Spreads and Ovoids of the Split Cayley Hexagon

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Preface to the second edition

No matter how many improvements one makes to a document, it seems that there always remain more that could still be made. At least this seems to be true of myself, and so it is that despite all my efforts to make a thesis that was just right, since its submission to the university I have felt the irresistible desire to make some modifications.

Most noticably, there are two new sections, both called "Remarks" and appearing at the ends of Chapters 3 and 4. These both contain additional material that has arisen since the thesis was originally completed, but nonetheless seem to belong here as they nicely round off a couple of jagged edges that existed before.

Next, there were two places in particular where I personally felt a little dissatisfied with the explanations that I had originally given, namely a part of the description of the triality from which $H(q)$ is constructed, and the determination of the compatibility conditions for line reguli. These have experienced substantial rewriting. The only other changes have been minor corrections.

For the purposes of cross-referencing, it is perhaps important for me to remark what has *not* changed. While page numbers, and to a smaller extent equation numbers, are different from those in the original submitted thesis, the numbers of chapters, sections, theorems and other theorem-like environments have remained unaffected. However, this does not apply, of course, to those that appear within the new "Remarks" sections.

Finally, I would like to thank my examiners, not only for examining the thesis, but also for the valuable comments that they passed on to me in their reports.

Preface

The geometries known as generalized polygons were introduced in the celebrated work of Tits [Tit59]. From there comes the focus of this thesis—the classical generalized hexagon $H(q)$, known as the split Cayley hexagon, which is taken here to be defined over a finite field.

Spreads and ovoids of $H(q)$ are sets of lines and points, respectively, with certain properties that essentially amount to saying that the elements of the set are, in some sense, scattered evenly over the entire geometry and yet are still quite well spaced out. Analogous objects in projective spaces and polar spaces, including generalized quadrangles, have long attracted interest and study, as they still do.

When attempting a study of structures of some type, like spreads and ovoids, one reasonably natural way to commence is to initially restrict one's attention to those that are reasonably symmetric—that is, ones with a lot of automorphisms—and so Bloemen, Thas and Van Maldeghem [BTVM98] introduced the notion of translation spreads and ovoids. These are the objects investigated in this thesis.

In Chapter 1, the concepts and results that are necessary for the rest of the work are provided. After an overview of some background material concerning finite fields and geometries, structures called m-systems are discussed and a characterization of 1-systems of the parabolic quadric $\mathcal{P}_6$ that lie in a hyperplane of $PG(6, q)$ is given. Next, the largest part of the chapter introduces the split Cayley hexagon $H(q)$ itself, its geometry, its coordinatization and its morphisms.

Chapter 2 introduces spreads and ovoids of generalized $2m$-gons and then the attention turns to $H(q)$. Spreads and ovoids of $H(q)$ are given coordinate representations and then known ones are described. In particular, all the known spreads and ovoids with the property of being locally hermitian appear—a property that turns out to be closely related to being translation ([BTVM98]).

The main results and investigations of this thesis appear in Chapter 3, where translation spreads and ovoids of $H(q)$ are defined. Almost immediately, translation ovoids are shown to exist only when $H(q)$ is self-dual, so spreads become the primary concern. In the paper [BTVM98], some general properties of translation spreads of $H(q)$ were proved, and this chapter extends on them, motivated by their analogy (dual in nature) with translation ovoids of $Q(4, q)$. In particular, spreads translation with respect to a line are shown to be characterized by the existence of an appropriate automorphism group that fixes one line and acts regularly on the rest, and as a consequence, the functions in the coordinate representation of such a spread are given by linearized polynomials. Then translation spreads with respect to a line are considered further, leading to their classification for $q = 3^h$, as well as the classification of those, for odd q, that have all the underlying field $GF(q)$ as their kernel. The final section of this chapter classifies the spreads of $H(q)$ that are translation with respect to two disjoint flags.

Finally, Chapter 4 aims to make connections with the generalized quadrangle $W(q)$ and its dual $Q(4, q)$. The construction of $W(q)$ is described in a manner parallel to that of $H(q)$ in order to highlight the analogy that exists between them, as well as to hopefully make the generalized hexagon $H(q)$ a little easier to imagine. As some of the work in the previous chapters has its motivation in this analogy, this is made explicit here with comparable results concerning translation ovoids of $Q(4, q)$ being quoted from [BTVM98]. Finally, ovoids of $Q(4, q)$ obtained from spreads of $H(q)$ are considered, and the semiclassical spreads of [BTVM98] arise as translation spreads with the entire underlying field $GF(q)$ as kernel.

Acknowledgments

I would like to thank my supervisors, Christine O'Keefe and Rey Casse, for their support and assistance, both academic and personal, throughout the creation of this thesis. That this has reached completion at all is due largely to them and I am truly deeply grateful to them both for their help, understanding and tolerance.

Without a doubt, the most valuable resources have always been people. For me, they have particularly included: Tim Penttila, who helped remove the mystery from seemingly complex things; Jef Thas, who introduced me to the paper [BTVM98] which, in the end, has been the main inspiration for most of this thesis; Hendrik Van Maldeghem, whose book [VM98], emails, and suggestions to me were invaluable and inspirational; and Matthew Brown, who was always very happy and able to explain matters of generalized quadrangles for me. I thank these people very much, for I have benefitted greatly from discussions with all of them. Thanks again also to my supervisor Christine, who introduced me to these people making these discussions possible, and thanks also to Burkard Polster, who possessed a copy of the draft of Hendrik's book [VM98] and realized that it would benefit me.

Many thanks indeed to both Christine and Hendrik for making possible the trip that I made to Ghent. My time there was certainly very informative and inspirational, and it was easily my most productive month of all, with all of Section 3.4 and a large part of Section 3.2 coming from that trip.

In addition to all that is academic, no less important over this period of time has been the personal support from people in my life. With this in mind, I would like to acknowledge with many thanks the friendship, love and support of David & Sonia Wansbrough, Silvia Manzanero Alonso, Claire Rivett, Georgina Graham, Keith Matthews, Ann Ross and, of course, my family.

Finally, I would like to acknowledge the financial support of an Australian Postgraduate Award.

Chapter 1

Introduction

In this chapter we give definitions, introduce notations and overview some results fundamental to the rest of the work. The reader is assumed to be familiar with finite fields and with projective spaces defined over them. Thorough references for these are the books [LN97] and [Hir98], respectively.

1.1 Finite fields

While a general familiarity with finite fields is assumed, here we will overview some of the more significant definitions and results that will be needed later.

The finite field, or Galois field, with $q = p^h$ elements, p a prime, is denoted $GF(q)$. The subfield $GF(p)$ of $GF(q)$ is the **prime field** and $GF(q)$ can be considered as an h dimensional vector space over it. For an element $a \in GF(q)$, the **trace** of a is given by

$$\mathrm{Tr}(a) = a + a^p + a^{p^2} + \cdots + a^{p^{h-1}}.$$

This is a linear functional from $GF(q)$ onto its prime field.

At times, we will be concerned with when some quadratic polynomial defined over $GF(q)$ is irreducible. This is covered by the following theorem. Notice that we shall be using the symbol $\not\square$ to represent an arbitrary nonsquare element in a field.

Theorem 1.1 (see [Hir98, Section 1.4])
Let $f(x) = ax^2 + bx + c$, $a \neq 0$, be a quadratic polynomial defined over $GF(q)$. If q is odd then f is irreducible if and only if the ***discriminant*** *Δ is a nonsquare; that is,*

$$\Delta = b^2 - 4ac = \not\square.$$

If q is even then f is irreducible if and only if $b \neq 0$ and the ***S-invariant*** *has trace 1; that is,*

$$\mathrm{Tr}\left(\frac{ac}{b^2}\right) = 1.$$

□

A consequence of this theorem is the following.

Corollary 1.2
The quadratic polynomial $x^2 - x + 1$ in $GF(q)$ is irreducible if and only if $q \equiv 2 \pmod 3$.

Proof For odd q, by the previous theorem the given quadratic is irreducible if and only if the discriminant $\Delta = -3$ is a nonsquare. This occurs precisely when h is odd and -3 is a nonsquare in the prime field $GF(p)$ (see, for example, the discussion at the beginning of [IR90, Section 11.3]). Now from the quadratic reciprocity theorem (see [IR90, Theorem 1, Section 5.2]), this is equivalent to the condition that h is odd and $p \equiv 2 \pmod 3$, or simply, that $q \equiv 2 \pmod 3$.

For even q, the given quadratic is irreducible precisely when $\mathrm{Tr}(1) = 1$. This corresponds to the exponent h being odd (see [Rom95, Theorem 7.1.3]), and this in turn is equivalent to $q \equiv 2 \pmod 3$. □

The following lemma proves to be useful from time to time.

Lemma 1.3 (see [BTVM98, Lemma 29])
Let $f(X)$ be a degree 4 polynomial over $GF(q)$, with q odd, such that $f(x)$ is a nonsquare for all $x \in GF(q)$. Then $f(X) = \gamma g(X)^2$ for some nonsquare $\gamma \in GF(q)$ and some monic irreducible quadratic $g(X)$.

Proof Since $f(x)$ takes only nonsquare values, in particular it is never zero, so $f(X)$ has no linear factor over $GF(q)$. Thus $f(X)$ has either two or four distinct roots in its splitting field. In the latter case, the curve $y^2 = f(x)$ is an elliptic curve (by [NZM91, Section 5.9], the genus is $\lfloor (4-1)/2 \rfloor = 1$) with no solutions in $GF(q)$. By the Hasse-Weil Theorem (see [HT91, Corollary 2.27]), we then have $q + 1 \leq 2\sqrt{q}$, which is impossible. Thus $f(X)$ has two distinct roots in the splitting field so $f(X) = \gamma g(X)^2$ for some nonzero $\gamma \in GF(q)$ and some monic irreducible quadratic $g(X)$. Finally, since $f(x)$ takes only nonsquare values, it follows that γ is a nonsquare. □

A **linearized polynomial** over $GF(q)$ is a polynomial $f(x)$ defined over $GF(q)$ that has the form $\sum_{i=0}^{m} \alpha_i x^{p^i}$, for some m (see [LN97, Chapter 3, Section 4]). Since for each polynomial over $GF(q)$ there is a unique polynomial of degree less than q to which it is equivalent as a function, and here we have $x^{p^{h+i}} = x^{p^i}$, each linearized polynomial is equivalent to a unique linearized polynomial in **standard form**, which is given by

$$f(x) = \sum_{i=0}^{h-1} \alpha_i x^{p^i},$$

where the coefficients α_i are in $GF(q)$. Since a linearized polynomial is a linear combination of the field automorphisms $x \mapsto x^{p^i}$ and the prime field is fixed by each of these, such a polynomial induces a linear transformation of $GF(q)$ considered as a vector space over $GF(p)$; that is, a linear operator on $GF(q)$ over $GF(p)$. The converse also holds, so we have the following theorem.

Theorem 1.4 (see [Rom95, Theorems 9.4.1 and 9.4.4])
Let $f(x)$ be a linearized polynomial over $GF(q)$. Then the function induced by $f(x)$ is a linear operator on $GF(q)$ over $GF(p)$. Conversely, every linear operator on $GF(q)$ over $GF(p)$ can be represented by a unique linearized polynomial in standard form. □

While having a linearized polynomial means that we have a significant restriction on the exponents that occur in the terms of the polynomial, there is another result that we will want to use that will enable us to restrict matters considerably further. This result originates in a paper of Carlitz ([Car60]) but generalizations and related results soon followed (see for instance [Car62], [McC63], [Gru81] and [Len90]). The form of the result stated here is essentially that of [Car62].

Theorem 1.5 (Carlitz [Car62])
Let χ be the multiplicative character of order two on $GF(q)$, where $q = p^h$ with p an odd prime, so $\chi(x) = 1$ if x is a square and $\chi(x) = -1$ if x is a nonsquare. Let f be a function in $GF(q)$ such that, for a fixed choice of $e = \pm 1$,

$$\chi\big(f(x) - f(y)\big) = e\chi(x - y)$$

whenever $x \neq y$. Then $f(x) = ax^{p^t} + b$ for some t in the range $0 \leq t < h$, and where $a, b \in GF(q)$ with $\chi(a) = e$. □

1.2 Geometry

In this section we define what we shall mean by geometries and some of the various maps that can exist on and between them. Then some of the more specific types of geometries that we will encounter are overviewed and related terminology is introduced. Familiarity with the theory of projective spaces is assumed; necessary information concerning these can be found in [Hir98].

1.2.1 Geometries and their maps

A **geometry** Γ is a set, whose elements are the **elements** of the geometry, together with an antireflexive symmetric binary relation I on Γ, called the **incidence relation**, and an ordered partition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_r$ of Γ into nonempty **type sets**. Elements are said to be of the same **type** precisely when they belong to the same type set. The incidence relation I satisfies the property that no two elements of the same type are incident; that is, if $x, y \in \Gamma_i$ then $x \not{\mathrm{I}}\, y$ (hence the antireflexivity). We will sometimes write $\Gamma = (\Gamma_1, \ldots, \Gamma_r, \mathrm{I})$, or just $\Gamma = (\Gamma_1, \ldots, \Gamma_r)$ when the incidence relation is clear. The number r of different types of elements is called the **rank** of the geometry. Also, the geometry is said to be **finite** if the set Γ is finite.

Incidence between elements is expressed with a variety of phrases such as "is on", "passes through", "is contained in", and so on. The context will keep matters clear so no confusion shall arise. Two elements x and y of Γ of the same type that are incident with a common element z are said to be **coincident** with z. If x and y are of a type called "points" and z is of a type called "lines" then the nomenclature is often emphasized by saying that x and y are **collinear**. Similarly, if x and y are lines and z is a point then x and y are **concurrent**. Finally, when x and y are two elements of Γ for which there is a unique element of a specified type that is incident with both of them, we refer to this element as the element xy of that type.

Let $\Gamma = (\Gamma_1, \ldots, \Gamma_r, \mathrm{I})$ and $\Sigma = (\Sigma_1, \ldots, \Sigma_t, \mathrm{I}')$ be two geometries. A **map**, or a **morphism**, between them is a function $\Phi : \Gamma \to \Sigma$ that preserves incidence and maps elements of the same type to elements of the same type; that is, $\mathrm{I}\,\Phi \subseteq \mathrm{I}'$ and for each i, $\Gamma_i\Phi \subseteq \Sigma_j$ for some j.

Now suppose Γ and Σ have the same rank r. An **isomorphism** between them is a bijective morphism $\Phi : \Gamma \to \Sigma$ that preserves the order of the type sets, so $\Gamma_i\Phi = \Sigma_i$, and whose inverse $\Phi^{-1} : \Sigma \to \Gamma$ is also a morphism of geometries. When an isomorphism exists, we say that the geometries are **isomorphic** and we write $\Gamma \cong \Sigma$. When Γ has a type called "lines" that is mapped to a type in Σ also called "lines", an isomorphism is also called a **collineation** to emphasize the fact that the property of collinearity is preserved. When $\Gamma = \Sigma$, an isomorphism is also called an **automorphism**. Under composition, the set of all automorphisms of a geometry Γ forms a group, $\mathrm{Aut}(\Gamma)$, called the **full automorphism group** of Γ. A subgroup of the full automorphism group is then just termed an **automorphism group**. An automorphism of order two is an **involution**.

An **anti-isomorphism** is a bijective morphism $\Phi : \Gamma \to \Sigma$ that does not preserve the order of the type sets, so $\Gamma_i\Phi = \Sigma_{i\sigma}$ for some nontrivial permutation σ of the set $\{1, \ldots, r\}$, and whose inverse is also a morphism. If the permutation σ has order two then Φ is called a **correlation**, or a **duality**. When a correlation exists, we say that the geometries are **dual**. When $\Gamma = \Sigma$, an anti-isomorphism Φ is also called an **anti-automorphism**. In this instance, an **absolute** element is an element x of Γ that is incident with its image under Φ; that is, such that $x \mathrel{\mathrm{I}} x\Phi$. If there is a correlation from Γ to itself then we say that Γ is **self-dual**. A correlation from Γ to itself that has order two is called a **polarity**, and when such a map exists, the geometry Γ is said to be **self-polar**.

An **incidence structure** is a rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. The choice of letters for the sets here reflects the fact that the two types are usually called "points" and "lines", as we shall do unless otherwise explicitly stated. Also, a line in $\mathcal{L}$ is often identified with the set of points in $\mathcal{P}$ that are incident with it. With this identification, incidence is then given by inclusion. A **flag** is a pair of elements $\{x, y\}$ in Γ with $x \mathrel{\mathrm{I}} y$. If there is a pair of integers (s, t) such that each line $x \in \mathcal{L}$ is incident with exactly $s + 1$ points and each point $y \in \mathcal{P}$ is incident with exactly $t + 1$ lines, then the incidence structure Γ is said to have **order** (s, t). If $s = t$ then we just say that Γ has order s.

For any given incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, there is a unique incidence structure up to isomorphism that is dual to Γ, namely $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathrm{I})$, so the points of one are the lines of the other, and vice versa. This incidence structure Γ^D is called the **dual** of Γ. Any statement that pertains to Γ has an equivalent form relating to Γ^D obtained by swapping references to points and lines. This new statement is called the **dual** of the original. When Γ is self-dual, so $\Gamma \cong \Gamma^D$, the dual of any statement about Γ is another statement that also pertains to Γ. This is the **principle of duality**.

1.2.2 Graphs

A **graph** G is an incidence structure (V, E) in which every element $e \in E$ is incident with exactly two elements[1] $u, v \in V$, which in turn uniquely determine the edge e. Thus we can consider E as being a set of unordered pairs from V. The elements of V are called **vertices** and the elements of E are called **edges**. Two vertices, u and v, that are coincident with a common edge are said to be **adjacent** and we write $u \sim v$.

Graphs are intimately connected with antireflexive symmetric binary relations. Indeed, the adjacency relation "$\sim$" is such a relation on V, and conversely, given an antireflexive symmetric binary relation ρ on a set V we obtain a graph (V, E) by taking E to be the set of unordered pairs $\{x, y\}$ with $x \rho y$. In particular, if Γ is a geometry then its incidence relation gives rise to a graph G with Γ as vertex set. This graph is called the **incidence graph** of Γ. Due to the close relationship between an incidence graph G and its corresponding geometry Γ, we often identify the two structures and so, where no confusion shall arise, we use the following terminology and notation as freely within the context of geometries as within that of graphs where they are explicitly defined.

In a graph, a **walk** is a sequence $v_0, v_1, \ldots, v_n$ of vertices in which the vertices of each successive pair, $v_i, v_{i+1}, 0 \leq i < n$, are adjacent. Since this walk starts at the vertex v_0 and ends at v_n, we call this a v_0–v_n walk and we say that the **length** of the walk is n as it has passed along n edges. A **path** is a walk in which no vertex is repeated and a **cycle** is a walk of length at least 3 in which $v_0 = v_n$ but no other repetition of vertices occurs. A graph G is said to be **connected** if there exists a u–v walk for every pair of vertices u, v in G. In the following sections it will always be assumed that our graphs, and so our geometries, are connected.

For a pair of vertices u, v in a graph G, the **distance** $d(u, v)$ between them is the length of the shortest u–v path. Notice that if $w \sim u$ then $|d(w, v) - d(u, v)| \leq 1$. In addition, if G is the incidence graph of an incidence structure then $d(u, v)$ is even or odd according to whether u and v are of the same type or not.[2] The **diameter** of a graph G is the greatest distance between two of its vertices. The **girth** of G is the length of the shortest cycle in G. If no cycles exist then the girth is considered to be infinite.

If there is a u–v path of length d then certainly $d(u, v) \leq d$. However, when d is sufficiently small we can be sure of equality here. This is something that we will use implicitly quite frequently so we state it here precisely in the form of a lemma.

Lemma 1.6
Let G be a graph with girth g and let u and v be two vertices of G for which there is a u–v path γ of length $d \leq g/2$. Then $d(u, v) = d$. Furthermore, if $d < g/2$ then γ is the unique minimum length u–v path.

Proof Suppose $d \leq g/2$ and let δ be another u–v path, this one with length $e \leq d$. Together with all or part of γ, this gives a cycle no longer than $d + e \leq 2d \leq g$. Since g is the length of the minimum cycle in G it follows that $d = e = g/2$. Notice that if $d < g/2$ then this is a contradiction, so the path γ is unique. Either way, we see that there are no u–v paths δ shorter than γ so $d(u, v) = d$. □

[1]Notice that our graphs consequently never contain loops; that is, they are **simple** graphs.
[2]The graph G is a **bipartite** graph.

Let u be a vertex of the graph $G = (V, E)$. For an integer i, the **sphere** with centre u and radius i is the set

$$G_i(u) = \{v \in V \mid d(u, v) = i\}$$

of vertices of G that are at distance i from u. When G is the incidence graph of a geometry Γ, we also write $\Gamma_i(u)$ for the set $G_i(u)$. Notice that if Γ is an incidence structure with order (s, t), then $|\Gamma_1(u)| = s+1$ or $t+1$ for each $u \in \mathcal{L}$ or $\mathcal{P}$, respectively.

For two vertices u and v of G, with $d(u, v) = d$, and an integer $0 \leq i \leq d$, the **distance-*i* trace** of v onto u is the set

$$u^v_{[i]} = G_i(u) \cap G_{d-i}(v)$$

of vertices w of G at distance i from u such that w is on some minimum length u–v path. The elements of the set $u^v_{[i]}$ are called **distance-*i* projections** of v onto u. When there is a unique minimum length u–v path, say $u = u_0, u_1, \ldots, u_d = v$, the set $u^v_{[i]}$ consists of only the single element u_i, and we denote this unique distance-i projection of v onto u by $v \triangleright_i u = u_i$. When $i = 1$, we just call the element u_1 the **projection** of v onto u and we write $v \triangleright u$.

1.2.3 Generalized polygons

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an incidence structure. A cycle of length $2k$ in Γ is called an **(ordinary) *k*-gon** as it is a cycle with k points and k lines. The geometry Γ is then a **generalized *n*-gon**, or **generalized polygon**, if the following axioms are satisfied:

(i) Γ contains no ordinary k-gon for $2 \leq k < n$.

(ii) Given any two elements $u, v \in \Gamma$, there is an ordinary n-gon Σ containing them.

(iii) There is an ordinary $(n+1)$-gon in Γ.

Equivalently, the incidence graph of Γ is a connected graph that has diameter n and girth $2n$ and is such that each vertex is on at least three edges [VM98, Lemma 1.3.6]. If we exclude the axiom (iii), or equivalently, allow that vertices in the incidence graph may be on only two edges, then any resulting incidence structure is a **weak generalized polygon**. An ordinary n-gon in a (weak) generalized n-gon is called an **apartment**. We will say that an apartment is **ordered** if there is a specific order placed on its elements. Generalized polygons are a generalization of projective planes, which are generalized triangles, and they were introduced in [Tit59, §11]. For a thorough introduction to generalized polygons, see [VM98, Chapter 1].

Let Γ be a finite generalized n-gon. It is a theorem of Feit and Higman [FH64] that then $n = 3, 4, 6$ or 8. Also, by [VM98, Corollary 1.5.3], Γ has an order (s, t) with $s, t \geq 2$. We will be primarily concerned with generalized hexagons, and occasionally generalized quadrangles, for which $s = t$, so for the sake of brevity we will specialize statements and results about generalized polygons to these instances. For more general information concerning generalized polygons, consult the reference [VM98].

In view of previous remarks, it is assumed for the rest of this section that Γ is a finite generalized n-gon of order s, where $n = 4$ or 6.

Let $u, v \in \Gamma$ be two distinct elements and let $d = d(u, v)$. Then $d \leq n$, the diameter of the incidence graph. When $d < n$, by Lemma 1.6 there is a unique minimum length u–v path and so, for each $i = 0, \ldots, d$, the distance-i projection $v \triangleright_i u$ is defined. In particular, there is the projection $v \triangleright u$ which is the unique element incident with u at distance $d - 1$ from v; for each of the other elements $w \mathrel{\mathrm{I}} u$ we have $d(w, v) = d + 1$.

If on the other hand $d(u, v) = n$, then the elements u and v are said to be **opposite** and, since n is even, the elements u and v are either both points or both lines. In either case, for each element $w \mathrel{\mathrm{I}} u$ we have $d(w, v) = n - 1$. Furthermore, through each of these elements $w \mathrel{\mathrm{I}} u$ there is exactly one minimum length u–v path, hence there are precisely $s + 1$ minimum length u–v paths in all and the distance-i traces $u^v_{[i]}$, with $0 < i < n$, all have size $\left|u^v_{[i]}\right| = s + 1$.

Let γ be a length $n - 2$ path $u_0, \ldots, u_{n-2}$. A collineation g of Γ that fixes each of the elements of γ elementwise—that is, the lines are fixed pointwise and the points are fixed linewise—is called an **elation** corresponding to the path γ, or a **γ-elation**. If the initial and final elements, u_0 and u_{n-2}, of the path γ are points then we may also say that g is a **point elation**. Similarly, if u_0 and u_{n-2} are lines then g is a **line elation**. Let $u \neq u_1$ be an element incident with u_0 and let $S = \{v \mathrel{\mathrm{I}} u \,|\, v \neq u_0\}$ be the set of elements incident with u but distinct from u_0. Then by [VM98, Proposition 4.4.3], the set $E(\gamma)$ of γ-elations is a group, called a **root group**, which acts semiregularly on the set S (this actually follows from the forthcoming Lemma 1.7). If in addition the root group $E(\gamma)$ acts transitively, and therefore regularly, on S then the path γ is called a **Moufang path**. If every path of length $n - 2$ is a Moufang path then the generalized polygon Γ is called a **Moufang polygon**.

The generalized polygon Γ is a **Tits polygon** if it has an automorphism group that acts transitively on ordered apartments. By [VM98, 5.2.9] and [BVM94], the finite generalized polygon Γ is a Tits polygon if and only if it is a Moufang polygon.

Now let u and v be two opposite elements of Γ. A collineation g that fixes both of these elements elementwise, or equivalently, fixes all apartments containing these two elements, is called a **homology** for the elements u and v.

Finally, for future reference we state the following result concerning collineations of a generalized n-gon.

Lemma 1.7 ([VM98, Theorem 4.4.2(v)–(vi)])
Let Σ an apartment in a generalized n-gon Γ and let u and v be two elements of Σ with $d(u, v)$ coprime to n. Let g be a collineation of Γ fixing Σ. If g also fixes the elements u and v elementwise then g is the identity. (See Figure 1.1 for an illustration.) □

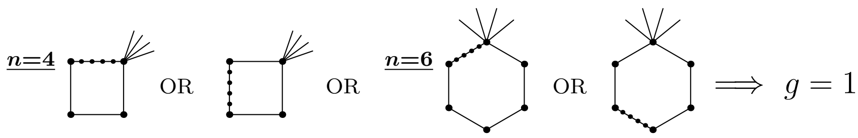


Figure 1.1: A collineation fixing so much must be the identity (Lemma 1.7).

1.2.4 Quadrics

This section reviews some of the essentials concerning quadrics in a projective space $PG(n, q)$. For comprehensive details, see the references [Hir98] and [HT91]. In the following, the notation Π_i always represents an i-dimensional subspace of $PG(n, q)$.

A **quadric** $\mathcal{Q}_n$ of $PG(n, q)$ is the set of points $x = (x_0, \ldots, x_n)$ that satisfy $f(x) = 0$, where f is a **quadratic form** $f(x) = \sum_{0\leq i,j\leq n} a_{ij}x_ix_j$. If f is not projectively equivalent to (that is, cannot be changed by a linear transformation into) a form in fewer than $n+1$ variables, then the form f and the quadric $\mathcal{Q}_n$ are said to be **nondegenerate**. Otherwise, they are **degenerate** and $\mathcal{Q}_n$ is a **cone** $\Pi_{n-k-1}\mathcal{Q}_k$, for some $k < n$, which is comprised of the following: the points of the subspace Π_{n-k-1}, called the **vertex**; the points of $\mathcal{Q}_k$, which is a nondegenerate quadric in some Π_k disjoint from Π_{n-k-1}; and all the points on lines that join points of Π_{n-k-1} to points of $\mathcal{Q}_k$.

When n is even, all nondegenerate quadrics $\mathcal{Q}_n$ are projectively equivalent and they are called **parabolic** quadrics, which we denote by $\mathcal{P}_n$. In the case that $n = 2$, a parabolic quadric $\mathcal{P}_2$ is also called a **conic**. When n is odd, there are exactly two classes of projectively equivalent nondegenerate quadrics, those that we call **hyperbolic** quadrics and denote by $\mathcal{H}_n$, and those that we call **elliptic** quadrics and denote by $\mathcal{E}_n$. Canonical forms for f that give representatives for each of these classes are:

$$\begin{aligned}
\mathcal{P}_n: &\quad x_0^2 + x_1x_2 + \cdots + x_{n-1}x_n,\\
\mathcal{H}_n: &\quad x_0x_1 + x_2x_3 + \cdots + x_{n-1}x_n,\\
\mathcal{E}_n: &\quad g(x_0, x_1) + x_2x_3 + \cdots + x_{n-1}x_n,
\end{aligned}$$

where g is an irreducible quadratic form (cf. Theorem 1.1).

Let $\mathcal{Q}_n$ be a nondegenerate quadric with corresponding form f. The **bilinear form** b **associated with** $\mathcal{Q}_n$ (or f) is given by $b(x, y) = f(x + y) - f(x) - f(y)$. Two points x and y of $PG(n, q)$ such that $b(x, y) = 0$ are said to be **conjugate** with respect to $\mathcal{Q}_n$. If x is such that $b(x, x) = 0$ then x is **self-conjugate**. In particular, the points of $\mathcal{Q}_n$ are all self-conjugate. A **totally isotropic space** is a subspace Π_k wholly contained in the quadric, so $f(x) = 0$ and $b(x, y) = 0$ for all $x, y \in \Pi_k$. The bilinear form b describes the geometry of the quadric in the sense that two points on $\mathcal{Q}_n$ are collinear in $\mathcal{Q}_n$ if and only if they are conjugate (see [HT91, Lemma 22.3.1]).

For a point x of $PG(n, q)$, the **perp** of x with respect to $\mathcal{Q}_n$ is the set $x^\perp = \{y \,|\, b(x, y) = 0\}$ of points that are conjugate to x. When, and only when, n and q are both even, there is a unique point N, called the **nucleus**, such that $N^\perp$ is the entire space $PG(n, q)$ (see [HT91, Section 22.3, Corollary 2]). For any other point x, its perp $x^\perp$ is a hyperplane, and when $x \in \mathcal{Q}_n$ this hyperplane is the **tangent prime** to $\mathcal{Q}_n$ at x.

For a subspace $X = \Pi_k$, its perp is the intersection $X^\perp = \bigcap_{x\in X} x^\perp$ of the perps of the points in X. So long as X does not contain the nucleus (when a nucleus exists), the perp $X^\perp$ is a subspace Π_{n-k-1} of dimension $n - k - 1$. If X is totally isotropic, then $X^\perp$ is the **tangent space** to $\mathcal{Q}_n$ at X and $X^\perp \cap \mathcal{Q}_n$ is a cone $X\mathcal{Q}_{n-2k-2}$, where $\mathcal{Q}_{n-2k-2}$ is the same type (parabolic, elliptic or hyperbolic) as $\mathcal{Q}_n$ (see [HT91, Lemma 22.4.5]).

When n and q are not both even, the map $X \mapsto X^\perp$ is a polarity of $PG(n, q)$ (see [HT91, Theorem 22.3.3]).

When $\mathcal{Q}_n$ is $\mathcal{P}_{2r}$, $\mathcal{E}_{2r+1}$ or $\mathcal{H}_{2r-1}$, the largest totally isotropic spaces are subspaces Π_{r-1} of dimension $r-1$. These subspaces Π_{r-1} are the **generators** of $\mathcal{Q}_n$. The geometry of all totally isotropic spaces on $\mathcal{Q}_n$ is then a rank r **polar space**.[3] In particular, if $x \in \mathcal{Q}_n$ is a point and Π_{r-1} is a generator not containing x, then there is a unique generator Π'_{r-1} such that $x \in \Pi'_{r-1}$ and the intersection $\Pi_{r-1} \cap \Pi'_{r-1}$ has dimension $r-2$. Furthermore, given a Π_{r-2} on such a $\mathcal{Q}_n$, there are exactly 2, $q+1$ or q^2+1 generators containing Π_{r-2}, depending upon whether $\mathcal{Q}_n$ is hyperbolic, parabolic or elliptic, respectively.

Consider a nondegenerate hyperbolic quadric $\mathcal{H}_{2r-1}$. It has already been noted that there are exactly two generators on any given Π_{r-2}, but further, the full set of generators divides into exactly two equivalence classes, where two generators Π_{r-1} and Π'_{r-1} are considered to be equivalent if the dimension k of their intersection $\Pi_{r-1} \cap \Pi'_{r-1} = \Pi_k$ has the same parity as their own dimension $r-1$ (see [HT91, Theorem 22.4.12]). In the case when $r = 2$, each equivalence class of lines on $\mathcal{H}_3$ is called a **regulus** and each is the **opposite regulus** of the other (see [Hir85, Section 15.1]).

The numbers of points and generators on a nondegenerate quadric are given by [HT91, Theorems 22.4.6 and 22.5.1]. Specifically, the numbers of points on the nondegenerate quadrics are

$$\begin{aligned}
|\mathcal{P}_{2s}| &= q^{2s-1} + q^{2s-2} + \cdots + q + 1, \\
|\mathcal{H}_{2s+1}| &= q^{2s} + \cdots + q^{s+1} + 2q^s + q^{s-1} + \cdots + q + 1, \\
|\mathcal{E}_{2s+1}| &= q^{2s} + \cdots + q^{s+1} + q^{s-1} + \cdots + q + 1,
\end{aligned}$$

and the number of generators

$$\begin{aligned}
&\text{on } \ \mathcal{P}_{2s} \ \text{ is } (q+1)(q^2+1)\ldots(q^s+1), \\
&\text{on } \mathcal{H}_{2s+1} \text{ is } 2(q+1)(q^2+1)\ldots(q^s+1), \\
&\text{on } \ \mathcal{E}_{2s+1} \text{ is } (q^2+1)(q^3+1)\ldots(q^{s+1}+1).
\end{aligned}$$

Finally, the number of points on a cone $x\mathcal{Q}_k$ with a point x as vertex is $q|\mathcal{Q}_k|+1$.

1.3 m-systems

Here we introduce m-systems on quadrics with particular emphasis on those in the quadric $\mathcal{P}_6$ as this is where, as we shall see, the generalized hexagon $H(q)$ lives. The basic idea behind m-systems is to have a set of m-dimensional spaces that are, in a sense, spread out apart from each other and all over the quadric. For more details on m-systems, see the paper [ST94].

Let $\mathcal{Q}$ be a nonsingular quadric of rank $r \geq 2$. In fact, while it will suffice here to suppose that $\mathcal{Q}$ is a nonsingular quadric, m-systems actually belong in the more general context of polar spaces (see the footnote [3]).

A **partial m-system** M of $\mathcal{Q}$, with $0 \leq m < r$, is a set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of totally isotropic spaces of dimension m such that no generator containing one meets any other.

[3] The classical reference for polar spaces is [BS74], and a very nice introduction can be found in the book [Bat97].

That is, if γ is a generator of $\mathcal{Q}$ such that $\pi_i \subseteq \gamma$ for some element π_i of M, then $\gamma \cap \pi_j = \emptyset$ for all $j \neq i$. This is the "spread out" part of the earlier description of where we are headed.

Upper bounds on the sizes of partial m-systems are found in [ST94], and interestingly, they are dependent only upon the choice of polar space, not on m. An **m-system** M is then defined to be a partial m-system that attains the appropriate upper bound. This ensures that the elements of M are sufficiently "all over", as suggested earlier. Thus when $\mathcal{Q}$ is $\mathcal{P}_{2n}$, $\mathcal{H}_{2n+1}$ or $\mathcal{E}_{2n-1}$, an m-system of $\mathcal{Q}$ is a partial m-system that contains $|M| = q^n + 1$ elements.

For a given m-system M, let $\widetilde{M} = \bigcup_{\pi \in M} \pi$ be the set of points that are incident with elements of M. Then a generator γ of $\mathcal{Q}$ that contains an element π of M meets $\widetilde{M}$ in exactly the $(q^{m+1} - 1)/(q - 1)$ points of π. In general, the number of points in which any generator γ meets $\widetilde{M}$ is independent of the choice of γ.

Theorem 1.8 (Shult and Thas, [ST94])
Let M be an m-system of the nondegenerate quadric $\mathcal{Q}$ and let γ be any generator of $\mathcal{Q}$. Then $|\gamma \cap \widetilde{M}| = (q^{m+1} - 1)/(q - 1)$. □

Notice that a 0-system is a set M of points such that every generator of $\mathcal{Q}$ contains exactly one point of M, and an $(r-1)$-system is a set of generators that partitions the set of points of $\mathcal{Q}$. A 0-system is also called an **ovoid** of $\mathcal{Q}$ and an $(r-1)$-system is also called a **spread** of $\mathcal{Q}$. The concept of m-system is actually a generalization of these spreads and ovoids.

We will be most interested in the quadric $\mathcal{P}_6$, which has rank 3. In addition to ovoids and spreads, or 0-systems and 2-systems, there are 1-systems, which are sets of lines on $\mathcal{P}_6$. Let Π be a hyperplane in $PG(6, q)$ that meets $\mathcal{P}_6$ in a nonsingular elliptic quadric $\mathcal{E}_5$. By [ST94, Theorem 9], any spread of this $\mathcal{E}_5$ is also a 1-system of the $\mathcal{P}_6$. In this way, we see that $\mathcal{P}_6$ always has 1-systems since the quadric $\mathcal{E}_5$ always has spreads [Tha83].

Let M be such a 1-system of $\mathcal{P}_6$ that is also a spread of an $\mathcal{E}_5$ and so is contained in a hyperplane Π. Then every generator γ of $\mathcal{P}_6$ meets Π, and therefore $\widetilde{M}$, in a set of $q + 1$ collinear points. This property characterizes 1-systems of $\mathcal{P}_6$ arising this way, as we demonstrate in our following theorem.

