BAER STRUCTURES, UNITALS AND ASSOCIATED FINITE GEOMETRIES

by

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Contents

Abstract

In this thesis we study the representation of finite translation planes in projective spaces introduced by André [1]. This theory was also developed by Bruck and Bose [21, 22] in a distinct but equivalent form. Throughout this thesis we refer to this representation as the *Bruck and Bose* representation or simply *Bruck-Bose*. Of particular importance is the representation of Baer subplanes of translation planes π_{q^2} of order q^2; the importance is due to the crucial role Baer subplanes have in the characterisation of various substructures, including unitals and maximal arcs, of projective planes, as will be evident in the text.

In Chapter 1 we present the necessary preliminary material required for the later chapters. In particular we present in detail the Bruck and Bose representation [21, 22] of the Desarguesian plane $PG(2, q^h)$ and the associated coordinatisation.

In Chapter 2 we begin by reviewing the known results concerning the representation of Baer subplanes of $PG(2, q^2)$ in the Bruck and Bose representation in $PG(4, q)$. We provide a new proof of the result of Vincenti [90] and Bose, Freeman and Glynn [19], that the non-affine Baer subplanes of $PG(2, q^2)$ are represented in Bruck-Bose by certain ruled cubic surfaces in $PG(4, q)$ which we term *Baer ruled cubic surfaces.* We characterise Baer ruled cubic surfaces in $PG(4, q)$ for a general fixed Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$. We determine that non-degenerate conics in Baer subplanes of $PG(2, q^2)$ are represented in Bruck-Bose by normal rational curves; a normal rational curve which arises in this way is of order 2, 3 or 4 and is therefore properly contained in a plane, hyperplane or no hyperplane of $PG(4, q)$ respectively. We apply these results to prove the existence of certain $(q^2 + 1)-$caps in $PG(4, q)$ which are not contained in any hyperplane of $PG(4, q)$ and which contain many normal rational curves of order 4. Further properties of these caps are determined in Chapter 3. We also include a discussion of the ruled cubic surface obtained as the projection from a point P of the Veronese Surface in $PG(5, q)$ onto a hyperplane not containing P; in this setting we determine some alternative proofs for our results and prove some extensions.

In Chapter 3 we investigate the Bruck and Bose representation in $PG(n, q)$ with $n > 4$. We prove various results concerning the regular $(h-1)-$spreads of $PG(2h-1, q)$ which determine the Bruck and Bose representation of $PG(2, q^h)$ in $PG(2h, q)$, treating the

case $h = 4$ in greater detail. In particular, we prove the existence of *induced* spreads and show how the induced spreads are closely related to the Bruck and Bose representation of the Baer substructures of $PG(2, q^h)$. To obtain further properties of the higher dimensional Bruck-Bose representation of the non-affine Baer substructures of the Desarguesian plane, we make use of the Bose representation [18] of $PG(2, q^2)$. In this chapter, we also prove results concerning the Bruck and Bose representation of non-degenerate conics in $PG(2, q^2)$ and we discuss the relationship between these results and the Bruck-Bose representation of non-affine Baer sublines of $PG(2, q^4)$ in $PG(8, q)$.

In Chapter 4 we investigate Baer subplanes and Buekenhout-Metz unitals in $PG(2, q^2)$. In particular we improve the known results by showing that in $PG(2, q^2)$, with $q > 13$, a Baer subplane and a Buekenhout-Metz unital with elliptic quadric as base have at least 1 point and at most $2q+1$ distinct points in their intersection. Our method of proof makes use of the Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$ and the properties of a certain irreducible sextic curve in $PG(4, q)$. We also prove that the non-classical Buekenhout-Metz unitals, with an elliptic quadric base, in $PG(2, q^2)$ are inherited from the classical unitals in $PG(2, q^2)$ by a certain procedure of swapping regular $1-$spreads of $PG(3, q)$ in the Bruck and Bose representation of $PG(2, q^2)$.

In Chapter 5 we prove that a unital in $PG(2, q^2)$ is a Buekenhout-Metz unital if and only if there exists a point T of the unital such that each secant line of the unital through T intersects the unital in a Baer subline. This is an improvement of the characterisation of Lefèvre-Percsy [56] and an improvement of the characterisation of Casse, O'Keefe and Penttila [26] for the cases $q > 3$.

In the final chapter we investigate the relationships between Thas maximal arcs, the generalized quadrangle $T_3(\mathcal{O})$ and egglike inversive planes. This work was motivated by the approach of Barwick and O'Keefe [13] in investigating the relationship between Buekenhout-Metz unitals and inversive planes (see also [6, Section 5.] and [92]). We attempt to characterise the Thas maximal arcs in those translation planes where they exist using two configurational properties; we do not succeed in this, but prove a characterisation of Thas maximal arcs in $PG(2, q^2)$ for certain values of q.

Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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Catherine Thérèse Quinn

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Chapter 1

Preliminary Results

In this chapter we collect together the main definitions, known results and constructions we require for our original work presented in later chapters.

1.1 Incidence Structures and Designs

In this section we follow Hughes and Piper [53].

An **incidence structure** $S = (\mathcal{V}, \mathcal{B}, \mathbf{I})$ is two sets $\mathcal{V}$ and $\mathcal{B}$ called **varieties** (or **points**) and **blocks** (or **lines**) respectively, with an incidence relation $\mathbf{I} \subseteq \mathcal{V} \times \mathcal{B}$; a point P is incident with a block ℓ if and only if $(P, \ell) \in \mathbf{I}$. An incidence structure S is **finite** if the sets $\mathcal{V}$ and $\mathcal{B}$ are both finite. From now on our incidence structures are finite incidence structures.

Given any block in an incidence structure S, there is a set of points incident with it and it will be convenient to identify the block with this pointset. An incidence structure has **repeated blocks** if there exist two blocks identified with the same pointset. If a point P is incident with a block ℓ then we shall write $P\mathbf{I}\ell$ or $P \in \ell$ and we shall use the expressions "P is on ℓ", "ℓ contains P", "ℓ passes through P" and similar convenient expressions.

A $\mathbf{t} - (\mathbf{v}, \mathbf{k}, \lambda)$ **design** is an incidence structure with exactly v points, no repeated blocks, each block is incident with exactly k distinct points and each subset of t distinct points is incident with exactly λ common blocks. A $t - (v, k, \lambda)$ design has **parameters** v, k, b, r, t, λ where, b equals the number of blocks, r equals the number of blocks incident with a point and v, k, t, λ are defined as above.

If S is a $t-(v,k,\lambda)$ design, then for any integer s satisfying $0 \leq s < t$, there are exactly λ_s blocks of S which are incident with any given subset of s distinct points of S, where

$$\lambda_s \;=\; \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Moreover we have the following identities for the parameters of S:

1. $b \;=\; \lambda \frac{\binom{v}{t}}{\binom{k}{t}}$;
2. if $t > 0$ then $bk \;=\; vr$;
3. if $t > 1$ then $r(k-1) \;=\; \lambda_2(v-1)$.

For an incidence structure $S(\mathcal{V}, \mathcal{B}, \mathbf{I})$ and P a point of S, we define the **internal structure**, S_P, of S at P to be the set of all blocks of S which contain P and the set of all points of S, except P, which lie on at least one of those blocks and the incidence in S_P is inherited from the incidence in S. In particular if $S = (\mathcal{V}, \mathcal{B}, \mathbf{I})$ is a $t-(v,k,\lambda)$ design, then for any point P of S the internal structure S_P of S at P is a $(t-1)-(v-1,k-1,\lambda)$ design with parameters

$$\begin{aligned} v' &= v-1 \\ k' &= k-1 \\ t' &= t-1 \\ \lambda' &= \lambda \\ b' &= r \\ r' &= \lambda_2 \end{aligned}$$

where v, k, t, λ, b, r are the parameters of S.

1.2 Projective, Affine and Translation Planes

In this section we briefly present some familiar results from [52]; for further detail concerning the material in this section consult [52].

A **projective plane** is a set of points and lines together with an incidence relation between the points and lines such that,

(i) Any two distinct points are incident with a unique line.

(ii) Any two distinct lines are incident with a unique point.

(iii) There exist four points such that no three are incident with one line.

If one line of a projective plane contains only a finite number of points, then every line in the plane contains a finite number of points. A projective plane with this property is called **finite**. We shall consider only finite projective planes in this thesis.

If π is a projective plane, then a set π^d of points and lines together with an incidence relation such that the points (lines) of π^d are the lines (points) of π and two elements of π^d are incident in π^d if and only if they are incident in π, is a projective plane. π^d is called the **dual plane** of π. Projective planes satisfy the **Principle of Duality**: Let A be any theorem about projective planes. If A^* is the statement obtained by interchanging the words *points* and *lines*, then A^* is a theorem about dual planes. Hence A^* is a theorem about projective planes (see [52, Theorem 3.2]).

Let π be a finite projective plane, then there exists a positive integer $n \geq 2$ such that each line of π is incident with exactly $n+1$ points and each point of π is incident with exactly $n+1$ lines. π contains exactly n^2+n+1 points and n^2+n+1 lines. The integer n is then called the **order** of π. All known finite projective planes have prime power order (see [52, Section III.2]).

For a fixed line ℓ_∞ of π, denote by *aff*(π) the set of points and lines of π obtained by deleting ℓ_∞ and all its points; the incidence in *aff*(π) is inherited from π. We write *aff*$(\pi) = \pi \backslash \ell_\infty$ or *aff*$(\pi) = \pi^{\ell_\infty}$. Then *aff*$(\pi)$ is an **affine plane of order n** and we call ℓ_∞ the **line at infinity** of *aff*(π) and call the points on ℓ_∞ the **points at infinity**. Two lines in *aff*(π) are **parallel** if they do not intersect in *aff*(π); **parallelism** in *aff*(π) is an equivalence relation and in this way each point at infinity in π corresponds to a unique parallel class of lines in *aff*(π).

A **collineation** of a finite projective plane π is a bijection from points to points and lines to lines; a collineation preserves collinearity. An **elation** with **axis** ℓ_∞ and **centre** X is a collineation of π which fixes all points of a line ℓ_∞ of π and fixes all lines through a point $X \in \ell_\infty$. If the group of elations with axis ℓ_∞ in π is transitive on the points of π not incident with ℓ_∞, then the finite projective plane π is a **translation plane** with **translation line** ℓ_∞. In this case the affine plane *aff*$(\pi) = \pi \backslash \ell_\infty$ is also referred to as a **translation plane**, however the context in which these terms are used in the text should make the meaning clear. (See [52, Chapter IV, section 5.] or [33, Section 3.1.22] for further details). Throughout this thesis we call the translation line ℓ_∞ of a translation plane π the **line at infinity**.

A finite projective plane π is called **Desarguesian** if it satisfies a certain configurational property for any choice of point V and line ℓ in π. A theorem of Baer, [52, Theorem 4.29], relates this configurational property of the plane to a collineation group property of the plane. Moreover, Baer's result states that a projective plane π is Desarguesian if and only if π is a translation plane with respect to a line ℓ for every choice of line ℓ in π. Consequently, for a Desarguesian projective plane π and for any line ℓ_∞ in π, the affine plane $\mathit{aff}(\pi) = \pi\backslash\ell_\infty$ is a translation plane.

For q a prime power, the Galois field plane $PG(2,q)$ is the unique Desarguesian finite projective plane of order q (see [33, Section 1.4]).

A **subplane** π_m of a finite projective plane π_n of order n is a subset of the elements of π_n which form a projective plane having the same incidence as π_n. A subplane π_m of π_n is called **proper** if $\pi_m \neq \pi_n$.

Bruck's theorem ([52, Theorem 3.7]) states that if a finite projective plane π_n of order n contains a proper subplane π_m of order m, then either $n = m^2$ or $n \geq m^2+m$. If $n = m^2$ then the subplane π_m of order m is called a **Baer subplane** of the finite projective plane π_{m^2} of order $n = m^2$. In this case each point in $\pi_{m^2}\backslash\pi_m$ is incident with a unique line of π_m and each line ℓ of π_{m^2} is either a line of π_m (and so intersects π_m in $m+1 < n+1$ points) or ℓ intersects π_m in a unique point. If a line ℓ intersects a Baer subplane B of π_{m^2} in $m+1$ points, then the intersection $\ell \cap B$ is called a **Baer subline** (of ℓ) in B.

Much of this thesis is devoted to examining Baer subplanes and utilising the properties of Baer subplanes; particularly in the case of the Desarguesian projective plane $PG(2,q^2)$ of square order q^2. We include the following results for later reference.

Theorem 1.2.1 [29] [33, Result 3.2.17] *In $PG(2,q^2)$, a quadrangle (four distinct points no three collinear) is contained in a unique Baer subplane of $PG(2,q^2)$.* $\square$

It follows from Theorem 1.2.1 that for a line ℓ in $PG(2,q^2)$, any three distinct points of ℓ are contained in a unique Baer subline of ℓ. Moreover, by Theorem 1.2.1, the Desarguesian plane $PG(2,q^2)$ contains exactly $q^3(q^3+1)(q^2+1)$ distinct Baer subplanes.

In the following characterisation of Baer subplanes of $PG(2,q^2)$, a **blocking k−set** in

$PG(2,q^2)$ is a subset of k points of $PG(2,q^2)$ which meets every line but contains no line completely (see [48, Section 13.1]).

Theorem 1.2.2 [48, Theorem 13.2.2] *In $PG(2,q^2)$, if B is a blocking $(q^2+q+1)-$set, then B is a Baer subplane of $PG(2,q^2)$.* $\square$

1.3 Projective Spaces

We briefly present some results of [48, Chapter 2] to establish terminology and notation.

Let $V = GF(q^{n+1})$ be the $(n+1)-$dimensional vector space over $GF(q)$ with origin $\mathbf{0}$. With respect to some basis of V, the elements of V are of the form $X = (x_0, x_1, \ldots, x_n)$, where $x_i \in GF(q)$. For $X = (x_0, x_1, \ldots, x_n)$, $Y = (y_0, y_1, \ldots, y_n)$ in V, the relation

$$X \equiv Y \quad \text{if and only if} \quad \begin{array}{l} x_i = \lambda y_i, \text{ for all } i = 0, 1, \ldots, n \\ \text{for some } \lambda \in GF(q)\backslash\{0\} \end{array}$$

is an equivalence relation on the vectors in $V\backslash\{0\}$ with equivalence classes the one-dimensional subspaces of V with the origin deleted. The set of equivalence classes is an **$\mathbf{n}-$dimensional projective space over $\mathbf{GF(q)}$** and is denoted by $\mathbf{PG(n,q)}$. For each $X \in V\backslash\{\mathbf{0}\}$, the equivalence class containing X is a **point** in $PG(n,q)$. Consequently, the number $\theta(n)$ of points in $PG(n,q)$ equals

$$|PG(n,q)| \;=\; \theta(n) \;=\; \frac{q^{n+1}-1}{q-1} \;=\; q^n + q^{n-1} + \ldots + q + 1.$$

It will be convenient to take the standard basis for V over $GF(q)$ and to use $X = (x_0, x_1, \ldots, x_n)$ to denote the point of $PG(n,q)$ which contains $X \in V$; we write X is a point of $PG(n,q)$ with **homogeneous coordinates** $X = (x_0, x_1, \ldots, x_n)$ to mean that the coordinates $\lambda(x_0, x_1, \ldots, x_n)$, $\lambda \in GF(q)\backslash\{0\}$, represent the same point X in $PG(n,q)$.

A **subspace of dimension m**, or **m$-$space**, of $PG(n,q)$ is a set Π_m of points all of whose coordinates form (together with the origin) a subspace of dimension $m+1$ of V. For $0 < m \leq n$, the number $\phi(m;n,q)$ of $m-$spaces of $PG(n,q)$ is given by,

$$\phi(m;n,q) \;=\; \frac{(q^{n+1}-1)(q^n-1)\ldots(q^{n-m+1}-1)}{(q^{m+1}-1)(q^m-1)\ldots(q-1)}.$$

An $(n-1)-$space of $PG(n,q)$ is called a **hyperplane** (or **prime**); the set of points $X = (x_0, x_1, \ldots, x_n)$ in $PG(n,q)$ in a hyperplane Σ_{n-1} of $PG(n,q)$ satisfy an equation

$$a_0x_0 + a_1x_1 + \ldots + a_nx_n \;=\; 0$$

where the coefficients $a_i \in GF(q)$ are not all zero. We shall call $[a_0, a_1, \ldots, a_n]$ the **(hyperlane) coordinates** of the hyperplane Σ_{n-1}.

If Π_r and Π_s are subspaces of $PG(n,q)$ of dimension r and s respectively, then:

1. the intersection of Π_r and Π_s is written $\Pi_r \cap \Pi_s$;
2. the **join** or **span** of Π_r and Π_s is written $\Pi_r\Pi_s$ or $\langle \Pi_r, \Pi_s \rangle$ and is the smallest subspace of $PG(n,q)$ which contains Π_r and Π_s;
3. **Dimension Theorem (Grassman's Identity)** Let dimΠ denote the dimension of a subspace Π of $PG(n,q)$, then

$$\mathrm{dim}\Pi_r + \mathrm{dim}\Pi_s \;=\; \mathrm{dim}(\Pi_r \cap \Pi_s) + \mathrm{dim}\langle \Pi_r, \Pi_s \rangle.$$

Note that distinct subspaces Π_r and Π_s are disjoint in $PG(n,q)$ if and only if as subspaces of V they intersect in the origin; by the definition of dimension of subspaces of $PG(n,q)$ we have $\Pi_r \cap \Pi_s = \emptyset$ implies $\mathrm{dim}(\Pi_r \cap \Pi_s) = -1$.

If S and S' are two subspaces in $PG(n,q)$ then a **collineation** $\sigma : S \to S'$ is a bijection which preserves incidence; that is, if $\Pi_r \subset \Pi_s$ then $\Pi_r^\sigma \subset \Pi_s^\sigma$.

A **projectivity** $\sigma : S \to S'$ is a bijection given by an $(n+1) \times (n+1)$ matrix $H \in PGL(n,q)$: for $X, Y \in PG(n,q)$, if $Y = X^\sigma$ then the corresponding (column) homogeneous coordinates satisfy $\lambda Y = HX$, for some $\lambda \in GF(q)\backslash\{0\}$. The matrix H is non-singular.

With respect to a fixed basis of V over $GF(q)$, an automorphism ϕ of $GF(q)$ induces an automorphism ϕ of $PG(n,q)$; this collineation is given by $X^\phi = (x_0^\phi, x_1^\phi, \ldots, x_n^\phi)$ for each point X in $PG(n,q)$. In particular, in $PG(n,q^2)$ the automorphism,

$$\begin{aligned} PG(n,q^2) &\longrightarrow PG(n,q^2) \\ X = (x_0, x_1, \ldots, x_n) &\mapsto \overline{X} = X^q = (x_0^q, x_1^q, \ldots, x_n^q) \end{aligned}$$

is called the **Fröbenius automorphism**. For each subspace Π_r of $PG(n,q^2)$ we call $\overline{\Pi}_r$, the image of Π_r under the Fröbenius automorphism, the **conjugate** space of Π_r with respect to the extension $GF(q^2)$ of $GF(q)$.

If Σ_{n-1} is any hyperplane in $PG(n,q)$, then $AG(n,q) = PG(n,q)\backslash\Sigma_{n-1}$ is an **n$-$dimensional affine space over GF(q)**. The subspaces of $AG(n,q)$ are the subspaces of $PG(n,q)$ with the points of Σ_{n-1} deleted. If the affine space $AG(n,q)$ is obtained from $PG(n,q)$ in this way, for a fixed hyperplane Σ_{n-1} of $PG(n,q)$, we call Σ_{n-1} the **hyperplane at infinity**.

For any projective space $PG(n,q)$ there is a **dual space** $PG(n,q)^d$ whose *points* and *hyperplanes* are respectively the hyperplanes and points of $PG(n,q)$; $PG(n,q)^d$ is an $n-$dimensional projective space over $GF(q)$, that is, $PG(n,q)^d$ is isomorphic to $PG(n,q)$. The projective space $PG(n,q)$ satisfies the **Principle of Duality**: For any theorem true in $PG(n,q)$, there is an equivalent theorem true in $PG(n,q)^d$; in particular, if T is a theorem in $PG(n,q)$ stated in terms of points, hyperplanes and incidence the same theorem is true in $PG(n,q)^d$ and gives a dual theorem T^d in $PG(n,q)$ by interchanging *point* and *hyperplane* whenever they occur. Thus *join* and *meet* are dual. Hence the dual of an $m-$space in $PG(n,q)$ is an $(n-m-1)-$space (see [48, Section 2.1]).

We now present results concerning subgeometries of $PG(n,q)$ which generalise the properties of subplanes of finite projective planes.

Since $GF(q)$ is a subfield of $GF(q^k)$ for $k>1$ a positive integer, the projective space $PG(n,q)$ is naturally embedded in $PG(n,q^k)$ once the coordinate system is fixed. Any $PG(n,q)$ embedded in $PG(n,q^k)$ is a **subgeometry** of $PG(n,q^k)$. We are particularly interested in the case $k=2$ and any $PG(n,q)$ embedded in $PG(n,q^2)$ is called a **Baer subgeometry** of $PG(n,q^2)$. Once the coordinate system is fixed, $PG(n,q)$ is called the **Baer n$-$space** or **real Baer n$-$space** of $PG(n,q^2)$.

As mentioned above, the Fröbenius automorphism in $PG(n,q^2)$ fixes $PG(n,q)$ pointwise.

A Baer subgeometry $PG(n,q)$ of $PG(n,q^2)$ has properties analagous to those of a Baer subplane of a finite projective plane, as follows.

Theorem 1.3.1 [73, Theorems 3.1, 3.2] *Let $B=PG(n,q)$ be embedded as a Baer subgeometry of $PG(n,q^2)$.*

(i) Each point P in $PG(n,q^2)\backslash B$ is incident with a unique line of B;

(ii) Each hyperplane Π_{n-1} of $PG(n,q^2)$ intersects B in either a $(n-1)-$space or an $(n-2)-$space of B. □

1.4 Quadrics

In this section we follow [50, Sections 22.1, 22.2].

A **quadric** $\mathcal{Q}_n$ in $PG(n,q)$ is any set of points $(x_0, x_1, \dots, x_n) \in PG(n,q)$ such that $F(x_0, x_1, \dots, x_n) = 0$ for some quadratic form $F \in GF(q)[x_0, x_1, \dots, x_n]$. We write $\mathcal{Q}_n = V(F)$ and F has form

$$F(x_0, x_1, \dots, x_n) \;=\; \sum_{i=0}^{n} a_i x_i^2 \;+\; \sum_{i<j} a_{ij} x_i x_j$$

where the $a_i, a_{ij} \in GF(q)$ are not all zero.

If there is no change in coordinate system which reduces the form F to one in fewer variables, then F is **non-degenerate** and $\mathcal{Q}_n$ is **non-singular**; otherwise F is **degenerate** and $\mathcal{Q}_n$ is **singular**.

The projective linear group $PGL(n+1,q)$ acting on all non-singular quadrics in $PG(n,q)$ has one or two orbits according as n is even or odd. For n even, the non-singular quadrics in $PG(n,q)$ are projectively equivalent and are called **parabolic**. For n odd, a non-singular quadric $\mathcal{Q}_n$ in $PG(n,q)$ is either **hyperbolic** or **elliptic**. See [50, Section 22.1] for the canonical forms of these quadrics.

Let $\mathcal{W}_n = V(F)$ be a quadric in $PG(n,q)$ with

$$F(x_0, x_1, \dots, x_n) \;=\; \sum_{i=0}^{n} a_i x_i^2 \;+\; \sum_{i<j} a_{ij} x_i x_j.$$

Define $A = [a_{ij}]$, where $a_{ii} = 2a_i$, $a_{ji} = a_{ij}$ for $i < j$. Let $B = [b_{ij}]$, where $b_{ii} = 0$, $b_{ji} = -b_{ij} = -a_{ij}$ for $i < j$.

If $q = 2^h$, then define **trace(t)**$=$ $t + t^2 + t^{2^2} + \dots + t^{2^{h-1}}$, $t \in GF(q)$. Let $\mathcal{C}_0 = \{t \in GF(q) \mid \mathrm{trace}(t) = 0\}$ and let $\mathcal{C}_1 = \{t \in GF(q) \mid \mathrm{trace}(t) = 1\}$. For q odd, $\mathcal{C}_0$ will denote the non-zero squares in $GF(q)$ and $\mathcal{C}_1$ will denote the non-squares in $GF(q)$. In the following theorem, for q even, Δ and α are evaluated as rational functions over the set of integers where a_i, a_{ij} are treated as indeterminates z_i, z_{ij}; then z_i, z_{ij} are specialised to a_i, a_{ij} to give the result.

Theorem 1.4.1 [50, Theorem 22.2.1]

(i) $\mathcal{W}_n$ *is singular or not according as* Δ *is zero or not, where*

$$\Delta \;=\; \begin{cases} \frac{1}{2}|A|, & n \text{ even} \\ |A|, & n \text{ odd} \end{cases}$$

(ii) *For n odd, the non-singular quadric $\mathcal{W}_n$ is hyperbolic or elliptic according as $\alpha \in \mathcal{C}_0$ or $\mathcal{C}_1$, where*

$$\alpha = \begin{cases} (-1)^{(n+1)/2}|A|, & q \text{ odd} \\ \{|B| - (-1)^{(n+1)/2}|A|\}/\{4|B|\}, & q \text{ even.} \end{cases}$$

□

For the quadric $\mathcal{W}_n = V(F)$ in $PG(n,q)$ a line ℓ is a **tangent** to $\mathcal{W}_n$ if ℓ contains a unique point of $\mathcal{W}_n$.

Let $X = (x_0, x_1, \ldots, x_n), Y = (y_0, y_1, \ldots, y_n) \in PG(n,q)$ with $X \neq Y$ and define

$$G(X,Y) = F(X+Y) - F(X) - F(Y).$$

If X is a point of the quadric and Y is not a point of the quadric, then XY is a tangent to $\mathcal{W}_n$ if and only if $G(X,Y) = 0$. If X, Y are both points of the quadric, then $G(X,Y) = 0$ if and only if the line XY lies on the quadric. Moreover for q even, if one of X and Y is not on the quadric, then XY is a tangent if and only if $G(X,Y) = 0$.

An alternative expression for $G(Y,X)$ is given by,

$$G(Y,X) = \sum \frac{\partial F}{\partial x_i}(Y)x_i.$$

For q even and n even, if $\mathcal{Q}_n$ is a non-singular (parabolic) quadric, then $\mathcal{Q}_n$ has a **nucleus**, that is, there exists a unique point $Y \notin \mathcal{Q}_n$ such that $G(Y,X) = 0$ for all points X; that is, a point Y for which $\frac{\partial F}{\partial x_i}(Y) = 0$, for all i.

Let $\mathcal{Q}_n$ be a non-singular quadric in $PG(n,q)$ and let P be a point of the quadric. The set of points X for which $G(P,X) = 0$ is the **tangent hyperplane** to $\mathcal{Q}_n$ at P. The tangent hyperplane at P contains any $m-$space on P which is contained in $\mathcal{Q}_n$.

If $\mathcal{W}_n$ is singular in $PG(n,q)$, then $\mathcal{W}_n$ is a **cone** $\Pi_k\mathcal{Q}_s$, the join of a **vertex** $k-$space Π_k to a non-singular quadric **base** $\mathcal{Q}_s$ contained in an $s-$space Π_s with $\Pi_k \cap \Pi_s = \Pi_{-1}$ and $k + s = n - 1$. Let P be a point of $\mathcal{W}_n$. The set of points X for which $G(P,X) = 0$ is the **tangent space** to $\mathcal{W}_n$ at P. The tangent space at P contains the vertex Π_k and if $P \in \Pi_k$, then the tangent space at P is the whole space $PG(n,q)$.

1.5 Arcs, Curves and Normal Rational Curves in $\mathbf{PG(n,q)}$

In this section, unless stated otherwise, the material can be found in [50, Sections 27.1, 27.5].

A $\mathbf{k_n}$**−arc** in $PG(n,q)$ is a set of k points not contained in a hyperplane with at most n points in any hyperplane of $PG(n,q)$.

A **rational curve** C^d of order d in $PG(n,q)$ is the set of points

$$\{P(t_0,t_1) = P(g_0(t_0,t_1),\ldots,g_n(t_0,t_1)) |\ t_0,t_1 \in GF(q)\ \}$$

where each g_i is a binary form of degree d and the highest common factor of the g_i is 1. The curve C^d may also be written

$$\{P(t) = P(f_0(t),\ldots,f_n(t)) |\ t \in GF(q) \cup \{\infty\}\ \}$$

where $f_i(t) = g_i(1,t)$. As the g_i have no non-trivial common factor, at least one of the f_i has degree d.

Also C^d is **normal** if it is not the projection of a rational curve C'^d, of order d, in $PG(n+1,q)$, where C'^d is not contained in a hyperplane.

Let C^d be a normal rational curve in $PG(n,q)$ not contained in a hyperplane. Then

(i) $q \geq n$;

(ii) $d = n$;

(iii) C^n is projectively equivalent to $\{P(t) = P(t^n, t^{n-1},\ldots,t,1) \mid t \in GF(q) \cup \{\infty\}\}$.

(iv) C^n consists of $q+1$ points no $n+1$ in a hyperplane.

Note that if C^n is a normal rational curve in $PG(n,q)$, then C^n has form

$$\{P(t) = P(f_0(t), f_1(t),\ldots,f_n(t)) \mid t \in GF(q) \cup \{\infty\}\ \}$$

where at least one of the polynomials $f_0, f_1, \ldots, f_n$ has degree n. Also, since $(f_0(t),\ldots,f_n(t))$ is the image of $(t^n, t^{n-1},\ldots,t,1)$ under some projectivity $H \in PGL(n+1,q)$, the polynomials $f_0(t),\ldots,f_n(t)$ are linearly independent.

A normal rational curve C^2 in $PG(2,q)$ is a non-degenerate conic and a normal rational curve C^3 in $PG(3,q)$ is called a **twisted cubic curve**.

By property (iv) above, a normal rational curve C^n of order n in $PG(n,q)$ is a $(q+1)_n-$arc. In certain cases the converse result is true; for example, for $(q+1)_4-$arcs in $PG(4,q)$ we have the following results.

Theorem 1.5.1 [25] *In $PG(4,q)$, $q=2^h$, every $(q+1)_4$-arc is the pointset of a normal rational curve.* □

Theorem 1.5.2 [50, Theorem 27.6.5] *In $PG(4,q)$, q odd, $q \le 7$, every $(q+1)_4$-arc is the pointset of a normal rational curve.* □

Theorem 1.5.3 [76] [81] *In $PG(n,q)$, q odd, with $q > (4n-23/4)^2$, every $(q+1)_n$-arc is the pointset of a normal rational curve.*
In particular, in $PG(4,q)$, q odd, $q > (10.25)^2$, every $(q+1)_4$-arc is the pointset of a normal rational curve. □

Note the following result due to Glynn [43].

Theorem 1.5.4 [43] *In $PG(4,9)$, there exists a 10_4-arc which is not the pointset of a normal rational curve.* □

1.6 Varieties and Plane Curves

In this section we follow Semple and Roth, *Introduction to Algebraic Geometry* [71].

The setting is $PG(n,q)$.

Definition 1.6.1 *A* **primal** *or* **hypersurface** *is the locus of points V whose coordinates $(x_0,\dots,x_n)$ satisfy an equation of the form:*

$$F(x_0,\dots,x_n) = \sum_{i_0+\dots+i_n=r} \rho_j x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} = 0, \quad j = 1,\dots,\binom{n+r}{n}$$

where the $\rho_j \in GF(q)$ are not all zero and F is a homogeneous polynomial. If F is of degree r, the primal is said to be of **order** *r and is denoted by $V = V_{n-1}^r$. If F is of degree 2, the primal is called a* **quadric**.

There are $\binom{n+r}{n}$ coefficients in such a polynomial F of degree r, therefore $\binom{n+r}{n}-1$ points of $PG(n,q)$ (in general position) determine a unique primal V_{n-1}^r.

Example: In $PG(n,q)$, $\binom{n+2}{n}-1=\binom{n+2}{2}-1=\frac{n(n+3)}{2}$ generic points determine a unique quadric. In other words, a quadric can be made to pass through $\frac{n(n+3)}{2}$ points; the quadric is unique if and only if these points result in $\frac{n(n+3)}{2}$ linearly independent conditions on the coefficients of the defining polynomial F.

Theorem 1.6.2 *An arbitrary line meets a V_{n-1}^r in r points (some may coincide, some may belong to an extension of the field $GF(q)$).* □

In $PG(n,q)$, n generic primals have, in general, points in common whose coordinates are found by solving their equations simultaneously for $x_0 : x_1 : \ldots : x_n$. When the primals are generally situated with respect to each other, we obtain the following result:

Bézout's Theorem 1.6.3 *In $PG(n,q)$, n generic irreducible primals $V_{n-1}^{r_i}$ $(i=1,2,\ldots,n)$, of orders $r_1,\ldots,r_n$ respectively, have $r_1r_2\ldots r_n$ common points.* □

Here **irreducible** means the polynomial defining a given primal is irreducible in the field and in any extension of the field.

Example: In $PG(2,q)$, two generic conics have 4 points in common; of these four points some may coincide, some may belong to an extension of the field. Similarly, in $PG(3,q)$, three generic quadrics have 8 points in common.

Note that when the primals are not in general position to one another the intersection need not be the number of points prescribed by Bézout's Theorem 1.6.3, for example:

Consider the three quadrics $x_0x_3-x_1x_2=0$, $x_1^2-x_2x_0=0$ and $x_1x_3-x_2^2=0$ in $PG(3,q)$; the complete intersection is the set $\zeta=\{(1,\theta,\theta^2,\theta^3);\theta\in GF(q)\cup\{\infty\}\}$ which contains more that 8 points if $q>7$. So here the quadrics are not generic, are not in general position with respect to one another.

Dimension of a variety

Definition 1.6.4 *A point-locus in $PG(n,q)$ is said to be an* **irreducible algebraic manifold** V_k **of dimension** k *if its points can be shown to be in algebraic 1-1 correspondence with the points of an irreducible primal $\mathcal{M}_k$ of a space $PG(k+1,q)$.*

Algebraically this means that if $\tilde{x} = (x_0, x_1, \ldots, x_n)$ is a general point in $PG(n,q)$ and if $\tilde{y} = (y_0, \ldots, y_{k+1})$ is a general point in $PG(k+1,q)$, then there exists a set of $n+1$ polynomials $F_0, \ldots, F_n$, homogeneous and of the same degree in $y_0, \ldots, y_{k+1}$, and a further irreducible homogeneous polynomial $M(y_0, \ldots, y_{k+1})$, such that as $\tilde{y}$ describes the primal $\mathcal{M}_k$ defined by $M(y_0, \ldots, y_{k+1}) = 0$, the point $\tilde{x} = (x_0, \ldots, x_n)$ given by $\rho x_i = F_i(y_0, \ldots, y_{k+1})$, $i = 0, \ldots, n$, describes V_k; and the correspondence is such that the generic point of V_k arises from only one point of $\mathcal{M}_k$.

The equations

$$\rho x_i = F_i(y_0, \ldots, y_{k+1}) \quad (i = 0, \ldots, n) \tag{1.1}$$

$$\text{and} \quad M(y_0, \ldots, y_{k+1}) = 0 \tag{1.2}$$

are called the **parametric equations** of V_k; the **parameters** are the $k+1$ ratios $y_i : y_0$ $(i = 1, \ldots, k+1)$.

A variety V_1 of dimension 1 is called a **curve** and a variety V_2 of dimension 2 is called a **surface**.

Example: In $PG(3,q)$ consider the plane $\mathcal{M}_2^1$ defined by $M(y_0, y_1, y_2, y_3) = y_3 = 0$. Consider:

$$\rho x_0 = F_0(y_0, y_1, y_2, y_3) = y_2^2,$$

$$\rho x_1 = F_1(y_0, y_1, y_2, y_3) = y_0 y_2,$$

$$\rho x_2 = F_2(y_0, y_1, y_2, y_3) = y_1 y_2,$$

$$\rho x_3 = F_3(y_0, y_1, y_2, y_3) = y_0 y_1.$$

So $(y_0, y_1, y_2, y_3) \longrightarrow (y_2^2, y_0 y_2, y_1 y_2, y_0 y_1)$ is an algebraic 1-1 correspondence and therefore the quadric with equation $x_1 x_2 - x_0 x_3 = 0$ in $PG(3,q)$ has dimension 2 and so is a (primal) variety V_2^2.

Order of a variety

An $(n-k)-$space in $PG(n,q)$ can be represented as the complete intersection of k hyperplanes in general position, that is, the solution set to a system of k linear equations of the form

$$\sum_{i=0}^{n} a_{ij} x_i = 0 \quad (j = 1, \ldots, k). \tag{1.3}$$

These k equations, combined with equations (1.1), represent k polynomials in $y_0, y_1, \ldots, y_{k+1}$. Together with equation (1.2) we have $k+1$ polynomials in $y_0, y_1, \ldots, y_{k+1}$ which represent $k+1$ primals in $PG(k+1, q)$ and by Bézout's Theorem these $k+1$ primals intersect in a given number, r say, of points. We have

Theorem 1.6.5 *A generic $(n-k)-$space in $PG(n,q)$ meets a variety V_k of dimension k in a fixed number r of points. We call r the* **order** *of the variety.* □

We shall write V_k^r for a **variety of dimension** k **and order** r in $PG(n,q)$.

Note: From above, V_{n-1}^r denotes a primal in $PG(n,q)$. Now $n-(n-1)=1$ and a subspace of dimension 1 is a line and by Theorem 1.6.2, a line intersects V_{n-1}^r in r points which is consistent with our definition of order.

Example: The points of a $h-$space in $PG(n,q)$ can be put in $1-1$ correspondence with those of a $h-$space in a $PG(h+1,q)$, with the latter being a primal defined by an equation order 1. Hence any projective subspace of dimension h is a variety of dimension h. By Grassman's identity (Dimension Theorem), a generic $(n-h)-$space in $PG(n,q)$ intersects a $h-$space in a unique point. Thus a $h-$space of $PG(n,q)$ is a variety of dimension h and order 1 and is denoted by S_h^1.

Intersections of varieties

Theorem 1.6.6 *In $PG(n,q)$, the intersection of two varieties V_k and V_h, of dimension k and h respectively, in general forms a manifold V_{k+h-n} of dimension $k+h-n$ (where $k+h \geq n$).* □

If two varieties V_k and V_h intersect properly in a variety V_{k+h-n} of dimension $k+h-n$ the intersection is called **normal**.

Theorem 1.6.7 ***(Generalized theorem of Bézout)*** *If two varieties V^r and V^s, of orders r and s respectively, intersect normally, then the intersection is a variety V^{rs} of order rs.* □

Examples:

(1) The intersection of a primal V_{n-1}^r and a $h-$space S_h^1 in $PG(n,q)$ is in general a primal, V_{h-1}^r say, of order r in the $h-$space.

(2) The intersection of a $(n-k)-$space S^1_{n-k} and a variety V^r_k in $PG(n,q)$ is in general $S^1_{n-k} \cap V^r_k = V^r_0$. In other words a generic $(n-k)-$space intersects a variety of dimension k in r points, where r is the order of the variety (as promised by our definition of order).

Consider a normal rational curve C^n_1 in $PG(n,q)$ as discussed in Section 1.5. Then by Section 1.5, the pointset of C^n_1 in canonical form is

$$\{P(t) = P(t^n, t^{n-1}, \ldots, t, 1) \mid t \in GF(q) \cup \{\infty\}\}.$$

The points of C^n_1 are in algebraic 1-1 correspondence with the points of the (primal) line in $PG(2,q)$ with points $\{(1,t,1) \mid t \in GF(q) \cup \{\infty\}\}$ and equation $M(y_0, y_1, y_2) = y_0 - y_2 = 0$, for example. Hence the normal rational curve indeed has dimension 1 as a variety.

Consider a generic hyperplane, that is an S^1_{n-1} in $PG(n,q)$, with equation $a_0x_0 + a_1x_1 + \ldots + a_nx_n = 0$, where $a_i \in GF(q)$ are not all zero. Then S^1_{h-1} intersects the curve in precisely n points, namely the points of C^n_1 with parameter t such that t satisfies $a_0t^n + a_1t^{n-1} + \ldots + a_n = 0$; note that some of these points may coincide or belong to an extension of $GF(q)$. Hence, by the definition of order of a variety, the normal rational curve is a variety C^n_1 of dimension 1 and order n.

We include some additional results for later reference. For a discussion of *genus*, the reader is referred to [71], [5] and [48]; note that the **genus** g of a curve is a non-negative integer. Here we state a few isolated results which we require in the proof of Theorem 4.1.4 in Chapter 4.

Theorem 1.6.8 *[5, Chapter VIII, Part VI]*

(i) An irreducible algebraic curve of order 6 *in* $PG(4,q)$, *which is not contained in a hyperplane, has maximum possible genus* 2, *that is,* $g \leq 2$.

(ii) An irreducible algebraic curve of order 4 *in* $PG(4,q)$ *which is not contained in a hyperplane has genus* $g = 0$ *(in fact this is a normal rational curve in* $PG(4,q)$*).*

Result 1.6.9 [5, Page 239, Example 8.] *Let l be a line in $PG(4,q)$ skew to the plane of a conic $\mathcal{C}$ in $PG(4,q)$. Let θ be a projectivity between the points P of the conic $\mathcal{C}$ and* *the points P^θ of the line l. The set of points on the lines PP^θ is a rational ruled cubic surface. The curve which is the intersection of this surface with a general quadric is a sextic curve, of genus 2.*

Theorem 1.6.10 Hasse-Weil Theorem [48, Theorem 10.2.1] [81] [2, Corollary 2.4] *If $\mathcal{C}^r$ is an absolutely irreducible curve of order r and genus g in $PG(n,q)$, $n \geq 2$ and if R is the number of points of $\mathcal{C}^r$ then,*

$$|R-(q+1)| \leq 2g\sqrt{q}$$

□

Finally we discuss **plane curves** specifically, that is, curves in $PG(2,q)$.

From above, in $PG(2,q)$, a **plane curve** C^n of order n is represented by an equation

$$f(x,y,z)=0$$

such that f is a polynomial of degree n, homogeneous in x,y,z.

The equation of a general C^n may be written in the form

$$f(x,y,z) \equiv u_0 z^n + u_1 z^{n-1} + \ldots + u_n = 0,$$

where $u_i \equiv u_i(x,y)$ is homogeneous of degree i in x and y.

A **multiple point of order** k (or k-fold point, $k>1$) of C^n is a point P of the curve such that a generic line through P meets the curve in only $n-k$ further points. A $k-$fold point is a **singular** point of the curve.

If $P'(0,0,1)$ is a k-fold point of C^n, the equation of C^n may be written in the form

$$z^{n-k}u_k(x,y) + z^{n-k-1}u_{k+1}(x,y) + \ldots + u_n(x,y) = 0 \tag{1.4}$$

where $u_k(x,y)=0$ is the equation of the k tangents of C^n at $P'(0,0,1)$; note that these tangents are not necessarily distinct or belonging to $GF(q)$.

1.7 The ruled cubic surface $\mathbf{V_2^3}$

The ruled cubic surface which we denote by V_2^3 is a variety which plays an important role in most of the work in this thesis; this was briefly introduced in Result 1.6.9. In this section we define this surface and briefly summarise the important properties required for later chapters.

The following material can be found in Bernasconi and Vincenti [15].

Consider the space $PG(4,q)$ and let C be a non-degenerate conic in a plane S_2 of $PG(4,q)$; let ℓ be a line of $PG(4,q)$ such that $\ell \cap S_2 = \emptyset$. Let ϕ be a projectivity between ℓ and C, that is if we denote the non-homogeneous coordinates of points of C (respectively ℓ) by θ (respectively λ) then for a fixed projectivity $\phi \in PGL(2,q)$ consider the one-to-one correspondence between points P of ℓ with points P^ϕ of C given by the relationship $\theta = \phi(\lambda)$, $\lambda \in GF(q) \cup \{\infty\}$. Note that since $\phi \in PGL(2,q)$, the projectivity is determined by the images of three distinct points of ℓ (see [48, page 119]).

Consider the set G of $q+1$ lines PP^ϕ where the point P varies over ℓ.

Theorem 1.7.1 [15] *The set of points incident with the lines of G is a rational ruled variety, of order 3 and dimension 2, of $PG(4,q)$.*

We call such a variety a **ruled cubic surface** and denote it by $\mathbf{V_2^3}$. The line ℓ shall be referred to as the **line directrix** of the ruled cubic surface, the conic C as the **base conic**, the projectivity ϕ as the **associated projectivity** and the $q+1$ lines in G shall be called **generators of** V_2^3.

Theorem 1.7.2 [15] *Let V_2^3 be a ruled cubic surface in $PG(4,q)$ with line directrix ℓ base conic C and associated projectivity ϕ. The following properties are satisfied by V_2^3:*

1. *Any three distinct generators of V_2^3 are not contained in a hyperplane of $PG(4,q)$,*
2. [15, Proposition 1.2] *For any hyperplane S_3^1 of $PG(4,q)$ the intersection $S_3^1 \cap V_2^3$ is a cubic curve $\mathcal{C}_1^3$; note that the cubic curve $\mathcal{C}_1^3$ may be reducible or may have some component(s) in $PG(4,F)$, where F is a field extension of $GF(q)$,*
3. (The proof of Theorem 2.1 in [15])
   - *(a) In $PG(4,q)$ there exist precisely q^2 conics on V_2^3 and each such conic is disjoint from the line directrix ℓ,*
   - *(b) Each conic on V_2^3 contains a unique point on each generator of V_2^3,*
   - *(c) Two distinct conics on V_2^3 intersect in a unique point (of V_2^3).*

Note that since any three distinct generators of V_2^3 are not contained in a hyperplane of $PG(4,q)$ the ruled cubic surface V_2^3 is not contained in a hyperplane of $PG(4,q)$. Finally we note that the ruled cubic surfaces in $PG(4,q)$ are projectively equivalent,

Theorem 1.7.3 [15, Theorem 1.2, Corollary] *If V and V' are two ruled cubic surfaces in $PG(4,q)$, then there exists at least one projectivity Φ of $PG(4,q)$ such that $V^{\Phi} = V'$.*

1.8 Segre varieties

In this section we follow Section 25.5 of [50]; we include some proofs to clarify the geometric properties of Segre varieties.

In [50, Section 25.5] a Segre variety is defined in terms of k projective spaces $PG(n_1,q)$, $PG(n_2,q), \ldots, PG(n_k,q)$ where $n_i \geq 1$; here we consider only the special case of a Segre variety defined in terms of two projective spaces $PG(n_1,q)$ and $PG(n_2,q)$ which according to [50] is the Segre variety most studied.

Consider two projective spaces $PG(n_1,q)$ and $PG(n_2,q)$, with $n_1, n_2 \geq 1$. Denote the points of $PG(n_i,q)$ by $P_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n_i}^{(i)})$, for $i = 1,2$.

Let $N_r = \{0,1,2,\ldots,r\}$ for any integer $r \geq 1$ and let η be a bijection of $N_{n_1} \times N_{n_2}$ onto N_m, where $m+1 = (n_1+1)(n_2+1)$.

Then the **Segre variety** of the two given projective spaces is the variety $\rho_{n_1;n_2}$ with pointset

$$\{P(x_0,\ldots,x_m) | x_j = x_{\eta(i_1,i_2)} = x_{i_1}^{(1)} x_{i_2}^{(2)} \text{ with } P_i = (x_0^{(i)},\ldots,x_{n_i}^{(i)}) \in PG(n_i,q),\ i = 1,2\}$$

of $PG(m,q) = PG(n_1n_2 + n_1 + n_2, q)$.

Thus, a typical point $P(x_0,x_1,\ldots,x_m)$ of $\rho_{n_1;n_2}$ is determined by a point $P_1(x_0^{(1)}, x_1^{(1)}, \ldots, x_{n_1}^{(1)})$ in $PG(n_1,q)$ and a point $P_2(x_0^{(2)}, x_1^{(2)}, \ldots, x_{n_2}^{(2)})$ in $PG(n_2,q)$. The $m+1 = (n_1+1)(n_2+1)$ components $x_0, x_1, \ldots, x_m$ of the coordinates of P are given by $x_j = x_{\eta(i_1,i_2)} = x_{i_1}^{(1)} x_{i_2}^{(2)}$ and therefore the components x_j are in one-to-one correspondence with *all* possible products of the form $x_{i_1}^{(1)} x_{i_2}^{(2)}$, where $i_1 \in \{0,1,\ldots,n_1\}$ and $i_2 \in \{0,1,\ldots,n_2\}$. Since $P_i(x_0^{(i)}, x_1^{(i)}, \ldots, x_{n_i}^{(i)}) \neq (0,0,\ldots,0)$, $i = 1,2$, each P_i contains at least one non-zero component; the product of these non-zero components (one from

P_1 and one from P_2) therefore occurs as a component of $P(x_0, x_1, \ldots, x_m)$. Hence for any point P in the Segre variety $\rho_{n_1;n_2}$ we have $P(x_0, x_1, \ldots, x_m) \neq (0, 0, \ldots, 0)$.

The integers n_1, n_2 are called the **indices** of the variety. It can be shown that this Segre variety is absolutely irreducible and non-singular, with order equal to

$$\frac{(n_1 + n_2)!}{n_1! n_2!}.$$

Any point $P(x_0, x_1, \ldots, x_m)$ of the Segre variety satisfies the following equations

$$x_{\eta(i_1,i_2)} x_{\eta(j_1,j_2)} - x_{\eta(i_1,j_2)} x_{\eta(j_1,i_2)} \;=\; 0. \tag{1.5}$$

Theorem 1.8.1 [50, Theorem 25.5.1] *The Segre variety $\rho_{n_1;n_2}$ is the intersection of all quadrics defined by the equations (1.5), and conversely any point of $PG(n_1n_2+n_1+n_2, q)$ satisfying the equations (1.5) corresponds to a unique element of $PG(n_1, q) \times PG(n_2, q)$.*

□

Let

$$\delta : PG(n_1, q) \times PG(n_2, q) \longrightarrow \rho_{n_1;n_2}$$

be defined by

$$(P_1(x_0^{(1)}, x_1^{(1)}, \ldots, x_{n_1}^{(1)}), P_2(x_0^{(2)}, x_1^{(2)}, \ldots, x_{n_2}^{(2)})) \longmapsto P(x_0, x_1, \ldots, x_m)$$

$$\text{with } x_j \;=\; x_{\eta(i_1,i_2)} \;=\; x_{i_1}^{(1)} x_{i_2}^{(2)}.$$

By Theorem 1.8.1 the mapping δ is a bijection.

Theorem 1.8.2 [50, Theorem 25.5.2] *For a given fixed point P_1 of $PG(n_1, q)$, the set of all points $\delta(P_1, P_2)$ with $P_2 \in PG(n_2, q)$, is an n_2-dimensional projective space contained in $\rho_{n_1;n_2}$.*

Similarly, for a given fixed point P_2 of $PG(n_2, q)$, the set of all points $\delta(P_1, P_2)$ with $P_1 \in PG(n_1, q)$, is an n_1-dimensional projective space contained in $\rho_{n_1;n_2}$.

Proof We prove the first statement; the second is proved analogously. $P_1(x_0^{(1)}, x_1^{(1)}, \ldots, x_{n_1}^{(1)}) \neq (0, 0, \ldots, 0)$ is a fixed point of $PG(n_1, q)$. For any point $P_2(x_0^{(2)}, x_1^{(2)}, \ldots, x_{n_2}^{(2)})$ in $PG(n_2, q)$, the components $x_0, x_1, \ldots, x_m$ of the coordinates of $P(x_0, x_1, \ldots, x_m) = \delta(P_1, P_2)$ are, up to order,

$$\begin{array}{llll} x_0^{(1)}x_0^{(2)}, & x_0^{(1)}, x_1^{(2)}, & \ldots, & x_0^{(1)}x_{n_2}^{(2)}, \\ x_1^{(1)}x_0^{(2)}, & x_1^{(1)}, x_1^{(2)}, & \ldots, & x_1^{(1)}x_{n_2}^{(2)}, \\ \vdots & & & \\ x_{n_1}^{(1)}x_0^{(2)}, & x_{n_1}^{(1)}, x_1^{(2)}, & \ldots, & x_{n_1}^{(1)}x_{n_2}^{(2)}, \end{array}$$

The set $\{P(x_0, x_1, \ldots, x_m) = \delta(P_1, P_2)\}$ is therefore a n_2-dimensional space. □

By Theorem 1.8.2, the Segre variety $\rho_{n_1;n_2}$ has a system Σ_1 of n_1-dimensional subspaces and a system Σ_2 of n_2-dimensional subspaces.

Theorem 1.8.3 [50, Theorem 25.5.3, Theorem 25.5.5] *Any two distinct elements of Σ_i are skew, for $i = 1, 2$. Each point of $\rho_{n_1;n_2}$ is contained in exactly one point of Σ_i, for $i = 1, 2$.*

Each element of Σ_i intersects each element of Σ_j, $i \neq j$, in exactly one point.

Proof Let $\Pi_{n_1} \in \Sigma_1$ correspond to the point $P_2 \in PG(n_2, q)$, that is $\Pi_{n_1} = \{\delta(P_1, P_2) \mid P_1 \in PG(n_1, q)\}$. Similarly, let $\Pi'_{n_1} \in \Sigma_1$ correspond to the point $P'_2 \in PG(n_2, q)$ and suppose $P_2 \neq P'_2$.

For any points $P_1, P'_1 \in PG(n_1, q)$, $(P_1, P_2) \neq (P'_1, P'_2)$ and therefore $\delta(P_1, P_2) \neq \delta(P'_1, P'_2)$ since δ is a bijection. It then follows from Theorem 1.8.2 that $\Pi_{n_1} \cap \Pi'_{n_1} = \emptyset$. Similarly for two distinct elements $\Pi_{n_2}, \Pi'_{n_2} \in \Sigma_2$, $\Pi_{n_2} \cap \Pi'_{n_2} = \emptyset$.

Consider the space $\Pi_{n_1} \in \Sigma_1$ which corresponds to the point $P_2 \in PG(n_2, q)$ and the space $\Pi_{n_2} \in \Sigma_2$ which corresponds to the point $P_1 \in PG(n_1, q)$, then $\delta(P_1, P_2)$ is the unique point contained in the intersection $\Pi_{n_1} \cap \Pi_{n_2}$. □

Corollary 1.8.4 [50, Theorem 25.5.4]

(i) The number of points of $\rho_{n_1;n_2}$ is $|\rho_{n_1;n_2}| = \theta(n_1)\theta(n_2) = |PG(n_1, q)|\,|PG(n_2, q)|$.

(ii) The number of n_1-dimensional subspaces in the system Σ_1 is $|\Sigma_1| = \theta(n_2) = |PG(n_2, q)|$. The number of n_2-dimensional subspaces in the system Σ_2 is $|\Sigma_2| = \theta(n_1) = |PG(n_1, q)|$. □

The main example of a Segre variety which we shall use in this thesis is the Segre variety $\rho_{1;n}$ in $PG(2n+1, q)$ with $n_1 = 1$ and $n_2 = n \geq 1$. The variety $\rho_{1;n}$ has order $n + 1$ and has $|\rho_{1;n}| = (q+1)\theta(n)$ points. The variety has a system Σ_1 of $|\Sigma_1| = \theta(n)$ lines and a system Σ_2 of $|\Sigma_2| = \theta(1) = q+1$ $n-$spaces. If $n=1$, then the variety $\rho_{1;1}$ in $PG(3,q)$ is a hyperbolic quadric. If $n=2$, then the variety $\rho_{1;2}$ in $PG(5,q)$ has a system Σ_1 of q^2+q+1 pairwise disjoint lines and a system Σ_2 of $q+1$ pairwise disjoint planes; in this case each line in Σ_1 intersects each plane in Σ_2 in exactly a point.

We include a few additional properties of Segre varieties, in particular, some results concerning the existence of Segre subvarieties of a Segre variety.

Theorem 1.8.5 [50, Theorem 25.5.6]
No hyperplane of $PG(m,q)$ contains the Segre variety $\rho_{n_1;n_2}$. □

It is convenient to use the following notation and to choose η as the following bijection in the definition of a Segre variety. The element $x_j = x_{\eta(i_1,i_2)}$ will be denoted by $x_{i_1 i_2}$. Let $\eta(i_1,i_2) = i_1(n_1+1)+i_2$. The equations (1.5) become

$$x_{i_1 i_2}x_{j_1 j_2} - x_{j_1 i_2}x_{i_1 j_2} = 0. \tag{1.6}$$

Theorem 1.8.6 [50, Theorem 25.5.7] *The Segre variety $\rho_{n_1;n_2}$ consists of all points $P(x_{00}, x_{01}, \ldots, x_{0n_2}, x_{10}, x_{11}, \ldots\ldots, x_{1n_2}, \ldots, x_{n_1 0}, x_{n_1 1}, \ldots, x_{n_1 n_2})$ of $PG(m,q)$ for which rank$[x_{ij}] = 1$.* □

By Theorem 1.8.6, for example, the Segre variety $\rho_{1;2}$ in $PG(5,q)$ consists of all points $(x_{00}, x_{01}, x_{10}, x_{11}, x_{20}, x_{21})$ for which

$$\text{rank}\begin{bmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \\ x_{20} & x_{21} \end{bmatrix} = 1$$

that is, for which

$$\begin{aligned} x_{00}x_{11} &- x_{10}x_{01} = 0 \\ x_{00}x_{21} &- x_{20}x_{01} = 0 \\ x_{10}x_{21} &- x_{20}x_{11} = 0. \end{aligned}$$

A Segre variety $\rho_{n_1;n_2}$ in $PG(m,q)$ is the intersection of quadrics with equations (1.6). Therefore any line ℓ of $PG(m,q)$ intersects $\rho_{n_1;n_2}$ in $0,1,2$ or $q+1$ points. By the following result, any line contained in the Segre variety $\rho_{n_1;n_2}$ must be contained in either an element of Σ_1 or an element of Σ_2.

