* Rwpri *geometries of rank* > 1 *with* $G_0 \cong$ *(q·q) : (q - 1).*

Proof: We start from the maximal subgroups of G_0. They have been determined in lemma 3.4.1. We see that the only subgroups of *Sz(q)* that contain one of the G_{01} candidates are thus isomorphic to G_0. Because the normalizer of each G_{01} candidate must be a subgroup of G_0, we obtain that G_1 must be equal to G_0 which is a contradiction with the fact that G_{01} is their intersection. □

Lemma 3.5.2 *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong$ Sz(q)*
If $2e + 1$ and $q - 1$ are prime numbers, the Rwpri *rank 2 geometries of G with $G_0 \cong D_{2(q-1)}$ are isomorphic to one of the geometries appearing in table 3.2.*
If $2e + 1$ is prime and $q - 1$ is not prime, the only Rwpri *rank 2 geometry of G with $G_0 \cong D_{2(q-1)}$ is obtained with $G_1 \cong$ q : (q-1).*
If $2e + 1$ is not prime, the Rwpri *rank 2 geometries of G with $G_0 \cong D_{2(q-1)}$ are isomorphic to one of the geometries appearing in table 3.3.*

Proof: By lemma 3.4.3, G_{01} can be isomorphic to either $q - 1$ of D_{2n} where n is a maximal divisor of $q - 1$.

Suppose first that $G_{01} = q - 1$.

The G_1 candidates are neither subgroups of α_q:4, nor of β_q:4 because of lemma 3.2.3. The normalizer of a $q - 1$ is G_0. Thus two dihedral groups $\mathrm{D}_{2(q-1)}$ never intersect on a cyclic group of order $q - 1$ in *Sz(q)*, and G_1 cannot be a $D_{2(q-1)}$.

G_1	G_{01}	$\#\Gamma$
$q:(q-1)$	$q-1$	*1*
$D_{2(q-1)}$	2	$\frac{(q-2)(q+4)}{4(2e+1)}+1$
D_{2n} *where* n *is a prime divisor of* α_q *or* β_q	2	$\frac{(q-2)(q+2)}{4(2e+1)}+1$
2^2	2	$\frac{q-2}{2(2e+1)}$
4	2	$\frac{q-2}{2(2e+1)}+1$

Table 3.2: $G_0 = D_{2(q-1)}$, and $2e+1$ and $q-1$ are prime numbers

G_1	G_{01}	$\#\Gamma$
$q:(q-1)$	$q-1$	*1*
Sz(s) iff s-1 is a maximal divisor of q-1	$D_{2(s-1)}$	*1*

Table 3.3: $G_0 = D_{2(q-1)}$ and $2e+1$ is not prime

So the only candidates must be subgroups of *(q:q):(q-1)* or *Sz(s)* with $q = s^a$ for some integer a.
Suppose $G_1 <$ *(q:q):(q-1)*. Then, in order to have G_{01} maximal in G_1 the only candidate G_1 is $q:(q-1)$. This gives rise to a RWPRI configuration. If we fix G_1 and G_{01} (this can be done because all subgroups isomorphic to G_{01} are conjugate in G_1), there is only one $D_{2(q-1)}$ intersecting G_1 in G_{01}. Thus we have only one such configuration up to isomorphism.
Suppose $G_1 <$ *Sz(s)*. As *Sz(s)* exists, it implies that $q-1$ is not a prime number. No maximal subgroup of *Sz(s)* contains a cyclic group of order $q-1$. Thus G_1 must be *Sz(s)*. But $q-1$ is not a maximal subgroup of *Sz(s)*.

Now suppose that $G_{01} = D_{2n}$ where n is a maximal divisor of $q-1$.

Let us first examine the case where $q-1$ is a prime number. Then G_{01} is a cyclic group of order 2. This gives rise to the following candidates for G_1.

- $D_{2(q-1)}$
- D_{2m} where m is a prime divisor of α_q or β_q.
- 2^2

- 4

All of these subgroups can be taken as G_1 to form a RWPRI configuration.
Suppose $G_1 = D_{2(q-1)}$. We can fix G_0 and G_{01} because all subgroups isomorphic to G_{01} are conjugate in G_0. We know that there are $\frac{q^2\cdot(q^2+1)}{2}$ subgroups isomorphic to G_0 in *Sz(q)*. We also know that G_0 contains $q-1$ subgroups of order 2. And the number of involutions in *Sz(q)* is equal to $(q^2+1)(q-1)$. Thus we can conclude that there are $\frac{q^2}{2}$ subgroups isomorphic to G_0 intersecting in the same G_{01}. If we want to fix the types of our geometries, G_{01} cannot transform one of these geometries into another. If we do not fix the types, then we have to count how many of our geometries are self-dual by conjugacy. Suppose we have a geometry $\Gamma(Sz(q); D_{2(q-1)} = G_0, D_{2(q-1)} = G_1)$. We know that the normalizer of G_{01} is a group of order q^2 isomorphic to $q{\cdot}q$ By looking at the action of $H = q{\cdot}q/G_{01}$ on the $\frac{q^2}{2}$ subgroups isomorphic to $D_{2(q-1)}$ containing G_{01}, we see than it is transitive and hence striclty transitive because its order is equal to its degree. Because H contains $q-1$ involutions, there are $q-1$ geometries that are self-dual, and the $\frac{q^2-2q}{2}$ elements of order 4 spread the $\frac{q^2}{2}$ subgroups into squares. Every square appears two times, that is one time in each way. And no two disjoint squares can have a common vertex. This means that through every of the $\frac{q^2}{2}$ points, there are $\frac{q^2-2q}{4}$ disjoint squares that fuse $\frac{q^2-2q}{2}$ geometries inside sets of 2 geometries. So finally, we have $q-1+\frac{q^2-2q}{4} = 1+\frac{(q-2)(q+4)}{4}$ non-conjugate geometries in this case. In $Aut(Sz(q))$, the stabilizer of G_0 and G_{01} is a cyclic group of order $2(2e+1)$. Its normalizer is two times bigger. The normalizer of G_1 is $G_1 : (2e+1)$. And each subgroup isomorphic to $G_1 : (2e+1)$ contains $q-1$ subgroups isomorphic to $Z_{2(2e+1)}$. This implies that there are two subgroups isomorphic to $G_1 : (2e+1)$ containing the same $Z_{2(2e+1)}$. Thus one geometry is fixed under the action of $Aut(Sz(q))$ and all the others are fused inside sets of $2e+1$. That is, we have $1+\frac{(q-2)(q+4)}{4(2e+1)}$ non-isomorphic RWPRI configurations and we observe that this number is indeed an integer thanks to lemma 3.2.7.

Suppose $G_1 = D_{2m}$ where m is a prime divisor of α_q or β_q. We can decide to fix it without loss of generality. All its subgroups isomorphic to G_{01} are conjugate in it. This implies that we can also fix G_{01}. The normalizer of G_1 in G is $m{:}4$. So the stabilizer of G_1 and G_{01} is a cyclic group of order 4. And there are $\frac{q^2}{2}$ subgroups isomorphic to G_0 that intersect G_1 in G_{01}. The cyclic group of order 4 fuses them inside sets of two. So there are $\frac{q^2}{4}$ non-conjugate geometries. The subgroup which stabilizes G_0 and G_{01} in $Aut(G)$ is isomorphic to $Z_{2(2e+1)}$. The subgroup stabilizing G_1 in $Aut(G)$ is isomorphic to $G_1 : (2e+1)$. Again, a simple counting argument permits us to conclude that there is only one subgroup isomorphic to G_1 which is fixed under the action of $Z_{2(2e+1)}$. Thus, from the $\frac{q^2}{4}$ non-conjugate geometries, one is fixed under the action of $Aut(G)$ and all other are fused inside sets of $2e+1$ geometries. This means we have $1+\frac{(q-2)(q+2)}{4(2e+1)}$ non-isomorphic RWPRI configurations of that kind. And, again, thanks to lemma 3.2.7, we observe that this number is indeed an integer.

Suppose $G_1 = 2^2$. Then G_1 contains three subgroups of order 2. Also, G_1 is contained in the stabilizer of some point. In each point-stabilizer, there are $q-1$ Z_2. We must combine two of them to obtain subgroups isomorphic to 2^2. We first choose one Z_2. Then, the second one can be chosen in $\frac{q-2}{2}$ ways. But there we have taken each possibility 3 times. So the number of 2^2 is $\frac{(q^2+1)(q-1)(q-2)}{6}$. From this, we can count the number of 2^2 containing the same Z_2. There are $\frac{q-2}{2}$ such subgroups. Thus, by fixing G_0 and G_{01}, we have $\frac{q-2}{2}$ possible G_1. These give $\frac{q-2}{2}$ non-conjugate geometries. The normalizer of G_1 in $Aut(G)$ is $G_1 : (2e+1)$ and this subgroup does not contain the stabilizer of G_0 and G_{01} in $Aut(G)$. Thus no configuration of that kind is fixed under the action of $Aut(G)$. This means that all of them are fused in sets of $2e+1$ geometries. Hence we have $\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations like that. Let us observe that in the smallest case (i.e. for $q = 8$), there is only one geometry of that kind up to isomorphism.

Suppose $G_1 = 4$. Let us count the number of cyclic groups of order 4 contained in a *Sz(q)*. Each of them is contained in the stabilizer of a point (there are q^2+1 such stabilizers). In each stabilizer, the $\tau(a, b)$ with $a \neq 0$ are elements of order 4. There are $(q-1)q$ such elements. In a cyclic group of order 4, there are 2 elements of order 4. Thus the number of cyclic groups of order 4 in *Sz(q)* equals $(q^2+1)(q-1)\frac{q}{2}$. We already know that the number of Z_2 in *Sz(q)* is $(q^2+1)(q-1)$. And it is well known that every Z_4 contains only one Z_2. Hence the number of Z_4 containing the same Z_2 is $\frac{q}{2}$. Now we fix G_0 and we can also fix G_{01} because all the subgroups that are isomorphic to G_{01} are conjugate in G_0. The stabilizer of G_0 and G_{01} is G_{01}. Thus it fixes G_1 also. This means we have $\frac{q}{2}$ non-conjugate geometries. The normalizer of G_1 in $Aut(G)$ being a subgroup isomorphic to $4 : (2e+1)$, it contains the stabilizer of G_0 and G_{01} in $Aut(G)$. This means that there is one configuration which is fixed under the action of $Aut(G)$ and all the others are fused under $Aut(G)$ inside sets of $2e+1$ geometries. Thus we have $1 + \frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations of that kind.

G_1	G_{01}	$\#\Gamma$
α_q: 4	4	$\frac{q-2)}{2(2e+1)}$
$n : 4$ *where* n *is a prime divisor of* β_q	4	$\frac{q-2}{2(2e+1)} + 1$
4×2	4	$\frac{q-2}{2(2e+1)}$

Table 3.4: The case $G_0 = \alpha_q$: 4 and α_q prime.

G_1	G_{01}	$\#\Gamma$
Sz(s) iff α_s *or* β_s *is a maximal divisor of* α_q	$\alpha_s : 4$ *or* $\beta_s : 4$	*1*

Table 3.5: The case $G_0 = \alpha_q$: 4 and α_q not prime.

Suppose now that $q - 1$ is not a prime number. Then G_{01} is a dihedral group D_{2n}. This means that the G_1 candidates will neither be subgroups of α_q:4, nor of β_q:4. Here we have two candidates for G_1. They are respectively $D_{2(q-1)}$ and *Sz(s)* if $s - 1 = n$. The case $G_1 = D_{2(q-1)}$ is not possible. This is due to the fact that $N_{Sz(q)}(D_{2n}) = D_{2n}$. While fixing G_0, we can also fix G_{01} because all the subgroups isomorphic to G_{01} in G_0 are conjugate. It is easy to see that there is only one subgroup isomorphic to *Sz(s)* which contains G_{01}. So the G_1 is uniquely determined. This gives us only one geometry up to conjugacy and thus up to isomorphism. □

An example of a configuration with $G_1 = Sz(s)$ can be obtained in *Sz(*512*)* by taking as G_0 (resp. G_1, G_{01}) a subgroup isomorphic to $D_{2(511)}$ (resp. *Sz(*8*)*, D_{14}).

Lemma 3.5.3 *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong Sz(q)$*
If α_q is a prime number, the RWPRI *rank 2 geometries of G with $G_0 \cong \alpha_q$: 4 are isomorphic to one of the geometries appearing in table 3.4.*
If α_q is not a prime number, the RWPRI *rank 2 geometries of G with $G_0 \cong \alpha_q$: 4 are isomorphic to one of the geometries appearing in table 3.5.*

Proof: By lemma 3.4.2, G_{01} can be isomorphic to either α_q: 2 or $n : 4$ where n is a maximal divisor of α_q.

Suppose first that $G_{01} = n : 4$.

Let us first consider the case where $n = 1$ (i.e. α_q is a prime number). In this case, several candidates appear for G_1.

- 4×2
- $m{:}4$ where $m \mid \beta_q$, and m is a prime number.
- $\alpha_q{:}4$

All these cases give rise to RWPRI candidates.
Suppose $G_1 = 4 \times 2$. Let us count the number of subgroups isomorphic to 4×2 in *Sz(q)*. Each of them is contained in the stabilizer of a point. In each point-stabilizer, there are $(q-1)\frac{q}{2}$ cyclic groups of order 4, as we saw before. The 4×2 are constructed by adding to these Z_4 an element of order 2. This element must be different from the one in the Z_4. Thus we have $q-2$ such possible elements. But a 4×2 contains 3 subgroups of order 2. This implies that there are $\frac{q-2}{2}$ subgroups isomorphic to 4×2 which contain the same Z_4. And every 4×2 contains 2 subgroups of order 4. Thus the number of 4×2 is $(q^2+1)(q-1)\frac{q}{2}\frac{q-2}{2}\frac{1}{2} = \frac{(q^2+1)q(q-1)(q-2)}{8}$. Now let us count the number of non-conjugate and non-isomorphic geometries. We can fix G_0. All the subgroups isomorphic to Z_4 are conjugate in G_0. This implies that we can also fix G_{01}. The stabilizer of G_0 and G_{01} being G_{01}, we have $\frac{q-2}{2}$ non-conjugate geometries. The $(2e+1)$ classes of subgroups isomorphic to 4×2 in G are fused under the action of $Aut(G)$. This means that the normalizer of a subgroup isomorphic to 4×2 will not contain elements of order $(2e+1)$. The stabilizer of G_0 and G_{01} in $Aut(G)$ is a subgroup isomorphic to $4 : (2e+1)$. Thus, there is no configuration which is fixed under the action of $Aut(G)$. Then, all the non-conjugate configurations are fused under the action of $Aut(G)$ inside sets of $2e+1$ geometries. Hence we have $\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations like that. Let us observe that in the smallest case (i.e. for $q = 8$), there is only one such geometry up to isomorphism.

Suppose $G_1 = m{:}4$. The subgroup G_1 has its normalizer in $\beta_q{:}4$. Thus it is self-normalizing. This gives us the number of subgroups isomorphic to $m{:}4$ in *Sz(q)*. There are $\frac{(q^2+1)q^2(q-1)}{4m}$ such subgroups. Each of them contains m cyclic subgroups of order 4. And we already know that there are $\frac{(q^2+1)q(q-1)}{2}$ cyclic groups of order 4 in *Sz(q)*. Thus every Z_4 is contained in $\frac{q}{2}$ subgroups $m{:}4$. We can fix G_0. All the subgroups isomorphic to Z_4 are conjugate in G_0. Thus we can also fix G_{01}. The stabilizer of G_0 and G_{01} is G_{01}. This implies that there are $\frac{q}{2}$ non-conjugate geometries here. The stabilizer of G_1 and G_{01} in $Aut(G)$ being isomorphic to $4 : (2e+1)$, there is only one configuration which is fixed under the action of $Aut(G)$ and all the others are fused inside sets of $2e+1$ geometries. Thus we have $1 + \frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations in that case.

Suppose $G_1 = \alpha_q{:}4$. Then there are $\frac{q}{2}$ subgroups isomorphic to G_1 that contain the same Z_4. One of them is G_0 so we have here $\frac{q}{2} - 1$ non-conjugate geometries if we decide to fix the types. It is easy to see that $N_G(Z_4)$ is a group which is of order $2q$ and which is isomorphic to $q : 2$. By looking at the action of $(q:2)/4$ on the set of $\frac{q}{2}$ subgroups isomorphic to $\alpha_q{:}4$, we see that this group is transitive and hence strictly transitive. It is also easy to see that all its elements (except the identity) are involutions. This means that all the $\frac{q}{2} - 1$ geometries we found are self-dual. Thus there are $\frac{q-2}{2}$ non-conjugate geometries. The stabilizer of G_0 and G_{01} in $Aut(G)$ is isomorphic to $4 : (2e+1)$. Thus there is no geometry fixed under the action of $Aut(G)$ and all of them are fused inside sets of $2e+1$ elements. This means we have $\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations of that kind. Let us observe that in the smallest case (i.e. for $q = 8$), there is only one geometry up to isomorphism.

Now suppose $n \neq 1$, which means that α_q is not a prime number. Then G_1 is not a subgroup of $\beta_q{:}4$ (resp. $D_{2(q-1)}$) because (α_q, β_q) (resp. $(\alpha_q, q-1)) = 1$. Two candidates arise for G_1. They are $\alpha_q{:}4$ and *Sz(s)* if and only if α_s or β_s is a maximal divisor of α_q.
The normalizer $N_{Sz(q)}(n{:}4)$ is $n{:}4$. Thus the case $G_1 = \alpha_q{:}4$ leads to a contradiction. We then have just one candidate left, that is the one with $G_1 =$ *Sz(s)*. We can fix G_0 and G_{01} because G_0 conjugates all its subgroups isomorphic to G_{01}, the stabilizer being G_{01}. And there is only one *Sz(s)* containing G_{01}. Thus there is only one such geometry up to conjugacy, and hence up to isomorphism.

Suppose that $G_{01} = \alpha_q{:}2$.

This implies that G_1 must be a subgroup of *Sz(s)*. So $q-1$ cannot be a prime number. The order of G_{01} must divide the order of G_1.
Suppose G_1 is a subgroup of *(s:s):(s-1)* or $D_{2(s-1)}$. Then α_q divides $s-1$ which is a contradiction with the fact that $(q-1, \alpha_q) = 1$ and $s-1 \mid q-1$.
Suppose G_1 is a subgroup of $\alpha_s{:}4$ (resp. $\beta_s{:}4$). Then the order of G_1 is smaller than the order of G_{01} which is also a contradiction. Finally, suppose $G_1 =$ *Sz(s)*. Then G_{01} must be a maximal subgroup of *Sz(s)*. This yields $s-1 = \alpha_q$ and $s-1 \mid q-1$, a contradiction. $\square$

G_1	G_{01}	$\#\Gamma$
β_q: 4	4	$\frac{q-2)}{2(2e+1)}$
$n : 4$ *where n is a prime divisor of* β_q	4	$\frac{q-2}{2(2e+1)}+1$
4×2	4	$\frac{q-2}{2(2e+1)}$

Table 3.6: The case $G_0 = \beta_q$: 4 and β_q prime.

G_1	G_{01}	$\#\Gamma$
Sz(s) iff α_s or β_s is a maximal divisor of β_q	$\alpha_s : 4$ *or* $\beta_s : 4$	*1*

Table 3.7: The case $G_0 = \beta_q$: 4 and β_q not prime.

Lemma 3.5.4 *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong$ Sz(q) If β_q is a prime number, the* Rwpri *rank 2 geometries of G with $G_0 \cong \beta_q$: 4 are isomorphic to one of the geometries appearing in table 3.6.*
If β_q is not a prime number, the only Rwpri *rank 2 geometries of G with $G_0 \cong \beta_q$: 4 are isomorphic to one of the geometries appearing in table 3.7.*

Proof: Same as proof of lemma 3.5.3 □

At this stage, we have determined the Rwpri configurations for the rank 2 geometries of the Suzuki groups *Sz(q)* where $2e + 1$ is a prime number. If $2e + 1$ is not a prime number, *Sz(q)* may have other maximal subgroups which are isomorphic to *Sz(s)*. Before starting the discussion about this case, we demonstrate a few lemmas that are used in the lemma giving the Rwpri rank 2 geometries of *Sz(q)* with $G_0 \cong$ *Sz(s)*.

Lemma 3.5.5 *For every subgroup isomorphic to (s.s):(s-1), there is only one subgroup isomorphic to Sz(s) containing it.*

Proof: Let P be a subgroup isomorphic to *(s.s):(s-1)*. All subgroups isomorphic to *Sz(s)* are conjugate in *Sz(q)*. We only have to show that $N_{Sz(q)}(P) \leq$ *Sz(s)* in order to prove that there exists only one *Sz(s)* containing P. *Sz(s)* has an orbit of length $s^2 + 1$. In this orbit, P fixes a point x and exchanges all of the other s^2

points. We know that $N_{Sz(q)}(P)$ fixes x. So, it is a subgroup of the stabilizer G_x of x which has structure *(q·q):(q-1)*. In G_x, there is a cyclic subgroup Z_{q-1} which has exactly two fixed points. And so does $Z_{s-1} \leq Z_{q-1}$. Thus the second fixed point is in the orbit of s^2 points. We also know that P has one orbit of length 1, one of length s^2 and all others of length $\geq s^2$. It can be easily seen that the other orbits have length $s^2(s-1)$.
Since $s \neq 2$, the normalizer $N_{Sz(q)}(P)$ must stabilize the orbit of 1 point and the orbit of s^2 points. Thus it must be a subgroup of *Sz(s)*. □

Lemma 3.5.6 *For every subgroup isomorphic to $D_{2(s-1)}$, there is only one subgroup isomorphic to Sz(s) containing it.*

Proof: The normalizer $N_{Sz(q)}(D_{2(s-1)})$ must be a subgroup of $D_{2(q-1)}$. Because $q-1$ and $s-1$ are odd numbers, $D_{2(s-1)}$ is self-normalizing in $D_{2(q-1)}$. □

Lemma 3.5.7 *[40] Let Π be the partition consisting of four classes of conjugate subgroups of Sz(q):*

- *the Sylow 2-subgroups of Sz(q),*
- *one class of cyclic groups of order q-1,*
- *one class of cyclic groups of order q+r+1,*
- *one class of cyclic groups of order q-r+1.*

If W is a subgroup of Sz(q), then there exists $X \in \Pi$ with $W \subseteq N_{Sz(q)}(X)$ or W is isomorphic to Sz(s) with $s \leq q$.