Theorem 1.9
Let M be a 1-system of $\mathcal{P}_6$ with the property that for every generator γ of $\mathcal{P}_6$, the intersection $\gamma \cap \widetilde{M}$ is a line of the quadric. Then M is contained in a hyperplane and so is a spread of an $\mathcal{E}_5$.

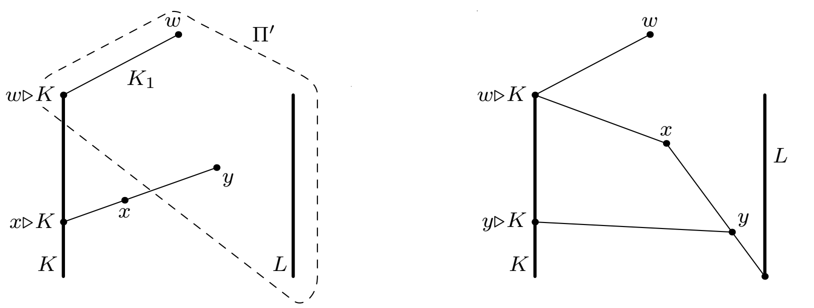
Proof Let L be a line on the quadric $\mathcal{P}_6$ and let γ be any generator containing L. Let $K = \gamma \cap \widetilde{M}$ be the set of points in which γ meets the lines of the 1-system M. By the given property of M, the set K is a line so L and K are two lines in a common plane γ. Hence either $L = K$, in which case we have $L \subseteq \widetilde{M}$, or $|L \cap \widetilde{M}| = |L \cap K| = 1$. Thus the set $\widetilde{M}$ satisfies the conditions of the theorem of [Pen92] which then asserts that $\widetilde{M} = \Pi \cap \mathcal{P}_6$ for some hyperplane Π. Now the result follows from the fact that the only hyperplane section of $\mathcal{P}_6$ with $|\widetilde{M}| = (q + 1)(q^3 + 1)$ points is an elliptic quadric. □

This characterizes 1-systems of $\mathcal{P}_6$ that are spreads of an $\mathcal{E}_5$ as those that meet every generator in $q+1$ collinear points. However, we actually only need M to have this property in relation to a much smaller collection of generators for us to be able to reach this same conclusion. Before giving this refinement of the previous theorem, we prove our following lemma.

Lemma 1.10
Let A be a set of points in $\mathcal{P}_4$ and let $K, L \subset A$ be two nonintersecting lines on the quadric. Suppose that for each point $x \in A$, the lines $K \triangleright x$ and $L \triangleright x$ of $\mathcal{P}_4$ through x and concurrent with K and L, respectively, are also contained within A. Then either $A = \mathcal{H}_3$ or $A = \mathcal{P}_4$.

Proof Since $K, L \subset A$ are nonintersecting lines, they generate a 3-dimensional space Π that meets the $\mathcal{P}_4$ in a nondegenerate hyperbolic quadric $\mathcal{H}_3$ (use [HT91, Theorem 22.8.3]). By the given property, all the lines of $\mathcal{H}_3$ that meet both K and L are contained within A as well, so the whole of $\mathcal{H}_3$ is contained in A. If there is no point of A outside of Π then $A = \mathcal{H}_3$.

Suppose now that there is a point $w \in A$ that is not contained in $\mathcal{H}_3$. Let x be any other point of $\mathcal{P}_4$ not in $\mathcal{H}_3$. We will show that x is in A as well.



(a) Case $x \triangleright K \neq w \triangleright K$ (b) Case $x \triangleright K = w \triangleright K$

Figure 1.2: Diagrams for proof of Lemma 1.10.

There are two cases to consider; see Figure 1.2 for diagrams. First suppose the points $x \triangleright K$ and $w \triangleright K$ are distinct. Let $K_1 = K \triangleright w$ be the unique line in $\mathcal{P}_4$ on w that meets the line K. Then by the given property, we have $K_1 \subset A$. Since w is not in Π, the lines K_1 and L are skew so they also generate a 3-dimensional space $\Pi' \neq \Pi$ that meets $\mathcal{P}_4$ in a nondegenerate hyperbolic quadric $\mathcal{H}'_3$. Since $K_1 \subset A$ and $L \subset A$, as with $\mathcal{H}_3$, the whole of $\mathcal{H}'_3$ is contained in A. Now the line $K \triangleright x$ meets the hyperplane Π' in a point y, which is then an element of $\mathcal{H}'_3$ and therefore of A. Since x lies on the line $K \triangleright x = K \triangleright y$, it follows that $x \in A$.

Now suppose $x \triangleright K = w \triangleright K$. This is illustrated in Figure 1.2(b). Let y be a point on the line $L \triangleright x$ such that $y \neq x$ and $y \neq x \triangleright L$. Then $y \triangleright K \neq w \triangleright K$, since otherwise we would have a triangle on the points x, y and $y \triangleright K = w \triangleright K = x \triangleright K$ in the generalized quadrangle $\mathcal{P}_4$. So y is a point of the type that we have already considered, and as such, we have $y \in A$. Since x is on the line $L \triangleright x = L \triangleright y$, it now follows that $x \in A$. □

We are now prepared to prove our promised refinement of Theorem 1.9.

Theorem 1.11
Let M be a 1-system of $\mathcal{P}_6$. Suppose there are two lines K and L of M such that for every generator γ of $\mathcal{P}_6$ meeting at least one of them, the intersection $\gamma \cap \widetilde{M}$ is a line of the quadric. Then M is contained in a hyperplane and so is a spread of an $\mathcal{E}_5$.

Proof Let π be the 3-dimensional space generated by K and L. Then $\pi \cap \mathcal{P}_6$ in a non-degenerate hyperbolic quadric $\mathcal{H}_3$ (consider the table of sections in [HT91, Section 22.8]). By the given property of M, all the points of this $\mathcal{H}_3$ belong to $\widetilde{M}$. In addition to K and L, there are $q^3 - 1$ more lines in M but only $(q+1)(q-1)$ more points in $\mathcal{H}_3$, so there is a line $N \in M$ that does not meet π. Let Π be the 5-dimensional space generated by N and π. Suppose there is a generator γ of $\mathcal{P}_6$ containing N and contained within Π. Then γ meets π in a point, which we have already seen belongs to $\widetilde{M}$. Thus γ is a generator on one element of M meeting another, contrary to M being a 1-system. So there are no generators of $\mathcal{P}_6$ on N contained within Π and we deduce that Π meets $\mathcal{P}_6$ in an elliptic quadric $\mathcal{E}_5$ (again, see the table in [HT91, Section 22.8]).

Let n be a point on N and let π_0 be the 4-dimensional space generated by π and n. Since $\pi_0 \cap \mathcal{E}_5$ contains the skew lines K and L, it follows that π_0 meets $\mathcal{E}_5$, and therefore $\mathcal{P}_6$, in a parabolic quadric $\mathcal{P}_4$. Let $A = \widetilde{M} \cap \pi_0$ be the set of points of $\mathcal{P}_4$ that are on lines of M. Then $K, L \subset A$ and $n \in A$, so $A \neq \mathcal{H}_3$. Consider a point $x \in A$ and let $L_1 = L \triangleright x$ be the line in $\mathcal{P}_4$ through x and intersecting L. Let γ be a generator of $\mathcal{P}_6$ on the line L_1. Since γ meets L, its intersection with $\widetilde{M}$ is a line. The point x is in $\widetilde{M}$ so the line in which γ meets $\widetilde{M}$ is the line L_1. Hence $L_1 \subset A$. Similarly, the line $K \triangleright x$ is contained in A. Thus the set A satisfies the conditions of Lemma 1.10 so A, and therefore $\widetilde{M}$ as well, contains all the points of $\mathcal{P}_4$.

Letting n range over all the points of N shows that all the points of $\mathcal{E}_5$ belong to $\widetilde{M}$. Finally, since $|\widetilde{M}| = |\mathcal{E}_5|$ it now follows that $\widetilde{M} = \mathcal{E}_5$ so M is contained in the hyperplane Π and it is a spread of $\mathcal{E}_5$. □

1.4 The generalized hexagon $H(q)$

In this section, the generalized hexagon $H(q)$ is described—its construction, some of its geometric properties and its automorphisms. For the most part, the following and further details can be found in the book [VM98].

1.4.1 Construction

Here we will describe in reasonable detail the construction due to Tits [Tit59] of the generalized hexagon $H(q)$ from a triality. The reader who is already familiar with generalized quadrangles may find it helpful to compare the construction in this section with the parallel construction of the symplectic quadrangle $W(q)$ described in Section 4.1.

To begin, let $\mathcal{H}_7$ be the nondegenerate hyperbolic quadric in $PG(7,q)$ given by[4]

$$X_0X_4 + X_1X_5 + X_2X_6 - X_3X_7 = 0. \tag{1.1}$$

We start by defining a rank 4 geometry $\Gamma = (\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \mathcal{L}, \mathrm{I})$ embedded in this quadric.[5] Let $\mathcal{L}$ be the set of lines on $\mathcal{H}_7$ and let $\mathcal{P}^{(0)}$ be the set of points on it. By [HT91, Theorem 22.4.12], the generators of $\mathcal{H}_7$, which are 3-dimensional projective subspaces, fall into two equivalence classes, where two generators x and y are considered to be equivalent if their intersection $x \cap y$ is empty, a line, or the whole subspace $x = y$. Let $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ be these equivalence classes. In this geometry Γ, the elements of the sets $\mathcal{P}^{(i)}$ are called "i-points", or just "points", and those of $\mathcal{L}$ are "lines". Incidence between elements of different types is given by inclusion except between elements of $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$, where a 1-point x and a 2-point y are incident if, as projective subspaces, their intersection $x \cap y$ is a plane.

The number of i-points is the same for each $i = 0, 1, 2$, and since the 0-points, as points of $PG(7,q)$, already have a natural labelling with homogeneous coordinates, we can label the elements of $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ with similar coordinates. This is done in such a way that incidence between points of different types is given by the trilinear form[6]

$$\begin{aligned} T(x,y,z) = &\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} + \begin{vmatrix} x_4 & x_5 & x_6 \\ y_4 & y_5 & y_6 \\ z_4 & z_5 & z_6 \end{vmatrix} \\ &- x_3(z_0y_4 + z_1y_5 + z_2y_6) + x_7(y_0z_4 + y_1z_5 + y_2z_6) \\ &- y_3(x_0z_4 + x_1z_5 + x_2z_6) + y_7(z_0x_4 + z_1x_5 + z_2x_6) \\ &- z_3(y_0x_4 + y_1x_5 + y_2x_6) + z_7(x_0y_4 + x_1y_5 + x_2y_6) \\ &+ x_3y_3z_3 - x_7y_7z_7. \end{aligned} \tag{1.2}$$

In particular, the 0-point $x = (x_0, x_1, \ldots, x_7)$ and the 1-point $y = (y_0, y_1, \ldots, y_7)$ are incident in Γ if and only if $T(x,y,z) \equiv 0$ as a function in the third parameter z; and similarly for other combinations of points of different types, where the first parameter of the trilinear form corresponds to 0-points, the second to 1-points and the third to 2-points. Notice that exactly which generator of $\mathcal{H}_7$ is the 1-point $y = (y_0, y_1, \ldots, y_7)$ is readily determined by putting this value of y into $T(x,y,z)$ and then requiring that all the coefficients of the z_i be zero. The resulting equations in the x_i then determine the corresponding generator. For a 2-point $z = (z_0, z_1, \ldots, z_7)$, the equations for the corresponding generator of $\mathcal{H}_7$ are found similarly.

Let θ be the map that sends each point of $\mathcal{P}^{(i)}$ to the point of $\mathcal{P}^{(i+1)}$ (addition modulo 3) that has the same coordinates (Figure 1.3). Since $T(x,y,z)$ is preserved by cyclic permutations of the three parameters, this map θ preserves incidence between different types of points. Now consider a 0-point x and two distinct collinear 1-points y and z. The 1-points y and z are incident with a uniquely determined common line yz,

[4]The choice of a minus sign before the last term is made in order to go directly to the desired representation of $H(q)$, unlike the standard references where a reflection is performed in the final step.

[5]This will actually be an instance of an **oriflamme** geometry (see [Asc86, Chapter 7]).

[6]This appears in [Car38, p51]. Alternatively, see [Tit59, 3.2] or [VM98, 2.4.6].

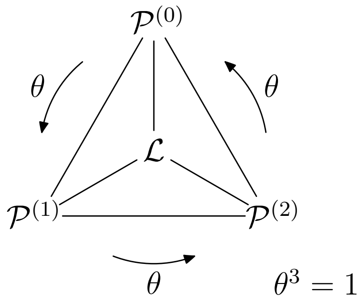


Figure 1.3: The triality θ

which is their intersection $y \cap z$ as projective subspaces. The 0-point x is then incident with the line yz if and only if $x \in y \cap z$, or equivalently, if and only if $x \mathrel{\mathrm{I}} y$ and $x \mathrel{\mathrm{I}} z$. In a similar manner, it can be verified that for any three points x, y and z, where y and z are of the same type as each other but of a different type from x, then $x \mathrel{\mathrm{I}} y$ and $x \mathrel{\mathrm{I}} z$ if and only if $x \mathrel{\mathrm{I}} yz$. It now follows from the fact that θ preserves incidence between different types of points that θ also preserves incidence between points and lines, where a line yz is mapped to the line $(yz)^\theta = y^\theta z^\theta$. Explicity, if $x \mathrel{\mathrm{I}} yz$ then $x \mathrel{\mathrm{I}} y$ and $x \mathrel{\mathrm{I}} z$, which after applying θ becomes $x^\theta \mathrel{\mathrm{I}} y^\theta$ and $x^\theta \mathrel{\mathrm{I}} z^\theta$, and so $x^\theta \mathrel{\mathrm{I}} (yz)^\theta$. Therefore θ is an anti-automorphism of the geometry Γ.

An anti-automorphism of Γ with order 3 is called a **triality**.[7] In particular, the map θ described above is a triality. Recall that an absolute element is one that is incident with its image, so here an absolute 0-point x is one that lies in the 3-dimensional projective subspace that is the 1-point x^θ, and the absolute lines are those that are fixed by θ. By [Tit59, 4.3.1–2] (see [VM98, Theorem 2.4.8]), the incidence structure obtained by taking these absolute 0-points and absolute lines of Γ with respect to θ is a generalized hexagon of order q. This is what we define to be the **split Cayley hexagon** $H(q)$. As explained in [VM98, 2.4.9], this name derives from an alternative method of construction using a split Cayley algebra (see [Sch62a] and [Sch62b]). To obtain the usual representation of $H(q)$ from here requires a little tedious work and so it is generally not done. For this reason, an independent demonstration of the steps involved is recorded here in detail.

In order to help keep things reasonably brief and to make use of the algebra of vectors of a three dimensional space (see for instance [Chi78]), for an ordered 8-tuple $x = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, let

$$\bar{x} = (x_0, x_1, x_2), \quad \tilde{x} = (x_4, x_5, x_6), \quad x' = x_3 \quad \text{and} \quad x'' = x_7.$$

Then the equation (1.1) of the quadric $\mathcal{H}_7$ can be rewritten

$$\bar{x} \cdot \tilde{x} - x'x'' = 0 \tag{1.3}$$

and two points x and y on the quadric are collinear if and only if

$$\bar{x} \cdot \tilde{y} + \tilde{x} \cdot \bar{y} - x'y'' - x''y' = 0. \tag{1.4}$$

[7]See [Car38, p54], where the principle of triality is introduced in analogy with the principle of duality as discussed in [Car25]. Also, the trialities that admit absolute points are classified in [Tit59].

Also, the trilinear form (1.2) can now be written as

$$\begin{aligned} T(x,y,z) =& \bar{z}\cdot(\bar{x}\times\bar{y} - x'\tilde{y} + \tilde{x}y'') + \tilde{z}\cdot(\tilde{x}\times\tilde{y} + x''\bar{y} - \bar{x}y') \\ & - z'(\tilde{x}\cdot\bar{y} - x'y') + z''(\bar{x}\cdot\tilde{y} - x''y''). \end{aligned} \tag{1.5}$$

A 0-point x is a point of $H(q)$ if and only if it is absolute with respect to θ; that is, if and only if $x \mathrel{\mathrm{I}} x^\theta$. Thus the points of $H(q)$ are precisely those points of $\mathcal{H}_7$ for which $T(x,x,z) \equiv 0$ as a function of z. From (1.3) and (1.5) we have

$$T(x,x,z) = (x' - x'')(-\bar{z}\cdot\tilde{x} - \tilde{z}\cdot\bar{x} + x'z' + x''z'')$$

and this is identically zero as a function of z if and only if[8]

$$x' = x''. \tag{1.6}$$

Notice that this equation determines a hyperplane of $PG(7,q)$ that intersects $\mathcal{H}_7$ in a nondegenerate parabolic quadric $\mathcal{P}_6$, so the points of $H(q)$ are exactly the points of this $\mathcal{P}_6$.

Now we identify the lines of $H(q)$. To begin, notice that the points on an absolute line ℓ are necessarily absolute themselves, for if $x \mathrel{\mathrm{I}} \ell$ then $x^\theta \mathrel{\mathrm{I}} \ell^\theta = \ell$, or as projective subspaces, $x \in \ell \subset x^\theta$, so $x \mathrel{\mathrm{I}} x^\theta$.

Consider a line xy where x and y are absolute points such that $x \mathrel{\mathrm{I}} y^\theta$. Notice that then $x \in x^\theta \cap y^\theta$. Since also $y \mathrel{\mathrm{I}} y^\theta$, we have $xy \subset y^\theta$, or equivalently $xy \mathrel{\mathrm{I}} y^\theta$. Then $(xy)^\theta \mathrel{\mathrm{I}} y^{\theta^2}$, and since $(xy)^\theta$ is a line and y^{θ^2} is a 3-dimensional projective subspace, we have $y^{\theta^2} \supset (xy)^\theta = x^\theta \cap y^\theta \ni x$. Hence $y^{\theta^2} \mathrel{\mathrm{I}} x$, from which we have $y \mathrel{\mathrm{I}} x^\theta$. Thus also $xy \subset x^\theta$ and it follows that $xy = (xy)^\theta$, so the line xy is absolute. Conversely, if the line xy is absolute then $xy = (xy)^\theta = x^\theta \cap y^\theta$, and in particular, $x \mathrel{\mathrm{I}} y^\theta$.

Hence the absolute lines are precisely those lines xy where x and y are absolute and $x \mathrel{\mathrm{I}} y^\theta$. In other words, they are the lines xy where x and y are in the hyperplane given by (1.6) and $T(x,y,z) \equiv 0$ as a function of z. For this latter condition to be satisfied, we see from (1.5) together with (1.6) that in particular we need

$$\bar{x}\times\bar{y} - x'\tilde{y} + \tilde{x}y' = 0 \quad\text{and}\quad \tilde{x}\times\tilde{y} + x'\bar{y} - \bar{x}y' = 0. \tag{1.7}$$

In fact, given a line xy where x and y are absolute, the conditions in (1.7) are sufficient to ensure that the line is absolute. To see this, we need only show that the two remaining coefficients, $\tilde{x}\cdot\bar{y} - x'y'$ and $\bar{x}\cdot\tilde{y} - x''y''$, in (1.5) vanish when these conditions are satisfied. Even further, we need only show that the former of these vanishes as then the other follows immediately from (1.4) and (1.6).[9]

Suppose first that $\bar{x} = 0$. Then from (1.3) and (1.6), we have $x' = x'' = 0$. Substitution into (1.4) now gives $\tilde{x}\cdot\bar{y} = 0$, hence $\tilde{x}\cdot\bar{y} - x'y' = 0$ as required.

Now suppose that $\bar{x} \neq 0$. To begin, notice that

$$(\tilde{x}\cdot\bar{y} - x'y')\bar{x} = (\tilde{x}\cdot\bar{y})\bar{x} - x'^2\bar{y} - x'(-x'\bar{y} + \bar{x}y'). \tag{1.8}$$

[8] See also [Tit59, 5.22].

[9] This is essentially the observation of Dickson [Dic01]; see [Tit59, 8.1.4].

By (1.3) and (1.6), we have $x'^2 = \tilde{x} \cdot \bar{x}$, and from (1.7) we have $-x'\bar{y} + \bar{x}y' = \tilde{x} \times \tilde{y}$. Thus, with the help of the identities $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ and $a \times a = 0$, the equation (1.8) becomes

$$\begin{aligned}(\tilde{x} \cdot \bar{y} - x'y')\bar{x} &= (\tilde{x} \cdot \bar{y})\bar{x} - (\tilde{x} \cdot \bar{x})\bar{y} - x'(\tilde{x} \times \tilde{y}) \\ &= \tilde{x} \times (\bar{x} \times \bar{y}) - x'(\tilde{x} \times \tilde{y}) + y'(\tilde{x} \times \tilde{x}) \\ &= \tilde{x} \times (\bar{x} \times \bar{y} - x'\tilde{y} + \tilde{x}y') \\ &= 0,\end{aligned}$$

where the final step has followed from (1.7). Since $\bar{x} \neq 0$, it follows that $\tilde{x} \cdot \bar{y} - x'y' = 0$.

Hence the absolute lines are precisely those lines xy of the quadric $\mathcal{P}_6$ that satisfy the conditions in (1.7). Finally, we remove from these conditions the reference to the points x and y on the line by rewriting them in terms of the **Plücker coordinates** of a line. In general, if $w = (w_0, \ldots, w_n)$ and $z = (z_0, \ldots, z_n)$ are two points in a projective space $PG(n, q)$, the Plücker coordinates of the line wz are the $\binom{n+1}{2}$ values $p_{ij} = \left|\begin{smallmatrix} w_i & w_j \\ z_i & z_j \end{smallmatrix}\right|$. These are a special case of Grassmann coordinates, which apply to subspaces in general. Details concerning these can be found in [HT91, Section 24.1].

In the former of the two conditions in (1.7), we have $\bar{x} \times \bar{y} = (p_{12}, p_{20}, p_{01})$ and $x'\tilde{y} - \tilde{x}y' = (p_{34}, p_{35}, p_{36})$, and in the latter, $-\tilde{x} \times \tilde{y} = (p_{65}, p_{46}, p_{54})$ and $x'\bar{y} - \bar{x}y' = (p_{30}, p_{31}, p_{32})$. Also, substituting (1.6) into (1.1) yields the equation of the quadric $\mathcal{P}_6$ in which $H(q)$ is embedded.

Thus the points of the generalized hexagon $H(q)$ are the points of the nondegenerate parabolic quadric $\mathcal{P}_6$ given by

$$X_0X_4 + X_1X_5 + X_2X_6 = {X_3}^2 \tag{1.9}$$

and the lines are the lines on this quadric whose Plücker coordinates satisfy

$$\begin{aligned} p_{12} &= p_{34}, & p_{20} &= p_{35}, & p_{01} &= p_{36}, \\ p_{65} &= p_{30}, & p_{46} &= p_{31}, & p_{54} &= p_{32}. \end{aligned} \tag{1.10}$$

1.4.2 Geometry

While we considered generalized polygons in Section 1.2.3, here we consider further this particular generalized hexagon, $H(q)$. Lemmas 1.12–1.18 are elementary results following from the natural embedding of $H(q)$ in $\mathcal{P}_6$ and belonging mostly to folklore. Explicitly or implicitly, they can mostly be found in [Tit59, §4], [VM98, Section 2.4] and [Ron80]. Throughout, we shall assume $\Gamma = H(q)$ is given by the standard representation in the quadric $\mathcal{P}_6$ as given at the end of the previous section. Thus points and lines of $H(q)$ will be regarded equally freely as points and lines of $\mathcal{P}_6$, although the context shall be made clear when confusion seems a possibility. In particular, since not all lines of $\mathcal{P}_6$ are lines of $H(q)$, the distinction will be made by referring to those that are as ***H*(*q*)-lines**.

Let x and y be two collinear points in $\mathcal{P}_6$ and let θ be the triality described in Section 1.4.1. Recall from page 15 that the line xy of $\mathcal{P}_6$ is an $H(q)$-line if and only if $y \in x^\theta$. But x^θ, as a 3-space on $\mathcal{H}_7$, meets $\mathcal{P}_6$ in a plane π which is then a generator of $\mathcal{P}_6$. So xy is an $H(q)$-line if and only if $y \in \pi$. We call this plane π the ***H*(*q*)-plane** of x. Since there are $q+1$ lines through x in π and all $q+1$ points on each of these lines are also points of $H(q)$, this demonstrates that $H(q)$ does indeed have order q. Notice that the same plane π cannot also be the $H(q)$-plane of y as then $H(q)$ would contain triangles on points x, y and z, where z is any point of π not on the line xy. Thus the points on an $H(q)$-line have distinct $H(q)$-planes. Since there are $q+1$ points on a line and there are also $q+1$ generators of $\mathcal{P}_6$ on a line, we have the following result.

Lemma 1.12
Let ℓ be an $H(q)$-line. Then the generators of $\mathcal{P}_6$ on ℓ are precisely the $H(q)$-planes of the points on ℓ. □

The following frequently used theorem also appears in the proof of [Yan76, Theorem 1.1], where the author shows that his "hyperbolic" and "singular" lines correspond to the lines of $\mathcal{P}_6$.

Theorem 1.13
Two points of $H(q)$ are opposite if and only if they are not collinear in the quadric $\mathcal{P}_6$.

Proof Let x and y be two points of $H(q)$. Then $d(x,y) = 2$, 4, or 6. Certainly, if $d(x,y) = 2$ then x and y are collinear in $H(q)$ and therefore also in $\mathcal{P}_6$. Also, if $d(x,y) = 4$, then x and y are both collinear in $H(q)$ with the point $z = x \triangleright_2 y$, so they are both in the $H(q)$-plane of z and are therefore collinear in $\mathcal{P}_6$.

Now suppose that x and y are collinear in $\mathcal{P}_6$. If the line xy is an $H(q)$-line then $d(x,y) = 2$. Suppose then that xy is not an $H(q)$-line, so then y does not lie in the $H(q)$-plane π_x of x. Recall from Section 1.2.4 that if w is a point of a nondegenerate quadric $\mathcal{Q}_n$ of rank r and Π is a generator not containing w, then there is a unique generator Π' containing w such that $\Pi' \cap \Pi$ has dimension $r-2$. Consequently, there is a unique generator π of $\mathcal{P}_6$ containing y and such that $\pi \cap \pi_x$ is a line ℓ. Since x and y are collinear in $\mathcal{P}_6$, we have $x \in \pi$, so ℓ is a line in π_x through x. It follows that ℓ is an $H(q)$-line so, by Lemma 1.12, the plane π is the $H(q)$-plane of some point $z \in \ell$. Hence yz is also an $H(q)$-line as it is a line through z contained in its $H(q)$-plane, and therefore $d(x,y) = 4$. □

Corollary 1.14
Let ℓ be a line of $\mathcal{P}_6$ that is not an $H(q)$-line. Then there is a unique $H(q)$-plane containing ℓ.

Proof Let $x, y \in \ell$ be two points. By the previous theorem, we have $d(x,y) = 4$ so the point $z = x \triangleright_2 y$ is the unique point whose $H(q)$-plane contains both x and y. □

Next is a theorem regarding lines that is comparable to Theorem 1.13.

Theorem 1.15
Two lines of $H(q)$ are opposite if and only if every generator of $\mathcal{P}_6$ containing one is disjoint from the other.

Proof Let ℓ and m be two $H(q)$-lines. Notice that the union of the generators of $\mathcal{P}_6$ containing ℓ is simply $\ell^\perp \cap \mathcal{P}_6$. By Lemma 1.12, these generators are the $H(q)$-planes of the points on ℓ, so $\ell^\perp \cap m$ is the set of points y on m with $d(y, x) \leq 2$ for some $x \mathrm{I} \ell$. Hence the points of $\ell^\perp \cap m$ are precisely those points on m with $d(y, \ell) \leq 3$. Thus every generator on ℓ is disjoint from m if and only if $d(y, \ell) = 5$ for every $y \mathrm{I} m$, or equivalently, the lines ℓ and m are opposite. □

Let x and y be opposite points in $H(q)$. Then the distance-1 trace $x^y_{[1]}$ is the sphere $\Gamma_1(x)$, and we have already seen the structure of this in $\mathcal{P}_6$—it is a pencil of lines in a generator, the $H(q)$-plane. Now we consider the structures in $\mathcal{P}_6$ of the distance-2 and distance-3 traces.

Lemma 1.16
Let x be a point of $H(q)$ and let π be its $H(q)$-plane. Then the distance-2 traces $x^y_{[2]}$ for points y opposite x, are precisely the lines of π not containing x.

Proof Let ℓ be a line of π not containing x, so $d(z, x) = 2$ for each $z \in \ell$. Let π' be another generator of $\mathcal{P}_6$ through ℓ and consider a point $y \in \pi' \setminus \ell$. By Corollary 1.14, the generator π' is not an $H(q)$-plane so, by Lemma 1.12, there are no $H(q)$-lines in π'. It now follows from Theorem 1.13 that $d(z, y) = 4$ for each $z \in \ell$. Thus $\ell = x^y_{[2]}$.

Suppose now that y is a point opposite x. Then certainly y is not in π so there is a unique generator π' containing y such that $\pi' \cap \pi$ is a line ℓ. By Theorem 1.13, the line ℓ does not contain x so from the previous paragraph we have $x^y_{[2]} = \ell$. □

Lemma 1.17
Let x and y be opposite points in $H(q)$. Then the distance-3 trace $x^y_{[3]}$ is a regulus on the quadric $\mathcal{P}_6$.

Proof By Lemma 1.16, the distance-2 traces $\ell = x^y_{[2]}$ and $m = y^x_{[2]}$ are lines of $\mathcal{P}_6$ and, since they have no point in common, they generate a 3-space Π_3. Also, each point $z \in \ell$ is collinear with some point, namely $y \triangleright_2 z$, in m. Thus the intersection $\Pi_3 \cap \mathcal{P}_6$ is a nondegenerate hyperbolic quadric $\mathcal{H}_3$. Finally, the lines of the distance-3 trace $x^y_{[3]}$ are the lines of $\mathcal{P}_6$, and hence of $\mathcal{H}_3$, that intersect ℓ and m, and so are the lines of a regulus. □

Lemma 1.18
Let ℓ and m be opposite lines in $H(q)$. Then the distance-3 trace $\ell^m_{[3]}$ is a conic on the quadric $\mathcal{P}_6$.

Proof Let Π_3 be the 3-space generated by the lines ℓ and m. Then $\Pi_3^\perp$ is a plane disjoint from Π_3 that meets $\mathcal{P}_6$ in a conic $\mathcal{C}$. From Theorem 1.13, the points of $\mathcal{C}$ are at distance at most 4 from the points of ℓ and m, so we conclude that they are all at distance 3 from each of these lines. □

From Lemma 1.17, the distance-3 trace $x^y_{[3]}$ for two opposite points x and y is a regulus on $\mathcal{P}_6$ and we distinguish it from others by calling this a **line regulus**, since it is comprised of lines of the hexagon. Also, a regulus is uniquely determined by two of its lines, and we denote the unique line regulus containing the opposite lines ℓ and m by $\mathcal{R}(\ell, m)$.

Similarly, a distance-3 trace $\ell^m_{[3]}$ for two opposite lines ℓ and m is uniquely determined by two of its points, say x and y, for, by the proof of Lemma 1.18, it is the conic contained in the plane $\Pi_3^\perp$, where Π_3 is the 3-space containing the line regulus $x^y_{[3]}$. Because of the dual relationship with line reguli, we call this a **point regulus** and we denote the unique point regulus containing two opposite points x and y by $\mathcal{R}(x, y)$. Also, a point regulus in a plane Π_2 and line regulus in a 3-space Π_3 such that $\Pi_2 = \Pi_3^\perp$ are called **complementary**.

Notice that there is a unique regulus on $\mathcal{P}_6$ containing two given opposite lines ℓ and m of $H(q)$ so there can never be any uncertainty about which regulus is the line regulus $\mathcal{R}(\ell, m)$. However, given two opposite points x and y of $H(q)$, there are numerous conics on $\mathcal{P}_6$ that contain x and y, while only one of them is the point regulus $\mathcal{R}(x, y)$. It is with this in mind that we provide our following geometric description of precisely which conic is the appropriate one.

Lemma 1.19
Let x and y be two opposite points of $H(q)$ and let π_x and π_y be their $H(q)$-planes. Then $\pi_x^\perp \cap \pi_y^\perp$ is a point P of $PG(6, q)$ not on $\mathcal{P}_6$ and the point regulus $\mathcal{R}(x, y)$ is contained in the plane determined by the three points x, y and P.

Proof Since x and y are opposite, the $H(q)$-planes π_x and π_y are disjoint, so their tangent spaces $\pi_x^\perp$ and $\pi_y^\perp$ meet in a point P of $PG(6, q)$ that is not on $\mathcal{P}_6$. In addition, the points x, y and P are not collinear since the line xP is tangent to $\mathcal{P}_6$. Let Π_3 be the 3-space that contains the complementary line regulus $x^y_{[3]}$, so then $\Pi_3^\perp$ is the plane containing the point regulus $\mathcal{R}(x, y)$. We have only to show that $P \in \Pi_3^\perp$.

Let ℓ and m be the distance-2 traces $x^y_{[2]}$ and $y^x_{[2]}$, respectively, from x and y onto each other. By Lemma 1.16, these are lines of $\mathcal{P}_6$ in the planes π_x and π_y. Furthermore, ℓ and m are lines of the regulus opposite to the line regulus in Π_3 so they generate this subspace. Thus, $\Pi_3^\perp = \ell^\perp \cap m^\perp \supset \pi_x^\perp \cap \pi_y^\perp = \{P\}$. □

By [VM98, Lemma 1.5.4], the numbers of points and lines in $H(q)$ are

$$|\mathcal{P}| = |\mathcal{L}| = q^5 + q^4 + q^3 + q^2 + q + 1, \tag{1.11}$$

which is indeed seen to be equal to the number of points on the quadric $\mathcal{P}_6$, and the number of elements opposite a given element x is

$$|\Gamma_6(x)| = q^5. \tag{1.12}$$

In view of Theorem 1.13, it is quite easy to see the truth of (1.12) for points. For a point x, the intersection $x^\perp \cap \mathcal{P}_6$ of the tangent prime at x with $\mathcal{P}_6$ is a quadric cone $x\mathcal{P}_4$, the points of which, by the theorem, are precisely the points y with $d(x, y) \leq 4$. The quadric cone has $q^4 + q^3 + q^2 + q + 1$ points and this, together with (1.11), readily gives the result.

Finally, we shall see that the case $q = 3^h$ plays a special rôle when dealing with $H(q)$. The reason for this lies essentially in the following very important theorem.

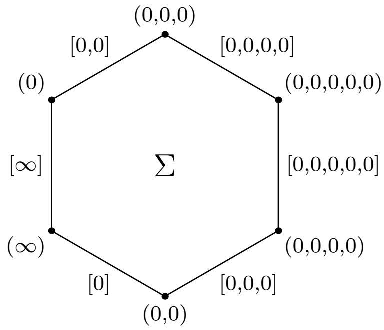


Figure 1.4: Labelling the hat-rack

Theorem 1.20 (see [VM98, Corollary 7.3.5])
The generalized hexagon $H(q)$ is self-dual if and only if $q = 3^h$ and it is self-polar if and only if $q = 3^{2e+1}$. □

Although we do not prove Theorem 1.20 here, a demonstration of the truth of the "if" parts, due originally to Tits [Tit61, Section 5], commences on page 28 in Section 1.4.4 after coordinatization and morphisms of $H(q)$ have been discussed.

1.4.3 Coordinates

Coordinatization of generalized polygons is done in a manner that generalizes the usual way in which projective planes are coordinatized [Hal43]. The general theory is discussed in [VM98, Chapter 3], and for generalized hexagons in particular, see also [DeSVM93]. Here we will only describe the essential points needed for us to be able to work with the coordinates.

To begin, let Σ be an apartment in the split Cayley hexagon $\Gamma = H(q)$. In order to distinguish between coordinates for points and coordinates for lines, we use parentheses "( )" for points and brackets "[ ]" for lines. With this in mind, label the elements of Σ as in Figure 1.4. This, now ordered, apartment Σ is called the **hat-rack** of the coordinatization. By [Tit73] (see also [VM98, Theorem 4.5.6]), the split Cayley hexagon is a Moufang hexagon. Hence it is also a Tits hexagon and the group of automorphisms of $H(q)$ acts transitively on ordered apartments. In particular, this means that the final coordinatization is dependent neither upon the choice of Σ nor on precisely where we start the labelling of the elements of Σ according to Figure 1.4. This will be a very convenient fact, analogous to being able to freely choose an origin and a pair of axes when working in the Euclidean plane.

Next, the points other than (∞) and (0) that are incident with the line $[\infty]$ are labelled (a), where a ranges over the nonzero elements of $GF(q)$. Likewise, the remaining lines incident with (∞) are labelled $[k]$, where k ranges over the nonzero elements of $GF(q)$. Now following the coordinatization process described in [VM98, Section 3.2], the coordinates of all the remaining elements of $H(q)$ are determined. The resulting coordinatization places the points, as well as the lines, in one-to-one correspondence with the set of all i-tuples of elements from $GF(q)$, where $0 \le i \le 5$ and we consider a 0-tuple to simply be one consisting of the single special symbol ∞. Notice that there are q^i i-tuples for each i, so in all, for $0 \leq i \leq 5$ there are $q^5 + q^4 + \cdots + q + 1$ i-tuples, which is indeed equal to the number of points and the number of lines in $H(q)$ as seen in (1.11).

