COMPUTATIONAL AND THEORETICAL ANALYSIS OF COHERENT CONFIGURATIONS AND RELATED INCIDENCE STRUCTURES

by

Sven Reichard

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Summer 2003

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ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Mikhail Klin, for his constant support. He agreed to help me prepare this thesis under time pressure, and I know it hasn't been easy. Many of the results presented here are based on his ideas and conjectures.

I would like to thank the members of the committee, Felix Lazebnik, Gary Ebert, and Qing Xiang, for helpful discussions and constructive criticism on the preliminary drafts. Without them, I would not have been able to bring the text into its final form. Further helpful comments were given by Andy Woldar and Stanley Payne.

The work over this text was started during a visit to Ben-Gurion University of the Negev in Beer Sheva, Israel, in the summer of 2002. I would like to thank the Department of Mathematics for their hospitality.

Many of the results presented here rely on combinatorial data gathered by Ted Spence, which is publicly available on his homepage [92]. His work on the classification of strongly regular graphs of small order was particularly helpful.

Finally, I want to thank my wife Franca for her constant love and support.

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ABSTRACT

In this thesis we investigate coherent configurations and their relation to incidence structures and diagram geometries. Coherent configurations, as introduced by Higman [52], form a generalization of association schemes (see Bannai-Ito [3]), and as such are central to Algebraic combinatorics.

In particular, we consider the following applications: Graphs with the t-vertex condition; an interesting new class of association schemes, so-called "Siamese schemes"; configurations arising from partial geometries, maximal arcs in projective spaces, and related geometries; and others.

The t-vertex condition was introduced by Hestenes and Higman [51] in order to give a combinatorial characterization of some properties of graphs related to rank 3 permutation groups ("rank 3 graphs"). This motivates the investigation of graphs which satisfy the t-vertex condition but which are not rank 3 graphs. We investigate the point graphs of generalized quadrangles. We show that they always satisfy the 5-vertex condition. If the order of the generalized quadrangle is of the form (s, s^2), they also satisfy the 6-vertex condition. This gives the first examples (in fact, an infinite series) of such graphs which are not rank 3 graphs.

Next, we consider Siamese color graphs, as introduced by Kharaghani and Torabi [60]. These are colorings of complete graphs which are related to distance regular graphs, Steiner systems and generalized quadrangles. In some cases they give rise to association schemes, which we call Siamese schemes. After developing axioms for these objects, we find that they impose strong restrictions on the possible parameters. We give a complete classification of these objects for small orders, as

well as a construction of an infinite series of Siamese schemes. As a byproduct, we obtain several new combinatorial objects, such as Steiner systems and distance regular graphs.

Given an incidence structure, or, more generally, a diagram geometry, one can define a set of natural binary relations on the elements (points, lines, etc.). If these relations form a coherent configuration, we speak of a *coherent incidence structure*. As examples we consider partial geometries, and diagram geometries arising from maximal arcs in projective planes. This leads to a new approach to the computer-aided construction of projective planes of order n containing maximal k-arcs. Special attention is given to the case $n = 12$, $k = 3$, since this is the smallest order for which the existence of a projective plane is still open.

Weakly neighborly maps are studied in Discrete Geometry as extremal objects. Here, we consider them under an algebraic point of view. In particular we classify small weakly neighborly maps admitting a sharply transitive automorphism group.

Part I

Preliminaries

Chapter 1

INTRODUCTION

1.1 Preamble

The material presented in this thesis combines three major research areas: Algebraic combinatorics, computer algebra, and the theory of incidence structures.

Its main goal is to bring together and to present in a unified manner problems related to several classes of combinatorial objects. These objects include: Coherent configurations, in particular association schemes; distance regular and strongly regular graphs; partial linear spaces and generalized quadrangles; two-graphs; symmetric designs; and others.

The main technique which unifies the various topics touched in this thesis is the use of the theory of coherent configurations. Therefore, we repeat some well-known axiomatics of various combinatorial and geometrical structures, and develop variations and generalizations in terms of coherent configurations.

An essential ingredient to our work are computer experiments. They stand at the beginning of all results presented here. Ideally, such results should be backed up (post-factum) by theoretical considerations. We consider this activity in two stages. Given an object constructed by a computer, an *model* of this object is its description on a theoretical level. From this model, we can elaborate *proofs* that the object satisfies certain properties.

We were able in some cases to give such proofs. In other places we at least got models, leading to nice new constructions.

All good things must come to an end; this holds for the work on this thesis. Therefore, in the last chapter we give a report of projects that couldn't be finished, and the current state of the affair.

1.2 Algebraic Combinatorics

Algebraic combinatorics is a relatively new branch of mathematics. It deals with highly symmetrical combinatorial objects, such as graphs and designs. Symmetry of an object is expressed in terms of the size and structure of its automorphism group.

The term "Algebraic Combinatorics" was coined by E. Bannai and I. Ito in their now classical monograph on the subject [3]. Originally, this term had a rather narrow definition, being applied to the study of association schemes, coherent configurations, and closely related objects. Subsequently, it was attributed to a much wider area of mathematics, including formal power series, partially ordered sets, etc. (see, e.g., Stanley [93]).

In what follows, we will use the term in the narrow, classical sense. In this context, following Bannai and Ito, Algebraic Combinatorics is sometimes described as "group theory without groups". In fact, association schemes can be considered as a combinatorial generalization of finite groups (see Zieschang [104]).

A more general notion is that of a coherent configuration [52]. This is a relational structure which satisfies certain axioms. A matrix analogue, which is called a coherent (cellular) algebra ([102], [31], [61]), is simply a matrix algebra (over $\mathbb{C}$) which is closed with respect to transposition and Schur-Hadamard (element-wise) multiplication.

In practice, we deal with highly symmetrical combinatorial objects. Here, we can replace the group theoretical definition of symmetry by a combinatorial one. For example, this step leads from vertex-transitive graphs (group theoretical) to regular graphs (combinatorial). In similar manner appear the following generalized notions of symmetrical objects: Strongly regular graphs, distance regular graphs, coherent configurations, block designs, etc.

There are several scientific schools which worked in the area of algebraic combinatorics: D.G. Higman, who created the notion of coherent configurations; R.C. Bose and others, who developed similar notions for statistical analysis; the Dutch school around J.J. Seidel; R.H. Bruck and his various students; the Soviet school starting with L.A. Kalužnin, who worked for a while independently from their Western colleagues; and the Japanese school around Bannai and Ito, who thoroughly developed the Theory of Association Schemes.

In the Soviet school there is a long tradition of computational activities, starting with the graph stabilization algorithm by Weisfeiler-Leman [103]. Later, I.A. Faradžev made important contributions to the constructive enumeration of combinatorial objects.

In this text, we try to continue these tradition, while at the same time using theoretical knowledge that has been accumulated over the last few decades.

1.3 Computer Algebra

In the early days of Algebraic Combinatorics, most of the work was done on a theoretical level. Computers were rarely used. Nowadays, combinatorialists rely more and more on computers and in particular on symbolic algebra systems, in order to search for interesting objects, to classify completely certain classes of objects, or to establish certain combinatorial properties,

At the heart of any computer investigation in Combinatorics lies a backtrack search. This technique allows us to search potentially vast problem spaces in relatively short time.

Whereas twenty years ago, every such search was hand-coded — sometimes in low-level languages — researchers today have the advantage of Computer Algebra Systems. These systems not only contain algorithms to perform the more common task without the need of additional programming. They also include in some cases libraries of combinatorial and algebraic data accumulated by generations of mathematicians.

We distinguish two kinds of such packages. General purpose systems are designed to deal with a broad range of mathematical problems (within reasonable bounds). Such systems, like GAP[38] and Maple often include their own programming language in order to be extensible. Specialized packages are designed for one specific task. Examples are COCO[30] for the investigation of coherent configurations, and Discreta for the construction of t-designs.

In the course of the preparation of this thesis, extensive use was made of both GAP and COCO. This is reflected in the text, where much attention is paid to details of the computation.

Let us give a brief overview of computations in Algebraic Combinatorics. Typically, we are concerned with classes of objects depending on some numerical parameters. Such parameters are, for example, the order and valency of a regular graph, or the number of points and blocks in a block design. The defining properties of the objects under consideration lead to certain feasibility constraints on the parameter set, arising from combinatorial identities, linear algebra, and number theory. Traditional numerical computation is used to determine feasible sets of parameters.

A concrete combinatorial object typically can be represented as a $(0, 1)$ matrix. In the case of graphs, we have the adjacency matrix (a_{ij}), where rows and columns correspond to vertices, and $a_{ij} = 1$ iff i and j are adjacent. For block designs we have the incidence matrix, where rows correspond to points, and columns correspond to blocks.

Typical computational tasks can be characterized as searches: Enumeration of objects, identification (isomorphism testing), and determining the automorphisms of an object. Here, we treat the entries of the matrix not as numbers, but only as distinguishable symbols. Thus the computations have a more symbolic character.

In the case of coherent configuration, we actually have two options of its algebraic representation [31]: The adjacency algebra W is an algebra generated by $(0, 1)$ matrices, thus lending itself to symbolic manipulations. It adequately reflects all information about the coherent configuration. The intersection algebra B is a general matrix algebra isomorphic to W, but consisting of matrices which are typically of much smaller size. It represents some important necessary conditions for the existence of a coherent configuration. Some computations, like isomorphism testing, are done on the symbolic level in W, whereas others, like the determination of eigenvalues, can be done numerically in the "smaller" algebra B.

1.4 Incidence structures

One of the classes of objects of interest in Algebraic Combinatorics are graphs, in particular graphs with interesting spectral properties. These include strongly regular graphs, distance regular graphs, and various generalizations.

Whereas in graphs we consider one type of objects, namely vertices, together with some binary relation of adjacency, incidence structures in the classical sense involve two kinds of elements, say, the points and lines of a geometry, and some incidence relation between elements of different kind.

In principle incidence structures can be considered as (bipartite) graphs, however, in most cases the use of geometrical language is more natural and appropriate.

Incidence structures and their relations to coherent configurations have been first studied by D.G. Higman. Here, we try to describe some new connections between these areas. In fact, we will use the term "incidence structure" loosely, allowing for more than two types of objects and thus including, e.g., diagram geometries.

For each incidence structure, we define an automorphism to be a permutation of the objects, preserving the type of each object, as well as incidence. The set of all automorphisms of a given structure forms a group; its study reveals a lot about the properties of the structure.

The notion of incidence structures gives rise to helpful correspondence between different kinds of combinatorial objects, such as graphs and two-graphs, or certain strongly regular graphs and symmetric designs.

We consider the use of incidence structures and their relation to coherent configurations as an innovation, in particular on a methodological and computational level.

1.5 Experimental mathematics

In my opinion, the effective use of computers for the solution of problems in algebraic combinatorics consists of several stages:

- Use existing general programs and packages; find their limitations.
- Extend and/or replace existing programs with ad hoc algorithms, customized for the problem at hand.
- Test any new algorithm using known combinatorial objects.
- Search for new interesting objects, or establish new properties of known objects.
- Interpret and generalize the new results on a theoretical level.

Let me illustrate these stages using examples from my own experience. Here, I would distinguish two typical cases. The essence of an experiment is a search for a concrete combinatorial object (case **A**), or the use of a computer in order to approach a proof for a purely theoretical result (case **B**).

A. During the preparation of my Master's Thesis, I spent a year with M. Klin at Beer Sheva, Israel. We were trying to find new partial geometries. The main idea was to construct such geometries from their point graphs. An algorithm for this task had been designed by A. Kasikova, a former student of Klin's, but it was never implemented.

We wrote a first implementation in C, and tested it on the point graphs of various known partial geometries. Using a graph constructed by Brouwer, Ivanov, Klin [13], we found several new geometries, but the search could not be completed. Finally, the algorithm was ported to GAP, extensive isomorph rejection was introduced, with the result that the search was completed in a matter of minutes. Four new geometries were found; they were constructed independently by Mathon-Street [72]; see also DeClerck [26]. Details about the algorithm can be found in [84].

B. An example of the second kind of experiment is more relevant to this thesis. It deals with the t-vertex condition for the point graphs of generalized quadrangles.

A. Pasini had given a sketch of the proof that some of these graphs satisfy the 5-vertex condition. A generalization of the classical theorem by Hestenes-Higman (Theorem 4.3) enabled us to simplify and complete the proof for all generalized quadrangles.

Thus the question arose what happens for $t = 6$. To get some data, we transformed the theoretical criterion into a working algorithm, and tested it for small values of t and for known graphs. Then, we established that in general, the 6-vertex condition is not satisfied. However, for certain non-classical generalized quadrangles of order $(5, 25)$, it did in fact hold.

This led to the conjecture that it holds for GQ's of order (s, s^2), which, using Pasini's original idea, was subsequently proven.

Taking the algorithm to the limit, it was established that some GQ's of order $(5, 25)$ with a non-rank 3 point graph in fact satisfy the 7-vertex condition, however, here we don't have any theoretical (that is, computer-free) justification yet.

1.6 Outline of the thesis and summary of new results

This thesis consists of two parts and seven chapters, followed by a list of references and several supplements.

The first part, **Preliminaries**, consists of the first three chapters. It contains definitions and known results for future reference.

Chapter 2 (A short thesaurus) contains a thesaurus of Algebraic Combinatorics. Its purpose is to collect classical, well-established material for easy reference. As a rule, things are presented rather briefly, without many examples or historical references.

Chapter 3 (Computer Algebra Techniques) contains a brief introduction to the area. It is accessible to a general audience, although in some places a certain "algorithmical culture" is assumed.

The second part, **Main Results**, consists of Chapters 4 through 7 and contains new results obtained during the preparation of this thesis.

Chapter 4 (Graphs with the t-vertex condition) is the most theoretical part of the text. The concept of the t-vertex condition goes back to Hestenes and Higman [51] and tries to capture combinatorial properties of highly symmetrical graph, so-called rank 3 graphs. Section 4.1 gives the necessary definitions, as well as a historical survey of known results.

The remainder of the chapter is motivated by a long-standing conjecture by M. Klin [31] that for sufficiently large t, the only graphs satisfying the t-vertex condition are rank 3 graphs. Hence, we consider graphs which are not rank 3 graphs, yet satisfy the t-vertex condition for $t = 4, 5, \ldots$

The three main classes of graphs under consideration here are: Graphs on 36 vertices, point graphs of generalized quadrangles in general, and points graphs of generalized quadrangles of order (s, s^2) in particular.

In Section 4.2, we describe the smallest examples strongly regular graphs satisfying the 4-vertex condition which are not rank 3 graphs. These graphs were discovered by Klin-Meszka-Rosa, based on computer catalogs by E. Spence.

In Section 4.3 we investigate point graphs of generalized quadrangles. The main result is Main Theorem 1 on page 59, which states that these graphs satisfy the 5-vertex condition.

In the next section we look at the special case of generalized quadrangles of order (s, s^2). Main Theorem 2 on page 74 states that these graphs satisfy the 6-vertex condition. This leads to the first known examples (in fact, infinitely many) of graphs which are not rank 3 graphs but satisfy the 6-vertex condition.

Finally we report the discovery of a graph with an intransitive automorphism group which satisfies the 7-vertex condition.

In conclusion we can say that if Klin's conjecture is true and there is a t_0 such that for any graph, the t_0-vertex condition implies a rank 3 automorphism group, then $t_0 \geq 8$.

In *Chapter 5 (Siamese association schemes)*, we define Siamese association schemes and Siamese color graphs as new interesting classes of combinatorial objects. We give some necessary conditions for their parameters, and link them to generalized quadrangles.

Investigating the smallest possible case on 15 points, we classify all Siamese color graphs of this order (Corollary 5.4); there are exactly two of them (up to isomorphism), and we give theoretical descriptions for both of them (Theorem 5.3, Proposition 5.16, Corollary 5.3)

In the process, we give a new combinatorial description of a Siamese Steiner system on 15 points (Subsection 5.2.5).

Investigating Siamese graphs on 40 points, we describe one association scheme (Section 6.3) as well as a few other color graphs. By computer, we construct a total of 9 such graphs (Section 6.4).

Here, we give new constructions for two "old" objects, namely a classical drg on 40 points (Subsection 5.3.5) and the generalized quadrangle $W(3)$ and its spread (Subsection 5.3.6).

In an attempt to generalize the given constructions to an infinite series, we investigate a certain transitive action of $PSL(2, q^2)$ (Section 5.7).

Chapter 6 (Coherent configurations) deals with a link between the theories of coherent configurations and incidence structures, or, more generally, diagram geometries.

As main examples of such coherent geometries, we investigate partial geometries (Proposition 6.3) and maximal arcs in projective planes (Theorem 6.2). The latter sheds new light on a construction of projective planes proposed by R.H. Bruck.

Applying this construction to the case of a projective plane of order 12, we obtain a few negative results (Proposition 6.13, Theorem 6.4).

As a further example of diagram geometries, we introduce the notion of a square root partial linear space (Section 6.3) as a rank 3 geometry. We give a complete analysis of small cases, and give a self-contained description of an example discovered by Brouwer, Koolen, and Klin.

Chapter 7 (Weakly neighborly polyhedral maps) is a completely self-contained chapter that is rather isolated from the rest of the text. It provides a new link between Algebraic Combinatorics and another area of Mathematics, namely Discrete Geometry. Although the new results presented here are rather modest, we feel that this new connection will be of interest.

References We tried to give all relevant references. However, it was not our goal to give a comprehensive historical picture.

Appendices In the course of the work, many combinatorial objects were constructed. Not for all of them nice computer-free models could be elaborated. In those cases, we provide enough data in the supplement to enable the interested reader to reconstruct these objects.

We also include some of the algorithms developed and used for this text. We describe how to check the t-vertex condition, and how to find spreads in incidence structures.

Chapter 2

A SHORT THESAURUS

2.1 Coherent configurations

Let Ω be a finite set, and let (G, Ω) be a permutation group, i.e., a subgroup of the symmetric group S_Ω of Ω. We denote by 2-orb(G, Ω) the orbits of G in its induced action on ordered pairs $(x, y) \in \Omega^2$. The 2-orbits can be considered as binary relations on Ω; in fact, they are the minimal G-invariant relations.

If the 2-orbits of (G, Ω) are $R_0, R_1, \ldots, R_{r-1}$, they satisfy the following properties:

1. The identity relation $Id_\Omega = \{(x, x)|x \in \Omega\}$ is the union of a set of 2-orbits.
2. Each pair $(x, y) \in \Omega^2$ is in exactly one 2-orbit, in other words, $\Omega^2 = \dot{\bigcup} R_i$.
3. If $R_i \in$ 2-orb(G, Ω), then $R_i^t = R_{i'} \in$ 2-orb(G, Ω). Here R_i^t denotes the transpose of R_i, i.e., $R_i^t = \{(y, x)|(x, y) \in R_i$.
4. There are numbers p_{ij}^k, $0 \leq i, j, k \leq r - 1$, such that for each $(x, y) \in R_k$ there are p_{ij}^k points z such that $(x, z) \in R_i$ and $(z, y) \in R_j$.

Example 2.1. Let $\Omega = \{1, 2, 3, 4, 5\}$, and let $G = D_5 = \langle(1, 2, 3, 4, 5), (2, 5)(3, 4)\rangle$ be the dihedral group of degree 5 and order 10. This group has three 2-orbits, namely $R_0 = (1, 1)^G$, $R_1 = (1, 2)^G$, $R_2 = (1, 3)^G$.

We have $Id_\Omega = R_0$. All three relations are symmetric, so $i' = i$ for $i = 0, 1, 2$.

Given a binary relation R on X, we define its adjacency matrix A as follows. The rows and columns of A are labelled by the elements of X, and for $i, j \in X$,

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in R \\ 0 & \text{otherwise} \end{cases}$$

The above properties of the 2-orbits of a permutation group, when translated to the adjacency matrices A_i, give the following:

1. The identity matrix I is the sum of some of the A_i;
2. $\sum_i A_i = J$, where J is the all-one matrix;
3. $A_i^t = A_{i'}$ for some i';
4. $A_i A_j = \sum_k p_{ij}^k A_k$.

The last of these properties says that the vector space spanned by $A_0, A_1, \ldots, A_{r-1}$ is closed under multiplication, hence it is a matrix algebra.

If we take these properties as axioms, we arrive at the following definition:

Definition 2.1. Let W be an algebra of matrices of order n over $\mathbb{C}$ which satisfies the following properties:

(CA1) W possesses a basis $\{A_0, A_1, \ldots, A_{r-1}\}$ of $(0,1)$-matrices.

(CA2) W contains I and J, i.e., the identity matrix and the all-one matrix.

(CA3) W is closed under transposition.

Then $W = \langle A_i \rangle$ is a *coherent algebra* of order n and rank r.

As before, we have $A_i^t = A_{i'}$ for some i'.

Definition 2.2. Let X be a finite set, and let $R_i, i = 0 \ldots r-1$ be binary relations on X. Let A_i be the adjacency matrix of R_i. If $W = \langle A_i \rangle$ is a coherent algebra, then $(X, \{R_i\})$ is a *coherent configuration.*

Definition 2.3. Given a coherent configuration $(X, \{R_i\})$, we refer to the relations R_i as *basis relations*. The graphs $\Gamma_i = (X, R_i)$ will be called *basis graphs*, whereas their adjacency matrices A_i are *basis matrices*.

This allows us to switch freely between the languages of matrices, relations, and graphs.

By the second axiom, the identity relation Id_Ω is the union of some basis relations $R_j = Id_{F_j}$. This gives a partition of Ω into sets F_j, which we call the *fibers* of the coherent configuration.

Proposition 2.1. *For each basis relation R_i, the set $\{x \in \Omega : \exists y : (x, y) \in R_i\}$ is a fiber, which we call the* initial fiber *of R_i. The initial fiber of the transpose $R_{i'}$ is the* terminal fiber *of R_i.*

Thus for each point x not in the initial fiber of R_i, the out-degree of x in Γ_i is 0. All points which are in the initial fiber of R_i have the same out-degree, which we denote by k_i; we will refer to it as the valency of R_i.

So far, we have seen coherent configurations arising from permutation groups.

Definition 2.4. Let (G, Ω) be a finite permutation group. The coherent algebra generated by the 2-orbits of G is called the *centralizer algebra* of (G, Ω), and is denoted by $V(G, \Omega)$. Any coherent algebra that is the centralizer algebra of some group is called *Schurian*.

The following is a collection of the simplest identities for the structure constants of a coherent configuration:

Proposition 2.2.

$$\begin{aligned} \sum_j p_{ij}^k &= k_i \quad \forall k \\ \sum_i p_{ij}^k &= k_{j'} \quad \forall k \end{aligned}$$

Definition 2.5. Given a coherent configuration $W = (X, \{R_i\})$, a permutation of X is a *(strong) automorphism* of W if it preserves all basis relations. It is called a weak automorphism if it induces a permutation on the basis relations.

The set of all automorphisms of W forms a group, the automorphism group $Aut(W)$.

The set of all weak automorphisms of W forms a group, which in the case of Schurian configurations is equal to the normalizer of $Aut(W)$ in the symmetric group $S(X)$. Therefore, we will call this group in general the *normalizer* $N(W)$ of W.

2.2 Association schemes

Here, we consider a special class of coherent configurations.

Definition 2.6. A coherent configuration is called an *association scheme* if all basis graphs are regular, or, equivalently, if it has exactly one fiber.

We will sometimes call an association scheme simply a scheme, if there is no danger of confusion.

Thus, in an association scheme, there is exactly one reflexive relation, usually denoted by R_0. Traditionally, the non-reflexive basis relations are called the *classes* of the association scheme; thus, a scheme with d classes is a coherent configuration of rank $d + 1$.

Among all association schemes, we distinguish two special types:

Definition 2.7. An association scheme is *symmetric* if all basis relations are symmetric. It is *commutative* if its adjacency algebra is commutative.

Proposition 2.3. *A symmetric association scheme is commutative.*

The adjacency algebra of a symmetric scheme is traditionally called its *Bose-Mesner algebra*, or BM-algebra for short.

Combinatorially, we can characterize an association scheme by its basis graphs.

Definition 2.8. An association scheme is *primitive* if all its non-reflexive basis graphs are connected. Otherwise, it is *imprimitive*.

Examples for association schemes come from distance regular graphs.

Definition 2.9. Let Γ be a simple graph of diameter d. Suppose that for any pair of vertices (x, y), the number of vertices z at distance i from x and at distance j from y depends only on the distance k of x and y. Then Γ is called *distance regular*.

Given such a graph, we can define relations $R_i, i = 0, \ldots, d$ with $(x, y) \in R_i$ if their distance in Γ is i.

Proposition 2.4. *The relations R_i defined above form an association scheme with d classes.*

Proposition 2.5. *The basis matrices of the association scheme defined by a distance regular graph have the property that for all $i = 0, \ldots, d$, $A_i = f_i(A_1)$, where f_i is a polynomial of degree i. Conversely, every association scheme with this property can be constructed from a distance regular graph.*

Association schemes coming from distance regular graphs are called *metrical* or sometimes *P-polynomial*.

We will use a number of other notions, in particular, intersection arrays, imprimitive drg's, antipodal covers, quotients. We refer to [16] for their discussion.

2.3 Strongly regular graphs

Let $\Gamma = (V, E)$ be a finite simple graph, that is, undirected, without loops or multiple edges. The *order* of Γ is $v = |V|$. For a vertex x, let $\Gamma(x)$ denote the set of neighbors of x. The number of neighbors is called the *valency* of x.

Definition 2.10. The graph Γ is called *regular* if all vertices have the same valency k. In this case, we say that Γ has valency k.

Given two vertices x and y of Γ, we can consider the set of common neighbors $\Gamma(x) \cap \Gamma(y)$.

Definition 2.11. Let Γ be a regular graph of order v and valency k. Suppose that there are numbers λ and μ such that all pairs of adjacent vertices have λ common neighbors, and all pairs of distinct non-adjacent vertices have μ common neighbors.

Then Γ is called a *strongly regular graph* with parameters (v, k, λ, μ).

We use the abbreviation "srg" for "strongly regular graph". We also use the notation $\mathrm{srg}(v, k, \lambda, \mu)$.

Example 2.2 (Pentagon). Let C_5 be the undirected cycle on 5 vertices, with vertex set $\{1, 2, 3, 4, 5\}$ and edges $\{1, 2\}$, $\{2, 3\}$ $\{3, 4\}$, $\{4, 5\}$, and $\{5, 1\}$. Then C_5 is regular of valency 2.

Moreover, any two adjacent vertices don't have any common neighbors, and any two distinct non-adjacent vertices have exactly one common neighbor. Thus, $\lambda = 0$ and $\mu = 1$, and we have an $\mathrm{srg}(5, 2, 0, 1)$.

In fact, if we consider the automorphism group of the pentagon, we see that it acts transitively on edges and on non-edges. Hence, clearly it is strongly regular. This motivates the following definition:

Definition 2.12. Let Γ be a graph, and let G be its automorphism group. Suppose that G has three orbits on ordered pairs of vertices. Then Γ is called a *rank 3 graph.*

In this case, there is one reflexive orbit, i.e., all pairs of the form (x, x), and two antireflexive orbits, namely, the edges and non-edges of Γ. Thus, Γ is a rank 3 graph iff its automorphism group acts transitively on vertices, edges, and non-edges. Clearly, a rank 3 graph is strongly regular.

We can describe any graph in terms of matrices.

Definition 2.13. Let $\Gamma = (V, E)$ be a finite graph. For $i, j \in V$, let $a_{ij} = 1$ if $i \sim j$, and $a_{ij} = 0$ otherwise.

Then the matrix $A = A(\Gamma) = (a_{ij})_{i,j \in V}$ is the $(0,1)$ adjacency matrix of Γ.

Usually, we will call this matrix just the adjacency matrix of Γ.

In the case of a simple graph, the adjacency matrix is symmetric with a zero diagonal. Let $\overline{\Gamma}$ be the complementary graph to Γ, and let $\overline{A} = J - I - A$ be its adjacency matrix.

Lemma 2.1. *Let Γ be a simple graph of order v. The following are equivalent:*

1. *Γ is a srg(v, k, λ, μ)*
2. *$A^2 = kI + \lambda A + \mu \overline{A}$.*

Definition 2.14. The $(0,1)$ spectrum of a graph Γ is defined as the spectrum of its $(0,1)$ adjacency matrix.

Again, we will omit the prefix $(0,1)$ if there is no danger of confusion.

Theorem 2.1. *A srg(v, k, λ, μ) has the eigenvalue k with multiplicity 1, and two other eigenvalues r, s satisfying*

$$r, s = \frac{1}{2}\left((\lambda - \mu) \pm \sqrt{\Delta}\right)$$

where $\Delta = 4(k - \mu) + (\lambda - \mu)^2$.

If we denote the multiplicities of r and s by f and g, respectively, we get the condition that $v = 1 + f + g$, and $trA = 0 = k + rf + gs$.

Thus, if s and r are the non-trivial eigenvalues of a strongly regular graph, we have that

$$\begin{aligned} B = (A - rI)(A - sI) &= A^2 - (r + s)A + rsI \\ B\mathbf{1} &= (k^2 - k(r + s) + rs)\mathbf{1} \\ &= \frac{k^2 - k(r + s) + rs}{v} J\mathbf{1} \end{aligned}$$

on the other hand, if w is an eigenvector orthogonal to $\mathbf{1}$,

$$\begin{aligned} Bw &= (A - rI)(A - sI)w \\ &= 0 \\ &= \frac{k^2 - k(r+s) + rs}{v} Jw \end{aligned}$$

Thus, we get that

$$\begin{aligned} A^2 &= -rsI + (r+s)A + \frac{k^2 - k(r+s) + rs}{v} J \\ &= kI + \lambda A + \mu(J - A - I) \\ &= (k - \mu)I + (\lambda - \mu)A + \mu J. \end{aligned}$$

Comparing coefficients we find that

$$\begin{aligned} \mu &= k + rs \\ \lambda &= r + s + \mu \end{aligned}$$

2.4 Incidence structures

Definition 2.15. Let P and B be sets, and let $I \subseteq P \times B$ be a relation, referred to as incidence. Then (P, B, I) is an *incidence structure*.

Usually, the elements of P will be called points, and the elements of B, blocks.

Definition 2.16. Let (P, B, I) be an incidence structure. The incidence structure (B, P, I^T) obtained by interchanging blocks and points, and reversing the incidence relation, is called its dual structure.

It is possible that in an incidence structure, two blocks are incident with the same set of points. In this case, we speak of *repeated blocks*. If this is not the case, we can identify each block with the set of points it is incident with, and replace I by the relation "$\in$". In this case, we usually omit mentioning the incidence relation altogether.

Given an incidence structure, we can naturally define three graphs related to it. The incidence graph is a bipartite graph defined on $P \cup B$, where a point and a block are adjacent if they are incident. The point graph is defined on P, where two points are adjacent if they are "collinear", i.e., are incident with a common block. Finally, the block graph is the point graph of the dual structure; vertices are the blocks, and two blocks are adjacent if they are incident to a common point.

Examples of incidence structures are partial geometries and generalized quadrangles, discussed below. Let us briefly mention other examples:

Definition 2.17. Let (P, B) be an incidence structure without repeated blocks, such that

- each block contains k points;
- each set of t points is contained in a unique block.

Then (P, B) is called a Steiner system $S(t, k, v)$, where $v = |P|$.

If $t = 2$, $k = 3$, we speak of a Steiner triple system, or $STS(v)$.

Definition 2.18. Let (P, B) be an incidence structure without repeated blocks, such that

- each block contains k points;
- each point is contained in k blocks;
- each pair of points is contained in λ blocks;
- each pair of blocks intersects in λ points.

Then (P, B) is a symmetric (v, k, λ)-design, with $v = |P|$.

This is not quite the standard definition (which involves Fisher's inequality), however, it can be shown to be equivalent (see, e.g., Cameron [23]). Note that the axioms above are self-dual, thus the dual of a symmetric design is a symmetric design with the same parameters.

We can generalize the notion of an incidence structure to the case where we have more than two types of elements.

Definition 2.19. Let $P_i, i = 1, \ldots, n$, be sets. For each pair $\{P_i, P_j\}$ let there be an incidence relation I_{ij}, such that (P_i, P_j, I_{ij}) is an incidence structure. Then $(\{P_i\}, \{I_{ij})$ is a *diagram geometry* of rank n.

In the usual definition of a diagram geometry, it is not required that there are incidences defined between any two types P_i, P_j of objects. However, we may always introduce the trivial relation $I_{ij} = P_i \times P_j$.

2.5 Partial linear spaces and partial geometries

Partial linear spaces and partial geometries are incidence structures satisfying certain regularity conditions:

Definition 2.20. A *partial linear space* with parameters K, R is an incidence structure of points and lines such that each line contains K points, each point lies on R lines, and any two lines intersect in at most one point.

We denote such a partial linear space by $pls(K, R)$.

Definition 2.21. A *partial geometry* is a partial linear space with the additional property that there is a constant T such that for any non-incident point-line pair (p, l) there are T lines through p which intersect l. In this case, K, R, T are the parameters of the partial geometry; we abbreviate this by $pg(K, R, T)$.

Theorem 2.2 (Bose[9]). *The point graph of a partial geometry $pg(K, R, T)$ is strongly regular, with parameters*

$$\begin{aligned}
v &= K\left(1 + \frac{(K-1)(R-1)}{T}\right) \\
k &= R(K-1) \\
l &= \frac{(K-T)(K-1)(R-1)}{T} \\
\lambda &= (K-2) + (R-1)(T-1) \\
\mu &= RT
\end{aligned}$$

We call a srg with these parameters a pseudo-geometric (K, R, T)-graph.

Theorem 2.3 (Bose[9]). *Let (P, B) be a partial linear space $pls(K, R)$ with a pseudo-geometric (K, R, T) point graph. Then (P, B) is a partial geometry $pg(K, R, T)$.*

Remark 2.1. The use of the parameters K, R, T is motivated by similar parameters from Design Theory. Another tradition, related to Finite Geometry, uses the parameters $s = K - 1$, $t = R - 1$, $\alpha = T$, and speaks e.g., of a partial linear space of order (s, t). For example, a projective plane of order q is a particular case of a partial linear space of order (q, q).

We will mostly use the combinatorial language, except for a few special cases where the geometrical language can definitely be regarded as the standard (see, e.g., Section 2.6).

2.6 Generalized quadrangles

Definition 2.22. A *generalized quadrangle* is a partial geometry with $T = 1$. The pair $(s = K - 1, t = R - 1)$ is the order of the generalized quadrangle. We will denote a generalized quadrangle of order (s, t) by $GQ(s, t)$.

In the case of $s = t$, we will simply speak of a GQ of order s.

The following is a special case of Theorem 2.2.

Theorem 2.4. *The point graph of a $GQ(s,t)$ is strongly regular, with parameters*

$$\begin{aligned} v &= (s+1)(st+1) \\ k &= s(t+1) \\ \lambda &= s-1 \\ \mu &= t+1 \end{aligned}$$

The following is a simple fact, but it will prove very useful in the subsequent analysis:

Theorem 2.5 (Cameron [22]). *The point graph of a $GQ(s,t)$ does not contain $K_4 - e$ as an induced subgraph.*

Here $K_4 - e$ denotes the graph obtained by removing an edge from the complete graph K_4 on 4 vertices.

Next, we describe the "classical" generalized quadrangles of order q, where q is a prime power. They were first constructed by J. Tits [99].

Theorem 2.6 (Tits). *Let P be the set of points of the projective space $P(3,q)$. Let B be the set of totally isotropic lines of symplectic polarity. Then (P,B) forms a generalized quadrangle of order q, denoted by $W(q)$.*

Theorem 2.7 (Tits). *Let Q be a non-singular quadric of projective index 1 in $PG(4,q)$. Let P be the the set of points of Q; let B be the set of absolute lines. Then (P,B) forms a generalized quadrangle of order q, denoted by $Q(4,q)$.*

In fact, these two constructions are related. We have

Theorem 2.8. *The dual to $W(q)$ is isomorphic to $Q(4,q)$. $W(q)$ is self-dual if and only if q is even.*

The automorphism groups of these generalized quadrangles are automorphism groups of classical projective groups.

Proposition 2.6. $Aut(W(q)) \cong P\Gamma Sp_4(q)$.

Proposition 2.7. $Aut(Q(4,q)) \cong P\Gamma O_5(q)$.

For any prime power q, there exist $GQ(q-1, q+1)$. This is a result by Ahrens-Szekeres [1] who give two constructions depending on the parity of q. The construction for q even was found independently by M. Hall, Jr. [46]. We follow here the exposition in Payne [79].

Let q be an odd prime power, and let $W(q)$ be the $GQ(q)$ arising from a symplectic polarity. Let x be a point of $W(q)$, and let π be the polar plane of x. Let P be the set of points of $PG(3,q)$ not on π. Let B be the set of lines of $W(q)$ not containing x, together with the lines in $PG(3,q)$ through x not in π.

Theorem 2.9 (Ahrens-Szekeres). *(P, B) is a generalized quadrangle of order $(q-1, q+1)$, denoted by $AS(q)$.*

Now let q be a power of 2. Let $\Pi = PG(3,q)$, π be a hyperplane in Π, and O be a hyperoval in π.

Let P be the set of points in $\Pi \setminus \pi$. Let B be the set of lines intersection O in exactly one point; hence these lines do not lie in π.

Theorem 2.10 (Ahrens-Szekeres, Hall). *(P, B) is a $GQ(q-1, q+1)$, denoted by $T_2^*(O)$.*

Remember that a spread in an incidence structure (P, B) is a partition of the points into blocks. If we have a generalized quadrangle with a spread, then this corresponds in the dual structure to a set of points which intersects each block in exactly one point. Such structures are called *ovoids.*

The classical GQ $Q(4,q)$ is known to contain ovoids. Thus, by duality, $W(q)$ contains spreads. On the other hand, $Q(4,q)$ contains spreads only if q is even, i.e., when it is self-dual.

Table 2.1: Spreads in small generalized quadrangles

GQ	s	t	points	lines	spreads	ovoids
$W(2)$	2	2	15	15	1	1
$AS(3)$	2	4	27	45	1	0
$W(3)$	3	3	40	40	1	0
$T_2^*(O)$	3	5	64	96	5	1
$W(4)$	4	4	85	85	1	1

In Table 2.1 we give a list of the number of non-isomorphic spreads and ovoids in generalized quadrangles on up to 100 points. We will later on come back this table in several different contexts.

A *triad* in a generalized quadrangle is a triple $\{x, y, z\}$ of pairwise non-collinear points. A center c of such a triad is a point collinear to all three points in the triad.

Centers of triads give the following characterizations of generalized quadrangles $GQ(s, s^2)$.

Proposition 2.8 (Bose-Shrikhande[10]). *In a generalized quadrangle of order (s, t), $s, t > 1$, the following are equivalent:*

- *Each triad has the same number of centers.*
- *Each triad has $s + 1$ centers.*
- $s^2 = t$.

2.7 Spreads in strongly regular graphs

2.7.1 Definitions

Definition 2.23. Let Γ be a graph. A *combinatorial spread* in Γ is a partition of the vertices of Γ into cliques of equal size.

Definition 2.24. Let Γ be a graph with vertex set V. A partition of the vertices $V = V_1 \cup \cdots \cup V_n$ is called *equitable* if for any two classes V_i and V_j and any two vertices x and y in V_i, x and y have the same number of neighbors in V_j.

Definition 2.25. An *equitable spread* in a graph Γ is a combinatorial spread which at the same time is an equitable partition.

Example 2.3. Consider the triangular prism $K_3 \times K_2$. It contains two disjoint 3-cliques, which form a combinatorial spread of size 2. It also contains a perfect matching, which can be considered as a combinatorial spread of size 3.

In particular, we are interested in spreads with large cliques. A result by Delsarte [27] shows that a clique in a strongly regular graph has at most $K = 1-k/s$ vertices.

Definition 2.26. A *Delsarte clique* in an $\mathrm{srg}(v, k, \lambda, \mu)$ is a K-clique with $K = 1 - k/s$.

A *Delsarte spread* in an srg is a partition of the vertices into Delsarte cliques.

Lemma 2.2. *A clique C is a Delsarte clique iff each vertex not in C has a constant number of neighbors in C.*

Corollary 2.1. *A Delsarte spread is equitable.*

In what follows, we will call a Delsarte spread in an srg simply a spread.

2.7.2 Spreads in srg's

Given a spread in an srg, we can consider it as an equivalence relation S on the set of vertices V, taking the cliques as equivalence classes. Thus we can define three symmetric binary relations on V:

- $(x, y) \in R_1$ if x and y are adjacent in Γ, but lie in different cliques;

- $(x, y) \in R_2$ if x and y are non-adjacent in Γ;
- $(x, y) \in R_3 = S$ if x and y lie in the same clique.

If we add the identity relation R_0 on V, we get the following:

Theorem 2.11 (Haemers, Tonchev [44]). *$\langle R_0, R_1, R_2, R_3\rangle$ is a three-class association scheme, with eigenmatrix*

$$\begin{pmatrix} 1 & k+k/s & n-k-1 & -k/s \\ 1 & r+1 & -r-1 & -1 \\ 1 & s+1 & -s-1 & -1 \\ 1 & r+k/s & -r-1 & -k/s \end{pmatrix}$$

Consider now a partial geometry $pg(K, R, T)$ and its point graph. In fact, here $K = 1 - k/s$, and so each line l is a Delsarte clique in the point graph. This also follows from Lemma 2.2, since each point not on l has exactly T neighbors on l.

