Graphs with few eigenvalues

An interplay between combinatorics and algebra

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Proefschrift

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Tilburg, August 1996 Edwin van Dam

Contents

Let me explain to you, Socrates, the order in which we have arranged our entertainment.

(Plato, *Timaeus*)

Hooggeleerde heer, ik heb deze proef zelf genomen en ook zelf de metingen verricht. Ik ben op een nacht naar een graanmolen geslopen met een graankorrel in de hand. Die graankorrel heb ik op de onderste molensteen gelegd. Toen heb ik met een zware takel de tweede molensteen op de graankorrel laten zakken, zodat ik een stenen sandwich had verkregen: *twee molenstenen met een graankorrel ertussen. Nu heeft een graankorrel de merkwaardige eigenschap dat zodra hij tussen twee stenen wordt geklemd hij de neiging krijgt om de losliggende steen, en dat is altijd de bovenste, in beweging te willen brengen. De graankorrel wil zich wentelen, hij wil als het ware gemalen worden. Meel*! *wil hij worden. Maar dat gaat zomaar niet. Daartoe moet hij zich verschrikkelijk inspannen om één van die stenen, de bovenste, te laten draaien. Maar op de lange duur heeft de korrel zich al een paar centimeter voortgewenteld. En dan gaat het proces steeds sneller en alléén maar door de wil en de macht van de graankorrel moet de bovenliggende molensteen de zware molenas in beweging brengen en door diezelfde wil worden de raderen in de kop van de molen in beweging gebracht, de as die naar buiten steekt en de wieken die eraan bevestigd zijn. Zo maken molens evenals ventilators wind. Met dit verschil dat de laatste op elektriciteit lopen en de eerste gewoon op graankorrels. U kunt zeker wel begrijpen hoe hard het gaat waaien als je een paar duizend korrels tegelijk gebruikt* ? ... *Mijne heren ... de enige die bij de proef aanwezig was, ben ik en toen de graankorrel meel was geworden hield het op met waaien. Het zal u niet eenvoudig vallen om mij snel en afdoend tegenbewijs voor mijn stelling te leveren. Per slot ben ik deskundig, ik ben expert, ik heb er jaren over nagedacht.*

(Biesheuvel, *In de bovenkooi*)

Chapter 1

Uw schip was niet bestemd door de heer van wouden en wateren om begerige bedriegers en koophandelaars over de zee te brengen. Mijn zusters woonden in de stammen die geveld werden om het schip te bouwen. Zij allen zingen. Hoor!

(Slauerhof, *Het lente-eiland*)

Introduction

1.1. Graphs, eigenvalues and applications

A graph essentially is a (simple) mathematical model of a network of, for example cities, computers, atoms, etc., but also of more abstract (mathematical) objects. Graphs are applied in numerous fields, like chemistry, management science, electrical engineering, architecture and computer science. Roughly speaking, a graph is a set of vertices representing the nodes of the network (in the examples these are the cities, computers or atoms), and between any two vertices there is what we call an edge, or not, representing whether there is a road between the cities, whether the computers are linked, or there are bonds between the atoms. These edges may have weights, representing distances, capacities, forces, and they can be directed (one-way traffic). Although the model is simple, that is, to the extend that we cannot see from a graph what kind of network it represents, the theory behind is very rich and diverse.

There is a large variety of problems in graph theory, for example the famous *traveling salesman problem*, the problem of finding a shortest tour through the graph visiting every vertex. For small graphs this problem may seem easy, but as the number of vertices increases, the problem can get very hard. The name of the problem indicates where the initial problem came from, but it is interesting to see that the problem is applied in quite different areas, for example in the design of very large scale integrated circuits (VLSI). Another kind of problem is the connectivity problem: how many edges can be deleted from the graph (due to road blocks, communication breakdowns) such that we can still get from any vertex to any other vertex by walking through the graph.

In this thesis we study special classes of graphs, which have a lot of structure. In the eye of the mathematical beholder graphs with much structure and symmetry are the most beautiful graphs. Important classes of beautiful graphs comprise the strongly regular graphs, and more general, distance-regular graphs or graphs in association schemes. The graphs of Plato's regular solids may be considered as ancient examples: the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron. Association schemes also occur in other fields of mathematics and their applications, like in the theory of coding of messages, to encounter errors during transmission or storage (on a CD for example), or to encrypt secret messages (like PIN-codes). Association schemes originally come from the design of statistical experiments, and they are also important in finite group theory.

Especially in the theory of graphs with much symmetry, but certainly also in other parts of graph theory, the use of (linear) algebra has proven to be very powerful. Depending on the specific problems and personal favor, graph theorists use different kinds of matrices to represent a graph, the most popular ones being the (0, 1)-adjacency matrix and the Laplace matrix. Often, the algebraic properties of the matrix are used as a bridge between different kinds of structural properties of the graph. The relation between the structural (combinatorial, topological) properties of the graph and the algebraic ones of the corresponding matrix is therefore a very interesting one. Sometimes the theory even goes further, for example, in theoretical chemistry, where the eigenvalues of the matrix of the graph corresponding to a hydrocarbon molecule are used to predict its stability.

Some examples of basic questions in algebraic graph theory are: can we see from the spectrum of the matrix whether a graph is regular (is every vertex the endpoint of a constant number of edges), or connected (can we get from any vertex to any other vertex), or bipartite (is it possible to split the vertices into two parts such that all edges go from one part to the other)? The answer depends on the specific matrix we used. Both adjacency and Laplace spectrum indicate whether a graph is regular, however, the adjacency spectrum recognizes bipartiteness, but not connectivity. For the Laplace spectrum it's just the other way around: it recognizes connectivity, but not bipartiteness.

A graph determines its spectrum, but certainly not the other way around. Thus it makes sense to investigate what structural properties can be derived from the eigenvalues, or more general, from some properties of the eigenvalues.

For example, is it possible to completely determine a graph from its adjacency spectrum $\{[6]^1, [2]^6, [-2]^9\}$? The answer is no, there are two different graphs with this spectrum, but they have similar combinatorial properties.

Other questions relate to the smallest adjacency eigenvalue of a graph. For example, there is a large class of graphs with all adjacency eigenvalues at least -2, the generalized line graphs. But there are more such graphs, and they have been characterized by means of so-called root lattices by Cameron, Goethals, Seidel and Shult [25]. Other type of results are bounds on special substructures in a graph in terms of (some of) the eigenvalues, like Hoffman's coclique bound.

Also if we wish to find graphs with special structural properties, it may be useful to first translate the properties into spectral properties, before trying our luck. For example, suppose we want to find all regular graphs for which any two vertices have precisely one common neighbour (that is, a vertex that is on edges with both these vertices). The Friendship theorem states that the only graph with this property is the triangle, and its proof relies on a simple algebraic property.

Of course, there are many more (type of) results in the field of spectral graph theory, and we refer the interested reader to the book by Cvetković, Doob and Sachs [33], for example.

1.2. Graphs with few eigenvalues - A summary

In general, most of the eigenvalues of a graph are distinct, but when many eigenvalues coincide, then it appears that we are in a very special case. If all eigenvalues are the same, then we must have an empty graph (a graph without edges). If we have only two eigenvalues, then essentially we must have a complete graph (a graph with edges between any two vertices). Here we study *graphs with few distinct eigenvalues*, where most of the times *few* means three or four. These graphs may be seen as algebraic generalizations of so-called strongly regular graphs. Strongly regular graphs (cf. [16, 95]) are defined in terms of combinatorial properties, but they have an easy algebraic characterization: roughly speaking they are the regular graphs with three (adjacency or Laplace) eigenvalues. By dropping regularity, and considering graphs with three adjacency eigenvalues, and graphs with three Laplace eigenvalues, we obtain two very natural generalizations. Seidel (cf. [94]) did a similar thing for the Seidel spectrum, and found graphs which are closely related to the combinatorial structures called regular two-graphs. Little is known about nonregular graphs with three adjacency eigenvalues. There are only two papers on the subject, by Bridges and Mena [10], and Muzychuk and Klin [85]. It turns out that things can get rather complicated, and there are many open questions. We shall have a closer look at the ones with least eigenvalue -2. Nonregular graphs with three Laplace eigenvalues seem to be unexplored (apart from geodetic graphs with diameter two, but these were not recognized as such), which may be surprising, as we find a rather easy combinatorial characterization of such graphs.

The fundamental problem of graphs with few (adjacency) eigenvalues has been raised by Doob [45]. In his view *few* is at most five, and he characterized a family of regular graphs with five eigenvalues related to Steiner triple systems. However, it seems too complex to study regular graphs with five eigenvalues in general. Doob [46] also studied regular graphs with four eigenvalues, the least one of which is -2. In the general case of four eigenvalues we derive some nice properties, like walk-regularity, but there is no easy combinatorial characterization, like in the case of three eigenvalues. Still we find many constructions.

Association schemes (cf. [3, 12, 15, 52]) form a combinatorial generalization of strongly regular graphs, and the next stage of investigation after strongly regular graphs would be to consider three-class association schemes. In such schemes all graphs are regular with at most four eigenvalues, so we can apply the results on such graphs. In this way we achieve more than by just applying the general theory of association schemes, and we find two rather surprising characterization theorems. The literature on three-class association schemes mainly consists of results on special constructions, and results on the special case of distance-regular graphs with diameter three. General results on three-class association schemes can be found in the early paper by Mathon [79], who gives many examples, and the thesis of Chang [26], although he restricts to the imprimitive case.

In the final chapter of this thesis we obtain bounds on the diameter of graphs and on the size of special subsets in graphs. The case of sharp bounds is investigated, and here distance-regular graphs and three-class association schemes show up. All bounds build on an interlacing tool and finding suitable polynomials. The diameter bounds are applied to error-correcting codes.

Appended to this thesis are lists of parameter sets for graphs with three Laplace eigenvalues, regular graphs with four eigenvalues, and three-class association schemes (on a bounded number of vertices). By combined efforts Spence and the author were able to find all graphs for almost all feasible parameter sets of regular graphs with four eigenvalues and at most 30 vertices, using both theoretical and computer results.

Parts of the results in this thesis have appeared elsewhere. The results on graphs with three Laplace eigenvalues in [38], regular graphs with four eigenvalues in [34] and [40], three-class association schemes in [35] and [39], bounds on the diameter in [37] and bounds on special subsets in [36].

1.3. Graphs, combinatorics and algebra - Preliminaries

In this section we give some preliminaries. Most of them are well-known results. For results on the *spectra of graphs* we refer to the books by Cvetković, Doob and Sachs [33] and by Cvetković, Doob, Gutman and Torgašev [32], for *interlacing* to the thesis [57] or paper [58] by Haemers, for *strongly regular graphs* to the papers by Brouwer and Van Lint [16] and Seidel [95], for *association schemes* and *distance-regular graphs* to the books by Bannai and Ito [3], Brouwer, Cohen and Neumaier [12], and Godsil [52], for *designs* to the book by Beth, Jungnickel and Lenz [4], for *codes* to the book by MacWilliams and Sloane [77], and for *switching* to the paper by Seidel [94].

1.3.1. Graphs

A *graph* G is a set V of so-called *vertices* with a subset E of the pairs of vertices, called the *edges* (throughout this thesis a graph is undirected, without loops and multiple edges, unless indicated otherwise). We say two vertices x and y are *adjacent* if the pair $\{x, y\}$ is an edge. Such vertices are also called *neighbours* of each other. We say the graph is *complete* if any two vertices are adjacent, and *empty* if no two vertices are adjacent. The *complement* $\overline{G}$ of a graph G is the graph on the same vertices, but with complementary edge set, that is, two vertices are adjacent in $\overline{G}$ if they are not adjacent in G. We can make nice pictures of graphs by drawing the vertices as dots (which we shall not label), and drawing edges as lines (or curves) between the vertices.

Two graphs are called *isomorphic* if there is a bijection between the respective vertex sets preserving edges (in Dutch: the graphs can be drawn in the same way, possibly after moving the vertices around). For example, the two graphs of Figure 1.3.1 are isomorphic. If two graphs are isomorphic, then we shall (in general) not distinguish between them, or even call them the same. An *automorphism* of a graph is a bijection from the vertex set to itself preserving edges. The set of automorphisms of a graph, with the composition operator, forms a group, called the *automorphism group*.

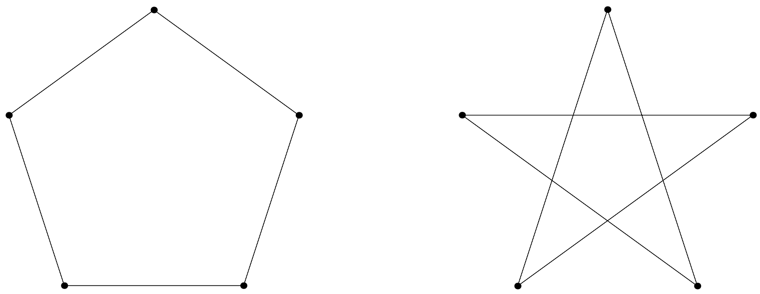

Figure 1.3.1. The 5-cycle (pentagon) and its complement

If X is a subset of V, then the induced subgraph of G on X is the graph with vertex set X, and with edges those of G that are contained in X. A *coclique* is an induced empty subgraph, and a *clique* is an induced complete subgraph. A graph is called *bipartite* if the vertices can be partitioned into two induced cocliques.

A *walk* of length l between two vertices x, y is a sequence of (not necessarily distinct) vertices $x = x_0, x_1, ..., x_l = y$, such that for any i the vertices x_i and x_{i+1} are adjacent. If all vertices are distinct then the walk is also called a *path*. If there is a path between any two vertices of the graph, then the graph is called *connected*. If not, then the graph has more than one (connected) *components*. The *distance* between two vertices is the length of the shortest path between these vertices. The maximal distance taken over all pairs of vertices is called the *diameter* of the graph.

The *degree* (or valency) of a vertex is its number of neighbours. If all vertices have the same degree then the graph is called *regular*.

1.3.2. Spectra of graphs

The *adjacency matrix* A of a graph is the matrix with rows and columns indexed by the vertices, with $A_{xy} = 1$ if x and y are adjacent, and $A_{xy} = 0$ otherwise. The *adjacency spectrum* of G is the spectrum of A, that is, its multiset of eigenvalues. A number λ is called an *eigenvalue* of a matrix M if there is a nonzero vector u such that $Mu = \lambda u$. The vector u is called an *eigenvector*. As the adjacency matrix is symmetric, the (adjacency) spectrum of a graph consists of real numbers. Moreover, for a connected graph, the adjacency matrix is nonnegative and irreducible, so the Perron-Frobenius theorem applies. This implies that the graph has a largest eigenvalue of multiplicity one, with a positive eigenvector, called *Perron-Frobenius eigenvector*. The largest eigenvalue can be seen as some "weighed average" of vertex degrees, and we have the following.

LEMMA 1.3.1. *Let G be a graph with largest eigenvalue λ_0, and denote by $k_{\max}$ and k_{ave} the largest and average vertex degree, respectively. Then $k_{\text{ave}} \le \lambda_0$ with equality if and only if G is regular. Moreover, $\lambda_0 \le k_{\max}$, and if G is connected, then we have equality if and only if G is regular.*

Throughout this thesis we shall denote by $\{[\lambda_0]^{m_0}, [\lambda_1]^{m_1},\ldots, [\lambda_r]^{m_r}\}$ the spectrum of a matrix with $r + 1$ distinct eigenvalues λ_i with multiplicities m_i. If the matrix is the adjacency matrix of a connected graph, then λ_0 denotes the largest eigenvalue, and has multiplicity $m_0 = 1$. By I, J and O we denote an identity matrix, a matrix consisting of ones, and a matrix consisting of zeros, respectively. By $\underline{1}$ and $\underline{0}$ we denote an all-one vector, and a zero vector, respectively. Besides the ordinary matrix multiplication we shall use two other matrix products. The product $\circ$ denotes *entrywise* (Hadamard, Schur) multiplication, i.e. $(A\circ B)_{ij} = A_{ij}B_{ij}$. The *Kronecker product* $\otimes$ is defined by

$$A\otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1m}B \\ A_{21}B & A_{22}B & \cdots & A_{2m}B \\ \vdots & \vdots & & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nm}B \end{pmatrix}.$$

The spectrum of a graph contains a lot of information on the graph, but in general the spectrum does not determine the graph (up to isomorphism). For example, we can see from the spectrum whether the graph is regular, or bipartite. In this introduction we shall only mention a few relations between the spectrum of a graph and its structural properties. For many more we refer to [32, 33], for example.

An important property of the adjacency matrix A is that A^l_{ij} counts the number of paths of length l from i to j. It is this property that relates many algebraic and combinatorial properties of graphs. For example, A^2_{ii} equals the degree d_i of vertex i, so that the number of edges of the graph equals

$$\tfrac{1}{2}\sum_i d_i = \tfrac{1}{2}\,Trace(A^2) = \tfrac{1}{2}\sum_i m_i\lambda_i^2 .$$

(Here we used the so-called *Handshaking lemma*: the sum of all vertex degrees equals twice the number of edges.) By counting the number of edges, we also derive the average vertex degree, so by Lemma 1.3.1 we find the following spectral characterization of regularity.

LEMMA 1.3.2. *Let G be a graph on v vertices, with eigenvalues* λ_i *and multiplicities* m_i, *with largest eigenvalue* λ_0, *then* $\sum_i m_i\lambda_i^2 \le v\lambda_0$ *with equality if and only if G is regular.*

The *Laplace matrix* Q of a graph is defined by $Q = D - A$, where D is the diagonal matrix of vertex degrees, and A is the adjacency matrix. The Laplace matrix is positive semidefinite, and has row sums zero, so it has a zero eigenvalue. Moreover, the multiplicity of the zero eigenvalue equals the number of connected components of the graph. Also here we have a characterization of regularity.

LEMMA 1.3.3. *Let G be a graph on v vertices, with Laplace eigenvalues* θ_i *and multiplicities* m_i, *then* $v\sum_i m_i(\theta_i^2 - \theta_i) \ge (\sum_i m_i\theta_i)^2$ *with equality if and only if G is regular.*

Proof. Suppose G has vertex degrees d_i and average degree k_{ave}. From the trace of the Laplace matrix Q it follows that $\sum_i d_i = \sum_i m_i\theta_i$. From the trace of Q^2 we derive that $\sum_i (d_i^2 + d_i) = \sum_i m_i\theta_i^2$. Thus it follows that $0 \le v\sum_i (d_i - k_{\text{ave}})^2 = v\sum_i m_i(\theta_i^2 - \theta_i) - (\sum_i m_i\theta_i)^2$, which proves the statement. □

For regular graphs, say of degree k, we have that $D = kI$, so the adjacency eigenvalues λ_i and the Laplace eigenvalues θ_i are related by $\theta_i = k - \lambda_i$. In general, however, the two spectra behave different. For example, we mentioned that the multiplicity of the zero Laplace eigenvalue indicates whether the graph is connected, but in general we cannot see

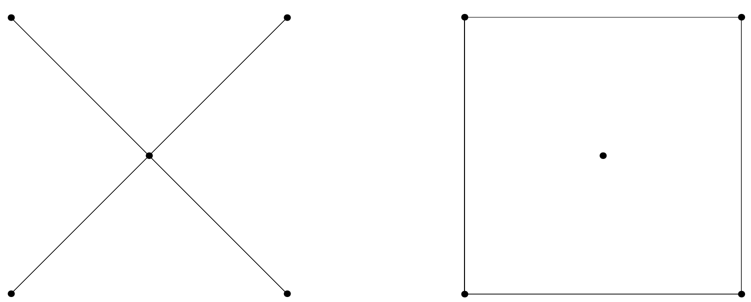

Figure 1.3.2. A connected and a disconnected graph with the same adjacency spectrum

from the adjacency spectrum whether a graph is connected. The graphs in Figure 1.3.2 (a standard example) have the same adjacency spectrum, but one is connected, while the other is not. On the other hand, we cannot see from the Laplace spectrum whether a graph is bipartite, while we can from the adjacency spectrum: a graph is bipartite if and only if its adjacency spectrum is symmetric around zero. (The graphs in Figure 1.3.2 have spectrum $\{[2]^1, [0]^3, [-2]^1\}$, and indeed, they are both bipartite.) The graphs in Figure 1.3.3 however have the same Laplace spectrum, and only one is bipartite.

