Substructures of Finite Geometries

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Abstract

This thesis deals with the study of unitals embedded in finite translation planes. In Chapter 1 we introduce the required mathematical background.

Chapter 2 looks at characterizations of unitals. We enumerate the Buekenhout unitals in a translation plane of dimension at most two over its kernel and prove a characterization of them. We show that all Buekenhout unitals in $PG(2,q^2)$ are classical and prove a characterization of the classical unital in terms of how the Baer sublines of $PG(2,q^2)$ meet it. We show that a non-classical Buekenhout-Metz unital in $PG(2,q^2)$ has a unique axis which provides a second characterization of the classical unital.

In Chapter 3, we give a new proof of the relationship between Buekenhout-Metz unitals and inversive planes that is more general than the known results, and gives a natural, geometric isomorphism between the resulting inversive plane and the egglike inversive plane arising from the ovoid involved in the construction of the Buekenhout-Metz unital. We then apply our results to investigate some parallel classes and partitions of the lines of any Buekenhout-Metz unital.

The unitals in the Hall plane are investigated in Chapter 4 by deriving the Desarguesian plane and observing the effect on the classical unital. Two new infinite classes of unitals are found in the Hall plane which are not isomorphic to the classical unital. The question of whether the classical unital is embeddable in the Hall plane is partially answered. We enumerate the Buekenhout and Buekenhout-Metz unitals in the Hall plane. The results are generalized to other translation planes.

Finally, an appendix is included that contains a brief historical survey of the study of unitals.

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Contents

List of Figures

Chapter 1

Introduction

This thesis deals with the study of unitals embedded in finite translation planes. In this chapter we give some of the necessary background. For more details, see Dembowski [47] or Hughes and Piper [68].

1.1 Incidence structures

An **incidence structure** is a set of points $\mathcal{P}$, blocks $\mathcal{B}$ and an incidence relation $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. We say a point P is incident with a block y if $(P, y) \in \mathcal{I}$. We only consider incidence structures with a finite number of points and blocks.

If we let the number of points be v and the number of blocks be b, then a t-(v, k, λ) **structure** (or t-structure) is an incidence structure with v points such that every block is incident with exactly k points and every subset of t points is incident with exactly λ common blocks. The numbers t, v, k, λ are called the parameters of the structure. We can deduce the following relationships between the parameters using simple counting arguments.

1. There are $b = \lambda \binom{v}{t} / \binom{k}{t}$ blocks.

2. Every point is incident with $r = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$ blocks.

3. Every set of n points, $0 \leq n \leq t$ is contained in exactly $n = \lambda\binom{v-n}{t-n}/\binom{k-n}{t-n}$ blocks.

4. $bk = vr$.

A t-structure may have repeated blocks, a t-(v, k, λ) **design** (or t-design) is a t-structure with no repeated blocks. See [69] for more details.

A **unital** is a 2-$(q^3 + 1, q + 1, 1)$ design, we call q the **parameter** of a unital and usually assume $q > 2$. A unital has $q^4 - q^3 + q^2$ blocks with q^2 blocks through every point. Section 1.2 introduces unitals as structures embedded in projective planes.

1.2 Projective planes

A **projective plane** is an incidence structure with points and lines that satisfies the following three axioms.

1. Any two points lie on a unique line.

2. Any two lines meet in a unique point.

3. There exists at least four points with no three collinear.

In this thesis we work solely with *finite* projective planes.

Associated with a finite projective plane $\mathcal{P}$ is a parameter q called the **order** of $\mathcal{P}$. $\mathcal{P}$ has $q^2 + q + 1$ lines and $q^2 + q + 1$ points. There are $q + 1$ points on every line of $\mathcal{P}$ and $q + 1$ lines through every point. In fact, the class of projective planes is exactly the class of 2-$(q^2 + q + 1, q + 1, 1)$ designs.

For any prime power q we can construct a projective plane $PG(2,q)$ whose points are coordinatized by the Galois field of order q, $GF(q)$. We call such a projective plane Desarguesian, as they are the only projective planes that satisfy Desargue's Theorem (see [47]). The first examples of finite projective planes in which Desargue's Theorem fails were constructed by Veblen and Wedderburn in 1907 [111]. Many others have been found since then, but all known examples of finite projective planes have prime power order.

There are two types of mappings in a projective plane that we use. A **collineation** of a projective plane is a bijective mapping of points onto points and lines onto lines that preserves incidence. A **correlation** of a projective plane is a bijective mapping of points onto lines and lines onto points that preserves incidence. A correlation of order two is called a **polarity**.

Let σ be a polarity of a projective plane $\mathcal{P}$. An **absolute** element of σ is a point or line of $\mathcal{P}$ that is incident with its image under σ. It is easy to show that each absolute point of σ lies on a unique absolute line and dually each absolute line contains a unique absolute point.

In 1946, Baer [8] showed that if $\mathcal{P}$ is a projective plane of order q, then a polarity σ of $\mathcal{P}$ has at least $q+1$ absolute points. A polarity with exactly $q+1$ absolute points is called an **orthogonal** polarity, the absolute points form a conic if q is odd and are collinear if q is even. In 1970, Seib [98] showed that σ has at most $q^{\frac{3}{2}}+1$ absolute points. A polarity with exactly $q^{\frac{3}{2}}+1$ absolute points is called **unitary** and the absolute points and non-absolute lines form a unital.

A **Hermitian curve** in $PG(2,q^2)$ is a set of points $x=(x_0,x_1,x_2)$ satisfying the equation $x^qHx^t=0$, where H is a matrix over $GF(q^2)$ satisfying $H^q=H^t$, see [61]. It has long been known that the absolute points and non-absolute lines of a unitary polarity in $PG(2,q^2)$, q a prime power, form a Hermitian curve, which we call the **classical unital**.

We say a unital is embedded in a projective plane $\mathcal{P}$ if there is a one to one mapping from the points of U onto q^3+1 points of $\mathcal{P}$ such that the lines of U extend to lines of $\mathcal{P}$. Equivalently, a **unital** U **embedded in a projective plane** $\mathcal{P}$ of order q^2 is a set of q^3+1 points of $\mathcal{P}$ such that every line of $\mathcal{P}$ meets U in 1 or $q+1$ points. A line of $\mathcal{P}$ is a **tangent line** or a **secant line** of U if it contains 1 or $q+1$ points of U respectively. The lines of U are the intersections of the secants of U with U, that is, if l is a secant line of U, then $\bar{l} = l \cap U$ is a line of U. A point of U lies on 1 tangent and q^2 secant lines of U. A point of $\mathcal{P}$ not in U lies on $q+1$ tangent lines and q^2-q secant lines of U.

Buekenhout [40] and Metz [87] give examples of non-classical unitals in the Desarguesian plane, that is, unitals that are not the absolute points of a unitary polarity.

An **affine plane** $\mathcal{A}$ is constructed from a projective plane $\mathcal{P}$ by removing a fixed line ℓ_∞ of $\mathcal{P}$, called the **line at infinity**, and the points of $\mathcal{P}$ on ℓ_∞, we write $\mathcal{A} = \mathcal{P}^{\ell_\infty}$. If $\mathcal{P} = PG(2,q)$, then $\mathcal{A}$ is denoted by $AG(2,q)$. If $\mathcal{A}$ is any affine plane, then there is (up to isomorphism) a unique projective plane containing $\mathcal{A}$.

1.3 Projective and affine spaces

Let $V = V(n+1,q)$ be the vector space of dimension $n+1$ over the field $GF(q)$. The set of subspaces of V together with the incidence relation of subspace containment is called the **projective space** $PG(n,q)$ of **dimension** n. If W is a subspace of V of dimension $m+1$, where $0 \le m \le n$, then the set of subspaces of W together with the incidence relation of subspace containment is a projective space isomorphic to $PG(m,q)$ and is called an m**-dimensional subspace** of $PG(n,q)$. Subspaces of $PG(n,q)$ of dimension 0, 1, 2 and $n-1$ are called **points, lines, planes** and **hyperplanes** respectively. The vector subspace of zero dimension is regarded as the empty projective space in $PG(n,q)$ and is defined to have dimension -1.

Let W and Y be two subspaces of $PG(n,q)$ and let $\langle W,Y\rangle$ denote the subspace spanned by W and Y. We can use **Grassman's identity** to calculate the dimension of the intersection or span of W and Y:

$$\dim W + \dim Y = \dim\langle W,Y\rangle + \dim(W \cap Y).$$

Every line of $PG(n,q)$ contains $q+1$ points and $PG(n,q)$ contains $q^n + \cdots + q + 1$ points, q is called the **order** of the projective space. In $PG(n,q)$ there are

$$\prod_{i=0}^{m} \frac{q^{n-i+1}-1}{q^{i+1}-1}$$

subspaces of dimension m, each contains $q^m + \cdots + q + 1$ points of $PG(n,q)$.

An **affine space** $AG(n,q)$ of **dimension** n is obtained by removing from $PG(n,q)$ a fixed hyperplane (called the **hyperplane at infinity**) together with all the subspaces incident with it.

1.4 Quadrics, k-arcs and ovoids

A **quadric** $\mathcal{Q}_n$ of $PG(n,q)$ is a set of points $(x_0,\ldots,x_n)$ of $PG(n,q)$ satisfying $F(x_0,\ldots,x_n)=0$ where F is a non-degenerate quadratic form with coefficients in $GF(q)$. That is,

$$F(x_0,\ldots,x_n) = \sum_{i,j=0}^{n} a_{ij}x_ix_j.$$

If n is even, there is exactly one projectively distinct non-singular quadric, called a **parabolic quadric** $\mathcal{P}_n$. If n is odd, there are two projectively distinct non-singular quadrics, a **hyperbolic quadric** $\mathcal{H}_n$ and an **elliptic quadric** $\mathcal{E}_n$.

In this thesis we will be working with the parabolic quadric of $PG(4,q)$, the hyperbolic and elliptic quadrics of $PG(3,q)$, and the parabolic quadric (or **conic**) of $PG(2,q)$. We briefly describe the structure of these quadrics.

A conic of $PG(2,q)$ contains $q+1$ points with no three of them collinear. An elliptic quadric of $PG(3,q)$ contains q^2+1 points with no three collinear.

In order to describe the structure of the hyperbolic quadric $\mathcal{H}_3$, we need to define a regulus of $PG(3,q)$.

A **regulus** of $PG(3,q)$ is a set $\mathcal{R}$ of $q+1$ skew lines such that any line that meets three lines of $\mathcal{R}$ meets them all. Hence there are $q+1$ skew lines of a regulus $\mathcal{R}'$ that each meet every line of $\mathcal{R}$. We call $\mathcal{R}'$ the **complementary** or **opposite** regulus of $\mathcal{R}$.

A hyperbolic quadric of $PG(3,q)$ consists of two complementary reguli and contains $(q+1)^2$ points. A parabolic quadric $\mathcal{P}_4$ of $PG(4,q)$ is a set of $(q+1)(q^2+1)$ points on $(q+1)(q^2+1)$ lines with $q+1$ lines through each point.

An m-dimensional subspace of $PG(n,q)$ meets a quadric $\mathcal{Q}_n$ in a quadric $\mathcal{Q}_m$. Thus any three-dimensional subspace of $PG(4,q)$ meets $\mathcal{P}_4$ in either a hyperbolic quadric, an elliptic quadric, or a conic cone. A plane meets a quadric in a point, a line, a conic or two lines. For more details on quadrics see [65, Chapter 22].

A k**-arc** in $PG(2,q)$ is a set of k points such that no three are collinear. It is easy to show that if q is odd, then $k \leq q+1$, and if q is even, then $k \leq q+2$. A $(q+1)$-arc is called an **oval** and a $(q+2)$-arc is called a **hyperoval**. A well known theorem of Segre [96] says that if q is odd, then every $(q+1)$-arc of $PG(2,q)$ is a conic. When q is even, a conic and its nucleus form a **regular** hyperoval, however there are examples of hyperovals in $PG(2,q)$ that are not regular, see [44].

We can extend the concept of k-arcs to higher dimensions in two ways. A k**-cap** in $PG(3,q)$ is a set of k points with no three collinear. The maximum value k can be is q^2+1, we call a (q^2+1)-cap an **ovoid**. Barlotti [14] showed that when q is odd, every ovoid is an elliptic quadric. Tits [108] gave an example of an ovoid, the **Suzuki-Tits ovoid**, that is not an elliptic quadric in $PG(3,q)$, $q=2^h$, h odd, $h>1$.

In $PG(4, q)$ an **ovoidal cone** is a cone whose base is an ovoid and whose vertex is a point not in the three dimensional subspace containing the ovoid. It is formed by joining each point of the base ovoid to the vertex. If the ovoid is an elliptic quadric, we call the ovoidal cone an **orthogonal cone**.

A k**-arc** in $PG(n, q)$ is a set of k points with no $n+1$ lying in a hyperplane. Thus a k-arc of $PG(3, q)$ is a set of k points, no four coplanar. The maximum size of a k-arc in $PG(3, q)$ is $k = q + 1$.

1.5 Baer subplanes

A **subplane** $\mathcal{P}_0$ of a projective plane $\mathcal{P}$ is a subset of the elements of $\mathcal{P}$ that form a projective plane with incidence inherited from $\mathcal{P}$. A subplane $\mathcal{P}_0$ of $\mathcal{P}$ is called **proper** if $\mathcal{P}_0 \neq \mathcal{P}$. If $\mathcal{P}$ is any finite projective plane, then the possible orders for subplanes of $\mathcal{P}$ are restricted by the following result.

Theorem 1.5.1 (Bruck)[34] *If $\mathcal{P}$ is a finite projective plane of order n with a proper subplane $\mathcal{P}_0$ of order m, then either $n = m^2$ or $n \geq m^2 + m$.*

If $n = m^2$, then every point of $\mathcal{P}$ is on a line of $\mathcal{P}_0$. We call $\mathcal{P}_0$ a **Baer subplane** of $\mathcal{P}$. Every line of $\mathcal{P}$ meets $\mathcal{P}_0$ in either 1 or $m+1$ points and are called **tangents** or **secants** of $\mathcal{P}_0$ respectively. If l is a secant of $\mathcal{P}_0$, then $l \cap \mathcal{P}_0$ is a line of $\mathcal{P}_0$ and is called a **Baer subline** of $\mathcal{P}$.

Any three collinear points of $PG(2, q^2)$ determine a unique Baer subline [35]. Hence any two Baer sublines meet in 0, 1, 2 or $q + 1$ points. Using this it is easy to show that any set of four points with no three collinear - a **quadrangle** - is contained in a unique Baer subplane.

1.6 Translation planes

This thesis investigates unitals embedded in finite translation planes. In order to define a translation plane we need to give some more details on collineations.

Recall that a collineation σ of a projective plane $\mathcal{P}$ is a bijection from points to points and lines to lines that preserves collinearity. We say a line l is **fixed pointwise** by σ if every point of l is fixed by σ. We say a point P is **fixed linewise** by σ if every line through P is fixed by σ.

If a collineation σ fixes a line l pointwise and a point V linewise, then σ is called a (V, l)-**perspectivity**. Let V be a point and l a line of the projective plane $\mathcal{P}$, then $\mathcal{P}$ is (V, l)-**transitive** if for all points A, B of $\mathcal{P}$ with A, B, V collinear and $A, B \notin l$, there exists a (V, l)-perspectivity σ with $A^\sigma = B$. If $\mathcal{P}$ is (X, l)-transitive for all $X \in l$, then l is a **translation line** of $\mathcal{P}$ and $\mathcal{P}$ is a **translation plane** with respect to l.

The literature often regards the affine plane $\mathcal{P}^l$ as a translation plane. The two statements are equivalent since $\mathcal{P}$ and l uniquely define $\mathcal{P}^l$ and conversely. In this thesis we will always refer to a translation plane as a projective plane with translation line ℓ_∞.

1.7 The Bruck and Bose representation

In this section we introduce the André [5] and Bruck and Bose [36, 37] representation of a finite translation plane $\mathcal{P}$ of dimension at most two over its kernel in $PG(4, q)$. The results of this section are discussed in [62, Section 17.7].

A **spread** of $PG(3, q)$ is a set of $q^2 + 1$ skew lines. The lines of a spread partition the points of $PG(3, q)$. Note that if $\mathcal{S}$ is a spread of $PG(3, q)$, then any plane of

$PG(3,q)$ contains exactly one line of $\mathcal{S}$ and meets all other lines of $\mathcal{S}$ in exactly one point.

Let Σ_∞ be a hyperplane of $PG(4,q)$ and let $\mathcal{S}$ be a spread of Σ_∞. We use the phrase **a subspace of** $PG(4,q)\backslash\Sigma_\infty$ to mean a subspace of $PG(4,q)$ that is not contained in Σ_∞.

The affine plane $\mathcal{P}^{\ell_\infty}$ can be represented by the affine space $PG(4,q)\backslash\Sigma_\infty$ as follows: the points of $\mathcal{P}^{\ell_\infty}$ are the points of $PG(4,q)\backslash\Sigma_\infty$; the lines of $\mathcal{P}^{\ell_\infty}$ are the planes of $PG(4,q)$ that meet Σ_∞ in a line of $\mathcal{S}$; and incidence is the natural inclusion. We complete this affine representation to a projective representation by letting points of ℓ_∞ correspond to lines of the spread $\mathcal{S}$. We say the spread $\mathcal{S}$ **generates** the translation plane $\mathcal{P}$.

A spread $\mathcal{S}$ is **regular** if for any three lines of $\mathcal{S}$, the regulus containing them is in $\mathcal{S}$. The projective plane $\mathcal{P}$ is Desarguesian if and only if the spread $\mathcal{S}$ is regular.

We call a Baer subplane that is secant to ℓ_∞ an **affine** Baer subplane of $\mathcal{P}$. The next lemma due to Grünig describes their representation in $PG(4,q)$.

Lemma 1.7.1 [60] *(i) Let $\mathcal{P}$ be a translation plane of order q^2. Any plane α of $PG(4,q)$ that intersects Σ_∞ in a line not in $\mathcal{S}$ can be regarded as a affine Baer subplane of $\mathcal{P}$.*

(ii) If $\mathcal{P}$ is Desarguesian, each Baer subplane secant to ℓ_∞ can be represented in this way.

Thus lines of $PG(4,q)\backslash\Sigma_\infty$ that meet Σ_∞ in a point of the line t of $\mathcal{S}$ correspond to Baer sublines of $\mathcal{P}$ meeting ℓ_∞ in the corresponding point T. In $PG(2,q^2)$, these are exactly the Baer sublines meeting ℓ_∞ in a point.

Vincenti [112] and Freeman [56] showed that certain translation planes of order q^4 contain affine Baer subplanes that do not correspond to planes of $PG(4,q)$.

Hence Lemma 1.7.1(ii) does not hold for all non-Desarguesian translation planes of dimension two over their kernel.

Let $\mathcal{B}$ be a Baer subplane of $\mathcal{P}$ that meets ℓ_∞ in the point T and let t be the line of the spread that corresponds to T. In $PG(4,q)$, $\mathcal{B}$ corresponds to a **ruled cubic surface** [92] that consists of $q+1$ disjoint lines of $PG(4,q)\setminus\Sigma_\infty$ each incident with t and such that no two are contained in a plane about t.

A Baer subline of $\mathcal{P}$ containing no point of ℓ_∞ is represented in $PG(4,q)$ by a conic in a plane through a line of $\mathcal{S}$. However there are more conics in a plane than Baer sublines of a line, so not all such conics correspond to Baer sublines of $\mathcal{P}$.

If U is a unital of a translation plane $\mathcal{P}$ of dimension at most two over its kernel, then the points of U correspond to a set of points in $PG(4,q)$, denoted by $\mathcal{U}$. In 1976, Buekenhout [40] investigated the set $\mathcal{U}$ of $PG(4,q)$ that corresponds to a classical unital U of $PG(2,q^2)$. The two cases of U tangent and secant to ℓ_∞ were looked at separately.

1. If U is a classical unital of $PG(2,q^2)$ meeting ℓ_∞ in the point T, then, in $PG(4,q)$, $\mathcal{U}$ is an orthogonal cone whose vertex lies on the line t of $\mathcal{S}$.

2. If U is a classical unital in $PG(2,q^2)$ meeting ℓ_∞ in the points $T_0,\ldots,T_q$, then, in $PG(4,q)$, $\mathcal{U}$ is a non-singular quadric that meets $\mathcal{S}$ in the regulus $t_0,\ldots,t_q$.

In the representation of $\mathcal{P}$ in $PG(4,q)$, any non-singular quadric that meets Σ_∞ in a regulus of the spread corresponds to a unital of $\mathcal{P}$ and any ovoidal cone whose vertex lies on a line of $\mathcal{S}$ corresponds to a unital of $\mathcal{P}$.

Let $\mathcal{P}$ be a translation plane of dimension at most two over its kernel and let U be a unital of $\mathcal{P}$. If U is tangent to ℓ_∞ and $\mathcal{U}$ corresponds to an ovoidal cone of $PG(4,q)$, then U is called a **Buekenhout-Metz unital with respect to ℓ_∞.** If $T=U\cap\ell_\infty$, then, since there is a unique tangent line through any point of U, we also call U Buekenhout-Metz with respect to the *point* T. If U is secant to ℓ_∞ and $\mathcal{U}$ corresponds to a non-singular quadric in $PG(4,q)$, then U is called a **Buekenhout unital with respect to ℓ_∞.** Note that if a Buekenhout unital exists in $\mathcal{P}$, then the underlying spread $\mathcal{S}$ contains a regulus.

By considering the case where $\mathcal{U}$ is an ovoidal cone based on a Suzuki-Tits ovoid, Buekenhout showed that there are non-classical unitals in $PG(2,q^2)$ where q is even and non-square. Metz [87] noted that this construction provided examples of non-classical unitals in $PG(2,q^2)$ for all q.

1.8 Derivation

Let $\mathcal{A}$ be a finite affine plane of order q^2 and let $\mathcal{A} = \mathcal{P}^{\ell_\infty}$ be its projective completion. Let $\mathcal{D}$ be a set of $q+1$ points of ℓ_∞ such that for any distinct points X, Y of $\mathcal{A}$ for which XY meets ℓ_∞ in a point of $\mathcal{D}$, there is a unique Baer subplane containing X, Y and $\mathcal{D}$. We call $\mathcal{D}$ a **derivation set** of $\mathcal{A}$ or $\mathcal{P}$.

We define a new incidence structure $\mathcal{A}'$ as follows: the *points* of $\mathcal{A}'$ are the points of $\mathcal{A}$; the *lines* of $\mathcal{A}'$ are the Baer subplanes of $\mathcal{A}$ that contain $\mathcal{D}$, and the lines of $\mathcal{A}$ that meet ℓ_∞ in a point not in $\mathcal{D}$; *incidence* is inclusion. The structure $\mathcal{A}'$ is an affine plane of order q^2, called the **derived plane** of $\mathcal{A}$.

We complete $\mathcal{A}'$ to a projective plane by adding the line ℓ'_∞, so $\mathcal{A}' = \mathcal{P}'^{\ell'_\infty}$. The points of $\ell_\infty \setminus \mathcal{D}$ have a natural correspondence to $q^2 - q$ of the points of ℓ'_∞. We denote the remaining $q+1$ points of ℓ'_∞ by $\mathcal{D}'$, they do not correspond in any natural way to the points of $\mathcal{D}$. Note that $\mathcal{D}'$ is a derivation set of $\mathcal{A}'$ and deriving with respect to $\mathcal{D}'$ gives the affine plane $\mathcal{A}$.

Derivation is an affine process, but since every affine plane has a unique (up to isomorphism) projective completion, we will talk about **deriving $\mathcal{P}$** and the **derived plane $\mathcal{P}'$**.

If $\mathcal{P}$ is a translation plane with respect to ℓ_∞, then so is $\mathcal{P}'$. If $\mathcal{P} = PG(2, q^2)$, then any Baer subline of ℓ_∞ is a derivation set and $\mathcal{P}'$ is the Hall plane of order q^2, $\mathcal{H}(q^2)$, regardless of the derivation set used. A Hall plane is a projective plane coordinatized by the Hall quasifield. See [68] for further details.

If $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are k disjoint derivation sets of ℓ_∞, then we can perform multiple derivations on the Desarguesian plane. The resulting plane is an André plane. See [68] for more details on non-Desarguesian translation planes.

Let $\mathcal{P}$ be a translation plane with respect to ℓ_∞ of dimension at most two over its kernel. We will be interested in the effect of derivation in the Bruck and Bose representation of $\mathcal{P}$ in $PG(4, q)$. Let Σ_∞ be a hyperplane of $PG(4, q)$ and let $\mathcal{S}$ be the spread of Σ_∞ that generates $\mathcal{P}$.

Let b be a Baer subline of ℓ_∞ that corresponds to a regulus $\mathcal{R}$ of $\mathcal{S}$. The translation plane $\mathcal{P}'$ whose spread $\mathcal{S}'$ is obtained from $\mathcal{S}$ by replacing the regulus $\mathcal{R}$ with its complementary regulus $\mathcal{R}'$ is the plane obtained by deriving $\mathcal{P}$ with respect to the derivation set b.

Chapter 2

Characterizations of unitals

2.1 Introduction

The problem of finding characterizations of the classical unital has received a great deal of attention. In 1979, Piper [91] conjectured that if a unital contains no O'Nan configurations (that is, a configuration containing four distinct lines meeting in six distinct points) then the unital is classical. This conjecture remains open. Wilbrink [113] showed that if a unital contains no O'Nan configurations and satisfies two additional configurational properties, then the unital is classical. Recently, Thas [105] characterized the classical unital in terms of its tangent lines, a property related to one of Wilbrink's configurations.

Another type of characterization of the classical unital is a group theoretic characterization. Hoffer [66] showed that if the automorphism group of a unital is the unitary group, then the unital is classical. Other group theoretic characterizations of the classical unital have been given by Biscarini [26], Abatangelo [1] and Biliotti and Korchmáros [24, 25]. There are also group theoretic characterizations of Buekenhout-Metz unitals, see [2] and [3].

A third way of characterizing the classical unital is in terms of Baer sublines of the projective plane $PG(2, q^2)$ that contains the unital. These techniques are also used to characterize Buekenhout-Metz unitals. In 1981 Lefèvre-Percsy [83] showed:

Theorem 2.1.1 [83] *Let U be a unital in the Desarguesian plane $PG(2, q^2)$ of order q^2, $q > 2$, and let l be a tangent line to U. If all Baer sublines having a point on l intersect U in $0, 1, 2$ or $q + 1$ points, then U is a Buekenhout-Metz unital with respect to l.*

Lefèvre-Percsy noted that the converse of this theorem is also true. Moreover, the theorem holds in translation planes of dimension two over their kernel. This characterization was improved in [43] and [92]. A further characterization of Buekenhout-Metz unitals in terms of their Baer sublines is given in [81]. Lefèvre-Percsy [84] and Faina and Korchmáros [54] independently proved the following characterization of classical unitals.

