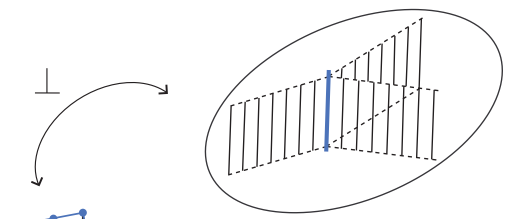
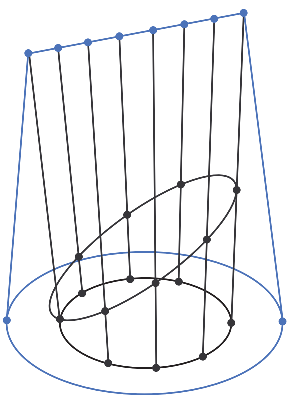
UNIVERSITEIT GENT

Vakgroep Zuivere Wiskunde
en Computeralgebra

m-Systems of Finite Classical Polar Spaces





Deirdre Luyckx

Promotor: Prof. Dr. J. A. Thas

Oktober 2002

Proefschrift voorgelegd
aan de Faculteit Wetenschappen
tot het behalen van de graad van
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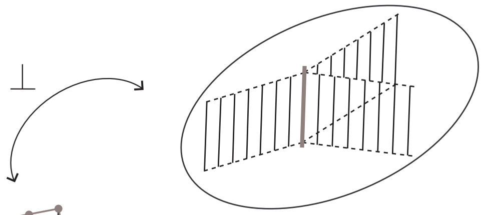

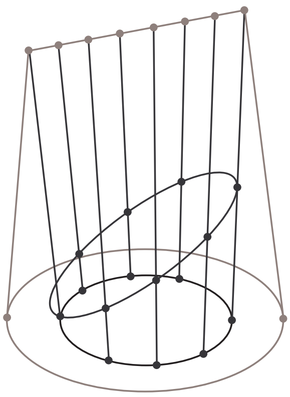

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Geometry

I prove a theorem and the house expands:
the windows jerk free to hover near the ceiling,
the ceiling floats away with a sigh.

As the walls clear themselves of everything
but transparency, the scent of carnations
leaves with them. I am out in the open

and above the windows have hinged into butterflies,
sunlight glinting where they've intersected.
They are going to some point true and unproven.

by Rita Dove[1]

1^3

Ontdaan van aanvang en vergaan,
moet gij mij tegenwoordig zijn;
val ik tesamen met uw lijn:
wij hebben twee punten gemeen:
geboorte en kennismaking.
Bij de derde aanraking,
die in het lichaam zich voltrok,
lig ik met u in hetzelfde vlak.
De vierde keer dat ik u vind
is bij mijn dood, dan zijn wij in 't
kubiek bijeen en nergens is
de een den ander een gemis.

door Gerrit Achterberg[2]

[1] from *Selected Poems*, New York, Vintage Books, 1993

[2] uit *Energie*, in *Verzamelde Gedichten*, Amsterdam, Querido, 1985, blz. 534

Prelude

The beauty of geometry inspires people. Not only those who have chosen to dedicate (part of) their lives to the study of geometrical structures, but also artists find in mathematics, and in particular in geometry, a source of inspiration. Examples are numerous: the geometric ornaments decorating Moresque architecture, the famous drawings by M. C. Escher — which are very popular among mathematicians —, the poems cited on the previous page, ...

Geometry tickles one's imagination. This is one of the reasons why I felt particularly attracted by this field of mathematics: it does not abound in long and ugly formulas, but leaves room to compose your own mental pictures of the objects you study. The drawing on the cover of this thesis illustrates this. It symbolizes the method of constructing a 1-system of $Q(6,q)$, q odd, from a rational normal cubic scroll as it is described in Chapter 4; it is in fact a representation of my own mental image of this method. Thus it also stands for what I like most in geometry: its graphicalness.

Conducting research in finite geometry, however, involves a lot more than drawing beautiful pictures. The thesis that lies in front of you is the fruit of a little more than three years of reading, listening, thinking, making calculations, trying things out, having to start all over again and finally achieving certain results, under the excellent guidance of my supervisor Prof. Dr. J. A. Thas. Let me briefly give an overview, more or less in chronological order, of the topics that came along my way.

Chapter 1 is meant as an introduction, to make the reader familiar with m-systems of finite classical polar spaces. These objects are still young; they were introduced in 1994 by Shult and Thas as a generalization of both ovoids and spreads of polar spaces. Being so recently introduced, they form a rich field of study, to which this thesis would like to add some knowledge.

In Chapter 2, an account is given of the very first small success of my three years of research. This involves the relation between m-systems and SPG reguli, for it is shown that all m-systems of $W_{2n+1}(q)$, $Q^-(2n+1,q)$

and $H(2n, q^2)$ are SPG reguli and hence give rise to semipartial geometries. After discussing the semipartial geometries arising from the known examples of m-systems, we pass on to the next topic.

A by-product of the results obtained in Chapter 2, which concerns the possible intersections of a 1-system of $Q^-(7, q)$ with a plane of this quadric, lies at the base of Chapter 3. This chapter is completely dedicated to the proof that the elliptic quadric $Q^-(7, q)$ has a unique 1-system if q is odd. This unique 1-system is the well-known classical one, which arises by applying the so-called "trace trick" to $Q^-(3, q^2)$, which is regarded as an ovoid of itself.

The next two chapters discuss the interesting topic of locally hermitian 1-systems of the parabolic quadric $Q(6, q)$, where Chapter 4 involves the odd values of q and Chapter 5 tackles the problem for q even. In both cases, a special kind of flock, which is called an "i-flock" for q odd, is associated with a locally hermitian 1-system of $Q(6, q)$. This technique is then used to study a known example of such a 1-system. It comes out that it is possible to construct this example geometrically, starting from its associated (i-)flock, and that this construction allows generalization. As such, a new class of locally hermitian 1-systems of $Q(6, q)$ is born, both for q odd and for q even. From further investigation, it follows that all elements of this class are semiclassical and for all values of q, a characterization result for locally hermitian, semiclassical 1-systems of $Q(6, q)$ in terms of their associated (i-)flock is obtained.

Guided by the success of the two previous chapters, we seek for generalization of the "flock method" to locally hermitian 1-systems of other quadrics. Therefore, in Chapter 6, a special type of flock is associated with a locally hermitian 1-system of the hyperbolic quadric $Q^+(7, q)$, regardless of the parity of q. Since no suitable 1-system is known to apply this technique to, a possible flock is constructed directly and the corresponding locally hermitian 1-system of $Q^+(7, q)$ is studied. It appears to be semiclassical and here as well, a characterization result for locally hermitian, semiclassical 1-systems of $Q^+(7, q)$ can be shown.

In Chapter 7, we change perspective and study the image of 1-systems of $Q^+(7, q)$ under a triality. It is found that trialities preserve the property of being locally hermitian and, moreover, map a locally hermitian, semiclassical 1-system of a $Q(6, q)$ on $Q^+(7, q)$ onto a locally hermitian, semiclassical 1-system of $Q^+(7, q)$. This latter 1-system is not contained in a hyperplane whenever the original one is not a spread of some generalized hexagon $\mathsf{H}(q)$ on $Q(6, q)$. This provides us with an alternative description of certain locally hermitian, semiclassical 1-systems of $Q^+(7, q)$.

The one but last chapter, Chapter 8, handles "derivation" of m-systems of polar spaces. This method is a generalization to m-systems of polar spaces of the well-known technique of replacing a regulus in a line spread of a projective or polar space by its opposite regulus, in order to obtain another line spread. Unfortunately, the only non-trivial m-systems to which derivation can be applied, are 1-systems containing reguli. Still, it can be shown that derivation with respect to a regulus yields new examples of 1-systems in several cases.

To round off this thesis, Chapter 9 links the results from the previous chapters with Chapter 2, by studying the semipartial geometries that arise from the newly discovered 1-systems of $W_5(2^r)$. On the one hand, it is shown that by applying derivation with respect to a regulus to a 1-system of $W_5(2^r)$, one also finds a new semipartial geometry. On the other hand, we obtain the result that non-isomorphic, locally hermitian 1-systems of $W_5(2^r)$ define non-isomorphic semipartial geometries. Hence several classes of new semipartial geometries can be constructed from the new classes of 1-systems of $W_5(2^r)$ which have been described in this thesis.

As a bonus, Appendix A gives an overview of all known m-systems of polar spaces at the time of this writing.

The work that is included in this thesis, could not have been accomplished without the help of many people. It is beyond doubt that I am greatly indebted to the *Fund for Scientific Research – Flanders (Belgium)/Fonds voor Wetenschappelijk Onderzoek – Vlaanderen (België)* for their financial support, which made it possible to conduct scientific research as a full-time job. But above all, I would like to express my deep gratitude towards my supervisor, Prof. Dr. J. A. Thas. He provided me with promising topics for research, suggested new ideas when it seemed to me I had tried all possible lines of thought, gave me the opportunity to report on our results during international conferences, and last but not least, he opened my eyes to a lot of techniques in finite geometry of which I was not aware previously.
I have also appreciated being surrounded by a lively maths department here in Ghent, with many fellow PhD students to share the pleasant sides, but also the small worries of working on a PhD with. In this context, I am very grateful to those who were always there to assist me in my everlasting struggle with the computer. It has been a great help that a knock on the door of my neighbours often sufficed to restore peace between my computer and me.

Finally, I value a lot the non-mathematical activities shared with my family and friends, the standing encouragement I received from my parents, and the patience and (TEXnical) support Marc had to offer.

Deirdre Luyckx
September 2002

Contents

Chapter 1

Definitions and basic results

This introductory chapter is meant to explain the basic notions that are needed to understand the problems that will be handled in this thesis, and the answers we will formulate to them.

The first part of this introduction puts forward the definitions of polarities, quadrics and hermitian varieties of projective spaces, followed by an axiomatic approach to the notion of a polar space. By a famous result of Tits [77], all finite polar spaces of rank at least three are classical, which implies that they arise from either a polarity, or a quadric, or a hermitian variety. Therefore polarities, quadrics and hermitian varieties are important building stones for the subsequent work.

The next few pages turn to the main subject itself, namely m-systems of finite classical polar spaces. Much attention will be paid to the definition of m-systems, and to their first properties. This part of the introductory chapter is to a large extent based on the paper "m-Systems of Polar Spaces" on (partial) m-systems by Shult and Thas [57], in which (partial) m-systems of polar spaces were introduced and a whole series of examples were given. We proceed with some additional definitions, each of them describing a class of m-systems with particular characteristics. Especially the definition of the locally hermitian, semiclassical 1-systems is important here, because these 1-systems will play a major role in Chapters 4, 5 and 6.

The third important topic are the generalized hexagons, with special attention for the split Cayley hexagon $\mathsf{H}(q)$ which can be defined on the parabolic quadric $Q(6,q)$. This generalized hexagon is particularly interesting for the study of m-systems, because every spread of $\mathsf{H}(q)$ is at the same time a 1-system of $Q(6,q)$. Therefore, also the coordinatization of $\mathsf{H}(q)$ will be explained in this preliminary chapter, since some examples of spreads of $\mathsf{H}(q)$ use this coordinatization explicitly.

As a last point, the definition of some other geometrical objects will be given, which are less important in this thesis but nevertheless appear on various spots.

1.1 Finite classical polar spaces

In this section, the finite classical polar spaces are introduced. It is mainly based on two books: the excellent treatise *"General Galois Geometries"* by Hirschfeld and Thas, and the standard work about generalized quadrangles, *"Finite generalized quadrangles"* by Payne and Thas.

1.1.1 Polarities

Consider the n-dimensional projective space $\mathsf{PG}(n,q)$ over the finite field $\mathsf{GF}(q)$. A *collineation* of $\mathsf{PG}(n,q)$, $n \geq 2$, is a permutation of the set of all subspaces of $\mathsf{PG}(n,q)$ which preserves inclusion between subspaces of $\mathsf{PG}(n,q)$. Every collineation is induced by a semilinear transformation of the underlying vector space V. For $n=1$, a collineation of $\mathsf{PG}(1,q)$ is defined as being induced by a semilinear transformation of the underlying vector space V. Hence, with respect to a given coordinate system in $\mathsf{PG}(n,q)$, a collineation of $\mathsf{PG}(n,q)$ is determined by a non-singular $(n+1)\times(n+1)$ matrix A over the finite field $\mathsf{GF}(q)$ and an automorphism θ of $\mathsf{GF}(q)$. The group of all collineations of $\mathsf{PG}(n,q)$ is denoted by $\mathsf{P\Gamma L}(n+1,q)$. The collineations of $\mathsf{PG}(n,q)$ for which θ is the identity, are called *projectivities* and they also form a group, denoted by $\mathsf{PGL}(n+1,q)$.

A *correlation* α of $\mathsf{PG}(n,q)$, $n \geq 2$, is a permutation of the set of all subspaces of $\mathsf{PG}(n,q)$ which reverses inclusion between subspaces of $\mathsf{PG}(n,q)$; that is, $\pi_i \subseteq \pi_j$ if and only if $\pi_j^\alpha \subseteq \pi_i^\alpha$ for all subspaces π_i, π_j of $\mathsf{PG}(n,q)$. Every correlation is induced by a semilinear mapping of the underlying vector space V onto its dual V^D. For $n=1$, a correlation is defined as being induced by a semilinear mapping of the underlying 2-dimensional vector space V onto its dual. If α is a correlation of $\mathsf{PG}(n,q)$ such that α^2 is the identity, then α is called a *polarity* of $\mathsf{PG}(n,q)$.
Suppose that for a given coordinate system of $\mathsf{PG}(n,q)$, the correlation α of $\mathsf{PG}(n,q)$ is determined by the non-singular matrix A and the automorphism θ of $\mathsf{GF}(q)$. Then α maps the point $p(x_0,x_1,\ldots,x_n)$ of $\mathsf{PG}(n,q)$ onto the

hyperplane with equation $a_0X_0 + a_1X_1 + \cdots + a_nX_n = 0$, where

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = A \begin{pmatrix} x_0^\theta \\ x_1^\theta \\ \vdots \\ x_n^\theta \end{pmatrix}.$$

Conversely, α maps the hyperplane $H \equiv a_0X_0 + a_1X_1 + \cdots + a_nX_n = 0$ onto the point $p(x_0, x_1, \ldots, x_n)$, determined by

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = (A^t)^{-1} \begin{pmatrix} a_0^\theta \\ a_1^\theta \\ \vdots \\ a_n^\theta \end{pmatrix}.$$

Here A^t stands for the transpose of the matrix A. It can be shown that α is a polarity if and only if θ^2 is the identity, which we will further denote by $\mathbb{1}$, and $(A^t)^\theta = kA$ for some non-zero scalar $k \in \mathsf{GF}(q)$ with the property that $k^{1+\theta} = 1$. If $\theta = \mathbb{1}$, then it holds that $A^t = \pm A$ and if θ is not the identity, it is possible to choose the coordinate system in such a way that $(A^t)^\theta = A$. Also, if $\theta^2 = \mathbb{1}$ and $\theta \neq \mathbb{1}$, than q must be a square and θ is the automorphism $\theta : x \mapsto x^{\sqrt{q}}$. Note finally that if $\theta = \mathbb{1}$, then a matrix satisfying $A^t = -A$, with the additional property that all of its diagonal elements are equal to 0, is always singular when n is even. Hence this situation does not occur in the study of polarities.
According to the conditions on θ and A, we distinguish different types of polarities.

1. For q odd, we have the following three types of polarities.
   (a) If $\theta = \mathbb{1}$, $A^t = -A$ and n is odd, we speak of a *symplectic polarity*.
   (b) If $\theta = \mathbb{1}$ and $A^t = A$, the corresponding polarity is an *orthogonal polarity*.
   (c) If $\theta \neq \mathbb{1}$, $\theta^2 = \mathbb{1}$ and $(A^t)^\theta = A$, one obtains a *unitary* or *hermitian polarity*.
2. For q even, there also exist three types of polarities.
   (a) If $\theta = \mathbb{1}$, $A^t = A$, n is odd and all diagonal elements of A are equal to 0, the polarity is called a *symplectic polarity*.
   (b) If $\theta = \mathbb{1}$, $A^t = A$ and not all diagonal elements of A are equal to 0, we speak of a *pseudo-polarity*.

(c) If $\theta \neq \mathbb{1}$, $\theta^2 = \mathbb{1}$ and $(A^t)^\theta = A$, we again have a *unitary* or *hermitian polarity*.

If α is a polarity of $\mathsf{PG}(n,q)$, then a point p of $\mathsf{PG}(n,q)$ is an *absolute point* for α if and only if p is incident with its image p^α under α. Similarly, a hyperplane H of $\mathsf{PG}(n,q)$ is an *absolute hyperplane* for α if and only if the point H^α is a point of H. In general, a subspace D of $\mathsf{PG}(n,q)$ is an *absolute subspace* for α if and only if one of D and D^α is contained in the other. By the definition of the distinct types of polarities, one can deduce that

- all points and hyperplanes are absolute for α if α is symplectic;
- the absolute points of an orthogonal polarity satisfy an equation of the form $\sum_{i,j=0}^{n} a_{ij}X_iX_j = 0$, with not all a_{ij} equal to 0 and $a_{ji} = a_{ij}$ for all $i, j = 0, 1, \ldots, n$;
- the absolute points of a hermitian polarity satisfy an equation of the form $\sum_{i,j=0}^{n} a_{ij}X_iX_j^\theta = 0$, with not all a_{ij} equal to 0 and $a_{ji} = a_{ij}^\theta$ for all $i, j = 0, 1, \ldots, n$;
- the absolute points of a pseudo-polarity are all points of some hyperplane, while its absolute hyperplanes are all hyperplanes, containing some common point.

As pseudo-polarities do not yield finite classical polar spaces, they will not be considered in the sequel.

1.1.2 Quadrics and hermitian varieties

A *quadric* in $\mathsf{PG}(n,q)$, $n \geq 1$, is a hypersurface of degree 2 which consists of all points, the coordinates of which satisfy a quadratic equation of the form

$$\sum_{\substack{i,j=0\\ i\leq j}}^{n} a_{ij}X_iX_j = 0,$$

with not all a_{ij} equal to 0. If $n = 2$, we also speak of a *conic*; for $n = 3$ the term *"quadratic surface"* is sometimes used.
A *hermitian variety* in $\mathsf{PG}(n,q^2)$, $n \geq 1$, is a set of points, determined by an equation of the form

$$\sum_{i,j=0}^{n} a_{ij}X_iX_j^q = 0,$$

where not all a_{ij} are equal to 0 and $a_{ji} = a_{ij}^q$ for all $i, j = 0, 1, \ldots, n$.
A quadric or a hermitian variety is *singular* if there exists a change of coordinate system which reduces its equation to one in fewer variables; otherwise the quadric or hermitian variety is said to be *non-singular*.

Every non-singular hermitian variety $\mathcal{H}$ in $\mathsf{PG}(n, q^2)$ is the set of absolute points of a hermitian polarity α of $\mathsf{PG}(n, q^2)$, where α is determined by the matrix $A = (a_{ij})$ of the coefficients a_{ij} in the equation of $\mathcal{H}$, and the automorphism $\theta : x \mapsto x^q$. As we have seen above, the converse also holds: the absolute points of a hermitian polarity in $\mathsf{PG}(n, q^2)$ form a non-singular hermitian variety.
If q is odd, every non-singular quadric $\mathcal{Q}$ in $\mathsf{PG}(n, q)$ consists of the absolute points of an orthogonal polarity α of $\mathsf{PG}(n, q)$, where α is determined by the matrix $A = (a_{ij})$ of the coefficients in the equation of $\mathcal{Q}$, with $a_{ji} := a_{ij}$ for all $i \leq j$. Here as well, the converse holds: the absolute points of an orthogonal polarity in $\mathsf{PG}(n, q)$, q odd, are exactly the points of a non-singular quadric.
Suppose now that q is even and consider a non-singular quadric $\mathcal{Q}$ with equation

$$\sum_{\substack{i,j=0 \\ i \leq j}}^{n} a_{ij} X_i X_j = 0$$

in $\mathsf{PG}(n, q)$. If n is odd, then the matrix $A' = (a'_{ij})$, with $a'_{ii} = 0$ and $a'_{ij} = a_{ij} = a'_{ji}$ for all $i, j \in \{0, 1, \ldots, n\}$ with $i \leq j$, is the matrix of a symplectic polarity of $\mathsf{PG}(n, q)$. Here the converse statement is not true, since the absolute points of a symplectic polarity in $\mathsf{PG}(n, q)$ are all points of $\mathsf{PG}(n, q)$. Finally, if n is even, no polarity is associated with a non-singular quadric $\mathcal{Q}$ in $\mathsf{PG}(n, q)$.

1.1.3 Finite polar spaces

We introduce finite polar spaces from an axiomatic point of view, as follows.

Definition
A *finite abstract polar space of rank* r, $r \geq 3$, consists of a set P of elements which we call "points", and a set of subsets of P, called "subspaces", which satisfy the following axioms.

P1 Every subspace, together with the subspaces that belong to it, is isomorphic to a projective space of dimension at most $r - 1$.

P2 The intersection of any family of subspaces is again a subspace.

P3 If π is a subspace of dimension $r-1$ and p is a point not in π, then there exists a unique subspace π' of dimension $r-1$, $p \in \pi'$, such that $\pi \cap \pi'$ has dimension $r-2$. The subspace $\pi \cap \pi'$ consists of all the points of π that are contained in a subspace of dimension 1, also containing p.

P4 There exist two disjoint subspaces of dimension $r-1$.

The integer $r-1$ is also referred to as the *projective index* of the polar space. The subspaces of a finite polar space will be called *totally singular* or *totally isotropic* subspaces in the sequel.

A finite polar space of rank $r = 2$ is by definition a *generalized quadrangle*, that is, an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are disjoint non-empty sets of objects which are called "points", respectively "lines", and where I is a symmetric incidence relation ($\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$), satisfying the following axioms:

GQ1 Every point is incident with $t+1$ lines, $t \geq 1$, and every two points are incident with at most one line.

GQ2 Every line is incident with $s+1$ points, $s \geq 1$, and every two lines are incident with at most one point.

GQ3 For every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$, which are not incident, there exist a unique point $y \in \mathcal{P}$ and a unique line $M \in \mathcal{L}$ such that $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.

Generalized quadrangles are very interesting objects and a lot of work has been done on the study of them, but this goes beyond the scope of this thesis.

We proceed with some examples of finite polar spaces.

Examples of finite polar spaces

1. Consider a symplectic polarity α of $\mathsf{PG}(n,q)$, n odd and $n \geq 3$. The point set of $\mathsf{PG}(n,q)$, together with the subspaces π of $\mathsf{PG}(n,q)$ which are absolute for α and such that $\pi \subseteq \pi^\alpha$, form a polar space of rank $r = \frac{n+1}{2}$. It is called a *symplectic polar space* and denoted by $W_n(q)$. For $n = 1$, we sometimes use the notation $W_1(q)$ for the $q+1$ points of a $\mathsf{PG}(1,q)$, which can also be seen as the set of absolute points of a symplectic polarity of $\mathsf{PG}(1,q)$.

2. Consider a non-singular quadric $\mathcal{Q}$ in $\mathsf{PG}(n, q)$, with $n \geq 3$. Then the points of $\mathcal{Q}$, together with the point sets of the projective subspaces of $\mathsf{PG}(n, q)$, contained in $\mathcal{Q}$, form an *orthogonal polar space*.
   If $n = 2k$ is even, then $\mathcal{Q}$ is projectively unique and it is called a *parabolic quadric*. In this case, the corresponding polar space has rank $r = k$ and the same notation $Q(2k, q)$ is used for both the quadric and the polar space it defines. For $n = 2$, the notation $Q(2, q)$ is also used for a conic.
   If $n = 2k + 1$ is odd, then there are two types of non-singular quadrics in $\mathsf{PG}(n, q)$. The first one is called a *hyperbolic quadric* and its corresponding polar space has rank $r = k + 1$. The quadric, as well as the polar space, is denoted by $Q^+(2k + 1, q)$. The second type of quadric yields a polar space of rank $r = k$ and to this quadric, one refers as an *elliptic quadric*. For elliptic quadrics and the polar spaces they determine, the same notation $Q^-(2k + 1, q)$ is used. For $n = 1$, the notations $Q^+(1, q)$ and $Q^-(1, q)$ may sometimes turn up to denote a hyperbolic quadric in $\mathsf{PG}(1, q)$, that is, two points on a line, respectively an elliptic quadric in $\mathsf{PG}(1, q)$, which is empty.

3. Consider a non-singular hermitian variety $\mathcal{H}$ in $\mathsf{PG}(n, q^2)$, $n \geq 3$. The points of $\mathcal{H}$, together with the projective subspaces consisting entirely of points of $\mathcal{H}$, form a *hermitian polar space*. If n is odd, it has rank $r = \frac{n+1}{2}$, and for n even, its rank equals $r = \frac{n}{2}$. Both the hermitian variety and the corresponding hermitian polar space are denoted by $H(n, q^2)$. Often the notation $H(2, q^2)$ is used for a hermitian curve, while $H(1, q^2)$ stands for a Baer subline of $\mathsf{PG}(1, q^2)$.

The above examples are called the *finite classical polar spaces* in the case where $r \geq 2$, respectively the *finite classical generalized quadrangles* for $r = 2$. There are several other examples of generalized quadrangles known, but for finite classical polar spaces of rank $r \geq 3$, a complete classification has been obtained by Tits.

Theorem 1.1.1 (Tits [77]) *All finite polar spaces of rank at least three are classical.*

If q is even, the polar spaces $Q(2n, q)$ and $W_{2n-1}(q)$ are isomorphic. This can be seen by projection of $Q(2n, q)$ from its nucleus onto a hyperplane $\mathsf{PG}(2n - 1, q)$ not containing the nucleus. Between the finite classical generalized quadrangles, there exist some additional isomorphisms, see Payne and Thas [50, 3.2]:

- $W_3(q)$ is isomorphic to the point-line dual of $Q(4,q)$; it is selfdual if and only if q is even,
- $Q^-(5,q)$ is isomorphic to the point-line dual of $H(3,q^2)$.

The finite classical polar spaces have the following numbers of points:

$$\begin{aligned}
|W_n(q)| &= \frac{q^{n+1}-1}{q-1}, \\
|Q(2n,q)| &= \frac{q^{2n}-1}{q-1}, \\
|Q^+(2n+1,q)| &= \frac{(q^n+1)(q^{n+1}-1)}{q-1}, \\
|Q^-(2n+1,q)| &= \frac{(q^n-1)(q^{n+1}+1)}{q-1}, \\
|H(n,q^2)| &= \frac{(q^{n+1}+(-1)^n)(q^n-(-1)^n)}{q^2-1}.
\end{aligned}$$

The subspaces of dimension $r-1$ of a finite classical polar space $\mathcal{P}$ of rank r, that is, the totally singular subspaces of maximal dimension, are also called the *generators* of $\mathcal{P}$. We denote the set of generators of $\mathcal{P}$ by $\Sigma(\mathcal{P})$. The numbers of generators of the finite classical polar spaces are as follows:

$$\begin{aligned}
|\Sigma(W_{2n+1}(q))| &= (q+1)(q^2+1)\cdots(q^{n+1}+1), \\
|\Sigma(Q(2n,q))| &= (q+1)(q^2+1)\cdots(q^n+1), \\
|\Sigma(Q^+(2n+1,q))| &= 2(q+1)(q^2+1)\cdots(q^n+1), \\
|\Sigma(Q^-(2n+1,q))| &= (q^2+1)(q^3+1)\cdots(q^{n+1}+1), \\
|\Sigma(H(2n,q^2))| &= (q^3+1)(q^5+1)\cdots(q^{2n+1}+1), \\
|\Sigma(H(2n+1,q^2))| &= (q+1)(q^3+1)\cdots(q^{2n+1}+1).
\end{aligned}$$

1.1.4 Groups of collineations and projectivities, stabilizing a quadric or hermitian variety

Suppose that $n = 2m$ is even and consider the projectively unique non-singular quadric $Q(2m,q)$ in $\mathsf{PG}(2m,q)$. Then the collineations, respectively projectivities, of $\mathsf{PG}(2m,q)$ which stabilize $Q(2m,q)$, form a group. In this thesis, that group will be denoted by $\mathsf{P\Gamma O}(2m+1,q)$, respectively $\mathsf{PGO}(2m+1,q)$. Similarly, if $n = 2m+1$ is odd and $Q^+(2m+1,q)$, respectively $Q^-(2m+1,q)$, is the projectively unique non-singular hyperbolic, respectively elliptic, quadric in $\mathsf{PG}(2m+1,q)$, one defines the stabilizers

$\mathsf{P\Gamma O}^+(2m+2,q)$ and $\mathsf{PGO}^+(2m+2,q)$, respectively $\mathsf{P\Gamma O}^-(2m+2,q)$ and $\mathsf{PGO}^-(2m+2,q)$, of these quadrics in $\mathsf{P\Gamma L}(2m+2,q)$ and $\mathsf{PGL}(2m+2,q)$. More information about these groups, their orders, etc., and also about other aspects of the theory of quadrics, hermitian varieties and symplectic polar spaces, can be found in the excellent treatise *"General Galois Geometries"* by Hirschfeld and Thas [26].

We want to draw the attention to one particular element of the stabilizer of a quadric $\mathcal{Q}_n$ in $\mathsf{PGO}(n+1,q)$. So let $\mathcal{Q}_n$ be a non-singular quadric in $\mathsf{PG}(n,q)$ and consider a point p not on $\mathcal{Q}_n$, where p is not the nucleus of $\mathcal{Q}_n$ if both q and n are even. We define a map

$$\mu_p : \mathcal{Q}_n \to \mathcal{Q}_n$$

as follows. For a point r of $\mathcal{Q}_n$, let r^{μ_p} be the point r itself whenever the line pr is tangent to $\mathcal{Q}_n$, and let r^{μ_p} be the unique point r' if pr is a secant of $\mathcal{Q}_n$ meeting $\mathcal{Q}_n$ in r and r'. Then by Lemma 22.6.3 of Hirschfeld and Thas [26], the map μ_p can be extended to an element of the stabilizer of $\mathcal{Q}_n$ in $\mathsf{PGO}(n+1,q)$. Moreover, this extension is unique and in the sequel it will also be denoted by μ_p.

1.1.5 Some conventions on notation

Let $\mathcal{P}$ be a non-singular quadric or hermitian variety in $\mathsf{PG}(n,q)$. It is well known that there exists a tangent hyperplane to $\mathcal{P}$ at every point of $\mathcal{P}$, which is $(n-1)$-dimensional. This tangent hyperplane will be denoted by $T_p(\mathcal{P})$ throughout this thesis. Similarly, for every m-dimensional totally singular subspace α of $\mathcal{P}$, the tangent space of $\mathcal{P}$ at α is by definition the intersection of all tangent hyperplanes at points of α. This subspace is $(n-m-1)$-dimensional and will be denoted by $T_\alpha(\mathcal{P})$. If a polarity $\perp$ is associated with $\mathcal{P}$, so if $\mathcal{P}$ is not a parabolic quadric for q even, then it is a well known fact that the tangent space $T_\alpha(\mathcal{P})$ and the image $\alpha^\perp$ of α under $\perp$ coincide. Finally, for every totally isotropic m-dimensional subspace α of $\mathcal{P}$, it holds that the tangent space $T_\alpha(\mathcal{P})$ intersects $\mathcal{P}$ in a singular quadric, respectively hermitian variety, with vertex α and base $\mathcal{P}'$, where $\mathcal{P}'$ is the projectively unique non-singular polar space in $\mathsf{PG}(n-2m-2,q)$ of the same type as $\mathcal{P}$. To the singular polar space $T_\alpha(\mathcal{P}) \cap \mathcal{P}$ we will further refer as the *cone* with vertex α and base $\mathcal{P}'$ and we will often use the notation $\alpha\mathcal{P}'$ for such a cone. In general, for a subspace α of $\mathsf{PG}(n,q)$ and a point set $\mathcal{P}'$ of some subspace β of $\mathsf{PG}(n,q)$, skew to α, the notation $\alpha\mathcal{P}'$ shall stand for the cone consisting of all points on all lines joining a point of the subspace α with a point of the point set $\mathcal{P}'$.

1.2 m-Systems and partial m-systems of polar spaces

1.2.1 Definition and first results

This subsection is for the most part based on the paper "m-Systems of Polar Spaces" by Shult and Thas [57]. In this paper, they introduce m-systems and partial m-systems of finite classical polar spaces, they prove some important properties and also give a list of examples. Here we have collected their main results, but for the examples, the reader is referred to Appendix A. We start with a definition.

Definition
Let $\mathcal{P}$ be a finite classical polar space of rank $r \geq 2$. A *partial m-system* $\mathcal{M}$ of $\mathcal{P}$, with $0 \leq m \leq r-1$, is a set of m-dimensional subspaces $\pi_1, \pi_2, \ldots, \pi_t$ of $\mathcal{P}$ such that each generator of $\mathcal{P}$ containing an element $\pi_i \in \mathcal{M}$ has an empty intersection with $(\pi_1 \cup \pi_2 \cup \cdots \cup \pi_t) \setminus \pi_i$. A partial 0-system of size t is also called a *partial ovoid*, or a *cap*, or a *t-cap*; a partial $(r-1)$-system is also called a *partial spread*.

Shult and Thas show that there exists an upper bound on the size of a partial m-system, which is, surprisingly, independent of m.

Theorem 1.2.1 ([57]) *Let $\mathcal{M}$ be a partial m-system of a finite classical polar space $\mathcal{P}$. Then we have the following upper bounds:*

$$\begin{aligned}
&\textit{if} \quad \mathcal{P} = W_{2n+1}(q), && \textit{then} \quad |\mathcal{M}| \leq q^{n+1}+1, && (1.1)\\
&\textit{if} \quad \mathcal{P} = Q(2n,q), && \textit{then} \quad |\mathcal{M}| \leq q^{n}+1, && (1.2)\\
&\textit{if} \quad \mathcal{P} = Q^+(2n+1,q), && \textit{then} \quad |\mathcal{M}| \leq q^{n}+1, && (1.3)\\
&\textit{if} \quad \mathcal{P} = Q^-(2n+1,q), && \textit{then} \quad |\mathcal{M}| \leq q^{n+1}+1, && (1.4)\\
&\textit{if} \quad \mathcal{P} = H(2n,q^2), && \textit{then} \quad |\mathcal{M}| \leq q^{2n+1}+1, && (1.5)\\
&\textit{if} \quad \mathcal{P} = H(2n+1,q^2), && \textit{then} \quad |\mathcal{M}| \leq q^{2n+1}+1. && (1.6)
\end{aligned}$$

These upper bounds inspire the following definition.

Definition
If a partial m-system of a finite classical polar space $\mathcal{P}$ meets the upper bound of Theorem 1.2.1, then it is called an *m-system*. A 0-system of $\mathcal{P}$ is also known as an *ovoid* of $\mathcal{P}$, while an $(r-1)$-system, with r the rank of $\mathcal{P}$, is called a *spread*.

For the set, consisting of all points in the elements of an m-system $\mathcal{M}$, we will use the notation $\tilde{\mathcal{M}}$; so $\tilde{\mathcal{M}}$ is the union of all elements of $\mathcal{M}$.

A first important property of m-systems is the fact that all m-systems of three types of finite classical polar spaces, have two intersection numbers with respect to hyperplanes.

Theorem 1.2.2 ([57]) *Every m-system of a polar space $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$ has two intersection numbers with respect to hyperplanes, namely:*

(a) If $\mathcal{P} = W_{2n+1}(q)$ and $H = p^\perp$, where $p^\perp$ denotes the image of p with respect to the symplectic polarity of $\mathsf{PG}(2n+1,q)$ defining $W_{2n+1}(q)$, then

- $|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n$ *if p is a point of $\tilde{\mathcal{M}}$,*
- $|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1}$ *if p is a point of $\mathsf{PG}(2n+1,q) \setminus \tilde{\mathcal{M}}$.*

(b) If $\mathcal{P} = Q^-(2n+1,q)$, then

- $|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n$ *if H is the tangent hyperplane of the quadric $Q^-(2n+1,q)$ at a point $p \in \tilde{\mathcal{M}}$,*
- $|\tilde{\mathcal{M}} \cap H| = \frac{(q^{m+1}-1)(q^n+1)}{q-1}$ *if H either is the tangent hyperplane of $Q^-(2n+1,q)$ at a point $p \notin \tilde{\mathcal{M}}$, or a non-tangent hyperplane of $Q^-(2n+1,q)$.*

(c) If $\mathcal{P} = H(2n,q^2)$, then

- $|\tilde{\mathcal{M}} \cap H| = \frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - q^{2n-1}$ *if H is the tangent hyperplane of the hermitian variety $H(2n,q^2)$ at a point $p \in \tilde{\mathcal{M}}$,*
- $|\tilde{\mathcal{M}} \cap H| = \frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1}$ *if H either is the tangent hyperplane of $H(2n,q^2)$ at a point $p \notin \tilde{\mathcal{M}}$, or a non-tangent hyperplane of $H(2n,q^2)$.*

The proof of this theorem involves counting arguments which are used to show that a hyperplane of the ambient space of $\mathcal{P}$ either contains exactly one element of $\mathcal{M}$, or contains some fixed number θ (>1) of m-spaces of $\mathcal{M}$. Clearly, the first possibility occurs exactly then, when the hyperplane either is tangent to $\mathcal{P}$ at a point of $\tilde{\mathcal{M}}$, or, in the case where $\mathcal{P} = W_{2n+1}(q)$, when it is of the form $p^\perp$ for some $p \in \tilde{\mathcal{M}}$. The value of θ in the respective cases is given in the following theorem.

Theorem 1.2.3 ([57]) *Suppose that $\mathcal{M}$ is an m-system of a polar space $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$. Then the number of elements of $\mathcal{M}$ contained in a hyperplane H, which is not the image $p^\perp$ of a point $p \in \tilde{\mathcal{M}}$ with respect to the polarity defining $\mathcal{P}$, is independent of the choice of H and equals θ, with*

(a) $\theta = q^{n-m} + 1$ if $\mathcal{P} = W_{2n+1}(q)$,

(b) $\theta = q^{n-m} + 1$ if $\mathcal{P} = Q^-(2n+1,q)$,

(c) $\theta = q^{2n-2m-1} + 1$ if $\mathcal{P} = H(2n,q^2)$.

By relying on a result by Calderbank and Kantor [8], one can deduce the following corollary from Theorem 1.2.2, which supplies us with a first interesting application of m-systems of (some) polar spaces.

Corollary 1.2.4 *Every m-system of a polar space $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $H(2n,q^2)$ defines a strongly regular graph and a projective linear two-weight code.*

One can also derive a property concerning the number of points of $\tilde{\mathcal{M}}$ in a generator G of $\mathcal{P}$. If this generator contains an element of $\mathcal{M}$, then it is obvious that $|\tilde{\mathcal{M}} \cap G| = \frac{q^{m+1}-1}{q-1}$, but this apparently holds in general as well.

Theorem 1.2.5 ([57]) *If $\mathcal{M}$ is an m-system of the finite classical polar space $\mathcal{P}$, then for any generator G of $\mathcal{P}$ we have*

$$|G \cap \tilde{\mathcal{M}}| = \frac{q^{m+1}-1}{q-1}.$$

Finally, there is an important existence result that limits the values of m for which an m-system of one of the polar spaces $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $H(2n,q^2)$ can exist. It was obtained by Hamilton and Mathon [21], by investigating the parameters of the strongly regular graph which can be constructed from $\mathcal{M}$ by Corollary 1.2.4.

Theorem 1.2.6 ([21]) *The finite polar spaces $W_{2n+1}(q)$, $Q^-(2n+1,q)$ and $H(2n,q^2)$ do not have m-systems for $n > 2m+1$. If q is even, then the parabolic quadric $Q(2n+2,q)$ has no m-systems for $n > 2m+1$.*

The second statement follows from the first one, since for q even, the polar spaces $W_{2n+1}(q)$ and $Q(2n+2,q)$ are isomorphic.

In [59], one can find many other results concerning bounds on the possible sizes of partial m-systems, but if applied to m-systems of $W_{2n+1}(q)$, $Q^-(2n+1,q)$, $H(2n,q^2)$, or $Q(2n+2,q)$ with q even, they are all weaker than the above result. An overview of these bounds is given in Appendix A.

1.2.2 Some more definitions

This section discusses some definitions of special features certain m-systems might have. It will mostly concern the case where $m = 1$, but the first definition is meaningful in general.

Definition
Let $\mathcal{M}$ be an m-system of a finite classical polar space $\mathcal{P}$ in $\mathsf{PG}(n, q)$. Then $\mathcal{M}$ is said to be a *proper* m-system of $\mathcal{P}$ if and only if $\mathcal{M}$ is not contained in a hyperplane of $\mathsf{PG}(n, q)$.

Let $\mathcal{P}_n$ either be a non-singular quadric, or a non-singular hermitian variety in $\mathsf{PG}(n, q)$. Then an m-system $\mathcal{M}$ of $\mathcal{P}_n$ can never be contained in the tangent hyperplane $T_x(\mathcal{P}_n)$ of $\mathcal{P}_n$ at a point x, for the following reason. If $\mathcal{M}$ either is a spread of $\mathcal{P}_n$ or if x is a point of $\tilde{\mathcal{M}}$, then this is clear. So suppose that $m < r - 1$ with r the rank of $\mathcal{P}_n$ and that $x \notin \tilde{\mathcal{M}}$, and assume that $\mathcal{M} \subseteq T_x(\mathcal{P}_n)$. Let γ be an $(n-2)$-dimensional subspace of $T_x(\mathcal{P}_n)$ such that $x \notin \gamma$. Then $\gamma \cap \mathcal{P}_n := \mathcal{P}_{n-2}$ is a non-singular quadric, respectively a non-singular hermitian variety, of the same type as $\mathcal{P}_n$. If we project the elements of $\mathcal{M}$ from x onto γ, we obtain a set $\mathcal{M}'$ of m-dimensional totally singular subspaces of $\mathcal{P}_{n-2}$ with the property that a generator of $\mathcal{P}_{n-2}$ which contains an element of $\mathcal{M}'$, is disjoint from all other elements of $\mathcal{M}'$. This property is an immediate consequence of the fact that $\mathcal{M}$ is an m-system of $\mathcal{P}_n$. So $\mathcal{M}'$ must be a partial m-system of $\mathcal{P}_{n-2}$. But $|\mathcal{M}| = |\mathcal{M}'|$ since all elements of $\mathcal{M}$ are contained in $T_x(\mathcal{P}_n)$ by assumption, and this is impossible by Theorem 1.2.1. We conclude that $\mathcal{M}$ cannot be contained in a tangent hyperplane of $\mathcal{P}_n$.
By a similar reasoning, one deduces that all m-systems of a symplectic polar space $W_{2n+1}(q)$ are proper, since in that case every hyperplane is of the form $p^\perp$, with $\perp$ the symplectic polarity defining $W_{2n+1}(q)$, and an analogous argument can be applied to $p^\perp$.

The definitions that follow now, will describe 1-systems with particular properties. We start with the concept of being locally hermitian at some line L.

Definition
Suppose that $\mathcal{M}$ is a 1-system of a non-singular quadric $\mathcal{Q}$ in $\mathsf{PG}(n, q)$. Then $\mathcal{M}$ is *locally hermitian* at the line $L \in \mathcal{M}$ if and only if for all lines $M \in \mathcal{M} \setminus \{L\}$, the regulus containing L and M of the hyperbolic quadric $\langle L, M\rangle \cap \mathcal{Q} = Q^+(3, q)$, is completely contained in $\mathcal{M}$.

One can extend the definition of being locally hermitian to 1-systems of the symplectic polar space $W_{2n+1}(q)$, $n \geq 1$. In order to obtain a meaningful definition, one must however rule out those reguli of lines of $W_{2n+1}(q)$, the opposite regulus of which does not entirely consist of totally isotropic lines. If q is odd, no regulus of lines of $W_{2n+1}(q)$ has an opposite regulus consisting of nothing but totally isotropic lines, so locally hermitian 1-systems will not be defined in this case. We therefore suppose that q is even. Then $W_{2n+1}(q)$ can be seen as the projection of a parabolic quadric $Q(2n+2,q)$ from its nucleus n onto a $\mathsf{PG}(2n+1,q)$ not through the nucleus. Let R be a regulus of totally isotropic lines of $W_{2n+1}(q)$, then $\langle R \rangle$ is 3-dimensional and there are two possibilities. If $\langle R \rangle$ is the projection from n onto $\mathsf{PG}(2n+1,q)$ of a 4-dimensional subspace δ containing n, then the lines of R correspond to $q+1$ lines of $Q(2n+2,q)$ which span δ and do not form a regulus in δ. In this case, the opposite regulus of R does not entirely consist of totally isotropic lines of $W_{2n+1}(q)$. On the other hand, if $\langle R \rangle$ is the projection from n onto $\mathsf{PG}(2n+1,q)$ of a 3-dimensional subspace γ not through n, then $\gamma \cap Q(2n+2,q)$ is a hyperbolic quadric $Q^+(3,q)$. Hence R corresponds to a regulus of lines of $Q(2n+2,q)$ and the opposite regulus $\overline{R}$ of R consists entirely of totally isotropic lines of $W_{2n+1}(q)$. In the sequel, such a regulus of $W_{2n+1}(q)$ will be called a *strong regulus* of $W_{2n+1}(q)$. Thus the strong reguli of $W_{2n+1}(q)$ are exactly those reguli which correspond to reguli of lines of $Q(2n+2,q)$ or, equivalently, those reguli the $q+1$ transversals of which are totally isotropic. Moreover, every two disjoint lines of $W_{2n+1}(q)$ determine a unique strong regulus containing both of them.

Definition

Let $\mathcal{M}$ be a 1-system of the symplectic polar space $W_{2n+1}(q)$, $n \geq 1$ and q even, in $\mathsf{PG}(2n+1,q)$. We say that $\mathcal{M}$ is *locally hermitian* at some line $L \in \mathcal{M}$ if and only if for every line $M \in \mathcal{M} \setminus \{L\}$, the unique strong regulus of $W_{2n+1}(q)$ which contains L and M, is completely contained in $\mathcal{M}$.

If $\mathcal{M}$ is locally hermitian at L and $|\mathcal{M}| = q^t+1$, it follows from the above definitions that $\mathcal{M}$ consists of q^{t-1} reguli, respectively strong reguli, through the line L.

In view of Theorem 1.2.6, it is clear that whenever $\mathcal{P}$ is one of $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $Q(2n,2^r)$, the definition of being locally hermitian is only meaningful for the small values of n. For parabolic quadrics with q odd and for hyperbolic quadrics, there may exist (locally hermitian) 1-systems for larger values of n too.

Suppose that $\mathcal{M}$ is a 1-system of a polar space $\mathcal{P}_n$ in $\mathsf{PG}(n,q)$, which is either a non-singular quadric or a symplectic polar space for q even, and assume furthermore that $\mathcal{M}$ is locally hermitian at some line $L \in \mathcal{M}$. For an arbitrary point x on L we consider an $(n-2)$-dimensional subspace γ in the tangent hyperplane $T_x(\mathcal{P}_n)$ of $\mathcal{P}_n$ at x, respectively in $x^\perp$ with $\perp$ the symplectic polarity defining $\mathcal{P}_n$ if $\mathcal{P}_n = W_{2k+1}(q)$, but such that x does not lie in γ. Then γ intersects $\mathcal{P}_n$ in a non-singular polar space $\mathcal{P}_{n-2}$ of the same type as $\mathcal{P}_n$. If $|\mathcal{M}| = q^t + 1$, then we denote the q^{t-1} reguli, respectively strong reguli if $\mathcal{P}_n = W_{2k+1}(q)$, of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^{t-1}}$. Each regulus R_i, $i = 1, 2, \ldots, q^{t-1}$, has a unique transversal through the point x, which is a line of $\mathcal{P}_n$ and intersects the subspace γ in a point n_i. Also the line L itself has a point in common with γ; denote the point $L \cap \gamma$ by l. As such we obtain q^{t-1} points n_i in γ, and the point l; this means that there arises a set of $q^{t-1}+1$ points of $\mathcal{P}_{n-2}$. Now suppose that two points n_i and n_j for $i \neq j$, respectively n_i and l, are collinear on $\mathcal{P}_{n-2}$. This implies that $\langle x, n_i, n_j \rangle$, respectively $\langle x, n_i, l \rangle$, is a totally singular plane of $\mathcal{P}_n$ which contains at least $2q+1$ points of $\mathcal{M}$. This is a contradiction to Theorem 1.2.5, so we conclude that no two points of the set $\{l, n_1, n_2, \ldots, n_{q^{t-1}}\}$ are collinear on $\mathcal{P}_{n-2}$. But since the size of a 1-system of $\mathcal{P}_n$ is $q^t + 1$, it holds that an m-system of $\mathcal{P}_{n-2}$ has $q^{t-1} + 1$ elements, regardless of the value of m. Therefore the set $\{l, n_1, n_2, \ldots, n_{q^{t-1}}\}$ is an ovoid of $\mathcal{P}_{n-2}$. This ovoid will be denoted by $\mathcal{O}_x$ in the sequel and it is called the *projection of $\mathcal{M}$ from x along reguli.*

As ovoids of $Q^-(5,q)$, $W_5(2^r)$ and $Q(6,2^r)$ do not exist, see Appendix A, there do not exist locally hermitian 1-systems of $Q^-(7,q)$, $W_7(2^r)$ or $Q(8,2^r)$ either. For $n > 3$, the existence of 1-systems of $W_{2n+1}(2^r)$, $Q^-(2n+1,q)$ and $Q(2n+2,2^r)$ has already been ruled out by Theorem 1.2.6, so that only the case $n \leq 2$ is meaningful here.

Definition

Let $\mathcal{M}$ be a 1-system of $\mathcal{Q}_n \in \{Q^-(5,q), Q(6,q), Q^+(7,q)\}$ and suppose that $\mathcal{M}$ is locally hermitian at some line $L \in \mathcal{M}$. Then $\mathcal{M}$ is said to be *semiclassical* if and only if for all points x on L, the ovoid $\mathcal{O}_x$ is the classical ovoid of $\mathcal{Q}_{n-2}$, that is, $\mathcal{O}_x \cong Q^-(3,q)$.

Clearly, every locally hermitian spread of the elliptic quadric $Q^-(5,q)$ is semiclassical, since for every point x on L, it holds that $T_x(Q^-(5,q)) \cap Q^-(5,q)$ is a cone $xQ^-(3,q)$.

Note also that it is possible to define a similar notion for locally hermitian 1-systems of other non-singular quadrics $\mathcal{Q}_n$, provided that there exists a classical ovoid of $\mathcal{Q}_{n-2}$. However, we will only encounter the concept of being semiclassical for 1-systems of the above mentioned quadrics.

Before we proceed to the last definition of this section, we need to introduce *hermitian spreads* of the elliptic quadric $Q^-(5,q)$ in $\mathsf{PG}(5,q)$. Consider the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ and the corresponding extension $Q^+(5,q^2)$ of $Q^-(5,q)$. One easily shows that there exist $2q^3(q^2+1)(q-1)$ planes on $Q^+(5,q^2)$ which are disjoint from $\mathsf{PG}(5,q)$. Let π be such a plane and denote its conjugate plane with respect to $\mathsf{GF}(q^2)$ by $\overline{\pi}$. For every point x of $Q^-(5,q)$, there exists exactly one line L_x of $\mathsf{PG}(5,q^2)$ which intersects π and $\overline{\pi}$ in a point. Since L_x contains three points of $Q^+(5,q^2)$, it must be a line of $Q^+(5,q^2)$, and consequently $L_x \cap Q^-(5,q)$ is a line of $Q^-(5,q)$. It follows that the set of lines $\{L_x \cap Q^-(5,q) \mid x \in Q^-(5,q)\}$ is a spread of $Q^-(5,q)$. This spread is called the *hermitian* or *classical* spread of $Q^-(5,q)$. By [35], one deduces that the set of points $\{L_x \cap \pi \mid x \in Q^-(5,q)\}$ is the point set of a hermitian curve $H(2,q^2)$ in π, which gives an explanation for the name of this spread.

Finally, let $\mathcal{M}$ be a 1-system of $\mathcal{Q}$, where $\mathcal{Q}$ is either the parabolic quadric $Q(6,q)$ or the hyperbolic quadric $Q^+(7,q)$. Suppose also that $\mathcal{M}$ is locally hermitian at the line $L \in \mathcal{M}$ and denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$. Consider two distinct reguli R_i, R_j of $\mathcal{M}$ through L. Then $\langle R_i, R_j \rangle \cap \mathcal{Q}$ is an elliptic quadric $Q^-_{ij}(5,q)$, for the following reason. If it is not an elliptic quadric $Q^-_{ij}(5,q)$, then it must be a (possibly singular) quadric of rank $r \geq 3$, which implies that for any line $K \neq L$ of R_i, there exists a totally singular plane α of $\langle R_i, R_j \rangle \cap \mathcal{Q}$, containing K. But in that case, the plane α and the 3-space $\langle R_j \rangle$ have at least a point in common, which is a point of some line of R_j. This contradicts the definition of a 1-system. Hence $\langle R_i, R_j \rangle \cap \mathcal{Q}$ is indeed an elliptic quadric $Q^-_{ij}(5,q)$. This quadric $Q^-_{ij}(5,q)$ defines a generalized quadrangle which is the point-line dual of the generalized quadrangle $H(3,q^2)$ arising from a non-singular hermitian variety in $\mathsf{PG}(3,q^2)$, as has already been mentioned. With R_i and R_j correspond point sets W_i and W_j on $H(3,q^2)$, which are Baer sublines of lines in $\mathsf{PG}(3,q^2)$ and have exactly one point of $H(3,q^2)$ in common. So W_i and W_j are contained in exactly one common plane which intersects $H(3,q^2)$ in a classical ovoid $\mathcal{O} \cong H(2,q^2)$ of $H(3,q^2)$. This implies that the reguli R_i and R_j uniquely define a hermitian spread $\mathcal{S}_{ij}$ of $Q^-_{ij}(5,q)$. In $\mathsf{PG}(5,q) = \langle Q^-_{ij}(5,q) \rangle$, consider a 3-dimensional subspace $\mathsf{PG}(3,q)$ skew to L. If $\mathcal{S}_{ij} = \{L, M_1, M_2, \ldots, M_{q^3}\}$ and $\tau := T_L(Q^-_{ij}(5,q))$ is the tangent space of $Q^-_{ij}(5,q)$ at L, then the lines $\langle L, M_i \rangle \cap \mathsf{PG}(3,q)$, $i = 1, 2, \ldots, q^3$, together with $\tau \cap \mathsf{PG}(3,q) := L'$ form a regular spread S of $\mathsf{PG}(3,q)$; see Bloemen, Thas and Van Maldeghem [4]. Let x be an arbitrary point on L and denote the

q^2 reguli of $\mathcal{S}_{ij}$ through L by $R'_1, R'_2, \ldots, R'_{q^2}$. Then each of these q^2 reguli has a unique transversal through x, which must be a generating line of the cone $T_x(Q^-_{ij}(5,q)) \cap Q^-_{ij}(5,q) := xQ^-_x(3,q)$. Together with the line L, this yields all q^2+1 lines containing x of the cone $xQ^-_x(3,q)$. For every point $z \in Q^-_x(3,q)$, $z \notin L$, the plane $\langle L, z\rangle$ intersects $\mathsf{PG}(3,q)$ in a point. In this way, we obtain q^2 points of $\mathsf{PG}(3,q)$ which form a plane π_x together with the line L'. This plane π_x is in fact the projection from L onto $\mathsf{PG}(3,q)$ of $\langle xQ^-_x(3,q)\rangle \setminus L$. Each conic of $Q^-_x(3,q)$ through $l' := L \cap Q^-_x(3,q)$ corresponds to a line of π_x, different from L'; this line belongs to the opposite regulus of a regulus of S through L'. Conversely, any such line of π_x corresponds to a conic of $Q^-_x(3,q)$ on l'. Hence each regulus of S containing L' defines a conic of $Q^-_x(3,q)$ through l', and conversely. The set of q reguli of $\mathcal{S}_{ij}$ defined by a regulus of S containing L' will be called an *$\mathcal{R}$-conic* of $\mathcal{Q}$. If $x' \in L \setminus \{x\}$, then the plane $\pi_{x'}$ defines the same $\mathcal{R}$-conics as the plane π_x. In particular, for every point y on L, the q transversals through y of the elements of an $\mathcal{R}$-conic, together with the line L, form a cone with vertex y and a non-singular conic $\mathcal{C}_y \subseteq Q^-_y(3,q)$ through the point $L \cap Q^-_y(3,q)$ as base.

1.3 Classical generalized hexagons

1.3.1 Definition

Classical generalized hexagons will emerge several times in this thesis. We start with a general definition of a generalized hexagon.

Definition
A *generalized hexagon* is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines such that the following three axioms are satisfied:

GH1 Γ contains no ordinary k-gon for $2 \leq k \leq 5$.

GH2 Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary hexagon in Γ.

GH3 There exists an ordinary heptagon in Γ.

If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ only satisfies conditions (i) and (ii), then it is called a *weak generalized hexagon*. A *thick line* of a weak generalized hexagon Γ is a line of Γ which contains at least three points of Γ.

It can be shown, see for instance [78, Corollary 1.5.3], that in a generalized hexagon, both the number of points on a line and the number of lines through a point are independent of the choice of the line, respectively point. For a finite generalized hexagon, these numbers are constants which we denote by $s+1$, respectively $t+1$, and we say that the generalized hexagon has *order* (s,t). The *dual* $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathrm{I}^{-1})$ of a generalized hexagon of order (s,t) is again a generalized hexagon, of order (t,s). Therefore, every statement about generalized hexagons has a dual form which is usually a new statement about generalized hexagons.

Up to duality, only two examples of generalized hexagons of order (s,t), $s \neq 1 \neq t$, are known: the *split Cayley hexagon* $\mathsf{H}(q)$, which has order (q,q), and the *twisted triality hexagon* $\mathsf{T}(q^3,q)$, which has order (q^3,q). In these examples, q is a prime power. Both of them arise as the geometry of absolute points and lines of a triality, an approach which is explained in more detail in Chapter 7. As we will not encounter the twisted triality hexagon $\mathsf{T}(q^3,q)$ later on, further particularities concerning this example will be omitted.

If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized hexagon, then one can introduce a *distance* function on the elements of the generalized hexagon. Two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are said to be at distance i if and only if i is the distance between the two elements in the incidence graph of the generalized hexagon. Clearly, the maximal distance between two elements is 6 and two elements at distance 6 are said to be *opposite*.

Tits' description of $\mathsf{H}(q)$

The following description of the split Cayley hexagon $\mathsf{H}(q)$ is due to Tits [76], and was established for an arbitrary field $\mathbb{K}$. Here, we restrict ourselves to the finite field $\mathsf{GF}(q)$. Consider in $\mathsf{PG}(6,q)$ the non-singular parabolic quadric $Q(6,q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. Then the points of $\mathsf{H}(q)$ are all points of $Q(6,q)$ and the lines of $\mathsf{H}(q)$ are exactly those lines of $Q(6,q)$, the Grassmann coordinates[3] of which satisfy the following six linear equations:

$$\begin{array}{lll} p_{12} = p_{34}, & p_{54} = p_{32}, & p_{20} = p_{35}, \\ p_{65} = p_{30}, & p_{01} = p_{36}, & p_{46} = p_{31}. \end{array}$$

This gives a complete and explicit description of $\mathsf{H}(q)$, embedded in the quadric $Q(6,q)$. In the literature, this embedding is called the *standard embedding* of $\mathsf{H}(q)$. If q is even, the polar space $Q(6,q)$ is isomorphic to the symplectic polar space $W_5(q)$, which yields an embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, where all lines of $\mathsf{H}(q)$ are totally isotropic for a symplectic polarity

[3] For the definition of Grassmann coordinates, the reader is referred to Chapter 24 of *"Generalized Galois Geometries"* by Hirschfeld and Thas [26].

in $\mathsf{PG}(5,q)$. In this representation, the split Cayley hexagon is sometimes called the *perfect symplectic hexagon*.
In its standard embedding, $\mathsf{H}(q)$ has the property that for any given point p, the points which are collinear with p in $\mathsf{H}(q)$ are exactly the points of some totally singular plane of $Q(6,q)$ through p. In other words, the $q+1$ lines of $\mathsf{H}(q)$ through p are exactly the lines of a flat pencil on p, contained in a totally singular plane of $Q(6,q)$.

The reason why the split Cayley hexagon $\mathsf{H}(q)$ is interesting in the field of m-systems of polar spaces, is that ovoids and spreads of $\mathsf{H}(q)$ yield m-systems of $Q(6,q)$. We first define ovoids and spreads of the generalized hexagon $\mathsf{H}(q)$.

Definitions
An *ovoid* of the generalized hexagon $\mathsf{H}(q)$ is a set of q^3+1 points of $\mathsf{H}(q)$ which are pairwise opposite in $\mathsf{H}(q)$. A *spread* of $\mathsf{H}(q)$ is a set of q^3+1 pairwise opposite lines of $\mathsf{H}(q)$.

The following theorems state some relations between ovoids and spreads of $\mathsf{H}(q)$ and m-systems of the parabolic quadric $Q(6,q)$.

Theorem 1.3.1 (Thas [67]) *In its standard embedding, all ovoids of $\mathsf{H}(q)$ are also ovoids of $Q(6,q)$, and conversely.*

Theorem 1.3.2 (Shult and Thas [57]) *In its standard embedding, every spread of $\mathsf{H}(q)$ is also a 1-system of $Q(6,q)$.*

An illustration of this last theorem is the following. Consider the generalized hexagon $\mathsf{H}(q)$ in its standard embedding on the parabolic quadric $Q(6,q)$ in $\mathsf{PG}(6,q)$. Let ε be a 5-dimensional subspace of $\mathsf{PG}(6,q)$, intersecting $Q(6,q)$ in an elliptic quadric $Q^-(5,q)$. Then the lines of $\mathsf{H}(q)$ contained in ε form a spread $\mathcal{S}$ of $\mathsf{H}(q)$, which is a hermitian spread of $Q^-(5,q)$. By Theorem 1.3.2, this spread is a 1-system of $Q(6,q)$, but it is obviously not a proper 1-system of $Q(6,q)$. If we consider this spread as a 1-system of $Q(6,q)$ or as a spread of $\mathsf{H}(q)$, it will also be called *hermitian*.
It is worth noticing that hermitian spreads of $Q^-(5,q)$ are characterized by the fact that they are locally hermitian at every line, see Bloemen, Thas and Van Maldeghem [4].

We mention a last theorem which enables us to construct a spread of $Q(6,q)$ from an ovoid of $\mathsf{H}(q)$, embedded in $Q(6,q)$.

Theorem 1.3.3 (Thas [66]) *If $\mathcal{O}$ is an ovoid of $\mathsf{H}(q)$ in its standard embedding, then the q^3+1 totally singular planes of $Q(6,q)$, containing all points collinear in $\mathsf{H}(q)$ with the respective points of $\mathcal{O}$, form a spread of $Q(6,q)$.*

1.3.2 Coordinatization

Similarly to the coordinatization of projective planes, it is possible to coordinatize generalized polygons, and in particular generalized hexagons. For a detailed description of the coordinatization of generalized polygons, we refer the reader to Van Maldeghem [78, Chapter 3]. Here we focus on generalized hexagons and we want to sketch roughly how the labelling of the elements works. But above all, it is our aim to discuss the coordinates of the split Cayley hexagon $\mathsf{H}(q)$, because they are a useful tool for many purposes. The following paragraphs are mainly based on De Smet and Van Maldeghem [15], in which the coordinatization of the split Cayley hexagon was first carried out.

The coordinatization in general

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized hexagon of order (s,t) and consider two sets R_1, respectively R_2, with cardinality s, respectively t, and both containing 0 and 1, but not ∞. The set R_1 will be used to label all but one points on a line; the set R_2 shall label all but one lines through a point. In the sequel, we denote the elements of R_1 by a and b, possibly enriched with sub- or superscripts, and similarly we use k and l to depict elements of R_2.

Consider a fixed ordinary subhexagon Σ of Γ and denote its elements by

$$x_0 \,\mathrm{I}\, L_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, L_2 \,\mathrm{I}\, x_3 \,\mathrm{I}\, L_4 \,\mathrm{I}\, x_5 \,\mathrm{I}\, L_5 \,\mathrm{I}\, x_4 \,\mathrm{I}\, L_3 \,\mathrm{I}\, x_2 \,\mathrm{I}\, L_1 \,\mathrm{I}\, x_0,$$

see Figure 1.1. Its elements are given coordinates as is indicated in Figure 1.1; note that the coordinates of points are written between round brackets, while square brackets are used for the coordinates of lines. The points on the line L_0, different from x_0, now get a label (a), $a \in R_1$, in such a way that there exists a bijection between R_1 and the points of L_0, different from x_0. Dually, the lines through x_0 and different from L_0 are labelled $[k]$, $k \in R_2$, again in a bijective manner. The unique point on L_5 at distance 4 from (a) is labelled $(a, 0, 0, 0, 0)$ and dually the line $[k, 0, 0, 0, 0]$ is the unique line through x_5 at distance 4 from $[k]$.
Next, let p_a be the point on $[1, 0, 0, 0, 0]$ at distance 4 from (a). Then to the unique point on L_1 at distance 4 from p_a we attach the label $(0, a)$ and the point on L_4 at distance 4 from $(0, a)$ is labelled $(0, 0, 0, 0, a)$. Dually, we get

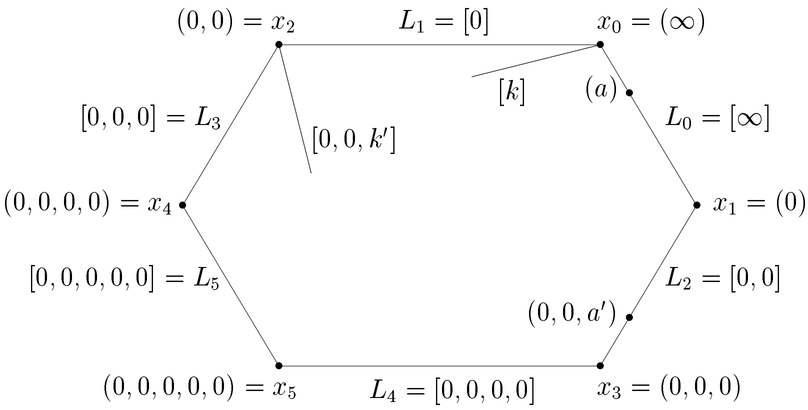

Figure 1.1: The ordinary subhexagon Σ

the lines $[0,k]$ through x_1 and $[0,0,0,0,k]$ through x_4.[4]
To each point of L_2, different from x_1, we now assign a coordinate $(0,0,a')$, $a' \in R_1$, again in a bijective manner; dually we assign the coordinates $[0,0,k']$, $k' \in R_2$, to the lines different from L_1 through the point x_2, such that a bijection exists between R_2 and the lines through x_2, except for L_1. The unique point on L_3 at distance 4 from $(0,0,a')$ is then labelled $(0,0,0,a')$ and dually we define the line $[0,0,0,k']$, incident with x_3. So far coordinates have been given to all elements of Γ which are incident with an element of Σ.
Finally we consider the elements of Γ, which are not incident with an element of Σ. The point on $[k]$ at distance 4 from $(0,0,0,0,b)$ gets the label (k,b); dually we obtain $[a,l]$. The point on $[a,l]$ at distance 4 from $(0,0,0,a')$ is labelled (a,l,a') and dually one defines $[k,b,k']$. To the point on $[k,b,k']$ at distance 4 from $(0,0,b')$ we attach the label (k,b,k',b'); dually this yields the line $[a,l,a',l']$. Eventually, the point on $[a,l,a',l']$ at distance 4 from $(0,a'')$ is given the label (a,l,a',l',a''), and dually for the line $[k,b,k',b',k'']$.

By the above procedure, every element of $\mathcal{P} \cup \mathcal{L}$ has been given an i-tuple, $i \in \{0,1,\ldots,5\}$ of coordinates, where $[\infty]$ and (∞) are said to be 0-tuples, and conversely to every i-tuple, $i \in \{0,1,\ldots,5\}$, there corresponds a unique element of $\mathcal{P} \cup \mathcal{L}$. If the sizes of the coordinate tuples of two elements of $\mathcal{P} \cup \mathcal{L}$ differ by exactly one, then they are incident if and only if the coordinate tuple of one element is obtained from that of the other element by deleting

[4]This subprocedure is called *"normalization"* in [78].

the last coordinate. Two elements of $\mathcal{P} \cup \mathcal{L}$ with an i-tuple, respectively j-tuple, as coordinates and such that $|i - j| \geq 2$ are never incident, nor are two elements with coordinate tuples of the same size $i \in \{1, 2, 3, 4\}$. Clearly, (∞) is incident with $[\infty]$. To complete the coordinatization, we only need to describe incidence between two elements which are labelled by a 5-tuple of coordinates. This can be done by introducing operations Ψ_i and Φ_i, $i = 1, 2, 3, 4$. Let p be a point with coordinates (a, l, a', l', a'') and L a line with coordinates $[k, b, k', b', k'']$. Then $\Psi_i(k, a, l, a', l', a'')$ gives the $(6 - i)$-th coordinate of the line through p at distance 4 from $[k]$. Dually, $\Phi_i(a, k, b, k', b', k'')$ gives the $(6-i)$-th coordinate of the point on L at distance 4 from (a). Using these operations, it can be expressed when the point p and the line L are incident, see Chapter 3 of [78]; we will only do this explicitly for the split Cayley hexagon $\mathsf{H}(q)$ in the next paragraph.

Coordinates for $\mathsf{H}(q)$

Consider the split Cayley hexagon $\mathsf{H}(q)$, embedded in the parabolic quadric $Q(6, q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. We choose $R_1 = R_2 = \mathsf{GF}(q)$ and describe the relation between the coordinates of $\mathsf{H}(q)$ and the coordinates of $\mathsf{PG}(6, q)$. The coordinates of the ordinary subhexagon Σ and the above mentioned bijections are chosen in the following way:

$$\begin{array}{rcl}
(1,0,0,0,0,0,0) & \rightarrow & (\infty), \\
(0,0,0,0,0,1,0) & \rightarrow & (0,0), \\
(0,1,0,0,0,0,0) & \rightarrow & (0,0,0), \\
(0,0,1,0,0,0,0) & \rightarrow & (0,0,0,0), \\
(0,0,0,0,1,0,0) & \rightarrow & (0,0,0,0,0), \\
(a,0,0,0,0,0,1) & \rightarrow & (a), \\
(0,1,0,0,0,0,-a') & \rightarrow & (0,0,a');
\end{array}$$

$$\begin{array}{rcl}
X_1 = X_2 = X_3 = X_4 = X_5 = 0 & \rightarrow & [\infty], \\
X_0 = X_2 = X_3 = X_4 = X_5 = 0 & \rightarrow & [0,0], \\
X_1 = X_3 = X_4 = X_6 = X_0 = 0 & \rightarrow & [0,0,0], \\
X_0 = X_2 = X_3 = X_5 = X_6 = 0 & \rightarrow & [0,0,0,0], \\
X_0 = X_1 = X_3 = X_5 = X_6 = 0 & \rightarrow & [0,0,0,0,0], \\
X_1 = X_2 = X_3 = X_4 = X_6 + kX_5 = 0 & \rightarrow & [k], \\
X_0 - k'X_2 = X_1 = X_3 = X_4 = X_6 = 0 & \rightarrow & [0,0,k'].
\end{array}$$

This determines the coordinates of all points and lines of $\mathsf{H}(q)$ by the procedure of the previous paragraph. The complete guide-book for translating coordinates of $\mathsf{H}(q)$ to coordinates of $\mathsf{PG}(6, q)$ and conversely, is given in Table 1.1.

POINTS	
Coordinates in $\mathsf{H}(q)$	Coordinates in $\mathsf{PG}(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(-al'+a'^2+a''l+aa'a'',-a'',-a,-a'+aa'',$ $1,l+2aa'-a^2a'',-l'+a'a'')$

LINES	
Coordinates in $\mathsf{H}(q)$	Coordinates in $\mathsf{PG}(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,l]$	$\langle(a,0,0,0,0,0,1),(-l,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,l,a',l']$	$\langle(-l-aa',1,0,-a,0,a^2,-a'),$ $(-al'+a'^2,0,-a,-a',1,l+2aa',-l')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-b'k),$ $(b'^2+k''b,-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 1.1: Coordinatization of $\mathsf{H}(q)$

One can calculate, by relying on the coordinates of $\mathsf{PG}(6,q)$, that a point (a,l,a',l',a'') is incident with a line $[k,b,k',b',k'']$ if and only if

$$\left\{\begin{array}{rcl} b & = & a''-ak, \\ a' & = & a^2k+b'+2ab, \\ l & = & k''-ka^3-3ba^2-3ab', \\ k' & = & k^2a^3+l'-kl-3a^2a''k-3a'a''+3aa''^2. \end{array}\right. \tag{1.7}$$

We do not give the explicit form of the operations Ψ_i and Φ_i, since this is not relevant for our purposes; they can be found in [78, Chapter 3].

An example

A *line regulus* of $\mathsf{H}(q)$ is defined as the set of lines at distance 3 from two opposite points of $\mathsf{H}(q)$. Consider as an example two opposite points (k,b) and (a,l,a') of $\mathsf{H}(q)$. Then, by taking the relations (1.7) into account, one can calculate that the line regulus defined by these two points is the following set of lines:

$$\{[\infty]\}\cup\{[k,b,\lambda,a'-a^2k-2ab,l-2ka^3-3a^2b+3aa']\mid\lambda\in\mathsf{GF}(q)\}.$$

Note that only the middle coordinate of the lines of this regulus varies in $\mathsf{GF}(q)$, thus yielding q different lines. Therefore a line regulus of this kind is sometimes denoted by $[[k, b, a' - a^2k - 2ab, l - 2ka^3 - 3a^2b + 3aa']]$.
By using Table 1.1, one easily calculates that a line regulus of $\mathsf{H}(q)$ is in fact a regulus of a 3-dimensional hyperbolic quadric $Q^+(3, q)$ on $Q(6, q)$. Hence a locally hermitian spread of $\mathsf{H}(q)$ can be defined in exactly the same way as a locally hermitian 1-system of $Q(6, q)$.

1.4 Some more geometries

Some other geometrical structures will appear in the course of this work. We give the definitions here.

1.4.1 (Semi)partial geometries

Definition
A *semipartial geometry* is an incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines satisfying the following axioms:

spg1 Each point is incident with $t+1$ ($t \geq 1$) lines and two distinct points are incident with at most one line.

spg2 Each line is incident with $s+1$ ($s \geq 1$) points and two distinct lines are incident with at most one point.

spg3 If two points are not collinear, then there are μ ($\mu > 0$) points collinear with both.

spg4 If a point x and a line L are not incident, then there are 0 or α ($\alpha \geq 1$) points which are collinear with x and incident with L.

A semipartial geometry with parameters s, t, α, μ has

$$|\mathcal{P}| := v = 1 + (t+1)s\left(1 + t\frac{s-\alpha+1}{\mu}\right)$$

points and the number $b := |\mathcal{L}|$ of lines is determined by $v(t+1) = b(s+1)$.

A *partial geometry* is a semipartial geometry such that for each non-incident point-line pair (x, L), there exist α ($\alpha \geq 1$) points which are collinear with x and incident with L. In this case, **spg3** is automatically satisfied, with $\mu = (t+1)\alpha$. Moreover, it can be shown that a semipartial geometry is a partial geometry if and only if $\mu = (t+1)\alpha$.

In the semipartial geometries we will encounter, there often exist a lot of *subnets* as a subgeometry. Therefore some words on nets of order $s+1$ and degree $t+1$, which are in fact partial geometries with $\alpha = t$, must be said.

1.4.2 Nets

Definition
A *net of order* $s+1$ *and degree* $t+1$ is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines satisfying the following axioms:

N1 Each point is incident with $t+1$ ($t \geq 1$) lines and two distinct points are incident with at most one line.

N2 Each line is incident with $s+1$ ($s \geq 1$) points and two distinct lines are incident with at most one point.

N3 If a point x and a line L are not incident, then there exists exactly one line which is incident with x and not concurrent with L.

In a net of order $s+1$ and degree $t+1$, two lines are called *parallel* if and only if they have no point in common. For every non-incident point-line pair (x, L), the unique line through x and not concurrent with L is then the unique line incident with x and parallel to L.

The number of points of a net of order $s+1$ and degree $t+1$ is $(s+1)^2$ and the number of lines equals $(s+1)(t+1)$. From the axioms, it immediately follows that a net of order $s+1$ and degree $t+1$ satisfies $t \leq s+1$. If $t = s+1$, this implies that every two distinct points are incident with exactly one line and consequently, the net is an affine plane of order $s+1$.

For later purposes, we mention two constructions of a net of order q^2 and degree $q+1$.

Consider a regulus R of lines in $\mathsf{PG}(3,q)$, embed $\mathsf{PG}(3,q)$ as a hyperplane in a projective space $\mathsf{PG}(4,q)$, and let $\mathsf{AG}(4,q)$ denote the affine space $\mathsf{PG}(4,q) \setminus \mathsf{PG}(3,q)$. Suppose that $\mathcal{P}$ is the set of all points of $\mathsf{AG}(4,q)$, let $\mathcal{L}$ consist of the planes of $\mathsf{AG}(4,q)$, the extensions of which to $\mathsf{PG}(4,q)$ meet $\mathsf{PG}(3,q)$ in a line of R, and let incidence be the incidence of $\mathsf{AG}(4,q)$. Then $\mathcal{N} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a net of order q^2 and degree $q+1$. In Johnson [30], a net of this kind is called a *regulus net*.

Consider the 3-dimensional projective space $\mathsf{PG}(3,q)$ and let N be a fixed line of $\mathsf{PG}(3,q)$. Define $\mathcal{P}$ to be the set of all lines of $\mathsf{PG}(3,q)$, skew to N, and let $\mathcal{L}$ be the set of points of $\mathsf{PG}(3,q)$, not on N. Then $\mathcal{N} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$, where

I is the incidence of $\mathsf{PG}(3, q)$, is a net of order q^2 and degree $q + 1$. A net of this kind is denoted by H_q^3 and called a *co-dimension 2 net* in Johnson [30].

Also, in the book *"Subplane covered nets"* by Johnson ([30]), in which the reader can find a wealth of information concerning nets and related topics, it is shown that every regulus net is isomorphic to a co-dimension 2 net and conversely.