Theorem 1.8.7 [50, Lemma 25.5.10]

Any line ℓ of $\rho_{n_1;n_2}$ is contained in an element of Σ_1 or Σ_2. $\square$

An $s-$space Π_s which is contained in $\rho_{n_1;n_2}$ and such that Π_s is contained in no $(s+1)-$space Π_{s+1} of $\rho_{n_1;n_2}$, is called a **maximal space** or **maximal subspace** of $\rho_{n_1;n_2}$. Using Theorem 1.8.7 it can be shown that,

Theorem 1.8.8 [50, Theorem 25.5.11, Corollary 1]

(i) The maximal spaces of the Segre variety $\rho_{n_1;n_2}$ are the elements of Σ_1 and Σ_2;

(ii) Each $s-$space of $\rho_{n_1;n_2}$, with $s > 0$, is contained in either a unique element of Σ_1 or a unique element of Σ_2. $\square$

By Theorems 1.8.4 and 1.8.8, it is possible to count the number of subspaces contained in $\rho_{n_1;n_2}$ as follows.

Corollary 1.8.9 [50, Theorem 25.5.11, Corollary 2] *Let $n_1 \leq n_2$. The number of $s-$spaces contained in $\rho_{n_1;n_2}$ is*

(i) $\theta(n_1)\phi(s;n_2,q) + \theta(n_2)\phi(s;n_1,q)$, for $0 < s \leq n_1$;

(ii) $\theta(n_1)\phi(s;n_2,q)$, for $n_1 < s \leq n_2$. $\square$

Finally, we present the results from [50, Section 25.5] on Segre subvarieties.

Theorem 1.8.10 [50, Theorem 25.5.12] *Let $P_i \in PG(n_i,q)$ and let $PG(d_i,q)$ be a d_i-space of $PG(n_i,q)$, $i = 1,2$. Then*

(i) $\delta(\{P_1\} \times PG(d_2,q))$ is a d_2-subspace and $\delta(PG(d_1,q) \times \{P_2\})$ is a d_1-subspace of $\rho_{n_1;n_2}$;

(ii) all subspaces of $\rho_{n_1;n_2}$ are obtained as in (i);

(iii) when $d_i > 0$, $i = 1,2$, $\delta(PG(d_1,q) \times PG(d_2,q))$ is a Segre variety $\rho_{d_1;d_2}$ contained in $\rho_{n_1;n_2}$;

(iv) $\rho_{d_1;d_2} = \rho_{n_1;n_2} \cap PG(m',q)$, where $m' = d_1d_2+d_1+d_2$ and $PG(m',q)$ is the $m'-$space generated by $\rho_{d_1;d_2}$;

(v) all Segre subvarieties of $\rho_{n_1;n_2}$ are obtained as in (iii). □

Note that by considering the number of subspaces of $PG(n_1, q)$ and $PG(n_2, q)$, by Theorem 1.8.10 the number of Segre subvarieties of $\rho_{n_1;n_2}$ can be calculated (see [50, Theorem 25.5.12, Corollary 1]).

Theorem 1.8.11 [50, Theorem 25.5.12, Corollary 2] *Let Π_s be an $s-$space of $\rho_{n_1;n_2}$, $s \geq 1$, contained in an element Π_{n_1} of Σ_1. Then the elements of Σ_2 meeting Π_s in a point are the elements of a system of maximal subspaces of a Segre subvariety $\rho_{s;n_2}$ of $\rho_{n_1;n_2}$.* □

1.9 Spreads and Reguli

The following definitions and results are found in [48, Chapter 4] and [50, Section 25.6]; for further detail the reader should refer to these texts.

A **spread** $\mathcal{S}_r$ of $r-$spaces of $PG(n, q)$ is a set of $r-$spaces which partitions $PG(n, q)$; that is, every point of $PG(n, q)$ lies in some $r-$space of $\mathcal{S}_r$ and every two $r-$spaces of $\mathcal{S}_r$ are disjoint. The $r-$spaces in $\mathcal{S}_r$ are the **elements** of $\mathcal{S}_r$.

A spread of $r-$spaces in $PG(n, q)$ will also be called an **r−spread** of $PG(n, q)$. In the case $r = 1$ and $n = 3$, a $1-$spread of $PG(3, q)$ will sometimes be called a **line spread** of $PG(3, q)$ or simply a **spread** of $PG(3, q)$.

Theorem 1.9.1 [48, Theorem 4.1.1] *The following are equivalent:*

(i) there exists a spread $\mathcal{S}_r$ of $r-$spaces of $PG(n, q)$;

(ii) $|PG(r, q)|$ divides $|PG(n, q)|$;

(iii) $(r + 1)$ divides $(n + 1)$. □

Consider a Segre variety $\rho_{1;n}$ in $PG(2n + 1, q)$. The system of maximal $n-$spaces of $\rho_{1;n}$ will be called an **n−regulus**. In the case $n = 1$, the Segre variety $\rho_{1;1}$ is a hyperbolic quadric in $PG(3, q)$; a **1−regulus** is also called a **regulus**.

Theorem 1.9.2 [50, Theorem 25.6.1, Corollary] *If $\Pi_n, \Pi_n^{'}, \Pi_n^{''}$ are mutually skew $n-$spaces in $PG(2n+1,q)$, $n \geq 1$, then the set of all lines having non-empty intersection with $\Pi_n, \Pi_n^{'}$ and $\Pi_n^{''}$ is a system of maximal spaces of a Segre variety $\rho_{1;n}$. Moreover, the $n-$regulus in $\rho_{1;n}$ which contains $\Pi_n, \Pi_n^{'}$ and $\Pi_n^{''}$ is the unique $n-$regulus in $PG(2n+1,q)$ which contains $\Pi_n, \Pi_n^{'}$ and $\Pi_n^{''}$.* □

For mutually skew $n-$spaces $\Pi_n, \Pi_n^{'}, \Pi_n^{''}$ in $PG(2n+1,q)$ the (unique) $n-$regulus containing $\Pi_n, \Pi_n^{'}$, and $\Pi_n^{''}$ is denoted by $R(\Pi_n, \Pi_n^{'}, \Pi_n^{''})$.

In this thesis we shall use the following definition of a *regular $n-$spread.*

Definition 1.9.3 *For $q > 2$ an $n-$spread $\mathcal{S}_n$ of $PG(2n+1,q)$ is called* **regular** *if for any three distinct elements $\Pi_n, \Pi_n^{'}, \Pi_n^{''}$ of $\mathcal{S}_n$ the whole $n-$regulus $R(\Pi_n, \Pi_n^{'}, \Pi_n^{''})$ is contained in $\mathcal{S}_n$.*

Theorem 1.9.4 [50, Theorems 25.6.4, 25.6.5] *For $q > 2$ an $n-$spread $\mathcal{S}_n$ of $PG(2n+1,q)$ is regular if and only if the $n-$spaces of $\mathcal{S}_n$ meeting any line not in an element of $\mathcal{S}_n$ form an $n-$regulus.* □

In the case $n = 1$, the above definition and theorem concerning regular $1-$spreads in $PG(3,q)$ are also valid in the case $q = 2$; by [49, Chapter 17] for $q = 2$, every $1-$spread in $PG(3,q)$ is regular. For $n > 1$ and $q = 2$ every $n-$spread in $PG(2n+1,2)$ satisfies the property that for any three distinct elements $\Pi_n, \Pi_n^{'}, \Pi_n^{''}$ of the spread the whole $n-$regulus $R(\Pi_n, \Pi_n^{'}, \Pi_n^{''})$ is contained in the spread. (See [50, Section 25.6] for more detail on the case $n > 1$ and $q = 2$.)

The regular $n-$spreads in $PG(2n+1,q)$ are projectively equivalent by the following theorem.

Theorem 1.9.5 [50, Theorem 25.6.7] *The group $PGL(2n+2,q)$ acts transitively on the set of all regular $n-$spreads of $PG(2n+1,q)$.* □

Finally we include the well known characterisation by Bruck of regular $1-$spreads of $PG(3,q)$.

Theorem 1.9.6 [20, Theorem 5.3] *Let $PG(3,q)$ be embedded as a subgeometry of $PG(3,q^2)$. Let $^-$ denote the Fröbenius automorphism of $PG(3,q^2)$ which fixes every* *point in $PG(3,q)$. Let g be any line of $PG(3,q^2)$ which contains no point of $PG(3,q)$. For each such line, g, let $\mathcal{S}_g$ denote the set of all lines of $PG(3,q)$ which meet g. Then, $\mathcal{S}_g = \mathcal{S}_{\overline{g}}$ is a regular spread of $PG(3,q)$. Every regular spread of $PG(3,q)$ can be represented in this manner for a unique pair of lines $g, \overline{g}$.* $\square$

1.10 The Bruck and Bose representation

In this section we present the results of Bruck and Bose ([21] and [22]) which provide us with a representation of translation planes of order q^h in the projective space $PG(2h,q)$. In particular, we obtain a representation of the Desarguesian plane $PG(2,q^h)$. We also obtain a convenient and natural coordinate system for $PG(2,q^h)$ in this Bruck and Bose representation.

1.10.1 The construction

In this section we follow [21, section 4.].

Let $\mathcal{S}$ be a $(h-1)-$spread of $\Sigma_\infty = PG(2h-1,q)$ and embed Σ_∞ as a hyperplane in $PG(2h,q)$.

Define an incidence structure *aff*$(\Pi) = (\mathbf{P}, \mathbf{B}, I)$ as follows:

The *points* of *aff*(Π) are the points of $PG(2h,q)\backslash\Sigma_\infty$.
The *lines* of *aff*(Π) are the $h-$spaces of $PG(2h,q)$ which intersect Σ_∞ in a unique element of $\mathcal{S}$. (Note that this implies that each such $h-$space is not contained in Σ_∞.)
The *incidence* relation of *aff*(Π) is that induced by the incidence relation of $PG(2h,q)$.

Theorem 1.10.1.1 [21, Theorem 4.1 and its Corollary] *aff*(Π) *is an affine plane of order q^h.* $\square$

The affine plane *aff*(Π) may be embedded in a projective plane Π by adjoining the spread $\mathcal{S}$ to *aff*(Π) as a *line at infinity* which we denote by ℓ_∞. Each element of $\mathcal{S}$ corresponds to a class of parallel lines of *aff*(Π), thus each element of $\mathcal{S}$ is adjoined to Π as a *point at infinity*.

Hence the corresponding projective plane Π has a perfectly concrete representation in terms of the above construction.

Theorem 1.10.1.2 [21, Theorem 7.1, Corollary] *aff*(Π) *is a translation plane with translation line the line at infinity. Moreover, every finite translation plane is isomorphic to at least one plane aff*(Π).

Theorem 1.10.1.3 [22, Theorem 12.1, Corollary] *The finite projective plane* Π *is Desarguesian if and only if the* $(h-1)-$*spread* $\mathcal{S}$ *of* Σ_∞ *is a regular spread.* □

Finally we note:

Theorem 1.10.1.4 [50, Theorem 25.6.7] *The group* $PGL(2h, q)$ *acts transitively on the set of all regular* $(h-1)-$*spreads of* $PG(2h-1, q)$. □

1.10.2 Some Galois theory

Before we present a coordinatisation for the projective plane Π, we review some well known Galois theory.

The following information can be found for example in chapter 7 of *A first course in Abstract Algebra,* [39], by John B. Fraleigh.

Let $GF(q)$ denote the (finite) Galois field of order q, where $q = p^r$, p is prime and $r \geq 1$ is an integer.

The integral domain of all polynomials in an indeterminate x with coefficients in the field $GF(q)$ is denoted by $GF(q)[x]$. Let h be a positive integer, $h > 1$; then there exists a monic polynomial of degree h in $GF(q)[x]$ which is irreducible over $GF(q)$. Denote this polynomial by,

$$p_\alpha(x) = x^h - c_{h-1}x^{h-1} - \ldots - c_1 x - c_0$$

where the c_i are in $GF(q)$.

There exists an extension field E of $GF(q)$ and an element $\alpha \in E$ such that $p_\alpha(\alpha) = 0$. Hence $\alpha \in E$ is algebraic over $GF(q)$ of degree h, and the polynomial $p_\alpha(x)$ is called the *minimal polynomial for α over $GF(q)$*. Each element b in $GF(q)(\alpha)$, a simple extension field of $GF(q)$, can be uniquely expressed in the form,

$$b = b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1}$$

where the b_i are in $GF(q)$. Thus, $GF(q)(\alpha)$ is a finite field extension of degree h over $GF(q)$ and therefore has q^h elements. It follows that $GF(q)(\alpha)$ is isomorphic to the unique finite field with q^h elements, $GF(q)(\alpha) \cong GF(q^h)$. We shall identify $GF(q)(\alpha)$ with $GF(q^h)$.

Consider the field extension $GF(q^h) = GF(q)(\alpha)$ of $GF(q)$, which from above is an algebraic extension of degree h; it is also a vector space of dimension h over $GF(q)$ with basis $\{1, \alpha, \alpha^2, \ldots, \alpha^{h-1}\}$ where addition of vectors is the usual addition in $GF(q^h)$ and scalar multiplication λb is the usual field multiplication in $GF(q^h)$ with $\lambda \in GF(q)$ and $b \in GF(q^h)$.

We shall often identify $GF(q^h)$ (as a vector space of dimension h over $GF(q)$) with the vector space $GF(q)^h$, since we have the following isomorphism of vector spaces,

$$\begin{array}{llcl} \phi : & GF(q^h) = GF(q)(\alpha) & \longrightarrow & GF(q)^h \\ & b = b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1} & \longmapsto & (b_0, b_1, \ldots, b_{h-1}) \end{array}$$

where the b_i are in $GF(q)$ and $\{1, \alpha, \ldots, \alpha^{h-1}\}$ is the basis, mentioned above, for $GF(q^h)$ as a vector space over $GF(q)$.

By the above theory, there exists an element $\beta \in GF(q^{2h})$, $\beta \notin GF(q^h)$, such that β is algebraic over $GF(q^h)$ and hence $GF(q^{2h})$ is a vector space of dimension 2 over $GF(q^h)$ with basis $\{1, \beta\}$.

The field $GF(q^{2h})$ is also a finite field extension of $GF(q)$ of dimension $2h$. Moreover, since $\{1, \alpha, \ldots, \alpha^{h-1}\}$ is a basis for $GF(q^h)$ as a vector space over $GF(q)$, and $\{1, \beta\}$ is a basis for $GF(q^{2h})$ as a vector space over $GF(q^h)$, the $2h$ elements,

$$\{1, \alpha, \ldots, \alpha^{h-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{h-1}\}$$

form a basis for $GF(q^{2h})$ as a vector space over $GF(q)$.

1.10.3 A regular spread for $\Sigma_\infty = \mathbf{PG(2h-1, q)}$

Our aim is to obtain a convenient coordinate representation of $PG(2, q^h)$ in the Bruck-Bose setting with construction Π as given in Section 1.10.1. By Theorems 1.10.1.1

and 1.10.1.3 we require a regular $(h-1)-$spread $\mathcal{S}$ of $\Sigma_\infty = PG(2h-1,q)$. The following determination of a regular spread $\mathcal{S}$ is a special case of the work of Bruck and Bose given in [21, section 5.].

Throughout this section we shall use the results of Section 1.10.2 and the notation introduced there.

Represent $\Sigma_\infty = PG(2h-1,q)$ as the $(2h)-$dimensional vector space $GF(q^{2h})$ over $GF(q)$; the points of $PG(2h-1,q)$ corresponding to the 1-dimensional vector subspaces of $GF(q^{2h})$. By Section 1.10.2 and the notation introduced there, $GF(q^{2h})$ has basis,

$$\{1, \alpha, \ldots, \alpha^{h-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{h-1}\}$$

as a vector space over $GF(q)$.

Let $J(\infty)$, $J(0)$, $J(1)$ be three distinct $(h-1)-$subspaces of $PG(2h-1,q)$, chosen so that as vector subspaces of $GF(q^{2h})$,

$J(\infty)$ has basis $\{1, \alpha, \ldots, \alpha^{h-1}\}$,

$J(0)$ has basis $\{\beta, \beta\alpha, \ldots, \beta\alpha^{h-1}\}$, and

$J(1)$ has basis $\{1+\beta, \alpha+\beta\alpha, \ldots, \alpha^{h-1}+\beta\alpha^{h-1}\}$.

Denote by $'$ the following linear transformation of $J(\infty)$ onto $J(0)$,

$$' \; : \; a \longmapsto a' = \beta a$$

and, consequently, the following linear transformation maps $J(\infty)$ onto $J(1)$,

$$a \longmapsto a + a'.$$

Note that the vector space $GF(q^{2h})$ is the direct sum of $J(\infty)$ and $J(0)$.

The three vector subspaces $J(\infty)$, $J(0)$, $J(1)$ intersect pairwise in the zero vector and hence, when considered as $(h-1)-$dimensional subspaces of $PG(2h-1,q)$, the three subspaces are pairwise disjoint.

Since $J(\infty)$ is the $h-$dimensional vector space $GF(q^h)$ over $GF(q)$ with basis $\{1, \alpha, \ldots, \alpha^{h-1}\}$, each element $a \in J(\infty)$ can be uniquely expressed in the form,

$$a \;=\; a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}$$

where the a_i are in $GF(q)$.

Note that α^h is an element of $GF(q^h)$ and, by Section 1.10.2,

$$\alpha^h \;=\; c_0 + c_1\alpha + \ldots + c_{h-1}\alpha^{h-1}. \tag{1.7}$$

since $\alpha \in GF(q^h)$ has minimal polynomial $p_\alpha(x) = x^h - c_{h-1}x^{h-1} - \ldots - c_0$, where the c_i are in $GF(q)$.

Similarly, for each power α^{h+i}, $i = 1, \ldots, h-2$, the element α^{h+i} is of course also an element of $GF(q^h)$ and therefore can be uniquely expressed as a linear combination of the basis elements $\{1, \alpha, \ldots, \alpha^{h-1}\}$. Hence, let

$$\alpha^{h+i} = g_{i,0} + g_{i,1}\alpha + \ldots + g_{i,h-1}\alpha^{h-1} \tag{1.8}$$

where the $g_{i,j}$ are in $GF(q)$.

Consider the product ba of two elements $b, a \in J(\infty)$. We have,

$$\begin{aligned} b &= b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1} \\ a &= a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1} \end{aligned}$$

where b_i and a_i are elements of $GF(q)$. Therefore ba is given by,

$$(b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1})(a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}) \tag{1.9}$$

and by substituting the expressions (1.7) and (1.8) into the product (1.9), we can simplify (1.9) and determine ba as a (unique) linear combination of $\{1, \alpha, \ldots, \alpha^{h-1}\}$. Denote this linear combination by,

$$\begin{aligned} ba &= (b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1})(a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}) \\ &= (d_0 + d_1\alpha + \ldots + d_{h-1}\alpha^{h-1}) \\ &= d \end{aligned}$$

where the d_i are in $GF(q)$ and $d \in J(\infty) = GF(q^h)$.

For convenience, we represent each element $a \in J(\infty)$ as a $h-$dimensional vector $(a_0, a_1, \ldots, a_{h-1})$, where $a = a_0 + a_1\alpha + \ldots + a_{h-1}\alpha^{h-1}$ with the $a_i \in GF(q)$ as usual. Then for each element $b \in J(\infty)$, $b = b_0 + b_1\alpha + \ldots + b_{h-1}\alpha^{h-1} = (b_0, b_1, \ldots, b_{h-1})$, the product (1.9) is equivalent to a linear transformation of $J(\infty)$ defined by a $h \times h$ matrix, which we shall denote by B_b, with entries in $GF(q)$, as follows,

$$\begin{array}{ccc} J(\infty) & \longrightarrow & J(\infty) \\ a = (a_0, a_1, \ldots, a_{h-1}) & \longmapsto & (a_0, a_1, \ldots, a_{h-1})B_b = (d_0, d_1, \ldots, d_{h-1}). \end{array}$$

and we use the convention that for a and B_b as above, the product aB_b is the element $d = d_0 + d_1\alpha + \ldots + d_{h-1}\alpha^{h-1}$ of $J(\infty) = GF(q^h)$.

For each of these $h \times h$ matrices B_b over $GF(q)$ defined above, let

$$J(b) \;=\; \{aB_b + a' \mid a \in J(\infty)\} \tag{1.10}$$

so that $J(b)$ is a $h-$dimensional vector subspace of $GF(q^{2h})$ and so represents a $(h-1)-$space in $\Sigma_\infty = PG(2h-1, q)$.

Let $\mathcal{C}$ denote the collection of the q^h matrices B_b over $GF(q)$, so that,

$$\mathcal{C} = \{B_b \mid b \in GF(q^h)\}.$$

Let $\mathcal{S}$ be the collection

$$\{J(\infty)\} \cup \{J(b) \mid b \in GF(q^h)\}$$

of $q^h + 1$ $(h-1)-$spaces in $PG(2h-1, q)$. Note that for $b = 0$ and $b = 1$ the definition of spaces $J(0)$ and $J(1)$ is consistent with our earlier definition of these spaces. We can also note by (1.9) and the following remarks, that $J(0)$ is defined by the zero matrix $B_0 = 0$ in $\mathcal{C}$ and $J(1)$ is defined by the identity matrix $B_1 = I$ in $\mathcal{C}$.

We now show that $\mathcal{S}$ is a regular $(h-1)-$spread of $\Sigma_\infty = PG(2h-1, q)$.

First we note that since $J(\infty)$ has basis $\{1, \alpha, \ldots, \alpha^{h-1}\}$ as a vector subspace of $GF(q^{2h})$ and given the Definition (1.10) of $J(b)$, the subspaces $J(\infty)$ and $J(b)$ have only the zero vector in common and hence as $(h-1)-$spaces in $PG(2h-1, q)$ they are disjoint.

Consider a matrix B_b in $\mathcal{C}$. For any element $a \in J(\infty)$ the product aB_b corresponds to the element ba in $J(\infty) = GF(q^h)$. Hence $aB_b = 0$, for $a \in J(\infty)$ and $a \neq 0$, if and only if $b = 0$. It follows that for every non-zero matrix B_b in $\mathcal{C}$, B_b is non-singular. Moreover we note that for distinct matrices B_{b_1}, B_{b_2} in $\mathcal{C}$,

$$B_{b_1} - B_{b_2} = B_{b_1 - b_2}$$

is an element of $\mathcal{C}$ since $b_1 - b_2 \in GF(q^h)$. Similarly, $\mathcal{C}$ is closed under matrix multiplication. In fact $(\mathcal{C}, +, \cdot)$ is isomorphic to the field $GF(q^h)$ under the isomorphism $B_b \mapsto b$ from $\mathcal{C}$ to $GF(q^h)$.

For distinct matrices B_{b_1}, B_{b_2}, since $B_{b_1} - B_{b_2}$ is an element of $\mathcal{C}$, by the above discussion $B_{b_1} - B_{b_2}$ is non-singular. Next suppose that the two vector subspaces $J(b_1)$ and $J(b_2)$ of $GF(q^{2h})$, corresponding to the distinct matrices $B_{b_1}, B_{b_2} \in \mathcal{C}$ respectively, have a non-zero vector x in common. By Definition (1.10), for some elements $a_1, a_2 \in J(\infty) = GF(q^h)$,

$$x \;=\; a_1 B_{b_1} + a_1' \;=\; a_2 B_{b_2} + a_2'$$

and by equating coefficients of the basis elements of $GF(q^{2h})$, we obtain $a_1^{'} = a_2^{'}$ and therefore $a_1 = a_2$. Hence we have the equality $a_1 B_{b_1} = a_1 B_{b_2}$ which implies $a_1(B_{b_1} - B_{b_2}) = 0$. Since $B_{b_1} - B_{b_2}$ is non-singular we have $a_1 = 0$ and so $x = 0$, a contradiction.

Hence $\mathcal{S}$ is a collection of $q^h + 1$ pairwise disjoint $(h-1)-$spaces in $\Sigma_\infty = PG(2h-1, q)$, that is , $\mathcal{S}$ is a $(h-1)-$spread of Σ_∞. Finally, by [22, Theorem 11.3] and since $(\mathcal{C}, +, \cdot)$ is a field, the spread $\mathcal{S}$ is a regular spread of Σ_∞.

By Theorem 1.10.1.3 and since $\mathcal{S}$ is a regular spread, the Bruck-Bose construction Π, of Section 1.10.1 with spread $\mathcal{S}$, is a Desarguesian projective plane of order q^h.

1.10.4 Coordinates for the projective plane $\Pi = \mathbf{PG(2, q^h)}$

Let Π be a finite projective plane with the construction of Section 1.10.1 with the notation introduced there. Let $\mathcal{S}$ be the regular $(h-1)-$spread of $\Sigma_\infty = PG(2h-1, q)$ determined in the previous section and with the notation introduced there. By Theorems 1.10.1.1 and 1.10.1.3, Π is the Desarguesian projective plane $PG(2, q^h)$ since $\mathcal{S}$ is a regular $(h-1)-$spread.

In this section we use the results of [21, section 6.] to obtain a coordinate system for this Desarguesian projective plane Π determined by $\mathcal{S}$. We shall utilise this coordinatisation in later chapters in examination of varieties, specified by their equations in $PG(2, q^h)$, in the Bruck-Bose setting.

First we recall a familiar coordinatisation of $PG(2, q^h)$. The points of $PG(2, q^h)$ have homogeneous coordinates (x, y, z), where $x, y, z \in GF(q^h)$ and x, y, z are not all equal to zero. Let ℓ_∞, the line at infinity, be the line with equation $z = 0$, or in line coordinates, ℓ_∞ is the line $[0, 0, 1]$. Let $AG(2, q^h) = PG(2, q^h) \backslash \ell_\infty$ be the affine plane obtained from $PG(2, q^h)$ by removing ℓ_∞ and all of its points. The points of $AG(2, q^h)$ have coordinates of the form $(x, y, 1)$ or occasionally for convenience we shall write these affine coordinates in the form (x, y).

The lines of $AG(2, q^h)$ may be divided into two types:

(i) Lines with equation $y = \gamma$ or, equivalently, with line coordinates $[0, 1, -\gamma]$, where $\gamma \in GF(q^h)$.

These lines constitute a parallel class of lines in $AG(2, q^h)$ with point at infinity $(1, 0, 0)$ in $PG(2, q^h)$.

(ii) Lines with equation $x = by + s$ or, equivalently, with line coordinates $[1, -b, -s]$, where $b, s \in GF(q^h)$.

For each $b \in GF(q^h)$ these lines constitute a parallel class of lines in $AG(2, q^h)$ with point at infinity $(b, 1, 0)$ in $PG(2, q^h)$.

We work in the Bruck-Bose setting to obtain a natural coordinatisation of the incidence structure Π, natural in the sense that the coordinatisation will correspond to the above coordinatisation of the plane $PG(2, q^h)$ in a convenient way.

We have $\Sigma_\infty = PG(2h-1, q)$ embedded as a hyperplane in the projective space $PG(2h, q)$. We represent $PG(2h-1, q)$ as a $2h-$dimensional vector space $GF(q^{2h})$ over the field $GF(q)$ with basis,

$$\{1, \alpha, \alpha^2, \ldots, \alpha^{h-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{h-1}\}.$$

Embed $GF(q^{2h})$ as a hyperplane in the $(2h+1)-$dimensional vector space $GF(q^{2h+1})$, and we only need to add a single element e^* say of $GF(q^{2h+1})$ which is not in $GF(q^{2h})$ in order to obtain a basis

$$\{1, \alpha, \alpha^2, \ldots, \alpha^{h-1}, \beta, \beta\alpha, \ldots, \beta\alpha^{h-1}, e^*\}$$

for $GF(q^{2h+1})$.

The regular $(h-1)$-spread $\mathcal{S}$ of $PG(2h-1, q)$ is the collection of $q^h + 1$ $h-$dimensional vector subspaces of $GF(q^{2h})$ defined in the previous section, with the notation introduced there,

$$\mathcal{S} = \{J(\infty)\} \cup \{J(b) \mid b \in GF(q^h)\}.$$

Consider the construction in Section 1.10.1 of the finite Desarguesian projective plane Π. Each affine point of Π is a 1-dimensional vector subspace of $GF(q^{2h+1})$ not contained in the hyperplane $GF(q^{2h})$ and so has a unique basis element of the form

$$x + y' + e^* \quad \text{or, equivalently,} \quad (x_0, x_1, \ldots, x_{h-1}, y_0, y_1, \ldots, y_{h-1}, 1)$$

where $y' \in J(0)$ so that $x, y \in J(\infty) = GF(q^h)$ and have unique representation in the form $x = \sum_{i=0}^{h-1} x_i\alpha^i$, $y = \sum_{i=0}^{h-1} y_i\alpha^i$, where the x_i, y_i are in $GF(q)$. (Note that we have used the fact that $GF(q^{2h})$ is the direct sum of $J(\infty)$ and $J(0)$.) Thus we define the coordinates of the affine point of Π with this basis element to be $(x, y, 1)$ for every ordered pair of elements $x, y \in J(\infty) = GF(q^h)$. We have defined,

$$\begin{aligned}(x, y, 1) &= \{x + y' + e^*\} \\ &= \{(x_0, x_1, \ldots, x_{h-1}, y_0, y_1, \ldots, y_{h-1}, 1)\}.\end{aligned}$$

A line of Π, distinct from the line at infinity, is a $(h+1)-$dimensional vector subspace of $GF(q^{2h+1})$ over $GF(q)$ which intersects $GF(q^{2h})$ in a unique element J of $\mathcal{S}$ and so has the form,

$$\begin{aligned}\langle J, (x, y, 1)\rangle &= \langle J, x + y' + e^*\rangle \\ &= \langle J, (x_0, x_1, \ldots, x_{h-1}, y_0, y_1, \ldots, y_{h-1}, 1)\rangle\end{aligned}$$

provided $(x, y, 1)$ is one of its points.

We divide these lines into two types:

(i) Lines with equation $y = \gamma$. If γ is in $J(\infty) = GF(q^h)$, the point $(x, y, 1)$ of Π lies on the line

$$\langle J(\infty), (0, \gamma, 1)\rangle$$

if and only if $y = \gamma$.

These lines constitute a parallel class of lines in *aff*(Π) with point at infinity $J(\infty)$ in Π.

(ii) Lines with equation $x = by + s$. If s is in $J(\infty) = GF(q^h)$ and $J(b)$ is in $\mathcal{S}$, the point $(x, y, 1)$ lies on the line

$$\langle J(b), (s, 0, 1)\rangle = \langle \{aB_b + a' \mid a \in J(\infty)\}, s + 0' + e^*\rangle$$

if and only if $(x - s) + y'$ is in $J(b)$, that is, if and only if

$$(x_0 - s_0, x_1 - s_1, \ldots, x_{h-1} - s_{h-1}) = (y_0, y_1, \ldots, y_{h-1})B_b$$

where $s = s_0 + s_1\alpha + \ldots + s_{h-1}\alpha^{h-1}$.

For each $b \in GF(q^h)$ these lines constitute a parallel class of lines in *aff*(Π) with point at infinity $J(b)$ in Π.

Now if we wish we can consider the line at infinity ℓ_∞ of Π as being the line with equation $z = 0$, or in line coordinates the line $[0, 0, 1]$. Each element of the regular spread $\mathcal{S} = \{J(\infty)\} \cup \{J(b) \mid b \in GF(q^h)\}$ is a point on the line at infinity and it is convenient to associate $J(b)$ with the coordinates $(b, 1, 0)$ for all $b \in GF(q) \cup \{\infty\}$, so that in particular $J(\infty)$ is associated with $(1, 0, 0)$.

1.11 Plane $\{\mathbf{k;n}\}-$arcs and sets of type $(\mathbf{m,n})$

A $\{k;n\}-$**arc** K in a finite projective plane π_q, of order q, is a set of $k = |K|$ points in the plane such that no $n+1$ are collinear but some n are collinear. Barlotti introduced this definition of a $\{k;n\}-$arc in 1956. If $n = 2$ we call K a $k-$**arc** and in Desarguesian projective planes of odd order the $(q+1)-$arcs are characterised in Segre's Theorem as follows.

Segre's Theorem 1.11.1 [69] *In $PG(2,q)$, q odd, every $(q+1)-$arc is a conic.* □

For later reference we include:

Theorem 1.11.2 [48, Theorem 12.2.5, Corollary 2] *Any $\{k;3\}-$arc in $PG(2,q)$, $q > 3$, satisfies $k \leq 2q+1$.* □

Let K be a $\{k;n\}-$arc in the finite projective plane π_q, of order q. If a line ℓ contains exactly s points of K we call ℓ an $s-$**secant** of K (0-secants are also called *external lines* and 1-secants are often called *tangents* of K). Denote by t_s $(s = 0, \ldots, n)$ the number of $s-$secants of K in π_q. The following identities are proved in [75].

$$\sum_{s=0}^{n} t_s = q^2 + q + 1 \tag{1.11}$$

$$\sum_{s=1}^{n} s t_s = k(q+1). \tag{1.12}$$

$$\sum_{s=2}^{n} s(s-1) t_s = k(k-1) \tag{1.13}$$

K is said to be of **type** $(m_1, m_2, \ldots, n)$, with $m_1 < m_2 < \ldots < n$, if $t_{m_1}, t_{m_2}, \ldots, t_n$ are non-zero, that is if every line of π_q intersects K in exactly $m_1, m_2, \ldots$ or n points.

Note that the points of a conic in $PG(2,q)$ is a set of type $(0,1,2)$ since each line of the plane is external, tangent or secant to the conic.

We now give some results concerning sets of type (m, n) in π_q which can be found in [75], [74].

Let K be a set of type (m, n), $0 \leq m < n \leq q+1$ in π_q; denote by k the number of points of K. Using the relationships (1.11), (1.12) and (1.13) we obtain $t_m + t_n = q^2 + q + 1$, $mt_m + nt_n = k(q+1)$ and $m(m-1)t_m + n(n-1)t_n = k(k-1)$. Note that some terms will vanish when $m = 0$ or $m = 1$. These equations are easily solved and the parameters t_m and t_n are given by,

$$t_m = \tfrac{1}{(n-m)}[n(q^2+q+1) - k(q+1)] \qquad t_n = \tfrac{1}{(n-m)}[k(q+1) - m(q^2+q+1)].$$

Moreover the integer k is found to satisfy

$$k^2 - k[(m+n)(q+1) - q] + mn(q^2+q+1) = 0.$$

By counting points of K on lines through a point $Q \notin K$, respectively a point $P \in K$, we obtain a bound for the cardinality k of a set of type (m, n),

$$\begin{array}{ll} \text{for } 1 \leq m, & mq + n \leq k \leq (n-1)q + m; \\ \text{for } m = 0, & k = (n-1)q + n. \end{array}$$

These bounds are best possible in the sense that there exist examples of sets in π_q, for some values of q, m, n, where the cardinality k takes the extremal values. Examples will be discussed in the following sections.

Let P be a point of K and denote by v_m, v_n the number $m-$secants, respectively $n-$secants, through P. Let Q be a point of π_q not in K and denote by u_m, u_n the number $m-$secants, respectively $n-$secants, through Q. Using the relationships,

$$\begin{array}{rclrcl} v_m + v_n & = & q+1 & u_m + u_n & = & q+1 \\ (m-1)v_m + (n-1)v_n & = & k-1 & mu_m + nu_n & = & k \end{array}$$

We can determine the parameters as,

$$\begin{array}{rclrcl} v_m & = & u_m - q/(n-m) & u_m & = & (n(q+1)-k)/(n-m) \\ v_n & = & u_n + q/(n-m) & u_n & = & (k - m(q+1))/(n-m). \end{array}$$

Since these parameters are all integer valued, it follows that a necessary condition for the existence of a set K of type (m, n), $0 \leq m < n \leq q+1$, in the plane π_q of order q, is that $(n-m)$ **divides** q.

Given a set K of type (m, n) of k points in a finite projective plane π_q, of order q, the following sets are related to K.

The **complement** $\bar{K}$ of K is the set of points of π_q not in K. $\bar{K}$ has $q^2 + q + 1 - k$ points and is a set of type $(q + 1 - n, q + 1 - m)$ in π_q.

In the dual plane π_q^d of π_q the set K^{d_1} of $m-$secants of K constitute a set of type (v_m, u_m) of t_m points; similarly the $n-$secants of K constitute a set K^{d_2} of type (u_n, v_n) in the dual plane with t_n points.

The trivial cases of sets of type (m, n) in π_q occur when $n - m = q$, where K is a line or the complement of a line, and when $n - m = 1$, where K is a point or the complement of a point. The non-trivial cases occur when $n - m$ is a proper divisor of q.

The sets of type $(0, n)$ in π_q (where n divides q is a necessary condition for existence) are called **maximal arcs**; such sets necessarily have cardinality $(n-1)q+n$ from above. We will leave the discussion of maximal arcs to a later section.

It remains to consider sets of type (m, n), $1 \leq m < n \leq q$ in a finite projective plane π_q of order q. We have already determined $(n - m)$ divides q is a necessary condition for existence. Using the parameters derived above of a set of type (m, n) as well as the associated sets in the dual plane and their parameters, Tallini-Scafati proved the following result in [75]:

Theorem 1.11.3 [75] *Suppose K is a set of type $(1, n)$, $n \leq q$, in a finite projective plane π_q, of prime power order, then q is a square and K is either a set of type $(1, \sqrt{q}+1)$ of $q\sqrt{q} + 1$ points OR a set of type $(1, \sqrt{q} + 1)$ of $q + \sqrt{q} + 1$ points.* □

The two sets in π_q identified in this characterisation are called a **unital of order** $\sqrt{q}$ and a **Baer subplane of order** $\sqrt{q}$ respectively (unitals will be defined and discussed in more detail in a later section).

Tallini improved Tallini-Scafati's result by removing the condition that the order of the plane must be a prime power.

Theorem 1.11.4 [74] *Suppose K is a set of type $(1, n)$, $n \leq q$, in a finite projective plane π_q, of order q, and $\dfrac{q}{(n-1)} = p^h$ p prime and $h > 0$ integer. Then $q = p^{2h}$, $n = \sqrt{q} + 1$ and K is either a Baer subplane or a unital of order $\sqrt{q}$.* □

1.12 Caps, Ovoids and Spreads of $\mathbf{PG(3,q)}$

For further detail regarding this section consult [33, 1.4.47 to 1.4.62], [49] and [61]; we restrict our attention to $PG(3,q)$.

A **k−cap** in $PG(3,q)$ is a set of k points no three of which are collinear. An **ovaloid** in $PG(3,q)$ is a k−cap of maximum size. For a k−cap $\mathcal{K}$ in $PG(3,q)$, each line ℓ in $PG(3,q)$ is called an **external**, **tangent** or **secant** line of $\mathcal{K}$ according as the intersection $\ell \cap \mathcal{K}$ contains 0, 1 or 2 points of $\mathcal{K}$.

An **ovoid** is a k−cap in $PG(3,q)$ such the tangent lines at each point form a plane. Moreover an ovoid in $PG(3,q)$ has exactly q^2+1 points. It is known that for $q>2$ an ovoid in $PG(3,q)$ is an ovaloid and conversely.

Let $\mathcal{O}$ be an ovoid in $PG(3,q)$. For each point $P \in \mathcal{O}$ there exist $q+1$ tangent lines to $\mathcal{O}$ at P; these tangent lines lie in a plane about P called the **tangent plane** to $\mathcal{O}$ at P. Each plane of $PG(3,q)$ intersects $\mathcal{O}$ in either 1 point or in a $(q+1)$−arc and is called a **tangent plane** or **secant plane** of $\mathcal{O}$ respectively.

For all values of q, the elliptic quadrics in $PG(3,q)$ form an infinite class of ovoids known as the **classical** ovoids of $PG(3,q)$. Each secant plane of an elliptic quadric in $PG(3,q)$ intersects the elliptic quadric in $q+1$ points of a non-degenerate conic. If q is odd, then every ovoid in $PG(3,q)$ is an elliptic quadric [8].

The only other known class of ovoids in $PG(3,q)$ are the **Tits Ovoids** which exist in $PG(3,2^{2r+1})$, $r \geq 1$ an integer. Their construction is given as follows.

In $PG(3,2^{2r+1})$, $r \geq 1$ an integer, consider the automorphism defined by

$$\sigma : x \longrightarrow x^{2^{r+1}}$$

so that $\sigma^2 : x \longrightarrow x^{2^{2r+2}} = x^2$. Let $\mathcal{O}_T$ be the set of points

$$\mathcal{O}_T = \{(1,z,y,x)|\ z = xy + x^{\sigma+2} + y^{\sigma}\} \cup \{(0,1,0,0)\}$$

(see [49, Theorem 16.4.5]). For $r=1$, $\mathcal{O}_T$ is the non-classical ovoid in $PG(3,8)$ discovered by Segre [69] in 1959. For all other r, the ovoid $\mathcal{O}_T$ was discovered by Tits [88] in 1960.

The Tits ovoids are the only known non-classical ovoids of $PG(3,q)$. Moreover for $q \leq 32$ the ovoids of $PG(3,q)$ are either classical or of Tits type (see [62], [63], [64]).

For σ the automorphism defined above, consider the following set of lines in $PG(3, 2^{2r+1})$:

$$g_\infty = \{(0, s, 0, t)|\ s, t \in GF(2^{2r+1})\}$$

$$g_{a,b} = \{(s, [ab + a^{\sigma+2} + b^\sigma]s + [a^{\sigma+1} + b]t, t + as, bs + a^\sigma t)|\ s, t \in GF(2^{2r+1})\}$$

where a, b are any two elements in $GF(2^{2r+1})$.

This set of $q^2 + 1$ lines forms a spread of $PG(3, 2^{2r+1})$ called the **Lüneburg Spread** [58, Section 23]. It can be proved for q odd that no set of $q^2 + 1$ tangents of an elliptic quadric in $PG(3, q)$ can form a spread of $PG(3, q)$; this is a recent result of Blokhuis and Wilbrink, a special case of which is a consequence of the main result of [7].

1.13 Unitals

A **unital** (or **unitary block design**) of order n is a $2-(n^3+1, n+1, 1)$ design, for some integer n (see [33, section 2.4.21]). A unital is therefore an incidence structure with $v = n^3 + 1$ points, $k = n + 1$ points on each block, such that any two distinct points are incident with a unique common block. A unital of order n has $b = n^2(n^2 - n + 1)$ blocks and each point is incident with exactly n^2 blocks.

The problem of determining for which values n a unital exists is only partially solved. The known examples of unitals are of order n where either n is a prime power or $n = 6$ (see [59] and [4]).

We now discuss some known examples of unitals.

1.13.1 The Classical Unitals

A **polarity** in a projective plane π is a one-to-one and onto map α from the points (respectively lines) of π to the lines (respectively points) of π of order 2 and which preserves incidence, that is,

$$\text{if } P\ \mathbf{I}\ \ell \text{ then } \ell^\alpha\ \mathbf{I}\ P^\alpha$$

for all points P and lines ℓ of π.

A point P (respectively a line ℓ) of π is called **absolute**, with respect to a polarity α, if P is incident with its image P^α under α (respectively $\ell^\alpha\ \mathbf{I}\ \ell$).

Let $a(\alpha)$ denote the number of absolute points of a polarity α in $PG(2,q)$. Since α has order 2, for a point P and a line ℓ,

$$P \mathbf{\ I\ } \ell \text{ if and only if } \ell^{\alpha} \mathbf{\ I\ } P^{\alpha}$$

and if $\ell = P^{\alpha}$ then

$$P \mathbf{\ I\ } P^{\alpha} \text{ if and only if } \ell^{\alpha} \mathbf{\ I\ } \ell$$

and therefore the absolute points are in one-to-one correspondence with the absolute lines. It also follows that each absolute line ℓ contains a unique absolute point, namely the point ℓ^{α}, and conversely, each absolute point lies on a unique absolute line. Thus $a(\alpha)$ is also equal to the number of absolute lines of the polarity α in $PG(2,q)$.

The polarities of $PG(2,q)$ are classified as follows:

Theorem 1.13.1.1 [48, Section 2.1(v)] *A* **polarity** *α of $PG(2,q)$, $q = p^h$ p prime, is of one of the following types:*

Name	*(also known as)*	$GF(q)$ $= GF(p^h)$	*Locus of Absolute points*
orthogonal	*(a)* **ordinary**	$p \neq 2$	*$q+1$ points X of a non-degenerate conic with equation $X^tAX = 0$, where A is a symmetric matrix in $GL(3,q)$*
	(b) **pseudo**	$p = 2$	*$q+1$ points of a line.*
unitary	**hermitian**	*p arbitrary; h must be even so that q is a square*	*$q\sqrt{q}+1$ points of a (***hermitian***) curve with equation* $$X^{\sqrt{q}}HX = 0,$$ *where H is a hermitian matrix (that is a matrix satisfying $H^t = H^{\sqrt{q}}$ and H non-singular) in $GL(3,q)$*

In $PG(2,q^2)$, the Desarguesian projective plane of square order q^2, the set $\overline{\mathcal{U}}$ of absolute points of a unitary (or *hermitian*) polarity is a set of q^3+1 points such that each line of the plane intersects $\overline{\mathcal{U}}$ in 1 or $q+1$ points. The lines are called **tangent** (absolute) or **secant** (non-absolute) lines respectively. The structure $\overline{\mathcal{U}}$ is a $2-(q^3+1, q+1, 1)$ design, that is, a unital of order q, where the blocks are the sets of $q+1$ absolute points on the secant (non-absolute) lines of the polarity and incidence is the natural point-line incidence of $PG(2, q^2)$.

A unital in $PG(2, q^2)$ which arises in this way from a unitary polarity is called a **classical unital** (or a **Hermitian unital**). The classical unitals are projectively equivalent under $PGL(3, q)$ (see [48, Theorem 7.3.1]) and therefore up to isomorphism a classical unital has the equation,

$$x^{q+1} + y^{q+1} + z^{q+1} = 0,$$

which is the canonical form of a non-singular Hermitian curve where the matrix H is taken as the identity matrix. The classical unitals in $PG(2, q^2)$ are also called **Hermitian curves**. For this reason in the literature a unital in $PG(2, q^2)$, which is not necessarily classical, is sometimes called a **Hermitian arc**.

The classical unitals have been characterised in a number of ways, for example:

Theorem 1.13.1.2 [57] [37] *In $PG(2, q^2)$, $q > 2$, a unital $\overline{\mathcal{U}}$ is classical if and only if each Baer subline in $PG(2, q^2)$ intersects $\overline{\mathcal{U}}$ in 0,1,2 or $q+1$ points .* □

1.13.2 Unitals embedded in Finite Projective planes

A unital $\overline{\mathcal{U}}$ of order n is said to be **embedded** in a finite projective plane π_q, of order q, if the points of $\overline{\mathcal{U}}$ are a subset of the points of π_q, each block of $\overline{\mathcal{U}}$ is a set of points collinear in π_q (with distinct blocks on distinct lines) and incidence in $\overline{\mathcal{U}}$ is induced by the point-line incidence in π_q. If $\overline{\mathcal{U}}$ is embedded in π_q we sometimes say $\overline{\mathcal{U}}$ is a unital **in** π_q.

The classical unitals are examples of unitals of order q (embedded) in the Desarguesian projective plane $PG(2, q^2)$.

Let $\overline{\mathcal{U}}$ be a unital of order s embedded in a finite projective plane π_q of order q. The points of $\overline{\mathcal{U}}$ are necessarily a set of type $(0, 1, s+1)$ or of type $(1, s+1)$ in π_q. Suppose $\overline{\mathcal{U}}$ is a set of type $(1, s+1)$ in π_q and if either q is a prime power or if q/s is a prime power then by Theorems 1.11.3 and 1.11.4 we have that q is a square and $\overline{\mathcal{U}}$ is a unital of order $\sqrt{q}$ in π_q.

Consider a classical unital $\overline{\mathcal{U}}$ of order q in $PG(2,q^2)$. Embed $PG(2,q^2)$ as a Baer subplane in $PG(2,q^4)$; then $\overline{\mathcal{U}}$ as a design has been embedded in $PG(2,q^4)$ and the unital is a set of type $(0,1,q+1)$ in $PG(2,q^4)$. So we have examples of unitals (as designs) which arise naturally as structures in projective planes, but which may have external lines. However if the embedded unital has no external lines then by the Tallini-Scafati and Tallini characterisations in Theorem 1.11.3 and Theorem 1.11.4, with the appropriate condition on the order of the plane, we may restrict our attention to finite projective planes of square order q^2 and (embedded) unitals of order q.

Thus a unital $\overline{\mathcal{U}}$ embedded in a finite projective plane π_{q^2} of order q^2 is a set of q^3+1 points of the plane such that each line intersects $\overline{\mathcal{U}}$ in exactly 1 or $q+1$ points; each line is called a **tangent** or **secant** line of $\overline{\mathcal{U}}$ respectively. Moreover, each point $P \in \overline{\mathcal{U}}$ is incident with a unique tangent line and q^2 secant lines of $\overline{\mathcal{U}}$. By the results of Section 1.11, since a unital $\overline{\mathcal{U}}$ in π_{q^2} is a $(q^3+1)-$set of type $(1,q+1)$, the set of q^3+1 tangent lines of $\overline{\mathcal{U}}$ are the points of a unital $\overline{\mathcal{U}}^d$ in the dual plane $\pi^d_{q^2}$ of π_{q^2}; the unital $\overline{\mathcal{U}}^d$ is called the **dual unital** of $\overline{\mathcal{U}}$ in $\pi^d_{q^2}$.

Unitals from unitary polarities

Above we defined the classical unitals in $PG(2,q^2)$ as those unitals (embedded in $PG(2,q^2)$) which arise as the set of absolute points of a unitary polarity in $PG(2,q^2)$.

Let π_q be a finite projective plane (not necessarily Desarguesian) of order q.

Due to the work of Baer [3] and Seib [70] we have the following results (statement taken from Hughes and Piper [52, Theorems 12.7, 12.11, 12.12]) concerning polarities in π_q,

Theorem 1.13.2.1 *Let σ be a polarity of a finite projective plane of order q. If q is not a square, then σ has $a(\sigma) = q+1$ absolute points and*

(a) if q is even, the absolute points are collinear

(b) if q is odd, the absolute points form a $(q+1)$-arc

If $q = s^2$ is a square, then σ has $a(\sigma) \leq s^3+1$ absolute points and if $a(\sigma) = s^3+1$ then the set of absolute points and non-absolute lines forms a unital of order $s = \sqrt{q}$. □

In π_q a polarity σ is called **orthogonal** if $a(\sigma) = q+1$ and **unitary** if $a(\sigma) = q^{3/2}+1$. By the classification of polarities in the Desarguesian plane $PG(2,q)$, given in Theorem 1.13.1.1, any polarity of $PG(2,q)$ is either orthogonal or unitary. Note that there exist examples of non-Desarguesian planes π_q of order q and polarities σ in π_q whose number of absolute points satisfy $q+1 < a(\sigma) < q^{3/2}+1$ (see for example [52, Exercise 12.16]).

Theorem 1.13.2.1 indicates one approach at finding new unitals, by finding unitary polarities in non-Desarguesian finite projective planes. See [32], [41], [42], [54], for example, for results concerning unitals constructed in this manner.

1.13.3 Buekenhout-Metz Unitals

The class of unitals known as **Buekenhout-Metz unitals** are defined in translation planes π_{q^2} of order q^2 with kernel of order q. In this thesis we shall not define the term *kernel*, but note by [33, 5.1.11], a translation plane π_{q^2} of order q^2 with kernel of order q is a translation plane which has a Bruck and Bose representation in $PG(4,q)$ defined by a $1-$spread $\mathcal{S}$ in a hyperplane $\Sigma_\infty = PG(3,q)$ of $PG(4,q)$.

In the Bruck and Bose representation, the line at infinity ℓ_∞ (with points the elements of the spread $\mathcal{S}$) is the translation line of the translation plane π_{q^2}.

The construction is as follows: Let $\mathcal{O}$ be an ovoid in a hyperplane of $PG(4,q)\backslash\Sigma_\infty$ intersecting Σ_∞ in a unique point X, where the tangent plane to $\mathcal{O}$ at X does not contain the unique line t of $\mathcal{S}$ incident with X. Let V be a point of t distinct from X. Let $\overline{\mathcal{U}}^*$ be the structure containing the spread line t and all points of $PG(4,q)\backslash\Sigma_\infty$ on the ovoidal cone with vertex V and base $\mathcal{O}$.

The ovoidal cone $\overline{\mathcal{U}}^*$ corresponds to a set $\overline{\mathcal{U}}$ of q^3+1 points in the translation plane π_{q^2} which is defined by the spread $\mathcal{S}$ of Σ_∞. The set $\overline{\mathcal{U}}$ is a unital in π_{q^2} tangent to ℓ_∞ at the point T which is represented by t in Bruck-Bose (see [24, Section 4. (4)]). We shall call a unital $\overline{\mathcal{U}}$ in π_{q^2} with the above construction a **Buekenhout-Metz Unital**, and we shall sometimes say $\overline{\mathcal{U}}$ is **Buekenhout-Metz re** $(\mathbf{T}, \ell_\infty)$. If a Buekenhout-Metz unital $\overline{\mathcal{U}}$ is constructed in a translation plane π_{q^2} as above, with the ovoid $\mathcal{O}$ an elliptic quadric, then we say $\overline{\mathcal{U}}$ is **Buekenhout-Metz with elliptic quadric as base**. (Note that we shall sometimes abbreviate *Buekenhout-Metz* to *B-M*.)

Buekenhout proved in [24] that each classical unital $\overline{\mathcal{U}}$ in the Desarguesian plane $PG(2,q^2)$ is Buekenhout-Metz re (T,ℓ_T) for any point $T\in\overline{\mathcal{U}}$ and ℓ_T the tangent line to $\overline{\mathcal{U}}$ at T. Moreover, Buekenhout showed that every classical unital in $PG(2,q^2)$ is

Buekenhout-Metz with elliptic quadric as base, that is, corresponds to an elliptic quadric cone in the Bruck-Bose representation of $PG(2, q^2)$.

Buekenhout constructed the first non-classical unitals in $PG(2, 2^{4r+2})$, $r \geq 1$, by taking $\mathcal{O}$ to be a Tits ovoid in $\Sigma_\infty = PG(3, q)$, $q = 2^{2r+1}$, in the above construction. Metz [60] extended this class of non-classical unitals in $PG(2, q^2)$ to all values of $q > 2$ by constructing Buekenhout-Metz unitals with base ovoid an elliptic quadric and such that the unitals did not arise from unitary polarities in $PG(2, q^2)$.

All known unitals in $PG(2, q^2)$ are Buekenhout-Metz unitals (see for example [26]).

Finally we state two characterisations; see Chapter 5 for a new characterisation of Buekenhout-Metz unitals in $PG(2, q^2)$, for $q > 3$.

Theorem 1.13.3.1 [56, Section 2., Theorem] *In $PG(2, q^2)$, $q > 2$, a unital $\overline{\mathcal{U}}$ is Buekenhout-Metz re (T, ℓ_∞) if and only if every Baer subline with a point on ℓ_∞ intersects $\overline{\mathcal{U}}$ in 0,1,2 or $q + 1$ points.* □

Theorem 1.13.3.2 [57, Proposition 1] *If $\overline{\mathcal{U}}$ is a Buekenhout-Metz unital re (T, ℓ_∞) in $PG(2, q^2)$, with base ovoid an elliptic quadric and if there exists a secant line l of $\overline{\mathcal{U}}$, not on T, such that $l \cap \overline{\mathcal{U}}$ is a Baer subline, then $\overline{\mathcal{U}}$ is a classical unital.* □

Unitals have been constructed in non-Desarguesian planes by using the construction of Buekenhout-Metz unitals given above, see for example [11, 12], [31].

1.14 Inversive Planes

A comprehensive introduction to inversive planes is given in Dembowski's *Finite Geometries* [33, Chapter 6]. Recent results concerning this topic can be found in [61], [82], [83], [84], for example.

Definition 1.14.1 (Statement from [53]) *An* **inversive plane** *I is a set of points with distinguished subsets of the points, called* **circles** *such that:*

(I1) *any three distinct points of I are in exactly one common circle;*

(I2) *if P, Q are points of I and ℓ is a circle with $P \in \ell$ but $Q \notin \ell$ then there is a unique circle of I which contains both P and Q and meets ℓ only in the point P;*

(I3) *I contains four points which are not on a common circle.*

Let I be an inversive plane and let P be a point of I. The set of points of I different from P together with the circles containing P (minus P) and with incidence given by inclusion, is called the **internal structure** I_P of I at P.

For every point P of I the internal structure I_P is an affine plane called the **internal plane** of I at P.

By (I1) for two distinct circles ℓ, m of I we have the number of points common to ℓ and m is 0,1 or 2 and in each case we say the circles ℓ and m are **disjoint**, **tangent** or **intersecting** respectively.