Thus, any spread in the partial geometry is also a spread of its point graph.

2.7.3 A. Brouwer's theorem

In the last subsection we have seen that a spread in an srg gives an association scheme W with 3 classes. One of the basis relations (R_2) is an srg, another one (the spread R_3) is an imprimitive srg.

Consider the graph Γ_1 defined by the basis relation R_1. Pairs of vertices (x, y) are adjacent in Γ_1 if they are in R_1; they are at distance 2 in Γ_1 if they are in R_2.

Thus we have the following:

Lemma 2.3. *If any two vertices in the same clique of S have distance 3 in Γ_1, then Γ_1 is distance regular, and W metric.*

Theorem 2.12 (A. Brouwer[15]). *W is metric iff Γ is a pseudo-geometric $GQ(s, t)$-graph.*

Proof. S is a spread, hence for a given clique C, each vertex not on C has the same number T of neighbors on C. If $T > 1$, then there are two vertices in C having distance 2 in Γ_1, hence the scheme is not metric.

□

The following is known on spreads in classical generalized quadrangles [97]:

- $W(q)$ of order q has spreads;
- Its dual, $Q(4, q)$ of order q, has spreads iff q is even; in this case, it is isomorphic to $W(4)$.
- $Q(5, q)$ of order (q, q^2) has spreads;
- Its dual, $H(3, q^2)$, doesn't have spreads.

This has led to the following question by Godsil-Hensel [40]: Are the pseudo-geometric $GQ(s, t)$-graphs which are not geometric, but which contain a spread?

This question was answered by Brouwer, Koolen, and Klin [17]. They constructed a non-geometric srg on 96 points (corresponding to a $GQ(5, 3)$) with a spread. Further examples were constructed by D.G. Fon-Der-Flaass [36].

The author used E. Spence's database of small srg's to find the smallest such examples. There are two such examples on 40 points; they will be described later in this thesis.

Chapter 3

COMPUTER ALGEBRA TECHNIQUES

3.1 Computers in combinatorics and algebra: A brief survey

3.1.1 In the beginning

Although the history of computers goes back to the 19th century, when C. Babbage designed his "Analytical machine", the first modern, programmable computers appeared in the 1940's. Soon after, mathematicians began using them to solve problems in Combinatorics with or without use of algebraic techniques.

Among the pioneers of Computer Combinatorics, we have to mention M. Hall, Jr. and J.D. Swift, who during the 1950's investigated projective planes, Steiner triple systems, and other objects. In particular, they used a computer to establish the uniqueness of the projective plane of order eight [50], to provide a computer confirmation of the classification of all Steiner triple systems on 15 points [48], and to classify projective planes of order nine satisfying certain conditions [49].

3.1.2 Backtracking

At the same time, research was carried out on the principles governing effective combinatorial search. A few people independently came up with what is now known as *backtrack search* or simply *backtracking*. The basic idea behind backtracking is that of "divide and conquer": Given a search space with possible solutions to a combinatorial problem, divide the space into several parts. If we can decide that one of the parts cannot possibly contain a solution, we can dismiss it altogether.

For each part that may contain solutions, we subdivide it into smaller parts, and examine those. Eventually, we arrive to a part that contains a single point of the search space, and we can check whether or not it is a solution.

We will give a more formal description of the backtracking algorithm later. Here, let us just note that the advantage of this algorithm is its capability to deal with large portions of the search space at once, without considering each point separately.

3.1.3 Enumeration

A typical problem in Combinatorics is that of the enumeration of combinatorial objects. Given a set of axioms, we are looking for all objects which satisfy these axioms. Usually one is interested in solutions which are inequivalent in a certain sense. This equivalence is often stated in terms of a group G acting on the set of all solutions.

In order to make a more precise explanation we need to concretize what a *combinatorial object* is. A suitable way of formalization which goes back to G. Polya and N.G. de Bruijn [19] is to consider the set Φ of all functions from a prescribed set Ω to another set D. In turn, Ω can be constructed with the aid of another initial set Ω_0 (for example, Ω is the power set of Ω_0, $\Omega = \Omega_0^k$ for some k, etc.), in particular, $\Omega = \Omega_0$. In all these cases we may speak about combinatorial objects over a basic set Ω_0. What is important for further consideration is that in this way we get a natural initial group G acting on Φ, in the simplest case this will be the symmetric group $S(\Omega_0)$, in more advanced cases it will be a group which is somehow composed from $S(\Omega_0)$ and other groups.

Finally, usually in the description of a combinatorial object a certain predicate f is defined on the set Φ such that a combinatorial object of a given type will be those elements of Φ satisfying f.

More formally, we have the following:

Problem 3.1 (Combinatorial Enumeration). Let Φ be a set of combinatorial objects with basic set Ω_0, and let G be a group acting on Ω_0, and thus acting naturally on Φ. Let f be a boolean predicate on Φ which is G-invariant. We call (G, Φ, f) a *combinatorial enumeration problem.*

Let $\omega = f^{-1}(\text{true})$ be the set of solutions of f. Then *analytical enumeration* is the determination of the number of orbits of (G, ω). We refer to the problem of determining representatives for these orbits as *constructive enumeration.*

In other words, analytical enumeration gives us the number of inequivalent solutions to a problem, where as constructive enumeration gives us a complete set of representatives for the equivalence classes of solutions.

Often, a class of combinatorial objects can be thought of as depending on an integer parameter i, like the undirected graphs on i vertices. Analytical enumeration results in the number n_i of non-equivalent solutions depending on i, in other words, an infinite sequence of integers. Instead of performing this enumeration for some small values of i, it is desirable to find a closed-form expression for n_i.

An alternative is the use of generating functions. Given a sequence of numbers $\{a_i\}$, we can consider the formal power series

$$g(x) = \sum_{i=0}^{\infty} a_i x^i$$

$g(x)$ is called the *(ordinary) generating function* for $\{a_i\}$. If we can find a closed-form expression for $g(x)$, this captures implicitly the information about all coefficients, and thus determines the sequence $\{a_i\}$ uniquely. In some cases, it is easier to express $g(x)$ than to express a_i in terms of i.

Usually, generating functions are considered in the case that one parameter n, e.g., the order of a graph, is fixed, and another parameter i, e.g., the number of edges, is bounded by some expression in n. In this case, we are dealing with a finite sequence $\{a_i\}$, and the generating function $g(x)$ is a polynomial.

3.1.4 Orderly generation

In ordinary backtracking an isomorphism check has to be performed for each newly found solution to compare it to previously found solutions. This requires repeated access to all solutions, as well as many isomorphism check.

R.C. Read [83] devised a method to avoid many of the tests, by introducing an ordering on the partial solutions. In this approach, known as "orderly generation", a partial solution is called *canonical* if it is the first in its isomorphism class. The ordering needs to be designed such that canonical solutions can be obtained from canonical partial solutions. Moreover, an effective test for canonicity is needed. Under these assumptions, the algorithm outputs canonical representatives for all isomorphism classes of solutions, with basically constant space requirements.

3.1.5 The Soviet school

Meanwhile, in the Soviet Union, techniques for the constructive enumeration of combinatorial objects were developed independently from the West. The origins of this development go back to the algorithmic school of A.S. Kronrod [66] and G.M. Adel'son-Vel'skiĭ.

During the 1960's and 1970's, it was I.A. Faradžev who made the most important input. He was trying to develop good algorithms for problems like the transitive closure of directed graphs. He also belonged to a group around Weisfeiler and Leman which worked on the graph isomorphism problem and in the process developed notions related to coherent configurations.

In the mid-1970's, his own interest shifted towards constructive enumeration. Independently from Read, he developed a version of orderly generation which used permutation groups extensively [29], [32].

The basic premise of his algorithm is the following: During the search, with each partial solution its automorphism group is calculated. This group is then used to simplify the test whether a partial solution is canonical. The determination of the automorphism group is simplified by the fact that partial solutions are composed element-wise, and adding an element to a partial solution can alter its group only in a very restricted way.

As a result, the algorithm yields not only the non-isomorphic solutions, but also their automorphism groups.

Several teams around Faradžev used this approach to enumerate various combinatorial objects: Regular graphs with given valencies, regular bipartite graphs, Ramsey graphs, 3-connected graphs, self-complementary graphs, 2-designs, Latin squares, etc. Their findings are summarized in [29].

During the early 1980's, A.V. Ivanov [56], a student of Faradžev's, combined these ideas with Read's algorithm to design his own approach for the enumeration of designs.

3.1.6 Computational group theory

Whereas above we concentrated on problems of a purely combinatorial nature, here we will look more closely at algorithmic solutions to problems in group theory.

The first problem for which an algorithmic solution has been described is the following:

Problem 3.2 (Coset Enumeration). Given an abstract group G (in terms of generators and relations) and a subgroup $H \leq G$ of finite index, determine the cosets G/H and the action of G on G/H.

An algorithm for this problem was described by Todd and Coxeter in 1936 [100]; variations of this algorithm are known as Todd-Coxeter algorithms.

The basic idea is the following: Each coset is represented by an integer. A list of known cosets is kept; in the beginning, it contains only the coset $H = He$.

For each coset Hg and each generator h, the algorithm tries to store the coset Hgh. Given a coset and a generator such that the product is not known yet, it is assumed that this represents a new coset.

During the search, the known relations reveal two numbers which actually represent the same coset. If such a collision occurs, these two elements are identified throughout the coset table, which may lead to additional collisions. When all products have assigned results, the process terminates.

Actually, the method described by Todd and Coxeter is not an algorithm in the strict sense of the word. During the execution a choice has to be made as to which product to consider next, and the termination of the process depends somewhat on these choices. Thus, certain heuristics are employed in order to ensure convergence in most cases.

Modern implementations of the Todd-Coxeter algorithms have been described by Faradžev [30], Cannon et.al. [24], Felsch et.al.[33], Neubüser [77].

As far as permutation groups are concerned, the single most important notion is that of a *base and strong generating set*, as introduced by C.C. Sims [90].

Definition 3.1. Let (G, Ω) be a permutation group. A *base* for G is a tuple $(v_1, \ldots, v_n)$ of elements of Ω such that the only element in G fixing each v_i is the identity.

In other words, $(v_1, \ldots, v_n)$ is a base for G if the point-wise stabilizer $G_{v_1,\ldots,v_n}$ is the trivial group $\{e\}$.

Given a base $(v_1, \ldots, v_n)$ for G, we can define a sequence of nested subgroups of G as follows: $G_0 = G$, $G_1 = G_{v_1} \leq G_0$, $G_2 = G_{v_1,v_2} \leq G_1$, $\ldots$, $G_n = G_{v_1,\ldots,v_n} = \{e\}$.

Definition 3.2. Let (G, Ω) be a permutation group, and let $B = (v_1, \ldots, v_n)$ be a base for G. A set $S \subseteq G$ is called a *strong generating set* for G (with respect to the base B) if it contains generators for all groups G_i, $0 \leq i \leq n$.

The following are a few observations:

- The points in the orbit of v_{i+1} under G_i are in one-to-one correspondence with the cosets G_i/G_{i+1}. Thus the index $[G_i : G_{i+1}]$ equals the length of this *basic orbit*.
- The size of G is the product of the lengths of all basic orbits.
- Each element of G is uniquely determined by its action on the base.

A base and strong generating set allow to use a "divide and conquer" approach for many problems related to permutation groups. Inductively, suppose that the group G_1 has been investigated. Then, in many cases, it suffices to consider certain coset representatives from G/G_1 in order to complete the investigation of G.

The remaining question is how to obtain a base and strong generating set from a set of generators. Sims devised a method using Schreier generators to find strong generators with respect to a given base. If no base is known, one can include all points of Ω into B; redundant base points can be removed when the strong generating set has been found.

Several implementations of the Schreier-Sims method exist. The implementations by Furst, Hopcroft, and Luks [37], Knuth [65], Jerrum [58], and Babai, Luks, and Seress [2] are known to take time polynomial in $|\Omega|$.

Theorem 3.1 (Babai, Luks, Seress). *The complexity of the Schreier-Sims method is bounded by* $O(|\Omega|^4 \log^c |\Omega|)$ *for some constant* c.

3.1.7 Development of packages

During the 1980, with computers becoming more and more available and powerful, there was a regular explosion of research connected to combinatorial enumeration.

It became apparent that for many problems in Combinatorics and other fields like Algebra, certain standard techniques were used again and again. In order to simplify the future development of programs, these algorithms were collected and presented in a unified fashion. This led to the development of computer algebra systems which emerged around 1990. Here we give an overview over some of these packages.

The most widely used computer algebra systems have their strengths in Calculus and Numerical Mathematics; much effort was invested in solutions of problems like symbolic integration of real-valued functions. Such systems include Maple and Mathematica.

For the rather specialized problem of investigating coherent configurations, the system COCO was developed in the Soviet Union.

For more general problems in group theory, combinatorics, and geometry, universities in Australia and Germany developed universal packages. The developers of these two systems, the Australian Cayley and the German GAP, first worked together, but later diverged. Cayley developed into Magma and was sold under a commercial license. GAP remained a free program, and is designed to be improved and enhanced by its users.

Today, there are thousands of people around the globe using one of Maple, Mathematica, Magma, and GAP. They are well organized, and exchange experience and information via Websites and mailing lists.

In the next section we will look more closely at those packages which have been used to obtain results contained in this thesis, in particular, COCO and GAP.

3.2 Available packages

Here, we describe a few computer algebra packages which are helpful in Algebraic Combinatorics. These include COCO, GAP, GRAPE, and Discreta.

3.2.1 COCO

COCO (Coherent Configurations) is a collection of programs for the investigation of coherent configurations. It was developed around 1990 by members of the Moscow Seminar on the Algebraic Theory of Combinatorial Objects, in particular, I.A. Faradžev and M.H. Klin [30], [31].

Originally written in Fortran-4 for IBM-370 computers, it has later been ported to C and adjusted for the use on personal computers. This version was slightly modified and ported to Unix by A.E. Brouwer; it is available from his homepage [14] [1].

The package provides the following facilities:

Inducing Given a permutation group (G, Ω) and a combinatorial structure X, compute the orbit $\Omega' = X^G$ and the action of G on Ω'.

Color Graph Given a permutation group (G, Ω), compute its centralizer algebra $V(G, \Omega)$.

Intersection Numbers Given a coherent configuration W, compute its intersection numbers, i.e., the structure constants p_{ij}^k of the corresponding coherent algebra.

Subschemes Given the structure constants of a coherent configuration, determine all mergings of classes which lead to homogeneous subconfigurations. Note that here the word subscheme is used jargonically.

Automorphism Groups Given a coherent configuration and a set of mergings, compute the automorphism groups of the resulting subschemes.

Typically, these five commands are used in this order; this allows to determine all association schemes invariant under a given permutation group (G, Ω) together with their automorphism groups; in particular, these groups are the 2-closed overgroups of G in S_Ω.

[1] This version doesn't compile with modern compilers. One workaround is to remove the prototypes of `malloc` and `callloc` from the file `def.h` in the `src` directory.

Let us add one more comment on the process of inducing. Suppose that G is a permutation group, and $H \leq G$ a subgroup. Let X be a combinatorial structure such that H is the stabilizer of X under G. If $\Omega = X^G$, we have a natural bijection $\Omega \leftrightarrow G/H$, and the actions of G on both sets are similar. Since orbits are easier to compute than coset representative, we get a space- and time-efficient way to represent permutation representations of finite groups.

In general, it may be difficult to verify that H is the full stabilizer of X under G. In the special case of a maximal subgroup H, we only need to check that X is invariant under H, and not under G. Thus it is easy to find representations for primitive permutation actions of G.

Beside the five programs outlined above, the package COCO also contains a library of the non-abelian finite simple groups of order less than 10^6 (except for $PSL(2, q)$) and their primitive permutation representations. The original goal was to determine the ranks of these groups, as well as the determination of all subschemes of their centralizer algebras. This goal has been accomplished, except for the subschemes of certain high-rank representations of $Sz(8)$, $PSU(3, 4)$, J_1, $PSL(3, 5)$, and J_2 (see [31] for details).

3.2.2 GAP

Whereas COCO is designed for a highly specialized task, GAP provides solutions for a wide range of problems in Discrete Mathematics, with a strong focus on group theory. GAP (Groups, Algorithms, Programs) was developed during the 1990's at the Rheinisch-Westfälische Hochschule in Aachen, Germany [88]. More recently, the main development has shifted to St. Andrews, Scotland.

The GAP package contains a relatively small kernel written in C, which makes it portable to all current operating systems. This kernel contains basic IO-capabilities, representations of primitive data types like arbitrary precision integers, finite permutations, and finite field elements, and an interpreter for the GAP language (a Pascal dialect). For the current version 4, this language has been enhanced with object-oriented features like data encapsulation and runtime polymorphism.

What makes GAP powerful is a vast library of routines and data written in the GAP language. It contains state-of-the-art algorithms for the investigation of permutation groups, matrix groups, rings, finite fields, etc.

One more feature is the support of so-called *share packages*. These are independent collections of routines, typically written by different authors, to deal with particular algebraic or combinatorial objects, like codes, graphs, crystallographic groups, etc.

3.2.3 GRAPE

One of these share packages is GRAPE (Graph Algorithms using Permutation Groups) developed by L. Soicher [91].

Its basic premise is that each graph is invariant under some group acting on the vertices (in the worst case, this is the trivial group). Such a group is stored together with each graph, and it is used to make the representation more compact, and to speed up calculations. In fact, many of the graph-theoretical algorithms rely on backtracking, and the known group is used to perform isomorph rejection.

Among the operations available for graphs are global invariants (diameter, girth, independence numbers, distance-regularity), questions related to subsets of vertices (induced subgraphs, cliques, connected components), and others. More involved algorithms are provided for example to enumerate partial linear spaces with a given point graph.

3.2.4 `nauty`

For the last decade, the standard program for graph isomorphism problems is B. McKay's `nauty` (*no automorphisms, yes?*) [75]. Its name reflects the fact that, statistically speaking, a random graph has a trivial automorphism group. This fact is used heuristically to find invariants which distinguish vertices of the graph; thus the algorithms is particularly successful when the full automorphism group is small.

Note that this premise of a small group is usually not satisfied for graphs of interest in Algebraic Combinatorics.

For a given graph, `nauty` computes the automorphism group together with a canonical labeling of the vertices. Since two canonically labeled graphs are isomorphic if and only if they are equal, this allows for a quick isomorphism check (quadratic in the number of vertices) of two graphs with known canonical labellings. The determination of the labelling is still computationally expensive, but it has to be performed only once for each graph under consideration.

`nauty` comes with its own language and file format for the description of graphs, permutations, and groups. Fortunately, GRAPE provides an interface to `nauty`, hence its most important capabilities are available directly from GAP, without the need to learn yet another language.

Part II

Main Results

Chapter 4

GRAPHS WITH THE T-VERTEX CONDITION

This chapter constitutes in a sense the most theoretical part of the thesis. In fact, the subject of the t-vertex condition was a starting point of the author's scientific interest.

A new attempt was undertaken two years ago. Here, we collect various results, trying to initiate a more systematical investigation of the t-vertex condition.

First, we give the necessary definitions and some basic known facts and constructions. Then, in Section 4.2, we give a complete description of the smallest non-rank 3 graphs with the 4-vertex condition.

Point graphs of generalized quadrangles appear to be good starting point, since we already know they satisfy the 4-vertex condition, and because the additional geometrical structure make many of the arguments more transparent. Their investigation is presented in Sections 4.3, 4.4, and 4.5.

The main new results presented here are the following: Point graphs of generalized quadrangles satisfy the 5-vertex condition (Main Theorem 1 on page 59); point graphs of generalized quadrangles of order (s, s^2) satisfy the 6-vertex condition (Main Theorem 2 on page 74). We describe a first example of a graph with an intransitive automorphism group which satisfies the 7-vertex condition (Section 4.5).

We also investigate small, self-complimentary strongly regular graphs (Section 4.6. All such graphs with up to 49 vertices wich satisfy the 4-vertex condition are rank 3 graphs.

4.1 The t-vertex condition

As we have seen, strong regularity is a necessary combinatorial condition for a graph to be of rank 3. Here we give a generalization of that concept giving stronger and stronger conditions.

4.1.1 Main definitions

One way to look at strong regularity is that for every pair (x, y) of vertices, we count certain subgraphs of order 3 containing x and y. The t-vertex condition generalizes this to subgraphs of arbitrary size.

Throughout, let Γ be the graph in question. Let $t \geq 3$ be an integer, and let Δ_1, Δ_2 be two induced subgraphs of Γ, each with two designated vertices x_i, y_i, $i = 1, 2$. These subgraphs are said to be of the same type w.r.t. x_i and y_i if there is an isomorphism $\Delta_1 \leftrightarrow \Delta_2$ that maps (x_1, y_1) to (x_2, y_2).

Now given any pair (x, y) of fixed vertices and any graph type, we can count how many graphs of this type w.r.t. x and y there are. If for any graph type of order up to t, that number doesn't depend on the particular choice of the fixed vertices but only on whether they are equal, adjacent, or non-adjacent, then we say that Γ satisfies the *t-vertex condition* [51].

The t-vertex condition in a way "interpolates" between rank 3 graphs and strongly regular graphs, since we have the following facts which are easily checked:

Lemma 4.1. *1. For $t > 3$, the t-vertex condition implies the $(t-1)$-vertex condition.*

2. A graph Γ of order v satisfies the 3-vertex condition iff it is strongly regular; it satisfies the v-vertex condition iff it is of rank 3.

Proof. 1. Follows directly from the definition.

2. For a given edge or non-edge (x, y), there are 4 different graph types to consider: the third vertex could be adjacent to both of fixed vertices, to one of

them, or to none. If we have the 3-vertex condition, then the number of graphs of the first type gives us λ (resp. μ), so the graph is strongly regular. If on the other hand we have an srg, then λ and μ give us the numbers for the first graph type; the other numbers can be deduced from those and the valency of the (regular) graph (see, e.g., [51] for more details).

The second part of the statement follows from the fact that the only v-vertex subgraph is the whole graph itself.

□

Since all rank 3 graphs are known, while the classification of strongly regular graphs appears to be without hope, it is interesting to investigate the substantially smaller class of graphs satisfying the t-vertex condition with $t \geq 4$.

4.1.2 Simple examples and basic results

The 4-vertex condition, as defined above, provides a nice heuristical method to prove that a given srg is not a rank 3 graph. In fact, it appears that most non-rank 3 srg's do not satisfy this condition.

Let us illustrate this with an example. The smallest srg which is not a rank 3 graph is Shrikhande's graph of order 16. One can construct this graph or rather its complement as follows . Consider the following 4×4 matrix:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array}$$

This is a Latin square, i.e., each "symbol" appears exactly once in every row and in every column. We define a graph that has the entries of this matrix as vertices, and two entries are joined by an edge if they are in the same row, in the same column, or they contain the same symbol. So, the vertex $(1,1)$ (the upper left entry) is connected to all vertices $(1, j)$ or $(i, 1)$, and to those vertices that contain a 1 ($(2, 4), (3, 3)$, and $(4, 2)$).

One can check that this graph is strongly regular, with parameters $v = 16, k = 9, \lambda = 4, \mu = 6$. (In fact, this construction yields a srg for any Latin square.) However, this is not a rank 3 graph; the automorphism group has two orbits on edges.

Let us check combinatorially that, say, the edges $((1, 1), (1, 2))$ and $((1, 1), (1, 3))$ do not lie in the same orbit. For a pair of vertices (x, y), we consider the set of all vertices not adjacent to either x or y. In the case of the first edge, we obtain the vertices $(2, 3)$ and $(4, 4)$, and we note that they are not adjacent, since they contain different symbols. In the second case, we get $(3, 2)$ and $(3, 4)$, which lie in the same row and hence are adjacent.

Hence, the two edges we considered can't lie in the same orbit. In fact, we showed that for two 4-vertex graph types, the numbers of graphs of those types with respect to an edge depends on the choice of the edge, hence the graph doesn't satisfy the 4-vertex condition, and thus is not a rank 3 graph.

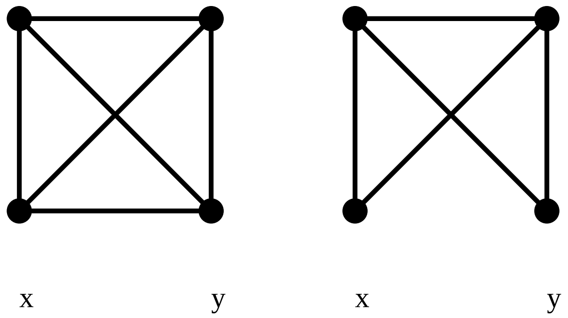
Note that in the case of the Shrikhande graph, the condition of a rank 3 group was relaxed in the slightest possible group-theoretical way (we have a rank 4 group), yet we were able to capture that relaxation combinatorially.

With growing t the number of different graph types grows very fast, making the naive count difficult. However it turns out that many of these counts are redundant. The following three results greatly reduce the number of cases to check.

Theorem 4.1 (A.V. Ivanov [57]). *If a graph is regular and satisfies the t-vertex condition on edges and non-edges, then it also satisfies it on vertices.*

Each graph type defines an invariant on the edges or non-edges of a graph. Usually, the numbers of graphs of the types in Table 4.1 are denoted by α and β. In

Table 4.1: Types to check for the 4-vertex condition



principle, many more such invariants could be defined. However, it turns out that they are not independent.

Theorem 4.2 (Hestenes-Higman[51]). *To check if a strongly regular graph satisfies the 4-vertex condition, it suffices to count for each edge or non-edge* (x, y) *graphs of types* K_4 *resp.* $K_4 - (x, y)$.

To generalize this, we need a characterization of the two graph types involved in the invariants α and β. Each graph type has two fixed vertices; we will call the remaining $t - 2$ vertices *additional vertices.* The minimum of the valencies of the additional vertices will be called the minimal valency of the graph type (hence, the valencies of the fixed vertices are ignored).

Now the graph types considered in Theorem 4.2 are those of minimal valency 3. This characterization allows the following generalization:

Theorem 4.3 ([85]). *If* Γ *is strongly regular and satisfies the* $(t-1)$*-vertex condition, then to check the* t *vertex condition it suffices to check graph types of minimal valency 3.*

We will refer to this result as the Generalized Hestenes-Higman Theorem.

4.1.3 Known examples

Now we come to the question of the existence of non-rank 3 graphs with the t-vertex condition, $t \geq 4$.

The first result in this direction is due to Higman and concerns point graphs of generalized quadrangles:

Theorem 4.4 (Higman[53]). *The point graph of a $GQ(s,t)$ satisfies the 4-vertex condition.*

Proof. This is a consequence of Theorems 2.5 and 4.2. The number β of copies of $K_4 - \{x, y\}$ is 0 for any non-edge $\{x, y\}$ since the graph doesn't contain $K_4 - e$.

If $\{x, y\}$ is an edge, then any vertex adjacent to both of them lies in the line connecting x and y, and thus any two such points are collinear. To obtain a K_4 containing x and y we have to pick two such points. Thus, the number α of such K_4 doesn't depend on x or y and is equal to $\binom{s-1}{2}$. $\square$

For a long time, point graphs were the only known examples of graphs satisfying the 4-vertex condition without being rank 3 graphs. A breakthrough was obtained by Brouwer, Ivanov, and Klin. First, Ivanov constructed a rank 4 graph of order 256 satisfying the 5-vertex condition [55]. It has the additional property that its subconstituents (subgraphs induced by neighbors resp. non-neighbors of a vertex) are strongly regular, and in fact satisfy the 4-vertex condition. One of the subconstituents has a rank 4 automorphism group, the other one an intransitive one.

Later it was shown by the three authors mentioned above that this is in fact the first member of an infinite series of graphs with the 4-vertex condition and strongly regular subconstituents; it was also shown that the first subconstituent satisfies the 4-vertex condition [13]. Their construction was later generalized by Ivanov [57], who found several other infinite series with similar properties.

This led to the following questions:

- Do these graphs in fact satisfy the 5-vertex condition?

- Do the second subconstituents satisfy the 4-vertex condition?

The author was able to answer the first question affirmatively using a more general form of Theorem 4.3; see [85] for details. The second question appears to be still open.

Until recently only one more sporadic example of such graphs was known [85]; its automorphism group is the Hall-Janko group J_2 in a rank 4 action on 280 points.

4.2 The smallest non-rank 3 graphs with the 4-vertex condition

Since the t-vertex condition is a "combinatorial approximation" of the properties of a rank 3 graph, it has for a long time been an interesting task to describe non-rank 3 graphs satisfying the t-vertex condition for $t = 4, 5, \ldots$. As stated above, the first such examples were described by D. Higman [53] and A.V. Ivanov [55].

Given that still little is known about such graphs in general, it is natural to concentrate on "extremal" questions. Two such questions are

- For given t, what is the smallest number of vertices of a non-rank 3 graph with the t-vertex condition?

- For given order v, what is the smallest t such that the t-vertex condition implies a rank 3 automorphism group?

Note that these questions are in a sense dual to each other. Here, we concentrate on the first question.

For $t = 3$, the smallest strongly regular graph which is not rank 3 is the Shrikhande graph on 16 vertices (see Section 4.1.2); it has a transitive automorphism group of rank 4.

Using E. Spence's online catalog of small srg's, Klin, Meszka, and Rosa investigated the case $t = 4$. Using the fact that this catalog is complete for orders up to 40, they got the following result:

Theorem 4.5 (Klin-Meszka-Rosa[62]). *The smallest non-rank 3 graphs satisfying the 4-vertex condition have order 36; for this order, there are 3 non-isomorphic examples.*

Since there is one rank 3 graph of order 36 (a rank 3 representation of the unitary group $U(3,3)$), we get a total of 4 graphs satisfying the 4-vertex condition.

Let us first of all collect some group theoretical information about these graphs. These are srg$(36, 14, 4, 6)$; hence they are in the switching classes of a regular two-graph Δ on 36 vertices. (For the definition and properties of regular two-graphs we refer the reader to [21].) Moreover, taking each vertex together with its neighbors as a block, we obtain a 2-design D.

Γ	$Aut(\Gamma)$	$Aut(\Delta)$	$Aut(D)$
Γ_1	12096	1451520	12096
Γ_2	64	1536	64
Γ_3	32	256	32
Γ_4	24	192	24

The groups of the rank 3 graphs are related to finite simple groups; in fact, we have that $Aut(\Gamma_1) = Aut(U(3,3))$, and $Aut(\Delta_1) = O(7,2) = Sp(6,2)$.

With all results obtained by computer, it is a goal to repeat them without the use of a computer. We distinguish here two phases in the interpretation of computer results: The first goal is to obtain a *model*, that is, a description of the objects under consideration in terms of known mathematical objects. Once such a model is obtained, we try to give a computer-free proof of the object's interesting properties.

In this case, the final goal would be to prove that the graphs really satisfy the 4-vertex condition. So far, this has not been accomplished; below we give models for three of these graphs. These models are related to certain Steiner triple systems, which are described below.

4.2.1 Pasch switching in $STS(15)$

In a projective space, consider three lines which pairwise intersect in three distinct points. Then these three lines are coplanar. Given a fourth line which intersects two of them in distinct points, it also lies in the same plane, hence all four lines have to intersect. This is known as the Axiom of Pasch.

This motivates the following definition:

Definition 4.1. In a Steiner triple system S, suppose there are four blocks which pairwise intersect, and all 6 intersections are different. Then this set of blocks is called a *Pasch configuration* in S.

Thus a Pasch configuration is of the general form

$$\{a, b, c\}, \{a, d, e\}, \{b, d, f\}, \{c, e, f\}$$

In this configuration, each point is collinear with all but one of the other points. In fact, the only pairs of non-collinear points are

$$\{a, f\}, \{b, e\}, \{c, d\}$$

Thus, if we interchange the points of one of these pairs, e.g., $\{c, d\}$, we get a second Pasch system, which we call the opposite of the given one.

$$\{a, b, d\}, \{a, c, e\}, \{b, c, f\}, \{d, e, f\}$$

Since the blocks in this configuration are exactly the triangles in the original one, the result doesn't depend on which pair of points we interchange.

The important thing to notice here is that collinearity is the same in both configurations. This leads to the following:

Theorem 4.6. *Let S be a Steiner triple system, and let P be a Pasch configuration of S. If we remove the blocks of P and replace them with the blocks of the opposite of P, the obtained incidence structure S' is again a Steiner triple system.*

Definition 4.2. The process of replacing a Pasch configuration by its opposite in a Steiner triple system is called *Pasch switching.*

There are exactly 80 isomorphism classes of $STS(15)$. A standard reference is the catalog by Mathon-Phelps-Rosa [73]. We now define a graph on these 80 systems, two systems being adjacent if one can be obtained from the other by a single application of Pasch switching. We call this the *switching graph.*

Note that the switching graph is undirected. In fact, if S' is obtained from S by replacing P with its opposite P', then we get back to S by replacing P' with its opposite P.

The following is a collection of facts about the switching graph:

- It has two connected components of sizes 1 and 79.
- It has vertices of valencies $0, 1, \dots, 14$.
- It contains 3 cliques of size 6, 5 maximal cliques of size 5, and 24 maximal cliques of size 4.

The isolated vertex is an $STS(15)$ with automorphism group of size 60; it doesn't contain any Pasch configurations.

4.2.2 The classical system

We now give two descriptions of the classical STS(15), a geometric one and a combinatorial one. This is the system #1 in the catalog by Mathon et.al.

Example 4.1 (Geometric $STS(15)$). Let S be the projective space $PG(3, 2)$. Let P be the set of projective points, and let B be the set of projective lines. Then the incidence structure (P, B) with natural incidence is a $STS(15)$. Its automorphism group is $PSL(4, 2)$.

Example 4.2 (Combinatorial $STS(15)$). Let $\Omega_0 = \{1, 2, \ldots, 8\}$ be a base set. Let B be the set of partitions of Ω_0 into $4+4$ elements. Then $|B| = \frac{1}{2}\binom{8}{2} = 35$.

Let D be a $3-(8, 4, 1)$ design defined on Ω_0. D is isomorphic to the affine space $AG(3, 2)$, and its automorphism group is $AGL(3, 2)$ of order 1344. Thus its orbit under the symmetric group S_8 is of length $8!/1344 = 30$.

Under the alternating group A_8, this orbit splits into two parts of length 15. Let P be one of these parts. We say that a point $p \in P$ and a block $b \in B$ are incident if $b \subset p$.

Proposition 4.1. *The incidence structure defined above is a Steiner triple system.*

Proof. Each affine design contains 7 parallel classes of planes; hence each point lies in 7 blocks.

Given two parallel classes such that any two planes intersect in two elements, there are two ways to complete them to an affine design; these two designs lie in different orbits under A_8. Hence, two blocks intersect in at most one point.

Given a plane, it contains 4 elements; hence there are 3 ways to partition it into $2+2$ elements. Given two parallel planes, any bijection between their partitions defines an affine design. There are $3! = 6$ such bijections; under the alternating group, they split into two orbits of length three. Hence, each block contains three points, and the proof is complete. $\square$

Proposition 4.2. *The automorphism group of the combinatorial $STS(15)$ is A_8.*

Theorem 4.7. *The Steiner triple systems described in Examples 4.1 and 4.2 are isomorphic.*

For a proof, see e.g., [95].

Using both the geometrical and the combinatorial face, we get the following properties of the system S:

Proposition 4.3. *S contains 15 subsystems of order 7; they all lie in the same orbit*

Proposition 4.4. *S contains 56 spreads (parallel classes); they all lie in the same orbit.*

Proposition 4.5. *S contains 240 resolutions in two orbits of length 120.*

4.2.3 First neighbors in the switching graph

If we consider the geometrical face of the classical $STS(15)$, we can easily investigate its Pasch configurations. Four projective lines which intersect pairwise have to be coplanar; conversely, any four coplanar lines form a Pasch system.

A Pasch system contains 4 lines and 6 points. If we take a whole two-dimensional subspace, it contains 7 lines and 7 points. Thus we get a Pasch system by deleting three concurrent lines from a plane. Hence, we can identify Pasch systems by incident point-plane pairs (P, Π).

The system S_1 contains 15 subplanes; each contains 7 points. Thus, there are altogether 105 Pasch systems, all lying in the same orbit under the automorphism group $PGL(4, 2)$. The group that fixes a given Pasch system thus has order $|A_8|/105 = 192$.

If we perform a Pasch switching, we obtain the system $STS(15)\#2$. It is certainly invariant under the group of order 192. We will see in the next section that this is indeed the full automorphism group.

If we fix one Pasch configuration (P, Π) of S_1, we can divide the remaining configurations (P', Π') as follows:

- $\Pi' = \Pi, P' \neq P$. There are 6 such configurations.

- $\Pi' \neq \Pi, P' = P$. There are 6 such configurations.
- $P \in \Pi', P \neq P', \Pi \neq \Pi'$. There are 6 additional planes through P, each containing 6 additional points. Thus there are 36 such configurations.
- $P' \in \Pi, P \neq P', \Pi \neq \Pi'$. There are 6 additional points in Π, each lying on 6 additional planes. There are 36 such configurations.
- $P' \notin \Pi, P \notin \Pi'$. Π' has to intersect Π in one of the lines of the initial configurations. Thus, there are $4 \cdot 2$ choices for Π'. Each contains 4 points not on Π, hence we get 32 configurations.

Two Pasch systems share a line unless they lie in different planes, and the point of one lies the plane of the other. Thus all but two of the above types of configurations are destroyed if we switch with respect to the original system. This switch transforms the plane to a new plane with different lines; this is the only place where new Pasch configurations appear.

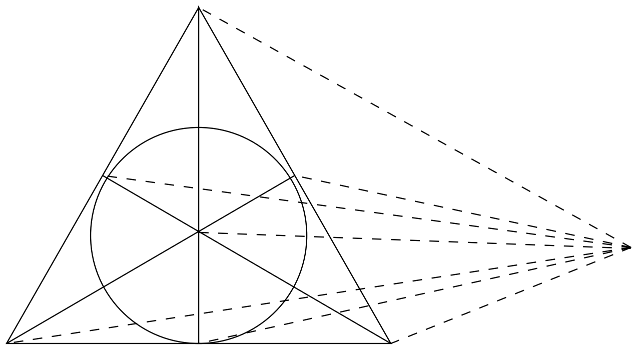
Thus, after one switch, we have three fundamentally different options of performing a second switch. This leads to three non-isomorphic Steiner systems. In particular, if the two Pasch systems are coplanar, we get the system S_3.

4.2.4 Anatomy of the second and third $STS(15)$

Here, we list a few facts about the Steiner triple systems 2 and 3 on 15 points.

The system S_2 is, as we have seen invariant under a group of order 192. In fact, this is the whole automorphism group. It has 3 orbits on points, of lengths 6, 1, 8, namely, the points of the Pasch system, the remaining point in the same plane, and the points not in that plane. During the switching, the planes through lines of the Pasch system are destroyed; the remaining 6 subsystems are preserved. The plane containing the Pasch system is changed, but it stays a plane. Thus we have

Table 4.2: Cone over a Fano plane



Proposition 4.6. *The system S_2 contains 7 subsystems, in orbits of lengths 1 and 6.*

The following fact is taken from Mathon-Phelps-Rosa:

Proposition 4.7. *The system S_2 contains 24 spreads. It is not resolvable.*

The third system S_3 is obtained from S_1 by switching with respect to two coplanar Pasch systems. It has an automorphism group of order 96. It contains 3 subsystems, 8 spreads, and no resolutions.

4.2.5 Descriptions of three graphs

Consider a regular two-graph with eigenvalues $\rho_1 = 5, \rho_2 = -7$. It has 36 vertices, and its descendant is an $\mathrm{srg}(35, 18, 9, 9)$. This is a pseudo-geometric graph, corresponding to the block graph of an STS(15).

Let S be a Steiner system, and let S' be a subsystem of order 7. S' is isomorphic to a projective plane of order 2. Let p be a point of S not in S', and let A be the set of blocks through p. Let B be the set of blocks in S', and let $C = A \cup B$. See Table 4.2, where the elements of A are represented as dashed lines, whereas the elements of B are solid lines.

Then we have the following facts:

- Both A and B induce complete graphs of order 7 in the block graph Δ.

- Each element of A is adjacent to 3 elements of B. Each element of B is adjacent to 3 elements of A.

- The complement of the subgraph of Δ induced by C is isomorphic to the distance 3 graph of the Heawood graph, the incidence graph of the projective plane of order 2.

- Let x be a point of S. If $x = p$, then x lies on 7 blocks from A, and on none from b. If $x \in S'$, then x lies on 1 block from A, and on 3 blocks from B. Otherwise, x lies on one block from A, and none from B.

  In particular, the blocks in C cover all points.

- Each block not in C contains exactly one point from S'.

- Each block not in C intersects exactly 6 blocks in C.

Now we define a graph Γ by adding an isolated vertex to Δ, and switching with respect to $V \setminus C$.

Proposition 4.8. *The graph Γ is regular, with valency 21.*

Proof. The added vertex is adjacent to each point in $V \setminus C$, thus its valency is 21.

The block graph $\Delta = (V, E)$ is a $\mathrm{srg}(35, 18, 9, 9)$. A vertex x in C is adjacent to 9 vertices in C, and to 9 vertices in $V \setminus C$. When switching with respect to $V \setminus C$, it will be adjacent to $9 + |V| - |C| - 9 = 35 - 14 = 21$ vertices in the switched graph Γ.