Theorem 2.1.2 [84, 54] *Let U be a unital of $PG(2, q^2)$, with $q > 2$. If each line of $PG(2, q^2)$ secant to U intersects it in a Baer subline, then U is classical.*

Let U be a unital in $PG(2, q^2)$. Lefèvre-Percsy [84] showed that the following two conditions are equivalent.

1. Every secant line of U meets U in a Baer subline.

2. Every Baer subline of $PG(2, q^2)$ meets U in 0, 1, 2 or $q + 1$ points.

Therefore, an equivalent statement to the above theorem is: *if each Baer subline of $PG(2, q^2)$ meets a unital U in 0, 1, 2 or $q + 1$ points, then U is classical.*

We give an improvement of this characterization in Section 2.2 by first generalizing Theorem 2.1.1 to provide a characterization of Buekenhout unitals in translation planes of dimension at most two over their kernel.

In $PG(2,q^2)$, the known classes of unitals are the classical unitals, the Buekenhout unitals, the Buekenhout-Metz unitals, and the duals of these unitals. We show, by counting, that in $PG(2,q^2)$, all Buekenhout unitals are classical. Casse, O'Keefe and Penttila [43] show that the dual of a Buekenhout-Metz unital in $PG(2,q^2)$ is Buekenhout-Metz. Together these two results imply that all the known unitals in $PG(2,q^2)$ are Buekenhout-Metz unitals.

Finally, we obtain a characterization of the classical unital in terms of the Baer sublines of $PG(2,q^2)$ that improves the characterization given in Theorem 2.1.2.

A set of points in $PG(n,q)$ is called a **set of class** $(0,1,2,q+1)$ if any line of $PG(n,q)$ meets the set in 0, 1, 2 or $q+1$ points. We will need the following characterization of these sets due to Tallini [102].

Theorem 2.1.3 [102] *Let T be a set of class $(0,1,2,q+1)$ in a projective space $\mathcal{P}$ of dimension $n \geq 3$ and order $q \neq 2$. If T contains at least as many points as a hyperplane of $\mathcal{P}$, then one of the following holds:*

1. *T is a quadric;*
2. *$q = 2^e$, n is even, and T is the union of a quadric $\mathcal{Q}$ and the unique point P common to all tangent hyperplanes of $\mathcal{Q}$;*
3. *T is the union of a hyperplane H and a point $P \notin H$.*

2.2 A characterization of the classical unital

We first prove a result that provides a characterization of Buekenhout unitals. Let $\mathcal{P}$ be a translation plane of order q^2, $q > 2$, that has dimension at most two over its kernel.

Let U be a unital of $\mathcal{P}$ that is secant to the translation line ℓ_∞ of $\mathcal{P}$. Suppose all Baer sublines having a point on ℓ_∞ intersect U in $0, 1, 2$ or $q+1$ points. We show that U is a Buekenhout unital with respect to ℓ_∞.

We will use the Bruck and Bose representation of $\mathcal{P}$ in $PG(4,q)$ described in Section 1.7. Let Σ_∞ be the hyperplane at infinity in $PG(4,q)$ and let $\mathcal{S}$ be the spread of Σ_∞ that generates $\mathcal{P}$. The point set of U corresponds to a set of points $\mathcal{U}$ in $PG(4,q)$ that meets Σ_∞ in $q+1$ lines of the spread $\mathcal{S}$. (This follows immediately from the correspondence between the points of ℓ_∞ and the lines of $\mathcal{S}$.) We begin by proving two lemmas about the structure of $\mathcal{U}$.

Lemma 2.2.1 *$\mathcal{U}$ is a set of class* $(0, 1, 2, q+1)$.

Proof We need to show that any line of $PG(4,q)$ meets $\mathcal{U}$ in 0, 1, 2 or $q+1$ points. Let P, Q be two points of $\mathcal{U}$. We show that either $PQ \cap \mathcal{U} = \{P, Q\}$ or the line PQ is contained in $\mathcal{U}$.

If both P and Q lie in Σ_∞, then either PQ is a line of $\mathcal{S}$, or P and Q lie in different lines of $\mathcal{S}$. If PQ is a line of $\mathcal{S}$, then PQ is contained in $\mathcal{U}$. If P and Q lie on different lines of $\mathcal{S}$, then PQ meets $q+1$ lines of $\mathcal{S}$ and corresponds to a Baer subline of ℓ_∞ (since, by Lemma 1.7.1(i), any plane of $PG(4,q)\backslash\Sigma_\infty$ through PQ corresponds to a Baer subplane of $\mathcal{P}$ that meets ℓ_∞ in a Baer subline). This Baer subline has a point on ℓ_∞ and so by our hypothesis, meets U in two or $q+1$ points.

We now suppose that at least one of the points P and Q is not in Σ_∞. Let X be the point $PQ \cap \Sigma_\infty$ (X may be equal to P or Q). Let m be the line of the spread $\mathcal{S}$ passing through X and let the plane containing PQ and m be α. Then α corresponds to a line l in $\mathcal{P}$ and the line PQ corresponds to a Baer subline of l with a point on ℓ_∞. Our hypothesis implies that this Baer subline intersects $\mathcal{U}$ in 0, 1, 2 or $q+1$ points. Hence in $PG(4,q)$, the line PQ intersects $\mathcal{U}$ in two or all of its points.

□

Lemma 2.2.2 *$\mathcal{U}$ contains no plane of $PG(4,q)$.*

Proof Let $\mathcal{U}$ meet Σ_∞ in the lines $t_0,\dots,t_q$ of the spread $\mathcal{S}$ and note that $\mathcal{U}$ can contain no plane of Σ_∞. So in $\mathcal{P}$, U meets ℓ_∞ in $T_0,\dots,T_q$, where the line t_i corresponds to the point T_i for $i=0,\dots,q$.

We first show that $\mathcal{U}$ does not contain a plane through any of the t_i. Let α be a plane of $PG(4,q)$ that meets Σ_∞ in the line t_i for some $i \in \{0\dots,q\}$. We show that either α meets $\mathcal{U}$ in t_i or α meets $\mathcal{U}$ in t_i and another line m. The plane α corresponds to a line l of $\mathcal{P}$ that contains T_i and is either tangent or secant to U.

If l is tangent to U, then $l \cap U = T_i$ and so $\alpha \cap \mathcal{U} = t_i$. If l is secant to U, then $l \cap U$ consists of $q+1$ points, one of which is T_i. Any three of these points lie on a unique Baer subline b, so by our hypothesis, all $q+1$ must lie on b. Now b corresponds to a line m of $PG(4,q)$ that meets Σ_∞ in a point of t_i. Thus α meets $\mathcal{U}$ in the lines m and t_i. Hence $\mathcal{U}$ contains no plane through the lines $t_0,\dots,t_q$ of the spread $\mathcal{S}$.

Suppose $\mathcal{U}$ contains a plane β not through any of the t_i. Then β plane intersects Σ_∞ in a line m contained in $\mathcal{U}$, $m \neq t_i$, $i=0,\dots,q$, so m meets each of the t_i in a single point. Thus β corresponds to a Baer subplane of $\mathcal{P}$ that is contained in U. A unital of $\mathcal{P}$ cannot contain a Baer subplane $\mathcal{B}$ of $\mathcal{P}$ as they are both sets of type $(1,q+1)$. Any point X of $\mathcal{P} \setminus \mathcal{B}$ lies on a secant of $\mathcal{B}$, so adding X to $\mathcal{B}$ gives a set with a $(q+2)$-secant. Thus we cannot add any subset of points to $\mathcal{B}$ and retain the property that every line meets our set in 1 or $q+1$ points. Hence $\mathcal{U}$ contains no plane of $PG(4,q)$.

□

We are now able to prove our characterization of Buekenhout unitals.

Theorem 2.2.3 *Let $\mathcal{P}$ be a translation plane of order q^2, $q>2$, and dimension at most two over its kernel. Let U be a unital of $\mathcal{P}$ that is secant to the translation line ℓ_∞ of $\mathcal{P}$. If all Baer sublines having a point on ℓ_∞ intersect U in $0,1,2$ or $q+1$ points, then U is a Buekenhout unital with respect to ℓ_∞.*

Proof Let $\mathcal{P}$ and U satisfy the hypothesis of the theorem. We will use the representation of $\mathcal{P}$ in $PG(4,q)$ given in Section 1.7. So U corresponds to a set of points $\mathcal{U}$ of $PG(4,q)$ that meets Σ_∞ in $q+1$ lines of the spread $\mathcal{S}$.

By Lemma 2.2.1, $\mathcal{U}$ is a set of class $(0,1,2,q+1)$ that contains

$$(q^3+1)-(q+1)+(q+1)^2 = q^3+q^2+q+1$$

points. A hyperplane in $PG(4,q)$ contains q^3+q^2+q+1 points so we can apply Theorem 2.1.3.

The third case of Theorem 2.1.3 cannot occur as it involves more points than $\mathcal{U}$ contains, so we examine the first two cases. There are seven possible quadrics in $PG(4,q)$. If we denote subspaces of dimension n by Π_n, parabolic quadrics of $PG(n,q)$ by $\mathcal{P}_n$, hyperbolic quadrics of $PG(n,q)$ by $\mathcal{H}_n$, and elliptic quadrics of $PG(n,q)$ by $\mathcal{E}_n$, then the quadrics of $PG(4,q)$ are: $\mathcal{P}_4$, $\Pi_0\mathcal{H}_3$, $\Pi_0\mathcal{E}_3$, $\Pi_1\mathcal{P}_2$, $\Pi_2\mathcal{H}_1$, $\Pi_2\mathcal{E}_1$ and $\Pi_3\mathcal{P}_3$. The last three contain a plane and so by Lemma 2.2.2, $\mathcal{U}$ cannot be one of them. The first four have q^3+q^2+q+1, q^3+2q^2+q+1, q^3+q+1 and q^3+q^2+q+1 points respectively. Thus the only possibilities for $\mathcal{U}$ are $\mathcal{P}_4$ and $\Pi_1\mathcal{P}_2$. But $\Pi_1\mathcal{P}_2$ is the join of a line and a conic and so consists of $q+1$ planes through a line, no three of which are contained in a hyperplane (see [65]). So by Lemma 2.2.2, $\mathcal{U}$ is not $\Pi_1\mathcal{P}_2$. Hence $\mathcal{U}$ is a non-singular quadric in $PG(4,q)$ that meets Σ_∞ in a regulus and so U is a Buekenhout unital of $\mathcal{P}$ with respect to ℓ_∞.

$\square$

Recall that a non-Desarguesian translation plane of dimension two over its kernel may contain affine Baer subplanes that do not correspond to a plane of $PG(4,q)$ (see Section 1.7). Consequently, non-Desarguesian translation planes of dimension two over their kernel may contain Baer sublines with a point on ℓ_∞ that do not correspond to lines of $PG(4,q)$. We note that this does not affect Theorem 2.2.3 since our hypothesis that all Baer sublines with a point on ℓ_∞ meet the unital in 0, 1, 2 or $q+1$ points means that all Baer sublines with a point on ℓ_∞ that correspond to lines of $PG(4,q)$ will meet the unital in 0, 1, 2 or $q+1$ points. However, while the converse of Theorem 2.2.3 is true in $PG(2,q^2)$, it is not necessarily true in non-Desarguesian translation planes of dimension two over their kernel.

Any line of $PG(2,q^2)$ that is secant to the classical unital U meets U in a Baer subline. By applying the above theorem, we deduce that the classical unital is Buekenhout with respect to any secant line. In a similar way we can use Theorem 2.1.1 to show that the classical unital is Buekenhout-Metz with respect to any tangent line.

Corollary 2.2.4 *The classical unital U of $PG(2,q^2)$ is Buekenhout with respect to every secant line of U, and Buekenhout-Metz with respect to every tangent line of U.*

We next use a counting argument to prove that all Buekenhout unitals in $PG(2,q^2)$ are classical.

Theorem 2.2.5 *If U is a unital in $PG(2,q^2)$ that is Buekenhout with respect to a secant of U, then U is classical.*

Proof First we count the number of classical unitals in $PG(2,q^2)$ that pass through a given Baer subline of ℓ_∞. By counting pairs l, U such that l is a secant line of the classical unital U in two ways we deduce that:

$$|\text{lines in } PG(2,q^2)|.|\text{classical unitals secant to a line}|$$
$$= |\text{classical unitals in } PG(2,q^2)|.|\text{secant lines of } U| \qquad \dagger$$

There are $q^3(q^2+1)(q^3-1)$ classical unitals in $PG(2,q^2)$ [47] and a unital has $q^4-q^3+q^2$ secant lines. Let t be the number of classical unitals secant to each line of $PG(2,q^2)$, then † becomes:

$$(q^4+q^2+1)t = q^3(q^2+1)(q^3-1)(q^4-q^3+q^2)$$

$$
\begin{aligned}
\frac{q^6-1}{q^2-1}t &= q^5(q^2+1)(q^3-1)(q^2-q+1) \\
t &= \frac{q^5(q^2+1)(q^2-1)(q^2-q+1)}{q^3+1} \\
&= q^5(q^2+1)(q-1).
\end{aligned}
$$

All the secant lines of a classical unital U meet U in a Baer subline. We count the number of classical unitals in $PG(2,q^2)$ that contain a given Baer subline of a secant line l. The number of Baer sublines of a line in $PG(2,q^2)$ is $\binom{q^2+1}{3}/\binom{q+1}{3} = q(q^2+1)$, since three distinct points are contained in a unique Baer subline.

The group of automorphisms of a line acts three transitively on its points, thus the number of classical unitals that contain a given Baer subline of l is constant. Hence the number of classical unitals that contain a given Baer subline of l in $PG(2,q^2)$ is:

$$
\begin{aligned}
&= \frac{|\text{classical unitals secant to } l|}{|\text{Baer sublines of } l|} \\
&= \frac{q^5(q^2+1)(q-1)}{q(q^2+1)} \\
&= q^4(q-1).
\end{aligned}
$$

Using [65, Theorem 22.8.2], we calculate the number of non-singular quadrics in $PG(4,q)$ to be $q^6(q^3-1)(q^5-1)$. We also calculate the number of three dimensional subspaces of $PG(4,q)$ that meet a given non-singular quadric in a hyperbolic quadric to be $\frac{q^4+q^2}{2}$.

We now count the number of non-singular quadrics in $PG(4,q)$ that contain a prescribed regulus of the hyperplane Σ_∞. That is, we count the number of non-singular quadrics that meet Σ_∞ in a given hyperbolic quadric.

The group of automorphisms of $PG(4,q)$ acts transitively on the hyperplanes of $PG(4,q)$. In addition, the hyperbolic quadrics of $PG(3,q)$ lie in a single orbit under the automorphism group of $PG(3,q)$. Thus the number of non-singular quadrics meeting Σ_∞ in a particular hyperbolic quadric is a constant.

There are $q^4+q^3+q^2+q+1$ hyperplanes in $PG(4,q)$, thus $\frac{\frac{1}{2}(q^4+q^2)}{q^4+q^3+q^2+q+1}$ of the non-singular quadrics in $PG(4,q)$ meet Σ_∞ in a hyperbolic quadric. That is, the number of non-singular quadrics that meet Σ_∞ in a regulus is:

$$\frac{\frac{1}{2}(q^4+q^2)}{q^4+q^3+q^2+q+1}(q^6(q^3-1)(q^5-1)) = \frac{1}{2}q^8(q^2+1)(q-1)(q^3-1). \qquad \ddagger$$

Note that these reguli do not all lie in the spread of Σ_∞ that generates $PG(2,q^2)$.

We now count the number of non-singular quadrics of $PG(4,q)$ that meet Σ_∞ in a specified regulus. That is, we want to divide $\ddagger$ by the number of hyperbolic quadrics in $PG(3,q)$ which we can calculate from [65, Theorem 22.8.2] to be $\frac{1}{2}q^4(q^2+1)(q^3-1)$. Thus the number of non-singular quadrics meeting Σ_∞ in a given regulus is:

$$\begin{aligned} &= \frac{|\text{non-singular quadrics meeting } \Sigma_\infty \text{ in a regulus}|}{|\text{hyperbolic quadrics in } PG(3,q)|} \\ &= \frac{\frac{1}{2}q^8(q^2+1)(q-1)(q^3-1)}{\frac{1}{2}q^4(q^2+1)(q^3-1)} \\ &= q^4(q-1). \end{aligned}$$

Since a non-singular quadric corresponds to a unique unital of $PG(2,q^2)$ we have shown that the number of Buekenhout unitals in $PG(2,q^2)$ that contain a given Baer subline of ℓ_∞ is $q^4(q-1)$. However, this is the number of classical unitals passing through a given Baer subline of ℓ_∞. Hence all the Buekenhout unitals in $PG(2,q^2)$ are classical.

□

As an immediate corollary to Theorems 2.2.3 and 2.2.5, we obtain our main result which is a characterization of the classical unital in terms of Baer sublines of $PG(2,q^2)$. We note that the characterization given by Lefèvre-Percsy in Theorem 2.1.2 is a direct consequence of this result.

Theorem 2.2.6 *Let U be a unital in $PG(2,q^2)$ and let l be a secant line of U. If every Baer subline with a point on l meets U in 0, 1, 2 or $q+1$ points, then U is classical.*

We also obtain a corollary on the number of Buekenhout unitals in translation planes of dimension at most two over their kernel.

Corollary 2.2.7 *Let $\mathcal{P}$ be a translation plane of dimension at most two over its kernel with line at infinity ℓ_∞, and let $\mathcal{S}$ be the spread of Σ_∞ in $PG(4,q)$ that generates $\mathcal{P}$. Let b be a Baer subline of ℓ_∞ that corresponds to a regulus of $\mathcal{S}$. There are $q^4(q-1)$ Buekenhout unitals with respect to ℓ_∞ in $\mathcal{P}$ that contain b.*

Proof This follows immediately from the proof of Theorem 2.2.5 where we counted the number of parabolic quadrics of $PG(4,q)$ that contain a given regulus $\mathcal{R}$ of $\mathcal{S}$. Each parabolic quadric corresponds to a unique Buekenhout unital of $\mathcal{P}$ that contains the Baer subline corresponding to $\mathcal{R}$.

$\square$

Note that we need to be careful in the statement of this corollary since although a regulus of $\mathcal{S}$ corresponds to a Baer subline of ℓ_∞, a Baer subline of ℓ_∞ does not necessarily correspond to a regulus of $\mathcal{S}$.

For example, let $\mathcal{B}$ be a Baer subplane of $PG(2,q^2)$ that meets ℓ_∞ in the Baer subline $b = \{B_0, \ldots, B_q\}$. In $PG(4,q)$, $\mathcal{B}$ corresponds to a plane α that meets $\mathcal{S}$ in $q+1$ points, one in each line of $\mathcal{R}_1 = \{b_0, \ldots, b_q\}$. Since $\mathcal{S}$ is regular, $\mathcal{R}_1$ forms a regulus. Let $\mathcal{R}_2$ be another regulus of $\mathcal{S}$ that contains b_0 and no other line of $\mathcal{R}_1$ (such a regulus exists as $\mathcal{S}$ is a regular spread) and let $\mathcal{R}_3$ be its complementary regulus. Consider the translation plane $\mathcal{P}$ whose spread is $\mathcal{S}' = \mathcal{S} \setminus \mathcal{R}_2 \cup \mathcal{R}_3$, then the plane α meets $q+1$ lines of $\mathcal{S}'$ namely $b_1, \ldots, b_q$ and a line t_0 of $\mathcal{R}_3$. Hence α corresponds to a Baer subplane of $\mathcal{P}$ that meets its line at infinity in the Baer subline b'. However the lines $b_1, \ldots, b_q$ are contained in the unique regulus $\mathcal{R}_1$, thus $b_1, \ldots, b_q, t_0$ do not form a regulus. Hence b' is a Baer subline of $\mathcal{P}$ that does not correspond to a regulus of $\mathcal{S}'$ and so there are no Buekenhout unitals containing b'.

2.3 Another characterization result

In this section we will work with a slightly less general definition of a Buekenhout-Metz unital in $PG(2,q^2)$. We will use the transitivity of the automorphism group of these unitals to obtain a characterization of the classical unital.

An **orthogonal** Buekenhout-Metz unital in a translation plane $\mathcal{P}$ of dimension at most two over its kernel is a Buekenhout-Metz unital that corresponds to an orthogonal cone in $PG(4,q)$, that is, the unital corresponds to a cone based on an elliptic quadric.

If U is a Buekenhout-Metz unital in $\mathcal{P}$ with respect to the line l and the point T, then we call l the **axis** and T the **vertex** of U.

Orthogonal Buekenhout-Metz unitals were studied by Baker and Ebert [11] and Ebert [50] who gave coordinates for them in $PG(2,q^2)$. They showed that the automorphism group fixing an orthogonal Buekenhout-Metz unital in $PG(2,q^2)$ with axis ℓ_∞ and vertex P is transitive on the points of $U \setminus P$ and transitive on the points of $\ell_\infty \setminus P$.

We now show that a non-classical orthogonal Buekenhout-Metz unital has a unique axis.

Theorem 2.3.1 *If U is a non-classical orthogonal Buekenhout-Metz unital with respect to some line l in $PG(2,q^2)$, then U is not an orthogonal Buekenhout-Metz unital with respect to any other tangent line.*

Proof Suppose U is an orthogonal Buekenhout-Metz unital with respect to two distinct axes l and m with vertices P and Q respectively. Baker and Ebert [11] and Ebert [50] show that the automorphism group of U fixing P is transitive on the points of $U \setminus P$. Since there is an axis through $Q \neq P$, there is an axis through every point of $U \setminus P$.

By the converse of Theorem 2.1.1, every Baer subline meets such an axis in 0, 1, 2 or $q+1$ points. Since every point of U lies on such an axis, every Baer subline of $PG(2,q^2)$ meets U in 0, 1, 2 or $q+1$ points. Thus by the characterization of classical unitals in Theorem 2.2.6, the unital U is classical. Hence if U is not classical, m cannot be an axis for U.

□

An equivalent result holds for Buekenhout unitals since, by Theorem 2.2.5, all Buekenhout unitals in $PG(2,q^2)$ are classical. Thus if a Buekenhout-Metz unital is Buekenhout with respect to some secant line, then it is classical. Alternatively we could use the same argument as above and apply the converse of Theorem 2.2.3 instead of Theorem 2.1.1.

We can now give a necessary and sufficient condition for a unital in $PG(2,q^2)$ to be classical.

Theorem 2.3.2 *A unital U in $PG(2,q^2)$ is classical if and only if U is orthogonal Buekenhout-Metz with respect to more than one of its tangent lines.*

Proof Buekenhout [40] showed that the classical unital is orthogonal Buekenhout-Metz with respect to some tangent line of $PG(2,q^2)$. The automorphism group of the classical unital is transitive on the lines of the unital, thus the classical unital is orthogonal Buekenhout-Metz with respect to all its tangent lines.

Conversely, the proof of Theorem 2.3.1 shows that if a unital is orthogonal Buekenhout-Metz with respect to two or more tangent lines, then it is classical.

□

The non-classical orthogonal Buekenhout-Metz unitals in $PG(2,q^2)$ constructed by Buekenhout and Metz provide examples of unitals that are orthogonal Buekenhout-Metz with respect to *exactly* one tangent line.

Chapter 3

Unitals and inversive planes

3.1 Introduction

In this chapter we look at the relationship between unitals and inversive planes. In Section 3.2 we define inversive planes, then describe the construction of an inversive plane from the classical unital and detail some of the known results.

In Section 3.3 we give a new proof of the relationship between Buekenhout-Metz unitals and inversive planes that is more general than the known results and gives a natural, geometric isomorphism between the resulting inversive plane and the egglike inversive plane arising from the ovoid involved in the construction of the Buekenhout-Metz unital. We then apply our results to investigate some parallel classes and partitions of the lines of any Buekenhout-Metz unital.

3.2 Preliminary results

An **inversive plane** is a 3-$(q^2+1, q+1, 1)$ design. We will be working with the point residual of an inversive plane. The **point residual** of a design is constructed by removing one point and all the blocks incident with that point from the design. The point residual of an inversive plane is a 2-$(q^2, q+1, q)$ design. An example of an inversive plane is as follows:

Let $\mathcal{O}$ be an ovoid of $PG(3,q)$. Consider the incidence structure $\mathcal{I}$ with *points* the points of $\mathcal{O}$, *blocks* the secant plane sections of $\mathcal{O}$, and *incidence* as inclusion. Then $\mathcal{I}$ is an inversive plane, called an **egglike** inversive plane. If the ovoid $\mathcal{O}$ is an elliptic quadric, then $\mathcal{I}$ is called a **Miquelian** inversive plane. Dembowski [47] showed that all inversive planes with even parameter q are egglike. To date, all known inversive planes are egglike.

To an egglike inversive plane there corresponds an ovoid in $PG(3,q)$ which can be used to construct a Buekenhout-Metz unital in any translation plane of order q^2 and dimension at most two over its kernel ([40], see Section 1.7). Several authors have investigated the converse of this construction with success in the case where the translation plane is $PG(2,q^2)$ and $\mathcal{O}$ is an elliptic quadric (see [113], [11] and [50]). We now describe two conditions that a unital must satisfy in order for this reverse construction to work.

Let U be a unital and P a point of U. We describe two conditions that U may satisfy, see Figure 3.1.

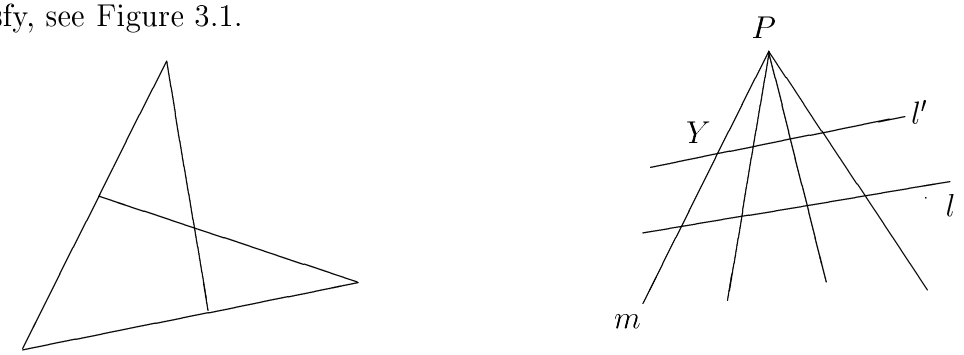

Figure 3.1: Conditions I_P and II_P.

Condition $\mathbf{I_P}$ *U contains no O'Nan configurations with P as vertex.*

Condition $\mathbf{II_P}$ *Let l be a line of U not through P, let m be a line of U through P meeting l, and let $Y \neq P$ be a point of U on m. There exists a line $l' \neq m$*

through Y intersecting all the lines through P that meet l.