Chapter 2

m-Systems of polar spaces and SPG reguli

m-Systems of polar spaces, the objects of study in this thesis, seem to be somehow related to SPG reguli. If one forgets for a moment the structure of the polar space on which an m-system lives, it is just a collection of pairwise disjoint m-dimensional subspaces of a projective space, satisfying some extra conditions. Considered in this way, the similarity to SPG reguli is apparent. SPG reguli were introduced by Thas in [69] as a means to construct new semipartial geometries. They consist of a set of m-dimensional subspaces of a projective space which satisfy certain well-chosen axioms. But the most striking is the fact that, except for some easy examples which yield a net or a dual net as semipartial geometry, all examples of SPG reguli that are given in [69], are spreads of quadrics or hermitian varieties. This appeared too surprising to be mere coincidence and a deeper relation between m-systems and SPG reguli was suspected.

In this chapter, we will see that indeed more can be said about the relation between m-systems and SPG reguli. As was briefly mentioned above, there exist SPG reguli which are not an m-system of some polar space. On the other hand, one can also easily find examples of m-systems of certain polar spaces which are not an SPG regulus. For some polar spaces however, it is possible to show that every m-system is an SPG regulus of the ambient space of the polar space. This will be the main result of this chapter; it appeared in [38].

2.1 SPG reguli

We start with the definition of an SPG regulus, following Thas [69].

Definition
An *SPG regulus* of $\mathsf{PG}(n,q)$ is a set $\mathcal{R}$ of m-dimensional subspaces π_1, π_2, $\ldots,\pi_r$, $r>1$, of $\mathsf{PG}(n,q)$, satisfying:

SPG1 $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

SPG2 If $\mathsf{PG}(m+1,q)$ contains $\pi_i \in \mathcal{R}$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $\mathcal{R} \setminus \{\pi_i\}$. If $\mathsf{PG}(m+1,q)$ has no point in common with $\pi_j \in \mathcal{R}$ for all $j \neq i$, then it is called a *tangent* $(m+1)$-space of $\mathcal{R}$ at π_i.

SPG3 If the point x of $\mathsf{PG}(n,q)$ is not contained in an element of $\mathcal{R}$, then it is contained in a constant number θ ($\theta \geq 0$) of tangent $(m+1)$-spaces of $\mathcal{R}$.

In some situations, it is easier to use an alternative set of axioms for some kinds of SPG reguli. These axioms are stricter, but they are fulfilled in the case we will consider. Therefore the following two axioms are introduced (see again [69]):

SPG3' The union of all tangent $(m+1)$-spaces is a $\mathsf{PG}(n-m-1,q)$, at each $\pi_i \in \mathcal{R}$.

SPG4' $r = q^{(n+1)/2}+1$.

It can then be shown that a set $\mathcal{R}$ of m-dimensional subspaces π_1, π_2, $\ldots,\pi_r$, $r>1$, of $\mathsf{PG}(n,q)$ with $n \neq 2m+1$, which satisfies **SPG1**, **SPG2**, **SPG3'** and **SPG4'**, is an SPG regulus.

The reason why SPG reguli are interesting, is the fact that a semipartial geometry can be constructed from every SPG regulus. The construction is again described in [69]; here we briefly sketch it.

Let $\mathcal{R}$ be an SPG regulus of $\mathsf{PG}(n,q)$, consisting of m-dimensional subspaces, and embed $\mathsf{PG}(n,q)$ as a hyperplane in a projective space $\mathsf{PG}(n+1,q)$. Define an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ of points and lines in the following way. The point set P is the set of all points of $\mathsf{PG}(n+1,q)$, not in $\mathsf{PG}(n,q)$. The line set B consists of all $(m+1)$-dimensional subspaces of $\mathsf{PG}(n+1,q)$ which contain an element of $\mathcal{R}$, but are not contained in $\mathsf{PG}(n,q)$. Incidence is the natural incidence of $\mathsf{PG}(n+1,q)$. Then it is shown in [69] that $\mathcal{S}$ is a semipartial geometry with parameters $s = q^{m+1}-1$, $t = r-1$, $\alpha = \alpha$ and $\mu = (r-\theta)\alpha$.

2.2 A connection between m-systems of polar spaces and SPG reguli

In this section, a simple but elegant combinatorial proof is given for the main result of this chapter, namely the fact that all m-systems of some polar spaces are SPG reguli. The result relies on the special property concerning intersection numbers with respect to hyperplanes of m-systems of these polar spaces, as it has been mentioned in Theorem 1.2.2.

Theorem 2.2.1 *If $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$, then all m-systems of $\mathcal{P}$ are SPG reguli of the ambient space of $\mathcal{P}$.*

Proof.
Let $\mathcal{M}$ be an m-system of a finite polar space $\mathcal{P}$, with $\mathcal{P} \in \{W_{2n+1}(q), Q^-(2n+1,q), H(2n,q^2)\}$, and denote its ambient space by $\mathsf{PG}(k,t)$, where $(k,t) \in \{(2n+1,q),(2n+1,q),(2n,q^2)\}$. Let π_m be an element of $\mathcal{M}$. For an arbitrary subspace $\pi_{m-1} \subseteq \pi_m$, consider an $(m+1)$-dimensional subspace π_{m+1} of $\mathsf{PG}(k,t)$ containing π_{m-1} and meeting $\mathcal{P}$ in $\pi_{m-1}\mathcal{P}_1$, where $\mathcal{P}_1$ is a $W_1(q)$, a $Q^+(1,q)$ or a $H(1,q^2)$ in the respective cases. Denote by X the number of points of $\tilde{\mathcal{M}}$ contained in $\pi_{m-1}\mathcal{P}_1$ and by Y the number of points of $\tilde{\mathcal{M}} \cap (\pi_{m+1}^\perp \setminus \pi_{m-1})$, with $\pi_{m+1}^\perp$ the image of π_{m+1} with respect to the polarity defining $\mathcal{P}$. We now use Theorem 1.2.2 to count the number of pairs (H,x) where H is a hyperplane containing π_{m+1} and x an arbitrary point of $(H \cap \tilde{\mathcal{M}}) \setminus \pi_{m+1}$. This yields the following in the respective cases.

(a) For $\mathcal{P} = W_{2n+1}(q)$ and $\mathcal{P} = Q^-(2n+1,q)$ we find the same result:

$$\left(Y + \frac{q^m-1}{q-1}\right)\left(\frac{(q^{m+1}-1)(q^n+1)}{q-1} - q^n - X\right) + \left(\frac{q^{2n-m}-1}{q-1} - \frac{q^m-1}{q-1} - Y\right)\left(\frac{(q^{m+1}-1)(q^n+1)}{q-1} - X\right) = \left((q^{n+1}+1)\frac{q^{m+1}-1}{q-1} - X\right)\frac{q^{2n-m-1}-1}{q-1},$$

from which we obtain

$$(q-1)Y + q^{n-m-1}(q-1)X - q^n + q^{n-m-1} - q^{m+1} + q^m = 0.$$

(b) For $\mathcal{P} = H(2n, q^2)$ the result is:

$$\begin{aligned}&\left(Y + \frac{q^{2m}-1}{q^2-1}\right)\left(\frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - q^{2n-1} - X\right)\\ &+\left(\frac{q^{4n-2m-2}-1}{q^2-1} - \frac{q^{2m}-1}{q^2-1} - Y\right)\left(\frac{(q^{2m+2}-1)(q^{2n-1}+1)}{q^2-1} - X\right)\\ &\qquad = \left((q^{2n+1}+1)\frac{q^{2m+2}-1}{q^2-1} - X\right)\frac{q^{4n-2m-4}-1}{q^2-1},\end{aligned}$$

which yields

$$(q^2-1)Y + q^{2n-2m-3}(q^2-1)X - q^{2n-1} + q^{2n-2m-3} - q^{2m+2} + q^{2m} = 0.$$

Now we consider the special case where $\pi_m = \langle \pi_{m-1}, y\rangle$ is an element of $\mathcal{M}$, for some $y \in \mathcal{P}_1$. In this case $Y = 0$ and we can determine X from the above equalities:

(a) For $\mathcal{P} = W_{2n+1}(q)$ and $\mathcal{P} = Q^-(2n+1, q)$ we obtain

$$X = \frac{q^{m+1}-1}{q-1} + q^{2m-n+1}; \tag{2.1}$$

here we put $\alpha := q^{2m-n+1}$.

(b) For $\mathcal{P} = H(2n, q^2)$ we find

$$X = \frac{q^{2m+2}-1}{q^2-1} + q^{4m-2n+3}; \tag{2.2}$$

and in this case we put $\alpha := q^{4m-2n+3}$.

The value of X tells us that every $(m+1)$-dimensional subspace of $\mathsf{PG}(k, t)$, containing $\pi_m \in \mathcal{M}$ and not contained in $\pi_m^\perp$, has exactly α points in common with $\tilde{\mathcal{M}} \setminus \pi_m$. From the definition of an m-system, it is known that every $(m+1)$-dimensional subspace of $\pi_m^\perp$ which contains π_m, has an empty intersection with all elements of $\mathcal{M} \setminus \{\pi_m\}$. Hence the union of all tangent $(m+1)$-spaces of $\mathcal{M}$ at π_m is exactly $\pi_m^\perp$ and thus has the dimension required in **SPG3'** of the alternative definition of an SPG regulus. As the number of elements of an m-system, see (1.1), (1.4) and (1.5), is exactly the value required in **SPG4'**, it follows that $\mathcal{M}$ satisfies **SPG1**, **SPG2**, **SPG3'** and **SPG4'**, so it is an SPG regulus in $\mathsf{PG}(k, t)$ with parameters

(a) for $\mathcal{P} = W_{2n+1}(q)$ or $\mathcal{P} = Q^-(2n+1, q)$:

$$r = q^{n+1} + 1, \quad \alpha = q^{2m-n+1} \quad \text{and} \quad \theta = q^{n-m} + 1;$$

(b) for $\mathcal{P} = H(2n, q^2)$:

$$r = q^{2n+1} + 1, \quad \alpha = q^{4m-2n+3} \quad \text{and} \quad \theta = q^{2n-2m-1} + 1. \qquad \square$$

We mention two interesting corollaries of the previous theorem.

Corollary 2.2.2 *It holds for all m-systems of the finite classical polar spaces $W_{2n+1}(q)$, $Q^-(2n+1, q)$ and $H(2n, q^2)$, that $2m + 1 \geq n$.*

Proof.
In (2.1) and (2.2), $X \geq |\pi_m|$ must hold. The result follows. $\square$

Remark
This inequality was first found by Hamilton and Mathon in [21], see also Theorem 1.2.6. However, their proof is distinct from this one.

The next corollary will be of special interest for the following chapter.

Corollary 2.2.3 *If $\mathcal{M}$ is a 1-system of $Q^-(7, q)$, then every line of $Q^-(7, q)$ meets $\tilde{\mathcal{M}}$ in 0, 1, 2 or $q+1$ points. If a line of $Q^-(7, q)$ contains $q+1$ points of $\tilde{\mathcal{M}}$, then it is necessarily a line of $\mathcal{M}$.*

Proof.
This follows immediately from the proof of Theorem 2.2.1, applied to 1-systems of the quadric $Q^-(7, q)$. $\square$

Remark
As a final remark, one may notice that a weak converse of Theorem 2.2.1 also holds. Suppose that $\mathcal{P}$ is an elliptic quadric $Q^-(n, q)$ with n odd or a hermitian variety $H(n, q^2)$ with n even and $\mathcal{R}$ is an SPG regulus of $\mathsf{PG}(n, q)$ satisfying **SPG3'** and **SPG4'**, with $n \neq 2m + 1$. If, moreover, all elements of $\mathcal{R}$ are totally singular subspaces of $\mathcal{P}$, then we show that $\mathcal{R}$ must be an m-system of $\mathcal{P}$. For $\pi_m \in \mathcal{R}$, the tangent space $T_{\pi_m}(\mathcal{P})$ is spanned by the $(m+1)$-spaces through π_m that intersect $\mathcal{P}$ exactly in π_m. By **SPG3'**, the tangent space $T_{\pi_m}(\mathcal{P})$ must hence coincide with the $(n-m-1)$-dimensional subspace of $\mathsf{PG}(n, q)$ which is the union of the tangent $(m+1)$-spaces of the SPG regulus $\mathcal{R}$ at π_m. Thus $T_{\pi_m}(\mathcal{P})$ meets $\tilde{\mathcal{M}}$ only in the points of π_m and $\mathcal{R}$ is an m-system of $\mathcal{P}$.

It might even be possible to prove that an SPG regulus $\mathcal{R}$ of $\mathsf{PG}(n, q)$ with $n \neq 2m + 1$, satisfying **SPG3'** and **SPG4'** and possibly some other extra conditions, always is an m-system of some polar space. This is, however, not easily shown, since one must reconstruct the polarity from the SPG regulus, which is a non-trivial task.

2.3 Semipartial geometries arising from the known m-systems

As has been mentioned earlier, every SPG regulus gives rise to a semipartial geometry by the construction of [69] which was recalled in Section 2.1. Hence, by the previous theorem, a semipartial geometry can be constructed from every m-system of $W_{2n+1}(q)$, $Q^-(2n+1,q)$ or $H(2n,q^2)$. For spreads of $H(2n,q^2)$ or $Q^-(2n+1,q)$, this was already observed by Thas in [69]. For arbitrary m-systems, the corresponding semipartial geometries have the following parameters:

(a) for $\mathcal{P} = W_{2n+1}(q)$ or $\mathcal{P} = Q^-(2n+1,q)$:

$$s = q^{m+1} - 1,\ t = q^{n+1},\ \alpha = q^{2m-n+1} \text{ and } \mu = q^{m+1}(q^{m+1} - 1);$$

(b) for $\mathcal{P} = H(2n,q^2)$:

$$s = q^{2m+2} - 1,\ t = q^{2n+1},\ \alpha = q^{4m-2n+3} \text{ and } \mu = q^{2m+2}(q^{2m+2} - 1).$$

For several values of m and n, these parameters are new. Unfortunately, most of the known m-systems of the considered polar spaces do not yield new semipartial geometries.

First we note that a lot of examples of m-systems arise from a known m-system in a small polar space by applying the so-called "trace trick". This means that the trace map is used to reduce the field while at the same time increasing the dimension; see [22] for an algebraic approach to the trace trick and [57] for a geometric explanation of this method; we will also come back to this technique in Appendix A. The corresponding semipartial geometry is clearly isomorphic to the one arising from the initial m-system in the small polar space, so m-systems which are constructed with the trace trick never yield new semipartial geometries. This observation highly reduces the number of candidates for new semipartial geometries.

Of the hermitian polar space $H(2n,q^2)$, only one m-system is known, apart from those obtained by the trace trick from this one, namely the point set of $H(2,q^2)$ considered as an ovoid (or a spread) of $H(2,q^2)$. The associated semipartial geometry is well known and was introduced by Debroey and Thas in [13]; it is often denoted by $T_2^*(\mathcal{U})$.

For the elliptic quadric $Q^-(2n+1,q)$, the situation is similar. First we observe that for q even, every m-system of $Q^-(2n+1,q)$ is also an m-system of $W_{2n+1}(q)$, which can be seen as follows. It is possible to embed $Q^-(2n+1,q)$ in a parabolic polar space $Q(2n+2,q)$ such that the nucleus of $Q(2n+2,q)$

is not contained in the ambient space $\mathsf{PG}(2n+1,q)$ of $Q^-(2n+1,q)$. Clearly, every m-system of $Q^-(2n+1,q)$ is an m-system of $Q(2n+2,q)$ as well. If we project $Q(2n+2,q)$ from its nucleus onto $\mathsf{PG}(2n+1,q)$, we obtain a symplectic polar space $W_{2n+1}(q)$. Now it is easily seen that the projection of the m-system of $Q(2n+2,q)$ is an m-system of $W_{2n+1}(q)$. As this m-system is completely contained in $\mathsf{PG}(2n+1,q)$, it is projected onto itself and this shows that every m-system of $Q^-(2n+1,q)$, q even, is an m-system of $W_{2n+1}(q)$. Hence we may omit the q even case.
If q is odd, m-systems of $Q^-(2n+1,q)$ are only known for the small dimensions, except for those which are constructed with the trace trick from the small ones. It is known that $Q^-(5,q)$ has several non-isomorphic spreads, but the case of spreads of elliptic quadrics has already been discussed in [69]. Moreover, $Q^-(5,q)$ has no ovoids and the point set of $Q^-(3,q)$, considered as an ovoid of $Q^-(3,q)$, yields the well-known semipartial geometry $T_3^*(\mathcal{O})$, with $\mathcal{O} \cong Q^-(3,q)$. Consequently, nothing new arises here.

Finally, we consider the known m-systems of $W_{2n+1}(q)$. The semipartial geometry corresponding to the regular spread of $W_{2n+1}(q)$, that is, a spread of $W_{2n+1}(q)$ which is regular considered as an n-spread of $\mathsf{PG}(2n+1,q)$, was given as an example in [69]. Other spreads of $W_{2n+1}(q)$ are known and they yield semipartial geometries with the same parameters.
Candidates for new semipartial geometries are given by the m-systems of $W_{2n+1}(2)$, $n \leq 4$, which were classified by computer by Hamilton and Mathon in [21]. By thoroughly viewing through their list, one notices that only few of the examples are new m-systems of $W_{2n+1}(2)$. Most examples are either previously known symplectic spreads of $\mathsf{PG}(2n+1,2)$, or known spreads of $Q^-(2n+1,2)$, or they arise by the trace trick applied to $Q^-(3,2^s)$, considered as an ovoid of itself. In the first two cases, the corresponding semipartial geometry was given as an example in [69], while in the latter case, the semipartial geometry is isomorphic to $T_3^*(\mathcal{O})$, with $\mathcal{O} \cong Q^-(3,2^s)$. The only new examples they find, belong to the ten existing non-isomorphic 3-systems of $W_9(2)$, one of which is a previously known spread of $Q^-(9,2)$. The other nine may indeed yield new semipartial geometries, but their parameters are not new.
Very recently, both Offer [46] and Cardinali, Lunardon, Polverino and Trombetti [10] independently discovered a new spread of the generalized hexagon $\mathsf{H}(2^{2h})$, which yields a new 1-system of the parabolic quadric $Q(6,2^{2h})$. By projection from the nucleus of $Q(6,2^{2h})$ onto a 5-dimensional subspace not containing the nucleus, a new 1-system of $W_5(2^{2h})$ is obtained. This 1-system is distinct from the only previously known 1-system of $W_5(q)$, which arises from $H(2,q^2)$ as described in Theorem A.1.6 of Appendix A, and the semipartial geometry of which is isomorphic to $T_2^*(\mathcal{U})$. Hence this new spread of

$\mathsf{H}(2^{2h})$ implies the existence of a semipartial geometry for $q = 2^{2h}$, for which we will prove in a later chapter that it is not isomorphic to a previously known example. But once again its parameters are not new.
All other known m-systems of $W_{2n+1}(q)$ give rise to known semipartial geometries, as they are always obtained by the trace trick from an m-system in a small polar space, the semipartial geometry of which is well known.

In the remaining chapters of this thesis, we will always consider the semipartial geometries that arise from m-systems of polar spaces, whenever a suitable m-system is constructed. Furthermore, we will pay attention to the question whether these semipartial geometries are new or not. However, this is often a non-trivial matter.

Chapter 3

The uniqueness of the 1-system of $Q^-(7,q)$, q odd

In the previous chapter, we came along a curious property concerning the possible intersections of a line of the elliptic quadric $Q^-(7,q)$ and the set of points $\tilde{\mathcal{M}}$, with $\mathcal{M}$ a 1-system of $Q^-(7,q)$. This property was formulated in Corollary 2.2.3 and limits the number of points of $\tilde{\mathcal{M}}$ on a line of $Q^-(7,q)$ to four possible values. At first sight, such a property might not seem very useful, but in the following pages we will see that for q odd, it has far-reaching consequences. The current chapter will be totally dedicated to proving that the elliptic quadric $Q^-(7,q)$, q odd, has a unique 1-system, namely the classical one. The starting point to come to this interesting result, will be the property of Corollary 2.2.3, however simple it yet may seem.

To obtain the desired uniqueness result, we need quite some machinery. The different steps of the proof will lead us to various topics of beautiful geometry which at first sight are not really linked to m-systems. Still, it is thanks to these results from less related fields that we will be able to draw our conlusions. This is why we start with a section explaining some necessary machinery, although we will not go into detail about these subjects.

The contents of this chapter has been published in the *Journal of Combinatorial Theory, Series A*, see [40].

3.1 Machinery

3.1.1 Some remarks on linear projectivities and the cross-ratio

We gather some results on the cross-ratio of points on a line and how this is linked to projectivities between lines. Most of these results are discussed on various spots in Hirschfeld's *"Projective Geometries over Finite Fields"* [24]. Here we limit ourselves to a selection of facts which will be useful tools in the sequel.

Definition
The *cross-ratio* of four distinct points x, y, z, u on a line, with homogeneous coordinates $x = (x_0, x_1)$, $y = (y_0, y_1)$, $z = (z_0, z_1)$ and $u = (u_0, u_1)$, is defined by

$$\{x, y; z, u\} := \frac{\begin{vmatrix} x_0 & z_0 \\ x_1 & z_1 \end{vmatrix} \cdot \begin{vmatrix} y_0 & u_0 \\ y_1 & u_1 \end{vmatrix}}{\begin{vmatrix} x_0 & u_0 \\ x_1 & u_1 \end{vmatrix} \cdot \begin{vmatrix} y_0 & z_0 \\ y_1 & z_1 \end{vmatrix}};$$

it holds that $\{x, y; z, u\} = \{y, x; u, z\} = \{z, u; x, y\} = \{u, z; y, x\}$.

Let $\mathcal{C}$ be a conic in $\mathsf{PG}(2, q)$, q odd, and r a point, not on $\mathcal{C}$. Denote the polar line of r with respect to $\mathcal{C}$ by N and consider a secant S of $\mathcal{C}$ through r. If we write $S \cap N := y$ and $S \cap \mathcal{C} := \{a_1, a_2\}$, then one easily checks, for instance by introducing coordinates and calculating the cross-ratio in a straightforward way, that $\{r, y; a_1, a_2\} = -1 = \{r, y; a_2, a_1\}$. This easy observation will be used repeatedly in this chapter.

Another useful property of the cross-ratio is that it is preserved by linear projectivities between lines over $\mathsf{GF}(q)$. If in particular the distinct collinear points p, q, r, s of the plane are in perspective from c with the respective collinear points $p^\alpha, q^\alpha, r^\alpha, s^\alpha$, then $\{p, q; r, s\} = \{p^\alpha, q^\alpha; r^\alpha, s^\alpha\}$, see Figure 3.1. Hence, one can say that if p, q, r and their images under a linear projectivity α between lines over $\mathsf{GF}(q)$ are known, the image under α of a point s of the first line is the unique point u of the second line such that $\{p, q; r, s\} = \{p^\alpha, q^\alpha; r^\alpha, u\}$. This allows to determine the image of points under a linear projectivity in terms of the cross-ratio.

If α is a linear projectivity between two skew lines L and M in $\mathsf{PG}(3, q)$, then the lines joining a point of L with its image under α on M, form a regulus of lines, where L and M clearly belong to the opposite regulus. Also, the cross-ratio of points is preserved along the lines of a regulus, so that in

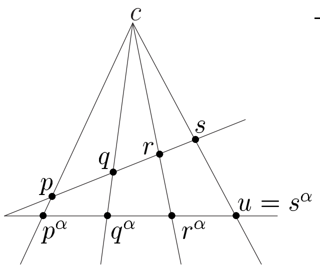

Figure 3.1: Cross-ratio in a pencil

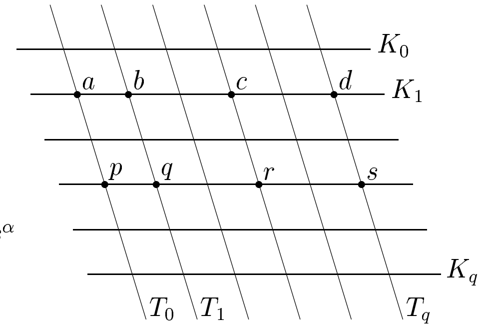

Figure 3.2: Cross-ratio on a regulus

Figure 3.2 it holds that $\{a, b; c, d\} = \{p, q; r, s\}$, where $\{K_0, K_1, \ldots, K_q\}$ and $\{T_0, T_1, \ldots, T_q\}$ form a pair of opposite reguli in $\mathsf{PG}(3, q)$.

By the above, one can extend the definition of cross-ratio of four collinear points in a straightforward way to the cross-ratio of four lines in a plane pencil or belonging to a regulus in $\mathsf{PG}(3, q)$. In both cases, the cross-ratio of lines is defined as the cross-ratio of the intersection points of these lines with a common secant line (not containing the vertex in the case of a pencil), since this value is independent of the choice of the secant line.

Note also that, more generally speaking, it is possible to define the cross-ratio of four distinct elements of any set of $q + 1$ objects, which can be described by one parameter $t \in \mathsf{GF}(q) \cup \{\infty\}$.

3.1.2 A property of quadrics in $\mathsf{PG}(2n - 1, q)$

Let Q_{2n-1} be a non-singular elliptic or hyperbolic quadric in $\mathsf{PG}(2n - 1, q)$, with $n \geq 1$ and q odd. Then by Thas [68] an equivalence relation can be defined on the points of $\mathsf{PG}(2n - 1, q)$, not on Q_{2n-1}, in the following way. Embed $\mathsf{PG}(2n - 1, q)$ as a hyperplane in $\mathsf{PG}(2n, q)$ and consider in $\mathsf{PG}(2n, q)$ a non-singular parabolic quadric $Q(2n, q)$ such that $\mathsf{PG}(2n - 1, q) \cap Q(2n, q) = Q_{2n-1}$. If the point p is the pole of $\mathsf{PG}(2n - 1, q)$ with respect to the polarity of $Q(2n, q)$, then equivalence is defined as follows. Two points x, y of $\mathsf{PG}(2n - 1, q) \backslash Q_{2n-1}$ are called equivalent if and only if px and py meet $Q(2n, q)$ in the same number of points. Hence we obtain two equivalence classes: the class consisting of the points x such that px is disjoint from $Q(2n, q)$ and the class of the points y such that py is a secant of $Q(2n, q)$.

If $n = 1$ and Q_1 is an elliptic quadric $Q^-(1, q)$ on some line L, so $Q^-(1, q)$ consists of two points defined over the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, we obtain an equivalence relation on the points of L. In this case, one can also show that the equivalence classes uniquely determine the quadric $Q^-(1,q)$. This result can be found in Fisher [17], where it is stated as a corollary of a more general theorem and using different terminology. We formulate the result in a slightly different way in the following lemma.

Lemma 3.1.1 (Fisher [17]) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two distinct conics in $\mathsf{PG}(2,q)$, q odd, and let r be an internal point of both conics. If every line through r is either a secant of both $\mathcal{C}$ and $\mathcal{C}'$, or disjoint from both $\mathcal{C}$ and $\mathcal{C}'$, then also the tangents through r of $\mathcal{C}$ and $\mathcal{C}'$, which are lines defined over $\mathsf{GF}(q^2)$, coincide.*

Next we turn our attention to the case $n = 2$, with Q_3 again an elliptic quadric $Q^-(3,q)$, q odd. It is well known that such an elliptic quadric can be interpreted as an inversive plane, where the conics on $Q^-(3,q)$ are the circles of the inversive plane. The equivalence relation on the points of $\mathsf{PG}(3,q) \setminus Q^-(3,q)$ then defines an equivalence relation on the circles of $Q^-(3,q)$ in a straightforward way: two circles are equivalent if and only if their poles with respect to the polarity of $Q^-(3,q)$ are equivalent.
In this context, we mention two lemmas which are important to us. Note that a *flock* of $Q^-(3,q)$ is a set of $q-1$ disjoint circles of $Q^-(3,q)$. As $Q^-(3,q)$ has q^2+1 points, there exist two points that are contained in no circle of the flock; these points are called the *carriers* of the flock. A flock of $Q^-(3,q)$ is *linear* if the planes of the $q-1$ circles have a line L in common; in this case the carriers are the two points of $L^\perp \cap Q^-(3,q)$, with $\perp$ the polarity of $Q^-(3,q)$.

Lemma 3.1.2 (Fisher and Thas [18]) *Let $\mathcal{F}$ be a flock of $Q^-(3,q)$. If n and s are the carriers of $\mathcal{F}$ and K is a circle of $\mathcal{F}$, then the circles of the pencil intersecting in n and s that contain a point of K, are all in the same equivalence class.*

Lemma 3.1.3 (Fisher and Thas [18]) *Let $\mathcal{F}$ be a linear flock of $Q^-(3,q)$, with carriers n and s. If K is any circle for which all the circles of the pencil intersecting in n and s that contain a point of K are in the same class, then K belongs to $\mathcal{F}$.*

Finally, we note that the reader can find a lot of information in the literature about the above defined equivalence relation and all related topics; we refer for instance to [6, §7], [14, 6.4.3], [18], [48, p. 24], [64, p. 84] or [68], where both geometric and algebraic approaches to the topic are explained.

3.1.3 Some words on eggs

It goes far beyond the scope of this thesis to elaborate a comprehensive theory on eggs of projective spaces. Eggs form a very rich field of study on their own, with links to flocks of cones, generalized quadrangles, translation planes, ... Here we restrict ourselves to giving some definitions and one particular result, which is sufficient for our objectives.

Definitions

An *egg* of $\mathsf{PG}(2n+m-1,q)$ is a set $O(n,m,q)$ of q^m+1 $(n-1)$-dimensional subspaces $\pi_0, \pi_1, \ldots, \pi_{q^m}$ of $\mathsf{PG}(2n+m-1,q)$, every three of which generate a $\mathsf{PG}(3n-1,q)$ and such that each element π_i of $O(n,m,q)$ is contained in a $\mathsf{PG}(n+m-1,q)$, having no point in common with $(\pi_0 \cup \pi_1 \cup \cdots \cup \pi_{q^m}) \setminus \pi_i$. The space $\mathsf{PG}(n+m-1,q)$ is called the *tangent space* of $O(n,m,q)$ at π_i.
An egg $O(n,m,q)$ is called *good* at an element π_i if for all distinct π_j, π_k, $j \neq i \neq k$, the space generated by π_i, π_j and π_k contains exactly $q^{m-n}+1$ elements of $O(n,m,q)$.

If $m = 2n$, one can construct an example of an egg in the following way. Consider the algebraic extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$ and the corresponding extension $\mathsf{PG}(4n-1,q^n)$ of $\mathsf{PG}(4n-1,q)$. In $\mathsf{PG}(4n-1,q^n)$, there exist n 3-dimensional subspaces $\mathsf{PG}^{(1)}(3,q^n), \mathsf{PG}^{(2)}(3,q^n), \ldots, \mathsf{PG}^{(n)}(3,q^n)$, which generate the space $\mathsf{PG}(4n-1,q^n)$ and constitute a conjugate n-tuple with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. Let $\mathcal{O}$ be an ovoid of $\mathsf{PG}^{(1)}(3,q^n)$. With every point $p^{(1)}$ of $\mathcal{O}$, there correspond $n-1$ points $p^{(i)} \in \mathsf{PG}^{(i)}(3,q^n)$, $i = 2,3,\ldots,n$, such that the points $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$ constitute a conjugate n-tuple with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. The points $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$ define an $(n-1)$-dimensional subspace of $\mathsf{PG}(4n-1,q)$. If we let $p^{(1)}$ vary in $\mathcal{O}$, we obtain a set of $q^{2n}+1$ such $(n-1)$-dimensional subspaces of $\mathsf{PG}(4n-1,q)$. It is easily seen that this set of $(n-1)$-dimensional subspaces forms an egg $O(n,2n,q)$ of $\mathsf{PG}(4n-1,q)$ and every egg constructed in this way, is called *regular*. In the special case where the ovoid $\mathcal{O}$ of $\mathsf{PG}^{(1)}(3,q^n)$ is an elliptic quadric $Q^-(3,q^n)$, the egg is said to be *classical*. Since for q odd every ovoid of $\mathsf{PG}(3,q^n)$ is an elliptic quadric by Barlotti [3] and Panella [49], every regular egg of $\mathsf{PG}(4n-1,q)$ with q odd, is classical.

From the construction of a regular egg, one immediately deduces that a regular egg is good at each of its elements. In [63], Thas shows that the converse also holds.

Theorem 3.1.4 (Thas [63]) *Every egg of* $\mathsf{PG}(4n-1,q)$ *which is good at each of its elements, is regular.*

3.2 Some words on 1-systems of $Q^-(7,q)$

By the definition in Section 1.2.1, applied to the special case of this chapter, a 1-system of $Q^-(7,q)$ is a set $\mathcal{M} := \{L_0, L_1, \ldots, L_{q^4}\}$ of q^4+1 lines of $Q^-(7,q)$ with the property that every plane of $Q^-(7,q)$ containing a line L_i of $\mathcal{M}$ has an empty intersection with $(L_0 \cup L_1 \cup \cdots \cup L_{q^4}) \setminus L_i$. From Corollary 2.2.3, we know that every line of $Q^-(7,q)$ meets $\tilde{\mathcal{M}}$ in 0, 1, 2 or $q+1$ points and the latter occurs if and only if the line belongs to $\mathcal{M}$. Together with Theorem 1.2.5, a fruitful observation concerning the planes of $Q^-(7,q)$ follows. If ζ is a plane of $Q^-(7,q)$, then it has $q+1$ points in common with $\tilde{\mathcal{M}}$ by Theorem 1.2.5. By Corollary 2.2.3, either no three points of $\zeta \cap \tilde{\mathcal{M}}$ are collinear, or $\zeta \cap \tilde{\mathcal{M}}$ consists of the $q+1$ points on a line of $\mathcal{M}$. The famous result of Segre [53, 54], which says that every $(q+1)$-arc of $\mathsf{PG}(2,q)$, q odd, is an irreducible conic, now implies that if q is odd, a plane of $Q^-(7,q)$ either contains a line of $\mathcal{M}$, or it meets $\tilde{\mathcal{M}}$ in a conic. This observation will play a crucial role in the sequel of this chapter.

Further, we recall that by Theorem 1.2.3, a hyperplane H of $\mathsf{PG}(7,q) = \langle Q^-(7,q)\rangle$ contains exactly one line of $\mathcal{M}$ if it is tangent to $Q^-(7,q)$ at a point $x \in \tilde{\mathcal{M}}$, and exactly q^2+1 lines of $\mathcal{M}$ otherwise. This result has an immediate corollary concerning 5-dimensional subspaces of $\mathsf{PG}(7,q)$.

Corollary 3.2.1 *If β is a 5-dimensional subspace of $\mathsf{PG}(7,q)$, intersecting $Q^-(7,q)$ in a non-singular elliptic or hyperbolic quadric, and β contains q^2+1 lines of $\mathcal{M}$, then $\beta \cap \tilde{\mathcal{M}}$ consists of the points on these q^2+1 lines of $\mathcal{M}$.*

Proof.
Suppose that there exists a point $x \in \beta \cap \tilde{\mathcal{M}}$, not on any of the q^2+1 lines of $\mathcal{M}$ in β and let $M \in \mathcal{M}$ be the line of $\mathcal{M}$ through x. Then $\langle \beta, M\rangle$ is a hyperplane of $\mathsf{PG}(7,q)$ which shares at least q^2+2 lines with $\mathcal{M}$, a contradiction to Theorem 1.2.3. It follows that $\beta \cap \tilde{\mathcal{M}}$ consists exactly of the points on the lines of $\mathcal{M}$ in β. □

By Section A.3.4 of Appendix A, there is one single 1-system of the elliptic quadric $Q^-(7,q)$ known, both for q even and for q odd. This is the classical 1-system of $Q^-(7,q)$, which arises from the trace trick applied to $Q^-(3,q^2)$, considered as an ovoid of itself. This means in this particular case that there exist two disjoint 3-spaces γ and $\overline{\gamma}$ in the extension $\mathsf{PG}(7,q^2)$ of $\mathsf{PG}(7,q) = \langle Q^-(7,q)\rangle$, which are conjugate with respect to $\mathsf{GF}(q^2)$ and polar with respect to the polarity of the extension $Q^+(7,q^2)$ of $Q^-(7,q)$. Moreover, $\gamma \cap Q^+(7,q^2)$ is an elliptic quadric $Q^-(3,q^2)$, and $\mathcal{M}$ consists of all lines $x\overline{x} \cap \mathsf{PG}(7,q)$, where x varies on $Q^-(3,q^2)$ and $\overline{x}$ is its conjugate, so it

is a point of $\overline{\gamma} \cap Q^+(7,q^2)$. In Shult and Thas [57], one can find a proof that the set of lines $\mathcal{M} = \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in Q^-(3,q^2)\}$ is indeed a 1-system of $Q^-(7,q)$.
Note that for $q = 2$, it follows from the classification result of Hamilton and Mathon [21], that the classical 1-system is the unique 1-system of $Q^-(7,2)$ up to isomorphism.

From now on, we will always assume that $\mathcal{M}$ is a 1-system of $Q^-(7,q)$, q odd.

3.3 On cones $rQ^+(3,q) \subseteq Q^-(7,q)$, containing two lines of $\mathcal{M}$

In this section, the structure of the points of $\tilde{\mathcal{M}}$ in a cone $rQ^+(3,q) \subseteq Q^-(7,q)$ containing two lines of $\mathcal{M}$ will be investigated.

Consider two lines $L_1, L_2 \in \mathcal{M}$ and a point $r \in Q^-(7,q)$ such that $\langle L_1, L_2\rangle \subseteq r^\perp$, with $\perp$ the polarity of $Q^-(7,q)$. Then $\langle L_1, L_2\rangle \cap Q^-(7,q)$ is a hyperbolic quadric $Q^+(3,q)$ and $\langle r, L_1, L_2\rangle \cap Q^-(7,q)$ is a cone $rQ^+(3,q)$. As each line of $Q^-(7,q)$ which does not belong to $\mathcal{M}$, has at most two points in common with $\tilde{\mathcal{M}}$, $\langle L_1, L_2\rangle \cap \tilde{\mathcal{M}}$ consists exactly of the points on L_1 and L_2. Suppose that there exists a plane of $rQ^+(3,q)$, different from $\langle r, L_1\rangle$ and $\langle r, L_2\rangle$, containing a line M of $\mathcal{M}$. Then M meets $Q^+(3,q)$ in a point of $\tilde{\mathcal{M}}$, not on L_1 or L_2, a contradiction. Consequently, every plane of $rQ^+(3,q)$, except for $\langle r, L_1\rangle$ and $\langle r, L_2\rangle$, meets $\tilde{\mathcal{M}}$ in an irreducible conic.

We introduce the following notation. The two reguli of lines of $Q^+(3,q)$ will be denoted by $\{L_1, L_2, S_1, S_2, \ldots, S_{q-1}\}$ and $\{N_1, N_2, \ldots, N_{q+1}\}$. The conics of points of $\tilde{\mathcal{M}}$ in the planes $\langle r, N_i\rangle$ will be denoted by $\mathcal{C}_i$ and the ones in the planes $\langle r, S_j\rangle$ by $\mathcal{C}'_j$. The plane $\langle r, L_1\rangle$, respectively $\langle r, L_2\rangle$, contains exactly one point of each $\mathcal{C}_i$, so $\langle r, L_1\rangle \cap \langle r, N_i\rangle$, respectively $\langle r, L_2\rangle \cap \langle r, N_i\rangle$, is the tangent line to $\mathcal{C}_i$ at $L_1 \cap N_i$, respectively $L_2 \cap N_i$. Hence N_i is the polar line of r with respect to $\mathcal{C}_i$. It follows that the line $\langle r, S_j\rangle \cap \langle r, N_i\rangle$ has either 0 or 2 points in common with $\mathcal{C}'_j$, and so r is an internal point of $\mathcal{C}'_j$ for all $j = 1, 2, \ldots, q-1$.

We show that also the tangents through r of the conics $\mathcal{C}'_j$, which are lines defined over the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, lie in a common plane; see figure 3.3 for a sketch of the situation.

Lemma 3.3.1 *There exist two conjugate lines A and $\overline{A}$ over $\mathsf{GF}(q^2)$, belonging to the regulus over $\mathsf{GF}(q^2)$ containing $\{N_1, N_2, \ldots, N_{q+1}\}$, such that, for each $j = 1, 2, \ldots, q-1$, the lines ra_j and $r\overline{a}_j$ are tangent to the conic $\mathcal{C}'_j$ at a_j and $\overline{a}_j$ respectively, where $a_j := A \cap S_j$ and $\overline{a}_j := \overline{A} \cap S_j$.*

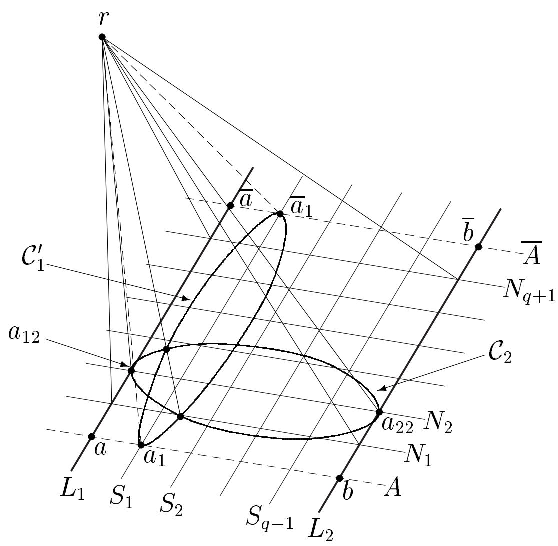

Figure 3.3: Notation for Section 3.3

Proof.
The tangents through r of the conic $\mathcal{C}_1$ are the lines $\langle r, N_1\rangle \cap \langle r, L_1\rangle$ and $\langle r, N_1\rangle \cap \langle r, L_2\rangle$, with tangent points $N_1 \cap L_1$ and $N_1 \cap L_2$. Assume that the secants through r of $\mathcal{C}_1$ are $\langle r, N_1\rangle \cap \langle r, S_j\rangle$, for $j = 1, 2, \ldots, \frac{q-1}{2}$. Consider a conic $\mathcal{C}_i \neq \mathcal{C}_1$. The lines $\langle r, N_i\rangle \cap \langle r, L_1\rangle$ and $\langle r, N_i\rangle \cap \langle r, L_2\rangle$ are the tangents through r of $\mathcal{C}_i$, with tangent points $N_i \cap L_1$ and $N_i \cap L_2$. Let β be the linear projectivity of $\langle r, L_1, L_2\rangle$ which fixes r and such that the restriction of β to N_i is the linear projectivity from N_i to N_1, defined by the regulus of $Q^+(3,q)$ containing L_1 and L_2. Then $\mathcal{C}_1$ and $\mathcal{C}_i^\beta$ are conics in the plane $\langle r, N_1\rangle$ with the same tangents through r and the same tangent points. By Section 3.1.2, these two tangent points on N_1 define an equivalence relation on the points of $N_1 \setminus \{L_1 \cap N_1, L_2 \cap N_1\}$, where the two equivalence classes correspond to the set of secants of $\mathcal{C}_1$ through r on the one hand and the set of lines on r missing $\mathcal{C}_1$ on the other hand. As $\mathcal{C}_i^\beta$ intersects N_1 in the same two points as $\mathcal{C}_1$ and r is the pole of N_1 with respect to both $\mathcal{C}_1$ and $\mathcal{C}_i^\beta$, it follows from Section 3.1.2 that these conics either have the same set of secants through r, or their sets of secants through r are complementary. This implies that either the secants through r of $\mathcal{C}_i$ are given by $\{\langle r, N_i\rangle \cap \langle r, S_j\rangle \mid j = 1, 2, \ldots, \frac{q-1}{2}\}$ or they are given by $\{\langle r, N_i\rangle \cap \langle r, S_j\rangle \mid j = \frac{q+1}{2}, \frac{q+3}{2}, \ldots, q-1\}$.

As r is an internal point of each conic $\mathcal{C}'_j$, there are exactly $\frac{q+1}{2}$ secants of $\tilde{\mathcal{M}}$ in each plane $\langle r, S_j\rangle$. We assume that the numbering of the lines N_i and S_j is such that for $i = 1, 2, \ldots, \frac{q+1}{2}$, the secants through r of $\mathcal{C}_i$ are contained in the planes $\langle r, S_j\rangle$, $j = 1, 2, \ldots, \frac{q-1}{2}$. Hence $|\mathcal{C}_i \cap \mathcal{C}'_j| = 2$ for $(i,j) \in \{1, 2, \ldots, \frac{q+1}{2}\} \times \{1, 2, \ldots, \frac{q-1}{2}\}$ and for $(i,j) \in \{\frac{q+3}{2}, \frac{q+5}{2}, \ldots, q+1\} \times \{\frac{q+1}{2}, \frac{q+3}{2}, \ldots, q-1\}$. If we put $L_s \cap \mathcal{C}_i := a_{si}$, then we obtain on L_s two sets of points $\{a_{si} \mid i = 1, 2, \ldots, \frac{q+1}{2}\}$ and $\{a_{si} \mid i = \frac{q+3}{2}, \frac{q+5}{2}, \ldots, q+1\}$ for $s = 1, 2$. It follows from Lemma 3.1.1 that these sets uniquely define points a, $\overline{a}$ on L_1 and b, $\overline{b}$ on L_2, where $ab := A$ and $\overline{ab} := \overline{A}$ belong to the regulus over $\mathsf{GF}(q^2)$ containing $\{N_1, N_2, \ldots, N_{q+1}\}$, such that the planes $\langle r, A\rangle$ and $\langle r, \overline{A}\rangle$ meet each plane $\langle r, S_j\rangle$ in the tangent through r of the extension to $\mathsf{GF}(q^2)$ of the conic $\mathcal{C}'_j$, $j = 1, 2, \ldots, q-1$.

Consider in $\mathsf{PG}(4,q) := \langle rQ^+(3,q)\rangle$ the involution θ with centre r and axis $\langle L_1, L_2\rangle$. For each point $x \in \langle L_1, L_2\rangle$, the line rx is fixed by θ, so θ induces an involution on rx with fixed points r and x. It is known (see Hirschfeld [24, p. 140]) that an involution on a line, which has two fixed points r and x, maps each point $y \notin \{r, x\}$ onto the unique point z such that $\{r, x; y, z\} = -1$. Let rx be a secant line of some conic $\mathcal{C}_i$, and hence also of some conic $\mathcal{C}'_j$. If $\mathcal{C}_i \cap \mathcal{C}'_j = \{h_{ij}, k_{ij}\}$, then clearly $\{r, x; h_{ij}, k_{ij}\} = -1$, so $h_{ij}^\theta = k_{ij}$. It follows that θ fixes each conic $\mathcal{C}_i$ and each conic $\mathcal{C}'_j$. Consequently, the tangent points on the tangents over $\mathsf{GF}(q^2)$ through r of the conics $\mathcal{C}'_j$ must be fixed by θ and hence they lie in $\langle L_1, L_2\rangle$. As these tangent points were already known to lie in the planes $\langle r, A\rangle$ and $\langle r, \overline{A}\rangle$, they must be points on the lines A and $\overline{A}$. □

Lemma 3.3.2 *The lines A and $\overline{A}$ are independent of the choice of $r \in Q^-(7,q)$ such that $\langle L_1, L_2\rangle \subseteq r^\perp$.*

Proof.
Let r and $\tilde{r}$ be two points on $Q^-(7,q)$, with $r, \tilde{r} \in \langle L_1, L_2\rangle^\perp$. The irreducible conics of points of $\tilde{\mathcal{M}}$ in the planes of the cone $\langle \tilde{r}, L_1, L_2\rangle \cap Q^-(7,q) := \tilde{r}Q^+(3,q)$, except for $\langle \tilde{r}, L_1\rangle$ and $\langle \tilde{r}, L_2\rangle$, will be denoted by $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}'_j$ in such manner that $\tilde{\mathcal{C}}_i = \langle \tilde{r}, N_i\rangle \cap \tilde{\mathcal{M}}$ and $\tilde{\mathcal{C}}'_j = \langle \tilde{r}, S_j\rangle \cap \tilde{\mathcal{M}}$, for $i = 1, 2, \ldots, q+1$ and $j = 1, 2, \ldots, q-1$. In $\mathsf{PG}(5,q) := \langle r, \tilde{r}, L_1, L_2\rangle$, consider the automorphism γ_i which fixes $\langle L_1, L_2\rangle$ pointwise, maps r onto $\tilde{r}$ and maps $\mathcal{C}_i$ onto $\tilde{\mathcal{C}}_i$, for $i = 1, 2, \ldots, q+1$. Let $\mathcal{C}_i$ and $\mathcal{C}'_j$ have two points in common. As $\langle \mathcal{C}'_j\rangle^{\gamma_i} = \langle \tilde{\mathcal{C}}'_j\rangle$ we have $|\langle \tilde{\mathcal{C}}'_j\rangle \cap \tilde{\mathcal{C}}_i| = 2$, and so $|\tilde{\mathcal{C}}'_j \cap \tilde{\mathcal{C}}_i| = 2$. This implies that the irreducible conics of points of $\tilde{\mathcal{M}}$ in $\tilde{r}Q^+(3,q)$ define the same sets of points of size $\frac{q+1}{2}$ on L_1 and L_2 and consequently also the same lines A and $\overline{A}$. This proves the lemma. □

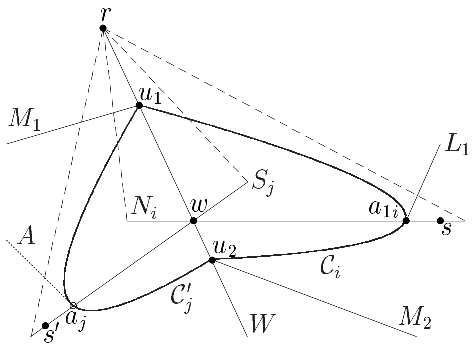

Figure 3.4: Notation for Section 3.4

3.4 On quadrics $Q^-(5,q) \subseteq Q^-(7,q)$

It will be shown in this section that there exist a lot of elliptic quadrics $Q^-(5,q)$ on $Q^-(7,q)$ which contain exactly q^2+1 elements of $\mathcal{M}$.

As in the previous section, we start with two lines $L_1, L_2 \in \mathcal{M}$ and a point r on $Q^-(7,q)$ such that $\langle L_1, L_2\rangle \subseteq r^\perp$. Consider two lines N_i and S_j such that $|\mathcal{C}_i \cap \mathcal{C}'_j| = 2$, which is the same as demanding that the line $\langle r, N_i\rangle \cap \langle r, S_j\rangle = rw := W$, with w the common point of N_i and S_j, is a secant of $\tilde{\mathcal{M}}$. Denote the points of $\mathcal{C}_i \cap \mathcal{C}'_j$ by u_1 and u_2, see Figure 3.4. In the plane $\langle r, N_i\rangle$, the line N_i is the polar line of r with respect to $\mathcal{C}_i$ and similarly, S_j is the polar line of r with respect to $\mathcal{C}'_j$ in $\langle r, S_j\rangle$. It follows that $\langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$ consists of the q^2+1 poles of N_i, respectively S_j, with respect to the conics $\tilde{\mathcal{M}} \cap \zeta$, with ζ any generator of $Q^-(7,q)$ through N_i, respectively S_j. Denote the pole of W with respect to $\mathcal{C}_i$ by s and the pole of W with respect to $\mathcal{C}'_j$ by s', so s is a point on N_i and s' a point on S_j. Next, we consider the unique lines $M_1, M_2 \in \mathcal{M}$ through u_1, respectively u_2. Then by interchanging the roles of L_1, L_2 and M_1, M_2, it is found that M_1 and M_2 must be contained in $s^\perp$ and $s'^\perp$, and so in $\langle s, s'\rangle^\perp$, since s and s' are poles of the line $W \in \langle M_1, M_2\rangle$ and hence play the same role for $\langle M_1, M_2\rangle$ as does r for $\langle L_1, L_2\rangle$.

We can repeat the above argument for all points $\tilde{r} \in \langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$ with the property that $|w\tilde{r} \cap \tilde{\mathcal{M}}| = 2$. Let, as before, a_{1i}, respectively a_{2i}, be the point $L_1 \cap N_i$, respectively $L_2 \cap N_i$, and a_j, respectively $\overline{a}_j$, the point $A \cap S_j$, respectively $\overline{A} \cap S_j$. Then it holds that $\{w, s; a_{1i}, a_{2i}\} = -1$ and $\{w, s'; a_j, \overline{a}_j\} = -1$, see Section 3.1.1. Since we work with the same point w for the point $\tilde{r}$ as we did for the point r, we also have the same lines N_i

and S_j and hence the same pairs $\{a_{1i}, a_{2i}\}$ and $\{a_j, \overline{a}_j\}$. So, the points s and s', which are uniquely determined by w, a_{1i}, a_{2i}, respectively by $w, a_j, \overline{a}_j$, are the same as well for the points r and $\tilde{r}$. Now, the two points of $w\tilde{r} \cap \tilde{\mathcal{M}}$ define two lines $\tilde{M}_1, \tilde{M}_2 \in \mathcal{M}$ which are contained in $\langle s, s'\rangle^\perp$ too. If $\tilde{r}$ varies in $\langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$, we thus obtain many lines $\tilde{M}_1, \tilde{M}_2$ of $\mathcal{M}$ in $\langle s, s'\rangle^\perp$. The following theorem states that there are q^2+1 such lines of $\mathcal{M}$ in $\langle s, s'\rangle^\perp$.

Theorem 3.4.1 *The elliptic quadric $\langle s, s'\rangle^\perp \cap Q^-(7,q) = Q^-(5,q)$ contains exactly q^2+1 elements of $\mathcal{M}$.*

Proof.
We shall count how many lines of $\mathcal{M}$ we find in $\langle s, s'\rangle^\perp$ by the above reasoning. First, observe that different choices for $\tilde{r}$ yield disjoint pairs $\{\tilde{M}_1, \tilde{M}_2\}$. For otherwise, some line $M \in \mathcal{M}$ would intersect wr and $w\tilde{r}$ for different points r and $\tilde{r}$ and $\langle w, r, \tilde{r}\rangle$ would be a plane of $Q^-(7,q)$. This is impossible because no two points of $\langle L_1, L_2\rangle^\perp \cap Q^-(7,q) = Q^-(3,q)$ are collinear.

If there exist α points r in $\langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$ such that $|wr \cap \tilde{\mathcal{M}}| = 2$, then this yields 2α lines of $\mathcal{M}$ in $\langle s, s'\rangle^\perp$. As there are at most q^2+1 lines of $\mathcal{M}$ in a hyperplane of $\mathsf{PG}(7,q)$, there are certainly at most q^2+1 lines of $\mathcal{M}$ in the 5-dimensional subspace $\langle s, s'\rangle^\perp$, so $2\alpha \leq q^2+1$ or $\alpha \leq \frac{q^2+1}{2}$.

Label the points of $\langle L_1, L_2\rangle \cap Q^-(7,q) = Q^+(3,q)$, not on L_1 or L_2, as $w_1, w_2, \ldots, w_{q^2-1}$ and let α_i be the number of points r of $\langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$ such that $|w_i r \cap \tilde{\mathcal{M}}| = 2$. By counting pairs (w_i, r) with $r \in \langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$ and $|w_i r \cap \tilde{\mathcal{M}}| = 2$, we obtain:

$$\sum_{i=1}^{q^2-1} \alpha_i = (q^2+1) \cdot \frac{1}{2}(q^2-1). \tag{3.1}$$

Suppose that there exists a point w_i with the property that $\alpha_i < \frac{q^2+1}{2}$. As $\frac{q^2+1}{2}$ is an upper bound for all α_i, this implies that

$$\sum_{i=1}^{q^2-1} \alpha_i < (q^2-1) \cdot \frac{q^2+1}{2},$$

a contradiction to (3.1). We conclude that $\alpha_i = \frac{q^2+1}{2}$ for each w_i and hence the elliptic quadric $\langle s, s'\rangle^\perp \cap Q^-(7,q) = Q^-(5,q)$ contains exactly q^2+1 lines of $\mathcal{M}$. $\square$

So, for each $w \in Q^+(3,q)$, not on L_1 or L_2, we have found an elliptic quadric $Q^-(5,q) = \langle s, s'\rangle^\perp \cap Q^-(7,q)$ which contains q^2+1 lines of $\mathcal{M}$. Note that there exists a second point w' of $Q^+(3,q)$, not on L_1 or L_2, which yields the same points s and s' and hence the same elliptic quadric, namely the point w' determined by $\langle s,s'\rangle^\perp \cap Q^+(3,q) = \{w,w'\}$. Moreover, if one starts with either the point s or the point s', then the elliptic quadric $Q^-(5,q)$ obtained is exactly $\langle w,w'\rangle^\perp \cap Q^-(7,q)$. We see that the points of $Q^+(3,q)$, not on L_1 or L_2, are divided into pairs (w,w') and that each such pair corresponds to a unique other pair (s,s'). We will refer to the set of these pairs as $\mathscr{P}(L_1,L_2)$; clearly $|\mathscr{P}(L_1,L_2)| = \frac{q^2-1}{2}$. The set of lines defined by the pairs of $\mathscr{P}(L_1,L_2)$ is denoted by $\mathscr{L}(L_1,L_2)$, that is,

$$\mathscr{L}(L_1,L_2) := \{ww' \mid (w,w') \in \mathscr{P}(L_1,L_2)\}.$$

The set of all elliptic quadrics $Q^-(5,q) = Q^-(7,q) \cap \langle s,s'\rangle^\perp$ with $(s,s') \in \mathscr{P}(L_1,L_2)$, will be denoted by $\mathscr{Q}(L_1,L_2)$.

For fixed lines L_1,L_2 of $\mathcal{M}$, each quadric $\langle s,s'\rangle^\perp \cap Q^-(7,q)$ with $(s,s') \in \mathscr{P}(L_1,L_2)$ contains the elliptic quadric $\langle L_1,L_2\rangle^\perp \cap Q^-(7,q) := Q^-(3,q)$, since s and s' are points of $\langle L_1,L_2\rangle$. Moreover, by Corollary 3.2.1, the 5-dimensional spaces of two distinct quadrics of $\mathscr{Q}(L_1,L_2)$ cannot have a 4-dimensional intersection, so they have exactly the 3-dimensional space of the elliptic quadric $Q^-(3,q)$ in common, that is, any two quadrics in $\mathscr{Q}(L_1,L_2)$ have exactly $Q^-(3,q)$ in common. Again by Corollary 3.2.1, the quadric $Q^-(3,q)$ is disjoint from $\tilde{\mathcal{M}}$. We will show that each elliptic quadric $Q^-(5,q)$ through $Q^-(3,q)$ which contains an element of $\mathcal{M}$, belongs to $\mathscr{Q}(L_1,L_2)$.

Lemma 3.4.2 *If $B \in \mathcal{M} \setminus \{L_1,L_2\}$ is not contained in any quadric of $\mathscr{Q}(L_1,L_2)$, then $\langle \gamma, B\rangle$, with $\gamma := \langle L_1,L_2\rangle^\perp$, intersects $Q^-(7,q)$ in a hyperbolic quadric $Q^+(5,q)$.*

Proof.
If $B \in \mathcal{M}\setminus\{L_1,L_2\}$ is not a line of any quadric of $\mathscr{Q}(L_1,L_2)$, then B is disjoint from all these quadrics, by Corollary 3.2.1. This implies that $\langle\gamma,B\rangle$ intersects each $\langle s,s'\rangle^\perp$, with $(s,s') \in \mathscr{P}(L_1,L_2)$, exactly in γ, or, equivalently, $\langle\gamma,B\rangle^\perp := K$ is disjoint from every line $ss' \in \mathscr{L}(L_1,L_2)$. If $K \cap L_i$ were non-empty for either $i=1$ or $i=2$, then $\langle\gamma,B\rangle$ and $L_i^\perp$ would be contained in a hyperplane of $\mathsf{PG}(7,q)$, so B would have a point in common with the singular quadric $L_iQ^-(3,q)$, a contradiction since $B \in \mathcal{M} \setminus \{L_1,L_2\}$. Thus K is a line in $\langle L_1,L_2\rangle$ which is disjoint from $Q^+(3,q) = \langle L_1,L_2\rangle \cap Q^-(7,q)$, that is, K is an external line of $Q^-(7,q)$. It follows that $\langle\gamma,B\rangle \cap Q^-(7,q)$ is a hyperbolic quadric $Q^+(5,q)$. $\square$

By this lemma and Theorem 3.4.1, all elliptic quadrics through $Q^-(3,q)$ and an arbitrary line of $\mathcal{M}$ belong to $\mathscr{Q}(L_1,L_2)$ and as such they all contain

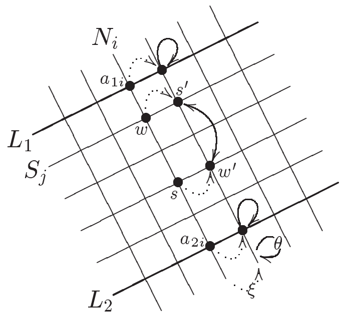

Figure 3.5: Lemma 3.5.1

exactly q^2+1 lines of $\mathcal{M}$. Apparently, it also holds that $\langle \gamma, B \rangle \cap Q^-(7,q)$, with $B \in \mathcal{M} \setminus \{L_1, L_2\}$, is always a non-singular quadric.

3.5 On quadrics $Q^+(5,q) \subseteq Q^-(7,q)$

It is the aim of this section to show that not only the elliptic quadrics $Q^-(5,q)$ through $Q^-(3,q)$ and some line of $\mathcal{M} \setminus \{L_1, L_2\}$, but also the non-singular hyperbolic quadrics containing $Q^-(3,q)$ and a line of $\mathcal{M} \setminus \{L_1, L_2\}$, meet $\mathcal{M}$ in exactly q^2+1 lines.

We start with some more notation; the one introduced before will still be used. The 3-dimensional subspace $\langle L_1, L_2 \rangle$ will henceforth be denoted by π. In $\pi \cap Q^-(7,q) = Q^+(3,q)$, we have already defined the lines L_1, L_2, A and $\overline{A}$, as well as the set of pairs $\mathscr{P}(L_1, L_2)$ and the set of lines $\mathscr{L}(L_1, L_2)$. Let T denote the line of π through the points $L_1 \cap A = a$ and $L_2 \cap \overline{A} = \overline{b}$; $\overline{T}$ denotes its conjugate, containing the points $L_1 \cap \overline{A} = \overline{a}$ and $L_2 \cap A = b$. As before, γ is the 3-dimensional subspace $\pi^\perp$, so that $\gamma \cap Q^-(7,q) = Q^-(3,q)$.

Lemma 3.5.1 *For each line $ww' \in \mathscr{L}(L_1, L_2)$, with $w = N_i \cap S_j$, the lines L_1, L_2 and ww' define a regulus $\mathcal{R}$ of π, consisting of L_1, L_2 and all lines $tt' \in \mathscr{L}(L_1, L_2)$, where t varies on $N_i \setminus \{L_1 \cap N_i, L_2 \cap N_i\}$.*

Proof.
The regulus of $Q^+(3,q)$ containing L_1 and L_2 defines a linear projectivity ξ from $N_i = ws$ to $w's'$. Let θ be the involution on $w's'$ with fixed points $L_1 \cap w's'$ and $L_2 \cap w's'$. Then $\xi\theta$ is a linear projectivity from N_i to $w's'$ and $w^{\xi\theta} = s'^{\theta} = w'$ since $\{L_1 \cap w's', L_2 \cap w's'; s', w'\} = -1$. Joining corresponding points of $\xi\theta$ yields a regulus containing L_1, L_2 and all lines $tt' \in \mathscr{L}(L_1, L_2)$ where t varies on $ws \setminus \{L_1 \cap ws, L_2 \cap ws\}$. $\square$

Remark
Similarly as in this lemma, one shows that the lines $ww' \in \mathscr{L}(L_1, L_2)$, where w varies on the line $S_j = ws'$, form a regulus which contains A and $\overline{A}$, if considered over $\mathsf{GF}(q^2)$.

Lemma 3.5.2 *All $\frac{q^2-1}{2}$ lines $ww' \in \mathscr{L}(L_1, L_2)$, as well as L_1 and L_2, belong to the regular spread of π, defined by T and $\overline{T}$.*

Proof.
Consider the involution σ in π with axes T and $\overline{T}$. The lines L_1 and L_2 are fixed by σ, because they intersect T and $\overline{T}$. Hence, σ induces on both L_i, $i = 1, 2$, an involution with fixed points $T \cap L_i$ and $\overline{T} \cap L_i$. From Hirschfeld [24, p. 140] and the fact that $\{T \cap L_i, \overline{T} \cap L_i; L_i \cap ws, L_i \cap w's'\} = -1$, it follows that $(L_i \cap ws)^\sigma = L_i \cap w's'$ for $i = 1, 2$. As a consequence, σ maps ws onto $w's'$. Similarly, A and $\overline{A}$ are also fixed by σ, with fixed points $A \cap T = L_1 \cap T = a$ and $A \cap \overline{T} = L_2 \cap \overline{T} = b$, respectively $\overline{A} \cap T = L_2 \cap T = \overline{b}$ and $\overline{A} \cap \overline{T} = L_1 \cap \overline{T} = \overline{a}$. Here, $\{a, b; ws' \cap A, w's \cap A\} = -1 = \{\overline{a}, \overline{b}; ws' \cap \overline{A}, w's \cap \overline{A}\}$, which implies that $(ws' \cap A)^\sigma = w's \cap A$ and $(ws' \cap \overline{A})^\sigma = w's \cap \overline{A}$. It follows that $(ws')^\sigma = w's$ and we may conclude that $w^\sigma = (ws' \cap ws)^\sigma = w's \cap w's' = w'$ and similarly $s^\sigma = s'$. It is now clear that the lines ww' and ss' are fixed by σ and consequently they intersect T and $\overline{T}$. As this argument holds for all lines ww' and ss' of $\mathscr{L}(L_1, L_2)$, the lemma follows. $\square$

We now use the Klein correspondence and consider the lines of π as points on the Klein quadric $\mathcal{K}$. We will use the same notation for lines of π and for the corresponding points on $\mathcal{K}$; this will not cause confusion. The lines of π belonging to the regular spread defined by T and $\overline{T}$, yield points on a 3-dimensional elliptic quadric $\mathcal{E}$ on $\mathcal{K}$, such that $\langle \mathcal{E} \rangle^\perp$, with $\perp$ the polarity of $\mathcal{K}$, is the external line $T\overline{T}$ of $\mathcal{K}$. We have seen that the lines $ww' \in \mathscr{L}(L_1, L_2)$ constitute $\frac{q+1}{2}$ reguli containing L_1 and L_2. On $\mathcal{K}$ each such regulus is a circle of $\mathcal{E}$, considered as an inversive plane, through the points L_1 and L_2. We denote the set of these $\frac{q+1}{2}$ circles by $\mathscr{C}$. Note that $(L_1L_2)^\perp = A\overline{A}$ with respect to $\mathcal{E}$, where $A\overline{A}$ is an external line of $\mathcal{E}$. Finally, the regulus consisting of the lines $ww' \in \mathscr{L}(L_1, L_2)$, where w varies on $S_j = ws'$, translates to a circle C of $\mathcal{E}$ in a plane through the line $A\overline{A}$.

This setting on the Klein quadric and the two lemmas of Fisher and Thas [18] which were mentioned in Section 3.1.2, are used to prove the following lemma.

Lemma 3.5.3 *If B is a line of $\mathcal{M} \setminus \{L_1, L_2\}$ such that $\langle \gamma, B \rangle \cap Q^-(7, q)$ is hyperbolic, then either $B' := \langle \gamma, B \rangle \cap \pi$ is a line of the regular spread defined by T and $\overline{T}$, or B' meets both A and $\overline{A}$.*

Proof.
Consider the line B' as a point on the Klein quadric. If B' does not belong to the regular spread defined by T and $\overline{T}$, then the point B' is not a point of $\mathcal{E}$. It is easily seen that $B'^{\perp} \cap \langle \mathcal{E} \rangle$ is a plane intersecting $\mathcal{E}$ in a circle C'. This circle C' is disjoint from the circles of $\mathscr{C}$, for otherwise the line B' would have a point in common with some line $ww' \in \mathscr{L}(L_1, L_2)$. This would imply that $\langle \gamma, B \rangle$ and $\langle s, s' \rangle^{\perp}$, where (s, s') is the pair corresponding to (w, w'), lie in a hyperplane of $\mathsf{PG}(7,q)$ and B would intersect $\langle s, s' \rangle^{\perp} \cap Q^-(7,q)$ in a point, a contradiction to Corollary 3.2.1.

From Lemma 3.1.2, applied to the linear flock of $\mathcal{E}$ consisting of the circles in the planes through $A\overline{A}$, which has carriers L_1 and L_2, it follows that the $\frac{q+1}{2}$ circles of $\mathscr{C}$ belong to the same equivalence class, because they all meet the circle C as we have seen in the remark following Lemma 3.5.2. Hence the $\frac{q+1}{2}$ other circles on L_1 and L_2 also belong to the same equivalence class. As C' is disjoint from the $\frac{q+1}{2}$ circles of $\mathscr{C}$ through L_1 and L_2, the other $\frac{q+1}{2}$ circles of the pencil through L_1 and L_2, which all belong to the same equivalence class, must all have two points in common with C'. From Lemma 3.1.3, it now follows that C' is a circle of the linear flock of $\mathcal{E}$ defined by $A\overline{A}$, which means that the line B' of π meets both A and $\overline{A}$. □

Note that if B' meets A and $\overline{A}$, it is a line of the regular spread defined by A and $\overline{A}$.

Theorem 3.5.4 *For every line $B \in \mathcal{M} \setminus \{L_1, L_2\}$ such that $\langle \gamma, B \rangle \cap Q^-(7,q)$ is a hyperbolic quadric $Q^+(5,q)$, the quadric $\langle \gamma, B \rangle \cap Q^-(7,q)$ contains q^2+1 lines of $\mathcal{M}$.*

Proof.
Consider two arbitrary lines $B_1, B_2 \in \mathcal{M} \setminus \{L_1, L_2\}$ such that $\langle \gamma, B_1 \rangle \cap Q^-(7,q)$ and $\langle \gamma, B_2 \rangle \cap Q^-(7,q)$ are distinct hyperbolic quadrics, which we denote by $Q_1^+(5,q)$, respectively $Q_2^+(5,q)$. Suppose that B_1 and B_2 are chosen in such manner that $Q_1^+(5,q)$ and $Q_2^+(5,q)$ are contained in some hyperplane H, with $H \cap Q^-(7,q) = Q(6,q)$. Then the lines $B_1' := \langle \gamma, B_1 \rangle \cap \pi$ and $B_2' := \langle \gamma, B_2 \rangle \cap \pi$ have a point p in common. It is impossible that H contains a third 5-dimensional subspace $\langle \gamma, B_3 \rangle$, $B_3 \in \mathcal{M} \setminus \{L_1, L_2, B_1, B_2\}$, with $\langle \gamma, B_3 \rangle \cap Q^-(7,q)$ hyperbolic, because in that case the three lines B_1', B_2' and $B_3' = \langle \gamma, B_3 \rangle \cap \pi$ would pairwise intersect in a point; so two lines B_i' and B_j', belonging to the same regular spread of π, would have a point in common, a contradiction. If $\langle \gamma, B_3 \rangle \cap Q^-(7,q)$ were elliptic, then by Theorem 3.4.1 and Lemma 3.4.2, $\langle \gamma, B_3 \rangle$ would contain all q^2+1 lines of $\mathcal{M}$ in $Q(6,q)$ and hence $\langle \gamma, B_3 \rangle = \langle \gamma, B_1 \rangle = \langle \gamma, B_2 \rangle$. This situation too contradicts

our assumptions and consequently, all q^2+1 lines of $\mathcal{M}$ in $Q(6,q)$ are either contained in $Q_1^+(5,q)$ or in $Q_2^+(5,q)$.

For each line $M \in \mathcal{M} \setminus \{L_1, L_2\}$, which is not contained in $Q(6,q)$, the plane $\langle H^\perp, M\rangle$ intersects $Q^-(7,q)$ in two lines, M and some other line K. In Chapter 2, it is shown that such planes $\langle H^\perp, M\rangle$ contain $q+2$ points of $\tilde{\mathcal{M}}$: the $q+1$ points on M and one other point m' on K. If $M' \in \mathcal{M}$ is the line of $\mathcal{M}$ on which m' lies, then $H^\perp$ is a point of $\langle M, M'\rangle$ and hence $\langle M, M'\rangle^\perp \subseteq H$. This means that H contains a second 3-dimensional subspace γ' playing the same role as γ and we can consider the 5-dimensional spaces $\langle \gamma', B_1\rangle$ and $\langle \gamma', B_2\rangle$. If one of them intersects $Q^-(7,q)$ in a $Q^-(5,q)$, then, by Theorem 3.4.1 and Lemma 3.4.2, this space contains q^2+1 elements of $\mathcal{M}$, so $\langle \gamma', B_1\rangle = \langle \gamma', B_2\rangle$. As at least one of $\langle \gamma, B_1\rangle$ or $\langle \gamma, B_2\rangle$ is generated by its lines in $\mathcal{M}$, this space, say $\langle \gamma, B_1\rangle$, coincides with $\langle \gamma', B_1\rangle = \langle \gamma', B_2\rangle$, and so $\langle \gamma, B_1\rangle \cap Q^-(7,q)$ is elliptic, a contradiction. We conclude that $\langle \gamma', B_i\rangle \cap Q^-(7,q) = Q_i^+(5,q)'$ for $i = 1,2$. Note that all lines of $\mathcal{M}$ in $Q(6,q)$ are contained in both $Q_1^+(5,q) \cup Q_2^+(5,q)$ and $Q_1^+(5,q)' \cup Q_2^+(5,q)'$. If $Q_1^+(5,q)' = Q_2^+(5,q)'$, then $\langle \gamma', B_1\rangle = \langle \gamma', B_2\rangle$ and this space contains all q^2+1 lines of $\mathcal{M}$ in $Q(6,q)$. As above, $\langle \gamma, B_1\rangle$ or $\langle \gamma, B_2\rangle$ must hence coincide with $\langle \gamma', B_1\rangle = \langle \gamma', B_2\rangle$, which implies that B_1, B_2 and γ are in the same 5-dimensional space, a contradiction to our assumption. We may thus assume that $\langle \gamma', B_1\rangle \neq \langle \gamma', B_2\rangle$.

If it is possible to choose γ' such that it is not contained in $\langle \gamma, B_1\rangle$, nor in $\langle \gamma, B_2\rangle$, then all four quadrics $Q_i^+(5,q)$, $Q_i^+(5,q)'$, $i = 1,2$, are distinct and so $Q_i^+(5,q) \cap Q_j^+(5,q)'$, $i,j \in \{1,2\}$, contains at most two lines of $\mathcal{M}$. It follows that $q^2+1 \leq 8$, so $q < 3$, a contradiction.

So suppose that all subspaces γ' are either contained in $\langle \gamma, B_1\rangle$ or in $\langle \gamma, B_2\rangle$. It follows from Theorem 1.2.5 and a simple counting argument that each hyperbolic quadric $Q^+(5,q)$ on $Q^-(7,q)$ contains

$$\frac{2(q+1)(q^2+1)\cdot(q+1)}{2(q+1)} = (q+1)(q^2+1)$$

points of $\tilde{\mathcal{M}}$. As there are lines of $\mathcal{M}$ in both $\langle \gamma, B_1\rangle$ and $\langle \gamma, B_2\rangle$ by assumption, this implies that there are also points of $\tilde{\mathcal{M}}$ in $\langle \gamma, B_i\rangle$, $i = 1,2$, the line of $\mathcal{M}$ through which is not contained in $\langle \gamma, B_i\rangle$. Moreover, each line of $\mathcal{M}$ in $\langle \gamma, B_i\rangle$ forces the existence of $q+1$ points of $\tilde{\mathcal{M}}$ in $\langle \gamma, B_j\rangle$, not on a line of $\mathcal{M}$ in $\langle \gamma, B_j\rangle$, $j \neq i$. So there exist points $x \in \tilde{\mathcal{M}}$, contained in $\langle \gamma, B_i\rangle$, $i = 1,2$, such that the line of $\mathcal{M}$ through x is not contained in H. Suppose that $x \in \tilde{\mathcal{M}}$ is such a point in $\langle \gamma, B_1\rangle$. Then, by using the same argument as above, we can find a 3-dimensional subspace γ_x ($\neq \gamma$) playing the same role as γ, which is contained in $x^\perp$ and in H. Now $x^\perp \cap Q_1^+(5,q)$ is a singular quadric $xQ_x^+(3,q)$. As $\gamma_x \cap Q^-(7,q)$ is an elliptic quadric $Q_x^-(3,q)$, γ_x

cannot be contained in $\langle \gamma, B_1 \rangle$, for there are no 3-dimensional subspaces in $\langle \gamma, B_1 \rangle \cap x^\perp$ intersecting $xQ_x^+(3,q)$ in an elliptic quadric $Q_x^-(3,q)$. Since we assume that γ_x must either be contained in $\langle \gamma, B_1 \rangle$ or in $\langle \gamma, B_2 \rangle$, it must be a subspace of $\langle \gamma, B_2 \rangle$. We have thus shown that it is possible to find a subspace γ' ($\neq \gamma$) in $\langle \gamma, B_1 \rangle$ and another subspace γ'' ($\neq \gamma$) in $\langle \gamma, B_2 \rangle$, both playing the same role as γ. As $\langle \gamma', B_1 \rangle = \langle \gamma, B_1 \rangle$ and the lines of $\mathcal{M}$ in $Q(6,q)$ are contained in $\langle \gamma, B_1 \rangle \cup \langle \gamma, B_2 \rangle$ on the one hand and in $\langle \gamma', B_1 \rangle \cup \langle \gamma', B_2 \rangle$ on the other hand, there are at most two lines of $\mathcal{M}$ in $\langle \gamma, B_2 \rangle$ and in $\langle \gamma', B_2 \rangle$, namely the ones in their intersection. Similarly there are at most two lines of $\mathcal{M}$ in $\langle \gamma, B_1 \rangle$ and in $\langle \gamma'', B_1 \rangle$, again the two possible lines in their intersection. This implies that $q^2 + 1 \leq 4$, so $q < 2$, a contradiction.

It follows from the above that no two distinct hyperbolic quadrics $\langle \gamma, B_1 \rangle \cap Q^-(7,q) = Q_1^+(5,q)$ and $\langle \gamma, B_2 \rangle \cap Q^-(7,q) = Q_2^+(5,q)$, $B_1, B_2 \in \mathcal{M} \setminus \{L_1, L_2\}$, are contained in a parabolic quadric $Q(6,q)$. Hence for every $Q(6,q) \subseteq Q^-(7,q)$, containing a hyperbolic quadric $Q^+(5,q) = \langle \gamma, B \rangle \cap Q^-(7,q)$ for some line $B \in \mathcal{M} \setminus \{L_1, L_2\}$, the q^2+1 lines of $\mathcal{M}$ in $Q(6,q)$ are contained in $\langle \gamma, B \rangle$. □

By putting together the results of Theorem 3.4.1, Lemma 3.4.2 and Theorem 3.5.4, we can now draw the following conclusion.