There are $9|C| = 9 \cdot 14 = 6 \cdot 21$ edges between C and $V \setminus C$. Thus, each point in $V \setminus C$ is adjacent to 6 vertices in C, and 12 vertices in $V \setminus C$. After switching, it will be adjacent to $1 + 12 + |C| - 6 = 21$ vertices in Γ. $\square$

Corollary 4.1. Γ *is a srg*$(36, 21, 12, 12)$*. Its complement is a srg*$(36, 14, 9, 9)$*.*

In Subsections 4.2.3 and 4.2.4, we described three systems $STS(15)$: The classical system S_1 related to $PG(3,2)$, the system S_2 with a group of order 192, and the system S_3 with a group of order 96.

Now, for each of the 3 Steiner systems S_i, we have to specify a subsystem s and a point p not on s such that switching the block graph with respect to the blocks which either lie on s or pass through p leads to a graph on 36 vertices which satisfies the 4-vertex condition.

In the case of the classical system S_1, things are quite simple. There is only one orbit of subsystems, so we can freely choose s. The stabilizer of s in the automorphism group G of the block graph of S_1 acts transitively on the points not on s, hence we can choose either point p. Switching results in an srg on 36 vertices which is invariant under the stabilizer of both s and p which his of size $|G|/(15 \cdot 8) = 336$. In fact, the graph is vertex transitive, with a group of size $336 \cdot 36 = 12096$.

The second system contains 2 orbits of subsystems of length 1 and 6. It has three orbits of points, of length 1, 6, and 8. If pick a subsystem from the longer orbit and a convenient point from the orbit of length 6, we get the graph Γ_2 by switching.

The third system has 3 orbits of points, with lengths 3, 4, and 8. It has two orbits of subsystems, with lengths 1 and 2. Picking a subsystem from the orbit of length 2, and a convenient point from the orbit of length 2, we obtain the graph Γ_4 by switching.

Theorem 4.8. *The switching construction above yields 3 of the 4 examples found by Klin-Meszka-Rosa.*

The remaining graph of order 36 satisfying the 4-vertex condition cannot be obtained by a similar construction.

4.3 The 5-condition for the point graphs of generalized quadrangles

4.3.1 Goals and strategy

In this section we investigate the 5-vertex condition in generalized quadrangles. In fact, we will prove the following new result:

Main Theorem 1. *The point graph of a generalized quadrangle satisfies the 5-vertex condition.*

For generalized quadrangles of order (s, s^2), this has been previously shown by A. Pasini (unpublished [78]). Although the proof presented here is independent, that result showed that generalized quadrangles yield a class of srg's which should be further investigated. Also, some of the techniques used by Pasini helped to obtain the result presented in Section 4.4.

By Theorem 4.3 we have to count graphs with minimal degree 3 and order 5. In other words, if x, y are the fixed vertices, and a, b, c the additional vertices, we need to consider the graphs where each of a, b, c is non-adjacent to at most one vertex.

If we take the complements of these graphs, we get that the valency of a, b, c is at most 1. If we discard a possible edge (x, y), that implies that the size of the graph, i.e., the number of its edges, is at most 3. Thus, we use the following strategy:

1. Enumerate all graphs of order 5 and size $i = 0, 1, 2, 3$ (Subsection 4.3.2);

2. Discard those graphs which cannot appear as subgraphs of generalized quadrangles (Subsection 4.3.3);

3. For the few remaining graph types, check that their numbers do not depend on the choice of the (non-) edge (x, y) (Subsection 4.3.4).

4.3.2 Graph types relevant to the 5-vertex condition

We start by determining all graph types which need to be considered in order to check whether a given graph satisfies the 5-vertex condition. As stated above, the size of the complements of these graph types is bounded by 3, in other words, the complements contain at most 3 edges.

In order to be sure not to miss anything, we will completely enumerate these complements. In the following we enumerate the relevant graphs up to isomorphism under the group $S(\{x,y\}) \times S(\{a,b,c\})$ of order 12. We will not consider the set $\{x,y\}$ as a possible edge. This leaves $\binom{5}{2} - 1 = 9$ possible edges.

We denote graphs by a list of its edges.

Graphs of size 0

The empty graph is uniquely determined by its size.

Graphs of size 1

a. $\{x,a\}$, $Aut = \langle (b,c) \rangle, |Aut| = 2$, length of orbit: 6.
b. $\{a,b\}$, $Aut = \langle (a,b), (x,y) \rangle, |Aut| = 4$, length of orbit: 3.

We accounted for $6 + 3 = 9 = \binom{9}{1}$ graphs.

Graphs of size 2

a. $\{x,a\}, \{x,b\}$, $Aut = \langle (a,b) \rangle, |Aut| = 2$, length of orbit: 6.
b. $\{x,a\}, \{y,b\}$, $Aut = \langle (a,b)(x,y) \rangle, |Aut| = 2$, length of orbit: 6.
c. $\{x,a\}, \{b,c\}$, $Aut = \langle (b,c) \rangle, |Aut| = 2$, length of orbit: 6.
d. $\{x,a\}, \{y,a\}$, $Aut = \langle (x,y), (b,c) \rangle, |Aut| = 4$, length of orbit: 3.
e. $\{x,a\}, \{a,b\}$, $Aut = \langle e \rangle, |Aut| = 1$, length of orbit: 12.
f. $\{a,b\}, \{b,c\}$, $Aut = \langle (a,c), (x,y) \rangle, |Aut| = 4$, length of orbit: 3.

We accounted for $6 + 6 + 3 + 12 + 6 + 3 = 36 = \binom{9}{2}$ graphs.

Graphs of size 3

a. $\{x,a\}, \{x,b\}, \{x,c\}$, $Aut = S(\{a,b,c\}$, $|Aut| = 6$, length of orbit: 2.
b. $\{x,a\}, \{x,b\}, \{y,c\}$, $Aut = \langle (a,b) \rangle$, $|Aut| = 2$, length of orbit: 6.

c. $\{x,a\},\{x,b\},\{y,a\}$, $Aut = \langle e \rangle$, $|Aut| = 1$, length of orbit: 12.
d. $\{x,a\},\{x,b\},\{a,b\}$, $Aut = \langle (a,b) \rangle$, $|Aut| = 2$, length of orbit: 6.
e. $\{x,a\},\{x,b\},\{a,c\}$, $Aut = \langle e \rangle$, $|Aut| = 1$, length of orbit: 12.
f. $\{x,a\},\{y,a\},\{a,b\}$, $Aut = \langle (x,y) \rangle$, $|Aut| = 2$, length of orbit: 6.
g. $\{x,a\},\{y,a\},\{b,c\}$, $Aut = \langle (x,y),(b,c) \rangle$, $|Aut| = 4$, length of orbit: 3.
h. $\{x,a\},\{y,b\},\{a,b\}$, $Aut = \langle (a,b)(x,y) \rangle$, $|Aut| = 2$, length of orbit: 6.
i. $\{x,a\},\{y,b\},\{a,c\}$, $Aut = \langle \rangle$, $|Aut| = 1$, length of orbit: 12.
j. $\{x,a\},\{a,b\},\{a,c\}$, $Aut = \langle (\{b,c) \rangle$, $|Aut| = 2$, length of orbit: 6.
k. $\{x,a\},\{a,b\},\{b,c\}$, $Aut = \langle \rangle$, $|Aut| = 1$, length of orbit: 12.
l. $\{a,b\},\{a,c\},\{b,c\}$, $Aut = \langle (x,y),(a,b),(a,b,c) \rangle$, $|Aut| = 12$, length of orbit: 1.

We accounted for $2 + 4 \cdot 12 + 5 \cdot 6 + 3 + 1 = 84 = \binom{9}{3}$ graphs.

The following graphs can be discarded because one of the additional vertices has degree greater than 1: 2.d–f, 3.c–l

This can be summarized as follows (recall that above we enumerated the complements of the relevant graphs):

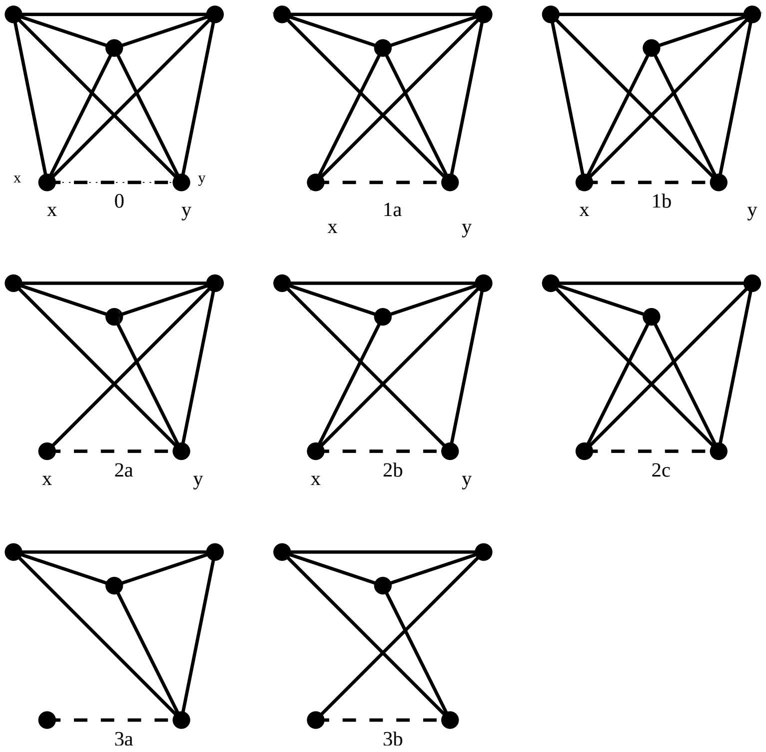
Theorem 4.9. *If a graph Γ satisfies the 4-vertex condition, then to check the 5-vertex condition it is sufficient to count the graphs of the types given in Table 4.3. Here, a dashed line denotes an optional edge connecting the fixed vertices.*

4.3.3 Easily discarded cases

We now proceed to apply the results obtained above to point graphs of generalized quadrangles. In this case, most of these graph types can be discarded since they contain a subgraph $K_4 - e$, which cannot happen in a generalized quadrangle (see Theorem 2.5).

Lemma 4.2. *In a generalized quadrangle, the following graph types do not appear: 1a, 1b, 2b, 2c, 3b.*

Table 4.3: Types to check for 5-vertex condition



Proof. In the pictures of Table 4.3, we denote the fixed vertices left to right by x, y, and the additional vertices left to right by a, b, c. For each of the graph types, we note a set of vertices which induces a $K_4 - e$:

1a: x, a, b, c

1b: x, a, b, c

2b: x, a, b, c

2c: y, a, b, c

3b: y, a, b, c

So these types do not occur, independently of whether x and y are adjacent. $\square$

4.3.4 Proof of the main theorem

This leaves us to check the following types: 0, 2a, 3a.

We will now show that the numbers for these subgraphs are uniquely determined by the axioms of generalized quadrangles. We will arrange these verifications in a three of lemmas, one for each type.

Lemma 4.3. *The number of graphs of type 0 w.r.t. x and y is $\binom{s-1}{3}$ if $x \sim y$, 0 otherwise.*

Proof. If $x \not\sim y$, then the graph contains a $K_4 - e$. Otherwise, each additional vertex is adjacent to both x and y and hence lies in the line connecting them. Each set of three points on this line yields a graph of type 0. $\square$

Lemma 4.4. *The number of graphs of type 2a is 0 if $x \sim y$. Otherwise, it is $(t+1)\binom{s-1}{2}$.*

Proof. If $x \sim y$, then $\{x, y, b, c\}$ induces a $K_4 - e$. Let $x \not\sim y$. The points $\{y, a, b, c\}$ form a clique, hence lie on a line. Thus to find such a graph, we choose a line l through y, with $t + 1$ possibilities. The point c is the unique neighbor of x on l; for a and b we can choose any two of the remaining points on l. $\square$

Lemma 4.5. *The number of graphs of type 3a is* $t\binom{s}{3}$ *if* $x \sim y$, $(t+1)\binom{s-1}{3}$ *otherwise.*

Proof. The vertices $\{y, a, b, c\}$ form a clique, hence are collinear. If $x \sim y$, we can choose any line through y but not through x; there are t such lines. We need to choose 3 other points on this line; this gives us $\binom{s}{3}$ choices.

If $x \not\sim y$, then we can choose any line through y ($t + 1$ possibilities) and then choose any three points on that line that are not adjacent to x, giving $\binom{s-1}{3}$ choices. □

We see that for all the graph types we have to check, the numbers do not depend on the particular choice of x and y. Together with the Theorem 4.3, this proves Main Theorem 1.

4.3.5 Discussion, further computational and theoretical results

The question remains whether Theorem 1 gives the best possible result. We will answer this by describing a GQ whose point graph doesn't satisfy the 6-vertex condition. Also, we will look what can be said under some restrictions on the parameters of the GQ.

For convenience, we repeat once more the complete list of the orders of generalized quadrangles with less than 100 points or lines. Here, v denotes the number of points, and l, the number of lines.

s	t	v	l
2	2	15	15
2	4	27	45
3	3	40	40
3	5	64	96
4	4	85	85

For all these sets of parameters, unique examples are known (up to isomorphism and duality, see, e.g., [20]). The groups occurring in all cases are rank 3 groups, except for the action of $Aut(GQ(3,5))$ on 96 lines. So, let us look at a construction of this generalized quadrangle (for more details, see [80]).

Let $\pi = PG(2,4)$ be embedded into $\Pi = PG(3,4)$. Note that $\Pi \setminus \pi$ has the structure of an affine space $AG(3,4)$, so we will talk of affine points, lines, etc. referring to objects that are not contained in π.

Choose a hyperoval $\mathcal{O}$ in π, i.e., a set of six points no three of which are collinear. Now we define an incidence structure by taking all the affine points, and those affine lines that meet π in a point of $\mathcal{O}$, with the incidence inherited from Π. For a given line, its intersection with π will also be called its *direction*.

It is not difficult to see that this incidence structure is a $GQ(3,5)$. Each line contains 4 affine points, and through each point there are 6 lines, one for each point of $\mathcal{O}$.

To check the third axiom, remember that any line in π intersects $\mathcal{O}$ in 2 points. In the affine space that means that every plane contains two parallel classes of chosen lines. Now, given a line l and a point P not on l, they span a plane. In this plane, there are two lines through P; one is parallel to l, and exactly one intersects l; hence the third axiom is satisfied.

The automorphism group of the line graph has two orbits on non-edges, namely pairs of parallel lines and pairs of skew lines. A computer search gives that these orbits can be distinguished using 6-vertex subgraphs, hence

Proposition 4.9. *The line graph of $GQ(3,5)$ doesn't satisfy the 6-vertex condition.*

Corollary 4.2. *Theorem 1 is as good as possible.*

So far, there is no transparent computer free proof of Proposition 4.9. However, we can prove the following:

Proposition 4.10. *The line graph of $GQ(3,5)$ doesn't satisfy the 8-vertex condition.*

Proof. We have to show that we can distinguish parallel and skew lines by using sets of 8 lines.

Two parallel lines lie in a common plane. If we take all lines in that plane, we obtain a 4×4 square lattice; in terms of the line graph this is a complete bipartite graph $K_{4,4}$. We want to show that no pair of skew lines can lie in such a grid.

Note that each plane contains only two parallel classes of lines. That means that if we have two parallel lines, they span a plane, and all lines intersecting both of them have to be parallel. On the other hand, given two skew lines, all lines intersecting both of them have to be skew.

Suppose now that there are two skew lines that lie in a 4×4 grid. Then all 4 lines intersecting both of them have to be pairwise skew; and the two remaining lines in the grid have to be skew to the lines we started with. Hence, no two lines in the grid are parallel, and we obtain 8 different directions. Since there are only lines of 6 different directions in the GQ, this is a contradiction.

Hence, parallel and skew lines can be distinguished using the 8-vertex condition. □

This can be easily modified to check that the 7-vertex condition isn't satisfied. However, for the computer-free check of the 6-vertex condition some new observations are needed.

4.4 The t-vertex condition for point graphs of $GQ(s, s^2)$

We have seen in Corollary 4.2 that the 6-vertex condition is not in general satisfied for point graphs of generalized quadrangles $GQ(s,t)$. Here, we show that it is satisfied in case of $t = s^2$. Using known constructions for $GQ(s, s^2)$, we thus obtain infinitely many examples of graphs satisfying the 6-vertex condition which do not admit a rank 3 automorphism group.

4.4.1 Graph types to check

According to the Generalized Hestenes-Higman Theorem, we have to check graph types of minimal degree at least 3. Thus the complements have maximal degree at most 2.

We list the induced subgraphs on the additional vertices. These are graphs on 4 vertices with maximal degree at most 2. Later we will see how the fixed vertices can be connected.

Size 0 There is only one graph of size 0, the empty graph $\emptyset$.

Size 1 There is only one graph of size 1.

1. $\{a, b\}$, denoted by K_2.

Size 2

1. $\{a, b\}, \{a, c\}$, denoted by P_3.
2. $\{a, b\}, \{c, d\}$, denoted by $K_2 \circ 2$

Size 3

1. $\{a, b\}, \{a, c\}, \{b, c\}$, denoted by K_3.
2. $\{a, b\}, \{b, c\}, \{c, d\}$, denoted by P_4.

Size 4

1. $\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}$, denoted by C_4.

We will call the corresponding graph types e.g., graph types of class P_3.

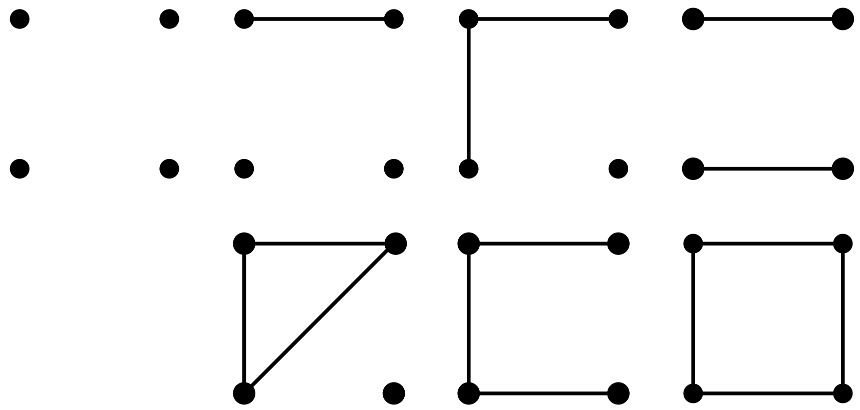
Throughout we will use that point graphs of generalized quadrangles do not contain $K_4 - e$.

4.4.2 Graphs of size 0

Theorem 4.10. *For any graph type T of class $\emptyset$, the number of subgraphs of type T w.r.t. x and y doesn't depend on the choice of the edge (non-edge) xy.*

Proof. The additional vertices $\{a, b, c, d\}$ form a clique, hence they lie on a common line l. The point x cannot be connected to exactly 2 or 3 of them, since otherwise

Table 4.4: Graphs of order 4 and maximal degree ≤ 2



we get a forbidden subgraph. So the possibilities for x are to be connected to 0, 1, or 4 of them; similarly for y.

Without loss of generality we may assume that x is connected to at least as many vertices as y.

1. If x isn't connected to any, neither is y. Thus we may choose any line l that doesn't pass through either.

   (a) If x and y are not collinear, then there are $2(t+1)$ passing through one of them, the other $(t+1)(st+1)-2(t+1) = (t+1)(st-1)$ do not pass through them. On $\mu(t+1)$ of them, x and y have a common neighbor, on the other $(t+1)(st-1-\mu)$, the neighbors are distinct. Thus the total number of graphs of this type w.r.t. x and y is $\mu(t+1)\binom{s}{4} + (t+1)(st-1-\mu)\binom{s-1}{4}$.

   (b) If x and y are collinear, then there are $(s-2)(t-1)$ lines not through x or y, but concurrent to xy. The remaining $(t+1)(st-1)-(s-2)(t-1)-1$ lines are skew to xy. If the lines are concurrent, the neighbors of x and y coincide, leaving $\binom{s}{4}$ choices for the additional vertices. Otherwise, the neighbors are distinct, leaving $\binom{s-1}{4}$ choices.

2. Suppose that $x \sim a$, but y is not connected to anything. The point x has $k = s(t+1)$ neighbors, one of which is y.

   If $x \sim y$, then x and y have $\lambda = s-1$ common neighbors, so there are $k-\lambda-1$ choices for a. If $x \not\sim y$, they have $\mu = t+1$ common neighbors, hence in this case there are $k-\mu-1$ choices for a.

   Once a is picked, we can choose any line l through a but not through x, giving t choices. On l we have to pick 3 points b, c, d distinct from a and the unique neighbor $z \neq a$ of y on l, giving $\binom{s-1}{3}$ choices.

   Altogether, there are $(k-\lambda-1)t\binom{s-1}{3}$ graphs of this type with respect to any edge, and $(k-\mu-1)t\binom{s-1}{3}$ with respect to any non-edge.

3. If both x and y are connected to a, then we have λ or μ choices for a, t or $t-1$ choices for the line through a depending on whether or not x and y are collinear, and $\binom{s}{3}$ choices for the remaining points.

4. If $x \sim a$, $y \sim b$ we have $k-\lambda-1$ resp. $k-\mu-1$ choices for a according as $x \sim y$ or not, t choices for the line through a, b is fixed, and $\binom{s-1}{2}$ choices for c and d.

5. If x is adjacent to all additional vertices, then these five points are collinear. If y is collinear to at least two of them, it lies on the same line; hence the number of graphs of that type is $\binom{s-1}{4}$.

   (a) If y is not adjacent to any other vertex, we may choose any line through x ($t+1$ choices), and pick 4 points on that line other than x and the neighbor of y, yielding $\binom{s-1}{4}$ choices.

   (b) If y is adjacent to x only, we have t choices for a line through x but not through y, and $\binom{s}{4}$ choices for the additional vertices on that line.

Table 4.5: Graph types of class 0

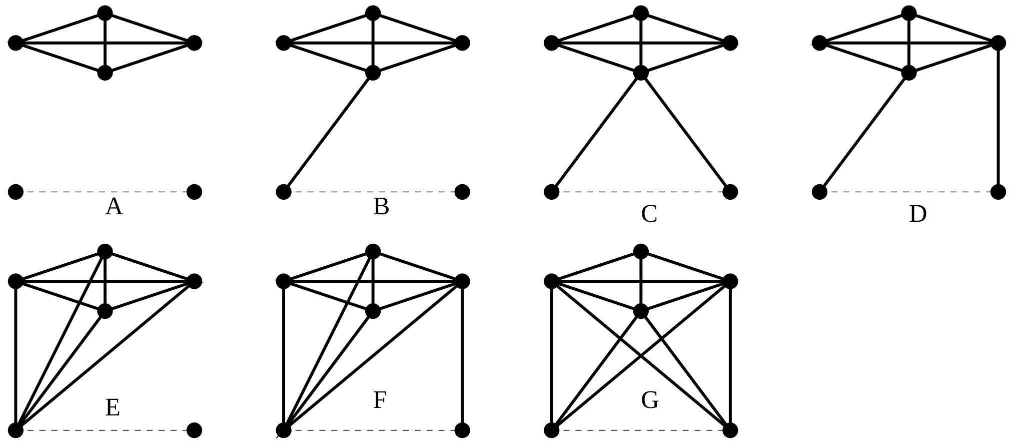
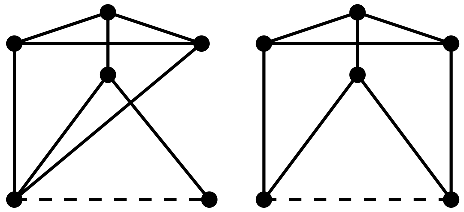


Table 4.6: Graph types of class 2



(c) If y is connected to a only, then we have $t+1$ choices for l, a is fixed, and $\binom{s-1}{3}$ for the remaining vertices.

Since we considered all possible cases, this finishes the proof. $\square$

4.4.3 Graphs of size 1

The only graph of size 1 is K_2; its complement on the 4 additional vertices is $K_4 - e$, which cannot occur in a generalized quadrangle.

4.4.4 Graphs of size 2

Theorem 4.11. *For any graph type T of class P_3, the number of subgraphs of minimal degree at least 3 and of type T w.r.t. x and y doesn't depend on the choice* *of the edge (non-edge) xy.*

Proof. In this case, the vertices b, c, d form a clique, and a is connected to d. Hence a is has to be connected to both x and y, and both b and c have to be connected to at least one of the fixed vertices.

If both b and c are connected to x, then d has to be connected to x to avoid a $K_4 - e$ on $\{b, c, d, x\}$. However, this leads to a $K_4 - e$ on $\{a, b, d, x\}$.

This leaves us with the case that $b \sim x, c \sim y$. If d is connected to either x or y, this leads to a $K_4 - e$ on either $\{x, a, b, d\}$ or $\{y, a, c, d\}$.

Since the graph satisfies the 5-condition, we know how many choices we have for picking a, b, and c. The point d is determined as the unique neighbor of a on the line bc, thus we get a fixed number of graphs of this type. □

Theorem 4.12. *Suppose that every triad has a constant number of centers. Then for any graph type T of class $K_2 \circ 2$, the number of subgraphs of minimal degree at least 3 and of type T w.r.t. x and y doesn't depend on the choice of the edge (non-edge) xy.*

Proof. In this case, we have a 4-cycle on $\{a, b, c, d\}$. Let us assume that a is connected to b and d.

Each vertex has to be connected to at least one of x and y. If x is connected to 3 of them, we get a $K_4 - e$. Thus two of the additional vertices are connected to x, and the other two to y.

1. If x is adjacent to a and b, then xab and ycd are lines. If they intersect in a point z, then adz forms a triangle in the geometry. Hence the two lines are skew.

   Pick any line through x and not through y, pick any line through y not intersecting the previous one. Pick any two points on the first line not adjacent to y. This fixes the other two points.

If $x \sim y$, then we have t choices for the first line, t choices for the second line, and $\binom{s}{2}$ choices for a and b. If $x \not\sim y$, then we have $t+1$ choices for the first line, t choices for the second line, and $\binom{s-1}{2}$ choices for a and b.

2. If x is adjacent to a and c, then we get a graph that actually fails to be an invariant for certain GQ's.

   The 5-vertex condition tells us how many choices we have for b, c, d. The possible choices for a are exactly the centers of the triad x, b, d. Since we assume that the number of centers is known, the number of graphs of this type is determined.

□

4.4.5 Graphs of size 3

Theorem 4.13. *Suppose that every triad has a constant number of centers. Then for any graph type T of class K_3, the number of subgraphs of minimal degree at least 3 and of type T w.r.t. x and y doesn't depend on the choice of the edge (non-edge) xy.*

Proof. In this case, d is connected to a, b, c, which in turn form a triad. Since we need a minimal degree of 3, these three points have to be connected to both x and y.

The point d cannot be connected to x or y, since this would lead to a $K_4 - e$ on, say, x, a, b, d.

From the 5-vertex condition, we know how many choices there are for a, b, c. This triad has a fixed number of centers, two of which are x and y; every other center is a valid choice for d. □

Theorem 4.14. *For any graph type T of class P_4, the number of subgraphs of minimal degree at least 3 and of type T w.r.t. x and y doesn't depend on the choice of the edge (non-edge) xy.*

Proof. In this case, a, b, c, d form a path. a and d have to be connected to both x and y; hence, $x \not\sim y$.

The points b and c are connected to at least one of x and y. If $b \sim x$ and $c \sim x$, then a, b, c, x induces a $K_4 - e$. Thus, w.l.o.g., $b \sim x, c \sim y$.

The lines xab and ycd are not concurrent to avoid triangles.

Pick any line l_1 through x. Pick any line l_2 through y not intersecting l_1. Each choice of points a, b on l_1 uniquely determines the points c, d.

Thus there are $(t+1)t\binom{s}{2}$ graph of this type. □

4.4.6 Graphs of size 4

Theorem 4.15. *For any graph type T of class C_4 and any pair of distinct vertices x, y, there are no subgraphs of minimal degree at least 3 and of type T w.r.t. x and y.*

Proof. The only edges between additional vertices are $\{a, c\}$ and $\{b, d\}$. Hence, to achieve a minimal degree of 3, all additional vertices need to be connected to both x and y. This forces x and y to be collinear. However, then, $acxy$ and $bdxy$ are distinct lines sharing two points. □

4.4.7 Main results

Combining the results of the previous section with Hestenes-Higman, we have got the following result:

Theorem 4.16. *To check whether the point graph of a generalized quadrangle satisfies the 6-vertex condition, it is sufficient to count graphs of two types (the ones mentioned in the theorems above).*

Theorem 4.17. *For any generalized quadrangle, if each triad has a constant number of centers, then the point graph satisfies the 6-vertex condition.*

Main Theorem 2. *The point graph of any $GQ(s, s^2)$ satisfies the 6-vertex condition.*

Proof. In such a GQ, every triad has $s + 1$ centers. $\square$

4.5 A non rank 3 graph satisfying the 7-vertex condition

The question remains if Theorem 2 is optimal in the sense that there are point graphs of $GQ(s, s^2)$ which do not satisfy the 7-vertex condition. This question is still open; here we describe one such graph which does satisfy the 7-vertex condition but which is not a rank 3 graph.

Below we give a construction of a non-classical $GQ(s, s^2)$. We follow the exposition of [98].

Let $F = GF(q)$. Let $\mathcal{C} = \{A_u : u \in F\}$ be a set of 2×2 matrices over F. Then $\mathcal{C}$ is called a q-clan if for $u \neq v$, $B = A_u - A_v$ is anisotropic, i.e., $\alpha B \alpha^T = 0$ has only the trivial solution $\alpha = (0, 0)$.

Example 4.3 (The classical q-clan). Let $x^2 + bx + c \in F[x]$ be irreducible. For $u \in F$, let

$$A_u = \begin{pmatrix} u & bu \\ 0 & cu \end{pmatrix}$$

Then $\mathcal{C} = \{A_u : u \in F\}$ is a q-clan.

Example 4.4 (A non-classical q-clan). Let $q \equiv 2 \mod 3$. Let

$$A_u = \begin{pmatrix} u & 3u^2 \\ 0 & 3u^3 \end{pmatrix}$$

Then $\mathcal{C} = \{A_u : u \in F\}$ is a q-clan.

Given such a q-clan $\mathcal{C}$, set $K_u = A_u + A_u^T$. Now we set $G = F^2 \times F \times F^2$, and define a multiplication via

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \alpha' \beta^T, \beta + \beta')$$

Under this multiplication, G is a group. Define subgroups of G as follows:

$$\begin{aligned} A(u) &= \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) : \alpha \in F^2\} \\ A(\infty) &= \{(0,0,\beta), \beta \in F^2\} \\ A^*(u) &= A(u)C \\ A^*(\infty) &= A(\infty)C \end{aligned}$$

where $C = \{(0, c, 0)\}$ is the center of G. Set

$$\begin{aligned} J &= \{A(u) : u \in F \cup \{\infty\}\} \\ J^* &= \{A^*(u) : u \in F \cup \{\infty\}\} \end{aligned}$$

Now we define an incidence structure as follows: Points are (i) the elements of G; (ii) the right cosets $A^*(u)g, A^*(u) \in J^*$; (iii) the symbol ∞. Lines are (a) the right cosets $A(u)g$; (b) the symbols $A(u)\infty$.

Points of types (i) are incident with each line of type (a) they are contained in; a point $A^*(u)g$ is incident with $A(u)\infty$ and with each line of type (a) contained in $A^*(u)g$; finally, the point ∞ is incident with each line $A(u)\infty$.

Proposition 4.11 (Kantor, Payne). *The incidence structure defined above is a $GQ(q, q^2)$.*

Proposition 4.12 (Payne). *If the q-clan used in the construction above is not classical, then the resulting generalized quadrangle has an intransitive automorphism group.*

Using the non-classical q-clan described above with $q = 5$, we obtained the following result:

Proposition 4.13 (Computer result). *Let Γ be the point graph of the generalized quadrangle obtained from the non-classical q-clan described in Example 4.4, for $q = 5$. Then Γ satisfies the 7-vertex condition.*

To our knowledge, this is the first description of such a graph which does not have a rank 3 automorphism group.

4.6 Self-complementary strongly regular graphs

As we have seen in the previous sections, one successful strategy in the search for graphs satisfying the t-vertex condition for some t is the following: Start with a rank 3 graph or a series of such graphs; capture some combinatorial properties of these graphs, and find other graphs satisfying these properties.

In Section 4.2 we started with a rank 3 graph related to the unitary group. In Section 4.3, we started with classical generalized quadrangles, i.e., rank 3 representations of orthogonal and symplectic groups.

Here, we will look at two more infinite series of rank 3 graphs. Unfortunately, this has not yet led to the discovery of new graphs satisfying the 4-vertex condition.

Definition 4.3. Let Γ be a finite, simple graph. Γ is called *self-complementary* if it is isomorphic to its complement, $\overline{\Gamma}$.

Example 4.5. The pentagon C_5 (cf. Example 2.2) is a self-complementary graph.

Here, we will investigate self-complementary, strongly regular graphs (sc-srg's).

Let Γ be a srg(v, k, λ, μ). Then its complement $\overline{\Gamma}$ is a srg$(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$.

Thus, for a scsrg, we get the equations

$$\begin{aligned} v-k-1 &= k \\ v-2k+\mu-2 &= \lambda \\ v-2k+\lambda &= \mu \end{aligned}$$

Hence, $v = 2k+1$, and $\mu = \lambda + 1$.

Proposition 4.14. *For a scsrg(v, k, λ, μ), we have $v = 4\mu + 1$, $k = 2\mu$, $\lambda = \mu - 1$.*

Thus, the parameters of a scsrg are completely determined by the value of μ.

Example 4.6 (Paley graphs). Let q be a prime power, $q \equiv 1 \mod 4$. Let $F = F_q$ be the field of order q. Let $G = F^* \cong C_{q-1}$ be the multiplicative group, and let X be its subgroup of index 2.

Consider the Cayley graph over the additive group $(F, +)$ with connection set X. This graph is called the *Paley graph* P_q.

Theorem 4.18 (Paley). *The Paley graph P_q is a rank 3 graph; moreover, it is a scsrg with $\mu = \frac{q-1}{4}$.*

Another infinite series of such graphs was rediscovered by W. Peisert [81]:

Example 4.7 (Peisert graphs). Let p be a prime, $p \equiv 3 \mod 4$. Let r be even, and let $q = p^r$. Let $F = F_q$, and let ω be a primitive element of F.

Let $X = \{\omega^j : j \equiv 0, 1 \mod 4\}$, and let P_q^* be the Cayley graph over $(F, +)$ with connection set X. We call P_q^* a *Peisert graph*.

Theorem 4.19 (Peisert). *The Peisert graphs are self-complementary rank 3 graphs.*

In fact, every rank 3 scsrg is isomorphic to either P_q or P_q^*, with one exception:

Theorem 4.20 (Peisert). *Let Γ be a self-complementary rank 3 graph. Then Γ is isomorphic either to a Paley graph, a Peisert graph, or an exceptional graph $G(23^2)$ of order 23^2.*

R. Mathon [71] classified all scsrg's with up to 49 vertices. There are exactly 27 of them, including 9 Paley graphs and one Peisert graph which is not a Paley graph. Investigating the remaining 17 graphs, we found that none of them satisfies the 4-vertex condition. Thus we have the following modest result:

Proposition 4.15 (Computer search). *Let Γ be a self-complementary srg with at most 49 vertices. Suppose that Γ satisfies the 4-vertex condition. Then Γ is a rank 3 graph; in particular it is isomorphic to either a Paley graph P_q or to P^*_{49}.*

Chapter 5

SIAMESE ASSOCIATION SCHEMES AND STEINER DESIGNS

We define a new class of association schemes with interesting combinatorial properties, and call them Siamese association schemes. They are linked to several kinds of combinatorial objects, such as strongly regular graphs, distance regular graphs, generalized quadrangles, and Steiner systems.

After giving the axioms, we discuss restrictions they impose on the possible parameters of such a scheme. In particular, the number of points has to be of the form $v = n^3 + n^2 + n + 1$, where n is an integer. We show that the resulting scheme is related to distance regular graphs, generalized quadrangles, and Steiner systems.

If we relax the axioms, we get the notion of a Siamese color graph, as previously introduced by Kharaghani and Torabi [60]. We show that on 15 points, there are exactly two non-isomorphic Siamese color graphs, one of which is an association scheme. For next possible order, on 40 points, we construct one association scheme and several color graphs.

Finally, we investigate how these constructions can be generalized to an infinite series of Siamese association schemes.

5.1 Axiomatics and main properties

5.1.1 Motivation

Here, we are looking for exceptional classes of association schemes which have nice properties. The best-known class of "nice" association schemes is the class of metrical (or P-polynomial) schemes; much research and literature is dedicated to this and the related area of distance regular graphs.

We will look at non-metrical schemes with other interesting properties. Our approach is motivated by the notion of *amorphic* association schemes. These are defined by the property that each merging of classes leads to a subscheme. In particular, each basis graph is strongly regular (possibly imprimitive).

The classical example of an amorphic scheme is defined by an affine plane, where the basis relations are the parallel classes. All basis graphs are imprimitive, and the union of basis graphs gives strongly regular graphs of Latin Square type.

It turns out that this classical example is typical. Amorphic schemes with more than 2 classes exist only on a square number $v = n^2$ of points. They are of one of two types:

- Latin square type, i.e., all valencies are multiples of $n - 1$;
- Negative Latin square type, i.e., all valencies are multiples of $n + 1$.

The smallest example of an amorphic scheme with 3 classes of negative Latin square type can exist on 16 points; all basis graphs have valency 5, and hence are isomorphic to the Clebsch graph. In fact, an edge-disjoint partition of the complete graph into 3 copies of the Clebsch graph exists.

We would like to develop a generalization of this notion. If in an amorphic scheme all basis graphs are srg's, i.e., distance regular of diameter 2, we could consider the case where all basis graphs are distance regular of diameter 3.

It follows from a theorem by Bannai-Bannai [4] that at most two basis graphs can be primitive drg's. In order to obtain a homogeneous structure, we require all drg's to be imprimitive. This leads us to ask for the following: Consider association schemes where as many basis graphs as possible are imprimitive distance regular graphs of diameter 3. We furthermore require that all these graphs share the same imprimitivity system.

Thus, so far each of the graphs is an antipodal drg of diameter 3. We now want that the graph and its imprimitivity system together form a nice structure, namely an srg. More accurately, if in the distance regular graph we add all edges joining vertices at distance 3, then we obtain a strongly regular graph.

This puts us in the situation where we can apply Brouwer's Theorem 2.12. We are now ready to formulate accurate axioms and to analyze the resulting combinatorial structures.

5.1.2 Axiomatics

Definition 5.1 (Siamese Color Graph). Let V be a finite set. Let $\{S, \Gamma_1, \Gamma_2, \ldots \Gamma_r\}$ be a partition of the edges of the complete graph on V such that

- S is an imprimitive strongly regular graph, i.e., a partition of V into cliques of equal size $s + 1$;
- Each Γ_i is an imprimitive distance regular graph of diameter 3 with antipodal system S;
- For each i, $\Gamma_i \cup S$ is a strongly regular graph.

Then $\Gamma = (V, \{S, \Gamma_1, \ldots, \Gamma_r\})$ is called a *Siamese color graph.* We call S the *spread* of Γ.

Siamese color graphs were first studied by Kharaghani et.al.[60]. The word "Siamese" comes from the observation that any two of the strongly regular graphs share the spread S, and thus recall the image of conjoined twins. However, after "surgical" removal of the spread, both can live an independent life as distance regular graphs.

We are interested in color graphs in particular if they form coherent configurations; this leads to the next definition.

Definition 5.2 (Siamese Association Scheme). If the distance regular graphs in a Siamese color graph Γ are basis graphs of an association scheme W, then W is called a *Siamese association scheme.*

Note that this definition allows the spreads to be a union of basis classes of the scheme. We now investigate possible parameters of Siamese color graphs and association schemes.

Proposition 5.1. *The strongly regular graphs in a Siamese color graph are pseudo-geometric $GQ(s,t)$-graphs.*

Proof. If we remove the spread from the srg, we obtain a distance regular graph of diameter 3. Thus, Theorem 2.12 applies. □

Proposition 5.2. *The parameters s and t do not depend on the choice of the srg in Γ.*

Proof. The size of a clique in the spread S is $s+1$. The number of points in a $GQ(s,t)$ is $v=(s+1)(st+1)$, thus t is uniquely determined by s and v. □

Thus we could speak of a Siamese color graph of order (s,t).

Proposition 5.3. *The valency of any drg Γ_i is st.*

Proof. Each vertex has $s(t+1)$ neighbors in the srg, and it has s neighbors in the spread. □

Proposition 5.4. *If r is the number of distance regular graphs of Γ, then $r=s+1$.*

Proof. Since the complete graph of valency $v-1$ is the union of a spread and r distance regular graphs of valency st, we have that $v-1=s+rst$. Now

$$\begin{aligned}
(s+1)(st+1) &= s+1+rst \\
(s+1)st &= rst \\
s+1 = r
\end{aligned}$$

□

Since the strongly regular graphs in a Siamese color graph are pseudo-geometric, it is natural to consider the case where they are actually geometric.

Definition 5.3. A Siamese color graph is called *geometric* if all its strongly regular graphs are point graphs of generalized quadrangles.

We will investigate geometric Siamese graphs further in the next subsection.