Condition II_P gives a configuration of q^2+q+1 points that has $q+1$ lines through P and q **cross** lines, where a cross line is a line not through P that meets each of the $q+1$ lines through P. We call this configuration a **configuration $\mathbf{II_P}$**, see Figure 3.2.

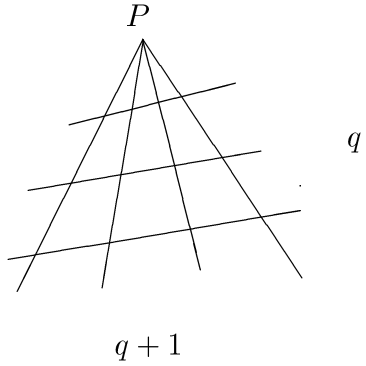

Figure 3.2: A configuration II_P.

These conditions are suggested by the conditions I and II of Wilbrink's [113]. In particular, Wilbrink shows that the classical unital in $PG(2,q)$ satisfies I_P and II_P for each of its points P and that if q is even, then a unital in $PG(2,q)$ satisfying I_P and II_P for each of its points P is a classical unital. For q odd he introduces a third condition to characterize the classical unital.

Let U be a unital of parameter q that satisfies I_P and II_P for a point $P \in U$. We use U to construct a new design K_P that has the parameters of a point residual of an inversive plane. We define an incidence structure K'_P as follows: the *points* of K'_P are the lines of U through the point P; the *blocks* of K'_P are the lines of U not through P; and a point of K'_P is *incident* with a block of K'_P if the corresponding lines of U meet. Let K_P be the reduced structure of K'_P, that is, the blocks of K_P are the equivalence classes of the blocks of K'_P, (II_P shows that all blocks of K'_P are repeated q times). It is easy to show, using I_P and II_P, that K_P is a 2-$(q^2, q+1, q)$ design.

Result *K_P is a 2-$(q^2, q+1, q)$ design.*

Proof There are q^2 secants of U through P, and $q+1$ points on every line of U, thus K_P has q^2 points and $q+1$ points on every line. It remains to show that any two points of K_P lie on exactly q blocks. We can see this by counting. Let l and m be two lines of U through P. Any two points of U lie on a unique line of U, thus there are q^2 lines of U (not through P) that meet both l and m. Hence there are q^2 blocks of K'_P that meet the points corresponding to l and m. Now each block of K'_P is repeated q times, so $q^2/q = q$ of these lines form blocks of K_P as required.

□

Thus K_P has the parameters of a point residual of an inversive plane. We wish to investigate the question: for which unitals U and points $P \in U$ is K_P the point residual of an inversive plane?

Wilbrink [113] shows that if U is a classical unital in $PG(2, q^2)$ and P is any point of U, then the design K_P constructed in the above way can be completed to an inversive plane. Baker and Ebert [11] show that if U is a Buekenhout-Metz unital with respect to P in $PG(2, q^2)$, q odd, then the design K_P constructed in the above way can be completed to a Miquelian inversive plane. Ebert [50] shows that if U is a Buekenhout-Metz unital with respect to the point P in $PG(2, q^2)$, q even, such that the ovoid used in the construction of U is an elliptic quadric (see Section 1.7), then the design K_P can be completed to a Miquelian inversive plane. Wilbrink's proof is synthetic and does not utilize the geometric structure inherent to a Buekenhout-Metz unital, while Baker and Ebert's proofs are coordinate based. We give a construction that uses the geometry inherent in the Buekenhout-Metz unital construction.

In the next section we show that if U is a Buekenhout-Metz unital with respect to the point P in any translation plane of dimension at most two over its kernel, then the associated 2-$(q^2, q+1, q)$ design K_P, constructed as above, is the point-residual of an inversive plane. Thus, this result is a generalization of the work of Wilbrink [113], Baker and Ebert [11] and Ebert [50]. Further, our proof gives a natural, geometric isomorphism between the resulting inversive plane and the egglike inversive plane arising from the ovoid involved in the construction of the Buekenhout-Metz unital.

We then apply our results to investigate some parallel classes and partitions of the lines of any Buekenhout-Metz unital.

3.3 Buekenhout-Metz unitals and inversive planes

We first show that if U is a unital which is Buekenhout-Metz with respect to a point P, then U satisfies I_P and II_P. Hence we can construct the design K_P from a Buekenhout-Metz unital. Throughout this section $\mathcal{P}$ is a translation plane of dimension at most two over its kernel.

Lemma 3.3.1 *If U is a Buekenhout-Metz unital with respect to the point P in $\mathcal{P}$, then U contains no O'Nan configuration with P as a vertex.*

Proof We use the Bruck and Bose representation of $\mathcal{P}$ in $PG(4,q)$ described in Section 1.7. Let $\mathcal{U}$ be the ovoidal cone of $PG(4,q)$ that corresponds to the unital U and let V be the vertex of $\mathcal{U}$. If U contains an O'Nan configuration with P as vertex, then in $PG(4,q)$, there are two lines l, m of $\mathcal{U}$ through V and two planes α, β containing different lines of the spread such that α and β meet l and m and α meets β in a point of $\mathcal{U}$ that is not in l or m.

Now $\langle \beta, l \rangle$ is a three-dimensional space that contains V and $\beta \cap m$ and hence contains m. Similarly, $\langle \alpha, l \rangle$ contains m. Thus $\langle \beta, l \rangle \cap \langle \alpha, l \rangle$ is a plane that contains l, m and $\alpha \cap \beta$, and so is $\langle l, m \rangle$. But $\langle l, m \rangle$ is a plane that meets $\mathcal{U}$ in the $2q+1$ points of l and m and so contains no further point of $\mathcal{U}$. This contradicts the assumption that $\alpha \cap \beta$ is in $\mathcal{U}$. Therefore U contains no O'Nan configuration with P as vertex.

□

Lemma 3.3.2 *Let U be a Buekenhout-Metz unital with respect to the point P in $\mathcal{P}$. Let $\overline{l}$ be a line of U not through P, let $\overline{m}$ be a line of U through P meeting $\overline{l}$, and let $Y \neq P$ be a point of U on $\overline{m}$. There exists a line $\overline{l'} \neq \overline{m}$ of U through Y that meets each line through P that $\overline{l}$ meets.*

Proof If l is a line of $\mathcal{P}$ that is secant to U, we denote $\overline{l} = l \cap U$. So in $\mathcal{P}$, the statement of the lemma becomes: let l be a line of $\mathcal{P}$ not through P that is secant to U, let m be a secant line of U through P that meets l in a point of U, and let $Y \neq P$ be a point of U on m. If the secants of U through P that meet l in a point of U are labelled $m, m_1, \ldots, m_q$, then there exists a line $l' \neq m$ through Y that meets each of the m_i in a point of U.

We use the representation of $\mathcal{P}$ in $PG(4,q)$. Let $\mathcal{U}$ be the ovoidal cone of $PG(4,q)$ with vertex V corresponding to the unital U, and let p be the line of $\mathcal{U}$ that belongs to the spread. The line l of $\mathcal{P}$ corresponds to a plane α on a line $t \neq p$ of the spread which meets $\mathcal{U}$ in an oval. The set $\overline{m} = m \cap U$ corresponds to a line $\overline{m} \neq p$ of $\mathcal{U}$ through V that meets α. Now $\langle t, \overline{m} \rangle = \langle \alpha, \overline{m} \rangle$ is a three dimensional space containing V and $\alpha \cap \mathcal{U}$ and hence meets $\mathcal{U}$ in an oval cone $V\mathcal{C}$ formed by joining V to each point of $\alpha \cap \mathcal{U}$. Let Y be a point of $\overline{m}$ not in α. The plane $\alpha' = \langle t, Y \rangle$ in $\langle t, \overline{m} \rangle$ meets every line of $V\mathcal{C}$, hence α' meets each line of $\mathcal{U}$ which meets α, and the result follows.

$\square$

Let U be a Buekenhout-Metz unital in $\mathcal{P}$ with respect to a point P. The preceding two lemmas show that U satisfies I_P and II_P. Therefore we can construct a 2-$(q^2, q+1, q)$ design K_P from U as in Section 3.2. We note that the blocks of K_P have a natural geometrical interpretation which we will use later. We present this interpretation in the next lemma which also demonstrates some parallel classes in the Buekenhout-Metz unital.

Lemma 3.3.3 *Let U be a Buekenhout-Metz unital with respect to ℓ_∞ at the point*

P in $\mathcal{P}$, and let T be a point of ℓ_∞, $T \neq P$. The $q^2 - q$ secant lines of U through T fall into $q-1$ classes of q lines. The lines in a given class meet a fixed set of $q+1$ lines of U through P, while lines in different classes do not meet a common line of U through P in a point of U. Each class corresponds to a block of the design K_P. The $q+1$ tangent lines to U through T meet U in $q+1$ collinear points of the remaining line through P.

Note that this means that if U is Buekenhout-Metz with respect to P in $\mathcal{P}$, then in any configuration II_P, the lines of the plane that form the *cross* lines meet ℓ_∞ in a point which is not in the unital.

Proof In the representation of $\mathcal{P}$ in $PG(4,q)$, let $\mathcal{U}$ be the ovoidal cone corresponding to the unital U and let V be the vertex of $\mathcal{U}$. Let α, β be secant planes of $\mathcal{U}$ through t, a line of the spread not containing V, that both meet the same line m through V. Now $\langle \alpha, \beta \rangle$ is a three dimensional space that contains m and hence V. Any line r of $\langle \alpha, \beta \rangle$ meets both α and β, thus any line of $\mathcal{U}$ through V that meets α also meets β. Hence α and β meet the same set of $q+1$ lines through V. These $q+1$ lines lie in the three dimensional subspace $\langle \alpha, \beta \rangle = \langle t, m \rangle$.

The plane $\langle t, V \rangle$ is a tangent plane of $\mathcal{U}$ in Σ_∞. There are $q+1$ three dimensional subspaces of $PG(4,q)$ containing $\langle t, V \rangle$, one is Σ_∞. A given class corresponds to q planes in one of these subspaces. The $q-1$ classes meet q^2-1 of the lines of $\mathcal{U}$ through V, thus the remaining three dimensional subspace through $\langle t, V \rangle$ meets $\mathcal{U}$ in the remaining line through V. Each of the q planes of $PG(4,q) \backslash \Sigma_\infty$ through t in this three-dimensional subspace meet this remaining line, these q planes correspond to q of the tangent lines to U through T, the remaining one is ℓ_∞. Thus the $q+1$ tangents of U through T meet U in $q+1$ collinear points of a Baer subline.

A given class meets $q+1$ lines of U through P, these lines correspond to $q+1$ points of K'_P. The q lines in a class correspond to q repeated blocks of K'_P. Thus the class corresponds to a single block of K_P and the $q+1$ points on it.

□

We now use these lemmas to show that we can always construct an inversive plane from a Buekenhout-Metz unital.

Lemma 3.3.4 *Let U be a Buekenhout-Metz unital with respect to a point P in $\mathcal{P}$, then the 2-$(q^2, q+1, q)$ design K_P constructed from U as in Section 3.2 is the point residual of an inversive plane.*

Proof In the representation of $\mathcal{P}$ in $PG(4,q)$, let V be the vertex of the ovoidal cone $\mathcal{U}$ that corresponds to U. We first explicitly construct the point-residual of an inversive plane in U, where the inversive plane corresponds to the ovoid used in the construction of the Buekenhout-Metz unital.

We work in the quotient space of $PG(4,q)$ by V. This is a three dimensional projective space $\Sigma = PG(3,q)$ such that the points of Σ are the lines of $PG(4,q)$ through V, the lines of Σ are the planes of $PG(4,q)$ containing V, and the planes of Σ are the hyperplanes of $PG(4,q)$ containing V. Incidence in Σ is containment. That is, an element $\widehat{a}$ of Σ is contained in an element $\widehat{b}$ of Σ if the corresponding elements a and b of $PG(4,q)$ satisfy $a \in b$.

Let $\widehat{\mathcal{O}}$ be the ovoid in Σ which is comprised of the lines of $\mathcal{U}$ through V. Let $\widehat{X}$ be the point of $\widehat{\mathcal{O}}$ corresponding to the line of the spread on V. Now consider the incidence structure L, where the *points* of L are the points of $\widehat{\mathcal{O}} \backslash \widehat{X}$, the *blocks* of L are the secant plane sections of $\widehat{\mathcal{O}}$ which do not pass through $\widehat{X}$, and *incidence* is containment. It is clear that L is the point-residual of an inversive plane, namely the egglike inversive plane corresponding to $\widehat{\mathcal{O}}$.

To complete the proof, we show that K_P is isomorphic to L. Clearly there is a natural bijection between the points of K_P and the points of L. We show that there is an incidence-preserving bijection on the blocks. Let t be a line of the spread not containing V, then $\langle t, V \rangle$ is a plane on V in Σ_∞. Thus in the quotient space, t gives rise to a line $\widehat{t}$ not on $\widehat{X}$ but contained in the tangent plane $\widehat{\pi}$ to $\widehat{\mathcal{O}}$ at $\widehat{X}$.

Conversely, such a line $\hat{t}$ in the quotient space corresponds to a plane in Σ_∞ and so contains a unique line t of the spread. By Lemma 3.3.3, the secant planes of $\mathcal{U}$ on t fall into $q-1$ classes of q planes each. The planes in a class meet the same $q+1$ lines through V, which form an oval cone. These sets of lines are the blocks of K_P and correspond in a one to one way with the blocks of L.

□

Hence we have proved the following result that a Buekenhout-Metz unital gives rise to an inversive plane.

Theorem 3.3.5 *Let $\mathcal{P}$ be a translation plane of dimension at most two over its kernel and let U be a Buekenhout-Metz unital with respect to the point P in $\mathcal{P}$. Let $\mathcal{U}$ be the corresponding ovoidal cone in $PG(4,q)$, with vertex V, so that the lines of $\mathcal{U}$ through V correspond to an ovoid $\hat{\mathcal{O}}$ in the quotient space $\Sigma = PG(3,q)$ of V in $PG(4,q)$. The 2-$(q^2, q+1, q)$ design with points the lines of U through P and blocks the lines of U not through P is the point residual of an egglike inversive plane; namely the inversive plane associated with the ovoid $\hat{\mathcal{O}}$.*

In particular, this Theorem includes the results of Wilbrink [113], Ebert [50] and Baker and Ebert [11]. Furthermore, it proves the natural isomorphism between the inversive planes they consider and the egglike inversive plane corresponding to the associated ovoid. In particular, a Buekenhout-Metz unital corresponding to an ovoidal cone based on an elliptic quadric naturally has a related Miquelian inversive plane, while a Buekenhout-Metz unital corresponding to an ovoidal cone based on a Suzuki-Tits ovoid has a related Suzuki-Tits inversive plane.

We can use this to generalize a result that Kestenband [77] proved for the classical unital.

Corollary 3.3.6 *Let K_P be a 2-$(q^2, q+1, q)$ design constructed from a Buekenhout-Metz unital as described in Section 3.2. The blocks of K_P can be partitioned into*

q^2 classes $\mathcal{B}_1, \ldots, \mathcal{B}_{q^2}$ of $q-1$ blocks. To each $\mathcal{B}_i$ there corresponds a different point X_i such that the blocks of $\mathcal{B}_i$ and the point X_i cover all the points of K_P.

Proof We use the notation of the proof of Lemma 3.3.4, and prove the result for the incidence structure L that is isomorphic to K_P. Let $\widehat{\pi}$ be the tangent plane to $\widehat{\mathcal{O}}$ at $\widehat{X}$. Through each line $\widehat{t_i}$ on $\widehat{\pi}$ not through $\widehat{X}$ there passes one plane distinct from $\widehat{\pi}$ and tangent to $\widehat{\mathcal{O}}$ at a point $\widehat{X_i}$, and $q-1$ secant planes to $\widehat{\mathcal{O}}$. Thus, for each i, the set $\mathcal{B}_i$ of blocks of L corresponding to the $q-1$ secant planes on $\widehat{t_i}$ together with the point $\widehat{X_i}$ cover the points of L.

It follows from this construction that the blocks within each class are distinct and are distinct from the blocks in different classes. It will follow that the sets $\mathcal{B}_1, \ldots, \mathcal{B}_{q^2}$ form a partition of the blocks of L if we show that they contain all the plane sections of $\widehat{\mathcal{O}}$ not through $\widehat{X}$. There are q^3+q^2+q+1 planes in Σ, they are either tangent or secant to $\widehat{\mathcal{O}}$. There is exactly one tangent plane through each point of $\widehat{\mathcal{O}}$, thus there are q^3+q secant planes of $\widehat{\mathcal{O}}$. There are q^2+q+1 planes through $\widehat{X}$, one is the tangent plane which we have already counted, thus there are $(q^3+q)-(q^2+q) = q^3-q^2$ planes that are secant to $\widehat{\mathcal{O}}$ and not through $\widehat{X}$. This is exactly the number of blocks in the classes $\mathcal{B}_i$ and the result follows.

□

A flock of an ovoid $\mathcal{O}$ is a set of $q-1$ planes that meet $\mathcal{O}$ in disjoint ovals. The planes in a given flock contain a common line (see [90] and [104]). We note that the classes in the above partition are flocks of the ovoid (or inversive plane).

We next aim to show that if an inversive plane can be constructed from a unital via the point P as in Section 3.2, then the unital is Buekenhout-Metz with respect to P. We will need the following lemma which tells us how a three dimensional space containing an ovoid of a Buekenhout-Metz unital meets the spread of Σ_∞ in $PG(4,q)$.

Lemma 3.3.7 *Let $\mathcal{O}$ be an ovoid in $PG(4, q)$ that meets Σ_∞ in exactly one point X that lies in the line p of the spread $\mathcal{S}$ of Σ_∞. Let $\Sigma_\mathcal{O}$ be the three dimensional subspace that contains the ovoid $\mathcal{O}$, and suppose $p \notin \Sigma_\mathcal{O}$, then there is exactly one line r of $\mathcal{S}$ that lies in $\Sigma_\mathcal{O}$.*

Proof The subspace $\Sigma_\mathcal{O} \cap \Sigma_\infty$ is a plane and so contains exactly one line of the spread r which is distinct from p as $p \notin \Sigma_\mathcal{O}$.

□

The planes through r that are contained in $\Sigma_\mathcal{O}$ are either tangent (two of them) or secant ($q - 1$ of them) to $\mathcal{O}$. There are q^2 planes of $PG(4, q)\backslash\Sigma_\infty$ through p, each contains $\mathcal{O} \cap p$ and exactly one other point of $\mathcal{O}$. A plane of $PG(4, q)\backslash\Sigma_\infty$ through any other line of the spread meets $\Sigma_\mathcal{O}$ in a line and so meets $\mathcal{O}$ in 0, 1 or 2 points.

Note that in $PG(2, q^2)$, the ovoid $\mathcal{O}$ corresponds to a set O which has some interesting properties. The set O consists of q^2 affine points together with the point P, such that through a fixed point R of ℓ_∞, there pass $q - 1$ lines that meet O in $q + 1$ points and two lines that are tangent to O. All lines through P meet O in two points. Lines through any other point of ℓ_∞ meet O in less than three points.

In order to prove our next result we need to introduce a slightly modified version of Condition II_P. We say a unital U in a projective plane satisfies Condition II_P^* for some point $P \in U$ if the following holds (see Figure 3.3).

Condition $\mathbf{II_P^*}$ *Let ℓ_∞ be the tangent line to U at P. Let l be a secant line of U not through P that meets ℓ_∞ in the point T. Let m be a secant line of U through P meeting l in a point of U and let $Y \neq P$ be a point of U on m. If $m, m_1, \ldots, m_q$ are the secant lines of U through P that meet l in a point of U, then the line $l' = YT$ intersects all the m_i in a point of U.*

So Condition II_P^* is Condition II_P with the requirement that U is in a projective plane and the extra property that the *cross* lines meet in a fixed point of ℓ_∞, the

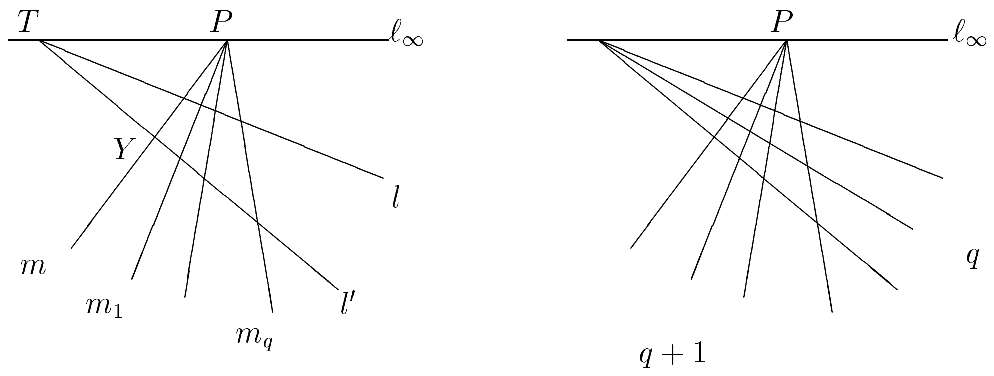

Figure 3.3: Condition II_P^* and a configuration II_P^*.

tangent line to U through P. Lemma 3.3.3 shows that all Buekenhout-Metz unitals satisfy Condition II_P^*. We call the configuration of $q+1$ lines through P and q *cross* lines which meet each of these $q+1$ lines in a point of U, with the cross lines containing a fixed point of ℓ_∞, a **configuration $\mathbf{II_P^*}$** (see Figure 3.3).

Theorem 3.3.8 *Let U be a parabolic unital in a translation plane $\mathcal{P}$ of dimension at most two over its kernel. Let P be the point of U on the translation line ℓ_∞ such that U satisfies I_P and II_P^*. In the Bruck and Bose representation of $\mathcal{P}$ in $PG(4,q)$, let $\mathcal{U}$ correspond to U, and let p be the line of the spread corresponding to P. Suppose $\mathcal{U}$ contains an elliptic quadric $\mathcal{O}$ which lies in a three dimensional subspace $\Sigma_{\mathcal{O}}$ that meets p in the point X. Then U is isomorphic to a Buekenhout-Metz unital with respect to P.*

Proof In $PG(4,q)$, U corresponds to the set $\mathcal{U}$ which consists of the line p of $\mathcal{S}$ and q affine points in each of the planes of $PG(4,q)\backslash\Sigma_\infty$ about p. Each of these planes contains exactly one point of $\mathcal{O}\backslash X$. We show that the points of $\mathcal{U}$ in each of these planes are collinear and meet p in a common vertex, that is, $\mathcal{U}$ is an ovoidal cone.

Let α, β, $\alpha_3,\ldots,\alpha_q$ be q planes about a line $t \neq p$ of the spread that all meet the same set of planes $\gamma_1,\ldots,\gamma_{q+1}$ about p in a point of $\mathcal{U}$. We can do this as U

satisfies Condition II_P^*. Let $\alpha \cap \gamma_i = A_i$, $\beta \cap \gamma_i = B_i$ and $\alpha_j \cap \gamma_i = C_{ji}$. We show that for each i, the points A_i, B_i, $C_{3i}, \ldots, C_{qi}$ are collinear in γ_i.

For each $i = 1, \ldots, q+1$ the plane γ_i meets $\mathcal{O}$ in the point $X \in p$ and another point X_i, so γ_i meets $\Sigma_{\mathcal{O}}$ in the line $l_i = XX_i$. Since the γ_i form a configuration II_P^* and the X_i lie in $\mathcal{O}$, the points $X_1, \ldots, X_{q+1}$ correspond to a block of K_P and so lie in a secant plane ε of $\mathcal{O}$. We let $\mathcal{C}_1 = \{X_1, \ldots, X_{q+1}\}$ and note that $\mathcal{C}_1$ is an oval of $\mathcal{O}$.

Let $\Sigma_{\alpha\beta} = \langle \alpha, \beta \rangle$, then $\Sigma_{\alpha\beta}$ is a three dimensional subspace that meets γ_i in exactly the line A_iB_i, $i = 1, \ldots, q+1$ (since $\Sigma_{\alpha\beta}$ contains A_i and B_i, but doesn't contain p). Hence the lines A_iB_i meet in a point of p. We will investigate $\Sigma_{\alpha\beta} \cap \Sigma_{\mathcal{O}}$. Suppose that $t \notin \Sigma_{\mathcal{O}}$, that is, t is not equal to the r of Lemma 3.3.7. Now $\Sigma_{\alpha\beta} \cap \Sigma_{\mathcal{O}}$ is a plane δ that is either tangent or secant to $\mathcal{O}$. Let $\mathcal{C}_2 = \delta \cap \mathcal{O}$, then $\mathcal{C}_2 = Y$ or $\mathcal{C}_2$ is an oval $\{Y_1, \ldots, Y_{q+1}\}$. Note that $\mathcal{C}_2$ contains at most two of the A_i, since if it contained three, then $\alpha = \delta$ which contradicts the fact that $t \notin \delta$. Similarly, $\mathcal{C}_2$ contains at most two of the B_i.

Each γ_i contains exactly one point of $\mathcal{O} \setminus X$, namely X_i. There are $q+1$ of the γ_i, and q of the *cross* planes, $\alpha, \beta, \alpha_3, \ldots, \alpha_q$, so by the pigeon hole principle one of the cross planes meets $\mathcal{O}$ in two points (it cannot meet it in any more than two points as $t \notin \Sigma_{\mathcal{O}}$). Without loss of generality, $\alpha \cap \mathcal{O} = \{A_1, A_2\}$. Thus $A_1, A_2 \in \delta$ and so δ cannot be tangent to $\mathcal{O}$. It follows that $\delta \cap \mathcal{O} = \mathcal{C}_2 = \{Y_1, \ldots, Y_{q+1}\}$, an oval of $\mathcal{O}$. Without loss of generality, let β contain at least one point B_3 of $\mathcal{O}$. Now A_1, A_2, B_3 are three non-collinear points in $\delta \cap \varepsilon$. Hence $\delta = \varepsilon$ and so $\mathcal{C}_1 = \mathcal{C}_2$ (since δ meets $\mathcal{O}$ in a unique oval).

Hence δ meets t in the point $W \in A_1A_2$. Through W there are $\frac{q+1}{2}$ or $\frac{q+3}{2}$ lines that meet $\mathcal{C}_1$ if q is odd (zero or two of these are tangents while the rest are bisecants of $\mathcal{C}_1$) and $\frac{q}{2}$ lines that meet $\mathcal{C}_1$ if q is even (these are all bisecants since W already lies on one bisecant A_1A_2). Thus the lines $m, n, m_3, \ldots, m_x$ through W meet $\mathcal{C}_1$ (where $x = \frac{q+1}{2}$ or $\frac{q+3}{2}$ if q is odd, and $x = \frac{q}{2}$ if q is even) where $m = \alpha \cap \delta$, $n = \beta \cap \delta$ and

$m_i = \alpha_i \cap \delta$, $i = 3, \ldots, x$. Hence the planes $\alpha, \beta, \alpha_3, \ldots, \alpha_x$ all lie in $\Sigma_{\alpha\beta}$ and since γ_i meets $\Sigma_{\alpha\beta}$ in exactly the line A_iB_i, the points $A_i, B_i, C_{3i}, \ldots, C_{xi}$ are collinear in γ_i.