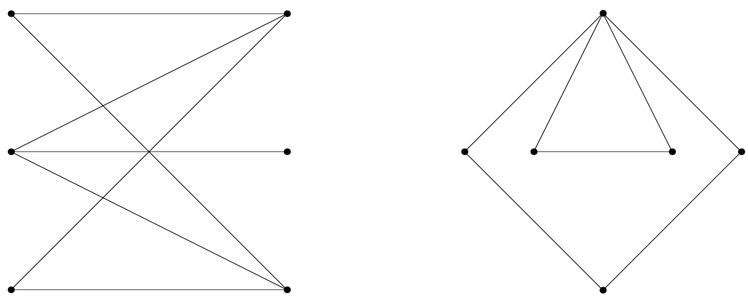

Figure 1.3.3. A bipartite and a nonbipartite graph with the same Laplace spectrum

The two spectra also have properties in common, like the following diameter bound.

LEMMA 1.3.4. *Let G be a connected graph with* $r + 1$ *distinct* (*adjacency or Laplace*) *eigenvalues. Then G has diameter at most r.*

As a consequence we find that the connected graphs with two eigenvalues are complete graphs. Note that if a graph has only one (adjacency or Laplace) eigenvalue, then this eigenvalue must be zero, and so the graph must be empty.

Both the adjacency and the Laplace eigenvalues are algebraic integers, as they are the roots of the respective characteristic polynomials, which are monic with integral coefficients. Here we shall state some more basic properties of the eigenvalues, using some elementary lemmas about polynomials with rational or integral coefficients (for example see [51]). By $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ we denote the rings of polynomials over the integers and rationals, respectively.

LEMMA 1.3.5. *If a monic polynomial* $p \in \mathbb{Z}[x]$ *has a monic divisor* $q \in \mathbb{Q}[x]$, *then also* $q \in \mathbb{Z}[x]$.

LEMMA 1.3.6. *Let* $p \in \mathbb{Q}[x]$. *If and only if* $a + \sqrt{b}$, *with* $a, b \in \mathbb{Q}$, *is an irrational root of p, then so is* $a - \sqrt{b}$, *with the same multiplicity.*

Applying these lemmas we now obtain the following results.

COROLLARY 1.3.7. *Every rational eigenvalue of a graph is integral.*

COROLLARY 1.3.8. *Let G be a graph. If and only if* $\frac{1}{2}(a + \sqrt{b})$ *is an irrational eigenvalue of* G*, for some* $a, b \in \mathbb{Q}$*, then so is* $\frac{1}{2}(a - \sqrt{b})$*, with the same multiplicity, and* $a, b \in \mathbb{Z}$.

The *minimal polynomial* m of the adjacency matrix A of a graph is the unique monic polynomial $m(x) = x^{r+1} + m_r x^r + \ldots + m_0$ of minimal degree such that $m(A) = O$. Similarly the minimal polynomial for the Laplace matrix is defined.

LEMMA 1.3.9. *The minimal polynomial m of a graph has integral coefficients.*

Proof. The following short argument was pointed out by Rowlinson [private communication]. The equation $m(A) = O$ can be seen as a system of v^2 (if v is the size of A) linear equations in the unknowns m_i, with integral coefficients. Since the system has a unique solution, this solution must be rational. (The solution can be found by Gaussian elimination, and during this algorithm all entries of the system remain rational.) So the minimal polynomial has rational coefficients, and since it divides the characteristic polynomial, we find $m \in \mathbb{Z}[x]$. □

1.3.3. Interlacing

A sequence of real numbers $b_1 \geq b_2 \geq \ldots \geq b_m$ is said to *interlace* another sequence of real numbers $a_1 \geq a_2 \geq \ldots \geq a_n$, $n > m$, if $a_i \geq b_i \geq a_{n-m+i}$ for $i = 1,\ldots, m$. The *interlacing* is called *tight* if there is an integer k, $0 \leq k \leq m$, such that $a_i = b_i$ for $i = 1,\ldots, k$, and $a_{n-m+i} = b_i$ for $i = k + 1,\ldots, m$.

An interesting property of symmetric matrices is that the eigenvalues of any principal submatrix interlace the eigenvalues of the whole matrix. When applied to graphs, we find that the adjacency eigenvalues of an induced subgraph interlace the adjacency eigenvalues of the whole graph. Interlacing of eigenvalues also occurs when we partition a matrix symmetrically and take the *quotient matrix*. This matrix is the matrix of average row sums of the blocks of the partitioned matrix. The eigenvalues of the quotient matrix interlace the eigenvalues of the original matrix. Moreover, if the interlacing is tight, then the matrix partition is *regular* (also called equitable), that is, in every block of the partitioned matrix the row sums are constant. This eigenvalue technique is used often, and turns out to be quite handy. For the proofs and many applications we refer to [57, 58]. Let's illustrate the technique by deriving Hoffman's coclique bound. Consider a regular graph G on v vertices of degree k with smallest eigenvalue $\lambda_{\min}$, with a coclique of size $\alpha(G)$. Partition the vertices of the graph into two parts: the coclique and the remaining vertices. Now partition the adjacency matrix A symmetrically according to the vertex partition. Then

$$A = \begin{pmatrix} O & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$$

with quotient matrix

$$B = \begin{pmatrix} 0 & k \\ k\dfrac{\alpha(G)}{v-\alpha(G)} & k - k\dfrac{\alpha(G)}{v-\alpha(G)} \end{pmatrix},$$

with eigenvalues k and $-k\alpha(G)/(v - \alpha(G))$. Since the eigenvalues of B interlace the eigenvalues of A, we find that $-k\alpha(G)/(v - \alpha(G)) \geq \lambda_{\min}$, from which the bound $\alpha(G) \leq v\lambda_{\min}/(\lambda_{\min} - k)$ follows. Cocliques meeting this so-called Hoffman bound are called *Hoffman cocliques*. Consequently for such cocliques we find that the interlacing of the eigenvalues of B and A is tight, and so the corresponding vertex partition is regular. So we find that any vertex outside a Hoffman coclique is adjacent to $k\alpha(G)/(v - \alpha(G))$ vertices of that coclique.

1.3.4. Graphs with least eigenvalue −2

Graphs with least (adjacency) eigenvalue −2 have been extensively studied. The characterization of Cameron, Goethals, Seidel and Shult [25] is of major importance in studying such graphs. If A is the adjacency matrix of such a graph, then $A + 2I$ is positive semidefinite, and thus it is the *Gram matrix* of a set of vectors in $\mathbb{R}^n$, i.e. $A + 2I = N^TN$, where the columns of N are the vectors *representing* the graph. These vectors must have length $\sqrt{2}$, and mutual inner products 1 or 0, so the vectors must have mutual angles 60° or 90°.

Examples are line graphs, cocktail party graphs and their common generalization, the generalized line graphs. If H is a graph, then the *line graph* $L(H)$ of H is obtained by taking the edges of H as vertices, any two of them being adjacent if the corresponding edges of H have a vertex of H in common. If N is the vertex-edge incidence matrix, that is, the matrix with rows indexed by the vertices of H and columns indexed by the edges of H, where $N_{xe} = 1$ if $x \in e$, and 0 otherwise, then $A + 2I = N^TN$, where A is the adjacency matrix of $L(H)$. A *cocktail party graph* $CP(n)$ is the complement of the disjoint union of n edges. A *generalized line graph* $L(H; a_1,..., a_m)$, where H is a graph on m vertices and a_i are nonnegative integers, is obtained by taking the line graph $L(H)$ of H and cocktail party graphs $CP(a_i)$ and adding edges between any vertex of $CP(a_i)$ and any vertex of $L(H)$ corresponding to an edge of H containing i.

Using the characterization of all irreducible root lattices, it is found that any connected graph with least eigenvalue −2 is represented by vectors in D_n or by vectors in E_8 (cf.

[25]). Furthermore, the graphs represented by vectors in D_n are precisely the generalized line graphs. The indices here denote the dimension of the space in which the graph is represented. Some of the graphs which are represented by vectors in E_8 may also be represented in the subsystems E_6 or E_7 of E_8. An example of a graph represented by vectors in E_6 is the Petersen graph.

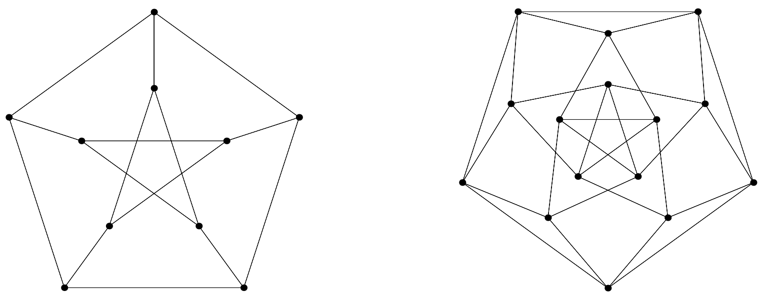

Figure 1.3.4. The Petersen graph and its line graph

1.3.5. Strongly regular graphs

A graph G is called *strongly regular* with parameters (v, k, λ, μ) if it has v vertices, is regular of degree k (with $0 < k < v - 1$), any two adjacent vertices have λ common neighbours and any two nonadjacent vertices have μ common neighbours. A connected strongly regular graph has three distinct (adjacency and Laplace) eigenvalues. In fact, any connected regular graph with three distinct (adjacency or Laplace) eigenvalues is strongly regular. Moreover, the eigenvalues determine the parameters, and the other way around. Note that the complement of a strongly regular graph is also strongly regular. We already saw some examples: the 5-cycle, the Petersen graph and the cocktail party graphs. Strongly regular graphs are very well investigated (cf. [16, 95]), and in this thesis we shall deviate from the concept of strongly regular graphs in a few different directions. In that way, strongly regular graphs play an important role in this thesis. They are often building blocks for the graphs that we are interested in. As such, we shall give the definitions of some important strongly regular graphs.

The *triangular graph* $T(n)$ is the line graph of the complete graph K_n, so we can represent the vertices by the unordered pairs $\{i, j\}$, $i, j = 1,..., n$, with two pairs adjacent if they intersect. The *Petersen graph* is the complement of $T(5)$. The triangular graph $T(n)$ is determined by its parameters (and spectrum) if $n \neq 8$. For $n = 8$, there are three other graphs with the same parameters, the so-called Chang graphs.

The *lattice graph* $L_2(n)$ is the line graph of the complete bipartite graph $K_{n,n}$, i.e. the vertices are the ordered pairs (i, j), $i, j = 1,..., n$, with two pairs adjacent if they coincide in one of the coordinates. The lattice graph $L_2(n)$ is determined by its spectrum unless $n = 4$. It that case there also is the Shrikhande graph. The *Latin square graphs* $L_m(n)$ are generalizations of the lattice graphs, and are constructed from mutually orthogonal Latin squares. Our definition uses the equivalent concept of orthogonal arrays. An *orthogonal array* is an $m \times n^2$ matrix (array) M with entries in $\{1,..., n\}$ such that for any two rows a, b we have that $\{(M_{ai}, M_{bi}) \mid i = 1,..., n^2\} = \{(i, j) \mid i, j = 1,..., n\}$. A graph $L_m(n)$ has vertices $1, 2,..., n^2$, and two vertices x, y are adjacent if $M_{ix} = M_{iy}$ for some i. This graph is strongly regular with spectrum $\{[mn-m]^1,\ [n-m]^{m(n-1)},\ [-m]^{(n-1)(n-m+1)}\}$.

For completeness, here we shall give some results on the numbers of nonisomorphic strongly regular graphs with parameters (v, k, λ, μ) on $v \le 40$ vertices, as they appear in the Appendix A.2. The 15 graphs with parameters (25, 12, 5, 6) and 10 graphs with parameters (26, 10, 3, 4) were found by Paulus [87]. An exhaustive computer search by Arlazarov, Lehman and Rosenfeld [1] showed that these are all the graphs with these parameters. In the same paper 41 graphs with parameters (29, 14, 6, 7) were found by an incomplete search (see also [19]). Independent exhaustive searches by Bussemaker and Spence (cf. [101]) showed that these are all. Bussemaker, Mathon and Seidel [19] also give 82 graphs with parameters (37, 18, 8, 9). According to Spence [101] there exist at least 3854 graphs with parameters (35, 16, 6, 8), 32548 graphs with parameters (36, 15, 6, 6) and 180 graphs with parameters (36, 14, 4, 6). Spence [97] also gives 27 graphs with parameters (40, 12, 2, 4). For other parameter sets occuring in Appendix A.2 we refer to [16].

1.3.6. Association schemes

Let V be a finite set of vertices. A *symmetric relation* on V is the same as a graph, if we allow loops (that is, edges between a vertex and itself). A d-class *association scheme* on V consists of a set of $d + 1$ symmetric relations $\{R_0, R_1,..., R_d\}$ on V, with identity (trivial) relation $R_0 = \{(x, x) \mid x \in V\}$, such that any pair of vertices is in precisely one relation. Furthermore, there are *intersection numbers* p_{ij}^k such that for any $(x, y) \in R_k$, the number of vertices z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k. If a pair of vertices is in relation R_i, then these vertices are called i-th *associates*. If the union of some relations is a nontrivial equivalence relation, then the scheme is called *imprimitive*, otherwise it is called *primitive*. By historical reasons the relations are called as such, but as the nontrivial relations are just ordinary graphs (these relations form a partition of the edge set of the complete graph), we shall both use the terminology of graphs and relations. For example, a 2-class association scheme is the same as a pair of complementary strongly regular graphs. In Figure 1.3.5 the three graphs of a 3-class association scheme are drawn.

Association schemes were introduced by Bose and Shimamoto [9]. Delsarte [42] applied association schemes to coding theory, and he used a slightly more general definition by not requiring symmetry for the relations, but for the total set of relations and for the intersection numbers. To study permutation groups, Higman (cf. [65]) introduced the even more general *coherent configurations*, for which the identity relation may be the union of some relations. In coherent configurations for which the identity relation is not one of its relations we must have at least 5 classes (6 relations).

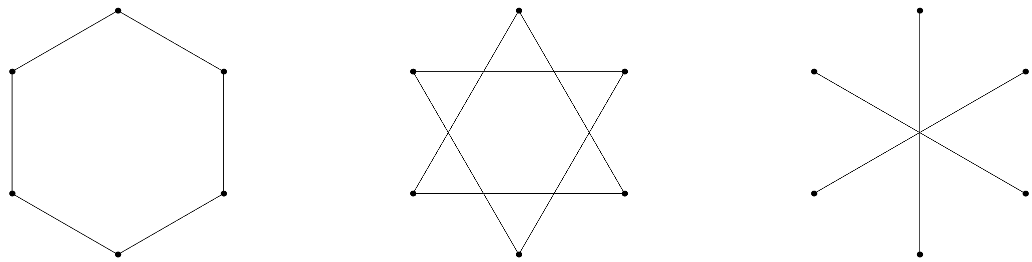

Figure 1.3.5. Three graphs forming a 3-class association scheme

There is a strong connection with group theory in the following way. If G is a permutation group acting on a vertex set V, then the *orbitals*, that is, the orbits of the action of G on V^2, form a coherent configuration. If G acts *generously transitive*, that is, for any two vertices there is a group element interchanging them, then we get an association scheme. If so, then we say the scheme is in the *group case*.

As general references for association schemes we use [3, 12, 15, 52].

1.3.6.1. The Bose-Mesner algebra

The nontrivial relations can be considered as graphs, which in our case are undirected. One immediately sees that the respective graphs are regular with degree $n_i = p_{ii}^0$. For the corresponding adjacency matrices A_i the axioms of the scheme are equivalent to

$$\sum_{i=0}^{d} A_i = J, \quad A_0 = I, \quad A_i = A_i^{\,T}, \quad A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \ .$$

It follows that the adjacency matrices generate a $(d + 1)$-dimensional commutative algebra **A** of symmetric matrices. This algebra was first studied by Bose and Mesner [8] and is called the *Bose-Mesner algebra* of the scheme. The corresponding algebra of a coherent configuration is called a *coherent algebra*, or by some authors a cellular algebra or cellular ring (with identity) (cf. [48]).

A very important property of the Bose-Mesner algebra is that it is not only closed under ordinary (matrix) multiplication, but also under entrywise (Hadamard, Schur) multiplication $\circ$. In fact, any vector space of symmetric matrices that contains the identity matrix I and the all-one matrix J, and that is closed under ordinary and entrywise multiplication is the Bose-Mesner algebra of an association scheme (cf. [12, Thm. 2.6.1]).

1.3.6.2. The spectrum of an association scheme

Since the adjacency matrices of the scheme commute, they can be diagonalized simultaneously, that is, the whole space can be written as a direct sum of common eigenspaces. In fact, **A** has a unique basis of minimal *idempotents* E_i, $i = 0,\ldots, d$. These are matrices such that

$$E_i E_j = \delta_{ij} E_i \;, \quad \text{and} \quad \sum_{i=0}^{d} E_i = I \;.$$

(The idempotents are projections on the eigenspaces.) Without loss of generality we may take $E_0 = v^{-1}J$. Now let P and Q be matrices such that

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \quad \text{and} \quad E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_i \;.$$

Thus $PQ = QP = vI$. It also follows that $A_j E_i = P_{ij} E_i$, so P_{ij} is an eigenvalue of A_j with multiplicity $m_i = \text{rank}(E_i)$. The matrices P and Q are called the *eigenmatrices* of the association scheme. The first row and column of these matrices are always given by $P_{i0} = Q_{i0} = 1$, $P_{0i} = n_i$ and $Q_{0i} = m_i$. Furthermore P and Q are related by $m_i P_{ij} = n_j Q_{ji}$. Other important properties of the eigenmatrices are given by the *orthogonality relations*

$$\sum_{i=0}^{d} m_i P_{ij} P_{ik} = v n_j \delta_{jk} \quad \text{and} \quad \sum_{i=0}^{d} n_i Q_{ij} Q_{ik} = v m_j \delta_{jk} \;.$$

The *intersection matrices* L_i defined by $(L_i)_{kj} = p_{ij}^k$ also have eigenvalues P_{ji}. In fact, the columns of Q are eigenvectors of L_i. Moreover, the algebra generated by the intersection matrices is isomorphic to the Bose-Mesner algebra.

An association scheme is called *self-dual* if $P = Q$ for some ordering of the idempotents.