Following [VM98, Section 3.2], the coordinates of an element are assigned in such a manner so as to essentially describe a path to it from the flag $\{(\infty), [\infty]\}$. To be precise, there is the path

$$\begin{gathered}[k,b,k',b',k''] \mathrm{~I~} (k,b,k',b') \mathrm{~I~} [k,b,k'] \mathrm{~I~} (k,b) \mathrm{~I~} [k] \mathrm{~I~} (\infty) \mathrm{~I~} \\ [\infty] \mathrm{~I~} (a) \mathrm{~I~} [a,\ell] \mathrm{~I~} (a,\ell,a') \mathrm{~I~} [a,\ell,a',\ell'] \mathrm{~I~} (a,\ell,a',\ell',a'')\end{gathered} \tag{1.13}$$

for any choice of $a, \ell, a', \ell', a'', k, b, k', b', k'' \in GF(q)$. Thus incidence between a point and a line when one has fewer than five coordinates is easily recognized; for example, the lines incident with the point (a, ℓ, a') are precisely the lines $[a, \ell, a', \ell']$, where ℓ' ranges over all of $GF(q)$, together with the line $[a, \ell]$. All that remains is to be able to identify when a point and a line are incident when they each have five coordinates. This depends upon exactly how the points incident with $[\infty]$ receive their coordinates (a) and how the lines $[k]$ are labelled, too. For the assignment of these coordinates that we choose to use, the resulting coordinatization is such that a point $(a, \ell, a', \ell', a'')$ and a line $[k, b, k', b', k'']$ are incident if and only if

$$\begin{aligned} b &= -ak + a'', \\ k' &= a^3k^2 + \ell' - \ell k - 3a^2a''k - 3a'a'' + 3aa''^2, \\ b' &= a^2k + a' - 2aa'', \\ k'' &= a^3k + \ell - 3a''a^2 + 3aa'; \end{aligned} \tag{1.14a}$$

or equivalently, they are incident if and only if

$$\begin{aligned} \ell &= -a^3k + k'' - 3a^2b - 3ab', \\ a' &= a^2k + b' + 2ab, \\ \ell' &= a^3k^2 + k' + kk'' + 3a^2kb + 3bb' + 3ab^2, \\ a'' &= ak + b. \end{aligned} \tag{1.14b}$$

These relations can be found in [VM98, 3.5.1].

The relations in (1.14a) and (1.14b) contain many terms with the coefficient 3, meaning that they become considerably more simple when the underlying field has characteristic 3, reflecting the special rôle played by these fields as indicated by Theorem 1.20. Explicitly stated, we have that when $q = 3^h$, a point $(a, \ell, a', \ell', a'')$ and a line $[k, b, k', b', k'']$ are incident if and only if

$$\begin{aligned} b &= -ak + a'', \\ k' &= a^3k^2 + \ell' - \ell k, \\ b' &= a^2k + a' + aa'', \\ k'' &= a^3k + \ell; \end{aligned} \tag{1.15a}$$

POINTS	
Coordinates in $H(q)$	Coordinates in $PG(6,q)$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,ℓ,a')	$(-\ell-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-kb')$
(a,ℓ,a',ℓ',a'')	$(-a\ell'+a'^2+\ell a''+aa'a'',-a'',-a,-a'+aa'',$ $1,\ell+2aa'-a^2a'',-\ell'+a'a'')$
LINES	
Coordinates in $H(q)$	Coordinates in $PG(6,q)$
$[\infty]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,0,1)\rangle$
$[k]$	$\langle(1,0,0,0,0,0,0),(0,0,0,0,0,1,-k)\rangle$
$[a,\ell]$	$\langle(a,0,0,0,0,0,1),(-\ell,1,0,-a,0,a^2,0)\rangle$
$[k,b,k']$	$\langle(b,0,0,0,0,1,-k),(k',k,1,b,0,0,b^2)\rangle$
$[a,\ell,a',\ell']$	$\langle(-\ell-aa',1,0,-a,0,a^2,-a'),$ $(-a\ell'+a'^2,0,-a,-a',1,\ell+2aa',-\ell')\rangle$
$[k,b,k',b',k'']$	$\langle(k'+bb',k,1,b,0,b',b^2-kb'),$ $(b'^2+bk'',-b,0,-b',1,k'',-kk''-k'-2bb')\rangle$

Table 1.5: Coordinatization of $H(q)$

or equivalently, they are incident if and only if

$$\begin{aligned}\ell &= -a^3k+k'',\\ a' &= a^2k+b'-ab,\\ \ell' &= a^3k^2+k'+kk'',\\ a'' &= ak+b.\end{aligned}\tag{1.15b}$$

Consider the standard representation of $H(q)$ in the quadric $\mathcal{P}_6$ as derived in Section 1.4.1. For a particular choice of hat-rack (without loss of generality) and a subsequent choice for the points (a) and lines $[k]$ that leads to the relations of (1.14a) and (1.14b) for determining incidence, the resulting correspondence between coordinates in $H(q)$ and coordinates in $PG(6,q)$ is given in Table 1.5. This table is quoted from [VM98, 3.5.1]. Notice that the line determined by points x and y in $PG(6,q)$ is indicated in the table by $\langle x,y\rangle$. Also, no confusion can arise between coordinates for points in $H(q)$ and coordinates in $PG(6,q)$ as in the coordinatization of $H(q)$, a point has at most five coordinates, while in $PG(6,q)$, coordinates are always seven in number.

From [VM98, 3.2.4], the relations in (1.14a) and (1.14b) for incidence determine the generalized hexagon $H(q)$ in a unique way. Consequently, since we will always be using a coordinatization for which these are the appropriate relations, we can, and will, always assume that the correspondence between coordinates in $H(q)$ and coordinates in $PG(6,q)$ is actually the one given in Table 1.5.

Now let us consider the problem of identifying when two elements are opposite. The points opposite (∞), and similarly the lines opposite $[\infty]$, are exactly those with five coordinates. Since we will be concerned primarily with sets of mutually opposite elements and we can always choose our hat-rack in such a manner that one of these elements is the one labelled with ∞, it will suffice for us to consider only the problem of identifying when two elements, each with five coordinates, are opposite.

Lemma 1.21
The two points $(a, \ell, a', \ell', a'')$ and (A, L, A', L', A'') of $H(q)$ are opposite if and only if

$$(a''\Delta a + \Delta a')(A''\Delta a + \Delta a') - \Delta a\Delta\ell' + \Delta a''\Delta\ell - 3\Delta a(a'A'' - a''A') \neq 0,$$

where $\Delta x = x - X$.

Proof We use coordinates in $PG(6, q)$ as given by Table 1.5. Then the corresponding coordinates for the points $(a, \ell, a', \ell', a'')$ and (A, L, A', L', A'') are

$$(-a\ell' + a'^2 + \ell a'' + aa'a'', -a'', -a, -a' + aa'', 1, \ell + 2aa' - a^2a'', -\ell' + a'a'')$$

and

$$(-AL' + A'^2 + LA'' + AA'A'', -A'', -A, -A' + AA'', 1, L + 2AA' - A^2A'', -L' + A'A''),$$

respectively. From the equation for $\mathcal{P}_6$ given in (1.9), its associated bilinear form is

$$b(x, y) = x_0y_4 + x_4y_0 + x_1y_5 + x_5y_1 + x_2y_6 + x_6y_2 - 2x_3y_3. \tag{1.16}$$

By Theorem 1.13, points are opposite in $H(q)$ if and only if they are not collinear in $\mathcal{P}_6$ and this, in turn, corresponds to the bilinear form b not being equal to zero. So substituting their coordinates into b, the given points are opposite if and only if

$$\begin{aligned} - a\ell' + a'^2 + \ell a'' + aa'a'' - AL' + A'^2 + LA'' + AA'A'' \\ - a''L - 2a''AA' + a''A^2A'' - \ell A'' - 2aa'A'' + a^2a''A'' \\ + aL' - aA'A'' + \ell'A - a'a''A - 2(a' - aa'')(A' - AA'') \neq 0 \end{aligned}$$

which, after rearrangement, is the inequality in the statement of the lemma. □

Lemma 1.22
The two lines $[k, b, k', b', k'']$ and $[K, B, K', B', K'']$ of $H(q)$ are opposite if and only if

$$\begin{aligned} (\Delta b^2 - \Delta k\Delta b')(\Delta b'^2 + \Delta b\Delta k'') - \\ (-k''\Delta k - \Delta k' + \Delta b\Delta b' - 3b'\Delta b)(K''\Delta k + \Delta k' + \Delta b\Delta b' + 3B'\Delta b) \neq 0, \end{aligned}$$

where $\Delta x = x - X$.

Proof We use coordinates in $PG(6,q)$ as given by Table 1.5. Then the line $[k,b,k',b',k'']$ is generated by the two points

$$(k'+bb',k,1,b,0,b',b^2-kb'),$$
$$(b'^2+bk'',-b,0,-b',1,k'',-kk''-k'-2bb').$$

Using the bilinear form associated with $\mathcal{P}_6$ as given in (1.16), the perp of this line with respect to $\mathcal{P}_6$ is then given by the equations

$$\begin{array}{rrrrrrrl} & b'X_1 & +(b^2-kb')X_2 & -2bX_3 & +(k'+bb')X_4 & +kX_5 & +X_6 & =0,\\ X_0 & +k''X_1 & +(-kk''-k'-2bb')X_2 & +2b'X_3 & +(b'^2+bk'')X_4 & -bX_5 & & =0. \end{array} \tag{1.17}$$

Similarly, the line $[K,B,K',B',K'']$ is generated by the two points

$$(K'+BB',K,1,B,0,B',B^2-KB'),$$
$$(B'^2+BK'',-B,0,-B',1,K'',-KK''-K'-2BB'),$$

or equivalently, is given by the system of equations

$$\begin{array}{rrrrrrrl} X_0 & & -(K'+BB')X_2 & & -(B'^2+BK'')X_4 & & & =0,\\ & X_1 & -KX_2 & & +BX_4 & & & =0,\\ & & -BX_2 & +X_3 & +B'X_4 & & & =0,\\ & & -B'X_2 & & -K''X_4 & +X_5 & & =0,\\ & & -(B^2-KB')X_2 & & -(-KK''-K'-2BB')X_4 & & +X_6 & =0. \end{array} \tag{1.18}$$

Now by Theorem 1.15, the given lines are opposite in $H(q)$ if and only if the equations in (1.17) and (1.18) have no nonzero solution in common. This is equivalent to the condition that the matrix

$$\begin{pmatrix} 0 & b' & b^2-kb' & -2b & k'+bb' & k & 1\\ 1 & k'' & -kk''-k'-2bb' & 2b' & b'^2+bk'' & -b & 0\\ 1 & 0 & -K'-BB' & 0 & -B'^2-BK'' & 0 & 0\\ 0 & 1 & -K & 0 & B & 0 & 0\\ 0 & 0 & -B & 1 & B' & 0 & 0\\ 0 & 0 & -B' & 0 & -K'' & 1 & 0\\ 0 & 0 & -B^2+KB' & 0 & KK''+K'+2BB' & 0 & 1 \end{pmatrix}$$

should be nonsingular. Performing the row operations $R_1 := R_1 - b'R_4 + 2bR_5 - kR_6 - R_7$ and $R_2 := R_2 - R_3 - k''R_4 - 2b'R_5 + bR_6$ reveals that this, in turn, is equivalent to the 2×2 determinant

$$\begin{vmatrix} \Delta b^2 - \Delta k\Delta b' & K''\Delta k + \Delta k' + \Delta b\Delta b' + 3B'\Delta b\\ -k''\Delta k - \Delta k' + \Delta b\Delta b' - 3b'\Delta b & \Delta b'^2 + \Delta b\Delta k'' \end{vmatrix}$$

being nonzero. Expanding this determinant gives the desired result. □

Now we identify, in terms of coordinates, the line reguli and point reguli of $H(q)$ that contain the line $[\infty]$ and the point (∞), respectively. Let us consider the line reguli first.

Let $L = [k, b, k', b', k'']$ be a line opposite $[\infty]$. From (1.13) and (1.14b), the points (k, b) and $(0, k'', b')$ are the distance-2 projections of L onto the points (∞) and (0), respectively, which are incident with $[\infty]$. Similarly, the distance-2 projections of $M = [K, B, K', B', K'']$ onto (∞) and (0) are (K, B) and $(0, K'', B')$. Now these two lines L and M determine the same line regulus with $[\infty]$ if and only if their distance-3 traces onto $[\infty]$, the complementary point reguli, are equal. Since a point regulus is uniquely determined by just two of its points, this in turn corresponds to $(k, b) = (K, B)$ and $(0, k'', b') = (0, K'', B')$. Thus, representing the line regulus determined by $[\infty]$ and L by $[[k, b, b', k'']]$, we have

$$[[k,b,b',k'']] = \mathcal{R}\big([\infty],[k,b,k',b',k'']\big) = \big\{[\infty]\big\} \cup \big\{[k,b,x,b',k''] \,|\, x \in GF(q)\big\}. \tag{1.19}$$

Now let $(a, \ell, a', \ell', a'')$ be a point opposite (∞). From (1.13) and (1.14a), the distance-2 projections of this point onto the lines $[\infty]$ and $[0]$, which are incident with (∞), are $[a, \ell]$ and $\big[0, a'', \ell' - 3a'a'' + 3aa''^2\big]$, respectively. Similarly to the case for lines, we then have that this point and another point $[A, L, A', L', A'']$ determine the same point regulus with (∞) if and only if $A = a$, $L = \ell$, $A'' = a''$ and $L' - 3A'A'' + 3AA''^2 = \ell' - 3a'a'' + 3aa''^2$. Substituting the first three of these into the fourth and solving for L', we have $L' = \ell' + 3(A' - a')a''$. Representing the point regulus determined by (∞) and $(a, \ell, 0, \ell', a'')$ by $(\!(a, \ell, \ell', a'')\!)$, we then have

$$(\!(a,\ell,\ell',a'')\!) = \big\{(\infty)\big\} \cup \big\{(a,\ell,x,\ell'+3a''x,a'') \,|\, x \in GF(q)\big\}, \tag{1.20}$$

and

$$\mathcal{R}\big((\infty),(a,\ell,a',\ell',a'')\big) = (\!(a,\ell,\ell'-3a'a'',a'')\!).$$

1.4.4 Morphisms

With the coordinatization of $H(q)$ introduced in the previous section, together with the natural embedding of $H(q)$ in the quadric $\mathcal{P}_6$ as seen in Section 1.4.1, we are now in a position to explicitly describe its automorphisms and anti-automorphisms.

To begin, since there is a unique minimum length x–y path when $d(x, y) \leq 5$, if we are given the images of the two elements x and y under the action of some automorphism or anti-automorphism g, then we know the images of all the other elements on the path. Consequently, if g fixes the flag $\{(\infty), [\infty]\}$, it suffices for us to just explicitly state the action of g on elements with five coordinates; the images of the remaining elements then follow from (1.13). When we give the action of an automorphism or anti-automorphism just by stating its action on elements with five coordinates, it will be understood that the flag $\{(\infty), [\infty]\}$ is fixed so the entire action does indeed follow.

By [VM98, Proposition 4.6.6], each collineation of $H(q)$ is induced by a unique semilinear transformation of the projective space $PG(6, q)$ via the natural embedding of $H(q)$ in $\mathcal{P}_6$ discussed in Section 1.4.1. Also, the group $\mathrm{Aut}(H(q)) \cap \mathrm{PGL}(7, q)$ of those collineations of $H(q)$ that are induced by linear transformations of $PG(6, q)$ is the Chevalley group $G_2(q)$, known as Dickson's group.[10] By [VM98, Theorem 8.3.2(i)], this automorphism group $G_2(q)$ is the group generated by all the elations of $H(q)$.

[10]In fact, the split Cayley hexagon belongs to the root system of type G_2, in the sense of [VM98, 5.4.1] (see [VM98, Theorem 5.4.6]), and in the literature it is also known as the G_2-hexagon.

Let Σ be an apartment of $H(q)$ and let G be the group generated by all elations corresponding to paths in Σ. Consider an arbitrary path γ of length 4 in $H(q)$ and let the set S be defined as in the definition of Moufang paths on page 7. By [VM98, Lemma 5.2.8], the group G contains a subgroup of γ-elations acting transitively on S. But we have seen that the group $E(\gamma)$ of *all* γ-elations acts semiregularly on S (this actually follows from Lemma 1.7), so $E(\gamma) \leqslant G$. Since γ was arbitrary, it follows that G contains all elations. Thus it is actually sufficient to take just the elations corresponding to the paths on a fixed apartment in order to generate the automorphism group $G_2(q)$.

Let g be a collineation of $H(q)$ in $G_2(q)$. Then g is induced by a unique linear transformation of $PG(6,q)$ and we denote the matrix that represents it by $[g]$ (remembering that we always assume the correspondence between coordinates of $H(q)$ and coordinates of $PG(6,q)$ to be the one given by Table 1.5). We will also refer to $[g]$ as the matrix corresponding to g. In the following, morphisms of $H(q)$ act on the right, while points of $PG(6,q)$ are considered as being represented by 7×1 matrices so the matrix multiplication is carried out on their left. Consequently, if $f, g \in G_2(q)$ are two collineations of $H(q)$ with corresponding matrices $[f]$ and $[g]$, respectively, then the composition fg of f followed by g is induced by the linear transformation represented by the matrix $[fg] = [g][f]$.

Given the action of a collineation g, to obtain the matrix $[g]$ we first identify the images of the six points of the hat-rack, which give all but the central column of $[g]$ up to multiplication by nonzero factors. The remaining column is then obtained by considering the image of some other point like $(-1,0,0)$, whose coordinates in $PG(6,q)$ are $(0,1,0,1,0,1,0)$. Finally, the actual factors by which the columns are multiplied is determined by considering the action on one more appropriately chosen[11] point, such as $(0,1)$, and keeping in mind that the equations in (1.9) and (1.10) must remain satisfied.

Conversely, if we are given a matrix $[g]$ corresponding to a collineation g of $H(q)$ that fixes the flag $\{(\infty),[\infty]\}$, then we can easily write down the action of g explicitly in terms of the coordinates of $H(q)$. As remarked earlier, since the flag $\{(\infty),[\infty]\}$ is fixed, it suffices for us to consider only the elements with five coordinates. For a point (a,ℓ,a',ℓ',a''), we multiply its coordinates in $PG(6,q)$ by the matrix $[g]$ to obtain the coordinates $(x_0,x_1,x_2,x_3,1,x_5,x_6)$ in $PG(6,q)$ of its image. Then in the coordinates of $H(q)$, the image is

$$(-x_2, x_5 + x_1x_2^2 - 2x_2x_3, -x_3 + x_1x_2, -x_6 - x_1^2x_2 + x_1x_3, -x_1).$$

For a line $[k,b,k',b',k'']$, we obtain the first four coordinates of the image by identifying the image of the point (k,b,k',b') as in the previous paragraph. Then to obtain the fifth coordinate, we consider the projection $[0,k'']$ of $[k,b,k',b',k'']$ onto the point (0), whose image we already know from the previous paragraph.

Considering the Plücker coordinate conditions in (1.10), the linear transformation of

[11] By "appropriately chosen" we mean that the two points considered in addition to the points of the hat-rack must be such that a collineation stabilizing the hat-rack elementwise is uniquely determined by the images of these two points. The explicit forms of such collineations is given in (1.21).

$PG(6,q)$ given by the matrix[12]

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is readily checked to preserve the set of lines of $H(q)$ and so induces a collineation μ. From Table 1.5, we see that μ fixes the apartment Σ that is the hat-rack of the coordinatization, while mapping the flag $\{(\infty), [0]\}$ to the flag $\{(0), [\infty]\}$. This is a 60° clockwise rotation of the hat-rack as illustrated in Figure 1.4. Now, together with μ, we will have a complete set of generators of $G_2(q)$ once we have the point elations corresponding to one path in Σ and the line elations corresponding to another, since all other elations corresponding to paths in Σ are obtained from these through conjugation by μ. This is what we do now, quoting the following elations and their corresponding matrices from [VM98, Appendix D].

Let γ_P be the path $(\infty), [\infty], (0), [0,0], (0,0,0)$ in Σ. The elations $E(\gamma_P, \delta)$ corresponding to this path are given by

$$E(\gamma_P, \delta) : \begin{cases} (a, \ell, a', \ell', a'') \mapsto (a, \ell - 3a^2\delta, a' + 2a\delta, \ell' + 3a\delta^2 + 3a'\delta, a'' + \delta) \\ [k, b, k', b', k''] \mapsto [k, b + \delta, k', b', k''], \end{cases}$$

where δ is any element of $GF(q)$. The corresponding matrices $[E(\gamma_P, \delta)]$ are

$$M_P(\delta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 1 & 0 & 0 & -\delta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \delta^2 & 2\delta & 0 & 0 & 1 \end{pmatrix}.$$

Let γ_L be the path $[\infty], (\infty), [0], (0,0), [0,0,0]$ in Σ. The elations $E(\gamma_L, \delta)$ corresponding to this path are given by

$$E(\gamma_L, \delta) : \begin{cases} (a, \ell, a', \ell', a'') \mapsto (a, \ell + \delta, a', \ell', a'') \\ [k, b, k', b', k''] \mapsto [k, b, k' - k\delta, b', k'' + \delta], \end{cases}$$

[12] This is a correction of the matrix provided in [VM98, Appendix D].

where δ is any element of $GF(q)$. The corresponding matrices $[E(\gamma_L, \delta)]$ are

$$M_L(\delta) = \begin{pmatrix} 1 & -\delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let γ be a length 4 path in Σ so $\gamma = \gamma_X \mu^k$, for some integer k and where X is either the symbol P or L. Then the γ-elations are $E(\gamma, \delta) = \mu^{-k} E(\gamma_X, \delta)\mu^k$, which have corresponding matrices $[E(\gamma, \delta)] = M^k M_X(\delta) M^{-k}$.

It will also be convenient to have the explicit form of homologies at our disposal. All the homologies for elements u and v, where $\{u, v\}$ ranges over all pairs of opposite elements in the hat-rack Σ, generate an automorphism group that leaves Σ fixed elementwise. In fact, by [VM98, Proposition 4.6.6(v)], this automorphism group is the elementwise stabilizer in $G_2(q)$ of Σ, also called the **torus** in $G_2(q)$ for Σ. Quoting again from [VM98, Appendix D], a general element of the torus in $G_2(q)$ for Σ is given by

$$\mathcal{T}(\alpha, \beta) : \begin{cases} (a, \ell, a', \ell', a'') \mapsto (\alpha a, \alpha^3\beta\ell, \alpha^2\beta a', \alpha^3\beta^2\ell', \alpha\beta a'') \\ [k, b, k', b', k''] \mapsto [\beta k, \alpha\beta b, \alpha^3\beta^2 k', \alpha^2\beta b', \alpha^3\beta k''], \end{cases} \tag{1.21}$$

where $\alpha, \beta \in GF(q)$ are nonzero elements of the field, and the corresponding matrix for $\mathcal{T}(\alpha, \beta)$ is

$$M_T(\alpha, \beta) = \begin{pmatrix} \alpha^4\beta^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^3\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha^3\beta^2 \end{pmatrix}.$$

While the collineations $E(\gamma_X, \delta)$ and μ generate the group $G_2(q)$ of collineations that are induced by projective linear transformations, we know from [VM98, Proposition 4.6.6] that we obtain the full automorphism group $\mathrm{Aut}(H(q))$ once we also include the collineations induced by automorphisms of the field $GF(q)$. Specifically, for each field automorphism ψ, there is a collineation of $H(q)$, which we shall also call ψ, whose action is given by

$$\psi : \begin{cases} (a, \ell, a', \ell', a'') \mapsto (a^\psi, \ell^\psi, a'^\psi, \ell'^\psi, a''^\psi) \\ [k, b, k', b', k''] \mapsto [k^\psi, b^\psi, k'^\psi, b'^\psi, k''^\psi]. \end{cases}$$

This is readily seen to preserve incidence as given in (1.13), (1.14a) and (1.14b), so this is indeed a collineation.

Now we are in a position to see the truth of the "if" parts of Theorem 1.20 by explicitly showing a duality and a polarity for the relevant cases. Suppose $q = 3^h$ so incidence in $H(q)$ is given by (1.13) together with (1.15a) and (1.15b). Also, the map $\sigma : x \mapsto x^3$ is an automorphism of $GF(q)$ called the **Frobenius automorphism**. Define a map φ of $H(q)$ by

$$\varphi : \begin{cases} (a, \ell, a', \ell', a'') \mapsto [a^3, \ell, a'^3, \ell', a''^3] \\ [k, b, k', b', k''] \mapsto (k, b^3, k', b'^3, k''). \end{cases} \tag{1.22}$$

Incidence as given in (1.13) is certainly preserved so all that remains is to consider incidence between points and lines with five coordinates. Now the point $(a, \ell, a', \ell', a'')$ and the line $[k, b, k', b', k'']$ are incident if and only if the conditions in (1.15a) are satisfied. Applying the Frobenius automorphism to the first and third equations there, we then have that these elements are incident if and only if

$$\begin{aligned} (b^3) &= -k^3(a^3) + (a''^3), \\ k' &= k^2(a^3) + \ell' - k\ell, \\ (b'^3) &= k^3(a^3)^2 + (a'^3) + (a^3)(a''^3), \\ k'' &= k(a^3) + \ell. \end{aligned}$$

But these are precisely the conditions in (1.15b) for the line $[a^3, \ell, a'^3, \ell', a''^3]$ and the point (k, b^3, k', b'^3, k'') to be incident. Thus the map φ preserves incidence and it is therefore a duality of $H(q)$. Notice that $\varphi^2 = \sigma$, the collineation induced by the Frobenius automorphism σ of $GF(q)$.

Now suppose that $q = 3^{2e+1}$. The map $\theta : x \mapsto x^{3^{e+1}}$ is then an automorphism of $GF(q)$, called the **Tits automorphism**, that satisfies $\theta^2 = \sigma$. The collineation of $H(q)$ induced by θ then satisfies $\theta^{-2}\varphi^2 = 1$. Thus we compose the inverse with φ to obtain the correlation

$$\rho = \varphi\theta^{-1} = \theta^{-1}\varphi : \begin{cases} (a, \ell, a', \ell', a'') \mapsto [a^\theta, \ell^{\theta^{-1}}, a'^\theta, \ell'^{\theta^{-1}}, a''^\theta] \\ [k, b, k', b', k''] \mapsto (k^{\theta^{-1}}, b^\theta, k'^{\theta^{-1}}, b'^\theta, k''^{\theta^{-1}}), \end{cases} \tag{1.23}$$

which then has order 2 and is therefore a polarity.

Finally, we identify certain subgroups of $G_2(q)$.

Let $\{x, y\}$ be a flag in $H(q)$. For each $z \neq x$ incident with y, let

$$U_z = \{w \in H(q) \mid d(w, x) = 6 \text{ and } w \triangleright y = z\}$$

be the set of elements opposite x and at distance 4 from z. We call this the ***z*-projection set** for the flag $\{x, y\}$. Since each element w opposite x has a unique projection onto y, the z-projection sets partition the set $\Gamma_6(x)$ of elements opposite x. In addition, since $H(q)$ is a Moufang hexagon, the root group corresponding to a length 4 path $x, y', \ldots$, with $y' \neq y$, acts transitively on the set of elements $z \neq x$ incident with y, so the z-projection sets all have the same size. Finally, since there are q elements $z \neq x$ incident with y and q^5 elements opposite x, we have $|U_z| = q^4$.

For a flag $\{x, y\}$ in $H(q)$, let $G^{\{x,y\}}$ be the group of collineations that fix both elements of the flag elementwise. For an element x of $H(q)$, let $G^x = \langle G^{\{x,y\}} \mid y \,\mathrm{I}\, x \rangle$ be the group generated by all collineations that fix x, as well as some element incident with x, elementwise.

The following result concerning these groups is due to Weiss. Here we state it in our current context.

Theorem 1.23 (Weiss [Wei79, Lemmas 1 and 2])
Let $\{x, y\}$ be a flag and let γ be a length 4 path $x, z, \dots$ with $z \neq y$. Then the groups $E(\gamma)$ and $G^{\{x,y'\}}$, with $y' \neq y$, and therefore G^x as well, all induce the same regular permutation group on the set $\Gamma_1(y) \setminus \{x\}$ of elements distinct from x that are incident with y. □

Let $x_0, x_1, \dots, x_8$ be a path with $(x_4, x_5) = (x, y)$, and for $0 \le i \le 4$, let γ_i be the path $x_i, x_{i+1}, \dots, x_{i+4}$. Then $G^{\{x,y\}} = E(\gamma_1)E(\gamma_2)E(\gamma_3)E(\gamma_4)$ by [VM98, Lemma 5.2.3(i)]. Also, by [VM98, Lemma 5.2.4(ii)], $H = E(\gamma_0)G^{\{x,y\}}$ is a group that acts regularly on the set $\Gamma_6(x)$ of elements opposite x. Notice that the root group $E(\gamma_0)$ is a subgroup of $G^{\{x,x_3\}}$, which is one of the generators of G^x, so $H \leqslant G^x$.

Now suppose $g \in G^x$ fixes some element w opposite x. Since x is fixed elementwise, every apartment on x and w is fixed. In particular, the element $w \triangleright y$ is fixed, so by Theorem 1.23, y is fixed elementwise, and then from Lemma 1.7, g is the identity. Thus G^x acts semiregularly on $\Gamma_6(x)$. It now follows that $G^x = H$ and $|G^x| = q^5$. In addition, the group $G^{\{x,y\}}$, as a subgroup of G^x, acts regularly on U_z and $|G^{\{x,y\}}| = q^4$.

Now we explicitly identify the groups when $\{x, y\}$ is the flag $\{(\infty), [\infty]\}$. To begin, we have just seen that

$$G^{(\infty)} = E(\gamma_P\mu^{-2})G^{\{(\infty),[\infty]\}} = E(\gamma_P\mu^{-2})E(\gamma_L)E(\gamma_P\mu^{-1})E(\gamma_L\mu)E(\gamma_P)$$

and

$$G^{[\infty]} = E(\gamma_L\mu^2)G^{\{(\infty),[\infty]\}} = E(\gamma_L\mu^2)E(\gamma_P)E(\gamma_L\mu)E(\gamma_P\mu^{-1})E(\gamma_L).$$

Thus a general element g of $G^{(\infty)}$ has corresponding matrix

$$[g] = M_P(w)\big(MM_L(y)M^{-1}\big)\big(M^{-1}M_P(v)M\big)M_L(x)\big(M^{-2}M_P(u)M^2\big)$$
$$= \begin{pmatrix} 1 & u^2w - x - 2uv & y + vw & 2uw - 2v & v^2 + uy + wx + uvw & w & -u \\ 0 & 1 & 0 & 0 & -w & 0 & 0 \\ 0 & 0 & 1 & 0 & u & 0 & 0 \\ 0 & u & w & 1 & uw - v & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & u^2 & v & 2u & x + uv & 1 & 0 \\ 0 & 2uw - v & w^2 & 2w & uw^2 - y - 2vw & 0 & 1 \end{pmatrix}$$

where u, v, w, x and y are arbitrary elements from $GF(q)$. From here, the action of g is found to be

$$\begin{aligned}(a, \ell, a', \ell', a'') \mapsto (&a - u, \ell + x - 3av + 3uv - 3a^2w - 3u^2w + 6auw,\\ &a' + v + 2aw - 2uw, \ell' + y + 3aw^2 - 3uw^2 + 3a'w + 3vw, a'' + w)\\ [k, b, k', b', k''] \mapsto [&k, b + w + ku, k' + y - kx - k^2u^3 - 3b^2u - 3bku^2 - 3bv - 3kuv,\\ &b' + v + ku^2 + 2bu, k'' + x - ku^3 - 3bu^2 - 3b'u].\end{aligned}$$

Since $G^{(\infty)}$ acts regularly on the set of points opposite (∞), for each point (A, L, A', L', A'') there is a unique collineation in $G^{(\infty)}$ that maps the point $(0,0,0,0,0)$ onto it. We denote this collineation by $\Psi(A, L, A', L', A'')$. From the action given above, we find $u = -A$, $w = A''$, $v = A' - 2AA''$, $x = L + 3AA' - 3A^2A''$ and $y = L' + 3AA''^2 - 3A'A''$ so the action of $\Psi(A, L, A', L', A'')$ is finally given by

$$\begin{aligned}
\big(a,\ell,a',\ell',a''\big) &\mapsto \big(a+A, \ell+L-3aA'-3a^2A'', a'+A'+2aA'', \\
&\qquad \ell'+L'+3a'A''+3aA''^2, a''+A''\big) \\
\big[k,b,k',b',k''\big] &\mapsto \big[k, b+A''-kA, k'+L'-kL+k^2A^3-3bA'+3b^2A-3bkA^2-3A'A'' \\
&\qquad -3kA^2A''+3AA''^2+6bAA'', b'+A'+kA^2-2bA-2AA'', \\
&\qquad k''+L+kA^3-3bA^2+3b'A-3A^2A''+3AA'\big].
\end{aligned} \tag{1.24}$$

Similarly, a general element h of $G^{[\infty]}$ has corresponding matrix

$$\begin{aligned}
[h] &= M_L(z)\big(M^{-1}M_P(v)M\big)\big(MM_L(y)M^{-1}\big)M_P(u)\big(M^2M_L(x)M^{-2}\big) \\
&= \begin{pmatrix}
1 & -z & y+xz-2uv & -2v & v^2+uz & u & 0 \\
0 & 1 & -x & 0 & -u & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & u & 1 & -v & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & v & 0 & z & 1 & 0 \\
0 & -v & u^2+vx & 2u & uv-y & x & 1
\end{pmatrix}
\end{aligned}$$

where u, v, x, y and z are arbitrary elements from $GF(q)$. Following the notation of [BTVM98], we denote the unique element of $G^{[\infty]}$ that maps the line $[0,0,0,0,0]$ to the line $[K, B, K', B', K'']$ by $\Theta[K, B, K', B', K'']$. Then, from the matrix above, we find $x = -K$, $u = B$, $v = B'$, $z = K''$ and $y = K' + KK'' + 3BB'$ and the action of $\Theta[K, B, K', B', K'']$ is given by

$$\begin{aligned}
\big(a,\ell,a',\ell',a''\big) &\mapsto \big(a, \ell+K''-a^3K-3a^2B-3aB', a'+B'+a^2K+2aB, \\
&\qquad \ell'+K'+KK''+\ell K+a^3K^2+3BB'+3aa'K+3aB^2 \\
&\qquad +3a'B+3a^2KB, a''+B+aK\big) \\
\big[k,b,k',b',k''\big] &\mapsto \big[k+K, b+B, k'+K'-kK''-3bB', b'+B', k''+K''\big].
\end{aligned} \tag{1.25}$$

Finally, the group $G^{\{(\infty),[\infty]\}}$ is

$$\begin{aligned}
G^{\{(\infty),[\infty]\}} &= \big\{\Psi(0, L, A', L', A'') \;\big|\; L, A', L', A'' \in GF(q)\big\} \\
&= \big\{\Theta[0, B, K', B', K''] \;\big|\; B, K', B', K'' \in GF(q)\big\},
\end{aligned}$$

where $\Psi(0, L, A', L', A'') = \Theta[0, A'', L' - 3A'A'', A', L]$.

Chapter 2

Spreads and Ovoids

In this chapter, spreads and ovoids of a generalized $2m$-gon are defined and some general facts about them are given. Following that, our attention is restricted to the case of $H(q)$ where there is a correspondence with certain m-systems of the embedding quadric. For ease of investigation, spreads and ovoids are represented in terms of coordinates and then certain known ones are described.

2.1 General introduction

The general idea behind spreads and ovoids is that of trying to select elements such that they are spread out over the geometry and as far apart from each other as possible. The farthest that two elements of a generalized $2m$-gon can be from each other is opposite each other, so this is what we require. Next, in order that these elements be, in some sense, spread out well over the whole geometry we also ask that no other element of the geometry should be too far away from the spread or ovoid. Recall that this motivation is the same as that which underlies m'-systems of polar spaces and, in fact, for appropriate values of m' these were called spreads and ovoids as well. It is perhaps not surprising then that there is a connection between the spreads and ovoids of $H(q)$ and the m'-systems of the underlying quadric $\mathcal{P}_6$. We now state matters more precisely.

A **spread** of a generalized $2m$-gon Γ is a set $\mathcal{S}$ of mutually opposite lines such that each element of Γ is at distance at most m from some line of $\mathcal{S}$. Dually, an **ovoid** of a generalized $2m$-gon Γ is a set $\mathcal{O}$ of mutually opposite points such that each element of Γ is at distance at most m from some point of $\mathcal{O}$.

Rather than always referring back to this definition when dealing with spreads and ovoids, the following two results provide useful characterizations of them. Although stated specifically for spreads, these apply just as well for ovoids by dualization.

Lemma 2.1 ([VM98, Lemma 7.2.2])
Let $\mathcal{S}$ be a set of lines of a generalized $2m$-gon Γ. Then $\mathcal{S}$ is a spread if and only if, for m even, every point, and for m odd, every line of Γ lies at distance less than m from a unique element of $\mathcal{S}$. Moreover, if the lines in $\mathcal{S}$ are mutually opposite, then we can replace "a unique" with "some". □

Lemma 2.2 ([VM98, **Proposition 7.2.3**])
A set $\mathcal{S}$ of mutually opposite lines in a finite generalized $2m$-gon of order s is a spread if and only if $|\mathcal{S}| = s^m + 1$. □

Let Γ be a generalized $2m$-gon of order s and let $\mathcal{S}$ be a spread of Γ. Let $L \in \mathcal{S}$ be some fixed line of $\mathcal{S}$ and let $x \mathrm{I} L$ be a point incident with it. We will write $\mathcal{S}^+$ for the set $\mathcal{S} \setminus \{L\}$. Now for elements z of Γ such that $d = d(x,z) \leq m$ and $z \triangleright x \neq L$, let

$$V_z = V_z(L,x) = \left\{K \in \mathcal{S}^+ \;\middle|\; K \triangleright_d x = z\right\} \tag{2.1}$$

be the set of lines in $\mathcal{S}$ whose minimum length paths to x pass via z. If $z = x$ then $V_z = \mathcal{S}^+$ and if $z \mathrm{I} x$ then $V_z = \mathcal{S} \cap U_z$, where U_z is the z-projection set for the flag $\{L, x\}$. We will call the set V_z the **spread projection set** for z. Dually, we define the **ovoid projection sets** V_z for an ovoid $\mathcal{O}$ and a flag $\{x, L\}$ where $x \in \mathcal{O}$. The following result and its proof, stated here for spreads, dualize for ovoids.

Lemma 2.3
Let $\mathcal{S}$ be a spread of a generalized $2m$-gon Γ of order s and let $\{L, x\}$ be a flag in Γ with $L \in \mathcal{S}$. Let z be an element of Γ with $d = d(x,z) \leq m$ and $z \triangleright x \neq L$. Then $|V_z(L,x)| = s^{m-d}$.