To complete the proof that the points of $\mathcal{U}$ in γ_i are collinear, we look at the other $q-1$ configurations II_P^* that meet γ_1 and γ_2. (There are q configurations II_P^* meeting γ_1 and γ_2 since any two points of K_P lie on q blocks of K_P and a block of K_P corresponds to a configuration II_P of U and so to a configuration II_P^* of U.)

Each configuration II_P^* of $\mathcal{U}$ that contains γ_1 and γ_2 consists of $q+1$ planes $\gamma_1, \gamma_2, \gamma_3^i, \ldots, \gamma_{q+1}^i$ about p that each meet q planes $\alpha_1^i, \ldots, \alpha_q^i$ in a point of $\mathcal{U}$, where $\alpha_1^i, \ldots, \alpha_q^i$ contain a line t^i of the spread, $i = 1, \ldots, q$, with $i = 1$ corresponding to the above case. Since U satisfies I_P and II_P, all the γ_j^i are distinct (since if a line meets three of the lines through P in a given configuration II_P, it must meet them all, otherwise we will have an O'Nan configuration on P which contravenes I_P).

Recall that $\gamma_1 \cap \mathcal{U} = \{A_1, B_1, C_{31}, \ldots, C_{q1}\}$, $\gamma_2 \cap \mathcal{U} = \{A_2, B_2, C_{32}, \ldots, C_{q2}\}$ and that $A_1, A_2 \in \mathcal{O}$. Let $\alpha_j^i \cap \gamma_k^i = C_{jk}^i$, $k = 1, \ldots, q+1$, $i, j = 1, \ldots, q$ and let $\alpha_1^i \cap \gamma_1 = A_1$, $\alpha_1^i \cap \gamma_2 = C_{i2}$, $i = 1, \ldots, q$. Note that $C_{i2} \neq A_2$ if $i \neq 1$, hence exactly one of these configurations contains the line A_1A_2, namely the one we worked with above when $i = 1$. Hence none of the configurations can contain the r of Lemma 3.3.7. (If a configuration contains r then exactly one of the planes α_j^i lies in $\Sigma_{\mathcal{O}}$, so this plane would contain A_1 and A_2. In the earlier argument, picking α as a plane that contained exactly two points of $\mathcal{O}$, $\gamma_1 \cap \mathcal{U}$ and $\gamma_2 \cap \mathcal{U}$, ensured that γ_1 and γ_2 are not contained in any configuration II_P^* that contains r.)

We repeat the above argument for each of the configurations; that is, for each i. We let $\Sigma_{\alpha_1^i\alpha_2^i} = \langle \alpha_1^i, \alpha_2^i \rangle$ where α_1^i and α_2^i contain at least three points of $\mathcal{O}$ between them (recall that α_1^i contains the point A_1 of $\mathcal{O}$) and deduce that $\alpha_1^i, \ldots, \alpha_x^i$ lie in $\Sigma_{\alpha_1^i\alpha_2^i}$. Thus the appropriate points of γ_1 and γ_2 are collinear. Since α_1^i meets γ_2 in C_{i2} and $A_2 \in \gamma_2 \cap \mathcal{O}$, two of the collinear points in γ_2 are A_2 and C_{i2}.

As we run through all the configurations (that is, as we repeat the argument for each i) we will get x points collinear in γ_2, two of which are A_2 and C_{i2}. Hence all of the points of $\gamma_2 \cap \mathcal{U}$ are covered and so they must all be collinear. Similarly all the affine points of $\mathcal{U}$ in γ_1 are collinear. In the above argument we showed that A_1B_1 and A_2B_2 meet in a point of p, thus the points of $\mathcal{U}$ in γ_1 and γ_2 form two lines that meet in a point of p.

If γ_i and γ_j are two planes of $PG(4,q)\backslash\Sigma_\infty$ about p, we can examine the II_P^* configurations containing them. Each plane γ_i is contained in exactly 1 configuration II_P^* that contains r (since there are q planes about r that contain an affine point of $\gamma_i \cap \mathcal{U}$ and they form a configuration II_P^*). If γ_i and γ_j are contained in such a configuration, then we consider γ_i and γ_k where γ_k is chosen so that γ_i and γ_k are not contained in a configuration II_P^* that contains r. We can now apply the above argument to deduce that the q affine points of $\mathcal{U}$ in γ_i and γ_k are collinear and that the two lines containing these points meet in a point of p.

We repeat this argument for all the planes of $PG(4,q)\backslash\Sigma_\infty$ about p and deduce that the q affine points of $\mathcal{U}$ in each plane about p meet in a common point V of p.

Thus $\mathcal{U}$ consists of q^2 lines of $PG(4,q)\backslash\Sigma_\infty$ through V that each contain exactly one point of $\mathcal{O}$; that is, $\mathcal{U}$ is an ovoidal cone whose vertex lies on p. Hence U is a isomorphic to a Buekenhout-Metz unital with respect to P.

□

3.4 Future work

Our original aim was to prove Theorem 3.3.8 without the hypothesis that Condition II_P^* holds; that is, "If a unital U in $\mathcal{P}$ gives rise to an inversive plane via K_P, then U is Buekenhout-Metz with respect to P". We are still trying to improve this theorem. It is unclear whether the Buekenhout-Metz unitals in a translation plane

$\mathcal{P}$ form an isomorphism class. That is, if U is isomorphic to a unital which is Buekenhout-Metz with respect to ℓ_∞, is U Buekenhout-Metz? We also need to answer this question.

When Wilbrink constructed an inversive plane from the classical unital he proved the more general result that if U is any unital satisfying I and II, then for any point P of U the associated design K_P is the point residual of an inversive plane. His proof does not generalize to show that "If U is a unital satisfying I_P and II_P, then U gives rise to an inversive plane" as it used the fact that a point $T \neq P$ of U is not contained in any O'Nan configurations. As yet we have been unable to prove this generalization.

If we are able to generalize these results, we will obtain as an immediate corollary the following characterization of Buekenhout-Metz unitals: "If U is a unital satisfying I_P and II_P, then U is Buekenhout-Metz with respect to P".

Chapter 4

Unitals in the Hall plane

4.1 Introduction

In this chapter we study the Buekenhout-Metz and Buekenhout unitals in the Hall plane. We take the classical unital in $PG(2, q^2)$ and derive the plane in order to observe the point set of $\mathcal{H}(q^2)$ (the Hall plane of order q^2) that corresponds to the classical unital. There are five different ways that the classical unital can meet the derivation set of $PG(2, q^2)$, each is investigated separately. In three of the cases the point set of $\mathcal{H}(q^2)$ that corresponds to the classical unital forms a unital in the Hall plane that is not isomorphic to the classical unital. In the other two cases the resulting point set does not form a unital of $\mathcal{H}(q^2)$. We also show that the classical unital is not embeddable in the Hall plane as a Buekenhout unital and that the Buekenhout unitals of $\mathcal{H}(q^2)$ are not embeddable as Buekenhout unitals in the Desarguesian plane. We discuss the number of Buekenhout-Metz and Buekenhout unitals in $\mathcal{H}(q^2)$. Finally, we generalize this technique to other translation planes.

In order to study the effect of derivation on the classical unital, we begin in Section 4.2 by looking at the effect of derivation on the Baer subplanes and Baer sublines of $PG(2, q^2)$. Section 4.3 deals with the effect of derivation on the classical unital and on Buekenhout-Metz unitals of $PG(2, q^2)$. In Section 4.4 we summarize our investigation of unitals in the Hall plane and look at the embeddability of the classical unital in $\mathcal{H}(q^2)$. We determine the number of known unitals in $\mathcal{H}(q^2)$. Finally, in Section 4.5, we generalize the techniques of this chapter to other translation planes.

4.2 Deriving PG(2,q^2) and its effect on the Baer sublines and Baer subplanes.

Derivation is defined in detail in Section 1.8, we briefly summarize it here in order to introduce the notation used in this chapter. Let $PG(2, q^2)$ be the Desarguesian plane of order q^2 and let $\mathcal{D} = \{T_0, \ldots T_q\}$ be a derivation set of ℓ_∞. Deriving $PG(2, q^2)$ with respect to $\mathcal{D}$ gives the Hall plane of order q^2, $\mathcal{H}(q^2)$ (see Section 1.8). The affine points of $\mathcal{H}(q^2)$ are the affine points of $PG(2, q^2)$. The affine lines of $\mathcal{H}(q^2)$ are the lines of $PG(2, q^2)$ not meeting $\mathcal{D}$ together with the Baer subplanes of $PG(2, q^2)$ that contain $\mathcal{D}$. The line at infinity of $\mathcal{H}(q^2)$ consists of the points of $\ell_\infty \setminus \mathcal{D}$ and $q+1$ new points, $\mathcal{D}' = \{R_0, \ldots, R_q\}$. If we derive $\mathcal{H}(q^2)$ with respect to $\mathcal{D}'$, we get $PG(2, q^2)$. The Hall plane contains other derivation sets of ℓ'_∞, we call $\mathcal{D}'$ the **classical derivation set of** $\mathcal{H}(q^2)$.

We will be using the Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$. The following notation will be used throughout this chapter. Let Σ_∞ be a hyperplane of $PG(4, q)$ and let $\mathcal{S}$ be the spread of Σ_∞ that generates $PG(2, q^2)$. Let $\mathcal{R} = \{t_0, \ldots, t_q\}$ be the regulus of $\mathcal{S}$ that corresponds to $\mathcal{D}$. The spread $\mathcal{S}'$ obtained from $\mathcal{S}$ by replacing the regulus $\mathcal{R}$ with its complementary regulus $\mathcal{R}' = \{r_0, \ldots, r_q\}$ (that is, $\mathcal{S}' = \mathcal{S} \setminus \mathcal{R} \cup \mathcal{R}'$) generates the Hall plane $\mathcal{H}(q^2)$.

In order to investigate the effect of derivation of $PG(2, q^2)$ on Baer subplanes we will need the following result due to Grünig.

Lemma 4.2.1 [60, Lemma 2.4] *Let $\mathcal{P}$ be a translation plane of order q^2. In the Bruck and Bose representation of $\mathcal{P}$ in $PG(4, q)$, let $\mathcal{S}$ be the spread of the hyper-*

plane Σ_∞ that generates $\mathcal{P}$.

(i) Any plane α of $PG(4,q)$ that intersects Σ_∞ in a line not in $\mathcal{S}$ can be regarded as a Baer subplane of $\mathcal{P}$.

(ii) If $\mathcal{P}$ is Desarguesian, each Baer subplane secant to ℓ_∞ can be represented in this way.

If $\mathcal{B}$ is a Baer subplane of a projective plane $\mathcal{P}$ that is secant to the line at infinity, then $\mathcal{B}$ is called an **affine Baer subplane** of $\mathcal{P}$. The next result describes the affine Baer subplanes of the Hall plane.

Theorem 4.2.2 *Let $PG(2,q^2)$ be the Desarguesian plane of order q^2 and derive with respect to the derivation set $\mathcal{D} = \{T_0, \ldots, T_q\}$ of ℓ_∞. The following subsets of $PG(2,q^2)$ derive to affine Baer subplanes of $\mathcal{H}(q^2)$.*

1. *Lines of $PG(2,q^2)$ that meet ℓ_∞ in a point of $\mathcal{D}$ derive to affine Baer subplanes of $\mathcal{H}(q^2)$.*
2. *Affine Baer subplanes of $PG(2,q^2)$ that are disjoint from the derivation set $\mathcal{D}$ derive to affine Baer subplanes of $\mathcal{H}(q^2)$.*
3. *Affine Baer subplanes of $PG(2,q^2)$ that contain greater than zero and less than $q+1$ points of $\mathcal{D}$ derive to affine Baer subplanes of $\mathcal{H}(q^2)$.*

Note The Hall plane of order q^2, $\mathcal{H}(q^2)$, may contain other affine Baer subplanes as Lemma 4.2.1(ii) is not necessarily true for non-Desarguesian translation planes (see Section 1.7). If these do exist, they do not have a natural correspondence to the lines or affine Baer subplanes of $PG(2,q^2)$.

Proof We work in $PG(4,q)$ using the Bruck and Bose representation described in Section 1.7. Let $\mathcal{S}$ and $\mathcal{S}'$ be the spreads of Σ_∞ that correspond to $PG(2,q^2)$

and $\mathcal{H}(q^2)$ respectively. So $\mathcal{S}'$ is obtained from $\mathcal{S}$ by replacing the regulus $\mathcal{R} = \{t_0, \ldots, t_q\}$ that corresponds to $\mathcal{D}$ with its opposite regulus $\mathcal{R}' = \{r_0, \ldots, r_q\}$.

1. Let l be a line of $PG(2, q^2)$ that meets the derivation set $\mathcal{D}$ in the point T_0. In $PG(4, q)$, l corresponds to a plane α through the line t_0 of the spread $\mathcal{S}$. Since $\mathcal{S}' = \mathcal{S} \setminus \mathcal{R} \cup \mathcal{R}'$, the line t_0 meets $q + 1$ lines of the spread $\mathcal{S}'$, namely those in $\mathcal{R}'$. Hence by Lemma 4.2.1(i), α corresponds to an affine Baer subplane of $\mathcal{H}(q^2)$. Therefore a line of $PG(2, q^2)$ that contains a point of $\mathcal{D}$ derives to an affine Baer subplane of $\mathcal{H}(q^2)$.

2. Let $\mathcal{B}$ be an affine Baer subplane of $PG(2, q^2)$. By Lemma 4.2.1(ii), $\mathcal{B}$ corresponds to a plane α of $PG(4, q)$ that meets Σ_∞ in a line not in $\mathcal{S}$. If $\mathcal{B}$ is disjoint from $\mathcal{D}$ in $PG(2, q^2)$, then α is disjoint from $t_0, \ldots, t_q$ in $PG(4, q)$. Thus α meets $q + 1$ lines of the spread $\mathcal{S}'$. Hence by Lemma 4.2.1(i) α corresponds to an affine Baer subplane of $\mathcal{H}(q^2)$. Therefore affine Baer subplanes of $PG(2, q^2)$ disjoint from $\mathcal{D}$ derive to affine Baer subplanes of $\mathcal{H}(q^2)$.

3. A derivation set $\mathcal{D}$ of ℓ_∞ in $PG(2, q^2)$ is a Baer subline of ℓ_∞. Thus an affine Baer subplane meets $\mathcal{D}$ in 0, 1, 2 or $q + 1$ points (as two Baer sublines meet in 0, 1, 2 or $q + 1$ points). Consequently a Baer subplane meeting $\mathcal{D}$ in less than $q + 1$ and greater than zero points must meet $\mathcal{D}$ in either one or two points. We consider these two cases separately.

Suppose $\mathcal{B}$ is an affine Baer subplane of $PG(2, q^2)$ that meets the derivation set $\mathcal{D}$ in exactly one point T_0. In $PG(4, q)$, $\mathcal{B}$ corresponds to a plane β that meets $q + 1$ lines of the spread $\mathcal{S}$, $s_0, \ldots, s_{q-1}, t_0$, where t_0 is the line of $\mathcal{S}$ that corresponds to T_0 and $s_i \in \mathcal{S} \setminus \mathcal{R} = \mathcal{S} \cap \mathcal{S}'$, $i = 0, \ldots, q - 1$. Now β meets t_0 in a unique point K which lies on a unique line r_i of $\mathcal{R}'$. Thus β meets $q + 1$ lines of the spread $\mathcal{S}'$, namely $s_0, \ldots, s_{q-1}, r_i$. Hence by Lemma 4.2.1(i), β corresponds to an affine Baer subplane of $\mathcal{H}(q^2)$.

Suppose $\mathcal{B}$ is an affine Baer subplane of $PG(2, q^2)$ that meets $\mathcal{D}$ in two distinct points, T_0 and T_1. In $PG(4, q)$, $\mathcal{B}$ corresponds to a plane β that meets $q + 1$

lines of the spread $\mathcal{S}$, $s_0, \ldots, s_{q-2}, t_0, t_1$ where $t_0, t_1 \in \mathcal{R}$, and $s_i \in \mathcal{S} \setminus \mathcal{R} = \mathcal{S} \cap \mathcal{S}'$, $i = 0, \ldots, q-2$. Let the points in which β meets t_0 and t_1 be A_0 and A_1 respectively. Suppose that A_0 and A_1 lie on the same line r_k of the regulus $\mathcal{R}'$. Then $r_k = \beta \cap \Sigma_\infty$ which is a contradiction as r_k is disjoint from the lines $s_0, \ldots, s_{q-2}$ of the spread $\mathcal{S}'$. Thus A_0 and A_1 lie on different lines of $\mathcal{R}'$, r_i and r_j, $i \neq j$. So β is a plane of $PG(4, q)$ that meets Σ_∞ in $q + 1$ lines of the spread $\mathcal{S}'$, namely $s_0, \ldots, s_{q-2}, r_i, r_j$. Hence by Lemma 4.2.1(i), β corresponds to an affine Baer subplane of $\mathcal{H}(q^2)$.

$\square$

We now use this result to show that derivation of $PG(2, q^2)$ leaves Baer sublines with a point on ℓ_∞ unchanged, that is a Baer subline of $PG(2, q^2)$ with a point on ℓ_∞ is also a Baer subline of the derived plane $\mathcal{H}(q^2)$.

Theorem 4.2.3 *Let b be a Baer subline of $PG(2, q^2)$ that contains a point of ℓ_∞. Let $\mathcal{D}$ be a derivation set of ℓ_∞ and let $\mathcal{H}(q^2)$ be the derived plane. Let $\mathcal{D}'$ be the classical derivation set of $\mathcal{H}(q^2)$.*

(i) If $b \cap \ell_\infty \notin \mathcal{D}$, then the points of b form a Baer subline of $\mathcal{H}(q^2)$.

(ii) If $b \cap \ell_\infty \in \mathcal{D}$, then the affine points of b together with the appropriate point of $\mathcal{D}'$ form a Baer subline of $\mathcal{H}(q^2)$.

Proof We use the notation of the previous theorem. Recall that a Baer subline of $PG(2, q^2)$ that contains a point of ℓ_∞ corresponds to a line of $PG(4, q)$ that meets Σ_∞ in a single point.

If $b \cap \ell_\infty = T \notin \mathcal{D}$, then b is the intersection of a line l of $PG(2, q^2)$ that meets ℓ_∞ in T and an affine Baer subplane of $PG(2, q^2)$ that contains zero, one or two points of $\mathcal{D}$. The line l is disjoint from $\mathcal{D}$ and so is a line of $\mathcal{H}(q^2)$. By Theorem 4.2.2(2),(3), a Baer subplane that meets $\mathcal{D}$ in zero, one or two points derives to a Baer subplane of $\mathcal{H}(q^2)$. Thus in $\mathcal{H}(q^2)$, b is the intersection of a line and a Baer subplane, that is, b is a Baer subline of $\mathcal{H}(q^2)$.

If $b \cap \ell_\infty = T_i \in \mathcal{D}$, then b is the intersection of a line of $PG(2, q^2)$ through T_i and a Baer subplane $\mathcal{B}$ that contains one, two or $q+1$ points of $\mathcal{D}$. By Theorem 4.2.2(1), such a line derives to an affine Baer subplane of $\mathcal{H}(q^2)$.

Suppose $\mathcal{B}$ contains $\mathcal{D}$, then $\mathcal{B}$ derives to a line of $\mathcal{H}(q^2)$ and so b is the intersection of a Baer subplane and a line of $\mathcal{H}(q^2)$. Hence b is a Baer subline of $\mathcal{H}(q^2)$.

Suppose $\mathcal{B}$ meets $\mathcal{D}$ in one or two points, then by Theorem 4.2.2(3), $\mathcal{B}$ is an affine Baer subplane of $\mathcal{H}(q^2)$. To show that b is a Baer subline of $\mathcal{H}(q^2)$ we show that it is the intersection of this Baer subplane with a line of $\mathcal{H}(q^2)$.

In $PG(4, q)$, b is a line that meets Σ_∞ in a point K of the line t_i of the spread $\mathcal{S}$. This point lies in a unique line r_j of the spread $\mathcal{S}'$. There is a unique plane α of $PG(4, q)$ that contains b and r_j. The plane α corresponds to a line of $\mathcal{H}(q^2)$ since it contains a line of $\mathcal{S}'$. So in $PG(4, q)$, b is the intersection of α and the plane corresponding to $\mathcal{B}$. Hence in $\mathcal{H}(q^2)$, b is the intersection of a line and a Baer subplane of $\mathcal{H}(q^2)$ and therefore is a Baer subline of $\mathcal{H}(q^2)$.

□

4.3 Deriving unitals of PG(2,q²)

Let U be a unital in $PG(2, q^2)$. Suppose we derive the plane with respect to a derivation set $\mathcal{D}$ of ℓ_∞. We get a set of points U' in $\mathcal{H}(q^2)$ that corresponds to the point set of U. If U is disjoint from $\mathcal{D}$, this correspondence is natural. If U contains some points of $\mathcal{D}$, then we need to be a little careful defining U' as derivation is an affine process and a unital is a projective object. We use the Bruck and Bose representation of $PG(2, q^2)$ and $\mathcal{H}(q^2)$ in $PG(4, q)$ to determine the points of U' on ℓ'_∞ in $\mathcal{H}(q^2)$ in this case. The points of U lie on a set $\mathcal{W}$ in $PG(4, q)$ which contains 1 or $q+1$ lines of the spread $\mathcal{S}$ that generates $PG(2, q^2)$. The set $\mathcal{W}$ will meet a subset of the lines of the spread $\mathcal{S}'$ that generates $\mathcal{H}(q^2)$. These lines correspond to the points of U' on ℓ'_∞ in $\mathcal{H}(q^2)$. We prove in the following subsections that in the cases when U' has the right number of points, it is a unital of $\mathcal{H}(q^2)$. In the cases when U' has too many points to be a unital, we show that the affine part of U' cannot be completed to a unital of $\mathcal{H}(q^2)$.

In this section we investigate the set U' of $\mathcal{H}(q^2)$: recall that all known unitals in $PG(2, q^2)$ are Buekenhout-Metz. If U is the classical unital of $PG(2, q^2)$, we determine when U' is a unital of $\mathcal{H}(q^2)$ and whether it is isomorphic to U. If U is a non-classical Buekenhout-Metz unital in $PG(2, q^2)$, we determine when U' is a unital of $\mathcal{H}(q^2)$ and discuss its isomorphism with other known unitals.

We consider the case when U is a classical unital first. In Chapter 2 we showed that the classical unital in $PG(2, q^2)$ is Buekenhout with respect to all its secant lines and Buekenhout-Metz with respect to all its tangent lines (Corollary 2.2.4). There are five possible relationships between $\mathcal{D}$ and U. We investigate these separately in the following subsections. The main results of these subsections are summarized in the following theorem.

Theorem 4.3.1 *Let U be the classical unital in $PG(2, q^2)$. Derive the plane with respect to a derivation set $\mathcal{D}$ of ℓ_∞ and denote by U' the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.*

1. [60]*Suppose U is secant to ℓ_∞ and $\mathcal{D} = U \cap \ell_\infty$, then U' is a Buekenhout unital of $\mathcal{H}(q^2)$ and U' contains O'Nan configurations with one vertex on ℓ_∞.*

2. *Suppose U is secant to ℓ_∞ and $\mathcal{D} \cap U = \emptyset$, then U' is a Buekenhout unital of $\mathcal{H}(q^2)$ and U' contains O'Nan configurations.*

3. *Suppose U is secant to ℓ_∞ and $0 < |\mathcal{D} \cap U| < q + 1$, then U' is not a unital in $\mathcal{H}(q^2)$. If $q > 3$, there are no Buekenhout unitals with respect to ℓ'_∞ in $\mathcal{H}(q^2)$ that contain less than $q+1$ and greater than zero points of the classical derivation set.*

4. *Suppose U is tangent to ℓ_∞ and $U \cap \ell_\infty \in \mathcal{D}$, then U' is not a unital of $\mathcal{H}(q^2)$.*

5. *Suppose U is tangent to ℓ_∞ and $U \cap \ell_\infty \notin \mathcal{D}$, then U' is a Buekenhout-Metz unital of $\mathcal{H}(q^2)$ and U' contains O'Nan configurations.*

We prove each case separately in the following subsections. We first describe the notation we will be using: we denote the lines at infinity of $PG(2, q^2)$ and $\mathcal{H}(q^2)$ by ℓ_∞ and ℓ'_∞ respectively; if $\mathcal{D}$ is the derivation set used to derive $PG(2, q^2)$ to give $\mathcal{H}(q^2)$, then we denote by $\mathcal{D}'$ the corresponding derivation set of ℓ'_∞ that is used to derive $\mathcal{H}(q^2)$ to give $PG(2, q^2)$ (that is, $\mathcal{D}'$ is the classical derivation set of $\mathcal{H}(q^2)$); in $PG(4, q)$, we denote the spreads of Σ_∞ that generate $PG(2, q^2)$ and $\mathcal{H}(q^2)$ by $\mathcal{S}$ and $\mathcal{S}'$ respectively. If T is a point of ℓ_∞, we denote the corresponding line of the spread in $PG(4, q)$ by t. Let $\mathcal{U}$ be the parabolic quadric or ovoidal cone of $PG(4, q)$ that corresponds to U and U'.

4.3.1 Case 1.

Let U be the classical unital in $PG(2, q^2)$ that is secant to ℓ_∞ and derive the plane with respect to the derivation set $\mathcal{D} = U \cap \ell_\infty$. Let U' be the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.

This case was investigated by Grünig [60]. He showed that U' is a unital of $\mathcal{H}(q^2)$ [Lemma 2.6] and that U' contains O'Nan configurations with exactly one point on ℓ'_∞ [Lemma 5.4]. More specifically, he showed that for any point $P \in U' \cap \ell'_\infty$, let l_1, l_2 be secant lines of U' through P and distinct from ℓ'_∞ and let l_3 be a secant line not through P intersecting l_1 and l_2 in points of U', then there exists an O'Nan configuration containing l_1, l_2 and l_3.

A useful way for determining whether two unitals are isomorphic is by counting the number of O'Nan configurations in each of them. If these numbers differ, then the two unitals are not isomorphic. The classical unital contains no O'Nan configurations [89], thus the unital U' is not isomorphic to the classical unital. We will use this technique again in the following cases.

As a corollary to Grünig's results we can partially answer the question: is the classical unital embeddable in the Hall plane?