A scheme is said to be *generated* by one of its relations R_i (or the corresponding graph) if this relation determines the other relations (immediately from the definitions of the scheme). In terms of the adjacency matrix A_i, this means that the powers of A_i, and J span the Bose-Mesner algebra. It easily follows that, if the corresponding graph is connected, then it generates the whole scheme if and only if it has $d + 1$ distinct eigenvalues. For example, a distance-regular graph generates the whole scheme.

1.3.6.3. The Krein parameters

As the Bose-Mesner algebra is closed under entrywise multiplication, we can write

$$E_i \circ E_j = \tfrac{1}{v} \sum_{k=0}^{d} q_{ij}^k E_k$$

for some real numbers q_{ij}^k, called the *Krein parameters* or dual intersection numbers. We can compute these parameters from the eigenvalues of the scheme by the equation

$$q_{ij}^k = \frac{m_i m_j}{v} \sum_{l=0}^{d} \frac{P_{il} P_{jl} P_{kl}}{n_l^2} \ .$$

The so-called *Krein conditions*, proven by Scott, state that the Krein parameters are nonnegative. Another restriction related to the Krein parameters is the so-called *absolute bound*, which states that for all i, j

$$\sum_{q_{ij}^k \neq 0} m_k \le \begin{cases} m_i m_j & \text{if } i \neq j\,, \\ \tfrac{1}{2} m_i (m_i + 1) & \text{if } i = j\,. \end{cases}$$

1.3.6.4. Distance-regular graphs

A *distance-regular graph* is a connected graph for which the distance relations (i.e. a pair of vertices is in R_i if their distance in the graph is i) form an association scheme. They were introduced by Biggs [6], and are widely investigated. As general reference we use [12]. A distance-regular graph with diameter two is a connected strongly regular graph and vice versa. Examples of distance-regular graphs with diameter three were already given by the line graph of the Petersen graph and the 6-cycle in Figures 1.3.4 and 1.3.5, respectively.

It is well known that an imprimitive distance-regular graph is bipartite or antipodal. *Antipodal* means that the union of the distance d relation and the trivial relation is an equivalence relation.

The property that one of the relations of a d-class association scheme forms a distance-regular graph with diameter d is equivalent to the scheme being *P-polynomial*, that is, the relations can be ordered such that the adjacency matrix A_i of relation R_i is a polynomial of degree i in A_1, for every i. In turn, this is equivalent to the conditions $p_{1i}^{i+1} > 0$ and $p_{1i}^k = 0$ for $k > i + 1$, $i = 0,..., d - 1$. For a 3-class association scheme the conditions are equivalent to $p_{11}^3 = 0$, $p_{11}^2 > 0$ and $p_{12}^3 > 0$ for some ordering of the relations.

Dually we say that the scheme is *Q-polynomial* if the idempotents can be ordered such that the idempotent E_i is a polynomial of degree i in E_1 with respect to entrywise multiplication, for every i. Equivalent conditions are that $q_{1i}^{i+1} > 0$ and $q_{1i}^{k} = 0$ for $k > i + 1$, $i = 0, \ldots, d - 1$. In the case of a 3-class association scheme these conditions are equivalent to $q_{11}^{3} = 0$, $q_{11}^{2} > 0$ and $q_{12}^{3} > 0$ for some ordering of the idempotents. (Here we say that the scheme has Q-polynomial ordering 123.)

1.3.7. Designs and codes

A t-(v, k, λ) *design* is a set of v points and a set of k-subsets of points, called blocks, such that any t-subset of points is contained in precisely λ blocks. In such a design the number of blocks equals $b = \lambda\binom{v}{t}/\binom{k}{t}$, and every point is in a constant number of blocks, called the *replication number* $r = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$. The *incidence matrix* N is the matrix with rows indexed by the points and columns indexed by the blocks such that $N_{xb} = 1$ if $x \in b$ (the point and block are called *incident*), and 0 otherwise. For a 2-design we have that $NN^T = rI + \lambda(J - I)$. A *symmetric* design is a 2-design with as many blocks as points. For such a design $k = r$, and $N^TN = kI + \lambda(J - I)$. A (finite) *projective plane* of order n is a 2-$(n^2 + n + 1, n + 1, 1)$ design. An *affine plane* of order n is a 2-$(n^2, n, 1)$ design. Here the blocks are also called *lines*. The *projective geometry* $PG(n, q)$ consists of all subspaces of the $(n + 1)$-dimensional vector space $GF(q)^{n+1}$ over the finite field $GF(q)$ of q elements. A projective point is a subspace of dimension one (a line through the origin in the vectorspace). The (projective) points and lines in $PG(2, q)$ form a projective plane, called the Desarguesian plane. The *Fano plane* is the (Desarguesian) projective plane of order 2. The *incidence graph* of a design is the bipartite graph with vertices the points and blocks of the design, where a point and a block are adjacent if and only if they are incident. A general reference for designs is [4].

A *code* C of length n is a subset of Q^n, where Q is some set, called the alphabet. If $Q = \{0, 1\}$ then the code is called *binary*. The (*Hamming*) *distance* between two codewords (elements of C) is the number of coordinates in which they differ. The *weight* of a codeword is the number of nonzero coordinates. The *minimum distance* of a code is the minimum Hamming distance between any two distinct codewords. The code is called e-error-correcting if the minimum distance is at least $2e + 1$. The *covering radius* of a code is the minimal number ρ such that any element of Q^n has Hamming distance at most ρ from some codeword.

If Q is a field (or a ring), so that Q^n is a vector space (a module), then the code is called *linear* if it is a linear subspace of Q^n. The *dimension* of the code is the dimension as a subspace. An $[n, k, d]$ code denotes a code of length n, dimension k and minimum distance d. The *dual* code $C^\perp$ of a code C consists of the words in Q^n that are orthogonal to all codewords of C. If C is linear with dimension k, then $C^\perp$ is linear with dimension $n - k$. A general reference for codes is [77].

1.3.8. Switching and the Seidel spectrum

Let G be a graph, and partition the vertices into two parts. (*Seidel*) *Switching* G according to this partition means that we change the graph by interchanging the edges and nonedges between the two parts. If A is the adjacency matrix of G which is partitioned according to the vertex partition as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

then the adjacency matrix A' of the switched graph equals

$$A' = \begin{pmatrix} A_{11} & J - A_{12} \\ J - A_{12}^T & A_{22} \end{pmatrix}.$$

If the partition is regular, then

$$\text{spectrum } A' = \text{spectrum } A \cup \text{spectrum } B' \setminus \text{spectrum } B,$$

where B and B' are the quotient matrices of A and A', respectively, according to the vertex partition.

A more natural matrix to study graphs and switching is the *Seidel matrix*. This matrix S is defined by $S = J - I - 2A$, where A is the adjacency matrix. For k-regular graphs on v vertices the Seidel eigenvalues ρ_i and the adjacency eigenvalues λ_i are related by $\rho_0 = v - 1 - 2k$, $\rho_i = -1 - 2\lambda_i$, $i \neq 0$. For nonregular graphs, like with the Laplace matrix, the two spectra behave different. An interesting property of the Seidel matrix is that switching does not change the Seidel spectrum. As switching defines an equivalence relation on graphs, it seems interesting to study switching classes of graphs. In Figure 1.3.6 a switching class of graphs is drawn.

1.3.9. A touch of flavour - Graphs with few Seidel eigenvalues

Rewriting the definition of the Seidel matrix gives $S_{xy} = -1$ if x and y are adjacent, $S_{xy} = 1$ if x and y are not adjacent, and zero diagonal. It follows that any graph on more than one vertex has at least two distinct Seidel eigenvalues. The empty and the complete graph have two distinct eigenvalues, but there is more. As a generalization of strongly regular graphs and conference matrices, Seidel (cf. [94]) introduced the so-called *strong graphs*. These are nonempty, noncomplete graphs whose Seidel matrix S satisfies the equation

$$(S - \rho_1 I)(S - \rho_2 I) = (v - 1 + \rho_1\rho_2)J,$$

for some ρ_1, ρ_2, and where v is the number of vertices. If $v - 1 + \rho_1\rho_2 \neq 0$, then it follows that S and J commute, and so they can be diagonalized simultaneously, and so the all-one vector is also an eigenvector of S, implying that the graph is regular, and so it follows that it is strongly regular. So if a strong graph is nonregular, then $v - 1 + \rho_1\rho_2 = 0$, and so it has two distinct Seidel eigenvalues. On the other hand, any graph with two distinct Seidel eigenvalues ρ_1, ρ_2 satisfies the equation $(S - \rho_1 I)(S - \rho_2 I) = O$, and by checking the diagonal it follows that $v - 1 + \rho_1\rho_2 = 0$, so we have a strong graph, unless the graph is empty or complete.

In order to give a combinatorial characterization of strong graphs, we define $p(x, y)$ as the number of neighbours of x which are not neighbours of y. Now a nonempty, noncomplete graph is strong if and only if the numbers $p(x, y) + p(y, x)$ only depend on whether x and y are adjacent or not. This follows from checking the quadratic equation for S, and moreover, it follows that for a strong graph we have

$p(x, y) + p(y, x) = -\frac{1}{2}(\rho_1 - 1)(\rho_2 - 1)$ if x and y are adjacent, and
$p(x, y) + p(y, x) = -\frac{1}{2}(\rho_1 + 1)(\rho_2 + 1)$ if x and y are not adjacent.

There are many examples of nonregular strong graphs, and we can find them in the switching classes of so-called regular two-graphs, as these are precisely the switching classes consisting of strong graphs with two Seidel eigenvalues (cf. [94]).

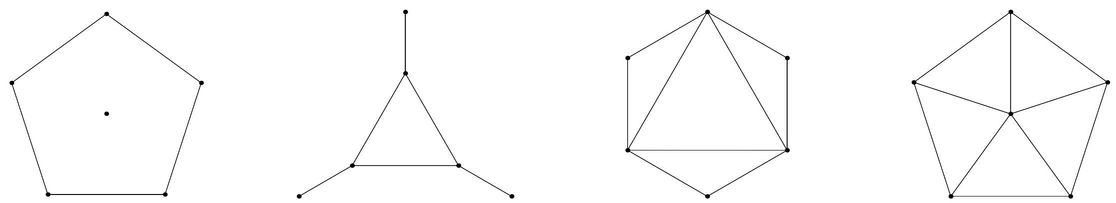

Figure 1.3.6. A switching class of strong graphs with Seidel eigenvalues $\pm\sqrt{5}$

Chapter 2

Ik ben met m'n voeten eerst geboren. M'n hoofd kwam er pas later uit! 't Zou mij trouwens verbazen dat m'n hoofd en m'n voeten van dezelfde vader zijn ... want ze lijken absoluut niet op elkaar!

(Kamagurka, *Bezige Bert*)

Graphs with three eigenvalues

In this chapter we have a look at the graphs that are generalizations of strongly regular graphs by dropping regularity. More precisely, we have a look at graphs with three distinct eigenvalues, for the adjacency and Laplace spectrum. Seidel (cf. [94]) already did a similar thing for the Seidel spectrum by introducing strong graphs, which turned out to have an easy combinatorial characterization, as we saw in Section 1.3.9.

When looking from the point of view of the adjacency spectrum, the combinatorial simplicity seems to disappear with the regularity. This all lies in the algebraic consequence that the all-one vector is no longer an eigenvector. This is what makes the adjacency matrix less appropriate for studying nonregular graphs. Here we should keep in mind that algebra and spectral techniques are tools in graph theory, although they have become subjects of their own. Still the fundamental question of few eigenvalues is interesting.

An alternative for studying (nonregular) graphs is the Laplace matrix. Roughly speaking, dropping regularity has no algebraic consequences. This enables us to show that graphs with three Laplace eigenvalues have an easy combinatorial characterization.

2.1. The adjacency spectrum

Connected graphs with only two distinct eigenvalues are easily proven to be complete graphs. Therefore the first nontrivial case consists of graphs with three distinct eigenvalues, with the regular ones being precisely the strongly regular graphs. A large family of (in general) nonregular examples is given by the complete bipartite graphs $K_{m,n}$ with spectrum $\{[\sqrt{mn}]^1, [0]^{m+n-2}, [-\sqrt{mn}]^1\}$. Other examples were found by Bridges and Mena [10] and Muzychuk and Klin [85], most of them being cones. A *cone* over a graph H is obtained by adding a vertex to H that is adjacent to all vertices of H. The cone can be obtained by switching an extra vertex in (see Figure 1.3.6 for the cone over the 5-cycle). If H is a strongly regular graph on v vertices, with degree k and smallest eigenvalue s, then it follows from Section 1.3.8 that the cone over H is a graph with three eigenvalues if and only if $s(k - s) = -v$ (cf. [85]). This condition is satisfied by infinitely many strongly regular graphs, which implies that there are infinitely many cones with three eigenvalues. A small example is given by the cone over the Petersen graph, with spectrum $\{[5]^1, [1]^5, [-2]^5\}$ (see Figure 2.1.1). Bridges and Mena [10] obtained results on cones with distinct eigenvalues λ_0, λ_1 and $-\lambda_1$. They proved that such graphs are cones over strongly regular (v, k, λ, λ) graphs with three possible exceptions (for the parameters). For two of the exceptions an example is given, the third exception is open.

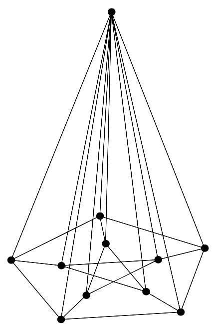

Figure 2.1.1. The cone over the Petersen graph

We know of four examples which are not cones that can be constructed by switching in a strongly regular graph. For example, switching in $T(9)$ with respect to the set of vertices $\{\{1, i\} \mid i = 2,..., 9\}$ (a 9-clique), gives a graph with spectrum $\{[21]^1, [5]^7, [-2]^{28}\}$ (cf. [85]). Similarly, switching with respect to an 8-clique in the strongly regular graph that is obtained from a polarity in Higman's symmetric 2-(176, 50, 14) design gives a nonregular graph with three eigenvalues, and so does switching with respect to three disjoint 6-cliques in the strongly regular Zara graph with parameters (126, 45, 12, 18). A more complicated example is constructed by Martin [private communication]. Take the strongly regular (105, 72, 51, 45) graph on the flags (incident point-line pairs) of $PG(2, 4)$, where two distinct flags (p_1, l_1) and (p_2, l_2) are adjacent if $p_1 = p_2$ or $l_1 = l_2$ or ($p_1 \notin l_2$ and $p_2 \notin l_1$). Now switching with respect to a set of 21 flags with the property that every point and every line is in precisely one flag (such a set exists by elementary combinatorial theory since it corresponds to a perfect matching in the incidence graph of $PG(2, 4)$) yields a nonregular graph with spectrum $\{[60]^1, [9]^{21}, [-3]^{83}\}$.

A new family of nonregular graphs with three eigenvalues, which are not cones, is constructed from symmetric 2-$(q^3 - q + 1, q^2, q)$ designs, which exist if q is a prime power and $q - 1$ is the order of a projective plane (cf. [4]). Take the incidence graph of such a design and add edges between all blocks. The resulting graph has spectrum $\{[q^3]^1, [q - 1]^{(q-1)q(q+1)}, [-q]^{(q-1)q(q+1)+1}\}$. The smallest example is derived from the complement of the Fano plane, and we find a graph with spectrum $\{[8]^1, [1]^6, [-2]^7\}$. The next case comes from 2-(25, 9, 3) designs. Denniston [43] found that there are exactly 78 such designs, and so there are at least 78 graphs with spectrum $\{[27]^1, [2]^{24}, [-3]^{25}\}$.

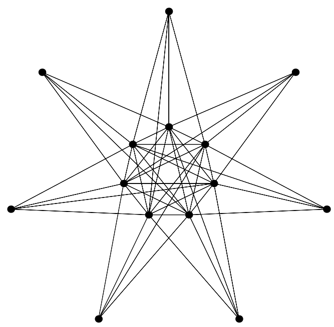

Figure 2.1.2. The graph derived from the complement of the Fano plane

Bridges and Mena [10] also mention a graph on 22 vertices with spectrum $\{[14]^1, [2]^7, [-2]^{14}\}$, which is not a cone. Muzychuk and Klin [85] descibed this graph in a way, similar to our construction: take the incidence graph of the unique quasi-symmetric 2-(8, 4, 3) design, and add an edge between two blocks if they intersect (in two points).

2.1.1. Nonintegral eigenvalues

For all known nonregular examples, except for the complete bipartite ones, all eigenvalues are integral. We shall prove that the only graphs with three eigenvalues for which the largest eigenvalue is not integral, are the complete bipartite graphs.

PROPOSITION 2.1.1. *Let G be a connected graph with three distinct eigenvalues of which the largest is not an integer. Then G is a complete bipartite graph.*

Proof. Suppose G has v vertices, adjacency matrix A with largest eigenvalue λ_0 and remaining eigenvalues λ_1 and λ_2. Since λ_0 is simple and not integral, it follows that at least one of these remaining eigenvalues is also simple and nonintegral. If we have only three vertices, then there is only one connected, noncomplete graph: $K_{1,2}$. So we may assume to have more than three vertices. In this case the remaining eigenvalue is of course not simple, and it follows that λ_0 and say λ_2 are of the form $\frac{1}{2}(a \pm \sqrt{b})$, with a, b integral, and then λ_1 is integral. Moreover, since $\lambda_2 \geq -\lambda_0$, we must have $a \geq 0$. Since the adjacency matrix of G has zero trace, it follows that $a + (v-2)\lambda_1 = 0$, so a is a multiple of $v-2$, and $\lambda_1 \leq 0$.

If $a = \lambda_1 = 0$, then G is bipartite. But G has diameter (at most) two, and so G must be a complete bipartite graph. If $\lambda_1 = -1$, then $a = v - 2$, and it follows that -1 is the smallest eigenvalue of G, otherwise we would have $\lambda_0 > v - 1$, which is a contradiction. But then $A + I$ is a positive semidefinite matrix of rank two, and we would have that G is the disjoint union of two cliques, which is again a contradiction. If $\lambda_1 = -2$, then $a = 2(v-2)$, and it follows that -2 is the smallest eigenvalue of G. Now $A + 2I$ is positive semidefinite of rank two, which cannot be the case. For the remaining case we have $\lambda_1 \le -3$, and then $a \ge 3(v-2)$. From $\lambda_0 \le v-1$, we now find that we can have at most three vertices, and so the proof is finished. □

So if we have a graph with three eigenvalues, which is not a complete bipartite graph, then we know that its largest eigenvalue is integral. The remaining two eigenvalues, however, can still be nonintegral, with many of the strongly regular conference graphs as examples. The following proposition reflects what is known as the "half-case" for strongly regular graphs.

PROPOSITION 2.1.2. *Let G be a connected graph on v vertices with three eigenvalues $\lambda_0 > \lambda_1 > \lambda_2$, which is not a complete bipartite graph. If not all eigenvalues are integral, then v is odd and $\lambda_0 = \frac{1}{2}(v-1)$, $\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{b}$, for some $b \equiv 1 \pmod 4$, $b \le v$, with equality if and only if G is strongly regular. Moreover, if $v \equiv 1 \pmod 4$ then all vertex degrees are even, and if $v \equiv 3 \pmod 4$ then $b \equiv 1 \pmod 8$.*

Proof. According to the previous proposition λ_0 is integral, so λ_1 and λ_2 must be of the form $\frac{1}{2}(a \pm \sqrt{b})$, with a, b integral, with the same multiplicity $\frac{1}{2}(v-1)$. Since the adjacency matrix has zero trace, we have $\lambda_0 + \frac{1}{2}a(v-1) = 0$. Since $0 < \lambda_0 < v-1$, and a is integral, we must have $\lambda_0 = \frac{1}{2}(v-1)$, $a = -1$. Now $\lambda_1\lambda_2$ is integral, and it follows that $b \equiv 1 \pmod 4$. Moreover, by Lemma 1.3.1 or Lemma 1.3.2 we have that the average vertex degree $k_{\text{ave}} = (\lambda_0^2 + \frac{1}{2}(v-1)(\lambda_1^2 + \lambda_2^2))/v = \frac{1}{2}\lambda_0(v+b)/v$ is at most λ_0, with equality if and only if G is strongly regular. This inequality reduces to $b \le v$.