Some subsets of circles in an inversive plane I are of particular importance and for reference later we have the following terminology:

A **bundle** of circles is the set of all circles through two distinct points P, Q of I. The points P and Q are called the **carriers** of the bundle.
A **pencil** is any maximal set of mutually tangent circles through a common point P, called the **carrier** of the pencil. (Note the pencils with given carrier P correspond to the parallel classes of lines in the affine plane I_P.)
A **flock** is a set of mutually disjoint circles in I such that, with the exception of precisely two points P, Q every point of I is on a (necessarily unique) circle of the flock. These points P, Q are called the **carriers** of the flock.

In the finite case an inversive plane can be defined in the following way.

Definition 1.14.2 *A* **finite inversive plane** *I is a 3-$(q^2+1, q+1, 1)$ design. We call q the* **order** *of I.*

For every point P of a finite inversive plane I of order q the internal plane I_P is an (finite) affine plane of order q (see [33, Section 6.1(4)]).

Up to isomorphism there is a unique inversive plane of order q, with $q \in \{2, 3, 4, 5, 7\}$. For $q = 7$ this was originally proved by R. F. Denniston with the aid of a computer;

in [84], as a corollary of a theorem we shall mention below, Thas gives a computer-free proof of the uniqueness of the inversive plane of order 7.

Let $\mathcal{O}$ be an ovoid in a 3-dimensional projective geometry. The points of $\mathcal{O}$ together with the intersections $\pi \cap \mathcal{O}$, with π a secant plane of $\mathcal{O}$, is an inversive plane $I(\mathcal{O})$. We call $I(\mathcal{O})$ the inversive plane associated with the ovoid $\mathcal{O}$. We call an inversive plane **egglike** if it is isomorphic to an $I(\mathcal{O})$ for some 3-dimensional ovoid $\mathcal{O}$ (see [33, Section 1.2] for *isomorphism* of incidence structures). If $\mathcal{O}$ is an ovoid of $PG(3,q)$, then the associated inversive plane $I(\mathcal{O})$ is a finite (egglike) inversive plane of order q.

Since the only known examples of ovoids in $PG(3,q)$, with $q > 2$, fall into two infinite classes there are consequently two known infinite families of finite egglike inversive planes. If the ovoid is an elliptic quadric of $PG(3,q)$, then the associated inversive plane is called **classical** or **Miquelian** since it satisfies the configurational condition known as the **Theorem of Miquel** (see [33, Chapter 6] for more detail.) The family of finite Miquelian inversive planes is denoted $\mathbf{M(q)}$. If the ovoid is a Tits ovoid in $PG(3, 2^{2r+1})$, with $r \geq 1$ an integer, then the associated inversive plane belongs to the second known family of finite egglike inversive planes which is denoted by $\mathbf{S(q)}$.

The only known finite inversive planes are the egglike inversive planes in the families $M(q)$ and $S(q)$. The problem of classification of ovoids of $PG(3,q)$, with $q > 2$, is equivalent to the classification of finite egglike inversive planes. As stated in an earlier section, the ovoids of $PG(3,q)$, with $q > 2$, have been classified for $q \leq 32$.

We now list some old and some recent important results concerning finite inversive planes.

1. [33, 6.1.3] For any point P of a finite egglike inversive plane $I(\mathcal{O})$ the affine plane I_P is the Desarguesian plane $AG(2,q)$.

2. [33, 6.2.14] Every (finite) inversive plane of even order q is egglike. Consequently q is a power of 2.

3. [33, 1.4.50] Every (finite) egglike inversive plane of odd order is Miquelian.

4. [82] [84] Let I be an inversive plane of odd order q, $q \notin \{11, 23, 59\}$. If for at least one point P of I the internal plane I_P is Desarguesian, then I is Miquelian.

5. [77] [65] If $\mathcal{F}$ is a flock of a finite egglike inversive plane $I = I(\mathcal{O})$, then $\mathcal{F}$ is **linear** (that is, the ovals of $\mathcal{O}$ in $PG(3,q)$, which correspond to the circles of the flock $\mathcal{F}$,

lie in planes of $PG(3,q)$ about a common line.)

6. [61] Let I be a finite egglike inversive plane of order $2 < q \leq 32$. If $q \in \{8, 32\}$ then I is Miquelian or of $S(q)$ type. If $q \notin \{8, 32\}$ then I is Miquelian.

Finally, we shall mention the **plane model of egglike inversive planes** which is given in [84] for example.

Let $\mathcal{O}$ be an ovoid of $PG(3,q)$ and let I denote the corresponding inversive plane. The circles of I are in one-to-one correspondence with the secant plane sections of the ovoid in $PG(3,q)$. As a consequence we shall interchange the setting between the incidence structure of the inversive plane and the geometry of the ovoid in $PG(3,q)$. We shall even abuse the terminology and refer to "circles" of $\mathcal{O}$ in $PG(3,q)$ when we mean secant plane sections of $\mathcal{O}$ which correspond to circles of the associated inversive plane. The context in which we do this should make our meaning clear.

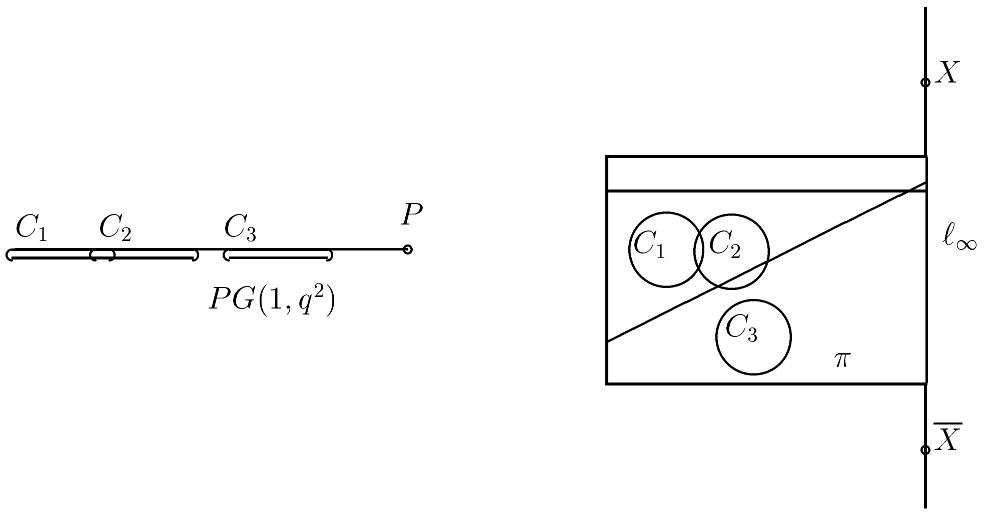
Let P be a point of $\mathcal{O}$ and let π be a plane of $PG(3,q)$, not containing P. The intersection of π and the tangent plane π_P of $\mathcal{O}$ at P is denoted by ℓ_∞. By projection ζ of $\mathcal{O} - \{P\}$ from P onto π, the points of $\mathcal{O} - \{P\}$ are mapped onto the q^2 points of $\pi \backslash \ell_\infty$, the circles of $\mathcal{O}$ through P (minus P) are mapped onto the q^2+q affine lines of π. The Desarguesian affine plane $\pi \backslash \ell_\infty$ is isomorphic to the internal plane I_P of I at P. Moreover the circles of $\mathcal{O}$ not through P are mapped by ζ onto $q^3 - q^2$ ovals of π; each such oval is disjoint from ℓ_∞.

If the ovoid $\mathcal{O}$ is an elliptic quadric, then the circles of $\mathcal{O}$ not through P are mapped by ζ onto the $q^3 - q^2$ non-degenerate conics of π containing two points $X, \overline{X} \in \ell_\infty$, which are the two points of $\mathcal{O}$ on ℓ_∞ belonging to the quadratic extension $GF(q^2)$ of $GF(q)$.

Example: Consider the projective line $PG(1,q^2)$. The points of $PG(1,q^2)$ together with the Baer sublines of $PG(1,q^2)$, with incidence given by inclusion, forms a Miquelian inversive plane I of order q (see [33, page 273]). Fix a point P of $PG(1,q^2)$ and consider the internal plane $I_P \cong AG(2,q)$ of I at P. By the above theory and using the same notation, denote by ℓ_∞ the line at infinity of I_P and let π denote the projective completion of I_P so that $I_P = \pi \backslash \ell_\infty$.

In the correspondence between the inversive plane I defined on $PG(1,q^2)$ and the internal plane I_P of I at P we have: The points of $PG(1,q^2) \backslash \{P\}$ are the q^2 points of $\pi \backslash \ell_\infty$. The Baer sublines of $PG(1,q^2)$ containing P (minus P) are the q^2+q lines of $\pi \backslash \ell_\infty$ and the Baer sublines of $PG(1,q^2)$ not containing P are the q^3-q^2 non-degenerate conics in π containing two fixed points $X, \overline{X} \in \ell_\infty$, with $X, \overline{X}$ conjugate with respect to the quadratic extension $GF(q^2)$ of $GF(q)$.

We represent the situation as follows, with C_1, C_2 and C_3 three Baer sublines of $PG(1,q^2)$ disjoint from P. In π, the Baer sublines are represented as non-degenerate conics containing two points $X, \overline{X}$ on ℓ_∞ in the quadratic extension.



1.15 Maximal Arcs

As discussed in Section 1.11, Barlotti [9] introduced the term **$\{k;n\}$-arc** for a set $\mathcal{K}$ of k points in a finite projective plane π_q of order q, where n, $n \neq 0$, is the greatest number of collinear points in the set. $\{k;2\}$-arcs are simply called **k-arcs**.

A $\{k;n\}$-arc is **complete** if it is not contained in a $\{k+1;n\}$-arc.

Let $\mathcal{K}$ be a $\{k;n\}$-arc in π_q. By considering the points of $\mathcal{K}$ on the $q+1$ lines through a point P of $\mathcal{K}$, it is easy to see that the number k of points of $\mathcal{K}$ satisfies:

$$\begin{aligned} k &\leq (q+1)(n-1)+1 \\ &= nq-q+n \\ &= (n-1)q+n. \end{aligned}$$

A $\{nq-q+n;n\}$-arc in π_q is called a **maximal arc**. Equivalently, a maximal arc may be defined as a non-empty set $\mathcal{K}$ of points in π_q such that every line of π_q meets $\mathcal{K}$ in either exactly n points or in none at all; the lines of π_q are called **secant** or **external** lines of $\mathcal{K}$.

Examples of maximal arcs:

For the given value of n a maximal $\{nq-q+n;n\}$-arc $\mathcal{K}$ in π_q is:

$n=1$,	A single point.
$n=2$,	A $(q+2)$-arc in π_q, q even, in other words $\mathcal{K}$ is a hyperoval in π_q, q even.
$n=q$,	The set of points $\pi_q \backslash \ell$ for ℓ a line of π_q.
$n=q+1$,	The set of points of the plane π_q.

Note that the study of hyperovals (maximal arcs with $n=2$) is an active field of study in its own right with an extensive literature.

Since the maximal arcs with $n=1$ or $q+1$ are determined we consider maximal arcs with $1<n<q+1$.

Let $\mathcal{K}$ be a (maximal) $\{(n-1)q+n;n\}$-arc in π_q, $n \leq q$. Let Q be a point of π_q not in $\mathcal{K}$. By definition every line through Q intersects $\mathcal{K}$ in 0 or n points therefore we have:

$$\begin{aligned} & n \text{ divides } (n-1)q+n \\ \text{hence} \quad & n \text{ divides } q. \end{aligned}$$

Hence Barlotti obtained *a necessary condition for the existence of a (maximal)* $\{nq-q+n;n\}$*-arc in* π_q*,* $n \leq q$*, is that* n *divides* q.

Also in [9] it was shown that if a $\{nq-q+n;n\}$-arc $\mathcal{K}$ exists in π_q then the set of external lines of $\mathcal{K}$ is a (maximal) $\{q(q-n+1)/n;q/n\}$-arc in the dual plane of π_q. It follows that a maximal $\{nq-q+n;n\}$-arc exists in $PG(2,q)$, $n \leq q$, if and only if a maximal $\{q(q-n+1)/n;q/n\}$-arc exists in $PG(2,q)$.

Denniston [34] proved that Barlotti's necessary condition for the existence of maximal arcs is sufficient in $PG(2,q)$, q even, by constructing infinite families of maximal arcs in Desarguesian planes of even order.

Cossu [30] showed the above necessary condition for existence of maximal arcs in π_q is not sufficient; he proved $PG(2,9)$ contains no $\{21;3\}$-arc. Thas [79] generalised Cossu's result by proving the following result.

Theorem 1.15.1 [79] *In* $PG(2,q)$*,* $q=3^h$ *and* $h>1$*, there are no* $\{2q+3;3\}$*-arcs and (hence) no* $\{q(q-2)/3;q/3\}$*-arcs.* □

In this 1987 paper Thas made the following conjecture:

Conjecture 1.15.1 [79] *In $PG(2,q)$, q odd, the only maximal arcs are $PG(2,q)$, $AG(2,q) = PG(2,q)\backslash\ell_\infty$ and the dual of $AG(2,q)$.*

This conjecture was recently proved by Ball, Blokhuis and Mazzocca [7].

Theorem 1.15.2 [7] *For q an odd prime power, and $1 < n < q$, the Desarguesian plane $PG(2,q)$ does not contain a $\{nq - q + n; n\}-$arc.* □

We now list some constructions and classes of maximal arcs.

In [78] Thas constructed an infinite family of maximal arcs in certain translation planes of even order. In the literature this family of maximal arcs has been referred to as the **Thas maximal arcs** and we shall do so here.

The Thas maximal arcs are defined in certain finite translation planes of order q^2 with kernel of order q; each such translation plane corresponds to a $1-$spread in a hyperplane Σ_∞ of $PG(4,q)$ by the $4-$dimensional Bruck and Bose representation of the translation plane (Section 1.10). The construction is as follows.

The construction of a Thas maximal arc: Let $\Sigma_\infty = PG(3,q)$ and consider an ovoid $\mathcal{O}$ and a spread $\mathcal{S}$ in Σ_∞ such that each line of $\mathcal{S}$ is incident with a unique point of $\mathcal{O}$. An ovoid $\mathcal{O}$ and a spread $\mathcal{S}$ in Σ_∞ with this property will be called a **Thas ovoid-spread pair** $(\mathcal{O}, \mathcal{S})$.

Let Σ_∞ be embedded as a hyperplane in $PG(4,q)$ and let X^* be a point of $PG(4,q)\backslash\Sigma_\infty$. Denote by $\mathcal{K}^*$ the set containing X^* and all points of $PG(4,q)\backslash\Sigma_\infty$ collinear with X^* and a point of $\mathcal{O}$.

The set of points $\mathcal{K}^*$ in $PG(4,q)$ represents a maximal $\{q^3 - q^2 + q; q\}$-arc $\mathcal{K}$ in the translation plane π_{q^2} of order q^2 with translation line ℓ_∞ corresponding to the spread $\mathcal{S}$. We call $\mathcal{K}$ a **Thas maximal arc** in π_{q^2} with **base point** X and **axis line** ℓ_∞.

Note that the axis line ℓ_∞ is an external line of $\mathcal{K}$ in π_{q^2}.

Existence of Thas maximal arcs: By the above construction, a Thas maximal arc exists in a translation plane π_{q^2} of order q^2 with translation line ℓ_∞ if and only if a Thas ovoid-spread pair $(\mathcal{O}, \mathcal{S})$ exists in $PG(3,q)$, for the spread $\mathcal{S}$ corresponding to π_{q^2} in the Bruck and Bose representation.

The known examples are [78]:

Translation plane	q	$(\mathcal{O}, \mathcal{S})$
$PG(2, q^2)$	even	(elliptic quadric, Regular spread)
	$q = 2^{2r+1}$, $r \geq 1$	(Tits ovoid, Regular spread)
Lüneburg plane	$q = 2^{2r+1}$, $r \geq 1$	(Tits ovoid, Lüneburg spread)
	$q = 2^{2r+1}$, $r \geq 1$	(elliptic quadric, Lüneburg spread)

Note that for q odd, an ovoid in $PG(3, q)$ is an elliptic quadric; by a recent result of Blokhuis and Wilbrink (see also Theorem 1.15.2 for certain cases) there exists no Thas ovoid-spread pair in $PG(3, q)$, q odd.

In [80] Thas generalised the above construction of a Thas maximal arc and constructed (maximal) $\{q^{2d-1} - q^d + q^{d-1}; q^{d-1}\}$-arcs in certain translation planes π_{q^d} of even order q^d.

For further constructions of maximal arcs in planes other than the Desarguesian plane see for example [45, 46].

1.16 Generalized Quadrangles

We present here some preliminary results concerning generalized quadrangles, for later reference. Unless stated otherwise the definitions and results of this section are from Payne and Thas [67].

A **(finite) generalized quadrangle ($\mathcal{GQ}$)** is an incidence structure $S = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called points and lines (respectively), and for which $\mathbf{I}$ is a symmetric point-line incidence relation satisfying the following three axioms:

$\mathcal{GQ}$ axiom (i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

$\mathcal{GQ}$ axiom (ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

$\mathcal{GQ}$ axiom (iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathbf{I} m \mathbf{I} Y \mathbf{I} \ell$.

The integers s and t are the parameters of the $\mathcal{GQ}$ and S is said to have **order** (s, t); if $s = t$, then S is said to have **order** s.

Let S be a $\mathcal{GQ}$ of order (s,t). Let X, Y be two (not necessarily distinct) points of S. We write $X \sim Y$ and say that X and Y are **collinear** if there exists a line ℓ of S such that $X\ \mathbf{I}\ \ell\ \mathbf{I}\ Y$. And $X \not\sim Y$ means X and Y are not collinear. Note that $X \sim X$ for any X in $\mathcal{P}$.

For $X \in \mathcal{P}$ the set $\{Y \in \mathcal{P}; X \sim Y\}$ of points in the $\mathcal{GQ}$ collinear with X is denoted $X^\perp$. The **trace** of a pair (X,Y) of distinct points is the set $X^\perp \cap Y^\perp$ and is denoted $\{X,Y\}^\perp$. More generally, if A is a subset of points in $\mathcal{P}$, then A "perp" is defined by $A^\perp = \cap\{X^\perp \mid X \in A\}$.

Result 1.16.1 *For distinct points X and Y in a $\mathcal{GQ}$ S of order (s,t), the cardinality of $\{X,Y\}^\perp$ is:*

$$|\{X,Y\}^\perp| = s+1 \quad \text{if } X \sim Y,$$
$$|\{X,Y\}^\perp| = t+1 \quad \text{if } X \not\sim Y.$$

$\square$

For distinct points X and Y in S, the **span** of the pair (X,Y) is

$$\{X,Y\}^{\perp\perp} = \{V \in \mathcal{P}; V \in Z^\perp \text{ for all } Z \in \{X,Y\}^\perp\}.$$

If $X \not\sim Y$, then $\{X,Y\}^{\perp\perp}$ is also called the **hyperbolic line** defined by X and Y.

A **triad** (of points) is a triple (X,Y,Z) of pairwise non-collinear points in $\mathcal{P}$. Given a triad $T = (X,Y,Z)$, a **center** of T is a point of $T^\perp$.

Result 1.16.2 [67, 1.2.4] *Let S be a $\mathcal{GQ}$ of order (s,t). If $s > 1$ and $t > 1$, then $s^2 = t$ if and only if each triad (of points) has a constant number of centers, in which case this constant number of centers is $s+1$.* $\square$

Let $s^2 = t > 1$, so that S is a $\mathcal{GQ}$ of order (s,s^2) and by Result 1.16.2, for any triad (X,Y,Z) we have $|\{X,Y,Z\}^\perp| = s+1$. If X',Y',Z' are three distinct points in $\{X,Y,Z\}^\perp$ then since (X',Y',Z') is necessarily a triad (by $\mathcal{GQ}$ axiom (iii)), we have $\{X,Y,Z\}^{\perp\perp} \subseteq \{X',Y',Z'\}^\perp$ and therefore $|\{X,Y,Z\}^{\perp\perp}| \leq |\{X',Y',Z'\}^\perp| = s+1$. We say a triad (X,Y,Z) is **3-regular** provided $|\{X,Y,Z\}^{\perp\perp}| = s+1$. A point X is called **3-regular** if and only if each triad containing X is 3-regular.

Result 1.16.3 [67, 1.3.3] *Let S be a $\mathcal{GQ}$ of order (s, s^2), $s \neq 1$, and suppose that any triad contained in $\{X,Y\}^{\perp}$, $X \not\sim Y$, is 3-regular. Then the incidence structure with pointset $\{X,Y\}^{\perp}$, with circleset the sets of elements $\{Z_1, Z_2, Z_3\}^{\perp\perp}$, where Z_1, Z_2, Z_3 are distinct points in $\{X,Y\}^{\perp}$, and with natural incidence, is an inversive plane of order s.*

We include a proof of Result 1.16.3 to clarify this case for later reference.

Proof: First note that by Result 1.16.1, $|\{X,Y\}^{\perp}| = s^2+1$ and so our incidence structure has s^2+1 points.

For distinct points $Z_1, Z_2, Z_3 \in \{X,Y\}^{\perp}$, (Z_1, Z_2, Z_3) is a triad (by GQ axiom (iii)) and $X, Y \in \{Z_1, Z_2, Z_3\}^{\perp}$. It follows that $\{Z_1, Z_2, Z_3\}^{\perp\perp} \subseteq \{X,Y\}^{\perp}$ and by the 3-regularity, each circle of our incidence structure is incident with exactly $s+1$ points. It also follows that any three distinct points in $\{X,Y\}^{\perp}$ determine a circle.

We now verify each property in Definition 1.14.1:

(I1): Let $c_1 = \{Z_1, Z_2, Z_3\}^{\perp\perp}$ and $c_2 = \{Z'_1, Z'_2, Z'_3\}^{\perp\perp}$ be two distinct circles. Suppose X_1, X_2, X_3 are three distinct points incident with both circles. Note that $\{Z_1, Z_2, Z_3\}^{\perp\perp}$ is determined uniquely by any three distinct points X'_1, X'_2, X'_3 in $\{Z_1, Z_2, Z_3\}^{\perp}$ since,

$$\{Z_1, Z_2, Z_3\}^{\perp\perp} \subseteq \{X'_1, X'_2, X'_3\}^{\perp}$$

and these two sets have the same cardinality $s+1$.

It follows that since c_1 and c_2 are distinct circles, the sets $\{Z_1, Z_2, Z_3\}^{\perp}$ and $\{Z'_1, Z'_2, Z'_3\}^{\perp}$ have at most two points in common. Points X_1, X_2, X_3 are each collinear to every point in $\{Z_1, Z_2, Z_3\}^{\perp}$ and to every point in $\{Z'_1, Z'_2, Z'_3\}^{\perp}$, therefore

$$\begin{aligned} |\{X_1, X_2, X_3\}^{\perp}| &\geq (s+1) + (s+1) - 2 \\ &= 2s \\ &> s+1 \text{ since } s > 1, \end{aligned}$$

a contradiction, since for the triad (X_1, X_2, X_3) we have $|\{X_1, X_2, X_3\}^{\perp}| = s+1$.

Therefore three distinct points determine a unique circle in our structure.

(I2): Distinct points $P, Q \in \{X,Y\}^{\perp}$ are contained in circles $\{P,Q,Z\}^{\perp\perp}$, where Z is any point in $\{X,Y\}^{\perp}$ distinct from P and Q. By *(I1)*, there are $\dfrac{s^2-1}{s-1} = s+1$ choices for Z and therefore there exist $s+1$ circles incident with P and Q, with no further point

in common. By counting, we obtain that each point of the structure, distinct from P and Q, is incident with (exactly) one circle which contains both P and Q.

By *(I1)* it follows that P is incident with $\dfrac{s^2(s^2-1)}{s(s-1)} = s^2+s$ circles. So there exist circles containing P and not Q, let ℓ_P be such a circle.

Each circle which contains P and Q intersects ℓ_P in at most one point besides P. There are s points of ℓ_P besides P and from above, each such point lies in some circle containing P and Q. Therefore there is one remaining circle, which contains P and Q but intersects ℓ_P only in the point P.

(I3): There are s^2+1 points in the structure and $s^2+1 \geq s+1$ since $s > 1$ and therefore there exist 4 points not on a common circle.

By Definition 1.14.1, the incidence structure is an inversive plane of order s. □

Result 1.16.4 [67, 3.1.2] *For each ovoid $\mathcal{O}$ in $PG(3,q)$ there is a $\mathcal{GQ}$ to be called* $\mathbf{T_3}(\mathcal{O})$ *constructed as follows:*
Let $\mathcal{O}$ be an ovoid in $PG(3,q)$. Further, let $PG(3,q) = \Sigma_\infty$ be embedded as a hyperplane in $PG(4,q)$.
Define **points** *as the following three types:*
- *Type (i)* *the points $PG(4,q)\backslash\Sigma_\infty$,*
- *Type (ii)* *the hyperplanes Π_3 of $PG(4,q)$ for which $|\Pi_3 \cap \mathcal{O}| = 1$,*
- *Type (iii)* *one new symbol (∞).*

Lines *are defined as the following two types:*
- *Type (a)* *the lines of $PG(4,q)$ which are not contained in Σ_∞ and meet $\mathcal{O}$ (necessarily in a unique point),*
- *Type (b)* *the points of $\mathcal{O}$.*

Incidence *is defined as follows: A point of type (i) is incident only with lines of type (a); here the incidence is that of $PG(4,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{O}$ in it. The point (∞) is incident with no line of type (a) and all lines of type (b).*

$T_3(\mathcal{O})$ **is a $\mathcal{GQ}$ of order** (q,q^2)**.** □

Result 1.16.5 [67, 3.3.2(ii)] *The point (∞) of the $\mathcal{GQ}$ $T_3(\mathcal{O})$ is 3-regular.* □

Definition of Property (G) **1.16.6** [66] *In a generalized quadrangle S of order (s,s^2),*

$s \neq 1$, let X, Y be distinct collinear points. We say that the pair $\{X, Y\}$ **has property** *(G) if every triad (X, X_1, X_2), with $Y \in \{X, X_1, X_2\}^{\perp}$, is 3-regular. Then also every triad (Y, Y_1, Y_2), with $X \in \{Y, Y_1, Y_2\}^{\perp}$, is 3-regular.*
We say that the generalized quadrangle S **has property** *(G)* **at a flag** *(X, ℓ), where X $\mathbf{I}$ ℓ, if every pair $\{X, Y\}$, $X \neq Y$, Y $\mathbf{I}$ ℓ has property (G).*
We say the generalized quadrangle S **has property** *(G)* **at the line** *ℓ, or the* **line** *ℓ* **has property** *(G), if each pair of points $\{X, Y\}$, $X \neq Y$, and X $\mathbf{I}$ ℓ $\mathbf{I}$ Y, has property (G).*

Result 1.16.7 [86, Section 2.4] *The point X of the generalized quadrangle S, of order (s, s^2), $s \neq 1$, is 3-regular if and only if each flag (X, ℓ), X $\mathbf{I}$ ℓ, has property (G).* $\square$

Chapter 2

The Bruck and Bose representation in $\mathbf{PG(4, q)}$

In this chapter we examine the Bruck and Bose representation of translation planes π_{q^2} of order q^2 with kernel of order q; that is, translation planes of order q^2 which are described by $1-$spreads of $PG(3, q)$ in the $4-$dimensional Bruck and Bose representation ([33, 5.1.11]). In particular we consider the $4-$dimensional Bruck and Bose representation of the Desarguesian plane $PG(2, q^2)$. In Section 2.1 we recall this special case of the general Bruck and Bose representation and establish notation for the chapter. When we wish to work with this representation we shall refer to it as the **Bruck-Bose setting**, or simply **Bruck-Bose**. Note that this Bruck-Bose representation of a translation plane is equivalent to the André group theoretic representation of a translation plane given in [1].

In this chapter we determine the representation in Bruck-Bose of Baer subplanes of $PG(2, q^2)$ and present characterisations of these structures. We also determine the representation in Bruck-Bose of conics contained in Baer subplanes of $PG(2, q^2)$; this work leads to results concerning the existence of certain $4-$dimensional caps which contain many normal rational curves.

2.1 Bruck-Bose in $\mathbf{PG(4, q)}$

Recall from Section 1.10 the following representation of $PG(2, q^2)$ in $PG(4, q)$ due to André [1] and Bruck and Bose [21] and [22];

Let ℓ_∞ denote a fixed line $PG(2,q^2)$ and call this line the *line at infinity* of $PG(2,q^2)$. The plane $AG(2,q^2) = PG(2,q^2)\backslash\ell_\infty$ is the Desarguesian affine plane of order q^2. Embed $\Sigma_\infty = PG(3,q)$ as a hyperplane in $PG(4,q)$. Let $\mathcal{S}$ be a fixed regular spread of Σ_∞. The affine plane $AG(2,q^2)$ is represented by the following incidence structure: the *points* are the points of $PG(4,q)\backslash\Sigma_\infty$, the *lines* are the planes of $PG(4,q)$, not contained in Σ_∞ and which meet Σ_∞ in a line of $\mathcal{S}$ and *incidence* is induced by the incidence in $PG(4,q)$. $AG(2,q^2)$ can be completed to the projective plane $PG(2,q^2)$ by the addition of ℓ_∞ whose points are the elements of the spread $\mathcal{S}$.

We shall use the phrase *a subspace of* $PG(4,q)\backslash\Sigma_\infty$ to mean a subspace of $PG(4,q)$ which is not contained in Σ_∞. The points $PG(4,q)\backslash\Sigma_\infty$ shall be referred to as *affine* points. Also if a line l of $PG(2,q^2)$ intersects a Baer subplane B of $PG(2,q^2)$ in a Baer subline m, we call l a *line of* B.

Note that $PG(2,q^2)$ is a translation plane with respect to any of its lines and therefore there is choice involved in fixing the line at infinity. Moreover, by Theorem 1.10.1.3, any regular $1-$spread in $PG(3,q)$ corresponds to a Bruck-Bose representation of $PG(2,q^2)$. Unless stated otherwise in this chapter, *the Bruck-Bose representation of* $PG(2,q^2)$ is the representation given above for a fixed line ℓ_∞ of $PG(2,q^2)$ and a fixed regular $1-$spread $\mathcal{S}$ of $\Sigma_\infty = PG(3,q)$.

Let π_{q^2} denote a translation plane of order q^2 with kernel of order q and with translation line ℓ_∞. Then by [33, 5.1.11] π_{q^2} is described by a $1-$spread $\mathcal{S}_\pi$ of $\Sigma_\infty = PG(3,q)$ in the Bruck-Bose setting in $PG(4,q)$.

For our discussion, we shall use the expression *the representation in* $PG(4,q)$ to mean the corresponding Bruck and Bose representation of the projective translation plane being discussed; moreover, the two representations coincide, that is $\pi_{q^2} = PG(2,q^2)$, if and only if $\mathcal{S}_\pi = \mathcal{S}$ is a regular spread of Σ_∞.

If X denotes a substructure in π_{q^2}, it will be convenient at times to denote by X^* the substructure in $PG(4,q)$ which is the Bruck-Bose representation of X. Conversely, if X^* is a substructure of $PG(4,q)$, we shall denote by X the subset of points and lines of π_{q^2} which is represented by X^* in Bruck-Bose.

2.2 Known Bruck-Bose representations of Baer substructures

The representation in $PG(4,q)$ of Baer subplanes of $PG(2,q^2)$ is of particular importance in our discussion. In this section we review the known results.

A **transversal plane** in $PG(4,q)$ is a plane in $PG(4,q)\backslash\Sigma_\infty$ which contains no line of $\mathcal{S}_\pi$. Let B^* be a transversal plane in $PG(4,q)$, then B^* is incident with $q+1$ distinct elements of $\mathcal{S}_\pi$; denote these spread elements by $\ell_1, \ell_2, \ldots, \ell_{q+1}$. The affine points of B^* together with these $q+1$ distinct elements of $\mathcal{S}_\pi$ incident with B^* correspond to a set B in π_{q^2} of q^2+q+1 points. For each $i = 1, 2, \ldots, q+1$, there exist q planes in $PG(4,q)\backslash\Sigma_\infty$ containing ℓ_i and such that each intersects B^* in q affine points. By the Bruck-Bose correspondence, each such plane represents a line of π_{q^2} incident with $q+1$ points of B. Furthermore the line at infinity intersects B in $q+1$ points. If we call these q^2+q+1 lines, *lines of* B, then it follows that B satisfies the definition of a finite projective plane and is therefore a Baer subplane of π_{q^2}. In this way, the transversal planes of $PG(4,q)$ represent Baer subplanes of π_{q^2} which contain the line at infinity as a line; a Baer subplane of π_{q^2} which contains the line at infinity as a line will be called an **affine** Baer subplane of π_{q^2}.

Theorem 2.2.1 [21, Section 9] *If B^* is transversal plane in $PG(4,q)$ then B^* is the Bruck-Bose representation of an affine Baer subplane B of π_{q^2}.* □

Since the number of transversal planes in $PG(4,q)$ equals the number of Baer subplanes of $PG(2,q^2)$ which contain the line at infinity as a line, we have:

Corollary 2.2.2 [21] *B is an affine Baer subplane of $PG(2,q^2)$ if and only if in Bruck-Bose, B^* is a transversal plane of $PG(4,q)$.*

Let B be an affine Baer subplane of $PG(2,q^2)$ and let $\ell \neq \ell_\infty$ be a line of B. In $PG(4,q)$, the plane ℓ^* intersects the transversal plane B^* in a line of $PG(4,q)\backslash\Sigma_\infty$ (which is not contained in Σ_∞). Therefore, by Corollary 2.2.2, any Baer subline of $PG(2,q^2)$ which intersects ℓ_∞ in a unique point is represented in $PG(4,q)$ by a line of $PG(4,q)\backslash\Sigma_\infty$; conversely, each line of $PG(4,q)\backslash\Sigma_\infty$ is the Bruck-Bose representation of a Baer subline of $PG(2,q^2)$ which contains a unique point of ℓ_∞.

We now present the representations in $PG(4,q)$ of Baer subplanes of $PG(2,q^2)$ which intersect ℓ_∞ in a unique point, and Baer sublines which are disjoint from ℓ_∞.

The following result is well known and is a consequence of the example given in Section 1.14.

Lemma 2.2.3 *A Baer subline b, containing no point on ℓ_∞, of a line a in $PG(2,q^2)$, is represented in $PG(4,q)$ by a non-degenerate conic $\mathcal{C}^*$ in the plane α representing a.* □

Definition 2.2.4 *The conics in $PG(4,q)\backslash\Sigma_\infty$ which represent Baer sublines of $PG(2,q^2)$ shall be called* **Baer conics***.*

A Baer conic in $PG(4,q)$ is necessarily disjoint from Σ_∞. Note that for the fixed regular spread $\mathcal{S}$ in Σ_∞ there exist non-degenerate conics disjoint from Σ_∞, in planes of $PG(4,q)\backslash\Sigma_\infty$ about spread elements, but which do not represent Baer sublines of $PG(2,q^2)$; that is, there exist non-Baer conics $PG(4,q)$. This result was proved by Metz [60] who showed that the number of Baer sublines of a line ℓ of $PG(2,q^2)$ disjoint from a fixed point $P \in \ell$ is strictly less than the number of non-degenerate conics in $PG(2,q)$ disjoint from a fixed line m in $PG(2,q)$.

For later reference, we consider the Bruck-Bose representation of some well known configurations of Baer sublines in $PG(2,q^2)$.

Lemma 2.2.5 *Let L_1 and L_2 be distinct affine points of a line a in $PG(2,q^2)$. Let $M = a \cap \ell_\infty$. There are q Baer sublines of a which contain L_1, L_2 and not M.*

Proof: The result follows from the fact that in $PG(2,q^2)$ there are $(q^2-1)/(q-1) = q+1$ Baer sublines containing L_1 and L_2 and there is a unique Baer subline of a containing the three distinct points L_1, L_2 and M (see Theorem 1.2.1 and the subsequent remarks). □

By interpreting the results of Lemma 2.2.3 and Lemma 2.2.5 in Bruck-Bose we obtain:

Lemma 2.2.6 *If L_1^* and L_2^* are distinct affine points in a plane α in $PG(4,q)\backslash\Sigma_\infty$ with $\alpha \cap \Sigma_\infty = m$, where m is an element of the spread $\mathcal{S}$ of Σ_∞, then there exist q Baer conics in α incident with both L_1^* and L_2^*.* □

We shall also make use of the following:

Lemma 2.2.7 [19] [89] [44] *In a projective plane* π_{q^2} *of order* q^2 *the number of points common to two Baer subplanes* B_1 *and* B_2 *of* π_{q^2} *is equal to the number of lines shared by* B_1 *and* B_2. □

Lemma 2.2.8 [72] *In* $PG(2, q^2)$ *two distinct Baer subplanes intersect in one of the following configurations:*

1. The empty set;

2. One point and one line : the point is either incident or non-incident with the line.

3. Two points and two lines: the point of intersection of the two lines plus a second point on one of the lines;

4. Three points and three lines forming a triangle configuration;

5. $q+1$ *points and* $q+1$ *lines: the* $q+1$ *points are collinear on one of the lines and the remaining* q *lines form a pencil through one of the points;*

6. $q+2$ *points and* $q+2$ *lines:* $q+1$ *of the points are collinear on one of the lines and the remaining* $q+1$ *lines form a pencil concurrent in the remaining point; each line contains* 2 *or* $q+1$ *of the points.* □

Above we recalled the representation in $PG(4, q)$ of the affine Baer subplanes of $PG(2, q^2)$, that is the Baer subplanes for which ℓ_∞ is a secant line. We now provide an alternative direct proof of a result obtained in [19] and also in [90], which determines the representation of Baer subplanes of $PG(2, q^2)$ which intersect ℓ_∞ in a unique point. The variety we call a **ruled cubic surface** V_2^3 is called a **twisted ladder** in [19]; its structure will be derived in the proof of the following Lemma, and will be used in the proof of Theorem 5.0.3 in a later chapter. (See Section 1.7 for more information on ruled cubic surfaces.)

Lemma 2.2.9 *Let* B *be a Baer subplane in* $PG(2, q^2)$ *such that* B *intersects* ℓ_∞ *in the unique point* P. *Then* B *corresponds to a ruled cubic surface* $\mathcal{B}$ *in* $PG(4, q)\backslash\Sigma_\infty$ *with* $\mathcal{B} \cap \Sigma_\infty = p$ *in* $PG(4, q)$, *where* p *is a line of the spread* $\mathcal{S}$ *of* Σ_∞.

Proof: Let $\mathcal{B}$ denote the structure in $PG(4, q)$ representing B. As B intersects ℓ_∞ in a unique point P, in $PG(4, q)$ $\mathcal{B}$ intersects Σ_∞ only in point(s) of the line p of $\mathcal{S}$ which represents P. The $q+1$ lines of B through the point $P \in \ell_\infty$ correspond to $q+1$ planes in $PG(4, q)\backslash\Sigma_\infty$ about p; each of these planes contain a line ℓ_i^* of $PG(4, q)\backslash\Sigma_\infty$ which represents a Baer subline of B incident with P. It follows that $\mathcal{B}$ contains $q+1$ lines $l_1^*, \ldots, l_{q+1}^*$ in $PG(4,q)\backslash\Sigma_\infty$ each incident with p and no two in a plane about p. It follows that no two of these lines intersect in a point not incident with p. Call the lines $l_1^*, \ldots, l_{q+1}^*$ **generators** of $\mathcal{B}$; we now prove that these lines are mutually skew.

Suppose $l_i^* \cap l_j^* \in p$ with $1 \leq i < j \leq q+1$. Then $\langle l_i^*, l_j^* \rangle$ is a transversal plane which represents a Baer subplane, distinct from B, and sharing with B a non-degenerate quadrangle, which by Theorem 1.2.1 is a contradiction. Therefore through each point of p there passes a unique generator of $\mathcal{B}$. It follows that points of $PG(4,q)$ on these generators are all the points of $\mathcal{B}$. The spread line p is therefore contained in $\mathcal{B}$ and p is called the line directrix of $\mathcal{B}$.

Let Q be any point of B, distinct from P; let $Q^* \in \mathcal{B}$ be its representative in $PG(4,q)$. In $PG(2,q^2)$, of the Baer sublines in B through Q, one is a subline of the line QP and the remaining q are disjoint from ℓ_∞. Therefore, by Lemma 2.2.3, the points of $\mathcal{B}$ lie on q distinct Baer conics $C_1^*, \ldots, C_q^*$ and one generator, l_{q+1}^* say, each through Q^*. Each conic C_i^* $(i = 1, \ldots, q)$ lies in a plane α_i which intersects Σ_∞ in a line m_i of the spread $\mathcal{S}$. Let α_{q+1} denote the plane $\langle Q^*, p \rangle$. In $PG(2,q^2)$, each subline of B through P intersects each subline of B through Q, and therefore each conic C_i^* intersects each generator of $\mathcal{B}$ in a unique point.

Consider the plane $\langle Q^*, l_1^* \rangle$; since $Q^* \notin l_1^*$, the plane $\langle Q^*, l_1^* \rangle$ is a transversal plane and therefore represents a Baer subplane B' of $PG(2,q^2)$, distinct from B. Now $|B \cap B'| \geq q+2$. It follows from Lemma 2.2.8 that the Baer subplanes B and B' intersect in $q+1$ lines of $PG(2,q^2)$ through Q and the line represented by $\langle l_1^*, p \rangle$. If l_1^* represents the Baer subline u of B, then the lines of $PG(2,q^2)$ joining Q to the $q+1$ points of u, common to B and B', are represented by the planes $\alpha_1, \ldots, \alpha_{q+1}$. As a transversal plane intersects Σ_∞ in a transversal line of a regulus in the regular spread $\mathcal{S}$, it follows that the spread lines $m_1, \ldots, m_q, p$, each of which intersects $\langle Q^*, l_1^* \rangle$, are generators of a hyperbolic quadric $\mathcal{Q}_2^2$ of Σ_∞. Thus the $q+1$ planes $\alpha_1, \ldots, \alpha_{q+1}$ constitute a quadric cone V_3^2 of $PG(4,q)$, with the point Q^* as vertex, and the quadric $\mathcal{Q}_2^2$ as base. Let $X_1^*, \ldots, X_{q+1}^*$ be the points of the Baer conic C_1^*. Then the $q+1$ planes $\langle X_i^*, p \rangle$ constitute a quadric cone V'^2_3 of $PG(4,q)$, with the line vertex p, and base C_1^*. Note that $Q^* = X_i^*$ for some i. These two quadric cones V_3^2 and V'^2_3 have the plane $\langle Q^*, p \rangle$ in common, and therefore residually intersect in a ruled cubic surface V_2^3. The planes of the two quadric cones represent lines of B, and therefore, by considering their intersection, it follows that $\mathcal{B}$ is precisely the ruled cubic surface V_2^3. $\square$

Since there exist conics in $PG(4,q)\backslash\Sigma_\infty$ which do not represent Baer sublines, it follows that there exist ruled cubic surfaces, with directrix a line of $\mathcal{S}$, which do not represent Baer subplanes B of $PG(2,q^2)$ intersecting ℓ_∞ in a unique point.

Definition 2.2.10 *The ruled cubic surfaces in $PG(4,q)\backslash\Sigma_\infty$, with line directrix a line of $\mathcal{S}$, which represent Baer subplanes of $PG(2,q^2)$ shall be called* **Baer ruled cubics***.*

It is well known that in $PG(2,q^2)$ there exist $q+1$ Baer subplanes containing a given point P and a Baer subline c of a line a not through P; if we let $P \in \ell_\infty$ and c be disjoint from ℓ_∞, then together with Lemma 2.2.3 and Lemma 2.2.9 this implies the following result, which we shall need in a later chapter.

Lemma 2.2.11 *Let α be a plane in $PG(4,q)\backslash\Sigma_\infty$, with $\alpha\cap\Sigma_\infty = m$ a line of the spread $\mathcal{S}$ of Σ_∞. If p is a line in $\mathcal{S}$ distinct from m and if $\mathcal{C}^*$ is a Baer conic in α, then there exist $q+1$ Baer ruled cubics containing p and $\mathcal{C}^*$.* $\square$

The representation in Bruck-Bose of Baer subplanes of $PG(2,q^2)$ is therefore completely determined. For translation planes π_{q^2}, with kernel of order q, the problem of determining the representation of Baer subplanes of π_{q^2} in Bruck-Bose is not completely solved. Freeman [40] gives examples of affine Baer subplanes of a translation plane π_{q^4}, of order q^4, which have a representation in $4-$dimensional Bruck-Bose which is distinct from those obtained above for the Desarguesian case (see also Foulser [38] for other examples.)

2.3 The Bruck-Bose representation of Conics in Baer subplanes of $\mathbf{PG(2,q^2)}$

For later work we shall need a classification of the possible intersections of a hyperplane of $PG(4,q)$ and a Baer ruled cubic surface in $PG(4,q)$. This is given in the next theorem.

Note: A Baer ruled cubic surface in $PG(4,q)$ is a variety of order 3 and dimension 2 properly contained in the 4-dimensional space. By the results of Section 1.6 a hyperplane of $PG(4,q)$ intersects the Baer ruled cubic surface in a cubic curve (see also

Theorem 1.7.2); a cubic curve on the Baer ruled cubic surface is one of the following:

(a) One line counted triply;

(b) Two lines, one counted doubly;

(c) Three lines;

(d) A conic and a line;

(e) A twisted cubic curve.

The only lines on a Baer ruled cubic surface are the generators and the line directrix. Apart from points and lines the ruled cubic surface contains no linear subspaces.

Also note that the intersection of a hyperplane with a ruled cubic surface may have components in some extension of the base field.

Theorem 2.3.1 *Let $\mathcal{B}$ be a Baer ruled cubic surface in $PG(4, q)$. Let $p \in \mathcal{S}$ denote the line directrix of $\mathcal{B}$, so that $\{p\} = \mathcal{B} \cap \Sigma_\infty$. Denote by Π_3 a hyperplane of $PG(4, q)$. The intersection $\mathcal{B} \cap \Pi_3$ in $PG(4, q)$ is one of the following:*

$\mathcal{B} \cap \Pi_3$	*The number of hyperplanes of $PG(4, q)$ which intersect $\mathcal{B}$ in such a configuration:*
(a) The line directrix p of $\mathcal{B}$	$(q^2 - q)/2$ *(Note: Σ_∞ is an example of such a hyperplane)*
(b) The union of a (unique) generator of $\mathcal{B}$ and the line directrix p of $\mathcal{B}$	$q + 1$
(c) The union of two generators of $\mathcal{B}$ and the line directrix p	$(q^2 + q)/2$
(d) The union of a Baer conic and a generator of $\mathcal{B}$ (Note: The Baer conic and generator intersect in a unique point)	$q^3 + q^2$
(e) A twisted cubic curve (Note: such a curve intersects the line directrix p in a unique point)	$q^4 - q^2$

Proof: By generating hyperplanes Π_3 from subsets of points of $\mathcal{B}$, the intersection sets $\Pi_3 \cap \mathcal{B}$ are determined. We proceed with this method until all $q^4 + q^3 + q^2 + q + 1$ hyperplanes of $PG(4, q)$ have been considered.

Let $P_1^*, \ldots, P_{q+1}^*$ denote the points of p and let $g_1^*, \ldots, g_{q+1}^*$ denote the generators of $\mathcal{B}$, such that $g_i^* \cap p = \{P_i^*\}$, $i = 1, \ldots, q+1$.

Let C^* be a Baer conic of $\mathcal{B}$ and let π_{C^*} be the plane in $PG(4, q)$ containing C^*. Hyperplanes $\langle \pi_{C^*}, g_i^* \rangle$, $i = 1, \ldots, q+1$, are the $q+1$ hyperplanes of $PG(4, q)$ about the plane π_{C^*}; each hyperplane contains both the Baer conic C^* and generator g_i^* respectively. The Baer conic C^* and the generator g_i^* constitute a cubic curve in the hyperplane $\langle \pi_{C^*}, g_i^* \rangle$ hence by the note preceding this theorem each hyperplane $\langle \pi_{C^*}, g_i^* \rangle$ contains exactly the Baer conic C^* and generator g_i^* of $\mathcal{B}$ for $i = 1, \ldots, q+1$ respectively.

There are q^2 Baer conics of $\mathcal{B}$ and $q+1$ generators of $\mathcal{B}$, thus there exist $q^2(q+1) = q^3+q^2$ hyperplanes of $PG(4, q)$ which intersect $\mathcal{B}$ in the union of a Baer conic and a generator of $\mathcal{B}$.

There exist $q^2 + q + 1$ hyperplanes about the line p, $q^2 + q$ distinct from Σ_∞. If ℓ is a line of π_{C^*}, a plane of $PG(4, q)\Sigma_\infty$ which contains a Baer conic C^* of $\mathcal{B}$, then $\langle \ell, p \rangle$ is a hyperplane containing the line directrix p. Depending on whether ℓ is an external, tangent or secant line of C^* in π_{C^*}, the hyperplane $\langle \ell, p \rangle$ intersects $\mathcal{B}$ in p plus 2, 1, or 0 generators of $\mathcal{B}$ in $PG(4, q)$ respectively. We consider these cases separately.

Two generators of $\mathcal{B}$ span a hyperplane about p. Such a hyperplane contains three lines of the Baer ruled cubic surface hence no further point of $\mathcal{B}$. Thus there exist $q(q+1)/2$ hyperplanes of $PG(4, q)$ which intersect $\mathcal{B}$ in the union of two generators of $\mathcal{B}$ and the line directrix p of $\mathcal{B}$.

About a plane $\langle p, g_i^* \rangle$, for fixed i, there exist $q+1$ hyperplanes; q contain a second generator of $\mathcal{B}$ and one intersects $\mathcal{B}$ in no further point; here the hyperplane intersects the ruled cubic surface doubly at g_i^*. By considering the $q+1$ generators in turn, we have that there exist $q+1$ hyperplanes of $PG(4, q)$ which intersect $\mathcal{B}$ in the union of a generator and the line directrix p of $\mathcal{B}$. The $q(q-1)/2$ remaining hyperplanes about p therefore each intersect $\mathcal{B}$ in exactly the line directrix p; each such hyperplane intersects the ruled cubic surface at p and two complex conjugate generators of the cubic surface in a quadratic extension extension of the base field.

Next consider a spread element m distinct from p. About m there is a unique plane π_m containing a Baer conic of $\mathcal{B}$. Hyperplanes $\langle \pi_m, P_i^* \rangle$ $i = 1, \ldots, q+1$ each intersect $\mathcal{B}$ in the union of the Baer conic in π_m and the generator g_i^* respectively; these hyperplanes have been counted above. About the plane $\langle m, P_i^* \rangle$, for fixed i, there are $q-1$ hyperplanes distinct from both Σ_∞ and $\langle \pi_m, P_i^* \rangle$; let Σ be one of these $q-1$ hyperlanes. Σ contains no Baer conic or generator of $\mathcal{B}$, and Σ does not contain the line directrix p of $\mathcal{B}$. The hyperplane Σ intersects each generator line of $\mathcal{B}$ in a unique point. As a hyperplane intersects a ruled cubic surface in a cubic curve, we conclude that Σ intersects $\mathcal{B}$ in an irreducible cubic curve, namely a twisted cubic curve.

The number of spread elements besides p is q^2; the number of points of p is $q+1$; from above, for a spread element $m \neq p$ and a point P_i^* of p there are $q-1$ hyperplanes about the plane $\langle m, P_i^* \rangle$ which each intersect $\mathcal{B}$ in a (distinct) twisted cubic curve. Thus there exist $q^2(q+1)(q-1) = q^4 - q^2$ hyperplanes of $PG(4,q)$ which intersect $\mathcal{B}$ in a twisted cubic curve.

We have considered $(q^3+q^2)+(q^2+q)/2+(q+1)+(q^2-q)/2+q^4-q^2 = q^4+q^3+q^2+q+1$ distinct hyperplanes of $PG(4,q)$, namely all hyperplanes of $PG(4,q)$. $\square$

In Theorem 2.3.1 the intersection sets (a), (b), (c) and (d) can be described in B, the Baer subplane of $PG(2,q^2)$ represented by $\mathcal{B}$, as respectively: a unique point P at infinity on B, a Baer subline in B containing P, the union of two distinct Baer sublines in B containing P and the union of a Baer subline in B through P and a Baer subline in B not through P. As a subset of points of B, the intersection set (e) has properties which are not so readily recognised. We now show that an intersection set of type (e) in B is a non-degenerate conic in B.

Lemma 2.3.2 *Let $\mathcal{B}$ be a Baer ruled cubic surface in the Bruck-Bose representation of $PG(2,q^2)$. If ζ^* is a twisted cubic curve on $\mathcal{B}$ then ζ^* is the Bruck-Bose representation of an oval ζ in the corresponding Baer subplane B of $PG(2,q^2)$.*

Proof: A twisted cubic curve ζ^* lies in a hyperplane Σ_{ζ^*} of $PG(4,q)$. Since ζ^* is contained in $\mathcal{B}$ and since $\Sigma_{\zeta^*} \cap \mathcal{B}$ is a cubic curve, we have $\Sigma_{\zeta^*} \cap \mathcal{B} = \zeta^*$. By Theorem 2.3.1 and its proof we have the following:

1. The hyperplane Σ_{ζ^*}, which contains the twisted cubic curve ζ^*, contains a unique spread element m distinct from the line directrix p of $\mathcal{B}$;
2. The planes about m in Σ_{ζ^*} each contain a unique point of ζ^*;
3. ζ^* has exactly $q+1$ points, one on each generator line of $\mathcal{B}$;
4. ζ^* contains a unique point of the line directrix p of $\mathcal{B}$.

Now we consider ζ^* as a set of points ζ in B, the Baer subplane of $PG(2,q^2)$ represented by $\mathcal{B}$, and show that no line of B contains three points of ζ.

Interpreting the properties (1) – (4) in the Baer subplane B we have by (4) ζ contains the point $\{P\} = B \cap l_\infty$. By (3) each line of B through P intersects ζ in at most one further point. Now suppose there exists a line l of B containing three distinct points of ζ; by the previous statement $P \notin l$. Also note that $l \neq l_\infty$ since l_∞ is not a line of B. In Bruck-Bose, l is a plane α_l containing 3 distinct points of the twisted cubic curve ζ^* and therefore by (1), and since no three points of a twisted cubic curve are collinear, the plane α_l is contained in the hyperplane Σ_{ζ^*}. Since α_l is necessarily a plane about a spread element (α_l is a Bruck-Bose representation of a line of $PG(2,q^2)$) α_l contains the unique spread element m in Σ_{ζ^*}. By (2) α_l therefore intersects ζ^* in a unique point, a contradiction to our assumption that α_l contains three distinct points of ζ^*. Thus in $PG(2,q^2)$ there exists no Baer subline in B which intersects ζ in more than two points. $\square$

Lemma 2.3.3 *Let $\mathcal{C}$ be a non-degenerate conic in $PG(2,q)$. Embed $PG(2,q)$ as a Baer subplane in $PG(2,q^2)$. Let $\mathcal{C}_{q^2}$ be the conic obtained by extending $\mathcal{C}$ to a conic in $PG(2,q^2)$. Let M be a point of $\mathcal{C}_{q^2}\backslash\mathcal{C}$. The $q+1$ lines joining M to the points of $\mathcal{C}$ are lines of a Baer subplane containing M.*

Proof: Without loss of generality let $\mathcal{C}$ be the conic with equation $xy = z^2$ and let M have coordinates $(\theta^2, 1, \theta)$ where $\theta \in GF(q^2)\backslash GF(q)$. Note that $(0,1,0)$ and $(1,0,0)$ are points of $\mathcal{C}$.