Lemma 3.5.8 *For every subgroup isomorphic to α_s:4, there is only one subgroup isomorphic to Sz(s) that contains it.*

Proof: The number of *Sz(s)*'s in *Sz(q)* is equal to [*Sz(q)* : *Sz(s)*] because $N_{Sz(q)}(Sz(s)) = Sz(s)$. By lemma 3.5.7, the normalizer $N_{Sz(q)}(\alpha_s{:}4)$ is either *Sz(s)* or a subgroup of α_q:4 or β_q:4. Suppose it is *Sz(s)*. Then *Sz(s)* is not simple, a contradiction. So it must be a subgroup of either α_q:4 or β_q:4. Anyway, α_s:4 is self-normalizing in the $Sz(s)$ that contains it. Thus $N_{Sz(q)}(\alpha_s{:}4) = \alpha_s{:}4$ and the

G_1	G_{01}	$\#\Gamma$
(q:s):(s-1)	*(s:s):(s-1)*	*1*
Sz(s') when it is a maximal subgroup and s' $\neq$ s	*Sz(t)*	*1*
D_{2n} *where s-1* $\vert_M$ *n* $\vert_P$ *q-1*	$D_{2(s-1)}$	*1*
n:4 where $\alpha_s \vert_M n \vert_P \alpha_q$ *or* β_q	α_s:4	*1*
n:4 where $\beta_s \vert_M n \vert_P \alpha_q$ *or* β_q	β_s:4	*1*

Table 3.8: The case G_0 = Sz(s).

number of α_s:4's in *Sz(q)* is equal to $[Sz(q) : \alpha_s{:}4]$. Every *Sz(s)* contains exactly $[Sz(s) : \alpha_s{:}4]$ subgroups isomorphic to α_s:4. Thus the number of *Sz(s)* containing the same α_s:4 is equal to 1. □

Lemma 3.5.9 *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong$ Sz(q) Let $G_0 \cong$ Sz(s) be a maximal subgroup of Sz(q) and suppose G_1 is not one of $D_{2(q-1)}$, α_q: 4, or β_q: 4 (these cases have been analysed in lemmas 3.5.2,3.5.3 and 3.5.4). Then the* RWPRI *rank 2 geometries of G with $G_0 \cong$ Sz(s) are isomorphic to one of the geometries appearing in table 3.8.*

Proof: Recall that the maximal subgroups of *Sz(s)* have one of the following structures.

- *(s:s):(s-1)*
- $D_{2(s-1)}$
- α_s:4
- β_s:4
- *Sz(t)* if and only if there exists a natural number b such that $s = t^b$.

Suppose G_{01} = *(s:s):(s-1)*.

Since G_{01} is the stabilizer of a point of the subovoid, G_1 must be either a subgroup of *Sz(s)* or *(q:q):(q-1)*. Lemma 3.5.5 implies that G_1 cannot be a subgroup

isomorphic to *Sz(s)*. Thus we have to look at $(q\hat{:}q):(q\text{-}1)$ and its subgroups to find some G_1 candidate.
All elements of G_{01} are of the form $\eta(k)\tau(a,b)$ where a, $b \in GF(s)$ and $k \in GF(s)^*$. The subgroup $< G_{01}, \tau(0,c) >$ with $c \in GF(q) - GF(s)$ has order $q \cdot s \cdot (s-1)$ because it gives elements of the form $\eta(z)\tau(x,y)$ with $x \in GF(s)$, $y \in GF(q)$ and $z \in GF(s)^*$. It contains G_{01} as a maximal subgroup because $GF(s)$ is a maximal subfield of $GF(q)$. It is easy to see that it has structure $(q\hat{:}s):(s\text{-}1)$.
The subgroup $< G_{01}, \tau(c,0) >$ with $c \in GF(q) - GF(s)$ has order $q^2(s-1)$ because it generates all elements of the form $\eta(z)\tau(x,y)$ with x, $y \in GF(q)$, $z \in GF(s)^*$. It contains $(q\hat{:}s):(s\text{-}1)$ and thus G_{01} is not maximal in it.
The subgroup $< G_{01}, \eta(l) >$ with $l \in GF(q) - GF(s)$ is the group $(q\hat{:}q):(q\text{-}1)$ which does not contain G_{01} as a maximal subgroup.
We can fix G_1 and G_{01} because G_1 conjugates all its subgroups isomorphic to G_{01}. Since G_{01} is self-normalizing as shown in the previous lemma, the stabilizer of G_1 and G_{01} is G_{01}. And there is only one subgroup isomorphic to *Sz(s)* which contains G_{01}. This means we have here only one geometry up to conjugacy, and hence up to isomorphism.

Suppose $G_{01} = D_{2(s-1)}$.

Since $s-1$ divides $q-1$ the only G_1 candidates are *Sz(s)* and D_{2n} where n is a divisor of $q-1$ and $s-1$ is a maximal divisor of n. In fact, we can suppose that n is a proper divisor of $q-1$ because the case $n = q-1$ has been dealt with while determining the configurations with $G_0 = D_{2(q-1)}$. Lemma 3.5.6 shows that two subgroups isomorphic to *Sz(s)* never intersect in a $D_{2(s-1)}$. Thus the only RWPRI candidates here are obtained by taking *Sz(s)* as G_0, and D_{2n} as G_1 where n is a proper divisor of $q-1$ and $s-1$ is a maximal divisor of n. It is easy to see that there is only one such geometry up to conjugacy and hence up to isomorphism, for every such n.

Suppose $G_{01} = \alpha_s : 4$.

This implies that G_1 will certainly not be a subgroup of $D_{2(q-1)}$ or $(q\hat{:}q):(q\text{-}1)$. The candidates are then another *Sz(s)* or n:4 where n is a proper divisor either of α_q or β_q and α_s is a maximal divisor of n. Lemma 3.5.8 implies that G_1 must be a n:4. We can fix G_1 because all subgroups isomorphic to G_1 in *Sz(q)* are conjugate. We can also fix G_{01} because all the subgroups isomorphic to G_{01} are conjugate in G_1. The stabilizer of G_1 and G_{01} is then G_{01}. And there is only one

Sz(s) containing G_{01}. Thus there is only one such geometry up to conjugacy and hence up to isomorphism.

Suppose $G_{01} = \beta_s : 4$.

Almost the same discussion as in the case $G_{01} = \beta_s : 4$ leads us to the conclusion that G_1 must be n:4 where n is a proper divisor either of α_q or of β_q and β_s is a maximal divisor of n, and that there is only one such geometry up to conjugacy and hence up to isomorphism.

Finally, suppose $G_{01} =$ *Sz(t)*.

The only candidates are subgroups isomorphic to *Sz(s')* for some s' thanks to the order of *Sz(t)*. The case $s = s'$ leads to a contradiction because, as *Sz(q)* is a simple group, and as the only subgroup between *Sz(q)* and *Sz(t)* is *Sz(s)* which is also simple, the normalizer of *Sz(t)* in *Sz(q)* is *Sz(t)*. Thus we have here $G_1 =$ *Sz(s')* with $s \neq s'$ that gives a geometry. We can fix G_0 and G_{01} here and there is only one subgroup isomorphic to G_1 that contains G_{01}. Thus there is only one such geometry up to conjugacy and hence up to isomorphism. □

3.6 Classification Theorem

The results of the preceding section provide a complete classification theorem giving all the firm, residually connected, flag-transitive and residually weakly primitive geometries of rank 2 for a Suzuki group *Sz(q)* up to isomorphism. The symbol $\#\Gamma$ in tables 3.9, 3.10 and 3.11 is the number of non-isomorphic geometries that arise from a RWPRI configuration. The last column of the tables tells whether or not the corresponding geometry satisfies $(2T)_1$. This condition is so easy to test in rank 2 that we do not give details about the results obtained.

Theorem 3.6.1 *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let Sz(q) be the Suzuki simple group over GF(q).*
If the numbers $2e+1$ and $q-1$ are prime, every F, RC, FT, RWPRI *rank 2 geometries of Sz(q) is isomorphic to one of the geometries appearing in table 3.9.*
If $2e+1$ is prime and $q-1$ is not prime, every F, RC, FT, RWPRI *rank 2 geometries of Sz(q) is isomorphic to one of the geometries appearing in table 3.10.*
If $2e+1$ is not a prime number, every F, RC, FT, RWPRI *rank 2 geometries of*

G_0	G_1	G_{01}	$\#\Gamma$	$(2T)_1$
$D_{2(q-1)}$	$D_{2(q-1)}$	2	$\frac{(q-2)(q+4)}{4(2e+1)}+1$	NO
	D_{2n} where n is a prime divisor of α_q or β_q	2	$\frac{(q-2)(q+2)}{4(2e+1)}+1$	NO
	2^2	2	$\frac{q-2}{2(2e+1)}$	NO
	4	2	$\frac{q-2}{2(2e+1)}+1$	NO
	$q:(q-1)$	$q-1$	1	YES
$\alpha_q$prime				
α_q:4	α_q:4	4	$\frac{q-2}{2(2e+1)}$	NO
	n:4 where n is a prime divisor of β_q	4	$\frac{q-2}{2(2e+1)}+1$	NO
	4×2	4	$\frac{q-2}{2(2e+1)}$	NO
$\beta_q$prime				
β_q:4	β_q:4	4	$\frac{q-2}{2(2e+1)}$	YES iff q=8
	n:4 where n is a prime divisor of α_q	4	$\frac{q-2}{2(2e+1)}+1$	NO
	4×2	4	$\frac{q-2}{2(2e+1)}$	YES iff q=8

Table 3.9: The case $2e+1$ prime and $q-1$ prime

Sz(q) is isomorphic to one of the geometries appearing in table 3.11.

Proof: This theorem is a compilation of the results obtained in lemmas 3.5.1, 3.5.2, 3.5.3, 3.5.4 and 3.5.9. Thanks to lemma 3.2.6, in the case $2e+1$ is not a prime number, we are sure that α_q and β_q are not prime numbers. □

Corollary 3.6.1 *If $2e+1$ is a prime number and q-1, α_q and β_q are not prime, then the group Sz(q) has only one geometry of rank 2.*

Proof: This is a direct consequence of theorem 3.6.1. □

Table 3.12 contains the number of prime divisors of $q-1$ ($\#(q-1)$), α_q ($\#\alpha_q$), β_q ($\#\beta_q$) and the number $\#\Gamma$ of non-isomorphic geometries obtained for every Suzuki group up to $e=60$. Thanks to corollary 3.6.1, we know that there is only one geometry for $e=11$, 18, 20, 21, 26, 29, 33, 35, 41, 48, 50, 51 and 54.

The smallest case, that is $Sz(8)$, has been studied completely in [30]. It is included in this thesis as Appendix A.

G_0	G_1	G_{01}	$\#\Gamma$	$(2T)_1$
$D_{2(q-1)}$	$q:(q-1)$	q-1	1	YES
$\underline{\alpha_q\text{prime}}$				
α_q:4	α_q:4	4	$\frac{q-2}{2(2e+1)}$	NO
	n:4 where n is a prime divisor of β_q	4	$\frac{q-2}{2(2e+1)}+1$	NO
	4×2	4	$\frac{q-2}{2(2e+1)}$	NO
$\underline{\beta_q\text{prime}}$				NO
β_q:4	β_q:4	4	$\frac{q-2}{2(2e+1)}$	NO
	n:4 where n is a prime divisor of α_q	4	$\frac{q-2}{2(2e+1)}+1$	NO
	4×2	4	$\frac{q-2}{2(2e+1)}$	NO

Table 3.10: The case $2e+1$ prime and $q-1$ not prime

G_0	G_1	G_{01}	$\#\Gamma$	$(2T)_1$
$D_{2(q-1)}$	$q:(q-1)$	q-1	1	YES
	Sz(s) if s-1 is a maximal divisor of q-1	$D_{2(s-1)}$	1	NO
α_q:4	*Sz(s)* if α_s or β_s is a maximal divisor of α_q	α_s:4 or β_s:4	1	NO
β_q:4	*Sz(s)* if α_s or β_s is a maximal divisor of β_q	α_s:4 or β_s:4	1	NO
Sz(s)	(q:s):(s-1)	(s:s):(s-1)	1	NO
	Sz(s') when it is a maximal subgroup and s' $\neq$ s	*Sz(t)*	1	NO
	D_{2n} where s-1 $\|_M$ n $\|_P$ q-1	$D_{2(s-1)}$	1	NO
	n:4 where α_s $\|_M$ n $\|_P$ α_q or β_q	α_s:4	1	NO
	n:4 where β_s $\|_M$ n $\|_P$ α_q or β_q	β_s:4	1	NO

Table 3.11: The case $2e+1$ not prime (and thus $q-1$ not prime)

e	**#(q-1)**	$\#\alpha_q$	$\#\beta_q$	$\#\Gamma$	**e**	**#(q-1)**	$\#\alpha_q$	$\#\beta_q$	$\#\Gamma$
1	1	1	1	29	2	1	1	1	177
3	1	2	1	2411	4	2	2	2	5
5	2	1	2	375	6	1	2	2	$\geq 6.10^6$
7	3	2	3	14	8	1	2	2	$\geq 1.10^9$
9	1	1	3	$\geq 1.10^{10}$	10	3	3	3	15
11	2	3	2	1	12	3	2	3	8
13	3	3	3	5	14	3	1	2	$\geq 3.10^7$
15	1	3	2	$\geq 2.10^{17}$	16	4	3	3	15
17	4	3	3	17	18	2	2	3	1
19	4	4	5	22	20	2	3	2	1
21	3	3	2	1	22	6	5	5	23
23	3	3	1	$\geq 7.10^{12}$	24	2	4	2	6
25	5	4	5	23	26	3	2	2	1
27	6	3	3	20	28	4	4	4	19
29	2	2	5	1	30	1	2	4	$\geq 1.10^{35}$
31	6	5	5	22	32	3	5	4	20
33	2	2	4	1	34	4	4	4	17
35	3	4	2	1	36	3	4	1	$\geq 3.10^{20}$
37	7	7	5	27	38	4	4	5	20
39	3	3	1	$\geq 1.10^{23}$	40	6	5	5	9
41	2	3	3	1	42	3	4	6	22
43	6	5	2	22	44	1	2	2	$\geq 5.10^{51}$
45	5	5	5	24	46	3	5	4	18
47	5	5	4	24	48	2	5	4	1
49	8	6	6	28	50	2	3	3	1
51	2	3	3	1	52	10	7	7	54
53	1	3	3	$\geq 4.10^{62}$	54	2	3	3	1
55	6	6	4	25	56	5	4	1	$\geq 2.10^{32}$
57	6	6	6	30	58	9	9	7	36
59	6	5	7	30	60	4	3	3	8

Table 3.12: The number of prime divisors of $q-1$, α_q and β_q, and the number of non-isomorphic geometries of rank 2

.

We also studied the thin geometries of $Sz(8)$ more deeply in [29]. This work is included in the thesis as Appendix B.

3.7 Observations on the results

While computing the exact number of geometries up to isomorphism for every $Sz(2^{2e+1})$ with $e \leq 60$, we observed the following facts.

- If $Sz(q)$ does not have proper subgroups isomorphic to a Suzuki group, then table 3.12 shows that its number of geometries up to isomorphism is either 1 or a number that grows very fast like 2411 for $e = 3$, more than 6 millions for $e = 6$, more than 10 billions for $e = 9$, etc.
- If $Sz(q)$ has proper subgroups isomorphic to some Suzuki simple group, then its number of geometries up to isomorphism is more than reasonable.

This seems a little bit intriguing. We could have thought that when a Suzuki ovoid has subovoids, its collineation group should have more geometries. This is not the case.

Maybe we should add some condition in order to restrict the number of geometries that appear. If we impose $(2T)_1$, we see that when $e > 2$, there is only one geometry that survives this test (the case $e = 1$ gives 3 such geometries). Thus the huge number of geometries we obtained in these cases disappear and the number of geometries obtained is quite small.

Anyway, one which seems to play a central role is the geometry given by the RWPRI configuration $(Sz(q), D_{2(q-1)}, q : (q-1), q-1)$. It is the geometry that appears in all cases. It satisfies $(2T)_1$.

Chapter 4

The rank 3 geometries of $Sz(q)$

This chapter is taken from [27].

4.1 Introduction

In the present chapter, we determine all RWPRI geometries of rank three, on which a Suzuki simple group $Sz(q)$, with q an odd power of 2, acts residually weakly primitively. Here we plan to determine the geometries that satisfy the RWPRI condition, be them flag-transitive or not. The main reason of this choice is that RWPRI is a property much easier to test than flag-transitivity. Also, it seems much more restrictive, so it is natural to test it first. Flag-transitivity will not be studied in this thesis.

Our main results are theorems 4.4.1, 4.4.2 and 4.4.3 which give all RWPRI geometries of rank 3 for any Suzuki simple group $Sz(q)$, with q an odd power of two. As in the rank 2 classification (see chapter 3), we subdivide the results into three cases, namely the case $q-1$ prime (and hence $2e+1$ prime), the case $q-1$ not prime and $2e+1$ prime, and the case $2e+1$ not prime (and hence $q-1$ not prime).

As a corollary of the classification theorems for the rank 3 RWPRI geometries of the Suzuki groups, we show in theorem 4.6.1 that there is no pre-geometry (and hence no geometry) of rank ≥ 4, satisfying both RWPRI and $(2T)_1$. Moreover, we show that the only rank three geometries satisfying that property are thin, and that they arise only for Suzuki groups $Sz(q)$ with $q-1$ a prime number. Thus, as in the rank 2 case, the $(2T)_1$ property seems to be a good candidate to enter our set of axioms. Anyway, we show in the next chapter that it is possible to classify all RWPRI pre-geometries of rank ≥ 4 for any Suzuki group $Sz(q)$.

The chapter is organised as follows. In section 2, we prove some preliminary lemmas used in the next sections for the classification process. In section 3, we prove some lemmas that tend to restrict the possible maximal parabolic subgroups of the geometries we want to classify. In section 4, we classify all rank 3, RWPRI geometries of a Suzuki group $Sz(q)$. In section 5, we show that for most of the geometries obtained, we can say whether the group from which they arise acts flag-transitively or not. Finally, in section 6, we prove that there is no RWPRI and $(2T)_1$ pre-geometry of rank strictly greater than 3 for any Suzuki simple group $Sz(q)$ and that rank 3 geometries appear only when $q-1$ is prime.

4.2 Preliminary lemmas

Lemma 4.2.1 *The group $Sz(q)$ has a unique class of maximal subgroups isomorphic to a Suzuki group if and only if $q = 2^{2e+1}$, $2e+1 = p^n$ with p a prime number, and $n > 1$ an integer.*

Proof: It suffices to apply the classification of maximal subgroups of $Sz(q)$ as given in [32]. □

Lemma 4.2.2 *Let $Sz(s)$ be a maximal subgroup of $Sz(q)$. Then $s-1 \mid_M q-1$ implies $s = 2^{p^n}$ and $q = s^p$, with p a prime number and n a positive integer.*

Proof: Suppose that there is no prime number p and positive integer n such that $q = 2^{p^n}$. Then, by lemma 4.2.1, there are at least two different classes of Suzuki groups in $Sz(q)$, say $Sz(s)$ and $Sz(t)$. Thus $q = s^p = t^{p'}$ with $p \neq p'$ two prime numbers. Also, we know that $s-1$ and $t-1$ divide $q-1$. And $(s-1)\cdot(t-1) < (q-1)$. This implies that $\frac{q-1}{s-1}$ cannot be prime because $q-1 = (s-1)\cdot\frac{t-1}{(s-1,t-1)}\cdot n$ with $n > 1$ an integer. □

Lemma 4.2.3 *If $Sz(q)$ has at least two different classes of maximal subgroups of Suzuki type, then $s-1$ is not a prime divisor of $q-1$.*

Proof: Obvious thanks to lemmas 4.2.1 and 4.2.2. □

Lemma 4.2.4 *Let $\Gamma(G; G_0, G_1, G_2)$ be a rank 3 geometry. If the diagram of Γ is linear, then Γ is a flag-transitive geometry.*

Proof: To check that Γ is a flag-transitive geometry, we have to see if $G_0(G_1 \cap G_2) = G_0G_1 \cap G_0G_2$. It is obvious that $G_0(G_1 \cap G_2) \subseteq G_0G_1 \cap G_0G_2$. We know that $\mid G_0(G_1 \cap G_2) \mid = \frac{|G_0| \cdot |G_1 \cap G_2|}{|G_0 \cap G_1 \cap G_2|}$. We also know that $\mid G_0G_1 \mid = \frac{|G_0| \cdot |G_1|}{|G_0 \cap G_1|}$. If the diagram of Γ is linear, it means that one of the rank 2 residues is a generalized digon. We may suppose that this residue is the one corresponding to G_1. This implies that $\frac{|G_1|}{|G_0 \cap G_1|} = \frac{|G_1 \cap G_2|}{|G_0 \cap G_1 \cap G_2|}$. Thus $\mid G_0(G_1 \cap G_2) \mid = \frac{|G_0| \cdot |G_1 \cap G_2|}{|G_0 \cap G_1 \cap G_2|} = \frac{|G_0| \cdot |G_1|}{|G_0 \cap G_1|} = \mid G_0G_1 \mid$. □

Remark that this lemma is a corollary of the main theorem proved in [1].