Corollary 4.3.2 *The classical unital of order q is not embeddable in the Hall plane of order q^2 as a Buekenhout unital with respect to ℓ'_∞ that contains the classical derivation set of ℓ'_∞.*

Proof Under derivation, a classical unital in $PG(2,q^2)$ containing $\mathcal{D}$ determines a Buekenhout unital in $\mathcal{H}(q^2)$ containing the classical derivation set $\mathcal{D}'$. In Corollary 2.2.7 we showed that the number of Buekenhout unitals secant to a Baer subline of the line at infinity in a translation plane of dimension at most two over its kernel is constant for all such planes. Therefore all the Buekenhout unitals in $\mathcal{H}(q^2)$ that contain $\mathcal{D}'$ are derived from the classical unital and so contain O'Nan configurations. Hence the classical unital cannot be embedded as a Buekenhout unital in $\mathcal{H}(q^2)$ that contains $\mathcal{D}'$.

□

Conversely we can show that any Buekenhout unital of $\mathcal{H}(q^2)$ containing the classical derivation set is not embeddable as a Buekenhout unital in $PG(2,q^2)$.

Corollary 4.3.3 *Let U' be a Buekenhout unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$ that contains the classical derivation set $\mathcal{D}'$, then U' is not embeddable in $PG(2,q^2)$ as a Buekenhout unital.*

Proof In Theorem 2.2.5 we showed that the only Buekenhout unitals in $PG(2,q^2)$ are the classical unitals. Thus if U' were embedded in $PG(2,q^2)$ as a Buekenhout unital, it would be isomorphic to the classical unital. However by the same argument used in the previous corollary, all the Buekenhout unitals in $\mathcal{H}(q^2)$ containing $\mathcal{D}'$ are derived from the classical unitals. Hence U' contains O'Nan configurations and so is not isomorphic to the classical unital.

□

Let U be a unital of a projective plane $\mathcal{P}$. The **affine part** of U, *aff*U is constructed by removing from U the points which lie on the line at infinity of $\mathcal{P}$. In the next lemma we show that although the unitals U and U' are not isomorphic as designs, their affine parts are isomorphic. This is because although the unitals look the same affinely, the secants that meet the line at infinity are joined in a different way in the two planes.

Lemma 4.3.4 *The affine part of U' is isomorphic to the affine part of U.*

Proof We identify the points of *aff*U' with those of *aff*U by identifying the affine points of $\mathcal{H}(q^2)$ with the affine points of $PG(2,q^2)$ by the derivation process.

There are two types of lines of $PG(2,q^2)$ that are secant to U. The first type is a line of $PG(2,q^2)$ that meets ℓ_∞ in a point not in $\mathcal{D}$. Such lines are unchanged by derivation, so they are still lines of $\mathcal{H}(q^2)$. Hence the $q+1$ points of *aff*U on such a line lie on a $(q+1)$-secant of *aff*U'.

The second type is a line of $PG(2,q^2)$ that meets ℓ_∞ in a point of the derivation set $\mathcal{D}$. Since U is classical, such a line meets U in a Baer subline. By Theorem 4.2.3, this Baer subline is unchanged by derivation since it has a point on ℓ_∞. Thus in $\mathcal{H}(q^2)$, there is a line that meets *aff*U' in the q affine points of this Baer subline, so both affine unitals contain these q-secants.

So we can identify the points of *aff*U' with those of *aff*U. The points on a $(q+1)$-secant of *aff*U lie on a $(q+1)$-secant of *aff*U'. The points on a q-secant of *aff*U lie on a q-secant of *aff*U'. Therefore the two structures are isomorphic.

□

The process of derivation is equivalent to removing some of the lines and points of the classical unital and putting them back in a different way. As a side effect of our investigations in this chapter, we obtain some results that describe which parts of the classical unital we can remove and put back in a different way in order to create a non-isomorphic unital. The next result shows that the incidence structure constructed from the classical unital by removing $q+1$ points of a secant line is not uniquely completable to a unital.

Corollary 4.3.5 *Let U be the classical unital in $PG(2,q^2)$. Remove from U a secant line and its $q+1$ points. There are at least two distinct ways of completing this set to a unital. One is the classical unital, the other is embeddable as a Buekenhout unital in $\mathcal{H}(q^2)$.*

Proof Removing a secant and its points from the classical unital gives us the affine part of the unital, *aff*U which is isomorphic to *aff*U' by Lemma 4.3.4. Thus there are at least two ways to complete *aff*U to a unital and they are non-isomorphic as only one contains O'Nan configurations.

$\square$

4.3.2 Case 2.

Let U be the classical unital in $PG(2,q^2)$ that is secant to ℓ_∞. We derive the plane using a derivation set $\mathcal{D}$ that is disjoint from U. Let U' be the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.

In this section we show that U' forms a unital of $\mathcal{H}(q^2)$ that is not isomorphic to the classical unital and is not isomorphic to the unital of $\mathcal{H}(q^2)$ described in Case 1. Note that since U is disjoint from $\mathcal{D}$, the points of $U' \cap \ell'_\infty$ have a natural correspondence to the points of $U \cap \ell_\infty$.

Lemma 4.3.6 *The set U' forms a unital in $\mathcal{H}(q^2)$.*

Proof The set U' contains q^3+1 points of $\mathcal{H}(q^2)$. We show that every line of $\mathcal{H}(q^2)$ meets U' in either 1 or $q+1$ points, from which it follows that U' is a unital. Clearly ℓ'_∞ meets U' in $q+1$ points since ℓ_∞ meets U in $q+1$ points. Let l be a line of $\mathcal{H}(q^2)$ that meets ℓ'_∞ in the point T.

If T is not in the derivation set $\mathcal{D}'$, then the points of l lie on a line of $PG(2,q^2)$ and so l contains 1 or $q+1$ points of U. Hence l contains 1 or $q+1$ points of U'.

Suppose T is in the derivation set $\mathcal{D}'$, then $T \notin U'$. Let $\mathcal{U}$ be the parabolic quadric of $PG(4,q)$ that corresponds to U, then $\mathcal{U}$ also corresponds to U'. Let α be the plane that corresponds to the line l, so $\alpha \cap \Sigma_\infty = t$, the line of the spread corresponding to T. Now α meets $\mathcal{U}$ in either a point, a line, a conic or two lines. If $\alpha \cap \mathcal{U}$ contains a line, then t contains a point of $\mathcal{U}$ which is a contradiction as $T \notin U'$. Thus α meets $\mathcal{U}$ in either 1 or $q+1$ points and so l meets U' in either 1 or $q+1$ points. Note that if l is secant to U', then the $q+1$ points $l \cap U'$ are not collinear in $PG(2,q^2)$; they form a conic in the Baer subplane that corresponds to l and a $(q+1)$-arc of $PG(2,q^2)$.

□

The proof of this lemma shows in fact that U' is a Buekenhout unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$. We will show that the designs U and U' are not isomorphic by constructing an O'Nan configuration in U'. We first investigate whether U' can contain an O'Nan configuration with a vertex on ℓ'_∞.

Lemma 4.3.7 *The unital U' contains no O'Nan configurations with two or three vertices on ℓ'_∞. If U' contains an O'Nan configuration l_1, l_2, l_3, l_4 with one vertex $T = l_1 \cap l_2$ on ℓ'_∞, then the lines $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are disjoint in $PG(4,q)$.*

Proof Suppose U' contains an O'Nan configuration with two or three vertices on ℓ'_∞. Such a configuration consists of four lines that each meet ℓ'_∞ in a point of U'. Since the derivation set is disjoint from the unital, these lines are also secants of U giving an O'Nan configuration in the classical unital U, which is a contradiction. Thus U' cannot contain an O'Nan configuration with two or three vertices on ℓ'_∞.

Suppose U' contains an O'Nan configuration with lines l_1, l_2, l_3, l_4 that has one vertex $T = l_1 \cap l_2$ on ℓ'_∞. Let $M = l_3 \cap \ell'_\infty$, $N = l_4 \cap \ell'_\infty$ and let the vertices of the O'Nan configuration be $A = l_1 \cap l_3, B = l_2 \cap l_3, C = l_3 \cap l_4, X = l_1 \cap l_4, Y = l_2 \cap l_4$, and T, see Figure 4.1. Note that at least one of M and N must lie in the derivation set $\mathcal{D}'$, otherwise l_3 and l_4 are also lines of $PG(2,q^2)$ giving an O'Nan configuration in the classical unital.

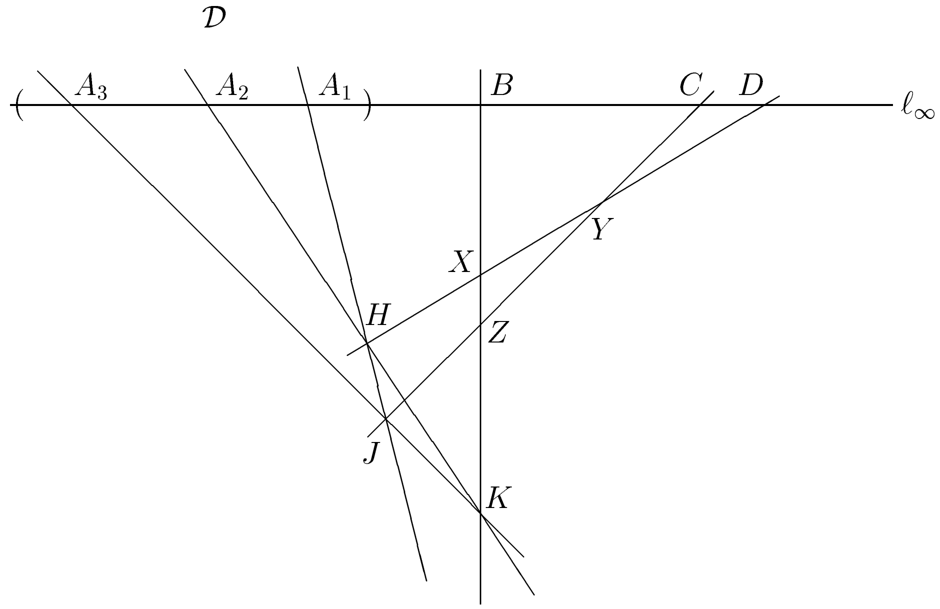

Figure 4.1: *An O'Nan configuration in U'.*

We use the representation of $\mathcal{H}(q^2)$ in $PG(4,q)$ and let $\mathcal{U}$ be the parabolic quadric corresponding to U'. Since U' is Buekenhout with respect to ℓ'_∞, the sets $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are Baer sublines of $\mathcal{H}(q^2)$ and correspond to lines of $PG(4,q)$ that meet Σ_∞ in a point of t (the line of $\mathcal{S}'$ that corresponds to T). These lines either meet in a point of t or they are disjoint in $PG(4,q)$.

Suppose the lines $\overline{l_1}$ and $\overline{l_2}$ meet in a point of t in $PG(4,q)$, then they are contained in a unique plane of $PG(4,q)$ that does not contain a line of the spread (if the plane contained t, it would meet $\mathcal{U}$ in $3q+1$ points which is impossible). This plane corresponds to an affine Baer subplane of $\mathcal{H}(q^2)$ (by Lemma 4.2.1(i)) that contains $\overline{l_1}$ and $\overline{l_2}$.

Let the lines l_3, l_4 of $\mathcal{H}(q^2)$ correspond to the planes α, β of $PG(4,q)$ that meet Σ_∞ in the lines m, n of the spread $\mathcal{S}'$ respectively. Let $\Sigma = \langle X, \alpha \rangle$ be the three dimensional subspace of $PG(4, q)$ that is spanned by X and α. Now $X, A \in \Sigma$ implies that $\overline{l_1} \in \Sigma$ and so $\overline{l_1} \cap t \in \Sigma$. Therefore $\overline{l_2} \in \Sigma$ (as $B \in \Sigma$) and so $Y \in \Sigma$.

Note that β is not contained in Σ since the three dimensional subspace spanned by m, n is Σ_∞ which is not equal to Σ, thus β meets Σ in a line which contains X, Y, C. This line contains at least three points of $\mathcal{U}$, but less than $q+1$ (as $m \notin \mathcal{U}$) which is a contradiction as a line of $PG(4, q)$ meets the quadric $\mathcal{U}$ in 0, 1, 2 or $q+1$ points. Therefore such a configuration cannot exist.

Hence if U' contains an O'Nan configuration with one vertex on ℓ'_∞, as in Figure 4.1, the lines $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are disjoint in $PG(4, q)$.

$\square$

In order to prove the existence of O'Nan configurations in U' we will need several preliminary lemmas.

We need to know what a conic in a Baer subplane $\mathcal{B}$ of $PG(2, q^2)$ looks like in the Bruck and Bose representation in $PG(4, q)$. If $\mathcal{B}$ is secant to ℓ_∞, then $\mathcal{B}$ corresponds to a plane α of $PG(4, q)$ and the points of a conic in $\mathcal{B}$ form a conic of α. If $\mathcal{B}$ is tangent to ℓ_∞, then $\mathcal{B}$ corresponds to a ruled cubic surface in $PG(4, q)$ (see Section 1.7). The following lemma shows that certain conics of these Baer subplanes correspond to $(q+1)$-arcs of $PG(3, q)$ in $PG(4, q)$ (that is, $q+1$ points lying in a three dimensional subspace of $PG(4, q)$, with no four points coplanar).

Lemma 4.3.8 *Let $\mathcal{B}$ be a Baer subplane of $PG(2, q^2)$ that meets ℓ_∞ in the point T. Let $\mathcal{C} = \{T, K_1, \ldots, K_q\}$ be a conic of $\mathcal{B}$. In the Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$, the points $K_1, \ldots, K_q$ form a q-arc of a three dimensional subspace of $PG(4, q)$.*

Proof Recall from Section 1.7 that $\mathcal{B}$ corresponds to a ruled cubic surface $\mathcal{V}$ of $PG(4, q)$ that meets Σ_∞ in the line t of the spread. The points of $\mathcal{V}$ lie on $q+1$ disjoint lines of $\mathcal{V}$, $l_1, \ldots, l_{q+1}$, called **generators**. Each generator meets Σ_∞ in a distinct point of t. The lines of $\mathcal{B}$ not through T correspond to conics of $\mathcal{V}$

in $PG(4,q)$. We label the points of $\mathcal{C}$ so that the point K_i lies on the line l_i, $i = 1, \ldots, q$ (since at most one point of $\mathcal{C} \setminus T$ lies on each l_i).

Suppose that l_1, l_2, l_3 span a three dimensional subspace Σ. Let X be any point of $\mathcal{B}$ that is not incident with l_1, l_2 or l_3. Now each line of $\mathcal{B}$ through X meets each of l_1, l_2, l_3. Such a line corresponds to a conic in $PG(4,q)$, and since three points of this conic lie in Σ, they all do. Thus every point of $\mathcal{V}$ lies in Σ, a contradiction as $\mathcal{V}$ spans four dimensional space. Hence no three of the l_i lie in a three dimensional subspace.

As a consequence of this, if A, B, C are points of $\mathcal{V}$ lying on different generators l_1, l_2, l_3 of $\mathcal{V}$, then A, B, C are not collinear in $PG(4,q)$. Since, suppose A, B, C lie on a line m of $PG(4,q)$, then m and t span a three dimensional subspace Σ which contains two points of each l_i, $i = 1, 2, 3$. Hence, Σ contains three generators l_1, l_2, l_3 of $\mathcal{V}$ which is not possible by the above.

We now show that in $PG(4,q)$, no four of the K_i lie in a plane. Suppose K_1, K_2, K_3, K_4 lie in a plane α, then α corresponds to an affine Baer subplane $\mathcal{B}'$ of $PG(2,q^2)$ (since no three of the K_i lie on a line of $PG(2,q^2)$). However K_1, K_2, K_3, K_4 form a quadrangle of $PG(2,q^2)$ and so are contained in a unique Baer subplane of $PG(2,q^2)$. This is a contradiction as $\mathcal{B} \neq \mathcal{B}'$. Hence no four of the K_i are coplanar.

Let the three dimensional subspace spanned by K_1, K_2, K_3, K_4 be Σ. Note that Σ does not contain t as no three of the l_i lie in a three dimensional subspace. Thus Σ meets t in one point and can contain at most one of the l_i.

Suppose one of l_1, l_2, l_3, l_4 lie in Σ, without loss of generality suppose $l_1 \in \Sigma$. Let $L_0 = l_1 \cap t$, $L_1 = K_1$, $L_i = l_i \cap \Sigma$, $i = 2, \ldots, q+1$ (so $L_i = K_i, i = 1, 2, 3, 4$).

Note that by the above, no three of the L_i, $i \geq 1$ are collinear in $PG(4,q)$. Since the only lines of $\mathcal{V}$ meeting t are the generators, no three of the L_i, $i \geq 0$ are collinear in $PG(4,q)$.

We now show that no three of the L_i are collinear in $PG(2,q^2)$. Suppose that

L_i, L_j and L_k, $i, j, k > 0$, are collinear in $PG(2, q^2)$, then the line l containing them corresponds to a plane β in $PG(4, q)$. Since L_i, L_j, L_k are not collinear in $PG(4, q)$, β is contained in Σ. Now in $PG(2, q^2)$, l contains three points of $\mathcal{B}$, and so l contains $q + 1$ points of $\mathcal{B}$. Hence l contains a point of each l_i.

Thus in $PG(4, q)$, β contains a point of each of l_i, hence β contains $L_2, \ldots, L_{q+1}$ as these are the only points of $l_2, \ldots, l_{q+1}$ respectively in Σ. However, $L_i = K_i$, $i = 2, 3, 4$, so in $PG(2, q^2)$, the points K_2, K_3, K_4 lie on the line l which is a contradiction as the K_i form a conic of $\mathcal{B}$. Therefore no three of the L_i, $i > 0$ are collinear in $PG(2, q^2)$.

If L_i, L_j and $L_0 = T$ are collinear in $PG(2, q^2)$, then the line containing them has three points in $\mathcal{B}$ and so has $q + 1$ points in $\mathcal{B}$. This is a contradiction as the only lines of $\mathcal{B}$ through T are the generators l_i, and the points L_i and L_j lie on different generators. Therefore, no three of the L_i, $i \geq 0$ are collinear in $PG(2, q^2)$.

We now show that no four of the L_i are coplanar in $PG(4, q)$. Suppose that four of the L_i lie in a plane α of $PG(4, q)$, then α corresponds to a line or an affine Baer subplane of $PG(2, q^2)$. If α corresponds to a line of $PG(2, q^2)$, then four of the L_i are collinear in $PG(2, q^2)$ which is not possible by the above.

If α corresponds to an affine Baer subplane of $PG(2, q^2)$, then the L_i cannot form a quadrangle of $PG(2, q^2)$ (as a quadrangle is contained in a unique Baer subplane). Hence three of the L_i must be collinear in $PG(2, q^2)$ which again contradicts the above. Therefore no four of the L_i are coplanar.

So we have shown that $L_0, L_1, \ldots, L_{q+1}$ form a set of $q + 2$ points of Σ, no four of them lying in a plane. This is impossible as the maximum size for a k-arc in $PG(3, q)$ is $k = q + 1$.

Hence l_1 cannot lie in Σ. Similarly $l_2, l_3, l_4 \notin \Sigma$. Thus if one of the l_i lie in Σ, then $i \neq 1, 2, 3, 4$.

We now let $l_i \cap \Sigma = L_i$ if $l_i \notin \Sigma$ (so $L_i = K_i$, $i = 1, 2, 3, 4$). If $l_i \in \Sigma$, we let

$L_i = l_i \cap t$. Using the same arguments as above, no three of the L_i are collinear in $PG(2, q^2)$ and consequently no four of the L_i are coplanar in $PG(4, q)$. Hence the set of points $\mathcal{C}' = \{L_1, \ldots, L_{q+1}\}$ satisfy the property that no four of them are coplanar and so $\mathcal{C}'$ is a $(q+1)$-arc of Σ.

Now the set $\mathcal{C}'$ corresponds to a set of $q+1$ points of $\mathcal{B}$ with no three of them collinear (since no three of the L_i are collinear in $PG(2, q^2)$). Moreover, $\mathcal{C} = \{T, K_1, \ldots, K_q\}$ and $\mathcal{C}'$ have five points in common, T, K_1, K_2, K_3, K_4, hence $\mathcal{C} = \mathcal{C}'$ (as there is a unique conic through five points). Thus $L_i = K_i$, $i = 1, \ldots, q$ and $L_{q+1} = l_{q+1} \cap t$. Hence in $PG(4, q)$, the K_i together with L_{q+1} form a $(q+1)$-arc of a three dimensional subspace Σ and the K_i form a q-arc of Σ.

□

Let $\mathcal{U}$ be a parabolic quadric in $PG(4, q)$. A **tangent hyperplane** of $\mathcal{U}$ is a hyperplane that meets $\mathcal{U}$ in a conic cone. Let G be the group of automorphisms of $PG(4, q)$ that fixes $\mathcal{U}$. There are $q^4(q^2+1)$ planes of $PG(4, q)$ that meet $\mathcal{U}$ in a conic. By [65, Theorem 22.6.6], the set of conics of $\mathcal{U}$ acted on by G has two orbits. If q is odd, one orbit contains **internal conics** and the other contains **external conics**. There are $\frac{1}{2}q^3(q-1)(q^2+1)$ internal conics and $\frac{1}{2}q^3(q+1)(q^2+1)$ external conics of $\mathcal{U}$ ([65, Theorem 22.9.1]). If q is even, one orbit consists of **nuclear conics** while the other contains **non-nuclear conics**. There are $q^2(q^2+1)$ nuclear conic and $q^2(q^4-1)$ non-nuclear conics of $\mathcal{U}$ ([65, Theorem 22.9.2]). The next lemma describes how many tangent hyperplanes of $\mathcal{U}$ contain a given conic of $\mathcal{U}$.

Lemma 4.3.9 *1. If q is odd, every internal conic of $\mathcal{U}$ is contained in zero tangent hyperplanes of $\mathcal{U}$, and every external conic of $\mathcal{U}$ is contained in two tangent hyperplanes of $\mathcal{U}$.*

2. If q is even, every nuclear conic of $\mathcal{U}$ is contained in $q+1$ tangent hyperplanes of $\mathcal{U}$, and every non-nuclear conic of $\mathcal{U}$ is contained in one tangent hyperplane of $\mathcal{U}$.

Proof There are q^3+q^2+q+1 tangent hyperplanes of $\mathcal{U}$ ([65, Theorem 22.8.2]) and $\mathcal{U}$ contains q^3+q^2+q+1 points. Since G is transitive on the points of $\mathcal{U}$ ([65, Theorem 22.6.4]), each point of $\mathcal{U}$ is the vertex of exactly one tangent hyperplane.

Let $\mathcal{U}$ meet the hyperplane Σ_∞ in a hyperbolic quadric $\mathcal{H}_3$. Every point V of $\mathcal{H}_3$ is the vertex of a conic cone of $\mathcal{U}$ that meets $\mathcal{H}_3$ in the two lines containing V. This accounts for $(q+1)^2$ of the tangent hyperplanes of $\mathcal{U}$, the remaining q^3-q meet Σ_∞ in a plane that contains a conic of $\mathcal{H}_3$.

Suppose q is odd. Let x be the number of tangent hyperplanes containing a given internal conic and let y be the number of tangent hyperplanes containing a given external conic (these numbers are constant as the conics of each type lie in a single orbit under G). By counting the number of conics of $\mathcal{U}$ in two ways we deduce that x times the number of internal conics of $\mathcal{U}$ plus y times the number of external conics of $\mathcal{U}$ is equal to the number of tangent hyperplanes of $\mathcal{U}$ times the number of conics of $\mathcal{U}$ in a tangent hyperplane. That is:

$$\begin{aligned}
\frac{1}{2}q^3(q-1)(q^2+1)x+\frac{1}{2}q^3(q+1)(q^2+1)y &= q^3(q^3+q^2+q+1)\\
\frac{1}{2}(q^2+1)q^3(x(q-1)+y(q+1)) &= q^3(q^2+1)(q+1)\\
x(q-1)+y(q+1) &= 2(q+1)\\
(y+x)q+(y-x) &= 2q+2.
\end{aligned}$$

Equating like powers of q implies that $y+x=2$ and $y-x=2$, hence $y=2$ and $x=0$. Therefore, every external conic in contained in two tangent hyperplanes of $\mathcal{U}$ and every internal conic is contained in zero tangent hyperplanes of $\mathcal{U}$.

Suppose q is even. Let x be the number of tangent hyperplanes containing a given nuclear conic and let y be the number of tangent hyperplanes containing a given non-nuclear conic. By counting the number of conics of $\mathcal{U}$ in two ways as above, we deduce that:

$$\begin{aligned}
q^2(q^2+1)x+q^2(q^4-1)y &= q^3(q^3+q^2+q+1)\\
x+y(q^2-1) &= q^2+q.
\end{aligned}$$

Both x and y are less than $q+1$, as a plane is contained in $q+1$ hyperplanes of $PG(4,q)$, so let $x = aq+b$. Equating like powers of q gives $y=1$, $a=1$ and $b-1=0$, hence $x=q+1$. Therefore, every nuclear conic in contained in $q+1$ tangent hyperplanes of $\mathcal{U}$ and every non-nuclear conic is contained in one tangent hyperplane of $\mathcal{U}$. Consequently, a nuclear conic is not contained in any hyperbolic quadrics or elliptic quadrics of $\mathcal{U}$, since the $q+1$ hyperplanes containing it are all tangent hyperplanes of $\mathcal{U}$.

$\square$

The final preliminary lemma describes the ways in which a Baer subplane can meet the classical unital in $PG(2,q^2)$.

Lemma 4.3.10 *Let U be the classical unital in $PG(2,q^2)$ and let $\mathcal{B}$ be a Baer subplane of $PG(2,q^2)$, then $\mathcal{B}$ meets U in one point, $q+1$ points of a conic or line of $\mathcal{B}$, or in $2q+1$ points of a line pair of $\mathcal{B}$.*

Proof We use the representation of $PG(2,q^2)$ in $PG(4,q)$ given in Section 1.7. By Corollary 2.2.4 the classical unital is Buekenhout with respect to any secant line and Buekenhout-Metz with respect to any tangent line. Let l be a secant line of U, then in $PG(4,q)$ with l as the line at infinity, U corresponds to a parabolic quadric. By Lemma 4.2.1(ii), all Baer subplanes secant to l correspond to planes of $PG(4,q)$. Any plane of $PG(4,q)$ meets a parabolic quadric in one point, $q+1$ points of a conic or line, or $2q+1$ points of two lines. So in $PG(2,q^2)$, a Baer subplane secant to l meets $\mathcal{U}$ in one point, a conic, a line, or two lines.

If, however, we take a tangent line of U to be our line at infinity, and work in $PG(4,q)$, then U corresponds to an orthogonal cone in $PG(4,q)$. By Lemma 4.2.1(ii) a Baer subplane secant to the line at infinity corresponds to a plane of $PG(4,q)$. Such a plane meets the orthogonal cone corresponding to the unital in either a point, a line, a conic, or two lines.