The lines in $PG(2,q^2)$ joining M to the points of $\mathcal{C}$ have line coordinates given by:

$$\{[1,0,-\theta] + \phi[0,\theta,-1] \mid \phi \in GF(q) \cup \{\infty\}\}.$$

The pencil of lines $\{[1,0,1]+\phi[0,1,1] \mid \phi \in GF(q) \cup \{\infty\}\}$ in $PG(2,q)$ is the pre-image of the above set of lines under the projectivity of $PG(2,q^2)$ given by the non-singular matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & -1+\theta & -\theta \end{bmatrix} \in PGL(3,q^2)$. We have therefore that the $q+1$ lines joining M to the points of $\mathcal{C}$ are $q+1$ lines of a Baer subplane containing M. $\square$

Theorem 2.3.4 *The q^4-q^2 twisted cubic curves on a Baer ruled cubic $\mathcal{B}$ with line directrix p in $PG(4,q)$ are the Bruck-Bose representations of the q^4-q^2 non-degenerate conics in B on the point P, where B is the Baer subplane of $PG(2,q^2)$ represented by $\mathcal{B}$ and P the point of B at infinity represented by p.*

Proof: For q odd the result follows from Lemma 2.3.2 and Segre's Theorem 1.11.1. For q even, let $\mathcal{C}$ be a non-degenerate conic in a Baer subplane B of $PG(2,q^2)$ such that $B \cap l_\infty = \{P\}$, a unique point, and let $P \in \mathcal{C}$. Conic $\mathcal{C}$ is a subconic of a conic $\mathcal{C}_{q^2}$ of $PG(2,q^2)$ and since l_∞ is not the tangent to $\mathcal{C}_{q^2}$ at the point P, l_∞ is a secant to the conic $\mathcal{C}_{q^2}$. Let M be the point of $\mathcal{C}_{q^2}$ distinct from P on the line l_∞. In B the points of $\mathcal{C}$ besides P lie on distinct lines of B on P. Thus in Bruck-Bose B is a Baer ruled cubic surface $\mathcal{B}$ with line directrix p (representing P) and the points $\mathcal{C}^*$ of the conic in Bruck-Bose besides p lie on distinct generator lines of $\mathcal{B}$; the point M is represented in Bruck-Bose by a spread element m. We now show that these points $\mathcal{C}^*$ lie in a hyperplane of $PG(4,q)$ so that by Theorem 2.3.1 the points of $\mathcal{C}^*$ are points of a twisted cubic curve on $\mathcal{B}$. In $PG(2,q^2)$, by Result 2.3.3, the $q+1$ lines on M joining M to the points of conic $\mathcal{C}$ are a pencil of lines in a Baer subplane B' of $PG(2,q^2)$ containing M. Since MP is a line of B', the line at infinity ℓ_∞ is secant to B' and therefore B' is represented in Bruck-Bose by a transversal plane $\mathcal{B}'^*$. The hyperplane $\langle m, \mathcal{B}'^* \rangle$ of $PG(4,q)$ therefore contains the points $\mathcal{C}^*$ representing the conic $\mathcal{C}$. Since the hyperplane $\langle m, \mathcal{B}'^* \rangle$ of $PG(4,q)$ contains the q affine points of $\mathcal{C}^*$ on $\mathcal{B}$ together with a unique point of $p \subseteq \mathcal{B}$ and since $\langle m, \mathcal{B}'^* \rangle$ contains no Baer conic on $\mathcal{B}$, by Theorem 2.3.1 $\langle m, \mathcal{B}'^* \rangle$ intersects $\mathcal{B}$ in a twisted cubic curve. It follows that the non-degenerate conic $\mathcal{C}$ is represented in Bruck-Bose by a twisted cubic curve on the Baer ruled cubic $\mathcal{B}$. $\square$

Corollary 2.3.5 *Let B be a Baer subplane of $PG(2,q^2)$ such that $|B \cap l_\infty| = 1$; let the unique point at infinity of B be P. The non-degenerate conics in B are represented in Bruck-Bose by either a twisted cubic curve (when the conic contains the point P) or a*

4-dimensional normal rational curve (when the conic does not contain the point P) on the Baer ruled cubic $\mathcal{B}$.

We shall give two proofs; the first is quite short but does not cover all cases and the second is a proof valid for all prime powers $q \geq 3$.

Proof of Corollary 2.3.5 (cases q even and q odd, $\mathbf{q \leq 7}$ or $\mathbf{(10.25)^2 \leq q}$): By Theorem 2.3.4 it remains to prove that a non-degenerate conic in B which does not contain the point at infinity P, is represented in Bruck-Bose by a 4-dimensional normal rational curve.

Let $\mathcal{C}$ be a non-degenerate conic in B which does not contain the point P. Each line of B intersects $\mathcal{C}$ in at most two points. Since non-degenerate conics on P in B are represented in Bruck-Bose by twisted cubic curves (see Theorem 2.3.4) and since a distinct non-degenerate conic in B intersects $\mathcal{C}$ in at most four points, by Theorem 2.3.1 a hyperplane of $PG(4,q)$ intersects the Bruck Bose representation of conic $\mathcal{C}$ in at most four points. We have therefore that the Bruck-Bose representation of conic $\mathcal{C}$ is a set of $q+1$ points $\mathcal{C}^*$ in $PG(4,q)$ with the property that no hyperplane intersects the set in more than four points; in other words we have a $(q+1)_4-$arc $\mathcal{C}^*$ in $PG(4,q)$ and by Theorems 1.5.1, 1.5.2 and 1.5.3, this arc is a $4-$dimensional normal rational curve for q even and q odd, $q \leq 7$ or $(10.25)^2 \leq q$. □

Lemma 2.3.6 *There exists $a, b \in GF(q^2) \backslash GF(q)$, $q \geq 3$, with the following properties:*

(i) $a \neq b, -b$,

(ii) $ab^{-1} + a^{-1}b \in GF(q^2) \backslash GF(q)$,

(iii) $ab^{-1} \in GF(q^2) \backslash GF(q)$,

and for such a, b we have $ab \neq 0$, $a^2 \neq 0$, $b^2 \neq 0$.

Proof: First we prove that there exists $x \in GF(q^2) \backslash GF(q)$, $q \geq 3$, such that $x + x^{-1} \notin GF(q)$:

For $q = 3$, $GF(3) = \{0, 1, 2\}$ and $GF(9) = \{0, 1, \omega, \omega^2, 2, \omega^5, \omega^6, \omega^7\}$ where $\omega^2 - \omega - 1 = 0$. Here $\omega + \omega^7 = \omega$ and $\omega^2 + \omega^5 = \omega^2$ as required.

For $q > 3$ consider

$$\begin{aligned} x + x^{-1} &= \lambda \\ \iff \quad x^2 - \lambda x + 1 &= 0 \end{aligned}$$

for $\lambda \in GF(q)$ and $x \neq 0$.

For each $\lambda \in GF(q)$, there exist at most two solutions $x, x^{-1} \in GF(q^2)\backslash GF(q)$ to this quadratic equation. Therefore there exist at least

$$q^2 - 2q - q \; = \; q(q-3) \; > \; 0$$

elements $x \in GF(q^2)\backslash GF(q)$ for which $x + x^{-1} \notin GF(q)$.

It remains to show that for $x \in GF(q^2)\backslash GF(q)$ for which $x + x^{-1} \notin GF(q)$, there exists $a, b \in GF(q^2)\backslash GF(q)$ such that $a \neq b, -b$ and $bx = a$:

By considering all $b \in GF(q^2)\backslash GF(q)$, we obtain $q^2 - q$ distinct elements bx and $bx \neq 0$ as neither x nor b is equal to 0.

Since

$$q^2 - q \; > \; q - 1 \; = \; |GF(q)\backslash\{0\}|$$

there exists a choice of $b \in GF(q^2)\backslash GF(q)$ for which $bx = a \notin GF(q)$.

If $a = b$ then $b(x-1) = 0$ implies that $x = 1 \in GF(q)$, a contradiction. If $a = -b$ then $b(x+1) = 0$ implies that $x = -1 \in GF(q)$, a contradiction. $\square$

Proof of Corollary 2.3.5 (case q $\geq$ 3): We investigate the representation in Bruck-Bose of a particular[1] non-degenerate conic $\mathcal{C}$ in a Baer subplane B of $PG(2, q^2)$ with $|B \cap \ell_\infty| = 1$ and $\mathcal{C} \cap \ell_\infty = \emptyset$. Let $\mathcal{C}'_{q^2}$ be the conic $\{(\theta^2, 1, \theta);\ \theta \in GF(q^2) \cup \{\infty\}\}$, that is, the conic in $PG(2, q^2)$ with equation $z^2 = xy$. The conic $\mathcal{C}'_{q^2}$ has nucleus $N(0, 0, 1)$ if q is even.

$\mathcal{C}'_{q^2}$ is fixed by projectivities of the plane defined by a matrix of the form:

$$H = \begin{bmatrix} a^2 & b^2 & 2ab \\ c^2 & d^2 & 2cd \\ ac & bd & bc + ad \end{bmatrix}$$

such that $ad - bc \neq 0$ (as $|H| = (ad-bc)^3$) (see [52, Theorem 2.37]).

The action of such a projectivity on the points of the conic $\mathcal{C}'_{q^2}$ is the map:

$$\begin{array}{rcl} (\theta^2, 1, \theta) & \longmapsto & \left(\left(\frac{a\theta+b}{c\theta+d}\right)^2, 1, \frac{a\theta+b}{c\theta+d} \right) \\ (1, 0, 0) & \longmapsto & \left(\left(\frac{a}{c}\right)^2, 1, \frac{a}{c} \right) \\ (\; N & \longmapsto & N, \quad \text{for } q \text{ even }). \end{array}$$

[1]For a non-degenerate conic $\mathcal{C}_1$ in a Baer subplane B_1, $B_1^\sigma = B$ for some collineation σ and $\mathcal{C}_1^\sigma$ is a non-degenerate conic in B and is therefore projectively equivalent to $\mathcal{C}$ via a collineation in B.

Let $\mathcal{C}' \subseteq \mathcal{C}'_{q^2}$ be the points of $\mathcal{C}'_{q^2}$ in the Baer subplane $PG(2,q)$; so $\mathcal{C}' = \{(\theta^2, 1, \theta);\ \theta \in GF(q) \cup \{\infty\}\}$.

We now find a projectivity with matrix H of the above form which maps $PG(2,q)$ to a Baer subplane B and maps $\mathcal{C}'$ to a conic $\mathcal{C}$ in B, with $|B \cap \ell_\infty| = 1$, and such that $B \cap \ell_\infty \neq (0,1,0), (1,0,0)$, that is, such that the unique point of B on ℓ_∞ is not a point of the conic. For our coordinate representation of $PG(2,q^2)$ the line at infinity ℓ_∞ is the line of $PG(2,q^2)$ with equation $z = 0$.

We will then represent $\mathcal{C}$ via coordinates in the Bruck-Bose setting and determine $\mathcal{C}$ as a set of points of a normal rational curve in $PG(4,q)$.

Consider the projectivity H defined by matrix:

$$\begin{bmatrix} a^2 & b^2 & 2ab \\ b^2 & a^2 & 2ab \\ ab & ab & a^2+b^2 \end{bmatrix}$$

where $a \neq b,\ -b,\ a, b, ab^{-1}, ab^{-1} + a^{-1}b \in GF(q^2)\backslash GF(q)$ (refer to Lemma 2.3.6).

The Baer subplane $PG(2,q)$ is mapped by H as follows:

For a point $(x,y,z) \in PG(2,q)$,

$$H \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} a^2x + b^2y + 2abz \\ b^2x + a^2y + 2abz \\ abx + aby + (a^2+b^2)z \end{bmatrix}. \tag{2.1}$$

The resulting set of points constitute a Baer subplane B whose intersection with ℓ_∞ is the set of points (2.1) with third coordinate zero, that is with

$$\begin{aligned} abx + aby + (a^2+b^2)z &= 0, \\ \text{that is,}\quad x + y + (ab^{-1} + a^{-1}b)z &= 0. \end{aligned} \tag{2.2}$$

Since $ab^{-1}+a^{-1}b \in GF(q^2)\backslash GF(q)$, equation (2.2) is the equation of a line not in $PG(2,q)$ and therefore the line (2.2) intersects $PG(2,q)$ in a unique point; that is, there exists a unique point $X = (x', y', z')$ in $PG(2,q)$ for which $x' + y' + (ab^{-1} + a^{-1}b)z' = 0$. Thus the Baer subplane B intersects ℓ_∞ in a unique point, namely the point with coordinates,

$$H \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} a^2x' + b^2y' + 2abz' \\ b^2x' + a^2y' + 2abz' \\ 0 \end{bmatrix} = X^H. \tag{2.3}$$

Also we need to show that X^H is not a point of the conic $\mathcal{C}$. The only points of $\mathcal{C}'_{q^2}$ on the line at infinity are $(1,0,0)$ and $(0,1,0)$ thus if $X^H \subseteq \mathcal{C} \cap \ell_\infty$ then X^H is $(1,0,0)$ or $(0,1,0)$. Now $X^H = (0,1,0)$ or $X^H = (1,0,0)$ if and only if $a^2x' + b^2y' + 2abz' = 0$ or $b^2x' + a^2y' + 2abz' = 0$ respectively.

We need to show $a^2x' + b^2y' + 2abz' \neq 0$ and $b^2x' + a^2y' + 2abz' \neq 0$.

Consider the lines $[a^2, b^2, 2ab]$, $[b^2, a^2, 2ab]$, $[1, 1, ab^{-1} + a^{-1}b]$. The point of intersection of lines $[a^2, b^2, 2ab]$ and $[1, 1, ab^{-1} + a^{-1}b]$ is $(b^2, a^2, -ab) \equiv (a^{-1}b, ab^{-1}, -1)$. The point of intersection of lines $[b^2, a^2, 2ab]$ and $[1, 1, ab^{-1} + a^{-1}b]$ is $(a^2, b^2, -ab) \equiv (ab^{-1}, a^{-1}b, -1)$. Since $X \in PG(2,q)$ and since $-1 \in GF(q)$ and $ab^{-1}, a^{-1}b \in GF(q^2)\backslash GF(q)$, it follows that $X \neq (a^{-1}b, ab^{-1}, -1)$ and $X \neq (ab^{-1}, a^{-1}b, -1)$. Hence we conclude that X^H is not a point of $\mathcal{C}$.

Hence $B = PG(2,q)^H$ is a Baer subplane with unique point X^H on ℓ_∞ and the Baer subplane B contains the non-degenerate conic $\mathcal{C} = \mathcal{C}'^H$ for which $\mathcal{C} \cap \ell_\infty = \emptyset$.

The coordinates of the points of $\mathcal{C} = \mathcal{C}'^H$ are given by:

$$H \begin{pmatrix} \theta^2 \\ 1 \\ \theta \end{pmatrix} = \begin{pmatrix} \left(\frac{a\theta+b}{b\theta+a}\right)^2 \\ 1 \\ \frac{a\theta+b}{b\theta+a} \end{pmatrix} \equiv \begin{pmatrix} \frac{a\theta+b}{b\theta+a} \\ \frac{b\theta+a}{a\theta+b} \\ 1 \end{pmatrix}$$

where $\theta \in GF(q) \cup \{\infty\}$.

(Note that if $a\theta + b = 0$ then $\theta = -a^{-1}b$ is not an element of $GF(q)$, a contradiction. Hence $a\theta + b \neq 0$ and similarly $b\theta + a \neq 0$.)

We now transform these plane coordinates to coordinates in $PG(4,q)$, that is the coordinates of the points $\mathcal{C}^*$ representing $\mathcal{C}$ in the Bruck-Bose setting in $PG(4,q)\backslash\Sigma_\infty$ using the results of Section 1.10.4.

Let α be an element of $GF(q^2)\backslash GF(q)$ with minimal polynomial

$$x^2 - \lambda x - \mu,$$

where $\lambda, \mu \in GF(q)$. Using $x \longmapsto \bar{x} = x^q$ to denote the Fröbenius map, we have

$$\begin{aligned} \alpha + \bar{\alpha} &= \lambda \\ \alpha\bar{\alpha} &= -\mu. \end{aligned}$$

Each element $x \in GF(q^2)$ can be written uniquely in the form $x = x_1 + \alpha x_2$ where $x_1, x_2 \in GF(q)$.

We have therefore:

$$\begin{aligned}
a = a_1 + \alpha a_2, &\quad a_1, a_2 \in GF(q),\\
b = b_1 + \alpha b_2, &\quad b_1, b_2 \in GF(q)\\
\text{and } \theta = \bar{\theta} &\quad \text{since } \theta \in GF(q) \text{ for points in } \mathcal{C}'.
\end{aligned}$$

Also $\bar{a} = a_1 + \bar{\alpha} a_2 = a_1 + (\lambda - \alpha) a_2 = (a_1 + \lambda a_2) - \alpha a_2$ and similarly $\bar{b} = (b_1 + \lambda b_2) - \alpha b_2$. For a point P in $\mathcal{C}$, with the coordinates $\left(\frac{a\theta+b}{b\theta+a}, \frac{b\theta+a}{a\theta+b}, 1\right)$ of P written as a row vector, and using the above representation we obtain:

$$\left(\frac{a\theta+b}{b\theta+a}, \frac{b\theta+a}{a\theta+b}, 1\right) \equiv \left(\frac{a\theta+b}{b\theta+a} \times \frac{\bar{b}\theta+\bar{a}}{\bar{b}\theta+\bar{a}}, \frac{b\theta+a}{a\theta+b} \times \frac{\bar{a}\theta+\bar{b}}{\bar{a}\theta+\bar{b}}, 1\right).$$

Now

$$\frac{a\theta+b}{b\theta+a} \times \frac{\bar{b}\theta+\bar{a}}{\bar{b}\theta+\bar{a}} = \frac{a\bar{b}\theta^2 + \theta(b\bar{b} + a\bar{a}) + b\bar{a}}{b\bar{b}\theta^2 + \theta(a\bar{b} + b\bar{a}) + a\bar{a}}$$

and note that the denominator is an element of $GF(q)$. Each term in the numerator is an element of $GF(q)$ except for $a\bar{b}$ and $b\bar{a}$, which we can write as follows,

$$\begin{aligned}
a\bar{b} &= (a_1 + \alpha a_2)(b_1 + \bar{\alpha} b_2)\\
&= a_1 b_1 + \alpha a_2 b_1 + \bar{\alpha} a_1 b_2 + \alpha\bar{\alpha} a_2 b_2\\
&= a_1 b_1 + \alpha a_2 b_1 + (\lambda - \alpha) a_1 b_2 - \mu a_2 b_2\\
&= a_1 b_1 - \mu a_2 b_2 + \lambda a_1 b_2 + \alpha(a_2 b_1 - a_1 b_2)\\
&= Q + \alpha L \;\; \text{(for ease of notation)}\\
\text{and } \; b\bar{a} &= a_1 b_1 - \mu a_2 b_2 + \lambda b_1 a_2 + \alpha(b_2 a_1 - b_1 a_2)\\
&= \tilde{Q} + \alpha \tilde{L} \;\; \text{(for ease of notation).}
\end{aligned}$$

(Note that $Q, \tilde{Q}, L, \tilde{L}$ are all elements of $GF(q)$.)

In Bruck-Bose the point P therefore corresponds to the point P^* in $PG(4,q)\backslash\Sigma_\infty$ with homogeneous coordinates:

$$\begin{pmatrix} \dfrac{Q\theta^2+\theta(b\bar{b}+a\bar{a})+\tilde{Q}}{b\bar{b}\theta^2+\theta(a\bar{b}+b\bar{a})+a\bar{a}} \\ \dfrac{L\theta^2+\tilde{L}}{b\bar{b}\theta^2+\theta(a\bar{b}+b\bar{a})+a\bar{a}} \\ \dfrac{\tilde{Q}\theta^2+\theta(b\bar{b}+a\bar{a})+Q}{a\bar{a}\theta^2+\theta(a\bar{b}+b\bar{a})+b\bar{b}} \\ \dfrac{\tilde{L}\theta^2+L}{a\bar{a}\theta^2+\theta(a\bar{b}+b\bar{a})+b\bar{b}} \\ 1 \end{pmatrix}$$

(in [6] Buekenhout applied a similar transformation to unitals in $PG(2,q^2)$ to find their image in Bruck-Bose via coordinates in $PG(4,q)$).

If we multiply through by

$$\left(b\bar{b}\theta^2+\theta(a\bar{b}+b\bar{a})+a\bar{a}\right)\left(a\bar{a}\theta^2+\theta(a\bar{b}+b\bar{a})+b\bar{b}\right),$$

which is a non-zero element of $GF(q)$, we obtain each component of the coordinate vector as a degree 4 polynomial in θ over $GF(q)$, where $\theta \in GF(q)\cup\{\infty\}$. On simplification, the points of $\mathcal{C}$ represented in Bruck-Bose are the points with coordinates given by:

$$\begin{bmatrix} a\bar{a}Q & \begin{matrix}(a\bar{b}+b\bar{a})Q\\ +a\bar{a}(b\bar{b}+a\bar{a})\end{matrix} & \begin{matrix}b\bar{b}Q+a\bar{a}\tilde{Q}\\ +(b\bar{b}+a\bar{a})(a\bar{b}+b\bar{a})\end{matrix} & \begin{matrix}(a\bar{b}+b\bar{a})\tilde{Q}\\ +b\bar{b}(a\bar{a}+b\bar{b})\end{matrix} & b\bar{b}\tilde{Q} \\ a\bar{a}L & (a\bar{b}+b\bar{a})L & b\bar{b}L+a\bar{a}\tilde{L} & (a\bar{b}+b\bar{a})\tilde{L} & b\bar{b}\tilde{L} \\ b\bar{b}\tilde{Q} & \begin{matrix}(a\bar{b}+b\bar{a})\tilde{Q}\\ +b\bar{b}(b\bar{b}+a\bar{a})\end{matrix} & \begin{matrix}b\bar{b}Q+a\bar{a}\tilde{Q}\\ +(b\bar{b}+a\bar{a})(a\bar{b}+b\bar{a})\end{matrix} & \begin{matrix}(a\bar{b}+b\bar{a})Q\\ +a\bar{a}(a\bar{a}+b\bar{b})\end{matrix} & a\bar{a}Q \\ b\bar{b}\tilde{L} & (a\bar{b}+b\bar{a})\tilde{L} & b\bar{b}L+a\bar{a}\tilde{L} & (a\bar{b}+b\bar{a})L & a\bar{a}L \\ a\bar{a}b\bar{b} & (b\bar{b}+a\bar{a})(a\bar{b}+b\bar{a}) & \begin{matrix}(a\bar{a})^2+(b\bar{b})^2\\ (a\bar{b}+b\bar{a})^2\end{matrix} & (a\bar{a}+b\bar{b})(a\bar{b}+b\bar{a}) & a\bar{a}b\bar{b} \end{bmatrix} \begin{pmatrix} \theta^4 \\ \theta^3 \\ \theta^2 \\ \theta \\ 1 \end{pmatrix} \tag{2.4}$$

for $\theta \in GF(q)\cup\{\infty\}$.

Denote by M the coefficient matrix in (2.4) and note that $M \in GL(5,q)$.

Here we have shown the set of points of $\mathcal{C}$ in Bruck-Bose is the set of images of points of a normal rational curve in $PG(4,q)$, in particular the image of

$\{(\theta^4, \theta^3, \theta^2, \theta, 1);\ \theta \in GF(q) \cup \{\infty\}\}$ under the projectivity determined by the matrix M. Matrix M is necessarily non-singular as $\mathcal{C}^*$ is a $(q+1)_4$-arc in $PG(4,q)$ (see the first proof of corollary 2.3.5) and hence $\mathcal{C}^*$ is not contained in a hyperplane of $PG(4,q)$. □

It remains to consider the representation in Bruck-Bose of non-degenerate conics which lie in Baer subplanes B of $PG(2,q^2)$ for which ℓ_∞ is a line of B. Let $\mathcal{C}$ be such a conic in a Baer subplane B of $PG(2,q^2)$ for which ℓ_∞ is a line of B. Coordinatise the plane so that ℓ_∞ is the line with equation $z=0$ and B is the subplane $PG(2,q)$, then the conic $\mathcal{C}$ is defined by a homogeneous quadratic equation $Q(x,y,z)$ in variables x,y,z and with coefficients in $GF(q)$, that is

$$\begin{aligned} \mathcal{C} &= \{(x,y,z)|Q(x,y,z)=0\} \\ &= \{(x,y,1)|Q(x,y,1)=0\} \cup \{(x,y,0)|Q(x,y,0)=0\}. \end{aligned}$$

In Bruck-Bose, B is therefore the transversal plane B^* defined by the equations $x_2 = y_2 = 0$ and $\mathcal{C}$ is represented by a non-degenerate conic $\mathcal{C}^*$, where

$$\begin{aligned} \mathcal{C}^* = &\ \{(x_1,0,y_1,0,1)|Q(x_1,y_1,1)=0\} \\ &\cup\{\text{spread elements corresponding to the points of } \mathcal{C} \text{ at infinity}\}. \end{aligned}$$

Thus $\mathcal{C}$ in Bruck-Bose is essentially a non-degenerate conic $\mathcal{C}^*$ in the transversal plane B^*.

We have therefore determined the representation in Bruck-Bose of any non-degenerate conic $\mathcal{C}$ in a Baer subplane of $PG(2,q^2)$. In summary, the Bruck-Bose representation $\mathcal{C}^*$ of $\mathcal{C}$ is determined and is one of the following:

- a non-degenerate conic in a (transversal) plane of $PG(4,q)$;
- a twisted cubic curve on a Baer ruled cubic surface of $PG(4,q)$;
- a 4-dimensional normal rational curve on a Baer ruled cubic surface of $PG(4,q)$.

2.4 A characterisation of Baer ruled cubic surfaces

In Section 2.2 we reviewed the Bruck-Bose representation of non-affine Baer subplanes of $PG(2,q^2)$; such Baer subplanes are represented in Bruck-Bose by certain ruled cubic surfaces which we call **Baer ruled cubics**. For a given Bruck-Bose representation of

$PG(2,q^2)$, that is for a fixed regular spread $\mathcal{S}$ of Σ_∞, let $\mathcal{R}$ be the set of all ruled cubic surfaces V_2^3 in $PG(4,q)$ with the property that the intersection $V_2^3 \cap \Sigma_\infty$ in $PG(4,q)$ is a line which is an element of the spread $\mathcal{S}$. There exist more ruled cubic surfaces in $\mathcal{R}$ than there exist non-affine Baer subplanes of $PG(2,q^2)$. That is, the Baer ruled cubic surfaces in Bruck-Bose constitute a proper subset of all the ruled cubic surfaces in $\mathcal{R}$. In this section we characterise the Baer ruled cubic surfaces, amongst all ruled cubic surfaces in $\mathcal{R}$, for a given regular spread in the Bruck-Bose representation of $PG(2,q^2)$ in $PG(4,q)$.

2.4.1 The extended ruled cubic surface

Consider a ruled cubic surface V_2^3 in $PG(4,q)$ as defined in Section 1.7 with the notation introduced there. So V_2^3 has base conic $\mathcal{C}$, line directrix ℓ and associated projectivity $\phi \in PGL(2,q)$.

Embed $PG(4,q)$ as a Baer subspace in $PG(4,q^2)$. Consider the ruled cubic surface V_2^3 over the extended field. Let $\underset{\sim}{\ell}$ be the (unique) line of $PG(4,q^2)$ such that $\underset{\sim}{\ell} \cap PG(4,q) = \ell$. Similarly let $\underset{\sim}{\mathcal{C}}$ be the (unique) conic in $PG(4,q^2)$ such that $\underset{\sim}{\mathcal{C}} \cap PG(4,q) = \mathcal{C}$. The plane of $\underset{\sim}{\mathcal{C}}$ is denoted by $\underset{\sim}{S_2}$ and $\underset{\sim}{S_2} \cap PG(4,q) = S_2$, where S_2 is the plane of the conic $\mathcal{C}$ in $PG(4,q)$.

Note that in $PG(4,q^2)$ the line $\underset{\sim}{\ell}$ is skew to the plane $\underset{\sim}{S_2}$. Since if not then $\underset{\sim}{\Sigma} = \langle \underset{\sim}{\ell}, \underset{\sim}{S_2} \rangle$ is at most a hyperplane of $PG(4,q^2)$ and by Theorem 1.3.1, the intersection $\Sigma = \underset{\sim}{\Sigma} \cap PG(4,q)$ is either a hyperplane or a plane of $PG(4,q)$. Since the line directrix ℓ and the base conic $\mathcal{C}$ of V_2^3 are contained in Σ, we have that the entire ruled cubic V_2^3 is contained in Σ. Recall from Section 1.7 that a ruled cubic surface V_2^3 in $PG(4,q)$ is not contained in any hyperplane of $PG(4,q)$ and so we obtain a contradiction.

The associated projectivity of V_2^3 between ℓ and $\mathcal{C}$ can be applied to the non-homogeneous coordinates $\underset{\sim}{\lambda}$ ($\underset{\sim}{\lambda} \in GF(q^2) \cup \{\infty\}$) of points on $\underset{\sim}{\ell}$. We denote this action by $\underset{\sim}{\phi}$ and obtain a projective correspondence between points $P(\underset{\sim}{\lambda})$ on $\underset{\sim}{\ell}$ and points $P(\underset{\sim}{\theta})$ on $\underset{\sim}{\mathcal{C}}$ given by,

$$\underset{\sim}{\theta} = \underset{\sim}{\phi}(\underset{\sim}{\lambda}),\ \underset{\sim}{\lambda} \in GF(q^2) \cup \{\infty\}.$$

Note that the projectivity $\underset{\sim}{\phi}$ restricted to points of ℓ (in $PG(4,q)$) is simply the projectivity $\phi \in PGL(2,q)$, that is

$$\underset{\sim}{\phi}|_\ell = \phi,$$

hence $\underset{\sim}{\phi}$ and ϕ are defined by the same 2×2 matrix over $GF(q)$. In other words $\underset{\sim}{\phi}$ is an element of $PGL(2,q)$.

Let $\underset{\sim}{G}$ be the set of lines $PP^{\underset{\sim}{\phi}}$ where the point P ranges over $\underset{\sim}{\ell}$. In this way we obtain a ruled cubic surface in $PG(4,q^2)$, which we shall denote by $\underset{\sim}{V_2^3}$, with line directrix $\underset{\sim}{\ell}$, base conic $\underset{\sim}{\mathcal{C}}$ and associated projectivity $\underset{\sim}{\phi} \in PGL(2,q)$ (see Section 1.7). Since $\underset{\sim}{\phi}|_\ell = \phi$, we note that

$$\underset{\sim}{V_2^3}|_{PG(4,q)} = V_2^3$$

and we call $\underset{\sim}{V_2^3}$ the **extended ruled cubic** of V_2^3.

We denote by $^{-}$ the *Fröbenius automorphism* of $GF(q^2)$,

$$\begin{array}{cccc} ^{-} : & GF(q^2) & \longrightarrow & GF(q^2) \\ & x & \longmapsto & x^q. \end{array}$$

We also denote by $^{-}$ the automorphic collineation of $PG(4,q^2)$ induced by the Fröbenius automorphism. The context in which this is done should make the meaning clear.

$$\begin{array}{cccc} ^{-} : & PG(4,q^2) & \longrightarrow & PG(4,q^2) \\ & P = P(x_0,x_1,x_2,x_3,x_4) & \longmapsto & \overline{P} = \overline{P}(x_0^q,x_1^q,x_2^q,x_3^q,x_4^q). \end{array}$$

Note that this collineation of $PG(4,q^2)$, which we call the *Fröbenius collineation*, fixes the Baer subspace $PG(4,q)$ pointwise and hence the ruled cubic surface V_2^3 is fixed pointwise also. Since $\underset{\sim}{\phi} \in PGL(2,q)$, the ruled cubic surface $\underset{\sim}{V_2^3}$ in $PG(4,q^2)$ is fixed by the Fröbenius collineation in the following way,

$$\begin{aligned} \underset{\sim}{V_2^3} &= \{PP^{\underset{\sim}{\phi}} | P \in \underset{\sim}{\ell}\} \\ &= \{P(\underset{\sim}{\lambda})P(\underset{\sim}{\phi}(\underset{\sim}{\lambda})) | \underset{\sim}{\lambda} \in GF(q^2) \cup \{\infty\} \ (P \in \underset{\sim}{\ell})\} \\ &= \{P(\underset{\sim}{\lambda}^q)P(\underset{\sim}{\phi}(\underset{\sim}{\lambda})^q) | \underset{\sim}{\lambda} \in GF(q^2) \cup \{\infty\} \ (P \in \underset{\sim}{\ell})\} \\ &\quad \text{(The Fröbenius automorphism permutes the elements of } GF(q^2)) \\ &= \{P(\underset{\sim}{\lambda}^q)P(\underset{\sim}{\phi}(\underset{\sim}{\lambda}^q)) | \underset{\sim}{\lambda} \in GF(q^2) \cup \{\infty\} \ (P \in \underset{\sim}{\ell})\} \\ &\quad (\text{since } \underset{\sim}{\phi} \in PGL(2,q)) \\ &= \{\overline{P}\overline{P^{\underset{\sim}{\phi}}} | P \in \underset{\sim}{\ell}\} \\ &= \overline{\underset{\sim}{V_2^3}}. \end{aligned}$$

Note that since $\phi \in PGL(2,q)$ the generators of V_2^3, and in particular the generator containing the points of ℓ with non-homogeneous coordinate ∞, are fixed as a set under the Fröbenius collineation.

We now use $PG(4,q^2)$ as our setting for a new proof of a result originally proved by Bernasconi and Vincenti [15, Theorem 2.4].

Theorem 2.4.1 [15] *A ruled cubic surface V_2^3 of $PG(4,q)$ represents a non-affine Baer subplane of a translation plane $\pi(\mathcal{S})$, defined by a $1-$spread $\mathcal{S}$ in a hyperplane Σ_∞ of $PG(4,q)$ in the usual way, if and only if $\pi(\mathcal{S})$ is Desarguesian.*

Proof: ($\Leftarrow$) The necessary result is Theorem 2.2.9 (see also [19] [90]).

($\Rightarrow$) To begin with, we establish the existence of a $1-$spread $\mathcal{S}$ in a hyperplane Σ_∞ of $PG(4,q)$, that is, the Bruck-Bose representation in of a translation plane. This is done using the method of Bernasconi and Vincenti [15], but the proof that the spread $\mathcal{S}$ is regular is new.

Let $V = V_2^3$ be a ruled cubic surface in $PG(4,q)$ with line directrix ℓ, base conic $\mathcal{C}$ and associated projectivity $\phi \in PGL(2,q)$ as defined in Section 1.7 with the notation introduced there. The conic $\mathcal{C}$ lies in a plane of $PG(4,q)$ which we denote by S_2. Let t be any external line of $\mathcal{C}$ in S_2. Put $\Sigma_\infty = \langle t, \ell \rangle$, that is Σ_∞ is the hyperplane of $PG(4,q)$ spanned by the pair of skew lines t and ℓ. Since each generator of V joins a point of ℓ and a point of $\mathcal{C}$, each generator g of V intersects Σ_∞ in a point of ℓ. thus in $PG(4,q)$ the ruled cubic V intersects the hyperplane Σ_∞ precisely in its line directrix ℓ.

By Theorem 1.7.2, since no three generators of V are contained in a hyperplane of $PG(4,q)$, the planes of two distinct conics on V are not contained in a hyperplane. Also, since any two distinct conics on V intersect in a unique point, the planes containing the q^2 conics on V meet the hyperplane Σ_∞ in q^2 distinct and pairwise skew lines. Denote the conics on V by $\mathcal{C} = \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q^2}$ and denote the planes of these conics by $S_2 = \alpha_1, \alpha_2, \ldots, \alpha_{q^2}$ respectively. Let $\alpha_i \cap \Sigma_\infty = \ell_i$ for $i = 1, \ldots, q^2$ and note that the set of lines,

$$\mathcal{S} = \{t = \ell_1, \ell_2, \ldots, \ell_{q^2}\} \cup \{\ell\}$$

is a set of q^2+1 pairwise skew lines which partition the points of Σ_∞. Thus $\mathcal{S}$ is a spread of Σ_∞ and it remains to show that $\mathcal{S}$ is a regular spread.

Embed $PG(4,q)$ as a Baer subspace in $PG(4,q^2)$. Consider the base conic $\mathcal{C}$ of V in the plane S_2. The spread element t in S_2 is an external line of $\mathcal{C}$ and so intersects $\mathcal{C}$ in two points $X, \overline{X}$ in the quadratic extension. The points $X, \overline{X}$ are *conjugate* with respect to the quadratic extension in the sense that the set $\{X, \overline{X}\}$ is fixed by the Fröbenius collineation. Note that $X, \overline{X}$ are points on the base conic of the extended ruled cubic $\underset{\sim}{V_2^3}$ in $PG(4, q^2)$. Consider the points on $\underset{\sim}{\ell}$ which correspond to $X = P(\theta), \overline{X} = P(\overline{\theta})$ via the associated projectivity $\underset{\sim}{\phi}$ of $\underset{\sim}{V_2^3}$. For some fixed $\lambda \in GF(q^2) \cup \{\infty\}$, we have

$$\begin{aligned} \theta &= \phi(\lambda) \\ \text{and therefore} \quad \overline{\theta} &= \overline{\phi(\lambda)} \\ &= \phi(\overline{\lambda}) \quad \text{since } \phi \in PGL(2, q). \end{aligned}$$

Hence the points $\{X, \overline{X}\}$ on $\underset{\sim}{\mathcal{C}}$ are in projective correspondence with points $\{A = P(\lambda), \overline{A} = P(\overline{\lambda})\}$ on $\underset{\sim}{\ell}$ and $A, \overline{A}$ are conjugate with respect to the quadratic extension.

Let $g, \overline{g}$ denote the pair of generators XA and $\overline{XA}$ of the ruled cubic surface $\underset{\sim}{V_2^3}$. The lines $g, \overline{g}$ lie in the quadratic extension $\underset{\sim}{\Sigma_\infty}$ of the hyperplane Σ_∞ and are disjoint from Σ_∞. By Theorem 1.9.6, $g, \overline{g}$ determine a unique regular spread of Σ_∞ consisting of the q^2+1 lines of Σ_∞ obtained by joining each point of g with its conjugate on $\overline{g}$. We denote this regular spread by $\mathcal{S}_{g\overline{g}}$. Note that $t = X\overline{X}$ and $\ell = A\overline{A}$ are elements of the regular spread $\mathcal{S}_{g\overline{g}}$.

For $i \neq 1$, consider the conic $\mathcal{C}_i$ on V. The conic lies in a plane α_i which contains the spread element ℓ_i of $\mathcal{S}$. Since ℓ_i is an external line of $\mathcal{C}_i$ in the plane of the conic, the intersection $\ell_i \cap \mathcal{C}_i$ is a pair $X_i, \overline{X_i}$ of points conjugate with respect to the quadratic extension. Thus in $PG(4, q^2)$ the spread line ℓ_i contains the points $X_i, \overline{X_i}$ of the extended ruled cubic $\underset{\sim}{V_2^3}$ and the conic $\mathcal{C}_i$ extends uniquely to a conic $\underset{\sim}{\mathcal{C}_i}$ contained in $\underset{\sim}{V_2^3}$. By Theorem 1.7.2, the conic $\underset{\sim}{\mathcal{C}_i}$ intersects each generator of the extended ruled cubic, in particular $\underset{\sim}{\mathcal{C}_i}$ contains a point of g and a point of $\overline{g}$. But since the plane of the conic $\underset{\sim}{\mathcal{C}_i}$ is not contained in $\underset{\sim}{\Sigma_\infty}$ (as α_i is not contained in Σ_∞) it follows that

$$\{g, \overline{g}\} \cap \mathcal{C}_i = \{g, \overline{g}\} \cap \ell_i = \{X_i, \overline{X_i}\}.$$

Thus the spread $\mathcal{S}$ is the unique regular spread $\mathcal{S}_{g\overline{g}}$ of Σ_∞ determined by lines $g, \overline{g}$ in the quadratic extension of Σ_∞. The Bruck-Bose incidence structure $\pi(\mathcal{S})$ is therefore a Desarguesian plane of order q^2 and the ruled cubic surface V is a Baer ruled cubic with respect to the regular spread $\mathcal{S}$. $\square$

In the following characterisation of Baer ruled cubic surfaces, $\mathcal{S}_{g\overline{g}}$ denotes a regular $1-$spread of $\Sigma_\infty = PG(3, q)$ determined in the usual way (see Theorem 1.9.6) by a pair of lines $g, \overline{g}$ in the quadratic extension of Σ_∞. Recall that a Baer ruled cubic surface in Bruck-Bose is a ruled cubic surface which represents via Bruck-Bose a non-affine Baer subplane of the corresponding Desarguesian plane.

Theorem 2.4.2 A characterisation of Baer ruled cubic surfaces: *Let $PG(2,q^2)$ have Bruck-Bose representation $\pi(\mathcal{S}) \subseteq PG(4,q)$, for a fixed regular spread $\mathcal{S} = \mathcal{S}_{g\overline{g}}$. A ruled cubic V_2^3 in $PG(4,q)$ is a Baer ruled cubic surface if and only if V_2^3 has line directrix an element of $\mathcal{S}$ and such that the extended ruled cubic of V_2^3 in $PG(4,q^2)$ contains the lines g and $\overline{g}$ as generators.*

Proof: ($\Rightarrow$) The necessary result follows from the proof of Theorem 2.4.1 and Theorem 1.9.6 which states that a regular spread of $PG(3,q)$ is determined by a unique pair of conjugate lines in the quadratic extension.

($\Leftarrow$) We count the number of ruled cubic surfaces in $PG(4,q)$ which have line directrix an element of $\mathcal{S}$ and such that the extended ruled cubic in $PG(4,q^2)$ contains g and $\overline{g}$ as generators. We show that the number of such ruled cubics equals the number of non-affine Baer subplanes of $PG(2,q^2)$.

Consider the Bruck-Bose representation $\pi(\mathcal{S})$ of $PG(2,q^2)$, where $\mathcal{S} = \mathcal{S}_{g\overline{g}}$. Let ℓ and t be two distinct elements of the spread $\mathcal{S}$. Let $\{X\} = g \cap t$, and $\{\overline{X}\} = \overline{g} \cap t$ denote the points of t on lines g and $\overline{g}$ respectively, in the quadratic extension. Let α be any plane of $PG(4,q)\backslash\Sigma_\infty$ which contains the line t. Let $\mathcal{C}$ be a non-degenerate conic in α such that $\mathcal{C} \cap t = \{X, \overline{X}\}$. In particular, note that t is an external line of $\mathcal{C}$ in the plane α in $PG(4,q)$.

Let m be a line of $PG(4,q)$ joining a point of ℓ with a point of $\mathcal{C}$. Consider the situation in the quadratic extension $PG(4,q^2)$: we have a line $\underset{\sim}{\ell}$ and a conic $\underset{\sim}{\mathcal{C}}$ such that the plane $\underset{\sim}{\alpha}$ of the conic is skew to the line $\underset{\sim}{\ell}$. The three lines $g, \overline{g}$ and m associate three distinct points of $\underset{\sim}{\ell}$ with three distinct points of $\underset{\sim}{\mathcal{C}}$ and so define a unique projectivity $\underset{\sim}{\phi}$ of $PGL(2,q^2)$ between $\underset{\sim}{\ell}$ and $\underset{\sim}{\mathcal{C}}$. By Section 1.7 we have a ruled cubic surface $\underset{\sim}{V_2^3}$ in $PG(4,q^2)$ with line directrix $\underset{\sim}{\ell}$, base conic $\underset{\sim}{\mathcal{C}}$ and associated projectivity $\underset{\sim}{\phi} \in PGL(2,q^2)$. Under the Fröbenius collineation $\underset{\sim}{\ell}$, $\underset{\sim}{\mathcal{C}}$ and the generator m of $\underset{\sim}{V_2^3}$ are fixed, since ℓ, $\mathcal{C}$ and m are contained in $PG(4,q)$. Also the pair of generators $\{g, \overline{g}\}$ of $\underset{\sim}{V_2^3}$ are fixed as a set. Thus $\underset{\sim}{V_2^3}$ and its image $\overline{\underset{\sim}{V_2^3}}$ under the Fröbenius collineation are a pair of ruled cubic surfaces in $PG(4,q^2)$ with the same line directrix, base conic and which share three distinct generators. Since the projectivity $\underset{\sim}{\phi}$ is determined uniquely by the three generators $g, \overline{g}$ and m, it follows that $\underset{\sim}{V_2^3}$ and $\overline{\underset{\sim}{V_2^3}}$ both have $\underset{\sim}{\phi}$ as associated projectivity. Hence,

$$\underset{\sim}{V_2^3} = \overline{\underset{\sim}{V_2^3}}$$

and therefore the points of ℓ (in $PG(4,q)$) are associated by $\underset{\sim}{\phi}$ to points of $\mathcal{C}$ (in $PG(4,q)$). Thus $\underset{\sim}{\phi}$ restricted to points of ℓ (in $PG(4,q)$) defines a ruled cubic surface V_2^3 in $PG(4,q)$ with line directrix ℓ and base conic $\mathcal{C}$. We have therefore determined that the projectivity $\underset{\sim}{\phi}$ is an element of $PGL(2,q) \subseteq PGL(2,q^2)$.

Moreover the ruled cubic V_2^3 in $PG(4,q)$ determined above has a line directrix an element of the spread $\mathcal{S}$ and has extended ruled cubic $\underset{\sim}{V_2^3}$ which contains lines g and $\overline{g}$ as generators. Since V_2^3 is determined uniquely by the choice of line directrix ℓ, base conic $\mathcal{C}$ and generator m, we count the number of such ruled cubic surfaces in $PG(4,q)$ as follows. In the following C denotes a non-degenerate conic in $AG(2,q)$ containing a fixed pair of special points, conjugate with respect to the quadratic extension.

$$\begin{aligned}
& |\mathcal{S}| \times \frac{(|\mathcal{S}|-1) \times |\{\text{ planes of } PG(4,q)\backslash\Sigma_\infty \text{ containing a spread element}\}| \times |\{C\}|}{|\{\text{conics on a ruled cubic surface in } PG(4,q)\}|} \times \frac{(q+1)^2}{q+1} \\
= \; & (q^2+1) \times \frac{(q^2)(q^2)(q^3-q^2)}{q^2} \times (q+1) \\
= \; & q^4(q^4-1).
\end{aligned}$$

Now $PG(2,q^2) = \pi(\mathcal{S})$ contains precisely this many non-affine Baer subplanes since $PG(2,q^2)$ contains $(q^2-q+1)(q^2+1)q^3(q+1)$ Baer subplanes of which $q^3(q^3+q^2+q+1)$ contain the line at infinity as a line. The result now follows. □

Corollary 2.4.3 A characterisation of Baer conics: *Let $PG(2,q^2)$ have Bruck-Bose representation $\pi(\mathcal{S}) \subseteq PG(4,q)$, for a fixed regular spread $\mathcal{S} = \mathcal{S}_{g\overline{g}}$. A non-degenerate conic $\mathcal{C}$ in $PG(4,q)$ is a Baer conic if and only if $\mathcal{C}$ is disjoint from Σ_∞ in $PG(4,q)$ and such that in the quadratic extension $\mathcal{C}$ contains a pair of conjugate points, $X, \overline{X}$ say, on the lines g and $\overline{g}$.* □

We have now completely determined the Bruck-Bose representation of the Baer substructures of $PG(2,q^2)$. The motivation for this work came from a paper by Jef Thas in which the plane model of a Miquelian inversive plane of order q is given. Consider the Miquelian inversive plane I with points the points of a line $PG(1,q^2)$ and with circles the Baer sublines of $PG(1,q^2)$. For a fixed point P of $PG(1,q^2)$ consider the internal plane $I_P \cong AG(2,q)$. By Section 1.14, the circles in I which do not contain P correspond precisely to the non-degenerate conics in I_P which contain a fixed pair $X, \overline{X}$ of of special points, conjugate with respect to the quadratic extension.

By the above characterisation of Baer conics, this plane model of Miquelian inversive planes is evident in the Bruck-Bose representation of $PG(2,q^2)$ for each affine line ℓ ($\cong PG(1,q^2)$) of $PG(2,q^2)$ and letting $\{P\} = \ell \cap \ell_\infty$.

2.5 Additional properties

In this section we present a result, valid for q even, which determines some properties of the q^2 nuclei associated with the q^2 Baer conics on a Baer ruled cubic surface in the Bruck-Bose representation of $PG(2,q^2)$ in $PG(4,q)$.

Result 2.5.1 *Let $\mathcal{B}$ be a Baer ruled cubic in the Bruck-Bose representation of $PG(2,q^2)$, q even. Let B be the Baer subplane of $PG(2,q^2)$ which is represented by $\mathcal{B}$ in Bruck-Bose. The q^2 nuclei associated with the q^2 Baer conics of $\mathcal{B}$ are distinct and lie in a plane of $PG(4,q)\backslash\Sigma_\infty$ about the line directrix p of $\mathcal{B}$.*

Proof: Let p denote the line directrix of $\mathcal{B}$. Let C_1^* be a Baer conic on $\mathcal{B}$ in plane α_1 of $PG(4,q)$. Let m_1 denote the unique spread element contained in α_1. Since q is even, we denote the nucleus of C_1^* in α_1 by N_1^*. Note that as m_1 is an external line to the conic C_1^*, the nucleus N_1^* is not incident with m_1. We have therefore that the nucleus of a Baer conic is an affine point of $PG(4,q)$, that is, a point in $PG(4,q)\backslash\Sigma_\infty$.

Let Q^* be a point on C_1^*; Q^* represents a point Q of B in $PG(2,q^2)$. In the proof of Theorem 2.2.9 it is shown that the Baer ruled cubic surface $\mathcal{B}$ is contained in the intersection of two quadric cones, namely V_3^2, with line vertex p and base the conic C_1^*, and $V_3'^2$, with point vertex Q^* and base the hyperbolic quadric in Σ_∞ determined by the regulus of spread elements which represent the points at infinity in $PG(2,q^2)$ of the lines of B incident with Q.

Let $P_1^*, P_2^*, \ldots, P_{q+1}^*$ denote the $q+1$ distinct points of the line directrix p of $\mathcal{B}$ and denote by $g_1^*, g_2^*, \ldots g_{q+1}^*$ the generators of $\mathcal{B}$, labelled so that P_i^* is incident with g_i^*, $i = 1, 2, \ldots, q+1$. Then $P_1^*C_1^*$ is a conic cone in a hyperplane, Π_3^1 say, and by Theorem 2.3.1 the intersection $\Pi_3^1 \cap \mathcal{B}$ is the union of the conic C_1^* and the generator g_1^*

of $\mathcal{B}$. In Π_3^1, the line $P_1^*N_1^*$ is the nuclear line of the conic cone $P_1^*C_1^*$, that is, every non-degenerate conic C in a plane α and such that C is the plane section $\alpha \cap P_1^*C_1^*$, has nucleus the point $\alpha \cap \{P_1^*N_1^*\}$. By considering the conic cones $P_i^*C_1^*$, $i = 1, 2, \ldots, q+1$, which are all contained in the quadric V_3^2, and by repeating the above argument, we have that each non-degenerate conic in V_3^2 has a nucleus incident with the plane $\langle N_1^*, p\rangle$ about p in $PG(4,q)$. Moreover, no such nucleus is incident with p.

Since $\mathcal{B} \subseteq V_3^2 \cap V_3^{'2}$ and since $\mathcal{B}$ contains q^2 distinct Baer conics, the q^2 associated nuclei of these conics lie in the plane $\langle N_1^*, p\rangle$. It remains to show that these nuclei are distinct, so that they constitute all q^2 points of $\langle N_1^*, p\rangle$ not incident with p.

Consider two distinct Baer conics C^* and $C^{'*}$ of $\mathcal{B}$ in planes α and α' respectively. Since α and α' represent lines of B in $PG(2,q^2)$ and since $\mathcal{B}$ is not contained in a hyperplane of $PG(4,q)$, the planes α and α' intersect in a unique point and this point of intersection is in $\mathcal{B}$ (see Theorem 1.7.2). Hence α, α' have no point in common which is not a point of $\mathcal{B}$, hence the Baer conics $C^*, C^{'*}$ have distinct nuclei. $\square$

2.6 An Alternative Approach

The Ruled Cubic Surface $\mathbf{R_2^3}$ as a model for $\mathbf{PG(2,q)}$

In this section we discuss the ruled cubic surface obtained as the projection of the Veronese Surface V_2^4 from any one of its points. The **Veronese Surface** is the variety

$$V_2^4 = \{P(x^2, xy, y^2, xz, yz, z^2) \mid (x,y,z) \text{ a point of } PG(2,q)\}$$

of $PG(5,q)$. It is of order 4 and dimension 2. (In [50, Section 25.1], the Veronese Surface is referred to as the quadric Veronesean of $PG(2,q)$.)

If we write $(x_0, x_1, x_2, x_3, x_4, x_5)$ for the coordinates of a general point in $PG(5,q)$, then V_2^4 is the complete intersection of the quadrics

$$\begin{array}{rclrcl} x_1^2 - x_0x_2 &=& 0, & x_0x_4 - x_1x_3 &=& 0,\\ x_3^2 - x_0x_5 &=& 0, & x_1x_5 - x_3x_4 &=& 0,\\ x_4^2 - x_2x_5 &=& 0, & x_2x_3 - x_1x_4 &=& 0. \end{array}$$

Moreover, the Veronese Surface contains no lines, that is, V_2^4 is a cap in $PG(5,q)$.

The map

$$\begin{array}{rrcl} \zeta: & PG(2,q) & \longrightarrow & PG(5,q) \\ \text{defined by} & (x,y,z) & \longmapsto & (x^2, xy, y^2, xz, yz, z^2) \end{array}$$

is a bijection of $PG(2,q)$ onto the Veronese Surface V_2^4; hence, V_2^4 contains exactly $|V_2^4| = q^2+q+1$ points. Also, under ζ, the points of the conic

$$ax^2 + by^2 + cz^2 + fyz + gzx + hxy \;=\; 0 \tag{2.5}$$

in $PG(2,q)$ correspond to the points of intersection of the hyperplane of $PG(5,q)$ with coordinates $[a,h,b,g,f,c]$ and the Veronese Surface. Since a curve C^r of order r in $PG(2,q)$ intersects a conic in $2r$ points (see Theorem 1.6.3), it follows that a curve C^r of $PG(2,q)$ maps by ζ into a curve of degree $2r$ on V_2^4. In particular a line of $PG(2,q)$ maps into an irreducible conic on V_2^4 and an irreducible conic of $PG(2,q)$ maps into an irreducible curve of order 4. By considering lines of $PG(2,q)$, we have

Theorem 2.6.1 Properties of the Veronese Surface:

1. [50, Theorems 25.1.7, 25.1.9] *Let ℓ be any line in $PG(2,q)$. Then $\zeta(\ell)$ is a non-degenerate conic on V_2^4. Moreover, each non-degenerate conic contained in V_2^4 is of the form $\zeta(\ell)$ for some line ℓ in $PG(2,q)$.*

   Each plane in $PG(5,q)$ which contains a non-degenerate conic on V_2^4 is called a **conic plane** *of V_2^4.*

2. [50, Theorem 25.1.11] *Any two conic planes of V_2^4 have exactly one point in common and this common point belongs to V_2^4.*

Thus V_2^4 contains q^2+q+1 non-degenerate conics, two distinct points of V_2^4 are contained in a unique non-degenerate conic on V_2^4 and two distinct non-degenerate conics on V_2^4 intersect in a unique point.

A degenerate conic which is a repeated line or two distinct lines in $PG(2,q)$ corresponds to a hyperplane section of V_2^4, where the hyperplane meets V_2^4 in a non-degenerate conic (counted doubly), or two conics with exactly one point in common, respectively.

At each point P of V_2^4, the $q+1$ tangent lines to the $q+1$ irreducible conics of V_2^4 at P span a plane $\pi(P)$; $\pi(P)$ is called the **tangent plane** of V_2^4 at P, and $\pi(P) \cap V_2^4 = \{P\}$. Also we note that by [50, Lemma 25.1.6 and Theorem 25.1.10] a projectivity of $PG(2,q)$

induces a permutation of the pointset of V_2^4 which is induced by a unique projectivity of $PG(5,q)$ which fixes V_2^4. Consult [50, Section 25.1] for further detail regarding the Veronese Surface V_2^4.

We now project V_2^4 from a point of V_2^4 to obtain a ruled cubic surface in a hyperplane of $PG(5,q)$ (see also [71, Section 3.22]).

Consider the point $P(0,0,0,0,0,1)$ on V_2^4 which is the image under ζ of the point $P'(0,0,1)$ of $PG(2,q)$. The tangent plane $\pi(P)$ to V_2^4 at P is given by the equations $x_0 = x_1 = x_2 = 0$. Project V_2^4 from P onto the hyperplane Π_4 with equation $x_5 = 0$. Since P is incident with $q+1$ conic planes of V_2^4, which pairwise meet in P, the given projection of V_2^4 from P yields q points of each of $q+1$ distinct lines $g_1, g_2, \ldots, g_{q+1}$ in Π_4. The $q+1$ remaining points on $g_1, g_2, \ldots, g_{q+1}$ (one on each line) are collinear in a line ℓ of Π_4, where ℓ is the projection from P of the tangent plane $\pi(P)$ of V_2^4 at P. The projection of V_2^4 from P onto the hyperplane Π_4 with equation $x_5 = 0$, is then the set

$$\{(x^2, xy, y^2, xz, xy, 0) \mid (x^2, xy, y^2, xz, yz, z^2) \text{ is a point of } V_2^4\}$$

of q^2+q points of Π_4. By Section 1.7, these points are q^2+q points of a ruled cubic surface with line directrix ℓ in Π_4. For the following discussion, we recall that a ruled cubic surface is defined as follows (see Section 1.7).

Definition: 2.6.1 *In $\Pi_4 = PG(4,q)$, consider a conic C^2 and a line ℓ skew to the plane of C^2. Set up a projective correspondence between them, and join corresponding points by lines. The ruled surface so obtained is of order 3, and is denoted R_2^3.*

By choosing the coordinate system in $PG(4,q)$, let point $(0,0,0,x,y)$ (where $x, y \in GF(q)$, $(x,y) \neq (0,0)$) on ℓ correspond to point $(x^2, xy, y^2, 0, 0)$ on C^2. Thus R_2^3 is

$$\{(x^2, xy, y^2, zx, zy);\ x, y \in GF(q),\ (x,y) \neq (0,0),\ z \in GF(q) \cup \{\infty\}\}$$

Define

$$\sigma : R_2^3 \to PG(2,q)$$

by

$$\begin{aligned} (x^2, xy, y^2, zx, zy) &\longmapsto (x, y, z) \\ \ell\ (z = \infty) &\longmapsto (0,0,1) \end{aligned}$$

Thus σ contracts ℓ into the point $(0,0,1)$, and

$$\sigma : R_2^3 \backslash \ell \to PG(2,q) \backslash \{(0,0,1)\}$$

is a bijection, by definition of σ. In an abuse of notation, we shall use σ to denote the map between $R_2^3 \backslash \ell$, in $PG(4,q)$, and $PG(2,q) \backslash \{(0,0,1)\}$, in both directions.