Lemma 4.2.5 *Let $Sz(s) < Sz(q)$. Then either $\alpha_s \mid \alpha_q$ and $\beta_s \mid \beta_q$ or $\alpha_s \mid \beta_q$ and $\beta_s \mid \alpha_q$.*

Proof: This is an immediate consequence of lemmas 3.2.4 and 3.2.5. □

It is possible to determine whether α_s divides α_q or not, as shown in the next lemma.

Lemma 4.2.6 *5 divides $q^2 + 1$ for all $q = 2^{2e+1}$ with e a positive integer.*

Proof: $q^2 + 1 = 2^{4e+2} + 1 = (2^2 + 1) \cdot \sum_{i=0}^{2e} (2^{4e-2i})(-1)^i$ □

Lemma 4.2.7 *If $q = 2^{2e+1}$, then $5 \mid \alpha_q$ (resp. β_q) if and only if $2e + 1 = 0$ or 3 (resp. 1 or 2) modulo 4.*

Proof: It is easy to see that $2^n = 2$ (resp. 4, 3, 1) modulo 5 if $n = 1$ (resp. 2, 3, 0) modulo 4. Then the following table completes the proof. □

e modulo 4	$2e+1$ modulo 4	$e+1$ modulo 4	α_q modulo 5	β_q modulo 5
0	1	1	0	1
1	3	2	3	0
2	1	3	1	0
3	3	0	0	3

Lemma 4.2.8 *Let m and n be odd positive integers.*
Then $(2^{2m}+1, 2^{2n}+1) = 2^{2(m,n)}+1$.

Proof: The following argument is due to Wieb Bosma.
It is well known that $(s^a-1, s^b-1) = s^{(a,b)}-1$ for all a and b positive integers.
Take $s=2$. Applying the latter result with $a=4m$ and $b=4n$, we get

$$(2^{4m}-1, 2^{4n}-1) = 2^{4(m,n)}-1$$

and with $a=2m$ and $b=4n$, we get

$$(2^{2m}-1, 2^{4n}-1) = 2^{2(m,n)}-1$$

Since $2^{2m}-1$ and $2^{2m}+1$ are coprime, we have

$$2^{4(m,n)}-1 = (2^{4m}-1, 2^{4n}-1) = (2^{2m}-1, 2^{4n}-1)\cdot(2^{2m}+1, 2^{4n}-1)$$
$$= (2^{2(m,n)}-1)\cdot(2^{2m}+1, 2^{2n}+1)\cdot(2^{2m}+1, 2^{2n}-1)$$

the latter because $2^{2n}-1$ and $2^{2n}+1$ are also coprime. Thus we still have to prove that $(2^{2m}+1, 2^{2n}-1) = 1$ to finish the proof. This must hold because n is odd. Indeed, suppose a prime number p divides $2^{2m}+1$ and $2^{2n}-1$. Then the order k of 2 *mod* p divides $4m$ but not $2m$ since $2^{2m} = -1$ *mod* p. Thus the number of factors 2 in k is 2. On the other hand, $2^{2n} = 1$ *mod* p yields that k divides $2n$ and the number of factors 2 in k cannot exceed 1 because n is odd. Thus we have a contradiction. □

4.3 Restricting the maximal parabolic subgroups

In this section, we prove lemmas that restrict the possible maximal parabolic subgroups to construct RWPRI pre-geometries of rank ≥ 3. Thanks to lemma 1.1.1, we know that every rank 3 pre-geometry is indeed a geometry, so we can use these lemmas for the classification of the rank 3 RWPRI geometries of $Sz(q)$. We prefer to talk about pre-geometries here since the results obtained in this section are used also in chapter 5 to construct all RWPRI pre-geometries of rank ≥ 4.

Since we want to construct RWPRI pre-geometries, we know that at least one of the maximal parabolic subgroups of the pre-geometry must be a maximal subgroup of $Sz(q)$. Not every maximal subgroup of $Sz(q)$ can give rise to RWPRI pre-geometries of rank ≥ 3. We give here lemmas that discard some of them.

We already saw in chapter 3 that there is no RWPRI pre-geometry of rank ≥ 2 with some G_i isomorphic to (q.q):(q-1) (see lemma 3.5.1).

Lemma 4.3.1 *There is no* RWPRI *pre-geometry of rank ≥ 3 with some G_i isomorphic to α_q:4 or β_q:4.*

Proof: We may assume without loss of generality that $G_0 = Z_n : 4$ where $n = \alpha_q$ or β_q. Knowing the structure of these groups, we can say that G_{01} must be either $n : 2$ or $n_p : 4$ where p is a prime divisor of n.
The case $G_{01} = n : 2$ implies that G_1 must be a subgroup of $n : 4$. This possibility is rejected by the fact that $n : 2$ is a normal subgroup of $n : 4$.
The case $G_{01} = n_p : 4$ is a little bit longer. Because $n_p : 4$ is self-normalized in $Sz(q)$, G_1 cannot be $n : 4$. Thus G_1 must be a $Sz(s)$ maximal in $Sz(q)$. And because we have supposed G_{01} maximal in G_0, it must also be maximal in G_1. Thus $n_p = \alpha_s$ or β_s. Because the case $G_{01} = n : 2$ leads to a dead end, $G_{02} = n_r : 4$ with r, a prime divisor of n and $r \neq p$. And this implies $G_2 = Sz(t)$. This already means that n cannot be a prime number and that $q-1$ must have at least two prime divisors. We also know that $G_{012} = n_{pr} : 4$ because it must be maximal in G_{01} and it must also be a subgroup of G_{02}.
Suppose $q = 2^{2e+1}$, $s = 2^{2f+1}$ and $t = 2^{2g+1}$. Because $Sz(s)$ and $Sz(t)$ are maximal subgroups of $Sz(q)$, we have $q = s^{p_s} = t^{p_t}$ with p_s and p_t two distinct prime numbers. Thus, $2e+1 = (2f+1)\cdot p_s = (2g+1)\cdot p_t$ and $(p_s, p_t) = 1$. This allows us to write $2e+1 = (2h+1)\cdot p_s \cdot p_t$. We know that n is either α_q or β_q and that n_p is either α_s or β_s. Then, $p = \frac{n}{n_p} > 2^{(2h+1)(p_s-1)p_t-1} > \alpha_t > \beta_t$. But p must divide α_t or β_t. So we have a contradiction. $\square$

We show that if G_0 is a $D_{2(q-1)}$, then the pre-geometry has rank at most 3.

Lemma 4.3.2 *Every* RWPRI *pre-geometry of rank $n > 3$ with $G_0 = D_{2(q-1)}$ implies that at least two G_{0i} are dihedral groups.*

Proof: We rely on the classification of the RWPRI geometries for the dihedral groups given in theorem 2.5.1, since all RWPRI pre-geometries are flag-transitive RWPRI geometries for these groups. The proof becomes obvious thanks to that classification. $\square$

Lemma 4.3.3 *Suppose $q-1$ is not prime. In a* RWPRI *pre-geometry of rank $n > 2$ with $G_0 = D_{2(q-1)}$, there can be at most one G_{0i} isomorphic to a dihedral group.*

Proof: Assume, by way of contradiction and without loss of generality, that G_{01} and G_{02} are dihedrical groups and that G_{01} is maximal in G_0. Then $G_{01} = D_{2x}$ with $x \,|_M\, q-1$. This implies $G_{02} = D_{2y}$ with $y \,|_M\, q-1$. Then G_1 (resp. G_2) must be a $Sz(s)$ (resp. $Sz(t)$). These two subgroups have to be maximal subgroups of $Sz(q)$ and G_{01} (resp. G_{02}) has to be maximal in $Sz(s)$ (resp. $Sz(t)$). We have $q = 2^{(2e+1)p_s p_t}$, $s = 2^{(2e+1)p_t}$ and $t = 2^{(2e+1)p_s}$. We also have $\frac{q-1}{s-1} = p$ a prime number, which must be a divisor of $t-1$. But

$$p = \frac{2^{(2e+1)p_s p_t} - 1}{2^{(2e+1)p_t} - 1} = \sum_{i=1}^{p_t} 2^{(2e+1)p_s(p_t-1)} > 2^{(2e+1)p_s} - 1$$

So p cannot divide $t-1$, a contradiction. $\square$

Lemma 4.3.4 *The maximal rank of a* RWPRI *pre-geometry with $G_0 = D_{2(q-1)}$ is three.*

Proof: Trivial by lemmas 4.3.2 and 4.3.3. $\square$

This is a fairly strong result. It permits us to concentrate only on the rank 3 when we construct RWPRI pre-geometries with $G_0 = D_{2(q-1)}$.

Lemma 4.3.5 *If Γ is a* RWPRI *pre-geometry of rank ≥ 4 of a Suzuki group $Sz(q)$, then all the maximal parabolic subgroups that are maximal in $Sz(q)$ must be isomorphic to a Suzuki group $Sz(s)$.*

Proof: Apply lemmas 3.5.1, 4.3.1 and 4.3.4. $\square$

4.4 The rank 3 geometries

Thanks to lemma 1.1.1, we know that any rank 3 pre-geometry constructed in this section is indeed a geometry. We determine now all the rank 3 RWPRI geometries of a Suzuki group $Sz(q)$ with $q = 2^{2e+1}$ and e a positive integer. As in the rank 2 classification given in theorem 3.6.1, we subdivide our discussion in three cases:

- $q-1$ is prime.
- $q-1$ is not prime and $2e+1$ is prime.
- $2e+1$ is not prime.

4.4.1 First case: $q-1$ is a prime number

If $q-1$ is a prime number, *Sz(q)* cannot have subgroups isomorphic to *Sz(s)*. Remark that lemma 4.3.4 tells us then that $Sz(q)$ does not have RWPRI pre-geometries of rank ≥ 4 in that case. The only G_0 candidate is $D_{2(q-1)}$ by lemmas 3.5.1 and 4.3.1. Thanks to theorem 2.4.1, we know that there are two possible residues for G_0.
The first one is with $G_{01} = q-1$, $G_{02} = 2$ and $G_{012} = 1$. The only subgroups of $Sz(q)$ containing G_{01} are then $D_{2(q-1)}$, $(q{\cdot}q) : (q-1)$ and $q : (q-1)$. Because $N_{Sz(q)}(q-1) = D_{2(q-1)}$, and $D_{2(q-1)}$ has only one subgroup isomorphic to $q-1$, the subgroup G_1 cannot be isomorphic to $D_{2(q-1)}$. Lemma 3.5.1 implies that G_1 cannot be $(q{\cdot}q) : (q-1)$. So we may assume $G_1 = q : (q-1)$. Because $B = G_{012} = 1$, we know that G_{12} must be a cyclic simple group. Here there are two possibilities. They are $q-1$ and 2.
Suppose first that $G_{12} = q-1$. Then G_2 must be either $q : (q-1)$ or $D_{2(q-1)}$. In both cases, G_{01} and G_{12} have two fixed points and it implies that G_{02} has a fixed point which is in the orbit of two points of G_0. This is a contradiction with the fact that all the Z_2's in a $D_{2(q-1)}$ have a fixed point which lies in the orbit of $q-1$ points of this subgroup.
Suppose that $G_{12} = 2$. It implies that G_2 contains two cyclic groups of order 2. Thus $G_2 = D_{2p}$ with $p \mid \alpha_q, \beta_q$ or $q-1$. The case $p = 2$ is not possible because G_{02} and G_{12} do not have the same fixed point.
The second possible residue is with G_{01} and G_{02} isomorphic to cyclic groups of order 2. There, either G_1 is one of $p : 2$ (with $p \mid \alpha_q$, β_q, 2 or $q-1$) and G_2 is one of $p' : 2$ (with $p' \mid \alpha_q$, $\beta_q$or $q-1$), or $G_1 = q : (q-1)$ and $G_2 = q : (q-1)$. Remark that there are no two maximal parabolic subgroups isomorphic to 2^2. Indeed, this should mean that all the G_{ij}'s have a common fixed point. Thus the group generated by G_0, G_1 and G_2 cannot be *Sz(q)* because, otherwise, it would be a subgroup of $q{\cdot}q : (q-1)$.

We resume the preceding discussion in a first theorem.

Theorem 4.4.1 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ and $q-1$ are prime numbers. Then all rank 3* RWPRI *geometries of $Sz(q)$ have a sublattice isomorphic to one of the following.*

-

G_0 $D_{2(q-1)}$	G_1 $q:(q-1)$	G_2 $q:(q-1)$
G_{01} 2	G_{02} 2	G_{12} $q-1$
	G_{012} 1	

-

G_0 $D_{2(q-1)}$	G_1 $q:(q-1)$	G_2 D_{2p}
G_{01} $q-1$	G_{02} 2	G_{12} 2
	G_{012} 1	

with $p \mid \alpha_q$, β_q or $q-1$, and p is a prime number.

-

G_0 $D_{2(q-1)}$	G_1 D_{2p}	G_2 $D_{2p'}$
G_{01} 2	G_{02} 2	G_{12} 2
	G_{012} 1	

with $p \mid \alpha_q$, β_q, 2 or $q-1$, and $p' \mid \alpha_q$, β_q, or $q-1$, and p, p' two prime numbers.

4.4.2 Second case: $q-1$ is not prime and $2e+1$ is prime

In this case again, we have $G_0 = D_{2(q-1)}$. Every possible rank 2 residue for G_0 has a dihedral subgroup as one of its maximal parabolic subgroups. Thus we may assume that $G_{01} = D_{2n}$ where $n \mid_M q-1$. As D_{2n} is self-normalized in $Sz(q)$, there is only one subgroup isomorphic to $D_{2(q-1)}$ that contains G_{01}. Thus we cannot find a subgroup G_1 to complete the geometry. Remark that in that case as in the preceding one, lemma 4.3.4 tells us that $Sz(q)$ does not have a RWPRI pre-geometry of rank ≥ 4. Thus we can now state the following theorem.

Theorem 4.4.2 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ is a prime number but $q-1$ is not. Then G does not possess any* RWPRI *pre-geometry of rank ≥ 3.*

4.4.3 Third case: $2e+1$ is not prime

This case is the most tricky one because the G_0 candidates are $D_{2(q-1)}$ and $Sz(s)$ where $s = 2^{2f+1}$ and $\frac{2e+1}{2f+1}$ is a prime number.

The case $G_0 = D_{2(q-1)}$

In this case, lemma 4.3.4 tells us that the maximal rank is 3. Also, lemma 4.3.3 and theorem 2.4.1 imply that one of G_{01} or G_{02} must be a cyclic group. We can choose $G_{01} = Z_{q-1}$ without restricting generality. The sublattice is then partially filled as below, with $n \mid_M q-1$.

G_0 $D_{2(q-1)}$	G_1	G_2
G_{01} $q-1$	G_{02} D_{2n}	G_{12}
	G_{012} n	

The only subgroups that can contain G_{02} are then G_0 and subgroups isomorphic to a Suzuki group. This implies G_2 is a Suzuki group and its residue is well known thanks to the classification theorem given in theorem 3.6.1. We know then that $G_2 = Sz(s)$ with $q = s^p$ and p a prime number. Also, we must have $n = s-1$. This means $\frac{q-1}{s-1}$ is a prime number. Lemma 4.2.2 yields then $2e+1 = p^a$. Now, looking at the rank 2 geometries of $Sz(s)$ as given in theorem 3.6.1, we can complete our geometry. Indeed, G_{12} must be $s:(s-1)$ and thus $G_1 = < G_{01}, G_{12} > = q:(q-1)$.

Lemma 4.4.1 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1$ is not a prime number. Then G has a unique* RWPRI *geometry of rank 3 with $G_0 = D_{2(q-1)}$ if and only if $2e+1 = p^a$ and p is a prime number. Its sublattice is as follows.*

No.	G_0	G_1	G_{01}	$(2T)_1$
1	$D_{2(s-1)}$	$D_{2(s-1)}$	2	NO
2		D_{2n} where n is a prime divisor of α_s or β_s	2	NO
3		2^2	2	NO
4		4	2	NO
5		$s:(s-1)$	$s-1$	YES
	$\underline{\alpha_s \text{prime}}$			
6	α_s:4	α_s:4	4	NO
7		n:4 where n is a prime divisor of β_s	4	NO
8		4×2	4	NO
	$\underline{\beta_s \text{prime}}$			
9	β_s:4	β_s:4	4	YES iff s=8
10		n:4 where n is a prime divisor of α_s	4	NO
11		4×2	4	YES iff s=8

Table 4.1: The case $s-1$ prime

G_0 $D_{2(q-1)}$	G_1 $q:(q-1)$	G_2 $Sz(s)$
G_{01} $q-1$	G_{02} $D_{2(s-1)}$	G_{12} $s:(s-1)$
	G_{012} $s-1$	

The case $G_0 = Sz(s)$

In order to have $Sz(s)$ maximal in $Sz(q)$, we must impose that $q = s^p$ with p a prime number.
The rank 2 residue of G_0 is given by theorem 3.6.1. We subdivided the classification of the rank 2 geometries of the Suzuki groups into three cases. Here, we use this subdivision again.

Suppose first that $s-1$ is a prime number. This means the residue of G_0 is one of those appearing in table 4.1. We take each of these 11 possible residues and try to extend them to construct RWPRI rank 3 geometries.

Residue 1:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} $D_{2(s-1)}$	G_{02} $D_{2(s-1)}$	G_{12}
	G_{012} 2	

Thus G_1 must contain a $D_{2(s-1)}$. This yields $G_1 = Sz(t)$ or $G_1 \leq D_{2(q-1)}$. The first case implies $t \neq s$ because $D_{2(s-1)}$ is self-normalized in $Sz(q)$. Again, the residue is then known and we must have $D_{2(s-1)}$ maximal in $Sz(t)$. This means $s-1 = t-1$ and thus $s = t$, a contradiction.
The second case yields $G_1 = D_{2(s-1)p}$ where $p \mid \frac{q-1}{s-1}$ is a prime number. The same argument shows that $G_2 = D_{2(s-1)p'}$ where $p' \mid \frac{q-1}{s-1}$ is also a prime number. Thanks to theorem 2.4.1, we know that $G_{12} = D_{2p}$ and $p = p'$. Thus $G_1 = G_2$, a contradiction.

Residue 2:

The sublattice looks as follows (with $n \mid_P \alpha_s$ or β_s).

G_0 $Sz(s)$	G_1	G_2
G_{01} $D_{2(s-1)}$	G_{02} D_{2n}	G_{12}
	G_{012} 2	

As in the preceding case, we know that $G_1 < D_{2(q-1)}$. This means $G_1 = D_{2(s-1)p}$ with $p \mid q-1$ a prime number and $G_{12} = D_{2p}$ thanks theorem 2.4.1. Thus $G_2 =< G_{02}, G_{12} >$ is a Suzuki simple group $Sz(t)$ with $t-1$ a prime number. Because of G_{02}, we must have $t \neq s$. This means $q = s^a = t^b$ and because $s-1$ (resp. $t-1$) is a prime number, we know that $s = 2^x$ (resp. $t = 2^y$) with x (resp. y) a prime number. Thus $q = 2^{xy}$, with x, y two different odd prime numbers. Thanks to lemma 4.2.8, we then get $n = 5$.

Residue 3:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} $D_{2(s-1)}$	G_{02} 2^2	G_{12}
	G_{012} 2	

Again G_1 must be a subgroup of $D_{2(q-1)}$. Thus $G_{12} = D_{2p}$ with $p \mid q-1$ a prime number as in the preceding case, and $G_2 =< 2^2, D_{2p} >$ must be a subgroup of Suzuki type. So $G_2 = Sz(t)$ with $t \neq s$ and $p = t-1$. Also, $q = 2^{xy}$, $s = 2^x$ and $t = 2^y$ and $s-1$ and $t-1$ must be prime numbers. Then the geometry is the following one.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} 2^2	G_{12} $D_{2(t-1)}$
	G_{012} 2	

Residue 4:

Almost the same discussion as in the preceding case leads to the following geometry.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} 4	G_{12} $D_{2(t-1)}$
	G_{012} 2	

Residue 5:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{12}
	G_{012} $s-1$	

Again, G_1 must be a subgroup of $D_{2(q-1)}$. Thus G_1 is a subgroup $D_{2(s-1)p}$ where $p \mid q-1$ is a prime number thanks to theorem 2.4.1. This implies that G_{12} may be isomorphic to either $Z_{(s-1)p}$ or $D_{2(s-1)}$. Suppose $G_{12} = D_{2(s-1)}$. This yields $G_2 = Sz(s)$. Then, $G_{02} = s:(s-1)$ must be contained in two distinct $Sz(s)$, a contradiction. So $G_{12} = Z_{(s-1)p}$. Then $G_2 =< s:(s-1), Z_{(s-1)p} >$ is a subgroup of $q:(q-1)$. In order to have a RWPRI geometry, we must impose $G_2 = q:((s-1)p)$. A priori, the number p can be any prime divisor of $\frac{q-1}{s-1}$. So we get more than one geometry most of the time. Remark that, when $p = \frac{q-1}{s-1}$, we get the geometry given in lemma 4.4.1.