Theorem 3.5.5 *For any three distinct lines $L_1, L_2, L_3 \in \mathcal{M}$, the 5-dimensional subspace $\langle L_1, L_2, L_3 \rangle$ intersects $Q^-(7,q)$ in either a $Q^+(5,q)$ or a $Q^-(5,q)$ and contains exactly q^2+1 lines of $\mathcal{M}$.*

Proof.
Let L_1, L_2, L_3 be three distinct lines of $\mathcal{M}$. Then $\langle L_1, L_2, L_3 \rangle$ is 5-dimensional and there exists a non-singular $Q(6,q) \subseteq Q^-(7,q)$ such that $\langle L_1, L_2, L_3 \rangle \subseteq \langle Q(6,q) \rangle$. Similarly as in the proof of Theorem 3.5.4, we can find a subspace $\gamma = \langle M, M' \rangle^\perp$, for some $M, M' \in \mathcal{M} \setminus \{L_1, L_2, L_3\}$, which is contained in $\langle Q(6,q) \rangle$. Now, Lemma 3.4.2 implies that $\langle \gamma, L_1 \rangle \cap Q^-(7,q)$ either is an elliptic quadric $Q^-(5,q)$ or a hyperbolic quadric $Q^+(5,q)$. In the respective cases, it follows from Theorem 3.4.1 or Theorem 3.5.4 that $\langle \gamma, L_1 \rangle$ meets $\mathcal{M}$ in q^2+1 lines, so that all q^2+1 lines of $\mathcal{M}$ in $Q(6,q)$ are contained in $\langle \gamma, L_1 \rangle$ and consequently $\langle \gamma, L_1 \rangle = \langle L_1, L_2, L_3 \rangle$. This proves the theorem. □

If $L_1, L_2, L_3 \in \mathcal{M}$ are such that $\langle L_1, L_2, L_3 \rangle \cap Q^-(7,q)$ is a hyperbolic quadric $Q^+(5,q)$, then the q^2+1 lines of $\mathcal{M}$ in $Q^+(5,q)$ constitute a 1-system $\mathcal{M}'$ of $Q^+(5,q)$. In Shult and Thas [57], it is shown that the 1-system of $Q^+(5,q)$, q odd, is unique up to a projectivity. It can be constructed as follows. There exist two disjoint and conjugate planes π and $\overline{\pi}$ in the extension $\mathsf{PG}(5,q^2)$ of $\mathsf{PG}(5,q) := \langle Q^+(5,q) \rangle$, which are polar with respect to the polarity of the extension $Q^+(5,q^2)$ of $Q^+(5,q)$ and such that π, respectively $\overline{\pi}$, intersects $Q^+(5,q^2)$ in an irreducible conic $\mathcal{C}$, respectively $\overline{\mathcal{C}}$. The lines of $\mathcal{M}'$ are then all lines $x\overline{x} \cap \mathsf{PG}(5,q)$, where x varies on $\mathcal{C}$ and $\overline{x} \in \overline{\mathcal{C}}$ is the conjugate of x.

Lemma 3.5.6 *If $\mathcal{M}'$ is a 1-system of $Q^+(5,q)$, q odd, then the planes π and $\overline{\pi}$ are the unique planes in the extension $\mathsf{PG}(5,q^2)$ of $\mathsf{PG}(5,q)$ which meet all lines of $\mathcal{M}'$.*

Proof.
Consider $Q^+(5,q)$ as the Klein quadric of the lines of $\mathsf{PG}(3,q)$. With the q^2+1 lines of $\mathcal{M}$ there correspond q^2+1 line pencils in $\mathsf{PG}(3,q)$, with respective vertices $p_0, p_1, \ldots, p_{q^2}$ and contained in the respective planes $\alpha_0, \alpha_1, \ldots, \alpha_{q^2}$. As $\mathcal{M}'$ is a 1-system, the set of points $\{p_0, p_1, \ldots, p_{q^2}\}$ is an ovoid of $\mathsf{PG}(3,q)$, which is an elliptic quadric $Q^-(3,q)$ by Barlotti [3] and Panella [49]; the planes $\alpha_0, \alpha_1, \ldots, \alpha_{q^2}$ are the tangent planes of $Q^-(3,q)$. We now consider the extension $Q^+(5,q^2)$ of $Q^+(5,q)$ and the corresponding extension $\mathsf{PG}(3,q^2)$ of $\mathsf{PG}(3,q)$. The conics $\mathcal{C}$ and $\overline{\mathcal{C}}$, being contained in polar planes with respect to $Q^+(5,q^2)$, correspond to the two reguli of a hyperbolic quadric $\mathcal{H}$ in $\mathsf{PG}(3,q^2)$. As the extensions to $\mathsf{GF}(q^2)$ of all lines of $\mathcal{M}'$ meet $\mathcal{C}$ and $\overline{\mathcal{C}}$ in a point, every line of $\mathcal{H}$ contains a unique point of $Q^-(3,q)$, so $Q^-(3,q)$ is contained in $\mathcal{H}$.

Suppose that there exists a third plane β in $\mathsf{PG}(5,q^2)$, meeting the extensions to $\mathsf{GF}(q^2)$ of all lines of $\mathcal{M}'$. If β is a plane of $Q^+(5,q^2)$, then it either corresponds to the set of lines in a plane of $\mathsf{PG}(3,q^2)$, or to the set of lines through a point of $\mathsf{PG}(3,q^2)$. In the first case, all points of $Q^-(3,q)$ must lie in a plane, while in the second case the q^2+1 tangent planes of $Q^-(3,q)$ must have a common point. In either case, a contradiction arises. Hence $\beta \cap Q^+(5,q^2)$ is a conic $\mathcal{C}'$; let $\mathcal{C}''$ denote the conic $\langle \mathcal{C}' \rangle^\perp \cap Q^+(5,q^2)$. Then $\mathcal{C}'$ and $\mathcal{C}''$ yield a hyperbolic quadric $\mathcal{H}'$ in $\mathsf{PG}(3,q^2)$, also containing $Q^-(3,q)$. As $\mathcal{H}$ and $\mathcal{H}'$ have common tangent planes at the points of $Q^-(3,q)$, the points of $Q^-(3,q)$ are contained in a plane, a contradiction. It follows that the planes π and $\overline{\pi}$ are indeed the unique planes meeting all lines of $\mathcal{M}'$. □

3.6 The uniqueness theorem

To prove the desired uniqueness result for 1-systems of $Q^-(7,q)$, q odd, we will rely on the results concerning eggs which were briefly discussed in Section 3.1.3.

Theorem 3.6.1 *For q odd, the quadric $Q^-(7,q)$ has a unique 1-system up to a projectivity.*

Proof.
It is easily seen that each 1-system $\mathcal{M}$ of $Q^-(7,q)$ is an egg $O(2,4,q)$ of $\mathsf{PG}(7,q)$, where the tangent space of the egg at a line $L \in \mathcal{M}$ is simply the tangent space at L of $Q^-(7,q)$. As we have seen in Theorem 3.5.5, $\mathcal{M}$, considered as an egg $O(2,4,q)$, is good at each of its elements. So, by Theorem 3.1.4, every 1-system of $Q^-(7,q)$ is a regular egg of $\mathsf{PG}(7,q) = \langle Q^-(7,q)\rangle$ and as q is odd, this egg is classical. Hence there exist two disjoint, conjugate 3-dimensional subspaces ρ and $\overline{\rho}$ in the extension $\mathsf{PG}(7,q^2)$ of $\mathsf{PG}(7,q)$, such that the lines of $\mathcal{M}$ are all lines $x\overline{x} \cap \mathsf{PG}(7,q)$, where x is a point of a classical ovoid $\mathcal{O} = Q^-(3,q^2)$ in ρ and $\overline{x}$ is its conjugate, belonging to the conjugate ovoid $\overline{\mathcal{O}}$ in $\overline{\rho}$. Since the lines of $\mathcal{M}$ are lines of $Q^-(7,q)$, their extensions to $\mathsf{GF}(q^2)$ are lines of the extension $Q^+(7,q^2)$ of $Q^-(7,q)$, which implies that $\mathcal{O} \subseteq \rho \cap Q^+(7,q^2)$ and $\overline{\mathcal{O}} \subseteq \overline{\rho} \cap Q^+(7,q^2)$. It remains to show that ρ and $\overline{\rho}$ are polar with respect to the polarity of $Q^+(7,q^2)$.

Consider two arbitrary lines $L_1, L_2 \in \mathcal{M}$. We determine how many hyperbolic quadrics $Q^+(5,q)$ there are, which contain L_1, L_2 and q^2-1 other lines of $\mathcal{M}$. Let ε_i, $i = 1,2$, denote two 5-dimensional subspaces through $\langle L_1, L_2\rangle$; then $\langle L_1, L_2\rangle^\perp := \gamma$ contains the lines $\varepsilon_i^\perp$, $i = 1,2$. Suppose that ε_1 and ε_2 intersect $Q^-(7,q)$ in elliptic quadrics $Q_1^-(5,q)$ and $Q_2^-(5,q)$ respectively, both containing q^2+1 lines of $\mathcal{M}$. In that case $\varepsilon_1^\perp$ and $\varepsilon_2^\perp$ are secants of $\gamma \cap Q^-(7,q) := Q^-(3,q)$. If $\varepsilon_1^\perp$ and $\varepsilon_2^\perp$ had a common point p, the hyperplane $p^\perp$ would contain both $Q_1^-(5,q)$ and $Q_2^-(5,q)$, which means that there would be $2q^2$ lines of $\mathcal{M}$ in $p^\perp$, a contradiction to Theorem 1.2.3. Hence $\varepsilon_1^\perp$ and $\varepsilon_2^\perp$ are disjoint and so there are at most $\frac{q^2+1}{2}$, the number of disjoint pairs of points of $Q^-(3,q)$, elliptic quadrics $Q^-(5,q)$ through L_1 and L_2, and having q^2+1 lines in common with $\mathcal{M}$. As every three lines of $\mathcal{M}$ span a 5-dimensional subspace which contains q^2+1 lines of $\mathcal{M}$ and which intersects $Q^-(7,q)$ in a non-singular quadric by Theorem 3.5.5, there exist at least $\frac{q^4-1}{q^2-1} - \frac{q^2+1}{2} = \frac{q^2+1}{2}$ hyperbolic quadrics $Q_i^+(5,q) \subseteq Q^-(7,q)$, containing L_1, L_2 and q^2-1 other lines of $\mathcal{M}$. Each of them yields two disjoint, conjugate planes π_i and $\overline{\pi}_i$ meeting the extensions to $\mathsf{GF}(q^2)$ of all lines of the 1-system $\mathcal{M}'_i$, induced by $\mathcal{M}$ in $Q_i^+(5,q)$. By Lemma 3.5.6, the planes π_i and $\overline{\pi}_i$ are the unique planes in $\langle Q_i^+(5,q^2)\rangle$ meeting the extensions to $\mathsf{GF}(q^2)$ of all lines of $\mathcal{M}'_i$. As ρ and $\overline{\rho}$ are not contained in any $\langle Q_i^+(5,q^2)\rangle$, this implies that for each $i = 1,2,\ldots,\frac{q^2+1}{2}$, one of π_i and $\overline{\pi}_i$, say π_i, is contained in ρ and the other one, $\overline{\pi}_i$, is contained in $\overline{\rho}$. The planes π_i share the line through the points l_1 and l_2, where l_j, $j = 1,2$, stands for the common point of ρ and the extension to $\mathsf{GF}(q^2)$ of L_j, and the planes $\overline{\pi}_i$ share the line $\overline{l_1 l_2}$. As every two planes π_i and $\overline{\pi}_i$ are polar with respect to the restriction of the polarity of $Q^+(7,q^2)$ to $\langle \pi_i, \overline{\pi}_i\rangle$, each plane $\overline{\pi}_i$ is contained in $l_1^\perp$ and $l_2^\perp$, with $\perp$ the polarity of $Q^+(7,q^2)$. This implies that $\langle \overline{\pi}_i \mid i = 1,2,\ldots,\frac{q^2+1}{2}\rangle$ is contained in both $l_1^\perp$ and $l_2^\perp$. But the subspace spanned by all planes $\overline{\pi}_i$ is exactly $\overline{\rho}$ and it follows that $\overline{\rho} \subseteq l_1^\perp \cap l_2^\perp$. As $L_1, L_2 \in \mathcal{M}$ were chosen arbitrarily, this means that

$$\overline{\rho} \subseteq \bigcap_{x \in \mathcal{O}} x^\perp,$$

thus $\overline{\rho} \subseteq \rho^\perp$. Since ρ and $\overline{\rho}$ are 3-dimensional, they must be polar with respect to the polarity of $Q^+(7,q^2)$ and it follows that $\mathcal{M}$ is a classical 1-system of $Q^-(7,q)$. This completes the proof of the theorem. $\square$

Chapter 4

Locally hermitian 1-systems of $Q(6, q)$, q odd

The ideas for this chapter were inspired by the close relation between flocks of quadratic cones in $\mathsf{PG}(3, q)$ and ovoids of the hyperbolic quadric $Q^+(5, q)$. A flock of a quadratic cone $\mathcal{K}$ in $\mathsf{PG}(3, q)$ is a set of q mutually disjoint conics on $\mathcal{K}$. Flocks of quadratic cones have been thoroughly studied by various authors; we refer for instance to [2], [18], [51] and [70]. The link between a flock of a quadratic cone and an ovoid of $Q^+(5, q)$ is easily understood in the following way.

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of a quadratic cone $\mathcal{K}$ with vertex p in $\mathsf{PG}(3, q)$. We embed the cone $\mathcal{K}$ in a non-singular hyperbolic quadric $Q^+(5, q)$ in such way that $\mathcal{K} \subseteq p^\perp$, with $\perp$ the polarity of $Q^+(5, q)$. For each conic $\mathcal{C}_i \in \mathcal{F}$, the plane $\langle \mathcal{C}_i \rangle$ will be denoted by π_i. It is clear that $\pi_i^\perp$ is a plane, intersecting $Q^+(5, q)$ in a conic $\mathcal{C}_i^*$ and as every $\mathcal{C}_i$ is contained in $p^\perp$, the point p belongs to all conics $\mathcal{C}_i^*$. Since every two planes π_i and π_j, $i \neq j$, have a line in common, disjoint from $\mathcal{K}$ and thus also from $Q^+(5, q)$, the 3-space $\langle \mathcal{C}_i^*, \mathcal{C}_j^* \rangle$ meets $Q^+(5, q)$ in an elliptic quadric $Q^-(3, q)$. Hence no two points of $\mathcal{C}_i^*$ and $\mathcal{C}_j^*$ are collinear on $Q^+(5, q)$, for all $i \neq j$, and it follows that $\cup_{i=1}^q \mathcal{C}_i^*$ is an ovoid of $Q^+(5, q)$.

By using the Klein correspondence, this ovoid of $Q^+(5, q)$ yields a spread of $\mathsf{PG}(3, q)$ consisting of q reguli sharing one line, which in turn gives rise to a translation plane of order q^2. This construction of translation planes is due independently to Thas and Walker; see for instance [70]. The process can also be reversed: starting from a spread of $\mathsf{PG}(3, q)$ consisting of q reguli through a common line, one can construct a flock of a quadratic cone, see [19].

Obviously, flocks of quadratic cones and the corresponding ovoids of $Q^+(5,q)$ are interesting objects with connections to a variety of other geometrical objects.

This chapter will be dedicated to locally hermitian 1-systems of $Q(6,q)$, q odd, for which there exists a similar connection to a peculiar kind of flock of a quadratic cone with line vertex in $\mathsf{PG}(4,q)$. As a first step, this connection will be explained in detail, then an algebraic description of the suitable flocks will be given and afterwards we apply the theory to known examples of locally hermitian 1-systems of $Q(6,q)$, q odd. The results we obtain from a particular known example, the so-called spread $\mathcal{S}_{[9]}$ of the hexagon $\mathsf{H}(q)$, admit to be generalized and as such they form the starting point for the discovery of a new class of locally hermitian 1-systems of $Q(6,q)$. All members of this class appear to be semiclassical in addition, which inspires a characterization result for proper locally hermitian, semiclassical 1-systems of $Q(6,q)$, q odd. Finally, some attention will be paid to possible generalizations of the developed theory, where it will become clear that there are only few possibilities for direct generalization.

The outcome of this research is contained in the *Proceedings of the Fourth Isle of Thorns Conference* on Finite Geometries, see [39].

4.1 Locally hermitian 1-systems of $Q(6,q)$ and i-flocks

In the sequel of this chapter, we assume that q is odd, unless mentioned otherwise. This implies that a polarity $\perp$ is defined by $Q(6,q)$. Let $\mathcal{M}$ be a 1-system of $Q(6,q)$ which is locally hermitian at some line L. Recall that by the definitions in Sections 1.2.1 and 1.2.2, $\mathcal{M}$ consists of q^3+1 lines, forming q^2 reguli through the line L. These q^2 reguli will be denoted by $R_1, R_2, \ldots, R_{q^2}$. We now have a closer look at the induced structure in $L^\perp \cap Q(6,q) = LQ(2,q)$. As each R_i contains the line L, it holds that $\mathcal{C}_i := \langle R_i \rangle^\perp \cap Q(6,q)$ is a conic lying on $LQ(2,q)$ for $i = 1, 2, \ldots, q^2$. The fact that $\mathcal{M}$ is a 1-system, implies that for $i \neq j$ the reguli R_i and R_j only share the line L and that $\langle R_i, R_j \rangle$ is 5-dimensional. By the argument which has been explained in detail in Section 1.2.2, see page 16, it follows that $\langle R_i, R_j \rangle \cap Q(6,q)$ is an elliptic quadric $Q^-_{ij}(5,q)$, for all $i \neq j$. This fact in turn implies that $\langle \mathcal{C}_i \rangle \cap \langle \mathcal{C}_j \rangle = \langle Q^-_{ij}(5,q) \rangle^\perp$ is an internal point of the parabolic quadric $Q(6,q)$, and hence also of the cone $LQ(2,q)$.

These observations justify the following definitions.

Definitions

(i) A *flock* of a quadratic cone $LQ(2,q)$ with line vertex L in $\mathsf{PG}(4,q)$ is a partition of $LQ(2,q) \setminus L$ in q^2 mutually disjoint conics, with the extra condition that any two disjoint conics of the partition generate $\mathsf{PG}(4,q)$. The planes containing the elements of the flock, will henceforth be called the *conic planes* of the flock.

(ii) If q is odd and the conic planes $\pi_1, \pi_2, \ldots, \pi_{q^2}$ of the flock pairwise intersect in an internal, respectively external, point of the cone $LQ(2,q)$, then we speak of an *i-flock*, respectively *e-flock*.

(iii) If q is even and the nuclei of the elements of the flock are all distinct, that is, together with L these nuclei form the nucleus plane of the cone, then the flock is called *general*.

In combination with the above construction, it is clear from these definitions that every locally hermitian 1-system of $Q(6,q)$, q odd, defines an i-flock of a cone $LQ(2,q)$ in $\mathsf{PG}(4,q)$.

Conversely, if we start from an i-flock of a cone $LQ(2,q)$ in $\mathsf{PG}(4,q)$, q odd, and embed the cone in a parabolic quadric $Q(6,q)$ in such way that $L^\perp \cap Q(6,q) = LQ(2,q)$, with $\perp$ the polarity of $Q(6,q)$, then we obtain a 1-system of this $Q(6,q)$ consisting of q^2 reguli through the line L, simply by reversing the above construction. Hence we have shown the following theorem.

Theorem 4.1.1 *For q odd, locally hermitian 1-systems of $Q(6,q)$ and i-flocks of $LQ(2,q)$ are equivalent objects.*

If we assume q to be even, then no polarity is associated with $Q(6,q)$. Still, by considering intersections of tangent spaces of $Q(6,q)$ instead of the polarity, any locally hermitian 1-system of $Q(6,q)$ also yields a flock of a quadratic cone $LQ(2,q)$. However, all planes of the elements of the flock share the nucleus of $Q(6,q)$, and, moreover, starting from such a flock, the construction cannot be reversed. Therefore we restrict ourselves to the case where q is odd.

Remark

Locally hermitian 1-systems of $Q(6,q)$ are known to exist. The simplest example is certainly the classical 1-system of $Q(6,q)$, which is the classical or hermitian spread of some elliptic quadric $Q^-(5,q) \subseteq Q(6,q)$. As has already been mentioned in Section 1.2.2, this 1-system even has a stronger property than being locally hermitian at some line L: it is hermitian, which means that the regulus defined by any two distinct lines of the 1-system is completely contained in the 1-system or, equivalently, that it is locally hermitian at any of its lines. It is clear that in this case $Q^-_{ij}(5,q) = Q^-(5,q)$ for all $i \neq j$ and consequently, for q odd, all conic planes of the associated i-flock share one point, namely the point $p = \langle Q^-(5,q)\rangle^\perp$, which is internal to $Q(6,q)$.

More examples are given in [4], where Bloemen, Thas and Van Maldeghem study locally hermitian spreads of the hexagon $\mathsf{H}(q)$. They describe among others the unique semiclassical, non-hermitian spread $\mathcal{S}_{[9]}$ of $\mathsf{H}(q)$, q odd and $q \equiv 1 \bmod 3$; this spread is locally hermitian at some line. As every spread of the hexagon $\mathsf{H}(q)$ is a 1-system of $Q(6,q)$, this spread yields another example of a 1-system of $Q(6,q)$ with an associated i-flock. This example will be studied in detail later on.

4.2 An algebraic description of flocks

Let L be the line $X_0 = X_1 = X_2 = 0$ of $\mathsf{PG}(4,q)$ and let $Q(2,q)$ be the conic $X_3 = X_4 = X_0X_2 - X_1^2 = 0$. Then the cone $LQ(2,q)$ has equation $X_0X_2 - X_1^2 = 0$. A flock $\mathcal{F}$ of $LQ(2,q)$ consists of q^2 disjoint conics $\mathcal{C}_i$ in planes π_i, disjoint from L. In general, a plane disjoint from L has equations of the form

$$\pi_i \equiv \begin{cases} a_iX_0 + b_iX_1 + c_iX_2 + X_3 = 0 \\ d_iX_0 + e_iX_1 + f_iX_2 + X_4 = 0, \end{cases}$$

with $a_i, b_i, c_i, d_i, e_i, f_i \in \mathsf{GF}(q)$. Let

$$\varphi_{ij} := (a_{ij}f_{ij} - c_{ij}d_{ij})^2 - (a_{ij}e_{ij} - b_{ij}d_{ij})(b_{ij}f_{ij} - c_{ij}e_{ij}),$$

where a_{ij} is a short notation for $a_i - a_j$ and similarly for $b_{ij}, \ldots, f_{ij}$. It is an easy exercise to show that $\langle \pi_i, \pi_j\rangle \neq \mathsf{PG}(4,q)$ implies that $\varphi_{ij} = 0$, for $i \neq j$. The following theorem describes the different kinds of flocks of $LQ(2,q)$ algebraically in terms of a condition on φ_{ij}.

Theorem 4.2.1

(i) With the above notation, the conics $\mathcal{C}_i = LQ(2,q) \cap \pi_i$, $i = 1, 2, \ldots, q^2$, form a flock of $LQ(2,q)$ if and only if for all $i \neq j$

$$\varphi_{ij} \neq 0.$$

(ii) For q odd, the conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q^2}$ form an i-flock, respectively an e-flock, of $LQ(2,q)$ if and only if for all $i \neq j$

φ_{ij} is a non-square, respectively a non-zero square, in $\mathsf{GF}(q)$.

(iii) For q even, the conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q^2}$ form a general flock of the cone $LQ(2,q)$ if and only if for all $i \neq j$

$$\varphi_{ij} \neq 0 \textit{ and } \operatorname{rank} \begin{bmatrix} 1 & b_i & e_i \\ 1 & b_j & e_j \end{bmatrix} = 2.$$

Proof.
For $i \neq j$, the intersection of the planes π_i and π_j is given by

$$\begin{cases} a_iX_0 + b_iX_1 + c_iX_2 + X_3 = 0 \\ d_iX_0 + e_iX_1 + f_iX_2 + X_4 = 0 \\ (a_i - a_j)X_0 + (b_i - b_j)X_1 + (c_i - c_j)X_2 = 0 \\ (d_i - d_j)X_0 + (e_i - e_j)X_1 + (f_i - f_j)X_2 = 0. \end{cases}$$

For $\langle \pi_i, \pi_j \rangle = \mathsf{PG}(4,q)$, the point $\langle \pi_i \cap \pi_j, L \rangle \cap \langle Q(2,q) \rangle := \{p_{ij}\}$ is the unique point, determined by

$$\begin{cases} a_{ij}X_0 + b_{ij}X_1 + c_{ij}X_2 = 0 \\ d_{ij}X_0 + e_{ij}X_1 + f_{ij}X_2 = 0 \\ X_3 = X_4 = 0. \end{cases}$$

Hence p_{ij} has coordinates $(r_{ij}, s_{ij}, t_{ij}, 0, 0)$, with

$$\begin{aligned} r_{ij} &= b_{ij}f_{ij} - c_{ij}e_{ij}, \\ s_{ij} &= c_{ij}d_{ij} - a_{ij}f_{ij}, \\ t_{ij} &= a_{ij}e_{ij} - b_{ij}d_{ij}. \end{aligned}$$

Clearly, under the assumption that $\langle \pi_i, \pi_j \rangle = \mathsf{PG}(4,q)$ for $i \neq j$, the set $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q^2}\} = \mathcal{F}$ is a flock of $LQ(2,q)$ if and only if $p_{ij} \notin Q(2,q)$ for all $i \neq j$, that is, if and only if $\varphi_{ij} \neq 0$ for all $i \neq j$.

The set $\mathcal{F}$ is an i-flock of $LQ(2,q)$, with q odd, if and only if p_{ij} is an internal point of $LQ(2,q)$ for all $i \neq j$, that is, if and only if φ_{ij} is a non-square for all $i \neq j$.

Similarly, $\mathcal{F}$ is an e-flock of $LQ(2,q)$, with q odd, if and only if φ_{ij} is a non-zero square for all $i \neq j$.

Finally, let q be even. The kernel of the conic $\mathcal{C}_i$ is the point $(0,1,0,b_i,e_i)$, $i = 1, 2, \ldots, q^2$. These nuclei are all distinct if and only if

$$\operatorname{rank} \begin{bmatrix} 1 & b_i & e_i \\ 1 & b_j & e_j \end{bmatrix} = 2$$

for all $i \neq j$. □

As an example, we determine algebraically the i-flock, corresponding to a hermitian spread of an induced $Q^-(5,q) \subseteq Q(6,q)$, q odd. Suppose that the parabolic quadric $Q(6,q)$ in $\mathsf{PG}(6,q)$ has the standard equation $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$. A hermitian spread of a $Q^-(5,q)$, embedded in $Q(6,q)$, yields a 1-system of $Q(6,q)$ which is locally hermitian at any line. As mentioned before, the conic planes of the associated i-flock all share one point p such that $p^\perp = \langle Q^-(5,q)\rangle$. To construct the i-flock of such a hermitian spread of $Q^-(5,q)$, we must first choose a line L of the spread and determine the cone $LQ(2,q) = L^\perp \cap Q(6,q)$. Here, the line L is chosen to be the line $X_0 = X_1 = X_2 = X_3 = X_4 = 0$; so $L^\perp$ is the subspace $X_1 = X_2 = 0$ and the cone $L^\perp \cap Q(6,q)$ has equations $X_1 = X_2 = X_0X_4 - X_3^2 = 0$. We will verify that an algebraic description of the i-flock of some hermitian spread $\mathcal{S}$ of a $Q^-(5,q) \subseteq Q(6,q)$, containing L, is given by:

$$\pi(k_i,b_i) \equiv \begin{cases} X_1 = X_2 = 0 \\ \gamma k_i X_0 + 2b_i X_3 + k_i X_4 + X_5 = 0 \\ \gamma b_i X_0 + 2\gamma k_i X_3 + b_i X_4 + X_6 = 0, \qquad \forall k_i, b_i \in \mathsf{GF}(q), \end{cases}$$

where γ denotes a non-square in $\mathsf{GF}(q)$. Indeed, the planes $\pi(k_i,b_i)$ have the point $p(-1,0,0,0,\gamma,0,0)$ in common and the polar space of p with respect to $Q(6,q)$ intersects $Q(6,q)$ in the elliptic quadric

$$Q^-(5,q) \equiv \begin{cases} \gamma X_0 - X_4 = 0 \\ \gamma X_0^2 + X_1X_5 + X_2X_6 - X_3^2 = 0. \end{cases} \tag{4.1}$$

By some basic computations, one can check that the corresponding spread $\mathcal{S}$ of $Q^-(5,q)$ consists of the reguli through L of the q^2 hyperbolic quadrics $R(k_i,b_i) \cap Q^-(5,q)$, where $R(k_i,b_i)$ denotes the 3-dimensional subspace with the following equations:

$$R(k_i,b_i) \equiv \begin{cases} b_i X_0 + (\gamma k_i^2 - b_i^2)X_2 + k_i X_3 = 0 \\ \gamma k_i X_0 + (b_i^2 - \gamma k_i^2)X_1 + b_i X_3 = 0 \\ \gamma X_0 - X_4 = 0. \end{cases}$$

Over the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, the elliptic quadric $Q^-(5,q)$ extends to a hyperbolic quadric $Q^+(5,q^2)$, with the same equations 4.1 but now considered over $\mathsf{GF}(q^2)$. Of course, γ is a square in $\mathsf{GF}(q^2)$. In $\mathsf{PG}(6,q^2)$, consider the planes β and $\overline{\beta}$, given by

$$\beta \equiv \begin{cases} \gamma X_0 - X_4 = 0 \\ X_1 + \sqrt{\gamma}X_2 = 0 \\ \sqrt{\gamma}X_0 - X_3 = 0 \\ \sqrt{\gamma}X_5 - X_6 = 0, \end{cases} \quad \text{and} \quad \overline{\beta} \equiv \begin{cases} \gamma X_0 - X_4 = 0 \\ X_1 - \sqrt{\gamma}X_2 = 0 \\ \sqrt{\gamma}X_0 + X_3 = 0 \\ \sqrt{\gamma}X_5 + X_6 = 0. \end{cases}$$

These planes are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ and they are generators of $Q^+(5,q^2)$. Moreover, one can calculate that for all $k_i, b_i \in \mathsf{GF}(q)$, the extension to $\mathsf{GF}(q^2)$ of the 3-space $R(k_i,b_i)$, which we denote by $\widehat{R}(k_i,b_i)$, intersects β, respectively $\overline{\beta}$, in the line $B(k_i,b_i)$, respectively $\overline{B}(k_i,b_i)$, with equations

$$B(k_i,b_i) \equiv \begin{cases} \gamma X_0 - X_4 = 0 \\ X_1 + \sqrt{\gamma}X_2 = 0 \\ \sqrt{\gamma}X_0 - X_3 = 0 \\ \sqrt{\gamma}X_5 - X_6 = 0 \\ X_0 + (\sqrt{\gamma}k_i - b_i)X_2 = 0, \end{cases}$$

respectively

$$\overline{B}(k_i,b_i) \equiv \begin{cases} \gamma X_0 - X_4 = 0 \\ X_1 - \sqrt{\gamma}X_2 = 0 \\ \sqrt{\gamma}X_0 + X_3 = 0 \\ \sqrt{\gamma}X_5 + X_6 = 0 \\ X_0 - (\sqrt{\gamma}k_i + b_i)X_2 = 0. \end{cases}$$

Since the lines $B(k_i,b_i)$ and $\overline{B}(k_i,b_i)$ belong to the regulus of $\widehat{R}(k_i,b_i) \cap Q^+(5,q^2)$, not containing L, it follows that $B(k_i,b_i)$ and $\overline{B}(k_i,b_i)$ have a point in common with the extensions to $\mathsf{GF}(q^2)$ of the $q+1$ lines of $\mathcal{S}$ in $R(k_i,b_i)$, for all $k_i,b_i \in \mathsf{GF}(q)$. As a consequence, the conjugate planes β and $\overline{\beta}$ meet the extensions to $\mathsf{GF}(q^2)$ of all lines of $\mathcal{S}$ in a point, so that $\mathcal{S}$ is indeed a hermitian spread of $Q^-(5,q)$.

For completeness, we check the condition on φ_{ij} from Theorem 4.2.1:

$$\varphi_{ij} = 4\gamma[\gamma(k_i-k_j)^2 - (b_i-b_j)^2]^2,$$

which is obviously a non-square in $\mathsf{GF}(q)$ since it cannot be equal to zero as long as $(k_i,b_i) \neq (k_j,b_j)$.

4.3 The i-flock associated with the semiclassical, non-hermitian spread of $\mathsf{H}(q)$, q odd and $q \equiv 1 \bmod 3$

4.3.1 Algebraic description

In [4], Bloemen, Thas and Van Maldeghem show that the classical generalized hexagon $\mathsf{H}(q)$, q odd and $q \equiv 1 \bmod 3$, admits a unique semiclassical, non-hermitian spread, which they call $\mathcal{S}_{[9]}$. They describe this spread explicitly by using the coordinatization of $\mathsf{H}(q)$ which has been discussed in Section 1.3.2, as follows:

$$\mathcal{S}_{[9]} = \{[\infty]\} \cup \{[9\gamma b', -\gamma k'', k', b', k''] \mid k', b', k'' \in \mathsf{GF}(q)\}.$$

Here, γ again denotes a fixed non-square in $\mathsf{GF}(q)$. If translated to the parabolic quadric $Q(6,q)$ with the same standard equation as in the previous section, this gives a 1-system consisting of the lines

$$L' = \langle (1,0,0,0,0,0,0), (0,0,0,0,0,0,1) \rangle,$$

and, for $k', b', k'' \in \mathsf{GF}(q)$:

$$\begin{aligned}\langle (k' - \gamma k'' b', 9\gamma b', 1, -\gamma k'', 0, b', \gamma^2 k''^2 - 9\gamma b'^2),& \\ (b'^2 - \gamma k''^2, \gamma k'', 0, -b', 1, k'', -7\gamma b' k'' - k') \rangle&.\end{aligned}$$

The spread $\mathcal{S}_{[9]}$ of $\mathsf{H}(q)$ is locally hermitian at the line $[\infty]$ of $\mathsf{H}(q)$, which corresponds to the line $L' = \langle (1,0,0,0,0,0,0), (0,0,0,0,0,0,1) \rangle$ of $Q(6,q)$. We aim at describing the i-flock associated with $\mathcal{S}_{[9]}$, considered as a 1-system of $Q(6,q)$. To that end we first determine the 3-spaces containing the reguli through L' which constitute $\mathcal{S}_{[9]}$; one can calculate that these 3-spaces have the following equations:

$$R(k'', b') \equiv \begin{cases} X_1 = 9\gamma b' X_2 + \gamma k'' X_4 \\ X_3 = -b' X_4 - \gamma k'' X_2 \\ X_5 = b' X_2 + k'' X_4, \qquad\qquad \forall k'', b' \in \mathsf{GF}(q). \end{cases}$$

The image of $R(k'', b')$ under the polarity of $Q(6,q)$ is the plane

$$\pi(k'', b') \equiv \begin{cases} X_2 = X_4 = 0 \\ b' X_1 + 2\gamma k'' X_3 + 9\gamma b' X_5 + X_6 = 0 \\ X_0 + k'' X_1 + 2b' X_3 + \gamma k'' X_5 = 0, \qquad \forall k'', b' \in \mathsf{GF}(q); \end{cases}$$

these planes are contained in $L'^{\perp}$, which has equations $X_2 = X_4 = 0$. The cone $L'^{\perp} \cap Q(6,q)$ has equations $X_2 = X_4 = X_1 X_5 - X_3^2 = 0$.
To have more similarities to the algebraic setting in Section 4.2, we apply a permutation of the coordinates: $X_0 \leftrightarrow X_5$, $X_1 \leftrightarrow X_4$, $X_i \leftrightarrow X_i$ for $i = 2, 3, 6$. This permutation stabilizes the quadric $Q(6,q)$, maps the line L' onto the line L with equations $X_0 = X_1 = X_2 = X_3 = X_4 = 0$ and the plane $\pi(k'', b')$ onto the following plane, where we now write k_i and b_i instead of k'' and b':

$$\pi(k_i, b_i) \equiv \begin{cases} X_1 = X_2 = 0 \\ \gamma k_i X_0 + 2b_i X_3 + k_i X_4 + X_5 = 0 \\ 9\gamma b_i X_0 + 2\gamma k_i X_3 + b_i X_4 + X_6 = 0, \quad \forall k_i, b_i \in \mathsf{GF}(q). \end{cases} \tag{4.2}$$

For completeness, we check the condition on φ_{ij} (see Theorem 4.2.1):

$$\varphi_{ij} = 4\gamma[3(b_i - b_j)^2 + \gamma(k_i - k_j)^2]^2.$$

As γ is a non-square in $\mathsf{GF}(q)$, φ_{ij} is a non-square provided that it is not equal to zero. This condition is satisfied for all $(k_i, b_i) \neq (k_j, b_j)$, because -3 is a square in $\mathsf{GF}(q)$ with $q \equiv 1 \bmod 3$.
We will denote the i-flock, determined by the planes $\pi(k_i, b_i)$ as in (4.2), by $\mathcal{F}_{[9]}$.

Remark
Taken into account the obvious resemblance between the algebraic descriptions of $\mathcal{F}_{[9]}$ and the flock associated with the hermitian spread of $Q^-(5, q)$, considered as a 1-system of $Q(6, q)$, it is a logical choice to denote this latter one by $\mathcal{F}_{[1]}$.

4.3.2 A geometric description of $\mathcal{F}_{[9]}$

Here we use the algebraic setting to describe the i-flock $\mathcal{F}_{[9]}$, and hence also the spread $\mathcal{S}_{[9]}$ of $\mathsf{H}(q)$, geometrically. We will study the properties of $\mathcal{F}_{[9]}$ step by step in a series of lemmas.

Lemma 4.3.1 *The intersection points of a fixed conic plane of the i-flock $\mathcal{F}_{[9]}$ with all other conic planes are exactly the $q + 1$ points of an irreducible conic lying in this plane.*

Proof.
First we prove the claim for the conic plane $\pi(0, 0)$. One can easily check that the intersection point of $\pi(0, 0)$ with a plane $\pi(k_i, b_i)$, for $(k_i, b_i) \in \mathsf{GF}(q)^2 \setminus \{(0, 0)\}$ is given by

$$(b_i^2 - \gamma k_i^2, 0, 0, 4\gamma k_i b_i, \gamma^2 k_i^2 - 9\gamma b_i^2, 0, 0).$$

Now suppose that two different planes $\pi(k_i, b_i)$ and $\pi(k_j, b_j)$, with $(0, 0) \neq (k_i, b_i) \neq (k_j, b_j) \neq (0, 0)$, intersect $\pi(0, 0)$ in the same point. Then there exists a non-zero constant $\mu \in \mathsf{GF}(q)$ such that

$$\begin{cases} b_j^2 - \gamma k_j^2 = \mu(b_i^2 - \gamma k_i^2) \\ 4\gamma k_j b_j = \mu(4\gamma k_i b_i) \\ \gamma^2 k_j^2 - 9\gamma b_j^2 = \mu(\gamma^2 k_i^2 - 9\gamma b_i^2), \end{cases}$$

which after some computation implies that μ is a square and $k_j = \sqrt{\mu} k_i$ as well as $b_j = \sqrt{\mu} b_i$. So two planes $\pi(k_i, b_i)$ and $\pi(k_j, b_j)$ share the same point with $\pi(0,0)$ if and only if there exists a constant $\lambda \in \mathsf{GF}(q)^*$ such that $(k_j, b_j) = (\lambda k_i, \lambda b_i)$. Hence every intersection point of $\pi(0,0)$ with some other conic plane is contained in q different conic planes, $\pi(0,0)$ included. This implies that the plane $\pi(0,0)$ contains exactly $\frac{q^2-1}{q-1} = q+1$ intersection points. We determine these $q+1$ points explicitly. By putting $b_i = 0$, one finds $q-1$ planes $\pi(k_i, 0)$, $k_i \in \mathsf{GF}(q)^*$, intersecting $\pi(0,0)$ in the same point $(-1,0,0,0,\gamma,0,0)$. For $b_i = 1$, there exist q planes $\pi(k_i, 1)$, $k_i \in \mathsf{GF}(q)$, which intersect $\pi(0,0)$ in q different points, namely the points

$$(1-\gamma k_i^2, 0, 0, 4\gamma k_i, \gamma^2 k_i^2 - 9\gamma, 0, 0), \qquad \forall k_i \in \mathsf{GF}(q).$$

It is straightforward to see that the $q+1$ intersection points we have found are indeed the points of the irreducible conic

$$\mathcal{C}' \equiv \begin{cases} X_1 = X_2 = X_5 = X_6 = 0 \\ 9\gamma^2 X_0^2 - 4\gamma X_3^2 + X_4^2 + 10\gamma X_0 X_4 = 0, \end{cases}$$

in the plane $\pi(0,0)$ with equations $X_1 = X_2 = X_5 = X_6 = 0$. This proves the lemma for the plane $\pi(0,0)$.

In the subspace $\mathsf{PG}(4,q) = L^\perp$ of $\mathsf{PG}(6,q)$, with equations $X_1 = X_2 = 0$, we consider the projectivity σ, determined by

$$\begin{cases} Y_i = X_i, & \text{for } i = 0,1,\ldots,4, \\ Y_5 = X_5 - \gamma k_i X_0 - 2b_i X_3 - k_i X_4, \\ Y_6 = X_6 - 9\gamma b_i X_0 - 2\gamma k_i X_3 - b_i X_4, \end{cases}$$

for some $(k_i, b_i) \in \mathsf{GF}(q)^2 \setminus \{(0,0)\}$. Then σ clearly stabilizes L, $L^\perp$ and the cone $LQ(2,q) = L^\perp \cap Q(6,q)$. In addition, $\pi(0,0)$ is mapped onto the conic plane $\pi(k_i, b_i)$ and for $(k_j, b_j) \in \mathsf{GF}(q)^2 \setminus \{(0,0)\}$, one easily deduces that $\pi(k_j, b_j)^\sigma = \pi(k_i + k_j, b_i + b_j)$. Hence the projectivity σ stabilizes the i-flock $\mathcal{F}_{[9]}$ and maps $\pi(0,0)$ onto $\pi(k_i, b_i)$. Since we can find such a projectivity for arbitrary $(k_i, b_i) \neq (0,0)$, the lemma is proved for any plane $\pi(k_i, b_i)$ with $(k_i, b_i) \in \mathsf{GF}(q)^2$. $\square$

The previous lemma implies that there are $\frac{q^2(q+1)}{q} = q(q+1)$ intersection points of conic planes, lying on an irreducible conic in each of these conic planes. In the following lemma we have a closer look at the configuration of these intersection points.

Note that in the sequel, the plane $\pi(0,0)$ will often be denoted by π.

Lemma 4.3.2 *For any point $x \in \mathcal{C}'$ the plane $\langle x, L\rangle$ has q points of some line in common with $\bigcup_{k_i,b_i \in \mathsf{GF}(q)} \pi(k_i, b_i)$. The $q+1$ lines obtained in this way, are disjoint.*

Proof.
If p_∞ denotes the point $(-1,0,0,0,\gamma,0,0)$ of $\mathcal{C}'$, then

$$\langle p_\infty, L\rangle \equiv \begin{cases} X_1 = X_2 = X_3 = 0 \\ \gamma X_0 + X_4 = 0. \end{cases}$$

It is easy to see that the plane $\pi(k_i,b_i)$, $k_i,b_i \in \mathsf{GF}(q)$, intersects $\langle p_\infty, L\rangle$ in the point $(1,0,0,0,-\gamma,0,-8\gamma b_i)$ and the q points one obtains in this way, obviously lie on the line

$$N_\infty \equiv \begin{cases} X_1 = X_2 = X_3 = X_5 = 0 \\ \gamma X_0 + X_4 = 0, \end{cases}$$

which shares the point $x_\infty := (0,0,0,0,0,0,1)$ with the line L.
For all $t \in \mathsf{GF}(q)$ we denote the point $(1-\gamma t^2, 0, 0, 4\gamma t, \gamma^2 t^2 - 9\gamma, 0, 0)$ of $\mathcal{C}'$ by p_t; thus

$$\langle p_t, L\rangle \equiv \begin{cases} X_1 = X_2 = 0 \\ 4\gamma t X_0 - (1-\gamma t^2)X_3 = 0 \\ (\gamma^2 t^2 - 9\gamma)X_0 - (1-\gamma t^2)X_4 = 0. \end{cases}$$

Now the plane $\pi(k_i,b_i)$, with $k_i,b_i \in \mathsf{GF}(q)$, has the point

$$(1-\gamma t^2, 0, 0, 4\gamma t, \gamma^2 t^2 - 9\gamma, 8\gamma(k_i - t b_i), 8\gamma^2 t(t b_i - k_i))$$

in common with $\langle p_t, L\rangle$. As a consequence, for every $t \in \mathsf{GF}(q)$ the plane $\langle p_t, L\rangle$ shares q points of the line

$$N_t \equiv \begin{cases} X_1 = X_2 = 0 \\ 4\gamma t X_0 - (1-\gamma t^2)X_3 = 0 \\ (\gamma^2 t^2 - 9\gamma)X_0 - (1-\gamma t^2)X_4 = 0 \\ \gamma t X_5 + X_6 = 0 \end{cases}$$

with $\bigcup_{k_i,b_i \in \mathsf{GF}(q)} \pi(k_i,b_i)$, and N_t intersects L in the point x_t with coordinates $(0,0,0,0,0,1,-\gamma t)$. We conclude that the lines $N_\infty, N_1, N_2, \ldots, N_q$ meet L in different points and since any two planes $\langle x, L\rangle$ and $\langle y, L\rangle$, $x,y \in \mathcal{C}'$ and $x \neq y$, only share the line L, the $q+1$ lines $N_\infty, N_1, N_2, \ldots, N_q$ are disjoint. $\square$

Until now, we know that every two conic planes intersect in exactly one point and the intersection points of a conic plane with the other ones form an irreducible conic. Moreover, every intersection point is contained in q such conics and exactly one line which meets L. This yields $q+1$ disjoint lines, one line on every point of L, projecting a conic. In the following we identify the configuration of the intersection points as the projection of the Veronese surface[5] $\mathscr{V}_2^4$ from a point on $\mathscr{V}_2^4$ onto a 4-dimensional space.

Lemma 4.3.3 *The $q(q+1)$ intersection points of the conic planes of $\mathcal{F}_{[9]}$, together with the line L, form a rational normal cubic scroll $\mathscr{R}^3$ having L as directrix line.*

Proof.
The $q(q+1)$ intersection points of conic planes, together with the line L, form a set which is the union of the $q+1$ lines joining the point $p_t = (1-\gamma t^2, 0, 0, 4\gamma t, \gamma^2 t^2 - 9\gamma, 0, 0)$ of $\mathcal{C}'$ to the respective point $x_t = (0,0,0,0,0,1,-\gamma t)$ of L, for all $t \in \mathsf{GF}(q) \cup \{\infty\}$. Now, by VII-3.22 of Semple and Roth [55], this set is a rational normal cubic scroll $\mathscr{R}^3$ having L as directrix line, since the map sending the point p_t to the point x_t is a linear projectivity from $\mathcal{C}'$ to L. $\square$

Remark
The variety $\mathscr{R}^3$ is the projection of a Veronese surface $\mathscr{V}_2^4$ from a point $u \in \mathscr{V}_2^4$ onto a hyperplane $\mathsf{PG}(4,q)$ of $\mathsf{PG}(5,q)$ not containing u, with $\mathscr{V}_2^4 \subset \mathsf{PG}(5,q)$; see VII-3.22 of [55]. The line L is the intersection of $\mathsf{PG}(4,q)$ with the tangent plane of $\mathscr{V}_2^4$ at u.

4.4 A new construction of 1-systems of $Q(6,q)$ for q odd

In this section, we investigate whether the above construction can be reversed. This means the following. If one starts from a rational normal cubic scroll $\mathscr{R}^3$ with directrix line L in $L^\perp$, is it always possible to construct a 1-system of $Q(6,q)$ in the same way as above? As we shall see now, the construction turns out to be reversible as long as we start from a rational normal cubic scroll, containing a conic all points of which are internal points of the conic

$$\mathcal{C} \equiv \begin{cases} X_1 = X_2 = X_5 = X_6 = 0 \\ X_0 X_4 = X_3^2, \end{cases}$$

which is the intersection of π with $Q(6,q)$.

[5]See Chapter 25 of *"Generalized Galois Geometries"* by Hirschfeld and Thas [26] for more information on Veronese varieties.

Consider a cone $L\mathcal{C}$ with vertex line L in $\mathsf{PG}(4,q)$, q odd, where $\mathcal{C}$ is a conic contained in some plane π skew to L. Let $\mathcal{C}'$ be another conic of π, disjoint from $\mathcal{C}$ and such that each point of $\mathcal{C}'$ is an internal point of $\mathcal{C}$. Then one easily sees that for every plane $\alpha \neq \pi$, which is also skew to L, the points of the conic $L\mathcal{C}' \cap \alpha$ are internal points of the conic $L\mathcal{C} \cap \alpha$. Hence, all points of the cone $L\mathcal{C}'$, not on its vertex L, are internal points of $L\mathcal{C}$.

In the following theorem, a way to construct i-flocks and consequently also locally hermitian 1-systems of $Q(6,q)$, is described.

Theorem 4.4.1 *Let the cone $LQ(2,q)$ be embedded as $L^{\perp}$ in $Q(6,q)$ and let π be a plane of $L^{\perp}$ skew to L, with $Q(6,q) \cap \pi := \mathcal{C}$. Assume that $\mathcal{C}' \subseteq \pi$ is a conic with the property that all of its points are internal points of $\mathcal{C}$. Then any rational normal cubic scroll $\mathscr{R}^3$ with directrix line L and such that $\mathcal{C}' \subseteq \mathscr{R}^3$, uniquely defines an i-flock of $LQ(2,q)$, and consequently also a locally hermitian 1-system of $Q(6,q)$.*

Proof.
It is known that the considered algebraic variety $\mathscr{R}^3$ consists of q^2 conics $\mathcal{C}'_i$, defining q^2 planes π_i, for $i = 1, 2, \ldots, q^2$, such that each two planes π_i and π_j, $i \neq j$, intersect in a unique point that belongs to both $\mathcal{C}'_i$ and $\mathcal{C}'_j$. Hence the planes π_i define a partition of $LQ(2,q) \setminus L$ into q^2 disjoint conics, as $LQ(2,q)$ and $L\mathcal{C}'$ only have the line L in common. By the choice of $\mathcal{C}'$ as a conic entirely consisting of internal points of $\mathcal{C}$, it holds that any two distinct planes π_i and π_j intersect in an internal point of $LQ(2,q)$. Consequently, the conics $\pi_i \cap Q(6,q)$ form an i-flock of $LQ(2,q)$, and hence they define a locally hermitian 1-system of $Q(6,q)$. $\square$

It is straightforward to check that the particular conics $\mathcal{C}$ and $\mathcal{C}'$, introduced in Section 4.3, are doubly tangent at two conjugate points with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, namely $r_1 = (1,0,0,\sqrt{-3\gamma},-3\gamma,0,0)$ and $r_2 = (1,0,0,-\sqrt{-3\gamma},-3\gamma,0,0)$. Let $\mathcal{B}$ be the pencil of conics defined by $\mathcal{C}$ and $\mathcal{C}'$. The singular elements of $\mathcal{B}$ are

$$\mathcal{C}_1 : (3\gamma X_0 + X_4)^2 = 0, \text{ and}$$
$$\mathcal{C}_2 : 9\gamma^2 X_0^2 + 12\gamma X_3^2 + X_4^2 - 6\gamma X_0 X_4 = 0.$$

Hence it is readily checked that the cross-ratio $\{\mathcal{C}_1, \mathcal{C}_2; \mathcal{C}, \mathcal{C}'\}$ is equal to 4.

In view of Theorem 4.4.1, we are interested in determining how we can distinguish the conics in the plane π, which consist entirely of internal points of $\mathcal{C}$. The observation that $\mathcal{C}$ and $\mathcal{C}'$ are doubly tangent at the two conjugate points r_1 and r_2, provides us with a clue to the solution of this question.

Note that the following lemma does not require the restriction that q is odd.

Lemma 4.4.2 *Let $\mathcal{C}$ be a non-singular conic in the plane $\mathsf{PG}(2,q)$ and let r_1 and r_2 be distinct points of the extension of $\mathcal{C}$ to $\mathsf{GF}(q^2)$, which are conjugate with respect to $\mathsf{GF}(q^2)$. Further, let G be the subgroup of $\mathsf{PGL}(3,q)$ which fixes $\mathcal{C}$ and also the line r_1r_2 (respectively the points r_1 and r_2). If y is a point of $\mathsf{PG}(2,q)$, not on $\mathcal{C}$, not on the line r_1r_2, and distinct from the pole k of the line r_1r_2 with respect to $\mathcal{C}$ for q odd, respectively distinct from the nucleus k of $\mathcal{C}$ for q even, then the orbit y^G is a non-singular conic doubly tangent to $\mathcal{C}$ at r_1 and r_2. Moreover, if G fixes the points r_1 and r_2, it acts regularly on the conic y^G.*

Proof.
Let $\mathcal{B}$ designate the pencil of conics in $\mathsf{PG}(2,q)$ defined by the line r_1r_2 counted twice and the non-singular conic $\mathcal{C}$. Then $\mathcal{B}$ also contains the singular conic consisting of the lines r_1k and r_2k, which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. Every element σ of G fixes $\mathcal{C}$ and the two singular conics in $\mathcal{B}$. Hence σ fixes each element of $\mathcal{B}$.

Now let σ be an element of $\mathsf{PGL}(3,q)$ which fixes $\mathcal{C}$ and also the points r_1 and r_2. If $y \in \mathsf{PG}(2,q)$ is not on $\mathcal{C}$, not on r_1r_2 and distinct from k, then $y^\sigma = y$ implies that $(r_1y \cap \mathcal{C}) \setminus \{r_1\}$ is fixed for σ, so it implies that $\mathcal{C}$ is pointwise fixed, and consequently σ is the identity. It follows that the group G consisting of all elements of $\mathsf{PGL}(3,q)$ fixing $\mathcal{C}$ and the points r_1 and r_2, acts regularly on any non-singular conic of $\mathcal{B}$. $\square$

Corollary 4.4.3 *Let $\mathcal{C}$ be a non-singular conic in the plane $\mathsf{PG}(2,q)$, q odd, and let r_1 and r_2 be distinct points of the extension of $\mathcal{C}$ to $\mathsf{GF}(q^2)$, which are conjugate with respect to $\mathsf{GF}(q^2)$. If $\mathcal{B}$ is the pencil of conics defined by $\mathcal{C}$ and the line r_1r_2 counted twice, then $\mathcal{B}$ contains $\frac{q-3}{2}$ non-singular conics consisting entirely of internal points of $\mathcal{C}$ and $\frac{q-1}{2}$ non-singular conics consisting entirely of external points of $\mathcal{C}$.*

Proof.
As any element of $\mathsf{PGL}(3,q)$ fixing $\mathcal{C}$ also fixes the set of all internal (respectively external) points of $\mathcal{C}$, this follows immediately from Lemma 4.4.2 and an easy counting argument. $\square$

The following theorem gives an answer to the question which non-singular conics consist entirely of internal points of the given conic $\mathcal{C}$ in π.

Theorem 4.4.4 *Let $\mathcal{C}$ be a non-singular conic in the plane $\mathsf{PG}(2,q)$, q odd. If $\mathcal{C}'$ is a non-singular conic consisting entirely of internal (respectively external) points of $\mathcal{C}$, then $\mathcal{C}'$ is doubly tangent to $\mathcal{C}$ at two distinct points r_1 and r_2, which are conjugate with respect to the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$.*

Proof.
Clearly $\mathcal{C}$ and $\mathcal{C}'$, considered as conics of $\mathsf{PG}(2,q)$, are disjoint. If considered over the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ however, $\mathcal{C}$ and $\mathcal{C}'$ have at least two conjugate points in common, say r_1 and r_2. Further, let k be the pole of the line r_1r_2 with respect to $\mathcal{C}$. Then k is a point of $\mathsf{PG}(2,q)$ which is internal for $\mathcal{C}$. The pencil of conics containing $\mathcal{C}$ and the line r_1r_2 counted twice, is denoted by $\mathcal{B}$. We must prove that $\mathcal{C}'$ belongs to $\mathcal{B}$.

Assume by way of contradiction that $\mathcal{C}' \notin \mathcal{B}$. Consider all non-singular conics of $\mathcal{B}$ having at least one point in common with $\mathcal{C}'$. At most two of these conics are tangent to $\mathcal{C}'$ and possibly $k \in \mathcal{C}'$. In any case there are at least $\frac{q+1}{2}$ such conics. By Lemma 4.4.2 all these conics of $\mathcal{B}$ consist entirely of internal points of $\mathcal{C}$, contradicting Corollary 4.4.3 which says that there exist exactly $\frac{q-3}{2}$ non-singular conics of $\mathcal{B}$ with this property. So we conclude that $\mathcal{C}' \in \mathcal{B}$. $\square$

According to Corollary 4.4.3, there are different possible choices for a conic $\mathcal{C}'$ consisting entirely of internal points of $\mathcal{C}$, which, by Theorem 4.4.1, all give rise to an i-flock of a quadratic cone $L\mathcal{C}$ with line vertex L. Moreover, to every such i-flock corresponds a locally hermitian 1-system of the parabolic quadric $Q(6,q)$. As the conic planes of an i-flock of this kind do not contain a common point, the corresponding 1-system of $Q(6,q)$ is always proper. We have also seen that one specific choice for $\mathcal{C}'$ yields the 1-system of $Q(6,q)$ which is isomorphic to the (unique) semiclassical, non-hermitian spread $\mathcal{S}_{[9]}$ of the generalized hexagon $\mathsf{H}(q)$. The obvious question is whether or not the 1-systems arising from different choices for $\mathcal{C}'$ can be isomorphic.

4.5 The isomorphism question

First of all we observe the following. In [4], it is shown by Bloemen, Thas and Van Maldeghem that every spread of the generalized hexagon $\mathsf{H}(q)$ which is locally hermitian for at least two distinct lines, is a hermitian spread, regardless of the parity of q. In fact, the proof they give does not use any specific properties of $\mathsf{H}(q)$, to the contrary, it explicitly uses the fact that a spread of $\mathsf{H}(q)$ is a 1-system of $Q(6,q)$. Consequently it remains valid for 1-systems of $Q(6,q)$ as well.

Theorem 4.5.1 (Bloemen, Thas and Van Maldeghem [4]) *If $\mathcal{M}$ is a 1-system of $Q(6,q)$ that is locally hermitian for at least two distinct of its elements, then $\mathcal{M}$ is a hermitian spread of an induced $Q^-(5,q) \subseteq Q(6,q)$.*

Let $\mathcal{M}_i$ be a proper 1-system of $Q(6,q)$, q odd, which is locally hermitian at the line L_i, $i=1,2$, and arising as described in the previous section from a rational normal cubic scroll. Choose a regulus R_i in $\mathcal{M}_i$ containing the line L_i, $i=1,2$. Then there is an element $\sigma \in \mathsf{PGL}(7,q)$ fixing $Q(6,q)$, for which $L_1^\sigma = L_2$ and $R_1^\sigma = R_2$. So, investigating the isomorphism question, we may restrict ourselves to proper 1-systems $\mathcal{M}_1$ and $\mathcal{M}_2$ of $Q(6,q)$, which are locally hermitian at a common line L and for which the corresponding i-flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ share a conic $\mathcal{C}$. If $\mathscr{R}_i^3$ is the rational normal cubic scroll corresponding to $\mathcal{F}_i$, then let $\mathcal{C}_i'$ be the conic $\mathscr{R}_i^3 \cap \pi$ with π the plane of $\mathcal{C}$, $i=1,2$. Without loss of generality we may assume that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ meet $\mathcal{C}$ in the same pair $\{r_1, r_2\}$ of conjugate points. Now let β be an element of $\mathsf{PGO}(7,q)$, for which $\mathcal{M}_1^\beta = \mathcal{M}_2$. By Theorem 4.5.1 we have $L^\beta = L$. Clearly $\mathcal{F}_1^\beta = \mathcal{F}_2$ and $(\mathscr{R}_1^3)^\beta = \mathscr{R}_2^3$. Let ${\mathcal{C}'}_1^\beta$ be the conic $\mathcal{C}_2''$, where $\mathcal{C}_2'' = \mathscr{R}_2^3 \cap \pi'$ for some conic plane $\pi' \neq \pi$ of $\mathcal{F}_2$. Then there exists an element $\theta \in \mathsf{PGO}(7,q)$ fixing $\mathscr{R}_2^3$ and mapping $\mathcal{C}_2''$ onto $\mathcal{C}_2'$. Hence $\beta\theta$ fixes the plane π, the conic $\mathcal{C}$, and maps $\mathcal{C}_1'$ onto $\mathcal{C}_2'$; as a consequence, Lemma 4.4.2 implies that $\mathcal{C}_1' = \mathcal{C}_2'$. Conversely, for $\mathcal{C}_1' = \mathcal{C}_2'$, there is always an element $\alpha \in \mathsf{PGL}(7,q)$ fixing $Q(6,q)$, L and $\mathcal{C}$ and mapping $\mathscr{R}_1^3$ onto $\mathscr{R}_2^3$. Consequently, under the action of the subgroup of $\mathsf{PGL}(7,q)$ fixing $Q(6,q)$, there arise as many orbits in the set of all proper locally hermitian 1-systems of $Q(6,q)$, constructed as described in Section 4.4, as there exist non-singular conics in π, doubly tangent to $\mathcal{C}$ at r_1 and r_2 and consisting of internal points of $\mathcal{C}$. This number equals $\frac{q-3}{2}$, as we have seen before.

Let $\mathcal{C}$ be a non-singular conic of $\mathsf{PG}(2,q)$ and let r_1 and r_2 be distinct points of the extension of $\mathcal{C}$ to $\mathsf{GF}(q^2)$, which are conjugate with respect to $\mathsf{GF}(q^2)$. Let $\mathcal{B}$ be the pencil of conics defined by $\mathcal{C}$ and the line r_1r_2 counted twice, and let $\mathcal{D}$ be the subset of size $\frac{q-3}{2}$ of $\mathcal{B}$, formed by the non-singular conics of $\mathcal{B}$ consisting entirely of internal points of $\mathcal{C}$. If H stands for the subgroup of $\mathsf{P\Gamma L}(3,q)$ formed by the elements fixing $\mathcal{B}$ and $\mathcal{C}$, then H fixes $\mathcal{D}$ and moreover $H \cong \mathrm{Aut}(\mathsf{GF}(q))$. Further, let d be the number of orbits of H in $\mathcal{D}$. Relying on the previous paragraph we have the following theorem.

Theorem 4.5.2 *Under the action of the subgroup of* $\mathsf{P\Gamma L}(7,q)$ *fixing* $Q(6,q)$*, there arise* d *orbits in the set of all proper locally hermitian 1-systems of* $Q(6,q)$*, constructed as described in Section 4.4 from a rational normal cubic scroll* $\mathscr{R}^3$*.*

Remarks

1. It is possible to determine the value of d, as follows. Each element of H fixes three distinct conics of the pencil $\mathcal{B}$, namely the two singular elements of $\mathcal{B}$ and the non-singular conic $\mathcal{C}$. In $\mathcal{B}$, let the singular

conic $r_1r_2 = \mathcal{C}_1$ correspond to the parameter value $t = \infty$, let the second singular conic $\mathcal{C}_2$ correspond to $t = 0$, and let $\mathcal{C}$ correspond to $t = 1$; then, if $\mathcal{C}' \in \mathcal{B}$ has parameter t, we have that $\{\mathcal{C}_1, \mathcal{C}_2; \mathcal{C}, \mathcal{C}'\} = t$. The action of an element $\sigma \in H$ with matrix A and automorphism $\theta \in \mathrm{Aut}(\mathsf{GF}(q))$ on an arbitrary conic of $\mathcal{B}$, can then be identified with the action of θ on the parameter $t \in \mathsf{GF}(q) \cup \{\infty\}$ of $\mathcal{B}$, which determines this conic. It is readily checked that the non-singular conics of $\mathcal{B}$, conisisting entirely of internal points of $\mathcal{C}$, correspond to the parameters $t \in \mathsf{GF}(q) \setminus \{0, 1\}$ for which $1 - t$ is a square in $\mathsf{GF}(q)$. Hence the number d is equal to the number of orbits of $\mathrm{Aut}(\mathsf{GF}(q))$ in the set $\{t \in \mathsf{GF}(q) \setminus \{0, 1\} \mid 1 - t \text{ is a square}\}$.

2. The orbit of H, corresponding to the spread $\mathcal{S}_{[9]}$ of $\mathsf{H}(q)$, q odd and $q \equiv 1 \bmod 3$, has size one. This follows immediately from the remark on the cross-ratio $\{\mathcal{C}_1, \mathcal{C}_2; \mathcal{C}, \mathcal{C}'\}$ preceding Lemma 4.4.2: this cross-ratio is equal to 4 and so it is fixed by the action of $\mathrm{Aut}(\mathsf{GF}(q))$ on $\mathsf{GF}(q) \cup \{\infty\}$.

4.6 A characterization result

4.6.1 Proper locally hermitian 1-systems of $Q(6, q)$, q odd

A theorem which characterizes all proper locally hermitian, semiclassical 1-systems of $Q(6, q)$, q odd, in terms of their i-flock, will be shown in this subsection. Together with the earlier results of this chapter and some results concerning spreads of $Q^-(5, q)$ which will be mentioned in a separate subsection, this will lead to a complete characterization and determination of the locally hermitian, semiclassical 1-systems of $Q(6, q)$, q odd.

For the definition of a 1-system being semiclassical, we refer to Section 1.2.2. One should also recall the definition of an $\mathcal{R}$-conic contained in a 1-system of $Q(6, q)$, which has been explained in Section 1.2.2 as well, as we shall immediately see that the locally hermitian 1-systems of $Q(6, q)$, arising from a rational normal cubic scroll, contain many $\mathcal{R}$-conics.

Before we turn to the proof of the characterization result, we show an interesting lemma which holds both for q odd and for q even, with the same proof.

Lemma 4.6.1 *Let $\mathcal{M}$ be a proper 1-system of $Q(6, q)$, which is locally hermitian at some line $L \in \mathcal{M}$ and in addition semiclassical. Denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$. Then for every $i \neq j$, the elliptic* *quadric $Q^-_{ij}(5,q) = \langle R_i, R_j\rangle \cap Q(6,q)$ contains exactly q reguli of $\mathcal{M}$ through L and these reguli form an $\mathcal{R}$-conic.*

Proof.
For $x \in L$ arbitrary, the projection $\mathcal{O}_x$ of $\mathcal{M}$ from x along reguli is an elliptic quadric $Q^-(3,q)$. On $Q(6,q)$, there are exactly q elliptic quadrics $Q^-_{xi}(5,q)$, $i = 1, 2, \ldots, q$, through $x\mathcal{O}_x$. Also, if R_j is an arbitrary regulus of $\mathcal{M}$ through L, then $\langle x\mathcal{O}_x, R_j\rangle \cap Q(6,q) = Q^-_{xk}(5,q)$ for some $k \in \{1, 2, \ldots, q\}$. As $\mathcal{M}$ consists of q^2 reguli through L, it is clear that the average number of reguli of $\mathcal{M}$ through L contained in a $Q^-_{xi}(5,q)$ equals $\frac{q^2.1}{q} = q$. Assume that there exists a $Q^-_{xl}(5,q)$ which contains at least $q+1$ reguli of $\mathcal{M}$ through L. Then, for $y \in L \setminus \{x\}$, the projection $\mathcal{O}_y$ of $\mathcal{M}$ from y along reguli meets $Q^-_{xl}(5,q)$ in at least $q+1$ points different from $l' := L \cap \mathcal{O}_y$. As l' together with these $q+1$ points necessarily spans $\langle \mathcal{O}_y\rangle$, it follows that the cone $y\mathcal{O}_y$ is also contained in $Q^-_{xl}(5,q)$. So $Q^-_{xl}(5,q)$ contains each of the cones $y\mathcal{O}_y$, $y \in L$. In particular, all transversals of all reguli of $\mathcal{M}$ through L are lines of $Q^-_{xl}(5,q)$ and consequently $\mathcal{M} \subseteq Q^-_{xl}(5,q)$. This is a contradiction as $\mathcal{M}$ is assumed to be a proper 1-system of $Q(6,q)$. Hence each $Q^-_{xi}(5,q)$ contains at most q reguli of $\mathcal{M}$ through L, that is, at most the average number of reguli of $\mathcal{M}$ through L in a $Q^-_{xi}(5,q)$, and so they must all contain exactly q reguli of $\mathcal{M}$ through L.

We still have to show that every $Q^-_{ij}(5,q) = \langle R_i, R_j\rangle \cap Q(6,q)$, with R_i and R_j distinct reguli of $\mathcal{M}$ through L, is obtained as $Q^-_{xi}(5,q)$ for some $x \in L$. Fix a regulus R_i of $\mathcal{M}$ through L. Then for each $x \in L$, R_i uniquely defines a $Q^-_{xi}(5,q)$ and every $Q^-_{xi}(5,q)$ contains $q-1$ reguli of $\mathcal{M} \setminus \{R_i\}$ through L. Hence there are at most $(q+1)(q-1) = q^2-1$ reguli of $\mathcal{M} \setminus \{R_i\}$ through L which span a $Q^-_{xi}(5,q)$, $x \in L$, together with R_i. If any of these reguli were counted twice, then $Q^-_{xi}(5,q)$ and $Q^-_{yi}(5,q)$ would coincide for some $x \neq y$, and so $\mathcal{M}$ would belong to $Q^-_{xi}(5,q)$, a contradiction. Hence we have counted exactly q^2-1 distinct reguli of $\mathcal{M} \setminus \{R_i\}$ through L and it follows that every $Q^-_{ij}(5,q)$ is indeed obtained as $Q^-_{xi}(5,q)$ for some $x \in L$.

Finally it remains to show that the q reguli in a $Q^-_{ij}(5,q)$ form an $\mathcal{R}$-conic, but this immediately follows from a similar argument as was used to show that $Q^-_{xi}(5,q)$ contains at most q reguli of $\mathcal{M}$ through L. □

This lemma will be used in the proof of the following characterization theorem.

Theorem 4.6.2 *A proper locally hermitian 1-system of $Q(6,q)$, q odd, is semiclassical if and only if its associated i-flock arises from a rational normal cubic scroll $\mathscr{R}^3$ as described in Section 4.4.*

Proof.
Suppose that $\mathcal{M}$ is a proper 1-system of $Q(6,q)$, q odd, which is locally hermitian at L, and denote its associated i-flock by $\mathcal{F}$. Throughout the proof, the reguli of $\mathcal{M}$ containing L will be denoted by $R_1, R_2, \ldots, R_{q^2}$. Further, we will use the notation $\mathcal{O}_x$ for the ovoid which is the projection of $\mathcal{M}$ from a point $x \in L$ along reguli onto some 4-space $\gamma \subseteq x^{\perp} \setminus \{x\}$, similarly as in Section 1.2.2.

First assume that $\mathcal{F}$ arises from a rational normal cubic scroll $\mathscr{R}^3$ with directrix line L as described in Section 4.4. For $x \in L$ arbitrary, we have to show that the projection $\mathcal{O}_x$ of $\mathcal{M}$ from x along reguli onto a 4-dimensional space $\gamma \subseteq x^{\perp} \setminus \{x\}$, is an elliptic quadric $Q^-(3,q)$. On $\mathscr{R}^3$, there exists a unique generator N through x. Denote the points of $N \backslash L$ by $s_1, s_2, \ldots, s_q$ and assume that $s_1 \in \gamma$. Then $N^{\perp} \cap Q(6,q) = xQ^-(3,q)$, where $Q^-(3,q)$ is the intersection with $Q(4,q) = \gamma \cap Q(6,q)$ of the image $s_1^{\perp}$ of s_1 under the polarity of $Q(6,q)$. As $s_i \in N$, it holds that $N^{\perp} \cap Q(6,q) = xQ^-(3,q)$ is contained in $s_i^{\perp} \cap Q(6,q) := Q_i^-(5,q)$, $i = 1,2,\ldots,q$. Each of the quadrics $Q_i^-(5,q)$, $i = 1,2,\ldots,q$, is the intersection of $\langle R_k, R_l \rangle$ with $Q(6,q)$ for some distinct reguli R_k, R_l of $\mathcal{M}$ through L, since s_i is contained in q distinct conic planes of the flock $\mathcal{F}$. Suppose that there exists a regulus $R_a \in \mathcal{M}$ containing L, the transversal of which through x meets γ in some point $n_a \notin Q^-(3,q)$. This regulus R_a defines a conic plane π of $\mathcal{F}$, which intersects each generator of $\mathscr{R}^3$ in a unique point. Hence $s_j \in \pi$ for some $j \in \{1,2,\ldots,q\}$, so $R_a \subseteq Q_j^-(5,q)$. This is a contradiction as $xQ^-(3,q) \subseteq Q_j^-(5,q)$ and $xn_a \subseteq Q_j^-(5,q)$ cannot hold at the same time if $n_a \notin Q^-(3,q)$. We conclude that the projection of $\mathcal{M}$ from x along reguli is completely contained in $Q^-(3,q)$, so $Q^-(3,q)$ must coincide with $\mathcal{O}_x$. As $x \in L$ was chosen arbitrarily, it follows that $\mathcal{M}$ is semiclassical.

Now suppose that $\mathcal{M}$ is a proper 1-system of $Q(6,q)$, which is semiclassical and locally hermitian at L. We will show in several steps that the i-flock $\mathcal{F}$, associated with $\mathcal{M}$, arises from a rational normal cubic scroll $\mathscr{R}^3$ with directrix line L.
By Lemma 4.6.1, a lot can already be said about the structure of the conic planes of the i-flock $\mathcal{F}$, associated with $\mathcal{M}$, and the intersection points of these conic planes. Clearly, there are q^2 conic planes, and since it has been shown in Lemma 4.6.1 that each $Q_{ij}^-(5,q) = \langle R_i, R_j \rangle \cap Q(6,q)$, $i \neq j$, contains exactly q reguli of $\mathcal{M}$ through L, each intersection point of two distinct conic planes is contained in q conic planes. Consequently, the number of intersection points equals $\frac{q^2(q^2-1)}{q(q-1)} = q(q+1)$ and each conic plane contains $\frac{q^2-1}{q-1} = q+1$ intersection points, as every regulus R_i is contained in $q+1$ elliptic quadrics $Q_{xi}^-(5,q)$, x on L.