Proof We use a backward induction on the value of d. Notice that since all the lines $K \in V_z$ are at distance $2m-1$ from x, we have $d(K,z) = 2m-1-d$.

Suppose $d = m$. Since z is at distance $m+1$ from L, if its distance from some other line of $\mathcal{S}$ is less than m, then it must in fact be $m-1$. Consequently, V_z is exactly the set of lines of $\mathcal{S}$ at distance less than m from z. By Lemma 2.1 we now have $|V_z| = 1$.

Now suppose $d < m$ and the result holds for $d+1$. Let $A = \Gamma_1(z) \setminus \{L \triangleright z\}$ be the set of q elements $u \neq L$ incident with z that are at distance $d+1$ from x. Then $V_z = \bigcup_{u \in A} V_u$. Also, for each $K \in V_z$ the projection $K \triangleright z$ is a uniquely determined element of A since $d(K,z) < 2m$. Hence the sets V_u, for $u \in A$, form a partition of the set V_z and therefore $|V_z| = \sum_{u \in A} |V_u| = s^{m-d}$. □

Notice that this has actually provided a proof of the "only if" part of Lemma 2.2.

Consider now the generalized hexagon $H(q)$. By Lemma 2.2, spreads and ovoids of $H(q)$ contain precisely $q^3 + 1$ elements, which is also the number of elements in an m-system of the underlying quadric $\mathcal{P}_6$. In fact, m-systems of $\mathcal{P}_6$ they are, as we shall see now. For the ovoid case $m = 0$, see [Tha81], and for the spread case $m = 1$, see [ST94].

First we consider an ovoid $\mathcal{O}$ of the generalized hexagon $H(q)$. The points of $\mathcal{O}$ are mutually opposite so by Theorem 1.13, no two of them are collinear in the quadric. Thus any generator of $\mathcal{P}_6$ containing a given point of $\mathcal{O}$ does not contain any other point of $\mathcal{O}$. Since by the dual of Lemma 2.2 the set $\mathcal{O}$ is the right size, it follows that $\mathcal{O}$ is a 0-system, or an ovoid, of the quadric $\mathcal{P}_6$. Conversely, if $\mathcal{O}$ is an ovoid of $\mathcal{P}_6$ then for each point x of $H(q)$, by Theorem 1.8 its $H(q)$-plane contains a unique point of $\mathcal{O}$, or equivalently, it is at distance at most 2 in $H(q)$ from a unique point of $\mathcal{O}$. Therefore, by the dual of Lemma 2.1, $\mathcal{O}$ is an ovoid of $H(q)$.

Notice also that the $H(q)$-planes of the points of an ovoid $\mathcal{O}$ of $H(q)$ are mutually disjoint as a point common to two $H(q)$-planes would be at distance 2 from the two corresponding points of $\mathcal{O}$. Since there is the right number of them, these $H(q)$-planes form a 2-system, or a spread, of the quadric $\mathcal{P}_6$.

Now consider a spread $\mathcal{S}$ of the generalized hexagon $H(q)$. The lines of $\mathcal{S}$ are mutually opposite, so by Theorem 1.15, no generator of $\mathcal{P}_6$ on one line of $\mathcal{S}$ meets any other line of $\mathcal{S}$. Since $\mathcal{S}$ contains the right number of lines by Lemma 2.2, it follows that $\mathcal{S}$ is a 1-system of $\mathcal{P}_6$.

We summarize this in the following lemma.

Lemma 2.4
Let $\mathcal{O}$ be a set of points of $H(q)$, or equivalently, of $\mathcal{P}_6$, and let $\mathcal{S}$ be a set of lines of $H(q)$. If $\mathcal{S}$ is a spread of $H(q)$ then it is also a 1-system of the quadric $\mathcal{P}_6$. The set $\mathcal{O}$ is an ovoid of $H(q)$ if and only if it is an ovoid of $\mathcal{P}_6$. If $\mathcal{O}$ is an ovoid of $H(q)$ then the $H(q)$-planes of its points form a spread of $\mathcal{P}_6$. □

2.2 Coordinates for spreads and ovoids

Let $\mathcal{S}$ be a spread of $H(q)$ and let the hat-rack of the coordinatization be chosen such that the lines $[\infty]$ and $[0,0,0,0,0]$ are lines of $\mathcal{S}$. The lines of $\mathcal{S}$ opposite $[\infty]$ then each have five coordinates and we write $\mathcal{S}^+$ for the set of these lines. Thus $\mathcal{S} = \mathcal{S}^+ \cup \{[\infty]\}$ and we have

$$\mathcal{S}^+ = \big\{[x,y,z,f,g] \;\big|\; x,y,z,f,g \in GF(q), \text{ subject to some constraints}\big\}. \tag{2.2}$$

We can endeavour to say something about the constraints mentioned above by noting that not only are the lines $[x,y,z,f,g]$ opposite $[\infty]$, but they are also opposite each other. By Lemma 1.22, the lines $[x,y,z,f,g]$ and $[x,y,Z,F,G]$ are opposite if and only if $z \neq Z$. Thus the ordered triples (x,y,z) in (2.2) are distinct for the lines of $\mathcal{S}^+$, or in other words, the lines of $\mathcal{S}^+$ are uniquely determined by their triples (x,y,z). There are exactly q^3 such triples and exactly q^3 lines in $\mathcal{S}^+$, so it follows that every ordered triple (x,y,z) determines a unique line of the spread. Thus the spread $\mathcal{S}$ takes the form

$$\mathcal{S} = \big\{[\infty]\big\} \cup \big\{[x,y,z,f(x,y,z),g(x,y,z)] \;\big|\; x,y,z \in GF(q)\big\}, \tag{2.3}$$

where f and g are functions dependent upon x, y and z, with $f(0,0,0) = g(0,0,0) = 0$. Notice that the representation (2.3) also follows from Lemma 2.3 which tells us that the spread projection set $V_{[x,y,z]}([\infty],(\infty))$ contains exactly one element for each ordered triple (x,y,z).

Now we repeat these steps for an ovoid $\mathcal{O}$ of $H(q)$. Suppose first, without loss of generality, that the hat-rack of the coordinatization is chosen such that the points (∞) and $(0,0,0,0,0)$ are points of $\mathcal{O}$. Then setting $\mathcal{O}^+ = \mathcal{O} \setminus \{(\infty)\}$, we have

$$\mathcal{O}^+ = \big\{(x,y,z,f,g) \;\big|\; x,y,z,f,g \in GF(q), \text{ subject to some constraints}\big\}. \tag{2.4}$$

By Lemma 1.21, the points (x,y,z,f,g) and (x,Y,z,F,G) are opposite if and only if $(g-G)(y-Y) \neq 0$, thus similarly to the case for spreads, we have that the ovoid $\mathcal{O}$ takes the form

$$\mathcal{O} = \big\{(\infty)\big\} \cup \big\{(x,y,z,f(x,y,z),g(x,y,z)) \;\big|\; x,y,z \in GF(q)\big\} \tag{2.5}$$

where $f(0,0,0) = g(0,0,0) = 0$. As in the case of spreads, this also follows from Lemma 2.3.

We have just used the fact that $(g-G)(y-Y) \neq 0$ implies $y \neq Y$, but this also means of course that $g \neq G$. Thus there is also a one-to-one correspondence between the points of $\mathcal{O}^+$ and the ordered triples (x, z, g) in (2.4). After renaming the variables, an alternative coordinate representation of the ovoid $\mathcal{O}$ is then

$$\mathcal{O} = \{(\infty)\} \cup \{(x, f(x,y,z), z, g(x,y,z), y) \mid x, y, z \in GF(q)\}, \tag{2.6}$$

where again f and g are functions such that $f(0,0,0) = g(0,0,0) = 0$.

The smallest split Cayley hexagon is $H(2)$ and one might expect that its structure is sufficiently trivial that it cannot have much variety in spreads or ovoids. In fact, we know from [Tha81] that $H(2)$ does not have any ovoids. For the case of spreads, we now use coordinates for our proof of the following result from folklore.

Theorem 2.5
The generalized hexagon $H(2)$ has a unique spread up to isomorphism.

Proof Let us suppose that $\mathcal{S}$ is a spread of $H(2)$ as given by (2.3) with $f(0,0,0) = g(0,0,0) = 0$. Applying the collineation $\Psi(1,0,0,0,0)$ (see page 31), we obtain another (possibly identical) spread

$$\mathcal{S}' = \{[\infty]\} \cup \{[x, y, z, F(x,y,z), G(x,y,z)] \mid x, y, z \in GF(2)\},$$

where $F(x,y,z) = f(x, x+y, z+x+y+xy) + x$ and $G(x,y,z) = f(x, x+y, z+x+y+xy) + g(x, x+y, z+x+y+xy) + y$. In particular, $F(0,0,0) = G(0,0,0) = 0$ also so whatever properties we find must be satisfied by $\mathcal{S}$, f and g must also be satisfied by $\mathcal{S}'$, F and G.

From Lemma 1.22 taken in the context of $GF(2)$, the condition for two lines $[k, b, k', b', k'']$ and $[K, B, K', B', K'']$ to be opposite is

$$(\Delta b + \Delta k \Delta b')(\Delta b' + \Delta b \Delta k'') + (\Delta k' + k'' \Delta k + B' \Delta b)(\Delta k' + K'' \Delta k + b' \Delta b) = 1. \tag{2.7}$$

Let us refer to the line $[x, y, z, f(x,y,z), g(x,y,z)]$ of $\mathcal{S}$ as $\ell(x,y,z)$. The lines $\ell(1,0,0)$, $\ell(0,1,0)$ and $\ell(1,1,0)$ are opposite $\ell(0,0,0)$ so we use the condition in (2.7) to obtain

$$f(1,0,0) = 1, \tag{2.8}$$
$$f(0,1,0) + g(0,1,0) = 1, \tag{2.9}$$
$$g(1,1,0) = 1. \tag{2.10}$$

Since these must also hold true for F and G, we then have

$$f(1,1,1) = 0, \tag{2.11}$$
$$g(0,1,1) = 0, \tag{2.12}$$
$$f(1,0,1) + g(1,0,1) = 0. \tag{2.13}$$

The lines $\ell(1,0,0)$ and $\ell(0,1,0)$ are opposite so, using (2.8) and (2.9),

$$f(0,1,0) = 1 \quad \text{and} \quad g(0,1,0) = 0, \tag{2.14}$$

and by considering $F(0,1,0) = 1$ we also have

$$f(0,1,1) = 1. \tag{2.15}$$

The lines $\ell(1,0,0)$ and $\ell(0,1,1)$ are opposite so, using (2.8), (2.12) and (2.15),

$$g(1,0,0) = 1, \tag{2.16}$$

and then also $G(1,0,0) = 1$ so, using (2.11), we have

$$g(1,1,1) = 1. \tag{2.17}$$

Considering the lines $\ell(1,1,0)$ and $\ell(0,1,1)$ and using (2.10), (2.12) and (2.15), we have

$$f(1,1,0) = 0, \tag{2.18}$$

which taken in the context of $\mathcal{S}'$, together with (2.13), leads to

$$f(1,0,1) = 1 \quad \text{and} \quad g(1,0,1) = 1. \tag{2.19}$$

Finally, since $\ell(0,0,1)$ is opposite $\ell(1,0,1)$, using (2.19) we have

$$f(0,0,1) + g(0,0,1) = 0,$$

and applying this relation to F and G gives us

$$f(0,0,1) = 0 \quad \text{and} \quad g(0,0,1) = 0. \tag{2.20}$$

x	y	z	$f(x,y,z)$	$g(x,y,z)$	by equations
0	0	0	0	0	initial assumption
0	0	1	0	0	(2.20)
0	1	0	1	0	(2.14)
0	1	1	1	0	(2.12) and (2.15)
1	0	0	1	1	(2.8) and (2.16)
1	0	1	1	1	(2.19)
1	1	0	0	1	(2.10) and (2.18)
1	1	1	0	1	(2.11) and (2.17)

Table 2.1: The functions f and g

Thus the functions f and g are uniquely determined for a spread $\mathcal{S}$ containing the lines $[\infty]$ and $[0,0,0,0,0]$. In particular, we have found that f and g necessarily take values as listed in Table 2.1. It is now easy to see that $f(x,y,z) = x+y$ and $g(x,y,z) = x$, so $\mathcal{S} = \big\{[\infty]\big\} \cup \big\{[x,y,z,x+y,x] \bigm| x,y,z \in GF(2)\big\}$. □

2.3 Hermitian spreads and ovoids

In this section we will describe the hermitian spreads of the generalized hexagon $H(q)$, which exist for all values of q, and then by dualizing when $q = 3^h$ we will obtain the hermitian ovoids.

2.3.1 Hermitian spreads

We describe here the construction of the hermitian spreads given in [Tha80]. Let $\mathcal{P}_6$ be the embedding quadric of the generalized hexagon $H(q)$ and let Π be a hyperplane of $PG(6,q)$ that meets $\mathcal{P}_6$ in an elliptic quadric $\Pi \cap \mathcal{P}_6 = \mathcal{E}_5$. Let $\mathcal{S}$ be the set of lines of $H(q)$ that lie within Π and so are on the quadric $\mathcal{E}_5$. We shall see that this set of lines is a spread of $H(q)$.

Since $\mathcal{E}_5$ does not contain any planes, it follows from Theorem 1.15 that $\mathcal{S}$ is a set of mutually opposite lines. Now let M be a line of $H(q)$ that is not in the set $\mathcal{S}$. Let x be the unique point in which this line meets the hyperplane Π and let π be the $H(q)$-plane of x. This plane π meets Π in a line L and since this is a line through x lying in the $H(q)$-plane of x it follows that L is a line of the generalized hexagon $H(q)$. Therefore L is a line of $\mathcal{S}$ with $d(L,M) = 2$. Hence every line of $H(q)$ is at distance at most 2 from some line of $\mathcal{S}$ so it follows from Lemma 2.1 that $\mathcal{S}$ is a spread of $H(q)$. Such a spread $\mathcal{S}$ that is comprised of the lines of $H(q)$ that lie in a hyperplane meeting $\mathcal{P}_6$ in an elliptic quadric $\mathcal{E}_5$ is called a **hermitian spread**.

The reason for the name of this spread is made clearer by an alternative construction that is described in [VM96]. The hexagon $H(q)$ is embedded in $H(q^2)$ and θ is taken to be an involution of $H(q^2)$ that fixes $H(q)$ pointwise. If L and M are two opposite lines of $H(q)$ and $p \mathrm{I} L$ is a point of $H(q^2)$ such that $p \neq p^\theta$, then by [Ron80, Theorem 6.12] there is a unique weak subhexagon Γ' of $H(q^2)$ with order $(1,q^2)$ that contains p and $p^\theta \triangleright M$. The set $\mathcal{S}$ of lines of Γ' that are fixed by θ then form the spread. But Γ' is isomorphic to the incidence graph of $PG(2,q^2)$, so for each line of $\mathcal{S}$ there corresponds a flag in $PG(2,q^2)$, and these are the points of a hermitian curve together with their corresponding tangents.

Consider two lines L and M of a hermitian spread $\mathcal{S}$ with defining hyperplane Π. Since the lines L and M are in Π, the 3-dimensional space that they generate, and therefore the line regulus $\mathcal{R}(L,M)$, is also contained within Π. Thus the lines of $\mathcal{R}(L,M)$ are all lines of the spread $\mathcal{S}$ as well.

Let π be a generator of the quadric $\mathcal{P}_6$. Letting $\tilde{\mathcal{S}}$ be the set of points that are incident with lines of $\mathcal{S}$, we then have $\pi \cap \tilde{\mathcal{S}} = \pi \cap \Pi$, which is a line. Notice that $\mathcal{S}$ is actually a 1-system of $\mathcal{E}_5$ so this is an instance of what we have already seen earlier in Theorem 1.9. Conversely, from that theorem we have that if $\mathcal{S}$ is a spread of $H(q)$ with the property that every generator of $\mathcal{P}_6$ meets the point set $\tilde{\mathcal{S}}$ in a line, then $\mathcal{S}$ lies in a hyperplane Π that meets the $\mathcal{P}_6$ in an elliptic quadric $\mathcal{E}_5$, so $\mathcal{S}$ is a hermitian spread. We state a slightly refined version of this in our next theorem.

Theorem 2.6
Let $\mathcal{S}$ be a spread of $H(q)$ and let $\tilde{\mathcal{S}}$ be the set of points incident with the lines of $\mathcal{S}$. Then the intersection $\pi \cap \tilde{\mathcal{S}}$ is a line for every $H(q)$-plane π if and only if $\mathcal{S}$ is a hermitian spread.

Proof We have already seen that if $\mathcal{S}$ is a hermitian spread then it has this property. Suppose then that $\mathcal{S}$ is a spread of $H(q)$ for which this intersection property holds. Let π be any generator of $\mathcal{P}_6$. By Lemma 2.4, $\mathcal{S}$ is a 1-system of $\mathcal{P}_6$ so from Theorem 1.8 the plane π meets $\tilde{\mathcal{S}}$ in exactly $q+1$ points. Let x and y be two of these points. By Lemma 1.12 and Corollary 1.14, there is an $H(q)$-plane π' on the line xy of $\mathcal{P}_6$. But π' meets $\tilde{\mathcal{S}}$ in a line and so, since the points x and y belong to $\tilde{\mathcal{S}}$, the intersection $\pi' \cap \tilde{\mathcal{S}}$ is the line xy. Hence the plane π, which also contains the points x and y, meets $\tilde{\mathcal{S}}$ in the line xy as well. It now follows from Theorem 1.9 that $\mathcal{S}$ is contained in a hyperplane and so is a hermitian spread. □

So we see that the hermitian spreads are characterized by the property of meeting the full set of $H(q)$-planes in lines. A natural way to try to improve on such a characterization is to reduce the number of $H(q)$-planes from all to just some.

Theorem 2.7
Let $\mathcal{S}$ be a spread of $H(q)$ and let $\tilde{\mathcal{S}}$ be the set of points on the lines of $\mathcal{S}$. Let L and M be two lines of $\mathcal{S}$. Then the intersection $\pi \cap \tilde{\mathcal{S}}$ is a line for every $H(q)$-plane π that meets either L or M if and only if $\mathcal{S}$ is a hermitian spread.

Proof The reverse implication, that hermitian spreads satisfy this property, is a consequence of the previous theorem. Suppose then that $\mathcal{S}$ is a spread with the given property. As in the proof of Theorem 2.6, the spread $\mathcal{S}$ then has this property not only for $H(q)$-planes meeting either L or M, but for all generators of $\mathcal{P}_6$ that meet one of these lines. It now follows from Theorem 1.11 that $\mathcal{S}$ lies in a hyperplane and so is a hermitian spread. □

2.3.2 Coordinates

Now we will find a coordinate respresentation for the hermitian spreads. Without loss of generality, we assume that the lines $[\infty]$ and $[0,0,0,0,0]$ are included in the spread. In particular, every hermitian spread of $H(q)$ is isomorphic in $H(q)$ to one that contains these two lines. The defining hyperplane Π then has the form $\nu X_1 - \mu X_3 + \lambda X_5 = 0$. Recall from (1.9) that the quadric $\mathcal{P}_6$ in which $H(q)$ is embedded is given by the equation $X_0X_4 + X_1X_5 + X_2X_6 - X_3^2 = 0$. If $\lambda = 0$, the hyperplane $\nu X_1 - \mu X_3 = 0$ intersects $\mathcal{P}_6$ in the quadric with equation $\nu X_0X_4 + \nu X_2X_6 + (\mu X_5 - \nu X_3)X_3 = 0$, which is hyperbolic, not elliptic. Thus $\lambda \neq 0$. Hence we may suppose that Π is given by the equation

$$\nu X_1 - \mu X_3 + X_5 = 0. \tag{2.21}$$

Let us denote the hermitian spread obtained from the hyperplane Π by $\mathcal{S}_H(\mu,\nu)$.

The intersection of Π with $\mathcal{P}_6$ is given by the equation

$$X_0X_4 + X_2X_6 - \left(\nu X_1^2 - \mu X_1X_3 + X_3^2\right) = 0,$$

which is an elliptic quadric if and only if the polynomial $f(x) = x^2 - \mu x + \nu$ is irreducible. If q is odd, this means that the discriminant $\mu^2 - 4\nu$ is a nonsquare, so for each choice of μ there are $(q-1)/2$ choices for ν. If q is even, then $f(x)$ is irreducible precisely when $\mu \neq 0$ and the trace of ν/μ^2 is 1. For each of the $q-1$ choices of μ there are then $q/2$ choices for ν. Either way, we see that there are $q(q-1)/2$ hermitian spreads containing two given opposite lines.

Let $[k, b, k', b', k'']$ be a line of $H(q)$. This line belongs to the spread $\mathcal{S}_H(\mu, \nu)$ if and only if it lies within the hyperplane Π. Using the coordinates in $PG(6, q)$, this is seen to be true precisely when the following two equations hold:

$$\begin{aligned} \nu k - \mu b + b' &= 0 \\ -\nu b + \mu b' + k'' &= 0. \end{aligned} \tag{2.22}$$

Solving these for b' and k'', we have from (2.3) that

$$\begin{aligned} \mathcal{S}_H(\mu, \nu) &= \{[\infty]\} \cup \{[x, y, z, -\nu x + \mu y, \mu\nu x - (\mu^2 - \nu)y] \mid x, y, z \in GF(q)\} \\ &= \bigcup_{x,y \in GF(q)} [[x, y, -\nu x + \mu y, \mu\nu x - (\mu^2 - \nu)y]], \end{aligned} \tag{2.23}$$

where the latter expresses the spread as a union of line reguli.

Notice that if $\gamma = \mu^2 - \nu \neq 0$ then we can solve the equations (2.22) for b and b'. This certainly is true when the polynomial $g(x) = x^2 - \mu x + \mu^2$ is reducible as then the irreducibility of $f(x) = x^2 - \mu x + \nu$ assures us that $g(x) \neq f(x)$ and so $\mu^2 \neq \nu$. Equivalently, we know $\gamma \neq 0$ when $(1/\mu^2)g(\mu x) = x^2 - x + 1$ is reducible, and by Corollary 1.2, this occurs precisely when $q \not\equiv 2 \pmod 3$. Consequently, whenever $\gamma = \mu^2 - \nu \neq 0$, we can also represent the spread as

$$\begin{aligned} \mathcal{S}_H(\mu, \nu) &= \{[\infty]\} \cup \{[x, \gamma^{-1}(\mu\nu x - y), z, \gamma^{-1}(\nu^2 x - \mu y), y] \mid x, y, z \in GF(q)\} \\ &= \bigcup_{x,y \in GF(q)} [[x, \gamma^{-1}(\mu\nu x - y), \gamma^{-1}(\nu^2 x - \mu y), y]], \end{aligned} \tag{2.24}$$

and in particular, this is always so when $q \not\equiv 2 \pmod 3$.

2.3.3 Uniqueness

We have already remarked that every hermitian spread of $H(q)$ is isomorphic in $H(q)$ to some $\mathcal{S}_H(\mu, \nu)$ where $f(x) = x^2 - \mu x + \nu$ is irreducible. Now we will explicitly describe a group of collineations of $H(q)$ that acts transitively on the set of all these spreads $\mathcal{S}_H(\mu, \nu)$ to show that, in fact, the hermitian spreads are all isomorphic in $H(q)$.

Let $\varphi(y, K) = E(\gamma_P \tilde{\mu}^{-2}, K)\mathcal{T}(y, 1)$ be the composition of the elation $E(\gamma_P \tilde{\mu}^{-2}, K)$ followed by the generalized homology $\mathcal{T}(y, 1)$, where here $\tilde{\mu}$ represents the collineation corresponding to the matrix M on page 27. The parameter K is then any element of $GF(q)$ and y is any nonzero element. This collineation fixes (∞) and $[\infty]$ and its action is given by

$$\begin{aligned} (a, \ell, a', \ell', a'') &\mapsto (ya - yK, y^3\ell, y^2a', y^3\ell', ya'') \\ [k, b, k', b', k''] &\mapsto [k, y(b + kK), y^3(k' - k^2K^3 - 3b^2K - 3bkK^2), \\ &\qquad y^2(b' + kK^2 + 2bK), y^3(k'' - kK^3 - 3bK^2 - 3b'K)]. \end{aligned}$$

These collineations form a group G with composition given by

$$\varphi(y,K)\varphi(z,L) = \varphi(yz, K + y^{-1}L).$$

By considering the action on the point $(0,0,0,0,1)$, it readily follows that we have $\varphi(y,K) = \varphi(z,L)$ if and only if $(y,K) = (z,L)$, so these collineations are distinct. Therefore $|G| = q(q-1)$.

Let Π be the hyperplane given by the equation (2.21), which then determines the hermitian spread $\mathcal{S}_H(\mu,\nu)$ with $f(x) = x^2 - \mu x + \nu$ the associated irreducible polynomial. Working in $PG(6,q)$, the hyperplane Π is mapped by $\varphi(y,K)$ to the hyperplane whose equation is

$$y^2 f(-K)X_1 - y(\mu + 2K)X_3 + X_5 = 0.$$

Thus $\varphi(y,K)$ fixes Π, and therefore $\mathcal{S}_H(\mu,\nu)$ as well, if and only if

$$\mu = y(\mu + 2K) \text{ and } \nu = y^2 f(-K). \tag{2.25}$$

Upon expanding $f(-K)$, eliminating y and then factoring again, this becomes

$$(\mu^2 - 4\nu)(\mu + K)K = 0.$$

When q is odd, the factor $\mu^2 - 4\nu$ is the discriminant of $f(x)$, and when q is even, it is just μ^2. In both cases, this factor is nonzero so we conclude that either $K = 0$ or $K = -\mu$. Substituting these values, in turn, into the former equation of (2.25) and solving for y we find that $\varphi(1,0)$ and $\varphi(-1,-\mu)$ are the only collineations in G that stabilize the hermitian ovoid $\mathcal{S}_H(\mu,\nu)$. These are certainly distinct, as when q is odd $-1 \neq 1$ and when q is even $\mu \neq 0$.

The number of spreads in the orbit of $\mathcal{S}_H(\mu,\nu)$ under the action of G is given by the quotient of the order of the group G and the order of the stabilizer in G of $\mathcal{S}_H(\mu,\nu)$, and thus it is $q(q-1)/2$. This is also the total number of hermitian spreads on the lines $[\infty]$ and $[0,0,0,0,0]$ so it follows that G acts transitively on the set of all these hermitian spreads. This proves the following fact which, although well known and often implicitly used, seems rarely to be stated or demonstrated explicitly.

Theorem 2.8
The hermitian spreads of the generalized hexagon $H(q)$ are isomorphic. □

In view of this theorem, we will frequently just write $\mathcal{S}_H$ for an arbitrary representative in the single equivalence class of all hermitian spreads. Hence we will take such expressions as $\mathcal{S} = \mathcal{S}_H$ to mean that $\mathcal{S}$ is a hermitian spread.

2.3.4 Hermitian ovoids

Let us now turn our attention to ovoids. The following only applies when $q = 3^h$ so that $H(q)$ is self-dual. We define a **hermitian ovoid** to be an ovoid that is the dual of a hermitian spread. Recall that for any two lines L and M of a hermitian spread $\mathcal{S}$, all the lines of the line regulus $\mathcal{R}(L,M)$ are also in $\mathcal{S}$. Dualizing, for any two points x and y

of a hermitian ovoid $\mathcal{O}$, all the points of the point regulus $\mathcal{R}(x, y)$ are also in $\mathcal{O}$. Also, the dual of Theorem 2.8 asserts that all hermitian ovoids are isomorphic. Similarly to the case of spreads above, we will use $\mathcal{O}_H$ to indicate an arbitrary hermitian ovoid and such expressions as $\mathcal{O} = \mathcal{O}_H$ will be interpreted to mean that $\mathcal{O}$ is a hermitian ovoid.

To obtain a coordinate representation for the hermitian ovoids, we apply the duality given in (1.22) to the representation of $\mathcal{S}_H(\mu, \nu)$ that we found in (2.24), which certainly does apply since $q \not\equiv 2 \pmod 3$. The resulting hermitian ovoid we denote by $\mathcal{O}_H(\mu^3, \nu^3)$. Notice that this duality maps the lines $[\infty]$ and $[0, 0, 0, 0, 0]$ to the points (∞) and $(0, 0, 0, 0, 0)$ respectively, so the hermitian ovoids $\mathcal{O}_H(\mu^3, \nu^3)$ are those containing these two points. Thus the hermitian ovoids containing the points (∞) and $(0, 0, 0, 0, 0)$ are

$$\begin{aligned}\mathcal{O}_H(\mu, \nu) &= \{(\infty)\} \cup \{(x, \gamma^{-1}(\mu\nu x^3 - y^3), z, \gamma^{-1}(\nu^2 x^3 - \mu y^3), y) \mid x, y, z \in GF(q)\} \\ &= \bigcup_{x,y \in GF(q)} \big((x, \gamma^{-1}(\mu\nu x^3 - y^3), \gamma^{-1}(\nu^2 x^3 - \mu y^3), y)\big), \end{aligned} \tag{2.26}$$

where $\gamma = \mu^2 - \nu$ is the discriminant of the irreducible polynomial $f(x) = x^2 - \mu x + \nu$ and so is a nonsquare, and the latter representation expresses the ovoid as the union of point reguli.

Finally, we apply the duality to the collineations $\varphi(y, K)$ to obtain collineations $\psi(y^3, K^3)$ which form a group of order $q(q-1)$ that acts transitively on the set of hermitian ovoids $\mathcal{O}_H(\mu, \nu)$. After relabelling, we then have $\psi(y, K) = E(\gamma_L \mu^2, K)\mathcal{T}(1, y)$ and its action is given by

$$\begin{aligned}(a, \ell, a', \ell', a'') &\mapsto \big(a, y(\ell + a^3 K), y(a' - a^2 K), y^2(\ell' + a^3 K^2 - \ell K), y(a'' - aK)\big) \\ [k, b, k', b', k''] &\mapsto [yk - yK, yb, y^2 k', yb', yk''].\end{aligned}$$

2.4 Ree-Tits spreads and ovoids

Let ρ be a polarity of the generalized hexagon $H(q)$, so by Theorem 1.20 the order is $q = 3^{2e+1}$. Let $\mathcal{S}$ be the set of absolute lines and $\mathcal{O}$ the set of absolute points of ρ. By [CPT76] (see also [VM98, Proposition 7.2.5]), the set $\mathcal{S}$ is a spread of $H(q)$ and this is known as a **Ree-Tits spread**. Similarly, the set $\mathcal{O}$ is an ovoid called a **Ree-Tits ovoid**. By [VM98, Lemma 7.7.1 and Theorem 7.7.2], all spreads obtained in this way are isomorphic in $H(q)$ and likewise all such ovoids are isomorphic. We will write $\mathcal{S}_R$ for an arbitrary Ree-Tits spread and $\mathcal{O}_R$ for a Ree-Tits ovoid. Thus such statements as $\mathcal{S} = \mathcal{S}_R$ will be interpreted as meaning that $\mathcal{S}$ is a Ree-Tits spread.

We now find a coordinate representation for a Ree-Tits spread $\mathcal{S}$. We use the polarity ρ arising from the Tits automorphism θ as given in (1.23). In particular, we then have $[k, b, k', b', k'']^\rho = (k^{\theta^{-1}}, b^\theta, k'^{\theta^{-1}}, b'^\theta, k''^{\theta^{-1}})$. Notice that $[\infty]^\rho = (\infty)$ so $[\infty]$ is a line of $\mathcal{S}$ and the other lines of $\mathcal{S}$ then each have five coordinates. Since the characteristic is 3, we use the relations for incidence given in (1.15a) to find that the line $[k, b, k', b', k'']$ is an absolute line precisely when

$$b = k''^{\theta^{-1}} - k^{1+\theta^{-1}} \quad \text{and} \quad b' = k'^{\theta^{-1}} + k^{\theta^{-1}} k''^{\theta^{-1}} + k^{1+2\theta^{-1}}.$$

Thus

$$\mathcal{S} = \{[\infty]\} \cup \{[k, k''^{\theta^{-1}} - k^{1+\theta^{-1}}, k', k'^{\theta^{-1}} + k^{\theta^{-1}} k''^{\theta^{-1}} + k^{1+2\theta^{-1}}, k''] \mid k, k', k'' \in GF(q)\}. \tag{2.27}$$

Either by repeating the same steps for ovoids or by applying the duality given in (1.22), we similarly find that a Ree-Tits ovoid $\mathcal{O}$ is given by

$$\mathcal{O} = \{(\infty)\} \cup \{(a, a''^{\theta} - a^{3+\theta}, a', a'^{\theta} + a^{\theta} a''^{\theta} + a^{3+2\theta}, a'') \mid a, a', a'' \in GF(q)\}.$$

This is identical to the representation given in [VM98, §7.7.5] of a Ree-Tits ovoid.

2.5 Locally hermitian spreads and ovoids

Here we will consider spreads and ovoids that, from the standpoint of a single line or point, seem somewhat similar to the hermitian spreads and ovoids. Since hermitian spreads exist for all values of q, whereas the hermitian ovoids only exist for $q = 3^h$, spreads seem to be the more natural context for exploring the concept of being locally hermitian, so it is with spreads that we shall start. After defining precisely what is meant by a locally hermitian spread, some general properties of these spreads will be given, including a coordinate condition to help us recognize when we have such a spread, and then the known ones for $q \neq 3^h$ will be described. Finally, locally hermitian ovoids will be discussed and the known ones will be given, and hence, the remaining known locally hermitian spreads will be given as well.

2.5.1 Locally hermitian spreads

Let $\mathcal{S}$ be a hermitian spread. Recall that for any pair of lines L and M in $\mathcal{S}$, all the lines of the line regulus $\mathcal{R}(L, M)$ are also included in $\mathcal{S}$. Also recall that for each line K in $\mathcal{R}(L, M)$, the line regulus $\mathcal{R}(L, K)$ is the same as $\mathcal{R}(L, M)$. Thus in particular, if we fix the line L then $\mathcal{S}$ is the union of q^2 distinct line reguli on L. This essentially describes what a hermitian spread "looks like" from a line.

Now let $\mathcal{S}$ be any spread of $H(q)$ and let L be one of its lines. We say that $\mathcal{S}$ is **locally hermitian** with respect to the line L if, for every other line M of $\mathcal{S}$, all the lines of $\mathcal{R}(L, M)$ are also in $\mathcal{S}$. Thus the spread $\mathcal{S}$ is the union of q^2 distinct line reguli on L and from this line it essentially looks like a hermitian spread.

In the above, while providing the motivation for the concept of locally hermitian, we have seen that a hermitian spread is locally hermitian with respect to each of its lines. In fact, hermitian spreads are characterized by this property of being locally hermitian with respect to every line, but one can do much better. Here we provide an independent proof of the following theorem that uses only the geometry of the embedding quadric of $H(q)$.

Theorem 2.9 (Bloemen, Thas, Van Maldeghem [BTVM98, Theorem 9]) *A spread of $H(q)$ that is locally hermitian with respect to two distinct lines is hermitian.*

Proof Let $\mathcal{S}$ be a spread that is locally hermitian with respect to the line L. Let π be any $H(q)$-plane that meets L in a point. By Lemma 2.4, $\mathcal{S}$ is a 1-system of $\mathcal{P}_6$ so by Theorem 1.8 there is another line M of $\mathcal{S}$ meeting π in a point. Let x and y be the points in which π meets these lines L and M. Since $\mathcal{S}$ is locally hermitian with respect to L, all the lines of the line regulus $\mathcal{R}(L, M)$ are in $\mathcal{S}$ as well. But $\mathcal{R}(L, M)$ is a regulus in $\mathcal{P}_6$ with xy a line of the opposite regulus, so for each point on the line xy there is a line of $\mathcal{R}(L, M)$, and hence $\mathcal{S}$, that is incident with it. Since π was arbitrary among $H(q)$-planes meeting L, we see that every such $H(q)$-plane meets the lines of $\mathcal{S}$ in a set of collinear points. The result now follows from Theorem 2.7. □

Just as this theorem characterizes hermitian spreads, the next result is a similar one that characterizes locally hermitian spreads. We say that a spread $\mathcal{S}$ is **point locally hermitian** with respect to a point x incident with a line K of $\mathcal{S}$ if, for every pair of lines L and M of $\mathcal{S} \setminus \{K\}$ with $L \triangleright_2 x = M \triangleright_2 x$, the points x, $x \triangleright L$ and $x \triangleright M$ are collinear in the quadric $\mathcal{P}_6$ (see Figure 2.2). For an equivalent description, notice first

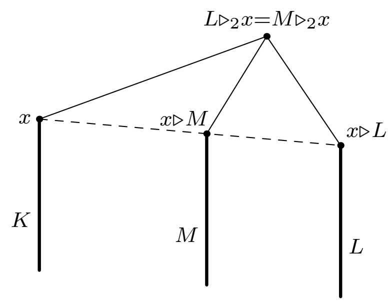


Figure 2.2: Point locally hermitian.

that for each line $L \neq K$ of $\mathcal{S}$ there is a unique line L_x in the opposite regulus of $\mathcal{R}(K, L)$ that passes through x. Although a line of $\mathcal{P}_6$, this line L_x is not a line of $H(q)$. Now $\mathcal{S}$ is point locally hermitian with respect to x if and only if for every line $L \neq K$ in $\mathcal{S}$, the points of L_x are all incident with lines of $\mathcal{S}$.

The following theorem and proof are a portion of the proof of [BTVM98, Theorem 6].