Residue 6:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} α_s: 4	G_{02} α_s: 4	G_{12}
	G_{012} 4	

Because α_s is self-normalized in $Sz(q)$, the subgroup G_1 must be a subgroup of α_q: 4 or β_q: 4 depending whether α_s divides α_q or β_q. This means $G_1 = (\alpha_s p):4$ and $G_{12} = p:4$ with p a prime number such that $p\alpha_s$ divides either α_q or β_q as α_s does, and $p \neq \alpha_s$. Thus $G_2 =<\alpha_s: 4, p:4>$ is isomorphic to G_1. This implies a contradiction because G_{12} is self-normalized in $Sz(q)$ and thus cannot be contained in two different $(\alpha_s p):4$.

Residue 7:

The sublattice looks as follows (with $n \mid_P \beta_s$).

G_0 $Sz(s)$	G_1	G_2
G_{01} α_s: 4	G_{02} $n:4$	G_{12}
	G_{012} 4	

The same argument as in the preceding case leads to $G_1 = \alpha_s p : 4$ and $G_{12} = p : 4$, with p a prime number such that $p\alpha_s$ divides either α_q or β_q as α_s does and $p \neq \alpha_s$. Then $G_2 =< n : 4, p : 4 >= Sz(q)$ because of lemma 4.2.5, which is not permitted.

Residue 8:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} α_s: 4	G_{02} 4×2	G_{12}
	G_{012} 4	

The same argument as in the preceding case leads to $G_1 = \alpha_s p : 4$ and $G_{12} = p : 4$, with p a prime number such that $p\alpha_s$ divides α_q or β_q and $p \neq \alpha_s$. Then $G_2 =< 4 \times 2, p : 4 >= Sz(t)$ with $t \neq s$ and we must have $p = \alpha_t$ or β_t. Such a geometry exists, for example, with $s = 2^3$, $t = 2^5$ and $q = 2^{15}$.

Residues 9 to 11:

The discussion for residue 9 (resp. 10, 11) is almost the same as for residue 6 (resp. 7, 8). We just have to read β_s instead of α_s in these cases.

A resume of the preceding discussion is given at the end of this section in theorem 4.4.3.

No.	G_0	G_1	G_{01}	$(2T)_1$
12	$D_{2(s-1)}$	$s : (s-1)$	s-1	YES
13		*Sz(t)* if t-1 is a maximal divisor of s-1	$D_{2(t-1)}$	NO
14	α_s:4	*Sz(t)* if α_t or β_t is a maximal divisor of α_s	α_t:4 or β_t:4	NO
15	β_s:4	*Sz(t)* if α_t or β_t is a maximal divisor of β_s	α_t:4 or β_t:4	NO
16	*Sz(t)*	(s:t):(t-1)	(t:t):(t-1)	NO
17		*Sz(t')* when it is a maximal subgroup and t' $\neq$ t	*Sz(u)*	NO
18		D_{2n} where t-1 $\|_M$ n $\|_P$ s-1	$D_{2(t-1)}$	NO
19		n:4 where α_t $\|_M$ n $\|_P$ α_s or β_s	α_t:4	NO
20		n:4 where β_t $\|_M$ n $\|_P$ α_s or β_s	β_t:4	NO

Table 4.2: The case $2e+1$ not prime (and thus $s-1$ not prime)

We now consider the case $s-1$ not prime and $s = 2^{2e+1}$ with $2e+1$ a prime number. This means that the residue of G_0 is one of numbers 5, 6, 7, 8, 9, 10 or 11 appearing in table 4.1. The discussions of the preceding cases corresponding to those residues is independent from the fact that $s-1$ is prime or not. So we can rely on the results obtained in the preceding discussion.

Finally, we have to consider the case where $s = 2^{2e+1}$ and $2e+1$ is not a prime number. In this case, the possible residues for G_0 are given in table 4.2.

Residue 12:

The same discussion as in the case of residue 5 leads to the following geometry.

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $q : ((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $s : (s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

Remark that, when $p = \frac{q-1}{s-1}$, we get the geometry given in lemma 4.4.1.

Residue 13:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} $D_{2(s-1)}$	G_{02} $Sz(t)$	G_{12}
	G_{012} $D_{2(t-1)}$	

Since $t-1 \mid_M s-1$ for this residue, we have $t = 2^{p^a}$ and $s = t^p$. Again, G_1 is a subgroup of $D_{2(q-1)}$, thus $G_1 = D_{2(s-1)n}$ and $G_{12} = D_{2n}$ with n a prime number that divides $\frac{q-1}{s-1}$. Then $G_2 =< Sz(t), D_{2n} >$ is a subgroup of Suzuki type. Let $G_2 = Sz(s')$. We must have $s' \neq s$, thus $q = 2^{(p^a \cdot p')}$ and $s' = 2^{(p^{a-1}p')}$. This implies that the residue of G_2 is of type 18, and $n \mid s'-1$. Here n can take more than one value. So there might be more than one geometry there.

Residue 14:

The sublattice looks as follows (where n is either α_t or β_t).

G_0 $Sz(s)$	G_1	G_2
G_{01} α_s: 4	G_{02} $Sz(t)$	G_{12}
	G_{012} $n:4$	

Because of lemma 4.2.5, we know that $Sz(t)$ is a maximal subgroup of $Sz(s)$. This implies that $G_2 = Sz(s')$ must also contain $Sz(t)$ as a maximal subgroup. Thus its residue is such that $G_{12} = n' : 4$ where n' divides s'^2+1 and $n \mid_M n'$. Also, $G_1 = \frac{\alpha_s n'}{n} : 4$.

Residue 15:

Almost the same discussion as the preceding one leads to the following geometry ($n = \alpha_t$ or β_t, $n \mid_M \beta_s$, $n' \mid s'^2 + 1$ and $n \mid_M n'$).

G_0 $Sz(s)$	G_1 $(\frac{\alpha_s n'}{n}) : 4$	G_2 $Sz(s')$
G_{01} β_s: 4	G_{02} $Sz(t)$	G_{12} $n' : 4$
	G_{012} $n : 4$	

Residue 16:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} $Sz(t)$	G_{02} $s{:}t : (t-1)$	G_{12}
	G_{012} $t{:}t : (t-1)$	

Because of G_{01}, the subgroup G_1 must be a subgroup of Suzuki type, say $Sz(s')$ with $s' \neq s$. Thus its residue is known. It implies that $G_{12} = s'{:}t : (t-1)$. Then $G_2 =< s{:}t : (t-1), s'{:}t : (t-1) >$ is a subgroup of $q{:}q : (q-1)$. In order to have a RWPRI geometry, we must be sure that either G_{02} or G_{12} is a maximal subgroup of G_2. This implies $G_2 = q{:}t : (t-1)$.

Residue 17:

The sublattice looks as follows.

G_0 $Sz(s)$	G_1	G_2
G_{01} $Sz(t)$	G_{02} $Sz(t')$	G_{12}
	G_{012} $Sz(u)$	

It is easy to see that the only geometry we can obtain here is with all subgroups of the sublattice isomorphic to subgroups of Suzuki type.

Residue 18:

The sublattice looks as follows (with $t-1 \mid_M n \mid_P s-1$).

G_0 $Sz(s)$	G_1	G_2
G_{01} $Sz(t)$	G_{02} D_{2n}	G_{12}
	G_{012} $D_{2(t-1)}$	

Because G_{01} is of Suzuki type, we have $G_1 = Sz(s')$ with $s' \neq s$. Thus the residue of G_1 is well known. Then $G_{12} = D_{2m}$ where $t-1 \mid_M m \mid_P s'-1$. And $G_2 =< D_{2n}, D_{2m} >= D_{2\frac{nm}{t-1}}$. Here $n \neq m$ thanks to theorem 2.4.1.

Residue 19:

The sublattice looks as follows (with $\alpha_t \mid_M n \mid_P \alpha_s$ or β_s).

G_0 $Sz(s)$	G_1	G_2
G_{01} $Sz(t)$	G_{02} $n:4$	G_{12}
	G_{012} $\alpha_t:4$	

Again, $G_1 = Sz(s')$ with $s' \neq s$. Thus $G_{12} = m:4$ with $\alpha_t \mid_M m \mid_P \alpha_{s'}$ or $\beta_{s'}$ and $G_2 = \frac{nm}{\alpha_t} : 4$.

Residue 20:

Almost the same discussion as in the preceding case leads to the following geometry.

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $\frac{nm}{\beta_t}:4$
G_{01} $Sz(t)$	G_{02} $n:4$	G_{12} $m:4$
	G_{012} $\beta_t:4$	

In that case, $\beta_t \mid_M n \mid_P \alpha_s$ or β_s and $\beta_t \mid_M m \mid_P \alpha_{s'}$ or $\beta_{s'}$

This ends the classification of all RWPRI geometries of rank 3 for the Suzuki groups.

We now state a theorem that summarizes the discussion of this section.

Theorem 4.4.3 *Let $G = Sz(q)$ with $q = 2^{2e+1}$ and suppose $2e+1 = p_1^{e_1}\dots p_n^{e_n}$ with $p_i \neq p_j$, $\forall i \neq j$, and $\sum_{i=1}^n e_i \geq 2$. Suppose $q = s^{p_i}$ for some $i \in \{1\dots n\}$. If $s-1$ is a prime number (which implies that $\sum_{i=1}^n e_i = 2$) then all rank 3* RWPRI *geometries of G have a sublattice isomorphic to one of the following.*

1.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(t)$
G_{01} $D_{2(s-1)}$	G_{02} D_{10}	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $t = 2^{p_2}$, $2e+1 = p_1 \cdot p_2$, and $p_1 \neq p_2$ two prime numbers.

2.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(s')$
G_{01} $D_{2(s-1)}$	G_{02} 2^2	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $s' = 2^{p_2}$, $2e+1 = p_1 \cdot p_2$, and $p_1 \neq p_2$ two prime numbers.

3.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(t-1)}$	G_2 $Sz(s')$
G_{01} $D_{2(s-1)}$	G_{02} 4	G_{12} $D_{2(t-1)}$
	G_{012} 2	

with $s = 2^{p_1}$, $s' = 2^{p_2}$, $2e+1 = p_1 \cdot p_2$, *are* $p_1 \neq p_2$ *two prime numbers.*

4.

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $q:((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

with $p \mid \frac{q-1}{s-1}$ *a prime number.*

5.

G_0 $Sz(s)$	G_1 $(ap):4$	G_2 $Sz(t)$
G_{01} $a:4$	G_{02} 4×2	G_{12} $p:4$
	G_{012} 4	

with $a = \alpha_s$ *or* β_s *(resp.* $p = \alpha_t$ *or* β_t*) prime numbers,* $n = 2$, $q = s^{p_1} = t^{p_2}$ *and* $p_1 \neq p_2$.

If $s-1$ *is not a prime number but* $\sum_{i=1}^n e_i = 2$ *then all rank 3* RWPRI *geometries of* G *have a sublattice isomorphic to sublattices number 4 or 5 of the case* $s-1$ *prime.*
Finally, if $\sum_{i=1}^n e_i > 2$ *then all rank 3* RWPRI *geometries of* G *have a sublattice isomorphic to one of the following.*

6.

G_0 $Sz(s)$	G_1 $D_{2(s-1)p}$	G_2 $q:((s-1)p)$
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{12} $(s-1)p$
	G_{012} $s-1$	

with $p \mid \frac{q-1}{s-1}$ *a prime number.*

7.

G_0 $Sz(s)$	G_1 $D_{2(t-1)pp'}$	G_2 $Sz(s')$
G_{01} $D_{2(t-1)p}$	G_{02} $Sz(t)$	G_{12} $D_{2(t-1)p'}$
	G_{012} $D_{2(t-1)}$	

with $p \mid \frac{s-1}{t-1}$, $p' \mid \frac{s'-1}{t-1}$ *two distinct prime numbers,* $s = t^{p_i}$ *and* $s' = t^{p_j}$, *with* $i \neq j$.

8.

G_0 $Sz(s)$	G_1 $(mpp'):4$	G_2 $Sz(s')$
G_{01} $mp:4$	G_{02} $Sz(t)$	G_{12} $mp':4$
	G_{012} $m:4$	

with $m = \alpha_t$ *or* β_t, $mp \mid \alpha_s$ *or* β_s, $mp' \mid \alpha_{s'}$ *or* $\beta_{s'}$, *and* p, p' *two prime numbers.*

9.

G_0 $Sz(s)$	G_1 $q{:}t:(t-1)$	G_2 $Sz(s')$
G_{01} $s{:}t:(t-1)$	G_{02} $Sz(t)$	G_{12} $s'{:}t:(t-1)$
	G_{012} $t{:}t:(t-1)$	

with $q = s^{p_i} = s'^{p_j} = t^{p_ip_j}$, *and* $i \neq j$.

10.

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $Sz(s")$
G_{01} $Sz(t)$	G_{02} $Sz(t')$	G_{12} $Sz(t")$
	G_{012} $Sz(u)$	

with $\sum_{i=1}^{n} e_i \geq 4$, $n \geq 3$, $q = s^{p_i} = s'^{p_j} = s"^{p_k} = t^{p_ip_j} = t'^{p_ip_k} = t"^{p_jp_k} = u^{p_ip_jp_k}$, *and* i, j, k *three pairwise distinct numbers.*

4.5 About the flag-transitivity

Most of the time, we are interested in knowing whether a RWPRI (pre-)geometry is a flag-transitive geometry or not. We do not intend to solve that problem in the present thesis. Anyway, lemma 4.2.4 permits us to give the answer for a lot of geometries obtained in the previous section.

For the case $q-1$ prime, looking at theorem 4.4.1, we see that geometries corresponding to the second type, and geometries corresponding to the third type with $p=2$ are indeed flag-transitive.

For the case $2e+1$ not prime, looking at theorem 4.4.3, we see that all geometries, except number 9 and 10, have a linear diagram and are thus surely flag-transitive.

We intend to study the flag-transitivity of those geometries that do not have a linear diagram in future work.

4.6 About the $(2T)_1$ property

If we look at the rank 3 RWPRI geometries that satisfy the $(2T)_1$ property, we see quite easily that in theorem 4.4.1, the only geometries that survive this test are the thin geometries, and in theorem 4.4.3, no geometry satisfies this test. This means that, if one wants to construct a rank 4 RWPRI pre-geometry satisfying $(2T)_1$ for a Suzuki group $Sz(q)$, one has to choose $q = 2^{pp'}$ with p, p' two prime numbers. Then, G_0 being a subgroup of Suzuki type because of lemma 4.3.5, it means that $G_0 = Sz(s)$ must have a rank 3 RWPRI geometry satisfying $(2T)_1$. This implies that $s-1$ is a prime number and we may assume that $G_{01} = D_{2(s-1)}$. Thus G_1 is either a group of Suzuki type or a dihedral group. It cannot be a dihedral group D_{2n} with n an odd number, because no such dihedral group has RWPRI pre-geometries of rank 3 or higher satisfying $(2T)_1$ thanks to the classification of the RWPRI geometries for these groups given in theorem 2.5.1 (we remind again here that all RWPRI pre-geometries of a dihedral group are flag-transitive geometries). So it must be a $Sz(t)$ with $t \neq s$. But then $s-1 = t-1$ and $s \neq t$, a contradiction. This permits to state the following theorem.

Theorem 4.6.1 *Let $G = Sz(q)$ with $q = 2^{2e+1}$.*
If $q-1$ is not prime, $Sz(q)$ has only one RWPRI *and $(2T)_1$ pre-geometry up to isomorphism. It is a rank 2 F, RC, FT geometry with $G_0 = D_{2(q-1)}$, $G_1 = q:(q-1)$ and $G_{01} = q-1$.*
If $q-1 \neq 7$ is prime, the RWPRI *and $(2T)_1$ pre-geometries of G are of rank ≤ 3, one of them only being of rank 2, the others being thin rank 3 geometries.*
Finally, if $q=8$, the RWPRI *and $(2T)_1$ pre-geometries of G are of rank ≤ 3, three of them only being of rank 2, the others being thin rank 3 geometries.*

Chapter 5

The pre-geometries of rank ≥ 4 of $Sz(q)$

This chapter is taken from [28].

5.1 Introduction

In this chapter, we determine all RWPRI pre-geometries of rank $\geq$ 4 for any Suzuki simple group $Sz(q)$. To do that, we start from the classification theorems stated in chapter 4. We get, as final result, eleven constructions leading to all RWPRI pre-geometries of rank ≥ 4 for any $Sz(q)$ (see theorem 5.3.1).
We plan to determine in future work which of these pre-geometries are indeed flag-transitive geometries. The results obtained in this chapter strenghten our belief that the RWPRI condition is an efficient condition to impose on the subgroups forming the geometries.

The chapter is organised as follows. In section 2, we determine the pre-geometries of rank 4. This section and the next one are there to show that even without imposing the $(2T)_1$ property (as suggested in chapter 4), it is possible to determine theoretically all the RWPRI pre-geometries of any rank. Also, nice families of pre-geometries arise, all with the same kind of diagram. In section 3, we resume the long discussion made in section 2 by giving eleven constructions that lead to all RWPRI pre-geometries of rank $\geq$ 4. Finally, in section 4, we give some concluding remarks about the results obtained in this chapter and mention some future work we plan to do in this area.

5.2 The rank ≥ 4 pre-geometries of $Sz(q)$

We remind the reader that lemma 1.1.1 permits us to rely on the rank 3 classification given in chapter 4 when we determine all rank ≥ 4 RWPRI pre-geometries since in the rank three case, every pre-geometry is a geometry. Thanks to lemma 4.3.5, we may assume $G_0 = Sz(s)$ with $q = s^p$ and p a prime number. Throughout this section we assume $s = 2^{2f+1}$ and $q = s^p = 2^{2e+1}$. We now concentrate on the rank 4. The higher ranks will appear clearly as generalizations of the extension process we use to construct the rank 4 pre-geometries from the rank 3 residues. These generalizations are summarized in the next section in eleven constructions of RWPRI pre-geometries.

5.2.1 $s - 1$ is prime

In this case, the rank 3 residues for G_0 are those given in theorem 4.4.1. We take each of the 3 different types of residue one by one and try to extend it to a rank 4 RWPRI pre-geometry.

Residue 1:

The sublattice is partly filled as in table 5.1.

Because $G_{01} = D_{2(s-1)}$, the subgroup G_1 is either a subgroup of $D_{2(q-1)}$ or a group of Suzuki type.
Suppose first that $G_1 < D_{2(q-1)}$. Then G_1 is a dihedral group. Thanks to theorem 2.5.1, we know how the G_1-residue looks like, and we also know that $G_1 = D_{2(s-1)n}$ with $n \mid \frac{q-1}{s-1}$ a prime number. Table 5.2 gives a more complete sublattice. Now, $G_2 =< s : (s-1), D_{2n}, G_{23} >$ implies that G_2 is a subgroup of Suzuki type, let's say $G_2 = Sz(s')$, with $q = s'^{p'}$ and $s' - 1$, p' two prime numbers. Thus the rank 3 residue of G_3 must be one of those appearing in theorem 4.4.1. Because all pre-geometries given in this theorem have at least two subgroups isomorphic to Z_2 in their sublattice, we see that the pre-geometry we have here is not RWPRI.

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{03} $s:(s-1)$	G_{12}	G_{13}	G_{23}
G_{012} 2	G_{013} 2	G_{023} $s-1$	G_{123}		
G_{0123} 1					

Table 5.1: The sublattice partly filled by residue 1

G_0 $Sz(s)$	G_1 $D_{2(s-1)n}$	G_2	G_3		
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{03} $s:(s-1)$	G_{12} D_{2n}	G_{13} D_{2n}	G_{23}
G_{012} 2	G_{013} 2	G_{023} $s-1$	G_{123} n		
G_{0123} 1					

Table 5.2: The residue 1 and G_1 a dihedral group

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{03} D_{2n}	G_{12}	G_{13}	G_{23}
G_{012} $s-1$	G_{013} 2	G_{023} 2	G_{123}		
G_{0123} 1					

Table 5.3: The sublattice partly filled by residue 2

We may thus assume $G_1 = Sz(s')$ with $q = s'^{p'}$ and $s'-1$ and p' two prime numbers. This yields $s'-1 = s-1$. Thus we have a contradiction because $D_{2(s-1)}$ is self-normalized in $Sz(q)$ and hence cannot be contained in two subgroups isomorphic to $Sz(s)$.

Residue 2:

The sublattice is partly filled as in table 5.3. Again, the subgroup G_1 is either a dihedral group or a Suzuki simple group. It cannot be a Suzuki group for the same reason as in the previous case. So we may assume $G_1 = D_{2(s-1)n'}$ with $n' \mid \frac{q-1}{s-1}$ a prime number. Thanks to theorem 2.5.1, we know that there are two possible residues for G_1.

The first one is with $G_{12} = D_{2(s-1)}$. It implies that $G_2 =< s:(s-1), D_{2(s-1)}, G_{23} >$ must be a subgroup of Suzuki type. Thus $G_2 = Sz(s')$ with $q = s'^{p'}$, and $s'-1$ and p' two prime numbers. Then $s-1 = s'-1$. Because $s:(s-1)$ is self-normalized in $Sz(q)$, we have $G_0 = G_2$, a contradiction.