First note that under σ our original conic

$$C^2 = \{(x^2, xy, y^2, 0, 0) \,|\, x, y \in GF(q),\ (x,y) \neq (0,0)\}$$

on R_2^3 is mapped to the line $z = 0$ of $PG(2,q)$.

We now consider the image in $PG(4,q)$, under σ, of lines and conics in $PG(2,q)$. One method is to use the bijection ζ of $PG(2,q)$ onto the Veronese Surface V_2^4 and then project V_2^4 from $P(0,0,0,0,0,1)$ onto the ruled cubic R_2^3; for clarity we explicitly determine these images using our bijection σ of $PG(2,q) \backslash \{(0,0,1)\}$ onto $R_2^3 \backslash \ell$.

First consider a line $ax + by + cz = 0$ $(a, b, c \in GF(q),\ c \neq 0)$ in $PG(2,q)$, not through $(0,0,1)$. A parametric form of this line is

$$(x, y, z) \equiv \left(1, t, \frac{-a - bt}{c}\right),$$

where $t \in GF(q) \cup \{\infty\}$. Using the map σ we have

$$\begin{aligned}(x^2, xy, y^2, zx, zy) &= (x_0, x_1, x_2, x_3, x_4) &\quad (2.6)\\ &= \left(1, t, t^2, \frac{-a - bt}{c}, \frac{-at - bt^2}{c}\right). &\quad (2.7)\end{aligned}$$

Thus the image in $PG(4,q)$ of the line of $PG(2,q)$ is the set of points $(x_0, x_1, x_2, x_3, x_4)$ on R_2^3 with the parametrisation (2.7) in quadratic functions of t. These points (2.7) also satisfy,

$$\begin{aligned} ax_0 + bx_1 + cx_3 &= 0 = \ x(ax + by + cz) \\ \text{and}\ \ ax_1 + bx_2 + cx_4 &= 0 = \ y(ax + by + cz); \end{aligned} \qquad (2.8)$$

the equations of two distinct hyperplanes, which intersect in a plane of $PG(4,q)$.

By Section 1.5, we have therefore that the image of the line $ax + by + cz = 0$ $(c \neq 0)$ is the set of points $(x_0, x_1, x_2, x_3, x_4)$ (2.7) of a conic on R_2^3 lying in the plane of $PG(4,q)$ defined by equations (2.8). Furthermore, the conic (2.7) is non-degenerate since it is the pre-image of the non-degenerate conic $\{(1,t,t^2,0,0)|\ t \in GF(q) \cup \{\infty\}\}$, in the plane $x_3 = x_4 = 0$ of $PG(4,q)$, under the projectivity of $PG(4,q)$ defined by the matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a & b & 0 & c & 0 \\ 0 & a & b & 0 & c \end{bmatrix}.$$

Note: 2.6.2 *In $PG(2,q)$, projectivities fixing $(0,0,1)$ induce a transformation on (x^2, xy, y^2, zx, zy) in $PG(4,q)$. As a consequence, the "conics" in $PG(4,q)$, which are the image under σ of lines $ax+by+cz=0$ $(a,b,c \in GF(q),\ c \neq 0)$, are projectively equivalent.*

Proof: Consider two lines ℓ_1 and ℓ_2 in $PG(2,q)$ with equations $ax+by+cz=0,\ c \neq 0$, and $a'x+b'y+c'z=0,\ c' \neq 0$, respectively; note that $(0,0,1)$ is not incident with either of these two lines. A projectivity fixing $(0,0,1)$ in $PG(2,q)$ and which maps ℓ_1 to ℓ_2 is given by

$$\begin{bmatrix} 1 \\ t \\ \frac{-a'-b't}{c'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{a}{c}-\frac{a'}{c'} & \frac{b}{c}-\frac{b'}{c'} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \frac{-a-bt}{c} \end{bmatrix}.$$

The corresponding projectivity in $PG(4,q)$ is therefore given by

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ \frac{-a'-b't}{c'} \\ \frac{-a't-b't^2}{c'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{a}{c}-\frac{a'}{c'} & \frac{b}{c}-\frac{b'}{c'} & 0 & 1 & 0 \\ 0 & \frac{a}{c}-\frac{a'}{c'} & \frac{b}{c}-\frac{b'}{c'} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \frac{-a-bt}{c} \\ \frac{-at-bt^2}{c} \end{bmatrix}.$$

□

Now consider a line $ax+by=0$ $(a,b \in GF(q),\ a,b$ not both zero) in $PG(2,q)$, that is, a line incident with $(0,0,1)$. Imposing the condition $ax+by=0$ on (2.6) gives

$$\begin{aligned} ax_0 + bx_1 &= 0, \\ ax_1 + bx_2 &= 0, \\ \text{and } ax_3 + bx_4 &= 0 \end{aligned} \tag{2.9}$$

which define the equation of a line in $PG(4,q)$ contained in R_2^3 and therefore incident with ℓ.

Thus the image of the line $ax + by = 0$ under σ is a (**generator**) line of R_2^3.

Consider the following linear combination of the equations (2.9),

$$\begin{aligned} a(ax_0 + bx_1) + b(ax_1 + bx_2) + c(ax_3 + bx_4) &= 0, \\ \text{that is} \quad a(ax_0 + bx_1 + cx_3) + b(ax_1 + bx_2 + cx_4) &= 0, \end{aligned}$$

which is a linear combination of the equations (2.8) and is therefore the equation of the hyperplane of $PG(4,q)$ which meets R_2^3 in the union of the conic (2.7) and the generator line (2.9).

A pencil of lines in $PG(2,q)$ through a fixed point $(x,y,z) \neq (0,0,1)$ corresponds to a collection of q conics on the surface R_2^3 together with a generator line of R_2^3. The planes containing these q conics generate a quadric cone in $PG(4,q)$ with point vertex (x^2, xy, y^2, zx, zy) in R_2^3; this is proved as follows, the plane (2.8), where a, b, c satisfy $ax + by + cz = 0$ for some fixed (x,y,z), passes through the point $(x_0, x_1, x_2, x_3, x_4)$ of $PG(4,q)$ if and only if

$$\begin{vmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \\ x & y & z \end{vmatrix} = 0$$

$$\text{that is, if and only if} \quad x(x_1x_4 - x_2x_3) + y(x_1x_3 - x_0x_4) + z(x_0x_2 - x_1^2) = 0. \tag{2.10}$$

It is easy to show that the quadric with equation (2.10) has the (fixed) point (x^2, xy, y^2, zx, zy) as a singular point, by showing the first partial derivatives of the defining polynomial are identically zero at (x^2, xy, y^2, zx, zy), hence by considering all possible quadrics in $PG(4,q)$, the quadric (2.10) in $PG(4,q)$ is a quadric cone with point vertex (x^2, xy, y^2, zx, zy) and base a hyperbolic quadric.

It also follows that R_2^3 is the complete intersection of the three quadrics given by,

$$x_1x_4 - x_2x_3 = 0, \quad x_1x_3 - x_0x_4 = 0, \quad \text{and} \quad x_0x_2 - x_1^2 = 0. \tag{2.11}$$

Through a point of $PG(4,q)$ not contained on the surface R_2^3, that is, not satisfying (2.11), there passes a unique plane of the system (2.8), namely the plane such that

$$a : b : c = x_1x_4 - x_2x_3 : x_1x_5 - x_0x_4 : x_0x_2 - x_1^2. \tag{2.12}$$

This is a plane of an irreducible conic unless $x_0x_2 - x_1^2 = 0$, in which case, the conic is the union of the line directrix ℓ and a generator line of R_2^3.

Hyperplane sections of R_2^3

As stated in Theorem 1.7.2 and in Section 1.6, the intersection of a hyperplane of $PG(4,q)$ with the ruled cubic surface R_2^3 is a cubic curve; such a cubic curve is possibly reducible, in which case the intersection is either the union of a conic and a generator line of R_2^3 or is the union of three lines of R_2^3 (not necessarily distinct or belonging to $GF(q)$).

By considering R_2^3 as the projection of the Veronese Surface V_2^4 in $PG(5,q)$ from a point of V_2^4, it is possible to determine the nature of the hyperplane sections of R_2^3 in $PG(4,q)$. We work through this explicitly using our bijection σ between $PG(2,q)\backslash\{(0,0,1)\}$ and $R_2^3\backslash\{\ell\}$.

Let $\underset{\sim}{\lambda}= [\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4]$, $\lambda_i \in GF(q)$ not all zero, be the coordinates of a hyperplane of $PG(4,q)$. The hyperplane intersects R_2^3 in points (x^2, xy, y^2, xz, yz) satisfying

$$\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2 + \lambda_3 zx + \lambda_4 zy = 0 \tag{2.13}$$

which corresponds to the points of a conic through $(0,0,1)$ in $PG(2,q)$.

The number of conics in $PG(2,q)$ through $(0,0,1)$ equals

$$q^4 - q^2 + \binom{q+1}{2} + \binom{q}{2} + q^2(q+1) + q + 1 = q^4 + q^3 + q^2 + q + 1,$$

counting non-degenerate conics and the four types of degenerate conics incident with $(0,0,1)$. Hence there is a one-to-one correspondence between the conics incident with $(0,0,1)$ in $PG(2,q)$ and the hyperplane sections of the cubic surface R_2^3 in $PG(4,q)$. Thus, R_2^3 is the projective model of the system of conics incident with $(0,0,1)$ in the plane.

We now consider each case in more detail.

Case 1: (2.13) is the equation of a non-degenerate conic in $PG(2,q)$.

By considering the first polar of $(0,0,1)$ with respect to (2.13), the tangent line to the conic at $(0,0,1)$ is given by

$$\lambda_3 x + \lambda_4 y = 0.$$

The condition for the hyperplane $\underset{\sim}{\lambda}$ to intersect ℓ in $(0,0,0,1,m)$, $m \in GF(q) \cup \{\infty\}$, is

$$\begin{aligned} \lambda_3 + \lambda_4 m &= 0 \\ \text{that is, } m &= -\tfrac{\lambda_3}{\lambda_4} \end{aligned}$$

and so in $PG(2,q)$, from above, $y = mx$ is the tangent line to the conic (2.13) at $(0,0,1)$.

Case 2: (2.13) is degenerate and $\lambda_3 = \lambda_4 = 0$.

The hyperplane $\underset{\sim}{\lambda} = [\lambda_0, \lambda_1, \lambda_2, 0, 0]$ in $PG(4, q)$ contains the line ℓ of R_2^3. The equation (2.13) becomes

$$\begin{aligned} \lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2 &= 0 \\ \text{that is, } (y - m_1 x)(y - m_2 x) &= 0 \end{aligned}$$

and the lines $y = m_1 x$ and $y = m_2 x$ may be distinct or coincident in $PG(2, q)$ or lie in an extension of the base field.

Case 2a: $m_1 = m_2 \in GF(q) \cup \{\infty\}$

The hyperplane $\underset{\sim}{\lambda}$ intersects R_2^3 in ℓ together with a unique generator line counted twice. In $PG(2, q)$, we have a degenerate conic on $(0, 0, 1)$ consisting of a (repeated) line $y = m_1 x$.

Case 2b: $m_1 \neq m_2$, $m_1, m_2 \in GF(q) \cup \{\infty\}$

The hyperplane $\underset{\sim}{\lambda}$ intersects R_2^3 in ℓ together with two generator lines. In $PG(2, q)$, we have a degenerate conic on $(0, 0, 1)$ consisting of two distinct lines $y = m_1 x$ and $y = m_2 x$.

Case 2c: m_1 and m_2 are conjugate elements of $GF(q^2)\backslash GF(q)$.

The hyperplane $\underset{\sim}{\lambda}$ intersects R_2^3 in ℓ together with two generator lines in the quadratic extension, that is, ℓ together with two generators of the extended ruled cubic surface (as discussed in Section 2.4.1). In $PG(2, q)$, we have a degenerate conic on $(0, 0, 1)$ consisting of two conjugate lines in the quadratic extension.

Case 3: (2.13) is degenerate and $\underset{\sim}{\lambda} = [-ma, a, b, -m, 1]$.

The equation (2.13) degenerates to become

$$(y - mx)(ax + by + z) = 0$$

for some $m \in GF(q) \cup \{\infty\}$, $a, b \in GF(q)$.

From above, the hyperplane $\underset{\sim}{\lambda}$ intersects R_2^3 in a generator line and a conic on R_2^3. In $PG(2, q)$, we have a degenerate conic on $(0, 0, 1)$ consisting of two lines, $y = mx$ (incident with $(0, 0, 1)$) and $ax + by + z = 0$ (not incident with $(0, 0, 1)$).

We now consider in more detail a non-degenerate conic through $P'(0, 0, 1)$ in the plane $PG(2, q)$. By Section 1.5, such a conic has points with parametric coordinates of the form $(f_0(t), f_1(t), f_2(t)) = (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t + c_3 t^2)$, $a_i, b_i \in GF(q)$, where P' is the point of the conic associated with the parameter $t = \infty$. Since the conic is non-degenerate it is a normal rational curve in the plane and therefore the polynomials f_i are linearly independent and have non non-trivial common factor. When the coordinates of the points of the conic are substituted into (2.6) we obtain a parametrisation of the corresponding points $(x_0, x_1, x_2, x_3, x_4)$ on R_2^3 in which x_0, x_1, x_2 are quadratic in t and x_3, x_4 are cubic in t, namely

$$\begin{aligned}
x_0 &= x^2 = & (a_1 + b_1 t)^2, \\
x_1 &= xy = & (a_1 + b_1 t)(a_2 + b_2 t), \\
x_2 &= y^2 = & (a_2 + b_2 t)^2, \\
x_3 &= zx = & (a_1 + b_1 t)(a_3 + b_3 t + c_3 t^2), \\
x_4 &= zy = & (a_2 + b_2 t)(a_3 + b_3 t + c_3 t^2).
\end{aligned}$$

Thus the conic corresponds to a rational cubic curve on the surface R_2^3. By case 1, these $q+1$ points lie in a hyperplane section of R_2^3. Note that all possible reducible cubic curves on R_2^3 have been considered in cases 2 and 3, with each such cubic curve obtained as a hyperplane section of R_2^3. Therefore by Section 1.5 and Theorem 1.7.2, the rational cubic curve above is an (irreducible) twisted cubic curve. Note that the point with parameter $t = \infty$ of this twisted cubic lies on the line directrix ℓ of R_2^3.

Alternatively, by the known results concerning the Veronese Surface V_2^4 in $PG(5, q)$ quoted above, we note that such a non-degenerate conic containing $P'(0, 0, 1)$ in $PG(2, q^2)$ corresponds to a rational quartic curve containing the point $P(0, 0, 0, 0, 0, 1)$ with points

$$(x_0, x_1, x_2, x_3, x_4, x_5) \; = \; (f_0^2(t), f_0(t)f_1(t), f_1^2(t), f_0(t)f_2(t), f_1(t)f_2(t), f_2^2(t))$$

and which constitute a hyperplane section of the Veronese Surface V_2^4 of $PG(5, q)$ containing P. On projection from P onto R_2^3, this quartic curve containing P projects to a cubic curve contained in a $3-$space.

Consider a non-degenerate conic in $PG(2, q)$ which does not contain $P'(0, 0, 1)$. For example $z^2 = xy$ has points with coordinates $(1, t^2, t)$, where $t \in GF(q) \cup \{\infty\}$. When we substitute these coordinates into (2.6) we obtain the coordinates

$$(x_0, x_1, x_2, x_3, x_4) \; = \; (1, t^2, t^4, t, t^3)$$

of points of a $4-$dimensional normal rational curve contained on the surface R_2^3 and disjoint from the line directrix ℓ.

We now show that every non-degenerate conic in $PG(2, q)$ which does not contain the point $P'(0, 0, 1)$ is mapped by σ to a $4-$dimensional normal rational curve on R_2^3 disjoint from the line directrix ℓ.

Curves on R_2^3

By Section 1.6, every curve C^{2m} in $PG(2,q)$, of order $2m$, with m-fold point at $(0,0,1)$ has equation

$$z^m u_m + z^{m-1} u_{m+1} + \ldots + u_{2m} = 0 \tag{2.14}$$

where $u_i \equiv u_i(x,y)$ $(i = m, \ldots, 2m)$ is homogeneous of degree i in x and y. As we shall show, the equation (2.14) can be expressed as a monomial of degree m in x^2, xy, y^2, zx, yz, so that the equation of the curve transforms by (2.6) into that of a primal (or hypersurface) V_3^m of order m in $PG(4,q)$. The image of the curve C^{2m} in $PG(2,q)$ under σ is $V_3^m \cap R_2^3 = C_1^{3m}$, a curve of order $3m$ in $PG(4,q)$ on the ruled cubic R_2^3.

For example, the equation of C^{2m} with m-fold point at $P'(0,0,1)$ is

$$z^m u_m + \ldots + z^0 u_{2m} = 0$$

where $z^{m-i} u_{m+i} = z^{m-i}(\rho_0 x^{m+i} + \rho_1 x^{m+i-1} y + \ldots + \rho_j x^{m+i-j} y^j + \ldots + \rho_{m+i} y^{m+i})$ for $i = 0, \ldots, m$ and $j = 0, \ldots, m+i$ for a fixed i and some $\rho_j \in GF(q)$.

Now

$$\begin{aligned} z^{m-i} x^{m+i-j} y^j &= (zx)^{m-i} x^{2(i-j)} (xy)^j \quad \text{for } i \geq j \\ &= (zx)^{m-j} (zy)^{j-i} (xy)^i \quad \text{for } i < j \end{aligned}$$

hence (2.14) can be expressed as a monomial of degree m in x^2, xy, y^2, zx, yz.

If a curve C^n in $PG(2,q)$ has an m-fold point at $(0,0,1)$, but is of order $n < 2m$, it can be turned into a curve of order $n' = 2m$ with an m-fold point at $(0,0,1)$ by adding $2m-n$ lines not through $(0,0,1)$. We then obtain a curve C^{2m} in $PG(2,q)$ with an equation given by the product of the equation defining C^n and the equations of the $2m-n$ lines. Since the lines do not pass through $(0,0,1)$, the multiplicity of $P'(0,0,1)$ does not change. As stated above, the equation of C^{2m} can be expressed as a monomial of degree m in x^2, xy, y^2, zx, zy, that is, a polynomial which defines the variety of intersection $V_3^m \cap R_2^3$ of a hypersurface V_3^m in $PG(4,q)$ and the ruled cubic surface R_2^3. Thus C^{2m} has image in $PG(4,q)$ given by this intersection of R_2^3 with the hypersurface V_3^m, namely, a variety V_1^{3m} which consists of the image of C^n together with $2m-n$ (Baer) conics.

If a curve C^n has a m-fold point at $(0,0,1)$, but is of order $n > 2m$, the addition of $n - 2m$ lines through $(0,0,1)$ makes it of order $n + n - 2m = 2(n-m)$ with a $(n - 2m + m = n - m)$-fold point at $(0,0,1)$. Thus yielding a curve $C^{2(n-m)}$ of order

$2(n-m)$ with a $(n-m)$-fold point at $(0,0,1)$. In $PG(4,q)$, the curve $C^{2(n-m)}$ has image $V_3^{n-m} \cap R_2^3 = C_1^{3(n-m)}$ which degenerates into the image of C^n and $n-2m$ generator lines.

We summarise these results in a theorem.

Theorem 2.6.3 *A curve C^n of order n in $PG(2,q)$ with $m-$fold point at $(0,0,1)$ has image in $PG(4,q)$ on the surface R_2^3, given by σ as follows*

$$\sigma(C^n) = \begin{cases} C_1^{3m} & \textit{if } n = 2m, \\ C_1^{2n-m} & \textit{if } n < 2m, \\ C_1^{2n-m} & \textit{if } n > 2m. \end{cases}$$

□

In particular, we note the following examples:

1. An irreducible conic C^2 through $(0,0,1)$, with $n=2$, $m=1$ and $n=2m$ in this case, has as image $V_3^1 \cap R_2^3 = V_1^3$, a twisted cubic curve on the surface R_2^3, as discussed above.

2. An irreducible conic C^2 not through $(0,0,1)$, with $n=2$, $m=0$ and $n>2m$ in this case, together with two lines through $(0,0,1)$ forms a quartic curve C_1^4 with 2-fold point at $(0,0,1)$. In $PG(4,q)$, the image is $V_3^2 \cap R_2^3 = V_1^6 = V_1^1 \cup V_1^1 \cup V_1^4$, that is two generator lines of R_2^3 together with a quartic curve on R_2^3. Thus $\sigma(C^2)$ is a quartic curve C_1^4 with $q+1$ points and this quartic curve C_1^4 is a normal rational curve for the following reasons:
   (a) C_1^4, being the image of a conic not through $(0,0,1)$, has no point on the directrix ℓ of R_2^3.
   (b) C_1^4 therefore cannot have a linear component n, since any line on R_2^3 has at least one point in common with ℓ.
   (c) If C_1^4 is reducible, then by (b) it can only be reducible to a pair of conics $\mathcal{C}_1, \mathcal{C}_2$. But these conics on R_2^3 are the image under σ of lines $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ of $PG(2,q)$; a contradiction, since our original conic C^2 in $PG(2,q)$ is irreducible.

(d) Consider any hyperplane S_3^1 of $PG(4,q)$. Then $C_1^4 \cap S_3^1 \subseteq R_2^3 \cap S_3^1$. But $R_2^3 \cap S_3^1$ is the image of a conic $\mathcal{C}$ in $PG(2,q)$ through $(0,0,1)$ (by our results above). Since $|\mathcal{C} \cap C^2| \leq 4$, we have $|S_3^1 \cap C_1^4| \leq 4$. Hence C_1^4 is properly contained in $PG(4,q)$ for all $q > 3$.

(e) If $q = 2$ or 3, the result follows.

We now prove that every 4−dimensional normal rational curve contained in R_2^3 and disjoint from the line directrix ℓ is the image under σ of a non-degenerate conic in $PG(2,q)$ which contains $P'(0,0,1)$

Consider a normal rational curve C^4 of $PG(4,q)$. Then, by Section 1.5, the curve C^4 is given by

$$\{P(t) = P(f_0(t), f_1(t), \dots, f_4(t)) \mid t \in GF(q) \cup \{\infty\}\}$$

where

(i) each polynomial f_i has degree at most 4, $i = 0,1,2,3,4$, with at least one of the polynomials having degree 4.

(ii) the polynomials f_0, f_1, f_2, f_3, f_4 are linearly independent with no non-trivial common factor.

(iii) C^4 is projectively equivalent to $\{(t^4, t^3, t^2, t, 1) \mid t \in GF(q) \cup \{\infty\}\}$.

Suppose that C^4 is a normal rational curve of $PG(4,q)$, lying on the ruled cubic R_2^3. Therefore C^4 is given by

$$\{(f(t)^2, f(t)g(t), g(t)^2, f(t)h(t), g(t)h(t));\ t \in GF(q) \cup \{\infty\}\}$$

where

(i) f, g are of degree at most 2 and h is of degree at most 3, since at least one of f, g is non-constant.

(ii) f^2, fg, g^2, fh, gh are linearly independent quartic polynomials, with at least one having degree 4, and have no non-constant polynomial as common divisor.

Note that f and g have no non-constant polynomial as a common divisor since otherwise f^2, fg, g^2, fh, gh have a common non-trivial divisor.

Define $\mathcal{C}$ in $PG(2,q)$ to be the image under σ of C^4 as follows

$$\sigma(C^4) = \mathcal{C} = \{(f(t), g(t), h(t));\ t \in GF(q) \cup \{\infty\}\}$$

Suppose $\mathcal{C}$ contains $P'(0,0,1)$. We consider each case separately.

Possibility 1: There exists $t_1 \in GF(q)$ such that $f(t_1) = g(t_1) = 0$. In that case $p = t - t_1$ divides both f and g, a contradiction, since by condition (ii) above, the polynomials f^2, fg, g^2, fh, gh have no non-trivial common factor.

Possibility 2: $t = \infty$ corresponds to (0,0,1), in which case the degree of h is strictly greater than the degree of f and strictly greater than the degree of g. By considering properties (i), (ii) above, this implies that h has degree 3, the degree of f is 0 or 1 and the degree of g is 1 or 0. In this case $\sigma(C^4) = \mathcal{C}$ is not a conic since it is not a normal rational curve (of order 2) in $PG(2,q)$.

Example: $f(t) = 1$, $g(t) = t$ and $h(t) = t^3$ so that

$$C^4 = \{(1, t, t^2, t^3, t^4) |\ t \in GF(q) \cup \{\infty\}\}$$

in which case $\mathcal{C} = \{(1, t, t^3) |\ t \in GF(q) \cup \{\infty\}\}$.

We have:

Theorem 2.6.4 *If a normal rational curve C^4 lies on R_2^3 and if, using the above notation for C^4, the degree of h is less than or equal to 2, then there is no point of C^4 on the line directrix ℓ of R_2^3.*

Suppose C^4 is the normal rational curve contained in R_2^3 with the above notation, and suppose h has degree less than or equal to 2. It follows that at least one of f, g has degree 2. Moreover, f, g, h are polynomials of degree at most 2, with at least one of degree 2. The polynomials have no non-trivial common factor, since f^2, fg, g^2, fh, gh have no non-trivial common factor, and therefore $\mathcal{C} = \{(f(t), g(t), h(t)) \mid t \in GF(q) \cup \{\infty\}\}$, by the definition of a normal rational curve in Section 1.5, is a non-degenerate conic in $PG(2,q)$.

Hence,

Theorem 2.6.5 *Every normal rational curve C^4 lying on R_2^3 and disjoint from the line directrix ℓ is mapped by σ to an irreducible conic of $PG(2,q)$ which does not contain $(0,0,1)$.*

Summarising our results in one theorem we obtain the following bijective correspondence.

Theorem 2.6.6 *There are as many normal rational curves of $PG(4,q)$ on R_2^3 which are disjoint from ℓ as there are irreducible conics of $PG(2,q)$ not through $(0,0,1)$. This number is $q^5 - q^4$.*

Proof: An irreducible conic in $PG(2,q)$ not containing $(0,0,1)$ maps under σ to a normal rational curve on R_2^3 with no point on ℓ (by the above Note 2.)

By Theorem 2.6.5 the converse is true.

The number of irreducible conics in $PG(2,q)$ is $q^5 - q^2$. The number of irreducible conics in $PG(2,q)$ on $(0,0,1)$ is $q^4 - q^2$; therefore the number of irreducible conics in $PG(2,q)$ not through $(0,0,1)$ is $q^5 - q^4$. $\square$

2.7 A look at $\mathbf{PG(2,q^4)}$ in Bruck-Bose

In this section we investigate more closely the Bruck-Bose representation of $PG(2,q^4)$. The plane $PG(2,q^4)$ has a 4-dimensional Bruck-Bose representation over $GF(q^2)$, which we shall denote by Π_{4,q^2}, and an 8-dimensional Bruck-Bose representation over $GF(q)$, which we denote by $\Pi_{8,q}$.

Consider a line ℓ, distinct from ℓ_∞, of $PG(2,q^4)$. As we have noted earlier in this chapter, a Baer subline b_{q^2+1} of ℓ which contains no point on ℓ_∞ is represented in Π_{4,q^2} by a non-degenerate conic $b^*_{q^2+1}$ in the plane ℓ^*, which represents ℓ. Furthermore, the conic $b^*_{q^2+1}$ is disjoint from Σ_∞, the hyperplane at infinity of Π_{4,q^2}.

In this section we show that the Baer sublines b_{q+1} of b_{q^2+1} are each represented in Π_{4,q^2} by a subconic of $b^*_{q^2+1}$ and each such subconic b^*_{q+1} is contained in a Baer subplane of the plane ℓ^* in Π_{4,q^2}. Each Baer subplane of this type intersects Σ_∞ in a unique point in Π_{4,q^2}.

Using this result together with Corollary 2.3.5 we obtain that such a Baer subline b_{q^2+1} in the 8-dimensional Bruck-Bose representation is a set of q^2+1 points in a 4-space ℓ^{**} (the representation in $\Pi_{8,q}$ of the line ℓ) of $\Pi_{8,q}$, which contains at least q^3+q 4-dimensional normal rational curves. We note that in 8-dimensional Bruck-Bose, b_{q^2+1} is in fact a $(q^2+1)-$cap in the $4-$space ℓ^{**} of $\Pi_{8,q}$ and since b_{q^2+1} is disjoint from ℓ_∞ in $PG(2,q^4)$ the corresponding $(q^2+1)-$cap is disjoint from the hyperplane at infinity in $\Pi_{8,q}$.

We begin by establishing a coordinate system for $PG(2,q^4)$ in 4-dimensional Bruck-Bose as in Section 1.10.4. Let ℓ_∞ be the line of $PG(2,q^4)$ with equation $z=0$. Let ℓ be the line $y=0$ and denote by $P=P(1,0,0)$ the point of intersection of the lines ℓ and ℓ_∞. Each point of $\ell-\{P\}$ has coordinates of the form $(a,0,1)$ where $a \in GF(q^4)$.

Let $GF(q^4)=GF(q^2)(\alpha)$, where $\alpha \in GF(q^4)\backslash GF(q^2)$ has minimal polynomial x^2+x+c for some fixed element c in $GF(q^2)$. Then every element b in $GF(q^4)$ can be uniquely expressed in the form $b=b_0+\alpha b_1$ where the b_i are in $GF(q^2)$.

Following Section 1.10.4, in Bruck-Bose ℓ is the plane, which we denote by ℓ^*, with affine points $(a_0,a_1,0,0,1)$, $a_i \in GF(q^2)$, where each point $(a=a_0+\alpha a_1,0,1)$ is a point of ℓ. The spread element in ℓ^* is $J(\infty)=\langle 1,\alpha\rangle \equiv \langle(1,0,0,0,0),(0,1,0,0,0)\rangle$.

For convenience, from now on we shall represent the coordinates of points of ℓ and ℓ^* as follows:

$$\begin{aligned} \ell &= \{P(1,0)\} \cup \{(a,1) \mid a \in GF(q^4)\} \\ \ell^* &= \{J(\infty)=\langle(1,0,0),(0,1,0)\rangle\} \cup \{(a_0,a_1,1) \mid a_0,a_1 \in GF(q^2)\}. \end{aligned}$$

The Baer sublines of ℓ which contain the point $P(1,0)$ are represented in Bruck-Bose by the lines of ℓ^* distinct from $J(\infty)$. We may divide these Baer sublines into two classes as follows.

Baer sublines of ℓ which contain P	Lines of $\ell^*-\{J(\infty)\}$
(i) $\{\theta(1,0)+(a,1)\} \cup \{(1,0)\}$, where $a \in GF(q^4)$ is fixed and $\theta \in GF(q^2)$.	(i) The lines $y=a_1$, where $a_1 \in GF(q^2)$ is fixed.
(ii) $\{\theta(b+\alpha,0)+(a,1)\} \cup \{(b+\alpha,0)\}$, where $b \in GF(q^2)$ is fixed, $a \in GF(q^4)$ is fixed and $\theta \in GF(q^2)$.	(ii) The lines $x=by+(a_0-ba_1)$, for all $b,a_0,a_1 \in GF(q^2)$

Note that there are q^2 distinct lines of type (i) and q^4 distinct lines of type (ii).

Let b_{q^2+1} denote the Baer subline of ℓ with the following points.

$$\{(1,1)\} \cup \{\begin{bmatrix} \alpha & 0 \\ \alpha & 1 \end{bmatrix}\begin{pmatrix} \theta \\ 1 \end{pmatrix} \mid \theta \in GF(q^2)\}.$$

$P(1,0)$ is not a point of b_{q^2+1} since if $(1,0) \equiv (\alpha\theta, \alpha\theta+1)$, then $\alpha\theta+1=0$ implies α is an element of $GF(q^2)$, a contradiction.

Consider a point $(\alpha\theta, \alpha\theta + 1) \equiv (\alpha\theta(\alpha\theta + 1)^{-1}, 1)$ of b_{q^2+1}. Suppose that $(\alpha\theta + 1)^{-1} = b_0 + \alpha b_1$ for a necessarily unique pair $b_0, b_1 \in GF(q^2)$. Then,

$$(b_0 + \alpha b_1)(1 + \alpha\theta) = 1$$

and on solving the system,

$$\begin{array}{rcrcl} b_0 & - & c\theta b_1 & = & 1 \\ \theta b_0 & + & b_1(1-\theta) & = & 0 \end{array}$$

we obtain $b_0 = (\theta - 1)(-c\theta^2 + \theta - 1)^{-1}$ and $b_1 = \theta(-c\theta^2 + \theta - 1)^{-1}$. Hence we may write the coordinates $(\alpha\theta(b_0 + \alpha b_1), 1)$ as

$$\begin{aligned} & (\alpha\theta((\theta - 1)(-c\theta^2 + \theta - 1)^{-1} + \alpha\theta(-c\theta^2 + \theta - 1)^{-1}), 1) \\ = \; & (-c\theta^2(-c\theta^2 + \theta - 1)^{-1} - \alpha\theta(-c\theta^2 + \theta - 1)^{-1}, 1) \end{aligned}$$

Therefore in Bruck-Bose, the Baer subline b_{q^2+1} is the set $b^*_{q^2+1}$ of points of ℓ^* with coordinates,

$$\{(1,0,1)\} \cup \{(-c\theta^2(-c\theta^2 + \theta - 1)^{-1}, -\theta(-c\theta^2 + \theta - 1)^{-1}, 1) \mid \theta \in GF(q^2)\}.$$

Therefore $b^*_{q^2+1}$ is the image of the conic $y^2 = xz$ in ℓ^* under a projectivity of ℓ^* since $b^*_{q^2+1}$ is given by,

$$\{(1,0,1)\} \cup \{ \begin{bmatrix} -c & 0 & 0 \\ 0 & -1 & 0 \\ -c & 1 & -1 \end{bmatrix} \begin{pmatrix} \theta^2 \\ \theta \\ 1 \end{pmatrix} \mid \theta \in GF(q^2)\}.$$

If we now let b_{q+1} be the subset of b_{q^2+1} of points with parameter $\theta \in GF(q) \cup \{\infty\}$, then b_{q+1} is a Baer subline of b_{q^2+1}. From the above calculations we have that b_{q+1} in Bruck-Bose is a subconic b^*_{q+1} of the conic $b^*_{q^2+1}$. Moreover, b^*_{q+1} is projectively related to the conic $\{(\theta^2, \theta, 1) \mid \theta \in GF(q) \cup \{\infty\} \}$ in the real Baer subplane $PG(2,q)$ of ℓ^*.

Since b_{q^2+1} does not contain P, $b^*_{q^2+1}$ and hence b^*_{q+1} has no point in Σ_∞. Suppose the Baer subplane B of ℓ^* which contains b^*_{q+1} intersects Σ_∞ in $q+1$ points. Then since ℓ^* is a quadratic extension of B, the conic $b^*_{q^2+1}$ would necessarily intersect Σ_∞ in two distinct and conjugate points, a contradiction. Thus, the Baer subplane B intersects Σ_∞ in a unique point. Another way to see this is to consider the projectivity which maps $PG(2,q)$ on to the Baer subplane B. A point (x,y,z) of $PG(2,q)$ is mapped by the projectivity to a point of Σ_∞ if and only if $-cx + y - z = 0$. Since c is an element of $GF(q^2)$, the line $[-c,1,-1]$ is not a line of $PG(2,q)$ and so intersects $PG(2,q)$ in a unique point. Thus B contains a unique point on Σ_∞.

Now consider another Baer subline $b^1 \neq b_{q+1}$ of b_{q^2+1}.

Any Baer subline of ℓ has stabiliser a subgroup of $PGL(2,q^4)$ isomorphic to $PGL(2,q^2)$. In particular the real Baer subline,

$$b^0 = \{(1,0)\} \cup \{(\theta,1) \mid \theta \in GF(q^2)\}$$

of ℓ has stabiliser $PGL(2,q^2)$. $PGL(2,q^2)$ acts triply transitively on the points of b^0 and so acts transitively on the Baer sublines of b^0. Hence any Baer subline of b^0 is given by,

$$\begin{pmatrix} \theta' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a\theta+b}{c\theta+d} \\ 1 \end{pmatrix}$$

for some choice of a,b,c,d in $GF(q^2)$ such that $ad-bc \neq 0$, where the parameter θ ranges over $GF(q) \cup \{\infty\}$. This Baer subline is projectively related to the Baer subline of b_{q^2+1} given by

$$b^1 = \begin{bmatrix} \alpha & 0 \\ \alpha & 1 \end{bmatrix} \begin{pmatrix} \theta' \\ 1 \end{pmatrix},$$

where $\theta' \in GF(q^2) \cup \{\infty\}$. In Bruck-Bose, this Baer subline is given by,

$$\begin{bmatrix} -c & 0 & 0 \\ 0 & -1 & 0 \\ -c & 1 & -1 \end{bmatrix} \begin{pmatrix} \theta'^2 \\ \theta' \\ 1 \end{pmatrix} = \begin{bmatrix} -c & 0 & 0 \\ 0 & -1 & 0 \\ -c & 1 & -1 \end{bmatrix} \begin{bmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{bmatrix} \begin{pmatrix} \theta^2 \\ \theta \\ 1 \end{pmatrix}$$

(See [52, Theorem 2.37]). Hence in Bruck-Bose, b^1 is the image b^{1*} of the subconic b^*_{q+1} under a projectivity of ℓ^* and so b^{1*} is a subconic of $b^*_{q^2+1}$ contained in a Baer subplane of ℓ^*. Again this Baer subplane intersects Σ_∞ in a unique point.

Hence every Baer subline of b_{q^2+1} in Bruck-Bose is a subconic of $b^*_{q^2+1}$ contained in some Baer subplane of ℓ^*, and the Baer subplane necessarily intersects Σ_∞ in a unique point.

We have concentrated our attention on a specific Baer subline b_{q^2+1} of ℓ, but any Baer subline of ℓ is the image of b_{q^2+1} under an element of $PGL(2,q^4)$. Let $b'_{q^2+1} \neq b_{q^2+1}$ be a Baer subline of ℓ which does not contain the point P, then

$$b'_{q^2+1} = \{H \begin{bmatrix} \alpha & 0 \\ \alpha & 1 \end{bmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} \mid \theta \in GF(q^2) \cup \{\infty\}\}$$

where H is some element of $PGL(2,q^4)$. Then $H' = H \begin{bmatrix} \alpha & 0 \\ \alpha & 1 \end{bmatrix}$ is an element of $PGL(2,q^4)$ and in Bruck-Bose, by repeating the arguments used for b_{q^2+1} above, $b'^{*}_{q^2+1}$ is a set of points of ℓ^* given by,

$$H'^{*} \begin{pmatrix} \theta^2 \\ \theta \\ 1 \end{pmatrix}$$

where H'^{*} is an element of $PGL(3,q^2)$ since $b'^{*}_{q^2+1}$ is a non-degenerate (Baer) conic in ℓ^*. Thus each Baer subline of b'_{q^2+1} in Bruck-Bose is projectively related to a subconic of $b^*_{q^2+1}$, which represents a Baer subline of b_{q^2+1}.

We therefore have the following theorem valid for any Baer subline b_{q^2+1} of an affine line ℓ of $PG(2,q^4)$ whose Bruck-Bose representation in $PG(4,q^2)$ is a non-degenerate (Baer) conic $b^*_{q^2+1}$ in the plane ℓ^* which represents ℓ.

Theorem 2.7.1 *If b_{q+1} is a Baer subline of b_{q^2+1}, then in 4-dimensional Bruck-Bose b_{q+1} is a subconic b^*_{q+1} of the conic $b^*_{q^2+1}$ such that b^*_{q+1} lies in a Baer subplane B of ℓ^*. Moreover, B intersects Σ_∞ in a unique point.* □

Note that since the plane ℓ^* is isomorphic to $PG(2,q^2)$, we can represent ℓ^* in 4-dimensional Bruck-Bose over $GF(q)$, using $\ell^* \cap \Sigma_\infty$ as the line at infinity of ℓ^*. Since $b^*_{q^2+1}$ is a non-degenerate conic in ℓ^* disjoint from the line at infinity of ℓ^*, in Bruck-Bose $b^*_{q^2+1}$ is a (q^2+1)-cap. Moreover this cap contains the Bruck-Bose image of the subconics of $b^*_{q^2+1}$ which each lie in non-affine Baer subplanes of ℓ^*. Since the subconics have no point on the line at infinity of ℓ^* and by Section 2.3 we have,

Theorem 2.7.2 *If b_{q^2+1} is a Baer subline of a line ℓ of $PG(2,q^4)$ such that b_{q^2+1} is disjoint from the line at infinity of $PG(2,q^4)$, then in 8-dimensional Bruck-Bose over $GF(q)$, b_{q^2+1} is a (q^2+1)-cap in the 4-space which represents ℓ. Moreover, this (q^2+1)-cap contains at least q^3+q 4-dimensional normal rational curves.* □

Note that a cap of the type of Theorem 2.7.2 is not a 3-dimensional ovoid. Since if the cap is contained in a hyperplane Σ of the 4-space of Π_8 which contains the cap, then Σ and the hyperplane at infinity intersect in a plane. It then follows that the cap contains points of Σ_∞, a contradiction. Hence such a cap is not contained in any hyperplane of the 4-space which represents ℓ in $\Pi_{8,q}$. Furthermore, such a cap is disjoint from at least one hyperplane of the 4−space in which it lies.

Chapter 3

The Bruck and Bose representation in $\mathbf{PG(n,q)}$, $\mathbf{n > 4}$

In this chapter we consider the Bruck-Bose representation in projective space of dimension greater than 4. In other words we consider the Bruck-Bose representation defined by spreads other than $1-$spreads of $PG(3,q)$.

3.1 Some properties concerning $\mathbf{(h-1)-}$spreads of $\mathbf{PG(2h-1,q)}$

In [50, page 206] a method for constructing spreads is given; a particular case of which is the following. Note that by Theorem 1.9.1 a $(2h-1)-$spread $\mathcal{S}_{2h-1,q^2}$ exists in $PG(4h-1,q^2)$ and since $\mathcal{S}_{2h-1,q^2}$ has more elements than there are points in $PG(4h-1,q)$, there exists an element of $\mathcal{S}_{2h-1,q^2}$ which is disjoint from $PG(4h-1,q)$; therefore, it is possible to embed $PG(2h-1,q^2)$ in the extension $PG(4h-1,q^2)$ of $PG(4h-1,q)$ in such a way that $PG(2h-1,q^2)$ does not contain a point of $PG(4h-1,q)$.

Construction 3.1.1 A construction of a $(2h-1)-$spread of $PG(4h-1,q)$ from a $(h-1)-$spread of $PG(2h-1,q^2)$:

Consider a projective space $PG(2h-1,q^2)$, $h \geq 1$. By Theorem 1.9.1, there exists an $(h-1)-$spread $\mathcal{S}'$ of $PG(2h-1,q^2)$ and $\mathcal{S}'$ contains $q^{2h}+1$ spread elements Π^j_{h-1,q^2}, $j=1,\dots,q^{2h}+1$, of dimension $h-1$ over $GF(q^2)$. Embed $PG(2h-1,q^2)$ in the extension $PG(4h-1,q^2)$ of $PG(4h-1,q)$ so that $PG(2h-1,q^2)$ does not contain a point of $PG(4h-1,q)$. The $(h-1)-$space Π^j_{h-1,q^2} and its conjugate $\overline{\Pi}^j_{h-1,q^2}$ generate a $(2h-1)-$space Π^j_{2h-1,q^2} of $PG(4h-1,q^2)$ and $\Pi^j_{2h-1,q^2} \cap PG(4h-1,q)$ is a $(2h-1)-$space $\Pi^j_{2h-1,q}$ of $PG(4h-1,q)$. The $q^{2h}+1$ spaces $\Pi^j_{2h-1,q}$ form a partition of $PG(4h-1,q)$ and we denote this $(2h-1)-$spread of $PG(4h-1,q)$ by $\mathcal{S}$.

We now prove:

Theorem 3.1.1 *In the Construction 3.1.1, if the $(h-1)-$spread $\mathcal{S}'$ of $PG(2h-1,q^2)$ is regular, then the $(2h-1)-$spread $\mathcal{S}$ of $PG(4h-1,q)$ is regular.*

Proof: Let Π^1_{h-1,q^2}, Π^2_{h-1,q^2}, Π^3_{h-1,q^2} be three distinct elements of $\mathcal{S}'$. Denote by $R' = R(\Pi^1_{h-1,q^2}, \Pi^2_{h-1,q^2}, \Pi^3_{h-1,q^2})$ the unique $(h-1)-$regulus of $PG(2h-1,q^2)$ containing these three spread elements. Let $\Pi^1_{2h-1,q}$, $\Pi^2_{2h-1,q}$, $\Pi^3_{2h-1,q}$ be the three distinct elements of $\mathcal{S}$ corresponding to Π^1_{h-1,q^2}, Π^2_{h-1,q^2}, Π^3_{h-1,q^2} respectively in the given construction. Let $R = R(\Pi^1_{2h-1,q}, \Pi^2_{2h-1,q}, \Pi^3_{2h-1,q})$ denote the unique $(2h-1)-$regulus of $PG(4h-1,q)$ containing $\Pi^1_{2h-1,q}$, $\Pi^2_{2h-1,q}$ and $\Pi^3_{2h-1,q}$. So R is a system of maximal $(2h-1)-$spaces of a Segre variety $\zeta_{1,2h-1}$ in $PG(4h-1,q)$. Over $GF(q^2)$, R becomes a $(2h-1)-$regulus R_{q^2} of $PG(4h-1,q^2)$. Due to the above construction of the spread $\mathcal{S}$ we have for $j=1,2,3$, Π^j_{h-1,q^2} is contained in Π^j_{2h-1,q^2} where Π^j_{2h-1,q^2} is the unique element of the regulus R_{q^2} which contains $\Pi^j_{2h-1,q}$. Thus the line transversals of R' in $PG(2h-1,q^2)$ are line transversals of R_{q^2} and therefore R' is a Segre subvariety $\zeta_{1,h-1}$ of R_{q^2} and by Theorem 1.8.10, the regulus R' is precisely the intersection $R_{q^2} \cap PG(2h-1,q^2)$.

It now follows that for any $(2h-1)-$space $\Pi^j_{2h-1,q}$ in R, where $\Pi^j_{2h-1,q}$ is distinct from $\Pi^1_{2h-1,q}, \Pi^2_{2h-1,q}$ and $\Pi^3_{2h-1,q}$, the unique element Π^j_{2h-1,q^2} of R_{q^2} which contains $\Pi^j_{2h-1,q}$ has the property $\Pi^j_{2h-1,q^2} \cap PG(2h-1,q^2) = \Pi^j_{h-1,q^2}$, for some element Π^j_{h-1,q^2} of R'. By the construction of $\mathcal{S}$ from $\mathcal{S}'$, if $\Pi^j_{2h-1,q}$ $(\in R)$ is an element of $\mathcal{S}$, then Π^j_{h-1,q^2} $(\in R')$ is an element of $\mathcal{S}'$. The converse of the preceding statement is true if Π^j_{h-1,q^2} $(\in R')$ is one of the $q+1$ elements of R' associated to the elements of R via the construction of the spread $\mathcal{S}$. (Note that R' has q^2+1 elements and R has $q+1$ elements).

If $\mathcal{S}'$ is a regular spread, then the regulus R' defined by $\Pi^1_{h-1,q^2}, \Pi^2_{h-1,q^2}, \Pi^3_{h-1,q^2}$ is contained in $\mathcal{S}'$ and therefore, by the preceding argument, the regulus R of $PG(4h-1,q)$ defined by $\Pi^1_{2h-1,q}, \Pi^2_{2h-1,q}, \Pi^3_{2h-1,q}$ is contained in $\mathcal{S}$. The result now follows. □

Consider a translation plane π of order q^{2h} defined by a Bruck-Bose construction with a $(h-1)-$spread $\mathcal{S}'$ of $\Sigma'_\infty = PG(2h-1, q^2)$. We now have a convenient correspondence between this Bruck-Bose representation of π and a second Bruck-Bose representation of π defined by a $(2h-1)-$spread $\mathcal{S}$ of $\Sigma_\infty = PG(4h-1, q)$, where $\mathcal{S}'$ and $\mathcal{S}$ are associated by the above construction.

For Desarguesian planes of certain orders which have a Bruck-Bose representation, the above Construction 3.1.1 and Theorem 3.1.1 provide us with a convenient method to obtain a Bruck-Bose representation of the plane in a space of higher dimension and lower order.

To illustrate this, we consider the Desarguesian plane $PG(2, q^4)$. The plane $PG(2, q^4)$ has a 4-dimensional Bruck-Bose representation defined by a regular line spread $\mathcal{S}'$ of $PG(3, q^2)$ and an $8-$dimensional Bruck-Bose representation defined by a regular $3-$spread $\mathcal{S}$ of $PG(7, q)$.

In the previous chapter we investigated the 4-dimensional Bruck-Bose representation of the Baer substructures of Desarguesian planes of square order. We now determine properties concerning the 8-dimensional Bruck-Bose representation of the Baer substructures of $PG(2, q^4)$ and some generalisations.

Theorem 3.1.2 *A regular $3-$spread $\mathcal{S}$ in $PG(7, q)$ has a well-defined and unique set of induced $1-$spreads, one in each element of $\mathcal{S}$.*

Proof: By Theorem 1.9.5, the regular $3-$spreads of $PG(7, q)$ are projectively equivalent. Therefore, we can assume that $\mathcal{S}$ is the regular $3-$spread of $PG(7, q)$ obtained from a regular $1-$spread $\mathcal{S}'$ of $PG(3, q^2)$ by the Construction 3.1.1 with $h = 2$. We repeat the construction for this special case to establish notation.

Embed $PG(7, q)$ in $PG(7, q^2)$ and let Σ_{3,q^2} be a $3-$space over $GF(q^2)$ in $PG(7, q^2)$ which has no point in common with $PG(7, q)$. Let $\mathcal{S}'$ be a regular $1-$spread of Σ_{3,q^2}. Consider the conjugate space $\overline{\Sigma}_{3,q^2}$ of Σ_{3,q^2}. For each element Π^j_{1,q^2} in $\mathcal{S}'$, $j = 1, \dots, q^4 + 1$, the $3-$space Π^j_{3,q^2} spanned by Π^j_{1,q^2} and its conjugate $\overline{\Pi}^j_{1,q^2}$ intersects $PG(7, q)$ in a $3-$space $\Pi^j_{3,q}$. These q^4+1 $3-$spaces $\Pi^j_{3,q}$ form a $3-$spread $\mathcal{S}$ of $PG(7, q)$ which by Theorem 3.1.1 is regular.

Each element $\Pi^j_{3,q}$ of $\mathcal{S}$ is the intersection $\langle \Pi^j_{1,q^2}, \overline{\Pi}^j_{1,q^2} \rangle \cap PG(7, q)$ for a unique line Π^j_{1,q^2} of $\mathcal{S}'$. For j fixed, the join of each point P of Π^j_{1,q^2} to its conjugate $\overline{P}$ yields a line of

$\Pi_{3,q}^j$ and the collection of these q^2+1 lines constitutes a regular $1-$spread $\mathcal{S}_1^j$ of $\Pi_{3,q}^j$ by Bruck's result (Theorem 1.9.6).

Hence each element $\Pi_{3,q}^j$ of the regular $3-$spread $\mathcal{S}$ of $PG(7,q)$ has a well defined induced regular $1-$spread which we denote by $\mathcal{S}_1^j$. $\square$

Consider the regular line spread $\mathcal{S}'$ of $\Sigma_{3,q^2} \cong PG(3,q^2)$ and the regular $3-$spread $\mathcal{S}$ of $PG(7,q)$ associated to $\mathcal{S}'$ by the Construction 3.1.1. By the Bruck-Bose construction of Section 1.10, these spreads correspond to a 4-dimensional Bruck-Bose representation of $PG(2,q^4)$ and an 8-dimensional Bruck-Bose representation of $PG(2,q^4)$ respectively. Denote these Bruck-Bose incidence structures by Π_{4,q^2} and $\Pi_{8,q}$ respectively.

By Theorem 3.1.2 and its proof, there exists a well defined 1-1 correspondence between the points of Σ_{3,q^2} and the (line) elements of the induced $1-$spreads $\{\mathcal{S}_1^j\}$ of $PG(7,q)$.

Definition 3.1.3 *For $\mathcal{S}$ a regular $3-$spread of $PG(7,q)$, the (line) elements of the q^4+1 induced regular $1-$spreads $\{\mathcal{S}_1^j\}$ shall be called* **partition lines**. *That is, for each $3-$space $\Sigma_j \in \mathcal{S}$, a line ℓ of Σ_j is a* **partition line** *if $\ell \in \mathcal{S}_1^j$, otherwise ℓ is a* **non-partition line**. *The remaining lines of $PG(7,q)$ are those not contained in any element of $\mathcal{S}$; these shall be called* **transversal lines**.

In Section 2.2, we discussed the representation in 4-dimensional Bruck-Bose of affine Baer subplanes and affine Baer sublines of Desarguesian planes of square order. By Corollary 2.2.2 an affine Baer subplane B of $PG(2,q^4)$ is represented in Π_{4,q^2} by a plane not contained in $\Sigma'_\infty = \Sigma_{3,q^2}$ and which meets Σ'_∞ in a line ℓ which is not an element of $\mathcal{S}'$. Consider such a line ℓ in Σ_{3,q^2}. The line ℓ and its conjugate $\overline{\ell}$ generate a $3-$space $\langle \ell, \overline{\ell} \rangle$ of $PG(7,q^2)$ and the intersection $\langle \ell, \overline{\ell} \rangle \cap PG(7,q)$ is a $3-$space Σ_ℓ of $PG(7,q)$. Since ℓ is incident with exactly q^2+1 $1-$spread elements in Σ_{3,q^2}, the $3-$space Σ_ℓ intersects exactly q^2+1 of the $3-$spaces in the spread $\mathcal{S}$ of $PG(7,q)$, meeting each in a partition line. So in particular Σ_ℓ is disjoint from the remaining spread elements in $\mathcal{S}$.

Consider the $8-$dimensional Bruck-Bose representation, $\Pi_{8,q}$, of $PG(2,q^4)$ defined by the regular $3-$spread $\mathcal{S}$ of $PG(7,q)$. Consider a $4-$dimensional subspace B^* of $\Pi_{8,q}$ which intersects $PG(7,q)$ in the $3-$space Σ_ℓ. Any $4-$space l^* in $\Pi_{8,q}$, not contained in $PG(7,q)$, and which intersects $PG(7,q)$ in a unique element of $\mathcal{S}$, either intersects B^* in a unique affine point, or the spread element contained in l^* is one of the q^2+1 incident with B^*. It follows by Theorem 1.2.2, that B^* represents an affine Baer subplane of

$PG(2,q^4)$, since B^* and the q^2+1 $3-$spread elements incident with B^* constitute a $(q^4+q^2+1)-$blocking set in $PG(2,q^4)$.

By considering all lines ℓ in Σ_{3,q^2} which are not elements of the $1-$spread $\mathcal{S}'$ and repeating the above procedure, we obtain the $8-$dimensional Bruck-Bose representation of all $q^4(q^8+q^6+q^4+q^2)$ affine Baer subplanes of $PG(2,q^4)$.

Intrinsic to this representation is the existence of $q^8+q^6+q^4+q^2$ $3-$spaces of $PG(7,q)$ which intersect precisely q^2+1 elements of the regular $3-$spread $\mathcal{S}$ of $PG(7,q)$ and such that the intersection in each case is a unique partition line, namely an element of the induced $1-$spread of that $3-$space.

Theorem 3.1.4 *Let $\mathcal{S}$ be a regular $3-$spread of $PG(7,q)$. For each $3-$space Σ of $PG(7,q)$ one of the following holds:*

(1) Σ is an element of $\mathcal{S}$ and therefore $\Sigma=\Sigma_j$ has a induced regular $1-$spread $\mathcal{S}_1^j$. By definition Σ contains exactly q^2+1 partition lines.

There are q^4+1 $3-$spaces Σ of this type in $PG(7,q)$.

(2) Σ intersects exactly q^2+1 elements of $\mathcal{S}$, meeting each in a partition line. This set of q^2+1 partition lines constitutes a regular $1-$spread of Σ which we shall call a **partition $1-$spread**.

Any two partition lines, contained in distinct elements of $\mathcal{S}$, span such a $3-$space. There are $q^8+q^6+q^4+q^2$ $3-$spaces Σ of this type in $PG(7,q)$.

(3) Σ intersects x elements of $\mathcal{S}$ where $x>q^2+1$. In this case either:

(i) $x=q^3+1$ and Σ intersects one element of $\mathcal{S}$ in a plane (which necessarily contains a partition line) and Σ intersects a further q^3 elements of $\mathcal{S}$, meeting each in a point,

or,

(ii) Σ intersects y elements of $\mathcal{S}$ in a line and Σ intersects a further $x-y=(q^3+q^2+q+1)-y(q+1)>0$ elements of $\mathcal{S}$ meeting each in a point.

In this case Σ contains at most one partition line.

If Π_1, Π_2, Π_3 are three distinct elements of $\mathcal{S}$ which each intersects Σ in a line, then Σ has a non-trivial intersection with each element of $\mathcal{S}$ in the 3−regulus $R(\Pi_1, \Pi_2, \Pi_3)$; indeed Σ intersects each such element of $\mathcal{S}$ in a line.

Proof:

By Theorem 3.1.2 the $q^4 + 1$ elements of $\mathcal{S}$ constitute the 3−spaces of $PG(7, q)$ of type (1).