5.1.3 Siamese partitions of Steiner designs

If we have a geometric Siamese color graph Γ, we can consider the lines of its generalized quadrangles. In this case, we will speak of the lines of Γ.

Proposition 5.5. *The lines of a geometric Siamese color graph Γ form a Steiner system $S(2, s+1, v)$.*

Proof. Each line is a line in a $GQ(s,t)$, hence it has size $s+1$. Each line is contained in either the spread, or in one of the distance regular graphs; hence it is monochromatic, i.e., all edges in the line have the same color. Thus we can talk about the color of a line.

It follows that two lines of different color can intersect in at most one point. Two distinct lines of the same color intersect in at most one point since they are contained in a GQ. Thus, each pair of vertices is contained in at most one line.

Finally, each pair of vertices is adjacent in at least one of the GQ's; thus there is a line connecting them. Hence, each pair of vertices is contained in exactly one line, and we have a Steiner system. □

Thus a geometric Siamese color graph provides a Steiner system with a spread, and a partition of the remaining blocks into sets which together with the spread form generalized quadrangles. We will call this a *Siamese partition* of the

Steiner system. We will say that the partition is *coherent* if the color graph is in fact a Siamese association scheme.

Reversely, given a Steiner system with a Siamese partition, we can define a color graph by assigning to each pair of vertices the color of the unique block that connects them.

Proposition 5.6. *The color graph defined by a Siamese partition of a Steiner system is Siamese.*

Proof. Clear. $\square$

Problem 5.1. What are necessary and sufficient conditions for a Siamese partition of a Steiner system, such that the resulting color graph is an association scheme?

5.1.4 Siamese graphs as simultaneous antipodal covers

Let Γ be a graph. Suppose that Γ has an equitable partition Π such that

- each class is a coclique;
- each point is adjacent to at most one point from each class.

In other words, the subgraph induced by any two classes is either empty or a matching, i.e., bipartite and regular of valency 1.

We can now define a graph Δ on Π by joining two classes if there are edges between them. Thus, Δ is the quotient graph Γ/Π. In this case we say that Γ is a *cover* of Δ. It is called an n-fold cover if each class in Π has n elements.

In the particular case that Γ is antipodal distance regular, and Π is its antipodal system, we can speak of an *antipodal cover* of Δ.

The distance regular graphs that occur in Siamese graphs are such antipodal covers:

Theorem 5.1. *Let Γ be a pseudo-geometric $GQ(s,t)$-graph containing the spread. Then the distance regular graph obtained from Γ by removing the spread is an $(s+1)$-fold cover of the complete graph K_{st+1}.*

A. Brouwer proved the following converse of this result:

Theorem 5.2. *Let Δ be an $(s+1)$-fold cover of the complete graph K_{st+1}. Then Δ is obtained by removing a spread from a pseudo-geometric $GQ(s,t)$-graph.*

This gives the following interpretation of Siamese graphs: The distance regular graphs in a Siamese graph form simultaneous antipodal covers of K_{st+1} which partition the edges of the complete multipartite graph $\bar{S}$, where S is the spread.

5.1.5 Pattern of investigation

Given a Siamese color graph, we can define many derived combinatorial objects. In order to investigate them, we try to describe their automorphism groups. There are quite a few groups of interest around:

- the automorphism group of the color graph;
- its normalizer;
- the automorphism groups of the distance regular graphs;
- the automorphism groups of the strongly regular graphs;

If Γ is geometric, we have in addition

- the automorphism groups of the GQ's;
- the automorphism group of the Steiner system;
- the automorphism group of its Siamese partition

Finally, if $s = t$ or $s = t + 2$ then each srg defines a symmetric design; this gives us one more group to consider.

In fact, not all of these groups are distinct. The automorphism group of a GQ is equal to that of its point graph; also, the automorphism groups of the color graph and the Siamese partition of the Steiner system coincide.

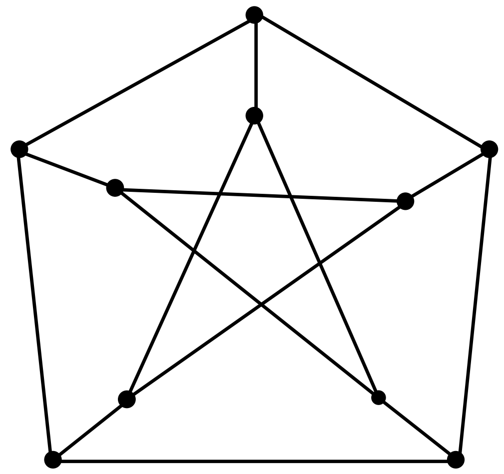
In what follows, we restrict ourselves to the case $s = t$.

5.2 Schemes on 15 points

5.2.1 Starting group

Consider the alternating group $G = A_5$ in its natural action on 5 points. It has order 60, and its Sylow 2-subgroup is the elementary abelian Klein group K_4 of order 4. If we take the action of G on the cosets of K_4, we obtain a transitive permutation group of degree 15.

We can describe this action in a more combinatorial way in terms of the Petersen graph. This graph has as vertices the 2-subsets of the 5-element set $\{1, 2, 3, 4, 5\}$. Two pairs are adjacent if they are disjoint. It is well known that this is an srg$(10, 3, 0, 1)$; in fact, it is a rank 3 representation of the symmetric group S_5.



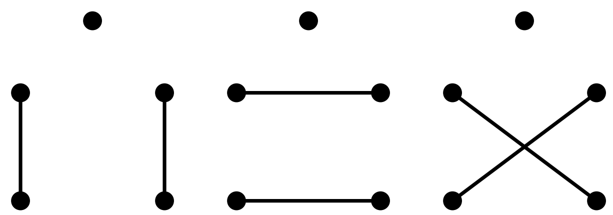
Thus we get an action of G on the 10 vertices of the Petersen graph. This graph has 15 edges, each being represented by two disjoint pairs. The stabilizer of the edge $\{\{1,2\},\{3,4\}\}$ in A_5 is the Klein group $\langle(1,2)(3,4),(1,3)(2,4)\rangle$. Thus we get that the action of G on the edges of the Petersen graph is similar to its action on the cosets of its Sylow-2 subgroup.

Thus, let Ω be the set of edges of the Petersen graph, and consider the permutation group (G,Ω) of degree 15. Let W be its centralizer algebra.

If we take the stabilizer of the point $\{\{1,2\},\{3,4\}\}$, we see that it also fixes the other 2 edges not involving the point 5. Every edge involving the point 5 shares a vertex with exactly one of them. Thus the remaining 12 edges are divided into 3 parts of 4 edges each. The point stabilizer acts transitively on each of these parts. Thus we have that the subdegrees of W are $1^3, 4^3$.

The problem with this description is that we cannot describe the basic relations naturally. The reason is that the Petersen graph, and hence its line graph, is invariant under the symmetric group S_5; under this group, some of the relations of W are merged.

Therefore, we give another description of W, which at first looks more complicated, but which allows us to define all basic relations naturally. We still start with the group A_5 acting naturally on $\Omega_0 = \{1,2,3,4,5\}$. The points are ordered triples of distinct matchings on a given set of 4 elements; in other words, they are ordered 1-factorizations of the complete graph K_4 with an additional isolated vertex. Thus, a typical point can be written as $(\{\{u,v\},\{x,y\}\},\{\{u,x\},\{v,y\}\},\{\{u,y\},\{v,y\}\})$, where u,v,x,y are distinct numbers between 1 and 5.



Each point thus defined has an isolated element; there are 5 choices for this element. For a given choice, there are $3! = 6$ ways to arrange the three matchings, hence there are 30 points altogether.

The automorphism group of such a factorization is the elementary abelian group E_4 in its regular action. Since $E_4 \leq A_5$, there are two orbits of length 15 under the alternating group. We pick one such orbit Ω; it contains three ordered factorizations for each isolated element.

In order to avoid confusion, we refer to elements of the basis set Ω_0 as *elements*, and the elements of Ω as *points*. Thus, we have 5 elements, and 15 points.

Consider one such ordered factorization. Each of the three components is a matching on the same set of 4 elements; the automorphism group of each such matching is the same copy of E_4. Thus we have that any element in A_5 which fixes one component of a given point also fixes the other two components. On the other hand, a 3-cycle that fixes the isolated element cyclically permutes the components.

Now consider two points x and y with distinct isolated elements. The first component of x contains one pair which does not involve the isolated element of y; this pair has to appear in exactly one component of y. For example, if we start with the point

$$x = (\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\})$$

which has 5 as its isolated element, and apply the cycle $(1,2,3,4,5)$, we get the point

$$y = (\{\{2,3\},\{4,5\}\},\{\{2,4\},\{3,5\}\},\{\{2,5\},\{3,4\}\})$$

with isolated element 1. The first component of x contains the pair $\{3,4\}$, which also appears in the third component of y.

Hence, we have the following binary relations for our points:

- $(x, y) \in R_0$ if $x_1 = y_1$, i.e., $x = y$;

- $(x, y) \in R_1$ if $x_1 = y_2$;

- $(x, y) \in R_2$ if $x_1 = y_3$;

- $(x, y) \in R_3$ if x_1 shares one pair with y_1;

- $(x, y) \in R_4$ if x_1 shares one pair with y_2;

- $(x, y) \in R_5$ if x_1 shares one pair with y_3.

Clearly, these relations form a partition of the cartesian square of the points. Also, they are G-invariant, where $G = A_5$. Below we list representatives for these relations.

$$
\begin{aligned}
R_0 &: \; y = x = (\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\}) \\
R_1 &: \; y = x^{(1,2,3)} = (\{\{1,4\},\{2,3\}\},\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\}) \\
R_2 &: \; y = x^{(1,3,2)} = (\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\},\{\{1,2\},\{3,4\}\}) \\
R_3 &: \; y = x^{(1,2)(3,5)} = (\{\{1,2\},\{4,5\}\},\{\{1,4\},\{2,5\}\},\{\{1,5\},\{2,4\}\}) \\
R_4 &: \; y = x^{(1,5)(2,3)} = (\{\{2,4\},\{3,5\}\},\{\{2,5\},\{3,4\}\},\{\{2,3\},\{4,5\}\}) \\
R_5 &: \; y = x^{(1,3)(2,5)} = (\{\{1,4\},\{3,5\}\},\{\{1,3\},\{4,5\}\},\{\{1,5\},\{3,4\}\})
\end{aligned}
$$

If we take the stabilizer of x, then the orbit of y has length 1 if $(x, y) \in R_i, i = 0, 1, 2$. It has length 4 if $(x, y) \in R_j, j = 3, 4, 5$, since it contains one point for each choice of the isolated element. Altogether, we get 6 orbits with a total length of 15, in other words, we described all the orbits. We have the following:

Proposition 5.7. *The relations $R_i, i = 0, \ldots, 5$ are the basis relation of the centralizer algebra W of A_5 acting on ordered factorizations of K_4.*

Proposition 5.8. *The action of A_5 is two-closed, i.e., $Aut(W) = A_5$.*

Proof. Taking the relations R_0, R_1, R_2 together, we get an equivalence relation; two points are equivalent if they have the same isolated point. Thus we can recover the

5 starting elements from the association scheme; any automorphism that fixes these 5 elements fixes every point of the scheme. Thus, the action on the 5 elements is faithful; since we saw that S_5 has an orbit of length 30 on the points, whereas A_5 has orbits of length 15, the automorphism group cannot be S_5. Thus, A_5 is the full automorphism group of the scheme. $\square$

Proposition 5.9. *The normalizer of W is $(S_5 \times S_3)^+$. It acts transitively on the non-reflexive relations of subdegree 1, and on the relations of subdegree 4.*

Here, for a permutation group G, G^+ denotes the set of even permutations in G, in other words, the intersection of G with the alternating group $A_\infty = \bigcup_{n=1}^{\infty} A_n$. The proof of this proposition will be postponed to Section 5.2.3.

5.2.2 Important mergings

Here, we will consider some mergings of the association scheme W which in turn lead to association schemes.

Proposition 5.10. *The graph $(\Omega, R_1 \cup R_2)$ is an imprimitive srg of valency 2, i.e., a spread with 5 classes.*

Proof. We have seen in the previous section that $R_0 \cup R_1 \cup R_2$ is an equivalence relation with 5 classes. $\square$

Proposition 5.11. *The graph (Ω, R_3) is isomorphic to the line graph of the Petersen graph; hence it is an antipodal drg of diameter 3. Its automorphism group is S_5.*

Since the normalizer acts transitively on the basic relations of valency 4, we get that also R_4 and R_5 give isomorphic drg's.

Proposition 5.12. *The graph $(\Omega, R_1 \cup R_2 \cup R_3)$ is a strongly regular graph of valency 6; it is isomorphic to the triangular graph $T(6) = L(K_6)$. Its automorphism group is S_6.*

Again, applying the normalizer, we get two more strongly regular graphs. Summarizing these results, we get

Theorem 5.3. *The centralizer algebra $W = V(A_5, \Omega)$ is a Siamese association scheme.*

5.2.3 A Siamese partition of STS(15)

Recall from the previous chapter the classical Steiner triple system on 15 points. There are two descriptions; geometrically speaking, this is the structure formed by points and lines in $PG(3,2)$, combinatorially, points are affine designs on 8 points, and lines are partitions of the 8 points into two parts of size 4.

We will now construct a subset of blocks which forms a generalized quadrangle. Consider a two-element subset of the base set Ω_0, say $\{1,2\}$. Let B_0 be the set of partitions of Ω_0 of the form $4+4$ which does not separate these two elements.

Proposition 5.13. *The incidence structure (Ω_0, B_0) is a generalized quadrangle of order 2.*

Proof. If we disregard the two selected base elements, we can identify each block in B_0 with a two-element subset of a six-element set, namely, the pair of elements that appears together with the selected pair. Points can be regarded as partitions of the form $2+2+2$. This is the dual of the classical, combinatorial model for $GQ(2)$. □

Since we can choose any two elements from Ω_0, we get a total of $\binom{8}{2} = 28$ generalized quadrangles embedded into the Steiner system S.

Now consider the generalized quadrangles defined by the pairs $\{1,2\}$ and $\{1,3\}$. The blocks that appear in both of them are the partitions of Ω_0 in which $1,2,3$ appear in the same class. There are 5 such partitions; all these blocks are disjoint. Thus we get

Proposition 5.14. *The generalized quadrangles embedded into S which are defined by two intersecting pairs $\{x, y\}, \{x, z\}$ from Ω_0 intersect in a spread; this spread depends only on the union $\{x, y, z\}$ of the pairs.*

Corollary 5.1. *The generalized quadrangles defined by the pairs $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $x, y, z \in \Omega_0$, intersect pairwise in a fixed spread.*

Corollary 5.2. *This construction provides a Siamese partition of the classical $STS(15)$.*

Thus, for each choice of three elements from Ω_0, we get a Siamese partition. Altogether, there are $\binom{8}{3} = 56$ such choices.

As we have seen before, a Siamese partition of a Steiner system defines a Siamese color graph. It turns out that the graph resulting from the construction above is the same as the graph arising from the association scheme defined in Section 5.2.1.

This can be seen as follows: If we fix each of the three generalized quadrangles, this means that each of the defining pairs is fixed in Ω_0, thus, the set $\{x, y, z\}$ is fixed pointwise. Thus, the color graph defined by this partition is invariant under A_5. This group acts transitively on the points, hence the stabilizer of a point is the elementary abelian group E_4. It follows that the two actions of A_5 are similar, hence we get isomorphic association schemes.

Proposition 5.15. *The group of automorphisms of the Steiner system which preserve the Siamese partition, while possibly interchanging the generalized quadrangles, is isomorphic to $(S_5 \times S_3)^+$.*

Proof. The generalized quadrangles correspond to pairs $\{x, y\}$, $\{x, z\}$, $\{y, z\}$ in Ω_0. The group which preserves them is the stabilizer of the set $\{x, y, z\}$ in A_8, which has the above stated structure. $\square$

This also gives a proof of Proposition 5.9. Summarizing this section, we have the following:

Proposition 5.16. *The classical Steiner system $STS(15)\#1 \cong PG(4,2)$ admits a coherent Siamese partition. Its automorphism group is A_5, its normalizer is $(S_5 \times S_3)^+$.*

5.2.4 A Siamese graph that is not an association scheme

To show the fact that the notion of a Siamese scheme is stronger than the notion of a Siamese graph, we give an example of a Siamese graph that is not an association scheme.

In the previous construction, we take the same points, but we replace A_5 by the stabilizer A_4 of one point. This group has 2 orbits of length 3 and 12; thus it acts regularly on one of the orbits, and it is more natural to consider the right regular action.

Let $G = A_4$. Let H be its Sylow-2-group. Let $V_1 = G$, $V_2 = \cup(G/H)$, and $V = V_1 \cup V_2$. Then $|V| = 12 + 3 = 15$, and G acts on V by right multiplication. Let $\bar{W} = (V, 2 - \mathrm{orb}(G, V))$.

Since G acts regularly on V_1, there are 12 2-orbits in $V_1 \times V_1$, and 3 2-orbits in $V_1 \times V_2$. The stabilizer of any point in V_2 is equal to H, thus there are 3 2-orbits in $V_2 \times V_1$, and 3 2-orbits in $V_2 \times V_2$. Thus we get a colored graph of rank $12 + 3 \cdot 3 = 21$.

We can choose the following 2-orbit representatives (e denotes the identity of G):

$$\begin{aligned}
\Psi_1 &: (e, e) \\
\Psi_2 &: (e, (1,3,2)) \\
\Psi_3 &: (e, (1,2,3)) \\
\Psi_4 &: (e, (0,3)(1,2)) \\
\Psi_5 &: (e, (0,2,3)) \\
\Psi_6 &: (e, (0,1,3)) \\
\Psi_7 &: (e, (0,1)(2,3)) \\
\Psi_8 &: (e, (0,2,1)) \\
\Psi_9 &: (e, (0,1,2)) \\
\Psi_{10} &: (e, (0,3,2)) \\
\Psi_{11} &: (e, (0,3,1)) \\
\Psi_{12} &: (e, (0,2)(1,3)) \\
\Psi_{13} &: (e, H) \\
\Psi_{14} &: (e, H \cdot (1,2,3))) \\
\Psi_{15} &: (e, H \cdot (1,3,2))) \\
\Psi_{16} &: (H, e) \\
\Psi_{17} &: (H, (1,3,2)) \\
\Psi_{18} &: (H, (1,2,3)) \\
\Psi_{19} &: (H, H) \\
\Psi_{20} &: (H, H \cdot (1,2,3)) \\
\Psi_{21} &: (H, H \cdot (1,3,2))
\end{aligned}$$

Now define the following binary relations on V:

$$\begin{aligned} S &= \Psi_8 \cup \Psi_9 \cup \Psi_2 0 \cup \Psi_{21} \\ \Phi_1 &= \Psi_5 \cup \Psi_{10} \cup \Psi_{12} \cup \Psi_{14} \cup \Psi_{18} \\ \Phi_2 &= \Psi_6 \cup \Psi_7 \cup \Psi_{11} \cup \Psi_{15} \cup \Psi_{17} \\ \Phi_3 &= \Psi_2 \cup \Psi_3 \cup \Psi_4 \cup \Psi_{13} \cup \Psi_{16} \end{aligned}$$

Finally, let $W = (V, \{Id_V, S, \Psi_1, \Psi_2, \Psi_3\})$.

Lemma 5.1. *$Aut(W) = A_4$.*

Lemma 5.2. *W is a Siamese graph of Siamese rank 3.*

Lemma 5.3. *W is not an association scheme.*

5.2.5 Anatomy of $STS(15)$ #7

We take a little detour to describe one more Steiner system on 15 points which we will use in the next subsection. It is the system number 7 (in the numeration of Mathon-Rosa-Phelps) which has an automorphism group of order 288.

We start with the group $H = S_4 \times S_4$ of order 576, in its natural action with orbits $O_1 = \{1, 2, 3, 4\}, O_2 = \{5, 6, 7, 8\}$. If we take the set of all even permutations in H, we obtain $G = H^+ = H \cap A_8$ of order 288, acting on $\Omega_0 = O_1 \cup O_2$.

There are two orbits of points. The first consists of partitions of O_2 of the form 2+2. An example is

$$p_1 = \{\{5, 6\}, \{7, 8\}\}.$$

There are three points in this orbit. The second orbit consists of partitions of Ω_0 into 4 pairs, each containing one point from each orbit. Hence, a typical point looks like

$$p_2 = \{\{1, 5\}\{2, 6\}, \{3, 7\}, \{4, 8\}\}$$

There are $4! = 24$ such points. However, under G they fall into two orbits of length 12; we pick one of them. Altogether, we get $3 + 12 = 15$ points.

There are three types of blocks. The first type consists of partitions of Ω_0 of the form $4+4$, such that each class contains 2 elements from each orbit. Thus, an example of such a block is

$$b_1 = \{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}\}.$$

The number of such blocks is $\frac{1}{2}\binom{4}{2}^2 = 18$.

The second type of blocks consists of pairs of elements, one from each orbit. Thus, a typical example is

$$b_2 = \{1, 5\}.$$

There are $4^2 = 16$ such blocks. Finally, there is one block b_∞. Altogether, we get $18 + 16 + 1 = 35$ blocks.

Incidence is defined as follows: b_∞ is incident to all three points of the first type. A block of the second type is incident to a point of the second type if the pair is contained in the partition. Finally, a block of the first kind contains one point of the first kind, namely the partition it induces on O_2, and two points of the second kind, namely those partitions of type $2+2+2+2$ which are a refinement of the partition of type $4+4$.

Proposition 5.17. *The structure thus defined is an $STS(15)$.*

Proof. By construction, each block contains 3 points. A point of the first kind is contained in b_∞. There are $\binom{4}{2} = 6$ ways to extend a partition of type $2+2$ to a partition of type $4+4$, thus it is contained in 6 blocks of the first kind. Altogether, the point is contained in $6+1 = 7$ blocks.

If we consider a point of the second kind, it is a partition of type $2+2+2+2$, hence it contains 4 pairs, and is incident to 4 blocks of the second kind. There are

3 ways to merge it into a partition of type $4+4$, hence it is incident with 3 blocks of the first kind. Altogether, it is incident to $3+4=7$ blocks.

It remains to show that any two points are collinear. All points of the first type are contained in b_∞. Given a point $\{\{w,x\},\{y,z\}\}$ of the first type, and a point $\{\{a,w\},\{b,x\},\{c,y\},\{d,z\}\}$ of the second kind, they are both contained in the block $\{\{a,b,w,x\},\{c,d,y,z\}\}$.

Finally, given two points of the second kind, one can be obtained by the other by applying an even permutation to O_2. If this permutation is an involution, they are contained in a block of the first kind; if it is of order 3, it has a fixed point, hence the two points are contained in a block of the second kind. $\square$

5.2.6 A non-coherent Siamese partition of $STS\#7$

Recall from Section 5.2.5 the description of an $STS(15)$ invariant under $(S_4\times S_4)^+$. This group acts naturally on two orbits $O_1=\{1,2,3,4\}, O_2=\{5,6,7,8\}$ (each copy of S_4 acts on one orbit).

Representatives of points are

$$\begin{aligned} p_1 &= \;=\{\{5,6\},\{7,8\}\} \\ p_2 &= \;=\{\{1,5\},\{2,6\},\{3,7\},\{4,8\}\} \end{aligned}$$

Representatives of blocks are

$$\begin{aligned} b_1 &= \;=\{\{1,2,5,6\},\{3,4,7,8\}\} \\ b_2 &= \{1,5\} \end{aligned}$$

as well as the block b_∞ which contains of all points of the first kind.

Now, select a pair $\{x,y\}$ of elements from O_1. Take all blocks of the first kind in which these two elements appear in the same class; take all blocks of the second kind which contain either x or y. Finally, add the block b_∞.

Proposition 5.18. *The chosen blocks form a generalized quadrangle of order 2.*

Proof. By definition, each block contains 3 points, and we have 15 points.

Since x and y from O_1 are given, we have to select two elements from O_2 to appear together with x, y to form a block of the first kind. We have $\binom{4}{2}$ possibilities, hence we get 6 blocks of the first kind.

For each of x and y, we can add any one point from O_2 to get a block of the second kind. Hence there are 8 such blocks. Together with b_∞ we get $6+8+1=15$ blocks.

Next, we need to check that each point is contained in 3 blocks. A point of the first kind is a partition of type $2+2$ of O_2. We have two choices of adding $\{x, y\}$ to one of the parts, and $O_1 \setminus \{x, y\}$ to the other, to obtain a block of the first kind. Moreover, such a point is contained in b_∞, thus altogether it is in 3 blocks.

A point of the second kind is a partition of type $2+2+2+2$; we have to merge the classes containing x and y, thus such a point is contained in exactly one block of the first kind. The two pairs which contain x and y describe two blocks of the second kind; hence, such a point is contained in 3 blocks.

Thus we have 15 points, 15 blocks, each point lies on 3 blocks, and each block contains 3 points. To finish the proof, it suffices to show that there are no triangles in this structure, i.e., no three blocks which intersect pairwise, but which do not intersect in a common point.

To summarize, we have

- The block b_∞ contains three points of type 1.
- A block of type 1 contains one point of type 1 and 2 points of type 2.
- A block of type 2 contains 3 points of type 2.
- A point of type 1 is contained in b_∞ and two blocks of type 1.
- A point of type 2 is contained in one block of type 1 and two blocks of type 2.

First, let us assume that the block b_∞ is contained in a triangle. Then two of the points are of the first kind; hence, the remaining two blocks are of the first kind. Then the remaining point needs to be of the second kind. This contradicts the fact that each such point is contained in a unique block of the first kind.

If a triangle contains a point of type 1, say $\{\{a,b\},\{c,d\}\}$, then the two blocks through this point are of type one; they are of the form $\{\{x,y,a,b\},\{*,*,c,d\}\}$ and $\{\{x,y,c,d\},\{*,*,a,b\}\}$. A block $\{x,z\}$ of the second kind intersects either the first of them if $z \in \{a,b\}$; it intersects the second if $z \in \{c,d\}$. Hence, it cannot intersect both of them, and we do not have a triangle.

Hence, all points need to be of the second kind. At most one of the blocks is of the first kind; thus there is a point incident with 2 blocks of the second kind. We may assume that these two blocks have the form $\{x,a\}$ and $\{y,b\}$. Then the third block cannot be of the second kind, since if it intersects the first block, it cannot contain x, and if it intersects the second block, it cannot contain y. But then one of the classes of this third block is $\{x,y,a,b\}$, which means it contains the initial point.

Hence, the blocks defined by a pair of elements from O_1 form a generalized quadrangle. $\square$

Proposition 5.19. *Two generalized quadrangles defined by pairs $\{x,y\}$ and $\{x,z\}$ intersect in a spread which depends only on x.*

Proof. Any generalized quadrangle contains b_∞. Since x, y, and z all come from O_1, the two GQ's cannot share a block of the first type. The blocks of the second kind common to both are of the form $\{x,a\}$ with $a \in O_2$. $\square$

Corollary 5.3. *The generalized quadrangles defined by three pairs intersecting in a common point form a Siamese partition of S.*

5.2.7 All geometric Siamese color graphs are obtained

Above we gave the construction of two geometric Siamese graphs on 15 points. To prove that there are no others, the following steps have been taken:

- A geometric Siamese graph on 15 points provides a Siamese partition of an $STS(15)$.
- There are 80 non-isomorphic $STS(15)$.
- For each $STS(15)$, a computer search was performed to enumerate all embedded $GQ(2)$.
- It was checked if three of these generalized quadrangles form a Siamese partition.

Thus, we get the following result:

Proposition 5.20 (Computer search). *The only $STS(15)$ admitting a Siamese partition are systems no. 1 and 7. In both cases, the partition is unique.*

Corollary 5.4. *There are exactly two non-isomorphic geometric Siamese color graphs on 15 vertices.*

5.2.8 Final summary

It is well-known that the triangular graph $T(n)$ is uniquely determined by its parameters, except for the case $n = 8$. In particular, $T(6)$ is the only $\mathrm{srg}(15, 6, 1, 3)$, and it is the point graph of the unique $GQ(2)$.

Therefore, all Siamese graphs on 15 points are geometrical. Together with the last section's results we finally get

Theorem 5.4. *There are up to isomorphism two Siamese graphs on 15 points, one of which is an association scheme.*

5.3 A classical Siamese scheme on 40 points

Here, we describe a Siamese association scheme W of order $(3,3)$ corresponding to the classical generalized quadrangle $W(3)$. This is why we call it "classical", in spite of the fact that it was not presented before.

5.3.1 Starting group

We start with the alternating group A_6 of order 360 in its natural action on $\Omega_0 = \{1,2,3,4,5,6\}$. Its Sylow 3-subgroup is elementary abelian of order 9; it is generated by two disjoint cycles of length 3, say $(1,2,3)$ and $(4,5,6)$.

The Sylow subgroup can be considered as the stabilizer in A_6 of the combinatorial object consisting of an ordered pair of directed cycles of length 3. Thus, if we use permutation notation for such directed graphs, we get the following reference point:

$$x = ((1,2,3),(4,5,6)).$$

Thus, we set $G = A_6$, $\Omega = x^G$, and $W = V(G,\Omega)$. We have that $|\Omega| = 360/9 = 40$. W has the following basis relations:

- $(x,y) \in R_0$ if $x = y$; subdegree 1, representative $y = ((1,2,3),(4,5,6))$.
- $(x,y) \in R_1$ if all arcs in x are reversed to obtain y; subdegree 1, representative $y = ((1,3,2),(4,6,5)) = (x_1^{-1}, x_2^{-1})$
- $(x,y) \in R_2$ if the components of x are interchanged, and the direction of the first component is reversed; subdegree 1, representative $y = ((4,6,5),(1,2,3)) = (x_2^{-1}, x_1)$.
- $(x,y) \in R_3$ if the components of x are interchanged, and the direction of the second component is reversed; subdegree 1, representative $y = ((4,5,6),(1,3,2)) = (x_2, x_1^{-1})$.

- $(x, y) \in R_4$ if the first components of x and y share an arc; subdegree 9, representative $y = ((2,3,4),(1,6,5))$.
- $(x, y) \in R_5$ if the second components of x and y share an arc; subdegree 9, representative $y = ((1,3,4),(2,5,6))$.
- $(x, y) \in R_6$ if the first component of x and the second component of y share an arc and vice versa; subdegree 9, representative $y = ((3,4,5),(1,2,6))$.
- $(x, y) \in R_7$ if the first component of x shares a reversed arc with the second component of y, and vice versa; subdegree 9, representative $y = ((3,4,6),(1,5,2))$.

If we add up the subdegrees, we get $4 \cdot 1 + 4 \cdot 9 = 40$. It can be checked easily that these relations are minimal G-invariant. This proves the following:

Proposition 5.21. *The centralizer algebra $W = V(G, \Omega)$ has rank 8, with subdegrees $1^4, 9^4$.*

We use COCO to investigate this scheme. The result will be explained in a few steps.

First let us remark that the group (G, Ω) is not 2-closed. For this reason we later on repeat our considerations with a more appropriate group, namely the 2-closure in some natural representation.

Also, we will find the normalizer of the 2-closure; this helps us to simplify the investigation of subschemes. Here, we just state that there are 10 non-isomorphic mergings; information about them is listed in Table 5.1.

Most important for our main goals are the mergings 6 (a drg) and 8 (an srg; as we will see later, it is the point graph of $W(3)$). Note that the scheme 6 is not Schurian. In other words, the basis graph of valency 9 is distance regular, but not distance transitive.

Table 5.1: Mergings of $V(A_6, \Omega)$

#	Rank	Subdegrees	Copies	$\|Aut\|$	Aut	Rank of Aut	Schurian
0	8	$1, 9^4, 1^{2\cdot 1+1}$	1	720	$A_6 \times S_2$	8	yes
1	6	$1, 18^2, 1^{2\cdot 1+1}$	1	1474560		6	yes
2	6	$1, 18, 9^2, 2, 1$	2	1440		6	yes
3	5	$1, 18^2, 2, 1$	1	1509949440		5	yes
4	5	$1, 36, 1^3$	1	$2^{20}10!$		5	yes
5	5	$1, 18^2, 2, 1$	2	1440		5	yes
6	4	$1, 3, 9, 27$	4	1440	$S_2 \times S_6$	6	no
7	4	$1, 36, 2, 1$	1	$2^{30}10!$		4	yes
8	3	$1, 27, 12$	4	51840	$Sp(4, 3)$	3	yes
9	3	$1, 36, 3$	1	$10!4!^{10}$	$S_4 \wr S_{10}$	3	yes
10	3	$1, 38, 1$	1	$2^{20}20!$	$S_2 \wr S_{20}$	3	yes

5.3.2 The 2-closure of the starting group

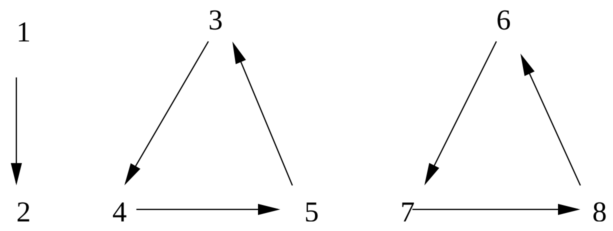
In the remainder of this section we will be once more involved in the consideration of certain classical material on the edge between permutation groups and geometry. In particular, the material presented here is related to the exceptional isomorphism of the simple groups A_6 and $PSL(2, 9)$.

There are several ways to establish this isomorphism. For reference, see, e.g., Hirschfeld [54] or Dye [28]. However, these methods are not directly related to the material here.

Instead, we will once more employ COCO to obtain the 2-closure of A_6.

Consider the following permutation on the set Ω of 40 points defined above: Given a point $((a, b, c), (d, e, f))$, reverse the direction of both cycles, to obtain $((a, c, b), (d, f, e))$. It is not hard to verify that all 8 relations are invariant under this permutation.

To get a more natural description of the full automorphism group, we consider the direct product $S_2 \times A_6$, acting naturally on $8 = 2 + 6$ points. Let it act on unordered pairs of directed graphs, where the first component is given below, and the second component is obtained by the first by reversing the directions of all the arcs.



Now, the transposition $(1, 2)$ in S_2 exactly realizes the additional automorphism described above, i.e., it reverses the direction of both cycles. Since W is invariant under $S_2 \times A_6$ of order 720, and the computations in COCO tell us that this is the order of the full automorphism group of W, we have that $Aut(W) = S_2 \times A_6$. We will be able to give a computer-free proof of the same fact later on.

5.3.3 The normalizer of the 2-closure

In order to find the normalizer of $Aut(W)$, we used GAP and its capabilities described in Chapter 3. The following is a list of facts for which no computer-free proof exists yet:

- The normalizer $N = N(Aut(W)$ is an extension of $S_2 \times A_6$ by the dihedral group D_4 of order 8. Hence, $|N| = 5760$.
- This extension is not a semidirect product.
- N acts transitively on the directed relations of subdegree 1, as well as on the relations of subdegree 9.

5.3.4 The automorphism group of the drg on 40 points

Here, we will describe the automorphism group of the distance regular graph which appears in the association scheme W.

Taking the basis relation R_4 in W, we get a distance regular graph of diameter 3. The association scheme generated by this graph is

$$\langle\langle R_4 \rangle\rangle = \langle R_0, R_4, R_1 \cup R_2 \cup R_3, R_5 \cup R_6 \cup R_7 \rangle .$$

We will show here that this scheme is invariant under $S_2 \times S_6$.

Thus, let $G = S_2 \times S_6$ act naturally on $8 = 2 + 6$ points. Generators for this groups are

$$G = \langle (1,2), (3,4), (4,5,6,7,8) \rangle$$

We will take a subgroup H of G isomorphic to $S_3 \times S_3$, such that $[G : H] = 1440/36 = 40$, and hence we get 40 cosets.

In order to define this group, let us do a short excursion to the topic of subdirect products of groups. Let H_1, H_2, and M be groups. Let $f_i : H_i \to M$ be homomorphisms. Consider the set $H \subset H_1 \times H_2$ defined by $(a,b) \in H$ if $f_1(a) = f_2(b)$ for $a \in H_1, b \in H_2$.

Proposition 5.22. *H is a group of order*

$$|H| = \frac{|H_1| \cdot |H_2|}{|M|}.$$

The group H is called a subdirect product of H_1 and H_2 (with respect to the given pair of homomorphisms to M). More information on this subject can be found in [89] and [86].

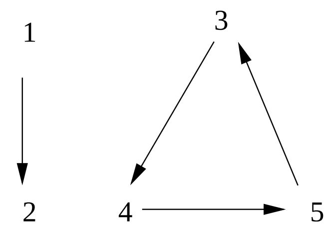
In our case we have $H_1 = S_3 \times S_3$, $H_2 = S_2$, and $M = S_2$. The homomorphism $f_2 : H_2 \to M$ is the identity. The homomorphism $f_1 : H_1 \to M$ has kernel $A_3 \times S_3$, i.e., it maps (a,b) to the identity if a is even.

The subdirect product $H \leq (S_3 \times S_3) \times S_2$ is in fact isomorphic to H_1, since for each element $(a,b) \in H_1$ there exists exactly one element $((a,b),c) \in H$, and multiplication in H, restricted to the first component, is exactly multiplication in H_1.

5.3.5 A new model of the classical antipodal drg on 40 points

We can use the information obtained in the previous section to give a combinatorial description of the distance regular graph.

We start with the group $G = S_2 \times S_6$ acting on 8 points. Consider the following directed graph.



We can consider a second graph obtained from this one by reversing the directions of all the arcs. We take these two graphs as an unordered pair, and consider it as one point of our future graph. Now we get all the vertices by taking the orbit under the group G.

Note that in each point one graph contains the arc $(1, 2)$, whereas the other contains the arc $(2, 1)$. We will refer to the former as the first component of a point.

Note further that that each point is uniquely determined by the cycle in its first component. Thus, using permutation notation, we can refer to the point depicted above as $(3, 4, 5)$.

The stabilizer of a point is exactly the subdirect product described in the previous section.

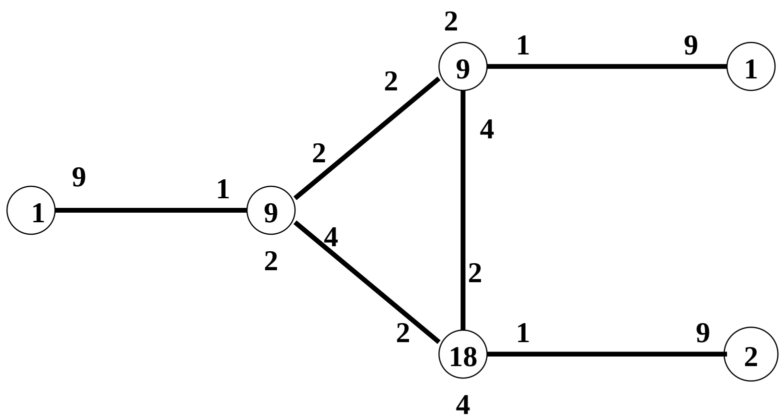
Now, we can describe all 2-orbits S_i of this action in terms of the cycles of the first components of points:

- $(x, y) \in S_0$ if $x = y$; subdegree 1, representative $(3, 4, 5)$;
- $(x, y) \in S_1$ if y is the cycle x reversed; subdegree 1, representative $(3, 4, 5)$;
- $(x, y) \in S_2$ if the two cycles do not share any points; subdegree 2, representative $(6, 7, 8)$;

- $(x, y) \in S_3$ if the two cycles share one arc; subdegree 9, representative $(3, 4, 6)$;
- $(x, y) \in S_4$ if the two cycles share one reversed arc; subdegree 9, representative $(3, 5, 6)$;
- $(x, y) \in S_5$ if the two cycles share one point; subdegree 18; representative $(3, 6, 7)$.

These are all minimal G-invariant relations; hence we have that the permutation rank of G is 6, and the subdegrees are $1, 1, 2, 9^2, 18$.

Now there are two relations with subdegree 9; we could choose either one to define the distance regular graph. Let us choose S_3. It is not hard to derive the following distribution diagram:



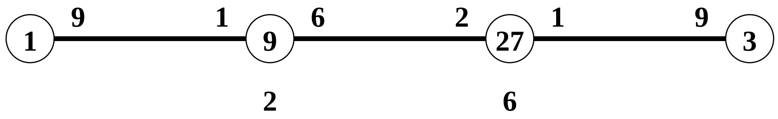
Let us just derive a few of the numbers appearing in the diagram. We still use $(3, 4, 5)$ as our reference point.

Let y be a point such that $(x, y) \in S_5$; e.g., $y = (3, 6, 7)$. Of the nine points adjacent to y, two share an arc with x, namely $(3, 6, 5)$ and $(3, 4, 7)$. Two points, $(3, 6, 4)$ and $(3, 5, 7)$, share a reversed arc with x. Only one point, $(5, 6, 7)$, doesn't share any point with x. Finally, the remaining 4 neighbors $(3, 6, 5)$, $(3, 6, 8)$, $(3, 5, 7)$, and $(3, 8, 7)$ share only one point with x.

The remaining constants in the diagram can be found by similar reasoning.