The second residue for G_1 completes a little more our sublattice (see table 5.4). Suppose $n \mid q-1$. Then $n = s-1$ and G_3 is a dihedral group. Thanks to theorem 2.5.1, we have $G_{23} = D_{2n'}$ and hence $G_2 =< s:(s-1), (s-1)n', D_{2n'} >$ must be a subgroup of Suzuki type. Then, $G_2 = Sz(s)$ and because $s:(s-1)$ is self-normalized in $Sz(q)$, we get $G_0 = G_1$, a contradiction.
Suppose then that $n \mid s^2+1$. Then $G_3 =< D_{2n}, D_{2n'}, G_{23} >$ must be a subgroup of Suzuki type, say $Sz(s')$ with $q = s'^{p'}$ and $s'-1$ and p' two prime numbers. Again, $p' \neq p$ otherwise, $s = s'$ and then $G_0 = G_2$ as in the preceding case.

G_0 $Sz(s)$	G_1 $D_{2(s-1)n'}$	G_2	G_3		
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{03} D_{2n}	G_{12} $(s-1)n'$	G_{13} $D_{2n'}$	G_{23}
G_{012} $s-1$	G_{013} 2	G_{023} 2	G_{123} n'		
G_{0123} 1					

Table 5.4: The residue 2 and G_1 a dihedral group.

G_0 $Sz(s)$	G_1 $D_{2(s-1)(s'-1)}$	G_2 $q:((s-1)(s'-1))$	G_3 $Sz(s')$		
G_{01} $D_{2(s-1)}$	G_{02} $s:(s-1)$	G_{03} D_{10}	G_{12} $(s-1)(s'-1)$	G_{13} $D_{2(s'-1)}$	G_{23} $s':(s'-1)$
G_{012} $s-1$	G_{013} 2	G_{023} 2	G_{123} s'-1		
G_{0123} 1					

Table 5.5: The only sublattice for residue 2.

This yields $G_{23} = s' : (s'-1)$ and $G_2 = q : ((s-1)(s'-1))$. Finally, thanks to lemma 4.2.8, we know that $n = 5$. The sublattice is then fully determined (see table 5.5). We don't know whether the group $Sz(q)$ acts flag-transitively on this pre-geometry, but if it does, the diagram of this pre-geometry looks as in figure 5.1.

The generalization to higher ranks is obvious. It is described in construction 5.3.1 of the next section.

Residue 3:

The sublattice is partly filled as in table 5.6. Again, we know that G_1 must be a dihedral group, and because $G_{012} \cong G_{013} \cong Z_2$, we have $G_{123} \cong Z_{n"}$ and $G_{12} \cong G_{13} \cong D_{2n"}$ thanks to theorem 2.5.1.

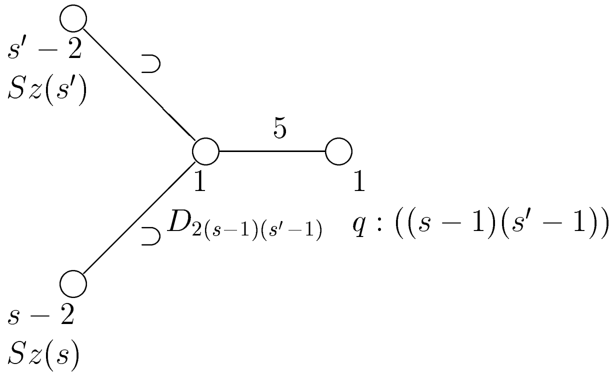


Figure 5.1: The diagram obtained for residue 2.

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $D_{2(s-1)}$	G_{02} D_{2n}	G_{03} $D_{2n'}$	G_{12}	G_{13}	G_{23}
G_{012} 2	G_{013} 2	G_{023} 2	G_{123}		
G_{0123} 1					

Table 5.6: The sublattice partly filled by residue 3

Suppose $n = s - 1$. Then $G_2 \cong G_1$ and because $N_{Sz(q)}(D_{2n"}) = D_{2n"}$, we get $G_1 = G_2$, a contradiction.
Suppose then that $n \neq s - 1 \neq n'$. Then $G_3 \cong Sz(s')$ (resp. $G_4 \cong Sz(s"))$, with $q = s'^{p'}$ (resp. $q = s"^{p"}$) and $s' - 1$ and p' (resp. $s" - 1$ and $p"$) two prime numbers. This yields $s' = s"$ and thus $G_2 = G_3$, a contradiction.

5.2.2 $s - 1$ not prime and $2f + 1$ prime

In this case, lemma 4.3.5 and theorem 4.4.2 show that there is no RWPRI pre-geometry of rank greater than 3.

5.2.3 $2f + 1$ not prime

This is the most complicated case to analyse. The rank 3 residues of G_0 are those given in theorem 4.4.3. They are subdivided into three cases depending of the primality of $t - 1$ and $2g + 1$, where $s = t^{p'}$ with p' a prime number and thus $2f + 1 = (2g + 1)p'$.

The case $t - 1$ is prime

In this case, the G_0-residues are numbers 1 to 5 of theorem 4.4.3. We analyse them one by one.

Residue 1:

The sublattice is partly filled as in table 5.7. In that table, $2f + 1 = p_1 \cdot p_2$ with $p_1 \neq p_2$, $s = t^{p_1} = t'^{p_2}$, and $t - 1$, $t' - 1$ two prime numbers. Because $Sz(s)$ is a maximal subgroup of $Sz(q)$, we have $2e + 1 = p_1p_2p_3$ with p_i prime for $i = 1$, 2, and 3. Also, G_1 (resp. G_3) contains a subgroup of Suzuki type. Thus G_1 (resp. G_3) is of Suzuki type. This yields $G_1 \cong Sz(s')$ (resp. $G_3 \cong Sz(s"))$ with $s \neq s'$ (resp. $s \neq s"$) and hence $p_3 \neq p_1$ (resp. $p_3 \neq p_2$). They both have a residue of type 1 as G_0. Now, G_2 must be a dihedral group and thanks to theorem 2.5.1, we can determine the full sublattice for this pre-geometry (see table 5.8). The generalization to higher ranks is obvious. It is described in construction 5.3.2 of the next section.

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)(t'-1)}$	G_{03} $Sz(t')$	G_{12}	G_{13}	G_{23}
G_{012} $D_{2(t-1)}$	G_{013} D_{10}	G_{023} $D_{2(t'-1)}$	G_{123}		
G_{0123} 2					

Table 5.7: The sublattice partly filled by residue 1

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $D_{2(t-1)(t'-1)(t"-1)}$	G_3 $Sz(s")$		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)(t'-1)}$	G_{03} $Sz(t')$	G_{12} $D_{2(t-1)(t"-1)}$	G_{13} $Sz(t")$	G_{23} $D_{2(t'-1)(t"-1)}$
G_{012} $D_{2(t-1)}$	G_{013} D_{10}	G_{023} $D_{2(t'-1)}$	G_{123} $D_{2(t"-1)}$		
G_{0123} 2					

Table 5.8: The only full sublattice for residue 1

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)n}$	G_{03} $s:((t-1)n)$	G_{12}	G_{13}	G_{23}
G_{012} $D_{2(t-1)}$	G_{013} $t:(t-1)$	G_{023} $(t-1)n$	G_{123}		
G_{0123} $t-1$					

Table 5.9: The sublattice partly filled by residue 4

Residue 2:

We can make the same analysis as in the previous case, just by changing D_{10} by 2^2.

The generalization to higher ranks is obvious. It is described in construction 5.3.3 of the next section.

Residue 3:

We can make the same analysis as in the previous case, just by changing 2^2 by Z_4.

The generalization to higher ranks is obvious. It is described in construction 5.3.4 of the next section.

Residue 4:

The sublattice is partly filled as in table 5.9. In that table, $n \mid \frac{s-1}{t-1}$ is a prime number, and $s = 2^{p_1 p_2}$ with p_1 and p_2 two prime numbers (not necessarily distinct). Because G_1 contains a subgroup of Suzuki type, it must be a subgroup of Suzuki type. Thus $G_1 = Sz(s')$ with $s' \neq s$ and $q = 2^{p_1 p_2 p_3}$ with $p_2 \neq p_3$, and $s' = 2^{p_1 p_3}$. The Borel subgroup of Γ_1 being $t-1$, the residue of G_1 is as the residue of G_0. The sublattice is then a little more complete (see table 5.10). Again here, $n' \mid \frac{q-1}{s'-1}$ is a prime number. Now, $G_2 =< D_{2(t-1)n}, D_{2(t-1)n'} >$ must be a dihedral group. This yields $G_{23} \cong (t-1)nn'$ and $G_2 = D_{2(t-1)nn'}$. Finally, $G_4 =< s : ((t-1)n), s' : ((t-1) : n'), (t-1)nn' >= q : ((t-1)nn')$. Thus the sublattice is fully known (see table 5.11).

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)n}$	G_{03} $s:((t-1)n)$	G_{12} $D_{2(t-1)n'}$	G_{13} $s':((t-1)n')$	G_{23} $(t-1)nn'$
G_{012} $D_{2(t-1)}$	G_{013} $t:(t-1)$	G_{023} $(t-1)n$	G_{123} $(t-1)n'$		
G_{0123} $t-1$					

Table 5.10: The residue 4 and G_1 known

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $D_{2(t-1)nn'}$	G_3 $q:((t-1)nn')$		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)n}$	G_{03} $s:((t-1)n)$	G_{12} $D_{2(t-1)n'}$	G_{13} $s':((t-1)n')$	G_{23} $(t-1)nn'$
G_{012} $D_{2(t-1)}$	G_{013} $t:(t-1)$	G_{023} $(t-1)n$	G_{123} $(t-1)n'$		
G_{0123} $t-1$					

Table 5.11: The only full sublattice for residue 4

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $(\gamma_t\gamma_{t'}):4$	G_{03} $Sz(t')$	G_{12}	G_{13}	G_{23}
G_{012} $\gamma_t:4$	G_{013} 4×2	G_{023} $\gamma_{t'}:4$	G_{123}		
G_{0123} 4					

Table 5.12: The sublattice partly filled by residue 5

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $\gamma_t\gamma_{t'}\gamma_{t"}:4$	G_3 $Sz(s")$		
G_{01} $Sz(t)$	G_{02} $(\gamma_t\gamma_{t'}):4$	G_{03} $Sz(t')$	G_{12} $(\gamma_t\gamma_{t"}):4$	G_{13} $Sz(t")$	G_{23} $(\gamma_{t'}\gamma_{t"}):4$
G_{012} $\gamma_t:4$	G_{013} 4×2	G_{023} $\gamma_{t'}:4$	G_{123} $\gamma_{t"}:4$		
G_{0123} 4					

Table 5.13: The completed sublattice for residue 5

The generalization to higher ranks is obvious. It is described in construction 5.3.5 of the next section.

Residue 5:

The sublattice is partly filled as in table 5.12. In that table, $\gamma_t = \alpha_t$ or β_t is prime and $\gamma_{t'} = \alpha_{t'}$ or $\beta_{t'}$ is also prime. Because G_{01} and G_{03} are Suzuki groups, G_1 and G_3 must also be Suzuki groups. Thus $q = 2^{p_1p_2p_3}$, $s = 2^{p_1p_2}$, $t = 2^{p_1}$, $t' = 2^{p_2}$, and $G_1 \cong Sz(s')$, $G_3 \cong Sz(s")$ with $s' = 2^{p_1p_3}$ and $s" = 2^{p_2p_3}$. The G_1 and G_3 residues are the same as the one of G_0. This yields $G_{13} \cong Sz(t")$ with $t" = 2^{p_3}$, $G_{123} \cong \gamma_{t"} : 4$ with $\gamma_{t"} = \alpha_{t"}$ or $\beta_{t"}$ a prime number, and $G_{23} \cong \gamma_{t'}\gamma_{t"} : 4$. Finally, $G_2 =< \gamma_t\gamma_{t'} : 4, \gamma_t\gamma_{t"} : 4 >$ yields $G_2 \cong \gamma_t\gamma_{t'}\gamma_{t"} : 4$. The sublattice is completely determined (see table 5.13).

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)n}$	G_{03} $s:((t-1)n)$	G_{12}	G_{13}	G_{23}
G_{012} $D_{2(t-1)}$	G_{013} $t:(t-1)$	G_{023} $(t-1)n$	G_{123}		
G_{0123} $t-1$					

Table 5.14: The sublattice partly filled by residue 6

The generalization to higher ranks is obvious. It is described in construction 5.3.6 of the next section.

The case $t-1$ not prime and $2g+1$ prime.

Theorem 4.4.3 says that residues 4 and 5 are the only possible G_0-residues. In the preceding discussion for residues 4 and 5, the primality of $t-1$ is not important. So these two cases are exactly the same, whether $t-1$ is prime or not.

The case $2g+1$ not prime.

In this case, the residues are numbers 6 to 10 of theorem 4.4.3. We analyse them one by one.

Residue 6:

The sublattice is partly filled as in table 5.14. In that table, $n \mid \frac{s-1}{t-1}$ is prime. Because of G_{01}, the subgroup G_1 must be a subgroup of Suzuki type. Its residue can only be of type 6. Also, $G_1 \cong Sz(s')$ with $s' \neq s$. The sublattice is then filled a little more (see table 5.15, where $n' \mid \frac{s'-1}{t-1}$ is prime). Now, because $G_2 =< G_{02}, G_{12}, G_{23} >$, we know that G_2 must be a dihedral group. Thanks to theorem 2.5.1, we then have $G_{23} \cong (t-1)nn'$. Thus $G_3 \cong q : ((t-1)nn')$ and the sublattice is completely determined (see table 5.16).

The generalization to higher ranks is obvious. It is described in construction 5.3.7

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)n}$	G_{03} $s:((t-1)n)$	G_{12} $D_{2(t-1)n'}$	G_{13} $s':((t-1)n')$	G_{23}
G_{012} $D_{2(t-1)}$	G_{013} $t:(t-1)$	G_{023} $(t-1)n$	G_{123} $(t-1)n'$		
G_{0123} $t-1$					

Table 5.15: Residue 6 and G_1 known

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $D_{2(t-1)nn'}$	G_3 $q:((t-1)nn')$		
G_{01} $Sz(t)$	G_{02} $D_{2(t-1)n}$	G_{03} $s:((t-1)n)$	G_{12} $D_{2(t-1)n'}$	G_{13} $s':((t-1)n')$	G_{23} $(t-1)nn'$
G_{012} $D_{2(t-1)}$	G_{013} $t:(t-1)$	G_{023} $(t-1)n$	G_{123} $(t-1)n'$		
G_{0123} $t-1$					

Table 5.16: The only sublattice for residue 6

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $D_{2(u-1)nn'}$	G_{03} $Sz(t')$	G_{12}	G_{13}	G_{23}
G_{012} $D_{2(u-1)n}$	G_{013} $Sz(u)$	G_{023} $D_{2(u-1)n'}$	G_{123}		
G_{0123} $D_{2(u-1)}$					

Table 5.17: The sublattice partly filled by residue 7

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $Sz(s")$	G_3 $D_{2(u-1)nn'n"}$		
G_{01} $Sz(t)$	G_{02} $D_{2(u-1)nn'}$	G_{03} $Sz(t')$	G_{12} $D_{2(u-1)nn"}$	G_{13} $Sz(t")$	G_{23} $D_{2(u-1)n'n"}$
G_{012} $D_{2(u-1)n}$	G_{013} $Sz(u)$	G_{023} $D_{2(u-1)n'}$	G_{123} $D_{2(u-1)n"}$		
G_{0123} $D_{2(u-1)}$					

Table 5.18: The only sublattice for residue 7

of the next section.

Residue 7:

The sublattice is partly filled as in table 5.17. In that table, $n \mid \frac{t-1}{u-1}$ and $n' \mid \frac{t'-1}{u-1}$ are prime. Thanks to subgroups G_{01} and G_{03}, we know that subgroups G_1 and G_3 are isomorphic to Suzuki simple groups. Their residue has the same form as that of G_0. Thus the sublattice is almost completed. And we readily see that G_2 must be a dihedral group. The sublattice is then completely determined (see table 5.18). There we have $t = u^{p_i}$, $t' = u^{p_j}$ and $t" = u^{p_k}$ with i, j, k three pairwise distinct integers in $\{1, \ldots, m\}$. Also, $n" \mid \frac{t"-1}{u-1}$ is a prime number.

The generalization to higher ranks is obvious. It is described in construction 5.3.8 of the next section.

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $(\gamma_u nn') : 4$	G_{03} $Sz(t')$	G_{12}	G_{13}	G_{23}
G_{012} $(\gamma_u n) : 4$	G_{013} $Sz(u)$	G_{023} $(\gamma_u n') : 4$	G_{123}		
G_{0123} $\gamma_u : 4$					

Table 5.19: The sublattice partly filled by residue 8

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $(\gamma_u nn'n") : 4$	G_3 $Sz(s")$		
G_{01} $Sz(t)$	G_{02} $(\gamma_u nn') : 4$	G_{03} $Sz(t')$	G_{12} $(\gamma_u nn") : 4$	G_{13} $Sz(t")$	G_{23} $(\gamma_u n'n") : 4$
G_{012} $(\gamma_u n) : 4$	G_{013} $Sz(u)$	G_{023} $(\gamma_u n') : 4$	G_{123} $(\gamma_u n") : 4$		
G_{0123} $\gamma_u : 4$					

Table 5.20: The only sublattice for residue 8

Residue 8:

The sublattice is partly filled as in table 5.19. In that table γ_u is either α_u or β_u and n (resp. n') is a prime number dividing $\frac{\alpha_t}{\gamma_u}$ or $\frac{\beta_t}{\gamma_u}$ (resp. $\frac{\alpha_{t'}}{\gamma_u}$ or $\frac{\beta_{t'}}{\gamma_u}$) depending whether γ_u divides α_t (resp. $\alpha_{t'}$) or β_t (resp. $\beta_{t'}$).
Because of G_{01} and G_{03}, we know that G_1 and G_3 must be Suzuki simple groups. So we may assume $G_1 \cong Sz(s')$ and $G_3 \cong Sz(s")$ with $q = s^p = s'^{p'} = s"^{p"}$ and $p, p', p"$ three pairwise distinct prime numbers. These two subgroups have residues of the same kind as G_0. Thus $G_{13} \cong Sz(t")$ with $t" = q^{\frac{1}{p'p"}}$, $G_{123} \cong (\gamma_u n") : 4$ with $n"$ a prime number dividing $\frac{\alpha_{t"}}{\gamma_u}$ or $\frac{\beta_{t"}}{\gamma_u}$. Also, $G_{12} \cong (\gamma_u nn") : 4$ and $G_{23} \cong (\gamma_u n'n") : 4$. Finally, because $G_2 =< G_{02}, G_{12}, G_{23} >$ we have $G_2 \cong (\gamma_u nn'n") : 4$. The sublattice is then fully known (see table 5.20).

The generalization to higher ranks is obvious. It is described in construction 5.3.9

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $s{\cdot}{:}u : (u-1)$	G_{03} $Sz(t')$	G_{12}	G_{13}	G_{23}
G_{012} $t{\cdot}{:}u : (u-1)$	G_{013} $Sz(u)$	G_{023} $t'{\cdot}{:}u : (u-1)$	G_{123}		
G_{0123} $u{\cdot}{:}u : (u-1)$					

Table 5.21: The sublattice partly filled by residue 9

G_0 $Sz(s)$	G_1 $Sz(s')$	G_2 $q{\cdot}{:}u : (u-1)$	G_3 $Sz(s'')$		
G_{01} $Sz(t)$	G_{02} $s{\cdot}{:}u : (u-1)$	G_{03} $Sz(t')$	G_{12} $s'{\cdot}{:}u : (u-1)$	G_{13} $Sz(t'')$	G_{23} $s''{\cdot}{:}u : (u-1)$
G_{012} $t{\cdot}{:}u : (u-1)$	G_{013} $Sz(u)$	G_{023} $t'{\cdot}{:}u : (u-1)$	G_{123} $t''{\cdot}{:}u : (u-1)$		
G_{0123} $u{\cdot}{:}u : (u-1)$					

Table 5.22: The only sublattice for residue 9

of the next section.

Residue 9:

The sublattice is partly filled as in table 5.21. Because of G_{01} (resp. G_{03}), the subgroup G_1 (resp. G_3) must be a Suzuki group, $Sz(s')$ (resp. $Sz(s'')$) with $q = s'^{p'} = s''^{p''}$ and $p' \neq p''$ two prime numbers. Also, thanks to G_{0123}, we know that their residues are as the one of G_0. This yields $G_{12} \cong s'{\cdot}{:}u : (u-1)$, $G_{13} \cong Sz(t'')$ with $q = t''^{p'p''}$, $G_{23} \cong s''{\cdot}{:}u : (u-1)$, $G_{123} \cong t''{\cdot}{:}u : (u-1)$. We then know that $G_2 \cong q{\cdot}{:}u : (u-1)$. Thus the sublattice is fully determined (see table 5.22).

The generalization to higher ranks is obvious. It is described in construction 5.3.10 of the next section.

G_0 $Sz(s)$	G_1	G_2	G_3		
G_{01} $Sz(t)$	G_{02} $Sz(t')$	G_{03} $Sz(t")$	G_{12}	G_{13}	G_{23}
G_{012} $Sz(u)$	G_{013} $Sz(u')$	G_{023} $Sz(u")$	G_{123}		
G_{0123} $Sz(v)$					

Table 5.23: The sublattice partly filled by residue 10

Residue 10:

The sublattice is partly filled as in table 5.23. This is the easiest pre-geometry to complete. It is obvious that all subgroups appearing in the sublattice must be Suzuki groups.

The generalization to higher ranks is obvious. It is described in construction 5.3.11 of the next section.

This ends the determination of all the Rwpri pre-geometries for a Suzuki simple group $Sz(q)$. We now resume the long discussion of this section in the next one.