Next, we show that there exist $q+1$ lines $N_1, N_2, \ldots, N_{q+1} \in L^\perp$ which are disjoint and such that the q points on $N_i \setminus L$ are intersection points of conic planes, for $i = 1, 2, \ldots, q+1$. From the proof of Lemma 4.6.1, we know that for each $x \in L$, $x\mathcal{O}_x$ is contained in exactly q elliptic quadrics $Q^-_{xi}(5,q)$, $i = 1, 2, \ldots, q$, which all contain an $\mathcal{R}$-conic of reguli of $\mathcal{M}$. Now $\langle x\mathcal{O}_x\rangle^\perp$ is a line $R_x \subseteq L^\perp$ and tangent to $Q(6,q)$ at x. As $x\mathcal{O}_x \subseteq Q^-_{xi}(5,q)$ for $i = 1, 2, \ldots, q$, the q intersection points of conic planes which correspond to the quadrics $Q^-_{xi}(5,q)$, $i = 1, 2, \ldots, q$, that is, the points $\langle Q^-_{xi}(5,q)\rangle^\perp$ for $i = 1, 2, \ldots, q$, are exactly the q points of $R_x \setminus L$. Consequently $\{R_x \mid x \in L\}$ is the desired set of lines $\{N_1, N_2, \ldots, N_{q+1}\}$.
Consider a fixed conic plane π of $\mathcal{F}$ and the set of $q+1$ intersection points contained in π. We show that no three of these intersection points are collinear, so that they form an irreducible conic in π. Suppose there exist three collinear intersection points $s_1, s_2, s_3 \in \pi$, with $s_i^\perp \cap Q(6,q) = Q^-_i(5,q)$, $i = 1, 2, 3$. Note that $Q^-_i(5,q)$, $i = 1, 2, 3$, contains the regulus corresponding to the conic plane π. Then $Q^-_1(5,q) \cap Q^-_2(5,q) = Q^-_2(5,q) \cap Q^-_3(5,q) = Q^-_3(5,q) \cap Q^-_1(5,q) = \langle s_1, s_2, s_3\rangle^\perp \cap Q(6,q)$. As s_1 is an internal point of $\pi \cap Q(6,q)$, the line s_1s_2 is not tangent to $Q(6,q)$ and consequently $\langle s_1, s_2, s_3\rangle^\perp \cap Q(6,q)$ is not singular, so $\langle s_1, s_2, s_3\rangle^\perp \cap Q(6,q) = Q(4,q)$. We have seen above that there exists a point x on L such that $x\mathcal{O}_x \subseteq Q^-_1(5,q)$; clearly for $i = 2, 3$, $x\mathcal{O}_x \nsubseteq Q^-_i(5,q)$, so $x\mathcal{O}_x \cap Q^-_i(5,q) = x\mathcal{C}_i$, for some $\mathcal{C}_i \subseteq \mathcal{O}_x$. The conic $\mathcal{C}_i$ defines an $\mathcal{R}$-conic of reguli of $\mathcal{M}$ through L on $Q^-_i(5,q)$, for $i = 2, 3$. Further, it holds that $x\mathcal{C}_i \subseteq Q^-_1(5,q) \cap Q^-_i(5,q)$ and consequently $x\mathcal{C}_i \subseteq Q(4,q)$ for $i = 2, 3$. But $\mathcal{C}_2 \neq \mathcal{C}_3$, as otherwise $Q^-_2(5,q) = Q^-_3(5,q)$, and a contradiction follows. We conclude that π contains an irreducible conic of intersection points of conic planes of $\mathcal{F}$.
Finally we prove that the intersection points of conic planes, together with the line L, form a rational normal cubic scroll $\mathscr{R}^3$ having L as directrix line. Let π be an arbitrary conic plane and p an arbitrary point of the conic $\mathcal{C}'$ of intersection points in π. Each of the $q+1$ intersection points in π lies on exactly one line N_i, $i \in \{1, 2, \ldots, q+1\}$; say $p \in N_{q+1}$. Consider a 3-dimensional subspace $\mathsf{PG}(3,q) \subseteq L^\perp$, not containing p, and consider the projection from p onto $\mathsf{PG}(3,q)$. Clearly, the q conic planes through p are projected onto q disjoint lines $L'_1, L'_2, \ldots, L'_q$ and L is projected onto a line L' disjoint from $L'_1, L'_2, \ldots, L'_q$. The lines $N_1, N_2, \ldots, N_q$ are projected onto q lines $N'_1, N'_2, \ldots, N'_q$ of $\mathsf{PG}(3,q)$ and as every conic plane has a unique point in common with each line N_i, every line N'_i meets $L'_1, L'_2, \ldots, L'_q$ in exactly one point. It is clear that each N'_i also meets L' in a unique point, and hence the lines $L', L'_1, L'_2, \ldots, L'_q$ form a regulus in $\mathsf{PG}(3,q)$. The tangent lines at p of the conics of intersection points lying in the conic planes through p, are projected onto q points on respectively $L'_1, L'_2, \ldots, L'_q$, lying on the

$(q+1)$st line N'_{q+1} of the opposite regulus of $\{L', L'_1, L'_2, \ldots, L'_q\}$. For all $i, j, k, l \in \{1, 2, \ldots, q+1\}$, the cross-ratio $\{\mathcal{C}' \cap N_i, \mathcal{C}' \cap N_j; \mathcal{C}' \cap N_k, \mathcal{C}' \cap N_l\}$ equals the cross-ratio $\{L'_1 \cap N'_i, L'_1 \cap N'_j; L'_1 \cap N'_k, L'_1 \cap N'_l\}$, where we assume that L'_1 is the projection of π onto $\mathsf{PG}(3,q)$. On the other hand, we have the following relations between cross-ratios:

$$\begin{aligned}\{L'_1 \cap N'_i, L'_1 \cap N'_j; L'_1 \cap N'_k, L'_1 \cap N'_l\} &= \{N'_i, N'_j; N'_k, N'_l\}\\ &= \{L' \cap N'_i, L' \cap N'_j; L' \cap N'_k, L' \cap N'_l\}\\ &= \{L \cap N_i, L \cap N_j; L \cap N_k, L \cap N_l\}.\end{aligned}$$

These equalities allow us to conclude that there exists a linear isomorphism which maps the points $N_i \cap L$ onto the points $N_i \cap \mathcal{C}$. It follows by Semple and Roth [55, p. 133] that the intersection points of the conic planes of $\mathcal{F}$, together with the line L, indeed form a rational normal cubic scroll $\mathscr{R}^3$ with directrix line L. This completes the proof of the theorem. □

4.6.2 Locally hermitian spreads of $Q^-(5,q)$

To obtain a complete picture of the locally hermitian, semiclassical 1-systems of $Q(6,q)$, q odd, it is necessary to consider the locally hermitian spreads of $Q^-(5,q)$. For this part, the restriction that q must be odd, does not apply. Note that a locally hermitian spread of $Q^-(5,q)$ always is semiclassical, since for any point x on $Q^-(5,q)$, the tangent hyperplane $T_x(Q^-(5,q))$ meets $Q^-(5,q)$ in a cone $xQ^-(3,q)$. In Section 4.2, we have already paid attention to the hermitian spread of $Q^-(5,q)$, q odd, which is locally hermitian at any line, and constructed its associated i-flock, if considered as a 1-system of $Q(6,q)$. Here we will discuss, for all q a construction producing locally hermitain spreads of $Q^-(5,q)$ from a line spread in $\mathsf{PG}(3,q)$ and we will see that all locally hermitian spreads of $Q^-(5,q)$ arise in this way. The construction is initially due to Thas [69], but here we use a slightly different approach, see [10].

Let $\mathcal{S}$ be a spread of an elliptic quadric $Q^-(5,q)$ in $\mathsf{PG}(5,q)$, which is locally hermitian at the line L. As always, we denote the reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$. If $\perp$ is the polarity of $Q^-(5,q)$, then each $L_i := \langle R_i\rangle^\perp$, $i = 1, 2, \ldots, q^2$, is a line of the 3-space $L^\perp = T_L(Q^-(5,q))$. Since $\langle R_i, R_j\rangle$ coincides with $\mathsf{PG}(5,q)$ for all $i \neq j$, any two distinct lines L_i and L_j are disjoint; moreover each line L_i is disjoint from L. Hence $S := \{L, L_1, L_2, \ldots, L_{q^2}\}$ is a line spread of $L^\perp = \mathsf{PG}(3,q)$. By reversing this construction, a locally hermitian spread of $Q^-(5,q)$ can be constructed from every line spread of $\mathsf{PG}(3,q)$, containing L. In the sequel, we will denote the locally hermitian spread $\mathcal{S}$ of $Q^-(5,q)$, arising from the line spread S in $\mathsf{PG}(3,q)$, by $\mathcal{S}(S)$.

If $\mathcal{S}(S)$ is the hermitian spread of $Q^-(5,q)$, then S is easily seen to be regular, but the converse is not true. In [10], Cardinali, Lunardon, Polverino and Trombetti show that there exist regular line spreads S in $\mathsf{PG}(3,q)$ such that $\mathcal{S}(S)$ is not hermitian. This subject is studied more deeply in [11], where Cardinali and Trombetti show that $\mathcal{S}(S)$ is hermitian if and only if S is a regular line spread in $\mathsf{PG}(3,q) = L^\perp$, defined by two conjugate lines T and $\overline{T}$ (with respect to the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$) which respectively belong to the two planes containing the line L, of the extension $Q^+(5,q^2)$ of $Q^-(5,q)$ to $\mathsf{GF}(q^2)$. They use a quite extensive algebraic argument to obtain this result. Here we will come to the same result, using geometric reasoning, in a much shorter way.

In the theorem and the proof that follow, the notation $\widehat{X}$ will stand for the extension to $\mathsf{GF}(q^2)$ of any object X, which is initially defined over $\mathsf{GF}(q)$.

Theorem 4.6.3 *A spread $\mathcal{S}(S)$ of $Q^-(5,q)$, which is locally hermitian at L, is hermitian if and only if S is a regular line spread in $\mathsf{PG}(3,q) = L^\perp$, defined by two conjugate lines T and $\overline{T}$ such that $\langle\widehat{L},T\rangle$ and $\langle\widehat{L},\overline{T}\rangle$ are the two generators through $\widehat{L}$ of the extension $Q^+(5,q^2)$ of $Q^-(5,q)$ to $\mathsf{GF}(q^2)$.*

Proof.
It is known, see for instance [69], that a spread $\mathcal{S}$ of $Q^-(5,q)$ is hermitian if and only if there exist two conjugate, disjoint planes π and $\overline{\pi}$, which are generators of the extension $Q^+(5,q^2)$ of $Q^-(5,q)$, meeting all lines of $\mathcal{S}$. The points of intersection of the extensions to $\mathsf{GF}(q^2)$ of the lines of $\mathcal{S}$ with π, form a hermitian curve $H(2,q^2)$ in π.

Suppose that $\mathcal{S}(S)$ is hermitian and consider it as being locally hermitian at some line L. Then $\pi\cap\widehat{L}$ and $\overline{\pi}\cap\widehat{L}$ are two conjugate points and π and $\overline{\pi}$ are not contained in $\widehat{L}^\perp$, with $\perp$ the polarity of $Q^+(5,q^2)$. If $R_1, R_2, \ldots, R_{q^2}$ are the reguli of $\mathcal{S}(S)$ through L, then $\widehat{\langle R_i\rangle}\cap\pi$ is a line and similarly for $\overline{\pi}$. Hence the extension to $\mathsf{GF}(q^2)$ of the line $L_i = \langle R_i\rangle^\perp$ of $L^\perp$ has a point in common with π and $\overline{\pi}$, for all $i = 1,2,\ldots,q^2$. It follows that $\pi\cap\widehat{L}^\perp$ and $\overline{\pi}\cap\widehat{L}^\perp$ must be conjugate lines T and $\overline{T}$, and $\{L, L_1, L_2, \ldots, L_{q^2}\}$ is the regular line spread in $L^\perp$, defined by T and $\overline{T}$. Since π and $\overline{\pi}$ are generators of $Q^+(5,q^2)$, T and $\overline{T}$ must be totally singular lines of $Q^+(5,q^2)$ and thus $\langle\widehat{L},T\rangle$ and $\langle\widehat{L},\overline{T}\rangle$ are the two generators through $\widehat{L}$ of $Q^+(5,q^2)$.

Conversely, if S is the regular line spread in $L^\perp$ defined by two conjugate lines T and $\overline{T}$, with $\langle\widehat{L},T\rangle$ and $\langle\widehat{L},\overline{T}\rangle$ the unique generators of $Q^+(5,q^2)$ on $\widehat{L}$, then every 3-space $\widehat{\langle R_i\rangle}$ has a plane α_i in common with $T^\perp$ and a plane $\overline{\alpha}_i$ with $\overline{T}^\perp$. This plane α_i contains $\widehat{L}$ and is not contained in $\widehat{L}^\perp$, so $\alpha_i\cap Q^+(5,q^2)$ consists of $\widehat{L}$ and a line in π, where π denotes the generator on T, different from $\langle \widehat{L}, T\rangle$. It follows that each $\widehat{\langle R_i\rangle}$ meets π in a line belonging to the regulus of $\widehat{\langle R_i\rangle} \cap Q^+(5,q^2)$, not containing $\widehat{L}$. In particular, the extension to $\mathsf{GF}(q^2)$ of every line of $\mathcal{S}(S)$ has a point in common with π. As the same holds for the unique generator $\overline{\pi}$ containing $\overline{T}$, different from $\langle \widehat{L}, \overline{T}\rangle$, the spread $\mathcal{S}(S)$ must be hermitian. □

Remark
If $\mathcal{S}(S)$ is a locally hermitian spread of $Q^-(5,q)$, q odd, and we consider $\mathcal{S}(S)$ as a 1-system of a $Q(6,q)$ in which $Q^-(5,q)$ is embedded, then the i-flock corresponding to $\mathcal{S}(S)$ clearly consists of the q^2 conics $\langle p, L_i\rangle \cap Q(6,q)$, where p is the pole of $\langle Q^-(5,q)\rangle^\perp$ with respect to $Q(6,q)$ and $S = \{L, L_1, L_2, \ldots, L_{q^2}\}$.

4.6.3 Conclusion

We conclude this section with a summary of the characterization results. If $\mathcal{M}$ is a proper locally hermitian, semiclassical 1-system of $Q(6,q)$, q odd, then the picture is complete. In Theorem 4.6.2, a characterization of these 1-systems as exactly the ones which arise from a rational normal cubic scroll $\mathscr{R}^3$ as described in Section 4.4, was obtained. Moreover, Theorem 4.4.4 determines the cubic scrolls that give rise to a proper locally hermitian, semiclassical 1-system of $Q(6,q)$. This solves the problem of the characterization and determination of all proper locally hermitian, semiclassical 1-systems of $Q(6,q)$, q odd, completely.

If the 1-system $\mathcal{M}$ of $Q(6,q)$ is not proper, it is a locally hermitian spread of an induced $Q^-(5,q) \subseteq Q(6,q)$. Such spreads can be constructed from a line spread S of $\mathsf{PG}(3,q)$ so that $\mathcal{M} = \mathcal{S}(S)$, as described above. A classification of these examples would involve a classification of all line spreads of $\mathsf{PG}(3,q)$, an endless task. Still, by Theorem 4.6.3, it is clear in which cases $\mathcal{M} = \mathcal{S}(S)$ is a hermitian spread of $Q^-(5,q)$.

4.7 Generalizations?

Given the nice results that arose from the method of associating an i-flock with a locally hermitian 1-system of $Q(6,q)$, q odd, it would be promising if it were possible to generalize this theory to parabolic quadrics in $\mathsf{PG}(2n,q)$, $n > 3$. However, it can easily be seen that there are only few possibilities for direct generalization.

Let $\mathcal{M}$ be an m-system of a parabolic quadric $Q(2n,q)$ in $\mathsf{PG}(2n,q)$, q odd and $n \geq 3$. Assume that $\mathcal{M}$ is the union of spreads of a collection of hyperbolic quadrics $Q_i^+(2m+1,q)$, $i = 1,2,\ldots,q^{n-m}$, each spread containing a fixed $\pi_0 \in \mathcal{M}$. Then m is necessarily odd. It also holds that $\langle Q_i^+(2m+1,q), Q_j^+(2m+1,q)\rangle$, $i \neq j$, is $(3m+2)$-dimensional and $\langle Q_i^+(2m+1,q), Q_j^+(2m+1,q)\rangle \cap Q(2n,q)$ is necessarily an elliptic quadric $Q_{ij}^-(3m+2,q)$. However, to obtain an m-system, we must also require that the generators of $Q_{ij}^-(3m+2,q)$ are m-dimensional, for otherwise any generator of $Q_{ij}^-(3m+2,q)$ through an element π_m of $\mathcal{M}$ which belongs to the spread in $Q_i^+(2m+1,q)$, certainly has at least a point in common with $Q_j^+(2m+1,q)$. Since all points of $Q_j^+(2m+1,q)$ are points of $\tilde{\mathcal{M}}$, this generator has a non-empty intersection with some element of $\mathcal{M} \setminus \{\pi_m\}$, a contradiction. Now the generators of $Q_{ij}^-(3m+2,q)$ are $(3\frac{m-1}{2}+1)$-dimensional, so that the condition $m = 3\frac{m-1}{2}+1$ arises. It readily follows from this condition that $m=1$ and consequently locally hermitian 1-systems of quadrics appear to be the only possible candidates for generalization in this way.

As the locally hermitian 1-systems of $Q(6,q)$, q odd, have already been discussed, it is natural to consider a generalization to $Q(2n,q)$, q odd and $n \geq 3$. So suppose that $\mathcal{M}$ is a 1-system of $Q(2n,q)$, q odd and $n \geq 3$, consisting of q^{n-1} reguli through a common line L; in other words, $\mathcal{M}$ is locally hermitian at L. Then for each point x on L, the projection of $\mathcal{M}$ from x along reguli onto a $(2n-2)$-space $\gamma \subseteq T_x(Q(2n,q)) \setminus \{x\}$ is an ovoid of $Q(2n-2,q) = \gamma \cap Q(2n,q)$. Now, by a result due to Gunawardena and Moorhouse [20], the parabolic quadric $Q(2n-2,q)$ has no ovoids for $n > 4$ and q odd. Since $n=3$ is the setting on $Q(6,q)$, the only meaningful other case is $n=4$, where $\mathcal{M}$ is a locally hermitian 1-system of $Q(8,q)$, q odd. Similarly as for $Q(6,q)$, one can associate an i-flock of the cone $T_L(Q(8,q)) \cap Q(8,q) = LQ(4,q)$ with $\mathcal{M}$. As $|\mathcal{M}| = q^4+1$, this i-flock must consist of q^3 parabolic quadrics $Q_i(4,q)$, $i = 1,2,\ldots,q^3$. In addition, for every two reguli R_i and R_j of $\mathcal{M}$ through L, we again have that $\langle R_i, R_j\rangle \cap Q(8,q)$ is an elliptic quadric $Q_{ij}^-(5,q)$. As such, $Q_i(4,q) \cap Q_j(4,q)$ must be a conic $\mathcal{C}_{ij}$, for all $i \neq j$, with the property that $\langle \mathcal{C}_{ij}\rangle^\perp$ intersects $Q(8,q)$ in an elliptic quadric $Q_{ij}^-(5,q)$. A conic $\mathcal{C}$ on $Q(8,q)$ for which $\langle \mathcal{C}\rangle^\perp \cap Q(8,q)$ is an elliptic quadric $Q^-(5,q)$, will be called *of elliptic type* in this context. We thus obtain an i-flock consisting of q^3 parabolic quadrics $Q_i(4,q)$, $i = 1,2,\ldots,q^3$, where every two elements of the i-flock have a conic of elliptic type in common. This whole setting is geometrically possible and hence it is an interesting question whether one can use this approach to achieve results concerning locally hermitian 1-systems of $Q(8,q)$, q odd.

Of course, there are probably other possible directions for generalization, and we will encounter one specific case in one of the following chapters.

However, attention will first be paid to the case of locally hermitian 1-systems of $Q(6, q)$ for q even, which is the topic of the next chapter.

Chapter 5

Locally hermitian 1-systems of $Q(6, q)$, q even

In the previous chapter, a method was described to study locally hermitian 1-systems of the quadric $Q(6, q)$, q odd, by associating an i-flock of a quadratic cone with a line vertex in $\mathsf{PG}(4, q)$ with them. This method gave rise to new 1-systems of $Q(6, q)$. As a natural continuation to that part, we now aim at a similar theory for q even. This cannot be achieved by straightforward generalization, as the method for q odd heavily relies on the polarity defined by the quadric $Q(6, q)$. For q even however, no polarity is associated with the quadric $Q(6, q)$, see for instance Hirschfeld [24, Theorem 5.28], so a new approach must be developed. The new approach again involves a kind of flock in $\mathsf{PG}(4, q)$, which will be linked to a locally hermitian 1-system of $Q(6, q)$, q even. This method will be applied to a known example of a locally hermitian 1-system of $Q(6, 2^{2e})$, which has been discovered as a spread of the classical generalized hexagon $\mathsf{H}(2^{2e})$. The results concerning this spread appear to be suitable for generalization and enable us to find a new class of 1-systems of $Q(6, q)$, q even. Similarly as for q odd, all new examples turn out to be semiclassical, which suggests that a characterization of the proper locally hermitian, semiclassical 1-systems of $Q(6, q)$, q even, might be within reach. Indeed, we are able to prove that a proper locally hermitian 1-system of $Q(6, q)$, q even, is semiclassical if and only if it belongs to the new class we describe.

It is also noticed that in combination with the results from Chapter 2, there arise semipartial geometries from our new class of 1-systems of $Q(6, q)$, but a further investigation of these semipartial geometries will be postponed to

a later chapter.
Eventually, the chapter is terminated by a description of a method to construct the classical generalized hexagon $\mathsf{H}(q)$ geometrically if $q \equiv 1 \bmod 3$.

The results obtained in this chapter will appear in the *Proceedings of the 2001 Oberwolfach Conference on Finite Geometries*, a special issue of *Designs, Codes and Cryptography*, see [41].

5.1 Associating a flock with a locally hermitian 1-system of $Q(6,q)$

For q even as well as for q odd, a locally hermitian 1-system $\mathcal{M}$ of $Q(6,q)$ consists of q^2 reguli through some line $\tilde{L} \in \mathcal{M}$; $\tilde{L}$ is the line at which $\mathcal{M}$ is locally hermitian. These q^2 reguli will be denoted by $R_1, R_2, \ldots, R_{q^2}$, as in the previous chapter. Recall that for $i \neq j$, the subspace $\langle R_i, R_j\rangle$ is 5-dimensional and intersects $Q(6,q)$ in an elliptic quadric $Q^-_{ij}(5,q)$, as has been explained in Section 1.2.2 of Chapter 1. In the present chapter, we always assume that q is even, unless otherwise mentioned.

Let $\mathcal{M}$ be a 1-system of the parabolic quadric $Q(6,q)$, which is locally hermitian at some line $\tilde{L}$. Consider a 4-dimensional subspace β of $\mathsf{PG}(6,q) = \langle Q(6,q)\rangle$ containing the nucleus n of $Q(6,q)$ and disjoint from $\tilde{L}$. We project $\mathsf{PG}(6,q) \setminus \tilde{L}$ from $\tilde{L}$ onto β. The tangent space $T_{\tilde{L}}(Q(6,q))$ of $Q(6,q)$ at $\tilde{L}$, minus $\tilde{L}$, is projected onto a plane π in β, containing an irreducible conic $\mathcal{C}'$ of points of $Q(6,q)$. Note that the nucleus of $Q(6,q)$ is also the nucleus of $\mathcal{C}'$. Each regulus R_i of $\mathcal{M}$ through $\tilde{L}$, minus $\tilde{L}$, is projected onto a line L_i in β. Clearly, any two lines L_i and L_j, $i \neq j$, are disjoint and in addition, $\langle L_i, L_j\rangle \cap \pi$ is a line, disjoint from $\mathcal{C}'$. For if $\langle L_i, L_j\rangle$ contained a point x of $\mathcal{C}'$, then the quadric $Q^-_{ij}(5,q)$ would contain the totally singular plane $\langle \tilde{L}, x\rangle$, a contradiction. Moreover, if some line L_i intersected the plane π in a point y, then the hyperbolic quadric $Q^+_i(3,q) := \langle R_i\rangle \cap Q(6,q)$ would either contain the whole totally singular plane $\langle \tilde{L}, y\rangle$ if $y \in \mathcal{C}'$, or it would meet the plane $\langle \tilde{L}, y\rangle$ exactly in the line $\tilde{L}$ if $y \notin \mathcal{C}'$. Both possibilities cannot occur, so we conclude that all lines L_i are disjoint from π.

We now consider the dual structure in β. The dual π^D of the plane π is a line L. To each point x of $\mathcal{C}'$, there corresponds a 3-space $D_x := x^D$ through L and the nucleus n of both $\mathcal{C}'$ and $Q(6,q)$ yields another 3-space $D_n := n^D$ through L. Any plane α in β and disjoint from L, meets each D_x in a line and these $q+1$ lines form a dual conic in α with nucleus line $D_n \cap \alpha$. Finally, the q^2 lines L_i are dualized to q^2 planes $\pi_i := L_i^D$, disjoint from L. As $\langle L_i, L_j\rangle$ is a 3-space for all $i \neq j$, any two planes π_i and π_j meet in a point. None of these intersection points $\pi_i \cap \pi_j$ belongs to any of the 3-spaces D_x or D_n, because the lines $\langle L_i, L_j \rangle \cap \pi$ are disjoint from $\mathcal{C}'$ for all $i \neq j$. Let $\mathcal{K}$ be the cone having L as vertex and the union of the $q+1$ lines $\pi_i \cap D_x$, $x \in \mathcal{C}'$ and i a given element of $\{1, 2, \ldots, q^2\}$, as base; so the base is a dual conic. Similarly, let $\overline{\mathcal{K}}$ be the cone having L as vertex and the union of the $q+2$ lines $\pi_i \cap D_x$, $x \in \mathcal{C}' \cup \{n\}$ and i a given element of $\{1, 2, \ldots, q^2\}$, as base; so here the base is a dual conic union its nucleus line. Then the q^2 planes π_i, $i = 1, 2, \ldots, q^2$, partition the cone $\mathcal{K}$, respectively $\overline{\mathcal{K}}$, minus its vertex L. Such a set of q^2 planes will be called a *flock* of the cone $\mathcal{K}$, respectively $\overline{\mathcal{K}}$. The same term will be used for the set of all dual conics, being the intersections of the cone with the q^2 planes.

Definitions

Consider in $\mathsf{PG}(4, q)$ a cone $\mathcal{K}$, respectively $\overline{\mathcal{K}}$, with a line L as vertex and a dual conic, respectively a dual conic union its nucleus line, as base. Such a cone $\mathcal{K}$ will be called a *dual conic cone* of $\mathsf{PG}(4, q)$.
A *flock* of a dual conic cone $\mathcal{K}$ is a partition of the cone without the vertex line into q^2 disjoint dual conics in planes π_i, $i = 1, 2, \ldots, q^2$. We also use the name *flock* for the set of the q^2 planes containing these dual conics. It is clear that a flock $\mathcal{F} = \{\pi_i \mid i = 1, 2, \ldots, q^2\}$ of a dual conic cone $\mathcal{K}$ is also a flock of the corresponding cone $\overline{\mathcal{K}}$, since all points on the nucleus line of the dual conic which is the base of $\mathcal{K}$, belong to exactly one line of the dual conic itself.

The preceding construction can also be reversed, as follows. Consider a flock $\mathcal{F} = \{\pi_i \mid i = 1, 2, \ldots, q^2\}$ of a dual conic cone $\mathcal{K}$ with vertex line L in $\mathsf{PG}(4, q)$. We first dualize everything in $\mathsf{PG}(4, q)$, which yields a plane $\pi = L^D$ containing an irreducible conic $\mathcal{C}'$, and a set of q^2 lines $L_i = \pi_i^D$ with the same properties as above. Now we embed $\mathsf{PG}(4, q)$ in a 6-dimensional projective space $\mathsf{PG}(6, q)$ and we consider in $\mathsf{PG}(6, q)$ a parabolic quadric $Q(6, q)$ such that $Q(6, q) \cap \pi = \mathcal{C}'$ and such that the nucleus of $\mathcal{C}'$ is also the nucleus of $Q(6, q)$. On this $Q(6, q)$ we choose a line $\tilde{L}$ disjoint from $\mathsf{PG}(4, q)$ but contained in the tangent space $T_x(Q(6, q))$ of $Q(6, q)$ at x, for all points x of $\mathcal{C}'$. For each line L_i, the space $\langle \tilde{L}, L_i \rangle$ is 3-dimensional and its intersection with $Q(6, q)$ contains lines. If $\langle \tilde{L}, L_i \rangle \cap Q(6, q)$ were singular, then $\langle \tilde{L}, L_i \rangle$ would contain a point z of π (not necessarily a point of $\mathcal{C}'$) and hence $z \in L_i$. This is a contradiction to the fact that π is disjoint from all lines L_i. So $\langle \tilde{L}, L_i \rangle \cap Q(6, q)$ is a hyperbolic quadric $Q_i^+(3, q)$ for all $i = 1, 2, \ldots, q^2$. We denote the regulus of $Q_i^+(3, q)$ containing $\tilde{L}$ by R_i. As any two lines L_i and L_j are disjoint, the subspace $\langle R_i, R_j \rangle$ is 5-dimensional for all $i \neq j$. Would it intersect $Q(6, q)$ in a quadric containing totally singular planes, then $\langle R_i, R_j \rangle \cap Q(6, q)$ would also contain a totally singular plane on $\tilde{L}$ and so $\langle L_i, L_j\rangle \cap \mathcal{C}'$ would not be empty. This is impossible because, by assumption, none of the points $\pi_i \cap \pi_j = \langle L_i, L_j\rangle^D$ belongs to the given dual conic cone $\mathcal{K}$ in $\mathsf{PG}(4,q)$. It follows that the union $\mathcal{M}$ of all reguli R_i, $i = 1, 2, \ldots, q^2$, is a partial 1-system of $Q(6,q)$. Finally, $|\mathcal{M}| = q^3 + 1$ and hence $\mathcal{M}$ is a 1-system of $Q(6,q)$, which is locally hermitian at $\tilde{L}$.

We have thus shown the following theorem.

Theorem 5.1.1 *For q even, each locally hermitian 1-system of $Q(6,q)$ yields a flock of a dual conic cone with a line as vertex in $\mathsf{PG}(4,q)$, and conversely from every such flock a locally hermitian 1-system of $Q(6,q)$ can be constructed.*

This equivalence opens new perspectives in the study of locally hermitian 1-systems of $Q(6,q)$, q even. In the next section, we will apply the method of associating a flock with a locally hermitian 1-system of $Q(6,q)$, to a known example, which is proper and was discovered by Offer as a spread of the classical generalized hexagon $\mathsf{H}(2^{2e})$, see [46] and [45]. Independently, Cardinali, Lunardon, Polverino and Trombetti [10] found a new spread of the hexagon $\mathsf{H}(2^{2e})$, which they discovered by using a representation of $\mathsf{H}(2^{2e})$ as a coset geometry. However, if one translates the spread of Offer to the setting of Cardinali, Lunardon, Polverino and Trombetti, it appears that both spreads are in fact equivalent. In even more recent work, see [36], Lunardon and Polverino draw the same conclusion in a different way, as they show that the spread of $\mathsf{H}(2^{2e})$ they have found in [10], is the unique semiclassical spread of $\mathsf{H}(2^{2e})$ which is not hermitian, and must hence be isomorphic to the example of Offer. They also prove that for $q = 2^{2e+1}$, the only semiclassical spread of the hexagon $\mathsf{H}(q)$ is the hermitian spread. Together with the similar classification result for q odd by Bloemen, Thas and Van Maldeghem in [4], all semiclassical spreads of the hexagon $\mathsf{H}(q)$ are thus known.
For the sequel of this chapter, we will rely on the work of Offer concerning this semiclassical, non-hermitian spread of $\mathsf{H}(2^{2e})$, since his description is easier to handle for our purposes.

One may notice that a 1-system of $Q(6,q)$, which is a hermitian spread of an induced $Q^-(5,q) \subseteq Q(6,q)$, has an associated flock, all planes of which contain a common point not belonging to the dual conic cone.

5.2 The spread $\mathcal{S}_{[\delta]}$ of Offer

In [46] and in [45], Offer describes a new locally hermitian spread of the hexagon $\mathsf{H}(2^{2e})$, which is non-hermitian. As every spread of the hexagon

$\mathsf{H}(q)$ is at the same time a 1-system of the parabolic quadric $Q(6,q)$, his construction provides us with an example of a proper locally hermitian 1-system of $Q(6,2^{2e})$. It is given explicitly in the following theorem.

Theorem 5.2.1 (Offer [46, 45]) *Let $q = 2^{2e}$ and let δ be some fixed element of $\mathsf{GF}(q)$ with $\mathrm{Tr}(\delta) = 1$. Then the set*

$$\mathcal{S}_{[\delta]} = \bigcup_{x,y\in\mathsf{GF}(q)} [[x, y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta}{\delta+1}y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta^2}{\delta+1}y]]$$

is a non-hermitian spread of $\mathsf{H}(q)$.

As has been pointed out in Section 1.3.2, $[[a,b,c,d]]$ is in this context a notation for the regulus $\{[\infty]\} \cup \{[a,b,\lambda,c,d] \mid \lambda \in \mathsf{GF}(q)\}$ of the hexagon $\mathsf{H}(q)$, where we use the well-known coordinatization of $\mathsf{H}(q)$ which was introduced by De Smet and Van Maldeghem in [15], see also Section 1.3.2. The spread $\mathcal{S}_{[\delta]}$ of Offer is locally hermitian at the line $[\infty]$ and we will investigate the corresponding flock.

In the following, we always work on the quadric $Q(6,q)$ with standard equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The line $[\infty]$, at which $\mathcal{S}_{[\delta]}$ is locally hermitian, is then the line $\tilde{L}$ on $Q(6,q)$ with equations $X_1 = X_2 = X_3 = X_4 = X_5 = 0$. It is straightforward to calculate the equations of the 3-dimensional subspace $R(x,y)$ generated by the regulus $\mathcal{R}(x,y)$ of $\mathcal{S}_{[\delta]}$ through $\tilde{L}$, with

$$\mathcal{R}(x,y) := [[x, y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta}{\delta+1}y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta^2}{\delta+1}y]].$$

One finds that

$$R(x,y) \equiv \begin{cases} X_1 + xX_2 + yX_4 = 0 \\ (\delta+1)^2yX_2 + (\delta+1)^2X_3 + (\delta^3x + \delta(\delta+1)y)X_4 = 0 \\ (\delta^3x + \delta(\delta+1)y)X_2 + (\delta^3x + \delta^2(\delta+1)y)X_4 \\ \qquad\qquad\qquad\qquad\qquad\qquad + (\delta+1)^2X_5 = 0, \end{cases}$$

for all $x, y \in \mathsf{GF}(q)$. We choose the 4-dimensional subspace $\beta \equiv X_0 = X_6 = 0$ to project upon from $\tilde{L}$, and in β we find q^2 lines $L(x,y) = R(x,y) \cap \beta$ with equations

$$L(x,y) \equiv \begin{cases} X_0 = X_6 = 0 \\ X_1 + xX_2 + yX_4 = 0 \\ (\delta+1)^2yX_2 + (\delta+1)^2X_3 + (\delta^3x + \delta(\delta+1)y)X_4 = 0 \\ (\delta^3x + \delta(\delta+1)y)X_2 + (\delta^3x + \delta^2(\delta+1)y)X_4 \\ \qquad\qquad\qquad\qquad\qquad\qquad + (\delta+1)^2X_5 = 0, \end{cases}$$

for all $x, y \in \mathsf{GF}(q)$. Further, the plane $\pi = T_{\tilde{L}}(Q(6,q)) \cap \beta$ is given by $\pi \equiv X_0 = X_2 = X_4 = X_6 = 0$ and the intersection of π with $Q(6,q)$ is the irreducible conic

$$\mathcal{C}' \equiv \begin{cases} X_0 = X_2 = X_4 = X_6 = 0 \\ X_1X_5 + X_3^2 = 0. \end{cases}$$

Finally, after taking the dual in β, we obtain a dual conic cone with vertex line $L := \pi^D \equiv X_0 = X_1 = X_3 = X_5 = X_6 = 0$ and consisting of the $q+1$ 3-dimensional subspaces $\{D_\infty\} \cup \{D_t \mid t \in \mathsf{GF}(q)\}$, with $D_\infty \equiv X_0 = X_5 = X_6 = 0$ and

$$D_t \equiv \begin{cases} X_0 = X_6 = 0 \\ X_1 + tX_3 + t^2X_5 = 0. \end{cases} \tag{5.1}$$

Here D_t stands for the dual of the point $x_t := (0,1,0,t,0,t^2,0)$ of $\mathcal{C}'$, for all $t \in \mathsf{GF}(q)$, and $D_\infty = x_\infty^D$ with x_∞ the point $(0,0,0,0,0,1,0)$ of $\mathcal{C}'$.
The flock associated with the spread $\mathcal{S}_{[\delta]}$ of $\mathsf{H}(2^{2e})$ thus consists of the q^2 planes $\pi(x,y) := L(x,y)^D$, given by

$$\pi(x,y) \equiv \begin{cases} X_0 = X_6 = 0 \\ y(\delta+1)^2X_1 + (\delta^3x + \delta(\delta+1)y)X_3 + (\delta+1)^2X_4 \\ \qquad\qquad + (\delta^3x + \delta^2(\delta+1)y)X_5 = 0 \\ x(\delta+1)^2X_1 + (\delta+1)^2X_2 + y(\delta+1)^2X_3 \\ \qquad\qquad + (\delta^3x + \delta(\delta+1)y)X_5 = 0, \end{cases}$$

for all $x, y \in \mathsf{GF}(q)$. We will denote this flock by $\mathcal{F}_{[\delta]}$.

5.3 A new description of the spread $\mathcal{S}_{[\delta]}$

In this section we will see that the flock $\mathcal{F}_{[\delta]}$, and hence also the spread $\mathcal{S}_{[\delta]}$, can be constructed geometrically. To that end, the structure of the intersection points of the planes of $\mathcal{F}_{[\delta]}$ is investigated.

Lemma 5.3.1 *The intersection points of a fixed plane $\pi(x,y) \in \mathcal{F}_{[\delta]}$ with the q^2-1 other planes of $\mathcal{F}_{[\delta]}$ are exactly the $q+1$ points of an irreducible conic in $\pi(x,y)$.*

Proof.
We first show that the plane $\pi(0,0) \in \mathcal{F}_{[\delta]}$ has this property. One easily calculates that the point of intersection of the planes $\pi(0,0)$ and $\pi(x,y)$, for some $x, y \in \mathsf{GF}(q)$, is given by

$$\begin{aligned}(0, \delta^3[\delta^3x^2 + y(x+y)(\delta+1)^2], 0,& \\ (\delta+1)^2[\delta^3x(x+y) + \delta(\delta+1)y(y+\delta x)], 0,& \\ (\delta+1)^2[\delta^3x^2 + (\delta+1)^2y^2 + \delta(\delta+1)xy], 0).&\end{aligned}$$

From these coordinates, it is obvious that $\pi(0,0)\cap\pi(x,y) = \pi(0,0)\cap\pi(\lambda x,\lambda y)$ for all $\lambda \in \mathsf{GF}(q) \setminus \{0\}$. As the flock $\mathcal{F}_{[\delta]}$ contains q^2-1 planes, different from $\pi(0,0)$, and every intersection point $\pi(0,0)\cap\pi(x,y)$, $x,y \in \mathsf{GF}(q)$, is contained in at least $q-1$ elements of $\mathcal{F}_{[\delta]} \setminus \{\pi(0,0)\}$, there are at most $q+1$ intersection points in $\pi(0,0)$. They are given by

$$
\begin{aligned}
\pi(0,0)\cap\pi(0,1) &= \{(0,\delta^3,0,\delta(\delta+1),0,(\delta+1)^2,0)\}, \text{ and}\\
\pi(0,0)\cap\pi(1,y) &= \{(0,\delta^3[\delta^3+y(y+1)(\delta+1)^2],0,\\
&\qquad(\delta+1)^2[\delta^3(y+1)+\delta(\delta+1)y(y+\delta)],0,\\
&\qquad(\delta+1)^2[\delta^3+(\delta+1)y(\delta y+y+\delta)],0)\}, \quad \forall y \in \mathsf{GF}(q).
\end{aligned} \tag{5.2}
$$

It is straightforward, but lengthy, to check that these $q+1$ points are all different, so $\pi(0,0)$ contains exactly $q+1$ intersection points. Further, one can calculate that the $q+1$ given points are the points of the following irreducible conic in $\pi(0,0)$:

$$
\mathcal{C} \equiv \begin{cases} X_0 = X_2 = X_4 = X_6 = 0\\ (\delta+1)^2X_1^2 + \delta^3X_3^2 + \delta^4X_5^2\\ \qquad + \delta(\delta+1)X_1X_3 + \delta^2X_1X_5 + \delta^3X_3X_5 = 0. \end{cases} \tag{5.3}
$$

Hence the intersection points of the plane $\pi(0,0)$ with all other elements of $\mathcal{F}_{[\delta]}$ satisfy the lemma.

In $\beta = \mathsf{PG}(4,q)$, consider for some fixed $x,y \in \mathsf{GF}(q)$ the following projectivity θ:

$$
\begin{cases}
Y_i = X_i \qquad\qquad \text{for } i = 1,3,5\\
(\delta+1)^2Y_2 = x(\delta+1)^2X_1 + (\delta+1)^2X_2 + y(\delta+1)^2X_3\\
\qquad\qquad + (\delta^3x + \delta(\delta+1)y)X_5\\
(\delta+1)^2Y_4 = y(\delta+1)^2X_1 + (\delta^3x+\delta(\delta+1)y)X_3\\
\qquad\qquad + (\delta+1)^2X_4 + (\delta^3x + \delta^2(\delta+1)y)X_5.
\end{cases}
$$

Then one easily sees that $L^\theta = L$ and that $D_t^\theta = D_t$ for all $t \in \mathsf{GF}(q) \cup \{\infty\}$, so θ stabilizes the given dual conic cone in β. Further, we have that $\pi(x,y)^\theta = \pi(0,0)$ and for all $(a,b) \in (\mathsf{GF}(q)\times\mathsf{GF}(q)) \setminus \{(x,y)\}$ it holds that $\pi(a,b)^\theta = \pi(x+a,y+b)$. Hence θ also stabilizes the flock $\mathcal{F}_{[\delta]}$ and maps $\pi(x,y)$ onto $\pi(0,0)$. As one can find such a projecivity θ for all $x,y \in \mathsf{GF}(q)$, all planes of $\mathcal{F}_{[\delta]}$ have the property that their intersection points with the other elements of $\mathcal{F}_{[\delta]}$ are the points of an irreducible conic. $\square$

From the previous lemma, one can deduce how many points of intersection there are. The flock $\mathcal{F}_{[\delta]}$ has q^2 elements, each one containing $q+1$ intersection points, and every intersection point is contained in q planes of $\mathcal{F}_{[\delta]}$. This yields $\frac{q^2(q+1)}{q} = q(q+1)$ points. We proceed to show that these $q(q+1)$ intersection points lie on a rational normal cubic scroll $\mathscr{R}^3$. This suggests a strong resemblance to the situation in Chapter 4.

Lemma 5.3.2 *The $q(q+1)$ intersection points of the planes $\pi(x,y)$, together with the line L, form a rational normal cubic scroll $\mathscr{R}^3$ with L as directrix line.*

Proof.
From the previous part, we know that the conic $\mathcal{C}$ consists of the points $s_\infty := \pi(0,0) \cap \pi(0,1)$ and $s_y := \pi(0,0) \cap \pi(1,y)$, $y \in \mathsf{GF}(q)$. The explicit coordinates of these points are given in (5.2). We first show that the planes $\langle L, s_y \rangle$, $y \in \mathsf{GF}(q) \cup \{\infty\}$, contain q intersection points, lying on some common line. This can be checked by straightforward calculation.

One can verify that the plane $\langle L, s_\infty \rangle$ is given by

$$\langle L, s_\infty \rangle \equiv \begin{cases} X_0 = X_6 = 0 \\ (\delta+1)X_1 + \delta^2 X_3 = 0 \\ (\delta+1)^2 X_1 + \delta^3 X_5 = 0 \end{cases}$$

and that a plane $\pi(x,y)$, $x, y \in \mathsf{GF}(q)$, intersects $\langle L, s_\infty \rangle$ in the point

$$(0, \delta^3(\delta+1), 0, \delta(\delta+1)^2, \delta^3 x, (\delta+1)^3, 0).$$

The coordinates of this point are independent of the parameter $y \in \mathsf{GF}(q)$, so that $\langle L, s_\infty \rangle$ shares q distinct points with $\bigcup_{x,y \in \mathsf{GF}(q)} \pi(x,y)$, all lying on the line

$$N_\infty \equiv \begin{cases} X_0 = X_2 = X_6 = 0 \\ (\delta+1)X_1 + \delta^2 X_3 = 0 \\ (\delta+1)^2 X_1 + \delta^3 X_5 = 0. \end{cases}$$

This line meets L in the point $p_\infty := (0,0,0,0,1,0,0)$.

For every $y \in \mathsf{GF}(q)$, we find that

$$\langle L, s_y \rangle \equiv \begin{cases} X_0 = X_6 = 0 \\ (\delta+1)^2[\delta^3(1+y) + \delta(\delta+1)y(y+\delta)]X_1 \\ \qquad\qquad + \delta^3[\delta^3 + y(y+1)(\delta+1)^2]X_3 = 0 \\ (\delta+1)^2[\delta^3 + (\delta+1)y(\delta y + y + \delta)]X_1 \\ \qquad\qquad + \delta^3[\delta^3 + y(y+1)(\delta+1)^2]X_5 = 0 \end{cases}$$

and one can check that $\langle L, s_y \rangle \cap \pi(x,z)$ is the point

$$(0, \delta^3[\delta^3+y(y+1)(\delta+1)^2], \delta^3(\delta+1)(z+xy), (\delta+1)^2[\delta^3(1+y)+\delta(\delta+1)y(y+\delta)],$$
$$\delta^3(y+\delta+\delta y)(z+xy), (\delta+1)^2[\delta^3 + (\delta+1)y(\delta y + y + \delta)], 0),$$

for all $x, z \in \mathsf{GF}(q)$. So it holds that $\pi(x,z) \cap \langle L, s_y \rangle = \pi(x',z') \cap \langle L, s_y \rangle$ for $x, x', z, z' \in \mathsf{GF}(q)$, if and only if $z + xy = z' + x'y$. This implies that the point $\pi(x,z) \cap \langle L, s_y \rangle$ is contained in the q planes $\pi(x', z + (x+x')y)$, $x' \in \mathsf{GF}(q)$. This yields q intersection points in the plane $\langle L, s_y \rangle$ and lying on the line

$$N_y \equiv \begin{cases} X_0 = X_6 = 0 \\ (y + \delta + \delta y)X_2 + (\delta+1)X_4 = 0 \\ (\delta+1)^2[\delta^3(1+y) + \delta(\delta+1)y(y+\delta)]X_1 \\ \qquad\qquad + \delta^3[\delta^3 + y(y+1)(\delta+1)^2]X_3 = 0 \\ (\delta+1)^2[\delta^3 + (\delta+1)y(\delta y + y + \delta)]X_1 \\ \qquad\quad + \delta^3[\delta^3 + y(y+1)(\delta+1)^2]X_5 = 0. \end{cases}$$

The intersection of L and N_y is the point $p_y := (0, 0, \delta+1, 0, y+\delta+\delta y, 0, 0)$, $y \in \mathsf{GF}(q)$; one also easily sees that $p_{y_1} \neq p_{y_2}$ for $y_1 \neq y_2$. Together with the point p_∞, we hence obtain all $q+1$ points of L.

We conclude that the $q(q+1)$ intersection points together with the line L form a set which is the union of $q+1$ lines joining the points p_y of L with the points s_y of $\mathcal{C}$, for $y \in \mathsf{GF}(q) \cup \{\infty\}$. As there exists a linear projectivity between the points of L and the points of $\mathcal{C}$ such that p_y and s_y are corresponding points for all $y \in \mathsf{GF}(q) \cup \{\infty\}$, it follows from VII-3.22 of Semple and Roth [55] that the set of intersection points of pairs of elements of $\mathcal{F}_{[\delta]}$, is a rational normal cubic scroll $\mathscr{R}^3$ with directrix line L. $\square$

Thanks to the above lemma, we have obtained a new description of the spread $\mathcal{S}_{[\delta]}$ of the hexagon $\mathsf{H}(2^{2e})$ starting from the cubic scroll $\mathscr{R}^3$ we have found. The conics on this cubic scroll define q^2 planes in $\mathsf{PG}(4,q)$ which yield a flock of the dual conic cone consisting of the 3-spaces D_t. From this flock we can construct the corresponding 1-system of $Q(6,q)$ by the method described in Section 5.1. This 1-system is then projectively equivalent to the spread $\mathcal{S}_{[\delta]}$ of the hexagon $\mathsf{H}(2^{2e})$, as we started from this spread. However, to have a complete geometric construction of this 1-system, we still need to describe the pair $(\mathcal{K}, \mathscr{R}^3)$, with $\mathcal{K}$ the dual conic cone and $\mathscr{R}^3$ the cubic scroll, geometrically. This problem will be solved in a later section.

5.4 A new class of 1-systems of $Q(6,q)$ for q even

In the previous section we have found a method to construct the 1-system $\mathcal{S}_{[\delta]}$ starting from the particular pair $(\mathcal{K}, \mathscr{R}^3)$. Here we will see that this same method can be used to construct new 1-systems of $Q(6,q)$, q even, starting from different pairs $(\mathcal{K}', \mathscr{R}^{3'})$ consisting of a dual conic cone $\mathcal{K}'$ and a rational normal cubic scroll $\mathscr{R}^{3'}$.

Theorem 5.4.1 *Consider a dual conic cone $\mathcal{K}$ with vertex line L in $\mathsf{PG}(4,q)$, q even, and let $\mathcal{C}^*$ be the dual conic which is the intersection of the cone with a plane π_0 skew to L. Let $\mathcal{C} \subseteq \pi_0$ be a conic, disjoint from $\mathcal{C}^*$. Then any rational normal cubic scroll $\mathscr{R}^3$ with L as directrix line and containing $\mathcal{C}$ defines a flock of the cone $\mathcal{K}$ and hence also a locally hermitian 1-system of a parabolic quadric $Q(6,q)$.*

Proof.
A rational normal cubic scroll $\mathscr{R}^3$ in $\mathsf{PG}(4,q)$ is known to contain exactly q^2 conics, defining q^2 planes π_i, $i = 1, 2, \ldots, q^2$, such that every two distinct planes meet in exactly one point which belongs to the cubic scroll. As $\mathcal{C}$ was chosen to be disjoint from $\mathcal{C}^*$, the cubic scroll and the cone $\mathcal{K}$ share nothing but the line L. Hence the planes π_i constitute a flock of the cone $\mathcal{K}$. By the method described in Section 5.1, this flock defines a locally hermitian 1-system of a parabolic quadric $Q(6,q)$. □

We would like to determine which pairs $(\mathcal{C}, \mathcal{C}^*)$ satisfy the above condition. To that end, we first have a closer look at the example in Sections 5.2 and 5.3, and consider again the particular conic $\mathcal{C}$ which was obtained there, in the plane $\pi(0,0)$ with equations $X_0 = X_2 = X_4 = X_6 = 0$. Recall that the coordinates of the points of the conic $\mathcal{C}$ have been given explicitly in (5.2) and that the conic has equation (5.3). The nucleus of $\mathcal{C}$ is the point $n = (0, \delta^2, 0, \delta, 0, \delta+1, 0)$. In $\pi(0,0)$ the dual conic cone given by (5.1) defines a dual conic $\mathcal{C}^*$, consisting of the $q+1$ lines $X_1 + tX_3 + t^2X_5 = 0 = X_0 = X_2 = X_4 = X_6$, $t \in \mathsf{GF}(q) \cup \{\infty\}$. The nucleus line of $\mathcal{C}^*$ is easily seen to be the line $N \equiv X_0 = X_2 = X_3 = X_4 = X_6 = 0$.

As $\mathcal{C}$ and $\mathcal{C}^*$ are known to be disjoint, it holds that N contains no point of $\mathcal{C}$ and dually no line of $\mathcal{C}^*$ contains n. Over the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ however, N has two points in common with $\mathcal{C}$ and n is contained in two lines of $\mathcal{C}^*$. If $a \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ is a solution of $\delta^2 + \delta X + (\delta+1)X^2 = 0$, so $\delta^2 + \delta a + (\delta+1)a^2 = 0$, then one can check that n lies on the following two lines of $\mathcal{C}^*$:

$$S_1 \equiv \begin{cases} X_1 + aX_3 + a^2X_5 = 0 \\ X_0 = X_2 = X_4 = X_6 = 0, \text{ and} \end{cases}$$

$$S_2 \equiv \begin{cases} X_1 + \left(a + \frac{\delta}{\delta+1}\right) X_3 + \left(a^2 + \frac{\delta^2}{(\delta+1)^2}\right) X_5 = 0 \\ X_0 = X_2 = X_4 = X_6 = 0. \end{cases}$$

These lines are tangent to $\mathcal{C}$ at the points $u_1 = (0, a^2, 0, 0, 0, 1, 0)$, respectively $u_2 = (0, a^2 + \frac{\delta^2}{(\delta+1)^2}, 0, 0, 0, 1, 0)$, which are both points of N. Hence $\mathcal{C}$ belongs to the pencil $\mathcal{B}$ of conics in $\pi(0,0)$, generated by the line N counted twice and the singular conic $S_1 \cup S_2$.

We now investigate such pencils of conics further. To that end, let $\mathcal{C}^*$ be any non-singular dual conic with nucleus line N in a plane π and let u_1 and u_2 be distinct points of N, where we allow u_1 and u_2 to be conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. Further, let n be the point of intersection of the lines of $\mathcal{C}^*$ containing respectively u_1 and u_2, and let $\mathcal{B}$ be the pencil of conics (over $\mathsf{GF}(q)$) in π, generated by N counted twice and the singular conic $S_1 \cup S_2$, with $S_i = nu_i$, $i = 1, 2$. Then we have the following lemma.

Lemma 5.4.2

(i) If an element $\gamma \in \mathsf{PGL}(3,q)$ stabilizes the pencil $\mathcal{B}$ and the dual conic $\mathcal{C}^*$, then γ stabilizes every element of $\mathcal{B}$. If H is the subgroup of such elements which fix S_1 and S_2, then for each non-singular $\mathcal{C}$ in $\mathcal{B}$ the group H acts regularly on the points of π on $\mathcal{C} \setminus N$.

(ii) If $\gamma \in \mathsf{P\Gamma L}(3,q)$, with corresponding automorphism θ of $\mathsf{GF}(q)$, stabilizes both $\mathcal{B}$ and $\mathcal{C}^*$, then the action of γ on the pencil $\mathcal{B}$, minus its two singular conics, can be identified with the action of θ on $\mathsf{GF}(q) \setminus \{0\}$.

Proof.
Without loss of generality, we may choose the coordinates as follows: $u_1 = (0, 1, a_1)$, $u_2 = (0, 1, a_2)$, $n = (1, 0, 0)$ and hence $S_1 \equiv a_1X_1 + X_2 = 0$ and $S_2 \equiv a_2X_1 + X_2 = 0$ with $a_1 \neq a_2$ as the points u_1 and u_2 are different. A general non-singular element $\mathcal{C}$ of the pencil $\mathcal{B}$ generated by $S_1 \cup S_2$ and $N = u_1u_2$ counted twice is given by

$$\mathcal{C} \equiv aX_0^2 + bX_1^2 + cX_2^2 + X_1X_2 = 0,$$

with

$$\begin{cases} b + ca_1^2 + a_1 = 0 \\ b + ca_2^2 + a_2 = 0, \end{cases} \tag{5.4}$$

and $a \in \mathsf{GF}(q) \setminus \{0\}$. Similarly, we find that the equation of $\mathcal{C}^*$ in dual coordinates U_0, U_1, U_2 has the following form

$$\mathcal{C}^* \equiv a'U_0^2 + b'U_1^2 + c'U_2^2 + U_1U_2 = 0,$$

with

$$\begin{cases} b'a_1^2 + c' + a_1 = 0 \\ b'a_2^2 + c' + a_2 = 0, \end{cases} \tag{5.5}$$

and $a' \in \mathsf{GF}(q) \setminus \{0\}$. Without loss of generality, we may assume that $a' = 1$. Also, by comparing (5.4) and (5.5), one notices that $b' = c$ and $b = c'$, so that

$$\mathcal{C}^* \equiv U_0^2 + cU_1^2 + bU_2^2 + U_1U_2 = 0.$$

Consider an element $\gamma \in \mathsf{P\Gamma L}(3,q)$ with matrix A and automorphism $\theta \in \mathrm{Aut}(\mathsf{GF}(q))$, which stabilizes $\mathcal{B}$. Then the matrix A is of the following form:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_1 & r_2 \\ 0 & r_3 & r_4 \end{pmatrix}.$$

One can calculate that the image of $\mathcal{C}^*$ under γ is given by:

$$\begin{aligned}(\mathcal{C}^*)^\gamma \equiv U_0^2 &+ (c^\theta r_1^2 + b^\theta r_2^2 + r_1r_2)U_1^2 + (c^\theta r_3^2 + b^\theta r_4^2 + r_3r_4)U_2^2 \\ &+ (r_1r_4 + r_2r_3)U_1U_2 = 0.\end{aligned}$$

So demanding that γ stabilizes $\mathcal{C}^*$ yields the following conditions:

$$\begin{cases} 1 = r_1r_4 + r_2r_3 \\ c^\theta r_1^2 + b^\theta r_2^2 + r_1r_2 = c \\ c^\theta r_3^2 + b^\theta r_4^2 + r_3r_4 = b. \end{cases} \tag{5.6}$$

Finally we determine the image of $\mathcal{C}$ under γ:

$$\begin{aligned}\mathcal{C}^\gamma \equiv\ & a^\theta(r_1r_4 + r_2r_3)^2X_0^2 + (c^\theta r_3^2 + b^\theta r_4^2 + r_3r_4)X_1^2 \\ &+ (c^\theta r_1^2 + b^\theta r_2^2 + r_1r_2)X_2^2 + (r_1r_4 + r_2r_3)X_1X_2 = 0,\end{aligned}$$

which, in view of the conditions (5.6), can be written as

$$\mathcal{C}^\gamma \equiv a^\theta X_0^2 + bX_1^2 + cX_2^2 + X_1X_2 = 0. \tag{5.7}$$

It is now clear that if $\gamma \in \mathsf{PGL}(3,q)$, so θ is the identity, then it follows from (5.7) that $\mathcal{C}^\gamma = \mathcal{C}$ and γ stabilizes $\mathcal{B}$ elementwise. Now, let H be the subgroup of $\mathsf{PGL}(3,q)$ consisting of such elements γ which fix S_1 and S_2. For $u_1, u_2 \in \pi$ this group H has size $q-1$, and for $u_1, u_2 \in \mathsf{PG}(2,q^2) \setminus \pi$ this group H has size $q+1$. If H does not act transitively on the points of $\mathcal{C} \setminus N$, then there exists a $\gamma_0 \in H$, γ_0 not the identity, which fixes a point v of $\mathcal{C} \setminus N$, so fixes the line vn. Hence γ_0 fixes three distinct lines containing n, so it must fix all lines on n and consequently all points of $\mathcal{C}$. This means that γ_0 is the identity, a contradiction. We conclude that H acts transitively on $\mathcal{C} \setminus N$ and taken the size of H into account, it follows that H acts regularly on the points of $\mathcal{C} \setminus N$. This shows (i).

Finally, if θ is not the identity, then it is clear from (5.7) that the action of γ on $\mathcal{B}$, minus its two singular conics, can be identified with the action of the automorphism θ on $\mathsf{GF}(q) \setminus \{0\}$. This proves the statement (ii). □

Corollary 5.4.3 *Every non-singular element of the pencil $\mathcal{B}$ either is disjoint from $\mathcal{C}^*$, or it consists entirely of points on lines of $\mathcal{C}^*$.*

Proof.
This is immediate from (i) of Lemma 5.4.2. □

From now on, we consider the case where u_1 and u_2 belong to $\mathsf{PG}(2,q^2)\backslash\pi$, because the example of the pair $(\mathcal{C},\mathcal{C}^*)$ from Sections 5.2 and 5.3 satisfies this condition.

In the plane π there are $\frac{q(q-1)}{2}$ points, not on a line of $\mathcal{C}^*$. One of them is the point n, which is in fact the element $S_1 \cup S_2$ of $\mathcal{B}$, if considered over $\mathsf{GF}(q)$. All other such points lie on exactly one non-singular element of $\mathcal{B}$, which, by taking Corollary 5.4.3 into account, is disjoint from $\mathcal{C}^*$. Hence there are exactly $\frac{q-2}{2}$ non-singular elements of $\mathcal{B}$, disjoint from $\mathcal{C}^*$. The remaining $\frac{q}{2}$ non-singular conics of $\mathcal{B}$ consist entirely of points on lines of $\mathcal{C}^*$.

Theorem 5.4.4 *Consider a projective plane $\mathsf{PG}(2,q)$, q even, let $\mathcal{C}^*$ be a dual conic in $\mathsf{PG}(2,q)$ and let $\mathcal{C}$ be a non-singular conic in $\mathsf{PG}(2,q)$ which is disjoint from $\mathcal{C}^*$. Then $\mathcal{C}$ is tangent to two lines of $\mathcal{C}^*$ which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, and the line joining the tangent points is the nucleus line of $\mathcal{C}^*$.*

Proof.
Denote the nucleus line of the dual conic $\mathcal{C}^*$ by N and the nucleus of the conic $\mathcal{C}$ by n. By assumption, $\mathcal{C}$ and $\mathcal{C}^*$ are disjoint, which implies that n does not lie on a line of $\mathcal{C}^*$ and N is disjoint from $\mathcal{C}$. Over the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, the line N has two conjugate points in common with $\mathcal{C}$, say u_1 and u_2. Denote by T_1, respectively T_2, the lines over $\mathsf{GF}(q^2)$ of $\mathcal{C}^*$ containing u_1, respectively u_2. We have to show that $\mathcal{C}$ is tangent to T_1 and T_2 at u_1, respectively u_2.

Let $\mathcal{B}$ be the pencil of conics generated by $T_1 \cup T_2$ and N counted twice. Then we want to prove that $\mathcal{C}$ belongs to $\mathcal{B}$. Suppose that this is not the case. Over $\mathsf{GF}(q)$, the conic $\mathcal{C}$ contains $q+1$ points, none of which lies on a line of $\mathcal{C}^*$. One of them might be the point $T_1 \cap T_2$, the nucleus of all non-singular conics of $\mathcal{B}$, but the remaining q points all lie on exactly one non-singular conic of $\mathcal{B}$. As each conic of $\mathcal{B}$ meets $\mathcal{C}$ in at most two points over $\mathsf{GF}(q)$, there are at least $\frac{q}{2}$ non-singular elements of $\mathcal{B}$ which meet $\mathcal{C}$. Since $\mathcal{C}$ is disjoint from $\mathcal{C}^*$ and all elements of $\mathcal{B}$ are either disjoint from $\mathcal{C}^*$ or consist entirely of points on lines of $\mathcal{C}^*$, $\mathcal{B}$ must contain at least $\frac{q}{2}$ non-singular conics which are disjoint from $\mathcal{C}^*$. This is a contradiction to the foregoing results and we conclude that $\mathcal{C}$ belongs to $\mathcal{B}$. □

In Theorem 5.4.4, we have determined all conics in π which yield cubic scrolls satisfying Theorem 5.4.1 and for a given $\mathcal{C}^*$, there are $\frac{q-2}{2}$ of them up to isomorphism. Each of these conics gives rise to a locally hermitian 1-system of $Q(6,q)$ and in the next section we will investigate whether these 1-systems are isomorphic or not.

Note that for $q=2$ no conic in π yields a cubic scroll satisfying Theorem 5.4.1, because $\frac{q-2}{2}=0$ in this case, so no 1-systems of $Q(6,2)$ are found by our method.

5.5 The isomorphism question

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two proper 1-systems of the parabolic quadric $Q(6,q)$, q even and $q>2$, which are constructed from cubic scrolls $\mathscr{R}_1^3$ and $\mathscr{R}_2^3$ respectively, using the construction method developed in Section 5.4. Let $\mathcal{M}_i$ be locally hermitian at L_i, $i=1,2$, and recall that by Theorem 4.5.1, the line L_i is unique with this property. For $i=1,2$, choose a regulus R_i in $\mathcal{M}_i$ containing the line L_i. Then there exists an element $\sigma \in \mathsf{PGL}(7,q)$ fixing $Q(6,q)$, for which $L_1^\sigma = L_2$ and $R_1^\sigma = R_2$. So we may assume without loss of generality that $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally hermitian at the same line $\tilde{L}$ and that, after projecting from this line $\tilde{L}$ onto the same 4-dimensional subspace $\mathsf{PG}(4,q)$ containing the nucleus of $Q(6,q)$ and dualizing, the cubic scrolls $\mathscr{R}_1^3$ and $\mathscr{R}_2^3$ associated with $\mathcal{M}_1$, respectively $\mathcal{M}_2$, have a conic plane π in common. Clearly $\mathscr{R}_1^3$ and $\mathscr{R}_2^3$ also have the same directrix line L. Denote by $\mathcal{C}_i$ the intersection of π with $\mathscr{R}_i^3$, $i=1,2$. Both conics $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint from a common dual conic $\mathcal{C}^*$ in π and by Theorem 5.4.4, we may assume that they belong to the same pencil $\mathcal{B}$ of conics, generated by the nucleus line N of $\mathcal{C}^*$ counted twice, and the singular conic $nu_1 \cup nu_2$ where u_1 and u_2 are distinct conjugate points on the extension of N over $\mathsf{GF}(q^2)$ and n is the common nucleus of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic under the subgroup $\mathsf{PGO}(7,q)$ of $\mathsf{PGL}(7,q)$ fixing $Q(6,q)$, so there exists an element $\zeta \in \mathsf{PGO}(7,q)$ such that $\mathcal{M}_1^\zeta = \mathcal{M}_2$. Then $\tilde{L}^\zeta = \tilde{L}$ by Theorem 4.5.1. In $\mathsf{PG}(4,q)$, ζ induces an element $\gamma \in \mathsf{PGL}(5,q)$ stabilizing the line L and the dual conic cone $L\mathcal{C}^*$, and further $(\mathscr{R}_1^3)^\gamma = \mathscr{R}_2^3$. Suppose that $\mathcal{C}_1^\gamma = \mathcal{C}_2'$, where $\mathcal{C}_2' = \pi' \cap \mathscr{R}_2^3$ for some conic plane π' of $\mathscr{R}_2^3$, different from π. Then there exists an element $\theta \in \mathsf{PGL}(5,q)$ stabilizing L, $\mathscr{R}_2^3$ and the dual conic cone $L\mathcal{C}^*$ in $\mathsf{PG}(4,q)$, and mapping $\mathcal{C}_2'$ onto $\mathcal{C}_2$. So $\gamma\theta$ fixes π, the dual conic cone $L\mathcal{C}^*$ and the line L, and maps $\mathcal{C}_1$ onto $\mathcal{C}_2$. The restriction of $\gamma\theta$ to π is thus an element of $\mathsf{PGL}(3,q)$ mapping $\mathcal{C}_1$ onto $\mathcal{C}_2$ while stabilizing $\mathcal{C}^*$ and the pencil $\mathcal{B}$, which implies by Theorem 5.4.2 that $\mathcal{C}_1 = \mathcal{C}_2$.

Conversely, if $\mathcal{C}_1 = \mathcal{C}_2$, then there exists an element $\alpha' \in \mathsf{PGL}(5,q)$ stabilizing L, the dual conic cone $L\mathcal{C}^*$ and mapping $\mathscr{R}_1^3$ onto $\mathscr{R}_2^3$. This α' defines an element $\alpha \in \mathsf{PGL}(7,q)$ stabilizing $Q(6,q)$ and $\tilde{L}$ and such that $\mathcal{M}_1^\alpha = \mathcal{M}_2$, so that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic under the action of $\mathsf{PGO}(7,q)$.

Let G be the subgroup of $\mathsf{P\Gamma L}(3,q)$ which stabilizes $\mathcal{C}^*$ and $\mathcal{B}$. By Lemma 5.4.2 the action of G on the set of all non-singular conics of $\mathcal{B}$ is equivalent to the action of $\mathrm{Aut}(\mathsf{GF}(q))$ on $\mathsf{GF}(q) \setminus \{0\}$. Let $\mathcal{D}$ be the subset of $\mathcal{B}$, consisting of all non-singular conics which are disjoint from $\mathcal{C}^*$, so $|\mathcal{D}| = \frac{q-2}{2}$, and let d denote the number of orbits of G in $\mathcal{D}$. By the foregoing, we obtain the following theorem.

Theorem 5.5.1 *Let $q > 2$ be even. Under the action of the subgroup of $\mathsf{PGL}(7,q)$ stabilizing $Q(6,q)$, there arise $\frac{q-2}{2}$ orbits in the set of proper locally hermitian 1-systems of $Q(6,q)$ constructed as described in Section 5.4. Under the action of the subgroup of $\mathsf{P\Gamma L}(7,q)$ stabilizing $Q(6,q)$, there arise d orbits in the set of proper locally hermitian 1-systems of $Q(6,q)$ constructed as described in Section 5.4.*

Remarks

1. The number d of orbits of G in $\mathcal{D}$ equals the number of orbits of $\mathrm{Aut}(\mathsf{GF}(q))$ in the set $\mathfrak{C}_0 \setminus \{0\}$, with $\mathfrak{C}_0$ the set of all trace zero elements in $\mathsf{GF}(q)$. This can be seen as follows, where we use the notation and the choice of the coordinates as introduced in the proof of Lemma 5.4.2. So a general non-singular element $\mathcal{C}$ of $\mathcal{B}$ is given by

$$aX_0^2 + bX_1^2 + cX_2^2 + X_1X_2 = 0,$$

$a \in \mathsf{GF}(q) \setminus \{0\}$, for some fixed b, c with $\mathrm{Tr}(bc) = 1$ since the nucleus line $N \equiv X_0 = 0$ of $\mathcal{C}^*$ is disjoint from $\mathcal{C}$ over $\mathsf{GF}(q)$.
In dual coordinates $\mathcal{C}^*$ is given by

$$U_0^2 + cU_1^2 + bU_2^2 + U_1U_2 = 0,$$

see again the proof of Lemma 5.4.2. Since we want to describe the set $\mathcal{D}$, we require that $\mathcal{C}$ is disjoint from $\mathcal{C}^*$. In particular the point $(\sqrt{b}, \sqrt{a}, 0)$ of $\mathcal{C}$ lies on no line of $\mathcal{C}^*$. This implies that

$$\sqrt{b}u_0 + \sqrt{a}u_1 \neq 0, \tag{5.8}$$

for all $(u_0, u_1, u_2) \in \mathsf{GF}(q)^3 \setminus \{(0,0,0)\}$ satisfying

$$u_0^2 + cu_1^2 + bu_2^2 + u_1u_2 = 0. \tag{5.9}$$

By eliminating u_0 from (5.8) and (5.9), it follows that

$$b(cu_1^2 + bu_2^2 + u_1u_2) + au_1^2 \neq 0,$$

for all $(u_1, u_2) \in \mathsf{GF}(q)^2 \setminus \{(0,0)\}$. Now, this quadratic expression has no solutions $(u_1, u_2) \neq (0,0)$ if and only if $\mathrm{Tr}(bc+a) = 1$, so if and only if $\mathrm{Tr}(a) = 0$. We conclude that the $\frac{q-2}{2}$ elements of the subset $\mathcal{D}$ of $\mathcal{B}$ correspond to the $\frac{q-2}{2}$ elements of $\mathcal{C}_0 \setminus \{0\}$. Now the result immediately follows from the proof of Lemma 5.4.2.

2. It is possible to determine geometrically which conic of the pencil $\mathcal{B}$ is associated with the spread $\mathcal{S}_{[\delta]}$ of Offer. To that end, notice that $\mathcal{B}$ can also be represented as follows, by considering the explicit equations of S_1, S_2 and N for this particular pencil (see Section 5.4):

$$\begin{aligned}\mathcal{B} \equiv\ & (\delta+1)^2X_1^2 + \delta^2 rX_3^2 + \delta^4 X_5^2 \\ & + \delta(\delta+1)X_1X_3 + \delta^2 X_1X_5 + \delta^3 X_3X_5 = 0.\end{aligned}$$

Here r is the parameter of $\mathcal{B}$ and varies in $\mathsf{GF}(q) \cup \{\infty\}$ and the conic $\mathcal{C}$ associated with $\mathcal{S}_{[\delta]}$ has parameter $r = \delta$. Remind that $S_1 \equiv X_1 + aX_3 + a^2X_5 = 0$ and $S_2 \equiv X_1 + (a + \frac{\delta}{\delta+1})X_3 + (a^2 + \frac{\delta^2}{(\delta+1)^2})X_5 = 0$, where a was a solution of the equation $(\delta+1)X^2 + \delta X + \delta^2 = 0$. For convenience, we will use the element $b \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ which is a solution of $X^2 + X + (\delta+1) = 0$ instead of a. The second solution of this last equation is then $b+1$ and a is related to b as $a = \frac{\delta b}{\delta+1}$. By using this element b, the equations of S_1 and S_2 can be written as

$$\begin{aligned} S_1 &\equiv (\delta+1)^2X_1 + \delta(\delta+1)bX_3 + \delta^2b^2X_5 = 0, \text{ and} \\ S_2 &\equiv (\delta+1)^2X_1 + \delta(\delta+1)(b+1)X_3 + \delta^2(b+1)^2X_5 = 0. \end{aligned}$$

We determine the intersection of the line $X_1 = 0$ of $\mathcal{C}^*$ with N, S_1 and S_2:

$$\begin{aligned} S_1 \cap (X_1 = 0) &:= p_1 = (0, \delta b, \delta+1), \\ S_2 \cap (X_1 = 0) &:= p_2 = (0, \delta(b+1), \delta+1), \\ N \cap (X_1 = 0) &:= p_3 = (0, 0, 1). \end{aligned}$$

Now consider an arbitrary element of the pencil $\mathcal{B}$ and determine the intersections of this element with the line $X_1 = 0$, over $\mathsf{GF}(q^2)$. Therefore we must introduce yet another element $f \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$, namely a solution of $X^2 + X + r = 0$; the second solution is again $f+1$. The intersection points are then given by $p_4 = (0, \delta, f)$ and $p_4' = (0, \delta, f+1)$.

One can calculate the cross-ratios $\{p_1, p_2; p_3, p_4\}$ and $\{p_1, p_2; p_3, p_4'\}$:

$$\begin{aligned}\{p_1, p_2; p_3, p_4\} &= \frac{f+b}{f+b+1}, \text{ and} \\ \{p_1, p_2; p_3, p_4'\} &= \frac{f+b+1}{f+b}.\end{aligned}$$

The sum of both can be seen to be equal to $\frac{1}{r+\delta+1}$ and hence the parameter $r = \delta$ corresponding to $\mathcal{C}$ is the unique parameter for which the sum of those two cross-ratios is equal to 1. Moreover, the group H introduced in Lemma 5.4.2 acts transitively on the $q+1$ lines of $\mathcal{C}^*$ defined over $\mathsf{GF}(q)$, which is similar to H acting transitively on the points of $\mathcal{C} \setminus N$, and fixes S_1, S_2, N and $\mathcal{C}$. Hence the above property of the cross-ratios is independent of the choice of the line of $\mathcal{C}^*$ we use to determine the points p_1, p_2, p_3, p_4 and p_4' and it provides us with a geometric way to distinguish the conic $\mathcal{C}$ of $\mathcal{S}_{[\delta]}$ from the other elements of $\mathcal{B}$. Together with the results of Section 5.4, we now have a complete geometric description of the 1-system $\mathcal{S}_{[\delta]}$.

5.6 Characterization of proper locally hermitian, semiclassical 1-systems of $Q(6, q)$, for $q > 2$ and even

In this section, we will show that every proper 1-system of $Q(6, q)$, $q > 2$ and even, which is locally hermitian at some line $\tilde{L}$ and semiclassical, can be constructed from a rational normal cubic scroll as described in Section 5.4, and conversely. We shall not repeat the notion of a semiclassical 1-system of $Q(6, q)$ since this has been explained in detail in Section 1.2.2 of Chapter 1. Moreover, the definition is the same as for q odd, so that by now, the reader certainly is familiar with it.

Theorem 5.6.1 *If $q > 2$ is even, a proper locally hermitian 1-system of $Q(6, q)$ is semiclassical if and only if its associated flock arises from a rational normal cubic scroll $\mathscr{R}^3$ as described in Section 5.4.*

Proof.
Let $\mathcal{M}$ be a proper 1-system of $Q(6, q)$, $q > 2$ and even, which is locally hermitian at a line $\tilde{L}$ and consists of the q^2 reguli $R_1, R_2, \ldots, R_{q^2}$ through $\tilde{L}$. Denote its associated flock by $\mathcal{F}$. In the proof, we will use the same notation as earlier in this chapter.

First we assume that the flock $\mathcal{F}$ can be constructed from a rational normal cubic scroll $\mathscr{R}^3$ as explained in Section 5.4. Consider an arbitrary point $\tilde{x} \in \tilde{L}$ and a 4-dimensional subspace $\gamma \in T_{\tilde{x}}(Q(6,q))$, such that $\tilde{x} \notin \gamma$. Then $\gamma \cap Q(6,q) = Q(4,q)$ and we must show that the projection of $\mathcal{M}$ from $\tilde{x}$ along reguli onto γ is a classical ovoid $Q^-(3,q) \subseteq Q(4,q)$. We project in $\mathsf{PG}(6,q)$ from $\tilde{L}$ onto the 4-space β as in Section 5.1. The subspace $\langle \tilde{L}, \gamma \rangle = T_{\tilde{x}}(Q(6,q))$ is 5-dimensional and consequently $\langle \tilde{L}, \gamma \rangle \setminus \tilde{L}$ is projected onto a 3-space $\eta \subseteq \beta$ containing the plane $\pi = T_{\tilde{L}}(Q(6,q)) \cap \beta$. So $\eta^D := x$ is a point of the line $L = \pi^D$, which is uniquely determined by the point $\tilde{x} \in \tilde{L}$. As x is a point of L, there exists a unique generator N_x of $\mathscr{R}^3$ through x. Then $N_x^D := \alpha$ is a plane in η and thus it intersects the plane $\pi \subseteq \eta$ in a line S. If S contains a point p_t of $\mathcal{C}$, then the plane $S^D = \langle L, N_x \rangle$ is contained in the 3-space $D_t = p_t^D$, which contradicts the fact that $\mathscr{R}^3$ meets every 3-space D_t only in the line L. So $S \cap \mathcal{C} = \emptyset$ and the 4-space $\langle \tilde{L}, \alpha \rangle$ is disjoint from $\mathcal{C}$. This implies that $\langle \tilde{L}, \alpha \rangle \cap Q(6,q)$ does not contain any totally singular planes through $\tilde{L}$ and thus contains no totally singular planes at all. Now, since $\alpha \subseteq \eta$, it holds that $\langle \tilde{L}, \alpha \rangle \cap Q(6,q) \subseteq \langle \tilde{L}, \eta \rangle \cap Q(6,q) = T_{\tilde{x}}(Q(6,q)) \cap Q(6,q) = \tilde{x}Q(4,q)$ and consequently $\langle \tilde{L}, \alpha \rangle \cap Q(6,q)$ must be a cone $\tilde{x}Q^-(3,q)$ for some $Q^-(3,q) \subseteq Q(4,q)$.

Now assume that there exists a regulus R_i of $\mathcal{M}$ through $\tilde{L}$, the transversal of which through $\tilde{x}$ meets γ in a point, not on $Q^-(3,q)$. Then $\langle R_i \rangle \cap \langle \tilde{x}Q^-(3,q) \rangle$ is the line $\tilde{L}$ and so L_i, the projection of $R_i \setminus \{\tilde{L}\}$ onto β, is disjoint from α. After dualizing, this means that $\langle L_i^D, \alpha^D \rangle = \langle \pi_i, N_x \rangle$ is the whole 4-space β, so that $N_x \cap \pi_i = \emptyset$. This contradicts the fact that each N_x meets each plane π_i of the flock $\mathcal{F}$ in a point and we conclude that all transversals through $\tilde{x}$ of all reguli of $\mathcal{M}$ containing $\tilde{L}$, meet γ in a point of $Q^-(3,q)$. Hence the projection of $\mathcal{M}$ from $\tilde{x}$ along reguli is the classical ovoid $Q^-(3,q) \subseteq Q(4,q)$. As $\tilde{x} \in \tilde{L}$ was chosen arbitrarily, we have shown that $\mathcal{M}$ is semiclassical.

To prove the converse, we assume that $\mathcal{M}$ is semiclassical and we will construct a suitable rational normal cubic scroll starting from $\mathcal{M}$ step by step.

By Lemma 4.6.1, which has been shown in the previous chapter regardless of the parity of q, it holds that for each two reguli R_i, R_j of $\mathcal{M}$ through $\tilde{L}$, the elliptic quadric $\langle R_i, R_j \rangle \cap Q(6,q) = Q^-_{ij}(5,q)$ contains an $\mathcal{R}$-conic of reguli of $\mathcal{M}$ through $\tilde{L}$. From the proof of this lemma, it also becomes clear that every regulus R_i of $\mathcal{M}$ through $\tilde{L}$ is contained in exactly $q+1$ such elliptic quadrics $Q^-_{ij}(5,q)$. After projecting onto β and dualizing, this implies that each of the planes π_i of $\mathcal{F}$ in β contains exactly $q+1$ intersection points with planes

$\pi_j \in \mathcal{F}$, $j \neq i$, and that each such intersection point is contained in q planes $\pi_k \in \mathcal{F}$, π_i included. As a consequence, there are in total $\frac{q^2(q+1)}{q} = q(q+1)$ intersection points of plane pairs (π_i, π_j), $i \neq j$, of elements of $\mathcal{F}$.