Thus all the Baer subplanes of $PG(2,q^2)$ meet the unital in one point, $q+1$ points of a conic or a line, or in a line pair.

□

Recall from Theorem 2.3.1 that a non-classical Buekenhout-Metz unital in $PG(2,q^2)$ is Buekenhout-Metz with respect to exactly one tangent line and is not Buekenhout with respect to any secant lines. Hence we cannot use this method to generalize the result to non-classical Buekenhout-Metz unitals in $PG(2,q^2)$.

We are now able to show that the unital U' does contain O'Nan configurations. We do this by constructing a configuration in the classical unital that derives to an O'Nan configuration of U'.

Lemma 4.3.11 *If $q > 5$, the unital U' contains O'Nan configurations. If H is a point of $U' \setminus \ell'_\infty$ and l a secant of U' through H that meets the classical derivation set, then there is an O'Nan configuration of U' that contains H and l.*

Proof We will show that U' contains an O'Nan configuration whose four lines meet ℓ'_∞ in the distinct points A, B, C, D where $A \in \mathcal{D}'$ and $B, C, D \notin \mathcal{D}'$. We prove this by constructing a configuration in the classical unital U in $PG(2,q^2)$ that will derive to an O'Nan configuration of U' in $\mathcal{H}(q^2)$.

The configuration that we construct in U is shown in Figure 4.2. It consists of six lines l_{A_1}, l_{A_2}, l_{A_3}, l_B, l_C, l_D and six points H, J, K, X, Y, Z of U with intersections as shown in Figure 4.2 and such that the line l_* meets ℓ_∞ in the point $*$ where $* \in \{A_1, A_2, A_3, B, C, D\}$ and with $A_1, A_2, A_3 \in \mathcal{D}$ and $B, C, D \notin \mathcal{D}$.

Note that J, K, H, A_1, A_2, A_3 form a quadrangle and so are contained in a unique Baer subplane which contains $\mathcal{D}$ (as A_1, A_2, A_3 are contained in the unique Baer subline $\mathcal{D}$). Hence derivation with respect to $\mathcal{D}$ leaves l_B, l_C and l_D unchanged in $\mathcal{H}(q^2)$ with H, J, K collinear in $\mathcal{H}(q^2)$, giving an O'Nan configuration in U'.

Let U be the classical unital in $PG(2,q^2)$ and $\mathcal{D}$ a derivation set of ℓ_∞ disjoint from U, as above. Let l_{A_1} be a secant line of U that meets $\mathcal{D}$ in the point A_1. Let H

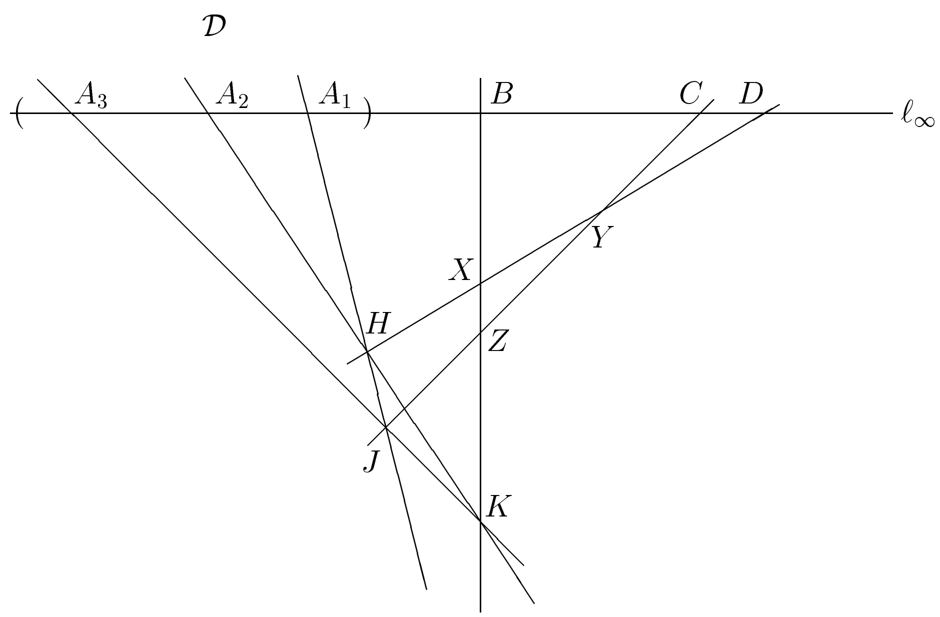

Figure 4.2: *The configuration in the classical unital that will derive to an O'Nan configuration in U'.*

and J be two points of U that lie on l_{A_1}. There is a unique Baer subplane $\mathcal{B}$ that contains H, J and $\mathcal{D}$ since a quadrangle is contained in a unique Baer subplane. Now the Baer subline $\mathcal{B} \cap l_{A_1}$ meets U in 0, 1, 2 or $q+1$ points and since $H, J \in U$ and $A_1 \notin U$ we have $\mathcal{B} \cap l_{A_1}$ meets U in two points. By Lemma 4.3.10, $\mathcal{B}$ meets U in $q+1$ points that form a conic in $\mathcal{B}$, as $\mathcal{D}$ is disjoint from U. Denote the points of the conic $\mathcal{B} \cap U$ by $H, K_1, K_2, \ldots, K_q$ (so $J = K_i$ for some i).

Through H there are q^2 secants of U, let l_D be a secant through H that meets ℓ_∞ in the point $D \notin \mathcal{D}$. Label the points of U on l_D by $H, Y_1, \ldots, Y_q$, then the lines K_jY_i, $i, j = 1, \ldots, q$, each contain two points of U and hence are secant to U.

We want to show that for some $i \neq j$ and $m \neq n$ the secants K_iY_m and K_jY_n meet in a point Z of U with $K_iY_m \cap \ell_\infty \notin \mathcal{D}$ and $K_jY_n \cap \ell_\infty \notin \mathcal{D}$. The configuration containing the points H, K_i, K_j, Y_m, Y_n, Z is the required configuration of U that will derive to an O'Nan configuration of U'.

In order to complete the construction of the configuration we will use the Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$ taking the line $HD = l_D$ as the line at infinity. Note that by Corollary 2.2.4, the classical unital is Buekenhout with respect to any secant line. So in $PG(4, q)$, U corresponds to a parabolic quadric $\mathcal{U}$ that meets the spread of Σ_∞ in the regulus $\mathcal{R} = \{h, y_1, \ldots, y_q\}$. If l is a secant of U that meets $l_D \cap U$, then $\bar{l} = l \cap U$ is a Baer subline of $PG(2, q^2)$ and corresponds to a line of $PG(4, q)$ that meets Σ_∞ in a point of $\mathcal{R}$.

The line l_D is tangent to $\mathcal{B}$ as $D \notin \mathcal{D}$, so in $PG(4, q)$, $\mathcal{B}$ corresponds to a ruled cubic surface (as in Section 1.7). By Lemma 4.3.8, the K_i form a q-arc of a three dimensional subspace Σ in $PG(4, q)$.

If l is a secant of U, let $\bar{l}$ denote the $q+1$ points of $l \cap U$. In $PG(4, q)$, let $\overline{K_i H} \cap h = H_i$, $i = 1, \ldots, q$, and let $\overline{K_i Y_j} \cap y_j = Y_{ji}$, $i, j = 1, \ldots, q$.

We now show that the set of points $\mathcal{C}_1 = \{H_1, Y_{11}, \ldots, Y_{q1}\}$ forms a conic in Σ_∞ and hence that $K_1\mathcal{C}_1$ is a conic cone. If the three points H_1, Y_{11}, Y_{21} are collinear, then the lines $K_1H_1, K_1Y_{11}, K_1Y_{21}$ are contained in a plane of $PG(4, q)$ that meets $\mathcal{U}$ in $3q + 1$ points which is not possible. Thus $\mathcal{C}_1$ is a set of $q + 1$ points, no three collinear.

Now consider the three dimensional subspace Σ_1 spanned by the lines K_1H_1, K_1Y_{11}, K_1Y_{21}. It meets $\mathcal{U}$ in either a hyperbolic quadric, an elliptic quadric or a conic cone, hence Σ_1 meets $\mathcal{U}$ in a conic cone whose vertex is K_1. The only lines of $PG(4, q)$ through K_1 that are secant lines of $\mathcal{U}$ are those that meet a line of $\mathcal{R}$. Hence $K_1\mathcal{C}_1$ is a conic cone and $\mathcal{C}_1$ forms a conic of the plane $\Sigma_1 \cap \Sigma_\infty$.

Similarly $\mathcal{C}_i = \{H_i, Y_{1i}, \ldots, Y_{qi}\}$ is a conic for each $i = 1, \ldots, q$, and $K_i\mathcal{C}_i$ forms a conic cone of $\mathcal{U}$. We denote the three dimensional subspace containing the conic cone $K_i\mathcal{C}_i$ by Σ_i, $i = 1, \ldots, q$.

Recall that Σ is the hyperplane of $PG(4, q)$ containing the K_i. Suppose $\Sigma_1 = \Sigma$, then $q - 1$ of the lines of the cone $K_1\mathcal{C}_1$ are $K_1K_2, \ldots, K_1K_q$. In $PG(2, q^2)$, this means that $q - 1$ of the lines $K_1Y_1, \ldots, K_1Y_q$ meet $\mathcal{D}$.

Let $\mathcal{D} = \{T_0, \ldots, T_q\}$ with $H \in K_1T_0$ and consider the lines $K_1T_1, \ldots, K_1T_q$. Suppose that $HD = l_1$ meets y of these lines $K_1T_1, \ldots, K_1T_y$ in a point of U, that is, y of the lines K_1Y_j meet ℓ_∞ in a point of $\mathcal{D}$. Now any other secant line l_2 of U through H can meet at most one of the lines $K_1T_1, \ldots, K_1T_y$ in a point of U, otherwise we would have an O'Nan configuration in U. So l_2 meets at most $q - y + 1$ of the lines $K_1T_1, \ldots, K_1T_q$ in a point of U.

So if we pick any other secant of U through H, we can ensure that $\Sigma_1 \neq \Sigma$. By excluding at most q secants of U through H we can ensure that $\Sigma_i \neq \Sigma$, $i = 1, \ldots, q$. There are $q^2 - q - 1$ possibilities for D, as $D \notin \mathcal{D}$, so there are enough choices left for D if $q^2 > 2q + 1$; that is, if $q > 2$.

So Σ_1 meets Σ in a plane that contains at most three of the K_i, since no four of the K_i are coplanar. By Lemma 4.3.9, if q is even, the $\mathcal{C}_i$ are all distinct and if q is odd, a given $\mathcal{C}_i$ is distinct from at least $q - 2$ of the $\mathcal{C}_i$'s. Hence if q is even, we can pick $K_i \notin \Sigma_1$ with $\mathcal{C}_i \neq \mathcal{C}_1$ for $i = 2, \ldots, q - 2$ (since two of the K_i may lie in Σ_1). If q is odd, we can pick $K_i \notin \Sigma_1$ with $\mathcal{C}_i \neq \mathcal{C}_1$ for $i = 2, \ldots, q - 3$ (since two of the K_i may lie in Σ_1 and one of the $\mathcal{C}_i$ may equal $\mathcal{C}_1$). Thus if $q \geq 4$, we can pick Σ_1 and Σ_2 so that $\mathcal{C}_1 \neq \mathcal{C}_2$ and $K_2 \notin \Sigma_1$.

Let α_{12} be the plane $\Sigma_1 \cap \Sigma_2$. We investigate how α_{12} meets the conic cones $K_1\mathcal{C}_1$ and $K_2\mathcal{C}_2$ by looking at how it meets $\mathcal{U}$. Since $\mathcal{U}$ and $K_i\mathcal{C}_i$ are quadrics, a plane must meet them in a quadric; that is, in a point, a line, a conic or two lines. We list the four possibilities explicitly for $\alpha_{12} \cap K_2\mathcal{C}_2$; the same possibilities occur for $\alpha_{12} \cap K_1\mathcal{C}_1$.

(a) α_{12} meets $K_2\mathcal{C}_2$ in the vertex K_2,

(b) α_{12} meets $K_2\mathcal{C}_2$ in a line through K_2,

(c) α_{12} meets $K_2\mathcal{C}_2$ in a conic and $K_2 \notin \alpha_{12}$,

(d) α_{12} meets $K_2\mathcal{C}_2$ in two lines through K_2.

Now since $K_2 \notin \Sigma_1$, possibilities (a), (b) and (d) cannot occur for $\alpha_{12} \cap K_2\mathcal{C}_2$, thus $\alpha_{12} \cap K_2\mathcal{C}_2$ is a conic and α_{12} meets $\mathcal{U}$ in a conic. Hence α_{12} meets $K_1\mathcal{C}_1$ in a conic or the vertex K_1. If $K_1 \in K_2\mathcal{C}_2$, then $K_1 \in K_2Y_{i2}$ for some i. However, there is only one line of $\mathcal{U}$ from K_1 to y_i, so $K_2Y_{i2} \in \alpha_{12}$, a contradiction. Thus α_{12} meets $K_1\mathcal{C}_1$ in the conic $\alpha_{12} \cap \mathcal{U}$.

We have deduced that every line of the cone $K_1\mathcal{C}_1$ meets a line of the cone $K_2\mathcal{C}_2$. At most two of these intersections occur in Σ_∞ since α_{12} meets Σ_∞ in a line which meets $\mathcal{C}_i$ in at most two points, hence $\mathcal{C}_1$ and $\mathcal{C}_2$ have at most two points in common. If $H_1 \neq H_2$, then one of the points of $\mathcal{U} \cap \alpha_{12}$ lies in K_1H_1 and one lies in K_2H_2, since K_1H_1 does not meet K_2H_2. Thus we have at least $q-3$ pairs of lines of $\mathcal{U}$, K_1Y_{n1} and K_2Y_{m2}, $n \neq m$, that meet in a point of $PG(4,q) \setminus \Sigma_\infty$.

The line K_1Y_{n1} is contained in a unique plane γ about y_m. Recall that y_m corresponds to the point Y_m in $PG(2,q^2)$ and γ corresponds to the line of $PG(2,q^2)$ through K_1 and Y_m. Therefore, in $PG(2,q^2)$, we have at least $q-3$ pairs of secants of U, K_1Y_n and K_2Y_m, $n \neq m$, that meet in a point Z of U. In order to complete the proof that we have constructed the required configuration in the classical unital of $PG(2,q^2)$, we need to ensure that for one of these pairs both the lines K_1Y_n and K_2Y_m are disjoint from $\mathcal{D}$.

Suppose that x of the lines $K_1Y_1, \ldots, K_1Y_q$ meet $\mathcal{D}$, that is, $HD = l_1$ meets x of the lines $K_1T_1, \ldots, K_1T_q$ in a point of U (recall that K_1H meets $\mathcal{D}$ in T_0). As before, if l_2 is a different secant line of U through H, then l_1 and l_2 can meet at most one common $K_1T_1, \ldots, K_1T_q$ in a point of U, otherwise we have an O'Nan configuration in U. If l_1 meets none of the $K_1T_1, \ldots, K_1T_q$, then we retain l_1. Otherwise l_1 meets K_1T_i for some i. There are q other secants of U through H that contain a point of $K_1T_i \cap U$, we label them $l_2, \ldots, l_{q+1}$. In the worst case, each l_k meets exactly one of the K_1T_j, $j \neq i$, in point of U. However, there are $q+1$ lines l_i and only $q-1$ lines K_1T_j, $j \neq i$, thus at least one of the l_k meets K_1T_i in a point of U, and no further K_1T_j, $j \neq i$, in a point of U. Thus by excluding at most $q-1$ choices of a secant line through H, we can ensure that at most one of the lines $K_1Y_1, \ldots, K_1Y_q$ meets $\mathcal{D}$. We have already excluded at most q choices for D, there are q^2-q-1 possibilities for D, so if $q^2 > 3q$, that is, $q > 3$, there are enough choices left for D.

In order to ensure that at least one of the above $q-3$ pairs K_1Y_n, K_2Y_m that meet in a point of U are disjoint from $\mathcal{D}$, it suffices to show that at least two of the $q-3$ lines K_2Y_m are disjoint from $\mathcal{D}$ (since at most one of the $q-3$ lines K_1Y_n meets $\mathcal{D}$). Thus we need at least five of the lines $K_2Y_1, \ldots, K_2Y_q$ disjoint from $\mathcal{D}$. Hence we need $q \geq 5$.

Suppose $x < 5$ of the lines $K_2Y_1, \ldots, K_2Y_q$ are disjoint from $\mathcal{D}$, so $q-x$ of them meet $\mathcal{D}$. If $q-3 > 2$, we can pick $K_3 \notin \Sigma_1$ with $\mathcal{C}_3 \neq \mathcal{C}_1$. Repeating the above argument gives $q-3$ pairs K_1Y_j, K_3Y_k, $j \neq k$, that meet in a point of U. Now if K_2Y_i meets $\mathcal{D}$, then K_3Y_i cannot meet $\mathcal{D}$, otherwise $Y_i \in \mathcal{B}$ which is a contradiction. Thus, in the worst case, exactly $\frac{q}{2}$ of the $K_2Y_1, \ldots, K_2Y_q$ meet $\mathcal{D}$ and $\frac{q}{2}$ of the $K_3Y_1, \ldots, K_3Y_q$ meet $\mathcal{D}$. So if $\frac{q}{2} \geq 5$, that is, $q \geq 10$, we have two lines that meet in a point of U and are disjoint from $\mathcal{D}$, hence we have constructed the required configuration in U when $q \geq 10$.

If $q-3 > 3$ with q odd, or $q-2 > 3$ with q even, that is $q \geq 6$, then we can pick $K_4 \notin \Sigma_1$ and $\mathcal{C}_4 \neq \mathcal{C}_1$. The above argument gives $q-3$ pairs K_1Y_j, K_4Y_k, $j \neq k$, that meet in a point of U. In the worst case, exactly $\frac{q}{3}$ of the $K_iY_1, \ldots, K_iY_q$, $i = 2, 3, 4$ meet $\mathcal{D}$. So if $q - \frac{q}{3} \geq 5$, that is, $q \geq 8$, we have a pair K_1Y_j, K_iY_k that meet in a point of U and are disjoint from $\mathcal{D}$. Hence if $q \geq 8$, we can construct the required configuration in the classical unital.

If $q = 7$, q is not divisible by three, so in the worst case, we can pick two of the $K_2Y_1, \ldots, K_2Y_q$ meeting $\mathcal{D}$ and so $7 - 2 = 5$ of them are disjoint from $\mathcal{D}$, giving the required configuration.

Therefore, if $q > 5$, we have constructed the required configuration in the classical unital that will derive to an O'Nan configuration of U.

We have shown that for any point H of $U \setminus \ell_\infty$ and for any Baer subplane $\mathcal{B}$ of $PG(2, q^2)$ containing $\mathcal{D}$, H and $q+1$ points of U, we can construct the required configuration in the classical unital through H and two other points of $U \cap \mathcal{B}$. Thus in U', for any point H of $U' \setminus \ell'_\infty$ and secant line l of U' through H that meets the classical derivation set $\mathcal{D}'$, the above configuration in the classical unital derives to an O'Nan configuration of U' containing H and l.

□

As U' contains O'Nan configurations and U does not contain O'Nan configurations, we obtain the immediate corollary that the designs U and U' are not isomorphic.

Corollary 4.3.12 *The unital U' is not isomorphic to the classical unital U.*

We can use our results in the same way as in Case 1 to discuss the embeddability of the classical unital in $\mathcal{H}(q^2)$ and the embeddability of our new unital in the Desarguesian plane. The proof of the next corollary follows that given for the corresponding corollaries in Case 1.

Corollary 4.3.13 *1. The classical unital of order $q > 5$ is not embeddable in $\mathcal{H}(q^2)$ as a Buekenhout unital with respect to ℓ'_∞ containing no point of the classical derivation set.*

2. Let U' be a Buekenhout unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$, $q > 5$, that contains no point of the classical derivation set $\mathcal{D}'$, then U' is not embeddable in $PG(2, q^2)$ as a Buekenhout unital.

By examining the occurrence of O'Nan configurations in our unital and Grünig's unital described in Case 1, we show that the two unitals are non-isomorphic. Therefore the class of unitals investigated in this case have not previously been studied in detail.

Theorem 4.3.14 *The Buekenhout unital obtained in Case 2 of Theorem* 4.3.1 *is not isomorphic to the Buekenhout unital obtained in Case 1 when $q \geq 3$*

Proof We show the two unitals are non-isomorphic by examining the frequency distribution of O'Nan configurations in them. Let U_1' be the unital of $\mathcal{H}(q^2)$ obtained in Case 1 of Theorem 4.3.1, and U_2' the unital of $\mathcal{H}(q^2)$ obtained in Case 2.

Let $\bar{l} = U_1' \cap \ell_\infty'$, Grünig showed that (i) U_1' contains no O'Nan configurations with two or more points on $\bar{l}$ and (ii) for any point $P \in \bar{l}$, if l_1, l_2 are lines of U_1' through P and l_3 a line of U that meets l_1 and l_2, then there exists an O'Nan configuration of U containing l_1, l_2 and l_3.

Let U be a unital and $\bar{l}$ a line of U. We call $\bar{l}$ a **G-O axis** (Grünig-O'Nan axis) of U if it satisfies (i) and (ii). We will show that U_2' does not contain a G-O axis.

There are three possibilities for such an axis in U_2': $U_2' \cap \ell_\infty'$, a secant line of U_2' that meets ℓ_∞' in a point of U_2', a secant line of U_2' that meets ℓ_∞' in a point not in U_2'.

Let P be a point of $U_2' \cap \ell_\infty'$. Let l_1, l_2 be secants of U_2' through P. In $PG(4,q)$, $\overline{l_1} = l_1 \cap U_2'$ and $\overline{l_2} = l_2 \cap U_2'$ are lines of $PG(4,q)$ that meet the line p of the spread $\mathcal{S}'$. Choose l_1 and l_2 such that $\overline{l_1}$ and $\overline{l_2}$ meet in a point of p in $PG(4,q)$. By Lemma 4.3.7, there is no O'Nan configuration that contains l_1 and l_2. This violates (ii), thus $U_2' \cap \ell_\infty'$ is not a G-O axis of U_2'. The same example shows that any other line of U_2' through P cannot be a G-O axis of U_1'.

Let l be a secant line of U_2' that meets ℓ_∞' in a point not in U_2'. Let Q be a point of U_2' on l and let l_1 and l_2 be secants of U_2' through Q that contain a point of $U_2' \cap \ell_\infty'$. There is no O'Nan configuration of U_2' containing l_1, l_2 and $U_2' \cap \ell_\infty'$ as U_2' contains no O'Nan configurations with three vertices on ℓ_∞' (Lemma 4.3.7). Therefore, l does not satisfy (ii) and cannot be a G-O axis of U_2'.

Hence no line of U_2' is a G-O axis and so U_2' is not isomorphic to U_1'.

□

As in Case 1, we obtain a result that describes what parts of the classical unital we can remove and put back in a different way so as to create a non-isomorphic unital. In this case we get the following corollary.

Corollary 4.3.15 *Let U be the classical unital. Consider the incidence structure that consists of a given secant line of U and all the lines of U that meet it. This structure can be completed to a unital in at least two distinct ways, one embeddable in $PG(2,q^2)$ and one embeddable in $\mathcal{H}(q^2)$.*

4.3.3 Case 3.

Let U be the classical unital secant to ℓ_∞ in $PG(2,q^2)$. Derive the plane with respect to a derivation set that contains one or two points of the unital. Let U' be the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.

Note that if $\mathcal{D}$ is a derivation set that contains greater than zero and less than $q+1$ points of U, then $\mathcal{D}$ contains exactly one or two points of U. This is because U meets ℓ_∞ in a Baer subline, $\mathcal{D}$ forms a Baer subline of ℓ_∞, and two Baer sublines meet in 0, 1, 2 or $q+1$ points.

Lemma 4.3.16 *The resulting set U' in $\mathcal{H}(q^2)$ is not a unital of $\mathcal{H}(q^2)$.*

Proof Suppose $\mathcal{D}$ contains exactly one point of U. Let $U \cap \ell_\infty = \{T_0, S_1, \ldots, S_q\}$ and suppose we derive with respect to the set $\mathcal{D} = \{T_0, \ldots, T_q\}$ where the T_i are distinct from the S_j. In $PG(4,q)$, let $\mathcal{S}$ be the spread of Σ_∞ that generates $PG(2,q^2)$ and let $\mathcal{R} = \{t_0, \ldots, t_q\}$ be the regulus corresponding to $\mathcal{D}$ with complementary regulus $\mathcal{R}' = \{r_0, \ldots, r_q\}$. Derivation of $PG(2,q^2)$ is equivalent to replacing the spread $\mathcal{S}$ by the spread $\mathcal{S}'$ that consists of the lines of $\mathcal{S}$ not in $\mathcal{R}$ and the lines of $\mathcal{R}'$.

Now t_0 meets each of the r_i, and t_0 is completely contained in $\mathcal{U}$, the parabolic quadric corresponding to U, so each of the r_i contains a point of $\mathcal{U}$. Hence U' meets ℓ'_∞ in $2q+1$ points, $R_0, \ldots, R_q, S_1, \ldots, S_q$, and U' is not a unital of $\mathcal{H}(q^2)$.

For any point V in $U' \cap \ell'_\infty$, there is a secant line of U' containing V. Hence removing one of the points of $U' \cap \ell'_\infty$ creates a q-secant in U'. Thus we cannot reduce the number of points of $U' \cap \ell'_\infty$ and get a unital of $\mathcal{H}(q^2)$.

Similarly, if U has two points in common with the derivation set, then U' has $2q$ points in common with ℓ'_∞ in $\mathcal{H}(q^2)$. Thus U' is not a unital of $\mathcal{H}(q^2)$.

□

For certain derivation sets of $PG(2, q^2)$, we can make a stronger statement. If we consider the incidence structure isomorphic to the affine part of U', *aff*U', then we cannot complete it to a unital. In $PG(4, q)$, $\mathcal{U}$ contains $q - 1$ lines that meet t_0 in a given point. Any two of these will span a plane β of $PG(4, q)$ that contains no further point of $\mathcal{U}$ and meets $q + 1$ lines $t_0, l_1, \ldots, l_q$ of the spread $\mathcal{S}$. The corresponding set of points $T_0, L_1, \ldots, L_q$ of $PG(2, q^2)$ form a derivation set of ℓ_∞ that meets U in greater than zero and less than $q+1$ points. If we derive $PG(2, q^2)$ with respect to this derivation set, then the line of $\mathcal{H}(q^2)$ corresponding to β meets *aff*U' in $2q$ points. Hence a structure isomorphic to *aff*U' cannot be completed to a unital.