5.3 Constructions and classification theorem

We give now the eleven constructions mentioned in the previous discussion. These are all generalizations of constructions of rank 4 pre-geometries described above. Of course, in order to use these constructions to build a Rwpri pre-geometry, the reader has to take care of the whole sublattice which is not completely detailed here, but which can be easily guessed by looking at the corresponding rank 4 pre-geometry from which it arises. After these constructions, we state a classification theorem for the Rwpri pre-geometries of rank greater than 3 for a Suzuki simple group $Sz(q)$.

Construction 5.3.1 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$. Suppose $n_i = 2^{p_i} - 1$ is prime for all $i = 1, \dots, m$.*

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ *for all* $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$;
$G_{m+1} \cong q : (n_1 \dots n_m)$,

with $\cap_{j=0}^{m-1} G_j = D_{10}$. Then $\Gamma(G; G_0, \dots, G_{m+1})$ is a rank $m+2$ RWPRI *pre-geometry.*

Construction 5.3.2 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$. Suppose $n_i = 2^{p_i} - 1$ is prime for all $i = 1, \dots, m$.*

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ *for all* $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$,

with $\cap_{j=0}^{m} G_j = D_{10}$. Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI *pre-geometry.*

Remark that the pre-geometries obtained in construction 5.3.2 are truncations of those obtained in construction 5.3.1.

Construction 5.3.3 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$. Suppose $n_i = 2^{p_i} - 1$ is prime for all $i = 1, \dots, m$.*

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ *for all* $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$,

with $\cap_{j=0}^{m} G_j = D_4$. Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI *pre-geometry.*

Construction 5.3.4 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$. Suppose $n_i = 2^{p_i} - 1$ is prime for all $i = 1, \dots, m$.*

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ *for all* $i = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$,

with $\cap_{j=0}^{m} G_j = Z_4$. Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m+1$ RWPRI *pre-geometry.*

Remark that the only difference between constructions 5.3.2, 5.3.3 and 5.3.4 is the intersection of the Suzuki subgroups.

Construction 5.3.5 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e + 1 = p_1 \dots p_{m-1} \cdot p_m^\epsilon$ with $p_i \neq p_j$ for all $i \neq j$ and $\epsilon = 1$ or 2. We distinguish the case $\epsilon = 1$ and the case $\epsilon = 2$.*

Case 1: $\epsilon = 1$

Suppose $t - 1 = 2^{p_i} - 1$ is prime for some $i \in \{1, \dots, m\}$. We construct a pre-geometry as follows.

Take $G_{j-1} \cong Sz(q^{\frac{1}{p_j}})$ *for all* $j = 1, \dots, m$, $j \neq i$;
$G_i \cong D_{2\Pi_{j\in\{1,\dots,m\}\setminus\{i\}}n_j}$ *where* $n_j \mid \frac{2^{p_ip_j}-1}{t-1}$ *are pairwise distinct prime numbers;*
$G_m \cong q : ((t-1)\Pi_{j\in\{1,\dots,m\}\setminus\{i\}}n_j)$.

Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m + 1$ RWPRI pre-geometry. The choice of i might lead to m pairwise distinct pre-geometries.

Case 2: $\epsilon = 2$

Suppose $t - 1 = 2^{p_m} - 1$ is prime. We construct a pre-geometry as follows.

Take $G_{j-1} \cong Sz(q^{\frac{1}{p_j}})$ *for all* $j = 1, \dots, m$;
$G_m \cong D_{2n_1 \dots n_m}$ *where* $n_j \mid \frac{2^{p_jp_m}-1}{t-1}$ *are pairwise distinct prime numbers;*
$G_{m+1} \cong q : ((t-1)\Pi_{j\in\{1,\dots,m\}\setminus\{i\}}n_j)$.

Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m + 2$ RWPRI pre-geometry.

Construction 5.3.6 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e + 1 = p_1 \dots p_m$ with $p_i \neq p_j$ for all $i \neq j$. Suppose that for all $i = 1, \dots, m$, at least one of $\alpha_{2^{p_i}}$ or $\beta_{2^{p_i}}$ is prime and denote that number by γ_i.*

Take $G_{i-1} \cong Sz(q^{\frac{1}{p_i}})$ *for all* $i = 1, \dots, m$;
$G_m \cong (\gamma_1 \dots \gamma_m) : 4$.

Then $\Gamma(G; G_0, \dots, G_m)$ is a rank $m + 1$ RWPRI pre-geometry.

Construction 5.3.7 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$. Let $\alpha \in \{3, \dots min(m+2, \sum_{i=1}^m e_i)\}$.*

Take $\alpha-2$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ $(i = 1, \dots, \alpha-2)$ and put $a_i = log_2(\frac{q}{s_i})$. Let $t = q^{\frac{1}{a_1 \cdots a_{\alpha-2}}}$ and take $n_i \mid \frac{t^{a_i}-1}{t-1}$ pairwise distinct prime numbers $(i = 1, \dots, \alpha-2)$. Take $G_{\alpha-2} \cong D_{2(t-1)n_1 \dots n_{\alpha-2}}$ and $G_{\alpha-1} \cong q : ((t-1)n_1 \dots n_{\alpha-2})$.

Then $\Gamma(G; G_0, \dots, G_{\alpha-1})$ is a rank α RWPRI *pre-geometry.*

Construction 5.3.8 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$. Let $\alpha \in \{3, \dots min(m+1, \sum_{i=1}^m e_i)\}$.*

Take $\alpha-1$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ with $i = 1, \dots, \alpha-1$. Put $a_i = log_2(\frac{q}{s_i})$. Let $t = q^{\frac{1}{a_1 \cdots a_{\alpha-1}}}$ and take $n_i \mid \frac{t^{a_i}-1}{t-1}$ pairwise distinct prime numbers $(i = 1, \dots, \alpha-1)$. Take $G_{\alpha-1} \cong D_{2(t-1)n_1 \dots n_{\alpha-1}}$.

Then $\Gamma(G; G_0, \dots, G_{\alpha-1})$ is a rank α RWPRI *pre-geometry.*

Construction 5.3.9 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$. Let $\alpha \in \{3, \dots min(m+1, \sum_{i=1}^m e_i)\}$.*

Take $\alpha-1$ pairwise non-isomorphic maximal subgroups $G_i \cong Sz(s_i)$ with $i = 1, \dots, \alpha-1$. Put $a_i = log_2(\frac{q}{s_i})$. Let $t = q^{\frac{1}{a_1 \cdots a_{\alpha-1}}}$. Take $\gamma = \alpha_t$ or β_t and take $n_i \mid \frac{\alpha_{t^{a_i}}}{\gamma}$ or $\frac{\beta_{t^{a_i}}}{\gamma}$pairwise distinct prime numbers $(i = 1, \dots, \alpha-1)$ such that $\gamma n_1 \dots n_{\alpha-1}$ divides either α_q of β_q. Take $G_{\alpha-1} \cong (\gamma n_1 \dots n_{\alpha-1}) : 4$.

Then $\Gamma(G; G_0, \dots, G_{\alpha-1})$ is a rank α RWPRI *pre-geometry.*

Construction 5.3.10 *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $e > 1$ a positive integer. Let $2e+1 = p_1^{e_1} \dots p_m^{e_m}$ with $p_i \neq p_j$ for all $i \neq j$. Let $\alpha \in \{3, \dots min(m+1, \sum_{i=1}^m e_i)\}$.*

Take $\alpha - 1$ *pairwise non-isomorphic maximal subgroups* $G_i \cong Sz(s_i)$ *with* $i = 1, \ldots, \alpha - 1$. *Put* $a_i = log_2(\frac{q}{s_i})$. *Let* $t = q^{\frac{1}{a_1 \cdots a_{\alpha-1}}}$.
Take $G_{\alpha-1} \cong q{:}t : (t-1)$.

Then $\Gamma(G; G_0, \ldots, G_{\alpha-1})$ *is a rank* α Rwpri *pre-geometry.*

Construction 5.3.11 *Let* $G \cong Sz(q)$ *with* $q = 2^{2e+1}$ *and* $e > 1$ *a positive integer.*
Let $2e + 1 = p_1^{e_1} \ldots p_m^{e_m}$ *with* $p_i \neq p_j$ *for all* $i \neq j$.
Let $\alpha \in \{2, \ldots m - 1\}$.

Take α *pairwise non-isomorphic maximal subgroups* $G_i \cong Sz(s_i)$ *with* $i = 1, \ldots, \alpha$.

Then $\Gamma(G; G_0, \ldots, G_{\alpha-1})$ *is a rank* α Rwpri *pre-geometry.*

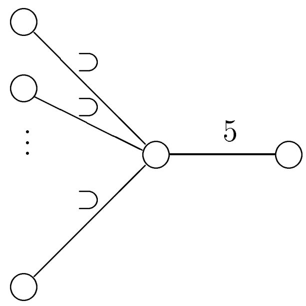
Theorem 5.3.1 *The* Rwpri *pre-geometries of rank* ≥ 4 *are those given by constructions 5.3.1 to 5.3.11.*

Proof: This is an immediate consequence of the discussion given in section 5.2 and the eleven constructions described in this section. $\square$

5.4 Observations and future work

A first observation is that for almost all Rwpri pre-geometries of rank greater than 3 of a Suzuki simple group $Sz(q)$, if they are flag-transitive geometries, then, their diagram is a star-diagram. Indeed, only those obtained by construction 5.3.11 will not have a star-diagram.

The pre-geometries given by construction 5.3.1 look very attractive. Indeed, if they are flag-transitive geometries, their diagram is fully known. It is a star diagram with one rank 2 residue being a generalized 5-gon and all other non-trivial rank 2 residues being circles. It looks as follows.



As future work, we plan to study the flag-transitivity of these Rwpri pre-geometries, and also of all those given by theorem 5.3.1.

Appendix A

The RWPRI geometries of Sz(8)

This appendix is taken from [30].

A.1 Introduction

The present appendix gives a complete classification of the firm and residually connected geometries on which the group $Sz(8)$ acts flag-transitively and residually weakly primitively.

This work continues a systematic investigation of groups that has started some years ago (see [7, 8, 9, 17, 18, 19, 20, 23]).

Here we study the smallest Suzuki simple group, namely $Sz(8)$. One of the reasons for this choice is that the infinite class of Suzuki groups $Sz(q)$, with $q = 2^{2e+1}$ and $e \geq 1$, looks particularly attractive for a general study and that a previous treatment of the smallest case may help in guessing the way to follow in general. Such a work is described in chapters 3, 4 and 5 of this thesis.

At the end of the appendix, we give the subgroup pattern of $Sz(8)$. This pattern has been computed by Michel Dehon in August 1994, using the help of the computer algebra package Cayley [10, 11].

For an introduction on the Suzuki groups, we refer to [33]. Facts about $Sz(8)$ can also be found in the Atlas of Finite Groups [14]. For a historical survey of the

Suzuki groups, we refer to [6].

Electronic Availability:
A MAGMA file containing the subgroups forming the 180 geometries obtained in this appendix, is available at the following address:
`http://cso.ulb.ac.be/~dleemans/abstracts/sz8.html`

A.2 The geometries of the Suzuki group $Sz(8)$

In this section we mention all the F, RC, FT, RWPRI geometries that can be constructed from $Sz(8)$.

Besides the numbering of each geometry, there is a number between parenthesis giving the number of non-conjugate geometries that are fused under the action of $Aut(G)$. We mention when a geometry satisfies the conditions PRI, QPRI, RPRI, $(2T)_1$. We also give diagrams of the geometries that do not satisfy the intersection property $(IP)_2$ as there are only 5 such geometries.

Up to isomorphism, there are 29 geometries of rank 2 and 151 of rank 3. There is no geometry of rank ≥ 4.

A.2.1 Rank 2 geometries

The rank 2 geometries have been determined using the classification theorem stated in chapter 3 (see theorem 3.6.1). Another way to obtain these results is the following. We look at the subgroup pattern of $Sz(8)$. We choose a subgroup G_0 maximal in G. Then we find all its maximal subgroups. Finally, all the subgroups containing a maximal subgroup of G_0 as a maximal subgroup are taken as G_1 subgroups. These couples of subgroups give us the rank 2 geometries of G. Then we still have to compute how many of them are pairwise non-isomorphic, which is a quite long (but not hard) task. And when we have all the geometries, we can compute their diagrams using the computer algebra package MAGMA. Table A.1 gives for each rank 2 geometry Γ, the rank 3 geometries containing it as a truncation, $Aut(\Gamma, G)$ and $Cor(\Gamma, G)$. The first entry is the number of the rank 2 geometry that is analysed. The second entry is a list of numbers corresponding to the rank 3 geometries that contain the corresponding rank 2 geometry as a truncation. When G_0 is not isomorphic to G_1, it is not possible to find non-type preserving automorphisms. Thus in these cases, we write nothing in the $Cor(\Gamma, G)$ column.

Theorem A.2.1 *Up to isomorphism, there are 29 firm, residually connected geometries of rank 2 on which Sz(8) acts flag-transitively and residually weakly primitively. Their diagrams are given below.*

Proof : This is a direct application of theorem 3.6.1. □

1. (3)	7 2 7		
	○———○		
	6	6	RPRI
	2080	2080	
	D_{14}	D_{14}	B = 2

2. (3)	7 3 7		
3. (3)	○———○		
4. (3)	6	6	RPRI
5. (3)	2080	2080	
6. (1)	D_{14}	D_{14}	B = 2

7. (3)	7 4 7		
	○———○		
	6	6	RPRI
	2080	2080	
	D_{14}	D_{14}	B = 2

8. (3)	6 2 6		
9. (3)	○———○		
10. (3)	12	6	QPRI
	2080	1120	
	D_{14}	D_{26}	B = 2

11. (3)	6 3 6		
12. (3)	○———○		
13. (1)	12	6	QPRI
	2080	1120	
	D_{14}	D_{26}	B = 2

14. (3)

8 2 8

4	6	QPRI
2080	2912	
D_{14}	D_{10}	B = 2

15. (3)

8 3 8

4	6	QPRI
2080	2912	
D_{14}	D_{10}	B = 2

16. (3)
17. (3)
18. (3)

8 4 8

4	6	QPRI
2080	2912	
D_{14}	D_{10}	B = 2

19. (1)

9 3 10

4	6	QPRI
2080	2912	
D_{14}	D_{10}	B = 2

20. (3)

13 5 14

1	6	
2080	7280	
D_{14}	2^2	B = 2

21. (3)

13 5 14

1	6	
2080	7280	
D_{14}	4	B = 2

22. (1)

12 6 12

1	6	
2080	7280	
D_{14}	4	B = 2

23. (1)

10 5 10

○————○

7	1	QPRI
2080	520	$(2T)_1$
D_{14}	$2^3{:}7$	B = 7

24. (3)

5 3 5

○————○

12	12	RPRI
560	560	
13:4	13:4	B = 4

25. (1)

5 3 6

○————○

12	4	RPRI
560	1456	
13:4	5:4	B = 4

26. (3)

6 4 6

○————○

12	4	RPRI
560	1456	
13:4	5:4	B = 4

27. (3)

8 5 8

○————○

1	12	QPRI
560	3640	
13:4	4×2	B = 4

28. (3)

8 5 8

○————○

4	4	RPRI
1456	1456	$(2T)_1$
5:4	5:4	B = 4

29. (3)

14 7 14

○————○

1	4	QPRI
1456	3640	$(2T)_1$
5:4	4×2	B = 4

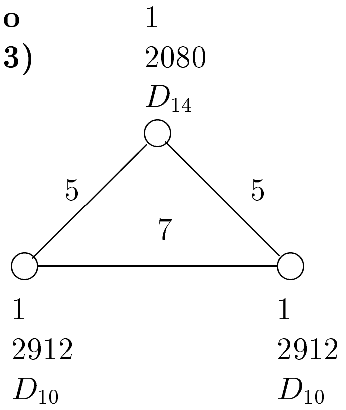
A.2.2 Rank 3 geometries

For the rank 3 case, we use the fact that all the rank 2 geometries of the subgroups of *Sz(*8*)* can easily be constructed using MAGMA.

Theorem A.2.2 *Up to isomorphism, there are 151 firm, residually connected geometries of rank 3 on which Sz(8) acts flag-transitively and residually weakly primitively. Their diagrams are given below.*

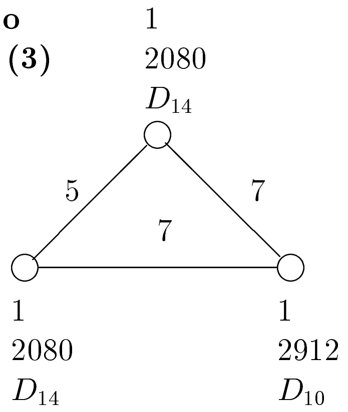
Sketch of the proof: We do not give a complete proof here since it is quite lenghty. The idea consists in looking at the rank 2 geometries of the maximal subgroups of *Sz(*8*)* which are respectively $2^{3+3} : 7$, $13 : 4$, $5 : 4$, $D_{14} \cong 7 : 2$ (see picture of the subgroup pattern at the end of the appendix).
It is easy to see that if we take $2^{3+3} : 7$ as one of the subgroups, say G_0, forming the geometry, we are not able to construct a geometry satisfying the RWPRI condition.
If we take $13 : 4$ (resp. $5 : 4$) as G_0, then there must be one of the G_{0i} isomorphic to $13 : 2 \cong D_{26}$ (resp. $5 : 2 \cong D_{10}$). By looking at the subgroup pattern we readily see that there is no possibility to find a subgroup G_i that does not contradict the RWPRI condition.
Thus the only maximal subgroup that can be taken as G_0 is $7 : 2 \cong D_{14}$. And the geometries of rank two of this group are known (see [9]). We then start from these geometries to construct configurations satisfying the RWPRI condition. And then, by using MAGMA, we test the firmness, the residual connectedness and the flag-transitivity on our configurations. The diagrams are computed with MAGMA while they could be computed by hand. $\square$

1. to 5. (3)



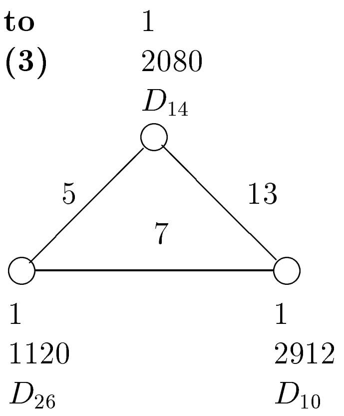
1-4: $Aut(\Gamma, G) = Cor(\Gamma, G) = Sz(8)$
5: $Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) \times 2$
QPRI
$(2T)_1$
B = 1

6. to
19. (3)



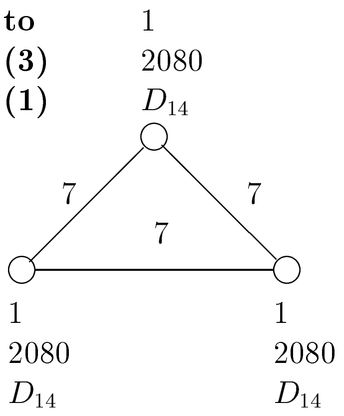
6-18: $Aut(\Gamma, G) = Cor(\Gamma, G) = Sz(8)$
19: $Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) \times 2$
QPRI
$(2T)_1$
B = 1

20. to
45. (3)



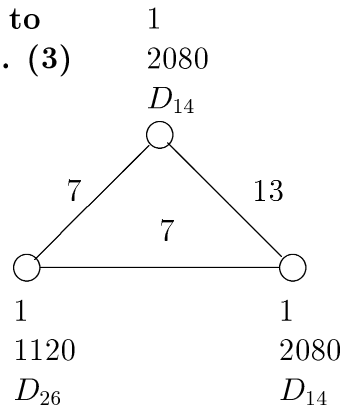
20-45: $Aut(\Gamma, G) = Sz(8)$
QPRI
$(2T)_1$
B = 1

46. to
58. (3)
59. (1)



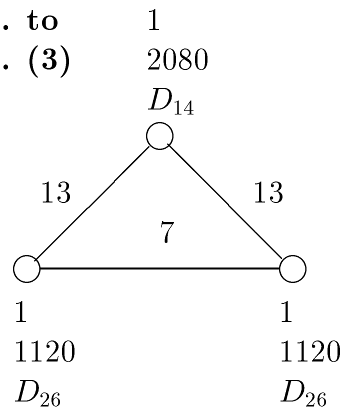
46-56: $Aut(\Gamma, G) = Cor(\Gamma, G) = Sz(8)$
57-58: $Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) \times 2$
59: $Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) : 3$
RPRI
$(2T)_1$
B = 1

60. to
101. (3)



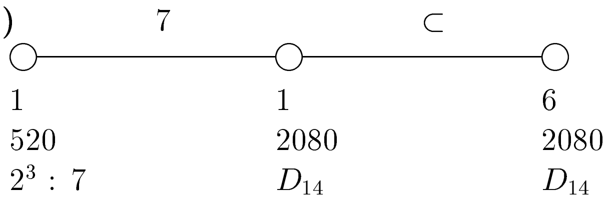
60-98: $Aut(\Gamma, G) = Cor(\Gamma, G) = Sz(8)$
99-101: $Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) \times 2$
QPRI
$(2T)_1$
B = 1

102. to
142. (3)



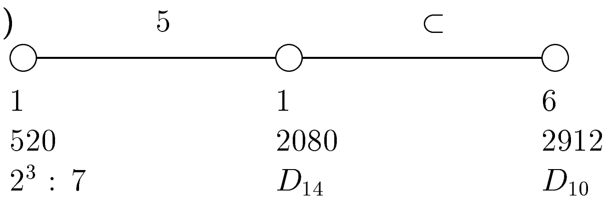
102-139: $Aut(\Gamma, G) = Cor(\Gamma, G) = Sz(8)$
140-142: $Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) \times 2$
QPRI
$(2T)_1$
B = 1