As a next step, we show that the $q+1$ intersection points in an arbitrary plane $\pi_i \in \mathcal{F}$ form a $(q+1)$-arc. So assume that this is not the case, that there exist three collinear intersection points s_1, s_2, s_3 in some plane π_i. Then $\langle s_1, s_2, s_3\rangle^D := \xi$ is a plane through the line $L_i = \pi_i^D$ and contained in three 3-spaces $\langle L_i, L_j\rangle$, $\langle L_i, L_k\rangle$, $\langle L_i, L_l\rangle$, which are the projections onto β of $\langle R_i, R_j\rangle \setminus \tilde{L}$, $\langle R_i, R_k\rangle \setminus \tilde{L}$ and $\langle R_i, R_l\rangle \setminus \tilde{L}$ respectively, for R_i, R_j, R_k, R_l distinct reguli of $\mathcal{M}$ through $\tilde{L}$. So $\langle \tilde{L}, \xi\rangle$ is a 4-dimensional subspace containing the regulus R_i and $\langle \tilde{L}, \xi\rangle \cap Q(6,q)$ is the intersection of the three elliptic quadrics $Q^-_{ij}(5,q)$, $Q^-_{ik}(5,q)$ and $Q^-_{il}(5,q)$. If $\xi \cap \pi$ were a line, then L_i would have a point in common with π, a contradiction. So $\xi \cap \pi$ is a point and consequently $\langle \tilde{L}, \xi\rangle$ is not contained in any tangent hyperplane $T_{\tilde{x}}(Q(6,q))$, with $\tilde{x} \in \tilde{L}$. Since $\langle \tilde{L}, \xi\rangle \cap Q(6,q)$ contains the line $\tilde{L}$ and is contained in an elliptic quadric $Q^-_{ij}(5,q)$, there remains only one possibility: $\langle \tilde{L}, \xi\rangle \cap Q(6,q)$ is a non-singular parabolic quadric $Q(4,q)$. Now there exists a point $\tilde{x} \in \tilde{L}$ such that $\tilde{x}\mathcal{O}_{\tilde{x}} \subseteq Q^-_{ij}(5,q)$. Clearly $\tilde{x}\mathcal{O}_{\tilde{x}} \not\subseteq Q^-_{ir}(5,q)$, for $r = k, l$, and thus $\tilde{x}\mathcal{O}_{\tilde{x}} \cap Q^-_{ir}(5,q) = \tilde{x}\mathcal{C}_r$ where $\mathcal{C}_k, \mathcal{C}_l \subseteq \mathcal{O}_{\tilde{x}}$ are irreducible conics through the point $\tilde{L} \cap \mathcal{O}_{\tilde{x}}$. The conics $\mathcal{C}_r$ define $\mathcal{R}$-conics in $Q^-_{ir}(5,q)$, $r = k, l$, and it also holds $\mathcal{C}_k \neq \mathcal{C}_l$ for otherwise the quadrics $Q^-_{ik}(5,q)$ and $Q^-_{il}(5,q)$ would contain the same $\mathcal{R}$-conic and hence they would coincide. On the other hand, $\tilde{x}\mathcal{C}_r$ is contained in $Q^-_{ij}(5,q) \cap Q^-_{ir}(5,q) = Q(4,q)$ for $r = k, l$. This is a contradiction as $\tilde{x}\mathcal{C}_k \neq \tilde{x}\mathcal{C}_l$ and we conclude that our initial assumption was wrong. So no three intersection points in a plane π_i are collinear.

For each point $x \in L$ we shall construct a line N_x such that all points of $N_x \setminus \{x\}$ are intersection points of plane pairs (π_i, π_j), $i \neq j$, and such that all these lines are disjoint. In the first part of this proof, we have already observed that to each point $\tilde{x} \in \tilde{L}$, there corresponds a unique point x of L. Now, one can see that the converse also holds. To that end, consider an arbitrary point x on L. Then this point x dualizes to a 3-space x^D in β containing $L^D = \pi$. So $\langle \tilde{L}, x^D\rangle$ is 5-dimensional and meets $Q(6,q)$ in a quadric containing the singular quadric $\langle \tilde{L}, \pi\rangle \cap Q(6,q) = T_{\tilde{L}}(Q(6,q)) \cap Q(6,q) = \tilde{L}Q(2,q)$. As such, $\langle \tilde{L}, x^D\rangle \cap Q(6,q)$ must be a cone $\tilde{x}Q(4,q)$ for a unique point $\tilde{x} \in \tilde{L}$; this point $\tilde{x}$ is the unique point on $\tilde{L}$, associated with x.

Consider a point $x \in L$ and the corresponding point $\tilde{x} \in \tilde{L}$. As $\mathcal{M}$ is semiclassical, there exists a unique cone $\tilde{x}\mathcal{O}_{\tilde{x}}$, where $\mathcal{O}_{\tilde{x}}$ is an elliptic quadric $Q^-(3,q)$, containing the transversals through $\tilde{x}$ of all reguli of $\mathcal{M}$ on $\tilde{L}$. The projection onto β of this $\langle \tilde{x}\mathcal{O}_{\tilde{x}}\rangle \setminus \tilde{L}$ is a plane $\alpha_{\tilde{x}}$, which meets π in a line, disjoint from $\mathcal{C}$. Every point of $\alpha_{\tilde{x}}$, not in π, lies on a unique line L_i, which is the projection onto β of a regulus R_i of $\mathcal{M}$ through $\tilde{L}$. The dual $\alpha_{\tilde{x}}^D$ of $\alpha_{\tilde{x}}$ in β is a line N_x through the point $x \in L$ and every plane $\pi_i = L_i^D$ meets N_x in a point, not on L. On $N_x \setminus L$ there can be at most q intersection points of plane pairs (π_j, π_k), $j \neq k$, and every such intersection point is known to lie in q different planes π_i. As there are q^2 planes π_i, every point of $N_x \setminus L$ must be an intersection point of a pair of planes of $\mathcal{F}$.

Now assume that $N_x \cap N_y \neq \emptyset$ for some distinct $x, y \in L$. Then $\langle \alpha_{\tilde{x}}, \alpha_{\tilde{y}} \rangle$ is 3-dimensional and the planes $\alpha_{\tilde{x}}$ and $\alpha_{\tilde{y}}$ meet in a line T. So $\langle \tilde{x}\mathcal{O}_{\tilde{x}} \rangle \cap \langle \tilde{y}\mathcal{O}_{\tilde{y}} \rangle$ is the 3-space $\langle \tilde{L}, T \rangle$ and $\tilde{x}\mathcal{O}_{\tilde{x}}$ and $\tilde{y}\mathcal{O}_{\tilde{y}}$ are contained in a common 5-space. But this means that $\mathcal{M}$ is completely contained in this 5-space, a contradiction. So the set of lines $\{N_x \mid x \in L\}$ has the desired properties.

Finally, we show that the intersection points of the plane pairs from $\mathcal{F}$ form a rational normal cubic scroll with directrix line L.
Consider an arbitrary regulus R_0 of $\mathcal{M}$ through $\tilde{L}$ and let $\pi_0 \in \mathcal{F}$ be the corresponding plane in β. In π_0, let x_0, r_0, s_0, t_0 and u_0 be arbitrary, but distinct, intersection points on the lines N_x, N_r, N_s, N_t and N_u respectively, with $x, r, s, t, u \in L$. As $q > 2$, such distinct intersection points exist. Further, choose a 3-dimensional subspace $\mathsf{PG}(3,q)$ of β not containing x_0. We project the set of the intersection points, minus x_0, and L from x_0 onto $\mathsf{PG}(3,q)$. There are q planes $\pi_j \in \mathcal{F}$ through x_0, which, in pairs, only share the point x_0; these planes, minus x_0, are projected onto q disjoint lines $L'_0, L'_1, \ldots, L'_{q-1}$ in $\mathsf{PG}(3,q)$, where L'_0 corresponds to π_0. The projection of L onto $\mathsf{PG}(3,q)$ is a line L', which is also disjoint from the lines $L'_0, L'_1, \ldots, L'_{q-1}$. Every line N_z, $z \in L \setminus \{x\}$, is also projected onto a line N'_z of $\mathsf{PG}(3,q)$ and as N_z meets all planes π_i in a point, N'_z meets every line of $\{L'_0, L'_1, \ldots, L'_{q-1}, L'\}$ in a point. Hence these $q+1$ lines form a regulus in $\mathsf{PG}(3,q)$ and the lines N'_z, $z \in L \setminus \{x\}$, belong to the opposite regulus. Now there exists a unique tangent line $V^i_{x_0}$ at x_0 to the $(q+1)$-arc of intersection points in every plane π_i through x_0. These tangent lines, minus x_0, are projected onto the points of the missing line N' of the opposite regulus of $\{L'_0, L'_1, \ldots, L'_{q-1}, L'\}$, where $N_x \setminus \{x_0\}$ yields the point $N' \cap L'$ after projection. So we obtain the following relations concerning cross-ratios:

$$\begin{aligned} \{x_0r_0, x_0s_0; x_0t_0, x_0u_0\} &= \{L'_0 \cap N'_r, L'_0 \cap N'_s; L'_0 \cap N'_t, L'_0 \cap N'_u\} \\ &= \{L' \cap N'_r, L' \cap N'_s; L' \cap N'_t, L' \cap N'_u\} \\ &= \{L \cap N_r, L \cap N_s; L \cap N_t, L \cap N_u\} \\ &= \{r, s; t, u\}. \end{aligned} \tag{5.10}$$

Similarly, it holds that

$$\{V^0_{x_0}, x_0s_0; x_0t_0, x_0u_0\} = \{x, s; t, u\}. \tag{5.11}$$

Let $y_0 \in \pi_0$ be one more intersection point, distinct from x_0, and repeat the previous argument, with the convention that $y_0 r_0 = V^0_{y_0}$ in case $y_0 = r_0$ and similarly for s_0, t_0, u_0. Then we find that

$$\{y_0 r_0, y_0 s_0; y_0 t_0, y_0 u_0\} = \{r, s; t, u\},$$

and thus $\{x_0 r_0, x_0 s_0; x_0 t_0, x_0 u_0\} = \{y_0 r_0, y_0 s_0; y_0 t_0, y_0 u_0\}$. So, in π_0, there arises a linear projectivity between the lines through x_0 and the lines through y_0; see for instance Hughes and Piper [28]. Now, by the theorem of Steiner (see for instance [27, Chapter 6]), it is immediately clear that the $(q+1)$-arc in π_0 is a conic. Since π_0 was chosen arbitrarily, the same holds for all planes π_i.

From (5.10) and (5.11), it also follows that there exists a linear isomorphism which maps the intersection point x_0 of N_x and π_0 onto the corresponding intersection point x of N_x and L, with x any point of L. By Semple and Roth [55, p. 133], this implies that the intersection points of the plane pairs (π_i, π_j), $i \neq j$, together with L, form a rational normal cubic scroll $\mathscr{R}^3$ with directrix line L. This proves the characterization result. $\square$

Remarks

For $q = 2$, the 1-systems of $Q(6, 2)$ have already been classified. This was done by computer by Hamilton and Mathon in [21]. In fact, they classify the 1-systems of $W_5(2)$, but as q is even, this is equivalent to classifying the 1-systems of $Q(6, 2)$. Up to isomorphism, they find two distinct 1-systems, namely the two non-isomorphic examples of spreads of an induced $Q^-(5, 2)$ on $Q(6, 2)$, one of which is the hermitian spread of $Q^-(5, 2)$.

If one combines the previous theorem with the result, achieved in Theorem 5.4.4, a complete charaterization and determination of all proper locally hermitian, semiclassical 1-systems of $Q(6, q)$, q even and $q > 2$ follows, for Theorem 5.4.4 provides us with a determination of all irreducible conics from which a suitable rational normal cubic scroll can be constructed.

The characterization results for q odd and q even compared

The characterization result we obtained in Theorem 5.6.1 strongly resembles to the one for q odd, as has been found in Theorem 4.6.2 of Chapter 4. In both cases, the proper locally hermitian 1-systems of $Q(6, q)$ which are semiclassical, can be constructed from a rational normal cubic scroll. This similarity might seem surprising at first sight, because the method to construct the 1-systems and the way a flock is associated with them, are different. There exists nevertheless a deeper link between the two construction methods, for a variation on the method applied in the even case, can also be used in the odd case and then the relation with the method used for q odd becomes clear. The approach for q even however, is to a certain extent artificial, while the method for the odd case, which uses the polarity of $Q(6,q)$, results from a more straightforward reasoning. This observation partly explains why we have chosen not to give a unified description for both cases. Another reason is that a unified theory would not reflect the natural course of the research, because the theory for q odd has been developed first and it was not until some months later that the even case was tackled, with less elegant techniques.

As a closure to this section, we try to give an explanation for the striking similarities between the results for q even and those for q odd.
Let q be odd and suppose that $\mathcal{M}$ is a 1-system of $Q(6,q)$, which is locally hermitian at some line $\tilde{L} \in \mathcal{M}$. Consider a 4-dimensional subspace β in $\mathsf{PG}(6,q)$, disjoint from $\tilde{L}$, and assume that $\beta \cap Q(6,q)$ is a non-singular quadric $Q(4,q)$; so $\beta = K^\perp$ for some line K of $\mathsf{PG}(6,q)$ with $|K \cap Q(6,q)| \in \{0,2\}$. Now we project the 1-system, minus $\tilde{L}$, from $\tilde{L}$ onto β. Then the q^2 reguli $R_1, R_2, \ldots, R_{q^2}$ of $\mathcal{M}$ through $\tilde{L}$, minus $\tilde{L}$, are projected onto q^2 lines $L_1, L_2, \ldots, L_{q^2}$ in β. The projection of $\tilde{L}^\perp \setminus \tilde{L}$ from $\tilde{L}$ onto β is a plane π, intersecting $Q(6,q)$ in a non-singular conic $\mathcal{C}$. The lines $L_1, L_2, \ldots, L_{q^2}$ have the property that they are pairwise disjoint, and in addition every line L_i is disjoint from π. Also, the 3-space $\langle L_i, L_j \rangle$ contains no point of $\mathcal{C}$, for all $i, j \in \{1, 2, \ldots, q^2\}$ with $i \neq j$. In β, we now consider the dual structure, but instead of the standard duality, which maps a point with coordinates $(u_0, u_1, \ldots, u_4)$ onto the hyperplane with equation $u_0X_0 + u_1X_1 + \cdots + u_4X_4 = 0$ and vice versa, we apply the restriction to β of the polarity $\perp$ of $Q(6,q)$, which is also a duality. This restriction is in fact the polarity defined by $Q(4,q)$ in β. As such, we obtain q^2 planes $\pi_i := L_i^\perp \cap \beta$; the plane π yields a line $L := \pi^\perp \cap \beta$ and the $q+1$ points $x_0, x_1, \ldots, x_q$ of $\mathcal{C}$ produce $q+1$ 3-dimensional subspaces $D_t := x_t^\perp \cap \beta$, $t = 0, 1, \ldots, q$, forming a dual conic cone $\mathcal{K}$ in β with vertex line L. By the properties of the lines $L_1, L_2, \ldots, L_{q^2}$, one deduces that $\pi_i \cap \pi_j$ is a point for all $i \neq j$, L is disjoint from all q^2 planes $\pi_1, \pi_2, \ldots, \pi_{q^2}$ and eventually, the point $\pi_i \cap \pi_j$ does not belong to any of the 3-spaces $D_0, D_1, \ldots, D_q$, for all $i, j \in \{1, 2, \ldots, q^2\}$ with $i \neq j$. Hence a flock $\mathcal{F} := \{\pi_1, \pi_2, \ldots, \pi_{q^2}\}$ of a dual conic cone $\mathcal{K}$ can be constructed from a locally hermitian 1-system of $Q(6,q)$ for q odd as well, by using an approach that is very similar to the one for q even. Note furthermore that this construction can be reversed.
But in addition, there exists the following relation between the flock of the dual conic cone $\mathcal{K}$ and the i-flock as constructed in Chapter 4. The dual conic cone $\mathcal{K}$ intersects π in a dual conic $\mathcal{C}^*$, consisting of the tangents of $\mathcal{C}$ at $x_0, x_1, \ldots, x_q$. The fact that the planes $\pi_1, \pi_2, \ldots, \pi_{q^2}$ define a flock of $\mathcal{K}$

now implies that they also define an i-flock of the quadratic cone with L as vertex and $\mathcal{C}$ as base. Finally, the planes π_i, $i = 1, 2, \ldots, q^2$, can be seen as the projections from K onto $K^\perp$ of the 4-spaces $L_i^\perp$, without K, and similarly the 3-spaces $D_0, D_1, \ldots, D_q$ are the projections of $x_i^\perp \setminus K$, $i = 0, 1, \ldots, q$, from K onto $K^\perp$.
On the other hand, one can also project from K onto the 4-dimensional subspace $\tilde{L}^\perp$. In this way, one obtains q^2 planes $\tilde{\pi}_i := L_i^\perp \cap \tilde{L}^\perp = \langle R_i \rangle^\perp$, constituting an i-flock of the quadratic cone $\tilde{L}^\perp \cap Q(6,q)$ with vertex $\tilde{L}$ and base $\mathcal{C}$. This is exactly the i-flock that has been constructed in Chapter 4. Moreover, the planes $\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_{q^2}$ form a flock of the dual conic cone consisting of the 3-spaces $x_i^\perp \cap \tilde{L}^\perp$, $i = 0, 1, \ldots, q$, which can also be considered as the projections of the 5-spaces $x_i^\perp$, without K, from K onto $\tilde{L}^\perp$.
From the above, it follows that the considered i-flocks in β and $\tilde{L}^\perp$, must be equivalent, because they are the projections from K of a structure of subspaces containing K, minus K, onto distinct 4-dimensional subspaces β and $\tilde{L}^\perp$, which are both disjoint from K.

These observations explain the similarities between the results for q even and q odd. In particular, it is now clear why the positions of the cubic scrolls with respect to the quadratic cone, respectively dual conic cone, are similar. In the even case the scroll contains a conic $\mathcal{C}'$ which is tangent to two distinct lines of the base $\mathcal{C}^*$ of the dual conic cone, where the tangent points as well as the tangent lines are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. In addition, the base $\mathcal{C}^*$ of the dual conic cone and $\mathcal{C}'$ do not have a point in common over $\mathsf{GF}(q)$. In the odd case the cubic scroll contains a conic $\mathcal{C}'$ which is doubly tangent to the base $\mathcal{C}$ of the quadratic cone at two distinct conjugate points with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ and consists entirely of internal points of $\mathcal{C}$. But if the i-flock is considered as a flock of the corresponding dual conic cone, with vertex L and as base the tangents of the conic $\mathcal{C}$ which form a dual conic $\mathcal{C}^*$ in π, then $\mathcal{C}'$ and $\mathcal{C}^*$ are in the same position as for q even. So in spite of the different approaches, it is not really surprising that the characterization results of Theorem 4.6.2 and Theorem 5.6.1 show a great analogy.

5.7 Applications

The new 1-systems of $Q(6,q)$ we have found for $q > 2$ even, have applications in a wide range of fields. As q is even, we can project $Q(6,q)$ from its nucleus onto a 5-dimensional subspace $\mathsf{PG}(5,q)$, not containing the nucleus, and in this way we obtain a symplectic polar space $W_5(q)$. A 1-system of $Q(6,q)$ is then projected onto a 1-system of $W_5(q)$, which implies that we have found new 1-systems of $W_5(q)$ as well. By [57], every 1-system of $W_5(q)$ defines a strongly regular graph and a linear projective 2-weight code, while it has been shown in Theorem 2.2.1 that every 1-system of $W_5(q)$ is an SPG regulus and hence a semipartial geometry can be constructed from it. So, our new class of 1-systems yields examples of strongly regular graphs, linear projective 2-weight codes, SPG reguli and semipartial geometries. In all cases, these objects have known parameters, since the parameters are independent of the choice of the 1-system of $W_5(q)$.

Concerning the semipartial geometries, we shall show in a later chapter that different 1-systems of $W_5(q)$, arising from our construction, yield non-isomorphic semipartial geometries if and only if they are not isomorphic under the subgroup of $\mathsf{P\Gamma L}(6,q)$ which stabilizes the symplectic polar space $W_5(q)$. From Theorem 5.5.1, it then follows that the class of semipartial geometries constructed from these 1-systems contains new examples.

Another application is of a completely different kind and involves the classical generalized hexagon $\mathsf{H}(q)$. If $q \equiv 1 \bmod 3$, then $\mathsf{H}(q)$ has a unique locally hermitian, semiclassical spread which is not hermitian. For q odd, this spread is denoted by $\mathcal{S}_{[9]}$ and was first described in Bloemen, Thas and Van Maldeghem [4], and for q even, it is the spread $\mathcal{S}_{[\delta]}$ which has been independently discovered by Offer [46] and Cardinali, Lunardon, Polverino and Trombetti [10]. In Chapter 4, we succeeded in giving a geometric description for $\mathcal{S}_{[9]}$ and in the current chapter, the same has been achieved for $\mathcal{S}_{[\delta]}$. Here we will explain that it is possible to reconstruct the hexagon $\mathsf{H}(q)$, $q \equiv 1 \bmod 3$, from its unique locally hermitian, semiclassical spread $\mathcal{S}$, which is not hermitian.

Suppose that $\mathcal{S}$ is a 1-system of the parabolic quadric $Q(6,q)$, $q \equiv 1 \bmod 3$, and assume that $\mathcal{S}$ is isomorphic to $\mathcal{S}_{[9]}$ for q odd or to $\mathcal{S}_{[\delta]}$ for q even. If $R_1, R_2, \ldots, R_{q^2}$ denote the q^2 reguli of $\mathcal{S}$ through the line L at which $\mathcal{S}$ is locally hermitian, then for every $i \neq j$, $\langle R_i, R_j\rangle \cap Q(6,q)$ is an elliptic quadric $Q^-_{ij}(5,q)$ containing an $\mathcal{R}$-conic of reguli of $\mathcal{S}$ through L, by Lemma 4.6.1. As has been explained in Section 1.2.2 when defining the notion of an $\mathcal{R}$-conic, the two reguli R_i and R_j uniquely define a hermitian spread $\mathcal{S}_{ij}$ of $Q^-_{ij}(5,q)$, containing the q reguli of the $\mathcal{R}$-conic in $Q^-_{ij}(5,q)$. As such we obtain $q(q+1)$ hermitian spreads $\mathcal{S}_{ij}$ of distinct elliptic quadrics $Q^-_{ij}(5,q)$, $i \neq j$. Now, by the foregoing theory, $\mathcal{S}$ arises from a rational normal cubic scroll $\mathscr{R}^3$ in $\mathsf{PG}(4,q)$ and every $Q^-_{ij}(5,q)$ corresponds to a point of $\mathscr{R}^3$. Also, if R_i, R_j, R_k, R_l are distinct reguli of $\mathcal{S}$ through L, such that $Q^-_{ij}(5,q)$ and $Q^-_{kl}(5,q)$ do not coincide, then these elliptic quadrics define distinct points on $\mathscr{R}^3$. The q^2 non-singular conics on $\mathscr{R}^3$ correspond to the q^2 reguli of $\mathcal{S}$ through L, and a conic on $\mathscr{R}^3$ contains a point p of $\mathscr{R}^3$ if and only if the corresponding regulus is contained in the elliptic quadric on $Q(6,q)$ defined by p. It is known that the cubic scroll $\mathscr{R}^3$ has the property that every pair of points on $\mathscr{R}^3$ is contained in exactly one non-singular conic of points of $\mathscr{R}^3$. This implies that any two distinct quadrics $Q^-_{ij}(5,q)$ and $Q^-_{kl}(5,q)$ have exactly one regulus of $\mathcal{S}$ through L in common. Also, $|\mathcal{S}_{ij} \cap \mathcal{S}_{kl}| = q+1$ as $\langle Q^-_{kl}(5,q)\rangle$ contains exactly $q+1$ lines of $\mathcal{S}_{ij}$. Thus the total number of lines, different from L, of the $q(q+1)$ hermitian spreads $\mathcal{S}_{ij}$ equals $q(q+1)(q^3-q^2)+q^3 = q^5$. The classical generalized hexagon $\mathsf{H}(q)$ we want to reconstruct, contains all lines of $\mathcal{S}$, and these lines form a spread of $\mathsf{H}(q)$. It is also known that every elliptic quadric $Q^-(5,q) \subseteq Q(6,q)$ contains exactly q^3+1 lines of $\mathsf{H}(q)$ and that these lines form a hermitian spread of $Q^-(5,q)$. This property applies to the elliptic quadrics $Q^-_{ij}(5,q)$, $i \neq j$, as well and in that case, the hermitian spread contains R_i and R_j, so it must be the unique hermitian spread $\mathcal{S}_{ij}$, defined by R_i and R_j. Consequently, all q^5 lines, different from L, of the $q(q+1)$ spreads $\mathcal{S}_{ij}$ belong to $\mathsf{H}(q)$ and in addition, they are all opposite L in $\mathsf{H}(q)$. Thus we have reconstructed from $\mathcal{S}$ the set of all q^5 lines of $\mathsf{H}(q)$ which are opposite L. Since it is possible to construct all lines of the hexagon from L and these q^5 lines, we can reconstruct $\mathsf{H}(q)$, starting from $\mathcal{S}$. As a geometric construction of the spread $\mathcal{S}$ has been elaborated in Chapters 4 and 5, this implies that we now dispose of a geometric construction of the classical generalized hexagon $\mathsf{H}(q)$ if $q \equiv 1 \bmod 3$.

Chapter 6

Locally hermitian 1-systems of $Q^+(7,q)$

As is obvious from Chapters 4 and 5, locally hermitian 1-systems of quadrics are very fruitful objects for research. In this chapter, we will leave the parabolic quadrics behind and turn our attention to locally hermitian 1-systems of the hyperbolic quadric $Q^+(7,q)$ in $\mathsf{PG}(7,q)$, where no distinction has to be made according to the parity of q. The theory that will be elaborated here, displays some similarities to what we have done for locally hermitian 1-systems of $Q(6,q)$. Again, we shall associate a kind of flock with the 1-system and use this flock as the base for further study. As in Chapters 4 and 5, special attention will be paid to the case of the locally hermitian 1-systems of $Q^+(7,q)$, which are in addition semiclassical. For 1-systems of $Q^+(7,q)$ as well as for 1-systems of $Q(6,q)$, a characterization of these examples in terms of their flock is possible. Moreover, we will see that there exists a wide variety in the class of proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$. An explicit determination of all examples however, seems out of reach. Finally, it is checked which locally hermitian, semiclassical 1-systems of $Q^+(7,q)$ were previously known and they are distinguished from the other examples appearing in this chapter, by the fact that their associated flock is of a special kind.

6.1 Flocks and locally hermitian 1-systems of $Q^+(7,q)$

6.1.1 The construction

Let $\mathcal{M}$ be a 1-system of the hyperbolic quadric $Q^+(7,q)$, which is locally hermitain at a line $L \in \mathcal{M}$. Then $\mathcal{M}$ consists of q^2 reguli $R_1, R_2, \ldots, R_{q^2}$ through L, since $|\mathcal{M}| = q^3 + 1$ by the definition in Section 1.2.2. Regardless the parity of q, a polarity $\perp$ is associated with $Q^+(7,q)$. This polarity maps each 3-space $\langle R_i \rangle$ onto a 3-space $\langle R_i \rangle^\perp := \gamma_i$, intersecting $Q^+(7,q)$ in a hyperbolic quadric $Q_i^+(3,q)$. In this way we obtain q^2 hyperbolic quadrics $Q_i^+(3,q)$, $i = 1, 2, \ldots, q^2$, in the 5-dimensional subspace $L^\perp$, where $L^\perp \cap Q^+(7,q)$ is a cone with vertex L and base a hyperbolic quadric $Q^+(3,q)$. It holds that L is disjoint from all q^2 quadrics $Q_i^+(3,q)$, $i = 1, 2, \ldots, q^2$.

As $\mathcal{M}$ is a 1-system of $Q^+(7,q)$, it is known that for all $i, j \in \{1, 2, \ldots, q^2\}$, $i \neq j$, $\langle R_i, R_j \rangle \cap Q^+(7,q)$ is an elliptic quadric $Q_{ij}^-(5,q)$, as has been explained in Section 1.2.2 for locally hermitian 1-systems of quadrics in general. This fact implies that $\langle R_i \rangle^\perp \cap \langle R_j \rangle^\perp$ is a line K_{ij}, external to $Q^+(7,q)$. So the q^2 hyperbolic quadrics $Q_i^+(3,q)$ are pairwise disjoint and hence they form a partition of the cone $LQ^+(3,q)$ without its vertex line L. This is an obvious generalization of a flock of a quadratic cone, and therefore we introduce the following definitions.

Definitions

A *flock* of a cone $LQ^+(3,q)$ in $\mathsf{PG}(5,q)$ with a line L as vertex and a hyperbolic quadric $Q^+(3,q)$ as base, is a partition of the cone without its vertex line into q^2 hyperbolic quadrics $Q_i^+(3,q)$, $i = 1, 2, \ldots, q^2$. We will also use the term *flock* for the set of q^2 3-dimensional subspaces $\gamma_i := \langle Q_i^+(3,q) \rangle$, $i = 1, 2, \ldots, q^2$.

The cone $LQ^+(3,q)$ itself will be called a *hyperbolic cone* in $\mathsf{PG}(5,q)$ *with line vertex* L.

Conversely, let $\mathcal{F} := \{\gamma_i \mid i = 1, 2, \ldots, q^2\}$ be a flock of a hyperbolic cone $LQ^+(3,q)$ in $\mathsf{PG}(5,q)$ with line vertex L. Embed $LQ^+(3,q)$ in a hyperbolic quadric $Q^+(7,q)$ with polarity $\perp$, and so that $L^\perp \cap Q^+(7,q) = LQ^+(3,q)$. Then for each element $\gamma_i \in \mathcal{F}$ it holds that $\gamma_i^\perp \cap Q^+(7,q)$ is a 3-dimensional hyperbolic quadric, say $Q_i^+(3,q)'$. Denote the regulus of $Q_i^+(3,q)'$ which contains the line L by R_i. Then we show that $\mathcal{M} := \bigcup_{i=1}^{q^2} R_i$ is a 1-system of $Q^+(7,q)$. The only property that has to be checked is that a generator G containing a line M of $\mathcal{M}$, is disjoint from all lines of $\mathcal{M} \setminus \{M\}$. Clearly, every generator through L is disjoint from all lines of $\mathcal{M} \setminus \{L\}$, so assume that $M \neq L$. If M belongs to the regulus R_i, then it is obvious that G has no point in common with the lines of $R_i \setminus \{M\}$. So suppose that $G \cap M'$ is at least a point, for some line $M' \in \mathcal{M}$, belonging to another regulus R_j, $j \neq i$. Then G intersects the subspace $\langle R_i, R_j \rangle$ at least in a totally singular plane. But by the definition of a flock $\mathcal{F}$ of $LQ^+(3, q)$, it holds that $\gamma_i \cap \gamma_j = \langle R_i, R_j \rangle^\perp$ is an external line K_{ij} of $Q^+(7, q)$. Thus $\langle R_i, R_j \rangle \cap Q^+(7, q) = K_{ij}^\perp \cap Q^+(7, q)$ must be an elliptic quadric $Q^-_{ij}(5, q)$, and consequently $G \cap \langle R_i, R_j \rangle$ cannot contain a totally singular plane. It follows that $\mathcal{M}$ is indeed a 1-system of $Q^+(7, q)$, which is, by construction, localy hermitian at the line L. Hence we have shown the following theorem.

Theorem 6.1.1 *With every locally hermitian 1-system of $Q^+(7, q)$, a flock of a hyperbolic cone $LQ^+(3, q)$ with line vertex L in $\mathsf{PG}(5, q)$ is associated, and conversely.*

In view of this theorem, it is meaningful to study locally hermitian 1-systems of $Q^+(7, q)$ in terms of their associated flock. Besides, one should bear in mind that there is an obvious resemblance between the foregoing construction and the construction of an i-flock from a locally hermitian 1-system of $Q(6, q)$, q odd. Since every locally hermitian 1-system $\mathcal{M}$ of $Q(6, q)$, q odd, can be considered as a locally hermitian 1-system of $Q^+(7, q)$, provided that $Q(6, q)$ is embedded in $Q^+(7, q)$, it must be possible to reconstruct the i-flock corresponding to $\mathcal{M}$ from its flock. This is indeed the case: if p is the polar point of $\langle Q(6, q) \rangle$ with respect to $Q^+(7, q)$, then all elements of the flock $\mathcal{F} := \{\gamma_1, \gamma_2, \ldots, \gamma_{q^2}\}$ of $\mathcal{M}$ contain the point p and the i-flock of $\mathcal{M}$ consists of all conics $\mathcal{C}_i := \gamma_i \cap Q(6, q)$, $i = 1, 2, \ldots, q^2$. The notion of a flock we have introduced here is thus a straightforward generalization of the i-flock from Chapter 4, but it will lead to interesting results for proper locally hermitian 1-systems of $Q^+(7, q)$ and it has the advantage that it can also be applied for q even.

6.1.2 A special case

We discuss an example of a flock $\mathcal{F}$ of a hyperbolic cone $LQ^+(3, q)$ with line vertex L in $\mathsf{PG}(5, q)$. Consider a 3-dimensional subspace γ_1 in $\mathsf{PG}(5, q)$, disjoint from L, and write $\gamma_1 \cap LQ^+(3, q) := Q_1^+(3, q)$. Let $\{K_0, K_1, \ldots, K_q\}$ be a regulus in γ_1 such that $K_i \cap Q_1^+(3, q) = \emptyset$ for all $i = 0, 1, \ldots, q$. Such a regulus always exists; in a regular spread of γ_1 containing one of the reguli of $Q_1^+(3, q)$ for instance, there are several suitable choices for the regulus $\{K_0, K_1, \ldots, K_q\}$. Let T_x, T_y and T_z be three transversals of the regulus

$\{K_0, K_1, \ldots, K_q\}$ and consider the planes $\langle x, T_x\rangle$, $\langle y, T_y\rangle$ and $\langle z, T_z\rangle$ for arbitrary, but distinct, points x, y, z on L. Then, by Hirschfeld and Thas [26, Theorem 25.6.1], the set of lines having a non-empty intersection with the mutually skew planes $\langle x, T_x\rangle$, $\langle y, T_y\rangle$ and $\langle z, T_z\rangle$ forms a system of maximal spaces of a Segre variety[6] $\mathscr{S}_{1;2}$. Clearly, the lines $L, K_0, K_1, \ldots, K_q$ all belong to this system of maximal spaces of $\mathscr{S}_{1;2}$. We denote the set of lines forming one system of maximal spaces of this Segre variety, without L, by $\mathcal{I}$, which implies that $|\mathcal{I}| = q^2 + q$. By the properties of the Segre variety $\mathscr{S}_{1;2}$ there exists for every point u on L exactly one plane $\langle u, T_u\rangle$, with T_u a transversal of $\{K_0, K_1, \ldots, K_q\}$, having a point in common with all lines of $\mathcal{I}$; these $q+1$ planes $\langle u, T_u\rangle$, u on L, constitute the second system of maximal spaces of $\mathscr{S}_{1;2}$. We will now use the set $\mathcal{I}$ to construct a flock $\mathcal{F}$ of $LQ^+(3,q)$.

Consider a fixed point x on L and denote the plane $\langle x, T_x\rangle$ by π. If $A \in \mathcal{I}$ is arbitrary, let a stand for the unique intersection point of A and π and let l similarly be the point $L \cap \pi$. Then for any two lines A and B of $\mathcal{I}$, the line ab is a line of π; consider a third line $C \in \mathcal{I} \cup \{L\}$ which meets ab in a point. If C were not contained in the 3-space $\langle A, B\rangle$, then $\langle A, B, C\rangle$ would be 4-dimensional and contain all planes $\langle u, T_u\rangle$, $u \in L \setminus \{x\}$. This is a contradiction and hence the 3-space $\langle A, B\rangle$ contains all $q+1$ lines of $\mathcal{I} \cup \{L\}$ which intersect π in a point of the line ab. This implies that the 3-space $\langle A, B\rangle$ contains exactly $q+1$ lines of $\mathcal{I}$ whenever the line ab does not contain the point l. We define $\mathcal{F}$ to be the set of 3-spaces $\{\langle A, B\rangle \mid A, B \in \mathcal{I} \text{ and } l \notin ab\}$ and we show that $\mathcal{F}$ is a flock of $LQ^+(3,q)$.

Firstly, the number of elements of $\mathcal{F}$ is q^2, which is the correct number of elements for a flock. For distinct $\langle A, B\rangle$ and $\langle C, D\rangle$ in $\mathcal{F}$, the lines ab and cd have a point in common, say f, which is the point $F \cap \pi$ for some line $F \in \mathcal{I}$. Clearly $\langle A, B, C, D\rangle$ is the whole $\mathsf{PG}(5,q)$. So the 3-spaces $\langle A, B\rangle$ and $\langle C, D\rangle$ intersect in a line, namely the line $F \in \mathcal{I}$. As the regulus $\{K_0, K_1, \ldots, K_q\}$ was chosen to be disjoint from $Q_1^+(3,q)$, any plane $\langle u, T_u\rangle$, with u on L, has exactly one point in common with $LQ^+(3,q)$, namely the point $u \in L$. Since all points on the lines of $\mathcal{I}$ are points of planes $\langle u, T_u\rangle$, but not on L, we find that all lines of $\mathcal{I}$ are disjoint from $LQ^+(3,q)$, so $\langle A, B\rangle$ and $\langle C, D\rangle$ meet in an external line of $LQ^+(3,q)$. Finally, every 3-space $\langle A, B\rangle \in \mathcal{F}$ is disjoint from L, and it follows that the elements of $\mathcal{F}$ form a partition of $LQ^+(3,q)\setminus L$ in q^2 hyperbolic quadrics $Q_i^+(3,q)$, $i = 1, 2, \ldots, q^2$, so that $\mathcal{F}$ is a flock of the hyperbolic cone $LQ^+(3,q)$.

In the sequel, we will always use the notation $\mathcal{I}$ for the set of lines of intersection of pairs of elements of the flock $\mathcal{F}$.

[6] For the definition and properties of Segre varieties, the reader is referred to Chapter 25 of *"Generalized Galois Geometries"* by Hirschfeld and Thas [26].

Remark
The system of lines $\mathcal{I} \cup \{L\}$, which are maximal subspaces of the Segre variety $\mathscr{S}_{1;2}$, contains a lot of reguli. If we consider two arbitrary lines of $\mathcal{I} \cup \{L\}$, then they span a 3-dimensional subspace of $\mathsf{PG}(5,q)$ which contains exactly $q+1$ lines of $\mathcal{I} \cup \{L\}$, as we have seen above, and meets every plane $\langle u, T_u \rangle$, u on L, in a line. Hence these two lines determine a unique regulus of lines of $\mathcal{I} \cup \{L\}$. In particular, the $q+1$ lines of $\mathcal{I}$ in an element $\gamma_i \in \mathcal{F}$, $i = 1, 2, \ldots, q^2$, form a regulus in γ_i, disjoint from $LQ^+(3,q)$.

The locally hermitian 1-system of $Q^+(7,q)$, associated with a flock $\mathcal{F} = \{\gamma_1, \gamma_2, \ldots, \gamma_{q^2}\}$ arising from a Segre variety $\mathscr{S}_{1;2}$ as described above, has special properties. Firstly, the lines of $\mathcal{I}$, which are also the lines of intersection of the elements of the flock $\mathcal{F}$, are pairwise disjoint. This implies that every two distinct elliptic quadrics $Q^-_{ij}(5,q) = (\gamma_i \cap \gamma_j)^\perp$ and $Q^-_{kl}(5,q) = (\gamma_k \cap \gamma_l)^\perp$ span the whole 7-dimensional projective space $\mathsf{PG}(7,q) = \langle Q^+(7,q) \rangle$. In particular, $\mathcal{M}$ is not contained in any hyperplane of $\mathsf{PG}(7,q)$, so it is a proper 1-system of $Q^+(7,q)$. Secondly, it follows from the construction of $\mathcal{F}$ that every line of $\mathcal{I}$ is contained in exactly q elements of $\mathcal{F}$. This in turn means that every elliptic quadric $Q^-_{ij}(5,q)$ contains exactly q reguli of $\mathcal{M}$ through L. In the next section we will discuss the 1-system, associated with the flock $\mathcal{F}$, in more detail.

6.2 A characterization result

In this section, it is our aim to characterize the locally hermitian, semiclassical 1-systems of $Q^+(7,q)$ in terms of their associated flock. For the definition of a semiclassical 1-system of $Q^+(7,q)$, we refer to Section 1.2.2. Of course, every locally hermitian, semiclassical 1-system of a $Q^-(5,q)$ or $Q(6,q)$ which is embedded in $Q^+(7,q)$, is also a locally hermitian, semiclassical 1-system of $Q^+(7,q)$, but these cases will be omitted here. For a discussion of the examples on $Q(6,q)$ we refer to Chapters 4 and 5, while the reader can learn more about the examples on $Q^-(5,q)$ in Section 4.6.2. Concerning the proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$, we will prove the following theorem.

Theorem 6.2.1 *A proper locally hermitian 1-system of $Q^+(7,q)$ is semiclassical if and only if its associated flock arises from a Segre variety $\mathscr{S}_{1;2}$ as described in Section 6.1.2.*

Proof.
Let $\mathcal{M}$ be a locally hermitian 1-system of $Q^+(7,q)$, with associated flock $\mathcal{F} := \{\gamma_1, \gamma_2, \ldots, \gamma_{q^2}\}$ which is constructed from a Segre variety $\mathscr{S}_{1;2}$ as described in Section 6.1.2. Denote the reguli through L that constitute $\mathcal{M}$ by $R_1, R_2, \ldots, R_{q^2}$ and let $\mathcal{I}$ be defined as before. Consider a transversal T_x of the regulus $\{K_0, K_1, \ldots, K_q\}$ of lines of $\mathcal{I}$ in γ_1 and denote the plane $\langle x, T_x\rangle$ of $\mathscr{S}_{1;2}$ by π_x. By construction, the plane π_x intersects each element γ_i of $\mathcal{F}$ in a line, so that $\pi_x^\perp$ and $\gamma_i^\perp = \langle R_i\rangle$ are contained in a 5-dimensional subspace and hence they meet in a plane. Now π_x has a unique point in common with $Q^+(7,q)$, namely the point x, which implies that $\pi_x^\perp \cap Q^+(7,q)$ is a cone $xQ_x^-(3,q)$. Since L is a line of $xQ_x^-(3,q)$, there is only one possibility for the plane $\pi_x^\perp \cap \langle R_i\rangle$: it must meet $Q^+(7,q)$ in two intersecting lines, one of which is the line L. The second line is thus the unique transversal through x of the regulus R_i. As this holds for all $\gamma_i \in \mathcal{F}$, it follows that $xQ_x^-(3,q)$ consists of L and the q^2 tranversals through x of the reguli R_i, $i = 1, 2, \ldots, q^2$, of $\mathcal{M}$ through L. Since we have this property for all points x on L, $\mathcal{M}$ is semiclassical.

To prove the converse, suppose that $\mathcal{M}$ is a proper locally hermitian, semiclassical 1-system of $Q^+(7,q)$, consisting of q^2 reguli $R_1, R_2, \ldots, R_{q^2}$ through the line $L \in \mathcal{M}$. Denote the flock, associated with $\mathcal{M}$, by $\mathcal{F}$. Then we must show that $\mathcal{F}$ arises from a Segre variety $\mathscr{S}_{1;2}$ as described in Section 6.1.2. First we show that for each two reguli R_i, R_j, $i \neq j$, the elliptic quadric $\langle R_i, R_j\rangle \cap Q^+(7,q) := Q_{ij}^-(5,q)$ contains exactly q reguli of $\mathcal{M}$ through L. As $\mathcal{M}$ is semiclassical, we know that for each point x on L, the q^2 transversals through x of the reguli $R_1, R_2, \ldots, R_{q^2}$, together with the line L, form a cone $xQ_x^-(3,q)$ on $Q^+(7,q)$. Consider a fixed point x on L, the corresponding cone $xQ_x^-(3,q)$ containing the transversals through x, and two arbitrary reguli R_i and R_j, $i \neq j$, of $\mathcal{M}$ through L. Then $\langle xQ_x^-(3,q), R_i, R_j\rangle := \eta_{xij}$ is at most 6-dimensional. For a point y of $L \setminus \{x\}$, it holds that $yQ_y^-(3,q) \cap \eta_{xij}$ contains at least three lines of $Q^+(7,q)$, namely L and the transversals of R_i and R_j through y. If $yQ_y^-(3,q)$ were contained in η_{xij}, this would imply that $\mathcal{M} \subseteq \eta_{xij}$, a contradiction to the assumption that $\mathcal{M}$ is not contained in a hyperplane of $\mathsf{PG}(7,q) = \langle Q^+(7,q)\rangle$. So we conclude that $yQ_y^-(3,q) \cap \eta_{xij}$ is a quadratic cone $y\mathcal{C}_y$, with $\mathcal{C}_y$ some conic on $Q_y^-(3,q)$ through the point $L \cap Q_y^-(3,q)$. But this implies that there exist exactly q reguli of $\mathcal{M}$ on L, the transversals of which through both x and y are contained in η_{xij}, so these q reguli must lie in η_{xij}. Now consider a third point $z \in L \setminus \{x, y\}$. Then similarly $zQ_z^-(3,q) \cap \eta_{xij} = z\mathcal{C}_z$, where $z\mathcal{C}_z$ consists of L and the transversals through z of the q reguli of $\mathcal{M}$ on L, which are already known to belong to η_{xij}. But then the subspace $\langle R_i, R_j\rangle$, which contains L and all transversals of R_i and R_j, must contain both $\langle y\mathcal{C}_y\rangle$ and $\langle z\mathcal{C}_z\rangle$, which forces the q reguli of

$\mathcal{M}$ through L in η_{xij} to be contained in the 5-dimensional subspace $\langle R_i, R_j\rangle$. So we have shown that there indeed lie exactly q reguli of $\mathcal{M}$ through L in the elliptic quadric $Q^-_{ij}(5,q)$.
Note that η_{xij} cannot be 5-dimensional, because in that case the subspace $\langle xQ^-_x(3,q), yQ^-_y(3,q)\rangle$ would be a subspace of $\mathsf{PG}(7,q)$ of dimension at most 6 and containing $\mathcal{M}$, a contradiction.
Let R_i, R_j, R_k and R_l be four distinct reguli of $\mathcal{M}$ containing L such that $\langle R_i, R_j\rangle \neq \langle R_k, R_l\rangle$. We want to show that these four reguli span the whole $\mathsf{PG}(7,q)$. So assume that this is not the case, which implies that the subspace $\langle R_i, R_j, R_k, R_l\rangle := \omega$ is 6-dimensional. For an arbitrary point x on L, it must then hold that ω contains two distinct quadratic cones $x\mathcal{C}_x = \langle R_i, R_j\rangle \cap xQ^-_x(3,q)$ and $x\mathcal{C}'_x = \langle R_k, R_l\rangle \cap xQ^-_x(3,q)$, and consequently $xQ^-_x(3,q)$ must be contained in ω. But the same holds for all points on L, that is, $yQ^-_y(3,q) \subseteq \omega$ for all points y on L. This implies that all lines of $\mathcal{M}$ lie in ω, a contradiction. We conclude that $\langle R_i, R_j, R_k, R_l\rangle = \langle Q^-_{ij}(5,q), Q^-_{kl}(5,q)\rangle$ is indeed the whole $\mathsf{PG}(7,q)$.
The results until now already allow us to draw some important conclusions concerning the flock $\mathcal{F}$ of $\mathcal{M}$. As before, let $\langle R_i\rangle^\perp$ be denoted by γ_i and $\langle R_i, R_j\rangle^\perp = \langle Q^-_{ij}(5,q)\rangle^\perp$, $i \neq j$, by K_{ij}. Then the line K_{ij} is the intersection of γ_i and γ_j and it is an external line of $Q^+(7,q)$. As every $Q^-_{ij}(5,q)$ contains exactly q reguli of $\mathcal{M}$ through L, every line K_{ij} is contained in exactly q elements γ_r of the flock $\mathcal{F}$. As a consequence, an element γ_r of $\mathcal{F}$ contains $\frac{q^2-1}{q-1} = q+1$ lines K_{ij}. From the fact that $\langle R_i, R_j, R_k, R_l\rangle$ is 7-dimensional whenever $\langle R_i, R_j\rangle$ and $\langle R_k, R_l\rangle$ are distinct, one also concludes that any two distinct lines K_{ij} and K_{kl} are disjoint. The set $\mathcal{I} := \{\langle R_i, R_j\rangle^\perp = K_{ij} \mid i,j \in \{1,2,\ldots,q^2\}, i \neq j\}$ thus consists of $q(q+1)$ pairwise disjoint lines.
Finally, consider an arbitrary point x on L and the cone $xQ^-_x(3,q)$ consisting of the transversals through x of the reguli $R_1, R_2, \ldots, R_{q^2}$. Then $\langle xQ^-_x(3,q)\rangle$ is 4-dimensional, so $\langle xQ^-_x(3,q)\rangle^\perp$ is a plane π_x containing x, and in particular, it only has the point x in common with $Q^+(7,q)$. Moreover, we have already shown that $\eta_{xij} = \langle xQ^-_x(3,q), R_i, R_j\rangle$ is a 6-dimensional subspace of $\mathsf{PG}(7,q)$ for all $i,j \in \{1,2,\ldots,q^2\}$, $i \neq j$, which implies that $\pi_x \cap K_{ij} = \eta^\perp_{xij}$ is a point. Consequently, the plane π_x shares a point with every line of $\mathcal{I}$ and it meets the line L in the point x. As x on L was chosen arbitrarily, it follows that there exist $q+1$ mutually skew planes π_x, one for each point x on L, which all have a point in common with every line of $\mathcal{I} \cup \{L\}$. By Hirschfeld and Thas [26, Lemma 25.6.1], we conclude that $\mathcal{I} \cup \{L\}$ is a system of generating lines of a Segre variety $\mathscr{S}_{1;2}$.
This proves the theorem. $\square$

Remark

From the proof of this theorem, one easily deduces the following property of $\mathcal{M}$. We know that for any two reguli R_i and R_j of $\mathcal{M}$ through L, the elliptic quadric $Q^-_{ij}(5,q)$ contains exactly q reguli of $\mathcal{M}$ through L. Also, for each point x on L, a quadratic cone $x\mathcal{C}_x$ consisting of L and the transversals through x of these q reguli, lies on $Q^-_{ij}(5,q)$. Let β be a 3-dimensional subspace of $\langle Q^-_{ij}(5,q)\rangle$, disjoint from L. Then the projection from L onto β of the q reguli of $\mathcal{M}$ through L and contained in $Q^-_{ij}(5,q)$, minus L, yields q pairwise disjoint lines in β; the projection of the tangent space $T_L(Q^-_{ij}(5,q))$, minus L, gives a $(q+1)$st line. These $q+1$ lines form a regulus in β the transversals of which are the $q+1$ projections of the 3-spaces $\langle x\mathcal{C}_x\rangle$, minus L, for all points x on L. In view of the discussion of $\mathcal{R}$-conics in Section 1.2.2, one concludes that the q reguli of $\mathcal{M}$ through L in $Q^-_{ij}(5,q)$ form an $\mathcal{R}$-conic.

The theorem we have just shown, characterizes the proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$ as exactly those 1-systems, the associated flock of which is of a special form. One must however notice that this does not at all imply that all such 1-systems have a similar structure. To the contrary, by some simple observations one is immediately convinced that there exists a lot of variety in the examples of proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$.

Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two locally hermitian, semiclassical 1-systems of $Q^+(7,q)$, with associated flocks $\mathcal{F}_1$, respectively $\mathcal{F}_2$, both arising from a Segre variety as described in Section 6.1.2. Let θ be an element of $\mathsf{P\Gamma O}^+(8,q)$ which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. Without loss of generality we may assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally hermitian at the same line L for which $L^\theta = L$ and that they share a regulus, say R_1, through L, with again $R_1^\theta = R_1$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are flocks in $\mathsf{PG}(5,q) = L^\perp$ of the same hyperbolic cone $LQ_1^+(3,q)$ with line vertex L. Moreover, $\mathcal{F}_1$ and $\mathcal{F}_2$ share at least one element $\gamma_1 = \langle R_1\rangle^\perp$, with $\gamma_1 \cap Q^+(7,q) = Q_1^+(3,q)$. In the 3-space γ_1, $\mathcal{F}_i$ defines a regulus $\mathcal{R}_i$, consisting of the lines of intersection of γ_1 with the other elements of $\mathcal{F}_i$, $i=1,2$. It must then hold that $\mathcal{R}_1^\theta = \mathcal{R}_2$ and θ stabilizes the hyperbolic quadric $Q_1^+(3,q)$. We will now see that there are several configurations of $Q_1^+(3,q)$, $\mathcal{R}_1$ and $\mathcal{R}_2$ for which this is not possible.

Consider a hyperbolic quadric $Q^+(3,q)$ in $\mathsf{PG}(3,q)$ and a regulus $\mathcal{R}$ in $\mathsf{PG}(3,q)$, which is disjoint from $Q^+(3,q)$. If translated to the Klein quadric $\mathcal{K} = Q^+(5,q)$ of lines of $\mathsf{PG}(3,q)$, the hyperbolic quadric $Q^+(3,q)$ corresponds to two conics $\mathcal{C}$ and $\mathcal{C}'$ on $\mathcal{K}$, where $\langle\mathcal{C}\rangle^\perp \cap \mathcal{K} = \mathcal{C}'$ with respect to the polarity $\perp$ of $\mathcal{K}$. The regulus $\mathcal{R}$ in turn yields a third conic $\tilde{\mathcal{C}}$ on $\mathcal{K}$. The fact that $\mathcal{R}$ and $Q^+(3,q)$ are disjoint, implies that no point of $\tilde{\mathcal{C}}$ is on a line of $\mathcal{K}$ which also contains a point of $\mathcal{C}$ or $\mathcal{C}'$. In other words, $\mathcal{C}$ and $\tilde{\mathcal{C}}$ belong to a partial ovoid of $\mathcal{K}$, as well as $\mathcal{C}'$ and $\tilde{\mathcal{C}}$. Taken this restriction into account, we can distinguish three possible cases.

(1) $\langle \mathcal{C}, \tilde{\mathcal{C}} \rangle = \mathsf{PG}(5,q)$ and $\langle \mathcal{C}', \tilde{\mathcal{C}} \rangle = \mathsf{PG}(5,q)$.

(2) $\langle \mathcal{C}, \tilde{\mathcal{C}} \rangle$ and $\langle \mathcal{C}', \tilde{\mathcal{C}} \rangle$ are at least 4-dimensional and they are not both of dimension 5. Note that $\langle \mathcal{C}, \tilde{\mathcal{C}} \rangle$ or $\langle \mathcal{C}', \tilde{\mathcal{C}} \rangle$ meets $\mathcal{K}$ in a $Q(4,q)$, which implies that there exists a symplectic polar space $W_3(q)$ in $\mathsf{PG}(3,q)$, having $\mathcal{R}$ and one of the two reguli of $Q^+(3,q)$ as sets of totally isotropic lines.

(3) Either $\langle \mathcal{C}, \tilde{\mathcal{C}} \rangle$ or $\langle \mathcal{C}', \tilde{\mathcal{C}} \rangle$ is 3-dimensional and thus intersects $\mathcal{K}$ in an elliptic quadric $Q^-(3,q)$. In this case, $\mathcal{R}$ and one of the two reguli of $Q^+(3,q)$ are contained in a common regular spread of $\mathsf{PG}(3,q)$.

In general, let $\mathcal{F} = \{\gamma_1, \gamma_2, \ldots, \gamma_{q^2}\}$ be a flock of $LQ^+(3,q)$ with the property that for some $i \in \{1,2,\ldots,q^2\}$, the regulus of lines of $\mathcal{I}$ in γ_i and one of the two reguli of $Q_i^+(3,q) := \gamma_i \cap LQ^+(3,q)$, belong to a common regular spread of γ_i. To this situation, we will further refer as the case of a 1-system with a flock *of regular spread type*. Note that for a flock $\mathcal{F} = \{\gamma_1, \gamma_2, \ldots, \gamma_{q^2}\}$ of $LQ^+(3,q)$ of regular spread type, the regulus of lines of $\mathcal{I}$ in γ_k belongs to a common regular spread of γ_k together with one of the two reguli of the hyperbolic quadric $Q_k^+(3,q) = \gamma_k \cap LQ^+(3,q)$, for all $k = 1,2,\ldots,q^2$. This can be seen as follows. Denote the Segre variety having the lines of $\mathcal{I}$ and L as maximal spaces by $\mathscr{S}_{1;2}$. Suppose, without loss of generality, that the regulus $\mathcal{R}_1$ of lines of $\mathcal{I}$ in γ_1 is part of a regular spread S_1 in γ_1 which also contains a regulus of $Q_1^+(3,q)$. Then there exist two conjugate lines T and $\overline{T}$ in the extension $\mathsf{PG}(3,q^2)$ of γ_1 which intersect the extension to $\mathsf{GF}(q^2)$ of every line of S_1 in a point. Now both T and $\overline{T}$ lie in a unique plane π, respectively $\overline{\pi}$, which is a maximal space of the extension of $\mathscr{S}_{1;2}$ to $\mathsf{GF}(q^2)$. For every $k \in \{2,3,\ldots,q^2\}$, it then holds that the extensions to $\mathsf{GF}(q^2)$ of the $q+1$ lines of $\mathcal{I}$ in γ_k meet each of the planes π and $\overline{\pi}$ in a point, so that $\pi \cap \widehat{\gamma}_k$ and $\overline{\pi} \cap \widehat{\gamma}_k$ are two conjugate lines with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, where $\widehat{\gamma}_k$ denotes the extension of γ_k to $\mathsf{GF}(q^2)$. On the other hand, the 3-spaces $\langle L, T \rangle$ and $\langle L, \overline{T} \rangle$ are generators of the extension of the cone $LQ^+(3,q)$ to $\mathsf{GF}(q^2)$, so that they both contain a line of the extension $Q_k^+(3,q^2)$ of $Q_k^+(3,q)$. These lines are exactly the lines $\pi \cap \widehat{\gamma}_k$ and $\overline{\pi} \cap \widehat{\gamma}_k$, for otherwise $\widehat{\gamma}_k$ would have at least a plane in common with $\langle L, T \rangle$ and $\langle L, \overline{T} \rangle$ and thus meet the extension of L in at least a point. This is a contradiction. Hence the regulus of lines of $\mathcal{I}$ in γ_k and one regulus of $Q^+_k(3,q)$ belong to the regular spread in γ_k , defined by the conjugate lines $\pi \cap \widehat{\gamma}_k$ and $\overline{\pi} \cap \widehat{\gamma}_k$.

Given two distinct reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\mathsf{PG}(3,q)$, both disjoint from $Q^+(3,q)$, it is clear that we cannot find a collineation $\alpha \in \mathsf{P\Gamma L}(4,q)$ which stabilizes $Q^+(3,q)$ and maps $\mathcal{R}_1$ onto $\mathcal{R}_2$ whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ belong to different classes (1), (2) or (3). Also, even if $\mathcal{R}_1$ and $\mathcal{R}_2$ do belong to the same class, it is not always so that the corresponding flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ yield 1-systems which are isomorphic under $\mathsf{P\Gamma O}^+(8,q)$. Suppose for instance that we are in case (3) for q odd, with $\langle \mathcal{C}, \tilde{\mathcal{C}} \rangle \cap \mathcal{K} = Q^-(3,q)$, $\mathcal{C} \cap \tilde{\mathcal{C}} = \emptyset$. Then automatically $\mathcal{C}'$ and $\tilde{\mathcal{C}}$ correspond to disjoint reguli. Recall that for q odd, an equivalence relation can be defined on the non-singular conics lying on the elliptic quadric $Q^-(3,q)$, see Section 3.1.2. It is then clear that a 1-system for which $\mathcal{C}$ and $\tilde{\mathcal{C}}$ belong to the same equivalence class, cannot be isomorphic to a 1-system for which $\mathcal{C}$ and $\tilde{\mathcal{C}}$ belong to different equivalence classes.

We conclude that within the class of the proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$, there still exist several non-isomorphic examples. In Chapter 7, we will encounter some of these examples again in another context.

6.3 The known examples of locally hermitian, semiclassical 1-systems of $Q^+(7,q)$

In Appendix A, Section A.3.3, it is explained that there are several examples of 1-systems of $Q^+(7,q)$ known. But which ones of them are locally hermitian and semiclassical? By having a thorough look at the different examples of 1-systems of $Q^+(7,q)$, one easily finds out that there is only one class known which contains locally hermitian elements. They are constructed from an ovoid of a hermitian variety $H(3,q^2)$ by the method described by Shult and Thas in [57], see also Theorem A.1.4. We briefly repeat the construction here and in the remainder of this chapter, we will refer to this kind of 1-system as a 1-system of $Q^+(7,q)$, *arising from an ovoid $\mathcal{O}$ of $H(3,q^2)$*.

Consider the quadratic extension $Q^+(7,q^2)$ of $Q^+(7,q)$ and let γ and $\overline{\gamma}$ be two disjoint generators of $Q^+(7,q^2)$, which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$ and contain no point of $\mathsf{PG}(7,q)$. Denote by T the set of lines of $Q^+(7,q)$, the extensions of which to $\mathsf{GF}(q^2)$ have a point in common with both γ and $\overline{\gamma}$. It is known that the set T constitutes a partition of the point set of $Q^+(7,q)$ and that the intersection points of the lines of T with γ form a hermitian variety $H(3,q^2)$ in γ, and similarly for $\overline{\gamma}$.

Let $\mathcal{O}$ be an ovoid of $H(3,q^2) \subseteq \gamma$ and let $\overline{\mathcal{O}} \subseteq \overline{\gamma}$ be its conjugate. Then by Shult and Thas [57], the set of lines $\mathcal{M} := \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in \mathcal{O}\}$ is a 1-system of $Q^+(7,q)$.

In this section, we will consider such 1-systems of $Q^+(7,q)$, arising from an ovoid $\mathcal{O}$ of $H(3,q^2)$, and investigate under what conditions they are locally hermitian and semiclassical. We start with an easy lemma.

Lemma 6.3.1 *A set of lines $R := \{x_i\overline{x}_i \cap \mathsf{PG}(7,q) \mid x_i \in \gamma, i = 0,1,\ldots,q\}$ is a regulus of lines in $\mathsf{PG}(7,q)$ if and only if the points $x_0, x_1, \ldots, x_q$ form a Baer subline of some line in γ.*

Proof.
The set R is a regulus in $\mathsf{PG}(7,q)$ if and only if $\langle R\rangle$ is 3-dimensional, that is, if and only if the extension of $\langle R\rangle$ to $\mathsf{PG}(7,q^2)$ intersects γ in a line A and $\overline{\gamma}$ in the conjugate line $\overline{A}$. These two lines A and $\overline{A}$ define a regular spread $\mathcal{S} := \{y\overline{y} \cap \mathsf{PG}(7,q) \mid y \in A\}$ in $\langle R\rangle$, containing all lines of R. Now a regulus in the regular spread $\mathcal{S}$ of $\mathsf{PG}(3,q) = \langle R\rangle$ is equivalent to a Baer subline of A, see [7], which shows the lemma. $\square$

Using this lemma, we can describe the requirement that $\mathcal{M}$ is locally hermitian and semiclassical in terms of properties of $\mathcal{O}$.

Theorem 6.3.2 *A 1-system $\mathcal{M}$ of $Q^+(7,q)$, arising from a non-classical ovoid $\mathcal{O}$ of $H(3,q^2)$, is locally hermitian and semiclassical if and only if $\mathcal{O}$ has the following properties:*

- **(P1)** *$\mathcal{O}$ consists of q^2 Baer sublines r_i, $i = 1,2,\ldots,q^2$, through a point p.*
- **(P2)** *For all $i,j \in \{1,2,\ldots,q^2\}$, $i \neq j$, the plane $\langle r_i, r_j\rangle$ contains exactly q Baer sublines of $\mathcal{O}$ through p. Moreover, for every line J in $\langle r_i, r_j\rangle$, not containing p, the projection from p onto J of these q Baer sublines, minus p, and the line $T_p(H(3,q^2)) \cap \langle r_i, r_j\rangle$, minus p, forms a Baer subline of J.*

Proof.
It is clear that $\mathcal{M}$ is locally hermitian at the line L if and only if the ovoid $\mathcal{O}$ consists of q^2 Baer sublines through the common point p of the extension of L to $\mathsf{PG}(7,q^2)$ and γ, which shows property **(P1)**. Denote the Baer subline of $\mathcal{O}$ through p, corresponding to the regulus R_i of $\mathcal{M}$ containing L, by r_i, $i = 1,2,\ldots,q^2$. The second property **(P2)** is a little more difficult to obtain.

First assume that $\mathcal{M}$ is locally hermitian at L and semiclassical. By the proof of Theorem 6.2.1, we then know that for all $i,j \in \{1,2,\ldots,q^2\}$, $i \neq j$, the elliptic quadric $Q^-_{ij}(5,q) = \langle R_i, R_j\rangle \cap Q^+(7,q)$ contains exactly q reguli of $\mathcal{M}$ through L. Let β be an arbitrary 3-dimensional subspace in $\langle R_i, R_j\rangle$, disjoint from L. As we have noticed in the remark following Theorem 6.2.1, the projections from L onto β of the reguli of $\mathcal{M}$ through L in $Q^-_{ij}(5,q)$, minus L, together with the projection of the 3-dimensional subspace $T_L(Q^-_{ij}(5,q))$, minus L, form a regulus in β. By Lemma 6.3.1, this implies that $\mathcal{O}$ has property **(P2)**.

Conversely, suppose that $\mathcal{O}$ has property **(P2)**. Then clearly every 5-dimensional elliptic quadric $Q^-_{ij}(5,q)$ contains q reguli of $\mathcal{M}$ through L. Without loss of generality, we give the proof for $i = 1$ and $j = 2$, assuming that $Q^-_{12}(5,q)$ contains the reguli $R_1, R_2, \ldots, R_q$ of $\mathcal{M}$ through L. Consider an arbitrary line J in $\langle r_1, r_2\rangle$, not containing the point p, and denote the 3-space $\langle J, \overline{J}\rangle \cap \mathsf{PG}(7,q)$ by β. Then by **(P2)** and Lemma 6.3.1, the projections from L onto β of $\langle R_1\rangle \setminus \{L\}, \langle R_2\rangle \setminus \{L\}, \ldots, \langle R_q\rangle \setminus \{L\}$ and $T_L(Q^-_{12}(5,q)) \setminus \{L\}$ form a regulus of lines in β. This implies that there exist $q+1$ 3-spaces $D_0, D_1, \ldots, D_q$ through L which meet each $\langle R_i\rangle$, $i = 1, 2, \ldots, q$, and $T_L(Q^-_{12}(5,q))$ in a plane. Since L is contained in each D_i, it holds that $D_i \cap Q^+(7,q)$ is either a $Q^+(3,q)$ or a quadratic cone $xQ(2,q)$ for some point x on L. But as D_i meets $T_L(Q^-_{12}(5,q))$ in a plane, the case $D_i \cap Q^+(7,q) = Q^+(3,q)$ is not possible, so for all $i = 0, 1, \ldots, q$, $D_i \cap Q^+(7,q)$ is a quadratic cone $xQ(2,q)$ for some point x on L. It also holds that $D_i \cap D_j = L$ for $i \neq j$, so that for all points x on L, we obtain a quadratic cone $xQ(2,q)$, the generators of which are L and the transversals through x of $R_1, R_2, \ldots, R_q$.
Next, let x be a fixed point of L and consider the projection along reguli of $\mathcal{M}$ from x onto a 5-space $\varepsilon \subseteq T_x(Q^+(7,q))$, x not in ε. This projection is an ovoid $\mathcal{O}_x$ of $Q^+(5,q) := \varepsilon \cap Q^+(7,q)$ through the point $l := L \cap \varepsilon$. If T_i denotes the transversal of R_i through x, then let n_i be the point $\varepsilon \cap T_i$, so $n_i \in \mathcal{O}_x$. From the previous paragraph, it follows that for all $i \neq j$, the points l, n_i and n_j define a conic $\mathcal{C}_{ij}$ of $Q^+(5,q)$, all points of which belong to $\mathcal{O}_x$. Consequently, $\mathcal{O}_x$ consists of $q(q+1)$ conics through the point l, which implies, by [19, Theorem 3.1], that $\mathcal{O}_x$ is classical, that is, $\mathcal{O}_x \cong Q^-(3,q)$. As this holds for all points x on L, we conclude that $\mathcal{M}$ is semiclassical. $\square$

Remark
With the above notation, the Baer subline we obtain on a line J of $\langle r_i, r_j\rangle$, not through p, contains at most two points of $H(3,q^2)$. In particular, if one chooses J to be the line joining a point x of r_i and a point y of r_j, then the points x and y are the only points of the Baer subline $\{\langle r_k\rangle \cap J \mid r_k \subseteq \langle r_i, r_j\rangle\} \cup \{T_p(H(3,q^2)) \cap J\}$ which belong to $H(3,q^2)$. This is clear since $T_p(H(3,q^2)) \cap \langle r_i, r_j\rangle$ is a tangent line of $H(3,q^2)$ through p, and as such

the point $T_p(H(3,q^2)) \cap J$ cannot be a point of $H(3,q^2)$. But this fact has deeper origins: it relies on the property of $H(3,q^2)$ that it does not contain any Baer subplanes consisting of q^2+q+1 hyperbolic lines, see for instance Dembowski [14].

As we have found out which properties the ovoid $\mathcal{O}$ of $H(3,q^2)$ must satisfy if we want $\mathcal{M}$ to be locally hermitian and semiclassical, we are now interested in determining which ovoids of $H(3,q^2)$ fulfil the conditions **(P1)** and **(P2)**. To that end, it is easier to study the dual situation, namely the spread $\mathcal{S} := \mathcal{O}^D$ in the point-line dual $Q^-(5,q)$ of $H(3,q^2)$. The duality between $H(3,q^2)$ and $Q^-(5,q)$ can be described as follows, see [74], or also [50, 3.2.3].
Let κ be the Klein correspondence between the lines of $\mathsf{PG}(3,q^2) = \langle H(3,q^2)\rangle$ and the points of the Klein quadric $Q^+(5,q^2)$ in $\mathsf{PG}(5,q^2)$. The image under κ of a point x of $\mathsf{PG}(3,q^2)$ is interpreted as $x^\kappa := \{A^\kappa \mid A \text{ a line through } x\}$, so x^κ is a Latin plane on $Q^+(5,q^2)$. For a plane π of $\mathsf{PG}(3,q^2)$, we define $\pi^\kappa := \{A^\kappa \mid A \text{ a line of } \pi\}$, which means that π^κ is a Greek plane on $Q^+(5,q^2)$. Next we define a map θ on the points and lines of $H(3,q^2)$ as follows. For all lines L of $H(3,q^2)$, we define $L^\theta := L^\kappa$. Then it holds that $\{L^\theta \mid L \text{ a line of } H(3,q^2)\}$ is the point set of an elliptic quadric $Q^-(5,q) = \Sigma \cap Q^+(5,q^2)$, for some 5-dimensional projective subspace $\Sigma = \mathsf{PG}(5,q)$ of $\mathsf{PG}(5,q^2)$. For the points of $H(3,q^2)$, we introduce the following definition: $x^\theta := x^\kappa \cap \Sigma$. Then x^θ is the line of $Q^-(5,q)$, consisting of the points $L^\theta = L^\kappa$ for all $q+1$ lines L of $H(3,q^2)$ through x. The map θ is a duality between $H(3,q^2)$ and $Q^-(5,q)$ and allows us to translate the properties of $\mathcal{O}$ to properties of $\mathcal{O}^D = \mathcal{S}$.
Let P be the line p^θ of $Q^-(5,q)$, then it is clear that $\mathcal{S}$ consists of q^2 reguli through the line P, say $R'_1, R'_2, \ldots, R'_{q^2}$. For all $i,j \in \{1,2,\ldots,q^2\}$, $i \neq j$, the plane $\pi := \langle r_i, r_j\rangle$ meets $H(3,q^2)$ in a classical ovoid $H(2,q^2)$ and we know that it contains q Baer sublines of $\mathcal{O}$ through the point p, say without loss of generality $r_1, r_2, \ldots, r_q$, so that i and j are also elements of $\{1,2,\ldots,q\}$. For the spread $\mathcal{S}$, this means that the reguli $R'_1, R'_2, \ldots, R'_q$ are contained in a hermitian spread $\mathcal{S}_{1q}$, or, generally speaking, every two reguli R'_i and R'_j are contained in a hermitian spread $\mathcal{S}_{ij}$ of $Q^-(5,q)$ which has q reguli through P in common with $\mathcal{S}$. Denote the line of $\mathsf{PG}(3,q^2)$ containing the Baer subline r_i by K_i, $i = 1,2,\ldots,q^2$. It holds for $i = 1,2,\ldots,q$ that K_i is a line of π through the point p, so that $K_i^\kappa := t_i$ is a point of $Q^+(5,q^2)$ in the intersection of the Latin plane p^κ and the Greek plane π^κ. So the q points $t_1, t_2, \ldots, t_q$ are on the line $T := p^\kappa \cap \pi^\kappa$. Note that the point $T \cap P := x$ is the point X^κ with X the line $T_p(H(3,q^2)) \cap \pi$. It also holds for all $i = 1,2,\ldots,q^2$, that t_i is a point of $\omega_i'^{\perp}$, where ω_i' denotes the extension to $\mathsf{GF}(q^2)$ of the 3-space

$\langle R_i' \rangle$ and $\perp$ stands for the polarity of $Q^+(5,q^2)$. Similarly, the conjugate point $\overline{t}_i$ that lies in the Greek plane α through P also belongs to $\omega_i'^{\perp}$. Thus the regular spread $\mathcal{S}_{1q}$ is the one which arises from the regular line spread in $T_P(Q^-(5,q))$ defined by T and $\overline{T}$, by the construction described in Section 4.6.2. Now, since the lines $\{K_1, K_2, \ldots, K_q, X\}$ intersect each line J of π, not through p, in a Baer subline, it follows that the points $\{t_1, t_2, \ldots, t_q, x\}$ form a Baer subline of the line T, so that the lines $t_1\overline{t}_1, t_2\overline{t}_2, \ldots, t_q\overline{t}_q$ and P define a regulus in $T_P(Q^-(5,q))$. As the above arguments hold for every two Baer sublines r_i and r_j of $\mathcal{O}$ through p, the locally hermitian spread $\mathcal{S}$ of $Q^-(5,q)$ is of the form $\mathcal{S}(S)$, where S is a line spread in $T_P(Q^-(5,q))$ containing P and with the property that any two distinct lines of S, together with P, define a regulus which is completely contained in S. By [19, Theorem 3.1], this implies that S is a regular line spread of the 3-space $T_P(Q^-(5,q))$. As $\mathcal{O}$ is not classical, no one of the two conjugate lines over $\mathsf{GF}(q^2)$ defining the regular spread S, which in turn determines $\mathcal{O}^D = \mathcal{S}(S)$, is contained in the plane p^κ. So $\mathcal{S}(S)$ is not hermitian.

Conversely, it is easily seen that we can return in the same way from a regular spread S of $\mathsf{PG}(3,q) = T_P(Q^-(5,q))$ containing P, but with the property that none of its defining lines over $\mathsf{GF}(q^2)$ belongs to p^κ, to the spread $\mathcal{S} = \mathcal{S}(S)$ of $Q^-(5,q)$ and its dual $\mathcal{S}^D = \mathcal{O}$, with $\mathcal{O}$ a non-classical ovoid of $H(3,q^2)$ satisfying **(P1)** and **(P2)**.

By combining this with Theorem 6.3.2, we find that the only 1-systems of $Q^+(7,q)$, arising from an ovoid $\mathcal{O}$ of $H(3,q^2)$, which are locally hermitian and semiclassical, are those for which $\mathcal{O}^D = \mathcal{S}$ is a locally hermitian spread of $Q^-(5,q)$, arising from a regular line spread S in $\mathsf{PG}(3,q)$ containing P, in such a way that $\mathcal{S} = \mathcal{S}(S)$, where the classical ovoid of $H(3,q^2)$ is now included.

Note that the spread S in the above is the regular spread in $T_P(Q^-(5,q))$, defined by T and $\overline{T}$, if and only if $\mathcal{S}$ coincides with $\mathcal{S}_{1q}$, so if and only if $\mathcal{O}$ is the classical ovoid $\pi \cap H(3,q^2) = H(2,q^2)$. This is the case if and only if $\mathcal{M}$ itself is a hermitian spread of the elliptic quadric $\langle \pi, \overline{\pi} \rangle \cap Q^+(7,q) = Q^-(5,q)'$.

Finally we try to find out what the flock of a proper locally hermitian, semiclassical 1-system of $Q^+(7,q)$ arising from an ovoid $\mathcal{O}$ of $H(3,q^2)$, looks like.

Theorem 6.3.3 *A proper locally hermitian, semiclassical 1-system of the hyperbolic quadric $Q^+(7,q)$ arises from an ovoid $\mathcal{O}$ of $H(3,q^2)$ if and only if its associated flock is of regular spread type.*

Proof.

In the course of this proof, we will distinguish between an object which is defined in $\mathsf{PG}(7,q)$ and its extension to $\mathsf{GF}(q^2)$ in the following way. If X is any object in $\mathsf{PG}(7,q)$, then the extension of X to $\mathsf{GF}(q^2)$ is denoted by $\widehat{X}$. Note that we do not make this distinction for objects, which are only considered over $\mathsf{GF}(q^2)$; in that case the notation with the "hat" ($\widehat{\ }$) will not be applied.