From the equation $(A - \lambda_0 I)(A - \lambda_1 I)(A - \lambda_2 I) = O$, we find that

$$A^3 = \tfrac{1}{2}(v-3)A^2 + (\tfrac{1}{2}(v-1) + \tfrac{1}{4}(b-1))A - \tfrac{1}{8}(v-1)(b-1)I.$$

The diagonal element of this matrix corresponding to vertex x counts twice the number of triangles Δ_x through x. Thus we find that

$$\Delta_x = \tfrac{1}{4}(v-3)d_x - \tfrac{1}{16}(v-1)(b-1),$$

where d_x is the vertex degree of x. Since Δ_x is integral we find that if $v \equiv 1 \pmod 4$, then d_x must be even, for every vertex x. If $v \equiv 3 \pmod 4$, then we must have $b \equiv 1 \pmod 8$. □

Of course, also if the eigenvalues are integral, we find restrictions for the degrees from the expression for Δ_x.

COROLLARY 2.1.3. *Let G be a graph with three integral eigenvalues. If all three eigenvalues are odd, then all vertex degrees are odd. If one of them is odd and two are even, then all vertex degrees are even.*

Returning to the graphs of Proposition 2.1.2, we should mention that although we do not know any nonregular example, we might consider the cone over the Petersen graph (with $v = 11$, $b = 9$) as one.

2.1.2. The Perron-Frobenius eigenvector

An important property of connected graphs with three eigenvalues is that $(A - \lambda_1 I)(A - \lambda_2 I)$ is a rank one matrix. It follows that we can write

$$(A - \lambda_1 I)(A - \lambda_2 I) = \alpha\alpha^T, \text{ with } A\alpha = \lambda_0\alpha.$$

Moreover, from the Perron-Frobenius theorem it follows that (the Perron-Frobenius eigenvector) α is a positive eigenvector, that is, all its components are positive. From the quadratic equation we derive that

$d_i = -\lambda_1\lambda_2 + \alpha_i^2$	is the degree of vertex i,
$\lambda_{ij} = \lambda_1 + \lambda_2 + \alpha_i\alpha_j$	is the number of common neighbours of i and j, if they are adjacent,
$\mu_{ij} = \alpha_i\alpha_j$	is the number of common neighbours of i and j, if they are not adjacent.

If we assume G not to be complete bipartite, so that $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ are integral, it follows that $\alpha_i\alpha_j$ is an integer for all i and j. We immediately see that this imposes strong restrictions for the possible degrees that can occur. We also see that if the graph is regular, then we have a strongly regular graph.

Now suppose that G has only two vertex degrees (which is the case in most known nonregular examples), say k_1 and k_2, with respective α_1 and α_2. Now fix a vertex x of degree k_1. Let k_{11} and k_{12} be the numbers of vertices of degree k_1 and k_2, respectively, that are adjacent to x. Then it follows that $k_{11} + k_{12} = k_1$ and since $A\alpha = \lambda_0\alpha$, it follows that $k_{11}\alpha_1 + k_{12}\alpha_2 = \lambda_0\alpha_1$. These two equations uniquely determine k_{11} and k_{12}, and the solution is independent of the chosen vertex x of degree k_1. It follows that the partition of the vertices according to their degrees is regular, which is a very strong restriction for the parameters.

If we assume that G is a cone, say over H, then we can prove that we have at most three distinct degrees. Indeed, take a vertex with degree $k_1 = v - 1$, and suppose we have another vertex of degree d_i. Now the common neighbours of these vertices are all neighbours of the latter except the first vertex. So $d_i - 1 = \lambda_{1i} = \alpha_1\alpha_i + \lambda_1 + \lambda_2$. But also

$d_i = -\lambda_1\lambda_2 + \alpha_i^2$, and so we get a quadratic equation for d_i, and so d_i can take at most two values. If d_i takes only one value, then we easily see that H must be strongly regular. If we have precisely two other degrees, say k_2, k_3, with respective α_2, α_3, then it follows from the quadratic equation that $\alpha_1 = \alpha_2 + \alpha_3$. Here it also follows quite easily that the partition of the vertices according to their degrees is regular. Bridges and Mena [10] used this to show that there are only three parameter sets for cones with eigenvalues λ_0, $\pm\lambda_1$ over a nonregular graph.

2.1.3. Graphs with least eigenvalue −2

The results from Section 2.1.1 imply that the only connected graphs with three eigenvalues, all greater than −2 are the complete bipartite graphs $K_{1,2}$ and $K_{1,3}$ and graphs from Proposition 2.1.2 with $b = 5$. However, here we can only have the strongly regular 5-cycle C_5, which is the unique graph with spectrum $\{[2]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{5}]^2, [-\frac{1}{2} - \frac{1}{2}\sqrt{5}]^2\}$.

PROPOSITION 2.1.4. *If G is a connected graph with three distinct eigenvalues, all greater than −2, then G is either $K_{1,2}$, $K_{1,3}$, or C_5.*

Proof. By the previous remarks, besides $K_{1,2}$ and $K_{1,3}$ we only have to check spectra from Proposition 2.1.2 with $b = 5$. First suppose $v > 9$ (Note that $v \equiv 1 \pmod 4$). Then for the average vertex degree we have $k_{\text{ave}} < \frac{1}{2}(v-1)$. Since the vertex degrees must be even, there must be a vertex x of degree $d_x \le \frac{1}{2}(v-1) - 2$. If $d_x > 2$, then for the number of triangles Δ_x through x we have

$$\Delta_x = \tfrac{1}{4}(v-3)(d_x-1) - \tfrac{1}{2} > \tfrac{1}{4}(v-5)(d_x-1) \ge \binom{d_x}{2},$$

which is a contradiction. Also if $d_x = 2$, then $\Delta_x = \frac{1}{4}(v-5) > 2$, a contradiction. The case $v = 9$ can be excluded by the following arguments, using the Perron-Frobenius eigenvector α. Here it follows that there must be a vertex x of degree 2, and so with $\alpha_x = 1$. Now α is an integral vector, implying that the vertex degrees can only take values 2, 5, 10,... But the vertex degrees must be even, and at most 8, so it follows that the graph is regular, which is a contradiction. □

Now it would be interesting to know all graphs with three eigenvalues, all of which are at least −2. By the characterization of Cameron, Goethals, Seidel and Shult [25], it follows that such a graph is a generalized line graph or can be represented by roots in the lattice E_8.

THEOREM 2.1.4. *If G is a connected graph with three distinct eigenvalues, all at least −2, then G is isomorphic to one of $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, C_5, $L_2(n)$, $n \ge 2$, $T(n)$, $n \ge 4$, or $CP(n)$, $n \ge 2$, or G is represented by a subset of E_8.*

Proof. First, suppose that G is a connected line graph, not C_5 or $K_{1,2}$, of some graph H, and G has three eigenvalues, say $\lambda_0 > \lambda_1 > \lambda_2 = -2$. Here we may assume that H is connected. Then the adjacency matrix A of G can be written as $A = N^TN - 2I$, where N is the vertex-edge incidence matrix of H. Now $NN^T = D + B$, where D is the diagonal matrix of vertex degrees in H, and B is the adjacency matrix of H. It follows that $D + B$ has eigenvalues $\lambda_0 + 2$, $\lambda_1 + 2$, and possibly 0. Suppose 0 is an eigenvalue with eigenvector u. Then $N^Tu = \underline{0}$. This implies that if i and j are adjacent in H, then $u_i = -u_j$. So H is bipartite. Moreover, since $D + B$ has three distinct eigenvalues, it follows that H has diameter at most two (the diameter is also smaller than the number of distinct eigenvalues of $D + B$), so H must be a complete bipartite graph $K_{m,n}$. Since the line graph of a nonregular complete bipartite graph has four distinct eigenvalues (unless m or n equals one, then we get a complete graph, see Chapter 3), H must be the complete bipartite graph $K_{n,n}$, $n \geq 2$, with the lattice graph $L_2(n)$ as line graph. Now suppose that 0 is not an eigenvalue. Then $D + B$ has only two distinct eigenvalues, and it follows that H is a complete graph K_n, with the triangular graph $T(n)$ as line graph.

Next, we assume that G is a generalized line graph $L(H; a_1,..., a_m)$ (where m is the number of vertices of H), which is not a line graph, so at least one of the a_i is nonzero. Now G can be represented in $\mathbb{R}^n$, where $n = m + \sum_i a_i$, as follows. Take $\{e_{i,j} \mid i = 1,..., m, j = 0,..., a_i\}$ as orthonormal basis of $\mathbb{R}^n$, then we represent the vertices of G by the vectors $e_{i,0} + e_{j,0}$ for all edges $\{i, j\}$ in H, and the vectors $e_{i,0} + e_{i,j}$ and $e_{i,0} - e_{i,j}$ for all $i = 1,..., m$, $j = 1,..., a_i$, any two of them being adjacent if and only if they have inner product one. In matrix form, if N is the generalized $(0, \pm1)$-incidence matrix, that is, with rows representing the basis of $\mathbb{R}^n$, and columns representing the vertices of G, then $A = N^TN - 2I$ is the adjacency matrix of G. Now suppose that G has distinct eigenvalues $\lambda_0 > \lambda_1 > \lambda_2 = -2$, then NN^T has eigenvalues $\lambda_0 + 2$, $\lambda_1 + 2$, and possibly 0. Suppose we have an eigenvalue 0, say with eigenvector u. Then $N^Tu = \underline{0}$, so if i and j are adjacent in H, then $u_{i,0} = -u_{j,0}$. For i with a_i nonzero, we have that $u_{i,0} = -u_{i,j}$, and $u_{i,0} = u_{i,j}$ for $j = 1,..., a_i$. So $u_{i,j} = 0$, and since we may assume H to be connected, it follows that $u = \underline{0}$, so 0 is not an eigenvalue of NN^T. Now let us have a closer look at this matrix. After rearrangement of the rows of N, it follows that

$$NN^T = \begin{pmatrix} D+B & O \\ O & 2I \end{pmatrix},$$

where D is the diagonal matrix with entries $D_{ii} = d_i + 2a_i$, where d_i is the vertex degree of i in H, and B is the adjacency matrix of H. Thus it follows that $\lambda_1 = 0$ and that $D + B$ has distinct eigenvalues $\lambda_0 + 2$ (with multiplicity one) and (possibly) 2. If H is a graph on one vertex, then there are no further restrictions, and G then is a cocktail party graph $CP(n)$. Otherwise, H is a complete graph (since the diameter is one), and since $D + B - 2I$ is a rank one matrix, it follows that $D = 3I$. Since a_i is nonzero for some i, it now follows that

$d_i = a_i = 1$ for all i. But then H is a single edge, and G is $K_{1,4}$. □

The strongly regular graphs with all eigenvalues at least −2 have already been classified by Seidel (cf. [25]). Besides the 5-cycle, the lattice graphs, the triangular graphs and the cocktail party graphs, there are the Petersen graph, the complement of the Clebsch graph, the Shrikhande graph, the complement of the Schläfli graph, and the three Chang graphs.

In the beginning of this section, we already saw some nonregular graphs with three eigenvalues, all of which are at least −2. Besides these, we also mention the cones over the lattice graph $L_2(4)$, the Shrikhande graph, the triangular graph $T(8)$ and the three Chang graphs. We should mention that a graph that is represented by a subset of E_8 has at most 36 vertices and vertex degrees at most 28, thus there are finitely many (see [20]). Both bounds are tight for the example (see the beginning of this section) obtained by switching in $T(9)$.

Now let's have a look at the graphs that are represented by roots in E_8 (and which are not generalized line graphs). First we shall find the ones that have a representation in the subsystem E_6.

PROPOSITION 2.1.5. *The only connected graphs with three eigenvalues that are represented by roots in E_6, and which are not generalized line graphs are the Petersen graph, the cone over the Petersen graph and the complement of the Clebsch graph.*

Proof. For graphs represented by roots in E_6 the multiplicity of the eigenvalue −2 is $v - 6$, where v is the number of vertices. Consequently such graphs have spectrum $\{[2(v-6)-5\lambda_1]^1, [\lambda_1]^5, [-2]^{v-6}\}$, and we also may assume that $\lambda_1 \geq 1$ (a connected graph with exactly one positive eigenvalue must be a complete multipartite graph).

Using the inequality $\sum_i m_i\lambda_i^2 = vk_{\text{ave}} \leq v\lambda_0$ (Lemma 1.3.2), we find that we must have $\lambda_1 = 1$ or $\lambda_1 = 2$, and we find possible spectra $\{[8]^1, [2]^5, [-2]^9\}$, $\{[10]^1, [2]^5, [-2]^{10}\}$, $\{[3]^1, [1]^5, [-2]^4\}$, $\{[5]^1, [1]^5, [-2]^5\}$, $\{[7]^1, [1]^5, [-2]^6\}$, $\{[9]^1, [1]^5, [-2]^7\}$ and $\{[11]^1, [1]^5, [-2]^8\}$. The first three of these spectra imply regularity (by Lemma 1.3.2), and it is well known that there are unique graphs with such spectra: the triangular graph $T(6)$, the complement of the Clebsch graph, and the Petersen graph, respectively.

The graphs with one of the remaining spectra are sure to be nonregular, and by using the Perron-Frobenius eigenvector, the vertex degrees in such graphs are either 3, 6 and 11, or 4 and 10. Now it immediately follows that a graph with spectrum $\{[11]^1, [1]^5, [-2]^8\}$ does not exist, since here we would have $\lambda_0 \geq k_{\max}$, which would imply regularity by Lemma 1.3.1. If only two vertex degrees occur, then we know that the partition according to the vertex degrees is regular. For degrees 4 and 10, we find, by checking the parameters, that this can only occur for a graph with spectrum $\{[5]^1, [1]^5, [-2]^5\}$, with one vertex of degree 10 and ten vertices of degree 4. Here it follows that we must have the cone over the Petersen graph. Moreover, a graph with this spectrum cannot have vertex degrees 3 and 6 (no regular partition), thus the cone over the Petersen graph is the unique graph with spectrum $\{[5]^1, [1]^5, [-2]^5\}$.

For a graph with spectrum $\{[7]^1, [1]^5, [-2]^6\}$ and n_i vertices of degree i, we find no solutions to the requirements $n_3 + n_6 + n_{11} = 12$, $3n_3 + 6n_6 + 11n_{11} = 78$ $(= \sum_i m_i\lambda_i^2)$, $n_{11} = 0$ or $n_{11} = 1$ (if $n_{11} \geq 1$ then we have a cone), so such a graph cannot exist. For a graph with spectrum $\{[9]^1, [1]^5, [-2]^7\}$ we find that we must have $n_3 = 3$, $n_6 = 1$ and $n_{11} = 9$, but it is easily seen that this is impossible, so such a graph cannot exist. □

Next, let's have a look at the graphs that are represented by roots in E_7.

PROPOSITION 2.1.6. *The only connected graphs with three eigenvalues that are represented by roots in E_7, and which are not generalized line graphs or represented by roots in E_6 are the graph derived from the complement of the Fano plane* (*see Figure* 2.1.2), *the Shrikhande graph, the cone over the Shrikhande graph, the cone over the lattice graph* $L_2(4)$, *and the complement of the Schläfli graph.*

Proof. For such graphs the multiplicity of the eigenvalue -2 is $v - 7$. Consequently they have spectrum $\{[2(v-7) - 6\lambda_1]^1, [\lambda_1]^6, [-2]^{v-7}\}$, and also here we may assume that $\lambda_1 \geq 1$.

Using the inequality $\sum_i m_i\lambda_i^2 = vk_{\text{ave}} \leq v\lambda_0$, we find that we must have $\lambda_1 = 1$, 2, 3 or 4. For $\lambda_1 = 4$, we find possible spectra $\{[16]^1, [4]^6, [-2]^{20}\}$ and $\{[18]^1, [4]^6, [-2]^{21}\}$, which both imply regularity. A graph with the latter spectrum does not exist, and there is a unique graph with the first spectrum: the complement of the Schläfli graph.

If $\lambda_1 = 3$, we find possible spectra $\{[10]^1, [3]^6, [-2]^{14}\}$, $\{[12]^1, [3]^6, [-2]^{15}\}$, $\{[14]^1, [3]^6, [-2]^{16}\}$, $\{[16]^1, [3]^6, [-2]^{17}\}$ and $\{[18]^1, [3]^6, [-2]^{18}\}$, and the first and last imply regularity. It is well known that there is no graph with the last spectrum, and a unique graph with the first spectrum: $T(7)$. By using the Perron-Frobenius eigenvector, and the fact that by Corollary 2.1.3 all vertex degrees must be even, we find that in a graph with one of the remaining three spectra, the vertex degrees are either 10 and 22, or 8 and 14. In any case the partition according to the vertex degrees must be regular, which gives a contradiction for any of the three spectra.

Also if $\lambda_1 = 1$, the vertex degrees must be even, and here we find possible spectra $\{[4]^1, [1]^6, [-2]^5\}$, $\{[6]^1, [1]^6, [-2]^6\}$, $\{[8]^1, [1]^6, [-2]^7\}$, $\{[10]^1, [1]^6, [-2]^8\}$ and $\{[12]^1, [1]^6, [-2]^9\}$, all of which imply nonregularity. Here we find that the possible vertex degrees are 4 and 10, and thus we always have a regular partition. The only spectrum which survives the constraints is $\{[8]^1, [1]^6, [-2]^7\}$, and we find that we must have seven vertices of degree 4 and seven vertices of degree 10. Moreover, the vertices of degree 4 induce a coclique, and the vertices of degree 10 induce a clique. Since any two vertices of degree 4 have two common neighbours (this follows from the Perron-Frobenius eigenvector), the edges between the vertices of degree 4 and the vertices of degree 10 form the incidence of the complement of the Fano plane. So here we find the graph of Figure 2.1.2, which is now proven to be the unique graph with spectrum $\{[8]^1, [1]^6, [-2]^7\}$.