Theorem 2.10 (Bloemen, Thas, Van Maldeghem [BTVM98, Theorem 6]) *A spread $\mathcal{S}$ of $H(q)$ that is point locally hermitian with respect to two distinct points on a common line K is locally hermitian with respect to K.*

Proof Suppose $\mathcal{S}$ is point locally hermitian with respect to the points x and y on K. Let L be any other line of $\mathcal{S}$, and let L_x and L_y be the lines of the opposite regulus of $\mathcal{R}(K, L)$ through the points x and y, respectively. Let w be a point on L_x and let M be the line in $\mathcal{R}(K, L)$ through w. This line M meets L_y in a point z. By the point locally hermitian property, there are lines M' and M'' of $\mathcal{S}$ on the points w and z, but $d(w, z) = 2$ so we conclude that $M' = M'' = M$, which is in the line regulus $\mathcal{R}(K, L)$. Since w was chosen arbitrarily from the points on L_x, all the lines of $\mathcal{R}(K, L)$ belong to $\mathcal{S}$. Also, L was chosen arbitrarily so it follows that $\mathcal{S}$ is locally hermitian with respect to K. □

Now let us consider the coordinates of the lines in a spread $\mathcal{S}$ that contains the line $[\infty]$ and is locally hermitian with respect to that line. Suppose $[k, b, k', b', k'']$ is a line of $\mathcal{S}$. Then all the lines of the line regulus (see (1.19))

$$[[k, b, b', k'']] = \mathcal{R}\big([\infty], [k, b, k', b', k'']\big) = \big\{[\infty]\big\} \cup \big\{[k, b, x, b', k''] \,|\, x \in GF(q)\big\}$$

are also in $\mathcal{S}$. In terms of the representation of a spread given in (2.3), this means that the functions $f(x, y, z)$ and $g(x, y, z)$ are independent of z. Conversely, if these functions are independent of z then $\mathcal{S}$ is locally hermitian. Thus $\mathcal{S}$ can be expressed in the form

$$\begin{aligned}\mathcal{S} &= \big\{[\infty]\big\} \cup \big\{[x, y, z, f(x, y), g(x, y)] \;\big|\; x, y, z \in GF(q)\big\} \\ &= \bigcup_{x,y \in GF(q)} [[x, y, f(x, y), g(x, y)]],\end{aligned} \tag{2.28}$$

where $f(0, 0) = g(0, 0) = 0$ and the second line expresses $\mathcal{S}$ as a union of line reguli.

Not any pair of functions f and g will serve to produce a spread and we will now find conditions under which they do. We say that two line reguli on a common line, say $\mathcal{R}(J, L)$ and $\mathcal{R}(J, M)$, are **compatible** if each line other than J from either is opposite all the lines of the other, or in other words, the two line reguli are distinct and the lines of the set $\mathcal{R}(J, L) \cup \mathcal{R}(J, M)$ are mutually opposite. Thus two line reguli are compatible when it seems that they could feasibly belong to a spread.

Let J, L and M be any three pairwise opposite lines in $H(q)$ and suppose that M is opposite every line of the line regulus $\mathcal{R}(J, L)$. Since M does not meet any line of $\mathcal{R}(J, L)$, these three lines generate a five dimensional subspace Π. Let π be a generator of $\mathcal{P}_6$ on M. By Theorem 1.15, the plane π does not meet any line of $\mathcal{R}(J, L)$ either and so does not meet the three dimensional subspace Π_3 generated by J and L at all. Since a plane in the subspace Π must meet Π_3, it follows that no generator of $\mathcal{P}_6$ on M is contained in Π, so $\Pi \cap \mathcal{P}_6$ is neither a hyperbolic quadric $\mathcal{H}_5$ nor a parabolic quadric cone $\Pi_0\mathcal{P}_4$. Consequently, Π meets $\mathcal{P}_6$ in a nondegenerate elliptic quadric $\mathcal{E}_5$. Now the line regulus $\mathcal{R}(J, M)$ is contained within Π so each of its lines also has the property that no generator of $\mathcal{P}_6$ on it meets any of the lines of $\mathcal{R}(J, L)$; that is, $\mathcal{R}(J, L)$ and $\mathcal{R}(J, M)$ are compatible. Thus, in order for two line reguli $\mathcal{R}(J, L)$ and $\mathcal{R}(J, M)$ to be compatible, it is sufficient that the one line M be opposite every line of $\mathcal{R}(J, L)$.

Let us suppose the common line J is the line $[\infty]$ and let L and M be $[k, b, k', b', k'']$ and $[K, B, K', B', K'']$, respectively. In view of the previous paragraph, to know when the line reguli $\mathcal{R}(J, L)$ and $\mathcal{R}(J, M)$ are compatible, it is sufficient to only determine when the line $[K, B, 0, B', K'']$ is opposite every line of $\mathcal{R}(J, L)$. In the following, we will always use the convention that when two variables are represented by the same letter, one lower case and one upper case, their difference is written as $\Delta x = x - X$. Now from the opposite line condition of Lemma 1.22, the lines $[K, B, 0, B', K'']$ and $[k, b, x, b', k'']$ are opposite when

$$\begin{aligned}x^2 &+ (k''\Delta k + K''\Delta k + 3b'\Delta b + 3B'\Delta b)x \\ &\quad + (k''\Delta k - \Delta b\Delta b' + 3b'\Delta b)(K''\Delta k + \Delta b\Delta b' + 3B'\Delta b) \\ &\qquad + (\Delta b^2 - \Delta k\Delta b')(\Delta b'^2 + \Delta b\Delta k'') \neq 0.\end{aligned} \tag{2.29}$$

As x ranges over all of $GF(q)$, the line $[k, b, x, b', k'']$ ranges over all the lines of $\mathcal{R}(J, L)$, so the line reguli are compatible if and only if this inequality holds for every value of x. Thus the given line reguli are compatible if and only if the quadratic in x in (2.29) is irreducible, and hence Theorem 1.1 applies. To help make the following steps a little clearer, let

$$A = k''\Delta k + 3b'\Delta b, \qquad B = K''\Delta k + 3B'\Delta b,$$
$$C = \Delta b\Delta b', \qquad D = (\Delta b^2 - \Delta k\Delta b')(\Delta b'^2 + \Delta b\Delta k'').$$

Then the quadratic in (2.29) can be rewritten as

$$x^2 + (A + B)x + (A - C)(B + C) + D. \tag{2.30}$$

For odd q, the discriminant is then

$$\begin{aligned} &(A + B)^2 - 4(A - C)(B + C) - 4D \\ =&(A + B)^2 - 4AB - 4(A - B)C + 4C^2 - 4D \\ =&(A - B)^2 - 4(A - B)C + 4C^2 - 4D \\ =&(A - B - 2C)^2 - 4D. \end{aligned}$$

For even q, notice that the coefficient of x in (2.30) is simply $A + B = \Delta b\Delta b' + \Delta k\Delta k''$. Next, since $x^2 + (A + B)x + AB = (x + A)(x + B)$ is reducible, the irreducibility of the quadratic in (2.30) is equivalent to the irreducibility of $x^2+(A+B)x+(A+B)C+C^2+D$, and the constant coefficient in this is

$$\begin{aligned} &(\Delta b\Delta b' + \Delta k\Delta k'')\Delta b\Delta b' + \Delta b^2\Delta b'^2 + (\Delta b^2 + \Delta k\Delta b')(\Delta b'^2 + \Delta b\Delta k'') \\ =&\Delta b^2\Delta b'^2 + \Delta b^3\Delta k'' + \Delta b'^3\Delta k. \end{aligned}$$

Thus, for odd q, the **compatibility condition** is

$$(\Delta b\Delta b' + \Delta k\Delta k'')^2 + 4(\Delta b'\Delta k - \Delta b^2)(\Delta b\Delta k'' + \Delta b'^2) = \not\square, \tag{2.31a}$$

and for even q, it is the trace equation

$$\operatorname{Tr}\left(\frac{\Delta b^2\Delta b'^2 + \Delta b^3\Delta k'' + \Delta b'^3\Delta k}{\Delta k^2\Delta k''^2 + \Delta b^2\Delta b'^2}\right) = 1 \tag{2.31b}$$

together with the requirement that the expression in the denominator in (2.31b) should be nonzero.

For an immediate application of these compatibility conditions, we now show that a locally hermitian spread can sometimes be given a coordinate representation similar to that for ovoids in (2.6).

Let $\mathcal{S}$ be a locally hermitian spread with respect to the line $[\infty]$ and let $[k, b, k', b', k'']$ and $[K, B, K', B', K'']$ be two lines of $\mathcal{S}$. Suppose $k = K$ and $k'' = K''$. For odd q, by the compatibility condition (2.31a) we have that if the two lines are not in the same line regulus, then

$$\Delta b^2\Delta b'^2 - 4\Delta b^2\Delta b'^2 = -3\Delta b^2\Delta b'^2 = \not\square.$$

When -3 is a square, that is, when $q \not\equiv 2 \pmod 3$ (see Corollary 1.2), this can never occur. It follows that for such q, if $(k, k'') = (K, K'')$ then the two lines are in the same line regulus. If instead q is even, then by (2.31b) we have $\mathrm{Tr}(1) = 1$. If we have further that $q = 2^{2e}$ then in fact $\mathrm{Tr}(1) = 0$ and so again the compatibility condition is not satisfied when $q \not\equiv 2 \pmod 3$. As in the case for q odd, we conclude that the pair (k, k'') uniquely determines the line regulus of the spread $\mathcal{S}$ to which the line $[k, b, k', b', k'']$ belongs.

It now follows that if $q \not\equiv 2 \pmod 3$ then a spread $\mathcal{S}$ that is locally hermitian with respect to the line $[\infty]$ can be represented by

$$\begin{aligned}\mathcal{S} &= \big\{[\infty]\big\} \cup \big\{[x, f(x,y), z, g(x,y), y] \;\big|\; x, y, z \in GF(q)\big\} \\ &= \bigcup_{x,y \in GF(q)} [[x, f(x,y), g(x,y), y]]\end{aligned} \tag{2.32}$$

where f and g are functions dependent only upon x and y, and the latter representation expresses $\mathcal{S}$ as a union of line reguli.

In addition to the hermitian spreads themselves, there are only two classes of locally hermitian spreads known and these were both found in [BTVM98]. The first class exists only for $q = 3^h$, when $H(q)$ is self-dual, and they were discovered by manipulating the hermitian ovoid in the quadric. To reflect this, we leave their description for the context of ovoids. It is the other class that we describe presently, although here we use an original approach different from that used in [BTVM98] thereby describing an independent rediscovery of these spreads.

Let q be odd and let $\mathcal{S}$ be a locally hermitian spread as represented in (2.28). Notice from (2.23) that the hermitian spread $\mathcal{S}_H(0, -\gamma)$, where γ is a nonsquare, corresponds to the functions $f(x,y) = \gamma x$ and $g(x,y) = -\gamma y$. We consider what happens when the coefficient in $f(x,y)$ is changed to another nonsquare, say $a^2\gamma$.

We have $f(x,y) = a^2\gamma x$ and $g(x,y) = -\gamma y$ with $a \neq 0$. Substituting into the compatibility condition (2.31a), we find that

$$\begin{aligned}&\big(a^2-1\big)^2\gamma^2x^2y^2 + 4\big(a^2\gamma x^2 - y^2\big)\big(-\gamma y^2 + a^4\gamma^2x^2\big) \\ &\quad = \big(1 - 6a^2 - 3a^4\big)\gamma^2x^2y^2 + 4a^6\gamma^3x^4 + 4\gamma y^4\end{aligned}$$

must be a nonsquare for all values of $(x,y) \neq (0,0)$. Putting $X = 2a^3\gamma x^2$ and $Y = 2y^2$, this becomes

$$\gamma\Big(\frac{1 - 6a^2 - 3a^4}{4a^3}XY + X^2 + Y^2\Big).$$

If the coefficient of XY inside the brackets is 2, then this is $\gamma(X+Y)^2$ so we are certainly guaranteed that this will always be a nonsquare for all $(x,y) \neq (0,0)$ so long as we never have $X = -Y$. But $X = -Y$ is equivalent to $\big(y/ax\big)^2 = -a\gamma$ and there are values of x and y for which this occurs precisely when $-a$ is a nonsquare. Thus we require that $-a$ be a nonzero square. Now setting the coefficient of XY to 2 and solving for a we have

$$\frac{1 - 6a^2 - 3a^4}{4a^3} = 2 \Leftrightarrow 3a^4 + 8a^3 + 6a^2 - 1 = 0 \Leftrightarrow (a+1)^3(3a-1) = 0.$$

The solution $a = -1$ is valid as $-a$ is a nonzero square, and this returns us to the hermitian spread. However, the solution $a = 1/3$, which is valid only for $q \equiv 1 \pmod 3$ when -3 is a nonzero square (see Corollary 1.2), gives a new spread. Setting the coefficient of XY to -2, in which case we require that we never have $X = Y$, or equivalently, that a is a nonzero square, leads to the solutions $a = 1$ and $a = -1/3$, and so ultimately to the same spreads.

Thus, apart from the hermitian spread, the only other spread of the sought form is obtained by setting $f(x, y) = \frac{1}{9}\gamma x$ when $q \equiv 1 \pmod 3$, and we call this spread $\mathcal{S}_{[9]}$ in accordance with [BTVM98] where it was first discovered; the derivation here, however, describes an independent rediscovery of the same. Notice that this spread contains the lines $[\infty]$ and $[0, 0, 0, 0, 0]$ and yet does not correspond to any $\mathcal{S}_H(\mu, \nu)$, so it certainly is not a hermitian spread. Also, it is easy to see from the representation given in (2.27) that that Ree-Tits spread is not locally hermitian with respect to the line $[\infty]$. Since its automorphism group acts transitively on its lines (in fact, doubly transitively—see [VM98, Theorem 7.7.6]), it follows that a Ree-Tits spread is not locally hermitian with respect to any of its lines, so $\mathcal{S}_{[9]}$ is not a Ree-Tits spread either.

Thus $\mathcal{S}_{[9]}$ is genuinely another spread and it is given by

$$\mathcal{S}_{[9]} = \bigcup_{x,y \in GF(q)} [[x, y, \tfrac{1}{9}\gamma x, -\gamma y]].$$

2.5.2 Locally hermitian ovoids

Let $\mathcal{O}$ be an ovoid of the generalized hexagon $H(q)$ and let x be some point of $\mathcal{O}$. Let $\mathcal{O}^+ = \mathcal{O} \setminus \{x\}$ be the points of $\mathcal{O}$ opposite x. Dualizing the notion of locally hermitian spread, we say that the ovoid $\mathcal{O}$ is **locally hermitian** with respect to the point x if for every point y in $\mathcal{O}^+$, all the points of the point regulus $\mathcal{R}(x, y)$ are also in $\mathcal{O}$. Thus $\mathcal{O}$ is the union of q^2 point reguli on x.

Two point reguli $\mathcal{R}(x, y)$ and $\mathcal{R}(x, z)$ on a common point x are **compatible** if for each point $u \in \mathcal{R}(x, y)$ and $v \in \mathcal{R}(x, z)$, with $v \neq x$, we have $d(u, v) = 6$. Thus q^2 point reguli on a common point form a locally hermitian ovoid if and only if they are mutually compatible. In analogy with the conditions for compatibility of line reguli given in (2.31a) and (2.31b), we now find conditions for the compatibility of point reguli.

Let $((a, \ell, \ell', a''))$ and $((A, L, L', A''))$ be two point reguli of $H(q)$ on (∞). Recall from (1.20) that the point regulus $((a, \ell, \ell', a''))$ is

$$((a, \ell, \ell', a'')) = \{(\infty)\} \cup \{(a, \ell, x, \ell' + 3a''x, a'') \mid x \in GF(q)\}.$$

Keeping the earlier convention that $\Delta x = x - X$, from the opposite point condition in Lemma 1.21, our point reguli are compatible if and only if

$$(a''\Delta a + \Delta x)(A''\Delta a + \Delta x) - \Delta a(\Delta\ell' + 3a''x - 3A''X) + \Delta a''\Delta\ell - 3\Delta a(xA'' - a''X)$$

is nonzero for all $x, X \in GF(q)$. After some rearrangement, this can be expressed as

$$\Delta x^2 - 2(a'' + A'')\Delta a\Delta x + a''A''\Delta a^2 - \Delta a\Delta\ell' + \Delta a''\Delta\ell,$$

which is a quadratic in Δx. That this must be nonzero for all $x, X \in GF(q)$ is now equivalent to this polynomial being irreducible. This certainly never occurs when q is even as the coefficient of Δx is zero, so this constitutes a proof that there are no locally hermitian ovoids in $H(q)$ when q is even, although we already know from [Tha81] that there are, in fact, no ovoids at all in this case. Therefore, we assume q is odd. By Theorem 1.1, the irreducibility of this polynomial, and hence the compatibility of the point reguli, corresponds to the discriminant being a nonsquare; that is, the **compatibility condition** is

$$\Delta a^2 \Delta a''^2 + \Delta a \Delta \ell' - \Delta a'' \Delta \ell + 3a'' A'' \Delta a^2 = \not\square. \tag{2.33}$$

Finally, we discuss the known locally hermitian ovoids. When $q = 3^h$, there are the hermitian ovoids, which are locally hermitian with respect to every one of their points. The only other known locally hermitian ovoids, discovered by Bloemen, Thas and Van Maldeghem in [BTVM98], are obtained from these and so exist only for these values of q. Also, the duals of these ovoids complete the list of known locally hermitian spreads.

The principle used in [BTVM98] for the construction of new ovoids of $H(q)$ is as follows. Let $\mathcal{O}$ be an ovoid of $H(q)$. Then $\mathcal{O}$ is also an ovoid of $\mathcal{P}_6$ by Lemma 2.4. Now if θ is an automorphism of the quadric $\mathcal{P}_6$, then the set $\mathcal{O}^\theta$ is another ovoid of $\mathcal{P}_6$ and so, by Lemma 2.4, it is also an ovoid of $H(q)$. But if θ is chosen so that it does not preserve the generalized hexagon, then $\mathcal{O}$ and $\mathcal{O}^\theta$ may not be isomorphic in $H(q)$. Applying this technique to the hermitian ovoids and the Ree-Tits ovoids gave ovoids that were new in $H(q)$, although not in the quadric. However, since $q = 3^h$ for these, a duality was applied and the new spreads of $H(q)$ thus obtained did provide new 1-systems of $\mathcal{P}_6$. See [BTVM98] for more details. Notice that the map θ might not map point reguli to point reguli, so many of the new ovoids $\mathcal{O}^\theta$ obtained from a hermitian ovoid $\mathcal{O}$ are not locally hermitian. Here however, we restrict our attention to those that are.

Let $\mathcal{O} = \mathcal{O}_H(\mu, \nu)$ be a hermitian ovoid and let θ be the automorphism of $\mathcal{P}_6$ whose matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.34}$$

with $\lambda \neq 0$. Considering Plücker coordinates, this map leaves p_{20} fixed while p_{35} is not, so by the definition of the lines of $H(q)$ given in (1.10), the hexagon is not fixed. The image of a point whose coordinates in the hexagon are $(a, \ell, a', \ell', a'')$ is $(a, \ell + a\lambda, a', \ell' + a''\lambda, a'')$, thus from the representation of the ovoid $\mathcal{O}_H(\mu, \nu)$ in (2.26), the locally hermitian ovoids obtained are

$$\mathcal{O}_\lambda(\mu, \nu) = \bigcup_{x,y \in GF(q)} \big((x, \gamma^{-1}(\mu\nu x^3 - y^3) + \lambda x, \gamma^{-1}(\nu^2 x^3 - \mu y^3) + \lambda y, y)\big), \tag{2.35}$$

where $f(x) = x^2 - \mu x + \nu$ is an irreducible polynomial with discriminant $\gamma = \mu^2 - \nu$.

As noted in the dual case in Section 2.5.1 on page 48, a Ree-Tits ovoid is not locally hermitian with respect to any of its points, so these ovoids $\mathcal{O}_\lambda(\mu,\nu)$ are certainly not Ree-Tits ovoids. Also, from the representation of the hermitian ovoids containing the points (∞) and $(0,0,0,0,0)$ in (2.26), one can see that these ovoids $\mathcal{O}_\lambda(\mu,\nu)$ are not hermitian ovoids as long as $h>1$ so that x and x^3 represent distinct functions in $GF(q)$. Thus these ovoids are indeed new ovoids for $q=3^h$ with $h>1$. We remark that when $h=1$, there is nothing new gained as the ovoid $\mathcal{O}_\lambda(\mu,\nu)$ is identical to the hermitian ovoid $\mathcal{O}_H(\mu+\lambda,\mu^2-\mu\lambda-1)$. Also, notice that if $\lambda=0$ then the matrix in (2.34) is the identity and $\mathcal{O}_0(\mu,\nu)$ is just the hermitian ovoid $\mathcal{O}_H(\mu,\nu)$. We shall refer to the new ovoids obtained in this way, with $\lambda\neq 0$ and $h>1$, as the $\mathcal{O}_\lambda$ ovoids.

That it is sufficient to have only considered maps θ as given by (2.34) is the following theorem from [BTVM98]. Although actually stated there for translation ovoids, the authors comment after the proof that the theorem also holds for locally hermitian ovoids.

Theorem 2.11 (Bloemen, Thas, Van Maldeghem, [BTVM98, Theorem 21]) *If $\mathcal{O}$ is a locally hermitian ovoid of $H(3^h)$ that is isomorphic in $\mathcal{P}_6$ to a hermitian ovoid, then either $\mathcal{O}$ is hermitian or $h>1$ and $\mathcal{O}$ is an $\mathcal{O}_\lambda$ ovoid.* □

Considering the actions of the maps φ and ψ in sections 2.3.3 and 2.3.4, but now observing their effects on the extra terms λx and λy in (2.35), we find

$$\mathcal{O}_\lambda(\mu,\nu)\psi(y,K)=\mathcal{O}_{y\lambda}\big(y(\mu+2K),y^2f(-K)\big),$$

where $f(x)=x^2-\mu x+\nu$ is the irreducible quadratic associated with $\mathcal{O}_\lambda(\mu,\nu)$. In particular, every $\mathcal{O}_\lambda$ ovoid is isomorphic to $\mathcal{O}_\lambda(0,-\gamma)$ for some nonzero λ and nonsquare γ.

The spreads dual to the $\mathcal{O}_\lambda$ ovoids we shall call $\mathcal{S}_\lambda$ spreads. Applying the duality given in (1.22), those containing the lines $[\infty]$ and $[0,0,0,0,0]$ are found to be

$$\mathcal{S}_\lambda(\mu,\nu)=\bigcup_{x,y\in GF(q)}\big[\big[x,\gamma^{-1}(\mu\nu x-y)+\lambda x^{1/3},\gamma^{-1}(\nu^2x-\mu y)+\lambda y^{1/3},y\big]\big],\qquad(2.36)$$

where again, $\gamma=\mu^2-\nu$ is a nonsquare. These, together with the $\mathcal{S}_{[9]}$ spreads and the hermitian spreads, are all the known locally hermitian spreads.

Chapter 3

Translation spreads and ovoids

3.1 Introduction

Let $\mathcal{T}$ be a spread or an ovoid of $H(q)$. We will define here what it means for $\mathcal{T}$ to be translation, firstly with respect to a flag $\{x, y\}$ and then with respect to an element x. In both cases, the element x is an element of $\mathcal{T}$, so x is a line when $\mathcal{T}$ is a spread and x is a point when $\mathcal{T}$ is an ovoid.

We begin by considering a partition of the elements of $\mathcal{T}$. Let x be an element of $\mathcal{T}$ and let y be an element of the hexagon that is incident with x, so $\{x, y\}$ is a flag. Let $\mathcal{T}^+ = \mathcal{T} \setminus \{x\}$. For each of the q elements $z \neq x$ on y, let $V_z = V_z(x, y)$ be the spread (or ovoid) projection set for z, whose elements are those elements of $\mathcal{T}$ that are at distance four from z, as introduced in Section 2.1 (see Figure 3.1). By Lemma 2.3, these sets V_z each contain q^2 elements and each element w of $\mathcal{T}^+$ belongs to exactly one of these sets, namely $V_{w \triangleright y}$. Thus the sets V_z partition $\mathcal{T}^+$ into q sets of size q^2.

Now we are prepared to define translation spreads and ovoids. The spread or ovoid $\mathcal{T}$ is said to be **translation with respect to the flag** $\{x, y\}$ if there is a group $G_{\{x,y\}}$ of collineations that stabilize $\mathcal{T}$, that fix each of x and y elementwise, and such that the group acts transitively on the set V_z for each $z \neq x$ on y. Since x and y are fixed elementwise by $G_{\{x,y\}}$, we have

$$G_{\{x,y\}} \leqslant G^{\{x,y\}} \leqslant G^x,$$

where $G^{\{x,y\}}$ and G^x are the groups described in Section 1.4.4 starting on page 29 where we saw that $G^{\{x,y\}}$ acts regularly on the z-projection sets U_z. In addition, from page 34, we have $V_z \subset U_z$. Thus the group $G_{\{x,y\}}$ is uniquely determined and it acts regularly on the sets V_z. This group $G_{\{x,y\}}$ is called the **associated group** of $\mathcal{T}$ with respect to the flag $\{x, y\}$. Notice that for two elements u and v of $\mathcal{T}^+$ there is a collineation g in $G_{\{x,y\}}$ such that $ug = v$ if and only if u and v belong to the same set V_z, and this is in turn equivalent to $u \triangleright y = v \triangleright y$.

By the regular action of G^x, the associated group $G_{\{x,y\}}$ is completely determined by its action on any one of the sets V_z. Thus if $\mathcal{T}$ is a spread given by the representation in (2.3), for instance, that is translation with respect to the flag $\{[\infty], (\infty)\}$, it is sufficient

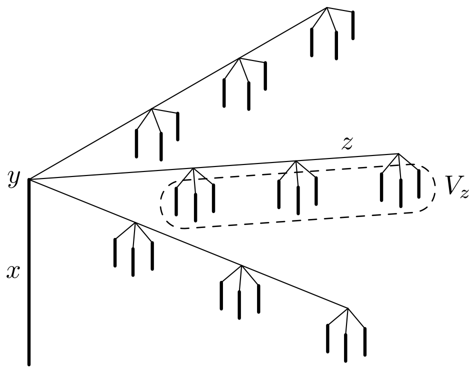


Figure 3.1: A spread projection set V_z.

to consider only the set $V_{[0]}$, so we have

$$G_{\{[\infty],(\infty)\}} = \left\{\Theta[0, y, z, f(0, y, z), g(0, y, z)] \;\middle|\; y, z \in GF(q)\right\}, \tag{3.1}$$

where the collineations Θ are as introduced in Section 1.4.4 on page 31. A similar statement holds for an ovoid translation with respect to the flag $\{(\infty), [\infty]\}$.

The spread or ovoid $\mathcal{T}$ is called **translation with respect to** x if it is translation with respect to the flag $\{x, y\}$ for each y that is incident with x. The **associated group** of $\mathcal{T}$ with respect to the element x is

$$G_x = \left\langle G_{\{x,y\}} \;\middle|\; y \mathrm{\ I\ } x \right\rangle \leqslant G^x, \tag{3.2}$$

the group generated by the associated groups with respect to the flags containing x. That this is a subgroup of G^x follows from the fact that each of the groups $G_{\{x,y\}}$ is, and as for those groups, G_x therefore acts semiregularly on the set $\mathcal{T}^+$ of elements of $\mathcal{T}$ opposite x.

Let us now consider the spreads and ovoids that were introduced in Chapter 2. Remember that the only known ovoids exist for $q = 3^h$ when $H(q)$ is self-dual, so it suffices to restrict our attention to the spreads. Using (3.1) and the action of the collineations Θ in $G^{[\infty]}$ as given in (1.25), it can be readily checked that all the spreads introduced, $\mathcal{S}_H$, $\mathcal{S}_R$, $\mathcal{S}_{[9]}$ and $\mathcal{S}_\lambda$, given by their coordinate representations in the previous chapter, are translation with respect to the flag $\{[\infty], (\infty)\}$.

While establishing the coordinate representation for the hermitian spreads, the hat-rack of the coordinatization was chosen so that $[\infty]$ and $[0, 0, 0, 0, 0]$ were any two arbitrarily chosen lines of the spread. Also, the point (∞) could have been chosen arbitrarily from the points incident with $[\infty]$, so a hermitian spread $\mathcal{S}_H$ is actually translation with respect to every flag $\{L, x\}$ where L is a line of $\mathcal{S}_H$, and therefore, translation with respect to every one of its lines.

For the Ree-Tits spreads, the coordinatization was chosen arbitrarily among those such that the flag $\{[\infty], (\infty)\}$ was an absolute point-line pair with respect to the polarity, so a Ree-Tits spread $\mathcal{S}_R$ is translation with respect to every flag $\{L, L^\rho\}$, where L is a line of $\mathcal{S}_R$ and ρ is the defining polarity. It is easily seen from the coordinate representation that the spread $\mathcal{S}_R$ is not locally hermitian, so it follows from the forthcoming Theorem 3.3 that it is not translation with respect to any other flag.

Finally, the spreads $\mathcal{S}_{[9]}$ and $\mathcal{S}_\lambda$, as given by their coordinate representations in Chapter 2, are translation with respect to the line $[\infty]$ (see [BTVM98]). This follows immediately from the forthcoming Theorem 3.7. Since these spreads are nonhermitian, yet are locally hermitian with respect to the line $[\infty]$, by Theorem 2.9 the automorphism groups of these spreads must leave $[\infty]$ invariant. Thus in particular, they are not translation with respect to any flag $\{K, y\}$, where $K \neq [\infty]$.

3.2 General results

Although translation spreads and ovoids were defined dually, the fact remains that arguments pertaining to one might not be able to be dualized to apply to the other. However, our next theorem lets us know that we will nonetheless not be needing to consider translation spreads and ovoids separately, as translation ovoids of $H(q)$ exist only when $H(q)$ is self-dual. In fact, it is conjectured that it is only when $H(q)$ is self-dual that it has ovoids at all (see the comment at the bottom of page 78), so this theorem supports the conjecture by putting the matter to rest for translation ovoids. Other results in this direction are the nonexistence of ovoids of $H(q)$ for q even [Tha81], and for $q = 5$ or 7 [O'KT95]. The following theorem appears in [Off01].

Theorem 3.1
There are no translation ovoids with respect to a flag, and so no translation ovoids with respect to a point, in $H(q)$ when q is not a power of 3.

Proof Let $\mathcal{O}$ be a translation ovoid with respect to a flag. Without loss of generality, we suppose that the hat-rack of the coordinatization is chosen such that $\mathcal{O}$ has the representation given in (2.6); that is,

$$\mathcal{O} = \big\{(\infty)\big\} \cup \big\{(x, f(x,y,z), z, g(x,y,z), y) \;\big|\; x, y, z \in GF(q)\big\},$$

where $f(0,0,0) = g(0,0,0) = 0$. The group associated with $\mathcal{O}$ with respect to the flag $\{(\infty), [\infty]\}$ is then

$$G_{\{(\infty),[\infty]\}} = \big\{\Psi(0, f(0,y,z), z, g(0,y,z), y) \;\big|\; y, z \in GF(q)\big\}.$$

The action of the collineations Ψ is given in (1.24), and from there we see that the element $\Psi(0, f(0,Y,Z), Z, g(0,Y,Z), Y)$ of $G_{\{(\infty),[\infty]\}}$ maps the point $(0, f(0,y,z), z, g(0,y,z), y)$ of $\mathcal{O}$ to

$$\big(0, f(0,y,z) + f(0,Y,Z), z + Z, g(0,y,z) + g(0,Y,Z) + 3zY, y + Y\big).$$

which must also be a point of $\mathcal{O}$. Therefore, for all $y, z, Y, Z \in GF(q)$,

$$g(0, y+Y, z+Z) = g(0,y,z) + g(0,Y,Z) + 3zY. \tag{3.3}$$

Putting $Y = 0$ in (3.3) we get

$$g(0, y, z+Z) = g(0,y,z) + g(0,0,Z) \tag{3.4}$$

and putting $y = 0$ in (3.4) gives

$$g(0, Y, z + Z) = g(0, 0, z) + g(0, Y, Z) + 3zY. \tag{3.5}$$

Considering (3.4) and (3.5) together now reveals that the term $3zY$ in (3.5) is identically zero and hence the characteristic of the field is three. □

By Theorem 3.1, translation ovoids can only exist when $H(q)$ is self-dual, so it suffices to consider only translation spreads. However, we will state and prove Theorem 3.11 in the context of ovoids. This is due essentially only to the fact that the functions $f(x, y)$ and $g(x, y)$ in the coordinate representation (2.35) of the $\mathcal{O}_\lambda$ ovoids seem nicer than the corresponding functions for the $\mathcal{S}_\lambda$ spreads in (2.36). All other results will be stated and proven in the context of spreads.

The rest of this section is primarily devoted to breaking Theorem 6 of [BTVM98] into its constituent components in order to elaborate and extend on them.

We have already noted that the known locally hermitian spreads are translation with respect to a line. This actually comes as no surprise in light of the following results which give a strong connection between translation and locally hermitian spreads. The next lemma and its proof are a portion of the proof of [BTVM98, Theorem 6].

Lemma 3.2 (Bloemen, Thas, Van Maldeghem [BTVM98, Theorem 6]) *Let $\mathcal{S}$ be a spread of $H(q)$, $q \neq 3^h$, that is translation with respect to a flag $\{L, x\}$, where L is a line of $\mathcal{S}$. Then $\mathcal{S}$ is point locally hermitian with respect to the point x.*

Proof Let $K \neq L$ be a line of $\mathcal{S}$. We need to show that for each point y on the unique line of the quadric that joins x to K, there is a line M of $\mathcal{S}$ incident with it. There is an apartment of $H(q)$ containing the lines L and K as well as the point x, and we suppose that this is taken to be the hat-rack of the coordinatization such that $\{L, x\}$ is the flag $\{[\infty], (\infty)\}$ and K is the line $[0, 0, 0, 0, 0]$. Then the points y are the points with coordinates $(0, 0, z, 0)$.

From the representation in (2.3), we have

$$\mathcal{S} = \big\{[\infty]\big\} \cup \big\{[x, y, z, f(x, y, z), g(x, y, z)] \;\big|\; x, y, z \in GF(q)\big\}, \tag{3.6}$$

where $f(0, 0, 0) = g(0, 0, 0) = 0$, and from (3.1),

$$G_{\{[\infty],(\infty)\}} = \big\{\Theta[0, y, z, f(0, y, z), g(0, y, z)] \;\big|\; y, z \in GF(q)\big\}. \tag{3.7}$$

What we need to show is that $f(0, 0, z) = 0$ for all values of z.

From (1.25), the collineation $\Theta[0, Y, Z, f(0, Y, Z), g(0, Y, Z)]$ of $G_{\{[\infty],(\infty)\}}$ maps the line $[0, y, z, f(0, y, z), g(0, y, z)]$ of $\mathcal{S}$ to

$$[0, y + Y, z + Z - 3yf(0, Y, Z), f(0, y, z) + f(0, Y, Z), g(0, y, z) + g(0, Y, Z)].$$

This must also be a line of $\mathcal{S}$, so we have

$$f\big(0, y + Y, z + Z - 3yf(0, Y, Z)\big) = f(0, y, z) + f(0, Y, Z).$$

Putting $Y = 0$ and $y = 0$ into this in turn, we obtain the two equations

$$f\big(0, y, z + Z - 3yf(0,0,Z)\big) = f(0, y, z) + f(0, 0, Z)$$

and

$$f(0, Y, z + Z) = f(0, 0, z) + f(0, Y, Z).$$

Using these two equations alternately, starting with the latter, we find

$$\begin{aligned} f(0, y, z) &= f(0, y, 0) + f(0, 0, z) \\ &= f\big(0, y, z - 3yf(0,0,z)\big) \\ &= f(0, y, z) + f\big(0, 0, -3yf(0,0,z)\big). \end{aligned}$$

Therefore, $f\big(0, 0, -3yf(0,0,z)\big) = 0$ for all $y, z \in GF(q)$. If $f(0,0,z) \neq 0$ for some z, then by choosing $y = \frac{-z}{3f(0,0,z)}$ we have, to the contrary, that $f(0,0,z) = 0$. Hence $f(0,0,z) = 0$ for all z and the result follows. □

Notice that the condition $q \neq 3^h$ cannot be entirely removable, as at least for the case when h is odd, the Ree-Tits spreads are examples of spreads that are translation with respect to a flag and yet are not point locally hermitian. Notice that it is a consequence of this lemma that if $q \neq 3^h$ and $\mathcal{S}$ is also translation with respect to another flag on the same line, then by Theorem 2.10, the spread $\mathcal{S}$ is locally hermitian. As it happens, this much does also hold for $q = 3^h$.

Theorem 3.3 (Bloemen, Thas, Van Maldeghem [BTVM98, Theorem 6]) *Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to two distinct flags $\{L, x\}$ and $\{L, y\}$ on a common line L. Then $\mathcal{S}$ is locally hermitian with respect to L.*

Proof For $q \neq 3^h$, the result follows from Theorem 2.10 and Lemma 3.2. Suppose then that $q = 3^h$. Again, this proof is essentially the one from [BTVM98].

Since the generalized hexagon $H(q)$ is self-dual when $q = 3^h$, we have from the coordinate representation of ovoids in (2.6) that we may suppose that $\mathcal{S}$ is given by

$$\mathcal{S} = \big\{[\infty]\big\} \cup \big\{[x, f(x,y,z), z, g(x,y,z), y] \;\big|\; x, y, z \in GF(q)\big\},$$

where $f(0,0,0) = g(0,0,0) = 0$. Furthermore, we may suppose that the hat-rack of the coordinatization has been chosen such that the two flags with respect to which $\mathcal{S}$ is translation are $\{[\infty], (\infty)\}$ and $\{[\infty], (0)\}$. From (1.25), the collineation $\Theta[k, b, k', b', k'']$ maps the line $[0, \ell]$ to the line $[0, \ell + k'']$, so the lines incident with (0) are fixed if and only if $k'' = 0$. Together with (3.1), we then have

$$G_{\{[\infty],(\infty)\}} = \big\{\Theta[0, f(0,y,z), z, g(0,y,z), y] \;\big|\; y, z \in GF(q)\big\}$$

and

$$G_{\{[\infty],(0)\}} = \big\{\Theta[x, f(x,0,z), z, g(x,0,z), 0] \;\big|\; x, z \in GF(q)\big\}.$$

We need to show that the functions f and g are independent of z. In the following, we will work only with the function f, but it will be observed that all the same conclusions apply equally to g.