First, let $\mathcal{M}$ be a 1-system of $Q^+(7,q)$ that is locally hermitian at the line L and semiclassical and assume that $\mathcal{M}$ arises from an ovoid $\mathcal{O}$ of $H(3,q^2)$. As before, denote the generator $\langle H(3,q^2)\rangle$ of the extension $Q^+(7,q^2)$ of $Q^+(7,q)$ to $\mathsf{GF}(q^2)$ by γ and its conjugate by $\overline{\gamma}$, so that the extensions to $\mathsf{GF}(q^2)$ of all lines of $\mathcal{M}$ have a point in common with γ and $\overline{\gamma}$. Since $\widehat{L}\cap\gamma$ is a point, it holds that $\widehat{L}^{\perp}\cap\gamma$ is a plane, say π. Similarly $\widehat{L}^{\perp}\cap\overline{\gamma}$ is the plane $\overline{\pi}$, conjugate to π. It also holds that γ intersects each $\widehat{\langle R_i\rangle}$, $i=1,2,\ldots,q^2$, in a line, and as a consequence, $\widehat{\langle R_i\rangle}^{\perp}\cap\gamma$ is also a line, belonging to π. But this implies that for all $i\neq j$, the line $\widehat{K}_{ij}$ of intersection of $\widehat{\langle R_i\rangle}^{\perp}$ and $\widehat{\langle R_j\rangle}^{\perp}$ has the point $\gamma\cap\widehat{\langle R_i\rangle}^{\perp}\cap\widehat{\langle R_j\rangle}^{\perp}$ in common with π and analogously the line $\widehat{K}_{ij}$ also meets $\overline{\pi}$ in a point. So π and $\overline{\pi}$ are two conjugate and disjoint planes having a point in common with the extension to $\mathsf{GF}(q^2)$ of each line of $\mathcal{I}$, where $\mathcal{I}$ is again the set of lines $\langle R_i\rangle^{\perp}\cap\langle R_j\rangle^{\perp}$, $i\neq j$. On the other hand, π and $\overline{\pi}$ are totally singular planes of the extension $Q^+(7,q^2)$ of $Q^+(7,q)$, so that the line $\widehat{\langle R_i\rangle}^{\perp}\cap\gamma=\widehat{\langle R_i\rangle}^{\perp}\cap\pi$ is a line of $Q_i^+(3,q^2):=\widehat{\langle R_i\rangle}^{\perp}\cap Q^+(7,q^2)$. It follows that the line $\widehat{\langle R_i\rangle}^{\perp}\cap\gamma$ is a transversal of one regulus of $Q_i^+(3,q^2)$, but also of the extension to $\mathsf{GF}(q^2)$ of the regulus of lines of $\mathcal{I}$ in $\widehat{\langle R_i\rangle}^{\perp}$. As the same holds for the line $\overline{\pi}\cap\widehat{\langle R_i\rangle}^{\perp}$, the regulus of lines of $\mathcal{I}$ in $\langle R_i\rangle^{\perp}$ and one of the two reguli of $Q_i^+(3,q)$ are contained in a common regular spread of $\langle R_i\rangle^{\perp}$. This holds for $i=1,2,\ldots,q^2$, so that the flock $\mathcal{F}$, associated with $\mathcal{M}$, is of regular spread type.

Conversely, assume that the flock $\mathcal{F}$, associated with the proper semiclassical 1-system $\mathcal{M}$ of $Q^+(7,q)$, which is locally hermitian at the line L, is of regular spread type and consider a regulus R_1 of $\mathcal{M}$ through L. Then there exist in the 3-space $\widehat{\langle R_1\rangle}^{\perp}$ two conjugate lines A and $\overline{A}$ having a point in common with the extensions to $\mathsf{GF}(q^2)$ of all $q+1$ lines of $\mathcal{I}$ in $\langle R_1\rangle^{\perp}$, and belonging to the hyperbolic quadric $Q_1^+(3,q^2)=\widehat{\langle R_1\rangle}^{\perp}\cap Q^+(7,q^2)$, where $Q^+(7,q^2)$ is again the extension of $Q^+(7,q)$ to $\mathsf{GF}(q^2)$. Since $\mathcal{I}\cup\{L\}$ is the set of lines of a Segre variety $\mathscr{S}_{1;2}$ in $L^{\perp}=\mathsf{PG}(5,q)$, there exists a unique plane π through A which belongs to the system of planes of the extension of $\mathscr{S}_{1;2}$ to $\mathsf{GF}(q^2)$. Clearly, π is disjoint from $\mathsf{PG}(5,q)$ and it has a point in common with the extension to $\mathsf{GF}(q^2)$ of each line of $\mathcal{I}\cup\{L\}$; the conjugate plane $\overline{\pi}$ of π through $\overline{A}$ has the same properties. Also, as A and $\overline{A}$ are lines of $Q^+(7,q^2)\cap\widehat{L}^{\perp}$, the planes π and $\overline{\pi}$ are totally singular planes of $Q^+(7,q^2)$.

Through π there exist two generators of $Q^+(7,q^2)$: the generator $\langle \widehat{L}, \pi\rangle$ and another one, say γ. Similarly, let $\overline{\gamma}$ be the generator of $Q^+(7,q^2)$ containing $\overline{\pi}$ but not $\widehat{L}$. Then γ and $\overline{\gamma}$ belong to the same family of generators and they are disjoint, for the following reason. The tangent subspaces $T_\pi(Q^+(7,q^2))$ and $T_{\overline{\pi}}(Q^+(7,q^2))$ of $Q^+(7,q^2)$ at π, respectively $\overline{\pi}$, meet in a line, which must hence be the line $\widehat{L}$. Moreover, $T_\pi(Q^+(7,q^2)) \cap Q^+(7,q^2) = \gamma \cup \langle \widehat{L}, \pi\rangle$ and similarly $T_{\overline{\pi}}(Q^+(7,q^2)) \cap Q^+(7,q^2) = \overline{\gamma} \cup \langle \widehat{L}, \overline{\pi}\rangle$, which implies that the intersection of γ and $\overline{\gamma}$ must be empty. The generators γ and $\overline{\gamma}$ are also conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$, because the conjugate of γ must contain $\overline{\pi}$ and cannot contain $\widehat{L}$, so it is indeed $\overline{\gamma}$. Finally, $\widehat{\langle R_i\rangle}^\perp \cap \gamma$ is a line for all $i = 1, 2, \ldots, q^2$, and as a consequence, $\widehat{\langle R_i\rangle}$ and γ also intersect in a line. As $\widehat{L}$ and π share a point x, it holds that $\gamma \cap \widehat{L} = \{x\}$. So the common line of γ and $\widehat{\langle R_i\rangle}$ must be a line of $Q^+(7,q^2)$ through x, but it cannot be $\widehat{L}$ itself. Hence it is a transversal over $\mathsf{GF}(q^2)$ of the regulus $\widehat{R}_i$, and consequently the extension to $\mathsf{GF}(q^2)$ of every line of R_i meets γ in a point. The same holds for $\overline{\gamma}$, and it now follows that $\mathcal{M}$ arises from an ovoid $\mathcal{O}$ of the hermitian variety $H(3,q^2)$ in γ. $\square$

As has already been mentioned before, the locally hermitian, semiclassical 1-systems of $Q^+(7,q)$ which arise from a non-classical ovoid $\mathcal{O}$ of $H(3,q^2)$, are the only previously known examples of proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$. By Theorem 6.3.3 and the discussion at the end of Section 6.2, we now know that, by using the construction described in Section 6.1.1, a lot of new examples are found, namely all examples for which the flock $\mathcal{F}$ is not of regular spread type.

Chapter 7

Trialities and 1-systems of $Q^+(7, q)$

A triality is a map of a D_4-geometry, which is defined on the hyperbolic quadric $Q^+(7, q)$ in $\mathsf{PG}(7, q)$. Trialities were introduced and first studied by Study in [60]. As trialities map lines of $Q^+(7, q)$ onto lines, it is meaningful to consider the image under a triality of a 1-system of $Q^+(7, q)$. In this chapter we will do so, but not for general 1-systems of $Q^+(7, q)$. In particular, we will try to formulate some answers to the following question. Given a triality τ of the D_4-geometry, attached to the 7-dimensional hyperbolic quadric $Q^+(7, q)$, and a 1-system $\mathcal{M}$ of an induced $Q(6, q) \subseteq Q^+(7, q)$, then what does the image $\mathcal{M}^\tau$ of $\mathcal{M}$ under τ look like? Following the leading thread of Chapters 4, 5 and 6, we will mainly focus on the by now well known proper locally hermitian, semiclassical 1-systems of $Q(6, q)$. It will turn out that they are transformed by a triality into proper 1-systems of $Q^+(7, q)$, which are nevertheless still locally hermitian and semiclassical. As such they must fit into the theory of Chapter 6 and in the one but final section, the link will be made between the results of both chapters. Eventually, the chapter is concluded by some isomorphism results concerning images of 1-systems of $Q^+(7, q)$ under a triality.

7.1 Trialities: definition and generalities

Consider the projectively unique hyperbolic quadric $Q^+(7, q)$ in $\mathsf{PG}(7, q)$. It is well known, see for instance [26, 22.4], that the generators of $Q^+(7, q)$,

which are 3-dimensional subspaces, can be subdivided in two subsets, often called *families*, in the following way. Two distinct generators are said to belong to the same family if and only if their intersection is empty or a line. Hence generators of different families intersect in a point or a plane and furthermore, every plane of $Q^+(7,q)$ is contained in exactly two generators, namely one from each family.
If we denote the two families of generators of $Q^+(7,q)$ by $\mathcal{F}_1$ and $\mathcal{F}_2$, the set of lines on $Q^+(7,q)$ by $\mathcal{L}$ and the set of points of $Q^+(7,q)$ by $\mathcal{P}$, then a D_4-geometry Ω can be attached to $Q^+(7,q)$, in the following way. The *0-points* of Ω are the points of $Q^+(7,q)$; the *lines* are just the lines of $Q^+(7,q)$; the *1-points* are the elements of one family of generators, say $\mathcal{F}_1$; and the *2-points* are the elements of the other family of generators. Incidence is symmetrized containment for i-points and lines, $i = 0,1,2$, and also for 0-points and j-points, $j = 1,2$. A 1-point $G_1 \in \mathcal{F}_1$ is said to be incident with a 2-point $G_2 \in \mathcal{F}_2$ if and only if the intersection $G_1 \cap G_2$ is a plane of $Q^+(7,q)$. An important property of the geometry Ω is the fact that every permutation of the set $\{\mathcal{P}, \mathcal{F}_1, \mathcal{F}_2\}$ defines a geometry which is isomorphic to Ω.

Definitions

A *triality* of the geometry Ω, attached to $Q^+(7,q)$ as above, is a map τ:

$$\tau : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{F}_1, \mathcal{F}_1 \to \mathcal{F}_2, \mathcal{F}_2 \to \mathcal{P}$$

preserving the incidence in Ω and such that τ^3 is the identity.
A point $p \in \mathcal{P}$ is called an *absolute* point of τ if p is incident with $p^\tau \in \mathcal{F}_1$. A 1- or 2-point, that is, a generator $G_i \in \mathcal{F}_i$, $i = 1,2$, is said to be *absolute* for τ if it is incident with its image G_i^τ. A line $L \in \mathcal{L}$ is called *absolute* for τ if $L = L^\tau$.

Although the condition "τ^3 is the identity" is an explicit part of the definition of a triality, most of the results that will be obtained in this chapter, hold under weaker conditions too. Often it is not necessary to require that τ^3 is the identity. Therefore we introduce another less common definition, which is due to Tits [76].

Definition

A *T-correspondence* of the geometry Ω is a map θ:

$$\theta : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{F}_1, \mathcal{F}_1 \to \mathcal{F}_2, \mathcal{F}_2 \to \mathcal{P}$$

which preserves incidence in Ω.

The definitions of absolute points and lines for a T-correspondence θ remain the same as for a triality τ.

Note that the assignment of the names $\mathcal{F}_1$ and $\mathcal{F}_2$ to the families of generators of $Q^+(7,q)$ is arbitrary, and hence the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$ may be interchanged in the above definitions. As such, $\tau^{-1} = \tau^2$ is also a triality and both θ^{-1} and θ^2 are T-correspondences. For convenience, we agree on the convention that a triality, respectively a T-correspondence, is always a map as in the definitions above, unless otherwise mentioned, for some choice of $\mathcal{F}_1$ and $\mathcal{F}_2$ which we consider to be fixed throughout this chapter.

Trialities are very interesting for the study of generalized hexagons, as some trialities produce generalized hexagons. This was shown by Tits in his celebrated paper "Sur la trialité et certains groupes qui s'en déduisent" [76]. It goes beyond the scope of this thesis to go into detail about this matter, so we restrict ourselves to mentioning the theorem that states the relation between trialities and generalized hexagons.

Theorem 7.1.1 (Tits [76]) *Let τ be a triality of the geometry Ω. Suppose that one of the following equivalent hypotheses is satisfied:*

(i) there exists at least one absolute i-point, for some $i \in \{0,1,2\}$, and every absolute i-point is incident with at least two absolute lines;

(ii) there exists a sequence of absolute lines $(L_1, L_2, \ldots, L_d)$, $d > 2$, such that L_i is concurrent with L_{i+1}, indices modulo d.

Then for every $i \in \{0,1,2\}$, the geometry $\Gamma^{(i)}$ with point set $\mathcal{P}^{(i)}_{\text{abs}}$ the set of absolute i-points, with line set $\mathcal{L}_{\text{abs}}$ the set of absolute lines and with the natural incidence, is a weak generalized hexagon with thick lines.

Over the finite field $\mathsf{GF}(q)$, there exist essentially two examples of generalized hexagons arising from a triality. The first one is the so-called *split Cayley hexagon*, denoted by $\mathsf{H}(q)$, which exists for every q. This hexagon has order (q,q) and an interesting fact is that all points and lines of $\mathsf{H}(q)$ are contained in a hyperplane $\mathsf{PG}(6,q)$ of $\mathsf{PG}(7,q)$. One should also keep in mind that two points of $\mathsf{H}(q)$ are opposite in $\mathsf{H}(q)$ if and only if they are not joined by a line of $Q^+(7,q)$. Moreover, the $q+1$ lines through a point of $\mathsf{H}(q)$ form a flat pencil of lines in a totally singular plane of $Q^+(7,q)$.
The other example is the *twisted triality hexagon* $\mathsf{T}(q'^3, q')$ and it exists only if $q = q'^3$ for some prime power q'. This example will not be important for the sequel.
For more information about trialities and their relation to generalized hexagons, the reader is referred to the original paper of Tits [76], or to Chapter 2

of the monograph *"Generalized Polygons"* [78] by Van Maldeghem.

To finish this section, we give two easy properties of T-correspondences.

Property 7.1.2 *Let θ be a T-correspondence of Ω. If L and M are two distinct concurrent lines, contained in a totally singular plane π on $Q^+(7, q)$, then the same holds for L^θ and M^θ.*

Proof.
If $L \cap M$ is the point x and $G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$ are the two generators of $Q^+(7, q)$ through π, then L^θ and M^θ are lines in the intersection of the two generators $x^\theta \in \mathcal{F}_1$ and $G_1^\theta \in \mathcal{F}_2$, and they contain the point G_2^θ. $\square$

Property 7.1.3 *Let θ be a T-correspondence of Ω and let π be a totally singular plane on $Q^+(7, q)$. Then the set of lines in π is mapped by θ onto the set of lines through a point in a generator of $Q^+(7, q)$.*

Proof.
Let again $G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$ denote the two generators through the plane π. Then all lines of π are mapped by θ onto all lines in the generator $G_1^\theta \in \mathcal{F}_2$ through the point G_2^θ. $\square$

7.2 T-correspondences and locally hermitian 1-systems of $Q(6, q)$

Before atention is paid to the image under a T-correspondence (and hence also a triality) of a locally hermitian 1-system of an induced $Q(6, q) \subseteq Q^+(7, q)$, it is first shown that T-correspondences map 1-systems of $Q^+(7, q)$ onto 1-systems, in the following easy, but basic, theorem.

Theorem 7.2.1 *Let θ be a T-correspondence of Ω and consider an arbitrary 1-system $\mathcal{M}$ of $Q^+(7, q)$. Then $\mathcal{M}^\theta$ is also a 1-system of $Q^+(7, q)$.*

Proof.
Suppose that the claim is false and $\mathcal{M}^\theta$ is not a 1-system of $Q^+(7, q)$. This implies that there exist a line $M \in \mathcal{M}^\theta$ and a generator G of $Q^+(7, q)$ on M such that G meets some line N of $\mathcal{M}^\theta \setminus \{M\}$ at least in a point x. In that case $\langle M, x\rangle$ is a totally singular plane on $Q^+(7, q)$ and as such contained in exactly two generators G_1 and G_2 of $Q^+(7, q)$; assume without loss of generality that $G = G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$. It then holds that $N^{\theta^{-1}}$ and $M^{\theta^{-1}}$ are lines of $\mathcal{M}$, the generator $x^{\theta^{-1}} \in \mathcal{F}_2$ contains $N^{\theta^{-1}}$, $G_1^{\theta^{-1}}$ is a point of $M^{\theta^{-1}}$

and as $x \in G_1$, it follows that the point $G_1^{\theta^{-1}}$ is contained in $x^{\theta^{-1}}$. In other words, $x^{\theta^{-1}}$ is a generator of $Q^+(7, q)$ through the line $N^{\theta^{-1}} \in \mathcal{M}$, which has the point $G_1^{\theta^{-1}}$ in common with the line $M^{\theta^{-1}} \in \mathcal{M} \setminus \{N^{\theta^{-1}}\}$. This is a contradiction to the fact that $\mathcal{M}$ is a 1-system of $Q^+(7, q)$, which proves the theorem. □

Since locally hermitian 1-systems are characterized by the fact that they consist of a number of reguli through some common line, it is important to understand the action of a T-correspondence on a regulus.

Lemma 7.2.2 *Let θ be a T-correspondence of Ω and consider a 3-dimensional subspace γ of $\mathsf{PG}(7, q)$ such that $\gamma \cap Q^+(7, q) = Q^+(3, q)$. Then θ maps each regulus of lines of $Q^+(3, q)$ onto a regulus of lines of some $Q^+(3, q)' = \gamma' \cap Q^+(7, q)$, with γ' also a 3-dimensional subspace of $\mathsf{PG}(7, q)$.*

Proof.
Denote the two reguli of the hyperbolic quadric $Q^+(3, q)$ by $\{K_0, K_1, \dots, K_q\}$ and $\{T_0, T_1, \dots, T_q\}$. Since for $i \neq j$, any two lines K_i and K_j, respectively T_i and T_j, are disjoint and not contained in a generator of $Q^+(7, q)$, the same holds for their images K_i^θ and K_j^θ, respectively T_i^θ and T_j^θ. On the other hand, every K_i meets every T_j in a point, which implies that K_i^θ and T_j^θ are contained in a generator of $Q^+(7, q)$, for $i, j \in \{0, 1, \dots, q\}$. If $K_i^\theta \cap T_j^\theta$ were a point y, then K_i and T_j would be contained in the generator $y^{\theta^{-1}}$ of $Q^+(7, q)$, clearly a contradiction. So K_i^θ is disjoint from T_j^θ, for all $i, j \in \{0, 1, \dots, q\}$ and the situation is such that every T_j^θ is contained in $(K_i^\theta)^\perp$, for all $i = 0, 1, \dots, q$. Consequently, the $q + 1$ lines T_j^θ, $j = 0, 1, \dots, q$, must form a regulus of $Q^+(3, q)'' := \langle K_0^\theta, K_1^\theta \rangle^\perp \cap Q^+(7, q)$ and this in turn implies that the lines $K_0^\theta, K_1^\theta, \dots, K_q^\theta$ form a regulus of $\langle Q^+(3, q)'' \rangle^\perp \cap Q^+(7, q) := Q^+(3, q)'$, where $Q^+(3, q)' = \langle K_0^\theta, K_1^\theta \rangle \cap Q^+(7, q)$. This proves the lemma. □

This lemma ensures that every T-correspondence θ maps a 1-system $\mathcal{M}$ of $Q^+(7, q)$ that is locally hermitian at some line L, onto a 1-system $\mathcal{M}^\theta$ that is locally hermitian at the line L^θ.
The next theorem we will prove, tells something about the dimension of $\langle \mathcal{M}^\theta \rangle$, but first we need another lemma.

Lemma 7.2.3 *Suppose that θ is a T-correspondence of Ω and let $\mathcal{A}$ be the line set of a generalized hexagon $\mathsf{H}(q)$ on $Q^+(7, q)$. Then the set $\mathcal{A}^\theta$ is the set of lines of a generalized hexagon $\mathsf{H}(q)'$ on $Q^+(7, q)$.*

Proof.
The generalized hexagon $\mathsf{H}(q)$ consists of the absolute points and lines of some triality τ of $Q^+(7,q)$, which satisfies the hypotheses of Theorem 7.1.1. Let L be a line of $\mathsf{H}(q)$, that is, L is absolute for τ. Then the line L^θ satisfies

$$(L^\theta)^{\theta^{-1}\tau\theta} = L^{\tau\theta} = L^\theta,$$

so L^θ coincides with its image under the map $\theta^{-1}\tau\theta$. It is obvious that $\theta^{-1}\tau\theta$ is also a triality of $Q^+(7,q)$, but to prove the claim completely, we must ensure that $\theta^{-1}\tau\theta$ satisfies one of the hypotheses of Theorem 7.1.1. We will check the second of the two equivalent conditions from that theorem.
On $\mathsf{H}(q)$ there certainly exists an ordinary hexagon, consisting of the sequence of lines $(L_1, L_2, L_3, L_4, L_5, L_6)$, where L_i is concurrent with L_{i+1}, indices modulo 6. Then $(L_1^\theta, L_2^\theta, L_3^\theta, L_4^\theta, L_5^\theta, L_6^\theta)$ is a sequence of absolute lines for $\theta^{-1}\tau\theta$. Since L_i is concurrent with L_{i+1} on $\mathsf{H}(q)$ for $i = 1, 2, \ldots, 6$ and indices taken modulo 6, $\langle L_i, L_{i+1}\rangle$ is a totally singular plane of $Q^+(7,q)$. Now by Property 7.1.2 this implies that L_i^θ and L_{i+1}^θ have an absolute point for $\theta^{-1}\tau\theta$ in common, for $i = 1, 2, \ldots, 6$ and indices taken modulo 6. Hence the triality $\theta^{-1}\tau\theta$ satisfies (ii) of Theorem 7.1.1 and so its absolute points and lines are the points and lines of a generalized hexagon.
Finally, the generalized hexagon of the absolute points and lines of $\theta^{-1}\tau\theta$ must be isomorphic to the one defined by τ, because we have that $M^{\theta^{-1}\tau\theta} = M$ if and only if $(M^{\theta^{-1}})^\tau = M^{\theta^{-1}}$. So a line M of $Q^+(7,q)$ is absolute for $\theta^{-1}\tau\theta$ if and only its inverse image $M^{\theta^{-1}}$ is absolute for τ. Hence both hexagons have the same order (q,q).
This proves the lemma completely. $\square$

Using this lemma, we can prove the following interesting theorem.

Theorem 7.2.4 *Let $\mathcal{A}$ be a set of lines on a $Q(6,q) \subseteq Q^+(7,q)$ and consider a T-correspondence θ of Ω. Then the lines of $\mathcal{A}^\theta$ lie in an induced $Q(6,q)' \subseteq Q^+(7,q)$ if and only if the lines of $\mathcal{A}$ are lines of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$.*

Proof.
Denote by $\mathcal{B}$ the set of all lines of $Q(6,q)$ and suppose that $\mathcal{A}^\theta \subseteq Q(6,q)'$ for some parabolic quadric $Q(6,q)' \subseteq Q^+(7,q)$. We investigate which lines of $\mathcal{B}^\theta$ are lines of this $Q(6,q)'$. If p is an arbitrary point of $Q(6,q)'$, then $p^{\theta^{-1}}$ is a generator of $Q^+(7,q)$. So $p^{\theta^{-1}}$ intersects $Q(6,q)$ in a totally singular plane, containing q^2+q+1 lines of $\mathcal{B}$. By Property 7.1.3, this implies that the q^2+q+1 lines of $\mathcal{B}^\theta$ through the point p are exactly the lines on p contained in some generator G of $Q^+(7,q)$ through p. But G meets $Q(6,q)'$ in a totally

singular plane α of $Q(6,q)'$, and hence the common lines of $\mathcal{B}^\theta$ and $Q(6,q)'$ through the point p are exactly the $q+1$ lines on p in the totally singular plane α. In other words, the common lines of $\mathcal{B}^\theta$ and $Q(6,q)'$ through an arbitrary point of $Q(6,q)'$ form a pencil of lines in some totally singular plane of $Q(6,q)'$ containing this point. Now, by Theorem 3.7 of [73], this implies that the common lines of $\mathcal{B}^\theta$ and $Q(6,q)'$ either form the set of all lines in the q^3+1 planes of a spread of $Q(6,q)'$, or they form the line set of a generalized hexagon $\mathsf{H}(q)'$ on $Q(6,q)'$. The first case cannot occur in the current situation, because in that case, $(\mathcal{B}^\theta)^{\theta^{-1}} = \mathcal{B}$ would contain q^2+q+1 lines through a point of $Q^+(7,q)$ in a generator of $Q^+(7,q)$. This is of course a contradiction, as the rank of $Q(6,q)$ is 3. So the lines of $\mathcal{B}^\theta$ on $Q(6,q)'$ form the set of lines of a generalized hexagon $\mathsf{H}(q)'$ on $Q(6,q)'$. Since $\mathcal{A}^\theta \subseteq \mathcal{B}^\theta$ and the lines of $\mathcal{A}^\theta$ lie on $Q(6,q)'$ by assumption, all lines of $\mathcal{A}^\theta$ must be lines of $\mathsf{H}(q)'$. If we now apply Lemma 7.2.3 to the T-correspondence θ^{-1}, we find that all lines of $\mathcal{A}$ are lines of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$.

As the converse has already been shown in Lemma 7.2.3, the proof of the theorem is complete. □

Suppose as a special case that $\mathcal{M}$ is a locally hermitian 1-system of $Q(6,q) \subseteq Q^+(7,q)$, not isomorphic to a spread of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$. Then by the above results, the image $\mathcal{M}^\theta$ of $\mathcal{M}$ under a T-correspondence θ is a locally hermitian 1-system of $Q^+(7,q)$ that is not contained in a non-tangent hyperplane of $\mathsf{PG}(7,q)$, while by the discussion in Section 1.2.2, it is neither contained in a tangent hyperplane of $\mathsf{PG}(7,q)$. We conclude that $\mathcal{M}^\theta$ is a proper locally hermitian 1-system of $Q^+(7,q)$. In the next section we will have a closer look at $\mathcal{M}^\theta$ if the 1-system $\mathcal{M}$ of $Q(6,q)$ is not only locally hermitian, but also semiclassical.

7.3 T-correspondences and locally hermitian, semiclassical 1-systems of $Q(6,q)$

Suppose again that θ is a T-correspondence of Ω. We now examine the special case where $\mathcal{M}$ is a locally hermitian 1-system of a $Q(6,q) \subseteq Q^+(7,q)$, which is in addition semiclassical. Let $L \in \mathcal{M}$ be a line at which $\mathcal{M}$ is locally hermitian and denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$, as usually. We already know that $\mathcal{M}^\theta$ is locally hermitian at the line L^θ, but one would also like to know whether $\mathcal{M}^\theta$ is semiclassical or not.

Let x be a point on L^θ. Then the q^2 transversals of the reguli of $\mathcal{M}^\theta$ through L^θ are the images under θ of q^2 lines in the generator $x^{\theta^{-1}} \in \mathcal{F}_2$

through L. From the proof of Lemma 7.2.2, it follows that if T is a transversal on x of a regulus R_i^θ, with R_i^θ the image under θ of some regulus R_i of $\mathcal{M}$ containing L, then $T^{\theta^{-1}}$ is a line of $\langle R_i\rangle^\perp$, with $\perp$ the polarity of $Q^+(7,q)$. As x is a point of T, the line $T^{\theta^{-1}}$ is thus the unique line in the intersection of $x^{\theta^{-1}}$ and $\langle Q_i^+(3,q)\rangle = \langle R_i\rangle^\perp$. So the transversals through x of the reguli of $\mathcal{M}^\theta$ containing L^θ, are the images under θ of the q^2 lines $\langle R_i\rangle^\perp \cap x^{\theta^{-1}}$, $i = 1,2,\ldots,q^2$. These q^2 lines in $x^{\theta^{-1}}$ are pairwise disjoint, since no two transversals through x of the reguli of $\mathcal{M}^\theta$ containing L^θ, are contained in a totally singular plane of $Q^+(7,q)$. For the same reason, all of them are also disjoint from L, so that $\{L\} \cup \{\langle R_i\rangle^\perp \cap x^{\theta^{-1}} \mid i = 1,2,\ldots,q^2\}$ is a spread of the 3-dimensional projective space $x^{\theta^{-1}}$. If $x^{\theta^{-1}} := G$, then we denote this spread by $\mathcal{S}_G$. Apparently, the structure of $\mathcal{S}_G$ determines the configuration of the transversals on x of the reguli of $\mathcal{M}^\theta$ through L^θ.

Lemma 7.3.1 *The 1-system $\mathcal{M}^\theta$ is semiclassical if and only if for every generator $G \in \mathcal{F}_2$ containing L, the spread $\mathcal{S}_G$ is regular.*

Proof.
Let x on L^θ be fixed and consider in $x^{\theta^{-1}} := G$ the spread $\mathcal{S}_G$. Suppose that $\{K_0, K_1, \ldots, K_q\}$ is a regulus of lines of $\mathcal{S}_G$ and denote its opposite regulus by $\{T_0, T_1, \ldots, T_q\}$. Then $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$ and $T_0^\theta, T_1^\theta, \ldots, T_q^\theta$ are lines through the point x. As K_i and K_j are disjoint for $i \neq j$, the lines K_i^θ and K_j^θ are not contained in a totally singular plane of $Q^+(7,q)$. On the other hand, $K_i \cap T_j$ is a point for all $i,j \in \{0,1,\ldots,q\}$, so that $\langle K_i^\theta, T_j^\theta\rangle$ must be a totally singular plane of $Q^+(7,q)$, by Property 7.1.2. Hence the only possible configuration is that $\langle K_0^\theta, K_1^\theta, \ldots, K_q^\theta\rangle \cap Q^+(7,q)$ is a quadratic cone $x\mathcal{C}$ with generating lines $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$ and similarly $\langle T_0^\theta, T_1^\theta, \ldots, T_q^\theta\rangle \cap Q^+(7,q)$ is a quadratic cone $x\mathcal{C}'$, but such that $\langle x\mathcal{C}\rangle^\perp \cap Q^+(7,q) = x\mathcal{C}'$. One immediately sees that the converse also holds: if $K_0^\theta, K_1^\theta, \ldots, K_q^\theta$, respectively $T_0^\theta, T_1^\theta, \ldots, T_q^\theta$, are the generators of a quadratic cone $x\mathcal{C}$, respectively $x\mathcal{C}'$, with $\langle x\mathcal{C}\rangle^\perp \cap Q^+(7,q) = x\mathcal{C}'$ as an additional property, then $\{K_0, K_1, \ldots, K_q\}$ is a regulus in G with opposite regulus $\{T_0, T_1, \ldots, T_q\}$.
It now easily follows that $\mathcal{M}^\theta$ is semiclassical if and only if the spread $\mathcal{S}_G$ is a regular spread of G, for all generators $G \in \mathcal{F}_2$ through the line L. □

Thanks to this lemma, we are able to show that $\mathcal{M}^\theta$ is semiclassical whenever $\mathcal{M}$ itself is semiclassical, by relying on the properties of locally hermitian, semiclassical 1-systems of $Q(6,q)$. We start with the case of a proper 1-system of $Q(6,q)$.

Theorem 7.3.2 *Let $\mathcal{M}$ be a proper 1-system of a $Q(6,q) \subseteq Q^+(7,q)$, which* *is locally hermitian at the line $L \in \mathcal{M}$, and consider a T-correspondence θ of Ω. If $\mathcal{M}$ is semiclassical, then $\mathcal{M}^\theta$ is semiclassical too.*

Proof.
Suppose that $\mathcal{M}$ is locally hermitian at the line L and furthermore semiclassical, and let the q^2 reguli of $\mathcal{M}$ through L be denoted by $R_1, R_2, \ldots, R_{q^2}$, as always. Consider two arbitrary reguli R_i and R_j of $\mathcal{M}$ through L. Then we know from Lemma 4.6.1 that the elliptic quadric $\langle R_i, R_j\rangle \cap Q^+(7,q) = Q^-_{ij}(5,q)$ contains exactly q reguli of $\mathcal{M}$ through L. To make the notation a little less complicated, let $i = 1$ and $j = 2$ and denote the q reguli of $\mathcal{M}$ through L in $Q^-_{12}(5,q)$ by $R_1, R_2, \ldots, R_q$. Let $G \in \mathcal{F}_2$ be an arbitrary generator of $Q^+(7,q)$ through the line L. We shall show that the lines $\langle R_i\rangle^\perp \cap G$, $i = 1, 2, \ldots, q$, together with L, form a regulus of lines of $\mathcal{S}_G$. Of course, the proof holds for all R_i, R_j, $i, j \in \{1, 2, \ldots, q^2\}$ with $i \neq j$, and not only for R_1 and R_2.

As the reguli $R_1, R_2, \ldots, R_q$ form an $\mathcal{R}$-conic of $Q^-_{12}(5,q)$ by Lemma 4.6.1, the transversals of $R_1, R_2, \ldots, R_q$ through x, together with the line L, form a quadratic cone $x\mathcal{C}$ on $Q^-_{12}(5,q)$ for every point x on L. Hence $\langle x\mathcal{C}\rangle^\perp \cap Q^+(7,q)$ is a quadratic cone $x\mathcal{C}'$ through x and contained in $L^\perp$, but not containing L. The 3-space $\langle x\mathcal{C}'\rangle$ meets every generator through L in a line, which is a generating line of $x\mathcal{C}'$. Denote the line $x\mathcal{C}' \cap G$ by A_x.
We also know that $\langle x\mathcal{C}\rangle$ meets each $\langle R_i\rangle$, $i = 1, 2, \ldots, q$, in a plane through L. As a consequence, $\langle x\mathcal{C}'\rangle \cap \langle Q^+_i(3,q)\rangle$ is a plane as well for all $i = 1, 2, \ldots, q$, where $Q^+_i(3,q) := \langle R_i\rangle^\perp \cap Q^+(7,q)$. Since every $Q^+_i(3,q)$ is disjoint from L and every line of $x\mathcal{C}'$ meets L in the point x, the quadratic cone $x\mathcal{C}'$ and $Q^+_i(3,q)$ have a non-singular conic $\mathcal{C}_i$ in common. But $\mathcal{C}_i$ shares exactly one point with the line A_x, which shows that A_x has a unique point in common with $Q^+_i(3,q)$, for $i = 1, 2, \ldots, q$. In other words: A_x meets L and the q lines $\langle R_i\rangle^\perp \cap G$, $i = 1, 2, \ldots, q$, in a point.
Suppose that two such lines A_x and A_y, for distinct points x and y on L, share a point u, say $u \in Q^+_1(3,q)$. Since $x \neq y$, $\langle A_x, A_y\rangle$ must then be a plane that contains L and the $q-1$ lines $\langle R_i\rangle^\perp \cap G$, $i = 2, 3, \ldots, q$. But this means that L and $\langle R_i\rangle^\perp$ are not disjoint for $i = 2, 3, \ldots, q$, a contradiction. So every two lines A_x and A_y, for distinct $x, y \in L$, are disjoint.
By the above, it follows that $\{L\} \cup \{\langle R_i\rangle^\perp \cap G \mid i = 1, 2, \ldots, q\}$ is a regulus of lines of $\mathcal{S}_G$, with opposite regulus $\{A_x \mid x \in L\}$. This property does not only hold for the q reguli $R_1, R_2, \ldots, R_q$, but for any q reguli of $\mathcal{M}$ through L which are contained in an elliptic quadric $Q^-_{ij}(5,q)$. As such, we have found that for any two lines $\langle R_i\rangle^\perp \cap G$ and $\langle R_j\rangle^\perp \cap G$, $i \neq j$, the regulus defined by these two lines and L completely consists of lines of $\mathcal{S}_G$. Consequently, there are $q(q+1)$ reguli through L in $\mathcal{S}_G$ and by Theorem 3.1 of Gevaert, Johnson

and Thas [19], it follows that $\mathcal{S}_G$ is regular.
Finally, since the generator $G \in \mathcal{F}_2$ through L was chosen arbitrarily, the desired result follows from Lemma 7.3.1. □

Note that the previous theorem is not valid for locally hermitian spreads of an elliptic quadric $Q^-(5,q)$, which is embedded in $Q^+(7,q)$. The following result solves the corresponding question in this case: it is possible to show that $\mathcal{M}^\theta$ is semiclassical if and only if the spread $\mathcal{M}$ of $Q^-(5,q)$ arises from a regular line spread of $\mathsf{PG}(3,q)$ by the construction, explained in Section 4.6.2.

Theorem 7.3.3 *Let $\mathcal{M}$ be a spread of an elliptic quadric $Q^-(5,q)$, embedded in $Q^+(7,q)$, which is locally hermitian at some line $L \in \mathcal{M}$, and consider a T-correspondence θ of Ω. Then the image $\mathcal{M}^\theta$ of $\mathcal{M}$ under θ is semiclassical if and only if $\mathcal{M} = \mathcal{S}(S)$ for a regular line spread S of $T_L(Q^-(5,q))$.*

Proof.
Suppose that $\mathcal{M}$ is locally hermitian at the line $L \in \mathcal{M}$ and denote the q^2 reguli of $\mathcal{M}$ through L by $R_1, R_2, \ldots, R_{q^2}$. By Lemma 7.3.1, $\mathcal{M}^\theta$ is semiclassical if and only if for every generator $G \in \mathcal{F}_2$ of $Q^+(7,q)$ through L, the spread $\mathcal{S}_G$ is a regular line spread of G. As $\mathcal{M}$ is locally hermitian, we can write $\mathcal{M} = \mathcal{S}(S)$ for some line spread S of $\beta := T_L(Q^-(5,q))$. Let T be the line $\langle Q^-(5,q)\rangle^\perp$, with $\perp$ the polarity of $Q^+(7,q)$. Then $\langle \beta, T\rangle$ is the 5-dimensional subspace $L^\perp$, with $\perp$ again the polarity of $Q^+(7,q)$. Also, both β and $\langle L, T\rangle = \beta^\perp$ only intersect $Q^+(7,q)$ in L. Now the spread S in β consists of L and the q^2 lines $\langle R_i\rangle^\perp \cap \beta$, and every subspace $\langle R_i\rangle^\perp$ contains the line T. So the spread S is in fact the projection from T onto β of the spaces $\langle L, T\rangle \setminus T$ and $\langle R_i\rangle^\perp \setminus T$, $i = 1, 2, \ldots, q^2$.
If $G \in \mathcal{F}_2$ is a generator of $Q^+(7,q)$ through L, then $\beta \cap G$ is the line L. It also holds that the spread $\mathcal{S}_G$ consists of L and the q^2 lines $\langle R_i\rangle^\perp \cap G$, so it is the projection from T onto G of $\langle L, T\rangle \setminus T$ and $\langle R_i\rangle^\perp \setminus T$, $i = 1, 2, \ldots, q^2$.
Hence it is clear that S is a regular line spread of β if and only if $\mathcal{S}_G$ is a regular line spread in G. This holds for all generators $G \in \mathcal{F}_2$ of $Q^+(7,q)$ containing the line L, so that the theorem follows. □

By now, we have quite extensive knowledge on what $\mathcal{M}^\theta$ looks like if $\mathcal{M} \subseteq Q(6,q)$ is locally hermitian and semiclassical and θ is a T-correspondence of Ω. If $\mathcal{M}$ is a spread of an induced elliptic quadric $Q^-(5,q) \subseteq Q(6,q)$, then Theorem 7.3.3 provides a criterion to decide whether $\mathcal{M}^\theta$ is semiclassical or not. Also, $\mathcal{M}^\theta$ spans the whole $\mathsf{PG}(7,q)$ whenever $\mathcal{M}$ is not hermitian.
If $\mathcal{M}$ is a proper locally hermitian, semiclassical 1-system of $Q(6,q)$, then $\mathcal{M}^\theta$ is also locally hermitian and semiclassical. Moreover, it is not contained

in a hyperplane of $\mathsf{PG}(7,q)$, provided that $\mathcal{M}$ is not a spread of some $\mathsf{H}(q)$ on $Q(6,q)$. In particular, it then follows that $\mathcal{M}$ and $\mathcal{M}^\theta$ are not isomorphic.

In the next section, we will have a look at $\mathcal{M}^\theta$ in view of the results of Chapter 6, since a characterization of the proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$ was obtained there.

7.4 $\mathcal{M}^\theta$ amongst the proper locally hermitian 1-systems of $Q^+(7,q)$

It is the aim of this section to compare the 1-systems $\mathcal{M}^\theta$, with $\mathcal{M}$ and θ as in the foregoing section, to the other previously known proper locally hermitian 1-systems of $Q^+(7,q)$, namely the ones that arise from an ovoid $\mathcal{O}$ of a hermitian polar space $H(3,q^2)$. The following theorem learns that both classes are essentially distinct.

Theorem 7.4.1 *Suppose that $\mathcal{M}$ is a locally hermitian, semiclassical 1-system of $Q(6,q) \subseteq Q^+(7,q)$ and let θ be a T-correspondence of Ω. Then $\mathcal{M}^\theta$ arises from an ovoid of $H(3,q^2)$ if and only if $\mathcal{M}$ is a spread of an induced $Q^-(5,q) \subseteq Q(6,q)$.*

Proof.
For this proof, let $\mathsf{PG}(7,q^2)$ be the extension of $\mathsf{PG}(7,q)$ to $\mathsf{GF}(q^2)$; the quadric $Q^+(7,q)$ extends to a hyperbolic quadric $Q^+(7,q^2)$.

First assume that $\mathcal{M}$ is a spread of an induced $Q^-(5,q) \subseteq Q(6,q)$. Then the extension of $Q^-(5,q)$ to $\mathsf{GF}(q^2)$ is a hyperbolic quadric $Q^+(5,q^2)$ and $\langle Q^+(5,q^2)\rangle^\perp \cap Q^+(7,q^2)$ is a secant line meeting $Q^+(7,q^2)$ in two points x and $\overline{x}$ which are conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. As every line of $\mathcal{M}$, considered as a line over $\mathsf{GF}(q^2)$, lies in a totally singular plane of $Q^+(7,q^2)$ through x, and in another one through $\overline{x}$, every line of $\mathcal{M}^\theta$, again considered over $\mathsf{GF}(q^2)$, has a point in common with the disjoint generators x^θ and $\overline{x}^\theta$ of $Q^+(7,q^2)$. This implies that $\mathcal{M}^\theta = \{y\overline{y} \cap \mathsf{PG}(7,q) \mid y \in \mathcal{O}\}$ for some ovoid $\mathcal{O}$ of the hermitian variety $H(3,q^2)$ which is defined in x^θ by the lines of $Q^+(7,q)$, the extensions of which meet both x^θ and $\overline{x}^\theta$.

Conversely, assume that there exist two generators G and $\overline{G}$ of $Q^+(7,q^2)$, which are disjoint and conjugate with respect to the extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. Further, suppose that $\mathcal{M}^\theta = \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in \mathcal{O}\}$ for some ovoid $\mathcal{O}$ of the hermitian variety $H(3,q^2)$, defined in G by the lines of $Q^+(7,q)$, the extensions of which to $\mathsf{GF}(q^2)$ meet both G and $\overline{G}$ in a point. There are two possibilities: either G and $\overline{G}$ are elements of the first family of generators of $Q^+(7,q^2)$, or they both belong to the second family of generators.

If G and $\overline{G}$ are elements of $\mathcal{F}_1$, then $G^{\theta^{-1}}$ and $\overline{G}^{\theta^{-1}}$ are points of $Q^+(7,q^2) \setminus Q^+(7,q)$. Moreover, every line of $\mathcal{M}^\theta$ has a point in common with G and $\overline{G}$, and consequently every line of $\mathcal{M}$, if considered over $\mathsf{GF}(q^2)$, lies in a totally singular plane of $Q^+(7,q^2)$ through $G^{\theta^{-1}}$ and in another one containing $\overline{G}^{\theta^{-1}}$. But this means that the line $\langle G^{\theta^{-1}}, \overline{G}^{\theta^{-1}} \rangle \cap \mathsf{PG}(7,q)$ is an external line of $Q^+(7,q)$, such that all lines of $\mathcal{M}$ are contained in the elliptic quadric $\langle G^{\theta^{-1}}, \overline{G}^{\theta^{-1}} \rangle^\perp \cap Q^+(7,q) = Q^-(5,q)$.
In the case where $G, \overline{G} \in \mathcal{F}_2$, the spaces $G^{\theta^{-1}}$ and $\overline{G}^{\theta^{-1}}$ are generators of $Q^+(7,q^2)$ of the first family. Consider an arbitrary line $M \in \mathcal{M}^\theta$ and denote the point $M \cap G$ by x. Then $M^{\theta^{-1}}$ is a line in the generator $x^{\theta^{-1}} \in \mathcal{F}_2$. But x is a point of G, which means that $x^{\theta^{-1}} \cap G^{\theta^{-1}}$ is a plane, so that the extension to $\mathsf{GF}(q^2)$ of $M^{\theta^{-1}}$ has a point y in common with $G^{\theta^{-1}}$. Similarly, the extension to $\mathsf{GF}(q^2)$ of $M^{\theta^{-1}}$ also has the point $\overline{y}$ in common with $\overline{G}^{\theta^{-1}}$. As $M \in \mathcal{M}^\theta$ was chosen arbitrarily, this implies that $\mathcal{M}$ too arises from an ovoid $\mathcal{O}'$ of the hermitian polar space $H(3,q^2)'$ in $G^{\theta^{-1}}$, by the usual construction. On the other hand, $\mathcal{M}$ is by assumption contained in a hyperplane of $\mathsf{PG}(7,q)$, so $\langle \mathcal{O}' \rangle$ cannot coincide with the 3-dimensional subspace $G^{\theta^{-1}}$. Thus $\langle \mathcal{O}' \rangle$ must be a plane in $G^{\theta^{-1}}$ and it follows that $\mathcal{M}$ is contained in the 5-dimensional subspace $\langle \mathcal{O}', \overline{\mathcal{O}'} \rangle$, considered over $\mathsf{GF}(q)$. We conclude that $\mathcal{M}$ is a 1-system of the elliptic quadric $\langle \mathcal{O}', \overline{\mathcal{O}'} \rangle \cap Q^+(7,q) = Q^-(5,q)$. $\square$

By the information of this theorem, it is clear that $\mathcal{M}^\theta$ is not isomorphic to an example of a locally hermitian 1-system of $Q^+(7,q)$ which was known before the discussion in Chapter 6, provided that $\langle \mathcal{M} \rangle = \mathsf{PG}(6,q)$ and $\mathcal{M}$ is not a spread of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$. For, as we have seen in the previous chapter, the only previously known proper locally hermitian 1-systems of $Q^+(7,q)$ arise from an ovoid $\mathcal{O}$ of the hermitian polar space $H(3,q^2)$. In combination with Theorem 6.3.3, the above theorem also implies that the flock, associated with $\mathcal{M}^\theta$, cannot be of regular spread type. Thus we have obtained an alternative description of a subclass in the class of all proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$, namely the subclass of all such 1-systems of the form $\mathcal{M}^\theta$, with $\mathcal{M}$ and θ as above.

7.5 Isomorphism results for $\mathcal{M}^\theta$

Another question that appears naturally, is the following. In Chapters 4 and 5, it has been shown that there exist several orbits under the action of $\mathsf{P\Gamma O}(7,q)$ in the set of the proper locally hermitian, semiclassical 1-systems of $Q(6,q)$. It seems obvious in this context to wonder whether two non-isomorphic 1-systems $\mathcal{M}_1$ and $\mathcal{M}_2$ of $Q(6,q)$ yield non-isomorphic 1-systems

$\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ of $Q^+(7,q)$, with θ a T-correspondence as is usual in this chapter. It will turn out that this is indeed the case.

We start with two easy cases; the first one is a corollary of Theorem 7.4.1.

Corollary 7.5.1 *Let θ be a T-correspondence of Ω. A spread $\mathcal{M}$ of an induced $Q^-(5,q) \subseteq Q^+(7,q)$ is a hermitian spread of $Q^-(5,q)$ if and only if $\mathcal{M}^\theta$ is a hermitian spread of some $Q^-(5,q)' \subseteq Q^+(7,q)$.*

Proof.
$\mathcal{M}$ is a hermitian spread of a $Q^-(5,q)$ on $Q^+(7,q)$ if and only if there hold two things: firstly $\mathcal{M}$ must be a spread of a $Q^-(5,q)$ and secondly $\mathcal{M}$ must arise from an ovoid $\mathcal{O}$ of $H(3,q^2)$, which is in this case the classical ovoid $\mathcal{O} \cong H(2,q^2)$. By applying Theorem 7.4.1 to θ, it follows from the first property that $\mathcal{M}^\theta$ arises from some ovoid $\mathcal{O}'$ of a hermitian variety $H(3,q^2)'$. On the other hand, $\mathcal{M} = (\mathcal{M}^\theta)^{\theta^{-1}}$ arises from an ovoid of $H(3,q^2)$, so that, again by Theorem 7.4.1 which we now apply to the T-correspondence θ^{-1}, $\mathcal{M}^\theta$ is a 1-system of a $Q^-(5,q)' \subseteq Q^+(7,q)$. By combining both results, the statement of the corollary follows. $\square$

Corollary 7.5.2 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be locally hermitian, semiclassical spreads of the hexagon $\mathsf{H}(q)$, embedded in a $Q(6,q) \subseteq Q^+(7,q)$. Then $\mathcal{M}_1 \cong \mathcal{M}_2$ under the action of $\mathsf{P\Gamma O}^+(8,q)$ if and only if $\mathcal{M}_1^\theta \cong \mathcal{M}_2^\theta$ under the action of $\mathsf{P\Gamma O}^+(8,q)$, with θ a T-correspondence of Ω.*

Proof.
This is an immediate consequence of Lemma 7.2.3, Corollary 7.5.1 and the fact that both for q odd and for q even, there exist at most two non-isomorphic locally hermitian, semiclassical spreads of $\mathsf{H}(q)$: the hermitian spread of a $Q^-(5,q)$ and $\mathcal{S}_{[9]}$ if q is odd and $q \equiv 1 \bmod 3$, see [4], respectively $\mathcal{S}_{[\delta]}$ if q is even and $q = 2^{2e}$, see [36]. $\square$

The next theorem handles the general case.

Theorem 7.5.3 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two proper locally hermitian, semiclassical 1-systems of a $Q(6,q) \subseteq Q^+(7,q)$ and let θ be a T-correspondence of Ω. Then $\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ are isomorphic under the action of $\mathsf{P\Gamma O}^+(8,q)$ if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic under the action of $\mathsf{P\Gamma O}^+(8,q)$.*

Proof.
In the proof, we will need the element $\mu_p \in \mathsf{PGO}^+(8,q)$, see Section 1.1.4, where p is the point $\langle Q(6,q)\rangle^\perp$ with $\perp$ the polarity of $Q^+(7,q)$. It is known that μ_p fixes $Q(6,q)$ pointwise, but it has another interesting property. Let

$G_1 \in \mathcal{F}_1$ be a generator of $Q^+(7,q)$. Then $G_1 \cap Q(6,q)$ is a totally singular plane, say π. As π is fixed by μ_p, the image of G_1 under μ_p must be a generator, different from G_1, through π. This generator is uniquely defined as the generator G_2 through π which belongs to $\mathcal{F}_2$. So $G_1^{\mu_p} = G_2$ and similarly $G_2^{\mu_p} = G_1$ and thus μ_p interchanges the families of generators $\mathcal{F}_1$ and $\mathcal{F}_2$.

First assume that $\mathcal{M}_1 \cong \mathcal{M}_2$ under the action of $\mathsf{P\Gamma O}^+(8,q)$, so there exists an element $\alpha \in \mathsf{P\Gamma O}^+(8,q)$ such that $\mathcal{M}_1^\alpha = \mathcal{M}_2$. If α interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\alpha\mu_p \in \mathsf{P\Gamma O}^+(8,q)$ fixes both families of generators and it also maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. So we may assume that α preserves the families $\mathcal{F}_1$ and $\mathcal{F}_2$. In that case we have that

$$(\mathcal{M}_1^\theta)^{\theta^{-1}\alpha\theta} = \mathcal{M}_1^{\alpha\theta} = \mathcal{M}_2^\theta,$$

so that $\theta^{-1}\alpha\theta$ maps $\mathcal{M}_1^\theta$ onto $\mathcal{M}_2^\theta$. But, as α preserves the families of generators, the map $\theta^{-1}\alpha\theta$ stabilizes the sets $\mathcal{P}$, $\mathcal{L}$, $\mathcal{F}_1$ and $\mathcal{F}_2$, preserves the incidence between these sets and stabilizes $Q^+(7,q)$. Hence $\theta^{-1}\alpha\theta$ is an element of $\mathsf{P\Gamma O}^+(8,q)$, which shows that $\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ are also isomorphic under $\mathsf{P\Gamma O}^+(8,q)$.

To prove the converse, assume that $\mathcal{M}_1^\theta$ and $\mathcal{M}_2^\theta$ are isomorphic under the action of $\mathsf{P\Gamma O}^+(8,q)$, with isomorphism $\alpha \in \mathsf{P\Gamma O}^+(8,q)$. Then it holds that $\mathcal{M}_1^{\theta\alpha} = \mathcal{M}_2^\theta$, from which one derives that $\mathcal{M}_1^{\theta\alpha\theta^{-1}} = \mathcal{M}_2$. Here one should distinguish between the case where α preseves $\mathcal{F}_1$ and $\mathcal{F}_2$ and the case where it interchanges both families of generators.
If α preseves the families $\mathcal{F}_1$ and $\mathcal{F}_2$, one easily sees that $\theta\alpha\theta^{-1}$ is itself an element of $\mathsf{P\Gamma L}(8,q)$, stabilizing $Q^+(7,q)$. So it immediately follows that $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ under $\mathsf{P\Gamma O}^+(8,q)$, with isomorphism $\theta\alpha\theta^{-1}$.
Suppose next that α interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$. It still holds that the map $\theta\alpha\theta^{-1}$ maps $\mathcal{M}_1$ onto $\mathcal{M}_2$, but now $\theta\alpha\theta^{-1}$ is not a collineation. It is easily seen that $\theta\alpha\theta^{-1}$ still maps lines onto lines, but unfortunately it interchanges points and generators of the first family, while preserving the second family of generators. Now the element $\mu_p \in \mathsf{PGO}^+(8,q)$ comes into play, as we consider the map $\theta\alpha\theta^{-1}\mu_p$. Clearly this map acts as follows on $\mathcal{M}_1$:

$$\mathcal{M}_1^{\theta\alpha\theta^{-1}\mu_p} = \mathcal{M}_2^{\mu_p} = \mathcal{M}_2,$$

so $\theta\alpha\theta^{-1}\mu_p$ also maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. But by the fact that μ_p interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$, $\theta\alpha\theta^{-1}\mu_p$ maps points onto elements of $\mathcal{F}_2$, elements of $\mathcal{F}_2$ onto elements of $\mathcal{F}_1$ and elements of $\mathcal{F}_1$ onto points. As it also preserves the incidence of Ω, $\theta\alpha\theta^{-1}\mu_p$ is a T-correspondence of Ω. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are 1-systems of a $Q(6,q) \subseteq Q^+(7,q)$, it now follows from Theorem 7.2.4 that

$\mathcal{M}_1$ and $\mathcal{M}_2$ are spreads of a generalized hexagon $\mathsf{H}(q)$ on $Q(6,q)$. Consequently Corollary 7.5.2 yields that $\mathcal{M}_1 \cong \mathcal{M}_2$ under $\mathsf{P\Gamma O}^+(8,q)$.
This proves the theorem. $\square$

Theorem 7.5.3 learns how many orbits under the action of $\mathsf{P\Gamma O}^+(8,q)$ there are in the set of all 1-systems of $Q^+(7,q)$ of the form $\mathcal{M}^\theta$, with $\mathcal{M}$ a proper locally hermitian, semiclassical 1-system of $Q(6,q) \subseteq Q^+(7,q)$ and θ a fixed T-correspondence or triality of the D_4-geometry Ω, attached to $Q^+(7,q)$. These numbers must be the same numbers as we have found in the remark following Theorem 4.5.2 for q odd and in the remark following Theorem 5.5.1 for q even.

Chapter 8

Derivation of m-systems

"Derivation" of m-systems refers to a method to construct new m-systems from known ones by replacing a number of elements of the m-system by a set of other subspaces, covering the same points. It is inspired by a very similar method which is known for line spreads of projective and polar spaces: if such a spread contains a regulus of lines, then one obtains another spread by deleting this regulus and replacing it by the regulus of its transversals. The method we discuss here, is a straightforward generalization of this technique and is also applicable to m-systems of polar spaces with $m > 1$, under certain conditions.

After an explanation of the concept of derivation of m-systems has been given, it will be investigated in which cases this method can be applied. The most fruitful case appears to be that of 1-systems containing reguli. In particular locally hermitian 1-systems, especially those of quadrics, are suitable for derivation and we will see that from the classes of 1-systems that have been constructed in Chapters 4, 5 and 6, other classes of 1-systems can be obtained. Unfortunately, it seems to be difficult to find m-systems with $m > 1$, which are appropriate for derivation.

Most of this chapter is contained in [42], which has been submitted for publication in the *European Journal of Combinatorics*.

8.1 Constructing new m-systems from known ones

In some cases, it is possible to construct new m-systems from known ones in a very simple way: by replacing a set of subspaces of the m-system by a suitably chosen other set of subspaces, such that the unions of the elements of both sets coincide.

Theorem 8.1.1 *Let $\mathcal{M} := \{\pi_1, \pi_2, \ldots, \pi_r\}$ be an m-system of a finite classical polar space $\mathcal{P}_n$ in $\mathsf{PG}(n,q)$. Suppose that there exists a $(2m+1)$-dimensional subspace $\mathsf{PG}(2m+1,q)$ of $\mathsf{PG}(n,q)$, containing s elements π_1, $\pi_2, \ldots, \pi_s$ of $\mathcal{M}$, with $1 \leq s \leq r$. If in $\mathsf{PG}(2m+1,q) \cap \mathcal{P}_n$ there exist s disjoint m-dimensional totally singular subspaces $\alpha_1, \alpha_2, \ldots, \alpha_s$ of $\mathcal{P}_n$ such that the point sets $\pi_1 \cup \pi_2 \cup \cdots \cup \pi_s$ and $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_s$ coincide, then $\mathcal{M}' := \{\alpha_1, \alpha_2, \ldots, \alpha_s, \pi_{s+1}, \pi_{s+2}, \ldots, \pi_r\}$ is also an m-system of $\mathcal{P}_n$.*

Proof.
If $\mathcal{M}$ is a spread of $\mathcal{P}_n$, then it is clear that $\mathcal{M}'$ also is a spread of $\mathcal{P}_n$, as $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$.

So assume that $\mathcal{M}$ is not a spread of $\mathcal{P}_n$. We must then show that every $(m+1)$-dimensional totally singular subspace of $\mathcal{P}_n$ containing an element of $\mathcal{M}'$, is disjoint from all other elements of $\mathcal{M}'$. Any $(m+1)$-dimensional totally singular subspace through an element π_i, $i \geq s+1$, clearly satisfies this condition, as $\mathcal{M}$ is an m-system and $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$. So, consider an $(m+1)$-dimensional totally singular subspace γ of $\mathcal{P}_n$ through an element α_i, $i \in \{1, 2, \ldots, s\}$. Since the elements α_i are generators of the (non-singular) polar space $\mathsf{PG}(2m+1,q) \cap \mathcal{P}_n$, γ is not contained in $\mathsf{PG}(2m+1,q)$ and thus disjoint from all elements α_j, $j \in \{1, 2, \ldots, s\} \setminus \{i\}$. Suppose that γ contains a point x of some π_j, $j \in \{s+1, s+2, \ldots, r\}$. If $\mathcal{P}_n$ is not a parabolic quadric for q even, let $\perp$ be the polarity defined by $\mathcal{P}_n$. Then $\langle \mathsf{PG}(2m+1,q) \rangle^{\perp}$ intersects $\mathcal{P}_n$ in a non-singular polar space $\mathcal{P}_{n-2m-2}$ and for each generator G of $\mathsf{PG}(2m+1,q) \cap \mathcal{P}_n$ it holds that $G^{\perp} \cap \mathcal{P}_n$ is the cone $G\mathcal{P}_{n-2m-2}$. Hence γ contains a point y of $\mathcal{P}_{n-2m-2}$; also, as $\mathcal{M}$ is an m-system, the point x does not belong to $\mathcal{P}_{n-2m-2}$. The line xy of γ meets α_i in some point z, which belongs to exactly one π_k with $k \in \{1, 2, \ldots, s\}$, because $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$. Now, as π_k is also a generator of $\mathsf{PG}(2m+1,q) \cap \mathcal{P}_n$, the subspace $\langle \pi_k, y \rangle$ is totally singular and meets the element π_j of $\mathcal{M}$ in the point x. This is a contradiction to the fact that $\mathcal{M}$ is an m-system and we conclude that γ must be disjoint from all elements π_j, $j \in \{s+1, s+2, \ldots, r\}$. This implies that $\mathcal{M}'$ is also an m-system of $\mathcal{P}_n$.

If q is even and $\mathcal{P}_n$ is a parabolic quadric $Q(n,q)$, no polarity is associated with $\mathcal{P}_n$. However, if we project $\mathcal{P}_n$ from its nucleus w onto a hyperplane $\mathsf{PG}(n-1,q)$ of $\mathsf{PG}(n,q)$ not containing w, then there arises a symplectic polar space $W_{n-1}(q)$. Hence $\mathcal{P}_n \cong W_{n-1}(q)$ and now the result follows from the foregoing paragraph. □

It is clear that Theorem 8.1.1 can be applied to any locally hermitian 1-system $\mathcal{M}$ of a non-singular quadric, in which case a regulus can be replaced by its opposite regulus to obtain another 1-system $\mathcal{M}'$. In this situation, it is possible to show that the 1-system $\mathcal{M}'$ arising in this way, is not isomorphic to the original one. This is an interesting result, because it implies that $\mathcal{M}'$ is new.

Theorem 8.1.2 *Consider a 1-system $\mathcal{M}$ of a non-singular quadric $\mathcal{Q}$ in $\mathsf{PG}(n,q)$, and assume that $\mathcal{M}$ is locally hermitian at some line $L \in \mathcal{M}$. Let $\mathcal{M}'$ be the 1-system of $\mathcal{Q}$, which is obtained by replacing a regulus R of $\mathcal{M}$ through L by its opposite regulus $\overline{R}$. Then $\mathcal{M}'$ is not locally hermitian and so $\mathcal{M}$ and $\mathcal{M}'$ are not isomorphic.*

Proof.
Denote the reguli of $\mathcal{M}$ through L by $R, R_1, R_2, \ldots, R_{s-1}$, with $s = \frac{|\mathcal{M}|-1}{q}$ and let $\mathcal{M}'$ consist of the lines of $R_i \setminus \{L\}$, $i = 1, 2, \ldots, s-1$, together with the lines of the opposite regulus $\overline{R}$ of R. We proceed to show that $\mathcal{M}'$ cannot be locally hermitian at any line $M \in \mathcal{M}'$, since for no line $M \in \mathcal{M}'$ it is possible to find s 3-spaces on M, each containing $q+1$ lines of $\mathcal{M}'$, while the other 3-spaces on M have no line in common with $\mathcal{M}'$ but M itself.

Consider a line M of $\overline{R}$ and the 3-space $\langle M, N\rangle$ with N an arbitrary line of a regulus R_i, $i \in \{1, 2, \ldots, s-1\}$, distinct from L. Then $\langle M, N\rangle$ contains at least two lines of $\mathcal{M}'$ and meets the $q-1$ lines of $R_i \setminus \{L, N\}$ in at least a point. Since $\mathcal{Q}$ is a non-singular quadric, a 3-dimensional subspace of $\mathsf{PG}(5,q)$ which contains $q+1$ lines of a 1-system of $\mathcal{Q}$, must be disjoint from all other lines of this 1-system, since a 3-dimensional subspace with this property must intersect $\mathcal{Q}$ in a hyperbolic quadric $Q^+(3,q)$. So, if $\langle M, N\rangle$ contained $q+1$ lines of $\mathcal{M}'$, then these $q+1$ lines would have to be M and the q lines of $R_i \setminus \{L\}$. But then $\langle M, N\rangle = \langle R_i\rangle$, so that the 3-dimensional subspace $\langle R_i\rangle$ would contain $q+1$ lines of the 1-system $\mathcal{M}$ and q additional points of $\tilde{\mathcal{M}}$, namely the points of M not on L. This is a contradiction. We conclude that the 3-space $\langle M, N\rangle$ contains at least two, but less than $q+1$ lines of $\mathcal{M}'$ and thus we cannot find a suitable collection of 3-spaces containing the line $M \in \overline{R}$.

Now suppose that $M \notin \overline{R}$. Then there exists a regulus R_i of $\mathcal{M}$ with $M \in R_i \setminus \{L\}$. The 3-space $\langle R_i\rangle$ meets $\mathcal{M}'$ in at least q lines and contains

$q+1$ points of $\tilde{\mathcal{M}}'$, not on these lines, namely the points of L. If $\langle R_i \rangle$ contained $q+1$ lines of $\mathcal{M}'$, then necessarily the line L would be one of these lines, a contradiction because $L \notin \mathcal{M}'$. As a consequence, $\langle R_i \rangle$ has exactly q lines in common with $\mathcal{M}'$ and again there is no suitable collection of 3-spaces containing M.

Thus we may conclude that $\mathcal{M}'$ is not locally hermitian and so it cannot be isomorphic to $\mathcal{M}$. □

Concerning the known examples of locally hermitian, semiclassical 1-systems of $Q(6,q)$ or $Q^+(7,q)$, which are non-hermitian (see Chapters 4, 5 and 6), one can even say more. In that case, we want to show that the 1-systems, arising from distinct, non-isomorphic, locally hermitian, semiclassical 1-systems of $Q(6,q)$ or $Q^+(7,q)$ by switching a regulus, are also non-isomorphic.

So suppose that $\mathcal{M}$ is a proper locally hermitian, semiclassical 1-system of $Q(6,q)$ or $Q^+(7,q)$. Then we first observe that $\mathcal{M}$ contains no reguli of lines, except for the ones through the line L at which $\mathcal{M}$ is locally hermitian. This can be seen as follows. Clearly, a hypothetical regulus R of lines of $\mathcal{M}$, not containing L, must consist of $q+1$ lines, belonging to $q+1$ different reguli of $\mathcal{M}$ through L. But this implies that the 5-dimensional subspace $\langle L, R \rangle$ contains at least $q+1$ reguli of $\mathcal{M}$ through L. This is a contradiction to Lemma 4.6.1 in case $\mathcal{M}$ is a 1-system of $Q(6,q)$, and to the proof of Theorem 6.2.1 if $\mathcal{M}$ is a 1-system of $Q^+(7,q)$. Hence we conclude that $\mathcal{M}$ contains exactly q^2 reguli, all of them containing the line L. By using this observation, one can show the following theorem.

Theorem 8.1.3 *Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two non-isomorphic, proper, locally hermitian, semiclassical 1-systems of $Q(6,q)$ or $Q^+(7,q)$. If $\mathcal{M}'_1$, respectively $\mathcal{M}'_2$, is obtained from $\mathcal{M}_1$, respectively $\mathcal{M}_2$, by replacing a regulus $R^{(1)}$, respectively $R^{(2)}$, by its opposite regulus $\overline{R}^{(1)}$, respectively $\overline{R}^{(2)}$, then $\mathcal{M}'_1$ and $\mathcal{M}'_2$ are not isomorphic either.*

Proof.
The 1-systems $\mathcal{M}'_i$, $i=1,2$, both contain exactly one regulus of lines, as can be seen as follows. Firstly, we have reasoned in the foregoing paragraph that they cannot contain a regulus which consists of lines of $\mathcal{M}_i$, so all possible reguli must contain a "new" line, that is, a line of $\overline{R}^{(i)}$. Now by the proof of Theorem 8.1.2, it holds that $\mathcal{M}'_i$, $i=1,2$, does not possess a regulus which contains as well lines of $\overline{R}^{(i)}$ as lines of $\mathcal{M}_i$. Thus the regulus $\overline{R}^{(i)}$ is the unique regulus in $\mathcal{M}'_i$, $i=1,2$.

As a consequence, an isomorphism $\theta \in \mathsf{P\Gamma O}(7,q)$, respectively $\mathsf{P\Gamma O}^+(8,q)$, which maps $\mathcal{M}'_1$ onto $\mathcal{M}'_2$, must also send $\overline{R}^{(1)}$ to $\overline{R}^{(2)}$. But this implies that

θ maps $R^{(1)}$ onto $R^{(2)}$ and hence $\mathcal{M}_1^\theta = \mathcal{M}_2$ must hold. This is a contradiction to the assumptions, so we conclude that $\mathcal{M}'_1 \ncong \mathcal{M}'_2$. $\square$

Remarks

1. Since for q even, the symplectic polar space $W_5(q)$ is isomorphic to the parabolic quadric $Q(6,q)$, the conclusions of both Theorem 8.1.2 and Theorem 8.1.3 also hold for locally hermitian 1-systems of $W_5(q)$, q even. Clearly, every statement about reguli then becomes a statement about strong reguli.

2. It is not obvious whether the result of Theorem 8.1.3 also holds in a more general setting, that is, for all locally hermitian 1-systems of an arbitrary non-singular quadric $\mathcal{Q}$. In this general case, we do not dispose of an analogue of Lemma 4.6.1, because the claim of that lemma is not true in general. This can be seen from the following counterexample.

   Consider an elliptic quadric $Q^-(5,q)$, $q \geq 3$, and an arbitrary line L on $Q^-(5,q)$. In the 3-space $T_L(Q^-(5,q))$, let S' be a regular spread of $T_L(Q^-(5,q))$ containing L and such that $\mathcal{S}' := \mathcal{S}(S')$ is a hermitian spread of $Q^-(5,q)$. Consider a regulus R' of S', not through L. Denote by S the spread of $T_L(Q^-(5,q))$, consisting of all lines of $S' \setminus R'$ and the lines of the opposite regulus $\overline{R}'$ of R'. Then S is not regular. Moreover, there certainly exists a regulus R in S', which contains no line of R' and does not contain L either. Hence $R \subseteq S$ and it follows that S is a spread of $T_L(Q^-(5,q))$ through L and containing a regulus of lines, not containing L. In addition, the lines of R, together with L, are contained in the regular spread S'.
   Denote the $q+1$ lines of R by $N_0, N_1, \ldots, N_q$ and let $\mathcal{M}$ be the locally hermitian spread $\mathcal{M} := \mathcal{S}(S)$ of $Q^-(5,q)$, arising as explained in Section 4.6.2 from the spread S of $T_L(Q^-(5,q))$. Then $\mathcal{M}$ is not hermitian, but the $q+1$ reguli $\langle L, N_i \rangle \cap Q^-(5,q)$, $i = 0,1,\ldots,q$, are part of the hermitian spread $\mathcal{S}' = \mathcal{S}(S')$ of $Q^-(5,q)$. Next, suppose that $Q^-(5,q)$ is embedded in a hyperbolic quadric $Q^+(7,q)$ and consider a triality τ of the D_4-geometry Ω, attached to $Q^+(7,q)$. Then the image $\mathcal{M}^\tau$ of $\mathcal{M}$ under τ is a locally hermitian 1-system, but it is not semiclassical by Theorem 7.3.3. Also, the images under τ of the $q+1$ reguli $\langle L, N_i \rangle \cap Q^-(5,q)$, $i = 0,1,\ldots,q$, belong to the 1-system $\mathcal{S}'^\tau$, which is a hermitian spread of some $Q^-(5,q)' \subseteq Q^+(7,q)$, by Corollary 7.5.1. It follows that $\mathcal{M}^\tau$ contains at least $q+1$ distinct reguli through L^τ, which lie in the 5-space $\langle Q^-(5,q)' \rangle$.

8.2 Examples

We first consider the most obvious situation in which Theorem 8.1.1 can be applied, namely that of a 1-system containing reguli. In this case, a regulus can be replaced by its opposite regulus, thus again yielding a 1-system, provided that the opposite regulus also consists of totally isotropic lines of the polar space. For 1-systems of $Q^-(5,q)$, which are in fact spreads, this is a well-known fact (see for instance [69]). In the following, we give an overview of other known 1-systems which are candidates for derivation with respect to a regulus.

For q odd, $Q^+(5,q)$ has a unique 1-system, see [57], which does not contain reguli. If q is even, every 1-system of $Q^+(5,q)$ is a spread of a $Q(4,q) \subseteq Q^+(5,q)$ and conversely, see again [57]. By 1.8.4 of Payne and Thas [50] an ovoid of $W_3(q)$ contains either 0 or 2 points of any given hyperbolic line of $W_3(q)$, and hence, as $W_3(q)$ is isomorphic to the point-line dual of $Q(4,q)$, a spread of $Q(4,q)$ contains at most 2 lines of any given regulus of $Q(4,q)$. So 1-systems of $Q^+(5,q)$, q even, do not satisfy the conditions of Theorem 8.1.1.

Of course, every 1-system of an induced elliptic quadric $Q^-(5,q)$ on $Q(6,q)$ also is a 1-system of $Q(6,q)$. But apart from these examples, $Q(6,q)$ has other classes of 1-systems containing reguli. Both for all q odd, see [4] and Chapter 4, and for all even q with $q \neq 2$, see [10], [36], [46] and Chapter 5, there exist locally hermitian 1-systems of $Q(6,q)$, which are not contained in a 5-dimensional subspace. In every such 1-system, which we assume to be locally hermitian at the line L, it is possible to replace any regulus containing L by its opposite regulus, which yields other 1-systems of $Q(6,q)$ for all $q > 2$. These 1-systems are not locally hermitian, as has been shown in Theorem 8.1.2, so they are new. For $q = 2$, 1-systems of $Q(6,2)$ have been classified by a computer result of Hamilton and Mathon [21]. They show that there exist two non-isomorphic examples: the regular line spread of a $Q^-(5,2) \subseteq Q(6,2)$ and the spread of $Q^-(5,2)$, obtained by replacing a regulus of the regular line spread by its opposite regulus.

The symplectic polar space $W_5(q)$ also has 1-systems containing reguli. For q odd however, the opposite regulus of a regulus of totally isotropic lines does not consist entirely of totally isotropic lines. Hence only the case of strong reguli if q is even, is interesting here. But for even q, the polar spaces $W_5(q)$ and $Q(6,q)$ are isomorphic, so here we find the same examples as for $Q(6,q)$.

By a result of Shult and Thas [57], see also Theorem A.1.4 in Appendix A, every ovoid $\mathcal{O}$ of $H(3,q^2)$ defines a 1-system of $Q^+(7,q)$. It is also known that a Baer subline the points of which belong to $\mathcal{O}$, corresponds to a regulus of lines in the 1-system of $Q^+(7,q)$. If such a regulus is replaced by its opposite

regulus, we obtain another 1-system of $Q^+(7,q)$.
The hermitian polar space $H(3,q^2)$ has several ovoids containing Baer sublines. Firstly, there is the classical ovoid $H(2,q^2) \subseteq H(3,q^2)$, which can be partitioned into q^2-q+1 disjoint Baer sublines. Hence, in the corresponding 1-system of $Q^+(7,q)$, we can replace any number of reguli, from 1 to q^2-q+1, by their opposite regulus. This yields a lot of 1-systems of $Q^+(7,q)$, including an example in which all lines of the classical 1-system have been replaced by new lines. Note that all these 1-systems are spreads of an induced $Q^-(5,q) \subseteq Q^+(7,q)$, so in fact these examples arising from $H(2,q^2)$ were already known.
As has been explained in detail in Section 6.3 of Chapter 6, every locally hermitian spread $\mathcal{S}(S)$ of $Q^-(5,q)$, which always arises from a line spread S in $\mathsf{PG}(3,q)$, dualizes to an ovoid $\mathcal{O}$ of $H(3,q^2)$, which consists of q^2 Baer sublines through a common point. Hence this ovoid yields a locally hermitian 1-system of $Q^+(7,q)$ in which at least one regulus can be replaced by its opposite regulus. Again by Theorem 8.1.2, a 1-system which is obtained by replacing a regulus of a locally hermitian 1-system of $Q^+(7,q)$ by its opposite regulus, cannot be locally hermitian, so it is in particular not isomorphic to the original 1-system.
Note that, by replacing a regulus of a locally hermitian spread of $Q^-(5,q)$ by its opposite regulus, we obtain another spread of $Q^-(5,q)$ containing at least one regulus. The corresponding ovoid of $H(3,q^2)$ then contains at least one Baer subline and hence gives rise to a 1-system of $Q^+(7,q)$ which also contains at least one regulus. By applying Theorem 8.1.1 to this regulus, yet another 1-system of $Q^+(7,q)$ is found.
Obviously, every 1-system of an induced $Q(6,q) \subseteq Q^+(7,q)$ is also a 1-system of $Q^+(7,q)$, but these examples have already been discussed.
Finally, from Chapters 6 and 7 it is known that there exist classes of proper locally hermitian, semiclassical 1-systems of $Q^+(7,q)$, which do not arise from an ovoid $\mathcal{O}$ of the hermitian variety $H(3,q^2)$. For these 1-systems too it is possible to replace at least one regulus by its opposite regulus, thus yielding a 1-system of $Q^+(7,q)$ that is new by Theorem 8.1.2.
We conclude that one can derive a lot of new 1-systems of $Q^+(7,q)$ from known ones by applying Theorem 8.1.1.

There is another case in which Theorem 8.1.1 can be applied, but unfortunately nothing new arises here. In [57], see also Theorem A.1.7, it is explained how every m-system of an elliptic quadric $Q^-(2n+1,q^2)$ gives rise to a $(2m+1)$-system of $Q^-(4n+3,q)$. For $n=2$ and $m=1$, every spread $\mathcal{S}$ of $Q^-(5,q^2)$ thus yields a 3-system $\mathcal{M}$ of $Q^-(11,q)$. If $\mathcal{S}$ contains reguli, then every regulus of $\mathcal{S}$ corresponds to a set of q^2+1 3-spaces $\pi_0, \pi_1, \ldots, \pi_{q^2}$ of $\mathcal{M}$, while the opposite regulus corresponds to q^2+1 other 3-spaces $\alpha_0, \alpha_1, \ldots, \alpha_{q^2}$ which can replace $\pi_0, \pi_1, \ldots, \pi_{q^2}$ to yield a different 3-system of $Q^-(11, q)$. This is, however, obviously the same 3-system as the one which is obtained by starting directly from the spread $\mathcal{S}'$ of $Q^-(5, q^2)$, where $\mathcal{S}'$ is the spread of $Q^-(5, q^2)$, obtained from $\mathcal{S}$ by replacing the considered regulus by its opposite regulus. So Theorem 8.1.1 can be applied here, but it gives no new examples.