Proposition 5.23. *The graph defined by the relation S_3 is a distance regular graph with the intersection array $\{1,2,9;9,6,1\}$.*

Proof. If we merge the relations S_1, S_2 and S_4, S_5, the diagram above collapses to



$\square$

Proposition 5.24. *The distance regular graph is arc-transitive.*

Proof. Clear, since all its arcs lie in the same 2-orbit S_3. $\square$

Proposition 5.25. *The association scheme defined by the drg is not Schurian.*

Proof. It has been checked by computer that the automorphism group of the drg has rank 6. $\square$

5.3.6 A new model of the classical generalized quadrangle $W(3)$

The graph described in the previous section is antipodal distance regular, with intersection array $\{1,2,9;9,6,1\}$. Thus, if we add the spread given by $S_1 \cup S_2$, we get a pseudo-geometric $GQ(3)$-graph.

In fact, this srg is geometric. To prove that, we need to find 40 4-cliques in this graph. Ten cliques are given by the spread.

To find the remaining cliques, we proceed as follows: Remember that the points of the graph can be identified with directed cycles of length 3 in a 6 element set Ω_0; two points being adjacent if they share an arc. There are a total of 30 ordered pairs of distinct elements in a 6-element set; each of them is contained in 4 cycles, and these cycles are pairwise adjacent. Thus, we get the 30 required additional cliques.

Since the srg is pseudo-geometric, these 40 cliques form a generalized quadrangle. However, we can also prove this directly.

Theorem 5.5. *Let $\Omega_0 = \{1, 2, 3, 4, 5, 6\}$. Let P be the set of directed cycles of length 3 in Ω_0. Let B_1 be the set of all partitions of Ω_0 into two triples. Let B_2 be the set of directed arcs from Ω_0. Let $B = B_1 \cup B_2$.*

Define incidence as follows: Let a cycle be incident to a partition if its set of vertices is one of the classes; let it be incident to an arc if it contains this arc. Then the incidence structure (P, B) is a generalized quadrangle of order 3, with spread B_1. This spread is invariant under $S_2 \times S_6$.

Proof. We have seen above that $|P| = 40$, $|B_1| = 10$, and $|B_2| = 30$, hence $|B| = 40$.

Each directed cycle is contained in one partition, and contains 3 arcs. Hence, each point is incident to 4 blocks.

Each partition contains two parts; for each of them, there are two possible orientations. Thus each block in B_1 is incident to 4 points.

Given an arc (x, y), there are 4 points remaining to complete the arc to a cycle of length 3. Hence, each block in B_2 is incident to 4 points.

Two distinct partitions cannot contain the same 3-set. In particular, two blocks in B_1 cannot be incident with the same point. Thus, B_1 forms a spread of the incidence structure.

Two arcs are incident with the same triangle if the endpoint of one is the starting point of the other. In this case, the triangle is unique determined; thus, two points in B_2 intersect in at most one point.

An arc and a partition are incident with the same triangle if the arc is contained in one of the classes of the partition. Also here the triangle is unique determined, thus two blocks, one from B_1 and one from B_2, share at most one points.

Altogether we have that two blocks intersect in at most one point; thus, (P, B) is a partial linear space of order $(3, 3)$.

Finally, we need to show that this is a partial geometry with $T = 1$. Thus, let p be a point, and b be a block not containing p.

If $b \in B_1$, then since $p \notin b$, not all three points of the triangle lie in the same class of b. Thus there is exactly one arc in p lying in one class; this is the unique block incident with p and intersecting b.

If $b \in B_2$, then the arc b is not contained in the triangle p. If b shares a point with p, then there is exactly one arc in p such that its endpoint is the starting point of b, or vice versa. If it doesn't share a point with p, then it intersect the unique block from B_1 containing p.

Altogether, we have that $T = 1$, hence we have a generalized quadrangle.

By construction, (P, B) is invariant under S_6. It is also invariant under reversal of each cycle. We have seen before that the structure of the generated group is $S_2 \times S_6$. □

Corollary 5.5. *We get the following:*

- *The basis relations of W of subdegree 9 are distance regular graphs.*
- *Adding the spread to the drg's yields srg's.*
- *W is a Siamese association scheme.*

Corollary 5.6. *The generalized quadrangle defined above is isomorphic to the classical model $W(3)$.*

Proof. There are exactly 2 non-isomorphic $GQ(3)$, namely $W(3)$ and its dual $Q(4, 3)$. The latter does not contain a spread. □

5.3.7 More about $W(3)$

The generalized quadrangle $W(3)$ is a classical object that has been thoroughly investigated. The following are well-known facts:

Proposition 5.26. $Aut(W(3)) = Aut(O(5, 3)) = O(5, 3).2$.

The group $O(5,3)$ is simple; as such it is also well understood. In fact, several classical groups are isomorphic to this group. In particular, we have that $O(5,3) \cong PSp_4(3) \cong PSU_4(2)$.

Proposition 5.27. *$W(3)$ has a unique spread.*

Proposition 5.28. *$O(5,3)$ has a maximal subgroup isomorphic to S_6. This is (up to conjugacy) the only maximal subgroup of index 36.*

Corollary 5.7. *$S_2 \times S_6$ is the complete automorphism group of the classical drg on 40 points.*

5.4 Siamese graphs and Siamese Steiner systems on 40 points

5.4.1 General picture

Most of the results in this section are of a computational nature. We will discuss the way they were obtained later on.

We have found altogether nine Steiner designs $S(2,4,40)$ which permit a Siamese partition; one of these is the classical design $PG(3,3)$.

It turns out that exactly three of them (including the classical one) have a point-transitive automorphism group, and that only the classical design has a block-transitive automorphism group.

Table 5.2 contains information about the various groups related to these Steiner systems. These groups are

G the automorphism group of the Steiner system.

H the automorphism group of the partition; this is the normalizer of the color graph.

K the automorphism group of the color graph.

For the groups G and H, we include the lengths of orbits in their actions on points and blocks. For K, we include the permutation rank.

Table 5.2: Some Siamese Steiner designs on 40 points

	$S(2,4,40)$			Partition		
#	$\lvert G\rvert$	P	B	$\lvert H\rvert$	P	B
1	24261120	40	130	11520	40	$10, 120$
2	160	40	$10, 40, 80$	160	40	$10, 40, 80$
3	480	40	$10, 20, 40, 60$	480	40	$10, 20, 40, 60$
4	1296	$4, 36$	$1, 24^2, 81$	144	$4, 36$	$1, 9, 24^2, 36^2$
5	648	$4, 36$	$1, 12^4, 81$	72	$4, 36$	$1, 9, 12^4, 36^2$
6	288	$4, 36$	$1, 9, 24^2, 36^2$	288	$4, 36$	$1, 9, 24^2, 36^2$
7	10368	$4, 36$	$1, 48, 81$	576	$4, 36$	$1, 9, 48, 72$
8	1296	$4, 36$	$1, 24^2, 81$	144	$4, 36$	$1, 9, 24^2, 36^2$
9	144	$4, 36$	$1, 9, 24^2, 36^2$	144	$4, 36$	$1, 9, 24^2, 36^2$

Recall that up to isomorphism there exist two geometrical srg$(40, 12, 2, 4)$, and only one of them has a spread, namely the point graph of $W(3)$. Moreover, this spread is unique up to isomorphism. Therefore, in a geometrical Siamese color graph on 40 points there is only one option for the drg and the srg. Note that all Siamese graphs described here are geometrical.

Besides the geometrical drg we have discovered two more drg's on 40 points; in fact, it gives the smallest possible answer to a question posed by Godsil-Hensel, see Section 2.7. This graph will be considered in the appendix. Currently we do not have any example of a Siamese graph involving this drg and thus is not geometrical.

Unfortunately we cannot describe all the Siamese partitions in detail. However, a few of them will be discussed below. Note also that for completeness details concerning all systems are included in Appendix A.1.

5.4.2 Vertex-transitive systems

As stated above, we have found three Siamese Steiner systems with an automorphism group acting transitively on points. One of them is the classical system $PG(3,3)$; the corresponding Siamese scheme has been described in Section 5.3.

Another interesting design has an automorphism group of order 480. Below we try to give a self-contained model of this object. We follow a similar line of action as before, however, here it is more convenient to start from the description of the Steiner system.

Consider the group $H = A_5 \times Z_4$. We assume that A_5 acts naturally on $\Omega_1 = \{1, \ldots, 5\}$, and that Z_4 is generated by the cycle $(6, 7, 8, 9)$, acting on $\Omega_2 = \{6, \ldots, 9\}$. Thus the basis set is $\Omega_0 = \Omega_1 \cup \Omega_2 = \{1, \ldots, 9\}$.

Let us first describe the points. Points are graphs with two disjoint edges, connecting two elements diagonally opposed on the cycle with elements in Ω_1. Thus, one possible point is $\{16, 28\}$ if we denote an edge from a to b by ab.. There are $\binom{5}{2} = 10$ pairs of elements in Ω_1; for each we can choose one of 4 positions of the cycle. Therefore, we have a total of 40 points.

We have the following basic blocks:

- $\{16, 28\}, \{19, 47\}, \{27, 59\}, \{48, 56\}$, with an orbit of length 60. These are points whose starting vertices form arcs of the cycle $(1, 2, 5, 4)$.
- $\{16, 28\}, \{18, 46\}, \{26, 48\}, \{37, 59\}$, with an orbit of length 40. This block contains the point $\{37, 59\}$ and all graphs which do not share a vertex with it.
- $\{16, 28\}, \{18, 26\}, \{17, 29\}, \{19, 27\}$, with an orbit of length 10. This block consists of all points involving 1 and 2.
- $\{16, 28\}, \{28, 36\}, \{28, 56\}, \{28, 46\}$, with an orbit of length 20. This block consists of all points containing the edge 28.

We give the following computer result without proof:

Proposition 5.29. *The structure defined above is a $S(2, 4, 40)$ containing exactly 4 generalized quadrangles; these GQ's form a Siamese partition. Its automorphism group is generated by $A_5 \times Z_4$ and the additional automorphism $(1, 2)(6, 8)$; hence it is a subgroup of $S_5 \times D_4$ of index 2.*

One more Steiner system with a transitive automorphism group (number 2 in Table 5.2) is cyclic; in other words, it is invariant under a cyclic group of order 40, acting regularly on the points. Thus, we identify the points with the set $\mathbb{Z}_{40}$ of integers modulo 40. We take the following basic blocks:

$$\begin{aligned} b_1 &= \{0, 10, 20, 30\} \\ b_2 &= \{0, 1, 6, 32\} \\ b_4 &= \{0, 3, 16, 18\} \\ b_3 &= \{0, 4, 11, 23\} \end{aligned}$$

For the action of $\mathbb{Z}_{40}$ on both points and blocks we use additive notation; thus, e.g., $b_1 + 1 = \{1, 11, 21, 31\}$, and $b_2 - 5 = \{35, 36, 1, 27\} = \{1, 27, 35, 36\}$.

Proposition 5.30. *The blocks obtained from b_i by applying the cyclic group of order 40 form an $S(2, 4, 40)$.*

Proof. The orbit of b_1 has length 10; all other orbits have length 40, thus we obtain a total of 130 blocks.

Two points whose difference is divisible by 10 are contained in a block in the first orbit. For all other differences, we have to check that they appear in a single block. In order to do this we list for each block the differences that appear between elements of this block. We can restrict ourselves to differences between 1 and 19.

$$\begin{aligned} b_2 &: 1, 5, 6, 8, 9, 14 \\ b_3 &: 2, 3, 13, 15, 16, 18 \\ b_4 &: 4, 7, 11, 12, 17, 19 \end{aligned}$$

We see that each difference appears exactly once; thus each pair of points is contained in exactly one block. Thus we have a Steiner system. $\square$

Corollary 5.8. *The system S described above is invariant under the cyclic group of order 40 generated by $x \mapsto x+1$.*

Proposition 5.31. *The system S is invariant under multiplication by 3.*

Proof. The block b_1 is invariant under this multiplication. Multiplying b_2 by 3 yields b_3. Multiplying b_3 by 3 yields $\{0, 8, 9, 14\} = b_2 + 8$. Finally, $b_4 \cdot 3 = \{0, 12, 29, 33\} = b_4 - 11$. $\square$

Corollary 5.9. *The system S is invariant under the group of transformations $x \mapsto ax + b$, $a \in \{1, 3, 9, 27\}$, $b \in \mathbb{Z}_{40}$, of order 160.*

It has been checked by computer that this is the full automorphism group.

Next, we describe just one of the generalized quadrangles. Take the subgroup $H = \langle x \mapsto ax + b \rangle$, $4|b$, $a \in \{1, 9\}$. Take the spread b_1^G together with the orbits of blocks b_4^H and $(b_2 - 1)^H$. Altogether we get 40 blocks, which form a generalized quadrangle. The other three GQ's are constructed in a similar manner.

5.4.3 Discussion

The Steiner designs and their Siamese partitions were obtained using at least one of two approaches. On 15 points, we took the point stabilizer in the group A_5 in its natural action, considered it as a subgroup of the automorphism group of the classical Siamese graph, and found one more graph invariant under this group. Similarly, here we could take a point stabilizer of A_6 and look for mergings. For reasons that will become clear below, we didn't take this subgroup of index 6, but a subgroup of index 10. This has led to some the Siamese graphs.

The other approach is to start from a Steiner system and to check if it permits a Siamese partition. In order to do that, a great number of $S(2, 4, 40)$ have been generated using the Kramer-Mesner method. In particular, all cyclic and 1-rotational Steiner systems were enumerated, as well as all systems invariant under groups of order 36, acting regularly on a subset of the blocks.

For each such system, all embedded GQ's were found, and subsequently all Siamese partitions. We don't know of any case were a system permits two non-isomorphic partitions.

In principle, it should be possible to enumerate all Siamese graphs on 40 points, or at least all geometrical ones.

The following is an outline of an algorithm to construct all geometrical Siamese graphs on 40 points: Let Δ be the point graph of $W(3)$; let S be a fixed spread in Δ, and let $\Gamma = \Delta \setminus S$ be the drg obtained by removing the spread.

Now each geometric srg intersecting Δ in S can be described by an embedding of Γ into $\overline{\Delta}$. Thus we need to find all inequivalent such embeddings which respect S. This gives us all non-isomorphic Siamese pairs.

We need to combine two such pairs in order to get a complete Siamese graph. Each embedding $g \in Aut(S)$ defines a regular graph $B = \Gamma \cup \Gamma^g$ of valency 18. If we take the complement of this graph and remove the spread, we get another such graph. For this graph to admit a partition into two drg's, it is necessary and sufficient that it be isomorphic to one of the graphs B constructed before. Thus, each isomorphism between a graph B and the complement (modulo the spread) of a graph B' yields a Siamese graph, and conversely, each geometric Siamese graph is obtained this way.

For an efficient implementation, some clever way has to be found to enumerate the double cosets $Aut(\Gamma)\backslash Aut(S)/Aut(\Gamma)$.

All Siamese Steiner systems constructed so far share the following properties: They are resolvable, and they do not contain maximal arcs. So far, there is no theoretical explanation for this.

5.5 Further computations

5.5.1 85 points

Since we found Siamese graphs related to $W(2)$ and $W(3)$, it is natural to consider $W(4)$ on 85 points. Several approaches were used to search for Siamese graphs related to $W(4)$; only one example was found; it is an association scheme, and will be discussed in the next section.

The most promising approach was to take a subgroup of small index in the automorphism group of this classical example, and to search for graphs invariant under this subgroup. Whereas this worked well on 15 and 40 points, there are no new examples on 85 points. However, this line of research is still in its beginning.

It might be reasonable to try and enumerate all geometric Siamese graphs on 85 points. However, no efficient algorithm for this task has been created yet.

It is reasonable to guess that there are Siamese graphs on 85 points which are not association schemes. However, no evidence for or against this conjecture has been found.

5.6 A computational approach to an infinite series

The classical examples on 15 and 40 points are in fact much more related than it appears from their description. We used the alternating groups A_5 and A_6 in order to get a nice combinatorial description of the association schemes. However, this language is not suitable for a generalization.

A different language arises from two classical exceptional isomorphisms between simple groups: $A_5 \cong PSL(2,4)$, and $A_6 \cong PSL(2,9)$. This leads to the conjecture on the existence of an infinite series.

Consider the group $G = PSL(2,q^2)$. It has a subgroup H of index $(q^2+1)(q+1)$, which is a subgroup of $AGL(1,q^2)$ of the form $E_{q^2}.S$, where S is the set of all non-zero squares in $\mathbb{F}_q$.

We consider the action of G on G/H. The corresponding centralizer algebras have been investigated for $q = 2,3,4,5,7$; in all cases we get a Siamese association scheme. Information about these schemes is presented below.

q	Subdegrees	$Aut(cgr)$	$Aut(drg)$	$Aut(srg)$
2	$1^3, 4^3$	$PSL(2,4)$	S_5	S_6
3	$1^4, 9^4$	$PSL(2,9).2$	$S_6 \times S_2$	$Aut(O(5,3))$
4	$1^5, 16^5$	$PSL(2,16)$	$PSL(2,16).Z_4$	$Aut(O(5,4))$
5	$1^6, 25^6$	$PSL(2,25)$	$PSL(2,25).E_4$	$Aut(O(5,5))$
7	$1^8, 49^8$	$PSL(2,49)$	$PSL(2,49).2$	$Aut(O(5,7))$

5.7 Towards an infinite series of Siamese association schemes

Here we present an outline of a proof of the existence of an infinite series of Siamese association schemes. We give sketches of proofs for a few initial steps.

We base our construction on the group $PSL(2,q^2)$ in a transitive action of degree $(q+1)(q^2+1)$. We consider the association scheme formed by the 2-orbits of this action. This scheme belongs to larger class of schemes previously considered by Mathon [70] and Biggs [8].

5.7.1 A first model

Throughout this section, let q be a prime power, and let $G = PSL(2,q^2)$. We could give a purely algebraic description of its action on the cosets of a certain subgroup. However, the structure of this subgroup depends slightly on q (in particular, on its parity). Therefore, we prefer a more geometrical description.

Construction 1 (Complete projective model). Let q be a prime power. Consider the projective line $PG(1,q^2)$ of order q^2. We label the points naturally by $\mathbb{F}_{q^2} \cup \{\infty\}$ and note that G acts naturally on this point set.

We define a (possibly directed) graph on the affine points of $PG(1,q^2)$, i.e., the elements of $K = \mathbb{F}_{q^2}$. Let $(x,y) \in K^2$ be an arc if $y-x$ is a non-zero square in $F = \mathbb{F}_q$. Denote this graph by γ.

Let $V = \gamma^G$. Let $\mathcal{M} = (V, 2-\mathrm{orb}(G,V))$.

Proposition 5.32. *The set $\mathcal{M}$ has cardinality* $(q+1)(q^2+1)$

Proof. The stabilizer of γ in G consists of the affine transformations of the form $x \mapsto ax+b$, where a is a non-zero square in F. The group formed by these transformations has the required index in $PSL(2,q^2)$. □

The picture becomes a bit clearer if we look at the connectivity components of γ. If two points x, y are connected, then $x - y \in F$. Thus the connectivity components are (possibly a refinement of) the additive cosets of F in K. The additive group $(K,+)$ acts transitively, so all components are isomorphic.

If we look at γ restricted to F, we find that it is in fact connected. Two points are connected if their difference is a square; this leads to well-known graphs. If q is even, this is the complete graph K_q. If $q \equiv 1 \mod 4$, then -1 is a square, so we get an undirected graph, the Paley graph P_q. If $q \equiv 3 \mod 4$, we get the so-called Paley tournament, a directed graph. So, γ has q connectivity components of size q, and one isolated point.

Remark 5.1. Note that in the case $q = 2$, γ consists of two non-incident edges. Note also that $PSL(2,4) = A_5$. Hence this construction is a generalization of the one described in Section 5.2.

Remark 5.2. The construction in Section 5.3 is different from the one given here; there we started from the natural action of A_6 on 6 points, here we start from the action of $PSL(2,9)$ on 10 points. Note, however, that $A_6 \cong PSL(2,9)$. It can be checked that the resulting color graphs are isomorphic.

5.7.2 The resulting association scheme

Our goal here is to investigate the association scheme $\mathcal{M} = (V, 2\text{-orb}(G,V))$. First, we give a second construction.

Construction 2 (Incomplete projective model). This goes back to a construction by Neumaier.

Let q be a prime power. Let $K = \mathbb{F}_{q^2}$, $F = \mathbb{F}_q$. Let $U = F^4 = K^2$. Let $V = PG(3, q)$. We can write the points of V in the form $\langle(a, b)\rangle$, where $a, b \in K$.

Let $C = F^* \leq K^*$. Let β be a primitive element of F, and for $i = 1, \ldots, q+1$, let $C_i = \beta^i C$ be the cosets of C in K^*. Let $C_0 = \{0\}$.

Define a color graph $W = (V, \phi)$ by setting

$$\phi(\langle(a,b)\rangle, \langle(c,d)\rangle) = i \quad \Longleftrightarrow \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in C_i$$

This gives a colored graph of order $q^3 + q^2 + q + 1$ and rank $q + 2$.

Lemma 5.4. *This color graph is well-defined.*

Proof. Let $x, y \in F$. Suppose, $\phi(\langle(a,b)\rangle, \langle(c,d)\rangle) = i$. Then

$$\begin{vmatrix} xa & xb \\ yc & yd \end{vmatrix} = xy \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in xyC_i = C_i,$$

so $\phi(\langle(xa, xb)\rangle, \langle(yc, yd)\rangle) = \phi(\langle(a,b)\rangle, \langle(c,d)\rangle)$ ☐

Lemma 5.5. *The color graph W is invariant under the group $SL(2, q^2)$.*

Proof. Let $A \in SL(2, q^2)$. Then

$$\begin{aligned} \begin{vmatrix} (a \quad b)A \\ (c \quad d)A \end{vmatrix} &= \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot A \right| \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} |A| \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

thus, $\phi(\langle(a,b)A\rangle, \langle(c,d)A\rangle) = \phi(\langle(a,b)\rangle, \langle(c,d)\rangle)$. ☐

Proposition 5.33. *The incomplete model is isomorphic to a merging of the complete model.*

Idea. Still needs some work. We need to take the WL-Stabilization of the incomplete model, since this is not an association scheme. Stabilizations splits the spread into relations of valency 1. □

Theorem 5.6. *$\mathcal{M}$ is an association scheme of rank $2q+2$, with subdegrees $1^{q+1}, (q^2)^{q+1}$. $\mathcal{M}$ is invariant under $PSL(2, q^2)$. The union of the basis relations of subdegree 1 constitutes a spread.*

Idea. The basis graph Γ_0 in W is a spread. To see this, note that $\phi(x, y) = 0$ if x and y are linearly dependent; this is an equivalence relation on the non-zero vectors. Each equivalence class contains $q^2 - 1$ vectors; they correspond to $q + 1$ projective points. Thus each class in the spread has $q+1$ points, and we have q^2+1 classes. □

5.7.3 Information about the normalizer

Let β be the primitive element of $K = \mathbb{F}_{q^2}$ defined above. Define a permutation g on V by $\langle(a, b)\rangle^g = \langle(\beta a, b)\rangle$. Let $M = \langle g\rangle$ be the cyclic group of order $q + 1$.

Lemma 5.6. *M is a subgroup of the normalizer of $\mathcal{M}$.*

Proof. Let $\langle(a, b)\rangle$ and $\langle(c, d)\rangle$ be points in V. Then

$$\begin{vmatrix} \beta a & b \\ \beta c & d \end{vmatrix} = \beta \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

thus $\phi(\langle(a, b)\rangle^g, \langle(c, d)\rangle^g)$ depends only on $\phi(\langle(a, b)\rangle, \langle(c, d)\rangle)$. Hence, g permutes the basis relations of valency q^2, and it is contained in the normalizer of $\mathcal{M}$. □

Lemma 5.7. *M acts transitively on the basis relations of valency q^2.*

Proof. The generator g permutes the cosets C_i cyclically. $\square$

Corollary 5.10. *The connected basis graphs are pairwise isomorphic.*

Lemma 5.8. *Each projective line is monochromatic.*

Proof. Follows from the properties of the determinant. $\square$

Corollary 5.11. *Each basis graph Γ_i is the point graph of a partial linear space of order $(q+1, q)$ formed by a set of projective lines.*

In summary, we have an association scheme which has the subdegrees of a Siamese scheme. In fact, $\mathcal{M}$ is Siamese.

5.7.4 Important embeddings

With this action, we have an embedding of $PSL(2, q^2)$ into the symmetric group of degree $v = (q^2+1)(q+1)$. In fact, we can get a chain of classical subgroups of S_v:

- $PSL(2, q^2) \leq PSp(4, q)$, since, by a result of Dye [28], $PSL(2, q^2)$ is contained in the stabilizer of a spread consisting of totally isotropic lines with respect to a symplectic polarity; in other words, a spread in the classical generalized quadrangle $W(q)$.
- $PSp(4, q) \leq P\Gamma L(4, q)$, by construction;
- $P\Gamma L(4, q) \leq S_v$.

Proposition 5.34. *The association scheme defined by $PSp(4, q)$, which is a merging of our scheme, gives a strongly regular graph, namely the point graph of $W(q)$.*

The embedding of $PSp(4, q)$ into $P\Gamma L(4, q)$ gives an embedding of $W(q)$ into $PG(3, q)$.

Now, applying the Singer cycle, we obtain $q+1$ copies of $W(q)$ which pairwise intersect in a spread. Thus we have

Theorem 5.7. *The color graph invariant under the reference action of $PSL(2, q^2)$, as it appears in Chapter 5, defines a Siamese association scheme on $v = (q^2+1)(q+1)$ points.*

5.8 Siamese graphs: Relaxation of axiomatics

5.8.1 A generalization

We can consider a Siamese color graph as a nice decomposition of the complete graph. We have seen that such graphs exist only for orders corresponding to graphs related to generalized quadrangles.

In order to get similar examples for other orders we need to relax our requirements. We want to keep the following properties of the decomposition:

- There are many srg's with the same parameters, intersecting pairwise in a common spread.
- Deleting the spread from any srg yields a 3-class association scheme (not necessarily metrical).
- There are as many srg's as possible.

More accurately, we have the following: Consider a partition of the complete graph K_n into relations $S, T_1, \ldots, T_r, D$ such that

- S is a spread;
- $T_i \cup S$ is an srg for $i = 1, \ldots, r$; the parameters of this srg do not depend on i;
- For each i, S and T_i are basis relations of a 3-class association scheme;
- D is a drg of diameter 3; its valency is smaller than the subdegree of T_i.

If each T_i is distance regular, and D is empty, we get exactly the definition of a Siamese color graph. We call any object satisfying this modified set of axioms a *weakly Siamese color graph.* In similar manner we can define weakly Siamese association schemes.

5.8.2 Discussion of the axioms

The idea of such relaxed axioms goes back to Kharaghani and Torabi [60], although they did not provide an explicit formulation.

5.8.3 Example on 63 points

We start with the group $G = P\Gamma L_2(8)$ of order 1512, acting naturally on 9 points. G contains one conjugacy class of involutions; the centralizer of an involution in G is $A_4 \times 2$ of order 24, hence, there are 63 involutions.

If we let G act on this conjugacy class, we obtain a rank 6 action. Its subdegrees are $1, 3^2, 8, 24^2$.

Investigating the corresponding association scheme with COCO, we find the following:

- The two relations of subdegree 3 together form a spread $9 \circ K_7$.
- There are two non-isomorphic srg's of valency 30.
- There are two association schemes with 3 classes and subdegrees $1, 6, 24, 32$.
- There is one antipodal drg of valency 8.

Since this association scheme is Schurian, we cannot claim that it is new. However, to our knowledge the information presented above has not been discussed.

There is one more link between this scheme and the known Siamese association schemes, since they all appear from actions of the group $PSL_2(q)$. A more systematical investigation is under way in collaboration with A.J. Woldar.

Chapter 6

COHERENT CONFIGURATIONS WITH A FEW FIBERS

In this chapter, we consider incidence structures, or more generally diagram geometries, which give rise to coherent configurations ("coherent incidence structures"). As examples, we investigate partial geometries and maximal arcs in projective planes.

This leads to a generalization of an idea by Bruck to construct projective planes of order n starting from a Steiner system. Special attention is given to the case $n = 12$.

As a further example of a diagram geometry we define square root partial linear spaces, which are related to generalized quadrangles. In particular, we give an accurate model of an example on 96 points, constructed recently by Brouwer, Koolen, and Klin [17].

6.1 More on coherent configurations

Let W be a coherent configuration with fibers $F_0, F_1, \ldots, F_{f-1}$. Each basic relation R_k is a subset of $F_a \times F_c$ for some a, c. We say in this case that R_k is of type (a, c).

Let $(x, y) \in R_k$, and let z be a point in F_b. Then $(x, z) \in R_i$, where R_i is of type (a, b), and $(z, y) \in R_j$, where R_j is of type (b, c).

We thus have the following:

Proposition 6.1. *If p_{ij}^k is a non-zero structure constant of W, then there are number a, b, c such that R_i is of type (a, b), R_j is of type (b, c), and R_k is of type (a, c).*

We call the triple (a, b, c) the *type* of the structure constant p_{ij}^k. Arranging structure constants by type saves us some work and space, since we can omit the consideration of constants that are trivially zero since they don't have a well-defined type.

We will use the following notation for the full tensor of structure constants of a coherent configuration. For each type (a, b, c), we give a table of the following form:

(a, b, c)		j_1	$\cdots$	j_m	
i_1	k_1	$p_{i_1 j_1}^{k_1}$	$\cdots$	$p_{i_1 j_m}^{k_1}$	n_{i_1}
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
	k_n	$p_{i_1 j_1}^{k_n}$	$\cdots$	$p_{i_1 j_m}^{k_n}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i_l	k_1	$p_{i_l j_1}^{k_1}$	$\cdots$	$p_{i_l j_m}^{k_1}$	n_{i_l}
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
	k_n	$p_{i_l j_1}^{k_n}$	$\cdots$	$p_{i_l j_m}^{k_n}$	
		$n_{j'_1}$	$\cdots$	$n_{j'_m}$	

where $i_1, \dots i_l$ are the colors of type (a, b), $j_1 \dots j_m$ are of type (b, c), and $k_1, \dots, k_n$ are of type (a, c). We include the subdegrees n_i and $n_{j'}$ as checksums (see Proposition 2.2).

6.2 Diagram geometries and incidence structures

Consider an incidence structure (P, B, I), where $I \subseteq P \times B$ is the incidence relation. As usual, we refer to the elements of P as points, and the elements of B as blocks. Let $X = P \cup B$.

The incidence relation I allows us to define naturally other binary relations on $X \times X$:

- I and $\bar{I} = (P \times B) \setminus I$ give a partition of $P \times B$.

- Taking the transposes I^t and $\bar{I}^t$, we get a partition of $B \times P$.
- We have the identity relations Id_P and Id_B on P and B.
- For each number i, we say that two distinct blocks (b_1, b_2) are in relation Q_i if $|b_1 \cap b_2| = i$. This gives a partition of $(B \times B) \setminus Id_B$ into symmetric antireflexive relations.
- Similarly, we say that two distinct points (p_1, p_2) are in relation S_j if there are exactly j blocks containing both of them.

Altogether, we obtain a partition of $X \times X$, i.e., a disjoint union

$$X \times X = Id_P \cup Id_B \cup I \cup \bar{I} \cup I^t \cup \bar{I}^t \cup \bigcup S_i \cup \bigcup Q_j \tag{6.1}$$

Note that some of these relations may be empty.

Definition 6.1 (Coherent incidence structure). We say that an incidence structure (P, B, I) is *coherent* if the non-empty relations defined in Equation 6.1 form a coherent configuration (with 2 fibers)

$$W = (X, \{Id_P, Id_b, I, \bar{I}, I^t, \bar{I}^t, S_0, S_1, \ldots, Q_0, Q_1, \ldots\}).$$

Example 6.1. Let $(P, B, P \times B)$ be the trivial incidence structure, i.e., all points are incident with all blocks. Let $v = |P|$, $b = |B|$. We get a coherent configurations with two fibers $F_1 = P$, $F_2 = B$. We label the structure constants as follows:

Type $(1,1)$: $R_0 = Id_P$, $R_1 = S_b = P \times P \setminus R_0$;

Type $(1,2)$: $R_2 = P \times B$;

Type $(2,1)$: $R_3 = B \times P$;

Type $(2,2)$: $R_4 = Id_B$, $R_5 = Q_v = B \times B \setminus R_4$.

Then the system of relations $(P \cup B, \{R_i\})$ is a coherent configuration with the non-trivial structure constants given in Table 6.1

Table 6.1: Structure constants of the trivial incidence structure

$(1,1,1)$	k	0	1	
0	0	1	0	1
	1	0	1	
1	0	0	$v-1$	$v-1$
	1	1	$v-2$	
		1	$v-1$	

There are two "classical" non-trivial examples of coherent incidence structures: symmetric designs and partial geometries.

Proposition 6.2. *A symmetric $2-(v,k,\lambda)$ design is a coherent incidence structure of rank 8.*

Proof. Let $F_0 = P$ be the set of points, $F_1 = B$ the set of blocks. We have the following 8 basic relations:

Type $(0,0)$: $R_0 = Id_P$, $R_1 = S_\lambda$;

Type $(0,1)$: $R_2 = I$, $R_3 = (P \times B) \setminus R_2$;

Type $(1,0)$: $R_4 = I^t$, $R_5 = (B \times P) \setminus R_4$;

Type $(1,1)$: $R_6 = Id_P$, $R_7 = Q_\lambda$.

The induced configurations on F_0 and F_1 are trivial of rank 2, hence we need not count the structure constants of types $(0,0,0)$ and $(1,1,1)$. Since the axioms of symmetric designs are self-dual, it suffices to consider types $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ — we get the other three types by switching points and blocks. Finally, we can use symmetry properties of the structure constants to obtain the constants of type $(1,0,0)$ from those of type $(0,0,1)$.

Thus we need only consider types $(0,0,1)$ and $(0,1,0)$.

Type $(0,0,1)$: Let x be a point, z be a block. If $x \in z$, then there are $k-1$ other points in z; the remaining $v-k$ are outside of z.

If $x \notin z$, then there are k points in z, and the remaining $v-k-1$ are outside z. We get the structure constants

$(0,0,1)$		2	3	
0	2	1	0	1
	3	0	1	
1	2	$k-1$	$v-k$	$v-1$
	3	k	$v-k-1$	
		k	$v-k$	

Type $(0,1,0)$: Let x and z be points. If $x = z$, then there are k blocks through x, and $v-k$ blocks not through x.

If $x \neq z$, then there are λ blocks through both of them, $k-\lambda$ blocks through x only, the same number of blocks through z only. The remaining $v-2k+\lambda$ blocks contain neither x nor z. Thus we get the following structure constants:

$(0,1,0)$		4	5	
2	0	k	0	k
	1	λ	$k-\lambda$	
3	0	0	$v-k$	$v-k$
	1	$k-\lambda$	$v-2k+\lambda$	
		k	$v-k$	

Since we determined all structure constants, this completes the proof. □

Proposition 6.3. *A partial geometry $pg(K,R,T)$ is a coherent incidence structure of rank 10.*

Proof. Let (P,B,I) be a partial geometry with parameters K,R,T. The resulting coherent configuration will have the fibers $F_0 = P$ and $F_1 = B$. We use the following labelling for the colors:

Type $(0,0)$: $R_0 = Id_P$. $R_1 = S_1$ is collinearity, $R_2 = S_0$ is non-collinearity.

Type $(0,1)$: $R_3 = I$ is incidence, R_4 is non-incidence.

Type $(1,0)$: $R_5 = R_3^t$, $R_6 = R_4^t$.

Type $(1,1)$: $R_7 = Id_B$, $R_8 = Q_1$ is concurrence, $R_9 = Q_0$ is non-concurrence.

We now determine the structure constants, organized by type. Again, by using the self-duality of the axioms and the symmetry properties of structure constants, it suffices to consider the types $(0,0,0)$, $(0,0,1)$, and $(0,1,0)$.

$(0,0,0)$: We have seen in Theorem 2.2 that the point graph is strongly regular, with parameters

$$\begin{aligned} v &= K\left(1 + \frac{(K-1)(R-1)}{T}\right) \\ k &= R(K-1) \\ l &= \frac{(K-T)(K-1)(R-1)}{T} \\ \lambda &= (K-2) + (R-1)(T-1) \\ \mu &= RT \end{aligned}$$

This gives us the following structure constants:

$(0,0,0)$		0	1	2	
0	0	1	0	0	1
	1	0	1	0	
	2	0	0	1	
1	0	0	k	0	k
	1	1	λ	$k-\lambda-1$	
	2	0	μ	$k-\mu$	
2	0	0	0	l	l
	1	0	$k-\lambda-1$	$l-k+\lambda+1$	
	2	1	$k-\mu$	$l-k+\mu-1$	
		1	k	l	

Type $(0,0,1)$: Let x be a point, z be a block. If x and z are incident, hence $(x,z) \in R_3$, then there are $R(K-1)$ points collinear with x, $K-1$ of which lie on z. None of the points not collinear with x lie on z.

If x and z are not incident, then z contains T points collinear with x, and $K-T$ points not collinear with x. Thus we get the structure constants

$(0,0,1)$		3	4	
0	3	1	0	1
	4	0	1	
1	3	$K-1$	$(R-1)(K-1)$	$R(K-1)$
	4	T	$RK-K-T$	
2	3	0	$(K-T)\frac{(K-1)(R-1)}{T}$	$(K-T)\frac{(K-1)(R-1)}{T}$
	4	$K-T$	$(K-T)\left(\frac{(K-1)(R-1)}{T}-1\right)$	
		K		

Type $(0,1,0)$: Let x, z be points. If $x = z$, then there are R blocks incident to them. If x and z are collinear, then there is exactly one block incident to both of them. There are $R-1$ blocks incident to x only, and the same number of blocks incident to z only. Since the total number of blocks is $R(1+(K-1)(R-1)/T)$, the remaining $R(K-1)(R-1)/T+1$ blocks contain neither x nor y.

If x and z are not collinear, then there are no blocks incident to both. There are R blocks to each of them; the remaining $R((K-1)(R-1)/T-1)$ contain neither.

$(0,1,0)$		5	6	
3	0	R	0	R
	1	1	$R-1$	
	2	0	R	
4	0	0	$K(R-1)(K-1)/T$	$K(R-1)(K-1)/T$
	1	$R-1$	$R(K-1)(R-1)/T+1$	
	2	R	$R\left(\frac{(K-1)(R-1)}{T}-1\right)$	
		R	$K(R-1)(K-1)/T$	

We get the structure constants of the remaining types by duality and symmetry. Hence, we get a coherent configuration. □

So far, we have considered incidence structures; these can be considered as special cases of diagram geometries (of rank 2). We can generalize our definition to diagram geometries of arbitrary rank.

Remember that a diagram geometry is a tuple $(F_0, F_1, \ldots, F_r)$ of sets (fibers), with incidence relations defined on each pair of fibers. Thus, (F_i, F_j) is an incidence structure for $i \neq j$; hence as before it defines natural binary relations on F_i. Letting j vary, and taking the intersections of all relations on F_i defined by F_j, we get a set of relations on F_i naturally defined by the diagram geometry $(F_0, F_1, \ldots, F_r)$.

Definition 6.2. A diagram geometry $\Gamma = (F_0, F_1, \ldots, F_r)$ is called *coherent* if the set of its natural relations forms a coherent configuration W. The rank d of W is called the *coherent rank* of Γ.

6.3 Square root partial linear spaces

In [17], several new srg$(96, 20, 4, 4)$ are constructed. It was later discovered that one of these graphs is the point graph of a partial linear space of order $(4, 4)$. No theoretical description of this graph was provided; moreover, there was a mistake in the identification of its automorphism group (the origin of this mistake will be discussed below).

Therefore, here we make a new attempt to describe this graph and its properties.

6.3.1 Axiomatics

Let P, B, and L be three finite sets. Define incidences on (P, L) and (P, B) such that

- (P, L) is a partial linear space of order (s, s).
- (P, B) is a symmetric (v, k, λ)-design.

We will refer to the elements of P, L, and B as points, lines, and blocks, respectively. We will identify the elements of L and B with $(s+1)$-subsets and k-subsets of P.

Now we define a third incidence structure on (L, B) by saying that a block and a line are incident if the intersection has size s.

We now require that (L, B) is a partial linear space of order (s, s). Moreover, given a block b and all lines incident with b, these lines intersect in a common point $p \notin b$. Finally, the intersections of these lines with b cover all of b.

6.3.2 Numerical constraints

Lemma 6.1. *The three sets P, B, L have the same size, namely v.*

Proof. Since (P, B) is a (v, k, λ) design, we see that $|P| = |B| = v$. Since (P, L) is a partial linear space with $s = t$, $|P| = |L|$. □

Lemma 6.2. *The block size k equals $s(s+1)$.*

Proof. Let b be a block. There are $s+1$ lines incident to b, and they intersect in a point not in b. Each of these lines intersects b in s points, and these intersections are disjoint and cover b. Hence, $|b| = s(s+1)$. □

Lemma 6.3. *The design (P, L) has a polarity without absolute points; i.e, there is a bijection $\phi : P \to L$ such that for all $p \in P$, $p \notin \phi(p)$.*

Proof. For each block b there is a point P such that the lines incident with b are also incident with P. This defines a mapping $\psi : B \to P$; moreover $\psi(b) \notin b$. Since any two points are incident with at most one common line, this mapping is one-to-one, and hence onto. Setting $\phi = \psi^{-1}$ gives the required mapping. □

Corollary 6.1. *The graph Γ defined on P by setting $x \sim y$ if $y \in \phi(x)$ is strongly regular, with parameters $v, k, \lambda, \mu = \lambda$.*

Corollary 6.2. *The point graph of the partial linear space (P, L) is equal to Γ.*

Proof. Both graphs are regular with valency $k = s(s+1)$. Suppose that x and y are adjacent in Γ. That means that $y \in \phi(x)$, and hence there is a line through both x and y. Thus x and y are collinear in the partial linear space. □

Corollary 6.3. $s - 1 \leq \lambda$.