Applying the collineation $\Theta[0, f(0,Y,Z), Z, g(0,Y,Z), Y]$ from the group $G_{\{[\infty],(\infty)\}}$ to the line $[x, f(x,y,z), z, g(x,y,z), y]$, we get

$$[x, f(x,y,z) + f(0,Y,Z), z + Z - xY, g(x,y,z) + g(0,Y,Z), y + Y].$$

Since this must also be a line of $\mathcal{S}$, this gives

$$f(x,y,z) + f(0,Y,Z) = f(x, y+Y, z+Z-xY).$$

Next we apply $\Theta[X, f(X,0,Z), Z, g(X,0,Z), 0]$, which belongs to the group $G_{\{[\infty],(0)\}}$, to the same line and get

$$[x+X, f(x,y,z) + f(X,0,Z), z+Z, g(x,y,z) + g(X,0,Z), y],$$

whence we have

$$f(x,y,z) + f(X,0,Z) = f(x+X, y, z+Z).$$

Now using these alternately, starting with the latter, we have

$$\begin{aligned} f(x,y,0) &= f(x,0,0) + f(0,y,0) \\ &= f(x,y,-xy) \\ &= f(x,y,0) + f(0,0,-xy). \end{aligned}$$

Thus $f(0,0,-xy) = 0$ for all $x, y \in GF(q)$; that is, $f(0,0,z) = 0$ for all $z \in GF(q)$. Finally, $f(x,y,z) = f(x,y,0) + f(0,0,z) = f(x,y,0)$, so the function f, and similarly g, is independent of z and therefore $\mathcal{S}$ is locally hermitian with respect to $[\infty]$. □

The groups $G_{\{L,x\}}$ are uniquely determined, they act regularly on the sets V_K and they have order q^2. On the whole, they are quite managable and we can get our hands on them fairly well if we need to. However, it is not quite so clear what the groups G_L are like. On their action on $\mathcal{S}^+$, we only know that it is semiregular, and as for their orders, we only know that they are powers of p lying somewhere between q^2 and q^3 inclusive. If $|G_L| = q^3$ then G_L acts regularly on $\mathcal{S}^+$ so, since we have $\mathcal{S}^+$ and we know $G_L \leqslant G^L$, we then have G_L at least as well as we have the groups $G_{\{L,x\}}$. It is with this in mind that we now proceed to get a firmer grip on these groups G_L.

Theorem 3.4 (Bloemen, Thas, Van Maldeghem [BTVM98, Lemma 5])
Let $\mathcal{S}$ be a spread of $H(q)$ and let L be a line of $\mathcal{S}$. If a subgroup of G^L of order q^3 stabilizes $\mathcal{S}$ then $\mathcal{S}$ is translation with respect to L.

Proof Let x be any point on L. We have only to show that $\mathcal{S}$ is translation with respect to the flag $\{L, x\}$.

Let G be the subgroup of G^L with $|G| = q^3$ that stabilizes $\mathcal{S}$. The action of G^L on the set of lines opposite L is semiregular, so the action of G is, too. Now since $|G|$ is equal to the number of lines of $\mathcal{S}$ opposite L, it follows that G acts transitively on them. Letting $K \neq L$ be a line incident with x, there is then a subgroup $H \leqslant G$ that acts transitively on the q^2 lines of the set V_K. By Theorem 1.23, since the collineations of $H \leqslant G^L$ fix the line K on x, they fix every line through x and hence H fixes every such set V_K. Noting that the action of H is semiregular since it is a subgroup of G^L, we see that H acts transitively on each of the sets V_K. Hence $\mathcal{S}$ is translation with respect to the flag $\{L, x\}$ with associated group $G_{\{L,x\}} = H$. □

The converse of the previous theorem is also true, as we shall see presently. While the case $q \not\equiv 2 \pmod 3$ is treated by [BTVM98, Theorem 6], we add the remaining values of q in our following theorem.

Theorem 3.5
Let $\mathcal{S}$ be a spread of $H(q)$, $q = p^h$, and let L be a line of $\mathcal{S}$. If

(i) $q \not\equiv 2 \pmod 3$ and $\mathcal{S}$ is translation with respect to two distinct flags on L;

(ii) $q \equiv 2 \pmod 3$, q odd, and $\mathcal{S}$ is translation with respect to $2 + \frac{2(q-1)}{p-1}$ distinct flags on L; or

(iii) $q = 2^{2e+1}$ and $\mathcal{S}$ is translation with respect to $2(q + 1)/3$ distinct flags on L;

then $\mathcal{S}$ is translation with respect to L. Moreover, the stabilizer $G^L_{\mathcal{S}}$ of $\mathcal{S}$ in G^L acts transitively on $\mathcal{S} \setminus \{L\}$ so $|G^L_{\mathcal{S}}| = q^3$, and either $G_L = G^L_{\mathcal{S}}$ or $q = 2^{2e+1}$ and $\left[G^L_{\mathcal{S}} : G_L\right] = 2$.

Proof We know from Theorem 3.3 that if $\mathcal{S}$ is translation with respect to two different flags on L then it is locally hermitian with respect to L. Thus we consider a locally hermitian spread $\mathcal{S}$ as given by (2.28) that is translation with respect to the flag $\{[\infty], (\infty)\}$. Thus

$$\mathcal{S} = \bigcup_{x,y \in GF(q)} [[x, y, f(x, y), g(x, y)]]$$

with $f(0, 0) = g(0, 0) = 0$, and from (3.1), the associated group for the flag $\{[\infty], (\infty)\}$ is

$$G_{\{[\infty],(\infty)\}} = \left\{\Theta[0, y, z, f(0, y), g(0, y)] \;\middle|\; y, z \in GF(q)\right\}.$$

What we aim to show is that $G_{\{[\infty],(\infty)\}}$ acts transitively on the set of lines not equal to $[\infty]$ that are incident with some point (a) on $[\infty]$, although we drop a little short of this goal when $q = 2^{2e+1}$. Then if $\mathcal{S}$ is also translation with respect to the flag $\{[\infty], (a)\}$, the group $G = \left\langle G_{\{[\infty],(\infty)\}}, G_{\{[\infty],(a)\}}\right\rangle$ acts transitively on the set of all q^3 lines of $\mathcal{S}$ opposite $[\infty]$, so $|G| = q^3$ and $\mathcal{S}$ is translation with respect to the line $[\infty]$ by Theorem 3.4. Also, since $G \leqslant G_{[\infty]} \leqslant G^{[\infty]}_{\mathcal{S}}$ and $\left|G^{[\infty]}_{\mathcal{S}}\right| \leq q^3$ by the regularity of $G^{[\infty]}$, we will also have $G_{[\infty]} = G^{[\infty]}_{\mathcal{S}} = G$.

Applying the collineation $\Theta[0, Y, Z, f(0, Y), g(0, Y)]$ in $G_{\{[\infty],(\infty)\}}$ to the line regulus $[[x, y, f(x, y), g(x, y)]]$ of $\mathcal{S}$ gives

$$[[x, y + Y, f(x, y) + f(0, Y), g(x, y) + g(0, Y)]],$$

which must also be a line regulus of $\mathcal{S}$. Thus $f(x, y) = f_1(x) + f_2(y)$ and $g(x, y) = g_1(x) + g_2(y)$, where f_2 and g_2 are linear operators on $GF(q)$ over $GF(p)$; that is

$$f_2(y + Y) = f_2(y) + f_2(Y) \quad \text{and} \quad g_2(y + Y) = g_2(y) + g_2(Y).$$

The projection of the line $[0, y, z, f_2(y), g_2(y)]$ of $\mathcal{S}$ onto a point (a) is the line $[a, h_a(y)]$, where

$$h_a(y) = -3a^2y - 3af_2(y) + g_2(y)$$

as given by the incidence equations in (1.14b). For each $a \in GF(q)$, let

$$K_a = \ker h_a = \{y \in GF(q) \mid h_a(y) = 0\}.$$

Notice that for each a, the function h_a is a linear operator of $GF(q)$ over $GF(p)$ since it is a linear combination of such functions. Thus K_a is a vector space over $GF(p)$ so $|K_a|$ is a power of p. Now as $G_{\{[\infty],(\infty)\}}$ acts transitively on the lines $[0, y, z, f_2(y), g_2(y)]$, it acts transitively on the lines $[a, h_a(y)]$. So what we want to show is that for some a, the function $h_a(y)$ is a bijection, or equivalently, that $K_a = \{0\}$.

First consider when $q = 3^h$ so then $h_a(y) = g_2(y)$. Applying the compatibility condition in (2.31a) to the line reguli $[[0, y, f_2(y), g_2(y)]]$ and $[[0, 0, 0, 0]]$, we have $-y^3g_2(y) = \not\square$ whenever $y \neq 0$, so in particular, $h_a(y) \neq 0$ whenever $y \neq 0$. Thus for all a we have $K_a = \{0\}$ and hence the result follows.

Suppose now that $3 \nmid q$. Treating $h_a(y) = 0$ with $y \neq 0$ as an equation in a, we have a quadratic equation. For odd q, the discriminant is

$$9f_2(y)^2 + 12yg_2(y) = -3\bigl(-3f_2(y)^2 - 4yg_2(y)\bigr), \tag{3.8}$$

and for even q, the S-invariant is

$$\frac{yg_2(y)}{f_2(y)^2}. \tag{3.9}$$

Since $\mathcal{S}$ is a spread, the line reguli $[[0, y, f_2(y), g_2(y)]]$ and $[[0, 0, 0, 0]]$ are compatible. From the compatibility condition in (2.31a), for odd q we have

$$y^2f_2(y)^2 - 4y^2\bigl(yg_2(y) + f_2(y)^2\bigr) = y^2\bigl(-3f_2(y)^2 - 4yg_2(y)\bigr) = \not\square,$$

and from (2.31b), for even q we have that $f_2(y) \neq 0$ and

$$\operatorname{Tr}\Bigl(\frac{y^2f_2(y)^2 + y^3g_2(y)}{y^2f_2(y)^2}\Bigr) = \operatorname{Tr}(1) + \operatorname{Tr}\Bigl(\frac{yg_2(y)}{f_2(y)^2}\Bigr) = 1.$$

Thus when $q \equiv 1 \pmod 3$, if q is odd then -3 is a square so the discriminant in (3.8) is nonsquare, and if q is even then $q = 2^{2e}$ and $\operatorname{Tr}(1) = 0$ so the S-invariant in (3.9) has trace one. In both cases, $h_a(y) = 0$ has no solutions in a for any $y \neq 0$, thus $K_a = \{0\}$ for all $a \in GF(q)$ and the result follows for $q \equiv 1 \pmod 3$.

Now suppose $q \equiv 2 \pmod 3$. Then the discriminant in (3.8) is a nonzero square and the S-invariant in (3.9) has trace zero while $f_2(y) \neq 0$. Therefore, for each $y \neq 0$ there are exactly two values, a and b, such that $y \in K_a$ and $y \in K_b$.

Let N_i be the number of values of a for which $|K_a| = p^i$, where $i = 0, 1, \ldots, h$. Then counting the nonzero elements in the sets K_a, remembering that each $y \neq 0$ belongs to exactly two of them, we have

$$(p-1)N_1 + (p^2-1)N_2 + \cdots + (p^h-1)N_h = 2(q-1). \tag{3.10}$$

Writing $N_{\geq a} = N_a + N_{a+1} + \cdots + N_h$, equation (3.10) gives us

$$(p-1)N_{\geq 1} \leq 2(q-1) \Rightarrow N_{\geq 1} \leq \frac{2(q-1)}{p-1}.$$

Hence if in addition to the flag $\{[\infty],(\infty)\}$ we have at least $\frac{2(q-1)}{p-1}+1$ more flags on $[\infty]$ with respect to which $\mathcal{S}$ is translation, then for at least one of these, say $\{[\infty],(a)\}$, we will have $K_a = \{0\}$. The result now follows for odd $q \equiv 2 \pmod 3$.

Finally, suppose $q = 2^{2e+1}$. From (3.10) we have $3N_{\geq 2} \leq 2(q-1)$. The greatest multiple of 3 not greater than $2(q-1)$ is $2(q-2)$ so we can tighten this inequality to give $N_{\geq 2} \leq 2(q-2)/3$. Thus if, including the flag $\{[\infty],(\infty)\}$, the spread $\mathcal{S}$ is translation with respect to at least $\frac{2(q-2)}{3}+2 = 2(q+1)/3$ flags on $[\infty]$, then for at least one of these, say $\{[\infty],(a)\}$, we will have $|K_a| = 1$ or 2. If the group $H = \langle G_{\{[\infty],(\infty)\}}, G_{\{[\infty],(a)\}}\rangle$ has order q^3, as certainly happens when $|K_a| = 1$, then the result follows as in the previous cases, so we suppose now that $|H| < q^3$ and that $|K_a| = 2$.

Notice that $G_{\{[\infty],(\infty)\}} \cap G_{\{[\infty],(a)\}}$ is the set of collineations $\Theta[0,y,z,f_2(y),g_2(y)]$, with $y \in K_a$ and $z \in GF(q)$, so this intersection contains $2q$ elements. Then

$$|H| \geq \left|G_{\{[\infty],(\infty)\}}G_{\{[\infty],(a)\}}\right| = \frac{\left|G_{\{[\infty],(\infty)\}}\right|\left|G_{\{[\infty],(a)\}}\right|}{\left|G_{\{[\infty],(\infty)\}} \cap G_{\{[\infty],(a)\}}\right|} = \frac{q^2q^2}{2q} = \tfrac{1}{2}q^3,$$

so we conclude that $|H| = q^3/2$. The orbit of $[0,0,0,0,0]$ under H is then

$$\mathcal{S}' = \big\{[x,y,z,f(x,y),g(x,y)] \;\big|\; x \in A \text{ and } y,z \in GF(q)\big\}$$

for some subset $A \subseteq GF(q)$ of order $q/2$, and the group H is

$$H = \big\{\Theta[x,y,z,f(x,y),g(x,y)] \;\big|\; x \in A \text{ and } y,z \in GF(q)\big\}.$$

The collineation $\Theta[X,0,0,f_1(X),g_1(X)] \in H$ sends the line regulus $[[x,0,f_1(x),g_1(x)]]$ of $\mathcal{S}$ to $[[x+X,0,f_1(x+X),g_1(x+X)]]$, which is then another line regulus of $\mathcal{S}$. Thus $f_1(x+X) = f_1(x)+f_1(X)$ for all $x \in GF(q)$ and $X \in A$. Also, since the set $\mathcal{S}'$ is fixed by H, we have that for $x \in A$ and $X \in A$, the sum $x+X$ belongs to A as well, so A is a subgroup of $\big(GF(q),+\big)$ of index 2. Consequently, if $x \notin A$ and $X \notin A$, then $x+X \in A$ so $f_1(x) = f_1\big((x+X)+X\big) = f_1(x+X)+f_1(X)$ and therefore $f_1(x+X) = f_1(x)+f_1(X)$. Hence f_1, and similarly g_1, are linear operators of $GF(q)$ over $GF(2)$. It now follows that the set $\big\{\Theta[x,y,z,f(x,y),g(x,y)] \;\big|\; x,y,z \in GF(q)\big\}$ of collineations forms a subgroup of $G^{[\infty]}$ fixing $\mathcal{S}$. Since its order is q^3, this subgroup is the stabilizer $G^{[\infty]}_{\mathcal{S}}$ of $\mathcal{S}$ in $G^{[\infty]}$, so by Theorem 3.4, the spread $\mathcal{S}$ is translation with respect to the line $[\infty]$. Also, we have $H \leqslant G_{[\infty]} \leqslant G^{[\infty]}_{\mathcal{S}}$ so either $G_{[\infty]} = G^{[\infty]}_{\mathcal{S}}$ or $G_{[\infty]} = H$, in which case $\big[G^{[\infty]}_{\mathcal{S}} : G_{[\infty]}\big] = 2$. □

In the previous theorem, the exceptional case when $q = 2^{2e+1}$ is certainly not entirely removable as at least when $e = 0$ it does indeed occur. Consider the spread $\mathcal{S}_H(1,1) = \bigcup_{x,y}[[x,y,x+y,x]]$. The collineation $\Theta[x,y,z,x+y,x] \in G^{[\infty]}_{\mathcal{S}}$ maps the line $[a,\ell]$ to the line $[a,\ell+x]$, so the point (a) is fixed linewise if and only if $x = 0$. Thus $G_{[\infty]} = G_{\{[\infty],(0)\}} = G_{\{[\infty],(1)\}} = G_{\{[\infty],(\infty)\}}$.

In view of Theorem 3.5, just as the associated group $G_{\{[\infty],(\infty)\}}$ of a spread that is translation with respect to the flag $\{[\infty],(\infty)\}$ is quite accessible, so too is the stabilizer $G_{\mathcal{S}}^{[\infty]}$ in $G^{[\infty]}$ of a spread that is translation with respect to the line $[\infty]$. Although this stabilizer is usually identical to the associated group $G_{[\infty]}$, the possibility that it is not when $q = 2^{2e+1}$ makes the associated group potentially bothersome, so the theorem suggests that the stabilizer in $G^{[\infty]}$ is perhaps the more convenient group to work with, and not the associated group after all. This is made a little clearer by our following theorem.

Theorem 3.6
A spread $\mathcal{S}$ of $H(q)$ is translation with respect to a line L if and only if the stabilizer $G_{\mathcal{S}}^{L}$ of $\mathcal{S}$ in G^{L} has order q^3.

Proof This follows from Theorems 3.4 and 3.5. □

The rather strong condition in Theorem 3.6 for a spread $\mathcal{S}$ to be translation with respect to a line L enables us to get a better picture of what $\mathcal{S}$ looks like, which we do in our next theorem.

Theorem 3.7
Let $\mathcal{S}$ be a spread of $H(q)$, $q = p^h$, containing the lines $[\infty]$ and $[0,0,0,0,0]$, that is represented as in either (2.28) or (2.32). Then $\mathcal{S}$ is translation with respect to the line $[\infty]$ if and only if the functions of the representation are of the form

$$f(x,y) = \sum_{i=0}^{h-1}\left(f_{1i}x^{p^i} + f_{2i}y^{p^i}\right) \quad \text{and} \quad g(x,y) = \sum_{i=0}^{h-1}\left(g_{1i}x^{p^i} + g_{2i}y^{p^i}\right),$$

with the coefficients $f_{ni}, g_{ni} \in GF(q)$.

Proof Let us assume that $\mathcal{S}$ is represented as in (2.28). If the representation of (2.32) is being used, then the same proof works, just swapping the second and last coordinates of lines and line reguli.

Suppose $\mathcal{S}$ is translation with respect to $[\infty]$. By Theorem 3.6, the stabilizer of $\mathcal{S}$ in $G^{[\infty]}$ is

$$G_{\mathcal{S}}^{[\infty]} = \left\{\Theta[x,y,z,f(x,y),g(x,y)] \;\middle|\; x,y,z \in GF(q)\right\}.$$

The collineation $\Theta[X,Y,Z,f(X,Y),g(X,Y)]$ from $G_{\mathcal{S}}^{[\infty]}$ when applied to the line regulus $[[x,y,f(x,y),g(x,y)]]$ of $\mathcal{S}$, gives the line regulus

$$\big[\big[x+X, y+Y, f(x,y)+f(X,Y), g(x,y)+g(X,Y)\big]\big].$$

From here we see that

$$f(x+X,y+Y) = f(x,y)+f(X,Y) \quad \text{and} \quad g(x+X,y+Y) = g(x,y)+g(X,Y).$$

Thus the functions $f(x,0)$, $f(0,y)$, $g(x,0)$ and $g(0,y)$ are all linear operators of $GF(q)$ over $GF(p)$, so by Theorem 1.4, the functions f and g take the claimed forms.

Suppose now that f and g are of the forms given in the statement of the theorem. Then the collineations $\Theta[x,y,z,f(x,y),g(x,y)]$ form a group of order q^3 fixing $\mathcal{S}$. The result now follows from Theorem 3.4. □

3.3 Spreads translation with respect to a line

In this section are two classifications of spreads translation with respect to a line subject to some condition. First, we consider those translation spreads of $H(q)$, with q odd, for which an object, which we shall call the kernel, is all of $GF(q)$. Then our attention turns to the translation spreads of $H(3^h)$. Since the hexagon $H(3^h)$ is self-dual, this is equivalent to considering ovoids translation with respect to a point, and this is exactly what we shall do, essentially because the resulting functions are a little tidier. Of these two cases, the latter appears in [Off01], and the former we shall see in Theorem 4.5 is equivalent to [BTVM98, Theorems 30–32].

Let $\mathcal{S}$ be a spread of $H(q)$ that contains the lines $[\infty]$ and $[0,0,0,0,0]$ and that is translation with respect to the line $[\infty]$. Then $\mathcal{S}$ can be represented as in (2.28), and possibly also as in (2.32). Given one of these representations, we define the **kernel** of $\mathcal{S}$, denoted $\ker\mathcal{S}$, to be the maximal subfield of $GF(q)$ such that for all $a \in \ker\mathcal{S}$ and all $x, y \in GF(q)$, the functions f and g of the representation satisfy

$$f(ax, ay) = af(x, y) \quad \text{and} \quad g(ax, ay) = ag(x, y).$$

From Theorem 3.7, the functions $f(x,0)$, $f(0,y)$, $g(x,0)$ and $g(0,y)$ are then all linear operators on $GF(q)$ over $\ker\mathcal{S}$.

The kernel, $\ker\mathcal{S}$, is well-defined in that it is independent of which of the representations from (2.28) and (2.32) is used (if the one in (2.32) applies), and also, it is independent of the choice of hat-rack for the coordinatization, up to the requirement that $\mathcal{S}$ should contain the line $[0,0,0,0,0]$ and be translation with respect to $[\infty]$. Let us make it clearer that this is indeed so.

Suppose $\mathcal{S}$ can be represented in both the form of (2.28) as well as that of (2.32), so for some functions f, g, F and G we have

$$\mathcal{S} = \bigcup_{x,y\in GF(q)} [[x, y, f(x,y), g(x,y)]] = \bigcup_{x,y\in GF(q)} [[x, F(x,y), G(x,y), y]].$$

Then from here we have

$$\begin{aligned} f\big(x, F(x,y)\big) &= G(x,y), & F\big(x, g(x,y)\big) &= y, \\ g\big(x, F(x,y)\big) &= y, & G\big(x, g(x,y)\big) &= f(x,y). \end{aligned}$$

Letting $\ker_1\mathcal{S}$ be the kernel of $\mathcal{S}$ as determined by the former representation of $\mathcal{S}$, which involves f and g, and letting $\ker_2\mathcal{S}$ be the kernel arising from the latter with F and G, we then have that for each $a \in \ker_1\mathcal{S}$,

$$F(ax, ay) = F\big(ax, ag(x, F(x,y))\big) = F\big(ax, g(ax, aF(x,y))\big) = aF(x,y).$$

Hence also,

$$G(ax, ay) = f\big(ax, F(ax, ay)\big) = f\big(ax, aF(x,y)\big) = af\big(x, F(x,y)\big) = aG(x,y).$$

Thus $\ker_1\mathcal{S} \subseteq \ker_2\mathcal{S}$. Similarly, the reverse inclusion holds, so $\ker_1\mathcal{S} = \ker_2\mathcal{S}$.

Now consider the collineations θ that fix $[\infty]$ and map some line of $\mathcal{S}$ to $[0,0,0,0,0]$. What we must show is that $\ker \mathcal{S}^\theta = \ker \mathcal{S}$.

If θ is induced by an automorphism ψ of the field $GF(q)$, then

$$\mathcal{S}^\theta = \bigcup_{x,y \in GF(q)} [[x^\psi, y^\psi, f(x,y)^\psi, g(x,y)^\psi]],$$

from which it is readily seen that $\ker \mathcal{S}^\theta = \ker \mathcal{S}$. By [VM98, Proposition 4.6.6(iv)], we now need only consider those collineations θ belonging to $G_2(q)$.

From (1.11), the number of points in $H(q)$ is $(q^6-1)/(q-1)$, and since each point has $q+1$ lines incident with it, the number of flags is $(q+1)(q^6-1)/(q-1)$. Since $H(q)$ is a Moufang polygon with respect to the group $G_2(q)$, this group acts transitively on the set of all flags, so the stabilizer H in $G_2(q)$ of the flag $\{[\infty], (\infty)\}$ has order

$$|H| = \frac{|G_2(q)|}{\#\text{ flags in } H(q)} = q^6(q-1)^2,$$

where the order of $G_2(q)$ is given in [VM98, Proposition 4.6.7]. It now follows from [VM98, Proposition 4.6.6(v) and Lemma 5.2.3(iii)] that the group H is generated by the torus for the hat-rack, together with the root groups for the paths γ_P, $\gamma_P\mu^{-1}$, $\gamma_P\mu^{-2}$, γ_L, $\gamma_L\mu$ and $\gamma_L\mu^2$. In addition, the collineation $E(\gamma_P\mu, \delta)$, with $\delta \neq 0$, fixes the lines $[\infty]$ and $[0,0,0,0,0]$ while mapping the point (∞) to the point (δ^{-1}). Since H contains $E(\gamma_P\mu^{-2})$, by the Moufang property of $H(q)$, the group H acts transitively on the set $\Gamma_1([\infty]) \setminus \{(\infty)\}$ of points distinct from (∞) that are incident with $[\infty]$. Thus H together with $E(\gamma_P\mu, \delta)$ generate the stabilizer H' of the line $[\infty]$ in $G_2(q)$, and we therefore need only consider collineations θ from this collection of generators for H'.

Let $\theta \in G^{[\infty]}$. From Theorem 3.5, the stabilizer of $\mathcal{S}$ in $G^{[\infty]}$ acts regularly on the lines of $\mathcal{S}$ opposite $[\infty]$, and since $\mathcal{S}^\theta$ is to contain the line $[0,0,0,0,0]$ and the action of $G^{[\infty]}$ is semiregular, it follows that θ fixes $\mathcal{S}$, so certainly, $\ker \mathcal{S}^\theta = \ker \mathcal{S}$.

Let $\theta \in E(\gamma_P\mu^{-2})$. Then for some $\delta \in GF(q)$, we have

$$\mathcal{S}^\theta = \bigcup_{x,y \in GF(q)} [[x, y+\delta x, f(x,y)+\delta^2 x + 2\delta y, g(x,y) - \delta^3 x - 3\delta^2 y - 3\delta f(x,y)]].$$

From here, we can see that multiplying the first two coordinates of a line regulus in $\mathcal{S}^\theta$ by an element $a \in \ker \mathcal{S}$ is equivalent to multiplying x and y by a, and thence the last two coordinates are multiplied by a as well. Thus $\ker \mathcal{S}^\theta \subseteq \ker \mathcal{S}$. Using the collineation θ^{-1}, this argument is reversible, so we conclude that $\ker \mathcal{S}^\theta = \ker \mathcal{S}$.

Let $\theta = \mathcal{T}(\alpha, \beta)$ be an element of the torus for the hat-rack. Then

$$\mathcal{S}^\theta = \bigcup_{x,y \in GF(q)} [[\beta x, \alpha\beta y, \alpha^2\beta f(x,y), \alpha^3\beta g(x,y)]],$$

from which we see again by the same reasoning that $\ker \mathcal{S}^\theta = \ker \mathcal{S}$.

Finally, let $\theta = E(\gamma_P\mu, \delta)$. Working with the matrices for the collineations given in Section 1.4.4, we find that

$$\mathcal{S}^\theta = \bigcup_{x,y \in GF(q)} [[x + \delta^3 g(x,y) - 3\delta y + 3\delta^2 f(x,y), \\ y - 2\delta f(x,y) - \delta^2 g(x,y), f(x,y) + \delta g(x,y), g(x,y)]],$$

and we can see once again that $\ker \mathcal{S}^{\theta} = \ker \mathcal{S}$ since the coefficients are all simply linear combinations of x, y, $f(x,y)$ and $g(x,y)$.

Thus we have demonstrated that the definition of kernel introduced here is well-defined in the sense mentioned above, and for the benefit of future reference we state this in the form of a theorem.

Theorem 3.8
Given a spread $\mathcal{S}$ that is translation with respect to a line, we may choose to represent it either in the form (2.28), or in the form (2.32) if possible, and then the kernel, $\ker \mathcal{S}$, is independent of which representation is chosen and of the choice of hat-rack, up to the requirement that $\mathcal{S}$ should contain the line $[0,0,0,0,0]$ and be translation with respect to the line $[\infty]$. □

Now we give our following classification of translation spreads for which $\ker \mathcal{S}$ is as large as possible.

Theorem 3.9
Let $\mathcal{S} = \bigcup_{x,y} \big[\big[x, y, f(x,y), g(x,y)\big]\big]$, where $f(0,0) = g(0,0) = 0$, be a translation spread of $H(q)$, q odd, with kernel $\ker \mathcal{S} = GF(q)$. Then either $\mathcal{S}$ is hermitian or $q \equiv 1 \pmod 3$ and $\mathcal{S}$ is isomorphic to $\mathcal{S}_{[9]}$.

Proof The collineation $\Psi(A,0,0,0,0)$, as described in (1.24) of Section 1.4.4, leaves the line regulus $[[0,0,0,0]]$ of $\mathcal{S}$ fixed, and putting $A = f(0,1)/2$, it sends the line regulus $\big[\big[0,1,f(0,1),g(0,1)\big]\big]$ of $\mathcal{S}$ to a line regulus $[[0,1,0,-\gamma]]$, for some $\gamma \in GF(q)$. Thus we may assume that $f(0,1) = 0$ and we set $\gamma = -g(0,1)$. Since $[[0,1,0,-\gamma]]$ is then a line regulus of $\mathcal{S}$, it is compatible with $[[0,0,0,0]]$, so from the compatibility condition in (2.31a) we have that γ is a nonsquare.

Consider the line regulus $[[1,0,a,b]]$ of $\mathcal{S}$, where $a = f(1,0)$ and $b = g(1,0)$. This line regulus is compatible with all the line reguli $\big[\big[0,y,f(0,y),g(0,y)\big]\big]$ which, since $\ker \mathcal{S} = GF(q)$, are precisely the line reguli $[[0,y,0,-\gamma y]]$. From the compatibility condition in (2.31a), we then have that

$$4\gamma y^4 + 4by^3 + (\gamma^2 - 3a^2 - 6a\gamma)y^2 + (2\gamma b - 6ab)y + (b^2 + 4a^3) \tag{3.11}$$

is a nonsquare for all $y \in GF(q)$. By Lemma 1.3, this quartic is then identical to $\gamma(2y^2 + uy + v)^2$ for some u and v. Equating coefficients gives a system of equations which leads to

$$\begin{aligned} b^2 + 4\gamma^2 v &= \gamma^3 - 3\gamma a^2 - 6\gamma^2 a \\ bv &= \gamma b - 3ab \\ \gamma v^2 &= b^2 + 4a^3. \end{aligned} \tag{3.12}$$

If $b \neq 0$, then from the second of these we have $v = \gamma - 3a$. Substituting this into the two remaining equations and eliminating b^2 then gives

$$-4\gamma^3 + 12a\gamma^2 - 12a^2\gamma + 4a^3 = 4(a-\gamma)^3 = 0$$

so $a = \gamma$. But then $v = -2\gamma$ and the third equation is $4\gamma^3 = b^2 + 4\gamma^3$, which leads to the contradiction $b = 0$.

Therefore $b = 0$. From the first and third equations in (3.12) we now have

$$64\gamma a^3 = 16\gamma(4a^3) = 16\gamma(\gamma v^2) = (4\gamma v)^2 = (\gamma^2 - 3a^2 - 6\gamma a)^2$$

whence we obtain

$$(\gamma^2 - 3a^2 - 6\gamma a)^2 - 64\gamma a^3 = 9a^4 - 28\gamma a^3 + 30\gamma^2 a^2 - 12\gamma^3 a + \gamma^4 = (a-\gamma)^3(9a-\gamma) = 0.$$

Hence either $a = \gamma$ or $3 \nmid q$ and $a = \gamma/9$. We already have $g(x,y) = xg(1,0) + yg(0,1) = -\gamma y$. If $a = \gamma$ then $f(x,y) = xf(1,0) + yf(0,1) = x\gamma$ so $\mathcal{S} = \mathcal{S}_H(0,-\gamma)$. Consider now the case when $a = \gamma/9$. The quartic in (3.11) is then

$$4\gamma y^4 + \frac{8\gamma^2 y^2}{27} + \frac{4\gamma^3}{27^2} = 4\gamma\Big(y^2 + \frac{\gamma}{27}\Big)^2$$

which is a nonsquare for all $y \in GF(q)$ so long as it is never zero. Thus $a = \gamma/9$ leads to a spread if and only if $-\gamma/27$ is a nonsquare, or equivalently, -3 is a nonzero square. From the proof of Corollary 1.2, this occurs precisely when $q \equiv 1 \pmod 3$. Finally, $f(x,y) = xf(1,0) + yf(0,1) = -\frac{1}{9}\gamma x$, so $\mathcal{S} = \mathcal{S}_{[9]}$. □

Corollary 3.10
If $\mathcal{S}$ is a spread of $H(p)$, p a prime, that is translation with respect to a line then either $\mathcal{S}$ is hermitian or $p \equiv 1 \pmod 3$ and $\mathcal{S}$ is isomorphic to $\mathcal{S}_{[9]}$.

Proof For odd p this follows from Theorem 3.9. For $p = 2$, use Theorem 2.5. □

Theorem 3.11
Let $\mathcal{O}$ be an ovoid of $H(3^h)$ that is translation with respect to a point. Then $\mathcal{O}$ is either hermitian or an $\mathcal{O}_\lambda$ ovoid.

Proof For $h = 1$, this follows from the dual of Corollary 3.10, so we suppose hereafter that $h > 1$.

By an appropriate choice of coordinates, we may suppose without loss of generality that $\mathcal{O}$ is translation with respect to (∞) and that it also contains the point $(0,0,0,0,0,0)$. By the dual of Theorem 3.3, the ovoid $\mathcal{O}$ is then locally hermitian with respect to (∞) so, using the dual of (2.32), we can express $\mathcal{O}$ as the union of point reguli

$$\mathcal{O} = \bigcup_{x,y \in GF(q)} \big(\!\big(x, f(x,y), g(x,y), y\big)\!\big), \tag{3.13}$$

where $f(0,0) = g(0,0) = 0$. We will show that f and g necessarily take the form of the functions in the representation (2.35) of the $\mathcal{O}_\lambda$ ovoids. Remember also that if $\lambda = 0$ there, then this corresponds to the representation of a hermitian ovoid, as shown in (2.26).

For each $y \neq 0$, the point regulus $((0, f(0,y), g(0,y), y))$ is compatible with the point regulus $((0,0,0,0))$, so from the compatibility condition in (2.33) (remembering that the characteristic is 3) we have

$$-yf(0,y) = \not\square.$$

Now from Theorem 1.5, it follows that $f(0,y) = -\alpha y^{3^s}$, where α is a nonsquare in $GF(q)$ and $0 \le s < h$. Similarly, the point regulus $((x, f(x,0), g(x,0), 0))$ is compatible with $((0,0,0,0))$, so

$$xg(x,0) = \not\square.$$

Again, by Theorem 1.5, it follows that $g(x,0) = \beta x^{3^t}$, where β is some nonsquare in $GF(q)$ and $0 \le t < h$. Now by the dual of Theorem 3.7, the functions f and g take the forms

$$f(x,y) = \tilde{f}(x) - \alpha y^{3^s} = \sum_{i=0}^{h-1} f_i x^{3^i} - \alpha y^{3^s} \tag{3.14}$$

and

$$g(x,y) = \beta x^{3^t} + \tilde{g}(y) = \beta x^{3^t} + \sum_{i=0}^{h-1} g_i y^{3^i}. \tag{3.15}$$

Consider the elation $E(\gamma_L \mu^2, -k) = \Theta[k,0,0,0,0]$, as described in Section 1.4.4. From the mapping in (1.25), its action on the points opposite (∞) is given by

$$\big(x, \ell, z, \ell', y\big) \longmapsto \big(x, \ell - kx^3, z + kx^2, \ell' + \ell k + k^2 x^3, y + kx\big).$$

In particular, this collineation fixes the points (∞) and $(0,0,0,0,0)$ so the ovoid $\mathcal{O}$ is mapped to another ovoid $\mathcal{O}_k$ that also contains these points and that is also translation with respect to (∞). Therefore $\mathcal{O}_k$ also has a representation as given by the dual of (2.32), say

$$\mathcal{O}_k = \bigcup_{x,y \in GF(q)} ((x, f_k(x,y), g_k(x,y), y))$$

where, exactly as for the functions f and g, we have

$$f_k(x,y) = \tilde{f}_k(x) - \alpha_k y^{3^{s(k)}} \quad \text{and} \quad g_k(x,y) = \beta_k x^{3^{t(k)}} + \tilde{g}_k(y),$$

and $\tilde{f}_k(x)$ and $\tilde{g}_k(y)$ are linearized polynomials in standard form.

To express these functions f_k and g_k in terms of f and g, we explicitly apply the collineation $\Theta[k,0,0,0,0]$ to $\mathcal{O}$. The point regulus $((x, f(x,y), g(x,y), y))$ of $\mathcal{O}$ is mapped to the point regulus

$$\big((x, f(x,y) - kx^3, g(x,y) + kf(x,y) + k^2x^3, y + kx)\big)$$

of $\mathcal{O}_k$. Changing y for $y - kx$ in this, we have a point regulus

$$\big((x, f(x, y-kx) - kx^3, g(x, y-kx) + kf(x, y-kx) + k^2x^3, y)\big)$$

of $\mathcal{O}_k$. Thus

$$\begin{aligned} g_k(x,y) &= g(x, y-kx) + kf(x, y-kx) + k^2x^3 \\ &= \beta x^{3^t} + \tilde{g}(y-kx) + k\tilde{f}(x) - \alpha k y^{3^s} + \alpha k^{1+3^s} x^{3^s} + k^2 x^3. \end{aligned} \tag{3.16}$$

Let $\delta_{i,j}$ be the function such that $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. Putting $y = 0$ in (3.16) and expanding $\tilde{f}$ and $\tilde{g}$ in accordance with (3.14) and (3.15), gives

$$\begin{aligned}\beta_k x^{3^{t(k)}} &= \beta x^{3^t} + \tilde{g}(-kx) + k\tilde{f}(x) + \alpha k^{1+3^s} x^{3^s} + k^2 x^3 \\ &= \beta x^{3^t} + \sum_{i=0}^{h-1} g_i(-k)^{3^i} x^{3^i} + \sum_{i=0}^{h-1} f_i k x^{3^i} + \alpha k^{1+3^s} x^{3^s} + k^2 x^3.\end{aligned}$$

Grouping together like powers of x, we then have

$$\beta_k x^{3^{t(k)}} = \sum_{i=0}^{h-1} \varphi_i(k) x^{3^i}, \tag{3.17}$$

where the coefficient of x^{3^i} is

$$\varphi_i(k) = \delta_{i,t}\beta - g_i k^{3^i} + f_i k + \delta_{i,s}\alpha k^{1+3^i} + \delta_{i,1}k^2, \tag{3.18}$$

which is a polynomial in k.