143. (3)



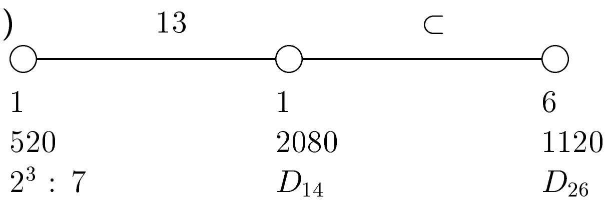
$Aut(\Gamma, G) = Sz(8)$
QPRI
B = 1

144. (1)



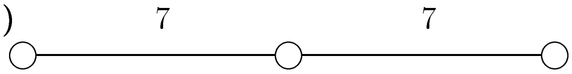
$Aut(\Gamma, G) = Sz(8) : 3$
QPRI
B = 1

145. (3)



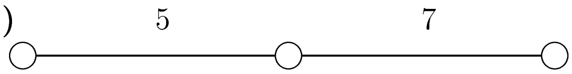
$Aut(\Gamma, G) = Sz(8)$
QPRI
B = 1

146. (3)



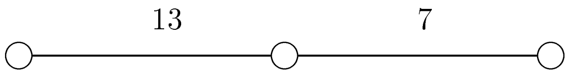
1	1	1	$Aut(\Gamma, G) = Cor(\Gamma, G) = Sz(8)$
2080	7280	2080	$(2T)_1$
D_{14}	2^2	D_{14}	B = 1

147. (3)



1	1	1	$Aut(\Gamma, G) = Sz(8)$
2080	7280	2912	$(2T)_1$
D_{14}	2^2	D_{10}	B = 1

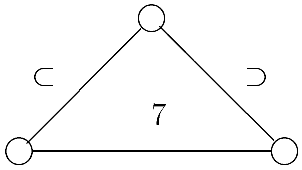
148. (3)
149. (3)
150. (3)



1	1	1	148-150: $Aut(\Gamma, G) = Sz(8)$
2080	7280	1120	$(2T)_1$
D_{14}	2^2	D_{26}	B = 1

151. (3)

6
2080
D_{14}



1	1	$Aut(\Gamma, G) = Sz(8)$ and $Cor(\Gamma, G) = Sz(8) \times 2$
520	520	QPRI
2^3:7	2^3:7	B = 1

A.2.3 Rank $\geq$ 4 geometries

Theorem A.2.3 *There is no geometry of rank ≥ 4 on which Sz(8) acts residually weakly primitively.*

Proof: The subgroup G_0 forming a geometry must be maximal in *Sz(8)*. Because 13:4, 5:4 and D_{14} do not have geometries of rank > 2, they cannot be taken as G_0. So the only candidate as G_0 is 2^{3+3}:7. It has two maximal subgroups which are 2^{3+3} and 2^3:7. But the only subgroup of *Sz(8)* containing either one of these two subgroups is G_0. So it is not possible to construct a geometry of rank > 1 with this subgroup. $\square$

No.	Rank 3 geometries containing it as a truncation	$Aut(\Gamma, G)$	$Cor(\Gamma, G)$
1	10,11,18,48,52,54,55,57(2),58(2),61,63,68,69,74,81,84,86,146	$Sz(8)$	$Sz(8)$
2	13-15,54,55,56(2),85,94	$Sz(8)$	$Sz(8) \times 2$
3	6,46-50,59(3),60,62,64,72,73,75,80,82,83,87,89	$Sz(8)$	$Sz(8)$
4	7,17,46,47,50(2),51,52(2),53,65,66,71,76,78,79,91,98,143	$Sz(8)$	$Sz(8)$
5	8,12,47,48,49(2),51,53,55,56,77,88,90,92,93,95,97	$Sz(8)$	$Sz(8)$
6	46,51,67	$Sz(8):3$	$Sz(8):3 \times 2$
7	9,16,19,53,54,57,58,70,96,99,100,101	$Sz(8)$	$Sz(8) \times 2$
8	20-27,60,62,63(2),64-71,99(2),102,104-106, 107(2),108-113,140(2),150	$Sz(8)$	
9	35-38,68,72,73(2),74(2),75-79,81,100(2),102,106,108,114, 116(2),117-126,141(2),148	$Sz(8)$	
10	39-41,61,67,75,77-79,80(2),81,84,85,90,95,97,98,101(2),110, 113,115,121,122,126,127,130,133,135,136,138,139,149	$Sz(8)$	
11	28-34,65,69,71,72,76,83,87,92,94,95,96(2),97,98,103,105, 109,111,119,123,128,131,134-136,137(2),138,139,142(2),145	$Sz(8)$	
12	42-44,60,64,66,70,82(2),83-85,86(2),87,88,89(2),90-93, 104,112,114,118,124,125,127-131,132(2),133,134	$Sz(8)$	
13	45,61,62,88,91,93,94,103,115,117,120,129	$Sz(8):3$	
14	4(2),5(2),11,17,18(2),19(2),22,25,31,147	$Sz(8)$	
15	6(2),7,8,21,24,33,35,38,39,42,43	$Sz(8)$	
16	1,2(2),3,14(2),15,16,20,26,32,36	$Sz(8)$	
17	7,8,10,12,15,16,28,29,40,41,45	$Sz(8)$	
18	3,9(2),10-13,23,27,30,37,44	$Sz(8)$	
19	1,13,17,34,144	$Sz(8):3$	
20	4,146(2),147-150	$Sz(8)$	
21		$Sz(8)$	
22		$Sz(8):3$	
23	143-145,151(2)	$Sz(8):3$	
24		$Sz(8)$	$Sz(8) \times 2$
25		$Sz(8):3$	
26		$Sz(8)$	
27		$Sz(8)$	
28		$Sz(8)$	$Sz(8) \times 2$
29		$Sz(8)$	

Table A.1: The rank 2 geometries seen as truncations of some rank 3 geometries.

Appendix B

Thin geometries for Sz(8)

This appendix is taken from [29].

B.1 Introduction

An observation that arises while classifying all firm, residually connected geometries on which a group G acts flag-transitively and fulfils some primitivity condition (as for example PRI, RPRI or RWPRI) is that when we get thin geometries, we generally get lots of them.
The classification of all the firm, residually connected geometries on which $Sz(8)$ acts flag-transitively and residually weakly primitively (RWPRI) (see Appendix A) gave us a lot of thin geometries (147 out of the 151 rank 3 ones).
The classification of all the thin geometries of $Sz(8)$ might help us to guess what is happening in the general case.

This work has first been done "by hand", that is by looking at the dihedral subgroups of $Sz(8)$ and seeing which of these subgroups could give rise to thin geometries, using MAGMA [12]. During a stay at the University of Sydney, we made a series of MAGMA programs that classify all thin residually connected geometries on which a group G acts flag-transitively. This permitted us to cross-check the results. And this is also the reason why we prefer to mention our results as a *fact* instead of a *theorem*, because no proof is given for the classification itself.

The appendix is organized as follows. In section 2, we list all the thin residually connected geometries on which $Sz(8)$ acts flag-transitively. We also give their correlation group and their rank 2 truncations. Some of these geometries can be derived from others by using a construction described in [31]. We mention when such relations occur. In section 3, we give some observations we made while looking at the results. We define a new diagram with more structure and we introduce a new property that seems interesting to impose. In section 4, we apply the Neumaier construction to obtain other geometries from those given in section 3. We mention whether they are chiral or regular. The geometries mentioned in this section can be seen as abstract polytopes which are a particular case of thin geometries. Much work has already been accomplished on abstract regular polytopes. The spherical case has been well detailed by Coxeter [15]. The toroidal case has been studied by McMullen and Shulte in a series of papers (see [35] for references on the subject). Chiral polytopes have been studied recently by Shulte and Weiss [38, 39] and Nostrand and Schulte [35].

This appendix is also a prelude to a project quite similar to [9], [7], that is, to build a new Atlas of thin geometries [24].

Electronic Availability: A MAGMA file, that contains the maximal parabolic subgroups of the 183 thin geometries given in this paper, is available at the following address.
`http://cso.ulb.ac.be/~dleemans/abstracts/sz8thin.html`

B.2 The thin geometries of $Sz(8)$

Fact B.2.1 *Up to isomorphism, there are 183 thin residually connected geometries on which the group $Sz(8)$ acts flag-transitively.*

Table B.1 lists, up to isomorphism, the thin geometries on which $Sz(8)$ acts flag-transitively. The columns a, b, and c correspond to those a, b and c appearing in the diagram given in figure B.1. The column "#CC" gives the number of conjugacy classes of geometries. These conjugacy classes are fused under $Aut(G)$. The correlation groups are given in the column $Cor(\Gamma, G)$. We do not mention the automorphism groups because $Aut(\Gamma, G) = G$ for every thin geometry Γ. We mention also when a geometry satisfies the RWPRI condition. If this condition is satisfied, the corresponding geometry was already mentioned in Appendix A.

Nr.	a	b	c	#CC	$Cor(\Gamma, G)$	Rwpri
1	2	5	7	3		Yes
2	2	5	13	3		No
3	2	7	7	3	$Sz(8)$	Yes
4-6	2	7	13	3		Yes
7	2	13	13	3	$Sz(8)$	No
8	5	5	5	1	$Sz(8){:}3$	No
9-12	5	5	7	3	$Sz(8)$	Yes
13	5	5	7	3	$Sz(8)\times 2$	Yes
14-17	5	5	13	3	$Sz(8)$	No
18	5	5	13	3	$Sz(8)\times 2$	No
19-31	5	7	7	3	$Sz(8)$	Yes
32	5	7	7	3	$Sz(8)\times 2$	Yes
33-58	5	7	13	3		Yes
59-71	5	13	13	3	$Sz(8)$	No
72	5	13	13	3	$Sz(8)\times 2$	No
73-83	7	7	7	3	$Sz(8)$	Yes
84-85	7	7	7	3	$Sz(8)\times 2$	Yes
86	7	7	7	1	$Sz(8){:}3$	Yes
87-125	7	7	13	3	$Sz(8)$	Yes
126-128	7	7	13	3	$Sz(8)\times 2$	Yes
129-166	7	13	13	3	$Sz(8)$	Yes
167-169	7	13	13	3	$Sz(8)\times 2$	Yes
170-179	13	13	13	3	$Sz(8)$	No
180-181	13	13	13	3	$Sz(8)\times 2$	No
182-183	13	13	13	1	$Sz(8){:}3$	No

Table B.1: The thin geometries of $Sz(8)$

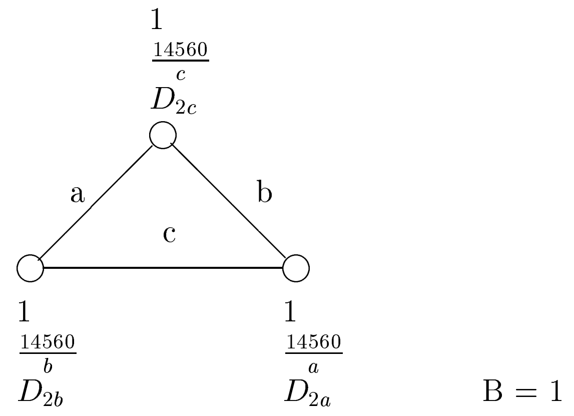


Figure B.1: The diagrams of the rank 3 thin geometries

Γ	Γ'
1	13 (replacing 2 by 5) and 32 (replacing 2 by 7)
2	18 (replacing 2 by 5) and 72 (replacing 2 by 13)
3	84 and 85 (replacing 2 by one of the two 7)
4-6	126-128 (replacing 2 by 7) and 167-169 (replacing 2 by 13)
7	180 and 181 (replacing 2 by one of the two 13)

Table B.2: The derived geometries

In [31], a concept of derived geometry is defined. It would be too long to recall that definition here. Roughly speaking, the construction is applyable to geometries having points and pairs of points as some of their varieties. If these geometries satisfy some additional conditions, we can replace the pairs of points by a copy of the points to obtain a new geometry. This process goes in the opposite way of the Neumaier construction (see section B.4) which is a linearization process. Table B.2 gives for each geometry having one of a, b or $c = 2$, the number of the derived geometry in the sense of [31].

If we decide to group geometries with their derived geometries and just count the "primitive" ones, then we have 176 geometries instead of 183.

Table B.3 lists the rank 2 geometries appearing as truncations of the thin geometries. The first column gives a number to each of them. The next two columns give

Nr.	G_0	G_1	d_{01}	g_{01}	d_{10}
1	D_4	D_{10}	16	7	15
2	D_4	D_{14}	14	5	13
3	D_4	D_{26}	10	5	10
4-5	D_{10}	D_{10}	9	4	10
6			10	5	10
7	D_{10}	D_{14}	8	2	8
8			8	3	8
9-11			8	4	8
12			10	3	9
13	D_{10}	D_{26}	6	3	6
14			8	2	8
15			8	3	8
16			8	4	8
17	D_{14}	D_{14}	7	2	7
18-22			7	3	7
23			7	4	7
24-26	D_{14}	D_{26}	6	2	6
27-29			6	3	6
30-31	D_{26}	D_{26}	6	2	6
32			6	3	6

Table B.3: Rank 2 geometries of $Sz(8)$ appearing as truncations of the thin geometries

the structure of the maximal parabolic subgroups. And the last three columns give the diameters and the gonality of these geometries.

Table B.4 gives, for each of the 183 thin geometries, its rank 2 truncations. The first column contains the number of the geometry that is analysed. The next three columns give the three truncations $\Gamma^{ij} = \Gamma(G; G_i, G_j)$. Sometimes, two truncations that are non-isomorphic have the same parameters. For these, when they appear as truncations in table B.4, we precise their number. For example, geometry 4 of table B.4 has a truncation Γ^{12} isomorphic to geometry number 25 of table B.3. A $*$ at the end of the information given in one of these three columns means that the corresponding truncation is isomorphic to the dual of the geometry given in table B.3. For example, geometry 3 of table B.4 has a truncation Γ^{12} which is isomorphic to the dual of geometry 17 appearing in table B.3.

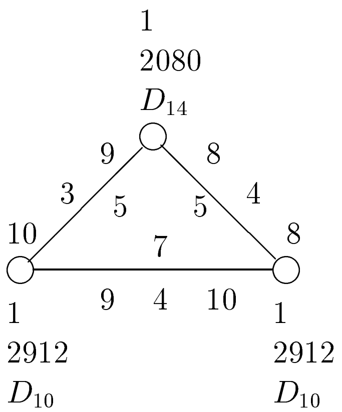
B.3 Some observations

We give in this section some observations arising from the tables given in the preceding section.

First, let us remark that we get only 15 distinct diagrams for 183 non-isomorphic geometries. It might be a good idea to add some information on our diagrams. We now decide to add three parameters on each edge of the diagram, that are the three parameters of the rank 2 truncation obtained by the two vertices of an edge.

Definition B.3.1 *The* extended diagram *of a firm, residually connected, flag-transitive geometry is a diagram (as defined in section 2), whose structure is completed by adding between every pair of vertices three parameters that are the diameters and the gonality of the corresponding rank 2 truncation.*

For example, geometry number 9 has an extended diagram as follows.



B = 1

Looking at the truncations of table B.4, we obtain 75 distinct extended diagrams.

It is of course not sufficient to look at the truncations in order to say whether two geometries are isomorphic or not. If two geometries have distinct rank 2

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
1	$16-7-15$	$14-5-13$	$8-2-8$
2	$16-7-15$	$10-5-10$	$8-2-8$
3	$14-5-13$	$14-5-13$	$7-2-7*$
4	$14-5-13$	$10-5-10$	$6-2-6^{(25)}$
5	$14-5-13$	$10-5-10$	$6-2-6^{(26)}$
6	$14-5-13$	$10-5-10$	$6-2-6^{(24)}$
7	$10-5-10$	$10-5-10$	$6-2-6^{(31)}$
8	$9-4-10^{(4)}*$	$9-4-10^{(4)}$	$9-4-10^{(4)}*$
9	$9-4-10^{(4)}*$	$8-4-8^{(9)}$	$10-3-9$
10	$9-4-10^{(4)}*$	$8-4-8^{(9)}$	$8-4-8^{(9)}$
11	$9-4-10^{(4)}*$	$8-4-8^{(9)}$	$8-4-8^{(11)}$
12	$9-4-10^{(4)}*$	$8-4-8^{(10)}$	$8-2-8$
13	$10-5-10$	$8-2-8$	$8-2-8$
14	$9-4-10^{(4)}$	$6-3-6$	$8-2-8$
15	$10-5-10$	$6-3-6$	$6-3-6$
16	$9-4-10^{(4)}*$	$6-3-6$	$8-2-8$
17	$9-4-10^{(4)}$	$8-2-8$	$8-2-8$
18	$10-5-10$	$8-2-8$	$8-2-8$
19	$8-3-8$	$8-3-8$	$7-3-7^{(19)}$
20	$8-4-8^{(10)}$	$8-3-8$	$7-3-7^{(20)}*$
21	$8-4-8^{(10)}$	$8-3-8$	$7-3-7^{(21)}$
22	$8-4-8^{(11)}$	$8-4-8^{(11)}$	$7-4-7$
23	$8-4-8^{(10)}$	$8-4-8^{(11)}$	$7-2-7*$
24	$8-2-8$	$8-4-8^{(11)}$	$7-2-7$
25	$8-4-8^{(10)}$	$8-4-8^{(11)}$	$7-3-7^{(21)}$
26	$10-3-9$	$8-4-8^{(11)}$	$7-3-7^{(18)}$
27	$8-4-8^{(9)}$	$8-4-8^{(9)}$	$7-3-7^{(18)}$
28	$8-4-8^{(10)}$	$8-4-8^{(9)}$	$7-3-7^{(18)}$
29	$8-4-8^{(10)}$	$8-4-8^{(9)}$	$7-4-7$
30	$10-3-9$	$8-2-8$	$7-3-7^{(20)}$
31	$8-2-8$	$8-2-8$	$7-2-7$
32	$8-2-8$	$8-2-8$	$7-4-7$

Table B.4: The truncations of the thin geometries

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
33	$8-4-8^{(9)}$	$8-2-8$	$6-2-6^{(24)}$
34	$8-3-8$	$6-3-6$	$6-2-6^{(24)}$
35	$8-2-8$	$6-3-6$	$6-2-6^{(24)}$
36	$8-4-8^{(11)}$	$8-2-8$	$6-2-6^{(24)}$
37	$8-3-8$	$8-2-8$	$6-2-6^{(24)}$
38	$8-2-8$	$8-2-8$	$6-2-6^{(24)}$
39	$8-4-8^{(9)}$	$8-2-8$	$6-2-6^{(24)}$
40	$8-4-8^{(11)}$	$8-3-8$	$6-2-6^{(24)}$
41	$8-4-8^{(10)}$	$8-2-8$	$6-3-6^{(27)}$
42	$8-4-8^{(10)}$	$8-3-8$	$6-3-6^{(27)}$
43	$8-4-8^{(11)}$	$6-3-6$	$6-3-6^{(27)}$
44	$8-2-8$	$6-3-6$	$6-3-6^{(27)}$
45	$8-4-8^{(9)}$	$6-3-6$	$6-3-6^{(27)}$
46	$8-3-8$	$6-3-6$	$6-3-6^{(27)}$
47	$10-3-9$	$8-2-8$	$6-3-6^{(27)}$
48	$8-3-8$	$6-3-6$	$6-2-6^{(25)}$
49	$8-4-8^{(9)}$	$8-2-8$	$6-2-6^{(25)}$
50	$8-4-8^{(11)}$	$6-3-6$	$6-2-6^{(25)}$
51	$8-3-8$	$8-4-8$	$6-2-6^{(25)}$
52	$8-3-8$	$8-3-8$	$6-2-6^{(26)}$
53	$8-4-8^{(10)}$	$6-3-6$	$6-2-6^{(26)}$
54	$8-4-8^{(10)}$	$6-3-6$	$6-2-6^{(26)}$
55	$8-3-8$	$8-3-8$	$6-3-6^{(28)}$
56	$8-3-8$	$8-3-8$	$6-3-6^{(28)}$
57	$8-4-8^{(11)}$	$8-3-8$	$6-3-6^{(28)}$
58	$8-4-8^{(10)}$	$8-3-8$	$6-3-6^{(29)}$
59	$6-3-6$	$8-3-8$	$6-2-6^{(31)}$
60	$8-2-8$	$6-3-6$	$6-2-6^{(31)}$
61	$8-2-8$	$8-2-8$	$6-2-6^{(31)}$
62	$6-3-6$	$8-2-8$	$6-2-6^{(31)}$
63	$8-3-8$	$8-2-8$	$6-2-6^{(31)}$
64	$6-3-6$	$8-2-8$	$6-2-6^{(31)}$