Proof. Any two collinear points in the partial linear space have at least $s-1$ common neighbors, namely the points contained in the line connecting them. Thus $\lambda \geq s - 1$. □

Lemma 6.4. *The case $\lambda = s - 1$ is not possible.*

Proof. In a strongly regular graph, we have that $\mu(v-k-1) = k(k-\lambda-1)$. In our case, this means that

$$\begin{aligned}
\lambda(v-k-1) &= k^2 - k\lambda - k \\
\lambda(v-1) &= k(k-1) \\
v-1 &= \frac{k(k-1)}{\lambda} \\
&= \frac{(s^2+s)(s^2+s-1)}{s-1} \\
&= \frac{s(s+1)(s^2+s-1)}{s-1} \\
&= s\frac{s^3+s^2+s^2+s-s-1}{s-1} \\
&= s\frac{s^3+2s^2-1}{s-1} \\
&= s^3+3s^2+3s+2+\frac{2}{s-1}
\end{aligned}$$

If $s-1$ divides 2, $s = 2, 3$. If $s = 2$, then $v = 31$. If $s = 3$, then $v = 67$. Strongly regular graphs of these orders do not exist. $\square$

Corollary 6.4. $\lambda \geq s$.

We will call the case $\lambda = s$ a *tight square root partial linear space.* Since we will only consider the tight case here, we will usually omit the word "tight".

The choice of the term "square root" stems from the fact that by our axioms, we have a mechanical combination of two partial linear spaces, which in fact are basically identical. Combining these two spaces yields a strongly regular graph, thus we can consider the srg as the square of the pls; conversely, the pls is the square root of the srg.

6.3.3 Parameterization of srpls(s,s)

We get the following expressions for the parameters of the point graph:

Proposition 6.4. *The point graph* Γ *of an srpls*(s,s) *is a strongly regular graph with parameters* $v = s^2(s+2)$, $k = s(s+1)$, $\lambda = \mu = s$

Proof. The expressions for λ and μ follow from the definition of a (tight) srpls. We have $k = s(s+1)$ for any point graph of a pls(s,s).

Above we have seen that $v - 1 = k(k-1)/\lambda$. Hence

$$\begin{aligned} v-1 &= \frac{k(k-1)}{\lambda} \\ &= \frac{(s^2+s)^2-(s^2+s)}{s} \\ &= s(s+1)^2-(s+1) \\ &= s^3+2s^2+s-s-1 \\ &= s^3+2s^2-1 = s^2(s+2)-1 \end{aligned}$$

□

Corollary 6.5. Γ *is a pseudo-geometric* $GQ(s+1,s-1)$*-graph.*

Thus we get an infinite, one-parametric series of possible parameters.

Proposition 6.5. *If* $s > 1$, *then* Γ *is not the point graph of a generalized quadrangle.*

Proof. Suppose that Γ is the point graph of a generalized quadrangle $GQ(s+1,s-1)$. By definition, Γ is the point graph of a $pls(s,s)$.

Since any two collinear points in a GQ determine a unique line, each line of the partial linear space is contained in a line of the GQ. Since the partial linear space contains $v = s^2(s+2)$ lines, whereas the GQ contains $s^3 < v$ lines, at least two lines of the pls are contained in the same line of the GQ. Hence, they share s points, which is a contradiction if $s > 1$. □

6.3.4 Small examples

If we consider the case $s = 1$, we have $v = 3$, $k = 2$, $\lambda = \mu = 1$. This is a degenerate case, where the graph is the triangle, the pls is formed by the edges of the triangle; the design does not appear.

For $s = 2$, we get $v = 16$, $k = 6$, $\lambda = \mu = 2$. There are exactly two strongly regular graphs with these parameters, the lattice graph $L_2(4)$, and the Shrikhande graph. The former is the point graph of a $GQ(3, 1)$, whereas the latter is the point graph of a pls$(2, 2)$.

The Shrikhande graph can be described as follows: Consider the cyclic Latin square of order 4:

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Vertices are the entries of this square. Two vertices are adjacent if they are in the same row, the same column, or if they contain the same entry. This graph can be embedded into the torus. Such an embedding is given in Table 6.2; here, any two vertices with the same label are identified.

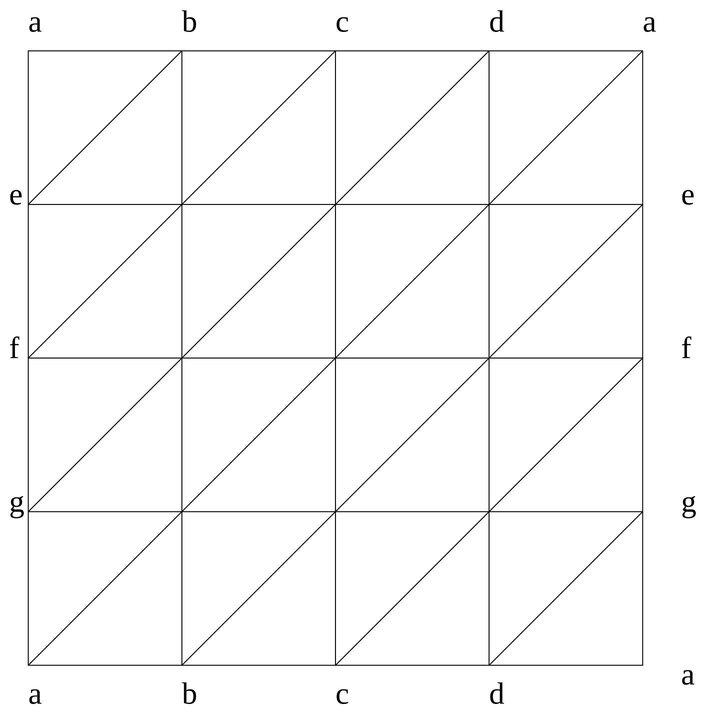
We can see in this picture that the graph contains 32 triangles, which are naturally divided into two parts, the "upper left" and the "lower right" part. Choosing either of these sets of 16 triangles as lines we obtain a $pls(2, 2)$.

In fact, we get a srpls$(2, 2)$; the corresponding design is a biplane of order 16.

For $s = 3$, we get the parameters of a $GQ(4, 2)$ on 45 points. The corresponding pseudo-geometric graphs were completely enumerated by E. Spence [92]; there are up to isomorphism 78 of them. Using GAP [38] and Grape [91], we checked that none of them is the point graph of a $pls(3, 3)$. Thus we have

Proposition 6.6. *An srpls$(3, 3)$ does not exist.*

Table 6.2: The Shrikhande graph



6.3.5 $s = 4$: A brief history

The results from Section 2.7 led Godsil and Hensel [40] to the following question: Are there pseudo-geometric $GQ(s,t)$ graphs with spreads, which are not geometric, and hence lead to new distance regular graphs?

M. Klin constructed two srg$(96, 20, 4, 4)$ and reported them at a meeting in Nahsholim as examples of partial difference sets [63]. He once more presented one of them during the Conference on Association Schemes, Codes and Designs in Pohang, Korea, July 2000, as a positive answer on the question by Godsil-Hensel.

During that conference, it was realized that the graph in question had been constructed before by Brouwer and Koolen, although this result had never been published. Joining efforts, they constructed further examples of srg's on 96 points (of valency 19 and 20), and discovered that one of the original graphs is the point graph of a $pls(4,4)$ [17].

6.3.6 Descriptions of the icosahedron

The icosahedron I is a distance regular graph of order 12, valency 5, and diameter 3. There are several ways to describe it (cf. [16])

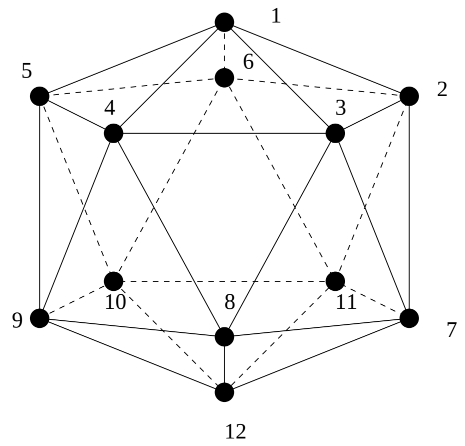
The classical way is related to the platonic solids. Take the unique regular polyhedron consisting of 20 triangles. Take its skeleton, i.e., its vertices and edges, to obtain the graph I. All symmetries of the graph are rotations and reflections of euclidean 3-space. However, to work with this model accurately, we would have to deal with twelve points in $\mathbb{R}^3$, most of whose coordinates are irrational.

Another construction can be characterized as algebraical. Consider the alternating group A_5 in its natural action on five elements. It contains 24 5-cycles, distributed to two conjugacy classes of size 12. Let C_1 be one such class, and let the elements of C_1 be the vertices of a graph.

Join two vertices x and y by an edge if xy^{-1} has order 3. We obtain a graph I. Evidently, I is invariant under A_5. It is moreover invariant under the mapping $x \mapsto x^{-1}$. In fact, $Aut(I) = A_5 \times S_2$.

6.3.7 Some properties

We start with a labeled copy I_0 of the icosahedron. We borrow the labelling from [64], where more information about its symmetries can be found.

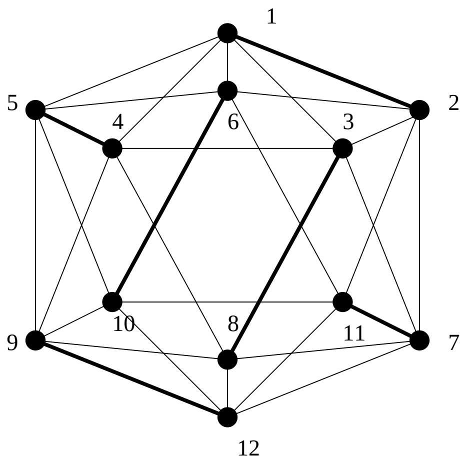


Pairs of antipodal vertices are $\{\{1, 12\}, \{2, 9\}, \{3, 10\}, \{4, 11\}, \{5, 7\}, \{6, 8\}\}$.

The icosahedron contains 27 orbits of Hamiltonian cycles, of lengths 20, 60, and 120 [34]. There is exactly one orbit of length 20, such that stabilizer of a representative in $G = Aut(I_0)$ is a cyclic group of order 6. We will call such a Hamiltonian cycle *face-antipodal.*

The icosahedron has 125 1-factors, forming orbits of lengths 5, 10, 20, 30, and 60. (This information has been obtained using GAP [38]). The orbit of length 5 can be described as follows:

Take a set of edges whose midpoints form a regular octahedron.



We will call these "good 1-factors".

6.3.8 Icosahedra with the same antipodal system

Altogether, there are many different labellings of the icosahedron by the numbers 1 through 12. We will restrict ourselves to those labelling which have the same antipodal system as I_0.

Proposition 6.7. *There are 384 labelled icosahedra with a given antipodal system.*

Proof. If we consider a given antipodal system as a graph $6 \circ K_2$, its automorphism group is $H = S_2 \wr S_6$ of order $64 \cdot 6!$. Each icosahedron has an automorphism group of size $5!$. Thus, the orbit of icosahedra under H has length $64 \cdot 6$. $\square$

Here, we chose a short group-theoretical proof; however this can also be shown directly and combinatorially.

If we take the distance two graph I_1 of I_0, we get another copy of the icosahedron with the same antipodal system. Moreover, the distance two graph of I_1 is I_0. We will call $\{I_0, I_1\}$ a pair of *twins*. Thus we get

Proposition 6.8. *The set of 384 labellings is partitioned into 192 pairs of twins.*

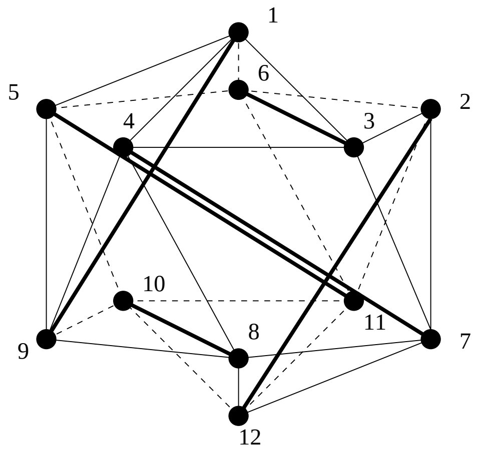
Let Ω be this set of pairs of twins. These 192 elements will be the points and lines of our future srpls$(4, 4)$, which we denote by D.

6.3.9 A natural model of D

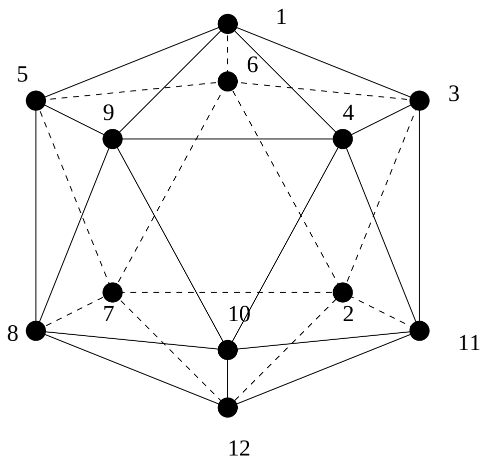
Given an icosahedron and a good 1-factor, we define a new graph as follows: Remove the edges of the 1-factor, and join each point to the antipodal mate of its lost neighbor.

In the tradition of Mathematical Chemistry, we call this process a *reaction* on the original icosahedron.

The result of this reaction is shown below.



If we rearrange the points of the result of the reaction, we see that we get again an icosahedron.



If we start with an icosahedron $\mathcal{I}$, our reaction gives a set N of new icosahedra. If we start with the twin $\mathcal{I}'$, we obtain a different set N'. It turns out that the elements of N' are the twins of the elements of N. Hence we can redefine the reaction as acting on pairs of twins.

We now define the reaction graph $\mathcal{R}$ as follows: Vertices are the 192 pairs of twins with a given antipodal system. Two vertices are joined by an edge if the above reaction moves one to the other.

If we investigate the obtained reaction graph $\mathcal{R}$, we can see the following:

- $\mathcal{R}$ is regular of valency 5;
- it is connected;
- it is bipartite;
- it is vertex-transitive;
- its girth is 6.

From the observations above we conclude the following: $\mathcal{R}$ is the Levi graph of a $pls(4,4)$. Its point graph Γ is strongly regular, and in fact it is a pseudo-$GQ(3,5)$-graph. Hence, we have an $spls(4,4)$.

6.3.10 Description of the group

If we take the set of all icosahedra with a given antipodal system the group $H = S_6 \wr S_2 = Aut(6 \circ K_2)$ of order $6!2^6$ acts naturally on them. If we let it act on sets of twins, the action is not faithful, and we get a group which is twice smaller. Thus we get H/S_2 as a group of automorphisms for the Levi graph of the partial linear space. In fact, this is the full automorphism group, which can be identified with $2^5 \rtimes S_6$, see below.

Since the pls is self-dual, the automorphism group of the point graph K'' is an index 2 subgroup. Thus we have that $|Aut(K'')| = 11520$. The structure of this group is $2^4.S_6$, where the extension is not a semidirect product.

6.3.11 Other ways

Consider a 4×4 lattice. Let B be the set of 6 points depicted below.

	+	+	+
+			
+			
+			

The group $G = Z_4 \times Z_4$ acts naturally on this lattice. Taking the orbit of B under G, we obtain a $2-(16, 6, 2)$ design, i.e., a biplane B_6.

The automorphism group of B_6 is known to be $E_{16} \rtimes S_6$.

We have a natural bijection between points and blocks (a polarity without absolute points). We call a set of points in B_6 admissible if it meets each block in 1 or 3 points.

	+	+	+
	+		
		+	
			+

Admissible sets have been studied for other reasons by Mathon [69].

All admissible sets are equivalent under $Aut(B_6)$, and there are exactly 192 of them.

Given an admissible set S, the set of blocks intersecting S in exactly 1 point forms an admissible set in the dual design.

Thus, S defines a partial 1-factor in the Levi graph of B_6.

The admissible sets can be identified with the points and lines of our $spls(4, 4)$.

There is no nice computer-free description of incidence for this model yet. However, it allows us to describe the whole automorphism group naturally (as opposed to a quotient group).

In fact, the automorphism group of $\mathcal{R}$ is isomorphic to the automorphism group of the Levi graph of B_6, which can be described as $E_{32} \rtimes A_6$.

Finally, we can describe the graph Γ in terms of two-graphs. Consider the regular two-graph on 6 points whose switching class contains the pentagon with an isolated point. This two-graph has automorphism group S_5. Thus there are 6 distinct copies.

The switching classes of this two-graph and its isomorphic complement together contain 4 non-isomorphic graphs. With each graph, it also contains its complement.

Each switching class contains $2^5 = 32$ graphs, hence we get 32 pairs of complementary graphs. Altogether we get $6 \cdot 32 = 192$ pairs of graphs, which can be identified with vertices of $\mathcal{R}$.

Adjacency in $\mathcal{R}$ can be described in terms of these graphs.

6.4 Maximal arcs in projective planes

6.4.1 Preliminaries

Let Π be a projective plane of order n. A k-arc in Π is a set A of points such that no line meets A in more than k points. k is the *degree* of A. It is clear that always $1 \leq k \leq n+1$. Examples:

- A single point is a 1-arc.
- A set of three non-collinear points is a 2-arc.
- Any set of points is an $n+1$-arc.

The size of a k-arc is limited. In fact we have

Proposition 6.9 (Tallini Scafati [94]). *A k-arc A in a projective plane of order n has at most $nk + k - n$ points. Equality holds if and only if each line meets A in 0 or k points.*

Proof. We may assume that A is not empty. Let $p \in A$ be a point. There are $n+1$ lines through p, each containing at most $k-1$ additional points of A. Each point in A (besides p) is contained in exactly one line through p. Thus,

$$|A| \leq (n+1)(k-1)+1$$

Equality holds if every line through p meets A in exactly k points. Since p was arbitrary, this holds for every line that intersects A. □

We will call a k-arc of size $nk+k-n$ a *maximal arc* in Π. Given a k-arc, we can distinguish lines to be intersecting or external, depending on whether or not they meet A.

Trivial cases of maximal arcs are singletons ($k=1$) and the set of all points ($k=n+1$). In what follows, we will exclude these cases.

Example 6.2. The complement of a line, i.e., an affine subplane, is a maximal n-arc.

Maximal 2-arcs are usually known as hyperovals. Constructions for hyperovals are known for all desarguesian planes of even order. For odd order, they do not exist, as can be seen from the following necessary conditions for the existence of a maximal k-arc:

Proposition 6.10. *Given a maximal k arc A, and a point $p \notin A$, there are $n+1-n/k$ intersecting lines through p.*

Proof. The intersecting lines meet A in k points each; this provides a partition of A. Hence the number of these lines is

$$\frac{nk+k-n}{k} = n+1-\frac{n}{k}$$

□

From this we get the following facts easily:

Corollary 6.6 (Cossu[25]).
- *k divides n.*
- *There are n/k external lines through each point not in A.*
- *The external lines form a maximal (n/k)-arc in the dual plane.*

We will denote the degree n/k of the dual arc by $\bar{k}$.

Given a maximal arc, we can consider the subgeometry formed by the points and intersecting lines.

Proposition 6.11. *The geometry formed by the points and intersecting lines of a maximal k-arc is a Steiner system $S(2, k, nk + k - n)$.*

Proof. Each intersecting line meets A in k points, thus the block size is k. Any two points in A, being points of a projective plane, are connected by exactly one line, which necessarily is intersecting. Hence we get a Steiner system. □

In order to avoid confusion, we will refer to the points of the Steiner systems as *vertices*. We also distinguish the blocks of the Steiner system and the lines of the projective plane.

Corollary 6.7. *The blocks corresponding to the lines through an exterior point form a spread of the Steiner system.*

Proof. We have seen above that the intersecting lines through an exterior point form a partition of the arc. □

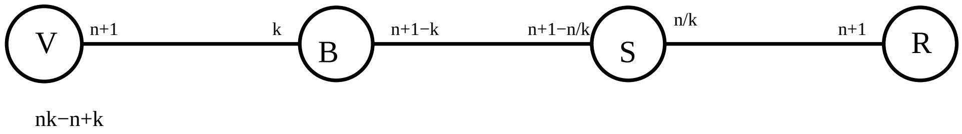
Hence we can identify the exterior points with some spreads of the Steiner system. Note, however, that there may be spreads which do not correspond to lines of the plane.

Proposition 6.12. *The spreads corresponding to the points on an external line form a resolution of the Steiner system.*

Proof. Let l be an external line. We can regard this as a set of external points, i.e., a set of spreads of the Steiner system. Every intersecting line meets l in a single point, thus every block of the Steiner system is contained in a single spread. Hence, we have a resolution. $\square$

Again, we identify external lines with some resolutions. Thus, a maximal k-arc gives a natural partition of the elements of Π:

- V: points on the arc; vertices of the Steiner system.
- B: intersecting lines; blocks of the Steiner system.
- S: points not on the arc; spreads.
- R: external lines; resolutions.



We have already seen that (V, B) is a $S(2, k, nk - n + k)$. Since R is a dual n/k-arc, we have that (S, R) is a dual $S(2, \bar{k}, n\bar{k} - n + \bar{k})$. What about (B, S)?

Theorem 6.1 (Thas [96]). *The incidence structure (B, S) is a partial geometry $pg(K, R, T)$, with*

$$\begin{aligned} K &= n + 1 - k \\ R &= n + 1 - \bar{k} \\ T &= n + 1 - k - \bar{k} \end{aligned}$$

Proof. Any two lines intersect, thus two blocks in B either intersect, or they lie in exactly one common spread from S. Each spread contains $n + 1 - \bar{k}$ blocks, and each block lies in $n + 1 - k$ spreads, thus we have a partial linear space.

Given a block b and a spread s not containing b, there are k blocks in s intersecting b. Thus, b is disjoint from $T = n + 1 - \bar{k} - k$ blocks in s, and each is contained in a spread through b. $\square$

In terms of the partial geometry (B, S), we can interpret the sets V and R as follows: R is a set of spreads of (B, S), such that each pair of skew lines is contained in exactly one spread of R. V is a set of ovoids of (B, S), such that each pair of non-collinear points is contained in exactly one ovoid from V. We will call such sets *complete sets of spreads (resp. ovoids)* for (B, S).

Question 6.1. What are necessary or sufficient conditions for a $pg(K, R, T)$ to contain a complete set of spreads?

In terms of the notions from Section 6.2, we can consider a projective plane with a maximal arc as a diagram geometry of rank 4, with fibers $F_0 = V$, $F_1 = B$, $F_2 = S$, $F_3 = R$.

Theorem 6.2. *The tuple (V, B, S, R) with natural incidences between all parts forms a coherent diagram geometry of coherent rank 28.*

Proof. Let as above, $F_0 = V$, $F_1 = B$, $F_2 = S$, $F_3 = R$. We have two colors of type $(0, 0)$ and $(3, 3)$, and three of type $(1, 1)$, and $(2, 2)$. Moreover, we have incidence and non-incidence for the types $(0, 1)$, $(1, 2)$, $(2, 3)$, and their transposes. Finally, we have one trivial color for types $(0, 2)$, $(2, 0)$, $(1, 3)$, $(3, 1)$, $(0, 3)$, $(3, 0)$.

Altogether, we have $2 \cdot 2 + 2 \cdot 3 + 6 \cdot 2 + 6 = 28$ colors.

Since (V, B), (B, S), and (S, R) are partial geometries, the induced structures on these sets are coherent configurations by Proposition 6.3.

Moreover, since (V, S), (V, R), and (B, R) are trivial incidence structures, we only need to consider those types of intersection numbers that involve three different fibers. There are 24 such types; using symmetry and duality, we can cut this down to $24/4 = 6$ types. The types we consider are $(0,1,2)$, $(0,1,3)$, $(0,2,1)$, $(0,2,3)$, $(0,3,1)$, $(1,2,0)$.

$(0,1,2)$: Let p be a point, s a spread. Then there is exactly one block through p in s, there are n blocks through p which are not in s, and $n - n/k$ blocks in s not through p. The remaining blocks are not in s and do not contain p.

$(0,1,3)$: Let p be a point, r a resolution. There are $n+1$ blocks incident with p, the other blocks are not incident with p. Since the incidence structures (V,S) and (V,R) are trivial, all blocks are in the same relation with r. Thus we get two non-trivial intersection numbers.

$(0,2,1)$: Let p be a point, b a block. Since the incidence structure (V,S) is trivial, all intersection numbers follow from the fact that there are $n+1-k$ spreads through b.

$(0,2,3)$: Again trivial.

$(0,3,1)$: Trivial, only one structure constant.

$(1,2,0)$: Trivial.

□

6.4.2 Bruck's dream

It is still an open question whether all projective planes have prime power order. It is known that no plane of order 6 or 10 exists; on the other hand, there are known planes for all prime power orders. Thus, the smallest open case is order 12.

Here, the possible degrees of a maximal k-arc are $k = 2, 3$ or dually $k = 6, 4$. The case of a hyperoval, $k = 2$, would lead to a Steiner system $S(2, 2, 14)$, i.e., the system formed by the vertices and edges of the complete graph K_{14}. The block graph of this system is just the line graph of K_{14}, i.e., the triangular graph T_{14}.

So, we would need a partial geometry whose point graph is the complement of T_{14}. In general, the graph $\overline{T_{2n}}$ is pseudo-geometric. It is known that $\overline{T_8}$ and $\overline{T_{12}}$ are not geometric. On the other hand, $\overline{T_6}$ is the point graph of the classical generalized quadrangle $GQ(2,2)$, and Mathon [68] showed that there are exactly two partial geometries with point graph $\overline{T_{10}}$. For $\overline{T_{14}}$, the question is still open.

R.H. Bruck [18] suggested a construction of a plane Π of order 12, which we reformulate here in terms of the material presented above. Suppose that Π contains a maximal 4-arc A. Then A induces a $S(2,4,40)$. The best known such Steiner system is formed by the points and lines of the projective space $PG(3,3)$. Thus, Bruck's idea was as follows:

- Let V and B be the points and lines of $PG(3,3)$; we have $|V| = 40$ and $|B| = 130$.
- Find a set S of 117 spreads and a set R of 27 resolutions consisting of those spreads, such that any two spreads have at most one line in common, and any two resolutions have at most one spread in common.
- Then $(V \cup S, B \cup R)$ with natural incidence forms a projective plane of order 12.

This has led to intense studies of the resolutions of $PG(3,3)$. In the end it was outlined by M. Hall and R. Roth [47] and later finally shown by A. Prince that this idea doesn't work.

Theorem 6.3 (Prince [82]). *The projective space $PG(3,3)$ cannot be extended to a plane of order 12.*

6.4.3 Confirmation and improvement of Prince's result

In fact, this can be decided without knowing all resolutions of $PG(3,3)$. If Π had such a "projective" maximal 4-arc, then (B,S) would be a $pg(9,10,6)$, and its point graph would be the complement of the block graph of $(V,S) = PG(3,3)$. Thus, we need to decide whether a given graph is the point graph of a partial geometry.

This is a standard task in algorithmic graph theory. There are good algorithms for this task. In the past, the author had devised his own algorithm [84]. Simultaneously, L. Soicher created a similar algorithm and included it in an updated version of his graph theoretical package GRAPE [91]. Here, we use the latter, since it is readily available.

The following is a brief outline of Soicher's algorithm. Let Γ be a possible point graph of a partial geometry $pg(K, R, T)$. According to Theorem 2.3, it is sufficient to find a partial linear space (with parameters K and R) which has Γ as a point graph. Lines in such a space form K-cliques in the graph. Thus, in a first step, all such cliques are constructed. Then, we have to find a set of cliques such that there are R cliques through each point, and no two cliques share more than one point. The algorithm heuristically starts constructing lines through the members of an independent set. If several solutions are found, they are tested for isomorphism.

Thus, we can give the following algorithmic confirmation and improvement of Prince's result:

Proposition 6.13 (Computer search). *There is no partial geometry with a point graph isomorphic to the complementary block graph of $PG(3,3)$.*

This is an improvement, since such a geometry could exist even if it doesn't extend to a plane (it doesn't have to possess a complete set of spreads R). On the other hand, the result still relies heavily on a computer search.

Of course, there might be a maximal 4-arc which induces a different structure. Equivalently, we might start with a 3-arc in the dual plane. This leads to the following algorithmic attempt to construct a plane of order 12 with a maximal 4-arc:

- Start with a Steiner system $S(2,4,40)$ or $S(2,3,27)$; let V be the set of points, B, the set of blocks.

- Let Γ be its block graph.
- Find a partial geometry with point graph $\overline{\Gamma}$. Let S be the set of its lines.
- Let Δ be its block graph.
- Find a Steiner system with $\overline{\Delta}$ as a block graph.
- Let R be the set of its points.

Then $(V \cup S, B \cup R)$ is a projective plane of order 12, with maximal arc V. There are obvious generalizations to different orders and arc degrees.

6.4.4 Completed searches for other Steiner systems

Whereas Steiner triple systems $S(2,3,v)$ have been classified for small orders v, there is little hope that this classification will ever be completed for systems of order 27, or even for $S(2,4,40)$. Hence we need to restrict ourselves to certain subclasses of Steiner systems.

Both for the construction of the systems themselves and for their investigation, it appears reasonable to require certain symmetry properties. The following classes of Steiner systems have been studied in the past:

- A system is cyclic if it admits a cyclic automorphism acting regularly on the points.
- A system is i-rotational, $i = 1, 2, \ldots$, if it admits a cyclic automorphism group with one fixed point, acting semiregular on the remaining $v - 1$ points, and having i orbits of length $(v-1)/i$.

We consider the following additional classes:

- A system is regular if it admits an automorphism group acting regularly on the points.

- A system is i-semiregular if it is invariant under a semiregular group of order v/i. In particular, it is i-cyclic if the semiregular group is cyclic.

The following is an outline of the algorithm to construct Steiner systems $S = S(2, k, v)$ invariant under a given permutation group (G, Ω) of degree v. For $1 \leq n \leq v$, let n-orb(G) denote the orbits of G on n-subsets of Ω. Clearly, the set of blocks of S is the union of a subset of k-orb(G).

Now we have to choose a subset of k-orbits such that each pair of points is covered exactly once. This allows us to eliminate such k-orbits which covers any pair more than once.

For the remaining set K of "good" k-orbits, we say that a 2-orbit O_2 and a k-orbit O_k are incident if a representative of O_2 is contained in an element of O_k. This defines an incidence structure (2-orb$(G), K)$. If we call the 2-orbits "points", and the k-orbits "lines", the task is now to find a set of lines which covers each point exactly once; in other words, we are looking for a spread in this structure.

A standard backtracking procedure is used to enumerate all spreads. Here, we use the fact that two spreads will give the same Steiner system if they are equivalent under the normalizer of G in the symmetric group $S(\Omega)$.

Below we list the classes of systems that have been examined. In no case a partial geometry was found. Note that there is some redundancy in this table, since a system can belong to several classes. For example, a 1-rotational system is also i-rotational for all feasible values of i. Moreover, each cyclic system is regular.

Steiner triple systems $S(2, 3, 27)$:

Property	Number	checked	existence
cyclic	8	(yes)	no
regular	71	yes	no
1-rotational	36	yes	no
2-rotational	1640	yes	no

Steiner triple systems $S(2,4,40)$:

Property	Number	checked	existence
cyclic	10	(yes)	no
regular	30	yes	no
1-rotational	2	yes	no

Summarizing, we get the following result:

Theorem 6.4. *Let $\Pi = (P, L)$ be a projective plane of order 12. Let V be a maximal 4-arc in Π, and let $S = P \setminus V$. Let B be the set of lines intersecting V, and let $R = L \setminus B$.*

1. *(V, B) is a $S(2,4,40)$ which is neither regular nor 1-rotational.*
2. *(R, S) is a $S(2,3,27)$ which is neither regular nor i-rotational for $i = 1, 2$.*

Chapter 7

WEAKLY NEIGHBORLY POLYHEDRAL MAPS

7.1 Preamble

The topic of weakly neighborly polyhedral map has a special place in this thesis, since it is not part of classical Algebraic Combinatorics. However, we believe that a new link between Discrete Geometry and Algebraic Combinatorics is established.

Thus the intended audience of this chapter is two-fold: For combinatorialists, this is an interesting application. For geometers, it serves as an exposition of algebraic and combinatorial techniques. This is why we tried to make this chapter as self-contained as possible; therefore, most definitions and basic facts appear here, as opposed to Chapter 2.

The results presented here include a confirmation of the uniqueness of a weakly neighborly, equivelar map on 16 points, as well as the non-existence of weakly neighborly, equivelar maps on $25 \leq v \leq 64$ points.

The material presented here is based on an indirect collaboration with U. Brehm. He posed the question of the existence of certain maps invariant under regular permutation groups, for which we give a partial answer below.

7.2 Polyhedral maps: A short tutorial

Here, we try to give a friendly introduction to the subject of polyhedral maps. We define these objects several times, in geometric, topological, and combinatorial language.

We start by looking at the platonic solids. These are 5 well-known, convex polytopes in euclidean 3-space. However, we will consider them as polyhedra by taking the boundaries of the polytopes.

Polyhedra can be thought of as consisting of three different types of objects: Faces, edges, and points. This leads us to our first, geometrical definition:

A *polyhedral map* [6],[11],[12] is a triple $(\mathcal{P}, \mathcal{E}, \mathcal{F})$, where $\mathcal{P}$, $\mathcal{E}$, and $\mathcal{F}$ are the points, edges, and faces, respectively, of a polyhedron in euclidean space.

The union of all the faces of a polyhedron forms a compact surface without boundaries in space. The faces form subsets of the surface, and any two faces, since they are planar, are either disjoint, or they intersect in a point or an edge. This leads to a reformulation and slight generalization of our definition.

Let S be a compact surface. A *polyhedral map on S* is a triple $(\mathcal{P}, \mathcal{E}, \mathcal{F})$, where

- $\mathcal{E}, \mathcal{F}$ are collections of subsets of S;
- each $f \in \mathcal{F}$ is a closed simply connected region;
- each $e \in \mathcal{E}$ is a curve;
- $\mathcal{P}$ is the set of endpoints of edges in $\mathcal{E}$;
- $\bigcup \mathcal{F} = S$;
- for $f_1, f_2 \in \mathcal{F}$, $f_1 \cap f_2 \in \{\emptyset\} \cup \mathcal{P} \cup \mathcal{E}$.

As before, we call the elements of $\mathcal{P}$, $\mathcal{E}$, and $\mathcal{F}$ points, edges, and faces respectively.

Each polyhedral map as defined earlier fits this new definition, but there are maps which cannot be realized with planar faces.

Going one step further, we can try and dispose of the surface, and look at maps from a purely combinatorial point of view. Points can be taken as the elements of some given set, and edges can be described by unordered pairs of points. The points and edges contained in a face form a cycle which completely determines the face. Thus we get to our final, combinatorial definition:

A *(combinatorial) polyhedral map* is a triple $(\mathcal{P}, \mathcal{E}, \mathcal{F})$, where $\mathcal{P}$ is a set, $\mathcal{F}$ is a set of undirected cycles on $\mathcal{P}$, and $\mathcal{E}$ is the set of edges in elements of $\mathcal{F}$, such that

- each cycle is a regular connected simple graph of valency 2;
- each edge is contained in exactly two cycles;
- each non-edge is contained in at most 1 cycle.

These axioms capture the requirement that two faces cannot share two non-adjacent vertices; in particular, they can not share more than one edge.

Given a polyhedral map $(\mathcal{P}, \mathcal{E}, \mathcal{F})$ we denote the numbers of points, edges, and faces, respectively by V, E, and F. For the platonic solids, the triples (V, E, F) are (sorted by the number of faces) $(4, 6, 4)$, $(8, 12, 6)$, $(6, 12, 8)$, $(20, 30, 12)$, and $(12, 30, 20)$.

In these five cases we have that $V + F = E + 2$. L. Euler observed that this holds for any polyhedral map on the sphere. We call $\chi = V - E + F$ the *Euler characteristic of the map.*

Euler's Theorem can be generalized as follows: Recall that any orientable surface S is equivalent to a sphere with g handles. Here, g is called the *genus* of S. Now, let M be a polyhedral map on S. Then $\chi = 2 - 2g$.

Conversely, we define the *genus* of an arbitrary polyhedral map as $g = 1 - \chi/2$. Then the genus of a map on an orientable surface coincides with the genus of the surface itself.

The platonic solids have been studied because of their symmetry and regularity. In fact, for each of them there exist numbers k and l such that each face has l vertices, and each point is contained in k faces. We call such a polyhedral map *of* *type* (k, l). If moreover, $k = l$, we speak of an *equivelar* polyhedral map of valency k.

For a polyhedral map of type (k, l), we have that $kV = 2E = lF$. In particular, an equivelar polyhedral map has as many faces as points. The Euler characteristic of such a map is given by

$$\begin{aligned} \chi &= V + F - E \\ &= 2V - \frac{kV}{2} = \left(2 - \frac{k}{2}\right) V, \end{aligned}$$

and thus the genus is

$$g = 2 - \frac{\chi}{2} = 2 + \left(\frac{k}{4} - 1\right) V.$$

Given a polyhedral map $(\mathcal{P}, \mathcal{E}, \mathcal{F})$, we can define a graph $\Gamma = (\mathcal{P}, \mathcal{E})$ by taking the points as vertices, and using the same set of edges. This graph is the *point graph* of the map. In fact, we also get an embedding of the graph into a surface S. In particular, point graphs of maps of genus 0 are planar.

The point graph of a polyhedral map of type (k, l) is regular of valency k. In particular, this holds for equivelar maps of valency k.

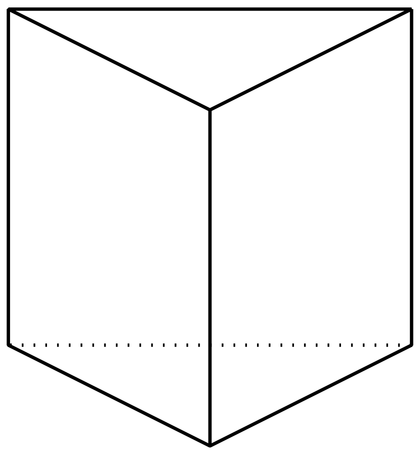
Reversely, any graph embedded in a surface defines naturally a polyhedral map. A graph may have non-isomorphic embeddings to several surfaces, thus leading to different polyhedral maps.

For the platonic solids, the point graphs are: the complete graph K_4 for the tetrahedron; the graph $\overline{3 \circ K_2}$ for the octahedron, and distance regular graphs of diameter 3 for the cube and icosahedron, and of diameter 5 for the dodecahedron.

7.3 Weakly neighborly polyhedral maps

In a map each edge appears in exactly two cycles, and each non-edge appears as a diagonal in at most one cycle. We denote the length of a cycle C by $|C|$. If we count all edges and diagonals of all the cycles, we get the following inequality.

Table 7.1: The triangular prism



Proposition 7.1. *For a polyhedral map we have*

$$\binom{V}{2} + E \geq \sum_{C \in \mathcal{C}} \binom{|C|}{2} \tag{7.1}$$

with equality holding iff each non-edge is a diagonal of some cycle.

Maps for which equality holds are called *weakly neighborly.* In other words, a map is weakly neighborly if each pair of points is contained in a common face. If each pair of points is actually an edge, we speak of *neighborly maps.*

Example 7.1. The smallest examples of neighborly resp. weakly neighborly maps are the tetrahedron and the triangular prism (see Table 7.1).

In the special case of equivelar maps of valency k, we know that $V = F$. Here, we get the following numerical constraint:

Proposition 7.2. *For an equivelar map of valency k,*

$$V \geq (k-1)^2$$

with equality if the map is weakly neighborly.

Proof. In this case, we have $2E = Vk$, and the inequality (7.1) becomes

$$\begin{aligned} V(V-1) + Vk &\geq Fk(k-1) \\ V - 1 &\geq k(k-2) \\ V &\geq (k-1)^2 \end{aligned}$$

□

We will call $d = V - (k-1)^2 \geq 0$ the deficiency of an equivelar map. We are interested in maps with small deficiency.

Example 7.2. The tetrahedron is an equivelar, neighborly map with $k = 3$ and $V = 4 = (k-1)^2$).

In what follows, we will mainly consider weakly neighborly, equivelar maps; we will abbreviate them as w.n.e. maps.

7.4 A naive graph-theoretical approach

Given an equivelar map of valency k, its point graph is regular, and the faces are cycles of length k. So, one naive approach to the construction of weakly neighborly equivelar maps starts with the point graph.

1. Start with a regular graph Γ of valency k and order $(k-1)^2$.
2. Find all cycles of length k in Γ.
3. Find a subset of cycles which contains each edge twice and each non-edge once.

These three steps deserve some comments:

In principle we can enumerate all weakly neighborly equivelar maps of a given order if we know all regular graphs of the required valency. For $k = 3$, there is a unique map with a complete point graph (see Example 7.1. For $k = 4$, all isomorphism classes of graphs (of size 9 and valency 4) have been constructed using B. McKay's `nauty` program; there are exactly 16 such classes ([5], [74]).