In the remaining case $\lambda_1 = 2$, we find possible spectra $\{[6]^1, [2]^6, [-2]^9\}$, which is realized only by the lattice graph $L_2(4)$ and the Shrikhande graph, $\{[8]^1, [2]^6, [-2]^{10}\}$,

$\{[10]^1, [2]^6, [-2]^{11}\}$, $\{[12]^1, [2]^6, [-2]^{12}\}$, $\{[14]^1, [2]^6, [-2]^{13}\}$ and $\{[16]^1, [2]^6, [-2]^{14}\}$. The last spectrum implies regularity, but no such strongly regular graph exists. In the other four possibilities we must have vertex degrees either 5, 8 and 13, or 6 and 12, or 7 and 16. First suppose that we have vertex degrees 5, 8 and 13. Consider a graph with spectrum $\{[8]^1, [2]^6, [-2]^{10}\}$, and n_5, n_8 and n_{13} vertices of degrees 5, 8 and 13, respectively. From the equations $n_5 + n_8 + n_{13} = 17$, $5n_5 + 8n_8 + 13n_{13} = 128$, we find either $n_5 = 11$, $n_8 = 1$, $n_{13} = 5$ or $n_5 = 6$, $n_8 = 9$, $n_{13} = 2$. In the first case, consider a vertex of degree 5, and let $k_{5,i}$ be the number of vertices of degree i adjacent to this vertex. Then from the equations $k_{5,5} + k_{5,8} + k_{5,13} = 5$ and $k_{5,5} + 2k_{5,8} + 3k_{5,13} = 8$ (which follows from the Perron-Frobenius eigenvector) we find that $k_{5,8} = 1$, so every vertex of degree 5 is adjacent to the unique vertex of degree 8, but there are 11 vertices of degree 5, which is a contradiction. So we are in the second case. Let $k_{i,j}$ be the number of vertices of degree j adjacent to a vertex of degree i. Here it follows that we must have $k_{13,5} = 3$, $k_{13,8} = 9$ and $k_{13,13} = 1$. So the two vertices of degree 13 have all vertices of degree 8 as (common) neighbours. But it follows from the Perron-Frobenius eigenvector that two vertices of degree 13 have precisely nine common neighbours, so it follows that every vertex of degree 5 is adjacent to precisely one of the vertices of degree 13, i.e. $k_{5,13} = 1$. Now it follows that every vertex of degree 5 is adjacent to one vertex of degree 8 ($k_{5,8} = 1$), while every vertex of degree 8 is adjacent to two vertices of degree 5 ($k_{8,5} = 2$), which is a contradiction.

Now suppose we have a graph with spectrum $\{[10]^1, [2]^6, [-2]^{11}\}$ and vertex degrees 5, 8 and 13. Then we must have either $n_5 = 7$, $n_8 = 2$, $n_{13} = 9$ or $n_5 = 2$, $n_8 = 10$, $n_{13} = 6$. In the first case it follows that $k_{5,8} = 1$ and $k_{8,5} = 2$, which is a contradiction. In the second case, consider the two vertices of degree 5. They have a unique common neighbour x, which must have degree 8, as a vertex of degree 13 can have at most one neighbour of degree 5. Besides the vertices of degree 5, x now can only have neighbours of degree 13. Now take a vertex of degree 13 which is not adjacent to any of the vertices of degree 5 (these exist since a vertex of degree 5 has at most one neighbour of degree 13). This vertex has four neighbours of degree 13, so it has four common neighbours with x, which is a contradiction, since it follows from the Perron-Frobenius eigenvector that they should have six common neighbours.

For a graph with spectrum $\{[12]^1, [2]^6, [-2]^{12}\}$ and vertex degrees 5, 8 and 13 we find $n_5 = 2$, $n_8 = 3$ and $n_{13} = 14$. Here it follows that $k_{8,5} = 0$, while $k_{5,8} \geq 1$, giving a contradiction. Finally, a graph with spectrum $\{[14]^1, [2]^6, [-2]^{13}\}$ cannot have vertex degrees 5, 8 and 13, since then we would have $\lambda_0 > k_{\max}$, which contradicts Lemma 1.3.1. So for none of the spectra we can have a graph with vertex degrees 5, 8 and 13.

In the remaining two cases for the vertex degrees we must have regular partitions, which can only occur for $\{[8]^1, [2]^6, [-2]^{10}\}$, with one vertex of degree 16 and sixteen vertices of degree 7, and we find that in this case we must have the cone over the lattice graph $L_2(4)$ or the cone over the Shrikhande graph. So we found all graphs with three eigenvalues that are represented by roots in E_7. □

For the graphs represented by roots in E_8, there are too many possibilities to do by hand.

Using the inequalities $\sum_i m_i\lambda_i^2 \le v\lambda_0$ and $v \le 36$, we find possible spectra $\{[2(v-8)-7\lambda_1]^1, [\lambda_1]^7, [-2]^{v-8}\}$ with $\lambda_1 = 1$ and $v = 13,\ldots, 19$, $\lambda_1 = 2$ and $v = 18,\ldots, 25$, $\lambda_1 = 3$ and $v = 23,\ldots, 30$, $\lambda_1 = 4$ and $v = 28,\ldots, 36$, or $\lambda_1 = 5$ and $v = 34,\ldots, 36$. The only spectra corresponding to regular graphs are $\{[12]^1, [4]^7, [-2]^{20}\}$, for which there are the triangular graph $T(8)$ and the three Chang graphs, and $\{[28]^1, [4]^7, [-2]^{28}\}$, for which no graph exists. The problem for the nonregular cases is that as the number of vertices increases, also the number of possible degrees increases. It is still possible to prove that no graph in the cases with at most twenty vertices exists, but (most of) the other cases are left (to a computer search?).

2.1.4. The small cases

As it turns out, we have found all nonregular graphs with three eigenvalues and at most twenty vertices. Of course, the regular ones are also known. The arguments we use in the next proposition are the same as those in the proofs of Propositions 2.1.5 and 2.1.6.

PROPOSITION 2.1.7. *The only connected nonregular graphs with three eigenvalues and at most twenty vertices, which are not complete bipartite are the cone over the Petersen graph, the graph derived from the complement of the Fano plane* (*see Figure* 2.1.2), *the cone over the Shrikhande graph and the cone over the lattice graph* $L_2(4)$.

Proof. Let's see which cases are left to check. For the cases with nonintegral eigenvalues, it follows from Proposition 2.1.2 that the only possible spectra are $\{[8]^1, [-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{13}]^8, [-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{13}]^8\}$ and $\{[9]^1, [-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{17}]^9, [-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{17}]^9\}$. Suppose we have a graph with the first spectrum. By Proposition 2.1.2, all vertex degrees must be even, and by use of the Perron-Frobenius eigenvector it then follows that the only possible vertex degrees are 4 and 12. But, if n_i denotes the number of vertices of degree i, then the equations $n_4 + n_{12} = 17$, $4n_4 + 12n_{12} = 120$ have no integral solution, so we have a contradiction. A graph with the latter spectrum must have vertex degrees either 5, 8 and 13, or 6 and 12, or 7 and 16. In the last two cases the partition according to the vertex degrees must be regular, which gives a contradiction. So we must have vertex degrees 5, 8 and 13, and either $n_5 = 0$, $n_8 = 17$, $n_{13} = 2$, or $n_5 = 5$, $n_8 = 9$, $n_{13} = 5$, or $n_5 = 10$, $n_8 = 1$, $n_{13} = 8$. In the first case, however, we have a regular partition, which cannot be the case. The last case can also be excluded, since every vertex of degree 5 must be adjacent to an even number of vertices of degree 8, and the unique vertex of degree 8 must be adjacent to three vertices of degree 5. In the remaining case, every vertex of degree 13 has at least two neighbours of degree 5. Since every vertex of degree 5 has at most two neighbours of degree 13, it follows that every vertex of degree 5 has precisely two neighbours of degree 13, and thus also three neighbours of degree 5. But two adjacent vertices of degree 5 have no common neighbours, and a vertex of degree 5 and a vertex of degree 13 which are adjacent must have two common neighbours, which gives a contradiction. Hence, also a graph with the second spectrum does not exist.

So we are left with integral spectra. First, consider the spectra with least eigenvalue $\lambda_2 < -2$. Here we find, by using the inequality $\sum_i m_i\lambda_i^2 < v\lambda_0$ of Lemma 1.3.2, that the only possible spectra with at most twenty vertices are $\{[8]^1, [1]^{10}, [-3]^6\}$, $\{[7]^1, [1]^{11}, [-3]^6\}$, $\{[11]^1, [1]^{10}, [-3]^7\}$, $\{[6]^1, [1]^{12}, [-3]^6\}$, $\{[10]^1, [1]^{11}, [-3]^7\}$, $\{[5]^1, [1]^{13}, [-3]^6\}$, $\{[9]^1, [1]^{12}, [-3]^7\}$ and $\{[13]^1, [1]^{11}, [-3]^8\}$, so all spectra have $\lambda_1 = 1$ and $\lambda_2 = -3$. As a consequence, the vertex degrees occuring in graphs with these spectra are either 4, 7, 12 and 19, or 5 and 11, or 6 and 15. Moreover, as $\lambda_1 + \lambda_2 = -2$, any two adjacent vertices of degree 4 have -1 common neighbours, hence the vertices of degree 4 form a coclique. First, consider the graphs on an odd number of vertices. In these cases not all vertex degrees can be odd (the Handshaking lemma), so degrees 5 and 11 cannot occur. Degrees 6 and 15 easily give contradictions in each of the cases, so we must have degrees 4, 7 and 12. Consider a graph with spectrum $\{[8]^1, [1]^{10}, [-3]^6\}$ on 17 vertices. From the fact that no vertex of degree 4 has a neighbour of degree 4, it follows (use the Perron-Frobenius eigenvector) that also no vertex of degree 4 has a neighbour of degree 12. This implies that every vertex of degree 12 has only neighbours of degree 12. Since there is at least one vertex of degree 12, it follows that the graph is disconnected, which is a contradiction. For spectrum $\{[6]^1, [1]^{12}, [-3]^6\}$ with 19 vertices, the fact that no vertex of degree 4 has a neighbour of degree 4 implies that such a vertex should have -2 neighbours of degree 12. So there cannot be a vertex of degree 4. But then $\lambda_0 < k_{\min}$, which contradicts Lemma 1.3.1. For spectrum $\{[10]^1, [1]^{11}, [-3]^7\}$ we find that we have three vertices of degree 4. Any vertex of degree 4 must have two neighbours of degree 7, while every vertex of degree 7 has no neighbours of degree 4, which gives a contradiction. Next, consider the graphs on an even number of vertices. In these cases, λ_0 is odd, so by Corollary 2.1.3 all vertex degrees must be odd. Thus we have vertex degrees 7 and 19 or 5 and 11. In the first case, it follows that we have (twenty vertices) a cone over a strongly regular graph on 19 vertices with degree 6, but such a graph does not exist. The latter case easily gives contradictions from the parameters for all the possible spectra.

Next, we have to check the graphs with $\lambda_2 = -2$, and so, by the previous section, what remains to be checked are the spectra $\{[2v-23]^1, [1]^7, [-2]^{v-8}\}$ for $v = 13,..., 19$, and spectra $\{[2v-30]^1, [2]^7, [-2]^{v-8}\}$ for $v = 18,..., 20$. For graphs with one of the spectra with $\lambda_1 = 1$, we find possible degrees either 3, 6, 11 and 18 or 4 and 10, or 5 and 14. Of course a graph with spectrum $\{[3]^1, [1]^7, [-2]^5\}$ cannot exist, since λ_0 is too small. The last two cases for the vertex degrees imply regular partitions, which give contradictions in all cases. Now suppose we have a graph with spectrum $\{[15]^1, [1]^7, [-2]^{11}\}$, on 19 vertices. Such a graph must have a vertex of degree 18, otherwise by Lemma 1.3.1 λ_0 is too large. Thus we have a cone, but this also leads to a contradiction. For the remaining spectra we must have degrees 3, 6 and 11. However, there are no graphs with spectra $\{[11]^1, [1]^7, [-2]^9\}$ and $\{[13]^1, [1]^7, [-2]^{10}\}$, again since λ_0 is too large. So suppose we have a graph with spectrum $\{[5]^1, [1]^7, [-2]^6\}$. Such a graph must have $n_3 = 11$, $n_6 = 2$ and $n_{11} = 1$. But now it follows that the vertex of degree 11 is adjacent to four vertices of degree 6, which gives a contradiction. Suppose we have a graph with spectrum

$\{[7]^1, [1]^7, [-2]^7\}$. Here it follows that (using that there is at least one vertex of degree 11) $n_3 = 7$, $n_6 = 5$ and $n_{11} = 3$. Let $k_{i,j}$ be the number of neighbours of degree j of some vertex of degree i. Then from the equations $k_{11,3} + k_{11,6} + k_{11,11} = 11$ and $k_{11,3} + 2k_{11,6} + 3k_{11,11} = 3\lambda_0$, we find that $k_{11,6} + 2k_{11,11} = 10$, which gives a contradiction, since $k_{11,6} \le 5$ and $k_{11,11} \le 2$. Finally, suppose we have a graph with spectrum $\{[9]^1, [1]^7, [-2]^8\}$. Then either $n_3 = 7$, $n_6 = 0$ and $n_{11} = 9$, or $n_3 = 2$, $n_6 = 8$ and $n_{11} = 6$. In the first case we have a regular partition, which gives a contradiction. In the latter case it follows that every vertex of degree 3 is adjacent to three vertices of degree 11, and since the two vertices of degree 3 must have one common neighbour, there is a vertex of degree 11 which is adjacent to two vertices of degree 3. But then this vertex must also be adjacent to seven vertices of degree 11, which cannot be the case.

Finally, we are left with the spectra with $\lambda_1 = 2$ and $\lambda_2 = -2$. Here we can have vertex degrees 5, 8 and 13, or 6 and 12, or 7 and 16. Suppose we have a graph with spectrum $\{[6]^1, [2]^7, [-2]^{10}\}$. Here we have vertex degrees 5, 8 and 13, otherwise $\lambda_0 \le k_{\min}$. It follows that $n_5 = 15$, $n_8 = 2$ and $n_{13} = 1$. However, this vertex of degree 13 must now be adjacent to five vertices of degree 8, which is a contradiction. So suppose we have a graph with spectrum $\{[8]^1, [2]^7, [-2]^{11}\}$. Here the regular partitions for degrees 6 and 12, or 7 and 16 give contradictions, so we must have degrees 5, 8 and 13, and then either $n_5 = 12$, $n_8 = 3$ and $n_{13} = 4$, or $n_5 = 7$, $n_8 = 11$ and $n_{13} = 1$. But we must also have $k_{13,8} + 2k_{13,13} = 11$, which gives a contradiction in the first case. In the second case it follows that the vertex of degree 13 is adjacent to all vertices of degree 8 and to two vertices of degree 5. Such a vertex x of degree 5 must then also be adjacent to one vertex of degree 8 and three vertices of degree 5. But x and the vertex of degree 13 must have three common neighbours, which gives a contradiction. Finally, suppose we have a graph with spectrum $\{[10]^1, [2]^7, [-2]^{12}\}$. Here degrees 6 and 12, and also 7 and 16 cannot occur. So also here we must have vertex degrees 5, 8 and 13. Now either $n_5 = 8$, $n_8 = 4$ and $n_{13} = 8$, or $n_5 = 3$, $n_8 = 12$ and $n_{13} = 5$. In the latter case it follows that every vertex of degree 13 has nine neighbours of degree 8, so there are 45 edges between vertices of degree 8 and vertices of degree 13. On the other hand, every vertex of degree 8 has at least four neighbours of degree 13, which gives at least 48 edges. So we are in the first case. Here every vertex of degree 13 must have three neighbours of degree 5, while every vertex of degree 5 has at most two neighbours of degree 13. But the number of vertices of degree 5 is the same as the number of vertices of degree 13, so we are finished. □

2.2. The Laplace spectrum - Graphs with constant μ and $\bar{\mu}$

As we have seen, the Laplace matrix of a graph is defined as the matrix $Q = D - A$, where D is the diagonal matrix of vertex degrees, and A is the adjacency matrix. This matrix is positive semidefinite, and the all-one vector is an eigenvector with eigenvalue 0. Connected graphs with two distinct Laplace eigenvalues are complete graphs, so the next step would be to consider connected graphs with three distinct Laplace eigenvalues.

However, it will be more convenient to consider graphs with two restricted Laplace eigenvalues. The *restricted eigenvalues* are those that have an eigenvector orthogonal to the all-one vector. The *restricted multiplicity* of an eigenvalue is the dimension of the eigenspace orthogonal to the all-one vector. Now the graphs with two restricted Laplace eigenvalues are precisely the connected graphs with three distinct Laplace eigenvalues and the disconnected graphs with two distinct Laplace eigenvalues.

It turns out that in such graphs only two vertex degrees can occur. Moreover, we shall prove that a graph has two restricted Laplace eigenvalues if and only if it has constant μ and $\overline{\mu}$. We say that a noncomplete graph G has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have μ common neighbours. A graph G has constant μ and $\overline{\mu}$ if G has constant $\mu = \mu(G)$, and its complement $\overline{G}$ has constant $\overline{\mu} = \mu(\overline{G})$. An example is given in Figure 2.2.1.

Graphs with constant μ and $\overline{\mu}$ form a common generalization of two known families of graphs. The regular ones are precisely the strongly regular graphs and for $\mu = 1$ we have the (nontrivial) geodetic graphs of diameter two.

Some similarities with so-called neighbourhood-regular or $\Gamma\Delta$-regular graphs (see [54, 75]) occur. These graphs can be defined as graphs G with constant λ and $\overline{\lambda}$, that is, in G any two adjacent vertices have λ common neighbours, and in $\overline{G}$ any two adjacent vertices have $\overline{\lambda}$ common neighbours. Here also only two vertex degrees can occur, but there is no easy algebraic characterization.

Using the results from the next section, we generated feasible parameter sets of graphs with three Laplace eigenvalues and at most 40 vertices (see Appendix A.2).

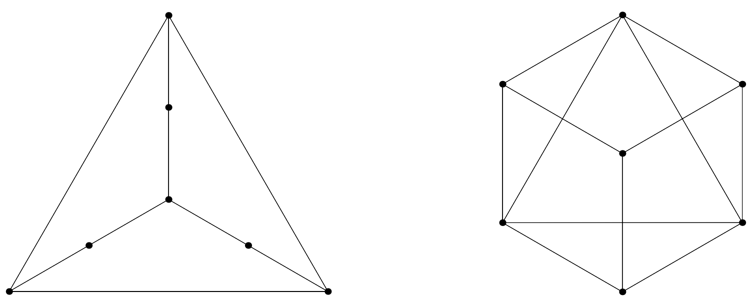

Figure 2.2.1. A graph with constant $\mu = 1$ and $\overline{\mu} = 2$ and its complement

2.2.1. The number of common neighbours and vertex degrees

In this section we shall derive some basic properties of graphs with two restricted Laplace eigenvalues. We start with a combinatorial characterization.

THEOREM 2.2.1. *Let G be a graph on v vertices. Then G has two distinct restricted Laplace eigenvalues θ_1 and θ_2 if and only if G has constant μ and $\overline{\mu}$. If so then only two vertex degrees k_1 and k_2 can occur, and $\theta_1 + \theta_2 = k_1 + k_2 + 1 = \mu + v - \overline{\mu}$ and $\theta_1\theta_2 = k_1k_2 + \mu = \mu v$.*

Proof. Let G have Laplace matrix $Q = D - A$. Suppose that G has two distinct restricted Laplace eigenvalues θ_1 and θ_2. Then $(Q - \theta_1 I)(Q - \theta_2 I)$ has spectrum $\{[\theta_1\theta_2]^1, [0]^{v-1}\}$ and row sums $\theta_1\theta_2$, so it follows that $(Q - \theta_1 I)(Q - \theta_2 I) = (\theta_1\theta_2/v)J$. If x is not adjacent to y, so $Q_{xy} = 0$ then $Q^2_{xy} = \theta_1\theta_2/v$, and so $\mu = \theta_1\theta_2/v$ is constant. Since the complement of G has distinct restricted Laplace eigenvalues $v - \theta_1$ and $v - \theta_2$ (it has Laplace matrix $vI - J - Q$), it follows that $\overline{\mu} = (v - \theta_1)(v - \theta_2)/v$ is also constant.