By the remarks preceding Theorem 3.1.4 there exist $q^8 + q^6 + q^4 + q^2$ 3−spaces of $PG(7, q)$ which each intersect $q^2 + 1$ distinct elements of $\mathcal{S}$ and which contain a partition 1−spread. We shall call these 3−spaces **partition** 3−**spaces** of $PG(7, q)$. We must show that these are the only 3−spaces of $PG(7, q)$ which intersect exactly $q^2 + 1$ distinct elements of $\mathcal{S}$.

$\Pi_{8,q}$ is the 8−dimensional Bruck-Bose representation of $PG(2, q^4)$. The line at infinity ℓ_∞ is the line with "points" the elements of $\mathcal{S}$ in $PG(7, q)$. As usual, the Baer subplanes of $PG(2, q^4)$ which are secant to ℓ_∞ are called *affine* Baer subplanes. There exist precisely $q^4(q^8 + q^6 + q^4 + q^2)$ affine Baer subplanes of $PG(2, q^4)$.

Consider a 4−space B^* in $PG(8, q)$ not contained in $PG(7, q)$ and which intersects $PG(7, q)$ in a 3−space Σ where Σ intersects exactly $q^2 + 1$ elements of $\mathcal{S}$. Necessarily, Σ intersects each of these $q^2 + 1$ elements of $\mathcal{S}$ in a line. By the incidence in $\Pi_{8,q}$, B^* intersects ℓ_∞ in exactly $q^2 + 1$ points. Each 4−space ℓ of $PG(8, q)$ which represents a line of $PG(2, q^4)$ distinct from ℓ_∞ is not contained in $PG(7, q)$ and meets $PG(7, q)$ in an element of $\mathcal{S}$. Such a 4−space ℓ either intersects B^* in a point of $PG(8, q)\backslash PG(7, q)$ or the element of $\mathcal{S}$ incident with ℓ is one of the $q^2 + 1$ 3−spread elements incident with B^*. It follows that B^* represents a $(q^4 + q^2 + 1)$−blocking set B in $PG(2, q^4)$. By Theorem 1.2.2, B is an affine Baer subplane of $PG(2, q^4)$.

Therefore any 4−space of $PG(8, q)$, not contained in $PG(7, q)$ and which meets $PG(7, q)$ in a partition 3−space represents an affine Baer subplane of $PG(2, q^4)$. There are $q^4(q^8 + q^6 + q^4 + q^2)$ such 4−spaces of $PG(8, q)$. Since this is also the number of affine Baer subplanes of $PG(2, q^4)$, there exist no further 3−spaces of $PG(7, q)$ (besides the partition 3−spaces) which intersect exactly $q^2 + 1$ elements of $\mathcal{S}$.

Let Σ be a 3−space of $PG(7, q)$ spanned by partition lines ℓ_1 and ℓ_2 where ℓ_1 and ℓ_2 lie in distinct elements of $\mathcal{S}$. In the quadratic extension, the lines ℓ_1 and ℓ_2 intersect

Σ_{3,q^2} in distinct points L_1 and L_2 respectively. The 3−space (over $GF(q^2)$) spanned by the line L_1L_2 and its conjugate $\overline{L_1L_2}$ is the quadratic extension Σ_{q^2} of Σ. By joining each point P on L_1L_2 to its conjugate $\overline{P}$ on $\overline{L_1L_2}$ we obtain a set of q^2+1 lines of Σ which by Bruck's Theorem 1.9.6 constitutes a regular 1−spread of Σ. The elements of this 1−spread are all partition lines and hence by definition, this regular 1−spread is a partition spread. Thus Σ is a partition 3−space. We have that two partition lines from distinct elements of $\mathcal{S}$ span a partition 3−space.

The 3−spaces of $PG(7,q)$ of type (1) and (2) have now been classified. The type (3) 3−spaces include all possible exceptions. It remains to prove the final remark regarding a 3−space of type (3)(ii).

Consider a 3−space Σ of $PG(7,q)$ which intersects strictly greater than q^2+1 elements of $\mathcal{S}$ but which meets no element of $\mathcal{S}$ in a plane. Suppose Σ intersects the 3−spread elements Π_1, Π_2, Π_3 each in a line ℓ_1, ℓ_2, ℓ_3 respectively. The lines ℓ_1, ℓ_2, ℓ_3 define a unique 1−regulus $R_1 = R(\ell_1, \ell_2, \ell_3)$ in Σ. The 3−spread elements Π_1, Π_2, Π_3 define a unique 3−regulus $R_3 = R(\Pi_1, \Pi_2, \Pi_3)$ which is contained in $\mathcal{S}$ since $\mathcal{S}$ is regular. The line transversals of R_1 are contained in Σ and are necessarily transversals of the regulus R_3. Hence each spread element in R_3 intersects Σ in a line, namely a maximal space of the Segre variety R_1. $\square$

Corollary 3.1.5 *Let $\Pi_{8,q}$ denote the Bruck-Bose representation of $PG(2,q^4)$ in $PG(8,q)$ and let Σ_∞ denote the hyperplane at infinity of $PG(8,q)$.*

B is an affine Baer subplane of $PG(2,q^4)$ if and only if in $\Pi_{8,q}$ B is a 4−space not contained in Σ_∞ and which intersects Σ_∞ in a partition 3−space.

b_ℓ is a Baer subline of $PG(2,q^4)$ that contains a point of ℓ_∞ if and only if in $\Pi_{8,q}$ b_ℓ is a plane not contained in Σ_∞ and which intersects Σ_∞ in a partition line.

Proof: The affine Baer subplane structure in $\Pi_{8,q}$ was determined in the proof of Theorem 3.1.4. A line ℓ, distinct from ℓ_∞, of an affine Baer subplane B of $PG(2,q^4)$ intersects B in a Baer subline b_ℓ that contains a point of ℓ_∞.

In $\Pi_{8,q}$, ℓ is a 4−space which intersects Σ_∞ in an element Σ_j of the 3−spread $\mathcal{S}$ and B is a 4−space which intersects Σ_∞ in a partition 3−space Σ. The 3−spaces Σ_j and Σ intersect in a partition line, hence in $\Pi_{8,q}$, the intersection $\ell \cap B$ is a plane, not contained in Σ_∞ and which contains a partition line. This plane is then the Bruck-Bose representation of the Baer subline b_ℓ. $\square$

Before considering the representation in $\Pi_{8,q}$ of the non-affine Baer subplanes of $PG(2,q^4)$ and the Baer sublines b_ℓ of $PG(2,q^4)$ which contain no point of ℓ_∞, we need to present some extra material. In the next section we recall the Bose representation of a plane $PG(2,q^2)$ in the projective space $PG(5,q)$ which was introduced in [18].

We conclude this section with a generalisation of the results determined for $PG(2,q^4)$ thus far.

Theorem 3.1.6 *Consider the Desarguesian plane $PG(2,q^{2^n})$ $(n \geq 1)$ and the n Bruck-Bose representations $\Pi_{2^{i+1}} = \Pi_{2^{i+1},q^{2^{n-i}}}$ $(1 \leq i \leq n)$ which are determined by a*

$$\begin{array}{rcl} \textit{regular } 1-\textit{spread} & \textit{in} & PG(3,q^{2^{n-1}}), \\ \textit{regular } 3-\textit{spread} & \textit{in} & PG(7,q^{2^{n-2}}), \\ & \vdots & \\ \textit{regular } (2^i-1)-\textit{spread} & \textit{in} & PG(2^{i+1}-1,q^{2^{n-i}}), \\ & \vdots & \\ \textit{regular } (2^n-1)-\textit{spread} & \textit{in} & PG(2^{n+1}-1,q) \textit{ respectively.} \end{array}$$

Then the regular (2^i-1)−spread in the hyperplane $PG(2^{i+1}-1,q^{2^{n-i}})$ at infinity of $\Pi_{2^{i+1}}$ has a set of induced regular $(2^{i-1}-1)$−spreads, one in each element of the (2^i-1)−spread. Furthermore, for each such induced regular $(2^{i-1}-1)$−spread, there exists a set of induced regular $(2^{i-2}-1)$−spreads, and so on, until finally there exist induced regular 1−spreads.

Proof Let $\mathcal{S}_1$ be regular 1−spread of $PG(3,q^{2^{n-1}})$ and embed $PG(3,q^{2^{n-1}})$ as a subspace in $PG(7,q^{2^{n-1}})$ in such a way that it is skew to $PG(7,q^{2^{n-2}})$. By Theorem 3.1.1 and the Construction 3.1.1, $\mathcal{S}_1$ determines a regular 3−spread $\mathcal{S}_3$ of $PG(7,q^{2^{n-2}})$ which has a set of induced regular 1−spreads, one in each element of $\mathcal{S}_3$, by Theorem 3.1.2. Embed $PG(7,q^{2^{n-2}})$ as a subspace in $PG(15,q^{2^{n-2}})$ in such a way that $PG(7,q^{2^{n-2}})$ is skew to the Baer subspace $PG(15,q^{2^{n-3}})$ of $PG(15,q^{2^{n-2}})$ and recursively repeat the above procedure using Construction 3.1.1. At the final stage we obtain a regular (2^n-1)−spread in $PG(2^{n+1}-1,q)$ which contains the nested induced regular spreads of each stage. If we stop the procedure before the final stage we have a regular (2^i-1)−spread in $PG(2^{i+1}-1,q^{2^{n-i}})$ with the nested induced regular spreads obtained up until that stage.

By Theorem 1.9.5, regular $(2^i-1)-$spreads in $PG(2^{i+1}-1, q^{2^{n-i}})$ are projectively equivalent, and so the regular spread we have constructed, which contains nested induced spreads, is representative. $\square$

Corollary 3.1.7 *For each $1 \leq i \leq n$, embed $PG(2^{i+1}-1, q^{2^{n-i}}) = \Sigma_\infty^{2^{i+1}-1}$ as a hyperplane in $PG(2^{i+1}, q^{2^{n-i}})$ and let $\Pi_{2^{i+1}}$ denote the Bruck-Bose representation of $PG(2, q^{2^n})$ in $PG(2^{i+1}, q^{2^{n-i}})$ determined by the regular $(2^i-1)-$spread $\mathcal{S}_{2^i-1}$ of $\Sigma_\infty^{2^{i+1}-1}$, as in Theorem 3.1.6.*

Then B is an affine Baer subplane of $PG(2, q^{2^n})$ if and only if in $\Pi_{2^{i+1}}$, B is a $(2^i)-$space B^* of $PG(2^{i+1}, q^{2^{n-i}})$ not contained in $\Sigma_\infty^{2^{i+1}-1}$ and which intersects $\Sigma_\infty^{2^{i+1}-1}$ in exactly $q^{2^{n-1}}+1$ elements of $\mathcal{S}_{2^i-1}$.

Furthermore, each element $\Lambda \in \mathcal{S}_{2^i-1}$ is either disjoint to B^* or intersects B^* in a unique element (a $(2^{i-1}-1)-$space of order $q^{2^{n-i}}$) of the induced regular $(2^{i-1}-1)-$spread $\mathcal{S}_j^{2^{i-1}-1}$ in Λ. $\square$

Note that the Bruck-Bose representation B^* of an affine Baer subplane B of $PG(2, q^{2^n})$ is determined in Corollary 3.1.7, regardless of which of the n possible Bruck-Bose representations of $PG(2, q^{2^n})$ is being considered. Moreover, implicit to Theorem 3.1.6 and its Corollary 3.1.7 is the Bruck-Bose representation of subplanes of order $q^{2^{n-j}}$ of $PG(2, q^{2^n})$ which contain the line at infinity as a line. Due to the existence of the induced spreads determined in Theorem 3.1.6, in a Bruck-Bose representation $\Pi_{2^{i+1}}$ of a Desarguesian plane $PG(2, q^{2^n})$ we have the Bruck-Bose representations of the subplanes, which contain the line at infinity as a line, nested in $\Pi_{2^{i+1}}$ as linear subspaces.

Finally, let ℓ be a subline of order $q^{2^{n-j}}$ $(1 \leq j \leq n)$ of a line L of $PG(2, q^{2^n})$ such that ℓ contains a unique point on the line at infinity. It follows from the above discussion that the representation of ℓ in any Bruck-Bose representation $\Pi_{2^{i+1}}$ of $PG(2, q^{2^n})$ is determined.

Corollary 3.1.8 *Let ℓ be a subline of order $q^{2^{n-j}}$ $(1 \leq j \leq n)$ of a line L of $PG(2, q^{2^n})$ such that ℓ contains a unique point on the line at infinity ℓ_∞ of $PG(2, q^{2^n})$. Let $\Pi_{2^{i+1}}$ $(1 \leq i \leq n)$ denote the Bruck-Bose representation of $PG(2, q^{2^n})$ defined by a regular $(2^i-1)-$spread of $PG(2^{i+1}-1, q^{2^{n-i}})$.*

Then the subline ℓ is represented by a $(2^{i-j})-$subspace ℓ^* of the $(2^i)-$space L^*, which represents L, in $\Pi_{2^{i+1}}$. Moreover, ℓ^* intersects the hyperplane at infinity $PG(2^{i+1}-1, q^{2^{n-i}})$

in exactly a unique induced spread element of dimension $2^{i-j}-1$ *and order* $q^{2^{n-i}}$. $\square$

In this way we obtain the Bruck-Bose representations of any (not just a Baer) subplane of $PG(2,q^{2^n})$ which contains the line at infinity as a line and any (not just a Baer) subline of a line of $PG(2,q^{2^n})$ such that the subline contains a unique point on the line at infinity.

3.2 The Bose representation of $\mathbf{PG(2,q^2)}$ in $\mathbf{PG(5,q)}$

The results of this section are well known and form part of the folklore of finite projective geometry. References are given where possible; however, it has been difficult to locate references for some of the well known results which are presented here. In order to provide a complete discussion of the Bose representation, we deemed it appropriate to prove these results.

In [18], Bose calls a $1-$spread of $PG(5,q)$ a **spread** (of lines), and we shall also in this section; that is, a **spread** of lines of $PG(5,q)$ is a set of lines in $PG(5,q)$ such that each point of $PG(5,q)$ is contained in one and only one line of the set. We shall also define a **dual spread** of $3-$spaces in $PG(5,q)$ to be a set of $3-$spaces such that each $4-$space of $PG(5,q)$ contains one and only one $3-$space of the set. Note that a spread in $PG(5,q)$ is equivalent to a dual spread in the dual space of $PG(5,q)$. Hence, the number of elements in a spread or dual spread of $PG(5,q)$ equals

$$\frac{q^6-1}{q-1}\times\frac{1}{q+1} \;=\; q^4+q^2+1.$$

The Bose representation of $PG(2,q^2)$ in $PG(5,q)$ relies on the existence of a spread $\mathcal{S}$ of $PG(5,q)$ of the following type.

Definition 3.2.1 *A spread* $\mathcal{S}$ *(of lines) of* $PG(5,q)$ *is a* **Bose spread** *if for any two distinct elements* ℓ_1,ℓ_2 *of* $\mathcal{S}$*, the* $3-$*space spanned by* ℓ_1 *and* ℓ_2 *contains exactly* q^2+1 *elements of* $\mathcal{S}$.

For a Bose spread $\mathcal{S}$ *in* $PG(5,q)$*, denote by* $\mathcal{H}_3$ *the collection of* $3-$*spaces*

$$\{\Sigma_{\ell_1,\ell_2}=\langle\ell_1,\ell_2\rangle \mid \ell_1,\ell_2\in\mathcal{S},\ \ell_1\neq\ell_2\}.$$

Note that there are precisely q^4+q^2+1 $3-$spaces in the set $\mathcal{H}_3$ for a given Bose spread $\mathcal{S}$ of $PG(5,q)$.

Given a Bose spread $\mathcal{S}$ in $PG(5,q)$, let $\pi_{q^2}(\mathcal{S})$ be the incidence structure with: *points* the elements of $\mathcal{S}$; *lines* the $3-$space elements of $\mathcal{H}_3$ and *incidence* given by containment.

Theorem 3.2.2 [18] *The incidence structure $\pi_{q^2}(\mathcal{S})$, where $\mathcal{S}$ is a Bose spread of $PG(5,q)$, is a projective plane of order q^2.*

Proof: Two distinct points of $\pi_{q^2}(\mathcal{S})$ correspond to two distinct elements ℓ_1 and ℓ_2 of $\mathcal{S}$. The lines ℓ_1, ℓ_2 span a unique $3-$space of $PG(5,q)$, which contains q^2-1 further elements of $\mathcal{S}$ since $\mathcal{S}$ is a Bose spread. Hence two points of $\pi_{q^2}(\mathcal{S})$ are contained in a unique line of $\pi_{q^2}(\mathcal{S})$.

Two lines of $\pi_{q^2}(\mathcal{S})$ correspond to two elements of $\mathcal{H}_3$ which we shall denote by Σ_1 and Σ_2. Suppose the $3-$spaces Σ_1, Σ_2 intersect in a plane σ of $PG(5,q)$. Each of Σ_1 and Σ_2 contains a subspread of $\mathcal{S}$, denote these subspreads by S_1 and S_2 respectively. The plane σ necessarily contains an element of S_1 (and S_2 respectively). Since two lines in σ intersect, and the spreads $S_1, S_2 \subseteq \mathcal{S}$, it follows that σ contains an element ℓ of $\mathcal{S}$ and ℓ is an element of both S_1 and S_2. The plane σ is incident with the remaining q^2 elements of S_1, meeting each such element in a point. Similarly σ is incident with the remaining q^2 elements of $S_2 \backslash \{\ell\}$, meeting each such element in a point. Thus each point of $\sigma \backslash \{\ell\}$ is incident with an element of S_1 and an element of S_2. Since $\mathcal{S}$ is a spread of $PG(5,q)$, we have a contradiction. Thus two distinct $3-$spaces in $\mathcal{H}_3$ intersect exactly in a line which is necessarily an element of $\mathcal{S}$. Therefore two distinct lines of $\pi_{q^2}(\mathcal{S})$ intersect in a unique point.

Since $\pi_{q^2}(\mathcal{S})$ has q^4+q^2+1 points, it follows that $\pi_{q^2}(\mathcal{S})$ is a projective plane of order q^2. □

Corollary 3.2.3 *If $\mathcal{S}$ is a Bose spread of $PG(5,q)$ then the associated collection of $3-$spaces $\mathcal{H}_3$ is a dual spread of $PG(5,q)$.*

Proof: In the proof of Theorem 3.2.2 we showed that two distinct elements of $\mathcal{H}_3$ intersect in a line, a line which is an element of $\mathcal{S}$. Therefore no hyperplane of $PG(5,q)$ contains two elements of $\mathcal{H}_3$. Since there are q^4+q^2+1 $3-$spaces in the set $\mathcal{H}_3$, and since each 3−space of $PG(5,q)$ is contained in $q+1$ distinct hyperplanes of $PG(5,q)$, it follows that each hyperplane of $PG(5,q)$ contains a unique element of $\mathcal{H}_3$. $\square$

We now prove a result of Thas which states that a projective plane defined by a Bose spread of $PG(5,q)$, in the manner defined above, is necessarily Desarguesian.

Theorem 3.2.4 [87] *The projective plane $\pi_{q^2}(\mathcal{S})$, defined above for a Bose spread $\mathcal{S}$ of $PG(5,q)$, is Desarguesian.*

Proof: Let $\mathcal{S}$ be a Bose spread of $PG(5,q)$ and let $\pi_{q^2}(\mathcal{S})$ be the incidence structure defined as above, which by Theorem 3.2.2 is a projective plane of order q^2. Embed $PG(5,q)$ as a hyperplane in $PG(6,q)$. Let Σ_{3,q^2} be the incidence structure with: *points* the points of $PG(6,q)\backslash PG(5,q)$; *lines* the planes of $PG(6,q)$ not contained in $PG(5,q)$ and which intersect $PG(5,q)$ in a unique element of $\mathcal{S}$ and *incidence* given by containment.

It can be shown that Σ_{3,q^2} is an affine 3−space and the plane $\pi_{q^2}(\mathcal{S})$ is then the plane at infinity of Σ_{3,q^2}. Any projective plane embedded in an affine projective 3−space is Desarguesian, hence $\pi_{q^2}(\mathcal{S})$ is Desarguesian as required.

This configuration also provides additional information about the Bose spread $\mathcal{S}$. Let $\overline{\Sigma}_{3,q^2}$ be the projective completion of Σ_{3,q^2}. Then the planes of $\overline{\Sigma}_{3,q^2}$ distinct from $\pi_{q^2}(\mathcal{S})$ are given by the 4−spaces of $PG(6,q)$ not contained in $PG(5,q)$ and which intersect $PG(5,q)$ in an element of $\mathcal{H}_3$. Each such 4−space is therefore a Bruck-Bose representation of a Desarguesian projective plane of order q^2 with hyperplane at infinity an element of $\mathcal{H}_3$. It follows that for each 3−space in $\mathcal{H}_3$, the q^2+1 elements of $\mathcal{S}$ contained in this 3−space constitute a regular spread of the space (see Section 1.10 and Theorem 1.10.1.3).
$\square$

Corollary 3.2.5 *If $\mathcal{S}$ is a Bose spread of $PG(5,q)$ and if $\mathcal{H}_3$ is the following collection of 3−spaces of $PG(5,q)$*

$$\{\Sigma_{\ell_1,\ell_2} = \langle \ell_1, \ell_2 \rangle \mid \ell_1, \ell_2 \in \mathcal{S},\ \ell_1 \neq \ell_2\},$$

then for every element Σ_{ℓ_1,ℓ_2} of $\mathcal{H}_3$ the subset of q^2+1 elements of $\mathcal{S}$ contained in Σ_{ℓ_1,ℓ_2} constitutes a regular 1− spread of Σ_{ℓ_1,ℓ_2}. $\square$

In summary, if $\mathcal{S}$ is Bose spread of $PG(5,q)$, then by definition any pair of elements of $\mathcal{S}$ spans a 3−space containing q^2+1 elements of $\mathcal{S}$. If a 3−space of $PG(5,q)$ contains q^2+1

elements of $\mathcal{S}$, then these spread elements necessarily constitute a regular 1−spread of the 3−space. The collection of such 3−spaces is a dual spread $\mathcal{H}_3$ of $PG(5,q)$ and the Desarguesian plane $PG(2,q^2)$ is isomorphic to the incidence structure with: *points* the elements of $\mathcal{S}$; *lines* the elements of $\mathcal{H}_3$ and *incidence* given by containment. We call such a representation of $PG(2,q^2)$ in $PG(5,q)$ a **Bose representation of $\mathbf{PG(2,q^2)}$**.

Lemma 3.2.6 [18] *For $\mathcal{S}$ a Bose spread in $PG(5,q)$: Each line m of $PG(5,q)$ is either (I) an element of $\mathcal{S}$, or (II) m is a transversal to a regulus of lines in $\mathcal{S}$ and m is contained in the unique element of $\mathcal{H}_3$ spanned by this regulus.*
Each plane of $PG(5,q)$ is either: (I) contained in an element of $\mathcal{H}_3$ and contains exactly one element of $\mathcal{S}$, or (II) contained in no element of $\mathcal{H}_3$ and is incident with q^2+q+1 distinct elements of $\mathcal{S}$.

Proof Let m be a line of $PG(5,q)$ which is not an element of $\mathcal{S}$. Since $\mathcal{S}$ is a spread of $PG(5,q)$, the line m is incident with exactly $q+1$ elements of $\mathcal{S}$. Any two elements of $\mathcal{S}$ incident with m span a 3−space $\Sigma \in \mathcal{H}_3$ which contains exactly q^2-1 further elements of $\mathcal{S}$. Since m is contained in Σ it follows that each of the elements of $\mathcal{S}$ incident with m is contained in Σ. By Corollary 3.2.5, the elements of $\mathcal{S}$ in Σ form a regular 1−spread and therefore m is a transversal to a regulus of elements of $\mathcal{S}$.

Each element ℓ of $\mathcal{S}$ is contained in q^3+q^2+q+1 planes of $PG(5,q)$ and each such plane is spanned by ℓ and a point on a distinct element of $\mathcal{S}$. Therefore, by Corollary 3.2.5, such a plane is contained in a unique element of $\mathcal{H}_3$. Moreover, in an element of $\mathcal{H}_3$ the q^2+1 elements of $\mathcal{S}$ constitute a 1−spread, therefore any plane contained in an element of $\mathcal{H}_3$ necessarily contains an element of $\mathcal{S}$. There are $(q^4+q^2+1)(q^3+q^2+q+1)$ planes of $PG(5,q)$ of this type and the remaining planes of $PG(5,q)$ therefore contain no element of $\mathcal{S}$.

Let σ be a plane in $PG(5,q)$ which contains no element of $\mathcal{S}$; σ is therefore not contained in any element of $\mathcal{H}_3$. As $\mathcal{S}$ is a spread, each point of σ is incident with an element of $\mathcal{S}$ and since σ contains no line which is an element of $\mathcal{S}$, σ is incident with q^2+q+1 distinct elements of $\mathcal{S}$. Each line in σ is a line of type (II) which is therefore contained in an element of $\mathcal{H}_3$. Since σ is not contained in any element of $\mathcal{H}_3$, σ is incident with q^2+q+1 elements of $\mathcal{H}_3$, meeting each in a line. □

Theorem 3.2.7 *If $\mathcal{S}$ is a Bose spread of $PG(5,q)$ and $\pi_{q^2}(\mathcal{S})$ is the Bose representation* *of $PG(2,q^2)$ defined by $\mathcal{S}$, then B is a Baer subplane of $PG(2,q^2)$ if and only if in the Bose representation, B is a $2-$regulus (or Segre variety $\rho_{1;2}$) whose q^2+q+1 transversal lines are elements of $\mathcal{S}$.*

Proof Let σ be a plane in $PG(5,q)$ of type (II) in the sense of Lemma 3.2.6. Let B be the set containing the q^2+q+1 elements of $\mathcal{S}$ incident with σ and the q^2+q+1 elements of $\mathcal{H}_3$ which each intersect σ in a (distinct) line. Since σ is a projective plane of order q, two elements ℓ_1, ℓ_2 of $\mathcal{S}$ in B define a line of type (II) in σ which is contained in a unique element of $\mathcal{H}_3$ in B and which contains ℓ_1 and ℓ_2. Conversely, two elements of $\mathcal{H}_3$ in B intersect in a unique element of $\mathcal{S}$ in B. Therefore B is a Baer subplane of $PG(2,q^2)$ in the Bose representation of $PG(2,q^2)$ in $PG(5,q)$ determined by $\mathcal{S}$. Counting shows there exists $q^3(q^3+1)(q^2+1)(q+1)$ planes σ of type (II) in $PG(5,q)$ and from above, each such plane corresponds to a Baer subplane of $PG(2,q^2)$. Since $PG(2,q^2)$ contains $q^3(q^2-q+1)(q^2+1)(q+1)$ distinct Baer subplanes, there exists a set B, as above, of q^2+q+1 elements of $\mathcal{S}$ which contains at least 3 planes of type (II). These planes must be pairwise disjoint, otherwise two intersecting planes of type (II) span at most a $4-$space of $PG(5,q)$ which would necessarily contain more than one element of $\mathcal{H}_3$; a contradiction. By Theorem 1.9.2, B is then a $2-$regulus in $PG(5,q)$. We continue in this way until all planes of type (II) have been considered. Each plane of type (II) therefore defines a $2-$regulus of planes of type (II) in $PG(5,q)$ with transversals all elements of $\mathcal{S}$; each such $2-$regulus corresponds to a Baer subplane of $PG(2,q^2)$ in the Bose representation and every Baer subplane of $PG(2,q^2)$ in the Bose representation is obtained in this way. □

We now prove the existence of Bose spreads in $PG(5,q)$.

Lemma 3.2.8 *It is possible to embed $\Pi_{2,q^2} = PG(2,q^2)$ in $PG(5,q^2)$ in such a way that $\Pi_{2,q^2} = PG(2,q^2)$ is disjoint from $PG(5,q)$, the real Baer $5-$space of $PG(5,q^2)$.*

Proof By Sved's result 1.3.1 a hyperplane of $PG(5,q^2)$ intersects $PG(5,q)$ in either a $4-$space or a $3-$space of $PG(5,q)$. Each subspace S_n of $PG(5,q)$ of dimension n extends uniquely to an $n-$space S_{n,q^2} over $GF(q^2)$, since a basis for S_n as a vector space over $GF(q)$ is a basis for S_{n,q^2} as a vector space over $GF(q^2)$. Furthermore, there are more hyperplanes in $PG(5,q^2)$ than in $PG(5,q)$. It follows that there exist hyperplanes of $PG(5,q^2)$ which intersect $PG(5,q)$ in a $3-$space of $PG(5,q)$.

Let Σ_{4,q^2} be a hyperplane of $PG(5,q^2)$ such that the intersection $\Sigma_{4,q^2} \cap PG(5,q)$ is a $3-$space; denote this $3-$space by Σ_3. In Σ_{4,q^2} the $3-$space Σ_3 extends uniquely to a $3-$space Σ_{3,q^2} over $GF(q^2)$. Let ℓ be a line of Σ_{3,q^2} which is skew to Σ_3. Let m be a line of $\Sigma_{4,q^2} \backslash \Sigma_{3,q^2}$ which intersects Σ_{3,q^2} in a unique point of ℓ. Note that m is therefore disjoint from $PG(5,q)$. The plane spanned by ℓ and m is disjoint from Σ_3 and is therefore disjoint from $PG(5,q)$. □

By Lemma 3.2.8 we can embed $\Pi_{2,q^2} = PG(2,q^2)$ in $PG(5,q^2)$ in such a way that Π_{2,q^2} is disjoint from the Baer $5-$space $PG(5,q)$ of $PG(5,q^2)$ and therefore no point of Π_{2,q^2} is fixed by the Fröbenius automorphism. The conjugate plane $\overline{\Pi}_{2,q^2}$ is then disjoint from Π_{2,q^2} and disjoint from $PG(5,q)$.

The join of each point P in Π_{2,q^2} to its conjugate point $\overline{P}$, with respect to the extension $GF(q^2)$ of $GF(q)$, is a line $P\overline{P}$ which intersects $PG(5,q)$ in a Baer subline of $P\overline{P}$. In this way we obtain a set $\mathcal{S}$ of q^4+q^2+1 lines of $PG(5,q)$, one through each point P of Π_{2,q^2} and its conjugate $\overline{P}$ in $\overline{\Pi}_{2,q^2}$.

Let ℓ_1 and ℓ_2 be two distinct lines of $\mathcal{S}$ in $PG(5,q)$. If ℓ_1 and ℓ_2 intersect, then since the plane spanned by ℓ_1 and ℓ_2 contains a line in Π_{2,q^2} and a line in $\overline{\Pi}_{2,q^2}$, the planes Π_{2,q^2} and $\overline{\Pi}_{2,q^2}$ intersect in a point; a contradiction. The lines of $\mathcal{S}$ are therefore pairwise disjoint and since $\mathcal{S}$ contains q^4+q^2+1 elements we have that $\mathcal{S}$ is a spread of lines in $PG(5,q)$.

It remains to prove that $\mathcal{S}$ is a Bose spread, that is, to prove that every $3-$space of $PG(5,q)$ spanned by two distinct elements of $\mathcal{S}$ contains exactly q^2+1 elements of $\mathcal{S}$.

For each point P_i of Π_{2,q^2} denote by ℓ_i the line in $\mathcal{S}$ incident with P_i. Let ℓ_i and ℓ_j be distinct elements of $\mathcal{S}$ in $PG(5,q)$. The $3-$space Σ_3 of $PG(5,q)$ spanned by ℓ_i and ℓ_j extends uniquely to a $3-$space Σ_{3,q^2} over $GF(q^2)$. The lines P_iP_j of Π_{2,q^2} and $\overline{P_iP_j}$ of $\overline{\Pi}_{2,q^2}$ are contained in Σ_{3,q^2} and the regular spread of Σ_3 determined by P_iP_j and $\overline{P_iP_j}$ consists of elements of $\mathcal{S}$ (see Theorem 1.9.6). Hence $\Sigma_3 = \langle \ell_i, \ell_j \rangle$ contains exactly q^2+1 elements of $\mathcal{S}$. We have therefore shown,

Lemma 3.2.9 *If $\Pi_{2,q^2} = PG(2,q^2)$ is embedded in $PG(5,q^2)$ in such a way that Π_{2,q^2} is disjoint from the Baer $5-$space $PG(5,q)$ of $PG(5,q^2)$, then if each point P of Π_{2,q^2} is joined to its conjugate point $\overline{P}$ with respect to the extension $GF(q^2)$ of $GF(q)$ we obtain a collection $\mathcal{S}$ of q^4+q^2+1 lines of $PG(5,q)$.*

The set $\mathcal{S}$ of lines of $PG(5,q)$ so constructed is a Bose spread of $PG(5,q)$ which we shall call a **canonical Bose spread***. $\square$

By this construction of a Bose spread $\mathcal{S}$, the isomorphism between $\Pi_{2,q^2} = PG(2,q^2)$ and the incidence structure $\pi_{q^2}(\mathcal{S})$, determined in Theorem 3.2.4, arises in a very natural way. Furthermore, for the Bose representation of $PG(2,q^2)$ defined by this construction of a Bose spread of $PG(5,q)$, the representation of the Baer subplanes of $PG(2,q^2)$ also arises in a natural way and is determined in the following theorem; a special case of [50, Lemma 25.6.8].

Theorem 3.2.10 [50, Lemma 25.6.8] *Let $PG(5,q^2)$ be an extension of the projective space $PG(5,q)$. In $PG(5,q^2)$, let $\Pi_{2,q}$ be a $2-$space over $GF(q)$ skew to $PG(5,q)$. If $P \in \Pi_{2,q}$ and if $\overline{P}$ is the conjugate point of P with respect to the extension $GF(q^2)$ of $GF(q)$, then the intersection of a line $P\overline{P}$ and the space $PG(5,q)$ is a line ℓ of $PG(5,q)$. These lines ℓ form a system of maximal spaces of a Segre variety $\rho_{1;2}$ of $PG(5,q)$.*

Finally we show that all Bose spreads of $PG(5,q)$ are equivalent to a Bose spread constructed as in Lemma 3.2.9.

Theorem 3.2.11 *If $\mathcal{S}$ is a Bose spread of $PG(5,q)$, then $\mathcal{S}$ is a canonical Bose spread of $PG(5,q)$.*

Proof Let $\mathcal{S}$ be a Bose spread of $PG(5,q)$ and let $\pi_{q^2}(\mathcal{S})$ be the Bose representation of $PG(2,q^2)$ in $PG(5,q)$ defined by $\mathcal{S}$. Embed $PG(5,q)$ as a Baer subspace in $PG(5,q^2)$. By Theorem 3.2.7 each Baer subplane of $\pi_{q^2}(\mathcal{S})$ is a $2-$regulus in $PG(5,q)$ with the property that the transversals of the $2-$regulus are all elements of $\mathcal{S}$. By Theorem 1.9.5, the $2-$reguli in $PG(5,q)$ are projectively equivalent. Choose a Baer subplane B in $\pi_{q^2}(\mathcal{S})$ and let $\rho_{1;2}$ be the corresponding $2-$regulus in $PG(5,q)$. The $2-$regulus $\rho_{1;2}$ in $PG(5,q)$ extends uniquely to a $2-$regulus in $PG(5,q^2)$ which we shall denote by $\rho_{(q^2)1;2}$. The q^2+q+1 transversals of $\rho_{1;2}$ in $PG(5,q)$ are therefore all elements of $\mathcal{S}$ and are transversals of $\rho_{(q^2)1;2}$ when considered as lines over $GF(q^2)$.

The Baer sublines of B are represented by the reguli of elements of $\mathcal{S}$ contained in $\rho_{1;2}$. Over $GF(q^2)$, these reguli have transversals, one in each axis plane of $\rho_{(q^2)1;2}$. Each element ℓ_i of $\mathcal{S}$ not in B is contained in a unique element Σ_i of $\mathcal{H}_3$ such that Σ_i is an element (a line) of B. In Σ_i, the line ℓ_i is disjoint from the regulus of elements of $\mathcal{S}$

which are the points of B in Σ_i. Therefore, over $GF(q^2)$, ℓ_i intersects this regulus, which is a 3−dimensional hyperbolic quadric, in two conjugate points P_i and $\overline{P}_i$, where P_i and $\overline{P_i}$ lie in distinct and conjugate axis planes of $\rho_{(q^2)1;2}$. Note that P_i and $\overline{P}_i$ are the only points of ℓ_i incident with the 2−regulus $\rho_{(q^2)1;2}$, since otherwise ℓ_i would be a transversal line of $\rho_{(q^2)1;2}$ and would then either be a point of B or be disjoint to $PG(5,q)$; in each case we have a contradiction.

Each of the q^2+q+1 elements of $\mathcal{H}_3$ in B contain a regular spread of elements of $\mathcal{S}$ and each such regular spread is defined by a line (and its conjugate line) over $GF(q^2)$ which is contained in an axis plane (and the conjugate axis plane respectively) of $\rho_{(q^2)1;2}$; denote these lines by $m_1, m_2, \ldots, m_{q^2+q+1}$ and the corresponding conjugate lines by $\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_{q^2+q+1}$. Note that every element ℓ_i in $\mathcal{S}$ is incident with at least one line m_j; if $\ell_i \in B$, then ℓ_i as a line over $GF(q^2)$ is incident with $q+1$ lines m_j and if $\ell_i \in \mathcal{S}\backslash B$, then ℓ_i is incident with a unique line m_j.

Since there are $(q^2-q)/2$ pairs of conjugate axis planes of $\rho_{(q^2)1;2}$ which are disjoint from $PG(5,q)$, at least one of these axis planes contains two of the lines m_j; denote this plane by Π_{2,q^2} and suppose m_1 and m_2 are contained in Π_{2,q^2}. Consider four distinct points of Π_{2,q^2}, two incident with m_1, two incident with m_2 and such that the four points form a quadrangle in Π_{2,q^2}. The elements of $\mathcal{S}$ incident with these four points correspond to a quadrangle of points in $\pi_{q^2}(\mathcal{S})$ which therefore defines a unique Baer subplane B' of $\pi_{q^2}(\mathcal{S})$. By Theorem 3.2.7, B' corresponds to a 2−regulus ρ of elements of $\mathcal{S}$. Over $GF(q^2)$ the 2−regulus ρ extends uniquely to a 2−regulus ρ_{q^2} and since Π_{2,q^2} intersects ρ_{q^2} in four distinct points, no three collinear, the plane Π_{2,q^2} is an axis plane of ρ_{q^2}. Therefore every element of $\mathcal{S}$ which is a point of B' is incident with Π_{2,q^2} and incident with the conjugate plane $\overline{\Pi}_{2,q^2}$. By considering all choices of a quadrangle of points incident with the lines m_1 and m_2 in Π_{2,q^2} and since every point of Π_{2,q^2} lies in at least one Baer subplane of Π_{2,q^2} which contains the lines m_1 and m_2, we obtain that every element of $\mathcal{S}$ is incident with Π_{2,q^2} and incident with the conjugate plane $\overline{\Pi}_{2,q^2}$.

Hence every element of the spread $\mathcal{S}$ of $PG(5,q)$ is obtained by joining a point P of Π_{2,q^2} to its conjugate $\overline{P}$, where Π_{2,q^2} is a plane of $PG(5,q^2)$ skew to $PG(5,q)$. The spread $\mathcal{S}$ is therefore a canonical Bose spread of $PG(5,q)$. $\square$

The original method of Bose in [18] was to obtain a coordinate representation of $PG(2,q^2)$ in $PG(5,q)$. Since we shall require this coordinate representation for some later calculations, we briefly present this work of Bose.

Let α be a primitive root of $GF(q^2)$. Then α satisfies an equation

$$\alpha^2 = \alpha + \gamma$$

where $x^2 - x - \gamma$ is irreducible over $GF(q)$.

Any point of $PG(2, q^2)$ has coordinates $(x, y, z) \neq (0, 0, 0)$, $x, y, z \in GF(q^2)$ where (x, y, z) and $(\rho x, \rho y, \rho z)$, $\rho \in GF(q^2)\backslash\{0\}$, represent the same point in homogeneous coordinates.

Since $\{1, \alpha\}$ is a basis for $GF(q^2)$ as a vector space over $GF(q)$ we can write $x = x_0 + \alpha x_1,\ y = y_0 + \alpha y_1,\ z = z_0 + \alpha z_1$ for a unique choice of $x_i, y_i, z_i \in GF(q)$.

We let the coordinates (x, y, z) in $PG(2, q^2)$ correspond to the coordinates $(x_0, x_1, y_0, y_1, z_0, z_1)$ in $PG(5, q)$. Note that $(x_0, x_1, y_0, y_1, z_0, z_1)$ and $(rx_0, rx_1, ry_0, ry_1, rz_0, rz_1)$, $r \in GF(q)\backslash\{0\}$, represent the same point in $PG(5, q)$, and they correspond to the points (x, y, z), (rx, ry, rz) respectively; the same point of $PG(2, q^2)$.

Consider three distinct elements $a, b, c \in GF(q^2)\backslash\{0\}$. As $GF(q^2)$ is a $2-$dimensional vector space over $GF(q)$, the three elements a, b, c are linearly dependent over $GF(q)$. Therefore there exist $\lambda_1, \lambda_2, \lambda_3 \in GF(q)$ not all zero and such that

$$\lambda_1 a + \lambda_2 b + \lambda_3 c = 0. \tag{3.1}$$

If $a = a_0 + \alpha a_1,\ b = b_0 + \alpha b_1,\ c = c_0 + \alpha c_1$, where $a_i, b_i, c_i \in GF(q)$, then from (3.1) we have,

$$\begin{aligned} \lambda_1 a_0 + \lambda_2 b_0 + \lambda_3 c_0 &= 0 \\ \text{and } \lambda_1 a_1 + \lambda_2 b_1 + \lambda_3 c_1 &= 0 \end{aligned} \tag{3.2}$$

Consider the triplets $(ax, ay, az),\ (bx, by, bz),\ (cx, cy, cz)$ which represent the same point (x, y, z) of $PG(2, q^2)$. Since for example,

$$(a_0 + \alpha a_1)(x_0 + \alpha x_1) = a_0 x_0 + \gamma a_1 x_1 + \alpha(a_1 x_0 + a_0 x_1 + a_1 x_1),$$

the triplets correspond to the sextuplets,

$$(a_0x_0 + \gamma a_1x_1, a_1x_0 + a_0x_1 + a_1x_1, a_0y_0 + \gamma a_1y_1, a_1y_0 + a_0y_1 + a_1y_1, a_0z_0 + \gamma a_1z_1, a_1z_0 + a_0z_1 + a_1z_1)$$
$$(b_0x_0 + \gamma b_1x_1, b_1x_0 + b_0x_1 + b_1x_1, b_0y_0 + \gamma b_1y_1, b_1y_0 + b_0y_1 + b_1y_1, b_0z_0 + \gamma b_1z_1, b_1z_0 + b_0z_1 + b_1z_1)$$
$$(c_0x_0 + \gamma c_1x_1, c_1x_0 + c_0x_1 + c_1x_1, c_0y_0 + \gamma c_1y_1, c_1y_0 + c_0y_1 + c_1y_1, c_0z_0 + \gamma c_1z_1, c_1z_0 + c_0z_1 + c_1z_1)$$

respectively.

By (3.2), these sextuplets are linearly dependent over $GF(q)$ and are therefore the coordinates of three collinear points in $PG(5,q)$ whenever a, b, c are from distinct cosets of $GF(q)\backslash\{0\}$ in the multiplicative group $GF(q^2)\backslash\{0\}$. Therefore as ρ varies over the q^2-1 non-zero elements of $GF(q^2)$ the sextuplets corresponding to $(\rho x, \rho y, \rho z)$ represent $q+1$ collinear points of $PG(5,q)$.

In this way, the q^4+q^2+1 points of $PG(2,q^2)$ can be made to correspond to a set of q^4+q^2+1 lines $\mathcal{S}_B$ of $PG(5,q)$, each point corresponding to one line.

Theorem 3.2.12 [18, Theorem1.1, Theorem 2.2] *$\mathcal{S}_B$ is a spread (of lines) of $PG(5,q)$. Moreover, for each pair of distinct elements ℓ_1, ℓ_2 of $\mathcal{S}_B$ the $3-$space determined by ℓ_1 and ℓ_2 contains exactly q^2+1 elements of $\mathcal{S}_B$; that is, $\mathcal{S}_B$ is a Bose spread of $PG(5,q)$.*

We call the Bose spread $\mathcal{S}_B$ the **coordinate Bose spread** *of $PG(5,q)$.* □

In this coordinate setting, Bose proved the representation of Baer subplanes of $PG(2,q^2)$ in $PG(5,q)$ as given in Theorem 3.2.7. For example, consider the Baer subplane $PG(2,q)$ of $PG(2,q^2)$, and let $\rho_i = r_{i0} + \alpha r_{i1},\ i = 1,2,\ldots,q+1$ be $q+1$ elements of $GF(q^2)$, one from each coset of $GF(q)\backslash\{0\}$ in the multiplicative group $GF(q^2)\backslash\{0\}$. We may choose $\rho_1 = 1$, the identity. Then each point $(x,y,z) \in PG(2,q)$ corresponds to a line of $\mathcal{S}_B$ in $PG(5,q)$ whose points are given by,

$$\{(xr_{i0}, xr_{i1}, yr_{i0}, yr_{i1}, zr_{i0}, zr_{i1}) \mid i = 1,2,\ldots,q+1\}.$$

The collection of q^2+q+1 lines of $\mathcal{S}_B$ obtained in this way are then the set of maximal spaces (transversals) of the Segre variety $\rho_{1;2}$ in $PG(5,q)$ defined by the following equations, for points with coordinates $(x_0, x_1, y_0, y_1, z_0, z_1)$,

$$\begin{aligned} x_0y_1 - x_1y_0 &= 0 \\ x_0z_1 - x_1z_0 &= 0 \\ y_0z_1 - y_1z_0 &= 0; \end{aligned}$$

(a Segre variety in $PG(5,q)$ by Theorem 1.8.6 and the subsequent remarks.)

The opposite system of maximal spaces of the Segre variety $\rho_{1;2}$, the $q+1$ axis planes contained in this Segre variety, are each defined by a set of equations of the form,

$$\begin{aligned} r_{i1}x_0 - r_{i0}x_1 &= 0 \\ r_{i1}y_0 - r_{i0}y_1 &= 0 \\ r_{i1}z_0 - r_{i0}z_1 &= 0, \end{aligned}$$

for a fixed $i \in \{1, 2, \ldots, q+1\}$.

Also, for each non-zero element $\rho \in GF(q^2)$, the transformation $(x, y, z) \mapsto (\rho x, \rho y, \rho z)$ of $PG(2, q^2)$ fixes every point in $PG(2, q^2)$ and therefore fixes the canonical Bose spread $\mathcal{S}_B$ of $PG(5, q)$.

Finally, we examine the relationship between the Bose representation of $PG(2, q^2)$ in $PG(5, q)$ and the Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$ with hyperplane Σ_∞ at infinity and $\mathcal{S}_\infty$ a regular $1-$spread of Σ_∞.

Consider the Bose representation of $PG(2, q^2)$ defined by a Bose spread $\mathcal{S}$ of $PG(5, q)$ as presented in this section and with the same notation. Let $PG(4, q)$ be a hyperplane of $PG(5, q)$. By Theorem 3.2.3, the set of $3-$spaces $\mathcal{H}_3$ associated to $\mathcal{S}$ is a dual spread of $PG(5, q)$ and hence the hyperplane $PG(4, q)$ contains a unique element of $\mathcal{H}_3$; denote this element of $\mathcal{H}_3$ by Σ_∞. By Corollary 3.2.5, the $3-$space Σ_∞ contains exactly $q^2 + 1$ elements of $\mathcal{S}$ and these $q^2 + 1$ lines constitute a regular $1-$spread of Σ_∞. Moreover, each element of $\mathcal{S}$ not in Σ_∞ is skew to Σ_∞ and therefore intersects $PG(4, q)$ in a unique point of $PG(4, q)\backslash\Sigma_\infty$. By the Bose correspondence between the points of $PG(2, q^2)$ and the elements of $\mathcal{S}$, each point of $PG(2, q^2)$ corresponds either to an element of $\mathcal{S}$ in Σ_∞ or to a unique point of $PG(4, q)\backslash\Sigma_\infty$. Also, as the lines of $PG(2, q^2)$ correspond to the elements of $\mathcal{H}_3$, and any two elements of $\mathcal{H}_3$ intersect exactly in a line of $\mathcal{S}$, each line of $PG(2, q^2)$ corresponds to either Σ_∞ or to a plane of $PG(4, q)$ not contained in Σ_∞ and which intersects Σ_∞ in an element of $\mathcal{S}$. We therefore have by Section 1.10,

Theorem 3.2.13 [19] *Given a Bose representation of $PG(2, q^2)$ in $PG(5, q)$, a Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$ is obtained by considering any fixed hyperplane $PG(4, q)$ of $PG(5, q)$ and redefining each point and line of $PG(2, q^2)$ to be the intersection of the corresponding subspace in the Bose representation with $PG(4, q)$.* $\square$

Corollary 3.2.14 [19] *In the Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$ defined by a regular spread $\mathcal{S}_\infty$ in a hyperplane Σ_∞: each Baer subplane of $PG(2, q^2)$ is either a plane of $PG(4, q)$ not contained in Σ_∞ and which intersects Σ_∞ in line not in $\mathcal{S}_\infty$, or a ruled cubic surface V_2^3 not contained in Σ_∞ and which intersects Σ_∞ in a line $\ell \in \mathcal{S}_\infty$ which is the line directrix of V_2^3.*

Proof Consider a Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$ obtained from a Bose representation of $PG(2, q^2)$ in $PG(5, q)$ as in Theorem 3.2.13 and with the notation introduced there. By Theorem 3.2.7, in the Bose representation of $PG(2,q^2)$ each Baer subplane of $PG(2,q^2)$ is a Segre variety $\rho_{1;2}$, with transversal lines all in the Bose spread $\mathcal{S}$. Each such variety has order 3 and dimension 3 and we shall denote it by R_3^3. A Segre variety R_3^3 intersects the hyperplane $PG(4,q)$ in a variety of order 3 and dimension 2; there are two cases to consider. Let R_3^3 be the Segre variety in $PG(5,q)$ which is the Bose representation of a Baer subplane B of $PG(2,q^2)$. Suppose the Baer subplane B of $PG(2,q^2)$ contains the line at infinity as a line, then in the Bose representation $\Sigma_\infty \in \mathcal{H}_3$ intersects the Segre variety R_3^3 in a $1-$regulus, a hyperbolic quadric H_2^2. In this case, the intersection $PG(4,q) \cap R_3^3$ is the variety $V_2^3 = H_2^2 \cup S_2^1$, the union of the hyperbolic quadric H_2^2 and an axis plane S_2^1 of the Segre variety R_3^3.

Alternatively, suppose the Baer subplane B of $PG(2,q^2)$ intersects the line at infinity in a unique point, then in the Bose representation $\Sigma_\infty \in \mathcal{H}_3$ intersects the Segre variety R_3^3 exactly in a single element ℓ of $\mathcal{S}$. In this case, the intersection $PG(4,q) \cap R_3^3$ is a variety V_2^3 containing $\ell \in \mathcal{S}$. Each element of $\mathcal{H}_3$ which contains ℓ and which represents a line of B in the Bose representation, intersects R_3^3 in a $1-$regulus of elements of $\mathcal{S}$. These $1-$reguli each intersect $PG(4,q)$ in a degenerate conic, namely the line ℓ and a transversal to the $1-$regulus. Hence the variety $V_2^3 = PG(4,q) \cap R_3^3$ consists of the line ℓ and $q+1$ lines of $PG(4,q)\backslash\Sigma_\infty$ which meet ℓ; such a variety is a ruled cubic surface with line directrix ℓ. $\square$

3.3 The Bose representation of Conics in $\mathbf{PG(2,q^2)}$

Consider the conic $\mathcal{C}$ in $PG(2,q^2)$ with points (x,y,z), $x,y,z \in GF(q^2)$ and not all zero, satisfying the equation $y^2 = xz$.

Let $GF(q^2) = GF(q)(\alpha)$ where $\alpha \in GF(q^2)\backslash GF(q)$ has minimal polynomial $p_\alpha(x) = x^2 - x - \gamma$ as in the previous section. Moreover, if q is even, then γ has trace$(\gamma) = 1$ and for q odd, γ has the property that $1+4\gamma$ is a non-square, since $x^2 - x - \gamma$ is irreducible in $GF(q)$ by [48, Section 1.4]. The element $\overline{\alpha} = \alpha^q$ is the second root of p_α and therefore

$$\begin{aligned} \alpha + \overline{\alpha} &= 1 \\ \text{and} \qquad \alpha\overline{\alpha} &= -\gamma. \end{aligned} \tag{3.3}$$

Consider the conic $\mathcal{C}$ in the Bose representation of $PG(2,q^2)$ in $PG(5,q)$, defined by the coordinate Bose spread $\mathcal{S}_B$ of the previous section. The points (x,y,z) of $\mathcal{C}$ in $PG(2,q^2)$ correspond to the points $(x_0,x_1,y_0,y_1,z_0,z_1)$ of $PG(5,q)$ which satisfy

$$(y_0+\alpha y_1)^2 \;=\; (x_0+\alpha x_1)(z_0+\alpha z_1) \tag{3.4}$$

where $x = x_0+\alpha x_1,\; y = y_0+\alpha y_1,\; z = z_0+\alpha z_1$ with $x_i, y_i, z_i \in GF(q)$. By expanding (3.4) and using the equations (3.3) and $\alpha^2 = \alpha+\gamma$ to simplify the expression, we obtain

$$y_0^2+\gamma y_1^2+\alpha(2y_0y_1+y_1^2) \;=\; x_0z_0+\gamma x_1z_1+\alpha(x_1z_0+x_0z_1+x_1z_1).$$

Thus the conic $\mathcal{C}$ in $PG(2,q^2)$ in the Bose representation is the subset of q^2+1 elements of the Bose spread $\mathcal{S}_B$, no three contained in the same $3-$space of $\mathcal{H}_3$, with points $(x_0,x_1,y_0,y_1,z_0,z_1)$ in $PG(5,q)$ contained in the intersection of the two quadrics Q_1 and Q_2 with equations

$$\begin{aligned} y_0^2+\gamma y_1^2-x_0z_0-\gamma x_1z_1 &= 0, && (3.5)\\ \text{and}\qquad y_1^2+2y_0y_1-x_1z_0-x_0z_1-x_1z_1 &= 0 && (3.6)\end{aligned}$$

respectively.

We now determine if each quadric Q_i is non-singular, and if so, we determine the characteristic (hyperbolic or elliptic) of the quadric. Our method and notation are consistent with that used in Section 1.4 ([50, Section 22.2] and in particular [50, Theorem 22.2.1]).

Let $\mathbf{x} = (x_0,x_1,y_0,y_1,z_0,z_1)$, then Q_1 and Q_2 are given by the quadratic forms defined by matrices

$$M_1 = \begin{bmatrix} 0&0&0&0&-1&0\\ 0&0&0&0&0&-\gamma\\ 0&0&1&0&0&0\\ 0&0&0&\gamma&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0&0&0&0&0&-1\\ 0&0&0&0&-1&-1\\ 0&0&0&2&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}$$

respectively. For the quadric Q_1, let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\gamma & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $|A| = 4\gamma^3 \neq 0$ for q odd and $|A| = 4\gamma^3 = 0$ for q even. When q is odd, let $a = -4\gamma^3$. By Theorem 1.4.1 for q odd, the quadric Q_1 is non-singular and Q_1 is elliptic if and only if a is a non-square; for q even, the quadric Q_1 is singular.

For the quadric Q_2, let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $|A| = 4 \neq 0$ for q odd and $|A| = 4 = 0$ for q even. When q is odd, let $a = -4$. By Theorem 1.4.1 for q odd, the quadric Q_2 is non-singular and Q_2 is elliptic if and only if a is a non-square; for q even, the quadric Q_2 is singular.

Thus for q odd, the quadrics Q_1 and Q_2 are both non-singular and for q even the quadrics Q_1 and Q_2 are both singular.

Consider the case when q is even. In this case the conic $\mathcal{C}$ in $PG(2, q^2)$ has nucleus $N(0, 1, 0)$. In the coordinate Bose representation of $PG(2, q^2)$ in $PG(5, q)$, the nucleus is represented by the line joining points $(0, 0, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 0, 0)$; denote this line by ℓ_N. The line ℓ_N intersects the quadric Q_1 in the point $P_1(0, 0, \sqrt{\gamma}, 1, 0, 0)$ (since q is even, γ is a square) and ℓ_N intersects Q_2 in the point $P_2(0, 0, 1, 0, 0, 0)$. Consider Π_{P_1} the tangent space to Q_1 at the point P_1. Following Section 1.4, the partial derivatives of the quadratic form defining Q_1 are,

$$\tfrac{\partial Q_1}{\partial x_0} = -z_0 \quad \tfrac{\partial Q_1}{\partial x_1} = -\gamma z_1 \quad \tfrac{\partial Q_1}{\partial y_0} = 2y_0 \quad \tfrac{\partial Q_1}{\partial y_1} = 2\gamma y_1 \quad \tfrac{\partial Q_1}{\partial z_0} = -x_0 \quad \tfrac{\partial Q_1}{\partial z_1} = -\gamma x_1,$$

which all equal zero at $P_1(0, 0, \sqrt{\gamma}, 1, 0, 0)$. Thus Π_{P_1} is the entire space and therefore P_1 is contained in the vertex of Q_1.