The next result shows that the only Buekenhout unitals in $\mathcal{H}(q^2)$, $q > 3$, are those that either contain the classical derivation set or are disjoint from it.

Lemma 4.3.17 *The Hall plane of order q^2, $\mathcal{H}(q^2)$, $q > 3$, contains no Buekenhout unitals with respect to ℓ'_∞ that meet the classical derivation set in less than $q + 1$ and greater than zero points.*

Proof Let $\mathcal{S} = \{s_1, \ldots, s_{q^2-q}, t_0, \ldots, t_q\}$ be the regular spread of Σ_∞ that generates $PG(2, q^2)$, then $\mathcal{S}' = \{s_1, \ldots, s_{q^2-q}, r_0, \ldots, r_q\}$ generates $\mathcal{H}(q^2)$, where $r_0, \ldots, r_q$ is the complementary regulus of $t_0, \ldots, t_q$. So $r_0, \ldots, r_q$ correspond to the classical derivation set of $\mathcal{H}(q^2)$. In order for a Buekenhout unital to exist in $\mathcal{H}(q^2)$ with less than $q + 1$ and greater than zero points of the classical derivation set there must be a regulus in the spread $\mathcal{S}'$ that contains less than $q + 1$ and greater than zero of the r_i. Two reguli meet in 0, 1, 2 or $q+1$ lines, so we require a regulus in $\mathcal{S}'$ that contains one or two of the r_i.

Suppose the lines $s_0, \ldots, s_{q-1}, r_0$ form a regulus in $\mathcal{S}'$. Any three skew lines of $PG(3,q)$ define a unique regulus, thus if $q-1 \geq 2$, that is, $q \geq 3$, then s_0, s_1, s_2 are contained in a unique regulus of Σ_∞. Since the spread $\mathcal{S}$ is regular, this regulus is contained in $\mathcal{S}$, thus r_0 is not in it. Hence there is no regulus containing three of the s_i and one of the r_i. Thus if $q > 2$, there are no Buekenhout unitals in $\mathcal{H}(q^2)$ that contain exactly one point of the classical derivation set.

Similarly, suppose the lines $s_0, \ldots, s_{q-2}, r_0, r_1$ form a regulus of $\mathcal{S}'$. If $q-2 \geq 2$, that is, $q \geq 4$, then s_0, s_1, s_2 are contained in a unique regulus which is in $\mathcal{S}$ and so does not contain any of the r_i. Hence if $q > 3$, there are no reguli in $\mathcal{S}'$ that contain exactly two of the r_i and so there are no Buekenhout unitals in $\mathcal{H}(q^2)$ that contain exactly two points of the classical derivation set. If $q = 3$, then s_0, s_1, r_0, r_1 may form a regulus of $\mathcal{S}'$. If so, then there is a parabolic quadric containing them which corresponds to a Buekenhout unital of $\mathcal{H}(q^2)$ that meets the classical derivation set in two points.

□

As in the previous cases, we obtain a corollary regarding the non-embeddability of the classical unital in $\mathcal{H}(q^2)$.

Corollary 4.3.18 *The classical unital is not embeddable as a Buekenhout unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$, $q > 3$, containing less than $q+1$ and greater than zero points of the classical derivation set.*

Proof This follows immediately from Lemma 4.3.17 which shows that the Hall plane of order q^2, $q > 3$, contains no Buekenhout unitals with respect to ℓ'_∞ that contain less than $q+1$ and greater than zero points of the classical derivation set.

□

4.3.4 Case 4.

Let U be the classical unital in $PG(2,q^2)$ that is tangent to ℓ_∞. Derive the plane with respect to a derivation set that contains the point $U \cap \ell_\infty$. Let U' be the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.

We begin by showing that the set U' is not a unital of $\mathcal{H}(q^2)$.

Lemma 4.3.19 *The point set U' does not form a unital of $\mathcal{H}(q^2)$.*

Proof Let the derivation set be $\mathcal{D} = \{T_0, \ldots, T_q\}$. In $PG(4,q)$, deriving $PG(2,q^2)$ with respect to $\mathcal{D}$ is equivalent to replacing the regulus $t_0, \ldots, t_q$ in $\mathcal{S}$ by its opposite regulus $r_0, \ldots, r_q$. Let U meet ℓ_∞ in the point T_0, so $\mathcal{U}$ contains the line t_0. In $PG(4,q)$, each of the r_i meets t_0, so in $\mathcal{H}(q^2)$, U' meets ℓ'_∞ in $q+1$ points $R_0, \ldots, R_q$. Hence U' has q^3+q+1 points and is not a unital of $\mathcal{H}(q^2)$.

If V is any point of $U' \cap \ell'_\infty$, then there is a secant line of U' containing V. Hence removing a point of $U' \cap \ell'_\infty$ creates a q-secant in U'. Thus we cannot reduce the number of points of $U' \cap \ell'_\infty$ and get a unital of $\mathcal{H}(q^2)$.

□

Although the point set U' is not a unital of $\mathcal{H}(q^2)$, there are Buekenhout-Metz unitals in $\mathcal{H}(q^2)$ with respect to a point of the classical derivation set. Such a unital can be based on the same ovoid as U in the following way. If $\mathcal{O}$ is an ovoid of $\mathcal{U}$, then $\mathcal{O}$ meets the line t_0 of $\mathcal{S}$ in exactly one point. It follows that $\mathcal{O}$ meets $\mathcal{S}'$ in exactly one point X of the line r_i, for some i. For any point $V \neq X$ on r_i, we can construct an ovoidal cone with base $\mathcal{O}$ and vertex V. This cone contains exactly one line of $\mathcal{S}'$ and corresponds to a Buekenhout-Metz unital of $\mathcal{H}(q^2)$ with respect to a point of the classical derivation set.

As in Case 3, for certain derivation sets of $PG(2,q^2)$, the design isomorphic to the affine part of U', *aff*U', cannot be completed to a unital. In $PG(4,q)$, $\mathcal{U}$ consists of q^2 lines of $PG(4,q)\backslash\Sigma_\infty$ through a vertex V of t_0. Any two of these lines span a plane β that meets $q+1$ lines of the spread $\mathcal{S}$. If we derive $PG(2,q^2)$ with respect to the derivation set corresponding to these $q+1$ lines, then β corresponds to a line of $\mathcal{H}(q^2)$ that meets $\mathit{aff}U'$ in $2q$ points.

4.3.5 Case 5.

Let U be the classical unital in $PG(2,q^2)$ that is tangent to ℓ_∞. Derive the plane using a derivation set that is disjoint from U. Let U' be the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.

In this section we show that the set U' forms a unital of $\mathcal{H}(q^2)$ that is not isomorphic to the classical unital or to the unital of $\mathcal{H}(q^2)$ described in Case 1. Note that since $\mathcal{D}$ is disjoint from U, the point $T = U \cap \ell_\infty$ in $PG(2,q^2)$ has a natural correspondence to the point $T = U' \cap \ell'_\infty$ in $\mathcal{H}(q^2)$.

Lemma 4.3.20 *The point set U' forms a unital of $\mathcal{H}(q^2)$.*

Proof Note that U' has the same number of points as U, q^3+1, and ℓ'_∞ meets U' in exactly one point. We work in $PG(4,q)$ and let $\mathcal{U}$ be the ovoidal cone that corresponds to U and to U'. A plane of $PG(4,q)$ meets an ovoidal cone in 1, $q+1$ or $2q+1$ points. Let α be a plane of $PG(4,q)$ that corresponds to a line of $\mathcal{H}(q^2)$. If α meets the ovoidal cone $\mathcal{U}$ in 1 or $q+1$ points, then the line of $\mathcal{H}(q^2)$ corresponding to α meets U' in 1 or $q+1$ points. If α meets $\mathcal{U}$ in $2q+1$ points, they must lie on two lines and so $\alpha \cap \Sigma_\infty$ contains a point of $\mathcal{U}$. Thus α contains t, the line of the spread $\mathcal{S}'$ that corresponds to T, and in $\mathcal{H}(q^2)$, the line corresponding to α meets the set U' in exactly $q+1$ points. Hence every line of $\mathcal{H}(q^2)$ meets U' in 1 or $q+1$ points and U' is a unital of $\mathcal{H}(q^2)$.

□

In fact we have shown that U' is a Buekenhout-Metz unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$. Note, this class of unitals was found independently by Blokhuis and Szőnyi [27]. We next show that the unital U' contains O'Nan configurations.

Lemma 4.3.21 *The unital U' contains O'Nan configurations if $q > 5$.*

Proof The proof of this involves constructing a configuration in the classical unital in $PG(2,q^2)$ that derives to an O'Nan configuration of U' in $\mathcal{H}(q^2)$. This configuration is identical to the one constructed in Lemma 4.3.11. The proof of this lemma used the fact that the classical unital is Buekenhout with respect to any secant line and did not use any properties of $U \cap \ell_\infty$. Therefore the proof of Lemma 4.3.11 applies directly to this case.

$\square$

Note that by Lemma 3.3.1, since U' is Buekenhout-Metz with respect to T, it contains no O'Nan configurations with T as vertex. However, the proof of Lemma 4.3.11 does not exclude the possibility of an O'Nan configuration with T as a vertex Suppose $T = Y_q = U \cap \ell_\infty$, we show that the construction of an O'Nan configuration in the proof of Lemma 4.3.11 does not give an O'Nan configuration in U' with T as a vertex in this case.

We use the same notation as in the proof of Lemma 4.3.11. Let $\mathcal{U}$ be the ovoidal cone with vertex V that corresponds to U. We use the Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$ with ℓ_∞ as the line at infinity. Now $\mathcal{B}$ is an affine Baer subplane and so corresponds to a plane of $PG(4,q) \setminus \Sigma_\infty$ that meets the $q+1$ lines $t_0, \ldots, t_q$ of $\mathcal{S}$ that correspond to $\mathcal{D}$. The points $H, K_1, \ldots, K_q$ form a conic $\mathcal{C}$ of $\mathcal{B}$ and so $V\mathcal{C}$ is a conic cone of $\mathcal{U}$. (Note that if U is Buekenhout with respect to ℓ_∞, then the lines of $\mathcal{U}$ from K_i to a given line of $\mathcal{S}$ do not form a conic cone; they span a four dimensional space. Therefore this argument does not exclude the possibility of an O'Nan configuration with a vertex on ℓ'_∞ in Case 2.)

The three dimensional subspace Σ that contains the conic cone $V\mathcal{C}$ meets Σ_∞ in a plane that contains $\mathcal{B} \cap \Sigma_\infty$ and so contains one of the lines t_i of $\mathcal{S}$. There are q planes of Σ through t_i that do not lie in Σ_∞. Each plane meets $V\mathcal{C}$ in a conic that contains exactly one of $H, Y_1, \ldots, Y_{q-1}$ and $0, 1$ or 2 of the $K_1, \ldots, K_q$. Hence if the plane containing the points K_j and Y_m and a line of $\mathcal{S}$ meets the line K_iV, then this plane contains the line t_i of $\mathcal{S}$.

Therefore, in $PG(2,q^2)$, if the line K_iY_q meets the line K_jY_m in a point of U, then $K_jY_m \cap \ell_\infty = T_i \in \mathcal{D}$. Hence this construction does not produce an O'Nan configuration with one vertex on ℓ'_∞.

As an immediate corollary to the fact that U' contains O'Nan configurations, the designs U and U' are not isomorphic.

Corollary 4.3.22 *The unital U' is not isomorphic to the classical unital if $q > 5$.*

We next show that the class of unitals obtained in this case is not isomorphic to the unitals obtained by Grünig and described in Case 1.

Lemma 4.3.23 *The unital U' of $\mathcal{H}(q^2)$, $q > 5$, is not isomorphic to Grünig's unital of $\mathcal{H}(q^2)$ described in Case 1.*

Proof Grünig constructed an O'Nan configuration through every point of his unital. Lemma 3.3.1 shows that there are no O'Nan configurations through the vertex T of U'. Therefore, the two unitals are not isomorphic.

$\square$

4.3.6 Deriving non-classical Buekenhout-Metz unitals of $\mathbf{PG(2,q^2)}$

The results obtained in Cases 4 and 5 can be generalized to the case where U is a non-classical Buekenhout-Metz unital with respect to ℓ_∞ in $PG(2,q^2)$.

Lemma 4.3.24 *Let U be a Buekenhout-Metz unital with respect to ℓ_∞ in $PG(2,q^2)$. Derive the plane with respect to the derivation set $\mathcal{D}$ of ℓ_∞ and let U' be the set of points in $\mathcal{H}(q^2)$ corresponding to the point set of U. If $U \cap \ell_\infty \notin \mathcal{D}$, then U' is a Buekenhout-Metz unital with respect to ℓ'_∞ of $\mathcal{H}(q^2)$. If $U \cap \ell_\infty \in \mathcal{D}$, then U' is not a unital of $\mathcal{H}(q^2)$.*

Proof The proofs of Lemmas 4.3.19 and 4.3.20 in Cases 4 and 5 did not rely on the unital being classical, thus they apply directly to the general case of U a non-classical Buekenhout-Metz unital with respect to ℓ_∞.

$\square$

The next result shows that if U' is a unital in $\mathcal{H}(q^2)$ derived from a non-classical orthogonal Buekenhout-Metz unital U in $PG(2,q^2)$, then U' contains an O'Nan configuration if U does.

Lemma 4.3.25 *Let U be a non-classical orthogonal Buekenhout-Metz unital in $PG(2,q^2)$, $q>4$, and derive the plane with respect to a derivation set $\mathcal{D}$ that is disjoint from U. If U contains an O'Nan configuration, then so does the resulting unital U' in $\mathcal{H}(q^2)$.*

Proof Let U be a non-classical orthogonal Buekenhout-Metz unital with respect to the point T of $PG(2,q^2)$ and suppose that U contains an O'Nan configuration. By Lemma 3.3.1, U cannot contain an O'Nan configuration with T as vertex, so let the lines of the O'Nan configuration meet $\ell_\infty \backslash T$ in the points A,B,C,D. Baker and Ebert [11] and Ebert [50] showed that the group of collineations fixing an orthogonal Buekenhout-Metz unital is transitive on the points of $\ell_\infty \setminus T$.

Hence if there exists one O'Nan configuration in U, then through each point of $\ell_\infty \setminus T$ there is a line contained in an O'Nan configuration of U. Hence U contains at least $\frac{q^2}{4}$ O'Nan configurations. We want to prove that one of these O'Nan configurations is disjoint from $\mathcal{D}$. This will be true if there are more O'Nan configurations than there are points of $\mathcal{D}$, that is, if $\frac{q^2}{4} > q+1$. So if $q>4$, U contains an O'Nan configuration that is disjoint from $\mathcal{D}$. The lines of such a configuration are unchanged by derivation with respect to $\mathcal{D}$ and so they are also secant lines of U' in $\mathcal{H}(q^2)$. Thus we have constructed an O'Nan configuration in U'.

$\square$

Hence if an orthogonal Buekenhout-Metz unital of $PG(2,q^2)$ contains an O'Nan configuration, then the derived orthogonal Buekenhout-Metz unitals of $\mathcal{H}(q^2)$ are not isomorphic to the classical unital. It is a long standing conjecture [91] that the only unital with no O'Nan configurations is the classical unital, so the assumption that a non-classical Buekenhout-Metz unital contains O'Nan configurations is reasonable.

We now show that under certain conditions the classical unital is not embeddable as a Buekenhout-Metz unital in $\mathcal{H}(q^2)$.

Corollary 4.3.26 *Suppose the non-classical orthogonal Buekenhout-Metz unitals in $PG(2,q^2)$, $q>4$, contain O'Nan configurations. Then the classical unital of $PG(2,q^2)$, $q>4$, is not embeddable in $\mathcal{H}(q^2)$ as an orthogonal Buekenhout-Metz unital with respect to a point of ℓ'_∞ not in the classical derivation set.*

Proof If the classical unital was embeddable in the Hall plane as an orthogonal Buekenhout-Metz unital disjoint from the classical derivation set, then it must be isomorphic to such a unital of $\mathcal{H}(q^2)$. The above arguments shows that all Buekenhout-Metz unitals in $\mathcal{H}(q^2)$ that are disjoint from the classical derivation set are derivable from a Buekenhout-Metz unital of $PG(2,q^2)$. In Section 4.3.5 we showed that deriving the classical unital tangent to ℓ_∞ and disjoint from the derivation set introduced O'Nan configurations, and in the above lemma we showed that when $q>4$, deriving a non-classical orthogonal Buekenhout-Metz unital preserves an O'Nan configuration. Thus all the orthogonal Buekenhout-Metz unitals in $\mathcal{H}(q^2)$ disjoint from the classical derivation set contain O'Nan configurations and so the classical unital cannot be isomorphic to any of them.

□

Baker and Ebert [11] and Hirschfeld and Szőnyi [65] showed that there are non-classical Buekenhout-Metz unitals in $PG(2,q^2)$ that are a pencil of conics. We can ask whether this pencil of conics can ever derive to a pencil of (q^2+1)-arcs of $\mathcal{H}(q^2)$. When q is odd, a conic of $PG(2,q^2)$ tangent to ℓ_∞ does not derive to an arc of $\mathcal{H}(q^2)$ regardless of the derivation set used, [80]. When q is even, there are cases when a derived conic is an arc of $\mathcal{H}(q^2)$ [88]. However, Ebert [50] showed that when q is even, orthogonal Buekenhout-Metz unitals contain no ovals of $PG(2, q^2)$. Thus an orthogonal Buekenhout-Metz unital that is a pencil of conics of $PG(2, q^2)$ does not derive to a pencil of arcs in $\mathcal{H}(q^2)$.

4.4 The unitals of the Hall plane

The preceding subsections provide a proof of Theorem 4.3.1 which, together with Lemma 4.3.24, describes the effect of derivation on all known unitals in $PG(2, q^2)$.

We proved that the classes of unitals obtained in Cases 2 and 5 are not isomorphic to the classical unital or to the unital described in Case 1 obtained by Grünig. The only infinite class of unitals studied in the Hall plane to date have been those of Case 1. Therefore, our unitals constitute new infinite classes of unitals in the Hall plane.

In addition to these infinite classes of unitals, the Hall plane contains Buekenhout-Metz unitals whose vertices lie in the classical derivation set that are not derivable from a unital of $PG(2, q^2)$, see Case 4. It is not known whether these are isomorphic to one of the above classes, or form another infinite class of unitals in the Hall plane.

We have also partially answered the question of whether the classical unital is embeddable in the Hall plane. In Corollary 4.3.26, we showed that under certain conditions the classical unital is not embeddable as a Buekenhout-Metz unital in $\mathcal{H}(q^2)$. The next corollary shows that the classical unital is not embeddable as a Buekenhout unital in $\mathcal{H}(q^2)$.

Corollary 4.4.1 *The classical unital of $PG(2, q^2)$ is not embeddable as a Buekenhout unital with respect to ℓ'_∞ in the Hall plane of order q^2, $q > 5$.*

Proof We investigated the three possible positions in Cases 1, 2 and 3. In Corollary 4.3.2, we showed that if $q \geq 3$, the classical unital is not embeddable in $\mathcal{H}(q^2)$

as a Buekenhout unital with respect to ℓ'_∞ that contains the classical derivation set. In Corollary 4.3.13, we showed that if $q > 5$, the classical unital is not embeddable in $\mathcal{H}(q^2)$ as a Buekenhout unital with respect to ℓ'_∞ that is disjoint from the classical derivation set.

In Lemma 4.3.17, we showed that $\mathcal{H}(q^2)$, $q > 3$, contains no Buekenhout unitals with respect to ℓ'_∞ that contain greater than zero and less than $q+1$ points of the classical derivation set. Therefore the classical unital is not embeddable in any of the three possible positions in $\mathcal{H}(q^2)$.

$\square$

We can also comment on whether the Buekenhout unitals of $\mathcal{H}(q^2)$ are embeddable in $PG(2, q^2)$.

Corollary 4.4.2 *The Buekenhout unitals with respect to ℓ'_∞ embedded in the Hall plane $\mathcal{H}(q^2)$, $q > 5$, are not embeddable as Buekenhout unitals in $PG(2, q^2)$.*

Proof We investigated the Buekenhout unitals of $\mathcal{H}(q^2)$ in Cases 1, 2 and 3. In Case 3 we showed that $\mathcal{H}(q^2)$, $q > 3$, contains no Buekenhout unitals with respect to ℓ'_∞ that contain greater than zero and less than $q+1$ points of the classical derivation set $\mathcal{D}'$. In Corollary 4.3.3 we showed that in $\mathcal{H}(q^2)$, $q \geq 3$, if U' is a Buekenhout unital with respect to ℓ'_∞ containing the classical derivation set, then U' is not embeddable as a Buekenhout unital in $PG(2, q^2)$. In Corollary 4.3.13 we showed that in $\mathcal{H}(q^2)$, $q > 5$, if U' is a Buekenhout unital with respect to ℓ'_∞ and disjoint from the classical derivation set, then U' is not embeddable as a Buekenhout unital in $PG(2, q^2)$. Hence no Buekenhout unitals of $\mathcal{H}(q^2)$ are embeddable as Buekenhout unitals in $PG(2, q^2)$.

$\square$

We are able to use the results of the previous section together with the results of Chapter 2 to comment on the number of Buekenhout-Metz and Buekenhout unitals in $\mathcal{H}(q^2)$ and their relationship to the unitals of $PG(2, q^2)$.

Corollary 4.4.3 *Let b be a Baer subline of ℓ'_∞ in $\mathcal{H}(q^2)$ that corresponds to a regulus of the spread in the representation of $\mathcal{H}(q^2)$ in $PG(4,q)$. There are $q^4(q-1)$ Buekenhout unitals with respect to ℓ'_∞ in $\mathcal{H}(q^2)$ that contain b. Those which contain zero or $q+1$ points of the classical derivation set are derivable from the classical unital in $PG(2,q^2)$. If $q > 3$, there are no Buekenhout unitals in $\mathcal{H}(q^2)$ that meet the classical derivation set in less than $q+1$ and greater than zero points.*

Through every point T of ℓ'_∞ in $\mathcal{H}(q^2)$ there are the same number of Buekenhout-Metz unitals with respect to T as there are with respect to a point of $PG(2,q^2)$. The Buekenhout-Metz unitals of $\mathcal{H}(q^2)$ that meet ℓ'_∞ in a point not in the classical derivation set are derivable from Buekenhout-Metz unitals in $PG(2,q^2)$, the others are not.

Proof We work in $PG(4,q)$, let the spreads $\mathcal{S}$ and $\mathcal{S}'$ generate $PG(2,q^2)$ and $\mathcal{H}(q^2)$ respectively. Let b be a Baer subline of ℓ'_∞ in $\mathcal{H}(q^2)$ that corresponds to a regulus of the spread $\mathcal{S}'$ in $PG(4,q)$. In Corollary 2.2.7 we showed that there are $q^4(q-1)$ Buekenhout unitals with respect to ℓ'_∞ in $\mathcal{H}(q^2)$ that contain b. Cases 1 and 2 showed that the Buekenhout unitals of $\mathcal{H}(q^2)$ that contain $q+1$ or zero points of the classical derivation set are derivable from the classical unital of $PG(2,q^2)$. Case 3 showed that if $q > 3$, there are no Buekenhout unitals in $\mathcal{H}(q^2)$ that meet the classical derivation set in less than $q+1$ and greater than zero points.

In Case 5, we showed that in $\mathcal{H}(q^2)$, every Buekenhout-Metz unital with respect to ℓ'_∞ disjoint from the classical derivation set is derivable from a Buekenhout-Metz unital of $PG(2,q^2)$. Thus there are the same number of Buekenhout-Metz unitals in $\mathcal{H}(q^2)$ with respect to a point disjoint from the classical derivation set as there are Buekenhout-Metz unitals with respect to a point of $PG(2,q^2)$.

In Case 4 we showed that if U in $\mathcal{H}(q^2)$ is a Buekenhout-Metz unital with respect to a point of the classical derivation set, then U is not derivable from a unital of $PG(2,q^2)$. However, there are the same number of orthogonal cones containing a line of $\mathcal{S}$ as there are containing a line of $\mathcal{S}'$. Hence there are the same number of

Buekenhout-Metz unitals in $\mathcal{H}(q^2)$ with respect to a point of the classical derivation set as there are Buekenhout-Metz unitals with respect to a point in $PG(2, q^2)$.

□

4.5 Unitals in other translation planes

Let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ be k disjoint derivation sets of ℓ_∞ in $PG(2, q^2)$. Deriving in sequence with respect to $\mathcal{D}_1, \ldots, \mathcal{D}_k$ gives another translation plane of dimension at most two over its kernel. By observing what happens to the classical and Buekenhout-Metz unitals of $PG(2, q^2)$ and repeating the arguments of this chapter, we get analogous results in these planes. In particular, a generalization of Corollary 4.4.3 describes the number of Buekenhout and Buekenhout-Metz unitals in such a plane and their relationship to the unitals of $PG(2, q^2)$.

Let $\mathcal{P}$ be a translation plane of dimension two over its kernel that contains a regulus in the underlying spread $\mathcal{S}$. We can generalize the arguments of this chapter to describe the effect of derivation on the Buekenhout and Buekenhout-Metz unitals of $\mathcal{P}$. Let U be a Buekenhout unital with respect to ℓ_∞ in $\mathcal{P}$. In the Bruck and Bose representation of $\mathcal{P}$ in $PG(4, q)$, U corresponds to a parabolic quadric $\mathcal{U}$ that meets Σ_∞ in a regulus of $\mathcal{S}$. Replacing this regulus by its opposite regulus is equivalent to deriving $\mathcal{P}$ with respect to the derivation set $U \cap \ell_\infty$, and gives a new translation plane $\mathcal{P}'$. The set $\mathcal{U}$ corresponds to a Buekenhout unital of $\mathcal{P}'$. In addition, if $\mathcal{R}$ is any other regulus of $\mathcal{S}$ that is disjoint from $\mathcal{U}$, then deriving $\mathcal{P}$ with respect to the derivation set corresponding to $\mathcal{R}$ produces another translation plane $\mathcal{P}''$, and $\mathcal{U}$ corresponds to a Buekenhout unital of $\mathcal{P}''$.

Similarly, let U^* be a Buekenhout-Metz unital with respect to ℓ_∞ in $\mathcal{P}$ and let $\mathcal{U}^*$ be the ovoidal cone of $PG(4, q)$ corresponding to U^*. Reversing a regulus of $\mathcal{S}$ that is disjoint from $\mathcal{U}^*$ is equivalent to deriving $\mathcal{P}$ with respect to the derivation set corresponding to the regulus. The set $\mathcal{U}^*$ corresponds to a Buekenhout-Metz unital of the derived plane. Determining whether these classes of unitals are isomorphic is more difficult in these cases.