Now suppose that μ and $\overline{\mu}$ are constant. If x and y are adjacent then $(vI - J - Q)^2_{xy} = \overline{\mu}$, so $\overline{\mu} = (v^2I + vJ + Q^2 - 2vJ - 2vQ)_{xy} = Q^2_{xy} + v$, and if x and y are not adjacent, then $Q^2_{xy} = \mu$. Furthermore $Q^2_{xx} = d_x^2 + d_x$, where d_x is the vertex degree of x. Now

$$Q^2 = (\overline{\mu} - v)(D - Q) + \mu(J - I - D + Q) + D^2 + D$$

$$= (\mu + v - \overline{\mu})Q + D^2 - (\mu + v - \overline{\mu} - 1)D - \mu I + \mu J.$$

Since Q and Q^2 have row sums zero, it follows that $d_x^2 - d_x(\mu + v - \overline{\mu} - 1) - \mu + \mu v = 0$ for every vertex x. So $Q^2 - (\mu + v - \overline{\mu})Q + \mu vI = \mu J$. Now let θ_1 and θ_2 be such that $\theta_1 + \theta_2 = \mu + v - \overline{\mu}$ and $\theta_1\theta_2 = \mu v$, then $(Q - \theta_1 I)(Q - \theta_2 I) = (\theta_1\theta_2/v)J$, so G has distinct restricted Laplace eigenvalues θ_1 and θ_2. As a side result we obtained that all vertex degrees d_x satisfy the same quadratic equation, thus d_x can only take two values k_1 and k_2, and the formulas readily follow. □

If the restricted Laplace eigenvalues are not integral, then they must have the same multiplicities $m_1 = m_2 = \frac{1}{2}(v - 1)$. If the Laplace eigenvalues are integral, then their multiplicities are not necessarily fixed by v, μ and $\overline{\mu}$. For example, there are graphs on 16 vertices with constant $\mu = 2$ and $\overline{\mu} = 6$ with Laplace spectrum $\{[8]^m, [4]^{15-m}, [0]^1\}$ for $m = 5, 6, 7, 8$ and 9 (see Section 5.3.3).

The following lemma implies that the numbers of vertices of the respective degrees follow from the Laplace spectrum.

LEMMA 2.2.2. *Let G be a graph on v vertices with two distinct restricted Laplace eigenvalues θ_1 and θ_2 with restricted multiplicities m_1 and m_2, respectively. Suppose there are n_1 vertices of degree k_1 and n_2 vertices of degree k_2. Then $m_1 + m_2 + 1 = n_1 + n_2 = v$ and $m_1\theta_1 + m_2\theta_2 = n_1k_1 + n_2k_2$.*

Proof. The first equation is trivial, the second follows from the trace of the Laplace matrix. □

The number of common neighbours of two adjacent vertices is in general not constant, but depends on the degrees of the vertices.

LEMMA 2.2.3. *Let G be a graph with constant μ and $\overline{\mu}$, and vertex degrees k_1 and k_2. Suppose x and y are two adjacent vertices. Then the number of common neighbours λ_{xy} of x and y equals*

$$\lambda_{xy} = \begin{cases} \lambda_{11} = \mu - 1 + k_1 - k_2 & \text{if } x \text{ and } y \text{ both have degree } k_1, \\ \lambda_{12} = \mu - 1 & \text{if } x \text{ and } y \text{ have distinct degrees,} \\ \lambda_{22} = \mu - 1 + k_2 - k_1 & \text{if } x \text{ and } y \text{ both have degree } k_2. \end{cases}$$

Proof. Suppose x and y have vertex degrees d_x and d_y, respectively. The number of vertices that are not adjacent to both x and y equals $\overline{\mu}$. The number of vertices adjacent to x but not to y equals $d_x - 1 - \lambda_{xy}$, and the number of vertices adjacent to y but not to x equals $d_y - 1 - \lambda_{xy}$. Now we have that $v = 2 + \lambda_{xy} + \overline{\mu} + d_x - 1 - \lambda_{xy} + d_y - 1 - \lambda_{xy}$. Thus $\lambda_{xy} = \overline{\mu} - v + d_x + d_y$. By using that $k_1 + k_2 = \mu + v - \overline{\mu} - 1$, the result follows. □

Both Theorem 2.2.1 and Lemma 2.2.3 imply the following.

COROLLARY 2.2.4. *A graph with constant μ and $\overline{\mu}$ is regular if and only if it is strongly regular.*

Observe that G is regular if and only if $(\mu + v - \overline{\mu} - 1)^2 = 4\mu(v - 1)$ or $n_1 = 0$ or $n_2 = 0$. Since we can express all parameters in terms of the Laplace spectrum, it follows that it can be recognized from the Laplace spectrum whether a graph is strongly regular or not. Of course, this also follows from the previous and Lemma 1.3.3.

Before proving the next lemma we first look at the disconnected graphs. Since the number of components of a graph equals the multiplicity of its Laplace eigenvalue 0, a graph with constant μ and $\overline{\mu}$ is disconnected if and only if one of its restricted Laplace eigenvalues equals 0. Consequently this is the case if and only if $\mu = 0$. So in a disconnected graph G with constant μ and $\overline{\mu}$ any two vertices that are not adjacent have no common neighbours. This implies that two vertices that are not adjacent are in distinct components of G. So G is a disjoint union of cliques. Since the only two vertex degrees that can occur are $v - \overline{\mu} - 1$ and 0, G is a disjoint union of $(v - \overline{\mu})$-cliques and isolated vertices.

LEMMA 2.2.5. *Let G be a graph with two restricted Laplace eigenvalues $\theta_1 > \theta_2$ and vertex degrees $k_1 \geq k_2$. Then $\theta_1 - 1 \geq k_1 \geq k_2 \geq \theta_2$, with $k_2 = \theta_2$ if and only if G or $\overline{G}$ is disconnected.*

Proof. Assume that G is not regular, otherwise G is strongly regular and the result easily follows. First, suppose that the induced graph on the vertices of degree k_1 is not a coclique. So there are two vertices of degree k_1 that are adjacent. Then the 2×2 submatrix of the Laplace matrix Q of G induced by these two vertices has eigenvalues $k_1 \pm 1$, and since these interlace the eigenvalues of Q, we have that $k_1 + 1 \leq \theta_1$. Since $k_1 + k_2 + 1 = \theta_1 + \theta_2$, then also $k_2 \geq \theta_2$.

Next, suppose that the induced graph on the vertices of degree k_2 is not a clique. So there are two vertices of degree k_2 that are not adjacent. Now the 2×2 submatrix of Q induced by these two vertices has two eigenvalues k_2, and since these also interlace the eigenvalues of Q, we have that $k_2 \geq \theta_2$, and then also $\theta_1 - 1 \geq k_1$.

The remaining case is that the induced graph on the vertices of degree k_1 is a coclique and the induced graph on the vertices of degree k_2 is a clique. Suppose we have such a graph. Since a vertex of degree k_1 only has neighbours of degree k_2, and $\lambda_{12} = \mu - 1$, we find that $k_1 = \mu$. Since any two vertices of degree k_1 have μ common neighbours, it follows that every vertex of degree k_1 is adjacent to every vertex of degree k_2, and we find that $k_2 \geq k_1$, which is a contradiction. So the remaining case cannot occur, and we have proven the inequalities.

Now suppose that G or $\overline{G}$ is disconnected. Then it follows from the observations before the lemma that $k_2 = \theta_2$. On the other hand, suppose that $k_2 = \theta_2$. Then it follows that $k_1 = \theta_1 - 1$ and from the equation $\theta_1\theta_2 = k_1k_2 + \mu$ it then follows that $k_2 = \mu$. Now take a vertex x_2 of degree k_2 that is adjacent to a vertex x_1 of degree k_1. If there are no such vertices then G is disconnected and we are done. It follows that every vertex that is not adjacent to x_2, is adjacent to all neighbours of x_2, so also to x_1. Since x_1 and x_2 have $\mu - 1$ common neighbours, x_1 is also adjacent to all neighbours of x_2. So x_1 is adjacent to all other vertices, and so $\overline{G}$ is disconnected. □

We conclude this section with so-called *Bruck-Ryser conditions*.

PROPOSITION 2.2.6. *Let G be a graph with constant μ and $\overline{\mu}$ on v vertices, with v odd, and with restricted Laplace eigenvalues θ_1 and θ_2. Then the Diophantine equation*

$$x^2 = (\theta_1 - \theta_2)^2 y^2 + (-1)^{\frac{1}{2}(v-1)} \mu z^2$$

has a nontrivial integral solution (x, y, z).

Proof. Let Q be the Laplace matrix of G, then

$$(Q - \tfrac{1}{2}(\theta_1 + \theta_2)I)(Q - \tfrac{1}{2}(\theta_1 + \theta_2)I)^T = Q^2 - (\theta_1 + \theta_2)Q + \tfrac{1}{4}(\theta_1 + \theta_2)^2 I$$

$$= \mu J + (\tfrac{1}{4}(\theta_1 + \theta_2)^2 - \theta_1\theta_2)I = \tfrac{1}{4}(\theta_1 - \theta_2)^2 I + \mu J.$$

Since $Q - \frac{1}{2}(\theta_1 + \theta_2)I$ is a rational matrix, it follows from a lemma by Bruck and Ryser (cf. [4]) that the Diophantine equation

$$x^2 = \tfrac{1}{4}(\theta_1 - \theta_2)^2 y^2 + (-1)^{\frac{1}{2}(v-1)} \mu z^2$$

has a nontrivial integral solution, which is equivalent to stating that the Diophantine equation above has a nontrivial integral solution. □

2.2.2. Cocliques

If $k_1 - k_2 > \mu - 1$, then the induced graph on the set of vertices of degree k_2 is a coclique, since two adjacent vertices of degree k_2 would have a negative number λ_{22} of common neighbours. It turns out (see Appendix A.2) that this is the case in many examples. Therefore we shall have a closer look at cocliques. If G is a graph, then we denote by $\alpha(G)$ the maximal size of a coclique in G.

LEMMA 2.2.7. *Let G be a graph on v vertices with largest Laplace eigenvalue θ_1 and smallest vertex degree k_2. Then $\alpha(G) \leq v(\theta_1 - k_2)/\theta_1$.*

Proof. Let C be a coclique of size $\alpha(G)$. Partition the vertices of G into C and the set of vertices not in C, and partition the Laplace matrix Q of G according to this partition of the vertices. Let B be the matrix of average row sums of the blocks of Q, then

$$B = \begin{pmatrix} k & -k \\ -k\dfrac{\alpha(G)}{v - \alpha(G)} & k\dfrac{\alpha(G)}{v - \alpha(G)} \end{pmatrix},$$

where k is the average degree of the vertices in C. Since B has eigenvalues 0 and $kv/(v - \alpha(G))$, and since these interlace the eigenvalues of Q, we have that $kv/(v - \alpha(G)) \leq \theta_1$. The result now follows from the fact that $k_2 \leq k$. □

Another bound is given by the multiplicities of the eigenvalues.

LEMMA 2.2.8. *Let G be a connected graph with Laplace spectrum $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, [0]^1\}$, where $\theta_1 > \theta_2 > 0$, such that $\overline{G}$ is also connected. Then $\alpha(G) \leq \min\{m_1, m_2 + 1\}$.*

Proof. Suppose C is a coclique with size greater than m_1. Consider the submatrix of the Laplace matrix Q induced by the vertices of C. This matrix only has eigenvalues k_1 and k_2, and since these interlace the eigenvalues of Q, we find that $k_2 \leq \theta_2$. This is in contradiction with Lemma 2.2.5, since G and $\overline{G}$ are connected. If C is a coclique of size greater than $m_2 + 1$, we find by interlacing that $k_1 \geq \theta_1$, which is again a contradiction. □

As remarked before, if G is a graph with constant μ and $\overline{\mu}$ with $\lambda_{22} < 0$, then the vertices of degree k_2 form a coclique. If this is the case, then $n_2 \le m_2$, and we know the adjacency spectrum of the induced graph on the vertices of degree k_1.

PROPOSITION 2.2.9. *Let G be a connected graph with Laplace spectrum $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, [0]^1\}$, where $\theta_1 > \theta_2 > 0$, such that $\bar{G}$ is also connected. Suppose that the n_2 vertices of degree k_2 induce a coclique, then $n_2 \le m_2$, and the n_1 vertices of degree k_1 induce a graph with adjacency spectrum $\{[\lambda_1]^1, [k_1 - \theta_2]^{m_2 - n_2}, [\lambda_2]^1, [-1]^{n_2 - 1}, [k_1 - \theta_1]^{m_1 - n_2}\}$, where λ_1 and λ_2 are determined by the equations $\lambda_1 + \lambda_2 = k_1 - 1$ and $\lambda_1^2 + \lambda_2^2 = n_1 k_1 - n_2 k_2 - (m_2 - n_2)(k_1 - \theta_2)^2 - (n_2 - 1) - (m_1 - n_2)(k_1 - \theta_1)^2$.*

Proof. The adjacency matrix A_1 of the graph induced by the vertices of degree k_1 is a submatrix of the matrix $k_1 I - Q$, where Q is the Laplace matrix of G. From interlacing it follows that A_1 has second largest eigenvalue $k_1 - \theta_2$ with multiplicity at least $m_2 - n_2$ and smallest eigenvalue $k_1 - \theta_1$ with multiplicity at least $m_1 - n_2$. Note that so far we didn't use that the vertices of degree k_2 induce a coclique. Now let

$$A = \begin{pmatrix} A_1 & N^T \\ N & O \end{pmatrix}$$

be the adjacency matrix of G, where the partition is induced by the degrees of the vertices. Two vertices of degrees k_2 have μ common neighbours, so $NN^T = k_2 I + \mu(J - I)$. A vertex of degree k_2 and a vertex of degree k_1 have $\mu - 1$ or μ common neighbours, depending on whether they are adjacent or not, so $NA_1 = \mu J - N$. Let $\{v_i \mid i = 0,\dots, n_2 - 1\}$ be an orthonormal set of eigenvectors of NN^T, with v_0 the constant vector, then $NN^T v_i = (k_2 - \mu)v_i$, $i = 1,\dots, n_2 - 1$. Now

$$A_1(N^T v_i) = (NA_1)^T v_i = (\mu J - N)^T v_i = -N^T v_i,\ i = 1,\dots, n_2 - 1.$$

Since $k_2 > \mu$ (otherwise G or $\bar{G}$ is disconnected), it follows that A_1 has -1 as an eigenvalue with multiplicity at least $n_2 - 1$.

By Lemma 2.2.8 we have $n_2 \le m_2 + 1$. Suppose that $n_2 = m_2 + 1$. Then $n_1 = m_1$ and it follows that A_1 has spectrum $\{[\lambda_1]^1, [-1]^{n_2 - 1}, [k_1 - \theta_1]^{m_1 - n_2}\}$ for some λ_1. Since A_1 has zero trace, and using Lemma 2.2.5, we have $\lambda_1 = n_2 - 1 + (m_1 - n_2)(\theta_1 - k_1) > n_1 - 1$, which is a contradiction. Hence $n_2 \le m_2$. Now let $\lambda_1 \ge \lambda_2$ be the remaining two eigenvalues of A_1. These eigenvalues (i.e., the equations in the statement) follow from the trace of A_1 and the trace of A_1^2. Since $\lambda_1 \le k_1$ (interlacing), it follows that $\lambda_2 \ge -1$. □

If the vertices of degree k_2 form a coclique, then Lemma 2.2.7 implies that

$$n_2 \le v(\theta_1 - k_2)/\theta_1.$$

If this bound is tight, then it follows from tight interlacing that the partition of the vertices into vertices of degree k_1 and vertices of degree k_2 is regular. So N is the incidence matrix of a 2-(n_2, κ, μ) design, where $\kappa = n_2k_2/n_1$. Furthermore, the adjacency matrix of the induced graph G_1 on the vertices of degree k_1 has spectrum

$$\{[k_1 - \kappa]^1,\ [k_1 - \theta_2]^{m_2+1-n_2},\ [-1]^{n_2-1},\ [k_1 - \theta_1]^{m_1-n_2}\},$$

so G_1 is a regular graph with at most four eigenvalues. It follows from the multiplicities that θ_1 and θ_2 must be integral. In this way it can be proven that there is no graph on 25 vertices with constant $\mu = 2$ and $\bar{\mu} = 12$, with 10 vertices of degree 6. These 10 vertices induce a coclique for which the bound is tight. The induced graph on the remaining 15 vertices has spectrum $\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$, but such a graph cannot exist, which follows from results in the next chapter (Section 3.3.5).

Examples for which the bound is tight are obtained by taking an affine plane for the design and a disjoint union of cliques for G_1. This is family *b* of the next section. Another example is constructed from a polarity with $q\sqrt{q} + 1$ absolute points in $PG(2, q)$ where q is the square of a prime power (cf. Section 2.2.4). In Section 2.2.5 we find a large family of graphs for which the bound of Lemma 2.2.8 is tight.

Also if $\lambda_{22} = 0$, so that the vertices of degree k_2 do not necessarily form a coclique, we find a bound on the number of vertices n_2 of degree k_2.

LEMMA 2.2.10. *If* $k_1 - k_2 \geq \mu - 1$, *then* $n_2 \leq v - \mu$.

Proof. Fix a vertex x_1 of degree k_1. If x_1 has no neighbours of degree k_2 then $n_1 \geq k_1 + 1 \geq \mu + k_2 \geq \mu$, and so $n_2 \leq v - \mu$. If x_1 has a neighbour x_2 of degree k_2, then x_1 and x_2 cannot have a common neighbour y_2 of degree k_2, since otherwise x_2 and y_2 have a common neighbour x_1, so that $0 \geq \mu - 1 + k_2 - k_1 = \lambda_{22} > 0$, which is a contradiction. So all common neighbours of x_1 and x_2 have degree k_1, so $n_1 \geq \lambda_{12} + 1 = \mu$, and so $n_2 \leq v - \mu$. □

2.2.3. Geodetic graphs of diameter two

A *geodetic graph* is a graph in which any two vertices are connected by a unique shortest path. Thus a geodetic graph of diameter two is a graph with constant $\mu = 1$. These graphs have been studied by several authors (cf. [12]), and it is known (see [12, Thm. 1.17.1]) that if G is a geodetic graph of diameter two, then

i. G is the cone (see Section 2.1) over a disjoint union of cliques, or
ii. G is strongly regular, or

iii. precisely two vertex degrees $k_1 > k_2$ occur. If X_1 and X_2 denote the sets of vertices with degrees k_1 and k_2, respectively, then X_2 induces a coclique, maximal cliques meeting both X_1 and X_2 have size two, and maximal cliques contained in X_1 have size $k_1 - k_2 + 2$. Moreover, $v = k_1 k_2 + 1$.