Put $i = 1$ in (3.18) and consider the polynomial

$$\varphi_1(k) = \delta_{1,t}\beta - g_1 k^3 + f_1 k + \delta_{1,s}\alpha k^4 + k^2.$$

The coefficient of k^2 is nonzero so $\varphi_1(k)$ is certainly not the zero polynomial. Similarly, putting $i = t$ in (3.18) we have

$$\varphi_t(k) = \beta - g_t k^{3^t} + f_t k + \delta_{t,s}\alpha k^{1+3^t} + \delta_{t,1}k^2,$$

which has a nonzero constant term and so is not the zero polynomial either.

Now consider the product $\varphi_1(k)\varphi_t(k)$. Since the individual polynomials are nonzero, this polynomial is also nonzero. In addition, the degree of this product is at most $4 + (3^t + 1)$, which is less than $q = 3^h$ since $t < h$ and $h > 1$. It follows that the product $\varphi_1(k)\varphi_t(k)$ is not identically zero as a function so we can choose k such that neither $\varphi_1(k)$ nor $\varphi_t(k)$ is zero. But comparing coefficients on each side of (3.17), we see that only one of the coefficients, namely $\beta_k = \varphi_{t(k)}(k)$, is nonzero. Hence $t(k) = t = 1$.

Similarly, the polynomial

$$\varphi_s(k) = \delta_{s,1}\beta - g_s k^{3^s} + f_s k + \alpha k^{1+3^s} + \delta_{s,1}k^2,$$

has a nonzero coefficient of k^{1+3^s} and hence is not the zero polynomial, so we can choose k such that neither $\varphi_1(k)$ nor $\varphi_s(k)$ is zero and we deduce from (3.17) that $t(k) = s = 1$.

For $i \neq 1$, we now have from (3.18) that

$$\varphi_i(k) = -g_i k^{3^i} + f_i k.$$

Equating coefficients in (3.17), these $\varphi_i(k)$ are all identically zero. For $i \neq 0$, so k^{3^i} and k are distinct powers of k, it follows that $g_i = f_i = 0$. When $i = 0$ however, we have

$\varphi_0(k) = (-g_0 + f_0)k \equiv 0$ and we conclude that $f_0 = g_0 = \lambda$, for some $\lambda \in GF(q)$. Hence the functions f and g take the forms

$$\begin{aligned} f(x,y) &= \lambda x + f_1 x^3 - \alpha y^3 \\ g(x,y) &= \beta x^3 + g_1 y^3 + \lambda y. \end{aligned} \tag{3.19}$$

Finally, since the point regulus $((1, f(1,y), g(1,y), y))$ is compatible with the point regulus $((0,0,0,0))$, we have from the compatibility condition in (2.33) that the quartic polynomial

$$p(y) = y^2 + g(1,y) - yf(1,y) = \alpha y^4 + g_1 y^3 + y^2 - f_1 y + \beta \tag{3.20}$$

takes a nonsquare value for every $x \in GF(q)$. By Lemma 1.3, it now follows that $p(y) = \gamma^{-1} p_0(y)^2$, where $\gamma = \alpha^{-1}$ is a nonsquare and $p_0(y) = y^2 + \mu y + \nu$ is an irreducible quadratic. Thus

$$p(y) = \gamma^{-1}\big(y^4 - \mu y^3 + (\mu^2 - \nu)y^2 - \mu\nu y + \nu^2\big)$$

and we equate coefficients with (3.20) to find $f_1 = \gamma^{-1}\mu\nu$, $\beta = \gamma^{-1}\nu^2$, $g_1 = -\gamma^{-1}\mu$ and $\gamma = \alpha^{-1} = \mu^2 - \nu$. Hence

$$\mathcal{O} = \bigcup_{x,y \in GF(q)} \big((x, \gamma^{-1}(\mu\nu x^3 - y^3) + \lambda x, \gamma^{-1}(\nu^2 x^3 - \mu y^3) + \lambda y, y)\big),$$

where $\gamma = \mu^2 - \nu$ is a nonsquare. When $\lambda = 0$, this is the hermitian ovoid $\mathcal{O}_H(\mu,\nu)$ as given in (2.26), and when $\lambda \neq 0$, this is the $\mathcal{O}_\lambda$ ovoid $\mathcal{O}_\lambda(\mu,\nu)$ as given in (2.35). □

We explicitly state the dual of this theorem for spreads.

Corollary 3.12
Let $\mathcal{S}$ be a spread of $H(3^h)$ that is translation with respect to a line. Then $\mathcal{S}$ is either hermitian or an $\mathcal{S}_\lambda$ spread. □

3.4 Spreads translation with respect to two flags

Theorem 3.5 gives some insight into spreads of the generalized hexagon $H(q)$ that are translation with respect to more than one flag, where the flags in question are all on a common line. In this section, we consider the situation where a spread $\mathcal{S}$ is translation with respect to flags on different lines, ultimately providing a classification of such spreads.

Since we will be concerned primarily with various flags in $H(q)$, let us first introduce the notion of distance between two flags. Remember that the distance between two elements of a geometry is simply the distance between the corresponding vertices of an appropriate graph, namely the incidence graph. In order to define the distance between two flags, we shall do much the same thing. Unless it is made clear to the contrary, in this section we shall use the convention that lower case letters represent points and upper case letters represent lines, while a point and a line represented by different cases of the same letter are incident.

Let Γ be an incidence structure. We define a graph G, called the **flag incidence graph**. The vertices of G are the flags of Γ and two vertices, $\mathcal{F}$ and $\mathcal{G}$, are adjacent if and only if they are distinct flags of Γ on a common element. Now we define the **distance** $d(\mathcal{F}, \mathcal{G})$ between two flags $\mathcal{F}$ and $\mathcal{G}$ of Γ to be the distance between the corresponding vertices in the flag incidence graph G. If the incidence structure Γ is a generalized n-gon, we say that two flags $\mathcal{F}$ and $\mathcal{G}$ are **opposite** if $d(\mathcal{F}, \mathcal{G}) = n$. For the case that Γ is a generalized hexagon, the relative positions of two flags given the distance between them is illustrated in Table 3.2.

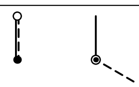
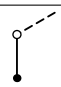
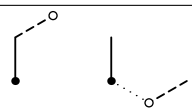
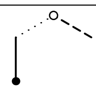
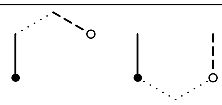
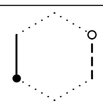
$d(\mathcal{F}, \mathcal{G})$	$\{d(x, X), d(y, Y)\}$	Configurations
0	$\{0\}$	
1	$\{0, 2\}$	
2	$\{2\}$	
3	$\{2, 4\}$	
4	$\{4\}$	
5	$\{4, 6\}$	
6	$\{6\}$	

Table 3.2: Distance between two flags $\mathcal{F} = \{x, X\}$ and $\mathcal{G} = \{y, Y\}$.

Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to two flags, $\mathcal{F} = \{x, X\}$ and $\mathcal{G} = \{y, Y\}$, on distinct lines so $X \neq Y$. Since the lines X and Y are lines of the spread $\mathcal{S}$, they are opposite, so the distance $d(\mathcal{F}, \mathcal{G})$ between the flags is either 5 or 6. First we address the case when $d(\mathcal{F}, \mathcal{G}) = 5$.

Theorem 3.13
Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to two flags $\mathcal{F} = \{x, X\}$ and $\mathcal{G} = \{y, Y\}$ with $d(\mathcal{F}, \mathcal{G}) = 5$. Then $\mathcal{S}$ is a hermitian spread.

Proof Since $d(\mathcal{F}, \mathcal{G}) = 5$, the distance between their points is $d(x, y) = 4$, so there is a unique point $k = x \triangleright_2 y$ with $d(x, k) = 2$ and $d(y, k) = 2$. See Figure 3.3 for a diagram. Let K be the line xk. Let w be any point on K distinct from both the points x and k, and then take W to be any line on w other than K. Since $d(W, X) = 4$, the line W does not belong to the spread $\mathcal{S}$, so by Lemma 2.1 there is a unique line U of $\mathcal{S}$ concurrent

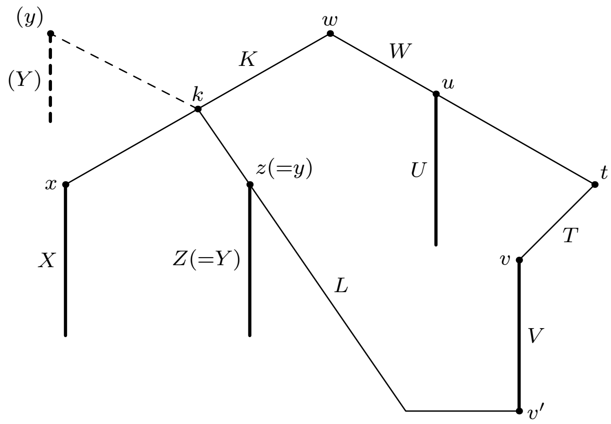


Figure 3.3: Diagram for the proof of Theorem 3.13.

with W. The spread line U is equal to neither X nor Y as these lines are at distances 4 and 6 respectively from W. Let u be the point in which the lines U and W meet. This point u is different from the point w since U and X both belong to the spread and are therefore opposite, while $d(w, X) = 3$.

Now we essentially repeat this with the rôle of the path (X, x, K, k, ky, y, Y) being played by the new path (U, u, W, w, K, x, X). So let t be a point on the line W distinct from the points u and w, and let T be some line other than W on this point. There is then a unique line V of the spread $\mathcal{S}$ concurrent with T and the point v that is common to the lines V and T is distinct from t. Just as U was seen to be different from both X and Y, so too is V different from both U and X. In addition, we have $V \neq Y$ as otherwise the hexagon $H(q)$ would contain the pentagon $(Y, v, T, t, W, w, K, k, ky, y)$.

So far, we have four distinct spread lines, X, Y, U and V, and the minimum length paths connecting each of the first three of these to the point k. Now we add the minimum length path from k to the line V.

The path (v, T, t, W, w, K, k) has length 6 so $d(v, k) = 6$ and consequently $d(V, k) = 5$. Let $v' = k \triangleright V$. Notice that $d(v', k) = 4$ so $v' \neq v$. Let $L = V \triangleright k$, which is then not a line of $\mathcal{S}$ since L and V are not opposite. Therefore there is a unique line Z of the spread $\mathcal{S}$ concurrent with L and we set z to be the point in which these two lines meet. The line Z is different from both X and V as otherwise there would be a triangle in the hexagon $H(q)$, however Z could possibly be equal to Y.

Now that we have our diagram established, we shall use the given translation properties of the spread $\mathcal{S}$ to find sufficiently many additional ones to be able to deduce the conclusion of the theorem.

Since $\mathcal{S}$ is translation with respect to $\{x, X\}$ and $Y \triangleright x = U \triangleright x$, there is an automorphism of the spread in the associated group $G_{\{x,X\}}$ that maps Y to the line U. Furthermore, since x is fixed by this collineation, $y = x \triangleright Y$ is mapped to $u = x \triangleright U$, so the flag $\{y, Y\}$ is sent to the flag $\{u, U\}$. It follows that $\mathcal{S}$ is translation with respect to $\{u, U\}$ as well. Similarly, by using an element of the group $G_{\{u,U\}}$ we can map the flag $\{x, X\}$ to $\{v, V\}$, so $\mathcal{S}$ is also translation with respect to the flag $\{v, V\}$.

If $Z = Y$ then $z = k \triangleright Z = k \triangleright Y = y$, so $\mathcal{S}$ is certainly translation with respect to the flag $\{z, Z\}$, and if on the other hand $Z \neq Y$, then using $G_{\{y,Y\}}$ we can map $\{x, X\}$ to $\{z, Z\}$ to see that $\mathcal{S}$ is translation with respect to the flag $\{z, Z\}$. Now with a collineation in $G_{\{z,Z\}}$ we can map $\{x, X\}$ to $\{v', V\}$, so $\mathcal{S}$ is also translation with respect to the flag $\{v', V\}$.

By Theorem 3.3, the spread $\mathcal{S}$ is locally hermitian with respect to V. Now select one of the flags not containing V with respect to which we know $\mathcal{S}$ is translation, say $\{x, X\}$. The associated group $G_{\{x,X\}}$ does not leave the line V fixed, so there are other lines with respect to which $\mathcal{S}$ is also locally hermitian. It now follows from Theorem 2.9 that $\mathcal{S}$ is a hermitian spread. □

Before we proceed with the case of spreads of $H(q)$ that are translation with respect to two opposite flags, we introduce some concepts in generalized $2m$-gons that we will want to use later.

Let Γ be a generalized $2m$-gon of order q, where $m = 2$ or 3, and let $\mathcal{S}$ and $\mathcal{O}$ be a spread and an ovoid, respectively, of Γ. The set $\mathcal{O} \cup \mathcal{S}$ is an **ovoid-spread pairing** if for each element $x \in \mathcal{O} \cup \mathcal{S}$ there is an element $y \in \mathcal{O} \cup \mathcal{S}$ such that $x \mathrel{\mathrm{I}} y$. By Lemma 2.1, such an element y is uniquely determined for a given x, so there is a well-defined function $\rho : \mathcal{O} \cup \mathcal{S} \to \mathcal{O} \cup \mathcal{S}$ that maps each element to the unique element of $\mathcal{O} \cup \mathcal{S}$ incident with it, thus $x \mathrel{\mathrm{I}} x^\rho$. We call this the **associated function** of the ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$.

Our following lemma is a partial analogy of Lemma 2.1 for ovoid-spread pairings.

Lemma 3.14
Let $\mathcal{O} \cup \mathcal{S}$ be an ovoid-spread pairing of a generalized $2m$-gon Γ. Then for every element $x \in \Gamma$, there is a unique element $y \in \mathcal{O} \cup \mathcal{S}$ such that $d(x, y) < m$ is minimal.

Proof Let $y \in \mathcal{O} \cup \mathcal{S}$ be such that $d(x, y)$ is minimal. Let $\mathcal{T} \in \{\mathcal{O}, \mathcal{S}\}$ be the set whose elements are, or are not, the same type as x according as m is odd or even, respectively. Then by Lemma 2.1, there is a unique element $z \in \mathcal{T}$ such that $d(x, z) < m$, so if $y \in \mathcal{T}$ then $y = z$ and we are done. Suppose then that $y \notin \mathcal{T}$. Then y and z are of different types so $d(x, y) < d(x, z)$ and hence $d(x, y^\rho) \leq d(x, z)$, where ρ is the associated function of $\mathcal{O} \cup \mathcal{S}$. Using Lemma 2.1 again, we then have $y^\rho = z$, and therefore $y = z^\rho$. □

Let θ be a polarity of Γ. By [VM98, Proposition 7.2.5] (see also [CPT76] for the case $m = 3$), the absolute lines of θ form a spread $\mathcal{S}$ of Γ and the absolute points form an ovoid $\mathcal{O}$. Furthermore, since the absolute elements are precisely those for which $x^\theta \mathrel{\mathrm{I}} x$, $\mathcal{O} \cup \mathcal{S}$ is an ovoid-spread pairing and the associated function is simply the restriction of the polarity θ. Such an ovoid-spread pairing is said to **arise from the polarity** $\boldsymbol{\theta}$. For $m = 2$, any spread $\mathcal{S}$ and ovoid $\mathcal{O}$ together form an ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ by virtue of the fact that the lines of the spread, considered as point sets, partition the set of points of the generalized quadrangle (a simple count reveals this—there are q^2+1 lines in the spread by Lemma 2.2, there are $q+1$ points on each line and there are q^3+q^2+q+1 points in the generalized quadrangle (see [VM98, 1.5.4])). However, for $m = 3$, the only known ovoid-spread pairings arise from polarities (see [VM98, 7.2.6]). In particular, the only known ovoid-spread pairings $\mathcal{O} \cup \mathcal{S}$ of $H(q)$ are those where $\mathcal{S}$ and $\mathcal{O}$ are a Ree-Tits spread and a Ree-Tits ovoid, respectively, arising from the same polarity.

Let $\mathcal{O} \cup \mathcal{S}$ be an ovoid-spread pairing of Γ and suppose that $x, y, z \in \mathcal{O} \cup \mathcal{S}$ are elements of the same type such that $y^\rho \triangleright_i x = z^\rho \triangleright_i x$, for some i. If $\mathcal{O} \cup \mathcal{S}$ arises from a polarity, then we apply this polarity and we also have that $y \triangleright_i x^\rho = z \triangleright_i x^\rho$. Motivated by this, an ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is **distance-i polar at x** if for all $y, z \in \mathcal{O} \cup \mathcal{S}$ of the same type as x, we have

$$y^\rho \triangleright_i x = z^\rho \triangleright_i x \Rightarrow y \triangleright_i x^\rho = z \triangleright_i x^\rho. \tag{3.21}$$

Since $y \triangleright_0 x = x$ for all x and y, it is clear that every ovoid-spread pairing is distance-0 polar at each of its elements. Now consider $i \geq m$. Let $x, y, z \in \mathcal{O} \cup \mathcal{S}$ be elements of the same type such that $y^\rho \triangleright_i x = z^\rho \triangleright_i x = u$. Then $d(y^\rho, u) = d(z^\rho, u) = 2m - i - 1$, so concatenating the minimum length y^ρ–u and u–z^ρ paths, we have a y^ρ–z^ρ walk of length $4m - 2i - 2 \leq 4m - 2m - 2 < 2m$. Since y^ρ and z^ρ belong to either a spread or an ovoid, it follows that $y = z$, so certainly $y \triangleright_i x^\rho = z \triangleright_i x^\rho$. Thus every ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at each of its elements for $i \geq m$. Therefore we only need to consider $1 \leq i < m$.

Lemma 3.15
If $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at x then $\mathcal{O} \cup \mathcal{S}$ is also distance-i polar at x^ρ.

Proof Let $y \in \mathcal{O} \cup \mathcal{S}$ be an element of the same type as x. Consider the spread and ovoid projection sets $A = V_{y^\rho \triangleright_i x}(x^\rho, x)$ and $B = V_{y \triangleright_i x^\rho}(x, x^\rho)$. Then A is the set of elements $w \in \mathcal{O} \cup \mathcal{S}$ of the same type as x^ρ such that $w \triangleright_i x = y^\rho \triangleright_i x$, and B is the set of elements $z \in \mathcal{O} \cup \mathcal{S}$ of the same type as x such that $z \triangleright_i x^\rho = y \triangleright_i x^\rho$.

Since $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at x, for each $w \in A$, the element w^ρ belongs to B. But $|A| = |B|$ by Lemma 2.3, so we also have then that for each $z \in B$, the element z^ρ is in A; that is, $\mathcal{O} \cup \mathcal{S}$ is distance-i polar at x^ρ. □

The ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ is **locally polar at** x if it is distance-i polar at x for each $1 \leq i < m$. By Lemma 3.15, $\mathcal{O} \cup \mathcal{S}$ is then also locally polar at x^ρ. We may also say that it is locally polar at the flag $\{x, x^\rho\}$.

Lemma 3.16
If an ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ of Γ is locally polar at each of its elements then it arises from a polarity.

Proof We extend the associated function ρ to all of Γ and show this is a polarity.

Let $w \in \Gamma$ and let x be the unique element of $\mathcal{O} \cup \mathcal{S}$, by Lemma 3.14, for which $d = d(x, w) < m$ is minimal. By Lemma 2.3, the set $V_w(x^\rho, x)$ is nonempty, so we can choose an element $y \in \mathcal{O} \cup \mathcal{S}$ such that y^ρ belongs to this set. Then $y^\rho \triangleright_d x = w$. Now we define $w^\rho = y \triangleright_d x^\rho$. To see that this defines a polarity we need to check that it is well-defined, that it has order two and that it preserves incidence.

Consider another element $z \in \mathcal{O} \cup \mathcal{S}$ such that $z \in V_w(x^\rho, x)$. Then $z^\rho \triangleright_d x = w = y^\rho \triangleright_d x$ so, since $\mathcal{O} \cup \mathcal{S}$ is locally polar at x, we also have $z \triangleright_d x^\rho = y \triangleright_d x^\rho$. Thus ρ is well-defined.

Next, consider the element $w^\rho = y \triangleright_d x^\rho$. We have $d(x^\rho, w^\rho) = d < m$, so from the proof of Lemma 3.14, the unique element of $\mathcal{O} \cup \mathcal{S}$ at minimum distance from w^ρ is either x or x^ρ. But since x and y are opposite, we have $w^\rho \triangleright x^\rho = y \triangleright x^\rho \neq x$, so x^ρ is the unique element of $\mathcal{O} \cup \mathcal{S}$ at minimum distance from w^ρ and hence $(w^\rho)^\rho = y^\rho \triangleright_d (x^\rho)^\rho = y^\rho \triangleright_d x = w$. Therefore ρ has order two.

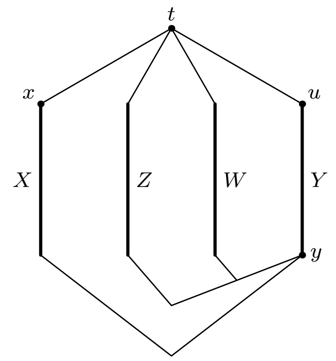


Figure 3.4: This configuration does not occur (Lemma 3.17).

All that remains is to check that incidence is preserved. Let $\{u, v\}$ be a flag of Γ and let x be the unique element of $\mathcal{O} \cup \mathcal{S}$, by Lemma 3.14, with $d(u, x)$ minimal. Relabelling if necessary, we may suppose $d = d(u, x) = d(v, x) + 1 \leq m$. By Lemma 2.3, we can choose an element $y \in \mathcal{O} \cup \mathcal{S}$ such that $y^\rho \in V_u(x^\rho, x)$. Then $y^\rho \triangleright_d x = u$ and also $y^\rho \triangleright_{d-1} x = v$.

Now if $d < m$ then $u^\rho = y \triangleright_d x^\rho$, and if $d = m$ then $u = y^\rho \triangleright_m x = x \triangleright_{m-1} y^\rho$ so $u^\rho = x^\rho \triangleright_{m-1} y = y \triangleright_m x^\rho$. Either way, we have $u^\rho = y \triangleright_d x^\rho$ and $v^\rho = y \triangleright_{d-1} x^\rho$, which are adjacent elements on the minimum length y–x^ρ path. Thus $u^\rho \mathrm{I}\, v^\rho$ and incidence is therefore preserved. □

We are now prepared to classify those spreads of $H(q)$ that are translation with respect to two opposite flags. The proof of the classification is broken into a few lemmas, which we give first.

Lemma 3.17
Let $\mathcal{S}$ be a nonhermitian spread of $H(q)$ that is translation with respect to two opposite flags $\mathcal{F} = \{X, x\}$ and $\mathcal{G} = \{Y, y\}$. Let $V \neq X \triangleright y$ be a line through y. Then there exists a unique line $Z \in \mathcal{S}$ such that $Z \triangleright y = V$ and $Z \triangleright_2 x = Y \triangleright_2 x$.

Proof By Lemma 2.3, there are exactly q lines Z of $\mathcal{S}$ such that $Z \triangleright_2 x = Y \triangleright_2 x$. Also, there are exactly q lines $V \neq X \triangleright y$ incident with y. Thus we need only show either the existence or the uniqueness property and the other will follow. We show uniqueness. For the following, see the illustration in Figure 3.4.

If $V = Y$ then $Z \triangleright y = Y$ implies that $d(Z, Y) < 6$, so necessarily $Z = Y$. Suppose then that $V \neq Y$. Let $Z, W \in \mathcal{S}$ be distinct lines such that $Z \triangleright_2 x = W \triangleright_2 x = Y \triangleright_2 x = t$ and $Z \triangleright y = W \triangleright y = V$, and let $u = x \triangleright Y$. Then $d(Z, t) = 3$ and $d(t, u) = d(t, Y) - 1 = 2$, so $Z \triangleright_2 u = t$. Similarly, $W \triangleright_2 u = t$. Now $Z \triangleright y = W \triangleright y$ so there is a collineation $g \in G_{\{Y,y\}}$ that maps W to Z. Since g fixes Y pointwise, the point u is fixed and so the point $t = Z \triangleright_2 u = W \triangleright_2 u$ is fixed also.

Consider the image of $\mathcal{F}$ under the action of g. Since $G_{\{Y,y\}}$ acts semiregularly on the lines of $\mathcal{S}$ opposite Y, the line X is certainly not fixed, so $\mathcal{F}g$ is another distinct flag with respect to which $\mathcal{S}$ is translation. The point $x \in \mathcal{F}$, which is at distance 2 from the fixed point t, is mapped to xg, which is then also at distance 2 from t. Thus $d(x, xg) = 4$. We now have that $\mathcal{S}$ is translation with respect to two flags $\mathcal{F}$ and $\mathcal{F}g$ with $d(\mathcal{F}, \mathcal{F}g) = 5$.

By Theorem 3.13, such a spread $\mathcal{S}$ is hermitian. It follows that if $\mathcal{S}$ is nonhermitian then two such distinct lines Z and W do not exist. This proves the result. □

Lemma 3.18
Let $\mathcal{S}$ be a nonhermitian spread of $H(q)$ that is translation with respect to two opposite flags $\mathcal{F} = \{X, x\}$ and $\mathcal{G} = \{Y, y\}$. Then for each line $W \in \mathcal{S}$ there is a unique point $w \mathrm{I} W$ such that $\mathcal{S}$ is also translation with respect to the flag $\{W, w\}$. Furthermore, the set $\mathcal{O}$ of these points is an ovoid so that $\mathcal{O} \cup \mathcal{S}$ is an ovoid-spread pairing.

Proof Let $W \in \mathcal{S}$ be a line of the spread. If $W \triangleright y = X \triangleright y$ then there is a collineation $g \in G_{\{Y,y\}}$ such that $W = Xg$ and $\mathcal{S}$ is then translation with respect to the flag $\{W, w\}$, where $w = xg$.

Now consider when $W \triangleright y \neq X \triangleright y$. By Lemma 3.17, there is a line $Z \in \mathcal{S}$ such that $Z \triangleright y = W \triangleright y$ and $Z \triangleright_2 x = Y \triangleright_2 x$. Then $Z \triangleright x = Y \triangleright x$, so there is a collineation $g \in G_{\{X,x\}}$ such that $Z = Yg$. Next, since $Z \triangleright y = W \triangleright y$, there is an $h \in G_{\{Y,y\}}$ such that $W = Zh = Ygh$. Now $\mathcal{S}$ is translation with respect to the flag $\{W, w\}$, where $w = ygh$. This shows the existence of an appropriate point w on each line W of $\mathcal{S}$.

Suppose now that there are two points w and w' on a line W such that $\mathcal{S}$ is translation with respect to both flags $\{W, w\}$ and $\{W, w'\}$. By Theorem 3.3, the spread $\mathcal{S}$ is then locally hermitian with respect to W. Furthermore, by their semiregular actions, not both $G_{\{X,x\}}$ and $G_{\{Y,y\}}$ fix W, so $\mathcal{S}$ is locally hermitian with respect to other lines as well. By Theorem 2.9, this implies that $\mathcal{S}$ is hermitian. But $\mathcal{S}$ is nonhermitian, so the uniqueness of the point $w \mathrm{I} W$ follows. Let $\mathcal{O}$ be the set of these points.

Finally, let $w, z \in \mathcal{O}$ be distinct points. Since they lie on opposite lines, we have either $d(w, z) = 4$ or $d(w, z) = 6$, but $\mathcal{S}$ is nonhermitian so from Theorem 3.13, the points w and z are opposite. Also, since there is exactly one point of $\mathcal{O}$ on each line of $\mathcal{S}$ we have $|\mathcal{O}| = |\mathcal{S}| = q^3 + 1$, so by Lemma 2.2, the set $\mathcal{O}$ is an ovoid. □

Let $\mathcal{S}$ be a nonhermitian spread of $H(q)$ that is translation with respect to two opposite flags. By Lemma 3.18, there is an ovoid $\mathcal{O}$ arising from $\mathcal{S}$ such that $\mathcal{O} \cup \mathcal{S}$ is an ovoid-spread pairing and $\mathcal{S}$ is translation with respect to each flag $\{x, x^\rho\} \in \mathcal{O} \cup \mathcal{S}$, where ρ is the associated function of $\mathcal{O} \cup \mathcal{S}$. Let g be a collineation of $H(q)$ that leaves $\mathcal{S}$ fixed and let x be a line of $\mathcal{S}$. Then $\mathcal{S}$ is translation with respect to the flag $\{xg, x^\rho g\}$ so $x^\rho g = (xg)^\rho \in \mathcal{O}$, since such a point incident with xg is unique by Lemma 3.18. Hence any collineation fixing $\mathcal{S}$ must also fix $\mathcal{O}$.

Lemma 3.19
Let $\mathcal{O} \cup \mathcal{S}$ be an ovoid-spread pairing of $H(q)$ with associated function ρ. Suppose that $\mathcal{S}$ is translation with respect to the flag $\{x, x^\rho\}$ and that the associated group $G_{\{x,x^\rho\}}$ of $\mathcal{S}$ with respect to this flag stabilizes $\mathcal{O}$. Then $\mathcal{O} \cup \mathcal{S}$ is distance-1 polar at x and at x^ρ.

Proof Let x be the point in the flag $\{x, x^\rho\}$ and let $y, z \in \mathcal{O}$ be such that $y^\rho \triangleright x = z^\rho \triangleright x$. Then there is a collineation $g \in G_{\{x,x^\rho\}}$ such that $y^\rho g = z^\rho$, and since g fixes $\mathcal{O}$, we then have $yg = z$. Since also g fixes x^ρ pointwise, the point $y \triangleright x^\rho$ is fixed and hence

$$y \triangleright x^\rho = (y \triangleright x^\rho)g = yg \triangleright x^\rho g = z \triangleright x^\rho.$$

Therefore $\mathcal{O} \cup \mathcal{S}$ is distance-1 polar at x, and by Lemma 3.15, it is also distance-1 polar at x^ρ. □

Corollary 3.20
The ovoid $\mathcal{O}$ in the statement of Lemma 3.19 is also translation with respect to the flag $\{x, x^\rho\}$, with the same associated group $G_{\{x,x^\rho\}}$ as $\mathcal{S}$. In addition, the order of $H(q)$ is $q = 3^h$.

Proof In proving distance-1 polarity at x, we have shown that the group $G_{\{x,x^\rho\}}$, which acts transtively on each of the spread projection sets V_K, with $K \mathrm{I}\, x$ and $K \neq x^\rho$, also acts transitively on the each of the ovoid projection sets V_k, with $k \mathrm{I}\, x^\rho$ and $k \neq x$. The claim concerning the order of $H(q)$ follows from Theorem 3.1. □

Lemma 3.21
Let $\mathcal{O} \cup \mathcal{S}$ be an ovoid-spread pairing of $H(q)$ with associated function ρ. Suppose that $\mathcal{S}$ is translation with respect to a flag $\{x, x^\rho\}$ and that the associated group $G_{\{x,x^\rho\}}$ of $\mathcal{S}$ with respect to this flag fixes $\mathcal{O}$. Then $\mathcal{O} \cup \mathcal{S}$ is distance-2 polar at x and at x^ρ.

Proof Let x be the point in the flag $\{x, x^\rho\}$ and let $y, z \in \mathcal{O}$ be such that $y^\rho \triangleright_2 x = z^\rho \triangleright_2 x$. By Lemma 3.15, it is sufficient to prove the claim for x. Without loss of generality, choose the hat-rack of the coordinatization such that $x = (\infty)$, $x^\rho = [\infty]$, $z = (0,0,0,0,0)$ and $z^\rho = [0,0,0,0,0]$. Then by the incidences given in (1.13), $y^\rho \triangleright_2 x = z^\rho \triangleright_2 x = (0,0)$ is equivalent to the condition that the first two coordinates of the line y^ρ are zero. What we need to show is that then $y \triangleright_2 x^\rho = z \triangleright_2 x^\rho = [0,0]$, or equivalently, that the first two coordinates of the point y are also zero. From here on, we work only with coordinates, so the symbols x, y and z are free to be used for elements of the field $GF(q)$.

Let $\mathcal{S}$ be given by the coordinate representation in (2.3), so

$$\mathcal{S} = \big\{[\infty]\big\} \cup \big\{[x, y, z, f(x,y,z), g(x,y,z)] \;\big|\; x, y, z \in GF(q)\big\},$$

and from (3.1), the associated group is

$$G_{\{[\infty],(\infty)\}} = \big\{\Theta[0, y, z, f(0,y,z), g(0,y,z)] \;\big|\; y, z \in GF(q)\big\}.$$

For brevity, let $\ell(x,y,z)$ be the line of $\mathcal{S}$ whose first three coordinates are (x,y,z).

Since $\mathcal{O}$ is fixed by the group $G_{\{[\infty],(\infty)\}}$, the point of $\mathcal{O}$ that is incident with the line $\ell(0,0,z)$ of $\mathcal{S}$ is the image of the point $(0,0,0,0,0)$ under the action of the collineation $\Theta[0,0,z,f(0,0,z),g(0,0,z)]$, and thus it is the point $(0, g(0,0,z), f(0,0,z), z, 0)$. Therefore, all we need to show is that $g(0,0,z) = 0$ for all z.

Suppose there is a $Z \in GF(q)$ such that $g(0,0,Z) \neq 0$. From Corollary 3.20 we have $q = 3^h$, so the collineation $g = \Theta[0,0,Z,f(0,0,Z),g(0,0,Z)]$ in $G_{\{[\infty],(\infty)\}}$ maps the line $\ell(x,0,0)$ of $\mathcal{S}$ to the line $\ell\big(x, 0, Z - xg(0,0,Z)\big)$. Choosing $x = \frac{Z}{g(0,0,Z)}$, we have a line $\ell(x,0,0)$ of $\mathcal{S}$ that is fixed by g, contradicting the semiregular action of the group $G_{\{[\infty],(\infty)\}}$ on the lines of $\mathcal{S}$ opposite $[\infty]$. Therefore $g(0,0,z) = 0$ for all $z \in GF(q)$ and the claim is proved. □

Together, these results that we have collected enable us to classify the spreads of $H(q)$ that are translation with respect to two opposite flags.

Theorem 3.22
Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to two opposite flags. Then $\mathcal{S}$ is either hermitian or a Ree-Tits spread.

Proof Suppose $\mathcal{S}$ is nonhermitian. By Lemma 3.18, there is an ovoid $\mathcal{O}$ such that $\mathcal{O} \cup \mathcal{S}$ is an ovoid-spread pairing and such that $\mathcal{S}$ is translation with respect to each flag $\{x, x^\rho\}$, where ρ is the associated function of $\mathcal{O} \cup \mathcal{S}$. By the comments following Lemma 3.18, the associated groups $G_{\{x,x^\rho\}}$ of $\mathcal{S}$ with respect to each of these flags stabilize the ovoid $\mathcal{O}$, so Lemmas 3.19 and 3.21 apply and therefore $\mathcal{O} \cup \mathcal{S}$ is locally polar at each of its elements. Finally, by Lemma 3.16, the ovoid-spread pairing $\mathcal{O} \cup \mathcal{S}$ arises from a polarity so $\mathcal{S}$ is a Ree-Tits spread and $\mathcal{O}$ is the corresponding Ree-Tits ovoid. □

We summarize the results that we have pertaining to spreads that are translation with respect to two flags in our following theorem.

Theorem 3.23
Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to two flags $\mathcal{F}$ and $\mathcal{G}$.

(i) If $d(\mathcal{F}, \mathcal{G}) = 5$ then $\mathcal{S} = \mathcal{S}_H$.

(ii) If $d(\mathcal{F}, \mathcal{G}) = 6$ then $\mathcal{S} = \mathcal{S}_H$ or $\mathcal{S} = \mathcal{S}_R$.

(iii) If $\mathcal{F}$ and $\mathcal{G}$ are on a common line L then $\mathcal{S}$ is locally hermitian with respect to L. If $q \not\equiv 2 \pmod 3$ then $\mathcal{S}$ is translation with respect L. If $q = 3^h$ then $\mathcal{S} = \mathcal{S}_H$ or $\mathcal{S} = \mathcal{S}_\lambda$.

Proof The cases when $d(\mathcal{F}, \mathcal{G}) = 5$ and $d(\mathcal{F}, \mathcal{G}) = 6$ are Theorems 3.13 and 3.22, respectively. For the case when $d(\mathcal{F}, \mathcal{G}) = 1$, the first part is Theorem 3.3, the second part is from Theorem 3.5, and the final part is Corollary 3.12. □

3.5 Remarks

In the previous section, the proof of Theorem 3.22 is broken into the sequence of Lemmas 3.14–3.21, and for the largest part the proof seems quite pleasing for its use of purely geometric arguments. However, it is only in Lemma 3.21 that this is a little spoiled by resorting to the use of coordinates. I am grateful to Hendrik Van Maldeghem for showing me how to avoid coordinates there as well, making for a completely coordinate free proof of Theorem 3.22. The key is in the following observation for which I provide a proof using the geometry of the quadric $\mathcal{P}_6$ in which the hexagon $H(q)$ is embedded. A counting argument can also be used, and this is outlined afterwards.

Theorem 3.24
Let x be a point of $H(q)$ and let y and z be two points opposite x such that their distance-2 traces $x^y_{[2]}$ and $x^z_{[2]}$ onto x have at least two points in common. Then their distance-3 traces onto x are either equal or have exactly one line in common; that is, $|x^y_{[3]} \cap x^z_{[3]}| = 1$ or $q + 1$.