For $k \geq 5$, this approach does not work. There are literally millions of non-isomorphic graphs of size 16 and valency 5, so there is no hope of investigating them all. We therefore need additional restrictions on the graphs to define a class that we can examine.

We decided to work with graphs whose automorphism group acts transitively on the vertices. For $k = 5, 6$ (orders 16 and 25), all such graphs are known (see G. Royle's homepage [87]) and there is a reasonable number of them.

Given a graph Γ, we need to find induced cycles of length k. Let us first consider the case that $k = 2m$ is even. Two vertices x, y of distance m on such a cycle have distance d, $2 \leq d \leq m$ in Γ. The cycle can be divided into two vertex-disjoint induced paths connecting x and y.

Thus we proceed as follows: For all pairs x, y of vertices of Γ at distance $2 \leq d \leq m$, find all induced paths of length m connecting x and y. For any two such paths which share only the vertices x and y, check if the union of their vertices induces a cycle.

To find induced paths of length m, we use backtracking [35], [43], [67], [101]. Suppose we have a partial path $(v_0 = x, v_1, \dots, v_n)$ of length n, where $d_\Gamma(v_n, y) \leq m - n$. We can extend this path by any neighbor v_{n+1} of v_n at distance at most $m - n + 1$ from y. Since we are looking for induced paths, we avoid any neighbors of v_i, $0 \leq i \leq n - 1$. Moreover, if $n < m - 1$, we also exclude neighbors of $v_m = y$.

Doing this for any choice of vertices x, y, we obtain all induced cycles of even length $k = 2m$.

If $k = 2m + 1$ is odd, we consider a point x on the cycle, and an edge (y, z), where both y and z are at distance m from x. Then we proceed as above.

Once we have candidates for the faces of our maps, we have to choose a suitable subset of them which actually forms a map. We know that each non-edge of the point graph needs to appear in exactly one cycle.

Suppose conversely that we have a system of cycles which contains each non-edge exactly once.

Finally, we get the following result:

Proposition 7.3. *The following are all weakly neighborly equivelar maps with* $3 \leq k \leq 6$*:*

- *For* $k = 3, v = 4$*, there is up to isomorphism exactly one weakly neighborly equivelar map, the tetrahedron.*
- *For* $k = 4, v = 9$*, there are three isomorphism classes of w.n.e. maps.*
- *For* $k = 5, v = 16$*, there is up to isomorphism one w.n.e. map invariant under a group acting transitively on points. (See Example 7.6 for a description of this map.)*
- *For* $k = 6, v = 25$*, there is no w.n.e. map invariant under a transitive group.*

7.5 Sharply transitive groups

Here we pay attention to a special class of vertex-transitive graphs, see Biggs [7] and Godsil-Royle [39] for more details.

Let (G, Ω) be a permutation group, i.e., Ω is a set, and G is a subgroup of the group $S(\Omega)$ of all permutations of Ω. We follow the convention that permutations are composed left-to-right, in other words, for $g, h \in G$ and $x \in \Omega$, $(gh)(x) = h(g(x))$. In order to make this clearer, we use an exponential notation for the images of permutations, i.e., we write x^g for $g(x)$. Then $x^{gh} = (x^g)^h$.

Two points $x, y \in \Omega$ are equivalent under G if there exists an element g such that $x^g = y$. That this is indeed an equivalence relation follows from the axioms of groups: The existence of an identity element gives us reflexivity; inverses give us symmetry, and if $x^g = y$ and $y^h = z$, then $x^{gh} = z$.

The equivalence classes of this relation are the *orbits* of G on Ω. We denote the orbit containing x by x^G. If $x \in \Omega$ is a point, we denote by G_x the stabilizer of x under G, i.e., the set of all elements in G which fix x. G_x is a subgroup of G, and its order is given by

$$|G| = |G_x| \cdot \left|x^G\right|$$

(G, Ω) is called *transitive* if there is only one orbit, in other words, for any $x, y \in \Omega$, there exists a $g \in G$ such that $x^g = y$. It is called *sharply transitive* (or sometimes *regular*) if for each x and y there exists exactly one element mapping x to y. In particular, the stabilizer of any point is trivial, and $|G| = |\Omega|$.

Let G be an (abstract) group. We define a permutation group (G, G) by setting $g^h = gh$. Then for any $g, h \in G$, there is a unique element mapping g to h, namely $g^{-1}h$. Thus we get a sharply transitive permutation group, which is called the right-regular action of G.

It turns out that in fact each sharply transitive group can be considered as a right-regular action. If (G, Ω) is such a group, we fix a point $x_0 \in \Omega$. Now, we can identify each point $y \in \Omega$ with the unique element $g \in G$ such that $x_0^g = y$. In particular, x_0 is identified with $e \in G$.Thus we identify Ω with G, and we get the right-regular action of G on itself.

Now suppose we have a graph Γ which admits a sharply transitive group G of automorphisms. We identify vertices with group elements. Since the group is transitive, we can describe the whole graph by giving the set $X \subset G$ of neighbors of e. In other words, the edges through e are of the form (e, x), with $x \in X$. Then, by transitivity, all other edges are of the form (y, xy), hence two points y, z are adjacent if $zy^{-1} \in X$.

Now suppose we have an equivelar, weakly neighborly map admitting a sharply transitive group G of automorphisms. In order to describe this map, we need to describe the point graph and the faces (cycles). The following fact shows that it suffices to give only one cycle.

Proposition 7.4. *G acts transitively on the cycles.*

Proof. Let C be a cycle, and let $H \leq G$ be the stabilizer of C in G, i.e., the subgroup consisting of all elements which fix C as a set. Since G is regular, all orbits of H on the points are of length $m = |H|$. We know that $m|v = |G|$.

On the other hand, C is fixed under H, and thus it is the union of H-orbits. Thus $m|k$.

However, since k and $v = (k-1)^2$ are mutually prime, we get that $m = 1$, and hence H is the trivial group. This implies that the orbit of C under G has length v, which completes the proof. □

Thus, if we give one cycle as a set $Y \subset G$ of points, we get all other cycles by transitivity.

Example 7.3. Let $G = Z_9$ be the cyclic group. Let $Y = \{0, 1, 3, 4\}$. Then, taking all translates, we get the twisted torus. Its point graph has the connection set $X = \{1, 3, 6, 8\}$.

Example 7.4. Let $G = E_9 = Z_3 \times Z_3$ be the elementary abelian group. We write elements as ij, $i, j = 0, 1, 2$. Let $Y = \{00, 01, 10, 11\}$. We get the torus; its graph has the connection set $X = \{01, 02, 10, 20\}$.

7.6 Incidence structures

Let G be a group, and let a polyhedral equivelar map be defined by a cycle $Y \subset G$, such that the point graph has connection set $X \subset G$.

If we consider the action of G on the ordered pairs $G \times G$, then each orbit (a 2-orbit in the sense of [31]) contains exactly one element of the form (e, x), where e is the identity element. Thus we can identify these 2-orbits with the elements of G. A pair (y, z) lies in orbit x if $yz^{-1} = x$.

Now we try to express the defining properties of a w.n.e. map algebraically. Let D be the multiset of differences of (not necessarily distinct) elements of Y. Then the multiplicity of d in D expresses the number of cycles that a representative of orbit d appears in.

Since X corresponds to the set of edge-orbits, we get that e appears k times in D, each $x \in X$ appears twice, and every other element appears once.

Example 7.5. Let us return to the cyclic example on 9 points, with $Y = \{0, 1, 3, 4\}$ and $X = \{1, 3, 6, 8\}$. If we consider the table of pairwise differences in Y, we get

	0	1	3	4
0	0	8	6	5
1	1	0	7	6
3	3	2	0	8
4	4	3	1	0

We see that 0 appears four times, $1, 3, 6, 8$ appear twice each, and all other elements appear exactly once.

A more elegant way to describe such relations of group elements is given by the notion of the group ring $\mathbb{Z}[G]$. Formally, this ring consists of all functions $f : G \to \mathbb{Z}$. Addition is defined pointwise.

We embed G into $\mathbb{Z}[G]$ by identifying g with the function that maps g to 1 and all other elements to 0. This allows us to express all elements of the group ring as linear combinations of group elements, $f = \sum_{g \in G} f(g)g$.

Now, we define the multiplication of the ring, restricted to G, to be just the group operation. By distributivity, this multiplication extends to all of $\mathbb{Z}[G]$, and we get indeed a ring.

Given a subset $Y \subseteq G$, we can define its characteristic function f_Y by setting $f_Y(x) = 1$ if $x \in Y$, and $f_Y(x) = 0$ otherwise. If there is no danger of confusion, we identify Y with f_Y.

In terms of the group ring, we can express the necessary condition for a weakly neighborly, equivelar map of valency k, cycle Y, and connection set X by the equation

$$\begin{aligned} YY^{-1} &= ke + 2X + (G - X - e) \\ &= G + (k-1)e + X \end{aligned}$$

This allows us to give our next example:

Example 7.6 (U. Brehm[11]). Let $G = Z_{16}$, $Y = \{0, 1, 2, 5, 8\}$. Then

$$YY^{-1} = G + 4e + X,$$

where $X = \{1, 3, 8, 13, 15\}$. This is in fact a weakly neighborly, equivelar map of valency 5.

Note that the equation does not give a sufficient condition for the existence of a map, since the subgraph induced by Y is not necessarily a cycle. Remember that $x, y \in G$ are adjacent if $xy^{-1} \in X$.

Proposition 7.5. *The following are necessary and sufficient for the existence of a weakly neighborly, equivelar map invariant under G with cycle $Y \subset G$ and connection set $X \subset G$:*

- *The equation*

  $$YY^{-1} = G + (k-1)e + X$$

  holds;

- *The graph defined on Y by joining x and y if $xy^{-1} \in X$ is a cycle of length k.*

In Example 7.6, we get the cycle $0 - 1 - 2 - 5 - 8 - 0$, so indeed we have a map.

Given a finite group G, we can now search for solutions (Y, X) of the above equation. We see that X is uniquely determined by Y. Hence, using a backtracking approach, we incrementally construct partial solutions Y' of sizes less than or equal to k. If a partial solution Y' of size k' extends to a full solution, we need that

$$Y'Y'^{-1} = k'e + 2X' + Z$$

where Z is disjoint from X', and $|X'| \leq |X| = k$. These restrictions allow us to discard some partial solutions early.

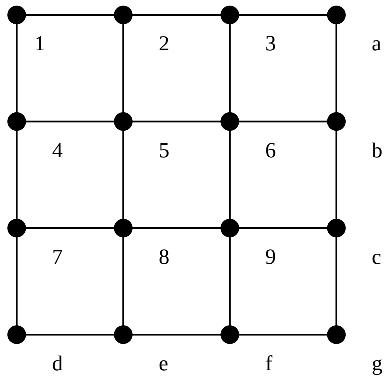
Further reduction is given by the fact that in a weakly neighborly map, any pair of distinct points is contained in at least one cycle. Thus we can pick two arbitrary group elements and assume without loss of generality that these elements are contained in Y.

Using GAP's library of small groups, we examined all possible candidates for $k = 3, \ldots, 9$. We got some solutions to the equation which did not come from maps, as well as one map on 4 points, 2 on 9 points, and one on 16 points. Thus we have

Proposition 7.6. *Let G be a group of order $v = (k-1)^2 \leq 64$, and let M be a weakly neighborly equivelar map of valency k invariant under a regular action of G. Then one of the following holds:*

- *$v = 4$, and M is the tetrahedron;*
- *$v = 9$, and M is either the torus or the twisted torus;*
- *$v = 16$, and M is the map described in Example 7.6.*

Table 7.2: Grid for w.n. map of degree 4



7.7 A topological approach

A different approach for the construction of w.n.e. maps with given valency k relies on the following. Consider a point on the map. It has k neighbors

Let M be a map of degree k. Fix one face f_0. Then there are k faces sharing one edge with f_0, and for each point in f_0, there are $k-3$ more faces through that point. All of these $1+k+k(k-3)=(k-1)^2$ faces are distinct, and hence we get all faces this way.

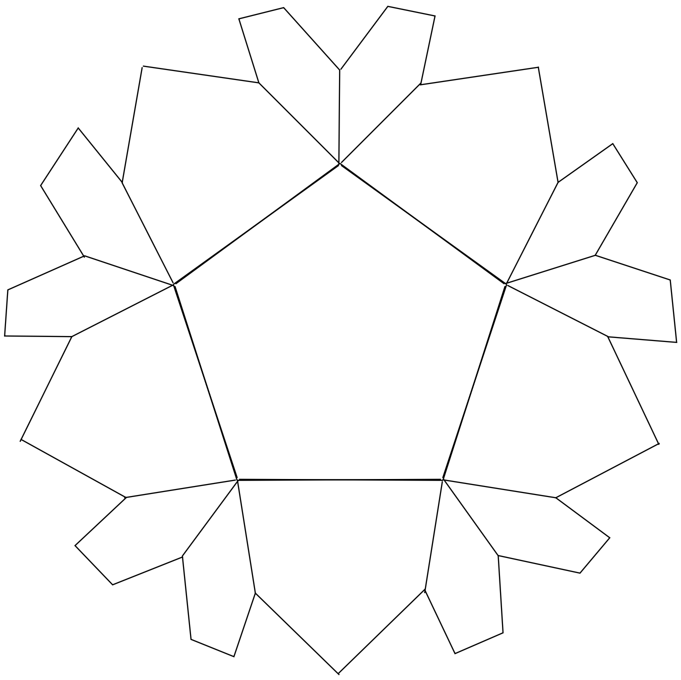
Hence, we can "cut" each such map along some edges to arrive at a planar map, which only depends on the degree, and not on the particular map. For $k=4$, this planar map is the 4×4 grid in Table 7.2. For $k=5$, it is shown in Table 7.3.

We will now try to invert this process. Given such a planar grid, we try to reconstruct a w.n. map by identifying certain points and edges. This has been implemented as a backtrack search. The following results were obtained:

Proposition 7.7. *Up to isomorphism, there are three equivelar w.n. maps of degree 4, and one equivelar w.n. map of degree 5, as described in Example 7.6.*

Remark 7.1. All these results were known, having been obtained by U. Brehm et.al. There are also computer-free proofs of all these facts.

Table 7.3: Grid for w.n. map of degree 5



Remark 7.2. The three solutions on 9 points can be described as follows (labelling as in Table 7.2)

1. "torus": Set $a = d = g = 1$, $b = 4$, $c = 7$, $e = 2$, $f = 3$.
2. "twisted torus": Set $a = g = 7$, $b = d = 1$, $c = 4$, $e = 2$, $f = 3$.
3. "Klein bottle": Set $a = d = g = 1$, $b = 4$, $c = 7$, $e = 3$, $f = 2$.

7.8 Implementing the topological approach

The approach outlined in the previous section has been implemented as a backtrack search. Given the valency k, first the topological map with $(k-1)^2$ faces is constructed. This map has too many points, so the problem is to identify some of them. In other words, we need to construct a mapping from the points to a set of $(k-1)^2$ labels, which will be the points of the resulting map.

This mapping has to satisfy the following restrictions:

- Each point has to lie on k cycles;
- Each edge has to lie on 2 cycles;
- Each non-edge has to lie on 1 cycle.

Any theoretical construction of such a mapping would start with the words "without loss of generality", meaning that we can arbitrarily label a few points, without losing any solutions. In order to eliminate one source of human error, this has been automated as follows:

We maintain a partial mapping, and the set of used labels. If for any point in the topological map the assignment of any used label leads to a contradiction, we can arbitrarily assign one of the unused labels to this point. In particular, since the set of used labels is initially empty, we can label the first point arbitrarily. In the cases we considered ($k = 4, 5, 6$) this led to $(k-1)^2$ assigned labels, which cuts down the search space considerably.

After that, no further isomorph rejection is used during the search. For $k = 4, 5$, a few tens of solutions were found, and the search was completed in a matter of seconds. For $k = 6$, the algorithm didn't terminate in reasonable time; no solution was found.

Chapter 8

CONCLUSION

Traditionally, in the enumeration of combinatorial objects there is a distinction of constructive enumeration and analytical enumeration. In the former case, the result is a list of all non-isomorphic objects, whereas the latter yields only the "length" of this list.

The reason for this distinction is that in certain cases, analytical enumeration can be performed rather quickly using Group Theory, and in particular the theory of Burnside marks. In a recent development, these techniques have been applied to constructive enumeration (see, e.g., Goldberg [41] [42] and Meringer [76]), leading to a revolution of orderly generation. According to the authors, the main bottleneck of these algorithms is the output routine.

Whereas these techniques have been successfully applied to regular graphs, unfortunately they are not (yet) applicable to the classes of objects considered here, like srg's and drg's.

Another development is the notion of "prolific" constructions. Without going into details of definition, we can regard a prolific construction as a meta-construction, or a way to obtain one type of combinatorial object from another one. For example, from a Steiner triple system we can obtain a strongly regular graph, namely its block graph.

If it is known that, like in the case of Steiner triple systems, there are many objects to start with (more accurately, hyperexponentially many with respect to some parameters), such a meta-construction is called prolific. Recently, D. Fon-Der-Flaass [36] developed new prolific constructions for strongly regular graphs with spreads, and for antipodal covers of complete graphs.

As mentioned above, we found all such covers on 40 points; there are exactly 3 of them. We also completed the search for 45 and 64 vertices; there are no examples on 45 points, and there are exactly 94 non-isomorphic graphs on 64 vertices. It is not clear yet how many of them can be obtained by Fon-Der-Flaass' constructions.

As far as coherent configurations are concerned, recent progress has been by S. Hobart et.al., who investigated spectral properties of their adjacency algebras. It is a natural step to see if these properties, when applied to the incidence structures considered here, lead to new feasibility conditions. However, unfortunately this is out of the scope of this thesis.

Below, we give a list of problems that arose from or are related to the results presented above. In the spirit of this thesis, many of these problems are of a computational nature. Note that these problems vary in difficulty.

We list the problems grouped thematically.

1. Problems related to the t-vertex condition:

   (a) Prove or disprove that the point graph of a $GQ(s, s^2)$ satisfies the 7-vertex condition.

   (b) Steiner triple systems: It is known that the block graph of an $STS(v)$ satisfies the 4-vertex condition if and only if it is rank 3, with the possible exception of $v = 25$. Fill this gap.

   (c) The following conjecture is due to Klin (e.g., [31]): There exists a t_0 such that the t_0-vertex condition implies a rank 3 automorphism group.

   (d) Are there self-complementary graphs which satisfy the 4-vertex condition, but which are not rank 3?

2. Problems related to Siamese graphs:

   (a) Complete the proof of the existence of an infinite series of Siamese association schemes.

   (b) Enumerate all Siamese graphs on 27 points.

   (c) Enumerate all Siamese graphs on 40 points.

   (d) Enumerate all (geometric) Siamese graphs on 64 and 96 vertices.

   (e) Find examples of Siamese association schemes besides the ones mentioned here (including the possible infinite series). In particular, do non-geometric schemes exist?

   (f) Classify all Siamese association schemes.

   (g) Classify all Siamese color graphs.

3. Problems related to coherent configurations:

   (a) Theoretical investigation of the coherent configurations arising from maximal arcs in projective planes and from partial geometries. In particular, use spectral properties of the configuration to find new feasibility conditions.

   (b) Existence of a projective plane of order 12 containing a hyperoval or a maximal 3-arc.

   (c) A positive answer of the previous problem would imply the existence of partial geometries $pg(7, 11, 5)$ resp. $pg(9, 10, 6)$. Do such geometries exist?

   (d) Another well-known problem in this area is the existence of a $pg(5, 28, 2)$. If it exists, its point graph would be the McLaughlin graph of order 275.

4. Problems related to polyhedral maps

(a) Existence of weakly neighborly equivelar maps on more than 16 points.

(b) Existence of new w.n.e. maps invariant under a regular group. Here, the use of Schur rings should be helpful.

5. Problems related to square root partial linear spaces

   (a) Classify all examples on 96 points.

   (b) Find further examples. The next possible orders are 175 (corresponding to a $GQ(6,4)$) and 288 ($GQ(7,5)$). For 288, no strongly regular graph is known.

   (c) Find an infinite series.

6. Problems related to strongly regular graphs

   (a) Describe all partial difference sets of sizes 19 and 20 on 96 points, and classify the corresponding Cayley graphs.

   (b) Describe all partial difference sets on 96 and 100 points, and classify the corresponding Cayley graphs.

   (c) (hard) Fill the gaps in E. van Dam's list of three-class association schemes on 96 and 100 points.

   (d) Are there three mutually orthogonal Latin squares of order 10?

   (e) Find a system of three linked designs with non-classical parameters.

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Appendix A

A CATALOG OF COMBINATORIAL OBJECTS

In this supplement, we give information about several combinatorial objects that were constructed during the preparation of this thesis. In some cases we cannot include all the information necessary to reconstruct the objects; these objects will be made available on the World Wide Web.

A.1 Siamese partitions on 40 points

Below is a list of Siamese partitions of Steiner systems on 40 points. This list is by no means complete. For each such system, we give the size of the automorphism group of the design, followed by the blocks. The blocks are divided into five parts: 10 blocks which form a spread, and 4 sets of 30 blocks each, which together with the spread form generalized quadrangles.

Note that System 1 is a partition of the lines of $PG(3,3)$. Systems 6 and 8 (with groups of size 480 and 160) have been discussed in the text.

```
System 1

automorphism group of size 24261120
 1  2  3  4   5  8 22 40   6 13 25 32   7 16 27 30   9 11 17 29
10 15 33 35  12 26 37 38  14 18 23 39  19 20 28 34  21 24 31 36

 1  5  6  7   1 17 18 19   1 35 36 37   2  8 23 26   2  9 24 27
 2 10 25 28   3 11 16 21   3 12 14 22   3 13 15 20   4 29 34 39
 4 30 32 40   4 31 33 38   5  9 20 38   5 10 21 39   6 11 23 33
 6 12 24 34   7 14 28 31   7 15 26 29   8 13 17 31   8 16 34 35
```

9 14 32 35	10 12 17 30	11 28 37 40	13 27 37 39	15 18 24 40
16 18 25 38	19 21 26 32	19 22 27 33	20 23 30 36	22 25 29 36

1 8 9 10	1 23 24 25	1 26 27 28	2 29 32 38	2 30 33 39
2 31 34 40	3 5 19 36	3 6 17 37	3 7 18 35	4 11 15 22
4 12 16 20	4 13 14 21	5 12 25 33	5 14 27 29	6 8 20 39
6 15 27 31	7 8 21 38	7 11 25 34	9 12 18 31	9 13 19 30
10 14 34 37	10 16 32 36	11 26 36 39	13 26 35 40	15 19 23 38
16 17 23 40	17 21 28 33	18 22 28 32	20 24 29 35	22 24 30 37

1 11 12 13	1 14 15 16	1 20 21 22	2 5 17 35	2 6 18 36
2 7 19 37	3 29 33 40	3 30 34 38	3 31 32 39	4 8 24 28
4 9 25 26	4 10 23 27	5 13 23 34	5 16 26 31	6 10 22 38
6 16 28 29	7 9 22 39	7 13 24 33	8 11 18 30	8 15 32 37
9 15 34 36	10 11 19 31	12 27 36 40	12 28 35 39	14 17 24 38
14 19 25 40	17 20 27 32	18 20 26 33	21 23 29 37	21 25 30 35

1 29 30 31	1 32 33 34	1 38 39 40	2 11 14 20	2 12 15 21
2 13 16 22	3 8 25 27	3 9 23 28	3 10 24 26	4 5 18 37
4 6 19 35	4 7 17 36	5 11 24 32	5 15 28 30	6 9 21 40
6 14 26 30	7 10 20 40	7 12 23 32	8 12 19 29	8 14 33 36
9 16 33 37	10 13 18 29	11 27 35 38	13 28 36 38	15 17 25 39
16 19 24 39	17 22 26 34	18 21 27 34	20 25 31 37	22 23 31 35

System 2

automorphism group of size 10368

1 2 3 4	5 11 17 26	6 20 33 39	7 9 23 36	8 15 18 21
10 13 30 32	12 16 24 40	14 27 34 35	19 31 37 38	22 25 28 29

1 5 6 7	1 14 15 16	1 29 30 31	2 8 26 38	2 10 27 39
2 28 36 40	3 17 23 33	3 18 24 34	3 19 25 32	4 9 12 22
4 11 21 37	4 13 20 35	5 12 19 27	5 13 18 28	6 21 32 40
6 22 34 38	7 8 25 35	7 10 24 37	8 12 30 33	9 14 26 32
9 18 31 39	10 15 17 22	11 16 25 39	11 30 34 36	13 16 23 38
14 28 33 37	15 19 20 36	17 31 35 40	20 24 26 29	21 23 27 29

1 8 9 10	1 26 27 28	1 38 39 40	2 7 19 34	2 17 24 32
2 18 25 33	3 11 22 35	3 12 20 37	3 13 21 36	4 5 16 30
4 6 15 29	4 14 23 31	5 9 25 37	5 21 34 39	6 9 24 35
6 13 19 26	7 11 15 40	7 20 28 30	8 14 17 20	8 16 19 22

```
10 11 31 33  10 12 29 34  12 18 23 26  13 14 25 40  15 27 32 37
16 27 33 36  17 29 36 38  18 30 35 38  21 24 28 31  22 23 32 39

 1 11 12 13   1 20 21 22   1 35 36 37   2  5 14 29   2  6 16 31
 2 15 23 30   3  8 28 39   3  9 27 38   3 10 26 40   4  7 18 32
 4 17 25 34   4 19 24 33   5  8 24 36   5 20 32 38   6 10 25 36
 6 11 18 27   7 12 14 39   7 22 26 31   8 13 31 34   9 13 29 33
 9 16 17 21  10 14 19 21  11 19 23 28  12 15 25 38  15 26 33 35
16 28 32 35  17 30 37 39  18 29 37 40  20 23 34 40  22 24 27 30

 1 17 18 19   1 23 24 25   1 32 33 34   2  9 11 20   2 12 21 35
 2 13 22 37   3  5 15 31   3  6 14 30   3  7 16 29   4  8 27 40
 4 10 28 38   4 26 36 39   5 10 23 35   5 22 33 40   6  8 23 37
 6 12 17 28   7 13 17 27   7 21 33 38   8 11 29 32   9 15 28 34
 9 19 30 40  10 16 18 20  11 14 24 38  12 31 32 36  13 15 24 39
14 18 22 36  16 26 34 37  19 29 35 39  20 25 27 31  21 25 26 30

System 3

automorphism group of size 1296
 1  2  3  4   5 16 30 36   6 18 25 34   7 10 20 38   8 19 27 31
 9 13 24 37  11 14 21 35  12 23 32 40  15 26 29 39  17 22 28 33

 1  5  6  7   1 11 12 13   1 26 27 28   2 14 20 33   2 16 31 40
 2 24 34 39   3 10 18 36   3 19 22 29   3 21 23 37   4  8 30 32
 4  9 15 25   4 17 35 38   5 15 23 33   5 19 24 35   6 14 29 32
 6 17 31 37   7  8 21 39   7  9 22 40   8 13 18 33   9 14 27 36
10 11 15 31  10 24 28 32  11 22 30 34  12 17 36 39  12 19 20 25
13 16 29 38  16 21 25 28  18 26 35 40  20 26 30 37  23 27 34 38

 1  8  9 10   1 14 15 16   1 32 33 34   2 12 26 38   2 13 22 36
 2 18 21 27   3  7 24 31   3 17 25 40   3 30 35 39   4  5 28 37
 4  6 11 19   4 20 23 29   5 10 21 40   5 26 31 34   6 10 22 39
 6 12 16 24   7 11 32 36   7 15 18 28   8 12 28 35   8 25 29 36
 9 11 17 26   9 18 23 30  13 15 19 40  13 20 34 35  14 22 23 31
14 25 37 38  16 17 20 27  19 30 33 38  21 24 29 33  27 32 37 39

 1 17 18 19   1 35 36 37   1 38 39 40   2  8 11 23   2  9 28 29
 2 10 25 30   3  5 13 14   3  6 27 33   3 15 20 32   4  7 16 34
 4 12 21 31   4 22 24 26   5  8 22 38   5 12 18 29   6  9 21 38
 6 23 26 36   7 13 17 23   7 27 29 35   8 15 34 37   9 16 19 32
```

```
10 12 33 37  10 14 19 26  11 16 33 39  11 18 20 24  13 25 31 39
14 28 34 40  15 17 21 30  20 28 31 36  22 25 32 35  24 27 30 40

 1 20 21 22   1 23 24 25   1 29 30 31   2  5 17 32   2  6 15 35
 2  7 19 37   3  8 16 26   3  9 12 34   3 11 28 38   4 10 13 27
 4 14 18 39   4 33 36 40   5  9 20 39   5 11 25 27   6  8 20 40
 6 13 28 30   7 12 14 30   7 25 26 33   8 14 17 24   9 31 33 35
10 16 23 35  10 17 29 34  11 29 37 40  12 15 22 27  13 21 26 32
15 24 36 38  16 18 22 37  18 31 32 38  19 21 34 36  19 23 28 39

System 4

automorphism group of size 1296
 1  2  3  4   5 14 31 36   6 12 17 26   7  9 23 39   8 28 29 32
10 11 15 21  13 25 33 37  16 18 34 38  19 20 24 30  22 27 35 40

 1  5  6  7   1 20 21 22   1 32 33 34   2  9 11 29   2 12 30 38
 2 13 31 40   3  8 18 37   3 10 19 35   3 17 23 36   4 14 25 27
 4 15 28 39   4 16 24 26   5 15 30 37   5 16 29 35   6 11 18 27
 6 13 19 28   7  8 24 40   7 10 25 38   8 12 14 21   9 14 19 34
 9 22 26 37  10 26 31 32  11 24 33 36  12 33 35 39  13 16 21 23
15 17 34 40  17 20 25 29  18 20 31 39  22 28 36 38  23 27 30 32

 1  8  9 10   1 17 18 19   1 35 36 37   2 14 23 26   2 15 24 27
 2 16 25 28   3  5 22 33   3  6 20 34   3 21 32 39   4  7 12 31
 4 11 30 40   4 13 29 38   5  9 24 38   5 11 17 28   6  9 25 40
 6 15 29 36   7 15 18 33   7 20 28 35   8 26 30 33   8 27 31 34
10 12 16 22  10 13 14 20  11 23 34 37  12 24 32 37  13 17 27 39
14 18 32 40  16 30 36 39  19 21 25 31  19 22 23 29  21 26 35 38

 1 11 12 13   1 29 30 31   1 38 39 40   2  7 19 37   2  8 17 35
 2 10 18 36   3  9 16 27   3 14 24 28   3 15 25 26   4  5 21 34
 4  6 22 32   4 20 23 33   5  8 25 39   5 12 19 27   6 10 24 39
 6 16 31 37   7 11 14 22   7 26 29 34   8 11 16 20   9 13 32 36
 9 17 21 30  10 27 29 33  12 18 23 28  13 24 34 35  14 17 33 38
15 19 32 38  15 23 31 35  18 22 25 30  20 26 36 40  21 28 37 40

 1 14 15 16   1 23 24 25   1 26 27 28   2  5 20 32   2  6 21 33
 2 22 34 39   3  7 13 30   3 11 31 38   3 12 29 40   4  8 19 36
 4  9 18 35   4 10 17 37   5 10 23 40   5 13 18 26   6  8 23 38
 6 14 30 35   7 16 17 32   7 21 27 36   8 13 15 22   9 12 15 20
```

```
 9 28 31 33  10 28 30 34  11 19 26 39  11 25 32 35  12 25 34 36
14 29 37 39  16 19 33 40  17 22 24 31  18 21 24 29  20 27 37 38
```

System 5

automorphism group of size 648

```
 1  2  3  4   5 11 19 27   6 16 24 40   7  9 22 32   8 12 15 29
10 17 25 36  13 26 31 34  14 20 30 39  18 21 28 35  23 33 37 38

 1  5  6  7   1 29 30 31   1 35 36 37   2 12 27 33   2 22 28 39
 2 25 34 40   3 13 15 20   3 16 19 32   3 17 21 23   4  8 11 38
 4  9 14 18   4 10 24 26   5 15 18 25   5 23 26 39   6 12 14 17
 6 13 28 38   7  8 21 34   7 10 20 33   8 16 36 39   9 13 27 36
 9 23 29 40  10 19 28 29  11 17 22 31  11 20 35 40  12 26 32 35
14 19 34 37  15 22 24 37  16 18 31 33  21 24 27 30  25 30 32 38

 1  8  9 10   1 26 27 28   1 38 39 40   2  7 31 37   2 16 21 29
 2 19 20 36   3 11 18 34   3 12 22 25   3 14 24 33   4  5 17 30
 4  6 15 35   4 13 23 32   5  9 21 33   5 12 31 40   6  9 20 34
 6 25 27 37   7 11 15 39   7 17 24 28   8 13 19 24   8 18 30 37
10 11 16 23  10 14 31 35  12 20 23 28  13 21 25 39  14 27 29 32
15 26 33 36  16 22 26 30  17 29 34 38  18 32 36 40  19 22 35 38

 1 11 12 13   1 20 21 22   1 23 24 25   2  5 14 38   2  6 18 26
 2 15 17 32   3  8 27 31   3  9 30 35   3 10 37 40   4  7 29 36
 4 16 28 34   4 19 33 39   5 10 22 34   5 24 29 35   6  8 22 33
 6 11 30 36   7 13 14 40   7 18 23 27   8 14 25 28   9 12 16 38
 9 19 25 26  10 12 18 39  11 28 32 37  13 17 33 35  15 19 21 40
15 23 30 34  16 17 20 27  20 26 29 37  21 31 36 38  24 31 32 39

 1 14 15 16   1 17 18 19   1 32 33 34   2  8 23 35   2  9 11 24
 2 10 13 30   3  5 28 36   3  6 29 39   3  7 26 38   4 12 21 37
 4 20 25 31   4 22 27 40   5  8 20 32   5 13 16 37   6 10 21 32
 6 19 23 31   7 12 19 30   7 16 25 35   8 17 26 40   9 15 28 31
 9 17 37 39  10 15 27 38  11 14 21 26  11 25 29 33  12 24 34 36
13 18 22 29  14 22 23 36  18 20 24 38  27 34 35 39  28 30 33 40
```

System 6

automorphism group of size 480

```
 1 23 24 25   2 15 19 22   3  5 27 40   4 12 29 34   6 20 31 37
```

7 8 13 16	9 21 26 32	10 17 30 39	11 18 28 36	14 33 35 38

1 2 3 4	1 8 9 10	1 35 36 37	2 13 18 21	2 17 20 38
3 6 26 39	3 7 14 28	4 11 31 32	4 16 30 33	5 8 11 38
5 19 25 34	5 21 30 37	6 8 12 22	6 18 25 33	7 15 29 37
7 17 25 32	9 15 27 33	9 20 28 34	10 14 19 31	10 18 29 40
11 15 24 39	12 14 21 24	12 17 27 36	13 23 27 31	13 34 35 39
16 19 26 36	16 20 24 40	22 23 28 30	22 32 35 40	23 26 29 38

1 5 6 7	1 17 18 19	1 32 33 34	2 5 14 26	2 10 25 36
2 16 31 34	3 8 25 37	3 11 30 34	3 19 21 38	4 5 28 39
4 9 25 35	4 13 19 20	6 15 30 35	6 21 29 36	7 9 11 22
7 10 12 38	8 14 17 29	8 15 28 32	9 17 31 40	10 20 27 32
11 14 20 23	12 15 23 40	12 18 26 37	13 24 26 30	13 33 36 40
16 18 27 35	16 21 23 39	22 24 27 29	22 33 37 39	24 28 31 38

1 11 12 13	1 20 21 22	1 38 39 40	2 8 24 35	2 9 12 30
2 28 37 40	3 9 23 36	3 13 15 17	3 20 29 35	4 8 21 27
4 10 23 37	4 15 18 38	5 10 16 22	5 12 31 33	5 18 24 32
6 9 16 38	6 11 19 27	6 17 24 34	7 18 20 30	7 19 23 33
7 26 34 40	8 31 36 39	10 11 26 35	13 14 32 37	14 22 34 36
14 25 27 30	15 25 26 31	16 25 28 29	17 21 28 33	19 29 32 39

1 14 15 16	1 26 27 28	1 29 30 31	2 6 23 32	2 7 27 39
2 11 29 33	3 10 24 33	3 12 16 32	3 18 22 31	4 6 14 40
4 7 24 36	4 17 22 26	5 9 13 29	5 15 20 36	5 17 23 35
6 10 13 28	7 21 31 35	8 18 23 34	8 19 30 40	8 20 26 33
9 14 18 39	9 19 24 37	10 15 21 34	11 16 17 37	11 21 25 40
12 19 28 35	12 20 25 39	13 22 25 38	27 34 37 38	30 32 36 38

System 7

automorphism group of size 288

1 2 3 4	5 12 15 24	6 17 31 40	7 9 20 36	8 14 18 27
10 13 30 33	11 21 26 39	16 34 35 38	19 22 25 32	23 28 29 37

1 5 6 7	1 26 27 28	1 32 33 34	2 13 37 40	2 18 24 35
2 25 36 39	3 14 21 29	3 15 20 31	3 16 22 30	4 8 12 38
4 9 11 19	4 10 17 23	5 13 14 19	5 16 23 39	6 11 18 30
6 25 29 38	7 8 22 37	7 10 21 35	8 31 33 39	9 16 27 40
9 24 29 33	10 15 25 27	11 15 34 37	12 21 32 40	12 28 30 36

```
13 20 26 38  14 17 34 36  17 22 24 26  18 20 23 32  19 28 31 35

 1  8  9 10   1 23 24 25   1 38 39 40   2  7 28 34   2 15 21 33
 2 22 27 31   3 11 17 35   3 12 19 37   3 13 18 36   4  5 26 30
 4  6 14 32   4 16 20 29   5  9 22 35   5 18 28 40   6  9 21 37
 6 13 24 34   7 12 14 39   7 17 25 30   8 15 17 29   8 19 26 34
10 11 28 32  10 12 16 31  11 20 24 27  13 22 29 39  14 23 33 35
15 32 36 38  16 18 21 25  19 20 33 40  23 26 31 36  27 30 37 38

 1 11 12 13   1 17 18 19   1 35 36 37   2  5 29 32   2  6 16 26
 2 14 20 30   3  8 25 40   3  9 23 38   3 10 24 39   4  7 27 33
 4 15 22 28   4 21 31 34   5  8 21 36   5 17 33 38   6 10 22 36
 6 12 23 27   7 11 29 40   7 16 19 24   8 13 16 28   9 13 31 32
 9 15 18 26  10 18 29 34  11 14 22 38  12 20 25 34  14 24 31 37
15 30 35 40  17 20 28 39  19 21 23 30  25 26 33 37  27 32 35 39

 1 14 15 16   1 20 21 22   1 29 30 31   2  8 11 23   2  9 12 17
 2 10 19 38   3  5 27 34   3  6 28 33   3  7 26 32   4 13 25 35
 4 18 37 39   4 24 36 40   5 10 20 37   5 11 25 31   6  8 20 35
 6 15 19 39   7 13 15 23   7 18 31 38   8 24 30 32   9 14 25 28
 9 30 34 39  10 14 26 40  11 16 33 36  12 18 22 33  12 26 29 35
13 17 21 27  16 17 32 37  19 27 29 36  21 24 28 38  22 23 34 40

System 8

automorphism group of size 160
 1  5  6  7   2 11 14 23   3  9 27 30   4 19 36 40   8 12 18 22
10 15 32 38  13 31 33 37  16 20 28 35  17 25 26 34  21 24 29 39

 1  2  3  4   1 20 21 22   1 32 33 34   2 12 15 24   2 16 25 37
 3  8 31 35   3 10 26 29   4 13 18 38   4 17 28 39   5  9 37 39
 5 12 26 36   5 23 35 38   6 14 19 30   6 15 17 31   6 16 18 29
 7  8 25 40   7 10 11 28   7 13 24 27   8 14 32 39   9 11 18 34
 9 15 20 40  10 19 22 37  11 21 31 36  12 28 30 33  13 14 20 26
16 27 32 36  17 22 23 27  19 24 34 35  21 25 30 38  23 29 33 40

 1  8  9 10   1 23 24 25   1 35 36 37   2  7 18 33   2 10 34 40
 2 21 27 35   3  7 17 20   3 11 32 37   3 22 24 40   4  7 15 29
 4  8 11 16   4 25 27 31   5 14 22 31   5 16 30 34   5 19 21 32
 6 11 26 39   6 12 27 38   6 13 28 40   8 13 17 21   9 14 28 29
 9 19 26 33  10 20 31 39  12 19 20 23  12 29 34 37  13 15 23 30
```

```
14 17 36 38  15 22 26 35  16 24 33 38  18 25 28 32  18 30 36 39

 1 14 15 16   1 17 18 19   1 29 30 31   2  6 20 32   2  8 26 30
 2 13 19 39   3  6 21 33   3 12 14 25   3 19 28 38   4  6 22 34
 4 10 30 35   4 14 24 37   5  8 24 28   5 10 13 25   5 11 27 40
 7  9 23 36   7 12 35 39   7 26 37 38   8 15 33 36   9 13 16 22
 9 17 24 32  10 18 21 23  11 17 33 35  11 22 29 38  12 31 32 40
15 27 34 39  16 21 26 40  18 20 27 37  20 25 29 36  23 28 31 34

 1 11 12 13   1 26 27 28   1 38 39 40   2  5 17 29   2  9 31 38
 2 22 28 36   3  5 15 18   3 13 34 36   3 16 23 39   4  5 20 33
 4  9 12 21   4 23 26 32   6  8 23 37   6  9 25 35   6 10 24 36
 7 14 21 34   7 16 19 31   7 22 30 32   8 19 27 29   8 20 34 38
10 12 16 17  10 14 27 33  11 15 19 25  11 20 24 30  13 29 32 35
14 18 35 40  15 21 28 37  17 30 37 40  18 24 26 31  22 25 33 39

System 9

automorphism group of size 144
 1 38 39 40   2 13 18 21   3 14 33 36   4  8 27 31   5 16 17 29
 6 10 22 35   7 12 28 34   9 19 23 32  11 24 30 37  15 20 25 26

 1  2  3  4   1  5  6  7   1 23 24 25   2 10 19 34   2 17 20 30
 3 12 29 37   3 15 22 32   4  9 11 28   4 16 26 35   5 11 15 18
 5 19 31 33   6  8 21 37   6  9 20 36   7 13 14 26   7 27 30 32
 8 14 17 23   8 15 34 40   9 13 29 40  10 11 14 39  10 25 27 29
12 18 23 35  12 20 31 39  13 22 24 31  16 21 32 39  16 24 34 36
17 22 28 38  18 27 36 38  19 26 37 38  21 25 28 33  30 33 35 40

 1  8  9 10   1 11 12 13   1 14 15 16   2  7 16 31   2  9 24 33
 2 22 25 39   3  7 19 40   3  8 25 30   3 13 17 35   4  6 12 33
 4 15 19 21   4 17 37 39   5 10 21 36   5 12 25 32   5 24 27 40
 6 16 23 30   6 18 26 40   7 10 20 37   8 18 28 29   9 17 26 34
11 19 22 29  11 26 31 36  13 20 23 27  14 18 32 37  14 22 27 34
15 24 28 35  20 29 33 38  21 30 34 38  23 28 36 39  31 32 35 38

 1 17 18 19   1 26 27 28   1 35 36 37   2  5 23 38   2  6 11 27
 2 12 15 36   3  9 16 27   3 10 24 38   3 18 20 34   4  7 25 38
 4 18 22 30   4 29 32 36   5  8 20 35   5 14 28 30   6 14 19 25
 6 29 34 39   7  9 21 35   7 11 17 33   8 12 19 24   8 13 33 39
 9 15 30 39  10 13 28 32  10 15 17 31  11 20 32 40  12 16 22 40
```

13 16 25 37	14 21 31 40	21 24 26 29	22 23 26 33	23 31 34 37

1 20 21 22	1 29 30 31	1 32 33 34	2 8 26 32	2 14 29 35
2 28 37 40	3 5 26 39	3 6 28 31	3 11 21 23	4 5 13 34
4 10 23 40	4 14 20 24	5 9 22 37	6 13 15 38	6 17 24 32
7 8 22 36	7 15 23 29	7 18 24 39	8 11 16 38	9 12 14 38
9 18 25 31	10 12 26 30	10 16 18 33	11 25 34 35	12 17 21 27
13 19 30 36	15 27 33 37	16 19 20 28	17 25 36 40	19 27 35 39

A.2 Strongly regular graphs on 96 points

In the already mentioned paper by Brouwer, Koolen, and Klin [17], several new srg's of order 96 were constructed. In particular, they constructed 32 srg$(96, 20, 4, 4)$ and 3 srg$(96, 19, 2, 4)$. Only few of them were known before.