Consider Π_{P_2} the tangent space to Q_2 at the point P_2. The partial derivatives of the quadratic form defining Q_2 are given by,

$$\frac{\partial Q_2}{\partial x_0} = -z_1 \quad \frac{\partial Q_2}{\partial x_1} = -z_1 - z_0 \quad \frac{\partial Q_2}{\partial y_0} = 2y_1 \quad \frac{\partial Q_2}{\partial y_1} = 2y_1 + 2y_0 \quad \frac{\partial Q_2}{\partial z_0} = -x_1 \quad \frac{\partial Q_2}{\partial z_1} = -x_1 - x_0,$$

which all equal zero at $P_2(0,0,1,0,0,0)$. Thus Π_{P_2} is the entire space and therefore P_2 is contained in the vertex of Q_2.

Moreover, for $i = 1,2$ respectively, P_i is the only point of Q_i such that $\Pi_{P_i} = PG(5,q)$ and therefore each of the quadrics Q_1, Q_2 has a point vertex for q even and the vertex is incident with ℓ_N. In the notation of [50], $Q_i = \Pi_0\mathcal{P}_4$, that is Q_i has a point vertex and base a parabolic quadric in a hyperplane disjoint from the vertex. A further verification that the vertex of Q_i is a point vertex is to check that a hyperplane of $PG(5,q)$ not incident with P_i intersects Q_i in a non-singular quadric. Consider the hyperplane Σ_0 of $PG(5,q)$ defined by the equation $y_0 = 0$ and which contains neither P_1 nor P_2. The intersection $\Sigma_0 \cap \ell_N$ is the point with coordinates $(0,0,0,1,0,0)$. For $i = 1,2$, let $Q_{i,0}$ be the quadric in Σ_0 such that $Q_{i,0} = Q_i \cap \Sigma_0$. Then each quadric $Q_{i,0}$ has points with coordinates $(x_0, x_1, 0, y_1, z_0, z_1)$ satisfying the equations

$$\begin{aligned} Q_{1,0}: \quad y_1^2 - x_1z_0 - x_0z_1 - x_1z_1 &= 0 \\ Q_{2,0}: \quad \gamma y_1^2 - x_0z_0 - \gamma x_1z_1 &= 0 \end{aligned}$$

respectively. For each of $Q_{1,0}$ and $Q_{2,0}$, $\frac{1}{2}|A| \neq 0$ and therefore the quadrics $Q_{1,0}$ and $Q_{2,0}$ are non-singular parabolic quadrics by Theorem 1.4.1. Furthermore, for $i = 1,2$, the point $(0,0,0,1,0,0)$ is the unique point in Σ_0 at which $\dfrac{\partial Q_{i,0}}{\partial x_j} = \dfrac{\partial Q_{i,0}}{\partial y_j} = \dfrac{\partial Q_{i,0}}{\partial z_j} = 0$, and therefore $(0,0,0,1,0,0)$ is the nucleus of $Q_{i,0}$. Hence for each $i = 1,2$, the quadric Q_i in $PG(5,q)$ in non-singular with point vertex and a parabolic quadric base.

Moreover, since the partial derivatives of both Q_1 and Q_2 all equal zero when evaluated at any point of the line ℓ_N, if Σ is a hyperplane of $PG(5,q)$ which intersects ℓ_N in a unique point distinct from P_1 and P_2, then the intersections $\Sigma \cap Q_1$ and $\Sigma \cap Q_2$ are both parabolic quadrics in Σ with common nucleus $\Sigma \cap \ell_N$. It also follows that if Σ_k is a hyperplane of $PG(5,q)$ such that the intersection $\Sigma_k \cap \ell_N$ is exactly the point P_1, then the intersection $\Sigma_k \cap Q_1$ is a singular quadric with point vertex P_1 and the intersection $\Sigma_k \cap Q_2$ is a non-singular quadric with nucleus P_1. Similarly, if Σ_k is a hyperplane of $PG(5,q)$ such that the intersection $\Sigma_k \cap \ell_N$ is exactly the point P_2, then the intersection $\Sigma_k \cap Q_2$ is a singular quadric with point vertex P_2 and the intersection $\Sigma_k \cap Q_1$ is a non-singular quadric with nucleus P_2.

For $\mathcal{S}$ a Bose spread of $PG(5,q)$ and $\pi_{q^2}(\mathcal{S})$ the Bose representation of $PG(2,q^2)$ in $PG(5,q)$ defined by $\mathcal{S}$ and $\mathcal{H}_3$ the dual spread associated with $\mathcal{S}$, we have therefore shown,

Theorem 3.3.1 *If $\mathcal{C}$ is a non-degenerate conic in $PG(2,q^2)$, then in the Bose representation of $PG(2,q^2)$ in $PG(5,q)$, the conic is a collection $\mathcal{C}_B$ of q^2+1 elements of $\mathcal{S}$, no three contained in the same $3-$space of $\mathcal{H}_3$. Furthermore, for q odd, the points of $\mathcal{C}_B$ lie in the intersection of two non-singular quadrics in $PG(5,q)$; for q even, the points of $\mathcal{C}_B$ lie in the intersection of two singular quadrics Q_1 and Q_2, each of which has a point vertex and a parabolic quadric base. In the q even case, the nucleus N of $\mathcal{C}$ is a line ℓ_N in the Bose representation and the point vertices of Q_1 and Q_2 are distinct points of ℓ_N.*
□

Consider a non-degenerate conic $\mathcal{C}$ in $PG(2,q^2)$ and let ℓ_∞ denote an external line of $\mathcal{C}$; call this line the line at infinity of $PG(2,q^2)$. If q is even, then the nucleus N of $\mathcal{C}$ is not incident with ℓ_∞. In the Bose representation of $PG(2,q^2)$ in $PG(5,q)$, the line at infinity is a $3-$space element Σ_∞ of $\mathcal{H}_3$ which is disjoint to the set of q^2+1 elements of $\mathcal{S}$ in the Bose representation $\mathcal{C}_B$ of the conic $\mathcal{C}$. Moreover Σ_∞ is disjoint from the element ℓ_N of $\mathcal{S}$ which corresponds to the nucleus N in the q even case. In Theorem 3.2.13 we presented the relationship between a Bose representation of $PG(2,q^2)$ in $PG(5,q)$ and a Bruck-Bose representation of $PG(2,q^2)$ in $PG(4,q)$. Let Π_4 be a hyperplane of $PG(5,q)$ which contains the $3-$space Σ_∞; note that for the q even case, the intersection $\Pi_4 \cap \ell_N$ is a unique point. By Theorems 3.2.13, 3.3.1 and the remarks preceding Theorem 3.3.1 we have,

Theorem 3.3.2 *If $\mathcal{C}$ is a non-degenerate conic in $PG(2,q^2)$ disjoint from the line at infinity ℓ_∞, then in the Bruck-Bose representation of $PG(2,q^2)$ in $PG(4,q)$ the conic $\mathcal{C}$ is a collection of q^2+1 affine points contained in the intersection of two quadrics Q_1 and Q_2. For q even the two quadrics are either both non-singular with a common nucleus or exactly one of the quadrics is singular with a point vertex which is then the nucleus of the second quadric.* □

From our work in the earlier chapters, a Baer subline of a line of $PG(2,q^4)$ which is disjoint from the line at infinity, is represented in $4-$dimensional Bruck-Bose by a non-degenerate conic in a plane about a spread element and such that the conic is disjoint from the hyperplane at infinity (see Section 2.2). If we then represent $PG(2,q^4)$ in 8−dimensional Bruck-Bose we obtain a representation of the non-degenerate conic, which in turn represents a Baer subline of a line of $PG(2,q^4)$. Theorem 3.3.2 provides additional information about this representation which was first discussed in Theorem 2.7.2 in a slightly different setting.

3.4 The Bruck-Bose representation of $\mathbf{PG(2,q^4)}$ in $\mathbf{PG(8,q)}$ revisited

The Bruck-Bose representation of $PG(2,q^4)$ in $PG(8,q)$ is determined by a regular 3−spread $\mathcal{S}_3$ in a fixed hyperplane $\Sigma_{7,q}$ of $PG(8,q)$. We denote this representation by $\Pi_{8,q}$ and we denote by ℓ_∞ the line of $PG(2,q^4)$ which corresponds to the spread $\mathcal{S}_3$ in $\Sigma_{7,q}$; we call ℓ_∞ the *line at infinity.*

In the first section of this chapter we investigated the representation of the affine Baer subplanes of $PG(2,q^4)$ in $\Pi_{8,q}$. In Corollary 3.1.5 we characterised the affine Baer subplanes of $PG(2,q^4)$ in terms of this representation. Moreover we determined how Baer sublines b_ℓ of lines of $PG(2,q^4)$, and such that b_ℓ is incident with ℓ_∞, are represented in $\Pi_{8,q}$. We now consider the non-affine Baer subplanes of $PG(2,q^4)$ and the Baer sublines which are disjoint from ℓ_∞.

Let ℓ be a line in $PG(2,q^4)$ distinct from ℓ_∞ and let P be the unique point of intersection of ℓ and ℓ_∞. Let b_ℓ be a Baer subline of ℓ such that b_ℓ is disjoint from ℓ_∞, so that P is not incident with b_ℓ. Also let Π_{4,q^2} be the Bruck-Bose representation of $PG(2,q^4)$ in $PG(4,q^2)$ defined by a regular 1−spread $\mathcal{S}_1$ of a hyperplane Σ_{3,q^2} of $PG(4,q^2)$. In Π_{4,q^2}, ℓ is represented by a plane ℓ^* of $PG(4,q^2)\backslash\Sigma_{3,q^2}$ and the intersection $\ell^*\cap\Sigma_{3,q^2}$ is a line P^* which is an element of the spread $\mathcal{S}_1$. By Theorem 2.2.3, the Baer subline b_ℓ is represented by a non-degenerate conic b_ℓ^* in the plane ℓ^* such that b_ℓ^* is disjoint from P^*; in Section 2.2 we called such a conic a *Baer conic*. Note that the plane $\ell^*\backslash\{P^*\}$, namely ℓ^* with the line P^* and all its points removed, is isomorphic to the affine plane $AG(2,q^2)$. By the results on internal structures of a Miquelian inversive planes discussed in Section 1.14 we have: the points of $\ell^*\backslash\{P^*\}$ correspond to the points of ℓ distinct from P; the lines of $\ell^*\backslash\{P^*\}$ correspond to the Baer sublines of ℓ which contain P; incidence is containment.

In $\Pi_{8,q}$, the line ℓ is represented by a 4−space ℓ^{**} of $PG(8,q)\backslash\Sigma_{7,q}$ and the intersection $\ell^{**}\cap\Sigma_{7,q}$ is a 3−space element P^{**} of the regular 3−spread $\mathcal{S}_3$ of $\Sigma_{7,q}$. By Theorem 3.1.2, there exists a fixed induced regular 1−spread $\mathcal{S}^1_\ell$ in P^{**}. By Corollary 3.1.5 the planes of $\ell^{**}\backslash P^{**}$ which intersect P^{**} in a unique line of $\mathcal{S}^1_\ell$ represent the Baer sublines of ℓ which contain the point P. Hence this regular 1−spread $\mathcal{S}^1_\ell$ in P^{**} defines a 4−dimensional Bruck-Bose representation of $PG(2,q^2)$ in the 4−space ℓ^{**}; this is the 4−dimensional Bruck-Bose representation of the plane ℓ^*.

Let b^{**}_ℓ denote the representation in $\Pi_{8,q}$ of the Baer subline b_ℓ of ℓ. In Π_{4,q^2}, since b^*_ℓ is a non-degenerate conic in ℓ^*, disjoint from the line P^* in ℓ^*, it follows from the preceding paragraph that $b^{**}_\ell\subseteq\ell^{**}$ is precisely a representation in 4−dimensional Bruck-Bose of a non-degenerate conic in the plane $\ell^*=PG(2,q^2)$ and disjoint from the line at infinity, P^*. This representation was explicitly determined in Section 3.3, in particular in Theorem 3.3.2.

By Theorem 2.2.9, a Baer subplane B of $PG(2,q^4)$ which intersects ℓ_∞ in a unique point R is represented in Π_{4,q^2} by a ruled cubic surface B^* with line directrix R^* where R^* is an element of $\mathcal{S}_1$. Moreover, the intersection $B^*\cap\Sigma_{3,q^2}$ in Π_{4,q^2} is exactly the line R^* and the points of B^* lie on q^2+1 distinct lines of $\Pi_{4,q^2}\backslash\Sigma_{3,q^2}$, one through each point of R^*. These lines represent the Baer sublines in B which are incident with R. The remaining Baer sublines in B are represented in Π_{4,q^2} by q^2 Baer conics on the ruled cubic surface B^*.

In $\Pi_{8,q}$, the Baer subplane B is represented by a structure B^{**} in $PG(8,q)$. The point R in $PG(2,q^4)$ is represented by an element R^{**} of the spread $\mathcal{S}_3$ of $\Sigma_{7,q}$. By Theorem 3.1.2 and Corollary 3.1.5 and by considering the situation in Π_{4,q^2} above, the Baer sublines in B incident with R are represented in $\Pi_{8,q}$ by q^2+1 distinct planes in $PG(8,q)\backslash\Sigma_{7,q}$ each of which intersect $\Sigma_{7,q}$ in a distinct line of the induced 1−spread $\mathcal{S}^1_j$ in R^{**}. Moreover, as B^* contains q^2 Baer conics in Π_{4,q^2}, the structure B^{**} contains q^2 representations of Baer conics in $\Pi_{8,q}$, where each has the structure determined in Theorem 3.3.2 for a given 4−space of $\Pi_{8,q}$, which corresponds to a line of $PG(2,q^4)$.

It is difficult to determine in more helpful detail the Bruck-Bose representation of the non-affine Baer subplanes of $PG(2,q^4)$.

To conclude the chapter we present one more geometric construction which may help to clarify some of the geometric properties of these representations. For the plane $PG(2,q^4)$

we have been moving back and forth between the 4−dimensional Bruck-Bose representation in $PG(4,q^2)$ and the 8−dimensional Bruck-Bose representation in $PG(8,q)$; the following construction provides a concrete link between the two representations.

Let $\mathcal{S}$ be a Bose spread of lines of $PG(5,q^2)$, so that by the results of Section 3.2, $\mathcal{S}$ defines a Bose representation of $PG(2,q^4)$. Let $\mathcal{H}_3$ denote the dual spread of $PG(5,q^2)$ associated with $\mathcal{S}$ in the usual way. Let $PG(4,q^2)$ denote a fixed hyperplane of $PG(5,q^2)$. By Theorem 3.2.13, $PG(4,q^2)$ determines a fixed 4−dimensional Bruck-Bose representation Π_{4,q^2} of $PG(2,q^4)$. Denote by Σ_{3,q^2} the unique 3−space in $\mathcal{H}_3$ which is contained in $PG(4,q^2)$. By Theorem 3.2.5, the set $\mathcal{S}_1$ of q^4+1 elements of $\mathcal{S}$ contained in Σ_{3,q^2} constitutes a regular 1−spread of Σ_{3,q^2}.

Embed $PG(5,q^2)$ as a subspace in $PG(11,q^2)$ in such a way that $PG(5,q^2)$ is skew to $PG(11,q)$; this is possible by Construction 3.1.1 with $h=3$. The 3−space Σ_ℓ spanned by an element ℓ of the Bose spread $\mathcal{S}$ and its conjugate $\overline{\ell}$, with respect to the extension $GF(q^2)$ of $GF(q)$, intersects $PG(11,q)$ in a 3−space; the join of each point $P\in\ell$ to its conjugate $\overline{P}$ yields a regular 1−spread of the 3−space $\Sigma_\ell\cap PG(11,q)$. The collection of such 3−spaces in $PG(11,q)$ constitutes a 3−spread of $PG(11,q)$. The 7−space of $PG(11,q^2)$ spanned by Σ_{3,q^2} and its conjugate space $\overline{\Sigma}_{3,q^2}$ intersects $PG(11,q)$ in a 7−space which we shall denote by $\Sigma_{7,q}$. Note that since Σ_{3,q^2} contains q^4+1 distinct elements of $\mathcal{S}$, $\Sigma_{7,q}$ contains exactly q^4+1 elements of the 3−spread of $PG(11,q)$ which therefore constitute a 3−spread of $\Sigma_{7,q}$; denote this 3−spread of $\Sigma_{7,q}$ by $\mathcal{S}_3$. The hyperplane $PG(4,q^2)$ of $PG(5,q^2)$ together with its conjugate $\overline{PG(4,q^2)}$ spans a 9−space of $PG(11,q^2)$ which intersects $PG(11,q)$ in a 9−space which we shall denote by $PG(9,q)$. Note that $\Sigma_{7,q}$ is a subspace of $PG(9,q)$. Each point $P\in PG(4,q^2)\backslash\Sigma_{3,q^2}$ is incident with a line of $PG(9,q)$, namely the line $P\overline{P}$. If we let $PG(8,q)$ be a hyperplane of $PG(9,q)$ which contains $\Sigma_{7,q}$, then $PG(8,q)$ intersects each line $P\overline{P}$ in a unique point, where $P\in PG(4,q^2)\backslash\Sigma_{3,q^2}$. By Theorem 3.1.1 the 3−spread $\mathcal{S}_3$ in $\Sigma_{7,q}$ is regular since the 1−spread $\mathcal{S}_1$ in Σ_{3,q^2} is regular. Therefore $\mathcal{S}_3$ defines an 8−dimensional Bruck-Bose representation $\Pi_{8,q}$ of $PG(2,q^4)$ in $PG(8,q)$. Moreover, in this construction the correspondence between Π_{4,q^2} and $\Pi_{8,q}$ arises in a natural way.

Chapter 4

Baer subplanes and Buekenhout-Metz Unitals

In this chapter we investigate the relationship between Baer subplanes and unitals in planes where both of these objects are defined. In particular, in a finite projective plane π_{q^2} of order q^2, we consider the problem of classifying the subsets of points of the plane, which are the set of points in the intersection of a Baer subplane and a unital. The earliest work on this problem is due to Seib [70] and the relevant paper is written in German; the following statement of Seib's result is taken from [16, Lemma 2.1, (1) (2)] [17, Result 2.4].

Theorem 4.0.1 [70] *Let σ be a Baer involution which leaves invariant a unital $\overline{\mathcal{U}}$ of a finite projective plane π_{q^2}, of square order q^2. Then B, the Baer subplane fixed pointwise by σ, contains exactly $q+1$ points of $\overline{\mathcal{U}}$, and exactly $q+1$ tangents of $\overline{\mathcal{U}}$ are lines of B.*
If q is even, then the $q+1$ points in $B \cap \overline{\mathcal{U}}$ are collinear in B.
If q is odd, then the $q+1$ points in $B \cap \overline{\mathcal{U}}$ form a $(q+1)$-arc in B. □

In [51] Hölz discussed classical unitals and Baer subplanes in $PG(2,q^2)$ and used his results to define two new designs. The results Hölz obtained on the intersection of a classical unital and a Baer subplane in $PG(2,q^2)$ are as follows. In the paper [51], Hölz defines,

Property (T): *For each point P in $PG(2,q^2)$, which lies in both a Baer subplane B and a classical unital $\overline{\mathcal{U}}$, the tangent line t_P to $\overline{\mathcal{U}}$ at P is a line of B.*

Theorem 4.0.2 [51, Lemma 2.2] *Let B be a Baer subplane of $PG(2, q^2)$ satisfying property (T), which contains at least three distinct points of a classical unital $\overline{\mathcal{U}}$. Then B has exactly $q + 1$ points in common with $\overline{\mathcal{U}}$.*

If q is even, these points are collinear. If q is odd, these points are either collinear or they form an oval in B. □

In [23] Bruen and Hirschfeld gave many combinatorial results for the intersection of a set of type (m, n) and a set of type (m', n') in a plane of order q, including specific results when the two sets concerned are a Baer subplane and a unital in a plane π_{q^2} of order q^2. In [44] Grüning gave similar results including the following result which has proved to be very useful in characterising unitals of $PG(2, q^2)$ (see the next chapter).

Theorem 4.0.3 [44] [23] *Let B be a Baer subplane and let $\overline{\mathcal{U}}$ be a unital in a projective plane π_{q^2} of order q^2. Denote by b_1 the number of lines of B which when extended are tangent lines of $\overline{\mathcal{U}}$ and let $|B \cap \overline{\mathcal{U}}|$ denote the number of points in the intersection of B and $\overline{\mathcal{U}}$. Then,*

$$|B \cap \overline{\mathcal{U}}| + b_1 = 2(q + 1)$$

□

For a projective plane π_{q^2} of order q^2 and a unital $\overline{\mathcal{U}}$ in π_{q^2}, the set of tangent lines to $\overline{\mathcal{U}}$ constitutes the set of points of a unital in the dual plane $\pi^d_{q^2}$ of π_{q^2} (see Section 1.13.2); this unital is the *dual unital* of $\overline{\mathcal{U}}$ and is denoted by $\overline{\mathcal{U}}^d$. Recall also, that for any Baer subplane B of π_{q^2} the set of lines of B constitutes a set of points of a Baer subplane B^d in the dual plane $\pi^d_{q^2}$ and B^d is the *dual Baer subplane* of B in $\pi^d_{q^2}$. By Theorem 4.0.3, we have

Corollary 4.0.4 *Let B be a Baer subplane and let $\overline{\mathcal{U}}$ be a unital in a projective plane π_{q^2} of order q^2. If we let $|B \cap \overline{\mathcal{U}}|$ denote the number of points in the intersection of B and $\overline{\mathcal{U}}$, then in the dual plane $\pi^d_{q^2}$ of π_{q^2} we have,*

$$|B^d \cap \overline{\mathcal{U}}^d| = 2(q + 1) - |B \cap \overline{\mathcal{U}}|$$

where B^d and $\overline{\mathcal{U}}^d$ are the dual structures of B and $\overline{\mathcal{U}}$ respectively. □

Theorem 4.0.3 also provides a bound on the maximum possible number of points in the intersection of a unital $\overline{\mathcal{U}}$ and Baer subplane B in a projective plane π_{q^2}, namely,

$$0 \leq |B \cap \overline{\mathcal{U}}| \leq 2(q+1).$$

The exact values of $|B \cap \overline{\mathcal{U}}|$ for which there exists a set of intersection of a Baer subplane and a unital in π_{q^2} of cardinality $|B \cap \overline{\mathcal{U}}|$, has been determined in [23] for $\overline{\mathcal{U}}$ a classical unital $\overline{\mathcal{U}}$ and a Baer subplane B in $PG(2, q^2)$. In [23] Bruen and Hirschfeld used the canonical equation of a classical unital in $PG(2, q^2)$ and an algebraic proof to obtain the following result.

Theorem 4.0.5 [23] *In $PG(2, q^2)$, for $\overline{\mathcal{U}}$ a classical unital and B a Baer subplane we have*

$$|B \cap \overline{\mathcal{U}}| = 1, q+1 \quad or \quad 2q+1$$

where the intersection sets are a unique point, $q+1$ points of a line of B or a conic in B, or a line pair in B respectively. □

We extend this work by giving a geometric proof of the above result which we obtain as a corollary to our results concerning the Buekenhout-Metz unitals in $PG(2, q^2)$.

4.1 The intersection of a Baer subplane and a Buekenhout-Metz unital in $\mathbf{PG(2, q^2)}$

We begin with a theorem which generalises Theorem 4.0.5 in certain cases. We acknowledge that recently in the literature some of the results in Theorem 4.1.1 have been proved independently in papers discussing derivation of Buekenhout-Metz unitals. See for example [11], [12], [31].

Theorem 4.1.1 *Let $\overline{\mathcal{U}}$ be a Buekenhout-Metz unital re (T, ℓ_∞) in $PG(2, q^2)$ and let B be a Baer subplane in $PG(2, q^2)$. Then,*

(i) if $|B \cap \ell_\infty| = q+1$ then $|B \cap \overline{\mathcal{U}}| = 1, q+1$ or $2q+1$ where the intersection sets are a unique point, $q+1$ points of a line of B or an oval in B, or a line pair in B respectively.

(ii) *if* $B \cap \ell_\infty = \{T\}$ *then* $|B \cap \overline{\mathcal{U}}| = q+1$ *or* $2q+1$ *where the intersection sets are a union of* q *points of* B *distinct from* T *and incident with distinct lines of* B *through* T *either the unique point* T *or* $q+1$ *points of a Baer subline in* B *containing* T *respectively.*

(Note: The remaining case $B \cap \ell_\infty = \{P\}$ *with* $P \neq T$ *is considered later in the chapter.)*

Proof: As the setting for our proof we use the $4-$dimensional Bruck-Bose representation Π_4 of $PG(2,q^2)$, defined by a regular $1-$spread $\mathcal{S}$ of a hyperplane Σ_∞ of $PG(4,q)$. As $\overline{\mathcal{U}}$ is a Buekenhout-Metz unital in $PG(2,q^2)$, in Bruck-Bose $\overline{\mathcal{U}}$ is an ovoidal cone $\overline{\mathcal{U}}^*$ with base ovoid $\mathcal{O}$ and vertex V, where V is incident with an element t of the spread $\mathcal{S}$, and t represents the unique point T of $\overline{\mathcal{U}}$ at infinity in $PG(2,q^2)$. The line at infinity ℓ_∞ of $PG(2,q^2)$ corresponds to the hyperplane Σ_∞ and in Bruck-Bose, $\overline{\mathcal{U}}^* \cap \Sigma_\infty = \{t\}$.

(i) In this case, the line at infinity is a line of the Baer subplane B and therefore B is represented in Bruck-Bose by a plane $\mathcal{B}$ of $PG(4,q)\backslash\Sigma_\infty$ which intersects Σ_∞ in a line which is not an element of the spread $\mathcal{S}$.

Suppose that $V \in \mathcal{B}$, then, since $\overline{\mathcal{U}}^*$ is an ovoidal cone with vertex V, the intersection $\mathcal{B} \cap \overline{\mathcal{U}}^*$ is the unique point V, a generator line of $\overline{\mathcal{U}}^*$ or a pair of distinct generator lines of $\overline{\mathcal{U}}^*$. In these cases the number $|B \cap \overline{\mathcal{U}}|$ of points in the intersection equals $1, q+1$ or $2q+1$ respectively.

Alternatively, suppose that $V \notin \mathcal{B}$. Note that the hyperplane Σ_∞ is the unique hyperplane of $PG(4,q)$ which intersects the ovoidal cone $\overline{\mathcal{U}}^*$ in exactly the line t; since in the quotient $3-$space determined by V, Σ_∞ corresponds to the unique tangent plane to the ovoid determined by $\overline{\mathcal{U}}^*$ at the ovoid point corresponding to t. Therefore in $PG(4,q)$ and since $\mathcal{B}$ is not contained in Σ_∞, the hyperplane $\langle V, \mathcal{B}\rangle$, spanned by the plane $\mathcal{B}$ and the vertex V of $\overline{\mathcal{U}}^*$ intersects $\overline{\mathcal{U}}^*$ in either an oval cone or in a unique line on $\overline{\mathcal{U}}^*$ distinct from t. In the latter case, $\mathcal{B} \cap \overline{\mathcal{U}}^*$ is a unique point of $PG(4,q)\backslash\Sigma_\infty$ and hence the Baer subplane B intersects $\overline{\mathcal{U}}$ in a unique point. Consider the case where the hyperplane $\langle V, \mathcal{B}\rangle$ intersects $\overline{\mathcal{U}}^*$ in an oval cone. Since $V \notin \mathcal{B}$, the transversal plane $\mathcal{B}$ is either tangent to this oval cone or intersects the oval cone in an oval of $q+1$ points of $\overline{\mathcal{U}}^*$. In these two cases the number $|B \cap \overline{\mathcal{U}}|$ of points in the intersection equals 1 or $q+1$ respectively.

(ii) In this case, the line at infinity intersects B in the unique point T and therefore in Bruck-Bose, B is a Baer ruled cubic surface $\mathcal{B}$ with line directrix t. Furthermore, since T is the unique point at infinity of $\overline{\mathcal{U}}$, in Bruck-Bose we have the intersection $\mathcal{B} \cap \Sigma_\infty = \overline{\mathcal{U}}^* \cap \Sigma_\infty = \{t\}$. Denote the generators of $\mathcal{B}$ by $g_1^*, \ldots, g_{q+1}^*$, where g_1^* denotes the unique generator line of $\mathcal{B}$ incident with the point V of t.

Recall that each generator line of $\overline{\mathcal{U}}^*$ passes through V and each plane of $PG(4,q)\backslash\Sigma_\infty$ about t contains a unique generator line of $\overline{\mathcal{U}}^*$. Thus each plane $\langle g_i^*, t\rangle$ $i = 1, \ldots, q+1$ contains a generator line, l_i^* say, of $\overline{\mathcal{U}}^*$. Note that as g_1^* passes through V, g_1^* is either the generator line l_1^* of $\overline{\mathcal{U}}^*$ or intersects l_1^* in the unique point V. Each line g_i^* $(i \neq 1)$ does not pass through V and therefore in the plane $\langle g_i^*, t\rangle$, the line g_i^* intersects the generator line l_i^* of $\overline{\mathcal{U}}^*$ in a unique point of $PG(4,q)\backslash\Sigma_\infty$. Therefore, for such a Baer subplane B, and for these two cases the number $|B \cap \overline{\mathcal{U}}|$ of points in the intersection of the Baer subplane and the Buekenhout-Metz unital equals $2q+1$ or $q+1$ respectively. $\square$

Note that by the proof of Theorem 4.1.1, if a Baer subplane contains an oval of points of a Buekenhout-Metz unital $\overline{\mathcal{U}}$ in $PG(2,q^2)$ as in case *(i)*, then the oval $B \cap \overline{\mathcal{U}}$ is related to an oval plane section of the $3-$dimensional base ovoid of $\overline{\mathcal{U}}$ in the following way.

Corollary 4.1.2 *Let $\overline{\mathcal{U}}$ be a Buekenhout-Metz unital re (T, ℓ_∞) in $PG(2,q^2)$ and let B be a Baer subplane in $PG(2,q^2)$ such that ℓ_∞ is a line of B. Let $\mathcal{O}$ denote the base ovoid of $\overline{\mathcal{U}}$.*

If the intersection $B \cap \overline{\mathcal{U}}$ is an oval O in B, then the oval O is projectively equivalent to an oval contained in a $3-$dimensional oval cone with base oval a plane section of the ovoid $\mathcal{O}$. $\square$

We now obtain the Bruen and Hirschfeld result (Theorem 4.0.5) as a corollary to Theorem 4.1.1 as follows.

Corollary 4.1.3 [23] *In $PG(2,q^2)$, for $\overline{\mathcal{U}}$ a classical unital and B a Baer subplane we have*

$$|B \cap \overline{\mathcal{U}}| = 1, q+1 \quad or \quad 2q+1$$

where the intersection sets are a unique point, $q+1$ points of a line of B or $q+1$ points of a conic in B and a line pair in B respectively.

Proof: By Section 1.13.3 the classical unital $\overline{\mathcal{U}}$ is Buekenhout-Metz re (T, l_T) for all points $T \in \overline{\mathcal{U}}$ and the corresponding tangent line l_T to $\overline{\mathcal{U}}$ at T. We begin by showing that for every Baer subplane B of $PG(2, q^2)$ there exists at least one line of B which when extended is a tangent line of $\overline{\mathcal{U}}$.

Suppose B is a Baer subplane of $PG(2, q^2)$ such that B contains no line tangent to $\overline{\mathcal{U}}$, then by Theorem 4.0.3, $|B \cap \overline{\mathcal{U}}| = 2q + 2$. As $\overline{\mathcal{U}}$ is classical, by Theorem 1.13.1.2 every Baer subline of $PG(2, q^2)$ intersects $\overline{\mathcal{U}}$ in $0, 1, 2$ or $q+1$ points; in particular, every Baer subline in B intersects $B \cap \overline{\mathcal{U}}$ in $0, 1, 2$ or $q + 1$ points. The set $B \cap \overline{\mathcal{U}}$ has too many points to be an oval in B. If $B \cap \overline{\mathcal{U}}$ is the disjoint union of a Baer subline in B and an oval in B, then a secant (Baer sub)line of the oval intersects $B \cap \overline{\mathcal{U}}$ in three points, contradicting Theorem 1.13.1.2. If $B \cap \overline{\mathcal{U}}$ is the union of two lines in B plus a further point Q say, then there exist q Baer sublines in B through Q which intersect $B \cap \overline{\mathcal{U}}$ in three points, contradicting Theorem 1.13.1.2. Alternatively, one could argue that by Theorem 1.13.1.2 $B \cap \overline{\mathcal{U}}$ is a Tallini set in B and no Tallini set in a plane of order q has cardinality $2q + 2$ (see [55].)

Therefore, the number of points $|B \cap \overline{\mathcal{U}}|$ in the intersection of B and $\overline{\mathcal{U}}$ is necessarily less than $2q + 2$ and by Theorem 4.0.3 this implies that B contains at least one line which when extended is a tangent line of $\overline{\mathcal{U}}$; denote this line by ℓ_∞. Since $\overline{\mathcal{U}}$ is Buekenhout-Metz with respect to the line ℓ_∞ and since B contains ℓ_∞ as a line, by Theorem 4.1.1, B intersects $\overline{\mathcal{U}}$ in $1, q + 1$ or $2q + 1$ points. Moreover since a classical unital has as base ovoid an elliptic quadric, by Corollary 4.1.2 if a Baer subplane B of $PG(2, q^2)$ intersects $\overline{\mathcal{U}}$ in an oval, then the oval is a non-degenerate conic in B. $\square$

Theorem 4.1.1 does not exhaust the possible intersections of a Buekenhout-Metz (B-M) unital $\overline{\mathcal{U}}$ and a Baer subplane B of $PG(2, q^2)$. It remains to consider the case when $\overline{\mathcal{U}}$ is B-M re (T, ℓ_∞) and B is a Baer subplane of $PG(2, q^2)$ such that $B \cap \ell_\infty$ is a unique point P on ℓ_∞ distinct from T. We partially solve the problem in this case, by improving the restriction on the number of points $|B \cap \overline{\mathcal{U}}|$ in the intersection of B and $\overline{\mathcal{U}}$ which was given in Theorem 4.0.3

Theorem 4.1.4 *Let B be a Baer subplane in $PG(2, q^2)$. Let $\overline{\mathcal{U}}$ be a Buekenhout-Metz unital re (T, ℓ_∞) in $PG(2, q^2)$. If the base ovoid of $\overline{\mathcal{U}}$ is an elliptic quadric, then for*

$q > 13$,

$$1 \leq |B \cap \overline{\mathcal{U}}| \leq 2q+1.$$

Proof: Consider a unital $\overline{\mathcal{U}}$ in $PG(2,q^2)$ which is Buekenhout-Metz re (T, ℓ_∞) and which has an elliptic quadric as base. By Theorem 4.0.3 and for B any Baer subplane of $PG(2,q^2)$,

$$0 \leq |B \cap \overline{\mathcal{U}}| \leq 2q+2.$$

If $\overline{\mathcal{U}}$ is a classical unital then by Theorem 4.0.5, $|B \cap \overline{\mathcal{U}}| = 1, q+1$ or $2q+1$ for all q as required.

If $\overline{\mathcal{U}}$ is a non-classical Buekenhout-Metz unital, then by Theorem 4.1.1 if B is a Baer subplane of $PG(2,q^2)$ which contains ℓ_∞ as a line or if B intersects ℓ_∞ in the unique point T, then $|B \cap \overline{\mathcal{U}}| = 1, q+1$ or $2q+1$ as required.

The remaining case to consider is the case where $\overline{\mathcal{U}}$ is a non-classical Buekenhout-Metz unital re (T, ℓ_∞) with elliptic quadric as base and B is a Baer subplane of $PG(2,q^2)$ such that B intersects the line at infinity in a unique point P distinct from T. It remains to prove for this case that $1 \leq |B \cap \overline{\mathcal{U}}| \leq 2q+1$ when $q > 13$. Our proof is by contradiction making use of several preliminary results. Suppose in this case the intersection $B \cap \overline{\mathcal{U}}$ contains $2q+2$ distinct points, then by Theorem 4.0.3 $B \cap \overline{\mathcal{U}}$ contains exactly $2q+2$ points. By Theorem 1.13.3.1 and since $P \in \ell_\infty$ is not a point of the unital, each Baer subline in B which contains the point P intersects $\overline{\mathcal{U}}$ in at most two points; as $|B \cap \overline{\mathcal{U}}| = 2q+2$, each Baer subline in B which contains P contains exactly two distinct points of $\overline{\mathcal{U}}$.

The unital $\overline{\mathcal{U}}$ is a set of points $\overline{\mathcal{U}}^*$ in Bruck-Bose, where $\overline{\mathcal{U}}^*$ is an elliptic quadric cone in $PG(4,q)$. In the Bruck-Bose setting, the Baer subplane B is a Baer ruled cubic surface $\mathcal{B}$ with line directrix p in $PG(4,q)$. The line at infinity of $PG(2,q^2)$ corresponds to a hyperplane Σ_∞ of $PG(4,q)$ and the line $\{p\} = \mathcal{B} \cap \Sigma_\infty$ is an element of the regular spread $\mathcal{S}$ of Σ_∞ which defines the Bruck-Bose representation. In Bruck-Bose, the element $p \in \mathcal{S}$ represents the unique point P of B on the line at infinity. The Baer sublines in B which contain P are, in Bruck-Bose, the generator lines of the ruled cubic surface $\mathcal{B}$; from above, each such generator line contains exactly two distinct points of $\overline{\mathcal{U}}^*$ in $PG(4,q) \backslash \Sigma_\infty$. In particular the line directrix p of $\mathcal{B}$ contains no point of $\overline{\mathcal{U}}^*$ in $PG(4,q)$. Denote these $q+1$ generator lines of $\mathcal{B}$ by $g_1^*, g_2^*, \ldots, g_{q+1}^*$. The q^2 Baer sublines in B

which do not contain P are represented in Bruck-Bose by q^2 distinct Baer conics on $\mathcal{B}$. For each secant line ℓ of $\overline{\mathcal{U}}$ not incident with T the intersection $\ell \cap \overline{\mathcal{U}}$ in Bruck-Bose is a non-degenerate conic, namely the plane section of the quadric $\overline{\mathcal{U}}^*$ by the plane ℓ^* of $PG(4,q)$, which corresponds to ℓ via Bruck-Bose. By Theorem 1.13.3.2 and since $\overline{\mathcal{U}}$ is non-classical with elliptic quadric as base, no such intersection $\ell \cap \overline{\mathcal{U}}$ is a Baer subline. It follows that no Baer subline in B is contained in $\overline{\mathcal{U}}$ and therefore, in Bruck-Bose, no Baer conic in $\mathcal{B}$ coincides with a conic $\ell^* \cap \overline{\mathcal{U}}^*$ (a plane section of the quadric $\overline{\mathcal{U}}^*$ in $PG(4,q)$). Moreover, since two distinct non-degenerate conics in $PG(2,q)$ intersect in at most four points, every Baer subline in B contains at most four points of $B \cap \overline{\mathcal{U}}$. Hence the $2q+2$ points in the intersection $B \cap \overline{\mathcal{U}}$ constitute a $\{2q+2;4\}-$arc in the Baer subplane B. Note that by Theorem 1.11.2 there exists a Baer subline in B which contains exactly four distinct points of $\overline{\mathcal{U}}$.

The quadric $\overline{\mathcal{U}}^*$ intersects the hyperplane Σ_∞ in the spread line t. Since the line directrix p of $\mathcal{B}$ is distinct from t and $\mathcal{B}$ contains no further point in Σ_∞, the line directrix p is disjoint from the quadric $\overline{\mathcal{U}}^*$ in $PG(4,q)$. In particular the point vertex V of $\overline{\mathcal{U}}^*$ is not a point of $\mathcal{B}$. Let γ denote the variety which is the intersection $\mathcal{B} \cap \overline{\mathcal{U}}^*$ in $PG(4,q)$. Note that the $2q+2$ points of γ in $PG(4,q)$ are disjoint from Σ_∞. Since the $2q+2$ points of γ lie two each on each generator of $\mathcal{B}$ and since $\mathcal{B}$ is not contained in any hyperplane of $PG(4,q)$, the variety γ is not contained in any hyperplane of $PG(4,q)$. Also from our above remarks and since γ does not contain p, any generator line of $\mathcal{B}$ or any Baer conic in $\mathcal{B}$ in $PG(4,q)$, we have that γ contains no lines or conics in $PG(4,q)$.

For the remainder of the proof, the points of $PG(4,q)$ will be called **rational** points; hence the variety γ has $2q+2$ rational points. Since γ is the intersection of a ruled cubic surface $\mathcal{B}$, of order 3 and dimension 2, and a quadric $\overline{\mathcal{U}}^*$, of order 2 and dimension 3, the variety γ has order 6 and dimension 1; γ is therefore a curve of order 6 in $PG(4,q)$. If γ is reducible, then the order of γ may be partitioned in the following ways: $6 = 1+5 = 1+1+4 = 1+1+1+3 = 1+1+1+1+2 = 1+1+1+1+1+1 = 2+4 = 2+3+1 = 2+2+2 = 3+3 = 1+1+1+3 = 1+1+2+2$. Hence if γ is reducible then one of the following holds:

(a) γ contains two twisted cubic curves.

(b) γ contains an irreducible conic and an irreducible quartic curve.

(c) γ contains three irreducible conics.

(d) γ is a curve with line components.

Note that the components of γ may coincide or may belong to some field extension of $GF(q)$.

We denote by $\hat{K}$ the algebraic closure of $K = GF(q)$. To show that γ is absolutely irreducible, by the above remarks it suffices to show that over $\hat{K}$, γ contains no lines, conics or twisted cubic curves.

By [15, Note 1.] each variety $\mathcal{B} = V_2^3$ of $PG(4,q)$ is the set of rational points of a variety $\hat{V}_2^3$ of $PG(4,\hat{K})$ obtained by a projectivity $\hat{\phi}$ between a line $\hat{p}$ and a conic $\hat{C}$ of $PG(4,\hat{K})$ where $\hat{\phi}|_p$ is a projectivity ϕ between the line p and a (Baer) conic C contained in $\mathcal{B}$ of $PG(4,q)$, that is $\phi \in PGL(2,q)$ (see also Section 2.4.1). So in $PG(4,\hat{K})$ the points of the ruled cubic surface $\hat{V}_2^3$ are partitioned by the generators of $\hat{V}_2^3$ and distinct generators of $\hat{V}_2^3$ intersect $\hat{p}$ in distinct points.

The quadric $\overline{\mathcal{U}}^*$ in $PG(4,q)$ extends to a quadric $\hat{Q}_{\mathcal{U}}$ in $PG(4,\hat{K})$, by considering the equation which defines $\overline{\mathcal{U}}^*$ in $PG(4,q)$ over the field $\hat{K}$. Each generator line of $\hat{V}_2^3$ intersects the quadric $\hat{Q}_{\mathcal{U}}$ in 1 or 2 points unless the generator is contained in $\hat{Q}_{\mathcal{U}}$. Consider the line directrix p of $\mathcal{B}$ in $PG(4,q)$. The line p is disjoint to $\overline{\mathcal{U}}^*$ in $PG(4,q)$, and therefore in the quadratic extension, the intersection $p \cap \overline{\mathcal{U}}^*$ is a pair of points A, A^q of p, conjugate with respect to the extension $GF(q^2)$ of $GF(q)$. Thus for the line $\hat{p}$ in $PG(4,\hat{K})$, the two points A, A^q are the only points of the quadric $\hat{Q}_{\mathcal{U}}$ incident with $\hat{p}$. In $PG(4,\hat{K})$ the sextic curve γ is the intersection of the ruled cubic surface $\hat{V}_2^3$ and the quadric $\hat{Q}_{\mathcal{U}}$.

Suppose the sextic curve γ contains a line component g. The only lines of $\hat{V}_2^3$ are the generators and the line directrix $\hat{p}$. Since no generator of $\mathcal{B}$ in $PG(4,q)$ is contained in $\overline{\mathcal{U}}^*$ and since $\hat{p}$ is not contained in γ, the line component g of γ is then a generator of $\hat{V}_2^3$ and is such that the points of g belong to some field extension of $GF(q)$. The lines g and $\hat{p}$ of $\hat{V}_2^3$ intersect in a unique point and since $g \subseteq \gamma \subseteq \hat{Q}_{\mathcal{U}}$ the point $g \cap \hat{p}$ is then a point of the quadric $\hat{Q}_{\mathcal{U}}$. Hence $g \cap \hat{p}$ is either the point A or A^q. Suppose, without loss of generality, that $g \cap \hat{p}$ is the point A; hence the generator g of $\hat{V}_2^3$ has one and therefore all of its points in the quadratic extension $PG(4,q^2)$ of $PG(4,q)$ (see Section 2.4.1). Every generator of $\hat{V}_2^3$ contains a unique point of the base conic $\hat{C}$ of $\hat{V}_2^3$, hence we denote by X the unique point of $\hat{C}$ incident with g. By definition of the ruled cubic $\hat{V}_2^3$, the points $A \in \hat{p}$ and $X \in \hat{C}$ of g are related by the projectivity $\hat{\phi}$. Since XX^q is a line of $PG(4,q)$ in the plane containing the base conic C of $\mathcal{B}$, the point X^q, conjugate to X

with respect to the quadratic extension, is therefore a point of the conic $\hat{C}$. The points $A^q \in \hat{p}$ and $X^q \in \hat{C}$ are therefore related by the projectivity $\phi \in PGL(2,q)$ and thus the line $g^q = X^q A^q$ is a generator line of the ruled cubic surface $\hat{V}_2^3$. Since g is a line component of γ, g is contained in the quadric $\hat{Q}_{\mathcal{U}}$ which is defined by a quadratic form with coefficients in $GF(q)$, namely the quadratic form which defines $\overline{\mathcal{U}}^*$ in $PG(4,q)$. It follows that the line g^q, conjugate to g with respect to the extension $GF(q^2)$ of $GF(q)$, is contained in $\hat{Q}_{\mathcal{U}}$ and hence is a line component of γ.

We have that the sextic curve γ contains two line components g, g^q, such that g and g^q contain no rational points, and therefore $\gamma \backslash \{g, g^q\}$ is residually a curve C_1^4 of order 4 which contains $2q+2$ distinct points in $PG(4,q)$. If C_1^4 is absolutely irreducible, then since the $2q+2$ rational points of C_1^4 are not contained in any hyperplane of $PG(4,q)$, we have by Theorem 1.6.8 that C_1^4 has genus $g=0$ and by Theorem 1.6.10 the curve C_1^4 has exactly $q+1$ rational points, a contradiction. Thus the curve C_1^4 must be reducible over some field extension of $GF(q)$. The curve C_1^4 has no line components, since by the above above arguments the lines g, g^q are the unique lines in $\hat{V}_2^3$ incident with $\hat{p}$ in the points $\{A, A^q\}$, the only two points of $\hat{p}$ in $\gamma = \hat{Q}_{\mathcal{U}} \cap \hat{V}_2^3$. The only possibility is that C_1^4 has a pair of conic components and therefore the rational points of C_1^4 are contained in conics of $PG(4,q)$. But the only conics in $PG(4,q)$ contained in $\mathcal{B}$ are Baer conics and since γ contains no Baer conic in $\mathcal{B}$, from the earlier comments in the proof, we have a contradiction.

We have established that the sextic curve γ has no line components.

Suppose the sextic curve γ contains an irreducible conic component C_1^2. Since γ contains no conics in $PG(4,q)$, the conic C_1^2 in γ is a conic on the surface $\hat{V}_2^3$ in $PG(4,\hat{K})$ and therefore contains at most one rational point. The remaining $2q+1$ rational points of γ are then contained in a curve $C_1^4 = \gamma \backslash \{C_1^2\}$ of order 4. By the argument presented above, the curve C_1^4 cannot be an absolutely irreducible component of γ, and since γ contains no line components, the curve C_1^4 must be the union of two conic components of γ. We obtain a contradiction as γ contains no conics in $PG(4,q)$ and yet γ has $2q+2$ rational points. Hence the sextic curve γ contains no conic components.

Suppose the sextic curve γ contains a twisted cubic component C_1^3. Since γ contains no line or conic components, γ must be the union of two irreducible cubic curve components. Suppose that C_1^3 is contained in $PG(4,q)$. By Theorem 2.3.1 the irreducible cubic curves on $\mathcal{B}$ contain a rational point of p. Since γ contains no rational point of p, an irreducible cubic component C_1^3 of γ is not contained in $PG(4,q)$. Thus if γ is the union of two irreducible cubic curve components and neither is contained in $PG(4,q)$, then we have a contradiction since γ has $2q+2$ rational points.

The sextic curve γ therefore contains no components of lower order over any field extension, hence γ is an absolutely irreducible sextic curve with $2q+2$ rational points.

By Theorem 1.6.10 for γ absolutely irreducible and with $2q+2$ points in $PG(4,q)$,

$$|2q+2-(q+1)| \leq 2g\sqrt{q}.$$

Since by Theorem 1.6.8, the sextic curve γ has genus g at most 2, we consider the possibilities $g = 0,1,2$. Both $g = 0,1$ give rise to a contradiction. For $g = 2$, on rearranging, we have $q - 4\sqrt{q} + 1 \leq 0$ and since q is a positive prime power, q must satisfy $2-\sqrt{3} \leq \sqrt{q} \leq 2+\sqrt{3}$; a contradiction if $q > 13$.

We now give a "dual" argument to show that if $|B \cap \overline{\mathcal{U}}| \neq 2q+2$ for any B-M unital and any Baer subplane of $PG(2,q^2)$, then $|B \cap \overline{\mathcal{U}}| \neq 0$ for any B-M unital and any Baer subplane of $PG(2,q^2)$. The same argument can be used to show that if there is no B-M unital and Baer subplane of $PG(2,q^2)$ with exactly $2q+2-m$ points in common for some fixed m satisfying $0 \leq m \leq 2q+2$ then there is no B-M unital and Baer subplane of $PG(2,q^2)$ with exactly m points in common.

Above we have shown for B a Baer subplane and $\overline{\mathcal{U}}$ a B-M unital, with base ovoid an elliptic quadric, in $PG(2,q^2)$, with $q > 13$, that $|B \cap \overline{\mathcal{U}}| \neq 2q+2$. Recall that the plane $PG(2,q^2)$ is isomorphic to the dual plane of $PG(2,q^2)$. The dual B^d of a Baer subplane B of $PG(2,q^2)$ is a Baer subplane of the dual plane. The dual of a B-M unital $\overline{\mathcal{U}}$, with base ovoid an elliptic quadric, is a B-M unital $\overline{\mathcal{U}}^d$, with base ovoid an elliptic quadric, in the dual plane (see [6], [26]). Using the definition of dual structures and Theorem 4.0.4,

$$\begin{aligned}
|B^d \cap \overline{\mathcal{U}}^d| &= (\text{The number of lines of } B \text{ which when extended are tangent lines of } \overline{\mathcal{U}}) \\
&= 2q+2-|B \cap \overline{\mathcal{U}}|
\end{aligned}$$

and since $|B \cap \overline{\mathcal{U}}| < 2q+2$, we have $|B^d \cap \overline{\mathcal{U}}^d| > 0$. This concludes our proof. □

So by Theorem 4.1.4, for a Baer subplane of $PG(2,q^2)$ and a B-M unital $\overline{\mathcal{U}}$ with elliptic quadric as base, there are restrictions on the values $|B \cap \overline{\mathcal{U}}|$ can take, for $q > 13$. In fact the arguments used in the proof of this theorem can be used to show that as q increases the bounds on $|B \cap \overline{\mathcal{U}}|$ further restrict the possible values, for a Baer subplane B and unital $\overline{\mathcal{U}}$ as in the statement of the theorem, as follows. Let B and $\overline{\mathcal{U}}$ be as in the statement of Theorem 4.1.4. Suppose that B is tangent to the line at infinity at a point P, where P is distinct from the unique point of $\overline{\mathcal{U}}$ on the line at infinity. In Bruck-Bose B is represented by a ruled cubic surface $\mathcal{B}$ and $\overline{\mathcal{U}}$ is a set of points $\overline{\mathcal{U}}^*$ in $PG(4, q)$. Let γ denote the intersection $\mathcal{B} \cap \overline{\mathcal{U}}^*$. If γ is an absolutely irreducible sextic curve then, by the arguments used in the proof of Theorem 4.1.4, the number of points R of γ in $PG(4, q)$ is restricted by q, according to the following table.

	Restriction on $R = \lvert\gamma\rvert = \lvert B \cap \overline{\mathcal{U}}\rvert$
$q > 13$	$0 < R < 2q + 2$
$q > 16$	$1 < R < 2q + 1$
$q > 17$	$2 < R < 2q$
$q > 19$	$3 < R < 2q - 1$
$q > 21$	$4 < R < 2q - 2$
$\vdots$	$\vdots$
etc.	etc.

Corollary 4.1.5 *Let $\overline{\mathcal{U}}$ be a unital in $PG(2, q^2)$, $q > 13$, and q odd. If there exists a Baer subplane B of $PG(2, q^2)$ with no point in common with $\overline{\mathcal{U}}$, then $\overline{\mathcal{U}}$ is not a Buekenhout-Metz unital.*

Proof: Suppose $\overline{\mathcal{U}}$ is a Buekenhout-Metz unital in $PG(2, q^2)$, then in Bruck-Bose $\overline{\mathcal{U}}$ is an ovoidal cone $\overline{\mathcal{U}}^*$ with elliptic quadric as base since q is odd. Then by Theorem 4.1.4 for any Baer subplane B of $PG(2, q^2)$ we have $|B \cap \overline{\mathcal{U}}| \geq 1$ and so no Baer subplane of $PG(2, q^2)$ is disjoint from $\overline{\mathcal{U}}$. The result now follows. □

Note that at present all known unitals in $PG(2, q^2)$ are Buekenhout-Metz unitals (see [26]).

Finally, we include the statement of the very recent result of Barwick, O'Keefe and Storme [14] which characterises Buekenhout-Metz unitals in translation planes π_{q^2} of order q^2 which can be represented in 4−dimensional Bruck-Bose. Note, in the following theorem, a *parabolic* unital in π_{q^2} is a unital for which the translation line ℓ_∞ of the plane is a tangent line of $\overline{\mathcal{U}}$; also, *linear* Baer subplanes of π_{q^2} are those Baer subplanes B of

π_{q^2} such that B is represented by a (transversal) plane in the 4−dimensional Bruck-Bose representation of π_{q^2}.

Theorem 4.1.6 [14] *Let $\overline{\mathcal{U}}$ be a parabolic unital in a translation plane π_{q^2} of order q^2 with kernel containing $GF(q)$.*

Then $\overline{\mathcal{U}}$ is a Buekenhout-Metz unital if and only if every linear Baer subplane of π_{q^2} meets $\overline{\mathcal{U}}$ in 1 modulo q points. □

4.2 Examples and quartic curves

Let $\overline{\mathcal{U}}$ be a Buekenhout-Metz unital re (T, ℓ_∞) and with an elliptic quadric as base in $PG(2, q^2)$. Let B be a Baer subplane of $PG(2, q^2)$ such that $|B \cap \overline{\mathcal{U}}| = 2q + 2$, the maximum possible number of points by Theorem 4.0.3. By the results of the previous section, the unital $\overline{\mathcal{U}}$ is necessarily non-classical and the Baer subplane B is necessarily tangent to the line at infinity ℓ_∞ at a point P distinct from the unique point T of $\overline{\mathcal{U}}$ on ℓ_∞.

In this section we shall give some specific examples in $PG(2, q^2)$, $q < 13$, of this situation; further, we show that in this case the points $B \cap \overline{\mathcal{U}}$ are points of a quartic curve in B with P a double point of the curve.

In Bruck-Bose, B is a Baer ruled cubic surface $\mathcal{B}$ with line directrix p a line of the regular 1−spread $\mathcal{S}$ in a hyperplane Σ_∞ of $PG(4, q)$, in the usual notation. The element p of $\mathcal{S}$ is the Bruck-Bose representation of the point $P = B \cap \ell_\infty$ in $PG(2, q^2)$.

The unital $\overline{\mathcal{U}}$ in Bruck-Bose is a quadric cone $\overline{\mathcal{U}}^*$ in $PG(4, q)$ and so has an associated quadratic form with coefficients in $GF(q)$; a point in $PG(4, q)$ has coordinates given by $(x_0, x_1, x_2, x_3, x_4)$ for some $x_i \in GF(q)$ and not all zero.