Remark

As we have seen, there arise new classes of 1-systems of $Q(6, q)$ by applying Theorem 8.1.1. In [57], see also Theorem A.1.7, it is shown that every 1-system of $Q(6, q)$, q odd, yields a 3-system of $Q^+(13, q)$. This means that we have also found several classes of new 3-systems of $Q^+(13, q)$. For even q, a 1-system of $Q(6, q)$ gives rise to a 3-system of $Q(12, q)$ (see again [57] or Theorem A.1.8) and, by embedding, also a 3-system of $Q^+(13, q)$. So, for even q, we have found new classes of 3-systems of both $Q(12, q)$ and $Q^+(13, q)$. Moreover, if q is even, we can continue in this way: the new 3-systems of $Q(12, q)$ yield new 7-systems of $Q(24, q)$ and of $Q^+(25, q)$. In general, for q even, we find new classes of $(2^{n+1} - 1)$-systems of $Q(6 \cdot 2^n, q)$ and $Q^+(6 \cdot 2^n + 1, q)$.

Chapter 9

m-Systems and semipartial geometries

This last chapter will be dedicated to the semipartial geometries arising from m-systems, thus establishing the link between Chapter 2 and the results from later chapters. In Chapter 5 it has already been mentioned briefly that the 1-systems that were discovered there, yield semipartial geometries. Here we will show that for non-isomorphic proper locally hermitian, semiclassical 1-systems of $Q(6,q)$, q even, the corresponding semipartial geometries are also non-isomorphic. But attention will also be paid to the semipartial geometries arising from the 1-systems, constructed in Chapter 8 by derivation with respect to a regulus. In that case as well, one can show that there arise new examples of semipartial geometries, a result which is also contained in [42]. However, none of these semipartial geometries has new parameters.

9.1 Introduction

For q even, every 1-system of $Q(6,q)$ is isomorphic to a 1-system of the symplectic polar space $W_5(q)$, which can be seen by projection from the nucleus of $Q(6,q)$ onto a 5-space, not containing the nucleus. As has been shown in Chapter 2, every 1-system of $W_5(q)$ is an SPG regulus of $\mathsf{PG}(5,q)$ and so a semipartial geometry can be constructed from it, see Section 2.1. Here we will investigate whether the new 1-systems of $W_5(q)$ we have discovered, on the one hand those which were constructed in Chapter 5 and on the other hand the ones arising from Theorem 8.1.1, also yield new semipartial geometries.

Let $\mathcal{M}$ be a 1-system of $W_5(q)$, and hence also an SPG regulus of the ambient space $\mathsf{PG}(5,q)$ of $W_5(q)$. The semipartial geometry, constructed from the SPG regulus $\mathcal{M}$ as explained in Section 2.1, will be denoted by $\mathsf{SPG}(\mathcal{M})$ and has parameters $s = q^2-1$, $t = q^3$, $\alpha = q$ and $\mu = q^2(q^2-1)$.

9.2 1-Systems, obtained by derivation with respect to a regulus

In this section, we study the case of 1-systems of $W_5(q)$, derived from other 1-systems following the results of Chapter 8. Note that derivation of 1-systems of $W_5(q)$ with respect to a regulus is not possible for q odd, since in this case, the opposite regulus of a regulus of totally isotropic lines contains only two totally isotropic lines. For this reason, we omit 1-systems of $W_5(q)$ for q odd.

Let $\mathcal{M}$ be a locally hermitian 1-system of $W_5(q)$, q even, and suppose that $\mathcal{M}'$ arises from $\mathcal{M}$ by replacing a strong regulus by its opposite regulus. We aim at showing that $\mathsf{SPG}(\mathcal{M})$ and $\mathsf{SPG}(\mathcal{M}')$ are not isomorphic in this case. We start with an easy lemma that will be used continually afterwards.

Lemma 9.2.1 *Let $\mathcal{G}$ be a semipartial geometry with parameters $s = q^2-1$, $t = q^3$, $\alpha = q$ and $\mu = q^2(q^2-1)$. If $\mathcal{N}$ is a subnet of order q^2 and degree $q+1$ of $\mathcal{G}$, then two points of $\mathcal{N}$ are collinear in $\mathcal{N}$ if and only if they are collinear in $\mathcal{G}$.*

Proof.
Suppose that x and y are two points of $\mathcal{N}$, which are collinear in $\mathcal{G}$, and consider an arbitrary line L of $\mathcal{N}$ through y, but not containing x. As $\alpha = q$, there exist exactly q lines of $\mathcal{G}$ through x and concurrent with L; one of them is the line through x and y. On the other hand, there are also exactly q lines of $\mathcal{N}$ through x and meeting L in a point, so these lines must coincide with the q lines of $\mathcal{G}$ with this property. This implies that x and y are also collinear in $\mathcal{N}$. The converse statement is trivially true: if two points of $\mathcal{N}$ are collinear in $\mathcal{N}$, then they are also collinear in $\mathcal{G}$. $\square$

The lemma that will be shown next, plays a key role in the theorem we want to prove eventually. Unfortunately, it requires a different proof for $q > 2$ and for $q = 2$. The proof in the case $q = 2$ is rather complicated, for which reason the lemma itself is split up in two parts.

Lemma 9.2.2 *Let $\mathcal{M}$ be a 1-system of a symplectic polar space $W_5(q)$, $q > 2$, in $\mathsf{PG}(5,q) := H$. A subnet of order q^2 and degree $q+1$ in $\mathsf{SPG}(\mathcal{M})$ is* *always the subgeometry induced by* $\mathsf{SPG}(\mathcal{M})$ *in a subspace* $\mathsf{PG}(4,q)$ *of the ambient space* $\mathsf{PG}(6,q)$ *of* $\mathsf{SPG}(\mathcal{M})$*, where* $\mathsf{PG}(4,q)$ *meets* H *in a* $\mathsf{PG}(3,q)$ *containing exactly* $q+1$ *lines of* $\mathcal{M}$*.*

Proof.
It is straightforward that the subgeometry, induced in such a $\mathsf{PG}(4,q)$, is a net of order q^2 and degree $q+1$.

To prove the converse, suppose that $\mathcal{N}$ is a subnet of order q^2 and degree $q+1$ of $\mathsf{SPG}(\mathcal{M})$, with $q>2$. Let x be a point of $\mathcal{N}$ and consider two lines π_0, π_1 of $\mathcal{N}$ on x, so π_i is a plane of $\mathsf{PG}(6,q)$ which meets H in a line $L_i \in \mathcal{M}$, $i=0,1$. Denote the 3-space $\langle L_0, L_1\rangle$ by β. We will show that β contains exactly $q+1$ elements of $\mathcal{M}$ and that $\mathcal{N}$ is the subgeometry induced in $\langle x,\beta\rangle$. Remember also that by Theorem 1.2.3, β cannot contain more than $q+1$ lines of $\mathcal{M}$.
Every plane in β on L_0 meets L_1 in a point and by the fact that $\mathcal{M}$ is an SPG regulus with $\alpha=q$, it holds that all planes on L_0 in β have q points in common with $\tilde{\mathcal{M}}$, not on L_0. A plane in β on L_0 with the property that these q points lie on a line will be called of type I; if this is not the case, it will be called of type II.

First we assume that there exist at least two distinct planes ρ and ρ' of type I on L_0 in β. Let the points of $\tilde{\mathcal{M}}\cap\rho$, respectively $\tilde{\mathcal{M}}\cap\rho'$, not on L_0, be on the line M, respectively M', and write $M\cap L_0 := y_0$ and $M\cap L_1 := y_1$. Now we consider the plane $\langle x,M\rangle$. All points of $xy_i\setminus\{y_i\}$, $i=0,1$, belong to $\mathcal{N}$. As every line of $\langle x,M\rangle$ not on x, y_0 or y_1 intersects both xy_0 and xy_1 and meets $\mathsf{PG}(5,q)$ in a point of $\tilde{\mathcal{M}}$, it follows that all points of $\langle x,M\rangle\setminus M$ are points of the subnet $\mathcal{N}$. Similarly, all points of the plane $\langle x,M'\rangle$, except for the ones on M', also belong to $\mathcal{N}$. This implies that the plane $\langle x,L_i\rangle$ is a line of $\mathcal{N}$ for all lines $L_i\in\mathcal{M}$ which have a point in common with M or M'. As there are only $q+1$ lines of $\mathcal{N}$ through the point x, each line $L_i\in\mathcal{M}$ which intersects M, must also meet M'. Hence $M\cap M'=\emptyset$ and β contains exactly $q+1$ lines $L_0,L_1,\ldots,L_q$ of $\mathcal{M}$. Now consider an arbitrary point z of $\langle x,\beta\rangle\setminus\beta$, not belonging to any of the planes $\langle x,L_i\rangle$, $i=0,1,\ldots,q$. Then $\langle z,L_0\rangle$ meets each plane $\langle x,L_i\rangle$, $i\neq 0$, in a point and so $\langle z,L_0\rangle$ contains at least q points of $\mathcal{N}$. Consequently $\langle z,L_0\rangle$ must be a line of $\mathcal{N}$ and in particular z is a point of $\mathcal{N}$. This implies that $\mathcal{N}$ is the subgeometry of $\mathsf{SPG}(\mathcal{M})$, induced in the 4-space $\langle x,\beta\rangle$.

Next suppose that there exists at most one plane of type I in β on L_0. Let ρ be a type II plane in β on L_0 with $\rho\cap L_1 := y_1$. Consider a point $z\neq y_1$ of $\tilde{\mathcal{M}}$ in the plane ρ, but not on L_0, and let y_0 be the point $y_1z\cap L_0$. Now we look at the plane $\langle x,y_0,y_1\rangle$. Every line K on z in $\langle x,y_0,y_1\rangle$, different from the lines y_0y_1 and xz, meets xy_0 and xy_1 in a point and thus all points

of $K \setminus \{z\}$ belong to $\mathcal{N}$. So all points of $\langle x, y_0, y_1\rangle$, not on y_0y_1 or xz, belong to $\mathcal{N}$. By similarly considering all lines on y_0 in the plane $\langle x, y_0, y_1\rangle$, we then find that the points of $xz \setminus \{z\}$ also belong to $\mathcal{N}$, which shows that $\mathcal{N}$ contains all points of $\langle x, y_0, y_1\rangle$ except for the ones on the line y_0y_1. If we repeat this reasoning for a point $z' \in \tilde{\mathcal{M}} \cap \rho$, z' not on L_0 or y_0y_1, we find a second plane $\langle x, z', y_1\rangle$ all points of which, except for the ones on $z'y_1$, belong to $\mathcal{N}$.
To proceed, consider the 3-dimensional subspace $\langle x, \rho\rangle$ and let u be a point of L_0, distinct from y_0 and $z'y_1 \cap L_0$. Then all lines of $\mathsf{PG}(6, q)$ in $\langle x, \rho\rangle \setminus \rho$ containing u and disjoint from the line xy_1, meet both planes $\langle x, y_0, y_1\rangle$ and $\langle x, z', y_1\rangle$ in a point and so the points on these lines, distinct from u, are points of $\mathcal{N}$. As a consequence, $\mathcal{N}$ contains all points of $\langle x, \rho\rangle \setminus (\rho \cup \langle x, y_1, u\rangle)$, which suffices in turn to conclude that also the points of $\langle x, y_1, u\rangle \setminus y_1u$ must be points of $\mathcal{N}$.
By assumption, there exists a second plane $\rho' \neq \rho$ of type II on L_0 in β, for which it similarly holds that $\mathcal{N}$ contains all points of $\langle x, \rho'\rangle \setminus \rho'$. By an analogous argument, now considering all lines in $\langle x, \beta\rangle \setminus \beta$ on a point $v \in L_1$, v not in ρ or ρ', we can deduce that all points of $\langle x, \beta\rangle \setminus \beta$ must belong to $\mathcal{N}$. As $\langle x, \beta\rangle \setminus \beta$ has q^4 points, which is exactly the number of points of $\mathcal{N}$, the subnet $\mathcal{N}$ must coincide with the subgeometry of $\mathsf{SPG}(\mathcal{M})$, induced in $\langle x, \beta\rangle$ and β itself must have exactly $q + 1$ lines in common with $\mathcal{M}$.

This proves the lemma. □

The above proof does not work if $q = 2$. Firstly, for $q = 2$ all planes through L_0 in β are of type I, so that the second part of the proof is superfluous. The first part of the proof, which handles the planes of type I, is not valid for $q = 2$: in this case one may not conclude that all points of $\langle x, M\rangle \setminus M$ are points of $\mathcal{N}$, as can be seen in Picture 9.1. The fact is that the reasoning from the proof does not allow us to conclude that the point v in Picture 9.1 is a point of $\mathcal{N}$.

Therefore we give a different proof for the case $q = 2$.

Lemma 9.2.3 *Let $\mathcal{M}$ be a 1-system of a symplectic polar space $W_5(2)$ in $\mathsf{PG}(5, 2) := H$. A subnet of order 4 and degree 3 in $\mathsf{SPG}(\mathcal{M})$ is always the subgeometry induced by $\mathsf{SPG}(\mathcal{M})$ in a subspace $\mathsf{PG}(4, 2)$ of the ambient space $\mathsf{PG}(6, 2)$ of $\mathsf{SPG}(\mathcal{M})$, where $\mathsf{PG}(4, 2)$ meets H in a $\mathsf{PG}(3, 2)$ containing exactly 3 lines of $\mathcal{M}$.*

Proof.
Let $\mathcal{N}$ be a subnet of order 4 and degree 3 of $\mathsf{SPG}(\mathcal{M})$. Consider a point x of $\mathcal{N}$ and denote the three lines of $\mathcal{N}$ on x by $\langle x, L_i\rangle$, with $L_i \in \mathcal{M}$ for

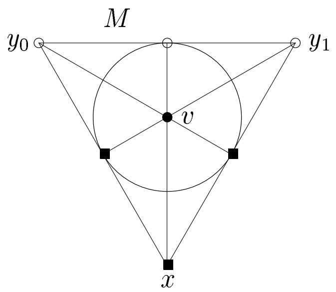

Figure 9.1: Why the proof of Lemma 9.2.2 does not work for $q = 2$.

$i = 0, 1, 2$. We want to show that $\langle L_0, L_1, L_2\rangle$ is 3-dimensional and $\mathcal{N}$ is the subgeometry, induced in the 4-space $\langle x, L_0, L_1\rangle$.

So suppose that $\langle L_0, L_1, L_2\rangle$ is at least 4-dimensional. We first show that the 3-space $\langle L_0, L_1\rangle$ contains exactly 3 points m, n and u of $\tilde{\mathcal{M}}$, not on L_0 or L_1, and that these points are on a line which forms a regulus together with L_0 and L_1. Let a_1 be a point of L_1 and consider the plane $\langle a_1, L_0\rangle$. This plane meets $\tilde{\mathcal{M}}$ in 2 points not on L_0, one of which is the point a_1. Denote the other point by m and define the point a_0 as the intersection point of the line a_1m with L_0. Similarly, if $b_1 \in L_1 \setminus \{a_1\}$, we obtain a point $n \in \tilde{\mathcal{M}}$ in the plane $\langle b_1, L_0\rangle$. If $b_1n \cap L_0 = a_0$, then the plane $\langle a_0, L_1\rangle$ has 3 distinct points, not on L_1, in common with $\tilde{\mathcal{M}}$, namely a_0, m and n. This is a contradiction, so $b_1n \cap L_0 := b_0 \neq a_0$. In the same way, let $u \in \tilde{\mathcal{M}}$ be the third point on the line c_0c_1, with c_i the remaining point of the line L_i for $i = 0, 1$. It then follows that mnu is a line and constitutes a regulus together with L_0 and L_1.

If the subspace $\langle L_0, L_1, L_2\rangle$ is 4-dimensional, then L_2 has one of the points $m, n, u \in \tilde{\mathcal{M}}$ in common with $\langle L_0, L_1\rangle$, say $m \in L_2$. Now consider the plane $\langle x, a_0, a_1\rangle$ and denote the third point of the line xa_i by d_i, $i = 0, 1$, see Picture 9.2. Then the lines d_0d_1 and a_0a_1 share the point m. As m is a point of L_2, this implies that $\langle d_0, L_2\rangle$ is a line of $\mathcal{N}$. Also, since $\langle x, L_2\rangle$ is a line of $\mathcal{N}$, the third point v on the line xm must be a point of $\mathcal{N}$. This in turn implies that both $\langle d_1, L_0\rangle$ and $\langle d_0, L_1\rangle$ are lines of $\mathcal{N}$, because $d_1v \cap H = \{a_0\}$ on the one hand and $d_0v \cap H = \{a_1\}$ on the other hand, and both a_0 and a_1 are points of $\tilde{\mathcal{M}}$.
We now turn our attention to the plane $\langle x, b_0, b_1\rangle$ and denote the third point on the line xb_i by e_i, $i = 0, 1$. Similarly as above, the line e_0e_1 meets H in the point $n \in \tilde{\mathcal{M}}$, so that $\langle e_0, N\rangle$ is a line of $\mathcal{N}$, where $N \in \mathcal{M}$ is the unique line of $\mathcal{M}$ containing n. Consider as a next step the subspace $\langle x, d_1, L_0\rangle$ of $\mathsf{PG}(6, 2)$,

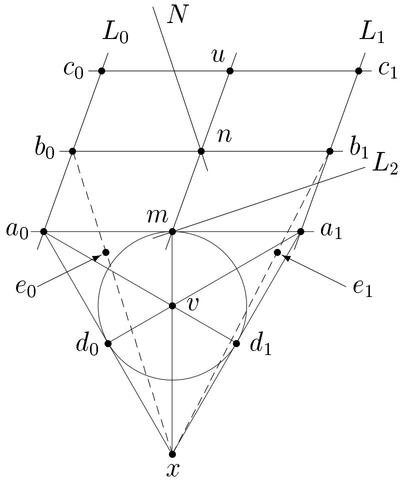

Figure 9.2: $\langle L_0, L_1, L_2\rangle$ is 4-dimensional in the proof of Lemma 9.2.3.

generated by the parallel lines $\langle x, L_0\rangle$ and $\langle d_1, L_0\rangle$ of $\mathcal{N}$. This subspace is 3-dimensional. Furthermore, the line $\langle e_0, N\rangle$ of $\mathcal{N}$ intersects $\langle x, L_0\rangle$ in the point e_0, which implies that $\langle e_0, N\rangle$ must also have a unique point in common with the line $\langle d_1, L_0\rangle$. As a consequence, $\langle e_0, N\rangle \cap \langle x, d_1, L_0\rangle$ is at least a line of $\mathsf{PG}(6,2)$ and $\langle x, d_1, L_0, N\rangle := \delta$ is at most 4-dimensional. By the previous observations, δ contains x and d_1, hence also a_1; in addition, both b_0 and n lie in δ, so that b_1 is a point of δ as well. It follows that $L_1 = a_1b_1$ is a line of δ. Hence the three lines L_0, L_1 and N of $\mathsf{PG}(6,2)$ belong to the 3-space $\delta \cap H$, or, in other words, the line N of $\mathcal{M}$ is contained in the 3-space $\langle L_0, L_1\rangle$. But the points of $\tilde{\mathcal{M}}$ in $\langle L_0, L_1\rangle$ are exactly m, n, u and the points of L_0 and L_1, and as a consequence N must coincide with the line mnu. This is a contradiction because m is a point of L_2 and we conclude that $\langle L_0, L_1, L_2\rangle$ cannot be 4-dimensional.

Next, assume that $\langle L_0, L_1, L_2\rangle$ is 5-dimensional and suppose that $M := mnu$, with m, n and u defined as above, is a line of $\mathcal{M}$. Let a_0, b_0, c_0, d_0, a_1, b_1, c_1 and d_1 also be defined as before. To keep track of the notation used in this part of the proof, one can keep Picture 9.3 in mind. By the previous part, we know that $\langle d_0, M\rangle$ is a line of $\mathcal{N}$. If e_i denotes the third point of the line xb_i, $i = 0, 1$, then the line e_0e_1 has the point n of M in common with the line b_0b_1 and thus $\langle e_0, M\rangle$ is also a line of $\mathcal{N}$. Similarly, if f_i is the third point on the line xc_i, $i = 0, 1$, then the line f_0f_1 intersects M in the point u, so that $\langle f_0, M\rangle$ is a line of $\mathcal{N}$ too. The three lines $\langle d_0, M\rangle$, $\langle e_0, M\rangle$ and $\langle f_0, M\rangle$

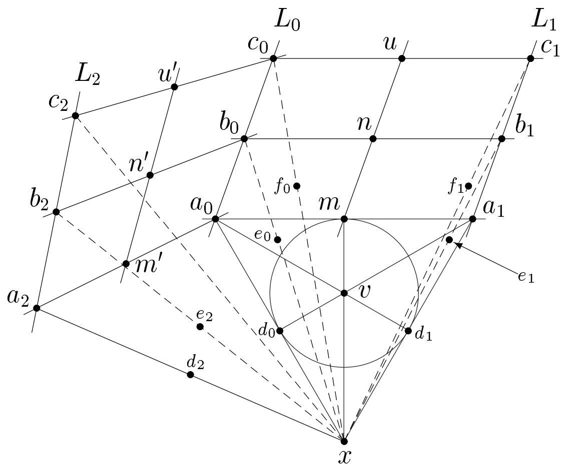

Figure 9.3: $\langle L_0, L_1, L_2\rangle$ is 5-dimensional in the proof of Lemma 9.2.3.

are parallel lines of $\mathcal{N}$ and contain together 12 points of $\mathcal{N}$. By assumption, the line $\langle x, L_2\rangle$ also belongs to $\mathcal{N}$ and yields 4 additional points of $\mathcal{N}$. This means that we have determined all 16 points of $\mathcal{N}$; note that x and the 12 points of $\mathcal{N}$ in $\langle d_0, M\rangle$, $\langle e_0, M\rangle$ and $\langle f_0, M\rangle$ lie in the 4-dimensional subspace $\langle x, L_0, L_1\rangle$. Note also that $\langle x, L_2\rangle$ is parallel to $\langle d_0, M\rangle$, $\langle e_0, M\rangle$ and $\langle f_0, M\rangle$. If $y \neq x$ is an arbitrary point of the line $\langle x, L_2\rangle$ of $\mathcal{N}$, then let π denote a line of $\mathcal{N}$, different from $\langle x, L_2\rangle$, through the point y. Since π cannot contain points of $\langle x, L_2\rangle$ except for y, it is clear that the three points of $\mathcal{N}$ in π and distinct from y, must be part of the 12 points of $\mathcal{N}$ in $\langle d_0, M\rangle$, $\langle e_0, M\rangle$ and $\langle f_0, M\rangle$. Consequently, π itself must be contained in the 4-space $\langle x, L_0, L_1\rangle$. But this implies that the line L_2 has the point $xy \cap L_2$ in common with the 3-space $\langle L_0, L_1\rangle$ and consequently $\langle L_0, L_1, L_2\rangle$ is at most 4-dimensional. This contradicts the assumptions and we may thus conclude that the points m, n and u belong to different lines M, N and U of $\mathcal{M}$.

We now define the points a_2, b_2, c_2 and $m' \in \tilde{\mathcal{M}}$ on a_0a_2, $n' \in \tilde{\mathcal{M}}$ on b_0b_2 and $u' \in \tilde{\mathcal{M}}$ on c_0c_2 in $\langle L_0, L_2\rangle$ similarly as the corresponding points in $\langle L_0, L_1\rangle$. Also, let d_2, respectively e_2, be the third point on the line xa_2, respectively xb_2. If the line M meets $\langle L_0, L_2\rangle$ in a point, then we show that this cannot be the point m'. If m' were a point of M, then the line $\langle d_0, M\rangle$ of $\mathcal{N}$ would have a point in common with all three lines $\langle x, L_i\rangle$, $i = 0, 1, 2$, of $\mathcal{N}$ on x, namely the point d_i. This is a contradiction to the axioms of a net of order 4 and degree 3. So assume that $M \cap \langle L_0, L_2\rangle = n'$. Then, similarly as before, both $\langle d_0, d_1, M\rangle$ and $\langle e_0, e_2, M\rangle$ are lines of $\mathcal{N}$. By assumption, the line $\langle x, L_2\rangle$

also belongs to $\mathcal{N}$ and it intersects $\langle e_0, M\rangle$ in the point e_2. Since $\langle d_0, M\rangle$ and $\langle e_0, M\rangle$ are parallel in $\mathcal{N}$, it follows that $\langle x, L_2\rangle$ must also intersect $\langle d_0, M\rangle$ in a point. This in turn implies that $\langle x, L_2\rangle$ has at least a line in common with the 3-space $\langle d_0, e_0, M\rangle$ and consequently $\delta := \langle d_0, e_0, x, M, L_2\rangle$ is at most 4-dimensional. But δ contains $b_2 \in L_2$ and $n' \in M$, so also the point b_0 lies in δ. On the other hand, the fact that x and d_0 are points of δ, implies that a_0 is a point of δ as well. Hence δ contains the line L_0. Finally, we now know that both $a_0 \in L_0$ and $m \in M$ lie in δ, so that also a_1 must be a point of δ. We conclude that the 3-space $\langle L_0, L_2\rangle = \delta \cap H$ meets L_1 in at least a point, so that $\langle L_0, L_1, L_2\rangle$ is at most 4-dimensional. This is again a contradiction to the assumptions and it follows that M is disjoint from $\langle L_0, L_2\rangle$ and similarly also from $\langle L_1, L_2\rangle$. Clearly, the same holds for N and U. By applying the above argument to the 3-spaces $\langle L_0, L_2\rangle$ and $\langle L_1, L_2\rangle$, it follows that there exist six distinct lines of $\mathcal{M} \setminus \{L_0, L_1, L_2, M, N, U\}$, three of which have a point in common with $\langle L_0, L_2\rangle$, and the other three with $\langle L_1, L_2\rangle$. Also by the above argument, no two of these lines coincide. All together this implies the existence of 12 lines of $\mathcal{M}$, which contradicts the fact that $|\mathcal{M}| = 9$. We conclude that $\langle L_0, L_1, L_2\rangle$ cannot be 5-dimensional either and thus the only possibility is that it is a 3-dimensional subspace of $H = \mathsf{PG}(5, 2)$.

It remains to show that $\mathcal{N}$ is the subgeometry, induced in the 4-space $\langle x, L_0, L_1\rangle$, but this is now easy. Let z be an arbitrary point of $\langle x, L_0, L_1\rangle \setminus \langle L_0, L_1\rangle$. If z lies in one of the planes $\langle x, L_i\rangle$, $i = 0, 1, 2$, then it is clear that z is a point of $\mathcal{N}$. So assume that z is not contained in any of the planes $\langle x, L_i\rangle$, $i = 0, 1, 2$. Then the plane $\langle z, L_0\rangle$ meets both $\langle x, L_1\rangle$ and $\langle x, L_2\rangle$ in a point not on L_1 or L_2. Hence $\langle z, L_0\rangle$ is a line of $\mathsf{SPG}(\mathcal{M})$ which contains two points of $\mathcal{N}$, which implies that $\langle z, L_0\rangle$ must be a line of $\mathcal{N}$. In particular, z is a point of $\mathcal{N}$ and it follows that all points of $\langle x, L_0, L_1\rangle \setminus \langle L_0, L_1\rangle$ belong to $\mathcal{N}$. This proves that $\mathcal{N}$ is the subgeometry of $\mathsf{SPG}(\mathcal{M})$, induced in $\langle x, L_0, L_1\rangle$. $\square$

Based on the previous lemma and Theorem 8.1.2, we can show that some of the new 1-systems of $W_5(q)$, q even, which are constructed from a known 1-system by replacing a strong regulus by its opposite regulus, also yield new semipartial geometries.

Theorem 9.2.4 *Let $\mathcal{M}$ be a 1-system of $W_5(q)$, q even, which is locally hermitian at some line $L \in \mathcal{M}$, and suppose that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by replacing a strong regulus R of $\mathcal{M}$ through L by its opposite regulus $\overline{R}$. Then the semipartial geometries $\mathsf{SPG}(\mathcal{M})$ and $\mathsf{SPG}(\mathcal{M}')$ are not isomorphic.*

Proof.
Since $\mathcal{M}$ is locally hermitian at the line L, it consists of q^2 strong reguli

$R, R_1, R_2, \ldots, R_{q^2-1}$ through L. If x is a fixed point of $\mathsf{SPG}(\mathcal{M})$, then each such strong regulus determines a unique subnet of $\mathsf{SPG}(\mathcal{M})$ of degree q^2 and order $q+1$ through x, and these q^2 subnets have exactly the line $\langle x, L\rangle$ of $\mathsf{SPG}(\mathcal{M})$ in common. A collection of q^2 such subnets exists through each point of $\mathsf{SPG}(\mathcal{M})$. In the proof of Theorem 8.1.2 however, it has been shown that there exists no line $M \in \mathcal{M}'$ with the property that for every line $N \in \mathcal{M}' \setminus \{M\}$, the 3-space $\langle M, N\rangle$ contains $q+1$ lines of $\mathcal{M}'$. Hence by Lemma 9.2.2 and Lemma 9.2.3 it follows that for no point x of $\mathsf{SPG}(\mathcal{M}')$ there can be found a collection of q^2 subnets through x of order q^2 and degree $q+1$, all containing some common line $\langle x, M\rangle$ of $\mathsf{SPG}(\mathcal{M}')$.

We may thus conclude that $\mathsf{SPG}(\mathcal{M}')$ is not isomorphic to $\mathsf{SPG}(\mathcal{M})$. $\square$

This theorem can be applied to the locally hermitian 1-systems of $W_5(q)$, q even, which are isomorphic to a locally hermitian, semiclassical 1-system of $Q(6,q)$ arising as described in Chapter 5. Let $\mathcal{M}$ be such a 1-system of $W_5(q)$, locally hermitian at the line $L \in \mathcal{M}$, and suppose that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by replacing one strong regulus through the line L by its opposite regulus. Then we have shown that $\mathsf{SPG}(\mathcal{M}')$ is not isomorphic to $\mathsf{SPG}(\mathcal{M})$, so from the new 1-systems new semipartial geometries arise.

Note that $\mathsf{SPG}(\mathcal{M})$ and $\mathsf{SPG}(\mathcal{M}')$ have the same parameters and, as $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$, they also have a common point graph. It follows that this point graph is not faithfully semigeometric.

9.3 Non-isomorphic, locally hermitian 1-systems of $W_5(q)$

Finally we focus on the question whether the semipartial geometries that arise from non-isomorphic locally hermitian, semiclassical 1-systems $\mathcal{M}_1$, respectively $\mathcal{M}_2$, of $W_5(q)$ with q even, are isomorphic or not. In order to come to a conclusion, we will first show that an isomorphism between $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ is induced by an element of $\mathsf{P\Gamma L}(7,q)$ which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$.

Theorem 9.3.1 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be locally hermitian 1-systems of a symplectic polar space $W_5(q)$ in $\mathsf{PG}(5,q) := H$, with q even and $q > 2$. If θ is an isomorphism between $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$, then θ is induced by an element $\vartheta \in \mathsf{P\Gamma L}(7,q)$ which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$.*

Proof.
As $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally hermitian, they both consist of q^2 strong reguli

through some line, say $L_1 \in \mathcal{M}_1$, respectively $L_2 \in \mathcal{M}_2$. Every such strong regulus defines a collection of subnets of order q^2 and degree $q+1$ in $\mathsf{SPG}(\mathcal{M}_i)$, $i = 1, 2$. Clearly, θ must map a subnet $\mathcal{N}_1$ of order q^2 and degree $q+1$ of $\mathsf{SPG}(\mathcal{M}_1)$ onto a subnet $\mathcal{N}_2$ of the same order and degree in $\mathsf{SPG}(\mathcal{M}_2)$. Suppose that $\mathcal{N}_1$ is a regulus net, determined by a regulus R of lines of $\mathcal{M}_1$; see Section 1.4.2 for the definition of a regulus net. Then by Lemma 9.2.2, the net $\mathcal{N}_2 = \mathcal{N}_1^\theta$ is the subgeometry of $\mathsf{SPG}(\mathcal{M}_2)$, induced in a $\mathsf{PG}(4,q)$ which intersects H in a $\mathsf{PG}(3,q)$, containing $q+1$ lines of $\mathcal{M}_2$. Now, since $\mathcal{N}_2$ is isomorphic to the regulus net $\mathcal{N}_1$ by assumption, a result of Johnson, see [29], implies that these $q+1$ lines of $\mathcal{M}_2$ must be the lines of a regulus R' in $\mathsf{PG}(3,q)$. Hence $\mathcal{N}_2$ is also a regulus net and we know that θ maps every regulus subnet of $\mathsf{SPG}(\mathcal{M}_1)$ onto a regulus subnet of $\mathsf{SPG}(\mathcal{M}_2)$.

Let $\mathcal{N}_1$ be an arbitrary regulus subnet of $\mathsf{SPG}(\mathcal{M}_1)$ and set $\mathcal{N}_2 := \mathcal{N}_1^\theta$. As has been mentioned in Section 1.4.2, the net $\mathcal{N}_1$ is isomorphic to a codimension 2 net H_q^3, which we consider to be embedded in a $\mathsf{PG}(3,q) \setminus N$ as described in Section 1.4.2. By Theorem 11.1 of Johnson [30], the full collineation group of H_q^3 is isomorphic to the stabilizer $\mathsf{P\Gamma L}(4,q)_N$ of the line N in $\mathsf{P\Gamma L}(4,q)$.

If $\mathcal{N}_i$, $i = 1, 2$, is the subgeometry of $\mathsf{SPG}(\mathcal{M}_i)$, induced in the 4-dimensional subspace δ_i of $\mathsf{PG}(6,q)$, where $\delta_i \cap H$ contains a regulus $R^{(i)}$ of lines of $\mathcal{M}_i$, then every element ζ of $\mathsf{P\Gamma L}(7,q)$ which maps δ_1 onto δ_2 and $R^{(1)}$ onto $R^{(2)}$, certainly induces an isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$. Now the number of such elements ζ equals the number of elements of $\mathsf{P\Gamma L}(7,q)$ stabilizing a given 4-dimensional subspace of $\mathsf{PG}(6,q)$, multiplied with the number of collineations of $\mathsf{PG}(4,q)$, stabilizing a regulus of lines in some hyperplane of $\mathsf{PG}(4,q)$. This last number can easily be calculated; it is equal to $hq^6(q-1)(q^2-1)^2$, where $q = p^h$ with p prime. But $hq^6(q-1)(q^2-1)^2$ is also the order of the group $\mathsf{P\Gamma L}(4,q)_N$, which implies that every isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$ must be induced by an element $\zeta \in \mathsf{P\Gamma L}(7,q)$ which maps δ_1 onto δ_2 and $R^{(1)}$ onto $R^{(2)}$.

Let $R_1, R_2, \ldots, R_{q^2}$ be the q^2 strong reguli of $\mathcal{M}_1$ through L_1. It then follows from the previous paragraph that θ maps all lines of $\mathsf{AG}(6,q) = \mathsf{PG}(6,q) \setminus H$, the extension of which to $\mathsf{PG}(6,q)$ meets H in a point of some $\langle R_i \rangle$, $i \in \{1, 2, \ldots, q^2\}$, onto lines of $\mathsf{AG}(6,q)$.

Next, we determine how many points of H are contained in some $\langle R_i \rangle$, $i \in \{1, 2, \ldots, q^2\}$. If two 3-spaces $\langle R_i \rangle$ and $\langle R_j \rangle$, $i \neq j$, have a plane in common, then this must be a plane through L_1. Hence it also contains a transversal of both R_i and R_j. In case these two transversals coincide, it follows that the elements of $\mathcal{M}_1$ are not pairwise disjoint, a contradiction. On the other hand, if the two transversals are distinct, then the plane $\langle R_i \rangle \cap \langle R_j \rangle$ contains a line of $\mathcal{M}_1$ and at least $2q-1$ points of $\tilde{\mathcal{M}}_1$, not on this line. This last situation is impossible as well, because $\mathcal{M}_1$ is an SPG regulus with $\alpha = q$. Consequently, $\langle R_i \rangle \cap \langle R_j \rangle$ is the line L_1 for all $i \neq j$. Thus the union of the 3-spaces $\langle R_i \rangle$, $i = 1, 2, \ldots, q^2$, contains exactly $q^5 + q^4 + q + 1$ points of H. The 3-space $L_1^\perp$, with $\perp$ the polarity of $W_5(q)$, intersects every $\langle R_i \rangle$, $i \in \{1, 2, \ldots, q^2\}$, in the line L_1, and as such it yields $q^3 + q^2$ additional points of H. We conclude that a point of H is either contained in some $\langle R_i \rangle$, $i \in \{1, 2, \ldots, q^2\}$, or it is a point of the 3-space $L_1^\perp$.

Let π be a plane of $\mathsf{PG}(6, q)$ not in H and assume that π intersects H in a line K which has exactly one point z in common with $L_1^\perp$. Then, by considering all lines of π through two distinct points of $\pi \setminus K$, and taken into account that the q affine points of every line of π not through z, are mapped by θ onto the q points of an affine line, it is evident that all points of $\pi \setminus K$ are mapped by θ onto the q^2 points of a plane of $\mathsf{AG}(6, q)$. Now, let M be any line of $\mathsf{PG}(6, q) \setminus H$ through a point $z \in L_1^\perp$. If π and π' are two distinct planes of $\mathsf{PG}(6, q)$ through M, which meet H in distinct lines K and K' through z, but not contained in $L_1^\perp$, then the points of $M \setminus \{z\}$ must be mapped by θ onto the points of the intersection of the affine planes $(\pi \setminus K)^\theta$ and $(\pi' \setminus K')^\theta$, which form a line of $\mathsf{AG}(6, q)$. It follows that θ maps all lines of $\mathsf{AG}(6, q)$ onto lines of $\mathsf{AG}(6, q)$.

From the foregoing and as $q > 2$, we conclude that θ is an element of $\mathsf{A\Gamma L}(7, q)$, which implies that it can be extended to an element $\vartheta \in \mathsf{P\Gamma L}(7, q)$. Obviously, ϑ must then map $\mathcal{M}_1$ onto $\mathcal{M}_2$. This proves the claim. $\square$

Remarks

1. There are several possible ways to show that θ preserves the collinearity of $\mathsf{AG}(6, q)$ within regulus subnets of $\mathsf{SPG}(\mathcal{M}_1)$.

   An alternative proof also relies on the fact that the full collineation group of H_q^3 is isomorphic to $\mathsf{P\Gamma L}(4, q)_N$, but it would also be valid over an infinite field. By investigating the isomorphism between a regulus net and the co-dimension 2 net H_q^3 in its representation in $\mathsf{PG}(3, q) \setminus N$, one easily sees that q collinear points in a regulus net which are also collinear in the affine space in which the regulus net is represented, correspond to q lines, disjoint from N, through a point p of $\mathsf{PG}(3, q) \setminus N$, and in a plane π of $\mathsf{PG}(3, q)$ not containing N. Also, q points of the regulus net which are collinear in the affine space but not in the net, correspond to the q lines of a regulus of $\mathsf{PG}(3, q)$ through the special line N. Since every element of $\mathsf{P\Gamma L}(4, q)_N$ preserves both types of configurations of q lines of $\mathsf{PG}(3, q)$, every isomorphism between two regulus nets must preserve the collinearity of the affine space containing the first regulus net.

A second possible proof does not use the collineation group of H_q^3, but relies on straightforward properties of a regulus net. Let $\mathcal{N}$ be a regulus net of order q^2 and degree $q+1$, embedded in an affine space $\mathsf{AG}(4,q)$ as usually. If a, b and c are three collinear points of $\mathcal{N}$, then one easily sees that there exist $2q^2-q-3$ points of $\mathcal{N}\setminus\{a,b,c\}$, which are collinear in $\mathcal{N}$ with a, b and c, provided that abc is a line of $\mathsf{AG}(4,q)$. If a, b and c are not collinear in $\mathsf{AG}(4,q)$ however, there exist only q^2-3 points of $\mathcal{N}\setminus\{a,b,c\}$, which are collinear in $\mathcal{N}$ with a, b and c. So if abc is a line of $\mathsf{AG}(4,q)$, then the same must hold for $a^\theta b^\theta c^\theta$.
Similarly, let a, b and c be distinct points of $\mathcal{N}$ and suppose that they are pairwise not collinear in $\mathcal{N}$. If abc is a line of $\mathsf{AG}(4,q)$, then it holds that no point of $\mathcal{N}\setminus\{a,b,c\}$ is collinear in $\mathcal{N}$ with a, b and c. If a, b and c are not collinear in $\mathsf{AG}(4,q)$, then it can be shown that there always exists at least one point of $\mathcal{N}\setminus\{a,b,c\}$, collinear in $\mathcal{N}$ with all three of a, b and c. Hence in this case as well, θ must map the line abc of $\mathsf{AG}(4,q)$ onto an affine line.

2. The proof of Theorem 9.3.1 is not valid if $q=2$. Still, we can draw some conclusions. By the classification of the 1-systems of $W_5(2)$, carried out by Hamilton and Mathon [21], the symplectic polar space $W_5(2)$ has exactly two non-isomorphic 1-systems. One of them is the hermitian spread of a $Q^-(5,2)$ and the other one is obtained from the hermitian spread by reversing a regulus, so it is not locally hermitian by Theorem 8.1.2. Since Theorem 9.3.1 deals with two distinct locally hermitian 1-systems of $W_5(q)$, it does not make sense for $q=2$. Moreover, the fact that the semipartial geometries, arising from the two non-isomorphic 1-systems of $W_5(2)$ are not isomorphic, follows from Theorem 9.2.4.

The result obtained in Theorem 9.3.1 will now be used to show that for non-isomorphic locally hermitian 1-systems $\mathcal{M}_1$ and $\mathcal{M}_2$ of $W_5(q)$, q even and $q>2$, it holds that the corresponding semipartial geometries $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ are also non-isomorphic.

Theorem 9.3.2 *Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two locally hermitian 1-systems of $W_5(q)$, q even and $q>2$. Then the corresponding semipartial geometries $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ are isomorphic if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic for the stabilizer of $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$.*

Proof.
Denote the line at which $\mathcal{M}_i$ is locally hermitian by L_i, for $i=1,2$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic for the stabilizer of $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$, with isomorphism α, then it is clear that α can be extended to an element $\beta \in \mathsf{P\Gamma L}(7,q)$ which induces an isomorphism between $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$.

Conversely, if θ is an isomorphism between the semipartial geometries $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$, then by Theorem 9.3.1, θ is induced by an element $\vartheta \in \mathsf{P\Gamma L}(7,q)$, which maps $\mathcal{M}_1$ onto $\mathcal{M}_2$. Without loss of generality, we may assume that $L_1^\vartheta = L_2$. If ϑ stabilizes the symplectic polar space $W_5(q)$, then the claim is obviously true. Therefore we assume that ϑ does not stabilize $W_5(q)$, which implies that $\mathcal{M}_2 = \mathcal{M}_1^\vartheta$ must be a 1-system of two distinct symplectic polar spaces $W_5(q)$ and $W_5(q)'$, with $W_5(q)'$ the image of $W_5(q)$ under ϑ. Denote the polarity of $W_5(q)$ by ζ and the polarity of $W_5(q)'$ by ξ. We shall prove that $\zeta = \xi$, so that $W_5(q)$ and $W_5(q)'$ coincide and the assumption is false.
For every line $M \in \mathcal{M}_2$, it holds that M^ξ is a 3-dimensional subspace and contains no points of $\tilde{\mathcal{M}}_2$, except for the ones on M. Since the union of the tangent planes at M of the SPG regulus $\mathcal{M}_2$ contains $\frac{q^4-1}{q-1}$ points, M^ξ must coincide with the union of these tangent planes. But the same holds for M^ζ, so that $M^\xi = M^\zeta$ for all lines $M \in \mathcal{M}_2$.
If x is a point of the line L_2 at which $\mathcal{M}_2$ is locally hermitian, then x^ξ is 4-dimensional and it must contain L_2^ξ and all totally isotropic lines of $W_5(q)'$ through x. Now the lines of $\mathcal{M}_2$ are totally isotropic for both ζ and ξ and as q is even, this implies that also the transversals of the q^2 strong reguli of $\mathcal{M}_2$ through L_2 are totally isotropic for ζ and ξ. Thus x^ξ, as well as x^ζ, contains $L_2^\xi = L_2^\zeta$ and all transversals on x of the q^2 strong reguli of $\mathcal{M}_2$ through L_2, and it follows that $x^\xi = x^\zeta$.
For a point $y \in \tilde{\mathcal{M}}_2$, y on some line $M \in \mathcal{M}_2 \setminus \{L_2\}$, it holds similarly that y^ξ contains $M^\xi = M^\zeta$ and the unique transversal through y of the strong regulus of lines of $\mathcal{M}_2$, determined by L_2 and M. But this transversal is also contained in y^ζ and does not lie in $M^\xi = M^\zeta$, and consequently we may again conclude that $y^\xi = y^\zeta$.
Finally, let r be a point of $\mathsf{PG}(5,q)$, not in $\tilde{\mathcal{M}}_2$. Then there exist $q+1$ tangent planes of the SPG regulus $\mathcal{M}_2$ through the point r. These $q+1$ tangent planes are totally isotropic for both ζ and ξ and must hence be contained in r^ζ and r^ξ. Two such tangent planes through r cannot have a line in common, because in that case the first tangent plane would meet the line of $\mathcal{M}_2$ in the second tangent plane in a point, and vice versa, a contradiction. As a consequence, these $q+1$ tangent planes span at least a 4-dimensional subspace of $\mathsf{PG}(5,q)$. On the other hand, all $q+1$ tangent planes must be contained in r^ζ and r^ξ, which are both 4-dimensional, so that the subspace generated by the $q+1$ tangent planes of the SPG regulus $\mathcal{M}_2$

through r must be 4-dimensional and coincide with r^ζ and r^ξ. This yields that $r^\xi = r^\zeta$ and we can now conclude that the symplectic polarities ζ and ξ are identical.

This proves the theorem. $\square$

In combination with the results from Chapter 5, Theorem 9.3.2 implies that there exist d non-isomorphic semipartial geometries $\mathsf{SPG}(\mathcal{M})$, with $\mathcal{M}$ a locally hermitian, semiclassical 1-system of $W_5(q)$, $q > 2$ and even, which is not hermitian. Here d stands again for the number of orbits of the automorphism group of $\mathsf{GF}(q)$ in $\mathcal{C}_0 \setminus \{0\}$. Note that none of these 1-systems is a spread of a $Q^-(5,q)$, since they are isomorphic to proper locally hermitian, semiclassical 1-systems of $Q(6,q)$, $q > 2$ and even. Thus the semipartial geometries they yield are new.

Appendix A

Which m-systems are known to exist?

When developing a mathematical theory about some object, the existence question is always a crucial point. For what is the use of building up a whole theory if no example is known to apply it to? Therefore mathematicians always seek for (new) examples of the objects they want to study. It is not otherwise for m-systems of polar spaces. For this reason, we try to give in this appendix an overview of all m-systems which are currently known to exist. This is however not an easy task, since it is hard not to overlook anything, especially because research is constantly in progress and a list which is complete today, might be incomplete tomorrow.

In the following pages, we will briefly discuss all examples of m-systems of polar spaces, and the non-existence results, which are known to us as of this writing. We start with what one might call "chains of m-systems". This refers to a series of theorems allowing the construction of m'-systems from given m-systems by making use of field reductions. Thanks to these theorems, the discovery of one example of an m-system often implies the existence of other examples and sometimes even an infinite class arises. Afterwards, the different finite classical polar spaces will be discussed separately and for each case an existence table will be given. In these tables it will be marked in blue whenever knowledge about the existence or non-existence of m-systems has been added by the previous chapters of this thesis.

A.1 Chains of m-systems

Field reductions are a powerful tool in the construction of m'-systems from given m-systems. There are two possible ways of approaching these field reductions: either geometrically, as has been the case in the course of this thesis, or algebraically by using the sesquilinear forms, associated with polar spaces; see Cameron [9, Chapter 6] for the definition of sesquilinear forms and how polar spaces are constructed from them. Here we opt for the algebraic approach, mostly because it is shorter to describe, but also because we have already discussed in detail several cases of field reductions in a purely geometric way in the preceding chapters. Also, the interested reader can find an excellent geometric explanation of the constructions of m'-systems from given m-systems which are enumerated in Section A.1.2, in Shult and Thas [57]. We start with a brief discussion of the algebraic background of field reductions, which is inspired by Shult and Thas [58].

A.1.1 Field reductions algebraically

Let V be a vector space of dimension d over $\mathsf{GF}(q^e)$, $e > 1$. Then V can also be regarded as a vector space $\overline{V}$ of dimension de over $\mathsf{GF}(q)$. We first explain two cases which use the trace map to pass from a form over $\mathsf{GF}(q^e)$ to a form over $\mathsf{GF}(q)$; therefore they are often referred to as the "trace trick". We also restrict ourselves to the case $e = 2$, but it is important to realize that the trace trick can be generalized to $e > 2$ as well.

Case 1: From a hermitian form to a quadratic form

Let $e = 2$ and consider a non-degenerate hermitian form $h : V \times V \to \mathsf{GF}(q^2)$. The usual trace map from $\mathsf{GF}(q^2)$ to $\mathsf{GF}(q)$ is denoted by $\mathrm{Tr} : \mathsf{GF}(q^2) \to \mathsf{GF}(q)$, so $\mathrm{Tr}(x) = x + x^q$ for all $x \in \mathsf{GF}(q^2)$. Then one can check that $\mathrm{Tr} \circ h : \overline{V} \times \overline{V} \to \mathsf{GF}(q)$ is a non-degenerate symmetric bilinear form, induced by a quadratic form $\overline{Q} : \overline{V} \to \mathsf{GF}(q)$ by setting $\overline{Q}(\mathbf{u}) := h(\mathbf{u}, \mathbf{u})$, for all $\mathbf{u} \in \overline{V}$. By investigating the dimension of the maximal totally singular spaces of $\overline{Q}$, one deduces that $\overline{Q}$ is elliptic for odd d and hyperbolic for even d.

Note that the construction of a 1-system of $Q^+(7, q)$ from an ovoid of $H(3, q^2)$, see Section 6.3, relies on a field reduction of this kind.

Case 2: From a quadratic form to a quadratic form

Suppose again that $e = 2$ and let $Q : V \to \mathsf{GF}(q^2)$ be a non-singular quadratic form. Define $\overline{Q} : \overline{V} \to \mathsf{GF}(q)$ by $\overline{Q}(\mathbf{u}) := \mathrm{Tr}(Q(\mathbf{u}))$; then $\overline{Q}$ is a quadratic form on $\overline{V}$, which is non-singular whenever not at the same time d is odd and q is even. Here, if d is even, the form $\overline{Q}$ is of the same type as Q. If d is odd, it is possible to obtain a form $\overline{Q}$ of either type (elliptic or hyperbolic), by replacing Q by a suitable scalar multiple.

By applying a field reduction of this kind, one finds the classical 1-system of $Q^-(7, q)$, which arises from the elliptic quadric $Q^-(3, q^2)$, regarded as an ovoid of itself; see Chapter 3 for more about this 1-system if q is odd.

The last field reduction we want to mention, does not use the trace map, but it is also interesting in view of m-systems of polar spaces.

Case 3: From a hermitian form to an alternating form

Once more we assume that $e = 2$ and start from a non-degenerate hermitian form $h : V \times V \to \mathsf{GF}(q^2)$. By considering the homomorphism $x \mapsto x + x^q$ of the additive group of $\mathsf{GF}(q)$, it becomes clear that there exists an element $a \in \mathsf{GF}(q^2)$ with the property that $a^q = -a$. Now consider the function $b : \overline{V} \times \overline{V} \to \mathsf{GF}(q)$, defined by $b(\mathbf{u}, \mathbf{v}) := a^{-1}(h(\mathbf{u}, \mathbf{v}) - h(\mathbf{v}, \mathbf{u}))$. Then b is easily seen to be a non-degenerate alternating bilinear form on $\overline{V}$; so in this case we pass from a hermitian polar space to a symplectic polar space.

An example of this kind can be found in Theorem A.1.6 below.

The field reductions described here, preserve the defining property of a partial m-system of a polar space, but, when starting with an m-system, the number of elements obtained in the new polar space over the smaller field does not always meet the upper bound from Theorem 1.2.1. Therefore we end up with partial m'-systems in some cases. In the next section, we explicitly mention the theorems concerning m-systems of polar spaces which are due to the above field reductions.

A.1.2 The results

The results in this section are taken from four papers: the very first paper on m-systems and partial m-systems of polar spaces by Shult and Thas [57], another paper by the same authors, namely [58], a paper by Hamilton and Quinn [22] and finally the survey paper [72] by Thas, in which some results have been reformulated.

The first three theorems arise from other techniques than the use of field reductions.

Theorem A.1.1 ([57]) *If $Q^-(2n+1,q)$ has an m-system, then $Q(2n+2,q)$ and $Q^+(2n+3,q)$ also have an m-system; if $Q(2n+2,q)$ has an m-system, then $Q^+(2n+3,q)$ also has an m-system.*[7]

Remark

Theorem A.1.1 can also be applied conversely in the case of spreads: if $Q^+(2n+3,q)$ admits a spread $\mathcal{S}$, then for any $Q(2n+2,q)$ which is embedded in $Q^+(2n+3,q)$, the intersections with $Q(2n+2,q)$ of the elements of $\mathcal{S}$ form a spread $\mathcal{S}'$ of $Q(2n+2,q)$. Similarly, if $Q(2n+2,q)$ has a spread $\mathcal{S}'$, then for any $Q^-(2n+1,q)$, embedded in $Q(2n+2,q)$, the intersections of the elements of $\mathcal{S}'$ with $Q^-(2n+1,q)$ form a spread of $Q^-(2n+1,q)$.

Theorem A.1.2 ([57]) *The polar space $Q(2n,q)$, q even, has an m-system if and only if the polar space $W_{2n-1}(q)$ has an m-system.*[8]

Theorem A.1.3 ([57]) *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be spreads of $Q^+(7,q)$, where the generators of $\mathcal{S}_1$ and the generators of $\mathcal{S}_2$ belong to different families. Then for each $\xi_i \in \mathcal{S}_1$ there is exactly one $\eta_j \in \mathcal{S}_2$ with $\xi_i \cap \eta_j = \pi_{ij}$ a plane. Also, the q^3+1 planes π_{ij} form a 2-system of $Q^+(7,q)$.*[9]

Remark

If $Q^+(4n+3,q)$, $n \geq 1$, admits a $2n$-system, then it admits a spread. This spread is obtained by considering all generators of a given family of generators of $Q^+(4n+3,q)$, containing an element of the $2n$-system.

To obtain the following results, the field reductions from Section A.1.1 are applied.

Theorem A.1.4 ([57])

(a) If $H(2n,q^2)$ admits an m-system, then $Q^-(4n+1,q)$ always admits a $(2m+1)$-system.

(b) If $H(2n+1,q^2)$ admits an m-system, then $Q^+(4n+3,q)$ always admits a $(2m+1)$-system.

Theorem A.1.5 ([58]) *If $H(l-1,q^{2n})$, n odd, admits an m-system, then $H(nl-1,q^2)$ admits a partial $(mn+n-1)$-system of size $q^{nl}+1$ when l is odd and $q^{(l-1)n}+1$ when l is even; for l odd this partial $(mn+n-1)$-system is an $(mn+n-1)$-system of $H(nl-1,q^2)$.*

[7] This theorem relies on simple embeddings.

[8] This result is an immediate consequence of the isomorphism between $Q(2n,q)$ and $W_{2n-1}(q)$ for q even.

[9] This theorem is shown by using counting arguments.

Theorem A.1.6 ([58]) *If $H(n, q^2)$, with $n \geq 2$, admits an m-system, then $W_{2n+1}(q)$ admits a partial $(2m+1)$-system of size q^n+1 if n is odd and $q^{n+1}+1$ if n is even; so for n even there arises a $(2m+1)$-system of $W_{2n+1}(q)$.*

The result of Theorem A.1.6 in the case where n is even, was first described in [57].

Theorem A.1.7 ([58]) *Assume that not at the same time q is even and n is odd.*

(a) If $Q^\epsilon(n-1, q^e)$, n even and $\epsilon \in \{+,-\}$, admits an m-system, then $Q^\epsilon(ne-1, q)$ admits a partial $(me+e-1)$-system of size $q^{e(n-2)/2}+1$ for $\epsilon = +$ and of size $q^{en/2}+1$ for $\epsilon = -$; hence for $\epsilon = -$ there arises an $(me+e-1)$-system of $Q^-(ne-1, q)$.

(b) If $Q(n-1, q^e)$, n odd, admits an m-system, then $Q(ne-1, q)$ admits a partial $(me+e-1)$-system of size $q^{e(n-1)/2}+1$ for e odd, and for e even $Q^\epsilon(ne-1, q)$ admits a partial $(me+e-1)$-system of size $q^{e(n-1)/2}+1$ for any $\epsilon \in \{+,-\}$; hence for $e = 2$ and $\epsilon = +$ there arises a $(2m+1)$-system of $Q^+(2n-1, q)$.

Part (a) of Theorem A.1.7, for the case where $\epsilon = -$ and $e = 2$, says that an m-system of $Q^-(2n+1, q^2)$, $n \geq 1$, yields a $(2m+1)$-system of $Q^-(4n+3, q)$. This result was first obtained in [57].
If part (b) of Theorem A.1.7 is applied for $\epsilon = +$ and $e = 2$, one sees that an m-system of $Q(2n, q^2)$, $n \geq 1$, gives rise to a $(2m+1)$-system of $Q^+(4n+1, q)$. This result can be found in [57] too, where one also finds the following stronger theorem for q even.

Theorem A.1.8 ([57]) *If $Q(2n, q^2)$, with $n \geq 1$ and q even, admits an m-system, then $Q(4n, q)$, and hence also $Q^+(4n+1, q)$, admits a $(2m+1)$-system.*

Theorem A.1.9 ([22]) *If $W_{2t-1}(q^s)$ admits an m-system, then $W_{2st-1}(q)$ admits an $(s(m+1)-1)$-system.*

A.2 Some non-existence results

By a technique which uses arguments concerning the p-rank of certain well-chosen incidence matrices, it is possible to formulate bounds on the number of elements of a partial m-system of a polar space. If applied to m-systems of polar spaces, these bounds imply the non-existence of several classes of

m-systems. In the cases where Theorem 1.2.6 holds however, the results obtained in this way seem to be weaker than the one of Theorem 1.2.6. Therefore, we do not mention these cases, although they were known before the discovery of Theorem 1.2.6. When Theorem 1.2.6 does not apply, the results that follow are in many cases all we know about the non-existence of m-systems.
The first Theorem is due to Blokhuis and Moorhouse [5], and to Moorhouse [44]. It only handles the case of ovoids.

Theorem A.2.1 ([5], [44])
(i) If the polar space $Q^+(2n+1,q)$, with $n \geq 1$ and $q = p^h$ for some prime p, admits an ovoid, then

$$p^n \leq \binom{2n+p}{2n+1} - \binom{2n+p-2}{2n+1}.$$

(ii) If the polar space $Q(2n,q)$, with $n \geq 2$ and $q = p^h$ for an odd prime p, admits an ovoid, then

$$p^n \leq \binom{2n+p-1}{2n} - \binom{2n+p-3}{2n}.$$

(iii) If the polar space $H(2n+1,q^2)$, with $n \geq 1$ and $q = p^h$ for a prime p, admits an ovoid, then

$$p^{2n+1} \leq \binom{2n+p}{2n+1}^2 - \binom{2n+p-1}{2n+1}^2.$$

A generalization to m-systems of polar spaces of Theorem A.2.1 has been obtained by Shult and Thas [59]. In the case of an ovoid, their bounds coincide with the ones, found by Blokhuis and Moorhouse.

Theorem A.2.2 ([59]) *If the finite classical polar space $\mathcal{P}$ admits an m-system, then*

(i) for $\mathcal{P} = Q^+(2n+1,2^h)$,

$$2^n \leq \binom{2n+2}{m+1};$$

(ii) for $\mathcal{P} = Q(2n,q)$ with $q = p^h$ and p an odd prime,

$$p^n \leq \binom{\binom{2n+1}{m+1}+p-2}{p-1} - \binom{\binom{2n+1}{m+1}+p-4}{p-3};$$

(iii) for $\mathcal{P} = Q^+(2n+1, q)$ with $q = p^h$ and p an odd prime,

$$p^n \leq \left(\begin{array}{c} \left(\begin{array}{c} 2n+2 \\ m+1 \end{array} \right) + p - 2 \\ p-1 \end{array} \right) - \left(\begin{array}{c} \left(\begin{array}{c} 2n+2 \\ m+1 \end{array} \right) + p - 4 \\ p-3 \end{array} \right);$$

(iv) for $\mathcal{P} = H(2n+1, q^2)$ with $q = p^h$ and p a prime,

$$p^{2n+1} \leq \left(\begin{array}{c} \left(\begin{array}{c} 2n+2 \\ m+1 \end{array} \right) + p - 2 \\ p-1 \end{array} \right)^2 - \left(\begin{array}{c} \left(\begin{array}{c} 2n+2 \\ m+1 \end{array} \right) + p - 3 \\ p-2 \end{array} \right)^2.$$

Recently, P. Sin[10] informed us that it is possible to improve the above bounds, by considering the p-rank of the incidence matrix of intersecting subspaces of $\mathsf{PG}(n, q)$ of arbitrary dimension.

To bring this section on non-existence results for m-systems to an end, we point out a result by A. Klein, see [33].

Theorem A.2.3 *The polar space $H(2n+1, q^2)$ does not admit ovoids for $n > q^3$.*

Although this bound is, generally speaking, less sharp than the one from Theorem A.2.1, it is still worth mentioning, because it has been obtained by relying on geometric and combinatorial arguments, while the proof of Theorem A.2.1 uses algebraic methods.

A.3 The existence of m-systems explicitly

Before we deal with the distinct polar spaces one by one, some words about the tables in general must be said. In the existence tables that follow, a "—" in a certain field means that the values of m and n of that field are not meaningful. This symbol appears for instance in the column of a 2-system and the row of a polar space of rank $r \leq 2$. If a field is left blank, this means that nothing is known for this case. In all other cases, we either write "YES" or "NO" and give a reference or a short explanation in the accompanying text.

[10] Private communication, see also "The p-rank of the incidence matrix of intersecting linear subspaces." on `http://www.math.ufl.edu/~sin`.

	ovoid	1-system	2-system	m-system, $m \geq 3$	spread
$W_3(q)$	q even: YES; q odd: NO see [62]	= spread	—	—	YES, see below
$W_5(q)$	NO, [67] or Th. 1.2.6	YES, see below	= spread	—	YES, see below
$W_7(q)$	NO, [67] or Th. 1.2.6	q even: YES Th. A.1.2 & Sect. A.3.2	q even: YES, see below	3-system = spread	YES, see below
$W_{2n+1}(q)$ $n > 3$	NO, [67] or Th. 1.2.6	NO, Th. 1.2.6	sometimes YES, see below; sometimes NO, Th. 1.2.6		YES, see below

Table A.1: Existence of m-systems of $W_{2n+1}(q)$

A.3.1 The symplectic polar space $W_{2n+1}(q)$

When giving an overview of the existence of m-systems of the symplectic polar space $W_{2n+1}(q)$, Theorem 1.2.6 is of great importance, since it restricts the possible values of m and n for which an m-system might exist to the cases where $n \leq 2m+1$. In Table A.1, the known results on the existence of m-systems of $W_{2n+1}(q)$ have been collected.

The symplectic polar space $W_{2n+1}(q)$ has at least one type of spread for all n. This spread is at the same time an n-spread of the projective space $\mathsf{PG}(2n+1,q)$ and it is also called the *regular n-spread* of $\mathsf{PG}(2n+1,q)$. For more about this spread (and in some cases other spreads) of $W_{2n+1}(q)$, we refer to [1], [31], [37], [52], [65] or [75].

If q is even, $W_3(q)$ always admits an ovoid, for instance the classical ovoid $\mathcal{O} \cong Q^-(3,q)$. But one can say more. For q even, the symplectic polar space $W_3(q)$ is isomorphic to the polar space $Q(4,q)$, so that every ovoid of $Q(4,q)$ also yields an ovoid of $W_3(q)$. Hence more details about ovoids of $W_3(q)$, q even, can be found in Section A.3.2.

Both for q even and q odd, there exists a 1-system of $W_5(q)$ that is sometimes called *classical*. It arises by Theorem A.1.6 from the point set of $H(2,q^2)$, which is regarded as an ovoid of $H(2,q^2)$. If q is even, various other 1-systems of $W_5(q)$ exist, since $W_5(q) \cong Q(6,q)$. For more about these examples, we refer to Section A.3.2.

In some cases, we can say that an m-system of $W_{2n+1}(q)$, with $m \geq 3$ and $n > 3$, always exists, often by applying Theorem A.1.9 to known examples. This yields the following results.

- For q even, $W_{4s-1}(q)$ admits an $(s-1)$-system, arising from an ovoid of $W_3(q^s)$ and Theorem A.1.9.
- $W_{6s-1}(q)$ always has $(2s-1)$-systems, namely the ones obtained from a 1-system of $W_5(q^s)$ and Theorem A.1.9. For q odd, only one class is known, which arises from the classical 1-system of $W_5(q^s)$. If q is even, several classes of 1-systems of $W_5(q^s)$ are known, thus yielding several classes $(2s-1)$-systems of $W_{6s-1}(q)$.
- For q even, $W_{2n+1}(q)$ admits an $(n-1)$-system. This can be seen as follows. If q is even, then $Q(2n+2, q)$ admits a spread, as we will explain in Section A.3.2. The intersections of the elements of this spread with an elliptic quadric $Q^-(2n+1, q) \subseteq Q(2n+2, q)$ form a spread $\mathcal{S}'$ of $Q^-(2n+1, q)$, which is at the same time an $(n-1)$-system of $Q(2n+2, q)$. Since $Q(2n+2, q) \cong W_{2n+1}(q)$ for q even, we find an $(n-1)$-system of $W_{2n+1}(q)$. For $n = 3$, this is a 2-system of $W_7(q)$, as has been marked in Table A.1.
- From the $(t-2)$-system of $W_{2t-1}(q^s)$, q even, we have just described, an $(st-s-1)$-system of $W_{2st-1}(q)$ arises by Theorem A.1.9.

For $q = 2$, the m-systems of $W_{2n+1}(q)$, for $n = 1, 2, 3, 4$, have been classified by computer by Hamilton and Mathon [21]. They obtain the following results.

- $W_3(2)$ has a unique ovoid, namely the classical ovoid $\mathcal{O} \cong Q^-(3, 2)$. It also has a unique spread, which is the regular line spread of $\mathsf{PG}(3, 2)$.
- $W_5(2)$ has no ovoids, by Theorem 1.2.6. It admits two distinct 1-systems, which are both spreads of $Q^-(5, 2)$. There also exists a projectively unique spread.
- $W_7(2)$ does not admit ovoids, by Theorem 1.2.6. It has projectively unique 1-, 2- and 3-systems.
- $W_9(2)$ has no ovoids or 1-systems, again by Theorem 1.2.6, while a computer search reveals that it neither has 2-systems. Concerning 3-systems, there exist exactly 10 non-isomorphic examples. Finally, $W_9(2)$ admits 3 non-isomorphic spreads.

	ovoid	1-system	2-system	m-system $m \geq 3$	spread
$Q(4,q)$	YES, [31], [37], [52] and [75]	= spread	—	—	q even: YES q odd: NO, [61], [71]
$Q(6,q)$	YES, for $q = 3^h$: [31] & Th. 1.3.2 NO, for – $q = 5, 7$ see [47] – q even: [67] or Th. 1.2.6	YES, Chapters 4, 5, 8; Th. 1.3.2 and [4]	= spread	—	YES, for – q even – q odd prime – q odd and $q \equiv 0 \bmod 3$, $q \equiv 2 \bmod 3$; see below
$Q(8,q)$	NO: [20] and [67]/Th. 1.2.6	YES, see below		3-system = spread	q even: YES
$Q(2n,q)$ $n > 4$	NO: [20] and [67]/Th. 1.2.6	q even: NO, Th. 1.2.6	sometimes YES, see below; q even, $n > 2m$: NO, Th. 1.2.6		q even: YES q odd and $n = 2k$: NO, [61], [71]

Table A.2: Existence of m-systems of $Q(2n,q)$

A.3.2 The parabolic quadric $Q(2n,q)$

Concerning the existence and non-existence of m-systems of the parabolic quadric $Q(2n,q)$, one can compose the existence table A.2.

For q even, $Q(2n,q)$ always has a spread since it is isomorphic to $W_{2n-1}(q)$ and this last polar space has a spread for all n. Thus there appears a "YES for q even" in the last column of Table A.2.

The parabolic quadric $Q(6,q)$ also admits spreads for most odd values of q. These spreads exist because $Q^+(7,q)$ has spreads for these values of q, see Section A.3.3. They simply consist of the intersections of the elements of a spread of $Q^+(7,q)$ with an embedded $Q(6,q) \subseteq Q^+(7,q)$, conform the remark following Theorem A.1.1.

By Theorem A.1.1, every 1-system of $Q^-(7,q)$ is also a 1-system of $Q(8,q)$. The elliptic quadric $Q^-(7,q)$ certainly has a 1-system, namely the classical 1-system, which arises from the point set of $Q^-(3,q^2)$, considered as an ovoid of $Q^-(3,q^2)$, by Theorem A.1.8. This 1-system exists both for q even and for q odd. If q is even, it also yields a 1-system of $W_7(q)$, which has been mentioned in Table A.1.

There are some general cases in which m-systems of $Q(2n, q)$ are known to exist. They all arise from Theorem A.1.1 and the known m-systems of $Q^-(2n-1, q)$, which will be discussed in Section A.3.4. Here we list the values of n and m to which this applies.

- $Q(4e, q)$, $e > 1$, admits an $(e-1)$-system.
- $Q(2st, q)$, $s > 1$ and $t > 1$, admits an $(st - s - 1)$-system.
- $Q(6s, q)$, $s > 1$ admits a $(2s-1)$-system.

Another series of values for which it is known that $Q(2n, q)$ admits an m-system, has been mentioned in the remark on page 146, where it is explained that $Q(6 \cdot 2^n, q)$ always admits several classes of $(2^{n+1} - 1)$-systems if q is even.

Finally, we refer to Section A.3.1 for a classification of the m-systems of $Q(2n, 2)$ for $n = 2, 3, 4, 5$, since $Q(2n, 2)$ is isomorphic to $W_{2n-1}(2)$.

A.3.3 The hyperbolic quadric $Q^+(2n+1, q)$

The hyperbolic quadric $Q^+(2n+1, q)$ is doubtlessly the polar space for which the most examples of m-systems are known. This is partly due to Theorem A.1.1 and the remark following Theorem A.1.3, which provide us with examples, coming from elliptic or parabolic quadrics. But there also exist other examples, which are proper m-systems of $Q^+(2n+1, q)$. This leads to the results, gathered in Table A.3.

Concerning spreads of $Q^+(2n+1, q)$, some easy observations already provide us with a lot of information. If $n = 2k$ is even, then the generators of $Q^+(4k+1, q)$ which belong to the same family, are never disjoint. Hence it is impossible to find more than two disjoint generators of $Q^+(4k+1, q)$, which implies that a spread never exists. If $n = 2k+1$ is odd, there exist disjoint generators of the same family and by the remark on page 164 following Theorem A.1.3, one can construct a spread of $Q^+(4k+3, q)$ from every spread of $Q(4k+2, q)$. For q even, such a spread always exists and as such, a spread of $Q^+(4k+3, q)$ always exists as well if q is even. These observations partly explain the last column of Table A.3.

The hyperbolic quadric $Q^+(5, q)$ has a projectively unique 1-system if q is odd, namely the classical one, which arises by Theorem A.1.8 from $Q(2, q^2)$, considered as an ovoid of itself. If q is even, every 1-system of $Q^+(5, q)$ is a spread of $Q(4, q)$ and conversely, so here we can refer to the results of Section A.3.2. Both of these properties have been shown by Shult and Thas

	ovoid	1-system	2-system	m-system, $m \geq 3$	spread
$Q^+(3,q)$	YES, e.g. conic	= spread	—	—	YES, one regulus
$Q^+(5,q)$	YES, [25, Chapter 17]	YES, see below	= spread	—	NO
$Q^+(7,q)$	YES, for – q odd prime – q odd and $q \equiv 0, 2 \bmod 3$ [12], [16], [31], [43], [56], [66], [67] – q even: triality & existence spreads	YES: – Sect. A.3.2 & Th. A.1.1 – Chapt. 6, 7, 8 – others: see below	YES, for – q even – q odd prime – q odd and $q \equiv 0, 2 \bmod 3$ Th. A.1.3 & existence spreads	3-system = spread	YES, for – q even p. 164 and Sect. A.3.2 – q odd prime – q odd and $q \equiv 0, 2 \bmod 3$ triality & existence ovoids
$Q^+(2n+1,q)$ $n > 3$	sometimes NO see below	YES, for – $n = 4$ – $n = 6$, $q = 3^h$ Th. A.1.8 & Sect. A.3.2	sometimes YES, see below; sometimes NO, Th. A.2.2		NO, for $n = 2k$ YES, for $n = 2k+1$ and q even

Table A.3: Existence of m-systems of $Q^+(2n+1,q)$

in [57] and together they constitute a classification of all proper 1-systems of $Q^+(5, q)$.

In the course of this work, several classes of 1-systems of $Q^+(7, q)$ have been discovered; most of them are locally hermitian examples. In Chapters 4 and 5, 1-systems of an induced $Q(6, q) \subseteq Q^+(7, q)$ have been constructed, while Chapters 6 and 7 were devoted to proper 1-systems of $Q^+(7, q)$ and in Chapter 8, new examples of both proper and non-proper 1-systems of $Q^+(7, q)$ have been obtained.
Here we want to draw the attention to one more example. N. Hamilton informed us about the fact that it is possible to construct a 1-system of $Q^+(7, q)$ by selecting q^3+1 suitable elements of an egg, arising from a Kantor–Knuth semifield flock (see for instance [32], [34]). This 1-system does not belong to any previously known class. As a consequence of the features of eggs, it has the property that every three of its lines span a 5-dimensional subspace of $\mathsf{PG}(7, q)$.

The spreads of $Q^+(7, q)$ which are known to exist for q odd, mostly arise from the ovoids of this same polar space. If $Q^+(7, q)$ admits an ovoid $\mathcal{O}$, then the image of $\mathcal{O}$ under a triality is a spread of $Q^+(7, q)$. Conversely, a triality also allows to transform a known spread into an ovoid, which explains the existence of ovoids of $Q^+(7, q)$ for q even.

Similarly as for parabolic quadrics $Q(2n, q)$, the existence of m-systems of $Q^-(2n-1, q)$ and Theorem A.1.1 imply the following results on m-systems of $Q^+(2n+1, q)$.

- $Q^+(4e+1, q)$, $e > 1$, admits an $(e-1)$-system.
- $Q^+(2st+1, q)$, $s > 1$ and $t > 1$, admits an $(st-s-1)$-system.
- $Q^+(6s+1, q)$, $s > 1$, admits a $(2s-1)$-system.

We also recall that in the remark on page 146, infinite classes of $(2^{n+1}-1)$-systems of $Q^+(6 \cdot 2^n + 1, q)$ have been found for q even, as well as several classes of 3-systems of $Q^+(13, q)$ for q odd.

Finally, the existence of m-systems of $Q^+(2n+1, 2)$ has been investigated by computer by Hamilton and Mathon [21] for $n = 1, 2, 3, 4$. We summarize their results.

- $Q^+(3, 2)$ has a unique ovoid, namely the classical ovoid $\mathcal{O} \cong Q(2, q)$. It also admits a unique 1-system, consisting of the 3 lines of one set of generators.
- $Q^+(5, 2)$ admits a unique ovoid, which is once more the classical example $\mathcal{O} \cong Q^-(3, 2)$. Since $W_3(2)$ has a projectively unique 1-system, see

	ovoid	1-system	m-system, $m \geq 2$	spread
$Q^-(5,q)$	NO, see [67] or Th. 1.2.6	= spread	—	YES, see below
$Q^-(7,q)$	NO, see [67] or Th. 1.2.6	YES, see below	2-system = spread	q even: YES p. 164 & Sect. A.3.2
$Q^-(2n+1,q)$ $n > 3$	NO, see [67] or Th. 1.2.6	NO, Th. 1.2.6	sometimes YES, see below sometimes NO, Th. 1.2.6	q even: YES p. 164 & Sect. A.3.2

Table A.4: Existence of m-systems of $Q^-(2n+1,q)$

Section A.3.1, this also holds for $Q(4,2)$ and hence for $Q^+(5,2)$. As mentioned above, a spread of $Q^+(5,2)$ does not exist.

- $Q^+(7,2)$ has a unique ovoid, and as a consequence also a unique spread, since a triality of $Q^+(7,2)$ transforms ovoids into spreads and conversely. From the computer results, one concludes that $Q^+(7,2)$ has a unique 2-system; so it must be the example, arising from the unique spread of $Q^+(7,2)$ and Theorem A.1.3. The computer search also reveals that there exist two non-isomorphic 1-systems of $Q^+(7,2)$, which are both spreads of a $Q^-(5,2) \subseteq Q^+(7,2)$.

- $Q^+(9,2)$ has no ovoids by the bound of Theorem A.2.2 and no spreads since $n = 4$ is even. It appears to have projectively unique 1-, 2- and 3-systems, none of which is proper.

A.3.4 The elliptic quadric $Q^-(2n+1,q)$

Not so many examples of m-systems of the elliptic quadric $Q^-(2n+1,q)$ are known. This is not surprising, since the scale of possible values for n and m for which an m-system of $Q^-(2n+1,q)$ might exist, is limited to $n \leq 2m+1$ by Theorem 1.2.6. The known results on the existence of m-systems of $Q^-(2n+1,q)$ have been collected in Table A.4.

The elliptic quadric $Q^-(5,q)$ has a spread for all q. The classical or hermitian example is the spread which arises by Theorem A.1.4 from the point set of $H(2,q^2)$, regarded as an ovoid of $H(2,q^2)$. Other examples can be obtained by switching a number of reguli of the classical spread, similarly as has been done in Chapter 8 for 1-systems of polar spaces in a more general context. But we have encountered still other spreads of $Q^-(5,q)$, namely all locally hermitian examples of the form $\mathcal{S}(S)$, with S an arbitrary line spread of $\mathsf{PG}(3,q)$. For more results about spreads of $Q^-(5,q)$, we refer to [50, 3.4], [69] or [75].