Here, we describe the construction of several more srg's with the same parameter sets. The following fact was used: If we take the vertices of such a srg as points, and the sets of neighbors of each vertex as blocks (including the vertex itself in the case of valency 19), we obtain a symmetric $2-(96, 20, 4)$ design. Conversely, given such a design, then an absolute polarity leads to an srg$(96, 19, 2, 4)$, whereas a polarity without absolute points leads to an srg$(96, 20, 4, 4)$.

We also used the following heuristical fact: The centralizer algebra of the automorphism group of a strongly regular graph is likely to contain other srg's with the same parameters, in particular, if the group is relatively small.

This led to the following heuristical construction:

- Start with a few known strongly regular graphs.
- For each graph, find the corresponding design.
- For each design, find all polarities with or without absolute points to get new graphs.

- For each graph, try to construct all srg's invariant under its automorphism group.

This has led to the construction of 8 non-isomorphic graphs of valency 19, and 92 non-isomorphic graphs of valency 20. Some of these graphs were previously known. Details about these graphs can be found below.

A.2.1 Graphs of valency 19

We constructed 8 non-isomorphic srg$(96, 19, 2, 4)$. Since $\lambda = 2$, the subgraph induced by the neighbors of a vertex is regular of valency 2, hence it is the union of disjoint cycles. The lengths of these cycles provides an invariant for the vertices, which we call the cycle structure.

i	$\|Aut\|$	Orbits	Cycle structures
1	9216	96	$3, 4^4$
2	1536	96	$3, 8^2$
3	1024	$64, 32$	$3, 4^4; 3, 8^2$
4	768	96	$3, 8^2$
5	768	$64, 24, 8$	$3, 4^4; 3, 4, 12$
6	768	96	$3, 4, 12$
7	768	96	$3, 4, 12$
8	768	96	$3, 4, 12$

The first graph with a group of order 9216 was first constructed by Haemers. In [17], two more graphs were constructed, with groups of size 192 and 144. We believe that the graphs 2 through 8 in the table above are new.

A.2.2 Graphs of valency 20

A total of 92 non-isomorphic srg$(96, 20, 4, 4)$ have been constructed. We list below the sizes of their automorphism groups, the number of its orbits, and the number of n-cliques for $n = 4, 5, 6$.

Some of these graphs appeared previously in [17]. There, a list of 24 non-isomorphic graphs appears, including the first three in our list. Note that the first graph is the point graph of the unique $GQ(5, 3)$. Furthermore, the list there includes three graphs with a group of order 768, one each with groups of the orders 1536, 576, 256, 144, and 128, as well as 13 graphs with smaller groups. Hence it is reasonable to assume that at least 81 of the graphs listed below are new.

Although several of the graphs contain 96 5-cliques, only the third graph in the list is the point graph of a partial linear space of order $(5, 5)$.

i	$\|Aut\|$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
1	138240	1	960	384	64
2	11520	1	480	96	16
3	11520	1	480	96	0
4	7680	2	320	64	0
5	7680	1	0	0	0
6	3072	2	576	128	0
7	3072	1	192	0	0
8	3072	1	192	0	0
9	3072	2	576	128	0
10	1536	1	192	0	0

i	$\|Aut\|$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
11	1536	1	192	0	0
12	1536	3	768	256	32
13	1536	2	576	128	0
14	1536	2	576	128	0
15	1536	3	768	256	32
16	1536	3	576	128	0
17	1152	3	672	192	16
18	1024	2	192	0	0
19	1024	2	320	0	0
20	1024	2	320	0	0

i	$\|Aut\|$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
21	1024	2	64	0	0
22	1024	2	64	0	0
23	768	2	480	96	16
24	768	1	192	0	0
25	768	2	480	96	0
26	768	1	192	0	0
27	768	1	96	0	0
28	768	2	480	96	0
29	768	1	192	0	0
30	768	2	96	0	0

i	$\lvert Aut \rvert$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
31	768	2	480	96	16
32	768	1	384	0	0
33	768	1	96	0	0
34	768	1	192	0	0
35	768	1	192	0	0
36	768	3	384	96	16
37	768	1	96	0	0
38	768	1	192	0	0
39	768	2	96	0	0
40	768	1	96	0	0

i	$\lvert Aut \rvert$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
41	768	1	192	0	0
42	576	1	288	0	0
43	512	3	256	0	0
44	512	3	192	0	0
45	512	3	128	0	0
46	512	3	448	64	0
47	512	2	192	0	0
48	512	3	64	0	0
49	512	3	128	0	0
50	512	2	64	0	0

i	$\|Aut\|$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
51	512	2	192	0	0
52	512	3	192	0	0
53	512	3	320	64	0
54	512	2	64	0	0
55	512	2	0	0	0
56	512	3	256	0	0
57	512	2	192	0	0
58	512	2	64	0	0
59	512	2	64	0	0
60	512	2	64	0	0

i	$\|Aut\|$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
61	512	2	64	0	0
62	512	2	192	0	0
63	512	3	256	0	0
64	512	3	64	0	0
65	512	2	64	0	0
66	512	2	256	0	0
67	512	2	192	0	0
68	512	2	320	0	0
69	384	2	384	48	0
70	384	2	96	0	0

i	$\lvert Aut\rvert$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
71	384	1	192	0	0
72	384	2	192	0	0
73	384	1	192	0	0
74	384	2	192	0	0
75	384	2	288	48	0
76	384	2	288	48	8
77	384	1	192	0	0
78	384	1	192	0	0
79	384	1	192	0	0
80	384	2	96	0	0

i	$\lvert Aut\rvert$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
81	384	2	96	0	0
82	384	1	192	0	0
83	384	1	192	0	0
84	384	2	192	0	0
85	384	2	192	0	0
86	384	2	384	48	8
87	384	2	96	0	0
88	256	2	192	0	0
89	256	3	192	0	0
90	256	3	192	0	0

i	$\lvert Aut\rvert$	orbits	$\#K_4$	$\#K_5$	$\#K_6$
91	256	3	64	0	0
92	128	4	192	0	0

A.3 Strongly regular graphs on 100 points

The next set of parameters we consider is that of an srg$(100, 36, 14, 12)$. These are the parameters of a rank 3 representation of the Hall-Janko group J_2. This graph is also pseudo-geometrical, corresponding to two orthogonal Latin squares of order 10.

Recently, Klin and Jørgensen constructed a further example of a non-geometric graph with these parameters, see [59].

If we take the subgraph induced by the neighbors of one vertex in such a graph, we get a regular graph of order 36 and valency 14. The best known such graph is strongly regular; it is the triangular graph $T(9)$. Thus the question arises whether there is an srg$(100, 36, 14, 12)$ such that the first subconstituent is isomorphic to $T(9)$. We will call such a graph a *combinatorial extension* of $T(9)$.

If we consider the graph $T(10)$ on 45 vertices, its complement is a pseudo-geometric $pg(5, 7, 3)$ graph. In fact, there are exactly two partial geometries with these parameters, see Haemers [45]. One of them has $H = P\Gamma L_2(8)$ as its automorphism group. This group has two orbits on points of lengths 36 and 9. The induced subgraph on the longer orbit is $T(9)$.

If we take the action of H on the lines and the longer orbit of points, we get a group of degree 99. Adding one isolated point gives us degree 100.

In this action, H is generated by the following permutations:

```
(2,16,17,31,6,14,35,28,26)(3,8,34,4,15,18,7,11,29)
(5,9,23,27,20,22,33,21,25)(10,24,30,36,12,32,37,13,19)
(38,94,55,72,39,71,48,77,45)(40,43,95,96,68,58,81,100,64)
(41,65,88,93,66,84,44,85,97)(42,91,73,83,67,50,60,80,57)
(46,76,56,54,62,70,59,79,51)(47,75,74,52,89,49,69,87,53)
(61,82,86,92,63,78,90,99,98)

(2,33,18,19,11,32)(3,35,31,12,24,10)(5,7,37,27,16,20)
(6,9)(8,15,17,13,30,23)(14,25,28,29,36,26)(21,22,34)
(38,54,55,53,52,57)(39,82)(40,58,70,66,79,65)(41,90,75,47,48,93)
(42,95,91,63,59,69)(43,76,62,73,87,99)(44,83,49)(45,81,68,80,67,78)
```

```
(46,94,96,61,74,86)(50,100,77,89,97,84)(51,60,72,88,92,85)(64,71,98)
```

Orbit representatives for this action are 1 (orbit length 1); 2 (orbit length 36), and 38 (orbit length 63).

The group H has 12 orbits on unordered pairs of points. Representatives are:

```
1:[  1,  2 ],  2:[  1, 38 ],  3:[  2,  3 ],  4:[  2,  4 ],
5:[  2, 38 ],  6:[  2, 45 ],  7:[  2, 46 ],  8:[  2, 47 ],
9:[ 38, 39 ], 10:[ 38, 45 ], 11:[ 38, 47 ], 12:[ 38, 61 ]
```

If we choose the orbits $1, 3, 8, 10$ as edges of a graph, we obtain an $srg(100, 36, 12, 14)$ with automorphism group $H = P\Gamma L(2, 8)$. This graph Γ_1 is the only strongly regular graph invariant under this action.

The group H has a maximal subgroup G of index 9, namely the point stabilizer in the natural action of $P\Gamma L_2(8)$ on 9 points. If we consider the action of G on 100 points, we obtain five more strongly regular graphs of valency 36.

Generators for G are as follows:

```
( 2,22,27,25,33,14,12)( 3,24, 4, 5,37,31,16)( 6,17,30,23, 8,13,15)
( 7,32,35,34,26,36,11)(10,29,18,19,21,28,20)(38,61,42,68,95,50,90)
(39,73,99,100,76,84,89)(40,78,79,70,82,81,80)(41,43,46,92,54,58,88)
(44,56,64,98,83,71,49)(45,97,52,59,69,93,94)(47,67,55,63,86,77,91)
(48,74,57,60,87,72,65)(51,75,53,62,66,96,85),

( 2,10,34)( 3, 4,31)( 6,14,33)( 7,27,36)( 8,11,26)( 9,24,37)
(12,19,21)(13,20,18)(15,22,30)(17,32,29)(23,28,35)(38,83,45)
(39,84,76)(40,75,87)(41,47,56)(42,78,74)(43,85,94)(44,86,80)
(46,52,72)(48,57,77)(49,53,96)(50,54,68)(51,55,93)(59,64,60)
(61,63,62)(65,71,95)(66,81,90)(67,88,82)(69,79,97)(70,92,98)
(73,99,89)
```

Orbit representatives are: 1 (isolated), 2 (orbit length 28), 3 (orbit length 8), 38 (orbit length 56), and 39 (orbit length 7). G has 43 orbits on unordered pairs. Representatives are

```
 1:[  1,  2 ],  2:[  1,  3 ],  3:[  1, 38 ],  4:[  1, 39 ],
 5:[  2,  3 ],  6:[  2,  4 ],  7:[  2,  6 ],  8:[  2,  7 ],
```

```
 9:[  2,  8 ], 10:[  2, 38 ], 11:[  2, 39 ], 12:[  2, 45 ],
13:[  2, 46 ], 14:[  2, 47 ], 15:[  2, 52 ], 16:[  2, 53 ],
17:[  2, 54 ], 17:[  2, 55 ], 19:[  2, 56 ], 20:[  2, 57 ],
21:[  2, 73 ], 21:[  2, 84 ], 23:[  3,  4 ], 24:[  3, 38 ],
25:[  3, 39 ], 26:[  3, 40 ], 27:[  3, 45 ], 28:[  3, 57 ],
29:[ 38, 39 ], 30:[ 38, 40 ], 31:[ 38, 41 ], 32:[ 38, 42 ],
33:[ 38, 45 ], 34:[ 38, 47 ], 35:[ 38, 48 ], 36:[ 38, 49 ],
37:[ 38, 52 ], 38:[ 38, 61 ], 39:[ 38, 62 ], 40:[ 38, 73 ],
41:[ 38, 87 ], 42:[ 38,100 ], 43:[ 39, 73 ]
```

There are 6 non-isomorphic $\mathrm{srg}(100, 36, 12, 14)$ which are invariant under this action. The representatives for these isomorphism classes that we give here all include the edges from the following orbits: $1, 2, 5, 8, 19, 23, 24, 31, 33$.

The remaining edge orbits we add to each graph are given below:

$$\begin{aligned}
\Gamma_1 &: 10, 20, 22, 32, 38, 40 \\
\Gamma_2 &: 10, 20, 21, 29, 32, 38 \\
\Gamma_3 &: 15, 16, 21, 29, 35, 39 \\
\Gamma_4 &: 18, 20, 21, 29, 35, 39 \\
\Gamma_5 &: 15, 16, 22, 35, 39, 40 \\
\Gamma_6 &: 18, 20, 22, 35, 39, 40
\end{aligned}$$

The graph Γ_1 is the same as considered before; hence it is invariant under $H = P\Gamma L(2, 8)$. All other graphs have G as their full automorphism groups.

A.4 Distance regular graphs on 40 points

Here, we describe two distance regular graphs with intersection array $(9, 6, 1; 1, 2, 9)$. These graphs are 4-fold antipodal covers of K_{10}. They do not come from a generalized quadrangle; hence they provide the smallest examples to the problem by Godsil-Hensel [40]. Using E. Spences database of strongly regular graphs, it has been checked that these are the only examples on 40 points.

Consider the complete bipartite graph $K_{3,4}$ with classes A, B, $|A| = 3$, $|B| = 4$. We define three kinds of vertices:

- Elements of B: 4 vertices;
- Edges of $K_{3,4}$: 12 vertices;
- Triples of non-incident edges: 24 vertices.

The group $S_3 \times S_4 = Aut(K_{3,4})$ acts transitively on each type of vertices. We get an intransitive group of degree 40.

Below we include the COCO protocol of the investigation of this action.

```
*** system CO-CO for construction of coherent configurations ***

output listing to file drg2.res

COCO>> ind * * drg2

* induced action of permutation group on structure *

input of group from <stdin>
input of structure for inducing from <stdin>
output of induced group to file drg2.gen

degree of group:
7
number of generators of group:
4
generators of group:
1.(0,1)
2.(0,1,2,3)
3.(4,5)
4.(4,5,6)

inducing on structure:
(0)/{0,4}/{{0,4},{1,5},{2,6}}

length of orbit 1 is 4
length of orbit 2 is 12
```

```
length of orbit 3 is 24

induced group on 40 points with 3 orbits of length 4,12,24

COCO>> cgr

* construction of 2-orbits of permutation group *

input of group from drg2.gen
input of options from <stdin>
output of colour graph to file drg2.cgr

orbit 1 has length 4
orbit 2 has length 12
orbit 3 has length 24

options(t,d,list or ranks):


stabilizer of point 0 of orbit 1 has
   2 suborbits on orbit 1
     number  length   representative
        0       1           0
        1       3           1
   2 suborbits on orbit 2
     number  length   representative
        2       3           4
        3       9           5
   2 suborbits on orbit 3
     number  length   representative
        4      18          16
        5       6          18
stabilizer of point 4 of orbit 2 has
   2 suborbits on orbit 1
     number  length   representative
        6       1           0
        7       3           1
   4 suborbits on orbit 2
     number  length   representative
        8       1           4
        9       3           5
       10       2           6
       11       6           8
```

```
   3 suborbits on orbit 3
     number  length   representative
       12       6          16
       13      12          17
       14       6          18
stabilizer of point 16 of orbit 3 has
   2 suborbits on orbit 1
     number  length   representative
       15       3           0
       16       1           3
   3 suborbits on orbit 2
     number  length   representative
       17       3           4
       18       6           5
       19       3          10
   7 suborbits on orbit 3
     number  length   representative
       20       1          16
       21       3          17
       22       6          18
       23       2          19
       24       6          20
       25       3          23
       26       3          24
1 row of blocks of colour graph is saved
2 row of blocks of colour graph is saved
3 row of blocks of colour graph is saved

reflexive suborbits:0,8,20
symmetrical suborbits:1,9,10,11,21,22,23,24,25,26
pairs of antisymmetrical suborbits:(2,6)(3,7)(4,15)(5,16)(12,17)(13,18)
(14,19)


COCO>> inm

* calculation of intersection numbers *

input of colour graph from file drg2.cgr
input of options from <stdin>
output of intersection numbers to file drg2.nrs

colour graph of rank 27 on 40 vertices
```

```
3 blocks of sizes 4,12,24

options (nrf, ndg or list):


there are 15 nonzero intersection numbers with i of type 1,1
there are 23 nonzero intersection numbers with i of type 1,2
there are 31 nonzero intersection numbers with i of type 1,3
there are 23 nonzero intersection numbers with i of type 2,1
there are 57 nonzero intersection numbers with i of type 2,2
there are 70 nonzero intersection numbers with i of type 2,3
there are 31 nonzero intersection numbers with i of type 3,1
there are 70 nonzero intersection numbers with i of type 3,2
there are 149 nonzero intersection numbers with i of type 3,3

COCO>> sub

* construction of subconfigurations *

input of intersection numbers from drg2.nrs
input of options from <stdin>
output of subconfigurations to file drg2.sub

configuration of rank 27 with 3 blocks

options(s,p,sp):
s

there exist 16 good symmetrical sets
1. subscheme of rank 4 by merging (0,8,20)(1,5,16,9,12,17,23,26)(2,6,10,25)*
(3,7,4,15,11,14,19,13,18,21,22,24)
2. subscheme of rank 4 by merging (0,8,20)(1,9,25)(2,6,5,16,10,12,17,23,26)*
(3,7,4,15,11,14,19,13,18,21,22,24)
3. subscheme of rank 4 by merging (0,8,20)(1,9,25)(2,6,5,16,10,14,19,23,21)*
(3,7,4,15,11,12,17,13,18,26,22,24)
4. subscheme of rank 3 with parameters (40,9,8) by merging (0,8,20)*
(1,3,7,4,15,9,11,12,17,13,18,25,26,22,24)(2,6,5,16,10,14,19,23,21)
5. subscheme of rank 3 with parameters (40,3,2) by merging (0,8,20)(1,9,25)*
(2,6,5,16,3,7,4,15,10,11,12,17,14,19,13,18,23,21,26,22,24)
6. subscheme of rank 3 with parameters (40,3,2) by merging (0,8,20)*
(1,5,16,3,7,4,15,9,11,12,17,14,19,13,18,23,21,26,22,24)(2,6,10,25)
7. subscheme of rank 3 with parameters (40,12,2) by merging (0,8,20)*
(1,2,6,5,16,10,9,12,17,23,25,26)(3,7,4,15,11,14,19,13,18,21,22,24)
```

```
7 symmetrical subschemes

COCO>> aut

* calculation of automorphism groups *

input of colour graph from file drg2.cgr
input of mergings from drg2.sub

colour graph of rank 27 on 40 vertices
3 blocks of size 4,12,24

colour graph of rank 27
intransitive automorphism group of order 144
with 3 orbits of lengths 4,12,24
base of length 3, 4 generators

colour graph of rank 4
intransitive automorphism group of order 144
with 3 orbits of lengths 4,12,24
base of length 3, 4 generators

colour graph of rank 4
intransitive automorphism group of order 432
with 2 orbits of lengths 4,36
base of length 4, 6 generators

colour graph of rank 4
transitive automorphism group of order 87091200
rank=4; subdegrees:1,3,9,27
base of length 10, 13 generators

colour graph of rank 3
transitive automorphism group of order 41616291150654603264000000000
rank=3; subdegrees:1,30,9
base of length 36, 41 generators

colour graph of rank 3
transitive automorphism group of order 230078188847156428800
rank=3; subdegrees:1,3,36
base of length 30, 38 generators
```

```
colour graph of rank 3
transitive automorphism group of order 230078188847156428800
rank=3; subdegrees:1,36,3
base of length 30, 33 generators

colour graph of rank 3
intransitive automorphism group of order 432
with 2 orbits of lengths 4,36
base of length 4, 7 generators


COCO>> end

* end of CO-CO *
```

Unfortunately, COCO is not able to identify distance regular graphs. However, Grape was used to determine that the mergings 1 and 2 of rank 4 are indeed distance regular graphs of valency 9. One of them has $G = S_3 \times S_4$ as its full automorphism group; the other admits a group of order 432.

Note that merging 7 is the strongly regular graph obtained by adding a spread to either drg.

Appendix B

SELECTED PATTERNS OF COMPUTATIONS

Here, we illustrate the computational techniques used to obtain some of the results by a few examples of the actual GAP code used. Note that a tutorial or an introduction into the GAP language is beyond the scope of this text; we refer the reader to the documentation [88] included in the package.

We also cannot include all the details; however we provide comments to clarify the meaning of functions which are given without the source code.

B.1 Checking the t-vertex condition

In Chapter 4, we needed to check for several graphs whether they satisfy the t-vertex condition for a given value of t. We assume that a graph Γ is given in Grape format.

According to Theorem 4.3, we need to count graph types whose additional vertices have a minimal valency of 3. The following is a function which calculates the types to check for a fixed t; their complements are returned as a list of undirected edges.

```
GraphsToCheck := function(t)
    local   g,  pairs,  gOnPairs,  sor,  i,
            graph, solutions, maxSize;
    if IsBound(graphTypes.(t)) then
        return graphTypes.(t);
    fi;
    g := Group(Concatenation([(1,2)],
                GeneratorsOfGroup(SymmetricGroup([3..t]))));
```

g is the group $S_2 \times S_{t-2}$ used for isomorphism check of types.

```
    pairs := Combinations([1..t], 2);
    maxSize := Length(pairs)-1;
```

A very conservative bound for the number of edges in the complement of a graph type.

```
    gOnPairs := Action(g, pairs, OnSets);
    sor := SetOrbitRepresentatives(gOnPairs, [2..Length(pairs)],
                   [1..maxSize]);
    Print("need to check ", Size(sor), " graphs\n");
```

`sor` contains all possible graph types on t vertices, represented as sets of undirected edges. Note that the function to construct set orbit representatives is not contained in the GAP library, but was written by the author based on code by Ch. Pech. Note also that the representatives are not stored explicitly, but are generated "on the fly" during the iteration below.

We now filter out those graphs which have a maximum valency of at most $t-4$; this forces the minimum valency of the complement to be at least 3.

```
    solutions := [[]];
    for i in sor do
        graph := List(i, x -> pairs[x]);
        if ForAll(numbersOfOccurences(Concatenation(graph),
                   [3..t]), y -> y <= t-4) then
            Add(solutions, graph);
        fi;
    od;
    Print(Length(solutions), " graphs found\n");
    return solutions;
end;
```

This completes the function. We have omitted some non-essential code, like progress reports (for large t the generation of all types takes quite some time), as well as a mechanism to cache results from previous calls.

Next, given a graph Γ, two fixed vertices x and y in Γ, and a graph type T, we need to count the number of subgraphs of Γ which are of type T with respect to x and y. In other words, we have to count all possible embeddings of T into Γ. In order to do this effectively, we use the automorphism group of Γ.

```
CountGraphType := function(gamma,t, x, y, type)
    local matrix, i, j, k, candidates, edge, Recursion, gamma_bar;
```

A few local variables deserve attention: `matrix` contains the adjacency matrix of `type`; `gamma_bar` is the complement of `gamma`.

We perform a backtrack search with full isomorph rejection for the embeddings. The search is performed by a recursive function, which takes the following parameters: `fixed`, the currently prescribed images; `level`, the preimage under consideration; and `candidates`, which is a list of lists, containing for each preimage the set of admissible images.

```
    Recursion := function(fixed, level, candidates)
        local stab, orb, i, result, orbits, newCandidates, j,
              neighbors, nonNeighbors;
```

If images for all but the last point have been prescribed, then we simply return the number of candidates for the remaining point.

```
        if level = t then
            return Length(candidates[level]);
        fi;
```

Otherwise, if there are no candidates left for any of the points, there are no solutions here.

```
        if ForAny([level..t], x -> Length(candidates[x]) = 0) then
            return 0;
        fi;
```

In all other cases, we have to do real work. We calculate the orbits of candidates for the current preimage under the stabilizer of the images that have already been fixed, and consider a representative of each in turn.

```
        stab := Stabilizer(gamma.group, fixed, OnTuples);
        orbits := Orbits(stab, candidates[level]);
        result := 0;
        for orb in orbits do
            j := orb[1];
            neighbors := Adjacency(gamma, j);
            nonNeighbors := Difference([1..gamma.order], neighbors);
            RemoveSet(nonNeighbors, j);
            newCandidates := ShallowCopy(candidates);
```

Depending on whether the preimages are adjacent or not, we update the candidates for the remaining points.

```
            for i in [level+1..t] do
                newCandidates[i] := StructuralCopy(candidates[i]);
                if matrix[level][i] = 1 then
                    newCandidates[i] :=
                      Intersection(newCandidates[i],
                              neighbors);
                else
                    newCandidates[i] :=
                      Intersection(newCandidates[i],
                              nonNeighbors);
                fi;
            od;
```

Finally, we call the function recursively, multiply the result by the length of the current orbit, and add it to the count.

```
            result := result + Recursion(Concatenation(fixed, [j]),
                              level+1,
                              newCandidates)*Length(orb);

        od;
        return result;
    end;
```

This finishes the recursive part of the algorithm. Before calling the recursion, we have to set up some variables, such as the adjacency matrix of the graph type, and the candidates for the images.

```
    matrix := NullMat(t,t);
    for i in type do
        matrix[i[1]][i[2]] := 1;
        matrix[i[2]][i[1]] := 1;
    od;
    matrix := 1-matrix;
    candidates := [0,0];
    for i in [3..t] do
        Add(candidates, [1..gamma.order]);
    od;
    gamma_bar := ComplementGraph(gamma);
```

Since x and y are known, we can use them to restrict the possible images.

```
    edge := [x,y];
    for i in [1,2] do
        for j in [3..t] do
            if matrix[i][j] = 1 then
                candidates[j] := Intersection(candidates[j],
                                      Adjacency(gamma, edge[i]));
            else
                candidates[j] := Intersection(candidates[j],
                                      Adjacency(gamma_bar, edge[i]));
                RemoveSet(candidates[j], edge[i]);
            fi;
        od;
    od;
```

Finally, we call the recursive part and return its result.

```
    return Recursion([x,y], 3, candidates);
end;
```

Now that we can count graphs of a given type, and we can generate the relevant graph types, it is straightforward to check the t-vertex condition. We perform the count for each type and each pair of vertices, up to equivalence under some part of the automorphism group of Γ that might be known. It is not always feasible to compute the full automorphism group, but often some subgroup can be obtained easily (e.g., when it follows from the construction of the graph).

The following function takes as arguments the graph Γ and the parameter t. It stops as soon as it finds a graph type T and two edges or non-edges such that the corresponding counts differ. This behavior can be changed by given a third argument equal to `true`, which causes a complete search for all graph types which can be used to distinguish edges or non-edges from different orbits.

```
CheckTCondition := function(arg)
    local stab, suborbs, edges, nonedges, graphs, results,i,
          gamma, t, complete, result, orbits;
    gamma := arg[1];
    t := arg[2];
    if Length(arg) > 2 then
        complete := arg[3];
    else
        complete := false;
    fi;
    result := true;
    orbits := Orbits(gamma.group, [1..gamma.order]);
    Print(Length(orbits), " orbits on vertices\n");
```

First, we determine representatives for all orbits on edges and non-edges.

```
    edges := [];
    nonedges := [];
    for i in orbits do
        stab := Stabilizer(gamma.group,i[1]);
        suborbs := Set(Orbits(stab, [1..gamma.order]));
        RemoveSet(suborbs, [i[1]]);
        suborbs := List(suborbs, x -> x[1]);
        Append(edges, List(Intersection(suborbs,
                                Adjacency(gamma,i[1])),
                x -> [i[1],x]));
        Append(nonedges, List(Difference(suborbs,
                                Adjacency(gamma,i[1])),
                x -> [i[1],x]));
    od;
```

Now we perform the count for all types and for all edges and non-edges, making sure that the numbers coincide.

```
    graphs := GraphsToCheck(t);
    if Length(edges) > 1 then
        Print("checking edges\n");
        for i in [1..Length(graphs)] do
            results := List(edges, x ->
                   CountGraphType(gamma, t, x[1], x[2], graphs[i]));
            Print(i, ": ", results, "\n");
            if Size(Set(results)) > 1 then
                Print("troubling graph: ", graphs[i], "\n");
                result := false;
                if not complete then
                    return false;
                fi;
            fi;
        od;
    fi;
    if Length(nonedges) > 1 then
        Print("checking nonedges\n");
        for i in [1..Length(graphs)] do
            results := List(nonedges, x ->
                    CountGraphType(gamma, t, x[1], x[2], graphs[i]));
            Print(i, ": ", results, "\n");
            if Size(Set(results)) > 1 then
                Print("troubling graph: ", graphs[i], "\n");
                result := false;
                if not complete then
                    return false;
                fi;
            fi;
        od;
    fi;
```

Finally, we just need to return the result.

```
    return result;
end;
```

This concludes the description of the algorithm to check the t-vertex condition of graphs. It has been tested with a number of known graphs. It was able to confirm the 7-vertex condition for a $GQ(5, 25)$-graph with an intransitive automorphism group in approximately 20 min. For smaller graphs and smaller values of t the result is usually obtained in a matter of seconds.

B.2 Finding spreads in incidence structures

Many problems in combinatorics can be reformulated in terms of spreads of incidence structures. Here, we describe a GAP function which determines all spreads up to isomorphism. Many ideas used in this implementation can be applied to backtrack search in general.

```
Spreads := function(n, sets, group)
```

The function takes three arguments: `n` is the number of points; `sets` contains the blocks of the incidence structure (as subsets of `[1..n]`; finally, `group` is a permutation group acting on `[1..n]` leaving the incidence structure invariant.

```
    local i, j, k,
          Initialize,
          chooseRecord,
          forbidRecord,
          CallRecord,
          available,
          decreaseAvailableRecord,
          chosenSet,
          chosen,
          forbidden,
          pointUsed,
          excluding, stack, stackPointer,
          Push, Pop,
          Recursion, solutions, inequivalent,
          groupOnSets, setsThroughPoint,
          sol, orbit,
          done,
          pointsCovered;
```

We maintain a number of global variables, such as the set of currently selected blocks and the set of blocks which is excluded. Changes to these variables are stored explicitly on a stack to enable us to revert them.

The following are functions to update the stack.

```
Push := function(what, arguments)
    stackPointer := stackPointer + 1;
    stack[stackPointer] :=
      rec(record := record, arguments := arguments);
end;
Pop := function(hook)
    local record, arguments, i, entry;
    while stackPointer > hook do
        entry := stack[stackPointer];
        arguments := entry.arguments;
        record := entry.record;
        record.undo(arguments);
        stackPointer := stackPointer - 1;
    od;
end;
```

We distinguish several operations on the global data. We can select a set into the possible spread (`choose`); we can exclude a set from further consideration.

For each point we keep track of the number of blocks through this point which are still available, i.e., not excluded. If a block is forbidden, we need to update this number for all the points it contains (`decreaseAvailable`).

We consider each operation as a set of several functions: Before performing the `action` itself, we check a `precondition` to see if the action is redundant (e.g., if we try to exclude a block that is already forbidden). We also include an `undo` function that reverts any changes done by `action`.

Performing one action possibly implies other changes. For example, if one block b is chosen, we need to exclude all blocks that intersect b non-trivially. These implications are captured in the function `implies`.

The following are the functions used to exclude a block.

```
Forbid := function(x)
    return CallRecord(forbidRecord, x);
end;
forbidRecord := rec(
```

```
  action := function(x)
    forbidden[x] := true;
  end,
  precondition := function(x)
    if forbidden[x] then
        return true;
    fi;
    if chosen[x] then
        return false;
    fi;
    return fail;
  end,
  implies := function(x)
    local i;
    return ForAll(sets[x],
           i -> CallRecord(decreaseAvailableRecord, i));
  end,
  undo := function(arguments)
    forbidden[arguments] := false;
  end,
  trace := function(x)
    Print("forbid ", x, ", ",sets[x], "\n");
  end);
```

Next, we describe how to select a new block.

```
Choose := function(x)
    return CallRecord(chooseRecord, x);
end;
chooseRecord :=
rec(action := function(x)
      local i;
      chosen[x] := true;
      AddSet(chosenSet, x);
      for i in sets[x] do
          pointUsed[i] := true;
      od;
      pointsCovered := pointsCovered + Size(sets[x]);
    end,
    precondition := function(x)
      if forbidden[x] then
```

```
          return false;
        fi;
        if chosen[x] then
          return true;
        fi;
        return fail;
      end,
      implies := function(x)
        return ForAll(sets[x], j ->
                ForAll(setsThroughPoint[j],
                 i -> (i = x) or CallRecord(forbidRecord, i)));
      end,
      undo := function(arguments)
        chosen[arguments] := false;
        ForEach(sets[arguments],x-> pointUsed[x] := false; end);
        RemoveSet(chosenSet, arguments);
        pointsCovered := pointsCovered - Size(sets[arguments]);
      end,
    trace := function(x)
      Print("choose ", x, "\n");
  end);
```

Finally, we keep track of the number of available blocks through each point. If this number goes down to 1 for any point p, we are forced to use the last remaining block in order to cover p.

```
  decreaseAvailableRecord :=
    rec(action := function(x)
      available[x] := available[x] - 1;
  end,
    precondition := function(x)
      if available[x] <= 1 then
          return false;
      fi;
      return fail;
  end,
    implies := function(point)
      local set;
      if available[point] = 1 and not pointUsed[point] then
          set := First(setsThroughPoint[point], i ->
```

```
                        not forbidden[i]);
            if set = fail then
                return false;
            fi;
            return CallRecord(chooseRecord, set);
        fi;
        return true;
    end,
      undo := function(x)
        available[x] := available[x] + 1;
    end,
      trace := function(x)
        Print("decrease ", x, "\n");
    end);
```

The following function is used to call these operations.

```
    CallRecord := function(record, x)
        local pre;
```

First, we check the precondition. If it fails, we can abort immediately.

```
        pre := record.precondition(x);
        if pre <> fail then
            return pre;
        fi;
```

Otherwise, we perform the action itself, push the operator onto the stack, and call any implied operations.

```
        record.action(x);
        Push(record, x);
        return record.implies(x);
    end;
```

Next, we describe the backtracking function itself.

```
    Recursion := function(useStabilizer)
        local i, j, k, min, point, cliques, hook, stab;
```

If all points are covered, then we have a solution. We save it and return.

```
if pointsCovered = n  then
    Add(solutions, ShallowCopy(chosenSet));
    return;
fi;
```

Otherwise, we look for a point which is not covered yet, and which has the minimum number of available blocks. This is done to reduce the branching factor of the search.

```
min := Length(sets);
for i in [1..n] do
    if not pointUsed[i] and (available[i] < min) then
        point := i;
        min := available[i];
    fi;
od;
cliques := Filtered(setsThroughPoint[point], x ->
                    not forbidden[x]);
```

We use isomorph rejection; therefore, we have to find the stabilizer of the currently selected blocks. It turns out that the computation of this group is expensive when many blocks are selected; hence we use a heuristic to decide if the use of groups is beneficial.

```
if useStabilizer then
    stab := Stabilizer(groupOnSets, chosenSet, OnSets);
    if Size(stab) <= 5 then
        useStabilizer := false;
    fi;
    cliques := ShallowCopy(Orbits(stab, cliques));
    Sort(cliques, function(x,y)
        return Size(Set(x)) < Size(Set(y));
    end);
    cliques := List(cliques, x -> x[1]);
else
    stab := Group(());
fi;
```

Finally, we loop through all possible blocks through the selected point and try to add them to our solution. If we succeed, we proceed with the recursion.

```
        for i in cliques do
            hook := stackPointer;
            if Choose(i) then
                Recursion(useStabilizer);
            fi;
            if done then
                return;
            fi;
            Pop(hook);
```

After dealing with one block we do not need to consider any other block in the same orbit.

```
            if not ForAll(Orbit(stab,i), x -> Forbid(x)) then
                return;
            fi;
        od;
    end;
```

The following function sets up the global data needed during the search.

```
    Initialize := function()
        local i, j;
        if not ForAll(sets, x -> IsSubset([1..n], x)) then
            Error("illegal input");
        fi;
        chosenSet := [];
        chosen := List(sets, x -> false);
        forbidden := List(sets, x -> false);
        available := List([1..n], x -> 0);
        for i in sets do
            for j in i do
                available[j] := available[j] + 1;
            od;
        od;
        #Print(available, "\n");
        stack := [];
        stackPointer := 0;
        pointUsed := List([1..n], x -> false);
        solutions := [];
        Print("calculating group\n");
```

```
        groupOnSets := Operation(group, sets, OnSets);
        StabChainOp(groupOnSets, rec(limit := Size(group)));
        Print("size of group: ", Size(groupOnSets), "\n");
        setsThroughPoint := List([1..n], x -> []);
        for i in [1..Length(sets)] do
            for j in sets[i] do
                Add(setsThroughPoint[j], i);
            od;
        od;
        pointsCovered := 0;
    end;
```

Finally, we get to the main function. After an quick sanity check, we initialize the data and perform the search.

```
    if Sum(List(sets, Length)) < n then
        return [];
    fi;
    Initialize();
    Recursion(true);
    Pop(0);
```

Since we use incomplete isomorph rejection, we need to check if any of the solutions found are equivalent.

```
    inequivalent := [];
    for  i in solutions do
        if not ForAny(inequivalent,
              x -> RepresentativeOperation(groupOnSets, i, x, OnSets)
              <> fail) then
            Add(inequivalent, i);
        fi;

    od;
    return inequivalent;
end;
```

Finally, we return all non-isomorphic spreads.

This function has been used successfully to find spreads in generalized quadrangles, and to find partial linear spaces with a given point graph. In the latter case we consider the edges of the point graph as points, and cliques as sets of edges; a spread in the resulting incidence structure is necessarily a partial linear space.