By Theorem 1.7.3, the ruled cubic surfaces in $PG(4, q)$ are projectively equivalent and hence we can choose a coordinate representation of $PG(4, q)$ such that the ruled cubic surface $\mathcal{B}$ is the ruled cubic surface R_2^3 whose points are given by,

$$\{(x^2, xy, y^2, zx, zy);\ x, y \in GF(q),\ (x, y) \neq (0, 0),\ z \in GF(q) \cup \{\infty\}\ \}.$$

By the results of Section 2.6, the Baer subplane B may be identified with $PG(2, q)$ with point coordinates

$$\{(x,y,z);\ x,y,z \in GF(q) \quad x,y,z \text{ not all zero } \}$$

and such that the line directrix of the ruled cubic surface corresponds to the point $P(0,0,1)$ in B. In an abuse of notation, we have chosen coordinates conveniently to represent B as the Baer subplane $PG(2,q)$. The unital $\overline{\mathcal{U}}$ and the line at infinity ℓ_∞ will therefore be given by new coordinates but we retain the same notation; thus $P(0,0,1)$ is the unique point of B on ℓ_∞. In Bruck-Bose, $\overline{\mathcal{U}}$ is still a quadric cone $\overline{\mathcal{U}}^*$, the image of the original quadric cone under a projectivity of $PG(4,q)$, and so denote its quadratic form by

$$Q(x_0,x_1,x_2,x_3,x_4).$$

The points of $PG(2,q^2)$ in the intersection $B\cap\overline{\mathcal{U}}$ are therefore the points (x,y,z) in $PG(2,q)$ satisfying,

$$Q(x^2,xy,y^2,zx,zy)=0,$$

this is a polynomial of degree 4, homogeneous in x,y,z and so represents a quartic curve C_1^4 in $B=PG(2,q)$. Moreover the highest degree of z in the polynomial is 2 and so the quartic curve has a double point at $P(0,0,1)$ by Section 1.6. This is consistent with the fact that in Bruck-Bose, the spread element p which represents $P(0,0,1)$, is a line and so intersects the quadric $\overline{\mathcal{U}}^*$ in two points; these two points lie in a quadratic extension $PG(4,q^2)$ of $PG(4,q)$, since in $PG(4,q)$, $\overline{\mathcal{U}}^*$ and p are disjoint. Thus in $PG(2,q^2)$, every line of $B=PG(2,q)$ through P intersects the quartic curve C_1^4 twice at P.

We note also that for small values of q computer searches have verified that there exist examples of Buekenhout-Metz unitals with elliptic quadric as base and Baer subplanes of $PG(2,q^2)$ such that the unital and Baer subplane are disjoint; hence there are also examples of the dual case where such a unital and Baer subplane intersect in $2q+2$ points. We include one such example.

An example in $\mathbf{PG(2,9)}$:

Consider the primitive polynomial x^2-x-1 with root (primitive element) ω. The elements of the fields $GF(3)$ and $GF(9)$ can be represented as follows:

$$\begin{aligned} GF(3) &= \{0,1,2\} \\ GF(9) &= \{0,1,\omega,\omega^2,\omega^3,\omega^4\equiv 2,\omega^5,\omega^6,\omega^7\} \end{aligned}$$

Multiplication in $GF(9)$ is the usual operation with $\omega^8 \equiv 1$. Field addition is given in the following table.

0	1	ω	ω^2	ω^3	ω^4	ω^5	ω^6	ω^7
1	ω^4	ω^2	ω^7	ω^6	0	ω^3	ω^5	ω
ω		ω^5	ω^3	1	ω^7	0	ω^4	ω^6
ω^2			ω^6	ω^4	ω	1	0	ω^5
ω^3				ω^7	ω^5	ω^2	ω	0
ω^4					1	ω^6	ω^3	ω^2
ω^5						ω	ω^7	ω^4
ω^6							ω^2	1
ω^7								ω^3

Consider the B-M unital $\overline{\mathcal{U}}_{\omega 0}$ with elliptic quadric as base with pointset given by,

$$\overline{\mathcal{U}}_{\omega 0} = \{(x, \omega x^2 + r, 1);\ r \in GF(3),\ x \in GF(9)\ \} \cup \{(0,1,0)\}.$$

The line at infinity is the line with equation $z = 0$, which is tangent to the unital at the point $(0,1,0)$ which we shall call the vertex of the unital. This form of a B-M unital with elliptic quadric as base in $PG(2, q^2)$, q odd, is given by Baker and Ebert in [6].

Let $PG(2,3)$ denote the Baer subplane of $PG(2,9)$ with points given by the coordinates $\{(x,y,z);\ x,y,z \in GF(3),\ x,y,z\ \text{ not all zero }\}$. Consider the following matrix with columns the coordinates of the 13 distinct points of $PG(2,3)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$

Consider the projectivity ϕ of $PG(2,9)$ associated with the matrix,

$$H_\phi = \begin{bmatrix} \omega^6 & \omega^4 & 0 \\ 1 & \omega^5 & 0 \\ \omega^6 & \omega^2 & 1 \end{bmatrix}$$

that is

$$\begin{array}{cccc} \phi\ : & PG(2,9) & \longrightarrow & PG(2,9) \\ & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & H_\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{array}$$

Let B be the Baer subplane which is the image of $PG(2,3)$ under the projectivity ϕ. The homogeneous coordinates of the 13 distinct points of B are given as columns on the following matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 & 1 & 1 & 1 & \omega^5 & \omega^5 & \omega^5 \\ 1 & \omega^6 & \omega^3 & \omega^5 & 1 & \omega^7 & \omega^5 & 0 & \omega^5 & \omega & \omega^5 & \omega^2 & \omega^4 \end{bmatrix}.$$

Of the above points of B, the following eight are points of $\overline{\mathcal{U}}_{\omega 0}$.

$$\begin{aligned} (0,0,1) &\equiv (0, \omega 0^2 + 0, 1) \\ (1,\omega,\omega^6) &\equiv (\omega^2, \omega\omega^4 + 1, 1)\omega^6 \\ (1,\omega,\omega^3) &\equiv (\omega^5, \omega\omega^2 + 1, 1)\omega^3 \\ (1,\omega^2,1) &\equiv (1, \omega + 1, 1) \\ (1,\omega^2,\omega^7) &\equiv (\omega, \omega\omega^2 + 0, 1)\omega^7 \\ (1,1,\omega) &\equiv (\omega^7, \omega\omega^6 + 0, 1)\omega \\ (1,\omega^5,\omega^2) &\equiv (\omega^6, \omega\omega^4 + 1, 1)\omega^2 \\ (1,\omega^5,\omega^4) &\equiv (\omega^4, \omega + 0, 1)\omega^4. \end{aligned}$$

We now analyse some properties of this set $B \cap \overline{\mathcal{U}}_{\omega 0}$ in the Baer subplane B of $PG(2,9)$. The unique point of B on the line at infinity is the point $P(1,1,0)$. Each line of B on P contains exactly two points of the set $B \cap \overline{\mathcal{U}}_{\omega 0}$, as expected, since we aim to prove that $B \cap \overline{\mathcal{U}}_{\omega 0}$ is the set of points in B of a quartic curve with double point P. The line $\omega x + z = 0$ in B is an external line of the set. The remaining eight lines of B are 3-secants of the set $B \cap \overline{\mathcal{U}}_{\omega 0}$.

Note that each line of B contains four points and therefore if $B \cap \overline{\mathcal{U}}_{\omega 0}$ had a 4-secant in B, disjoint from the point P, the unital $\overline{\mathcal{U}}_{\omega 0}$ would be classical (by Lefèvre-Percsy Theorem 1.13.3.2), a contradiction to Theorem 4.0.5 since B contains $2q + 2 = 8$ points of the unital.

We want to verify in this case that $B \cap \overline{\mathcal{U}}_{\omega 0}$ is a quartic curve in B with double point P.

Let θ be the projectivity associated with the matrix,

$$H_\theta = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

that is

$$\begin{array}{rcl} \theta \ : \ PG(2,9) & \longrightarrow & PG(2,9) \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & H_\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{array}$$

Now consider the map $\sigma = \theta\phi^{-1}$ which maps B to the real Baer subplane $PG(2,3)$ and maps P to the point $(0,0,1)$. The image of the pointset $B \cap \overline{\mathcal{U}}_{\omega 0}$ under σ is the pointset in $PG(2,3)$ whose coordinates are given by the columns in the following matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 \end{bmatrix}.$$

(Note that by [48, Section 2.6(iii)] a projectivity of the plane does not change the order (degree) of a curve in the plane.)

It is now a brief exercise to verify that the above set of points in $PG(2,3)$ lies on the quartic curve with equation

$$z^2(x^2+xy+2y^2)+z(x^3+xy^2+y^3)+x^2y^2+xy^3=0$$

which has the point $(0,0,1)$ as a double point (see Section 1.6).

4.3 Concerning Classical Unitals in $\mathbf{PG(2,q^2)}$

In [24] Buekenhout showed that a classical unital $\overline{\mathcal{U}}$ in $PG(2,q^2)$ is Buekenhout-Metz with respect to any tangent line of $\overline{\mathcal{U}}$; denote by ℓ_∞ a tangent line of $\overline{\mathcal{U}}$. Moreover, in the 4−dimensional Bruck-Bose representation of $PG(2,q^2)$ with respect to ℓ_∞, the unital $\overline{\mathcal{U}}$ is represented by an ovoidal cone $\overline{\mathcal{U}}^*$ in $PG(4,q)$ with an elliptic quadric as base. Since the plane is the Desarguesian plane $PG(2,q^2)$, the spread $\mathcal{S}$ in the Bruck-Bose representation is a regular 1−spread of a fixed hyperplane Σ_∞ of $PG(4,q)$, in the usual notation. In [60], Metz showed that for a given regular spread in Σ_∞ there exist non-Baer conics in $PG(4,q)$ in planes about the spread elements. Metz used this fact to construct Buekenhout-Metz unitals with elliptic quadric as base and which are non-classical unitals in $PG(2,q^2)$.

Let $\overline{\mathcal{U}}$ be a non-classical unital with the above Metz construction in $PG(2,q^2)$. So in Bruck-Bose with respect to a given fixed spread $\mathcal{S}$, the unital is an ovoidal cone $\overline{\mathcal{U}}^*$

with elliptic quadric as base and such that there exists a plane ℓ^* of $PG(4,q)$ about an element p of the spread $\mathcal{S}$, for which $\ell^* \cap \overline{\mathcal{U}}^*$ is a non-degenerate non-Baer conic C^*. In $PG(2,q^2)$, ℓ^* corresponds to a secant line ℓ of $\overline{\mathcal{U}}$ for which $\ell \cap \overline{\mathcal{U}}$ is not a Baer subline.

Denote by t the spread element of $\mathcal{S}$ in the ovoidal cone $\overline{\mathcal{U}}^*$; note that t is distinct from p. By setting up a projectivity ϕ between t and C^* in $PG(4,q)$ we obtain a set V_2^3 of $q+1$ lines XX^ϕ where X ranges over the $q+1$ points of t. The set so obtained is a ruled cubic surface V_2^3 with line directrix t and a conic directrix C^* by Section 1.7. Bernasconi and Vincenti [15, Section 2] proved that there exists a regular spread $\mathcal{S}'$ of Σ_∞ for which V_2^3 is a Baer ruled cubic for the Desarguesian plane $\pi = \pi(\mathcal{S}')$ (see Theorem 2.4.1). Note that the line p is a spread element of $\mathcal{S}'$ and C^* is then a Baer conic representing a Baer subline C of $\pi(\mathcal{S}')$. Consider $\overline{\mathcal{U}}^*$ in $PG(4,q)$ with respect to this new spread $\mathcal{S}'$; $\overline{\mathcal{U}}^*$ corresponds to a set of points $\overline{\mathcal{U}}'$ in the Desarguesian plane $\pi(\mathcal{S}')$, which is by definition a Buekenhout-Metz unital (with elliptic quadric as base) in $\pi(\mathcal{S}')$. Moreover, there exists a secant line of $\overline{\mathcal{U}}'$, not on the vertex point of $\overline{\mathcal{U}}'$ and which intersects $\overline{\mathcal{U}}'$ in a Baer subline C. Hence by Theorem 1.13.3.2, $\overline{\mathcal{U}}'$ is a classical unital in the Desarguesian plane $\pi(\mathcal{S}')$.

Hence for any non-classical B-M unital $\overline{\mathcal{U}}$, with base ovoid an elliptic quadric, in $PG(2,q^2)$, in Bruck-Bose it is easy to construct, by the above procedure, a new regular spread $\mathcal{S}'$ in Σ_∞, for which $\overline{\mathcal{U}}$ is a classical unital in the Desarguesian plane $\pi(\mathcal{S}')$.

We have shown,

Theorem 4.3.1 [27] *In $PG(2,q^2)$, every non-classical Buekenhout-Metz unital $\overline{\mathcal{U}}$ with elliptic quadric as base is inherited from a classical unital $\overline{\mathcal{U}}_C$ in $PG(2,q^2)$, by a procedure of switching regular $1-$spreads of $\Sigma_\infty = PG(3,q)$ in the $4-$dimensional Bruck-Bose representation of $PG(2,q^2)$ in $PG(4,q)$.* □

Chapter 5

A characterisation of Buekenhout-Metz unitals

The known unitals in $PG(2,q^2)$ are to within collineation Buekenhout-Metz unitals, namely a unital whose representation in Bruck-Bose is isomorphic to that given in Section 1.13.3. In [24] the classical unitals were shown to be B-M and it was proved that for certain q even that there exist non-classical (B-M) unitals in translation planes of dimension 2 over their kernel. Metz showed in [60] this class of unitals contained non-classical unitals for all $q > 2$. The Buekenhout unitals, that is the unitals with construction given in [24, Section 3., Theorem 4] were shown in [10] to be classical and therefore B-M. In [26], see also [6], the dual of a Buekenhout-Metz unital in $PG(2,q^2)$ was shown to be B-M in $PG(2,q^2)$ and thus all known unitals in $PG(2,q^2)$ are B-M unitals.

There exist many characterisations of B-M unitals and classical unitals, see for example Theorems 1.13.3.1, 1.13.3.2.

The B-M unitals in $PG(2,q^2)$ were characterised by Lefèvre-Percsy as follows (a variant of Theorem 1.13.3.1):

Theorem 5.0.1 [56] *Let $\overline{\mathcal{U}}$ be a unital in $PG(2,q^2)$ where $q > 2$ and let ℓ_∞ be some tangent line to $\overline{\mathcal{U}}$. If all Baer sublines having a point on ℓ_∞ intersect $\overline{\mathcal{U}}$ in $0, 1, 2$ or $q+1$ points, then $\overline{\mathcal{U}}$ is a B-M unital re (T, ℓ_∞) for T the unique point of $\overline{\mathcal{U}}$ on ℓ_∞.* □

In [26] the Lefèvre-Percsy (Theorem 5.0.1) characterisation in $PG(2,q^2)$ was improved in the cases q even and $q = 3$ by weakening the hypotheses to give the result:

Theorem 5.0.2 [26, Theorem 1.3] *Let $\overline{\mathcal{U}}$ be a unital in $PG(2,q^2)$, where $q>2$ is even or $q=3$. Then $\overline{\mathcal{U}}$ is a B-M unital if and only if there exists a point T of $\overline{\mathcal{U}}$ such that the points of $\overline{\mathcal{U}}$ on each of the q^2 secant lines to $\overline{\mathcal{U}}$ through T form a Baer subline.* □

In this chapter, we extend the result of Theorem 5.0.2 by proving it for $q>3$; we therefore obtain

Theorem 5.0.3 [68] *Let $\overline{\mathcal{U}}$ be a unital in $PG(2,q^2)$, $q>2$. Then $\overline{\mathcal{U}}$ is a B-M unital if and only if there exists a point T of $\overline{\mathcal{U}}$ such that the points of $\overline{\mathcal{U}}$ on each of the q^2 secant lines to $\overline{\mathcal{U}}$ through T form a Baer subline.*

5.1 Proof of theorem

Let $\overline{\mathcal{U}}$ be a unital in $PG(2,q^2)$, where $q>3$, with the line at infinity ℓ_∞ a tangent line of $\overline{\mathcal{U}}$. Let $T=\ell_\infty\cap\overline{\mathcal{U}}$ and suppose that the points of $\overline{\mathcal{U}}$ on each line of $PG(2,q^2)$ through T, and distinct from ℓ_∞, form a Baer subline.

Represent $PG(2,q^2)$ in $PG(4,q)$ as in Section 2.1 with the notation introduced there. The unital $\overline{\mathcal{U}}$ corresponds to a set of points $\overline{\mathcal{U}}^*$ in $PG(4,q)$. As observed in [26], the above hypothesis is equivalent to the hypothesis that $\overline{\mathcal{U}}^*$ consists of a spread element t together with a union of q^2 lines $l_1^*,l_2^*,\dots,l_{q^2}^*$ of $PG(4,q)\backslash\Sigma_\infty$, each meeting t but pairwise having no common point in $PG(4,q)\backslash\Sigma_\infty$. We call l_i^* $(i=1,\dots,q^2)$ a *generator line of* $\overline{\mathcal{U}}^*$. In [26], with a sequence of lemmata, the following result is obtained:

Lemma 5.1.1 [26] *$\overline{\mathcal{U}}$ is either a B-M unital or $\overline{\mathcal{U}}^*$ has the following structure:*
The generator lines of $\overline{\mathcal{U}}^*$ fall into q oval cones $\mathcal{C}_1,\dots,\mathcal{C}_q$, with distinct vertices $V_1,\dots,V_q$ respectively. Each cone has $q+1$ generators, namely the line t and q generator lines of $\overline{\mathcal{U}}^*$. The cones pairwise intersect in t and have a common tangent plane π (about t) which is contained in Σ_∞ . Cone $\mathcal{C}_i$ lies in a 3-dimensional space Σ_i $(i=1,\dots,q)$ and the spaces $\Sigma_\infty,\Sigma_1,\dots,\Sigma_q$ have the plane π as common intersection. We call each Σ_i $(i=1,\dots,q)$ a **conespace**. □

We now prove Theorem 5.0.3

Theorem 5.1.2 *Let $\overline{\mathcal{U}}$ be a unital in $PG(2,q^2)$, where $q>3$. Then $\overline{\mathcal{U}}$ is a B-M unital if and only if there exists a point T of $\overline{\mathcal{U}}$ such that the points of $\overline{\mathcal{U}}$ on each of the q^2 secant lines to $\overline{\mathcal{U}}$ through T form a Baer subline.*

Proof: The "necessary" result is well known; it follows from the construction of a B-M unital, see Theorem 1.13.3.1. We now prove "sufficiency".

In $PG(2,q^2)$, $q>3$, let T be a point of a unital $\overline{\mathcal{U}}$ such that the points of $\overline{\mathcal{U}}$ on each of the q^2 secant lines to $\overline{\mathcal{U}}$ through T form a Baer subline. We may assume the tangent line to $\overline{\mathcal{U}}$ at T is ℓ_∞.

Suppose $\overline{\mathcal{U}}$ is not a B-M unital. By Lemma 5.1.1, $\overline{\mathcal{U}}^*$ has the q-cone structure defined there.

Let α be a plane of $PG(4,q)\backslash\Sigma_\infty$ representing a secant line of $\overline{\mathcal{U}}$, not through T, in $PG(2,q^2)$. Then α is a plane about a line m of the spread $\mathcal{S}$, m being distinct from t.

Choose two generator lines l_1^*, l_2^* from distinct cones of $\overline{\mathcal{U}}^*$ and incident with α. Let

$$L_1^* = l_1^* \cap \alpha$$

$$L_2^* = l_2^* \cap \alpha.$$

By Lemma 2.2.6, there exist q Baer conics in $\alpha\backslash m$ containing the points L_1^* and L_2^*. By Lemma 2.2.11, for each such Baer conic there exist $q+1$ Baer ruled cubics containing the Baer conic and t; each Baer ruled cubic is determined completely by joining L_1^* to a point of t. Thus, there exist q Baer ruled cubics containing the line l_1^* and the point L_2^*. By Theorem 1.2.1, no two Baer ruled cubics containing l_1^* and L_2^* contain the same generator line through L_2^* and therefore there exists a unique Baer ruled cubic $\mathcal{B}$ containing the generator lines l_1^* and l_2^* of $\overline{\mathcal{U}}^*$. Let C^* be the Baer conic $\mathcal{B}\cap\alpha$ and let B be the Baer subplane of $PG(2,q^2)$ represented in $PG(4,q)$ by $\mathcal{B}$.

Let $g_3^*,\dots,g_{q+1}^*$ together with l_1^* and l_2^* be the generators of $\mathcal{B}$; let $l_1^*, l_2^*, g_3^*,\dots,g_q^*$ pass through vertices $V_1, V_2, V_3, \dots, V_q$ respectively on t and g_{q+1}^* through the unique non-vertex point of t (see Lemma 5.1.1). Clearly g_{q+1}^* is not a generator line of $\overline{\mathcal{U}}^*$. None of $g_3^*,\dots,g_q^*$ are generator lines of $\overline{\mathcal{U}}^*$ as by Lemma 4.0.3, B can intersect $\overline{\mathcal{U}}$ in at most $2q+2$ points.

The plane $\langle g_{q+1}^*, t\rangle$ is a plane of $PG(4,q)\backslash\Sigma_\infty$ about t and therefore contains exactly one generator line l^* of $\overline{\mathcal{U}}^*$. As $l^*\cap g_{q+1}^*$ is necessarily an affine point, by counting the number of affine points in $\mathcal{B} \cap \overline{\mathcal{U}}^*$ together with t we obtain,

$$|B \cap \overline{\mathcal{U}}| \geq 2q + 2.$$

By Lemma 4.0.3 we have,

$$|B \cap \overline{\mathcal{U}}| = 2q + 2.$$

Hence no line g_i^* $(i = 3, \ldots, q)$ can intersect $\overline{\mathcal{U}}^*$ in an affine point. The plane $\langle g_i^*, t\rangle$ $(i = 3, \ldots, q)$ is a plane of $PG(4, q)\backslash\Sigma_\infty$ about t and therefore contains a line l_i^* of $\overline{\mathcal{U}}^*$ which, from above, intersects t in the vertex V_i. Since the generators of cone $\mathcal{C}_i$ are contained in the 3-dimensional space Σ_i we have that $\mathcal{B}$ has a generator in each conespace Σ_i $(i = 1, \ldots, q)$.

Let Q be the unique point of intersection of the plane α and the plane π, the common tangent plane to the cones $\mathcal{C}_i$ $(i = 1, \ldots, q)$. Note that $Q = \alpha \cap \pi = m \cap \pi$, and m is not a tangent to the Baer conic C^* in α. Hence, if q is even Q is not the nucleus of the C^*. Since $q > 3$, there exists a secant line of the Baer conic C^* through Q, incident with C^* in two (distinct) points M_i^*, M_j^* say, such that neither point is on the line g_{q+1}^*. But then M_i^*, M_j^* belong to two distinct generators of $\mathcal{B}$ belonging to distinct conespaces Σ_i, Σ_j say. Hence both M_i^* and M_j^* belong to π, a contradiction. Hence, by Lemma 5.1.1, $\overline{\mathcal{U}}$ is a B-M unital. $\square$

Note that in the case $q = 3$, the possibility exists that there is no secant line of C^* through Q which does not intersect g_{q+1}^* and in that case the secant line may lie in a unique conespace. For this reason the above proof is not valid when $q = 3$.

Chapter 6

Maximal Arcs, Inversive planes and $\mathbf{T_3}(\mathcal{O})$

In this chapter we investigate the relationship between Thas maximal arcs and egglike inversive planes. We show that a Thas maximal arc has an associated egglike inversive plane isomorphic in a natural way to an inversive plane obtained from the generalized quadrangle $T_3(\mathcal{O})$, by the method given in Theorem 1.16.3. We also show the inversive plane obtained from a Thas maximal arc is isomorphic in a natural way to an inversive plane obtained from a certain Buekenhout-Metz unital. The relationship between inversive planes and Buekenhout-Metz unitals was recently explored by Barwick and O'Keefe in [13]; see also [6, Section 5.] and [92].

6.1 Maximal arcs and Thas maximal arcs

Let $\mathcal{K}$ be a Thas maximal arc in a translation plane π_{q^2} of order q^2 with associated Bruck-Bose construction as given in Section 1.15 and the notation introduced there. Thus π_{q^2} has a Bruck-Bose representation Π_4 in $PG(4,q)$, where Σ_∞ denotes a fixed hyperplane of $PG(4,q)$ and $\mathcal{S}$ is a fixed $1-$spread of Σ_∞. The translation line of π_{q^2}, which we denote by ℓ_∞ and call the *line at infinity*, is represented in Bruck-Bose by the hyperplane Σ_∞; the points of ℓ_∞ corresponding to the elements of the spread $\mathcal{S}$ of Σ_∞. The Thas maximal arc $\mathcal{K}$ is defined by a $3-$dimensional ovoid $\mathcal{O}$ in Σ_∞ with the property that each element of $\mathcal{S}$ contains exactly one point of $\mathcal{O}$. The points of $\mathcal{K}$ in Bruck-Bose, are the points of $PG(4,q)\backslash\Sigma_\infty$ contained in an ovoidal cone with base ovoid $\mathcal{O}$ and vertex a point X in

$PG(4,q)\backslash\Sigma_\infty$. Note that by definition ℓ_∞ is an external line of the Thas Maximal arc $\mathcal{K}$. In an abuse of notation we shall use X to denote the base point of $\mathcal{K}$ in π_{q^2} and also to denote the image of this point in the Bruck-Bose representation.

Denote by $o_1, \dots, o_{q^2+1}$ the points of the ovoid $\mathcal{O}$ in Σ_∞ which defines the Thas maximal arc $\mathcal{K}$. Call the lines Xo_i, $i = 1, \dots, q^2+1$, in $PG(4,q)$, **generator lines of $\mathcal{K}$.** Let π_{o_i} denote the unique tangent plane to $\mathcal{O}$ in Σ_∞ at the point o_i, $i = 1, \dots, q^2+1$. Recall that the unique spread line through a point o_i of $\mathcal{O}$ is contained in the tangent plane π_{o_i} at o_i, since the plane π_{o_i} contains a spread line and each spread line contains a unique (and therefore at least one) point of $\mathcal{O}$. Denote by s_i the spread line incident with the point o_i of $\mathcal{O}$.

There exist $q+1$ hyperplanes of $PG(4,q)$ which contain the plane $\langle X, s_i\rangle$, for a fixed point o_i of the ovoid $\mathcal{O}$. The hyperplane $\langle X, \pi_{o_i}\rangle$ contains the unique generator line Xo_i of $\mathcal{K}$ and therefore the $q-1$ planes in $\langle X, \pi_{o_i}\rangle$ about the spread line s_i, besides π_{o_i} and the plane $\langle X, s_i\rangle$, represent the $q-1$ external lines of $\mathcal{K}$ on the point at infinity represented by s_i. These $q-1$ external lines together with ℓ_∞ are all the external lines to $\mathcal{K}$ on the point at infinity of π_{q^2} represented by s_i.

The remaining q hyperplanes on $\langle X, s_i\rangle$ each intersect the ovoidal cone in an oval cone. Let Σ be such a hyperplane, so that Σ contains q generator lines of $\mathcal{K}$ besides Xo_i. Planes about s_i in Σ, besides $\Sigma \cap \Sigma_\infty$, intersect the oval cone in q points of $\mathcal{K}$ and of these planes all, except $\langle X, s_i\rangle$, intersect the same q generator lines of $\mathcal{K}$. We have the following well known result:

Result 6.1.1 *Let $\mathcal{K}$ be a Thas maximal arc with base point X and axis line ℓ_∞ in a translation plane π_{q^2} of order q^2, where ℓ_∞ is the translation line of π_{q^2}. Let P be a point of ℓ_∞, then the secant lines of $\mathcal{K}$ incident with P besides XP are partitioned into q classes of $q-1$ lines such that the lines in a class intersect the same generator lines of $\mathcal{K}$.*

6.2 Excursion into $\mathbf{T_3}(\mathcal{O})$

In this section we recall the definition of the generalized quadrangle $T_3(\mathcal{O})$ and some of the well known properties of this generalized quadrangle.

Consider the generalized quadrangle $T_3(\mathcal{O})$, defined with the ovoid $\mathcal{O}$ in $PG(3,q)$ as per Result 1.16.4 with the notation introduced there. Let X be a point of type (i) in $T_3(\mathcal{O})$ and let Y be the point (∞) and so $X \not\sim Y$. Now

$$\{X,Y\}^{\perp} = \{\langle X, \pi_{o_1}\rangle, \dots, \langle X, \pi_{o_{q^2+1}}\rangle\}$$

where $o_1, \dots, o_{q^2+1}$ are the points of the ovoid $\mathcal{O}$ and π_{o_i} is the tangent plane to $\mathcal{O}$ in $PG(3,q)$ at the point o_i, $i = 1, \dots, q^2 = 1$. The points in $\{X,Y\}^{\perp}$ are points of type (ii) in $T_3(\mathcal{O})$.

Let Z_1, Z_2, Z_3 be three distinct points in $\{X,Y\}^{\perp}$, then $\{Z_1, Z_2, Z_3\}$ is necessarily a triad by $\mathcal{GQ}$ axiom (iii). Considering Z_1, Z_2, Z_3 as hyperplanes of $PG(4,q)$, $Z_1 \cap Z_2 \cap Z_3$ is a line XQ on X which intersects Σ_∞ in a unique point Q and $Q \notin \mathcal{O}$. Thus in $T_3(\mathcal{O})$,

$$\{Z_1, Z_2, Z_3\}^{\perp} = \{\text{points of the line } XQ \backslash \{Q\}\} \cup \{Y\}.$$

A point of $\{Z_1, Z_2, Z_3\}^{\perp\perp}$ is therefore a point of type (ii), since such a point is collinear with Y, contains X and contains all points o_i of $\mathcal{O}$ such that Q is incident with the plane π_{o_i}. For any ovoid $\mathcal{O}$ in $PG(3,q)$, a point $Q \notin \mathcal{O}$ lies in the tangent planes of exactly $q+1$ points of $\mathcal{O}$ and the corresponding $q+1$ points of $\mathcal{O}$ constitute an oval in $\mathcal{O}$ [69]. We have therefore that $|\{Z_1, Z_2, Z_3\}^{\perp\perp}| = q+1$ and since Z_1, Z_2, Z_3 are three arbitrary and distinct points in $\{X, Y, \}^{\perp}$ we have that every triad in $\{X,Y\}^{\perp}$ is 3-regular.

Alternatively one could argue that since $Y \in \{Z_1, Z_2, Z_3\}^{\perp}$ and since Y is 3-regular, the flag (Y, ℓ) has property (G) , for all lines ℓ of type (b) and therefore every triad $\{Z_1, Z_2, Z_3\}$ is 3-regular.

If we let X be a point of type (i) in $T_3(\mathcal{O})$ and let Y be the point (∞), then from the above discussion we can apply Theorem 1.16.3 and construct an inversive plane $I_{X3}(\mathcal{O})$ from $T_3(\mathcal{O})$ as follows (see also the proof of [67, Theorem 5.3.1]).

Result 6.2.1 *For the generalized quadrangle $T_3(\mathcal{O})$ let X be a point of type (i) and let Y be the point (∞). The associated inversive plane $I_{X3}(\mathcal{O})$ is defined as follows:*

Points: *Hyperplanes $\langle X, \pi_{o_i}\rangle$, $i = 1, \dots, q^2+1$.*

Circles: *All sets $\{\langle X, \pi_{o'_1}\rangle, \dots, \langle X, \pi_{o'_{q+1}}\rangle\}$ where $\{o'_1, \dots, o'_{q+1}\}$ are the points of an oval (plane section) of $\mathcal{O}$.*

6.3 Thas maximal arcs and Inversive planes

Motivated by [92] we have the following definition.

Definition 6.3.1 *An* **O'Nan** *configuration is a set of six distinct points with the following properties. The set contains four distinct points A, B, C, D of which no three are collinear and the remaining two points E, F are such that $\{E\} = AC \cap BD$ and $\{F\} = AB \cap CD$. The six points A, B, C, D, E, F are called the* **vertices** *of the configuration.*

Let $\mathcal{K}$ be a maximal arc in a projective plane π_q of order q. Let X be a point of $\mathcal{K}$. We say $\mathcal{K}$ satisfies property

I_X: If $\mathcal{K}$ contains no O'Nan configurations with X a vertex.

II_X: If l is a secant line of $\mathcal{K}$ not through X, m a secant line of $\mathcal{K}$ through X meeting l in a point of $\mathcal{K}$ and $Y(\neq X)$ a point of $\mathcal{K}$ on m, then there exists a line $l' \neq m$ incident with Y and meeting every line through X that meets l and such that l' intersects each such line in a point of $\mathcal{K}$.)

We now show that a Thas maximal arc $\mathcal{K}$ with base point X satisfies I_X and II_X and these properties lead to defining an inversive plane associated to the Thas maximal arc.

Let $\mathcal{K}$ be a Thas maximal arc with base point X in a translation plane π_{q^2} of order q^2 with translation line ℓ_∞. Note that π_{q^2} has a Bruck-Bose representation in $PG(4, q)$ with the usual notation.

Lemma 6.3.1 *$\mathcal{K}$ satisfies I_X.*

Proof: Suppose there exists an O'Nan configuration in $\mathcal{K}$ with X a vertex. Let m_1 and m_2 be the two secant lines of $\mathcal{K}$ not incident with X in the configuration. Let P_i be the point of intersection of m_i and ℓ_∞, $i = 1, 2$. The three points of $\mathcal{K}$ on m_1 in the O'Nan configuration correspond to three generator lines l_1, l_2, l_3 of $\mathcal{K}$ and m_2 intersects these same generator lines of $\mathcal{K}$ in the O'Nan configuration. By Result 6.1.1 and the comments preceding it, in the Bruck-Bose representation of π_{q^2}, l_1, l_2, l_3 generate a hyperplane of $PG(4, q)\backslash\Sigma_\infty$ which contains the spread lines corresponding to $P_1, P_2 \in \ell_\infty$, a contradiction since Σ_∞ is the only hyperplane of $PG(4, q)$ which contains two distinct elements of the spread $\mathcal{S}$. Therefore there exist no O'Nan configurations in $\mathcal{K}$ with X a vertex. $\square$

Lemma 6.3.2 $\mathcal{K}$ *satisfies* II_X

Proof: Let l be a secant line of $\mathcal{K}$ not on X and let $l \cap \ell_\infty = \{P\}$. The result now follows from Result 6.1.1. $\square$.

Consider the incidence structure $I'_{\mathcal{K}}$ defined by:

Points:	generator lines of $\mathcal{K}$
Blocks:	secant lines of $\mathcal{K}$ not incident with X; identifying blocks with their points and using the property II_X to eliminate repeated blocks
Incidence:	is inherited from the translation plane

Lemma 6.3.3 $I'_{\mathcal{K}}$ *is a 2-*$(q^2+1, q, q-1)$ *design.*

Proof: There are q^2+1 generator lines of $\mathcal{K}$, corresponding to the points of the ovoid in the construction of $\mathcal{K}$, therefore the number v' of points of $I'_{\mathcal{K}}$ is q^2+1. A secant line of $\mathcal{K}$ which is not incident with X intersects q generator lines of $\mathcal{K}$, hence the number k' of points in a block is q.

By Result 6.1.1, each point of ℓ_∞ corresponds q distinct blocks of $I'_{\mathcal{K}}$ and since each secant line of $\mathcal{K}$ intersects ℓ_∞ in a unique point, blocks corresponding to distinct points of ℓ_∞ are distinct. Therefore the number b' of blocks of $I'_{\mathcal{K}}$ is therefore $q(q^2+1) = q^3+q$.

By Result 6.1.1 there exist $q-1$ secants on a point $P \in \ell_\infty$ which define the same block of $I'_{\mathcal{K}}$. A generator line of $\mathcal{K}$ has $q-1$ points of $\mathcal{K}$ besides X and there exist q^2 secant lines not containing X through each such point. Therefore in $I'_{\mathcal{K}}$, the number r' of blocks containing a point is $q^2(q-1)/(q-1) = q^2$.

Consider two generator lines of $\mathcal{K}$; they each have $q-1$ points besides X. From above a block is defined by $q-1$ distinct secant lines of $\mathcal{K}$ and therefore the number λ'_2 of blocks containing two fixed points is $(q-1)^2/(q-1) = q-1$.

It follows that $I'_{\mathcal{K}}$ is a 2-$(q^2+1, q, q-1)$ design. $\square$

We have that a block, B_P say, of $I'_{\mathcal{K}}$ is determined by $q-1$ distinct secant lines of $\mathcal{K}$ each incident with a common point $P \in \ell_\infty$. Thus to each block B_P in $I'_{\mathcal{K}}$ is associated a unique point not incident with the block, namely, the generator line of $\mathcal{K}$ on the line XP. We use this fact to define a new incidence structure as follows.

Definition 6.3.4 *Let $I_{\mathcal{K}}$ be the incidence structure defined by:*

Points:	*generator lines of $\mathcal{K}$*
Circles:	$\{\{$*Block* B_P *of* $I'_{\mathcal{K}}\} \cup \{$ *the generator line of* $\mathcal{K}$ *in* $XP\}$ *; for all blocks* B_P *in* $I'_{\mathcal{K}}\}$
Incidence:	*containment*

Lemma 6.3.5 *The incidence structure $I_{\mathcal{K}}$ is a 3-$(q^2+1, q+1, 1)$ design, namely a finite inversive plane of order q.*

Proof: $I_{\mathcal{K}}$ has the same number of points and blocks as $I'_{\mathcal{K}}$ therefore $v = v' = q^2+1$ and $b = b' = q^3+q$. The number k of points in a block of $I_{\mathcal{K}}$ is $b = b'+1 = q+1$.

The number r of blocks on a fixed point of $I_{\mathcal{K}}$ is given by

$r = r' + \{$ the number of blocks of $I'_{\mathcal{K}}$ determined by secant lines on a fixed point of $\ell_\infty\}$

Using the definition of blocks of $I'_{\mathcal{K}}$ and Result 6.1.1 we have $r = r' + q = q^2 + q$.

It remains to show that for any three distinct points of $I_{\mathcal{K}}$ there exists a unique block containing them.

Let l_1, l_2, l_3 be three distinct points of $I_{\mathcal{K}}$, that is, l_1, l_2, l_3 are three generator lines of $\mathcal{K}$ in the Bruck-Bose representation of the translation plane. The three lines span a hyperplane Σ in $PG(4,q)$ which intersects Σ_∞ in a plane containing a unique spread element; denote this spread element by P. Since the hyperplane Σ intersects the ovoidal cone of the Thas maximal arc in three generator lines, Σ contains an oval cone of generator lines. Thus the planes in Σ about P represent secant lines of $\mathcal{K}$ and define a unique block of $I_{\mathcal{K}}$ containing the points l_1, l_2, l_3.

We have shown therefore that $I_{\mathcal{K}}$ is an inversive plane. $\square$

Theorem 6.3.6 *The inversive plane $I_{\mathcal{K}}$ associated to a Thas maximal arc $\mathcal{K}$ with base point X in a translation plane π_{q^2} is isomorphic to the inversive plane $I_{X3}(\mathcal{O})$ obtained from the generalized quadrangle $T_3(\mathcal{O})$ (defined in the $PG(4,q)$ with ovoid $\mathcal{O}$ of the construction of $\mathcal{K}$.)*
The inversive planes are egglike.

Proof: The result follows from the above discussion of the construction in $PG(4,q)$ of $I_{\mathcal{K}}$ and Result 6.2.1. $\square$

Remark: The inversive plane associated to a Buekenhout-Metz unital (see Barwick and O'Keefe [13]) is isomorphic in a natural way to the inversive planes of Theorem 6.3.6 defined with the same ovoid $\mathcal{O}$ of $PG(3,q)$, since both Thas maximal arcs and Buekenhout-Metz unitals are defined using a $4-$dimensional ovoidal cone with base ovoid a $3-$dimensional ovoid $\mathcal{O}$.

6.4 A characterisation of Thas maximal arcs

In this section we endeavour to find a converse to the main result of Section 6.3. We attempt to characterise Thas Maximal Arcs with the configurational properties I_X and II_X. We weaken our hypothesis and obtain a partial converse.

6.4.1 A sequence of lemmata

Let $\mathcal{K}$ be a (maximal) $\{q^3-q^2+q;q\}$-arc in a translation plane π_{q^2} of order q^2 with kernel $GF(q)$, so that π_{q^2} has a Bruck and Bose representation in $PG(4,q)$ defined by a spread in the hyperplane Σ_∞ of $PG(4,q)$. Denote by ℓ_∞ the translation line of π_{q^2} and suppose ℓ_∞ is an external line of $\mathcal{K}$.

Let X be a fixed point of $\mathcal{K}$.

We say $\mathcal{K}$ satisfies:

I_X: (As in Section 6.3.)

II_X^*: If l is a secant line of $\mathcal{K}$ not through X and P is the point of intersection of lines l and ℓ_∞, then there exist $q-2$ further secant lines of $\mathcal{K}$ incident with P and which intersect every line through X that meets l (these intersections are all in $\mathcal{K}$).

Suppose $\mathcal{K}$ satisfies properties I_X and II_X^*.

We proceed with a sequence of lemmata determining some properties of $\mathcal{K}$, but first we introduce some terminology.

Each line on X contains $q-1$ points of $\mathcal{K}$ besides X; call such a set of $q-1$ points of $\mathcal{K}$ on a line through X a **variety**. For a variety V (on a line l through X), label the point at infinity of l, namely $l \cap \ell_\infty$, by P_V. We shall sometimes refer to P_V as the **point at infinity of the variety** V.

Let l be a secant line of $\mathcal{K}$ not on X. Then l is incident with q varieties and by II_X^* there exist $q-2$ further secants of $\mathcal{K}$ incident with these same q varieties and concurrent with l in a point P on ℓ_∞. Call such a collection of q varieties a **block** b and call the associated point P on ℓ_∞ the **point at infinity of the block** b and say b **is a block of** P.

Lemma 6.4.1.1 *For a point $P \in \ell_\infty$,*

(i) Distinct blocks of P are disjoint (they have no varieties in common.)

(ii) P is the point at infinity of exactly q blocks.

Proof: Let P be a point on ℓ_∞.

(i) Let b_1 and b_2 be two blocks of P. Suppose b_1 and b_2 intersect in a variety V_1. Let l_1 be a secant line of $\mathcal{K}$ on P incident with b_1 (and therefore incident with every variety in b_1). Since l_1 is incident with the variety V_1 of block b_2 and l_1 passes through P, then l_1 must be one of the $q-1$ secant lines of $\mathcal{K}$ on P incident with every variety in b_2 by II_X^*. Since l_1 intersects $\mathcal{K}$ in exactly q points, blocks b_1 and b_2 must coincide. We have shown therefore that distinct blocks of P are disjoint.

(ii) There exist q^2-q secant lines of $\mathcal{K}$ on P besides the line XP. For each block of P there exist $q-1$ secant lines of $\mathcal{K}$ on P which determine that block and since by (i) distinct blocks of P are disjoint, there are exactly q blocks of P. □

Lemma 6.4.1.2 *Let P and Q be two points on ℓ_∞ and let b_P, b_Q be a block of P,Q respectively. Then the blocks b_P and b_Q intersect in exactly $0, 1, 2$ or q varieties.*

Proof: If $P = Q$ then by Lemma 6.4.1.1 b_P intersects b_Q in 0 or q varieties.

If $P \neq Q$, suppose b_P and b_Q have three varieties V_1, V_2, V_3 in common; V_i contained in line l_i, $i = 1, 2, 3$, incident with X. Let R be a point of $\mathcal{K}$ in V_1. By II_X^*, the line RP is a secant line of $\mathcal{K}$ incident with P and incident with the varieties in b_P; also the line RQ is a secant line of $\mathcal{K}$ on Q incident with b_Q. The lines RP, RQ, l_2 and l_3 are four lines of an O'Nan configuration in $\mathcal{K}$ with X as a vertex; a contradiction, as $\mathcal{K}$ satisfies I_X, thus in this case b_P and b_Q have at most 2 varieties in common. □

Lemma 6.4.1.3 *There are exactly $q^3 + q$ blocks in $\mathcal{K}$.*

Proof: By Lemma 6.4.1.1 there are q blocks corresponding to each of the q^2+1 points of ℓ_∞ and by definition (or the proof of Lemma 6.4.1.2) a block corresponds to a unique point at infinity. The result follows. $\square$

Lemma 6.4.1.4 *Let V_1 and V_2 be two distinct varieties. There exist exactly $q-1$ blocks containing both V_1 and V_2.*

Proof: Let V_1 be on line l_1 through X and let V_2 be on line l_2 through X. Let R be a point (of $\mathcal{K}$) in V_1. The join of R to each point of V_2 defines $q-1$ secant lines m_i $(i = 1, \ldots, q-1)$ of $\mathcal{K}$, not on X and with distinct points $P_1, \ldots P_{q-1}$ on the line at infinity. The line m_i defines block B_i, containing both varieties V_1 and V_2, and with point at infinity P_i (for $i = 1, \ldots, q-1$). Thus there exist at least $q-1$ blocks containing both V_1 and V_2.

By II_X^*, for each block B_i there exist $q-2$ further lines through P_i incident with both V_1 and V_2, thus giving all the possible lines joining a point of V_1 and a point of V_2. Thus there exist exactly $q-1$ blocks containing both V_1 and V_2. $\square$

Lemma 6.4.1.5 *There are exactly q^2 blocks containing a given variety V.*

Proof: Let P_V be the point at infinity of a fixed variety V. For each point P on the line at infinity besides P_V, V lies in a block of P, since there exist secant lines of $\mathcal{K}$ on P incident with points in V. Therefore by Lemmata 6.4.1.1 and 6.4.1.2, V lies in exactly one block of P $(P \in \ell_\infty \backslash \{P_V\})$, with no two distinct points at infinity determining the same block containing V. Since there are q^2 points on ℓ_∞ besides P_V, there exist exactly q^2 blocks containing the variety V. $\square$

Let $\mathcal{V}$ be the set of varieties and $\mathcal{B}$ be the set of blocks and with incidence $\mathbf{I}$ the natural containment relation. We define an incidence structure $\mathcal{I}' = (\mathcal{V}, \mathcal{B},\ \mathbf{I})$.

Lemma 6.4.1.6 *The incidence structure $\mathcal{I}' = (\mathcal{V}, \mathcal{B}, \mathbf{I})$ is a 2-$(q^2+1, q, q-1)$ design with parameters:*

$$\begin{aligned} v' &= q^2+1 \\ k' &= q \\ b' &= q^3+q \\ r' &= q^2 \\ \lambda_2' &= q-1 \end{aligned}$$

Proof: Lemmata 6.4.1.1, 6.4.1.2, 6.4.1.3, 6.4.1.3, 6.4.1.4 and 6.4.1.5 determine the parameters of $\mathcal{I}'$. $\square$

Next we define a new incidence structure $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathbf{I})$ based on $\mathcal{I}'$. Let the set of varieties $\mathcal{V}$ of $\mathcal{I}'$ be the points $\mathcal{P}$ of $\mathcal{I}$ and let

$$\begin{aligned} \mathcal{C} &= \{\{\text{varieties in a block } B_P \text{ of a point } P\} \cup \{\text{the variety contained in the line } XP\} \\ &\quad : \text{for all blocks } B_P \text{ of a point } P, \text{ for all points } P \text{ on } \ell_\infty\}. \end{aligned}$$

Call the elements of $\mathcal{C}$ **circles** and call $\mathcal{C}$ the **set of circles** in $\mathcal{I}$.

There is a natural one-to-one correspondence between blocks of $\mathcal{I}'$ and circles of $\mathcal{I}$ since each block of $\mathcal{I}'$ is contained in a unique circle and conversely each circle of $\mathcal{I}$ contains a unique block of $\mathcal{I}'$.

Lemma 6.4.1.7 *The incidence structure $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathbf{I})$ is a 2-$(q^2+1, q+1, q+1)$ design with parameters*

$$\begin{aligned} v &= q^2+1 \\ k &= q+1 \\ b &= q^3+q \\ r &= q^2+q \\ \lambda_2 &= q+1 \end{aligned}$$

Proof: Now $v = v' = q^2+1$ and $b = b' = q^3+q$ using the definition of $\mathcal{I}$ and the natural one-to-one correspondence between circles and blocks. The number k of varieties in a circle is one more than the number k' of varieties in a block, therefore $k = k'+1 = q+1$.

For a variety V with point at infinity P, the number of circles containing V equals the number of blocks containing V plus the number of blocks of P, therefore $r = r' + q = q^2 + q$.

Lastly, consider two varieties V_1 and V_2 with points at infinity P_1 and P_2 respectively. Variety V_1 lies in a unique block of P_2 and similarly variety V_2 lies in a unique block of P_1 and there are $q-1$ blocks containing both V_1 and V_2. Therefore the number λ_2 of circles containing both V_1 and V_2 is $q+1$. $\square$

Corollary 6.4.1.8 *The following four statements are equivalent for the incidence structure $\mathcal{I}$.*

(i) *three distinct varieties are contained in at least one circle*

(ii) *three distinct varieties are contained in at most one circle*

(iii) *the design $\mathcal{I}$ has parameter $\lambda_3 = 1$*

(iv) *the design $\mathcal{I}$ is a finite inversive plane*

Proof: If three distinct varieties are contained in a unique circle, for any choice of three distinct varieties, then $\mathcal{I}$ is a 3-$(q^2+1, q+1, 1)$ design with the parameters given in Lemma 6.4.1.7 together with $\lambda_3 = 1$, that is, $\mathcal{I}$ is a finite inversive plane.

Let λ_{3_i}, $i = 1, \dots, \binom{v}{3}$, be the number of circles containing three given (distinct) varieties V_1, V_2, V_3, for all $\binom{v}{3}$ possible choices of V_1, V_2, V_3. We now count in two ways the number of 3-flags of $\mathcal{I}$

$$\sum_{i=1}^{\binom{v}{3}} \lambda_{3_i} = b\binom{k}{3}.$$

Thus the average number $\lambda_{3_i}ave$ of circles on three varieties is given by

$$\begin{aligned} \lambda_{3_i}ave &= b\tbinom{k}{3}/\tbinom{v}{3} \\ &= 1. \end{aligned}$$

Therefore if $\lambda_{3_i} \geq 1$ for all i then $\lambda_{3_i} = 1$ for all i. Similarly if $\lambda_{3_i} \leq 1$ for all i then $\lambda_{3_i} = 1$ for all i. □

Lemma 6.4.1.9 *Let V_1 and V_2 be two distinct varieties in a block b_P of a point P ($P \in \ell_\infty$). Let l_i be the lines on X containing V_i, with the point at infinity of l_i denoted by Q_i, $i = 1, 2$.*

If a Baer subplane B of π_{q^2} contains P, Q_1, Q_2 and X then
either *B contains no points of V_1 or V_2*
or *B contains the same number of points of V_1 as of V_2.*

Proof: Let R be a point of V_1 in B. Since PR and l_2 are lines of B, the point $PR \cap l_2$ is a point of B. Since l_1 and l_2 lie in the block b_P of P, by II^*_X, the point $PR \cap l_2$ of B is a point on l_2 of the maximal arc $\mathcal{K}$, that is $PR \cap l_2$ is a point of V_2. The same argument holds if we suppose R is a point of V_2 in B.

It follows that either B contains no points of V_1 and V_2 or B contains the same number of points of V_1 as of V_2. □

We now use the results of Section 2.1 and Section 2.2 concerning the Bruck-Bose representation in $PG(4,q)$ of some of the Baer subplanes and Baer sublines of π_{q^2}. In the following lemmata, a *linear* Baer subplane of π_{q^2} is a Baer subplane of π_{q^2} which is represented in Bruck-Bose by a (transversal) plane of $PG(4,q)\backslash\Sigma_\infty$ which intersects Σ_∞ in a line which is not a line of the spread $\mathcal{S}$ of Σ_∞; a *linear* Baer subline is a Baer subline of a line of π_{q^2} which is represented in Bruck-Bose by a line of $PG(4,q)\backslash\Sigma_\infty$.

Lemma 6.4.1.10 *There exists a linear Baer subline in π_{q^2} containing X and which contains at least one further point of $\mathcal{K}$. If a linear Baer subline which contains X contains exactly n further points of $\mathcal{K}$, then every linear Baer subline which contains X and which contains further points of $\mathcal{K}$ contains exactly n points of $\mathcal{K}$ besides X.*

Proof: Let l_1 be a line on X containing a linear Baer subline l_{B1}, where l_{B1} contains X and contains n points of $\mathcal{K}$ besides X. Let $l_2(\neq l_1)$ be any other line containing a linear Baer subline l_{B2}, with $X \in l_{B2}$, and such that l_{B2} contains further points of $\mathcal{K}$. There exists a linear Baer subplane B of π_{q^2} containing l_{B1} and l_{B2} and since l_{B1} and l_{B2} both have points at infinity, the line at infinity is a line of B.

Let l be a line not through X and such that l contains a point of $\mathcal{K}$ in l_{B1} and a point of $\mathcal{K}$ in l_{B2}, then l is a line of B and intersects ℓ_∞ in a point P of B. Thus, as l is a secant line of $\mathcal{K}$ on P and hence the varieties in l_1 and l_2 lie together in a block of P. Now by Lemma 6.4.1.9, Baer sublines l_{B1} and l_{B2} contain the same number of points of $\mathcal{K}$ besides X. The result now follows. □

By Lemma 6.4.1.10, the linear Baer sublines of π_{q^2} which contain X contain either 0 or n further points of $\mathcal{K}$, where $1 \leq n \leq q-1$ is a fixed integer. Moreover, since each secant line of $\mathcal{K}$ incident with X contains exactly $q-1$ points of $\mathcal{K}$ distinct from X, the integer n divides $q-1$.

Lemma 6.4.1.11 *If π_{q^2} is the Desarguesian plane $PG(2,q^2)$, then each linear Baer subline of π_{q^2} which contains X contains either 0 or n further points of $\mathcal{K}$, where $1 < n \leq q-1$ is a fixed integer such that n divides $q-1$.*

Proof: If π_{q^2} is the Desarguesian plane $PG(2,q^2)$, then by 1.15.2 and since π_{q^2} contains a maximal arc $\mathcal{K}$ we have that q is even. Moreover in the Bruck-Bose representation of π_{q^2} in $PG(4,q)$ the $1-$spread $\mathcal{S}$ of $\Sigma_\infty = PG(3,q)$ is then a regular spread. By the remarks preceding this lemma we have that each linear Baer subline of π_{q^2} which contains X contains exactly 0 or n further points of $\mathcal{K}$, where $1 \leq n \leq q-1$ is a fixed integer and n divides $q-1$. Suppose that $n = 1$. Firstly if $q = 2$, then $\mathcal{K}$ is necessarily a Thas maximal arc in π_{q^2}, so we consider the case $q \geq 4$. Consider two distinct varieties V_1 and V_2 of $\mathcal{I}'$ contained in lines ℓ_1, ℓ_2 of π_{q^2} respectively. By definition ℓ_1 and ℓ_2 intersect in the point X of $\mathcal{K}$. Denote by P_1 and P_2 the points at infinity of ℓ_1 and ℓ_2 respectively. In Bruck-Bose, the points P_1, P_2 on ℓ_∞ correspond to distinct elements P_1^*, P_2^* of the regular spread $\mathcal{S}$ of Σ_∞. In $\mathcal{I}'$, there exist $q-1$ distinct blocks which contain the varieties V_1 and V_2; denote the points at infinity of these blocks by $Q_1, Q_2, \ldots, Q_{q-1}$. In Bruck-Bose the points Q_i correspond to $q-1$ distinct elements of the spread $\mathcal{S}$; denote these spread elements by Q_i^*, $i = 1, \ldots, q-1$. There exist $q+1$ reguli in $\mathcal{S}$ containing P_1^* and P_2^*, therefore there exists at least one regulus $\mathcal{R}$ of lines of $\mathcal{S}$ which contains P_1^* and P_2^* but which contains no spread element Q_i^*. Let $\mathcal{R}'$ denote the opposite regulus of $\mathcal{R}$ in Σ_∞. In Bruck-Bose, the lines ℓ_1 and ℓ_2 correspond to planes ℓ_1^* and ℓ_2^* in $PG(4, q)$ respectively; both planes contain X and a line P_1^*, P_2^* respectively of $\mathcal{S}$.

Since $n = 1$, the q points of $\mathcal{K}$ in ℓ_1 are represented in Bruck-Bose by the point X and $q-1$ further points of $\ell_1^* \backslash \{P_1^*\}$ on distinct lines of ℓ_1^* through X. Similarly for the points of $\mathcal{K}$ incident with ℓ_2. In Bruck-Bose, since $q \geq 4$ there exists a line m in the opposite regulus of $\mathcal{R}$ such that the plane $B = \langle m, X \rangle$ contains a point of $\mathcal{K}$ in ℓ_1^* besides X and a point of $\mathcal{K}$ in ℓ_2^* besides X; denote these two points of $\mathcal{K}$ in B, which are distinct from X, by Y_1^*, Y_2^* respectively. Each point Y_i^* corresponds to a point Y_i in π_{q^2} incident with the variety V_i for $i = 1, 2$. The line $Y_1 Y_2$ is distinct from ℓ_∞ and intersects ℓ_∞ in a point Q which is necessarily the point at infinity of a block containing both varieties V_1 and V_2. In Bruck-Bose Q corresponds to a spread element Q^* contained in the regulus $\mathcal{R}$ of $\mathcal{S}$; a contradiction, since the regulus $\mathcal{R}$ contains no element which is the Bruck-Bose representation of a point of infinity of a block containing the varieties V_1 and V_2. Hence $n \neq 1$ and therefore $n > 1$ as required. $\square$

Note that a *Mersenne prime* is a prime number which can be written in the form $2^p - 1$ for some positive integer p which is necessarily prime (see [47, Theorem 18]). There are 31 known Mersenne primes and it is conjectured that there exist an infinite number of Mersenne primes.

Corollary 6.4.1.12 *Suppose $\mathcal{K}$ is a maximal $\{q^3 - q^2 + q; q\}-$arc in the Desarguesian plane $PG(2, q^2)$ satisfying properties I_X and II_X^* for some point X in $\mathcal{K}$. If $q - 1$ is a Mersenne prime, then $\mathcal{K}$ is a Thas maximal arc with base point X and axis line ℓ_∞.* $\Box$

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