Another interesting question that arises from the work in this chapter is whether Buekenhout and Buekenhout-Metz unitals are embeddable in unique translation planes of dimension at most two over their kernel. I conjecture that a Buekenhout unital is uniquely embeddable as a Buekenhout unital, and that a Buekenhout-Metz unital is uniquely embeddable as a Buekenhout-Metz unital.

Conjecture *Let $\mathcal{P}$ and $\mathcal{P}'$ be translation planes of order q^2 and dimension at most two over their kernels, with translation lines ℓ_∞ and ℓ'_∞ respectively. Let U be a unital of order q.*

1. *If U is embeddable in $\mathcal{P}$ as a Buekenhout unital with respect to ℓ_∞, then U is not embeddable in $\mathcal{P}'$ as a Buekenhout unital with respect to ℓ'_∞.*
2. *If U is embeddable in $\mathcal{P}$ as a Buekenhout-Metz unital with respect to ℓ_∞, then U is not embeddable in $\mathcal{P}'$ as a Buekenhout-Metz unital with respect to ℓ'_∞.*

Appendix

A brief historical survey of unitals

1966

Lüneburg [85]

There exists a unital admitting a collineation group isomorphic to the Ree group of order $(q^3+1)q^3(q-1)$, where $q = 3^{2r+1}$, which cannot be embedded in a projective plane such that the group carries over. The Ree group acts 2-transitively on this unital called the *Ree unital*.

Tallini-Scafati [100]

A study of $\{k, n\}$-arcs of type (m, n) in a projective plane is presented. It is shown that a set of type $(1, n)$ in a projective plane of prime power order q is either a unital or a Baer subplane while a set of type (m, q) must be the complement of one of these sets. It is also shown that if U is a unital in a projective plane of order q^2 and P is a point not in U, then there are $q+1$ tangents to U through P. The dual of a unital embedded in a projective plane is a unital in the dual plane and the classical unital is self dual. A characterization of the classical unital is given: a unital in $PG(2, q^2)$ that satisfies the *condition of reciprocity* is classical.

1967

Tallini-Scafati [101]

A set of type $(1, k, n+1)$ in a projective plane of prime power order n that contains no line is either a unital or a Baer subplane and so is of type $(1, \sqrt{n}+1)$.

1970

Seib [98]

If π is a polarity of a finite projective plane of order q with n absolute points, then $n \leq q^{3/2}+1$. If $n = q^{3/2}+1$, then the absolute points and non-absolute lines of π form a unital with parameter $\sqrt{q}$. It is also shown that if $\mathcal{P}$ is a finite projective plane of order q^2, U a unital in $\mathcal{P}$, π a planar involution of $\mathcal{P}$ that fixes U and B a Baer subplane of $\mathcal{P}$ fixed pointwise by π, then exactly $q+1$ points and q^2 lines of U lie in B. If q is even, the points are collinear, if q is odd, the points form an oval of B.

1971

Kantor [72]

Let $\mathcal{P}$ be a projective plane of order q^2 and U a unital that is formed from a unitary polarity of $\mathcal{P}$. If G is a subgroup of collineations of $\mathcal{P}$ that leaves U invariant and acts transitively on the flags (X, l) where X is a point of U and l is a line of U through X, then $\mathcal{P}$ is Desarguesian, U is classical and $PSU(3, q^2) \leq G$. Camina & Gagen [42] note that transitivity of G on the secants of U implies the flag transitivity of G thus improving Kantor's result.

Dickey [48]

If U is a unital embedded in a projective plane $\mathcal{P}$, then the complement of U is formed by removing from $\mathcal{P}$ all the points of U. A structure with these parameters can also be constructed from a hermitian variety of $PG(3, q^2)$ by defining points as secant lines through a given point P of the variety and the blocks are the planes through P. It is shown that any incidence structure with the parameters of the complement of a unital can be embedded in a projective plane of square order.

A slightly generalized version of the incidence structure is classified by Beutelspacher in [22] and [23], namely if S is an incidence structure with line degrees q^2 or $q(q-1)$, $q > 3$, then S is the complement of a unital or a Baer subplane in a projective plane of order q^2.

1972

O'Nan [89]

It is well known that the unitary groups $PSU(3, q^2)$ and $PGU(3, q^2)$ act on the point set of the classical unital in $PG(2, q^2)$ as doubly transitive permutation groups. The main result is that the full automorphism group of the classical unital is $P\Gamma U(3, q^2)$. As a corollary it is shown that any group of automorphisms of the classical unital U that acts 2-transitively on the points of U contains $PSU(3, q^2)$. The classical unital contains no O'Nan configurations.

Hoffer [66]

If $\mathcal{P}$ is a projective plane of order q^2 and G a collineation group of $\mathcal{P}$ that is isomorphic to $PSU(3, q^2)$, then $\mathcal{P}$ fixes a unital U of $\mathcal{P}$, $\mathcal{P}$ is Desarguesian, and U is classical.

Ganley [57]

This paper investigates polarities in planes coordinatized by commutative semifields. If D is a commutative semifield and has a non-trivial involutory automorphism, then the plane coordinatized by D admits a unitary polarity and so contains a unital.

Ganley [58]

A study of unitals formed by the absolute points of a unitary polarity in a projective plane coordinatized by a Dickson semifield of odd order q^2 is presented. There are infinitely many of these unitals that do not have a transitive automorphism group. In addition these unitals contain O'Nan configurations.

1974

Taylor [103]

Let $\mathcal{P}$ be a projective plane, π a unitary polarity of $\mathcal{P}$ and U the unital whose points are the absolute points of π. A relation of perpendicularity is defined between the blocks of U and it is shown that this relation can be described in terms of the incidence structure of U. The projective plane $\mathcal{P}$ together with the polarity π can be reconstructed from U in such a way that any automorphism of U extends to a collineation of $\mathcal{P}$ that commutes with π.

1976

Buekenhout [40]

Let U be the classical unital in $PG(2,q^2)$ tangent to ℓ_∞, then in the Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$, U corresponds to an orthogonal cone. If U is classical and secant to ℓ_∞, then U corresponds to a non-singular quadric. This is used to show the existence of unitals in every translation plane of order q^2 with kernel $GF(q)$. Furthermore, there are non-classical unitals in $PG(2,q^2)$, q even and non-square.

1978

Baker [9]

Polarities of elliptic semiplane are defined. An upper bound on the number of absolute points of such a polarity is given. If this bound is achieved, the absolute points form a design called a "unital" of the elliptic semiplane.

1979

Barlotti & Lunardon [15]

The Δ-planes are those semitranslation planes that are derived from the dual of a translation plane of dimension two over its kernel. Working in the Bruck and Bose representation of these planes in $PG(4,q)$, a unital is constructed that does not arise from a polarity. A survey of these results is given in English in [16].

Barnabei & Bonetti [17]

Let U be the classical unital in $PG(2, q^2)$ and let l be a tangent of U at the point P. The incidence structure obtained by removing the points of U and the points of l from $PG(2, q^2)$ is a Bolyai-Lobachevsky plane. Let Γ be the unitary group, then the collineation group of this plane is isomorphic to Γ_P and so is transitive on the points.

Metz [87]

It is noted that Buekenhout's construction [40] provides examples of non-classical unitals in $PG(2, q^2)$ for all prime power q.

Piper [91]

A survey of the historical development of unitals is given. Three questions are posed on the relationship between unitals and projective planes. The isomorphism of unitals is discussed. A combinatorial method of derivation is presented which allows the construction of new unitals from existing ones. Finally, an incidence structure is constructed from a derivable unital which under certain conditions is a generalized quadrangle. If the unital is classical, the quadrangle is regular.

1981

Brouwer [31, 32]

The three questions posed by Piper are answered by constructing unitals of parameter $q = 3$. Unitals are constructed that (a) are not embeddable in a projective plane, (b) are embeddable but do not arise from a polarity, and (c) are embeddable in two different planes. Using a computer, 138 non-isomorphic unitals of parameter $q = 3$ are constructed, 11 of which are embeddable in projective planes of order $q^2 = 9$, four of these in the Hughes plane. Other unitals of parameter $q = 3$ have been constructed in [29, 97] (they construct a unital isomorphic to the classical unital) and in [94], [46] and [30].

deFinis [45]

A $s(q^3+1)$-set of type $(s(q+1)-q, s(q+1))$ is constructed in projective plane $\mathcal{P}$ of order q^2 from s pairwise skew unitals. In $PG(2,q^2)$ there are no skew classical unitals. The existence of skew unitals in general is unknown.

Hölz [67]

Let U be the classical unital in $PG(2,q^2)$. New blocks are defined on the point set of U in order to get two new 2-designs, U_1 and U_2, such that the block set of U_1 is the disjoint union of the block sets of U and U_2. The new blocks result from intersections of U with appropriate Baer subplanes of $PG(2,q^2)$. U_1 is a 2-$(q^3+1, q+1, q+2)$ design, U_2 is a 2-$(q^3+1, q+1, q+1)$ design. A number of lemmas on the intersection of Baer subplanes with the classical unital are given.

Kestenband [74]

It is shown that two classical unitals in $PG(2,q^2)$ can intersect in $q+1$, q^2-q+1, q^2+q+1, $(q+1)^2$, q^2+1 or 1 points. The intersection configuration is illustrated in each case, the first three are a line, an arc and a configuration II_P (see Section 3.2) respectively.

In [75] Kestenband generalizes this to describe all possible intersections between degenerate Hermitian unitals in $PG(2,q^2)$. A degenerate Hermitian unital is a curve $x^qHx^t=0$ in $PG(2,q^2)$ in which the Hermitian matrix H has rank less than three.

Lefèvre-Percsy [83]

Let U be a unital of $PG(2,q^2)$ and l a tangent line to U. If all Baer sublines having a point on l intersects U in 0, 1, 2 or $q+1$ points, then U is a Buekenhout-Metz unital. This result also holds for parabolic unitals in translation planes of order q^2 with a kernel of order q.

1982

Biscarini [26]

Let G be a group of collineations acting on a unital U of $PG(2,q^2)$. If G is transitive on the $q+1$ secants of U and G is generated by involutions, then U is classical.

Lefèvre-Percsy [84]

The following characterization of the classical unital is proved: let U be a unital of $PG(2, q^2)$, if each line secant to U meets it in a Baer subline, then U is classical. If q is odd, the assumption of this result may be weakened as follows: let U be a Buekenhout-Metz unital with respect to P in $PG(2, q^2)$, q odd. If there is a secant line not through P that meets U in a Baer subline, then U is classical.

1983

Faina [52]

Let U be the classical unital in $PG(2, q^2)$, $q \geq 3$. If S is a set of q^2 points not in U such that the line joining any two points of S is tangent to U, then the points of S all lie on the same tangent to U.

Faina & Korchmáros [54]

The characterization of classical unitals given by Lefèvre-Percsy [84] is proved in a different way. The proof uses group theory to show that U is reciprocal and hence classical.

Larato [81]

Let U be a parabolic unital in $\mathcal{P}$, a translation plane of odd order q^2. U is Buekenhout-Metz if and only if for any two affine lines l, m through $U \cap \ell_\infty$ there is an affine Baer subplane B of $\mathcal{P}$ such that $B \cap U = (l \cap U) \cup (m \cap U)$.

Wilbrink [113]

Three geometric conditions satisfied by the classical unital are given (the first states that a unital contains no O'Nan configurations). A characterization of the classical unital in terms of these conditions is proved. If a unital satisfies all three conditions, it is classical. If a unital of even parameter satisfies the first two conditions, it is classical. Wilbrink constructs an inversive plane from the classical unital and shows that if a unital U satisfies the first two conditions and the generalized quadrangle described by Piper [91] is regular for every line of U, then U is classical.

1984

Abatangelo [1]

The following group theoretic characterization of the classical unital is proved: a Buekenhout-Metz unital U of $PG(2, q^2)$, q odd, is classical if and only if there is a cyclic linear collineation group of order $q^2 - 1$ that maps U onto itself and fixes two distinct points of U. By applying this characterization a new proof of Metz's result [87] is given.

Larato [82]

An incidence structure with $(q^2 - 1)^2$ points, $q^2 - 1$ blocks and a transitive automorphism group is constructed from a family of Hermitian curves of $PG(2, q^2)$. A parallel relationship is defined on the blocks.

1985

Hirschfeld [62]

An equation in $PG(4, q)$ is given that represents a Buekenhout-Metz unital in $PG(2, q^2)$. This provides an affine equation for non-classical Buekenhout-Metz unitals in $PG(2, q^2)$.

Kantor [73]

The classification of all finite simple groups is used to show that the only unitals with an automorphism group that is 2-transitive on its points are the classical unitals and the Ree unitals.

1986

Bruen & Hirschfeld [39]

A number of results on intersections of Hermitian varieties and quadrics are given with corollaries relating to unitals and Baer subplanes in $PG(2, q^2)$. For example: the number of points common to two unitals in $PG(2, q^2)$ is equal to the number of their common tangents; if U, V are unitals and U^*, V^* their sets of non-tangent lines, then $|U^* \cap V^*| - |U \cap V| = (q^2 - q + 1)(q^2 - q - 1)$; and if U is the classical unital and B a Baer subplane of $PG(2,q^2)$ with U^* and B^* their set of tangents, then the possible values for $|U \cap B|$ and $|U^* \cap B^*|$ are: 1 and q^3-2q; $q+1$ and q^3-q; or $2q+1$ and q^3. They also determine the number of secants that U and B have in common, and the number of lines that are tangent to one and secant to the other.

Fisher, Hirschfeld & Thas [55]
A cyclic projective plane can be partitioned into q^2+q+1 complete (q^2-q+1)-arcs K_i. A corollary of their results shows that the classical unital is the disjoint union of $q+1$ complete (q^2-q+1)-arcs and the K_i are the intersection of two classical unitals. Also the tangents to any of the complete (q^2-q+1) -arcs K_i in $PG(2,q^2)$ form a dual classical unital if and only if q is even.

Grünig [59]
A geometric proof that the Ree unital $R(3)$ cannot be embedded in a projective plane of order 9 is given.

1987

Grünig [60]
By deriving the Desarguesian plane of order q^2, for every prime power q, a unital of order q is constructed that can be embedded in two non-isomorphic planes; the Hall plane of order q^2 and its dual plane. These unitals contain O'Nan configurations whose positions are described. The full automorphism group of these unitals is the collineation groups inherited from the classical unital, that is, $P\Gamma U(2,q^2)$.

A useful lemma shows that the number of points a unital U and a Baer subplane B have in common is $|B \cap U| = 2(q+1) - b$, where b is the number of lines of B that are tangents of U.

Kleidman [79]
Let S be a finite linear space and G a flag transitive group of automorphisms of S. If G is almost simple and the socle of G is an exceptional group of Lie type, then S is the Ree unital.

Tiersma [107]

Parameters and generator matrices are given for the codes obtained from Hermitian curves in $PG(2, 2^{2s})$. Automorphisms of these codes are discussed and some results on self duality are given.

Van Lint and Springer [110]

The parameters of codes constructed from Hermitian curves in $PG(2, q^2)$ are calculated. They note that the codes obtained from Hermitian curves are usually better than the corresponding Reed-Solomon codes with the same rate.

1988

Key & Rostom [78]

A regular set for a design D is a set of points Q such that only the identity automorphism of D fixes Q. Both the classical unital and the Ree unital with parameter $q \geq 3$ contain regular 5-sets. It is noted that the classical unital contains no regular 3-sets but does contain regular 6-sets.

Rosati [95]

A unital of parameter q is constructed in the Hughes plane of order q^2. An algebraic description of the unital is given.

1989

Assmus & Key [6]

The classical unitals $U(q)$ and the Ree unitals $RU(q)$ are examined for the existence of ovals. The classical unital does not contain ovals when $q > 2$. Arcs of size $(3q+1)$ are constructed in the Ree unitals $RU(q)$ for any $q = 3^{2n+1}$, however, the existence of ovals in the Ree unitals is unsettled. If $q = 3$, these arcs are ovals and the classical and Ree unitals are embeddable in the same Hölz design [67]. The coding theory applications are discussed.

Bagchi & Bagchi [12]

A general construction for Steiner 2-designs with prime power block size is presented. Analysis on small block sizes shows that they have constructed a cyclic unital with a non-prime power order, that is, a 2-$(217, 7, 1)$ design. These unitals were also constructed by Mathon [86].

Biliotti & Korchmáros [24]
A study of collineation groups preserving unitals embedded in projective planes of odd order is presented. A strong result on the structure of non-abelian simple groups that preserve an embedded unital is given. The main result is: let G be a collineation group of a projective plane $\mathcal{P}$ of order q^2, $q > 5$, that fixes a unital U of $\mathcal{P}$, is transitive on the points of U, and has a socle of even order, then $\mathcal{P}$ is Desarguesian and U is classical. A corollary on primitive groups preserving a unital is given.

Biliotti & Korchmáros [25]
The structure of a collineation group G fixing a unital U embedded in a projective plane $\mathcal{P}$ of even order q^2 is investigated. If G is transitive on the points of U and the socle of G has even order, then $\mathcal{P}$ is Desarguesian, U is classical and $PSU(3, q^2) \leq G \leq P\Gamma U(3, q^2)$. The primitive case follows as a corollary.

Brouwer [33]
Rosati's unital [95] with $q = 3$ does not coincide with any of the unitals found in [31]. Brouwer gives an algebraic description of one of his unitals in the Hughes plane of order 9 and asks if this can be generalized to describe a unital in the Hughes plane of order q^2, q odd. So far there are 154 non-isomorphic unitals with parameter $q = 3$, 12 are embeddable in a projective plane of order $q^2 = 9$.

Doyen [49]
An investigation is carried out into flag transitive automorphism groups of 2-$(v, k, 1)$ designs. As an application it is shown that the only line transitive unitals are the Hermitian unitals and the Ree unitals.

1990

Abatangelo & Larato & Rosati [4]

It is shown that the unital constructed by Rosati [95] in the Hughes plane is a unital in the plane derived from the Hughes plane. The collineation group fixing each unital is described.

Assmus & Key [7]

An examination of the codewords associated with Baer subplanes, unitals and ovals in $PG(2, q^2)$ is given A characterization of the classical unital in terms of its codewords is conjectured.

Baker & Ebert [10]

A family of non-classical unitals U in $PG(2, q^2)$, q odd is constructed as a pencil of q conics. If H is the classical unital, then $|H \cap U| \equiv 1 \bmod p$. They conjecture that $|H \cap U|$ must be one of $q^2 - 2q + 1$, $q^2 - q + 1$, $q^2 + 1$, $q^2 + q + 1$ or $q^2 + 2q + 1$. The intersection configurations appear to be at random, unlike Kestenband's result for classical unitals [74].

Jackson, O'Keefe & Penttila [70]

An attempt is made to embed the Mathon [86], and Bagchi & Bagchi [12] unitals $U(u)$ of non-prime power parameter u in a projective plane of order u^2. It is shown that the unital with parameter $u = 4$ is not embeddable in a projective plane of order 16. It is still an open question whether $U(6)$ is embeddable in a projective plane of order 36.

Kestenband [76]

A family of unitals is constructed in the Hughes plane. A particular instance of this construction is Rosati's unital [95]. It is shown that this family of unitals is not isomorphic to the classical unital by exhibiting a configuration that exists in the latter but not in the former.

1991

Abatangelo & Larato [3]

The projective group Γ fixing the non-classical unital U in $PG(2, q^2)$, q odd, constructed by Hirschfeld and Szőnyi [65] is determined. They prove that this group characterizes U.

They note that the group fixing the classical unital in $PG(2, q^2)$ satisfies the following property: let $A \neq B$ be points of a Hermitian curve, then the stabilizer of both A and B is a cyclic group of order $q^2 - 1$ and preserves any Baer subline containing A and B. It is also known that non-classical Buekenhout-Metz unitals do not satisfy this property.

Batten [21]

Let S be a blocking set in a finite projective plane $\mathcal{P}$. Let G be an automorphism group of $\mathcal{P}$ that fixes S and is flag transitive on S, then S is a Baer subplane of $\mathcal{P}$ or S is the classical unital and $\mathcal{P} = PG(2, q^2)$.

Blokhuis, Brouwer & Wilbrink [28]

A unital U in $PG(2, q^2)$ is classical if and only if it is in the code generated by the lines of $PG(2, q^2)$. As a consequence, $|H \cap U| \equiv 1 \bmod p$ where H is the classical unital and U is any unital of $PG(2, q^2)$, a conjecture of Assmus and Key [7].

Faina [53]

Let U be a unital embedded in a finite projective plane and let $\mathcal{F}$ be a family of partitions on U. Certain conditions are given on $\mathcal{F}$ in order to characterize U as a classical unital in $PG(2, q^2)$.

Hirschfeld, Storme, Thas & Voloch [63]

A characterization of the classical unital is given: in $PG(2, q^2)$, $q \neq 2$, any algebraic curve of degree $q + 1$ with at least $q^3 + 1$ points in $PG(2, q^2)$ and without linear components, is a Hermitian curve.

Hirschfeld & Szőnyi [64]

Pencils of conics and pencils of Hermitian curves with a single base point are investigated. A key result that emerges is that for q odd there exists a unital in $PG(2,q^2)$ that is a pencil of q conics and further, it is a non-classical Buekenhout-Metz unital. It is shown that $PG(2,q^2)$ without q^2 points of one line can be partitioned into q Hermitian curves with one point in common.

Jackson, Piper & Wild [71]

They investigate the embeddability of the unitals of non-prime power parameter admitting a point regular automorphism constructed by Bagchi and Bagchi [12] and Mathon [86]. It is shown that any unital constructed by this method cannot be embedded in a projective plane $\mathcal{P}$ in such a way that the unital arises from a polarity and the point regular group of the unital is induced by an automorphism group of $\mathcal{P}$.

Tonchev [109]

Let $H(3)$ be the 2-$(28,4,5)$ Hölz design constructed from the classical unital of order 3 as in [67]. It is shown that any unital contained in $H(3)$ is either classical or of Ree type. If B is any block of $H(3)$, then there are 24 classical unitals and 192 Ree unitals of $H(3)$ that contain B.

1992

Abatangelo [2]

Let U be Buekenhout-Metz with respect to the point P in $PG(2,q^2)$, q even. The collineation group G that fixes U fixes P, has a normal subgroup that acts on the points of $U \setminus P$ as a sharply transitive group, and the one point stabilizer of G has order $q-1$ for any point of $U \setminus P$. This property characterizes the Buekenhout-Metz unitals.

Baker & Ebert [11]

Coordinates for the Buekenhout-Metz unitals in $PG(2,q^2)$, q odd, are given. They are used to show that certain Buekenhout-Metz unitals are partial pencils of conics.

Furthermore, these conics are the only possible conics contained in a Buekenhout-Metz unital embedded in $PG(2, q^2)$, q odd. The number of projectively inequivalent Buekenhout-Metz unitals in $PG(2, q^2)$, q odd is given. It is shown that the group of collineations fixing a non-classical Buekenhout-Metz unital also fixes its vertex. This group is computed and its point orbits are given. Finally, they use their coordinates of a Buekenhout-Metz unital to prove the existence of an associated inversive plane.

Baker, Ebert, Korchmáros & Szőnyi [13]
Let U be the classical unital in $PG(2, q^2)$ and P a point not in U, then the $q^2 - q$ secants of U through P together with $P^{\perp}$ is a collection of $q^2 - q + 1$ lines of U that partition the points, called a *spread* of U. An orthogonally divergent spread is one in which no line of the spread contains the pole of any other line of the spread. The authors show that an orthogonally divergent spread of U exists for q even, $q \equiv 1 \bmod 3$. They construct maximal orthogonal divergent spreads for other q and determine the automorphism groups of these partial spreads.

Bruen & Fisher [38]
Let U be the classical unital in $PG(2, q^2)$ and let S be a set of points not in U that is maximal with respect to the property that the line joining any two points of S is tangent to U. Either S consists of q^2 points that lie on a line tangent to U, or S consists of $q + 2$ points, $q + 1$ of which form a Baer subline of a tangent l of U with the other point off l.

Ebert [50]
Coordinates are given for *orthogonal* Buekenhout-Metz unitals in $PG(2, q^2)$, q even. They are used to compute the collineation group fixing U and to describe its point orbits. The number of projectively inequivalent orthogonal unitals in $PG(2, q^2)$, q even, is given. It is shown that the orthogonal Buekenhout-Metz unital contains no ovals of $PG(2, q^2)$. The coordinates are used to construct an associated inversive plane.

Kestenband [77]

A 2-$(q^2, q+1, q)$ design is constructed from the classical unital. It is used to construct 2-$(q^{2m}, q+1, q)$ designs for all $m \geq 1$ with no two blocks having more than 2 points in common and satisfying a certain partition property.

Thas [105]

Proves the following well known conjecture: let U be a unital in $PG(2,q^2)$, if the tangents of U at collinear points of U are concurrent, then U is classical.

Thas, Cameron & Blokhuis [106]

An extension is given of Segre's generalization of Menlaus' Theorem to an arbitrary collection of lines. As an application, a new proof of Thas' characterization [105] is given using algebraic geometry.

1994

Barwick [18]

It is shown that all Buekenhout unitals in $PG(2,q^2)$ are classical. The following characterization of the classical unitals is proved: let U be a unital in $PG(2,q^2)$ and l a secant of U. If every Baer subline with a point on l meets U in 0, 1, 2 or $q+1$ points, then U is classical.

Barwick [19]

The unitals in the Hall plane are investigated by deriving the Desarguesian plane and observing the effect on the classical unital. Two new infinite classes of unitals are found in $\mathcal{H}(q^2)$, they are not isomorphic to the classical unital. The question of whether the classical unital is embeddable in the Hall plane is partially answered.

Barwick & O'Keefe [20]

A new proof is given of the relationship between Buekenhout-Metz unitals and inversive planes that is more general than the known results and gives a natural, geometric isomorphism between the resulting inversive plane and the egglike inversive plane arising from the ovoid involved in the construction of the Buekenhout-Metz unital. The results are applied to demonstrate some parallel classes and partitions of the set of blocks of any Buekenhout-Metz unital.

Casse, O'Keefe & Penttila [43]

A characterization of the Buekenhout-Metz unitals of $PG(2, q^2)$, q even, is given in terms of the secant lines through a single point of the unital. With the addition of extra conditions further characterizations of Buekenhout-Metz unitals are obtained for all q. As an application, it is shown that the dual of a Buekenhout-Metz unital in $PG(2, q^2)$ is a Buekenhout-Metz unital, and all known unitals in $PG(2, q^2)$ are Buekenhout-Metz unitals. Penttila and Royle have used a computer to check that all unitals in $PG(2, 9)$ are Buekenhout-Metz.

Quinn & Casse [92]

The characterization of Buekenhout-Metz unitals given in [43] for q even is generalized to q odd. Let U be a unital in $PG(2, q^2)$, $q > 2$, U is Buekenhout-Metz if and only if there exists a point T of U such that the points of U on each of the q^2 secant lines to U through T form a Baer subline.

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