For $n > 2$ as well, the elliptic quadric $Q^-(2n+1,q)$ always admits spreads if q is even, namely the examples consisting of the intersections with $Q^-(2n+1,q)$ of the elements of a spread of a $Q(2n+2,q)$ in which $Q^-(2n+1,q)$ is embedded; see also the remark on page 164, following Theorem A.1.1.

There certainly exists a 1-system of $Q^-(7,q)$, both for q even and q odd, namely the classical example. It results from the application of Theorem A.1.8 to the elliptic quadric $Q^-(3,q^2)$, which is considered as an ovoid of itself. In Chapter 3, we have shown that this example is unique if q is odd; for q even, nothing more is known about 1-systems of $Q^-(7,q)$.

Concerning m-systems of $Q^-(2n+1,q)$ in general, some infinite classes arise from the chains of m-systems, described in Section A.1.2. We list these examples and also indicate which chain they are due to.

- $Q^-(4e-1,q)$, $e > 1$, admits an $(e-1)$-system, which is obtained from $Q^-(3,q^e)$, seen as on ovoid of itself, and Theorem A.1.7 (a).
- $Q^-(2st-1,q)$, $s > 1$ and $t > 1$, admits an $(st-s-1)$-system for q even. This example is constructed from a spread of $Q^-(2t-1,q^s)$, q even, and Theorem A.1.7 (a).
- $Q^-(6s-1,q)$, $s > 1$, admits several classes of $(2s-1)$-systems for all q, which arise from a spread of $Q^-(5,q^s)$ and Theorem A.1.7 (a). If q is even, this case is contained in the previous one.

Finally, Hamilton and Mathon [21] have also studied the m-systems of $Q^-(2n+1,2)$ for $n = 1,2,3,4$ by computer. They find that for $m < n \leq 4$, the m-systems of $Q^-(2n+1,2)$ are exactly the m-systems of the symplectic polar space $W_{2n+1}(2)$ defined by $Q^-(2n+1,2)$. Hence we refer to Section A.3.1 for an overview of these examples.

A.3.5 The hermitian varieties $H(l,q^2)$

The knowledge concerning m-systems of the hermitian varieties is very poor. If $l = 2n$ is even, we dispose of the bound of Theorem 1.2.6, which implies that m-systems of $H(2n,q^2)$ do not exist for $n > 2m+1$. An overview of

	ovoid	1-system	m-system, $m \geq 2$	spread
$H(3,q^2)$	YES, see below	= spread	—	NO, [67], [71]
$H(4,q^2)$	NO, [67] or Th. 1.2.6	= spread	—	$H(4,4)$: NO, Brouwer[11]
$H(2n+1,q^2)$ $n>1$	sometimes NO, Th. A.2.1 - A.2.2 or Th. A.2.3	sometimes NO, Th. A.2.2		NO, [67], [71]
$H(2n,q^2)$ $n>2$	NO, [67] or Th. 1.2.6	sometimes YES, see below sometimes NO, Th. 1.2.6		

Table A.5: Existence of m-systems of $H(l,q^2)$

other known results is given in Table A.5.

The generalized quadrangle $H(3,q^2)$ is isomorphic to the point-line dual of $Q^-(5,q)$, so the existence question for ovoids of $H(3,q^2)$ is equivalent to the existence question for spreads of $Q^-(5,q)$. Hence many examples of ovoids of $H(3,q^2)$ are known and we refer to the discussion of spreads of $Q^-(5,q)$ in Section A.3.4 for more details on this topic.

There is one case in which an m-system of $H(2n,q^2)$ can be obtained from the chains, described in Section A.1.2. From Theorem A.1.5, applied to the point set of $H(2,q^{2r})$ which is regarded as an ovoid of itself, it follows that $H(3r-1,q^2)$ has an $(r-1)$-system for all odd r.

Concerning the existence of m-systems of $H(l,q^2)$, the current knowledge is limited to these results.

[11] Computer result, unpublished.

Bijlage B

m-Systemen van eindige polaire ruimten

In deze bijlage wordt een overzicht in vogelvlucht gegeven van de bijdrage die door deze thesis wordt geleverd tot de theorie van m-systemen van eindige polaire ruimten. Van de belangrijkste stellingen vindt de lezer hier een schets van het bewijs, die niet bedoeld is om alle stappen in detail te verklaren, maar wel om een idee te geven van de gevolgde gedachtengang.

B.1 Definities

B.1.1 Polaire ruimten

m-Systemen van eindige polaire ruimten zijn het centrale thema van deze thesis. Om te begrijpen wat een m-systeem is, moet men echter eerst op de hoogte zijn van de definitie van een polaire ruimte.

Een *eindige abstracte polaire ruimte* van *rang* $r \geq 3$ is een incidentiestructuur die bestaat uit een eindige verzameling $\mathcal{P}$ van objecten die we *punten* noemen, en een verzameling van deelverzamelingen van $\mathcal{P}$, *deelruimten* genaamd, met de volgende eigenschappen.

P1 Elke deelruimte, samen met de deelruimten die ertoe behoren, is isomorf met een projectieve ruimte $\mathsf{PG}(d, q)$, $-1 \leq d \leq r - 1$. Het getal d heet de *dimensie* van de deelruimte.

P2 De doorsnede van elke twee deelruimten is een deelruimte.

P3 Zij π een deelruimte van dimensie $r-1$ en $p \in \mathcal{P}$ een punt dat niet in π ligt. Dan bestaat er juist één deelruimte π' van dimensie $r-1$, die p bevat en π snijdt in een deelruimte van dimensie $r-2$. De deelruimte $\pi \cap \pi'$ bestaat uit alle punten van π die samen met p in een 1-dimensionale deelruimte liggen.

P4 Er bestaan twee disjuncte deelruimten van dimensie $r-1$.

De deelruimten van dimensie $r-1$ noemen we ook de *generatoren* van de polaire ruimte.

Uit een belangrijke stelling van Tits [77] volgt dat alle eindige polaire ruimten van rang $r \geq 3$ *klassiek* zijn. Dat wil zeggen dat ze gedefinieerd worden door een symplectische polariteit, een parabolische, hyperbolische of elliptische kwadriek, of een hermitische variëteit. In de respectieve gevallen worden de volgende notaties gebruikt: $W_{2n+1}(q)$, $Q(2n,q)$, $Q^+(2n+1,q)$, $Q^-(2n+1,q)$ of $H(n,q^2)$. De laatste vier notaties staan zowel voor de polaire ruimte als voor de kwadriek of hermitische variëteit.
De klassieke polaire ruimten kunnen ingebed worden in projectieve ruimten. Dit betekent dat hun punten en deelruimten sommige punten en deelruimten zijn van een zekere projectieve ruimte. In dit geval wordt de projectieve ruimte die voortgebracht wordt door de punten van een polaire ruimte, ook wel de *omhullende* ruimte van deze polaire ruimte genoemd.

B.1.2 m-Systemen

Ovoïden en spreads van de klassieke polaire ruimten worden reeds lang bestudeerd. De *m-systemen* zijn in feite veralgemeningen van deze begrippen en werden voor het eerst ingevoerd in 1994 door Shult en Thas [57], op de volgende wijze.

Een *partieel m-systeem* van een eindige polaire ruimte $\mathcal{P}$ is een verzameling $\mathcal{M} = \{\pi_1, \pi_2, \ldots, \pi_r\}$ van m-dimensionale deelruimten van $\mathcal{P}$ met de eigenschap dat elke generator van $\mathcal{P}$ die een element $\pi_i \in \mathcal{M}$ bevat, disjunct is van alle elementen van $\mathcal{M} \setminus \{\pi_i\}$. Er bestaat een bovengrens voor het aantal elementen van een partieel m-systeem, die afhankelijk is van de polaire ruimte waarop het m-systeem leeft, maar — verrassend genoeg — onafhankelijk van de waarde van m. Deze bovengrens werd op zuiver combinatorische wijze afgeleid in [57] en de waarde luidt in de verschillende gevallen als volgt.

Stelling B.1.1 *Zij $\mathcal{M}$ een partieel m-systeem van een eindige polaire ruimte $\mathcal{P}$. Dan gelden de volgende uitspraken:*

$$\begin{array}{llll}
\textit{als} & \mathcal{P} = W_{2n+1}(q) \textit{ of } Q^-(2n+1,q), & \textit{dan} & |\mathcal{M}| \leq q^{n+1}+1; \\
\textit{als} & \mathcal{P} = Q(2n,q) \textit{ of } Q^+(2n+1,q), & \textit{dan} & |\mathcal{M}| \leq q^{n}+1; \\
\textit{als} & \mathcal{P} = H(2n,q^2) \textit{ of } H(2n+1,q^2), & \textit{dan} & |\mathcal{M}| \leq q^{2n+1}+1.
\end{array}$$

Een partieel m-systeem waarvan het aantal elementen precies gelijk is aan de bovengrens uit Stelling B.1.1, wordt een *m-systeem* genoemd. Voor $m = 0$ komt deze definitie overeen met de in de literatuur gebruikelijke definitie van een *ovoïde*, terwijl de waarde $m = r-1$, met r de rang van de polaire ruimte, de definitie van een *spread* oplevert.

Als $\mathcal{M}$ een m-systeem is van een zekere polaire ruimte $\mathcal{P}$, dan noteren we met $\tilde{\mathcal{M}}$ de verzameling van alle punten die tot de elementen van $\mathcal{M}$ behoren.

Tot slot van deze paragraaf vestigen we de aandacht op een laatste definitie. Beschouw een eindige polaire ruimte $\mathcal{P}$, die ingebed is in $\mathsf{PG}(n,q)$, en een m-systeem $\mathcal{M}$ van $\mathcal{P}$. Dan zegt men dat $\mathcal{M}$ een *echt* m-systeem is van $\mathcal{P}$ als en slechts als $\mathcal{M}$ niet bevat is in een hypervlak van $\mathsf{PG}(n,q)$. Deze definitie is zeer nuttig omdat de echte m-systemen van polaire ruimten zich vaak anders gedragen dan de voorbeelden die niet echt zijn (in het vervolg ook *onecht* genoemd). In het bijzonder wat betreft karakteriserende stellingen is het onderscheid tussen echte en onechte m-systemen cruciaal.

B.1.3 Overige meetkunden

Een aantal andere meetkundige structuren komen meermaals aan bod in deze thesis. We vermelden de definitie van een (semi-)partiële meetkunde en een net.

Een *semi-partiële meetkunde* is een incidentiestructuur $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ van punten en rechten, waarvoor aan de volgende voorwaarden voldaan is:

spg1 Door elk punt gaan $t+1$ $(t \geq 1)$ rechten en door twee verschillende punten gaat hoogstens één rechte.

spg2 Op elke rechte liggen $s+1$ $(s \geq 1)$ punten en twee verschillende rechten hebben hoogstens één punt gemeen.

spg3 Als twee punten niet op een zelfde rechte liggen, dan bestaan er μ $(\mu > 0)$ punten die collineair zijn met beide punten.

spg4 Zij x een punt en L een rechte, niet door x. Dan bestaan er 0 of α $(\alpha \geq 1)$ punten op L die collineair zijn met x.

Indien geëist wordt dat er voor elk niet-incident punt–rechtepaar (x, L) juist α $(\alpha \geq 1)$ punten van L bestaan die collineair zijn met x, dan bekomt men de definitie van een *partiële meetkunde*. In dit geval is aan voorwaarde **spg3** automatisch voldaan, met $\mu = (t+1)\alpha$. Een partiële meetkunde waarvoor $\alpha = t$, wordt ook wel een *net van de orde* $s+1$ *en de graad* $t+1$ genoemd.

B.2 Sommige m-systemen zijn SPG-reguli

Een *SPG-regulus* in $\mathsf{PG}(n, q)$ is een verzameling $\mathcal{R}$ van m-dimensionale deelruimten $\pi_1, \pi_2, \ldots, \pi_r$, $r \geq 1$, van $\mathsf{PG}(n, q)$, die aan de volgende voorwaarden voldoen:

SPG1 $\pi_i \cap \pi_j = \emptyset$ voor alle $i \neq j$.

SPG2 Indien $\mathsf{PG}(m+1, q)$ een zekere $\pi_i \in \mathcal{R}$ bevat, dan heeft het een punt gemeen met 0 of α, $\alpha > 0$, elementen van $\mathcal{R} \setminus \{\pi_i\}$. Als een dergelijke $\mathsf{PG}(m+1, q)$ geen enkel punt gemeen heeft met π_j, voor alle $\pi_j \in \mathcal{R} \setminus \{\pi_i\}$, dan noemt men $\mathsf{PG}(m+1, q)$ een $(m+1)$-dimensionale *raakruimte* aan $\mathcal{R}$ in π_i.

SPG3 Indien een punt x van $\mathsf{PG}(n, q)$ tot geen enkel element van $\mathcal{R}$ behoort, dan is het bevat in een constant aantal θ, $\theta \geq 0$, $(m+1)$-dimensionale raakruimten aan $\mathcal{R}$.

In dit hoofdstuk wordt op een combinatorische wijze, en enkel steunend op elementaire eigenschappen van m-systemen, aangetoond dat sommige m-systemen tevens SPG-reguli zijn, wat aanleiding geeft tot de volgende stelling.

Stelling B.2.1 *Zij $\mathcal{M}$ een willekeurig m-systeem van een polaire ruimte $W_{2n+1}(q)$, $Q^-(2n+1, q)$ of $H(2n, q^2)$. Dan is $\mathcal{M}$ een SPG-regulus van de projectieve ruimte die deze polaire ruimte omhult.*

Schets van het bewijs.
Beschouw een m-systeem $\mathcal{M}$ van een eindige polaire ruimte $\mathcal{P} \in \{W_{2n+1}(q)$, $Q^-(2n+1, q)$, $H(2n, q^2)\}$ en duid de omhullende projectieve ruimte van $\mathcal{P}$ aan met $\mathsf{PG}(k, t)$, zodat $(k, t) \in \{(2n+1, q), (2n+1, q), (2n, q^2)\}$. Zij $\pi_m \in \mathcal{M}$ willekeurig en beschouw een $(m+1)$-dimensionale deelruimte $\pi_{m+1} \supseteq \pi_m$, die $\mathcal{P}$ snijdt in een ontaarde polaire ruimte $\pi_{m-1}\mathcal{P}_1$, voor een zekere $(m-1)$-dimensionale deelruimte π_{m-1} van π_m. Hierbij staat $\mathcal{P}_1$ voor $W_1(q)$, $Q^+(1, q)$ of $H(1, q^2)$ in de respectieve gevallen, en uiteraard is $\pi_m = \langle \pi_{m-1}, p \rangle$ voor een zeker punt $p \in \mathcal{P}_1$. Door op twee manieren de koppels (H, x) te tellen,

waarbij H een hypervlak door π_{m+1} is en x een punt van $(H \cap \tilde{\mathcal{M}}) \setminus \pi_{m+1}$, vinden we dat het aantal punten van $\tilde{\mathcal{M}}$ in $\pi_{m-1}\mathcal{P}_1$ constant is. Aangezien elke $(m+1)$-dimensionale deelruimte van $\mathcal{P}$ die π_m bevat, een lege doorsnede heeft met alle elementen van $\mathcal{M} \setminus \{\pi_m\}$ en men **SPG3** gemakkelijk kan afleiden uit de elementaire eigenschappen van m-systemen van de betrokken polaire ruimten, laat bovenstaande telling toe te besluiten dat het m-systeem in alledrie de gevallen een SPG-regulus is. □

Dankzij een constructie van Thas [69], kan een semi-partiële meetkunde geassocieerd worden aan elke SPG-regulus. Dit gebeurt als volgt. Veronderstel dat $\mathcal{R}$ een SPG-regulus is in $\mathsf{PG}(n,q)$, waarvan de elementen m-dimensionaal zijn. Bed $\mathsf{PG}(n,q)$ als een hypervlak in een $\mathsf{PG}(n+1,q)$ in en beschouw de volgende incidentiestructuur $\mathcal{S} = (P, B, \mathrm{I})$ van punten en rechten. De punten van P zijn per definitie alle punten van $\mathsf{PG}(n+1,q)$ die niet in $\mathsf{PG}(n,q)$ liggen; de rechtenverzameling B bestaat uit alle $(m+1)$-dimensionale deelruimten van $\mathsf{PG}(n+1,q)$ die $\mathsf{PG}(n,q)$ snijden in een element van $\mathcal{R}$; en incidentie is gewoon de incidentie van $\mathsf{PG}(n+1,q)$. De aldus geconstrueerde structuur is een semi-partiële meetkunde met parameters $s = q^{m+1}-1$, $t = r-1$, $\alpha = \alpha$ en $\mu = (r-\theta)\alpha$ en ze wordt in het vervolg aangeduid met de notatie $\mathsf{SPG}(\mathcal{R})$.

In combinatie met Stelling B.2.1 vinden we dus dat sommige m-systemen aan de basis liggen van een semi-partiële meetkunde. Uit een grondige studie van de gekende voorbeelden blijkt dat een aantal van de semi-partiële meetkunden, opgebouwd vanuit een m-systeem van $W_{2n+1}(q)$, nieuw zijn, maar helaas geldt dit niet voor hun parameters.

B.3 Uniciteit van het 1-systeem van $Q^-(7,q)$, q oneven

Zowel voor q oneven als voor q even is er slechts één 1-systeem van de elliptische kwadriek $Q^-(7,q)$ gekend, dat als volgt geconstrueerd wordt. Beschouw de kwadratische extensie $\mathsf{PG}(7,q^2)$ van $\mathsf{PG}(7,q)$ en de bijhorende extensie $Q^+(7,q^2)$ van $Q^-(7,q)$. Het is mogelijk twee disjuncte 3-ruimten γ en $\overline{\gamma}$ te vinden, die toegevoegd zijn ten opzichte van $\mathsf{GF}(q^2)$, elkaars beeld zijn onder de polariteit van $Q^+(7,q^2)$ en zó dat $\gamma \cap Q^+(7,q^2)$ een elliptische kwadriek $Q^-(3,q^2)$ is. Uit Shult en Thas [57] volgt dat $\mathcal{M} := \{x\overline{x} \cap \mathsf{PG}(7,q) \mid x \in Q^-(3,q^2)\}$ een 1-systeem is van $Q^-(7,q)$. Dit 1-systeem wordt het *klassieke* 1-systeem van $Q^-(7,q)$ genoemd.

Dit hoofdstuk is volledig gewijd aan de vraag of het klassieke 1-systeem van $Q^-(7,q)$, q oneven, uniek is. Het antwoord blijkt „ja" te zijn, waarbij de sleutel tot de oplossing in het vorige hoofdstuk ligt. Daar bekwamen we namelijk als bijproduct de volgende interessante eigenschap van 1-systemen van $Q^-(7,q)$ voor q oneven.

Eigenschap B.3.1 *Zij $\mathcal{M}$ een willekeurig 1-systeem van de elliptische kwadriek $Q^-(7,q)$, q oneven. Dan heeft elke rechte van $Q^-(7,q)$ die geen rechte van $\mathcal{M}$ is, hoogstens twee punten gemeen met $\tilde{\mathcal{M}}$. Een vlak van $Q^-(7,q)$ bevat ofwel een rechte van $\mathcal{M}$, ofwel een irreducibele kegelsnede van punten van $\tilde{\mathcal{M}}$.*

Deze observatie ligt aan de basis van een meetkundig bewijs voor de volgende stelling.

Stelling B.3.2 *Als q oneven is, heeft de elliptische kwadriek $Q^-(7,q)$ een uniek 1-systeem op een projectiviteit na, namelijk het klassieke 1-systeem.*

Schets van het bewijs.
Zij $\mathcal{M}$ een willekeurig 1-systeem van $Q^-(7,q)$, q oneven, en duid de polariteit van $Q^-(7,q)$ aan met $\perp$.

In eerste instantie beschouwen we twee rechten $L_1, L_2 \in \mathcal{M}$, en een punt $r \in \langle L_1, L_2\rangle^\perp$. Dan is $\langle r, L_1, L_2\rangle \cap Q^-(7,q)$ een kegel $rQ^+(3,q)$, waarbij we overeenkomen dat $\langle L_1, L_2\rangle \cap Q^-(7,q) := Q^+(3,q)$. Wegens Eigenschap B.3.1 geldt dat alle vlakken van $rQ^+(3,q)$, behalve $\langle r, L_1\rangle$ en $\langle r, L_2\rangle$, een irreducibele kegelsnede van punten van $\tilde{\mathcal{M}}$ bevatten. Door de structuur van deze kegelsneden nauwgezet onder de loep te nemen, kan aangetoond worden dat er in de extensie $Q^+(3,q^2)$ van $Q^+(3,q)$ twee toegevoegde rechten A en $\overline{A}$ bestaan, die de extensies van L_1 en L_2 tot $\mathsf{GF}(q^2)$ in een punt snijden en de volgende eigenschap hebben. Veronderstel dat de regulus van $Q^+(3,q)$ die L_1 en L_2 bevat, gegeven wordt door $\{L_1, L_2, S_1, S_2, \ldots, S_{q-1}\}$. Dan is elke kegelsnede van punten van $\tilde{\mathcal{M}}$ in een vlak van de vorm $\langle r, S_j\rangle$, $j \in \{1, 2, \ldots, q-1\}$, rakend aan de rechten ra_j, respectievelijk $r\overline{a}_j$, in het punt $a_j := A \cap S_j$, respectievelijk $\overline{a}_j := \overline{A} \cap S_j$. Bovendien kan men bewijzen dat de rechten A en $\overline{A}$ onafhankelijk zijn van de keuze van het punt r van $Q^-(3,q) := \langle L_1, L_2\rangle^\perp \cap Q^-(7,q)$.

Zij vervolgens L_1, L_2, $Q^+(3,q)$ en $Q^-(3,q)$ zoals hierboven. Door te steunen op de inmiddels goed gekende structuur van de punten van $\tilde{\mathcal{M}}$ in kegels van de vorm $rQ^+(3,q)$, kunnen we aantonen dat elke elliptische kwadriek $Q^-(5,q)$ op $Q^-(7,q)$ die $Q^-(3,q)$ bevat én minstens één rechte gemeen heeft met $\mathcal{M} \setminus \{L_1, L_2\}$, precies q^2+1 rechten van $\mathcal{M}$ bevat. Voorts wordt beredeneerd dat voor elke rechte $B \in \mathcal{M} \setminus \{L_1, L_2\}$, de kwadriek

$\langle Q^-(3,q), B\rangle \cap Q^-(7,q)$ steeds niet-ontaard is en bijgevolg ofwel een $Q^-(5,q)$, ofwel een $Q^+(5,q)$ is.

De volgende stap in het bewijs betreft hyperbolische kwadrieken van de vorm $Q^+(5,q) = \langle Q^-(3,q), B\rangle \cap Q^-(7,q)$ voor een zekere $B \in \mathcal{M} \setminus \{L_1, L_2\}$. Ook voor zulke kwadrieken wordt bewezen dat ze steeds juist q^2+1 rechten van $\mathcal{M}$ bevatten. Dit gedeelte van het bewijs maakt gebruik van de theorie van inversieve vlakken en flocks van de elliptische kwadriek $Q^-(3,q)$, zie bijvoorbeeld Fisher en Thas [18]. Merk op dat in deze situatie de q^2+1 rechten van $\mathcal{M}$ in $Q^+(5,q)$ een 1-systeem vormen van $Q^+(5,q)$ en dat het 1-systeem van $Q^+(5,q)$, q oneven, uniek (en klassiek) is wegens Shult en Thas [57].

Daarna worden de voorgaande resultaten samengevoegd, zodat uiteindelijk kan besloten worden dat voor elke drie verschillende rechten L_1, L_2, L_3 van $\mathcal{M}$, de 5-dimensionale deelruimte $\langle L_1, L_2, L_3\rangle$ de kwadriek $Q^-(7,q)$ snijdt in een $Q^-(5,q)$ of een $Q^+(5,q)$ die precies q^2+1 rechten van $\mathcal{M}$ bevat. Hiertoe volstaat het in $\langle L_1, L_2, L_3\rangle$ een 3-ruimte van de vorm $\langle M, M'\rangle^\perp$ te vinden, voor zekere $M, M' \in \mathcal{M} \setminus \{L_1, L_2, L_3\}$, wat steeds mogelijk blijkt.

Tenslotte wordt het bewijs afgerond door de observatie dat $\mathcal{M}$ een ei $O(2,4,q)$ is, dat wegens het bovenstaande goed is in elk van zijn elementen. Dankzij een resultaat van Thas [63], impliceert dit dat $\mathcal{M}$ (als ei) regulier is. Het is nu een koud kunstje om deze structurele informatie omtrent $\mathcal{M}$ te combineren met de structurele informatie waarover we al beschikten, namelijk dat de q^2+1 rechten van $\mathcal{M}$ in een $Q^+(5,q)$ een klassiek 1-systeem vormen van deze $Q^+(5,q)$. Deze tactiek bewijst ruimschoots dienst, want ze leidt ons uiteindelijk tot het besluit dat $\mathcal{M}$ klassiek is. $\square$

B.4 Lokaal hermitische 1-systemen van de parabolische kwadriek $Q(6,q)$, q oneven

Een 1-systeem $\mathcal{M}$ van een niet-ontaarde kwadriek $\mathcal{Q}$ wordt *lokaal hermitisch* in een rechte $L \in \mathcal{M}$ genoemd als en slechts als voor elke rechte $M \in \mathcal{M}\setminus\{L\}$, de regulus die L en M bevat van de hyperbolische kwadriek $Q^+(3,q) = \langle L, M\rangle \cap \mathcal{Q}$, volledig bestaat uit rechten van $\mathcal{M}$.

In dit hoofdstuk buigen we ons in het bijzonder over lokaal hermitische 1-systemen van de parabolische kwadriek $Q(6,q)$ voor q oneven. Het best gekende voorbeeld hiervan is het zogenaamde *hermitische* 1-systeem, dat een hermitische spread is van een elliptische kwadriek $Q^-(5,q)$, ingebed in $Q(6,q)$. Deze spread is het duale van de klassieke ovoïde $H(2,q^2)$ van $H(3,q^2)$, gelet op het feit dat $H(3,q^2) = Q^-(5,q)^D$, waarbij de notatie X^D

het duale object van een zeker object X aanduidt. De hermitische spread van $Q^-(5,q)$ is lokaal hermitisch in elk van zijn rechten. Het omgekeerde geldt ook, en zelfs sterker: elk 1-systeem van $Q(6,q)$ dat lokaal hermitisch is in tenminste twee rechten, is hermitisch.

Als q oneven is, definieert $Q(6,q)$ een polariteit $\perp$. Beschouw een willekeurig 1-systeem $\mathcal{M}$ van $Q(6,q)$ dat lokaal hermitisch is in een zekere rechte $L \in \mathcal{M}$ en noteer voortaan de q^2 reguli van $\mathcal{M}$ door L met $R_1, R_2, \ldots, R_{q^2}$. Het beeld onder $\perp$ van een 3-ruimte $\langle R_i \rangle$, $i \in \{1, 2, \ldots, q^2\}$, is een vlak dat $Q(6,q)$ snijdt in een irreducibele kegelsnede $\mathcal{C}_i$, bestaande uit punten van de kegel $LQ(2,q) := L^\perp \cap Q(6,q)$. Door toepassing van de polariteit bekomt men q^2 dergelijke kegelsneden uit het 1-systeem $\mathcal{M}$. Uit de eigenschappen van 1-systemen leidt men bovendien gemakkelijk af dat $\langle \mathcal{C}_i \rangle \cap \langle \mathcal{C}_j \rangle$ een inwendig punt is van $Q(6,q)$, en dus ook van $LQ(2,q)$, voor alle $i, j \in \{1, 2, \ldots, q^2\}$ met $i \neq j$. De volgende definitie ligt dan ook voor de hand.

Een partitie van de punten van een kegel $LQ(2,q)$, niet op L, in q^2 disjuncte kegelsneden met de bijkomende eigenschap dat de vlakken van elke twee zulke kegelsneden — kortweg *kegelsnedevlakken* genoemd — elkaar snijden in een inwendig punt van $LQ(2,q)$, wordt een *i-flock* van $LQ(2,q)$ genoemd.

Aangezien bovenstaande constructie van een i-flock uit een lokaal hermitisch 1-systeem van $Q(6,q)$ duidelijk omkeerbaar is, zijn lokaal hermitische 1-systemen van $Q(6,q)$, q oneven, en i-flocks van $LQ(2,q)$ dus equivalente objecten.

Met deze constructie in het achterhoofd vatten we de studie van een interessant 1-systeem van $Q(6,q)$ aan, namelijk de spread $\mathcal{S}_{[9]}$ van de veralgemeende zeshoek $\mathsf{H}(q)$ op $Q(6,q)$, met $q \equiv 1 \bmod 3$, die beschreven wordt door Bloemen, Thas en Van Maldeghem in [4]. Deze spread is, zoals alle spreads van $\mathsf{H}(q)$, tevens een 1-systeem van $Q(6,q)$ en is lokaal hermitisch in precies één rechte (en dus niet hermitisch). De gekende coördinaten van de elementen van $\mathcal{S}_{[9]}$, opgevat als 1-systeem van $Q(6,q)$, worden vervolgens gebruikt om de bijhorende i-flock expliciet te bepalen. Het daartoe vereiste rekenwerk loont zeker de moeite: we vinden namelijk dat de snijpunten van de kegelsnedevlakken van de i-flock, samen met L, een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$ met directrixrechte L vormen (zie [55, VII-3.22] voor meer informatie over dit object). Bovendien kan men, vertrekkend van dit regeloppervlak, de i-flock opnieuw opbouwen, doordat de kegelsneden op het regeloppervlak precies de vlakken van de i-flock bepalen. Deze observatie voorziet ons van een meetkundige constructie van de i-flock, en dus ook van het bijhorende 1-systeem.

Vervolgens veralgemenen we de constructie van het 1-systeem uit dit specifieke rationale normale kubische regeloppervlak door te onderzoeken welke andere rationale normale kubische regeloppervlakken als uitgangspunt kunnen genomen worden. Er blijkt dat elk dergelijk regeloppervlak met L als directrixrechte en dat volledig uit inwendige punten van $LQ(2, q)$ bestaat, geschikt is. Uit een studie van dubbelrakende bundels kegelsneden in $\mathsf{PG}(2, q)$ kunnen we dan besluiten dat er onder de actie van de groep $\mathsf{PGO}(5, q)$ $\frac{q-3}{2}$ niet-isomorfe dergelijke regeloppervlakken bestaan. Bovendien zijn de meeste van de $\frac{q-3}{2}$ bijhorende lokaal hermitische 1-systemen van $Q(6, q)$ evenmin isomorf onder $\mathsf{P\Gamma O}(7, q)$, zodat een nieuwe klasse van 1-systemen van $Q(6, q)$, q oneven, het levenslicht ziet.

Tenslotte beogen we een karakterisering van deze klasse van 1-systemen, maar om die te begrijpen, moet eerst een bijkomende definitie ingevoerd worden. Met het oog op later gebruik formuleren we ze in een iets algemenere context.

Zij $\mathcal{M}$ een 1-systeem van $\mathcal{Q}_n \in \{Q(6, q), Q^+(7, q)\}$, met $n = 6$ of $n = 7$ in de respectieve gevallen, dat lokaal hermitisch is in de rechte $L \in \mathcal{M}$. Kies een punt x op L willekeurig en zij γ een $(n-2)$-dimensionale deelruimte van het raakhypervlak $T_x(\mathcal{Q}_n)$ aan $\mathcal{Q}_n$ in x, met $x \notin \gamma$. Elke regulus R_i van $\mathcal{M}$ door L heeft een unieke transversale door x, die γ snijdt in een punt. Samen met het punt $L \cap \gamma$ bekomen we dus $q^2 + 1$ punten van $\gamma \cap \mathcal{Q}_n := \mathcal{Q}_{n-2}$, waarbij $\mathcal{Q}_{n-2}$ een $Q(4, q)$, respectievelijk een $Q^+(5, q)$, is. Doordat $\mathcal{M}$ een 1-systeem is van $\mathcal{Q}_n$, ziet men gemakkelijk in dat deze $q^2 + 1$ punten een ovoïde vormen van $\mathcal{Q}_{n-2}$. Deze ovoïde wordt ook wel de *projectie volgens reguli van $\mathcal{M}$ vanuit x op γ* genoemd en aangeduid met $\mathcal{O}_x$. Indien de ovoïde $\mathcal{O}_x$ de klassieke ovoïde $Q^-(3, q)$ is voor elk punt x op L, dan zegt men dat $\mathcal{M}$ *semi-klassiek* is.

Een opmerkelijke eigenschap van semi-klassieke 1-systemen van $Q(6, q)$ en $Q^+(7, q)$ is dat voor alle $i \neq j$, de elliptische kwadriek $Q^-_{ij}(5, q) := \langle R_i, R_j \rangle \cap Q(6, q)$, respectievelijk $\langle R_i, R_j \rangle \cap Q^+(7, q)$, precies q reguli van $\mathcal{M}$ door L bevat. Deze bemerking zal ons verderop meermaals van nut zijn.

Aan de hand van de bovenstaande definitie, kunnen we de volgende karakteriserende stelling formuleren.

Stelling B.4.1 *Een echt lokaal hermitisch 1-systeem van $Q(6, q)$, q oneven, is semi-klassiek als en slechts als zijn bijhorende i-flock kan geconstrueerd worden uit een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$ zoals hierboven beschreven.*

Schets van het bewijs.
We gebruiken dezelfde notaties als voorheen: L is de rechte waarin $\mathcal{M}$ lokaal hermitisch is en $R_1, R_2, \ldots, R_{q^2}$ zijn de q^2 reguli van $\mathcal{M}$ door L.

Als de i-flock van $\mathcal{M}$ geconstrueerd kan worden uit een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$, gaan we als volgt te werk. Zij x een willekeurig punt van L en γ een hypervlak van $T_x(Q(6,q))$, met $x \notin \gamma$. Dan ligt er op het regeloppervlak $\mathscr{R}^3$ een unieke beschrijvende rechte N_x die x bevat. Er geldt dat $N_x^\perp \cap Q(6,q)$ een zekere kegel $xQ^-(3,q)$ is, waarbij we overeenkomen dat $Q^-(3,q) \subseteq \gamma$. Een eenvoudige redenering leert nu dat deze $Q^-(3,q)$ precies de projectie volgens reguli $\mathcal{O}_x$ van $\mathcal{M}$ vanuit x op γ is, zodat $\mathcal{M}$ semi-klassiek is.

Veronderstel omgekeerd dat $\mathcal{M}$ semi-klassiek is. Uit de bemerking vlak voor deze stelling volgt meteen dat de i-flock van $\mathcal{M}$ precies $q(q+1)$ doorsnedepunten van kegelsnedevlakken telt, dat elk doorsnedepunt in q kegelsnedevlakken ligt en dat elk kegelsnedevlak $q+1$ doorsnedepunten bevat. Men kan dan aantonen dat geen drie van de $q+1$ doorsnedepunten in een kegelsnedevlak collineair zijn, zodat deze punten een irreducibele kegelsnede vormen. Vervolgens blijkt dat de beschrijvende rechten van het gezochte regeloppervlak precies de $q+1$ rechten $\langle x\mathcal{O}_x\rangle^\perp$ zijn, waarbij x op L varieert en $\mathcal{O}_x$ de projectie volgens reguli van $\mathcal{M}$ vanuit x op een zekere γ voorstelt. Tenslotte leert een argument betreffende dubbelverhoudingen ons dat de beschouwde doorsnedepunten, samen met L, inderdaad een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$ vormen, met L als directrixrechte. $\square$

Als laatste onderdeel in dit hoofdstuk dient nog een woordje gezegd te worden over lokaal hermitische spreads van $Q^-(5,q)$, omdat die tevens (onechte) 1-systemen zijn van elke $Q(6,q)$ waarin men $Q^-(5,q)$ inbedt. Dergelijke spreads zijn altijd van de vorm $\mathcal{S}(S)$ voor een zekere rechtenspread S van $\mathsf{PG}(3,q)$. Dit betekent het volgende.

Zij $\mathcal{S}$ een spread van $Q^-(5,q)$ die lokaal hermitisch is in L, met reguli $R_1, R_2, \ldots, R_{q^2}$ door L zoals gebruikelijk. Dan is de raakruimte $L^\perp = T_L(Q^-(5,q))$ een $\mathsf{PG}(3,q)$ die L en de q^2 rechten $\langle R_i\rangle^\perp$ bevat. Men ziet eenvoudig in dat deze q^2+1 rechten twee aan twee disjunct zijn en dus een rechtenspread S van $\mathsf{PG}(3,q)$ vormen. Deze constructie werkt ook omgekeerd: met elke rechtenspread S' van $\mathsf{PG}(3,q)$ die L bevat, komt een spread $\mathcal{S}(S')$ van $Q^-(5,q)$ overeen, die lokaal hermitisch is in de rechte L.
Een belangrijk resultaat in deze context is dat $\mathcal{S}(S')$ hermitisch is als en slechts als S' een reguliere rechtenspread is in $\mathsf{PG}(3,q)$, waarvan alle elementen, indien ze beschouwd worden als rechten over $\mathsf{GF}(q^2)$, steunen op twee toegevoegde rechten T en $\overline{T}$ zó dat $\langle L,T\rangle$ en $\langle L,\overline{T}\rangle$ de generatoren door L zijn van de extensie $Q^+(5,q^2)$ van $Q^-(5,q)$ tot $\mathsf{GF}(q^2)$.

B.5 Lokaal hermitische 1-systemen van de parabolische kwadriek $Q(6,q)$, q even

Gezien het succes van de methode om lokaal hermitische 1-systemen van $Q(6,q)$, q oneven, te bestuderen aan de hand van een i-flock die ermee geassocieerd werd, pogen we deze techniek uit te breiden tot de even waarden van q. Dit is niet triviaal, aangezien $Q(6,q)$ voor q even geen polariteit definieert en directe veralgemening dus niet mogelijk is. Daarom ontwikkelen we een andere methode.

Beschouw een 1-systeem $\mathcal{M}$ van $Q(6,q)$, q even, dat lokaal hermitisch is in de rechte $\tilde{L} \in \mathcal{M}$, waarbij we de q^2 reguli van $\mathcal{M}$ door $\tilde{L}$ weer aanduiden met $R_1, R_2, \ldots, R_{q^2}$. Kies een 4-dimensionale deelruimte β, disjunct van $\tilde{L}$, die de kern n van $Q(6,q)$ bevat en beschouw de projectie vanuit $\tilde{L}$ op β. De projectie van $T_{\tilde{L}}(Q(6,q)) \setminus \tilde{L}$ vanuit $\tilde{L}$ op β is een vlak π, met $\pi \cap Q(6,q)$ een irreducibele kegelsnede $\mathcal{C}'$ met kern n, en elke $\langle R_i \rangle \setminus \tilde{L}$ wordt geprojecteerd op een rechte L_i, disjunct van π. Doordat $\mathcal{M}$ een 1-systeem is, kan men bovendien afleiden dat voor alle $i \neq j$, de rechten L_i en L_j disjunct zijn en dat de 3-ruimte $\langle L_i, L_j \rangle$ geen enkel punt van $\mathcal{C}'$ bevat. Vervolgens beschouwen we de duale situatie in β. Elk punt x van $\mathcal{C}'$ dualiseert tot een 3-ruimte $D_x = x^D$ die de rechte $L := \pi^D$ bevat; het is duidelijk dat voor elk vlak α van β dat geen punt gemeen heeft met L, de rechten $\{D_x \cap \alpha \mid x \in \mathcal{C}'\}$ een duale kegelsnede vormen. De q^2 rechten L_i leveren in de duale situatie q^2 vlakken $\pi_i = L_i^D$, die disjunct zijn van L. Voorts snijden elke twee vlakken π_i en π_j, $i \neq j$, elkaar in juist één punt dat tot geen van de ruimten D_x behoort. We voeren nu de volgende definities in.

Een kegel in $\mathsf{PG}(4,q)$ met een rechte als top en een duale kegelsnede als basis, noemen we een *duale-kegelsnedekegel.* Een *flock* van een duale-kegelsnedekegel $\mathcal{K}$ is een partitie van $\mathcal{K}$, minus de toprechte, in q^2 duale kegelsneden.

Gezien deze definities hebben we dus gevonden dat men met elk lokaal hermitisch 1-systeem van $Q(6,q)$, q even, een flock van een duale-kegelsnedekegel kan associëren. Men ziet bovendien gemakkelijk in dat deze constructie omkeerbaar is, zodat het 1-systeem en de flock gelijkwaardig zijn.

We toetsen de zopas beschreven methode aan het enige gekende voorbeeld van een lokaal hermitisch 1-systeem van $Q(6,q)$, q even, dat niet hermitisch is, namelijk de spread $\mathcal{S}_{[\delta]}$ van $\mathsf{H}(2^{2e})$, zie Offer [46] of Cardinali, Lunardon, Polverino en Trombetti [10] voor een beschrijving van $\mathcal{S}_{[\delta]}$. Net als in het vorige hoofdstuk berekenen we expliciet de vergelijkingen van de vlakken die de elementen van de flock geassocieerd met $\mathcal{S}_{[\delta]}$ bevatten, waarbij $\mathcal{S}_{[\delta]}$ opgevat wordt als 1-systeem van $Q(6,q)$. We vinden een analoog resultaat als voor q oneven: de vlakken die de flockelementen bevatten, snijden elkaar twee aan twee in een punt en de aldus gevonden doorsnedepunten vormen, samen met L, een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$ met L als directrixrechte. Dit betekent dat de flock, en dus ook het 1-systeem, ondubbelzinnig kan geconstrueerd worden vanuit dit regeloppervlak, zodat we ook over een meetkundige constructie van $\mathcal{S}_{[\delta]}$, opgevat als 1-systeem van $Q(6,q)$, beschikken. Het vervolg is zeer gelijklopend met het vorige hoofdstuk. Door de onderlinge ligging van de duale-kegelsnedekegel $\mathcal{K}$ en het regeloppervlak $\mathscr{R}^3$ te wijzigen, verrijst een nieuwe klasse van lokaal hermitische 1-systemen van $Q(6,q)$, q even, die $\frac{q-2}{2}$ elementen telt, waarvan de meeste niet isomorf zijn onder $\mathsf{P\Gamma O}(7,q)$. Deze 1-systemen hebben eveneens als kenmerkende eigenschap dat ze semi-klassiek zijn, zoals uit de volgende stelling blijkt.

Stelling B.5.1 *Voor q even en $q>2$ is een echt lokaal hermitisch 1-systeem van $Q(6,q)$ semi-klassiek als en slechts als de bijhorende flock kan geconstrueerd worden uit een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$, zoals hierboven besproken werd.*

Schets van het bewijs.
Zij $\mathcal{M}$ een 1-systeem van $Q(6,q)$, $q>2$ en even, dat lokaal hermitisch is in de rechte $\tilde{L}$, waarbij $R_1, R_2, \ldots, R_{q^2}$ de reguli van $\mathcal{M}$ door $\tilde{L}$ zijn.

Veronderstel eerst dat de flock van $\mathcal{M}$ geconstrueerd wordt uit een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$ met directrixrechte L, waarbij $L^D = \pi = T_{\tilde{L}}(Q(6,q)) \cap \beta$. Voor elk punt x op L heeft $\mathscr{R}^3$ een unieke beschrijvende rechte N_x door x. Nu is N_x^D een zeker vlak α in β en er geldt dat $\langle \tilde{L}, \alpha \rangle$ een 4-dimensionale ruimte is die $Q(6,q)$ snijdt in een kegel $\tilde{x}Q^-(3,q)$, voor een welbepaalde $\tilde{x}$ op $\tilde{L}$. Men kan vervolgens beredeneren dat $\tilde{L}$ en de transversalen door $\tilde{x}$ van de q^2 reguli van $\mathcal{M}$ door $\tilde{L}$, precies de q^2+1 rechten door $\tilde{x}$ van $\tilde{x}Q^-(3,q)$ zijn. Bijgevolg is $\mathcal{M}$ semi-klassiek.

Veronderstel omgekeerd dat $\mathcal{M}$ semi-klassiek is. Uit de bemerking vóór Stelling B.4.1 halen we de volgende informatie. De flock die bij $\mathcal{M}$ hoort, bepaalt q^2 vlakken die elkaar twee aan twee in een punt snijden, elk dergelijk vlak bevat $q+1$ doorsnedepunten met andere vlakken en elk doorsnedepunt ligt zelf in q verschillende vlakken. In het totaal zijn er dus $q(q+1)$ doorsnedepunten. Men kan voorts aantonen dat de $q+1$ doorsnedepunten in het vlak van een flockelement een $(q+1)$-boog vormen.
Vervolgens construeren we de beschrijvende rechten van het gezochte regeloppervlak $\mathscr{R}^3$. Zij $\tilde{x} \in \tilde{L}$ willekeurig, dan is $\mathcal{O}_{\tilde{x}} \cong Q^-(3,q)$ omdat $\mathcal{M}$ semi-klassiek is. Dus is $\langle \tilde{x}\mathcal{O}_{\tilde{x}} \rangle$ 4-dimensionaal en na projectie op β en dualiseren, vinden we een rechte N_x door een zeker punt x van L. Doen we dit voor

alle punten van $\tilde{L}$, dan bekomen we $q+1$ rechten in β die twee aan twee disjunct zijn en elk een verschillend punt van L bevatten. Deze rechten zijn de beschrijvenden van het gezochte regeloppervlak $\mathscr{R}^3$.
Tenslotte wordt een argument in verband met dubbelverhoudingen gebruikt om enerzijds aan te tonen dat de $(q+1)$-boog in het vlak van een flockelement een kegelsnede is, en anderzijds dat de doorsnedepunten van de vlakken die de flockelementen bevatten, samen met L een rationaal normaal kubisch regeloppervlak $\mathscr{R}^3$ vormen. Dit bewijst het gestelde. □

Als toemaatje in dit hoofdstuk wordt nog een methode geschetst om de veralgemeende zeshoek $\mathsf{H}(q)$ meetkundig te construeren als $q \equiv 1 \bmod 3$, die vertrekt van de gevonden meetkundige constructie van $\mathcal{S}_{[9]}$ voor q oneven, respectievelijk $\mathcal{S}_{[\delta]}$ voor q even.

B.6 Lokaal hermitische 1-systemen van de hyperbolische kwadriek $Q^+(7,q)$

Als logisch vervolg op de twee vorige hoofdstukken streven we ernaar de mooie en vruchtbare theorie van lokaal hermitische 1-systemen van $Q(6,q)$ te veralgemenen naar ander kwadrieken. Dit lukt voor de hyperbolische kwadriek $Q^+(7,q)$. Net zoals voor $Q(6,q)$, q oneven, ligt de polariteit van $Q^+(7,q)$ aan de basis van een methode om op eenduidige wijze een flock te hechten aan een lokaal hermitisch 1-systeem van $Q^+(7,q)$, en wel als volgt.

Beschouw een lokaal hermitisch 1-systeem $\mathcal{M}$ van $Q^+(7,q)$, dat bestaat uit q^2 reguli $R_1, R_2, \ldots, R_{q^2}$ door de rechte $L \in \mathcal{M}$. Dan is $L^\perp \cap Q^+(7,q)$, met $\perp$ de polariteit van $Q^+(7,q)$, een kegel $LQ^+(3,q)$. Voor alle $i = 1,2,\ldots,q^2$ is $\langle R_i\rangle^\perp \cap Q^+(7,q)$ een hyperbolische kwadriek $Q_i^+(3,q)$, die bestaat uit punten van $LQ^+(3,q) \setminus L$. Bovendien geldt voor alle $i \neq j$, dat $\langle R_i\rangle^\perp \cap \langle R_j\rangle^\perp$ een uitwendige rechte K_{ij} van $Q^+(7,q)$ is, zodat $Q_i^+(3,q)$ en $Q_j^+(3,q)$ in het bijzonder disjunct zijn. Definiëren we nu een *flock* van een kegel $LQ^+(3,q)$ als een partitie van $LQ^+(3,q) \setminus L$ in q^2 hyperbolische kwadrieken $Q_i^+(3,q)$, $i = 1,2,\ldots,q^2$, dan wordt uit het voorgaande duidelijk dat lokaal hermitische 1-systemen van $Q^+(7,q)$ en flocks van $LQ^+(3,q)$ gelijkwaardig zijn.

Waar we voor de parabolische kwadriek $Q(6,q)$ over een gedroomd voorbeeld beschikten om de methode van de (i-)flock op toe te passen, is dat voor $Q^+(7,q)$ helaas niet het geval. Daarom bedenken we zelf een manier om uit een goedgekozen Segre-variëteit $\mathscr{S}_{1;2}$ een flock, en dus een lokaal hermitisch 1-systeem van $Q^+(7,q)$ te construeren, op de volgende wijze.

Beschouw de kegel $LQ^+(3,q)$ in $\mathsf{PG}(5,q)$, met $L \subseteq Q^+(7,q)$ en zo dat $LQ^+(3,q) = L^\perp \cap Q^+(7,q)$ ($\perp$ staat voor de polariteit van $Q^+(7,q)$), en een 3-dimensionale deelruimte γ_1 van $\mathsf{PG}(5,q)$, waarbij $\gamma_1 \cap LQ^+(3,q)$ een hyperbolische kwadriek $Q_1^+(3,q)$ is. In γ_1 kiezen we nu een rechtenregulus $\{K_0, K_1, \ldots, K_q\}$ waarvan alle elementen disjunct zijn van $Q_1^+(3,q)$. Zij vervolgens $\mathscr{S}_{1;2}$ een Segre-variëteit van een rechte en een vlak, met de eigenschap dat de rechten $K_0, K_1, \ldots, K_q$ en L tot één van de beide systemen van maximale deelruimten van $\mathscr{S}_{1;2}$ behoren. Het is geweten dat het systeem van rechten van $\mathscr{S}_{1;2}$ precies q^2 reguli bevat die niet door L gaan, waarvan $\{K_0, K_1, \ldots, K_q\}$ er één is. Deze q^2 reguli bepalen q^2 3-ruimten in $\mathsf{PG}(5,q)$, die $LQ^+(3,q)$ snijden in de elementen van een flock en vertrekkend van deze flock kan een lokaal hermitisch 1-systeem van $Q^+(7,q)$ geconstrueerd worden. Door de onderlinge ligging van $Q_1^+(3,q)$ en $\{K_0, K_1, \ldots, K_q\}$ te laten variëren, ontstaat nu een hele klasse van lokaal hermitische 1-systemen van $Q^+(7,q)$. Bovendien blijkt dat deze 1-systemen niet alleen lokaal hermitisch zijn, maar ook semi-klassiek en dat deze eigenschap kenmerkend is voor de klasse.

Stelling B.6.1 *Een echt lokaal hermitisch 1-systeem van $Q^+(7,q)$ is semi-klassiek als en slechts als zijn bijhorende flock kan geconstrueerd worden uit een Segre-variëteit $\mathscr{S}_{1;2}$, zoals hierboven besproken werd.*

Schets van het bewijs.
Zij $\mathcal{M}$ een 1-systeem van $Q^+(7,q)$ dat lokaal hermitisch is in de rechte $L \in \mathcal{M}$ en noteer de q^2 reguli van $\mathcal{M}$ door L met $R_1, R_2, \ldots, R_{q^2}$.

Veronderstel eerst dat de flock van $\mathcal{M}$ uit een Segre-variëteit $\mathscr{S}_{1;2}$ geconstrueerd kan worden, zoals hierboven beschreven. Voor elk punt x van L bestaat er precies één vlak π_x dat tot het tweede systeem van maximale deelruimten van $\mathscr{S}_{1;2}$ behoort; er geldt dat $\pi_x^\perp \cap Q^+(7,q) = xQ^-(3,q)$. Het is nu mogelijk te bewijzen dat deze kegel $xQ^-(3,q)$ de rechte L en de transversalen door x van $R_1, R_2, \ldots, R_{q^2}$ bevat, zodat $\mathcal{M}$ semi-klassiek is.

Omgekeerd gaan we uit van de veronderstelling dat $\mathcal{M}$ semi-klassiek is. Uit de opmerking voorafgaand aan Stelling B.4.1 weten we dat voor alle $i \neq j$, de kwadriek $\langle R_i, R_j \rangle \cap Q^+(7,q) = Q_{ij}^-(5,q)$ precies q reguli van $\mathcal{M}$ door L bevat. Vervolgens wordt bewezen dat elke twee verschillende elliptische kwadrieken $Q_{ij}^-(5,q)$ en $Q_{kl}^-(5,q)$ de hele $\mathsf{PG}(7,q)$ opspannen. Deze informatie laat toe te besluiten dat de verzameling $\mathcal{I}$ van de rechten $\langle R_i \rangle^\perp \cap \langle R_j \rangle^\perp$, $i \neq j$, precies $q(q+1)$ elementen bevat, dat elke $\langle R_i \rangle^\perp$ (met andere woorden, elke ruimte opgespannen door een flockelement) precies $q+1$ rechten van $\mathcal{I}$ bevat en dat elke rechte van $\mathcal{I}$ bevat is in q flockelementen.
Tenslotte werd $\mathcal{M}$ semi-klassiek verondersteld, zodat $\mathcal{O}_x \cong Q^-(3,q)$ voor

elk punt x van L. Bijgevolg geldt voor alle $x \in L$ dat $\langle x\mathcal{O}_x\rangle^\perp$ een vlak is en dit vlak snijdt elke rechte van $\mathcal{I}$ in een punt, waaruit volgt dat $\mathcal{I} \cup \{L\}$ een systeem van maximale deelruimten is van een Segre-variëteit $\mathscr{S}_{1;2}$. Dit vervolledigt het bewijs. $\square$

Het lijkt onbegonnen werk om voor de aldus bekomen klasse van lokaal hermitische, semi-klassieke 1-systemen van $Q^+(7,q)$ te bepalen hoeveel niet-isomorfe voorbeelden ze bevat, want dit probleem houdt verband met de ligging van twee disjuncte reguli in $\mathsf{PG}(3,q)$. Wel kunnen we aantonen dat alle voorheen gekende elementen van deze klasse zich van de nieuwe voorbeelden onderscheiden door het feit dat de twee disjuncte reguli in deze gevallen steeds tot een reguliere spread van $\mathsf{PG}(3,q)$ behoren.

B.7 1-Systemen van $Q^+(7,q)$ en trialiteiten

De kwadriek $Q^+(7,q)$ heeft twee *families* van generatoren, $\mathcal{F}_1$ en $\mathcal{F}_2$, gedefinieerd door de voorwaarde dat twee generatoren tot dezelfde familie behoren als en slechts als ze ofwel disjunct zijn, ofwel precies een rechte gemeen hebben. Twee generatoren van verschillende families snijden dus in een punt of in een vlak. Beschouw nu de volgende incidentiemeetkunde Ω. De *0-punten* van Ω zijn de punten van $Q^+(7,q)$, de *i-punten* zijn de elementen van $\mathcal{F}_i$, $i = 1,2$, en de *rechten* van Ω zijn gewoon de rechten van $Q^+(7,q)$. De verzameling van de 0-punten noteren we met $\mathcal{P}$; de verzameling van de rechten met $\mathcal{L}$. Incidentie tussen i-punten, $i = 0,1,2$, en rechten enerzijds en tussen 0-punten en j-punten, $j = 1,2$, anderzijds, is „bevatten of bevat zijn in". Een 1-punt en een 2-punt zijn incident als en slechts als ze mekaar snijden in een vlak. Een *trialiteit* van Ω is een afbeelding $\tau : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{F}_1, \mathcal{F}_1 \to \mathcal{F}_2, \mathcal{F}_2 \to \mathcal{P}$, die de incidentie van Ω bewaart en zó dat τ^3 de identieke afbeelding ($\mathbb{1}$) is. Eisen we niet dat $\tau^3 = \mathbb{1}$, dan noemen we τ een *T-correspondentie*.

In dit hoofdstuk wordt het beeld van een 1-systeem onder een T-correspondentie (en dus ook onder een trialiteit) behandeld, waarbij we ons toespitsen op onze favoriete 1-systemen, namelijk de lokaal hermitische, semi-klassieke 1-systemen van een $Q(6,q)$, ingebed in $Q^+(7,q)$. Een eerste belangrijke bemerking is het feit dat elke T-correspondentie lokaal hermitische 1-systemen van $Q^+(7,q)$ afbeeldt op lokaal hermitische 1-systemen, doordat het beeld van een regulus onder een T-correspondentie opnieuw een regulus is. Voorts bekomen we de volgende resultaten.

Stelling B.7.1 *Zij $\mathcal{M}$ een 1-systeem van $Q(6,q) \subseteq Q^+(7,q)$ en θ een T-correspondentie van de meetkunde Ω. Dan is $\mathcal{M}^\theta$ bevat in een hypervlak van* $\mathsf{PG}(7,q)$ *als en slechts als $\mathcal{M}$ een spread is van een zekere* $\mathsf{H}(q)$ *op $Q(6,q)$.*

Schets van het bewijs.
Zij $Q(6,q) \subseteq Q^+(7,q)$ en noteer de verzameling van de rechten van $Q(6,q)$ met $\mathcal{B}$. Voor een willekeurige $Q(6,q)' \subseteq Q^+(7,q)$ onderzoeken we de verzameling $\mathcal{B}^\theta \cap Q(6,q)'$. Er blijkt dat voor elk punt p van $Q(6,q)'$, de rechten door p van $\mathcal{B}^\theta \cap Q(6,q)'$ de $q+1$ rechten door p in een totaal singulier vlak van $Q(6,q)'$ zijn. Een steling uit Thas [73] impliceert nu dat de rechten van $\mathcal{B}^\theta \cap Q(6,q)'$ ofwel alle rechten in de vlakken van een spread van $Q(6,q)'$ zijn, ofwel de rechten van een $\mathsf{H}(q)'$ op $Q(6,q)'$. De eerste mogelijkheid kunnen we uitsluiten, terwijl het gestelde uit de tweede volgt. $\square$

Stelling B.7.2 *Veronderstel dat $\mathcal{M}$ een echt 1-systeem is van $Q(6,q) \subseteq Q^+(7,q)$, dat lokaal hermitisch is in een rechte $L \in \mathcal{M}$, en beschouw een willekeurige T-correspondentie θ van Ω. Als $\mathcal{M}$ semi-klassiek is, dan is ook $\mathcal{M}^\theta$ semi-klassiek.*

Schets van het bewijs.
De q^2 reguli van $\mathcal{M}$ door L noteren we, zoals gewoonlijk, met $R_1, R_2, \ldots, R_{q^2}$. Voor een willekeurige generator $G \in \mathcal{F}_2$ door L, vormen de rechten $\langle R_i \rangle^\perp \cap G$, samen met L, een spread van G, die we $\mathcal{S}_G$ noemen. In eerste instantie tonen we aan dat het beeld $\mathcal{M}^\theta$ van $\mathcal{M}$ onder θ semi-klassiek is als en slechts als de spread $\mathcal{S}_G$ regulier is voor alle generatoren $G \in \mathcal{F}_2$ die L bevatten. Vervolgens veronderstellen we dat $\mathcal{M}$ semi-klassiek is. Dan is geweten dat $\langle R_i, R_j \rangle \cap Q^+(7,q) = Q^-_{ij}(5,q)$ precies q reguli van $\mathcal{M}$ door L bevat, voor alle $i \neq j$. Door de goed gekende structurele eigenschappen van deze q reguli in een $Q^-_{ij}(5,q)$ te interpreteren, volgt dat deze q reguli in $\mathcal{S}_G$ een rechtenregulus door L bepalen. Aangezien dit geldt voor alle $i \neq j$, moet $\mathcal{S}_G$ regulier zijn, wat het gestelde bewijst. $\square$

Stellingen B.7.1 en B.7.2, gecombineerd met de voorgaande resultaten omtrent lokaal hermitische, semi-klassieke 1-systemen van $Q(6,q)$, leveren een alternatieve beschrijving voor een subklasse in de klasse van de lokaal hermitische, semi-klassieke 1-systemen van $Q^+(7,q)$ die niet bevat zijn in een hypervlak van $\mathsf{PG}(7,q)$.

Voor de volledigheid vermelden we, zonder schets van het bewijs, nog een stelling die zich uitspreekt over het al dan niet semi-klassiek zijn van $\mathcal{M}^\theta$ indien $\mathcal{M}$ een lokaal hermitische spread is van een $Q^-(5,q) \subseteq Q^+(7,q)$.

Stelling B.7.3 *Zij $\mathcal{M}$ een spread van een elliptische kwadriek $Q^-(5,q) \subseteq Q^+(7,q)$, die lokaal hermitisch is in een rechte $L \in \mathcal{M}$. Beschouw een T-correspondentie θ van Ω. Dan is het beeld $\mathcal{M}^\theta$ van $\mathcal{M}$ onder θ semi-klassiek als en slechts als $\mathcal{M} = \mathcal{S}(S)$ voor een reguliere rechtenspread S van $T_L(Q^-(5,q))$.*

Als afsluiting van dit onderwerp worden nog een aantal stellingen bewezen, waaruit volgt dat de beelden onder een T-correspondentie van echte, niet-isomorfe, lokaal hermitische, semi-klassieke 1-systemen van een $Q(6,q)$ die ingebed is in $Q^+(7,q)$, evenmin isomorf zijn.

B.8 Afleiden van m-systemen

Sommige m-systemen hebben een bijzondere structuur, die toelaat op een eenvoudige wijze nieuwe m-systemen te construeren uit gekende m-systemen, een techniek die we „afleiden" noemen. Het resultaat wordt samengevat in de volgende stelling.

Stelling B.8.1 *Zij $\mathcal{M} := \{\pi_1, \pi_2, \ldots, \pi_r\}$ een m-systeem van een eindige polaire ruimte $\mathcal{P}_n$ in $\mathsf{PG}(n,q)$. Veronderstel dat er een deelruimte $\mathsf{PG}(2m+1,q)$ van $\mathsf{PG}(n,q)$ bestaat, die s elementen $\pi_1, \pi_2, \ldots, \pi_s$ van $\mathcal{M}$ bevat, met $1 \leq s \leq r$. Indien er in $\mathsf{PG}(2m+1,q) \cap \mathcal{P}_n$ precies s disjuncte m-dimensionale deelruimten $\alpha_1, \alpha_2, \ldots, \alpha_s$ van $\mathcal{P}_n$ bestaan, met de eigenschap dat de puntenverzamelingen $\pi_1 \cup \pi_2 \cup \cdots \cup \pi_s$ en $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_s$ samenvallen, dan is $\mathcal{M}' := \{\alpha_1, \alpha_2, \ldots, \alpha_s, \pi_{s+1}, \pi_{s+2}, \ldots, \pi_r\}$ eveneens een m-systeem van $\mathcal{P}_n$.*

Schets van het bewijs.
Als $\mathcal{M}$ een spread is van $\mathcal{P}_n$, dan is het gestelde triviaal aangezien $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$. Veronderstel dus dat $\mathcal{M}$ geen spread is van $\mathcal{P}_n$. Zij γ een $(m+1)$-dimensionale deelruimte van $\mathcal{P}_n$ die een element $\alpha_i \in \mathcal{M}'$ bevat, $1 \leq i \leq s$. Veronderstel dat γ een punt $x \in \pi_j$ bevat, voor een zekere $j \in \{s+1, s+2, \ldots, r\}$. In γ bestaat er een punt $y \neq x$ dat collineair is in $\mathcal{P}_n$ met alle punten van $\mathsf{PG}(2m+1,q) \cap \mathcal{P}_n$. Beschouw nu de rechte xy; deze snijdt α_i in een punt z. Maar $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}'$ bij veronderstelling, zodat z tot een zekere $\pi_k \in \mathcal{M}$ behoort, met $1 \leq k \leq s$. Dus is $\langle \pi_k, x \rangle$ een totaal singuliere deelruimte van $\mathcal{P}_n$, wat strijdig is met het feit dat $\mathcal{M}$ een m-systeem van $\mathcal{P}_n$ is. Er volgt dat γ disjunct is van $\pi_{s+1}, \pi_{s+2}, \ldots, \pi_r$.

Men ziet eenvoudig in dat de andere voorwaarden waaraan $\mathcal{M}'$ moet voldoen om een m-systeem van $\mathcal{P}_n$ te zijn, eveneens vervuld zijn, zodat het gestelde volgt. $\square$

Tot nog toe zijn er voor $m > 1$ geen niet-triviale voorbeelden gekend waarop de voorgaande stelling kan worden toegepast. Daarom gaan we hier niet dieper op in. In het bijzondere geval van 1-systemen van polaire ruimten kent Stelling B.8.1 echter vele toepassingen. Om die reden herformuleren we ze in dit geval als volgt.

Stelling B.8.2 *Zij $\mathcal{M}$ een 1-systeem van een eindige polaire ruimte $\mathcal{P}_n$. Als $\mathcal{M}$ een regulus R bevat, dan is ook $\mathcal{M}' := (\mathcal{M} \setminus R) \cup \overline{R}$, met $\overline{R}$ de regulus van de transversalen van R, een 1-systeem van $\mathcal{P}_n$.*

Indien deze stelling wordt toegepast op een lokaal hermitisch 1-systeem $\mathcal{M}$ van een kwadriek of van een symplectische polaire ruimte met q even[12], dan kan men aantonen dat $\mathcal{M}'$ niet lokaal hermitisch is, en bijgevolg zijn $\mathcal{M}$ en $\mathcal{M}'$ niet isomorf. Bij de bespreking van alle voorbeelden waarop Stelling B.8.1 kan toegepast worden, blijkt dan ook dat vele nieuwe 1-systemen ontstaan uit de gekende lokaal hermitische 1-systemen.

B.9 m-Systemen en semi-partiële meetkunden

In dit laatste hoofdstuk wordt de link gelegd met het tweede. We onderzoeken namelijk de semi-partiële meetkunden nader, die kunnen geconstrueerd worden uit de nieuwe 1-systemen die in de vorige hoofdstukken ontdekt werden. Doordat $W_5(2^r)$ isomorf is met $Q(6, 2^r)$, correspondeert elk 1-systeem van $Q(6, 2^r)$ met een 1-systeem van $W_5(2^r)$ en uit Paragraaf B.2 weten we dat zulke 1-systemen van $W_5(2^r)$ SPG-reguli zijn en dus aan de basis liggen van een semi-partiële meetkunde.

Gezien de nieuwe voorbeelden van 1-systemen van $Q(6, 2^r)$ uit Paragraaf B.5 lokaal hermitisch zijn, is er nood aan een overeenkomstige definitie voor 1-systemen van $W_5(2^r)$.
Een regulus van rechten van $W_5(2^r)$ met de eigenschap dat alle rechten van zijn transversale regulus eveneens totaal isotroop zijn voor de symplectische polariteit die $W_5(2^r)$ definieert, noemen we een *sterke regulus* van $W_5(2^r)$. Een 1-systeem $\mathcal{M}$ van $W_5(2^r)$ heet *lokaal hermitisch* in een rechte $L \in \mathcal{M}$ als en slechts als voor elke rechte $M \in \mathcal{M} \setminus \{L\}$, de (unieke) sterke regulus van $W_5(2^r)$ die L en M bevat, volledig bestaat uit rechten van $\mathcal{M}$.

Zij $\mathcal{M}$ een lokaal hermitisch 1-systeem van $W_5(q)$, q even, met bijhorende semi-partiële meetkunde $\mathsf{SPG}(\mathcal{M})$. Dan komt elke sterke regulus van rechten van $\mathcal{M}$ overeen met tal van deelnetten in $\mathsf{SPG}(\mathcal{M})$ van de orde q^2 en de graad $q+1$. Een voor dit hoofdstuk cruciale stap wordt geleverd door het volgende lemma, dat zegt dat elk deelnet van $\mathsf{SPG}(\mathcal{M})$ van de orde q^2 en de graad $q+1$ precies $q+1$ rechten van $\mathcal{M}$ bepaalt, die in een 3-dimensionale deelruimte liggen.

[12] De definitie van een lokaal hermitisch 1-systeem van $W_5(2^r)$ wordt gegeven in Paragraaf B.9.

Lemma B.9.1 *Zij $\mathcal{M}$ een 1-systeem van een symplectische polaire ruimte $W_5(q)$ in $\mathsf{PG}(5,q) := H$. Elk deelnet van $\mathsf{SPG}(\mathcal{M})$ van de orde q^2 en de graad $q+1$ is de deelmeetkunde van $\mathsf{SPG}(\mathcal{M})$, geïnduceerd in een deelruimte $\mathsf{PG}(4,q)$ van de omhullende ruimte $\mathsf{PG}(6,q)$ van $\mathsf{SPG}(\mathcal{M})$, die H snijdt in een $\mathsf{PG}(3,q)$ die precies $q+1$ rechten van $\mathcal{M}$ bevat.*

Schets van het bewijs.
Dit lemma kent een verschillend bewijs voor $q = 2$ en voor $q > 2$. Voor het geval $q > 2$ wordt, uitgaande van twee rechten van het deelnet, stap voor stap het hele deelnet geconstrueerd. Er blijkt dat dit deelnet inderdaad een deelmeetkunde is zoals beschreven in de stellingopgave.

Voor $q = 2$ liggen de zaken iets ingewikkelder. In dit geval worden drie rechten van het deelnet door een vast punt van het deelnet beschouwd, die drie rechten $L_0, L_1, L_2 \in \mathcal{M}$ bepalen. Uit het ongerijmde wordt eerst aangetoond dat de deelruimte $\langle L_0, L_1, L_2\rangle$ niet 4-dimensionaal kan zijn en vervolgens dat ze evenmin 5-dimensionaal kan zijn. Als enige mogelijkheid blijft dus over dat $\langle L_0, L_1, L_2\rangle$ 3-dimensionaal is, waarna het gestelde gemakkelijk volgt. □

In het bijzondere geval dat een deelruimte $\mathsf{PG}(4,q)$ van $\mathsf{PG}(6,q)$ het hypervlak H snijdt in een $\mathsf{PG}(3,q)$ die een regulus (al dan niet sterk) van rechten van $\mathcal{M}$ bevat, noemen we het in $\mathsf{PG}(4,q)$ geïnduceerde deelnet een *regulusdeelnet* van $\mathsf{SPG}(\mathcal{M})$.

Door te steunen op het vorige lemma, kan de volgende stelling relatief eenvoudig bewezen worden.

Stelling B.9.2 *Beschouw een 1-systeem $\mathcal{M}$ van $W_5(q)$, q even, dat lokaal hermitisch is in een rechte $L \in \mathcal{M}$, en zij $\mathcal{M}'$ het 1-systeem dat bekomen wordt door een sterke regulus R van $\mathcal{M}$ door L te vervangen door de sterke regulus $\overline{R}$ van zijn transversalen. Dan zijn de semi-partiële meetkunden $\mathsf{SPG}(\mathcal{M})$ en $\mathsf{SPG}(\mathcal{M}')$ niet isomorf.*

Schets van het bewijs.
Uit de structuur van $\mathcal{M}'$ kan men afleiden dat geen enkele rechte $M \in \mathcal{M}'$ de eigenschap heeft dat voor elke rechte $N \in \mathcal{M}' \setminus \{M\}$, de 3-ruimte $\langle N, M\rangle$ precies $q+1$ rechten van $\mathcal{M}'$ bevat. De rechte $L \in \mathcal{M}$ heeft deze eigenschap daarentegen wel, want $\mathcal{M}$ is lokaal hermitisch in L. In combinatie met het vorige lemma wordt nu duidelijk dat $\mathsf{SPG}(\mathcal{M})$ veel meer deelnetten van de orde q^2 en de graad $q+1$ bevat dan $\mathsf{SPG}(\mathcal{M}')$, waaruit volgt dat beide semi-partiële meetkunden niet isomorf zijn. □

Vervolgens richten we onze aandacht op niet-isomorfe, lokaal hermitische 1-systemen van $W_5(q)$, $q > 2$ en even. Een belangrijk resultaat luidt als volgt.

Stelling B.9.3 *Veronderstel dat $\mathcal{M}_1$ en $\mathcal{M}_2$ lokaal hermitische 1-systemen zijn van een symplectische polaire ruimte $W_5(q)$ in $\mathsf{PG}(5,q) := H$, q even en $q > 2$. Als θ een isomorfisme is tussen $\mathsf{SPG}(\mathcal{M}_1)$ en $\mathsf{SPG}(\mathcal{M}_2)$, dan wordt θ geïnduceerd door een element $\vartheta \in \mathsf{P\Gamma L}(7,q)$ dat $\mathcal{M}_1$ afbeeldt op $\mathcal{M}_2$.*

Schets van het bewijs.
Zij $\mathsf{AG}(6,q)$ de 6-dimensionale affiene ruimte waarin $\mathsf{SPG}(\mathcal{M}_1)$ ingebed is. Uit een stelling van Johnson [29] volgt dat θ een regulusdeelnet van $\mathsf{SPG}(\mathcal{M}_1)$ steeds afbeeldt op een regulusdeelnet van $\mathsf{SPG}(\mathcal{M}_2)$. Aan de hand van dit resultaat kan aangetoond worden dat θ de collineariteit van $\mathsf{AG}(6,q)$ behoudt binnen regulusdeelnetten van $\mathsf{SPG}(\mathcal{M}_1)$, bijvoorbeeld door de orde van de collineatiegroep van een regulusdeelnet te vergelijken met de orde van de deelgroep van $\mathsf{P\Gamma L}(7,q)$ die een gegeven regulusdeelnet van $\mathsf{SPG}(\mathcal{M}_1)$ afbeeldt op een zeker regulusdeelnet van $\mathsf{SPG}(\mathcal{M}_2)$.

Noteer de q^2 reguli van $\mathcal{M}_1$ door de rechte L_1 waarin $\mathcal{M}_1$ lokaal hermitisch is, als $R_1, R_2, \ldots, R_{q^2}$. Twee verschillende punten x en y van $\mathsf{SPG}(\mathcal{M}_1)$ kunnen slechts in de volgende twee posities liggen: ofwel snijdt de rechte xy het hypervlak H in een punt dat tot een zekere $\langle R_i \rangle$ behoort, $i \in \{1, 2, \ldots, q^2\}$, ofwel behoort het snijpunt $z := xy \cap H$ tot de 3-ruimte $L_1^\perp$. In het eerste geval weten we al dat θ de collineariteit van $\mathsf{AG}(6,q)$ bewaart voor punten op de rechte xy. Veronderstel dus dat $z \in L_1^\perp$. Door twee verschillende projectieve vlakken π en π' door xy te kiezen, die enkel het punt z gemeen hebben met $L_1^\perp$, volgt onmiddellijk dat θ ook in dit geval de collineariteit van $\mathsf{AG}(6,q)$ behoudt voor de punten op de rechte xy. Hieruit volgt het gestelde. $\square$

Tenslotte wordt deze stelling toegepast om het laatste resultaat van dit proefschrift te bewijzen.

Stelling B.9.4 *Veronderstel dat $\mathcal{M}_1$ en $\mathcal{M}_2$ twee lokaal hermitische 1-systemen zijn van $W_5(q)$, q even en $q > 2$. Dan zijn de bijhorende semi-partiële meetkunden $\mathsf{SPG}(\mathcal{M}_1)$ en $\mathsf{SPG}(\mathcal{M}_2)$ isomorf als en slechts als $\mathcal{M}_1$ en $\mathcal{M}_2$ isomorf zijn voor de stabilisator van $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$.*

Schets van het bewijs.
Veronderstel dat $\mathcal{M}_i$ lokaal hermitisch is in de rechte L_i, $i = 1, 2$, en zij θ een isomorfisme tussen $\mathsf{SPG}(\mathcal{M}_1)$ en $\mathsf{SPG}(\mathcal{M}_2)$, geïnduceerd door het element $\vartheta \in \mathsf{P\Gamma L}(7,q)$ dat $\mathcal{M}_1$ op $\mathcal{M}_2$ afbeeldt. Als ϑ de symplectische polaire

ruimte $W_5(q)$ stabiliseert, is het gestelde waar. Indien dit niet zo is, geldt dat $\mathcal{M}_2 = \mathcal{M}_1^\vartheta$ een lokaal hermitisch 1-systeem is van twee verschillende symplectische polaire ruimten, namelijk $W_5(q)$ en zijn beeld onder ϑ. Uit de eigenschappen van $\mathcal{M}_2$, opgevat als SPG-regulus van $\mathsf{PG}(5, q)$, wordt nu duidelijk dat deze situatie onmogelijk is. $\square$

Uit Stellingen B.9.2 en B.9.4 mogen we besluiten dat de nieuwe voorbeelden van 1-systemen van $W_5(q)$, q even, die in de loop van deze thesis gevonden werden, dus ook nieuwe voorbeelden van semi-partiële meetkunden opleveren.

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Index of notation

$\mathsf{GF}(q)$	the Galois field of q elements, q a prime power
$\mathsf{PG}(n,q)$	the n-dimensional projective space over $\mathsf{GF}(q)$
$\mathsf{AG}(n,q)$	the n-dimensional affine space over $\mathsf{GF}(q)$
$W_{2n+1}(q)$	the polar space, arising from a non-singular symplectic polarity in $\mathsf{PG}(2n+1,q)$
$Q(2n,q)$	the non-singular parabolic quadric in $\mathsf{PG}(2n,q)$
$Q^+(2n+1,q)$	the non-singular hyperbolic quadric in $\mathsf{PG}(2n+1,q)$
$Q^-(2n+1,q)$	the non-singular elliptic quadric in $\mathsf{PG}(2n+1,q)$
$H(n,q^2)$	the non-singular hermitian variety in $\mathsf{PG}(n,q^2)$
$\mathsf{P\Gamma L}(n+1,q)$	the group of collineations of $\mathsf{PG}(n,q)$
$\mathsf{PGL}(n+1,q)$	the group of projectivities of $\mathsf{PG}(n,q)$
$\mathsf{P\Gamma O}(2n+1,q)$	the group of collineations fixing $Q(2n,q)$
$\mathsf{PGO}(2n+1,q)$	the group of projectivities fixing $Q(2n,q)$
$\mathsf{P\Gamma O}^+(2n+2,q)$	the group of collineations fixing $Q^+(2n+1,q)$
$\mathsf{PGO}^+(2n+2,q)$	the group of projectivities fixing $Q^+(2n+1,q)$
$\mathsf{P\Gamma O}^-(2n+2,q)$	the group of collineations fixing $Q^-(2n+1,q)$
$\mathsf{PGO}^-(2n+2,q)$	the group of projectivities fixing $Q^-(2n+1,q)$
$\mathsf{A\Gamma L}(n+1,q)$	the group of collineations of $\mathsf{AG}(n,q)$
$\mathsf{H}(q)$	the finite split Cayley hexagon of order (q,q)
$\mathsf{T}(q^3,q)$	the finite twisted triality hexagon of order (q^3,q)
$T_\alpha(\mathcal{P})$	the tangent space of the polar space $\mathcal{P}$ at the totally singular subspace α of $\mathcal{P}$

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