Proof Notice that if the distance-3 traces $x^y_{[3]}$ and $x^z_{[3]}$ have two lines in common, then by Lemma 1.17 they are equal, so we have really only to show that they are not disjoint.

Let $\ell = x^y_{[2]}$ and $\mathcal{R} = x^y_{[3]}$. Then ℓ is a line of $\mathcal{P}_6$ by Lemma 1.16, and $\mathcal{R}$ is a line regulus. By Lemma 1.18, the complementary point regulus to $\mathcal{R}$ is a conic $\mathcal{C}$ in a plane π, and $\pi \cap \ell = \emptyset$ since no element of ℓ is at distance 3 from the lines of $\mathcal{R}$. Also, the points of $\mathcal{C}$ are at distance at most 4 from the points of ℓ so $\pi \subset \ell^\perp$. It follows that $\ell^\perp$ is spanned by ℓ and π so that $\ell^\perp \cap \mathcal{P}_6$ is the conic cone $\ell\mathcal{C}$ with vertex ℓ and base $\mathcal{C}$. Thus for any given generator γ of $\mathcal{P}_6$ on ℓ, there is a unique point w of $\mathcal{C}$ in γ, and this point then satisfies $x^w_{[2]} = x^y_{[2]} = \ell$ and $x^w_{[3]} = x^y_{[3]} = \mathcal{R}$.

Now consider the point z as in the statement of the theorem. Then $x^z_{[2]} = \ell$ and we let $\mathcal{R}' = x^z_{[3]}$. As we are really only concerned with the distance-3 trace $\mathcal{R}'$ and not the point z itself, we may, by the previous paragraph, assume that y and z are in a common generator γ of $\mathcal{P}_6$ on ℓ. Consider a line of γ passing through y and z and let u be the point in which it meets the line ℓ. Let $L \in \mathcal{R}$ be the line of the distance-3 trace of y onto x that is incident with the point u. Then there is a generator γ' of $\mathcal{P}_6$ on L that contains the points y and u, and so therefore also the point z. This implies that $d(z, L) = 3$ and we conclude that $L \in \mathcal{R}'$ so the distance-3 traces $\mathcal{R}$ and $\mathcal{R}'$ are not disjoint. □

As mentioned previously, Theorem 3.24 can also be demonstrated with a counting argument, which we outline now. First, one verifies that of the q^5 points opposite x, exactly q^3 of them have the same distance-2 trace onto x as does y. Of these, the number for which a given line $L \in x^y_{[3]}$ belongs to the distance-3 trace is q^2. Now noting that a distance-3 trace onto x is a line regulus which is completely determined by two of its $q+1$ lines, and that in addition to x there are q points in the complementary point regulus, we determine that the number of points w for which $x^w_{[2]} = x^y_{[2]}$ and $|x^w_{[3]} \cap x^y_{[3]}| > 0$ is $(q+1)q^2 - q^2 = q^3$, which is precisely the number of points z such that $x^z_{[2]} = x^y_{[2]}$. Thus for all such points z we must have $|x^z_{[3]} \cap x^y_{[3]}| > 0$, as required.

With the help of this result, we can now complete the proof of Theorem 3.22 without the use of coordinates.

Alternative proof of Lemma 3.21 As in the original proof, let x be the point in the flag $\{x, x^\rho\}$ and let $y, z \in \mathcal{O}$ be distinct points such that $y^\rho \triangleright_2 x = z^\rho \triangleright_2 x = u$. We must show that $y \triangleright_2 x^\rho = z \triangleright_2 x^\rho$.

By Lemma 3.19, the distance-2 traces $x^y_{[2]}$ and $x^z_{[2]}$ have a common point $y \triangleright x^\rho = z \triangleright x^\rho$. Since u is also a point in common to these two sets, Theorem 3.24 applies so there is a line ℓ in $x^y_{[3]} \cap x^z_{[3]}$.

Let $k = \ell \triangleright x$ and let m be the unique line of the spread $\mathcal{S}$ that is concurrent with ℓ. Let g be the unique collineation in $G_{\{x,x^\rho\}}$ that maps y^ρ to z^ρ. Since g fixes the point x linewise and k is incident with x, the line $\ell = y \triangleright_2 k$ is mapped to $z \triangleright_2 k = \ell$, so the line ℓ is fixed. Since g stabilizes the spread $\mathcal{S}$, it follows then that g also fixes the spread line m. Now by the semiregular action of $G_{\{x,x^\rho\}}$ on $\mathcal{S} \setminus \{x^\rho\}$ together with the fact that $g \neq 1$, it follows that $m = x^\rho$ and $\ell = y \triangleright_2 x^\rho = z \triangleright_2 x^\rho$, as required. □

Chapter 4

Connections with Generalized Quadrangles

Generalized quadrangles are, on the whole, generally more familiar objects and in this chapter we hope to make the generalized hexagon $H(q)$ a little less mysterious than it might at first seem by showing some connections that exist between it and generalized quadrangles. Indeed, much of what we have learnt about $H(q)$ has its motivations lying in the strong analogy with the symplectic quadrangle $W(q)$. It is this analogy that we focus on first, and then we consider a connection between spreads of $H(q)$ and ovoids of the generalized quadrangle $Q(4,q)$, or equivalently, spreads of $W(q)$ since these generalized quadrangles are dual. In this chapter, a familiarity with generalized quadrangles is assumed. For details, see either of the references [PT84] or [VM98].

4.1 Construction of the Symplectic Quadrangle $W(q)$

In Section 1.4.1, we described the construction of the split Cayley hexagon $H(q)$. With the aim of making that construction seem a little more natural than it might otherwise, here we draw on the analogy with the symplectic quadrangle $W(q)$ to provide a parallel description of its construction.

The essential principle in the construction of $H(q)$ given in Section 1.4.1, is to define a rank 4 geometry, with three types of points, that then has a three-way symmetry giving rise to a triality as illustrated in Figure 1.3. The absolute elements of the triality then make the generalized hexagon $H(q)$. Here, we use a rank 3 geometry, with two types of points, that has a two-way symmetry and a corresponding polarity. Then $W(q)$ is the geometry of the absolute elements of the polarity.

To begin, define the rank 3 geometry $\Gamma = (\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \mathcal{L}, \mathrm{I})$, where $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(1)}$ are the sets of points and planes, respectively, of the projective space $PG(3,q)$, the set $\mathcal{L}$ is the set of lines of $PG(3,q)$, and incidence is given by the usual incidence in $PG(3,q)$. Then Γ is really just $PG(3,q)$, but with the planes being considered as a type of point in order to emphasize the similarity with Section 1.4.1.

As in Section 1.4.1, we label the 0-points (that is, the elements of the set $\mathcal{P}^{(0)}$) with homogeneous coordinates by simply using their coordinates as points of $PG(3,q)$, and

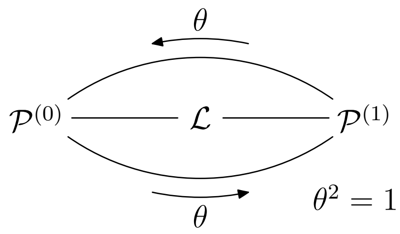


Figure 4.1: The polarity θ

we want to assign similar coordinates to the 1-points. While previously the assignment was made such that incidence between points of different types was given by the trilinear form (1.2), here we do it such that incidence between 0-points and 1-points is given by the bilinear form

$$B(x,y) = \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} - \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}. \tag{4.1}$$

Specifically, the 0-point x and the 1-point y are incident if and only if $B(x,y) = 0$. Thus the 1-point with coordinates (a,b,c,d) is the plane of $PG(3,q)$ given by the equation $bX_0 - aX_1 - dX_2 + cX_3 = 0$.

Now the map θ that takes a point of one type to the point of the other with the same coordinates is a polarity since the bilinear form is preserved (up to sign); that is, if $B(x,y) = 0$ then $B(y,x) = 0$, so if $x \mathrm{~I~} y$ then $x^\theta \mathrm{~I~} y^\theta$. See Figure 4.1 and compare with Figure 1.3 of Section 1.4.1.

An absolute 0-point is one for which $B(x,x) = 0$, and it is readily seen from (4.1) that all 0-points are absolute. Now let x be a point and y another such that $x \mathrm{~I~} y^\theta$. Then also $y \mathrm{~I~} x^\theta$. Also, since all points are absolute, we have $x \mathrm{~I~} x^\theta$ and $y \mathrm{~I~} y^\theta$. Thus, in terms of projective subspaces of $PG(3,q)$, we have $x \in x^\theta \cap y^\theta$ and $y \in x^\theta \cap y^\theta$, so the line xy is absolute since $xy = x^\theta \cap y^\theta = x^\theta y^\theta$. Conversely, if xy is absolute then $xy = x^\theta y^\theta$ so in particular we have $x \in y^\theta$, which in Γ means $x \mathrm{~I~} y^\theta$. Thus, similarly to that which occurs in Section 1.4.1, the absolute lines are precisely those lines xy for which $B(x,y) = 0$.

The absolute 0-points and absolute lines of Γ form a generalized quadrangle of order q (see [VM98, 2.3.17]) and this is what we know as the **symplectic quadrangle** $W(q)$. From the preceding discussion, we have a description of $W(q)$, analogous to that of $H(q)$ at the end of Section 1.4.1, as being comprised of all the points of $PG(3,q)$ together with the lines of $PG(3,q)$ whose Plücker coordinates satisfy

$$p_{01} = p_{23}.$$

One can also identify numerous similarities between the geometry of $W(q)$ and that of $H(q)$ as discussed in Section 1.4.2. For instance, the lines incident with a point x of $W(q)$ form a pencil in a plane of $PG(3,q)$, namely the plane that is the 1-point x^θ of Γ. Also, Lemmas 1.12 and 1.16, for instance, translate almost unchanged to the case of $W(q)$. Finally, while the number 3 seems to play a special rôle in the context of $H(q)$, for the generalized quadrangle $W(q)$ it is the value 2. Indeed, by [Tit62] and [Tha72] (see also [VM98, Corollary 7.3.5]), $W(q)$ is self-dual if and only if $q = 2^h$ and it is self-polar if and only if $q = 2^{2e+1}$ (cf. Theorem 1.20). In fact, the conjecture referred to on page 53

POINTS	
Coordinates in $Q(4,q)$	Coordinates in $PG(4,q)$
(∞)	$(1,0,0,0,0)$
(a)	$(a,0,0,1,0)$
(k,b)	$(-b,1,k,-k^2,0)$
(a,ℓ,a')	$(-\ell^2+aa',-a,\ell,a',1)$
LINES	
Coordinates in $Q(4,q)$	Coordinates in $PG(4,q)$
$[\infty]$	$\langle(1,0,0,0,0),(0,0,0,1,0)\rangle$
$[k]$	$\langle(1,0,0,0,0),(0,1,k,-k^2,0)\rangle$
$[a,\ell]$	$\langle(a,0,0,1,0),(-\ell^2,-a,\ell,0,1)\rangle$
$[k,b,k']$	$\langle(-b,1,k,-k^2,0),(-k'^2,0,k',b-2kk',1)\rangle$

Table 4.2: Coordinatization of $Q(4, q)$

concerning the nonexistence of ovoids of $H(q)$ when $3 \nmid q$ arises from the nonexistence of ovoids of $W(q)$ when $2 \nmid q$ [Tha73].

In all, the analogy between $W(q)$ and $H(q)$ is quite strong. For further details, see [VM98] where the analogy is drawn upon and made fairly clear wherever possible. In particular, see [VM98, 2.4.18] where polarities of $PG(3, q)$ and trialities of the geometry Γ in Section 1.4.1 are compared.

4.2 Ovoids of $Q(4, q)$

Now we turn our attention to the generalized quadrangle $Q(4, q)$, whose points and lines are the totally isotropic spaces on a nondegenerate parabolic quadric $\mathcal{P}_4$. This quadrangle is nothing other than the dual of the symplectic quadrangle $W(q)$ (see [VM98, Proposition 3.4.13]), so the analogy between $W(q)$ and $H(q)$ carries over to one between $Q(4, q)$ and $H(q)$ that is dual in nature; we only choose to consider $Q(4, q)$ rather than $W(q)$ for reasons of convenience.

In this section, we take the quadric $\mathcal{P}_4$ of $Q(4, q)$ to be given by the equation

$$X_0X_4 + X_1X_3 + X_2^2 = 0.$$

Also, we use a coordinatization of $Q(4, q)$ that is similar to the coordinatization of $H(q)$ described in Section 1.4.3 (see [VM98, 3.4.7]). Specifically, the elements of $Q(4, q)$ are assigned coordinates according to Table 4.2. Then incidence is given by the paths

$$[k, b, k'] \text{ I } (k, b) \text{ I } [k] \text{ I } (\infty) \text{ I } [\infty] \text{ I } (a) \text{ I } [a, \ell] \text{ I } (a, \ell, a')$$

together with $(a, \ell, a') \text{ I } [k, b, k']$ if and only if

$$b = a' + ak^2 + 2\ell k$$
$$k' = \ell + ak.$$

Spreads and ovoids of generalized $2m$-gons were defined in Section 2.1, and the definition of translation spreads and ovoids in Section 3.1 is readily generalized to generalized $2m$-gons (see [BTVM98] for details), so we assume a familiarity with these concepts for $Q(4,q)$. Translation ovoids of $Q(4,q)$ are discussed in [BTVM98, Section 3.2] and here we essentially repeat some of that, while making comparisons with what we have learnt about spreads of $H(q)$.

Let $\mathcal{O}$ be an ovoid of $Q(4,q)$. By an appropriate choice of coordinates, it may be supposed that the points (∞) and $(0,0,0)$ belong to $\mathcal{O}$ so then the ovoid is given by

$$\mathcal{O} = \{(\infty)\} \cup \{(x, y, f(x,y)) \mid x, y \in GF(q)\}, \tag{4.2}$$

where $f(0,0) = 0$ and $\Delta y^2 \neq \Delta x \Delta f$ whenever $(\Delta x, \Delta y) \neq (0,0)$—a condition comparable to the opposite line condition in Lemma 1.22. Compare this with the representation of spreads of $H(q)$ in (2.3). Concerning translation ovoids, we have the following results.

Theorem 4.1 (Bloemen, Thas, Van Maldeghem [BTVM98, Theorem 12]) *If $\mathcal{O}$ is a translation ovoid of $Q(4,q)$ with respect to a point x, then the associated group G_x acts regularly on the set of points $\mathcal{O} \setminus \{x\}$.* □

Lemma 4.2 (Bloemen, Thas, Van Maldeghem [BTVM98, Lemma 13]) *Let $\mathcal{O}$ be an ovoid of $Q(4,q)$ and let x be a point of $\mathcal{O}$. If a subgroup of G^x of order q^2 stabilizes $\mathcal{O}$ then $\mathcal{O}$ is translation with respect to x.* □

Corollary 4.3 (Bloemen, Thas, Van Maldeghem [BTVM98, Corollary 14]) *If $\mathcal{O}$ is an ovoid of $Q(4,q)$ containing (∞) and $(0,0,0)$ that is represented as in (4.2), with $q = p^h$, then $\mathcal{O}$ is translation with respect to (∞) if and only if*

$$f(x,y) = \sum_{i=0}^{h-1} \left(f_{1i} x^{p^i} + f_{2i} y^{p^i}\right),$$

with the coefficients $f_{ni} \in GF(q)$. □

Lemma 4.2 has a partner in Theorem 3.4 for $H(q)$. One objective in Section 3.2 has been to likewise identify results for $H(q)$ comparable to Theorem 4.1 and Corollary 4.3. This has been achieved in Theorems 3.6 and 3.7, where the rôle of the associated group G_x in Theorem 4.1 has been played instead by the stabilizer G_S^L in Theorem 3.6.

Examples of translation ovoids of $Q(4,q)$ are the classical ovoids $\mathcal{O}_E$, which are elliptic quadrics $\mathcal{E}_3$. Notice that these are then determined by a hyperplane of $PG(4,q)$ that meets $\mathcal{P}_4$ in an elliptic quadric, which is analogous to the construction of the hermitian spreads $\mathcal{S}_H$ of $H(q)$ described in Section 2.3.1. Thus the classical ovoids $\mathcal{O}_E$ and the hermitian spreads $\mathcal{S}_H$ appear to be analogous objects, and indeed, they do share numerous properties; for instance, both are translation with respect to each of their elements. Just as we found all hermitian spreads containing the lines $[\infty]$ and $[0,0,0,0,0]$ in Section 2.3.2, we also find all classical ovoids of $Q(4,q)$ containing the points (∞) and $(0,0,0)$, which are then

$$\mathcal{O}_E(\mu,\nu) = \{(\infty)\} \cup \{(x,y,-\nu x+\mu y) \mid x,y \in GF(q)\}, \tag{4.3}$$

where $f(x) = x^2 - \mu x + \nu$ is an irreducible quadratic.

Let $\mathcal{O}$, represented as in (4.2), be translation with respect to the point (∞). Similar to the definition in Section 3.3 of the kernel of a translation spread of $H(q)$, the **kernel** of $\mathcal{O}$, denoted $\ker\mathcal{O}$, is defined to be the maximal subfield of $GF(q)$ such that $f(ax,ay) = af(x,y)$, for all $a \in \ker\mathcal{O}$ and $x,y \in GF(q)$. We then have the following classification of translation ovoids $\mathcal{O}$ for which $\ker\mathcal{O}$ is as large as possible.

Theorem 4.4 (Bloemen, Thas, Van Maldeghem [BTVM98, Corollary 17]) *Let $\mathcal{O}$ be a translation ovoid of $Q(4,q)$ containing (∞) and $(0,0,0)$, represented as in (4.2), that has kernel $\ker\mathcal{O} = GF(q)$. Then $\mathcal{O}$ is a classical ovoid.* □

Illustrating further the relationship between the classical ovoids $\mathcal{O}_E$ and the hermitian spreads $\mathcal{S}_H$, compare this theorem with the classification in Theorem 3.9 of translation spreads of $H(q)$ for which the kernel is as large as possible. There however, we also see the $\mathcal{S}_{[9]}$ spreads appear. It was remarked in Section 3.3 that Theorem 3.9 is equivalent to [BTVM98, Theorems 30–32]. The remainder of this section is devoted to this fact.

Let $\mathcal{S}$ be a spread of $H(q)$ that is point locally hermitian with respect to some point x on a line $K \in \mathcal{S}$. For each line $L \neq K$ in $\mathcal{S}$, let L_x be the unique line of $\mathcal{P}_6$ that passes through x and that is concurrent with L. Since K and L are opposite, these lines L_x are not $H(q)$-lines. The tangent prime $x^\perp$ meets the quadric $\mathcal{P}_6$ in a quadric cone $x\mathcal{P}_4$, and since $\mathcal{S}$ is point locally hermitian at x, the line K and the lines L_x meet $\mathcal{P}_4$ in a set $\mathcal{O}_{\mathcal{S}}(x)$ of exactly q^2+1 points. By Lemma 2.4, $\mathcal{S}$ is a 1-system of $\mathcal{P}_6$, so from Theorem 1.8, each generator contains precisely $q+1$ points that are on lines of $\mathcal{S}$. Since every point on each line L_x is incident with some line of $\mathcal{S}$, it follows that no two distinct lines L_x and M_x are coplanar in $\mathcal{P}_6$. Similarly, no generator on K meets any other line of $\mathcal{S}$. Consequently, no two points of $\mathcal{O}_{\mathcal{S}}(x)$ are collinear. By Lemma 2.2, the set $\mathcal{O}_{\mathcal{S}}(x)$ is therefore an ovoid of the generalized quadrangle $Q(4,q)$. In the case that the spread $\mathcal{S}$ is locally hermitian with respect to K, this process was introduced in [BTVM98] as "projection along reguli".

Now suppose, without loss of generality, that $\mathcal{S}$ contains the lines $[\infty]$ and $[0,0,0,0,0]$, and that the point at which $\mathcal{S}$ is point locally hermitian is (∞). Then $\mathcal{S}$ is given by

$$\mathcal{S} = \{[\infty]\} \cup \{[x,y,z,f(x,y),g(x,y,z)] \mid x,y,z \in GF(q)\},$$

for some functions f and g with $f(0,0) = g(0,0,0) = 0$. Notice that f is independent of z by the point locally hermitian property.

The tangent prime to $\mathcal{P}_6$ at x is given by the equation $X_4 = 0$, and this meets $\mathcal{P}_6$ in the quadric cone $x\mathcal{P}_4$, where we may take $\mathcal{P}_4$ to be contained in the 4-space Π_4 given by $X_0 = X_4 = 0$. Let us choose coordinates within Π_4 such that the correspondence with the coordinates in $PG(6,q)$ is

$$(0,x_1,x_2,x_3,0,x_5,x_6) \longleftrightarrow (-x_6,-x_1,x_3,x_5,x_2).$$

Then the equation for $\mathcal{P}_4$ in terms of the coordinates in Π_4 is $X_0X_4 + X_1X_3 + X_2^2 = 0$, so we use the coordinatization of $Q(4,q)$ given in Table 4.2.

Notice that the line $[\infty]$ of $\mathcal{S}$ meets $\mathcal{P}_4$ in the point $(0,0,0,0,0,0,1)$, which in the coordinates of Π_4 is $(1,0,0,0,0)$, and this is the point (∞) of $Q(4,q)$. Thus the ovoid $\mathcal{O}_{\mathcal{S}}(\infty)$ of $Q(4,q)$ contains (∞). For a line $L = \big[x, y, z, f(x,y), g(x,y,z)\big]$ of $\mathcal{S}$, the line L_x of $\mathcal{P}_6$ is the line passing through the two points $(1,0,0,0,0,0,0)$ and $\big(z + yf(x,y), x, 1, y, 0, f(x,y), y^2 - xf(x,y)\big)$, which correspond to the points (∞) and $\big(x, y, z, f(x,y)\big)$ of $H(q)$, respectively. This line meets Π_4 in the point whose coordinates in Π_4 are $\big(-y^2 + xf(x,y), -x, y, f(x,y), 1\big)$, which from Table 4.2 is seen to correspond to the point $\big(x, y, f(x,y)\big)$ of $Q(4,q)$. Thus the ovoid $\mathcal{O}_{\mathcal{S}}(\infty)$ is given by

$$\mathcal{O}_{\mathcal{S}}(\infty) = \big\{(\infty)\big\} \cup \big\{(x, y, f(x,y)) \bigm| x, y \in GF(q)\big\}. \tag{4.4}$$

Suppose now that the spread $\mathcal{S}$ is translation with respect to the line $[\infty]$. Then by Theorem 3.3, $\mathcal{S}$ is locally hermitian with respect to $[\infty]$, and in particular, it is point locally hermitian with respect to the point (∞) of $H(q)$. Also, by Theorem 3.7, the function f has the form

$$f(x,y) = \sum_{i=0}^{h-1} \big(f_{1i}x^{p^i} + f_{2i}y^{p^i}\big),$$

with the coefficients $f_{ni} \in GF(q)$, so by Corollary 4.3, the ovoid $\mathcal{O}_{\mathcal{S}}(\infty)$ of $Q(4,q)$ is translation with respect to (∞). Furthermore, we have $\ker\mathcal{S} \leqslant \ker\mathcal{O}_{\mathcal{S}}(\infty)$.

Consider the case that $\ker\mathcal{S} = GF(q)$. Then also $\ker\mathcal{O}_{\mathcal{S}}(\infty) = GF(q)$ so by Theorem 4.4, the ovoid $\mathcal{O}_{\mathcal{S}}(\infty)$ is a classical ovoid. Since the point (∞) of $H(q)$ could have been taken to be any point on $[\infty]$ and we would still have $\ker\mathcal{S} = GF(q)$ (see Theorem 3.8), it follows that $\mathcal{O}_{\mathcal{S}}(x)$ is a classical ovoid for every point x incident with $[\infty]$. In the terminology of [BTVM98], a locally hermitian spread $\mathcal{S}$ with respect to a line L, for which $\mathcal{O}_{\mathcal{S}}(x)$ is a classical ovoid for every $x \mathrel{\mathrm{I}} L$, is called **semiclassical**. We are part of the way to having proved our next theorem.

Theorem 4.5
Let $\mathcal{S}$ be a locally hermitian spread of $H(q)$ with respect to the line L. Then the following are equivalent:

(i) $\mathcal{S}$ is semiclassical;

(ii) $\mathcal{O}_{\mathcal{S}}(x)$ and $\mathcal{O}_{\mathcal{S}}(y)$ are classical ovoids for two points $x \neq y$ on L;

(iii) $\mathcal{S}$ is translation with respect to L and $\ker\mathcal{S} = GF(q)$.

Proof In the preceding discussion, we have already seen that (iii) implies (i). Also, (i) implies (ii) by the definition of semiclassical. We have only to show now that (iii) is implied by (ii).

Without loss of generality, we suppose that the hat-rack of the coordinatization is chosen such that the two points referred to in (ii) are the points (∞) and (0), and that $\mathcal{S}$ is given by

$$\mathcal{S} = \bigcup_{x,y \in GF(q)} \big[\!\big[x, y, f(x,y), g(x,y)\big]\!\big],$$

for some functions f and g with $f(0,0) = g(0,0) = 0$. The ovoid $\mathcal{O}_{\mathcal{S}}(\infty)$ is then as in (4.4). Now we determine the ovoid $\mathcal{O}_{\mathcal{S}}(0)$ of $Q(4, q)$.

Let Π_4 be the 4-space given by $X_2 = X_6 = 0$, which is then in the tangent prime to $\mathcal{P}_6$ at the point (0), and let $\mathcal{P}_4 = \Pi_4 \cap \mathcal{P}_6$ be the parabolic quadric for the generalized quadrangle $Q(4, q)$. Choose coordinates in Π_4 such that the correspondence with the coordinates in $PG(6, q)$ is given by

$$(x_0, x_1, 0, x_3, x_4, x_5, 0) \longleftrightarrow (-x_0, x_1, -x_3, -x_5, x_4).$$

Then the equation of $\mathcal{P}_4$ in terms of the coordinates of Π_4 is $X_0X_4 + X_1X_3 + X_2^2 = 0$, so Table 4.2 is used to assign coordinates in the generalized quadrangle $Q(4, q)$. Notice that the line $[\infty]$ of $H(q)$ meets Π_4 in the point $(1, 0, 0, 0, 0, 0, 0)$, which then corresponds to the point (∞) of $Q(4, q)$.

Next, the projection in $H(q)$ of (0) onto the line $\big[x, y, 0, f(x, y), g(x, y)\big]$ of $\mathcal{S}$ is the point $\big(0, g(x, y), f(x, y), xg(x, y)+3yf(x, y), y\big)$. The line of $\mathcal{P}_6$ through this point and (0) meets Π_4 in the point whose coordinates in $Q(4, q)$ are $\big(y, f(x, y), -g(x, y)\big)$. Thus

$$\mathcal{O}_{\mathcal{S}}(0) = \big\{(\infty)\big\} \cup \big\{(y, f(x, y), -g(x, y)) \;\big|\; x, y \in GF(q)\big\}. \tag{4.5}$$

Since $\mathcal{O}_{\mathcal{S}}(\infty)$ is a classical ovoid, from its representation in (4.4) and the representation of the classical ovoids in (4.3), the function $f(x, y)$ is linear in x and y. Similarly, the ovoid $\mathcal{O}_{\mathcal{S}}(0)$ is classical so the function $g(x, y)$ is linear in y and $f(x, y)$, and therefore, in x and y. It now follows from Theorem 3.7 that the spread $\mathcal{S}$ of $H(q)$ is translation with respect to the line $[\infty]$ and, by the linearity of the functions f and g, the kernel of $\mathcal{S}$ is all of $GF(q)$. □

In view of this theorem, our classification in Theorem 3.9 of the translation spreads $\mathcal{S}$ of $H(q)$ with $\ker \mathcal{S} = GF(q)$, for q odd, is equivalent to the classification in [BTVM98, Theorems 30–32] of semiclassical spreads of $H(q)$ for q odd.

When the spread $\mathcal{S}$ is the hermitian spread $\mathcal{S}_H(\mu, \nu)$, as given in (2.23), the ovoids $\mathcal{O}_{\mathcal{S}}(\infty)$ and $\mathcal{O}_{\mathcal{S}}(0)$ of $Q(4, q)$, as shown in (4.4) and (4.5), are both the classical ovoid $\mathcal{O}_E(\mu, \nu)$. Thus, since the hermitian spreads are all isomorphic, if $\mathcal{S}$ is any spread of $H(q)$ that is point locally hermitian with respect to the point (0) and for which $\mathcal{O}_{\mathcal{S}}(0)$ is a classical ovoid, then we may suppose that this is any particular classical ovoid $\mathcal{O}_E(\mu, \nu)$ that we choose by applying an appropriate collineation of $H(q)$. This enables us to give our following necessary condition for a spread $\mathcal{S}$ of $H(2^h)$ to be translation with respect to $[\infty]$ and have $\ker \mathcal{S} = GF(2^h)$.

Theorem 4.6
Let $q = 2^h$ and let δ be some fixed element of $GF(q)$ with $\mathrm{Tr}(\delta) = 1$. Let $\mathcal{S}$ be a spread of $H(q)$ that is translation with respect to $[\infty]$ and such that $\ker \mathcal{S} = GF(q)$. Then $\mathcal{S}$ is isomorphic to a spread of the form

$$\bigcup_{x, y \in GF(q)} \big[\![x, y, \nu x + \mu y, \nu x + (\mu + \delta)y]\!\big],$$

for some μ and ν such that $x^2 + \mu x + \nu$ is irreducible.

Proof By Theorem 4.5, $\mathcal{S}$ is semiclassical, and by our previous comments, we may suppose that the ovoid $\mathcal{O}_{\mathcal{S}}(0)$ is the classical ovoid $\mathcal{O}_E(1,\delta)$. From the representation of $\mathcal{O}_{\mathcal{S}}(0)$ in (4.5) and that of the classical ovoids in (4.3), this amounts to $g(x,y) = f(x,y)+\delta y$. The ovoid $\mathcal{O}_{\mathcal{S}}(\infty)$ is also a classical ovoid $\mathcal{O}_E(\mu,\nu)$, for some μ and ν such that $x^2+\mu x+\nu$ is irreducible, and from (4.4) and (4.3), this gives us $f(x,y)=\nu x+\mu y$. The result follows. □

4.3 Remarks

As a result of Theorem 4.6, a computer search for semiclassical spreads of $H(2^h)$ becomes feasible for the first few values of the exponent h. Doing this for $h \leq 6$ revealed that the hermitian spread is the unique semiclassical spread when $h = 1$, 3 or 5; however, for the even values $h = 2$, 4, and 6, exactly one extra spread was discovered. From these three examples, I have identified a new infinite class of spreads which I describe now.

Theorem 4.7
Let $q = 2^{2e}$ and let δ be some fixed element of $GF(q)$ with $\mathrm{Tr}(\delta) = 1$. Then the set

$$\mathcal{S} = \bigcup_{x,y\in GF(q)} [[x, y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta}{\delta+1}y, \frac{\delta^3}{(\delta+1)^2}x + \frac{\delta^2}{\delta+1}y]] \tag{4.6}$$

is a nonhermitian semiclassical spread of $H(q)$.

Proof Notice that this set corresponds to the set described in Theorem 4.6 with $\mu = \delta/(\delta+1)$ and $\nu = \delta^3/(\delta+1)^2$. From there, if this set $\mathcal{S}$ were a hermitian spread then we would have $(\mu,\nu) = (1,\delta)$, and in particular, this gives $\mu = 1$, which cannot be true. So $\mathcal{S}$ is certainly not a hermitian spread. Also, it is clear from the representation in (4.6) that if $\mathcal{S}$ is a spread then its kernel is $\ker\mathcal{S} = GF(q)$ so it is semiclassical by Theorem 4.5.

For the line regulus of $\mathcal{S}$ corresponding to the pair $(x,y)\in GF(q)^2$ in (4.6), let $T(x,y)$ be the expression in the compatibility condition in (2.31b) on page 46. To show that $\mathcal{S}$ is a spread, we have then only to show that $\mathrm{Tr}\big(T(x,y)\big) = 1$ for all pairs $(x,y)\neq(0,0)$.

To begin, since $\mathcal{S}_H(1,\delta)$ certainly is a spread, we have from the compatibility condition in (2.31b) that

$$S(x,y) = \frac{A(x,y)}{B(x,y)^2} \tag{4.7}$$

has trace equal to 1 for all $(x,y)\neq(0,0)$, where

$$A(x,y) = \delta^3x^4 + \delta^2x^3y + (\delta^2+\delta)x^2y^2 + (\delta+1)xy^3 + \delta y^4,$$

and

$$B(x,y) = \delta x^2 + xy + y^2.$$

After a few messy lines, the function $T(x,y)$ in terms of $A(x,y)$ and $B(x,y)$ is found to be

$$T\Big(\frac{x}{\delta}, \frac{y}{\delta+1}\Big) = \frac{A(x,y) + xy^3}{\big(B(x,y)+x^2\big)^2}\ . \tag{4.8}$$

In order to compare $T(x,y)$ more readily with $S(x,y)$, we make a change of variables so that the denominator in (4.8) is identical to that which appears in (4.7). Since $q = 2^{2e}$, there is an element $\theta \in GF(q)$ such that $\theta^2 = \theta + 1$. Then $B(x, \theta x + y) + x^2 = B(x,y)$, so we let

$$T'(x,y) = T\Big(\frac{x}{\delta}, \frac{\theta x + y}{\delta+1}\Big).$$

To show that $T(x,y)$ has trace equal to 1 for all nonzero pairs (x,y), we have only to show the same for $T'(x,y)$, and since we know that this is so for $S(x,y)$, we can instead endeavour to show that $\mathrm{Tr}\big(S(x,y) + T'(x,y)\big) = 0$ for all nonzero pairs (x,y). Now

$$S(x,y) + T'(x,y) = \frac{C(x,y)}{B(x,y)^2}$$

where

$$\begin{aligned} C(x,y) &= A(x,y) + A(x, \theta x + y) + x(\theta x + y)^3 \\ &= \delta^2 x^4 + (\delta + \delta\theta)x^3 y + \delta\theta x^2 y^2 + xy^3, \end{aligned}$$

and it can be checked by direct substitution that $X = \delta\theta x^2 + xy$ is a solution of the quadratic equation $X^2 + B(x,y)X + C(x,y) = 0$ for all nonzero pairs (x,y). Thus this quadratic is always reducible and so $\mathrm{Tr}\big(C(x,y)/B(x,y)^2\big) = \mathrm{Tr}\big(S(x,y) + T'(x,y)\big) = 0$. It now follows that $\mathcal{S}$ is a spread of $H(q)$. □

These new spreads exist only when $q \equiv 1 \pmod 3$ and the original computer search indicated that these are the only nonhermitian semiclassical spreads for $q \leq 2^6$. So perhaps there could be a classification of semiclassical spreads for even q that is similar to the one in Theorem 3.9 for odd q, where these new spreads would appear as an analogue of the $\mathcal{S}_{[9]}$ spreads. Perhaps, further, these spreads and the $\mathcal{S}_{[9]}$ spreads can be given a common description.

Final Remarks

For spreads and ovoids of generalized quadrangles and generalized hexagons, the notion of being translation with respect to an element or a flag was introduced in the paper [BTVM98] of Bloemen, Thas and Van Maldeghem. The objective in this thesis has been to investigate these objects further in the context of the split Cayley hexagon $H(q)$.

Firstly, I have obtained classification results. Theorem 3.1 proves the nonexistence of translation ovoids of $H(q)$ when $3 \nmid q$. The spreads of $H(q)$ that are translation with respect to a line are classified in Theorem 3.9 subject to the condition that the kernel is all of $GF(q)$, and in Corollary 3.12 subject to $q = 3^h$. Spreads of $H(q)$ that are translation with respect to two disjoint flags are classified by Theorems 3.13 and 3.22.

Next, I have developed a number of useful tools. Lemmas 1.21 and 1.22 give coordinate conditions for elements of $H(q)$ to be opposite (the former is also stated and used in [BTVM98]). Similarly, there are the compatibility conditions for line reguli and point reguli on pages 46 and 49, respectively. Theorems 3.5 and 3.6 assure us that the stabilizer $G^L_{\mathcal{S}}$ in G^L of a spread $\mathcal{S}$ that is translation with respect to a line L is certainly always accessible, whereas the associated group G_L might misbehave when q is even. Consequently, the stabilizer $G^L_{\mathcal{S}}$ is perhaps the more convenient group to work with, and indeed, it is by using this that Theorem 3.7 is obtained, giving the forms of the functions in a coordinate representation of $\mathcal{S}$. Also, the notion of kernel of a translation spread of $H(q)$ is defined in analogy with a similar definition in [BTVM98] for translation ovoids of $Q(4, q)$, followed by the tedious verification that it is in fact well-defined, as stated in Theorem 3.8.

In addition, many already known things have been worked independently here, in some cases for completeness, in some others to present a new approach, and in others for the reason that their explicit demonstrations are difficult or impossible to locate. For instance, a characterization of 1-systems of $\mathcal{P}_6$ that lie in a hyperplane is given in Theorem 1.11, thereby leading to a new geometric proof of Theorem 2.9. In the reverse direction, by Theorem 4.5 it follows that the proof of Theorem 3.9 provides an algebraic proof of results of [BTVM98] that were previously dealt with geometrically. Also included in this category are the derivations of the representation of $H(q)$ in $\mathcal{P}_6$ given in equations (1.9) and (1.10), of the collineations Θ (and Ψ), and of the spread $\mathcal{S}_{[9]}$, as well as the demonstrations of the uniqueness of spreads of $H(2)$ in Theorem 2.5 and of the hermitian spreads in Section 2.3.3.

As with many things, the successes so far by no means suggest completion and there are indeed further investigations to be carried out in this direction. These could include: classifications along the lines of Theorem 3.9, but where the kernel is allowed to be smaller than $GF(q)$; bringing Theorem 4.6 to a close by classifying the spreads of $H(2^h)$ with kernel as large as possible; strengthening Theorem 3.1 by loosening the requirement that the ovoid be translation with respect to a flag, and thereby continue working towards the proof of the conjecture that $H(q)$ has no ovoids at all when $3 \nmid q$; and investigation of the significance of the notion of local polarity for ovoid-spread pairings in generalized quadrangles.

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