If G is of type *i*, then G need not have constant $\bar{\mu}$. Indeed, its complement is disconnected, so if $\bar{\mu}$ is constant, then $\bar{\mu} = 0$, and it follows that G is the cone over a coclique (a disjoint union of 1-cliques), so G is isomorphic to $K_{1,n}$, $n \geq 2$. If G is of type *ii* then clearly $\bar{\mu}$ is constant. Now suppose that G is of type *iii*. Since $\mu = 1$, every edge is in a unique maximal clique. Let x and y be two adjacent vertices, then x and y cannot both be in X_2. If one is in X_1, and the other in X_2, then they have no common neighbour, since maximal cliques meeting both X_1 and X_2 have size two. So $\lambda_{12} = 0$ and then $\bar{\mu}_{12} = v - k_1 - k_2$. If both x and y are in X_1, then by the previous argument they have no common neighbours in X_2, and since every maximal clique contained in X_1 has size $k_1 - k_2 + 2$, they have $k_1 - k_2$ common neighbours in X_1. So $\lambda_{11} = k_1 - k_2$, and then also $\bar{\mu}_{11} = v - k_1 - k_2$. So G has constant $\bar{\mu}$. The following four families of graphs are all known examples of type *iii*.

a. Take a clique and a coclique of size k_1, and an extra vertex. Add k_1 disjoint edges between the clique and the coclique, and add edges between the extra vertex and every vertex of the coclique (see also Section 2.2.5, an example is given in Figure 2.2.1).

b. Take an affine plane. Take as vertices the points and lines of the plane. A point is adjacent to a line if it is on the line, and two lines are adjacent if they are parallel (disjoint).

c. Take the previous example and add the parallel classes to the vertices. Add edges between each line and the parallel class it is in, and add edges between all parallel classes.

d. Take a projective plane with a polarity σ. Take as vertices the points of the plane. Two points x and y are adjacent if x is on the line y^σ (cf. Section 2.2.4).

2.2.4. Symmetric designs with a polarity

Let D be a symmetric design. A *polarity* of D is a one-one correspondence σ between its points and blocks such that for any point p and any block b we have that $p \in b$ if and only if $b^\sigma \in p^\sigma$. A point is called *absolute* (with respect to σ) if $p \in p^\sigma$. Now D has a polarity if and only if it has a symmetric incidence matrix A. An absolute point corresponds to a one on the diagonal of A.

Suppose that D is a symmetric 2-(v, k, λ) design with a polarity σ. Let $G = P(D)$ be the graph on the points of D, where two distinct points x and y are adjacent if $x \in y^\sigma$. Then the only vertex degrees that can occur are k and $k - 1$. The number of vertices with degree $k-1$ is the number of absolute points of σ. Let A be the corresponding symmetric incidence matrix, then $Q = kI - A$ is the Laplace matrix of G. Since A is a symmetric incidence matrix of D, we find that $(kI - Q)^2 = A^2 = AA^T = (k-\lambda)I + \lambda J$, so $Q^2 - 2kQ + (k^2 - k + \lambda)I = \lambda J$. Thus Q has two distinct restricted eigenvalues $k \pm \sqrt{k-\lambda}$. The converse is also true.

THEOREM 2.2.11. *Let G be a graph with constant μ and $\bar{\mu}$ on v vertices, with vertex degrees k and $k-1$. Then G comes from a symmetric* 2-(v, k, μ) *design with a polarity.*

Proof. Let G have restricted Laplace eigenvalues θ_1 and θ_2, then $\theta_1 + \theta_2 = 2k$ and $\mu = \theta_1\theta_2/v = k(k-1)/(v-1)$. Hence we have that $Q^2 - 2kQ + v\mu I = \mu J$. Now let $A = kI - Q$, then A is a symmetric (0, 1)-matrix with row sums k, and $AA^T = A^2 = k^2I - 2kQ + Q^2 = (k^2 - v\mu)I + \mu J = (k-\mu)I + \mu J$, so A is the incidence matrix of a symmetric 2-(v, k, μ) design with a polarity. □

Since the polarities in the unique 2-(7, 3, 1), 2-(11, 5, 2) and 2-(13, 4, 1) designs are unique, the graphs we obtain from these designs are also uniquely determined by their parameters. For the graph from the polarity in the 2-(11, 5, 2) and its complement, see Figure 2.2.2. The graph in Figure 2.2.1 comes from the polarity in the Fano plane.

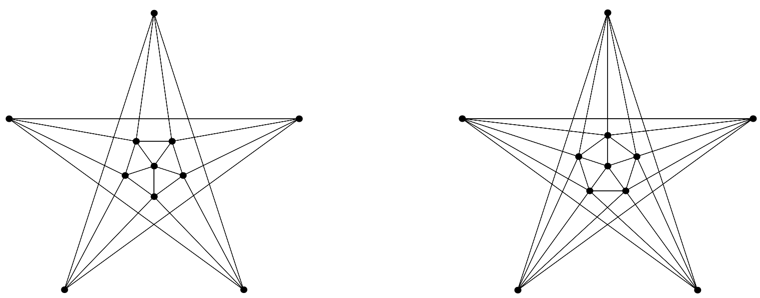

Figure 2.2.2. A graph with constant $\mu = 2$ and $\bar{\mu} = 3$ and its complement

In a projective plane of order n, where n is not a square, any polarity has $n+1$ absolute points. If n is a square, then the number of absolute points in a polarity lies between $n+1$ and $n\sqrt{n}+1$. The projective plane $PG(2, q)$ admits a polarity with $q+1$ absolute points for every prime power q and a polarity with $q\sqrt{q}+1$ absolute points for every square q of a prime power. If a polarity in a projective plane of order n has $n+1$ absolute points then the set of absolute points forms a line if n is even, and an oval if n is odd, that is, no three points are on one line (cf. [4, § VIII.9]). Using this, we find that there is precisely one graph from a polarity with 5 absolute points in the projective plane of order 4, and precisely one graph from a polarity with 6 absolute points in the projective plane of order

5. Using the remarks in Section 2.2.2 we also find precisely one graph from a polarity with 9 absolute points in the projective plane of order 4.

By Paley's construction of Hadamard matrices (cf. [4, Thm. I.9.11]) we obtain symmetric 2-$(2^e(q + 1) - 1,\ 2^{e-1}(q + 1) - 1,\ 2^{e-2}(q + 1) - 1)$ designs with a polarity with $2^{e-1}(q + 1) - 1$ absolute points, for every odd prime power q and every $e > 0$.

Furthermore, we found polarities with 0, 4, 8, 12 and 16 absolute points in a 2-(16, 6, 2) design (see Section 5.3.3), a polarity in the 2-(37, 9, 2) design from the difference set (cf. [4, Ex. VI.4.3]) and a polarity with 16 absolute points in the 2-(40, 13, 4) design of points and planes in $PG(3, 3)$. Spence [private communication] found polarities with 3, 7, 11 and 15 absolute points in 2-(15, 7, 3) designs, polarities in 2-(25, 9, 3) and 2-(30, 13, 3) designs, polarities with 5, 11, 17, 23, 29 and 35 absolute points in 2-(35, 17, 8) designs, polarities with 0, 6, 12, 18, 24, 30 and 36 absolute points in 2-(36, 15, 6) designs and polarities with 10, 16, 22, 28, 34 and 40 absolute points in 2-(40, 13, 4) designs.

2.2.5. Other graphs from symmetric designs.

Let D be a symmetric 2-(w, k, λ) design. Fix a point x. We shall construct a graph $G = G(D)$ that has constant μ and $\overline{\mu}$. The vertices of G are the points and the blocks of D, except for the point x. Between the points there are no edges. A point y and a block b will be adjacent if and only if precisely one of x and y is incident with b. Two blocks will be adjacent if and only if both blocks are incident with x or both blocks are not incident with x. It is not hard to show that the resulting graph G has constant $\mu = k - \lambda$ and constant $\overline{\mu} = w - k - 1 + \lambda$. In G the $n_1 = w$ blocks have degrees $k_1 = w - 1$, and the $n_2 = w - 1$ points have degrees $k_2 = 2(k - \lambda)$. Note that D and the complement of D give rise to the same graph G. We have the following characterization of $G(D)$.

THEOREM 2.2.12. *Let G be a graph with constant μ and $\overline{\mu}$ on $2w - 1$ vertices, such that both G and $\overline{G}$ are connected. Suppose G has w vertices of degree k_1, and $w - 1$ vertices of degree k_2, and suppose that the vertices of degree k_2 induce a coclique. Then $k_1 = w - 1$, $k_2 = 2\mu$, and $G = G(D)$, where D is a symmetric 2-$(w, k, k - \mu)$ design.*

Proof. Let

$$A = \begin{pmatrix} A_1 & N^T \\ N & O \end{pmatrix}$$

be the adjacency matrix of G, where the partition is induced by the degrees of the vertices. It follows from Lemma 2.2.8 and Proposition 2.2.9 that $m_1 = m_2 = n_2$, and that A_1 has spectrum $\{[\lambda_1]^1, [\lambda_2]^1, [-1]^{w-2}\}$, with $\lambda_1 + \lambda_2 = k_1 - 1$, and $\lambda_1 \geq \lambda_2 \geq -1$. On the other hand, it follows from the trace of A_1 that $\lambda_1 + \lambda_2 = w - 2$, so that $k_1 = w - 1$. Since $k_1k_2 = \mu(v - 1)$, we then find that $k_2 = 2\mu$.

Suppose that $\lambda_2 = -1$, then $\lambda_1 = w - 2 - \lambda_2 = w - 1$, so $A_1 = J - I$. But then G is disconnected, which is a contradiction. Now $A_1 + I$ is positive semidefinite of rank two with diagonal 1, and so it is the Gram matrix of a set of vectors of length 1 in $\mathbb{R}^2$, with mutual inner products 0 or 1. It follows that there can only be two distinct vectors, and A_1 is the adjacency matrix of a disjoint union of two cliques, say of sizes k and $w - k$. Let $N = (N_1\ N_2)$ be partitioned according to the partition of A_1 into two cliques, where N_1 has k columns and N_2 has $w - k$ columns. From the equation $NA_1 = \mu J - N$ we derive that $N_1J = N_2J = \mu J$, so both N_1 as N_2 have row sums μ. Now let

$$M = \begin{pmatrix} \underline{1}^T & \underline{0}^T \\ J - N_1 & N_2 \end{pmatrix},$$

then M is square of size w, with row sums k. Furthermore, we find that $(J - N_1)(J - N_1)^T + N_2N_2^T = (k - 2\mu)J + NN^T = (k - 2\mu)J + (k_2 - \mu)I + \mu J = \mu I + (k - \mu)J$, and so we have that $MM^T = \mu I + (k - \mu)J$, so M is the incidence matrix of a symmetric 2-$(w, k, k - \mu)$ design D, and $G = G(D)$. □

The matrix N that appears in the proof above is the incidence matrix of a structure, that is called a pseudo design by Marrero and Butson [78] and a "near-square" λ-linked design by Woodall [104]. An alternative proof of Theorem 2.2.12 uses Theorem 3.4 of [78] which states that a pseudo $(w \neq 4\mu, k_2 = 2\mu, \mu)$-design comes from a symmetric design in the way described above. The problem then is to prove the case $w = 4\mu$, however.

For every orbit of the action of the automorphism group of the design D on its points, we get a different graph $G(D)$ by taking the fixed point x from that orbit. Since the trivial 2-$(k_1 + 1, 1, 0)$ (here we get family a of geodetic graphs given in Section 2.2.4), the 2-(7, 3, 1), the 2-(11, 5, 2) and the 2-(13, 4, 1) designs are unique and have an automorphism group that acts transitively on the points, the graphs we obtain are uniquely determined by their parameters. According to Spence [private communication], the five 2-(15, 7, 3) designs have respectively 1, 2, 3, 2 and 2 orbits, the three 2-(16, 6, 2) designs all have a transitive automorphism group, and the six 2-(19, 9, 4) designs have respectively 7, 5, 3, 3, 3 and 1 orbits. Thus we get precisely ten graphs on 29 vertices with constant $\mu = 4$ and $\bar{\mu} = 10$, three graphs on 31 vertices with constant $\mu = 4$ and $\bar{\mu} = 11$, and 22 graphs on 37 vertices with constant $\mu = 5$ and $\bar{\mu} = 13$.

2.2.6. Switching in strongly regular graphs

Let G be a strongly regular graph with parameters $(v = 2k + 1, k, \lambda, \mu^*)$. Fix a vertex x and "switch" between the set of neighbours of x and the set of vertices (distinct from x)

that are not neighbours of x, that is, a vertex that is adjacent to x and a vertex that is not adjacent to x are adjacent if and only if they are not adjacent in G. All other adjacencies remain the same (note that this is not ordinary Seidel switching). If the (adjacency) eigenvalues of G are k, r and s, then we obtain a graph with restricted Laplace eigenvalues $2(\lambda + 1) - s$ and $2(\lambda + 1) - r$. The graph has constant $\mu = k - \mu^* = \lambda + 1$ and $\overline{\mu} = \mu^*$, and there is one vertex of degree k and $2k$ vertices of degree $2(\lambda + 1)$. Almost all examples have $k = 2(\lambda + 1) = 2\mu^*$, so that we get a (strongly) regular graph. The only known (to us) examples for which $k \neq 2(\lambda + 1)$ are the triangular graph $T(7)$ and its complement. (Note that from one pair of complementary graphs we get another pair of complementary graphs.) From the complement of $T(7)$ we get a graph with constant $\mu = 4$ and $\overline{\mu} = 6$ on 21 vertices with one vertex of degree 10 and 20 vertices of degree 8. The subgraph induced by the neighbours of the vertex x of degree 10 is the Petersen graph.

This construction can be reversed, that is, if G is a graph on v vertices with constant μ and $\overline{\mu}$, such that there is precisely one vertex of degree $k = \frac{1}{2}(v - 1)$ and $2k$ vertices of degree 2μ, then it must be constructed from a strongly regular graph in the above way. Since $T(7)$ is uniquely determined by its parameters, and it has a transitive automorphism group it follows that there is precisely one graph with constant $\mu = 4$ and $\overline{\mu} = 6$ on 21 vertices with one vertex of degree 10 and 20 vertices of degree 8.

Next, let G be a strongly regular graph with parameters $(v^* = 2k + 1, k, \lambda, \mu^*)$ with a regular partition into two parts, where one part has k_2 vertices and the induced graph is regular of degree $k_2 - \mu^* - 1$, and the other part has $v^* - k_2$ vertices and the induced graph is regular of degree $k - \mu^*$. (Then $k_2(k - k_2 + \mu^* + 1) = (v^* - k_2)\mu^*$.) Add an isolated vertex to the second part and then switch (see Section 1.3.8) with respect to this partition. The obtained graph has one vertex of degree k_2 and v^* vertices of degree $k_1 = k_2 + k - 2\mu^*$. If the (adjacency) eigenvalues of G are k, r and s, then we obtain a graph with restricted Laplace eigenvalues $k_1 - s$ and $k_1 - r$, and it has constant $\mu = k_2 - \mu^*$ and $\overline{\mu} = k + 1 - k_2 + \mu$. Again, we obtain a (strongly) regular graph if $k = 2\mu^*$.

Also here the construction can be reversed. A graph on v vertices with constant μ and $\overline{\mu}$, such that $\mu + \overline{\mu} = \frac{1}{2}v$ and with precisely one vertex of degree k_2 must be constructed from a strongly regular graph in the above way.

If we take $T(7)$ and take for one part of the partition a 7-cycle or the disjoint union of a 3-cycle and a 4-cycle, then we find that there are precisely two nonisomorphic graphs on 22 vertices with constant $\mu = 3$ and $\overline{\mu} = 8$, with 21 vertices of degree 9 and one vertex of degree 7. In $T(7)$ there cannot be a regular partition with $k_2 = 12$ (which is the other value satisfying the quadratic equation) since this would give a graph which is the complement of a graph with $\lambda_{22} = 0$ and $n_1 < \mu$, contradicting Lemma 2.2.10.

2.3. Yet another spectrum

During the investigation of graphs with least eigenvalue -2 in Section 2.1.3, we came up with another interesting matrix related to graphs, namely the matrix $NN^T = D + A$, where

N is the incidence matrix, D the diagonal matrix of vertex degrees, and A the adjacency matrix of the graph (see also [33]). A natural step would be to investigate the connected graphs G for which this matrix has three eigenvalues. (Also here the regular ones are precisely the strongly regular graphs, and a graph with two eigenvalues is complete). Let's have a short look at these graphs. As we have seen before, we have an eigenvalue 0 if and only if the graph is bipartite. Moreover, $N^TN = 2I + L(A)$, where $L(A)$ is the adjacency matrix of the line graph of G, has the same nonzero eigenvalues. So the line graph can have two, three or four distinct (adjacency) eigenvalues. If it has two, then it is a complete graph, and G must be $K_{1,n}$ for some $n \geq 2$. If it has three, then it follows from the results in Section 2.1.3 that the line graph must be one of $K_{1,2}$, C_5, $T(n)$ or $L_2(n)$, so that G (with $D + A$ having three eigenvalues) must be C_5 or $K_{n,n}$, hence strongly regular. If the line graph has four eigenvalues, then G is not bipartite. Using the results of Chapter 3 (see Section 3.3.5) we find that if the line graph of G is regular, then G must be strongly regular. So besides the graphs $K_{1,n}$, $n \geq 2$, the only connected nonregular graphs for which $D + A$ has three eigenvalues are the connected nonbipartite graphs for which the line graph is nonregular with four (adjacency) eigenvalues. Here we shall finish our short investigation of this spectrum, and have a look at regular graphs with four eigenvalues.

Chapter 3

A pleasant faced man steps up to greet you
He smiles and says he's pleased to meet you
Beneath his hat the strangeness lies
Take it off, he's got three eyes
Truth is false and logic lost
Now the fourth dimension is crossed

You have entered the Twilight Zone
Beyond this world strange things are known
Use the key, unlock the door
See what your fate might have in store
Come explore your dreams' creation
Enter this world of imagination

(Rush, 2112)

Regular graphs with four eigenvalues

As we have seen, connected regular graphs having at most three distinct eigenvalues are very well classified by means of combinatorial properties: they are the complete and the strongly regular graphs. By dropping regularity in the previous chapter, we have already seen some possible generalizations. When keeping regularity, distance-regular graphs of diameter d (or more generally, d-class association schemes) are generalizations of complete ($d = 1$) and strongly regular ($d = 2$) graphs from a combinatorial point of view. The adjacency matrices of these graphs have $d + 1$ distinct eigenvalues, but for $d > 2$ the converse is not true, in fact most regular graphs with $d + 1$ distinct eigenvalues are not distance-regular with diameter d (and do not come from d-class association schemes). When looking at the number of distinct eigenvalues, the next step after strongly regular graphs would be to look at the regular graphs with four distinct eigenvalues. Already for those graphs, many examples exist that are not distance-regular or from three-class association schemes (in the next chapter we shall have a closer look to decide which ones are). Still we can deduce some nice properties. An important observation is that regular graphs with four eigenvalues are walk-regular, which implies rather strong conditions for the possible spectra. Furthermore we shall give several constructions, some characterizations, and uniqueness and nonexistence results. Many of the constructions use strongly regular graphs. We also generate a list of feasible spectra for regular graphs with four eigenvalues and at most 30 vertices. By combining the theoretic results with computer results, Spence and the author were able to find all graphs for most of the feasible spectra thus found (see Appendix A.3, for more see [40]).

3.1. Properties of the eigenvalues

In this section we shall derive some properties of the eigenvalues of graphs with four distinct eigenvalues. To obtain these we shall use Lemmas 1.3.5-9 about polynomials with rational or integral coefficients (see Chapter 1).