Table B.4: The truncations of the thin geometries *(continued)*

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
65	$8-2-8$	$8-4-8$	$6-2-6^{(31)}$
66	$6-3-6$	$8-2-8$	$6-2-6^{(31)}$
67	$6-3-6$	$6-3-6$	$6-2-6^{(31)}$
68	$8-4-8$	$8-3-8$	$6-2-6^{(31)}$
69	$6-3-6$	$8-3-8$	$6-2-6^{(31)}$
70	$8-4-8$	$6-3-6$	$6-3-6$
71	$6-3-6$	$8-3-8$	$6-3-6$
72	$8-2-8$	$8-2-8$	$6-3-6$
73	$7-3-7^{(22)}$	$7-3-7^{(19)}*$	$7-3-7^{(20)}$
74	$7-3-7^{(20)}$	$7-3-7^{(19)}*$	$7-3-7^{(21)}$
75	$7-3-7^{(21)}$	$7-3-7^{(19)}*$	$7-2-7*$
76	$7-3-7^{(21)}*$	$7-3-7^{(19)}*$	$7-3-7^{(21)}$
77	$7-3-7^{(20)}*$	$7-3-7^{(19)}*$	$7-3-7^{(20)}*$
78	$7-3-7^{(22)}$	$7-3-7^{(20)}*$	$7-3-7^{(21)}$
79	$7-3-7^{(20)}*$	$7-2-7*$	$7-3-7^{(20)}$
80	$7-3-7^{(21)}*$	$7-4-7$	$7-3-7^{(20)}$
81	$7-2-7*$	$7-4-7$	$7-3-7^{(18)}$
82	$7-3-7^{(21)}$	$7-3-7^{(18)}$	$7-2-7$
83	$7-3-7^{(21)}*$	$7-3-7^{(18)}$	$7-3-7^{(18)}$
84	$7-2-7$	$7-4-7$	$7-2-7*$
85	$7-2-7*$	$7-4-7$	$7-2-7$
86	$7-3-7^{(19)}$	$7-3-7^{(19)}*$	$7-3-7^{(19)}$
87	$7-3-7^{(19)}*$	$6-2-6^{(26)}$	$6-2-6^{(24)}$
88	$7-2-7*$	$6-2-6^{(26)}$	$6-3-6^{(29)}$
89	$7-3-7^{(19)}$	$6-3-6^{(29)}$	$6-2-6^{(24)}$
90	$7-3-7^{(21)}*$	$6-2-6^{(24)}$	$6-2-6^{(24)}$
91	$7-3-7^{(19)}$	$6-3-6^{(28)}$	$6-2-6^{(24)}$
92	$7-3-7^{(20)}$	$6-3-6^{(27)}$	$6-2-6^{(24)}$
93	$7-3-7^{(20)}*$	$6-3-6^{(28)}$	$6-2-6^{(24)}$
94	$7-3-7^{(22)}$	$6-2-6^{(26)}$	$6-2-6^{(24)}$
95	$7-2-7*$	$6-2-6^{(25)}$	$6-2-6^{(24)}$
96	$7-2-7*$	$6-3-6^{(27)}$	$6-2-6^{(24)}$

Table B.4: The truncations of the thin geometries *(continued)*

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
97	$7-4-7$	$6-3-6^{(28)}$	$6-2-6^{(24)}$
98	$7-3-7^{(20)}*$	$6-3-6^{(27)}$	$6-2-6^{(24)}$
99	$7-3-7^{(19)}*$	$6-3-6^{(27)}$	$6-2-6^{(25)}$
100	$7-3-7^{(19)}*$	$6-2-6^{(25)}$	$6-2-6^{(25)}$
101	$7-2-7*$	$6-2-6^{(25)}$	$6-2-6^{(25)}$
102	$7-3-7^{(19)}*$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
103	$7-3-7^{(20)}$	$6-3-6^{(27)}$	$6-2-6^{(25)}$
104	$7-3-7^{(21)}$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
105	$7-3-7^{(20)}*$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
106	$7-3-7^{(20)}$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
107	$7-3-7^{(19)}$	$6-2-6^{(26)}$	$6-2-6^{(26)}$
108	$7-2-7*$	$6-2-6^{(26)}$	$6-2-6^{(25)}$
109	$7-3-7^{(19)}*$	$6-3-6^{(28)}$	$6-3-6^{(28)}$
110	$7-3-7^{(19)}$	$6-3-6^{(27)}$	$6-3-6^{(28)}$
111	$7-2-7*$	$6-2-6^{(26)}$	$6-3-6^{(28)}$
112	$7-3-7^{(18)}$	$6-2-6^{(26)}$	$6-3-6^{(28)}$
113	$7-2-7$	$6-3-6^{(28)}$	$6-3-6^{(28)}$
114	$7-3-7^{(19)}*$	$6-3-6^{(27)}$	$6-3-6^{(28)}$
115	$7-3-7^{(21)}$	$6-3-6^{(29)}$	$6-3-6^{(28)}$
116	$7-3-7^{(19)}$	$6-3-6^{(28)}$	$6-3-6^{(28)}$
117	$7-3-7^{(21)}$	$6-2-6^{(26)}$	$6-3-6^{(28)}$
118	$7-3-7^{(20)}*$	$6-3-6^{(29)}$	$6-3-6^{(28)}$
119	$7-3-7^{(21)}$	$6-3-6^{(27)}$	$6-3-6^{(28)}$
120	$7-3-7^{(21)}$	$6-3-6^{(29)}$	$6-3-6^{(28)}$
121	$7-3-7^{(18)}$	$6-3-6^{(29)}$	$6-3-6^{(27)}$
122	$7-3-7^{(21)}$	$6-2-6^{(26)}$	$6-3-6^{(27)}$
123	$7-4-7$	$6-3-6^{(27)}$	$6-3-6^{(27)}$
124	$7-3-7^{(21)}*$	$6-2-6^{(26)}$	$6-3-6^{(27)}$
125	$7-3-7^{(20)}*$	$6-2-6^{(26)}$	$6-3-6^{(27)}$
126	$7-4-7$	$6-2-6^{(24)}$	$6-2-6^{(24)}$
127	$7-4-7$	$6-2-6^{(25)}$	$6-2-6^{(25)}$
128	$7-4-7$	$6-2-6^{(26)}$	$6-2-6^{(26)}$

Table B.4: The truncations of the thin geometries *(continued)*

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
129	$6-2-6^{(25)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
130	$6-3-6^{(29)}$	$6-3-6^{(27)}$	$6-2-6^{(30)}$
131	$6-3-6^{(28)}$	$6-2-6^{(24)}$	$6-2-6^{(30)}$
132	$6-3-6^{(27)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
133	$6-2-6^{(25)}$	$6-2-6^{(24)}$	$6-2-6^{(30)}$
134	$6-2-6^{(24)}$	$6-2-6^{(24)}$	$6-2-6^{(30)}$
135	$6-2-6^{(25)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
136	$6-3-6^{(27)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
137	$6-2-6^{(26)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
138	$6-3-6^{(27)}$	$6-2-6^{(24)}$	$6-3-6$
139	$6-3-6^{(28)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
140	$6-2-6^{(26)}$	$6-2-6^{(24)}$	$6-2-6^{(31)}$
141	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
142	$6-2-6^{(26)}$	$6-3-6^{(29)}$	$6-2-6^{(31)}$
143	$6-2-6^{(25)}$	$6-2-6^{(25)}$	$6-2-6^{(31)}$
144	$6-3-6^{(29)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
145	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-3-6$
146	$6-3-6^{(27)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
147	$6-3-6^{(29)}$	$6-2-6^{(25)}$	$6-3-6$
148	$6-2-6^{(26)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
149	$6-2-6^{(26)}$	$6-2-6^{(25)}$	$6-2-6^{(31)}$
150	$6-3-6^{(27)}$	$6-2-6^{(25)}$	$6-3-6$
151	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
152	$6-3-6^{(28)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
153	$6-2-6^{(26)}$	$6-2-6^{(25)}$	$6-2-6^{(30)}$
154	$6-2-6^{(26)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
155	$6-3-6^{(27)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
156	$6-3-6^{(29)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
157	$6-2-6^{(26)}$	$6-3-6^{(28)}$	$6-3-6$
158	$6-3-6^{(27)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
159	$6-3-6^{(28)}$	$6-3-6^{(28)}$	$6-2-6^{(31)}$
160	$6-2-6^{(26)}$	$6-3-6^{(28)}$	$6-2-6^{(30)}$

Table B.4: The truncations of the thin geometries *(continued)*

Nr.	Γ^{01}	Γ^{02}	Γ^{12}
161	$6-3-6^{(27)}$	$6-3-6^{(28)}$	$6-3-6$
162	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(31)}$
163	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(31)}$
164	$6-3-6^{(27)}$	$6-3-6^{(27)}$	$6-2-6^{(30)}$
165	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(31)}$
166	$6-2-6^{(26)}$	$6-3-6^{(27)}$	$6-2-6^{(30)}$
167	$6-2-6^{(24)}$	$6-2-6^{(24)}$	$6-3-6$
168	$6-2-6^{(25)}$	$6-2-6^{(25)}$	$6-3-6$
169	$6-2-6^{(26)}$	$6-2-6^{(26)}$	$6-3-6$
170	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-3-6$
171	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-2-6^{(30)}$
172	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
173	$6-3-6$	$6-2-6^{(31)}$	$6-2-6^{(30)}$
174	$6-3-6$	$6-2-6^{(31)}$	$6-3-6$
175	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-3-6$
176	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
177	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
178	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-3-6$
179	$6-3-6$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
180	$6-2-6^{(31)}$	$6-2-6^{(31)}$	$6-3-6$
181	$6-3-6$	$6-2-6^{(31)}$	$6-2-6^{(31)}$
182	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-2-6^{(30)}$
183	$6-2-6^{(30)}$	$6-2-6^{(31)}$	$6-2-6^{(30)}$

Table B.4: The truncations of the thin geometries *(continued)*

truncations, they are non-isomorphic. But the converse is not necessarily true. There are several examples appearing here. The geometries having three rank 2 truncations in common are (33,39), (53,54), (55,56), (59,69), (62,64,66), (84,85), (105,106), (109,116), (115,120), (129,135), (132,136), (137,140), (141,151,152), (148,153), (155,158), (162,163,165).

If we observe table B.4, we see that a lot of truncations do not satisfy the $(IP)_2$ condition. A new condition to impose on our geometries is the following.

Condition B.3.1 *Γ satisfies the* strong intersection property of rank 2 $(SIP)_2$ *if Γ is $(IP)_2$ and for every edge of the diagram of Γ, the corresponding truncation satisfies the $(IP)_2$ condition.*

Remark that when a geometry has a rank 2 residue which is a generalized digon, the rank 2 truncation obtained by the two vertices that are not joined by an edge does not satisfy $(IP)_2$ or is a generalized digon. This is why we restrict ourselves to truncations corresponding to edges of the diagram. Out of the 183 thin geometries of $Sz(8)$, only 50 are $(SIP)_2$. This new condition seems attractive because the intersection property (IP) implies it for every geometry of finite rank.

Theorem B.3.1 *Let Γ be a geometry of rank $n < \infty$. If Γ satisfies (IP), then Γ satisfies $(SIP)_2$.*

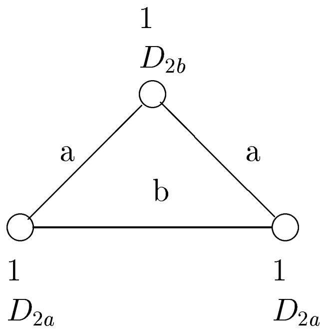
Proof: Let I be the set of types of Γ. Suppose Γ is not $(SIP)_2$. To show that Γ cannot be (IP), we suppose it is (IP) and we derive a contradiction. Because Γ is not $(SIP)_2$, there exist two types 0 and 1 in I such that the residue of a flag of type $I \setminus \{0,1\}$ has gonality at least 3 and the truncation $\Gamma^{\{0,1\}}$ has a circuit of length 4. Hence there are two 1-elements, say e and e', such that $\mid \sigma_0(e) \cap \sigma_0(e') \mid \geq 2$. Because Γ satisfies (IP), it must have a flag F such that $e * F * e'$ and $\sigma_0(F) = \sigma_0(e) \cap \sigma_0(e')$. Thus $\sigma_0(F)$ has at least two 0-elements. It implies that F is a flag of type $t(F) \subseteq I \setminus \{0,1\}$.
If $t(F) = I \setminus \{0,1\}$, then the geometry Γ_F is not $(IP)_2$ and thus Γ cannot be (IP), a contradiction.
Otherwise, the geometry Γ_F, whose rank is smaller than the rank of Γ, satisfies (IP) and does not satisfy $(SIP)_2$. We may then suppose $\Gamma = \Gamma_F$ and start the discussion again until we get the contradiction. $\square$

In a first version of this paper, we stated theorem B.3.1 for flag-transitive geometries. Antonio Pasini showed us that it was not necessary to impose the flag-transitivity in the hypotheses of this theorem.

Because buildings satisfy (IP), it might be interesting to impose this new property. Also, theorem B.3.1 shows that $(SIP)_2$ is a necessary condition for (IP). This is very helpful because (IP) is not easy to test with a computer whereas $(SIP)_2$ is. For example, the second rank 4 geometry mentionned in [23] for the Janko group J_1 cannot satisfy (IP) because one of its rank 2 truncations, namely the one with $PSL(2,11)$ and $S_3 \times D_{10}$ is not $(IP)_2$ and hence this geometry is not $(SIP)_2$.

B.4 Applying the Neumaier construction

This construction is detailed in [34]. It would be too long to recall it here. We just give a rough idea of how it works. We start from a diagram as the one given below.

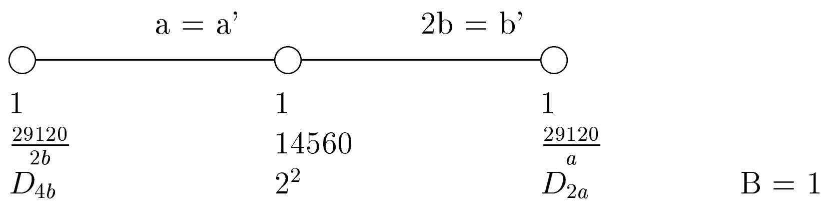


B = 1

Applying the Neumaier construction to a geometry corresponding to the diagram given above, we obtain another geometry whose diagram is given below, for the group $Sz(8)\times 2$. This group acts chirally or regularly on the geometry, depending on the correlation group of the starting geometry. Roughly speaking, if the starting geometry has a duality involving the two D_{2a}, then $Sz(8)\times 2$ acts regularly on the new geometry. Otherwise, it acts chirally. Remark that the starting diagram was a triangle whereas the diagram of the geometry obtained by the Neumaier construction is linear. This construction may be seen as a linearisation process of the diagram of a geometry.

Nr.	a	b	c	a’	b’	Chiral or Regular
3	2	7	7	4	7	Chiral
7	2	13	13	4	13	Chiral
8	5	5	5	3	5	Chiral
9-12	5	5	7	5	14	Chiral
13	5	5	7	5	14	Regular
14-17	5	5	13	5	26	Chiral
18	5	5	13	5	26	Regular
19-31	5	7	7	7	10	Chiral
32	5	7	7	7	10	Regular
59-71	5	13	13	10	13	Chiral
72	5	13	13	10	13	Regular
73-83	7	7	7	7	14	Chiral
84-85	7	7	7	7	14	Regular
86	7	7	7	3	7	Chiral
87-125	7	7	13	7	26	Chiral
126-128	7	7	13	7	26	Regular
129-166	7	13	13	13	14	Chiral
167-169	7	13	13	13	14	Regular
170-179	13	13	13	13	26	Chiral
180-181	13	13	13	13	26	Regular
182-183	13	13	13	3	13	Chiral

Table B.5: The geometries obtained by the Neumaier construction



In table B.5 we give for each thin geometry obtained in section B.2 the geometry obtained by applying this construction (when the construction is applyable) and we mention whether the group $Sz(8)\times 2$ acts chirally or regularly.

Remark that, according to Francis Buekenhout, Jacques Tits arrived at the same construction independently.

Bibliography

[1] M. Aschbacher. Flag structures on Tits geometries. *Geom. Dedicata*, 14:21–31, 1983.

[2] W. Bosma, J. Cannon, and C. Playoust. The Magma Algebra System I: the user language. *J. Symbolic Comput.*, (3/4):235–265, 1997.

[3] F. Buekenhout. Diagrams for geometries and groups. *J. Combin. Theory Ser. A*, 27:121–151, 1979.

[4] F. Buekenhout. (g, d, d*)-gons. In Johnson N.L., Kallaher M.J., and Long C.T., editors, *Finite Geometries*, pages 93–102, Marcel Dekker, New York, 1983.

[5] F. Buekenhout, editor. *Handbook of Incidence Geometry.* Elsevier, Amsterdam, 1995.

[6] F. Buekenhout. About the history of the Suzuki Groups. Preprint, 1996.

[7] F. Buekenhout, P. Cara, and M. Dehon. Geometries of small almost simple groups based on maximal subgroups. *Bull. Belg. Math. Soc. - Simon Stevin Suppl.*, 1998.

[8] F. Buekenhout, M. Dehon, and D. Leemans. All geometries of the Mathieu group M_{11} based on maximal subgroups. *Experiment. Math.*, 5:101–110, 1996.

[9] F. Buekenhout, M. Dehon, and D. Leemans. An Atlas of residually weakly primitive geometries for small groups. *Mém. Acad. Royale Belg., Classe des Sciences*, 1996. To appear.

[10] J. Cannon. CAYLEY*: A language for group theory.* Department of Pure Mathematics, University of Sydney, December 1982.

[11] J. Cannon and W. Bosma. CAYLEY*: quick reference guide*. Department of Pure Mathematics, University of Sydney, 1991.

[12] J. Cannon and W. Bosma. *Handbook of* MAGMA *functions*. Department of Pure Mathematics, University of Sydney, November 1994.

[13] J. Cannon and C. Playoust. *An introduction to* MAGMA. Department of Pure Mathematics, University of Sydney, 1993.

[14] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson. *Atlas of Finite Groups*. Oxford U.P., 1985.

[15] H.S.M. Coxeter. *Regular Polytopes*. 3rd. Ed. Dover, New York, 1973.

[16] H.S.M. Coxeter and W.O.J. Moser. *Generators and relations for discrete groups*. Springer-Verlag, second edition, 1965.

[17] M. Dehon. Classifying geometries with Cayley. *J. Symbolic Comput.*, 17:259–276, 1994.

[18] M. Dehon and X. Miller. The residually weakly primitive and $(\mathrm{IP})_2$ geometries of M_{11}. In preparation.

[19] M. Dehon and X. Miller. The residually weakly primitive and $(\mathrm{IP})_2$ geometries of U(4,2). In preparation.

[20] H. Gottschalk. *A classification of geometries associated with PSL(3,4)*. Diplomarbeit, Giessen, 1995.

[21] H. Gottschalk. A connection between six distinguished diagrams. *Atti Sem. Mat. Fis. Univ. Modena*, 1996.

[22] H. Gottschalk. Rank three geometries associated with PSL(3,4). *Bull. Belg. Math. Soc. Simon Stevin*, 3:147–160, 1996.

[23] H. Gottschalk and D. Leemans. The residually weakly primitive geometries of the Janko group J_1. In A. Pasini et al., editor, *Groups and Geometries*, pages 65–79. Birkhäuser, 1998.

[24] D. Leemans. An atlas of regular thin geometries for small groups. Preprint, 1997.

[25] D. Leemans. The residually weakly primitive geometries of the dihedral groups. Preprint, 1997.

[26] D. Leemans. The rank 2 geometries of the simple Suzuki groups Sz(q). *Beiträge Algebra Geom.*, 39(1):97–120, 1998.

[27] D. Leemans. The rank 3 geometries of the simple Suzuki groups Sz(q). Preprint, 1998.

[28] D. Leemans. The residually weakly primitive pre-geometries of the simple Suzuki groups Sz(q). Preprint, 1998.

[29] D. Leemans. Thin geometries for the Suzuki simple group Sz(8). In F. De Clerck et al., editor, *Proc. Third Int. Conf. on Finite Geometries and Combinatorics*, volume 5, pages 373–387. Bull. Belg. Math. Soc. - Simon Stevin, 1998.

[30] D. Leemans. The residually weakly primitive geometries of the Suzuki simple group Sz(8). In Campbell C.M., Robertson E.F, Ruskuc N., and Smith G.C., editors, *Proceedings of Groups St Andrews 1997 in Bath.* CUP, To appear.

[31] C. Lefèvre-Percsy, N. Percsy, and D. Leemans. New geometries for finite groups and polytopes. Preprint, 1998.

[32] H. Lüneburg. *Die Suzukigruppen und ihre Geometrien.* Lecture Notes in Math., Vol.10, Springer, Berlin, 1965.

[33] H. Lüneburg. *Translation Planes.* Springer-Verlag, New York, 1980.

[34] A. Neumaier. Rectagraphs, diagrams, and Suzuki's sporadic simple group. In E. Mendelsohn, editor, *Algebraic and Geometric Combinatorics*, volume 15 of *Ann. Discrete Math.*, pages 305–318, 1982.

[35] B. Nostrand and E. Schulte. Chiral polytopes from hyperbolic honeycombs. *Discrete Comput. Geom.*, 13:17–39, 1995.

[36] K. Rosenbaum. Die Untergruppen der Diedergruppe. *Math. Semesterber.*, 42:133–137, 1995.

[37] P. Samuel. *Théorie algébrique des nombres.* Hermann, Paris, second edition, 1971.

[38] E. Schulte and A.I.Weiss. Chiral polytopes. *In Applied Geometry and Discrete Mathematics. The Victor Klee Festschrift. Editors: P. Gritzmann and B. Sturmfels, DIMACS Series, AMS-ACM*, 4:493–516, 1991.

[39] E. Schulte and A.I.Weiss. Chirality and projective linear groups. *Discrete Math.*, 131:221–261, 1994.

[40] M. Suzuki. On a class of doubly transitive groups. *Ann. of Math.*, 75:105–145, 1962.

[41] J. Tits. Géométries polyédriques et groupes simples. *Atti 2a Riunione Groupem. Math. Express. Lat. Firenze*, pages 66–88, 1962.

[42] J. Tits. Ovoïdes et groupes de Suzuki. *Arch. Math.*, 13:187–198, 1962.