Let G be a connected k-regular graph on v vertices with adjacency spectrum $\{[k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$. Now Lemma 1.3.5 implies that the polynomials p and q defined by

$$p(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = \frac{m(x)}{x-k} ,$$

$$q(x) = (x-\lambda_1)^{m_1-1}(x-\lambda_2)^{m_2-1}(x-\lambda_3)^{m_3-1} = \frac{c(x)}{m(x)} ,$$

where c is the characteristic polynomial and m is the minimal polynomial, have integral coefficients. We shall use these polynomials in the proof of the following proposition.

PROPOSITION 3.1.1. *Let G be a connected k-regular graph on v vertices with spectrum* $\{[k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$. *Then* (*i*) *$G$ has four integral eigenvalues, or* (*ii*) *G has two integral eigenvalues, and the other two have the same multiplicities and are of the form* $\frac{1}{2}(a \pm \sqrt{b})$, *with a, b integral, or* (*iii*) *G has one integral eigenvalue, and* $m_1 = m_2 = m_3 = \frac{1}{3}(v-1)$ *and* $k = \frac{1}{3}(v-1)$ *or* $k = \frac{2}{3}(v-1)$.

Proof. Without loss of generality we may assume $m_1 \le m_2 \le m_3$. If $m_1 = m_2 < m_3$, then $(x-\lambda_3)^{m_3-m_1} = q(x)/p(x)^{m_1-1} \in \mathbb{Z}[x]$, so λ_3 is integral. Now it follows that $(x-\lambda_1)(x-\lambda_2) \in \mathbb{Z}[x]$, so λ_1 and λ_2 are both integral or of the form $\frac{1}{2}(a \pm \sqrt{b})$, with a, b integral.

If $m_1 < m_2$, then $(x-\lambda_2)^{m_2-m_1}(x-\lambda_3)^{m_3-m_1} = q(x)/p(x)^{m_1-1} \in \mathbb{Z}[x]$. Now it follows that λ_2 and λ_3 are both integral or of the form $\frac{1}{2}(a \pm \sqrt{b})$, with a, b integral, and if λ_2 and λ_3 are irrational, then $m_2 = m_3$. In both cases it follows that λ_1 is integral.

So, if G has only one integral eigenvalue, then all three multiplicities must be the same. In that case they must be equal to $\frac{1}{3}(v-1)$, and $k + \frac{1}{3}(v-1)(\lambda_1 + \lambda_2 + \lambda_3) = Trace(A) = 0$, where A is the adjacency matrix of G. Since $p \in \mathbb{Z}[x]$, we have that $\lambda_1 + \lambda_2 + \lambda_3$ is integral, so k is a multiple of $\frac{1}{3}(v-1)$. Now it follows that $k = \frac{1}{3}(v-1)$ or $k = \frac{2}{3}(v-1)$. □

Each of the three cases of Proposition 3.1.1 can occur. Small examples are given by the 6-cycle C_6 with spectrum $\{[2]^1, [1]^2, [-1]^2, [-2]^1\}$, the complement of the union of two 5-cycles $\overline{2C_5}$ with spectrum $\{[7]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{5}]^4, [-\frac{1}{2} - \frac{1}{2}\sqrt{5}]^4, [-3]^1\}$, and the 7-cycle C_7 with spectrum $\{[2]^1, [2\cos\frac{2\pi}{7}]^2, [2\cos\frac{4\pi}{7}]^2, [2\cos\frac{6\pi}{7}]^2\}$.

Another important property of connected regular graphs with four distinct eigenvalues is that the multiplicities of the eigenvalues follow from the eigenvalues and the number of vertices (cf. [33, p. 161]). This follows from the following three equations, which uniquely determine m_1, m_2 and m_3 from v, $k = \lambda_0$, λ_1, λ_2 and λ_3.

$$\begin{aligned} 1 + m_1 + m_2 + m_3 &= v, \\ \lambda_0 + m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 &= 0, \\ \lambda_0^2 + m_1\lambda_1^2 + m_2\lambda_2^2 + m_3\lambda_3^2 &= vk. \end{aligned}$$

The second equation follows from the trace of A (the adjacency matrix of G), and the third from the trace of A^2. Note that the eigenvalues alone do not determine the multiplicities. For example, the complement of the Cube has spectrum $\{[4]^1, [2]^1, [0]^3, [-2]^3\}$, while the line graph of the Cube has spectrum $\{[4]^1, [2]^3, [0]^3, [-2]^5\}$. This example is the smallest of an infinite class given by Doob [45, 46].

Using the above conditions we were able to generate all possible spectra for regular graphs with four eigenvalues and at most 30 vertices. Different algorithms were used in each of the three cases of Proposition 3.1.1 and they in turn were checked for some of the feasibility conditions of Section 3.2.1.

3.2. Walk-regular graphs and feasibility conditions

A *walk-regular graph* is a graph G for which the number of walks of length l from a given vertex x to itself (closed walks) is independent of the choice of x, for all l (cf. [55]). If A is the adjacency matrix of G, then this number equals A^l_{xx}, so G is walk-regular whenever A^l has constant diagonal for all l. Note that a walk-regular graph is regular.

If G has v vertices and is connected k-regular with four distinct eigenvalues k, λ_1, λ_2 and λ_3, then $(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = \frac{1}{v}(k - \lambda_1)(k - \lambda_2)(k - \lambda_3)J$ (i.e. $h(A) = J$, where h is the Hoffman polynomial (cf. [70])). Since A^2, A, I, and J all have constant diagonal, we see that A^l has constant diagonal for every l. So a connected regular graph with four distinct eigenvalues is walk-regular.

3.2.1. Feasibility conditions and feasible spectra

If G is walk-regular on v vertices with degree k and spectrum $\{[\lambda_0]^{m_0}, [\lambda_1]^{m_1}, \ldots, [\lambda_r]^{m_r}\}$, the number of triangles through a given vertex x is independent of x, and equals

$$\Delta = \tfrac{1}{2}A^3_{xx} = \frac{Trace(A^3)}{2v} = \frac{1}{2v}\sum_{i=0}^{r} m_i\lambda_i^3 .$$

This expression gives a feasibility condition for the spectrum of G, since Δ should be a nonnegative integer. In general, it follows that

$$\theta_l = \frac{1}{v}\sum_{i=0}^{r} m_i \lambda_i^l$$

is a nonnegative integer. Since the number of closed walks of odd length l is even, θ_l should be even if l is odd. For even l, we can also sharpen the condition, since then the number of nontrivial closed walks (that is, those containing a cycle) is even. For example, when $l = 4$, the number of trivial closed walks through a given vertex (i.e. passing only one or two other vertices) equals $2k^2 - k$, so

$$\Xi = \frac{\theta_4 - 2k^2 + k}{2}$$

is a nonnegative integer, and it equals the number of quadrangles through a vertex. Here we allow the quadrangles to have diagonals. When $l = 6$, the number of nontrivial closed walks through a vertex equals $\theta_6 - k(5k^2 - 6k + 2)$, which should be even. In case we have four distinct eigenvalues the following lemma on the number of quadrangles through an edge will also be useful.

LEMMA 3.2.1. *If G is a connected regular graph with four distinct eigenvalues, such that the number of triangles through an edge is constant, then the number of quadrangles through an edge is also constant.*

Proof. Since G is connected and regular, say of degree k, with four distinct eigenvalues, its adjacency matrix A satisfies the equation $A^3 + p_2A^2 + p_1A + p_0I = pJ$, for some p_2, p_1, p_0 and p. Now $A^3_{xy} + p_2\lambda_{xy} + p_1 = p$, for any two adjacent x, y with λ_{xy} common neighbours. Since the number of triangles through an edge is constant, say λ, we have $\lambda_{xy} = \lambda$, and so the number of walks of length three from x to y is equal to $A^3_{xy} = p - p_1 - p_2\lambda$. Since there are $2k - 1$ walks which are trivial, the number of quadrangles containing edge $\{x, y\}$ equals $p - p_1 - p_2\lambda - 2k + 1$, which is independent of the given edge. □

Note that if ξ is the (constant) number of quadrangles through an edge, and if Ξ is the number of quadrangles through a vertex, then $\xi = 2\Xi/k$.

The complement of a connected regular graph with four eigenvalues is also such a graph unless it is disconnected. In the algorithms to generate feasible spectra, we only generated those spectra for which $k \geq v - 1 - k$, thus ensuring connectivity. In the Appendix A.3 however, we printed the complementary spectrum, unless it implied disconnectivity.

In the algorithm to generate spectra with four integral eigenvalues we checked that θ_l

was an even nonnegative integer for $l = 3, 5, 7, 9$ and 11 and that it was a nonnegative integer in the cases $l = 8$, 10 and 12. In addition we tested to see that both $\theta_4 - 2k^2 + k$ and $\theta_6 - k(5k^2 - 6k + 2)$ were even nonnegative integers, and that the complementary spectrum gave rise to numbers of triangles and quadrangles through a vertex that were also nonnegative integers. For technical reasons we checked different conditions in the case of two integral eigenvalues, namely the conditions on θ_l for $l = 3,..., 6$ and the complementary θ_l for $l = 3,..., 8$. Finally, in the remaining case of one integral eigenvalue it was not necessary to implement so many conditions. Here we checked only the conditions on θ_3 and θ_4. The spectra thus generated are termed feasible.

3.2.2. Simple eigenvalues

If a walk-regular graph has a simple eigenvalue $\lambda \neq k$, then we can say more about the structure of the graph. We shall prove that the graph admits a *regular partition into halves with degrees* $(\frac{1}{2}(k + \lambda), \frac{1}{2}(k - \lambda))$, that is, we can partition the vertices into two parts of equal size such that every vertex has $\frac{1}{2}(k + \lambda)$ neighbours in its own part and $\frac{1}{2}(k - \lambda)$ neighbours in the other part. As a consequence we obtain that $k - \lambda$ is even, a condition which was proven by Godsil and McKay [55]. This condition eliminates, for example, the existence of a graph with spectrum $\{[14]^1, [2]^9, [-1]^{19}, [-13]^1\}$. We also find other divisibility conditions.

LEMMA 3.2.2. *Let B be a symmetric matrix of size v, having constant diagonal and constant row sums t, and spectrum* $\{[t]^1, [s]^1, [0]^{v-2}\}$, *then v is even* (*unless* $s = 0$), *and* (*possibly after permuting rows and columns*) *B can be partitioned into four equally large parts as*

$$B = \begin{pmatrix} \frac{t+s}{v}J & \frac{t-s}{v}J \\ \frac{t-s}{v}J & \frac{t+s}{v}J \end{pmatrix}.$$

Proof. Consider the matrix $C = B - \frac{t}{v}J$, then C is symmetric, has constant diagonal, say a, row sums zero and spectrum $\{[s]^1, [0]^{v-1}\}$. So C has rank 1. By noticing that the determinant of all principal submatrices of size two must be zero, and using that C is symmetric and has constant diagonal, it follows that C only has entries $\pm a$. Since C has row sums zero, it follows that v is even (unless $s = 0$) and that we can partition C into four equally large parts as

$$C = \begin{pmatrix} aJ & -aJ \\ -aJ & aJ \end{pmatrix}.$$

Now B has nontrivial eigenvalues t and va, so $s = va$, and the result follows. □

THEOREM 3.2.3. *Let G be a connected walk-regular graph on v vertices and degree k, having distinct eigenvalues k, $\lambda_1, \lambda_2,..., \lambda_r$, of which at least one eigenvalue unequal to k, say λ_j, has multiplicity one. Then v is even and G admits a regular partition into halves with degrees $(\frac{1}{2}(k + \lambda_j), \frac{1}{2}(k - \lambda_j))$. Moreover, v is a divisor of*

$$\prod_{i \neq j} (k - \lambda_i) + \prod_{i \neq j} (\lambda_j - \lambda_i) \quad \textit{and} \quad \prod_{i \neq j} (k - \lambda_i) - \prod_{i \neq j} (\lambda_j - \lambda_i).$$

Proof. Let $b(x) = \prod_{i \neq j} (x - \lambda_i)$, and let $B = b(A)$, then it follows from Lemma 3.2.2 (B has constant diagonal since G is walk-regular) that v is even and we can partition B in four equally large parts as

$$B = \begin{pmatrix} \frac{t+s}{v} J & \frac{t-s}{v} J \\ \frac{t-s}{v} J & \frac{t+s}{v} J \end{pmatrix}, \text{ where } t = \prod_{i \neq j} (k - \lambda_i) \text{ and } s = \prod_{i \neq j} (\lambda_j - \lambda_i).$$

Now $(\underline{1}, -\underline{1})^T$ is an eigenvector of B with eigenvalue s, and since this eigenvalue is simple, and A and B commute, it follows that $(\underline{1}, -\underline{1})^T$ is also an eigenvector of A, and the corresponding eigenvalue must then be λ_j. This implies that if we partition A the same way as we partitioned B, with

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}, \text{ then } A_{11}\underline{1} = A_{22}\underline{1} = \tfrac{1}{2}(k + \lambda_j)\underline{1} \text{ and } A_{12}\underline{1} = A_{12}^T\underline{1} = \tfrac{1}{2}(k - \lambda_j)\underline{1}.$$

Since λ_j must be an integer, and $b(x) = m(x)/(x - k)(x - \lambda_j)$, where m is the minimal polynomial of G, it follows from Lemma 1.3.5 that b has integral coefficients, and so B is an integral matrix. But then $v | t + s$ and $v | t - s$. □

COROLLARY 3.2.4. *If G is a connected walk-regular graph with degree k, and λ is a simple eigenvalue, then $k - \lambda$ is even.*

As a consequence of the divisibility conditions in Theorem 3.2.3 we derive that there are no graphs with feasible spectrum $\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$ (on 18 vertices), or $\{[13]^1, [5]^1, [1]^{22}, [-5]^8\}$ (on 32 vertices).

3.2.3. A useful idea

Let G be a connected k-regular graph on v vertices with four eigenvalues k, λ_1, λ_2 and λ_3, and adjacency matrix A. The idea of Theorem 3.2.3 also turns out to be very useful when we do not have a simple eigenvalue. The matrix $C = C(\lambda_1, \lambda_2)$ defined by

$$C(\lambda_1, \lambda_2) = (A - \lambda_1 I)(A - \lambda_2 I) - \frac{(k - \lambda_1)(k - \lambda_2)}{v} J$$

is a symmetric matrix with row sums zero and one nonzero eigenvalue $(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ with multiplicity m_3 (the multiplicity of λ_3 as an eigenvalue of G). Now C or $-C$ is a positive semidefinite matrix of rank m_3, and C has constant diagonal $k + \lambda_1\lambda_2 - (k - \lambda_1)(k - \lambda_2)/v$. Of course, as A^2 is a matrix with nonnegative integral entries and A is a (0, 1)-matrix, the other entries of C are very restricted. Especially when m_3 is small we get strong restrictions on the structure of G. This enables us to show uniqueness of the graph in Proposition 3.3.5, and prove the nonexistence of graphs in a substantial number of cases in Section 3.4. It also proved to be a powerful tool in the computer search by Spence and the author (cf. [40]).

3.3. Examples, constructions and characterizations

3.3.1. Distance-regular graphs and association schemes

Several examples of graphs with four distinct eigenvalues can be contructed from distance-regular graphs, or more generally, association schemes. The graphs are obtained by taking the union of some classes (or just one class) as adjacency relation. In general, graphs from d-class association schemes have $d + 1$ eigenvalues, but sometimes some eigenvalues coincide. Many examples come from three-class association schemes (see the next chapter), such as the Johnson scheme $J(n, 3)$ and the Hamming scheme $H(3, q)$. An example coming from a five-class association scheme is obtained by taking distance 3 and 5 in the Dodecahedron as adjacency relation. The resulting graph has spectrum $\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$.

In general distance-regularity is not determined by the spectrum of the graph. Haemers [59] proved that it is, provided that some additional conditions are satisfied. Haemers and Spence [61] found (almost) all graphs with the spectrum of a distance-regular graph with at most 30 vertices. Most of these graphs have four distinct eigenvalues.

In the case that we only have one integral eigenvalue, all known examples come from pseudocyclic three-class association schemes. A large family of examples comes from the so-called cyclotomic schemes. We can define the associated graphs Cycl(v) as follows. Let $v \equiv 1 \pmod 3$ be a prime power. Take as vertices the elements of the field $GF(v)$. Two vertices are adjacent if their difference is a cube in the field. The smallest example is the 7-cycle C_7. It is determined by its spectrum, as are Cycl(13) and Cycl(19), which can be proven by hand.

More of the examples to come will turn out to be one of the relations in a three-class association scheme. In this chapter, however, we shall focus on the number of distinct eigenvalues.

3.3.2. Bipartite graphs

Examples of bipartite graphs with four distinct eigenvalues are the incidence graphs of symmetric 2-(v, k, λ) designs. We shall denote such graphs by $IG(v, k, \lambda)$. It is proven by Cvetković, Doob and Sachs [33, p. 166] that these are the only examples, i.e. a connected bipartite regular graph with four distinct eigenvalues must be the incidence graph of a symmetric 2-(v, k, λ) design. Moreover, it is distance-regular and its spectrum is

$$\{[k]^1,\ [\sqrt{k-\lambda}]^{v-1},\ [-\sqrt{k-\lambda}]^{v-1},\ [-k]^1\}.$$

3.3.3. The complement of the union of strongly regular graphs

Suppose G is regular with tv vertices and spectrum $\{[k]^t, [r]^{tf}, [s]^{tg}\}$. Then G is the union of t strongly regular graphs (all with the same spectrum and hence the same parameters), and the complement of G is a connected regular graph with spectrum

$$\{[tv-k-1]^1, [-s-1]^{tg}, [-r-1]^{tf}, [-k-1]^{t-1}\},$$

so it has four distinct eigenvalues (if $t > 1$). In general, if a connected regular graph has four distinct eigenvalues, then its complement is also connected and regular with four distinct eigenvalues, or it is disconnected, and then it is the union of strongly regular graphs, all having the same spectrum.

3.3.4. Product constructions

If G is a graph with adjacency matrix A, then we denote by $G \otimes J_n$ the graph with adjacency matrix $A \otimes J_n$, and by $G \circledast J_n$ we denote the graph with adjacency matrix $(A+I) \otimes J_n - I$. If G is connected and regular, then so are $G \otimes J_n$ and $G \circledast J_n$. Note that $\overline{G \otimes J_n} = \overline{G} \circledast J_n$, where $\overline{G}$ is the complement of G. If G has v vertices and spectrum $\{[k]^1, [r]^f, [0]^m, [s]^g\}$, where m is possibly zero, then $G \otimes J_n$ has vn vertices and spectrum

$$\{[kn]^1, [rn]^f, [0]^{m+vn-v}, [sn]^g\}.$$

Similarly, if G has v vertices and spectrum $\{[k]^1, [r]^f, [-1]^m, [s]^g\}$, where m is possibly zero, then $G \circledast J_n$ has vn vertices and spectrum

$$\{[kn+n-1]^1, [rn+n-1]^f, [-1]^{m+vn-v}, [sn+n-1]^g\}.$$

So, if we have a strongly regular graph or a connected regular graph with four distinct eigenvalues of which one is 0 or −1, then this construction produces a bigger graph with four distinct eigenvalues. The following proposition is a characterization of $C_5 \otimes J_n$, from which its uniqueness and the uniqueness of its complement $C_5 \circledast J_n$ follows.

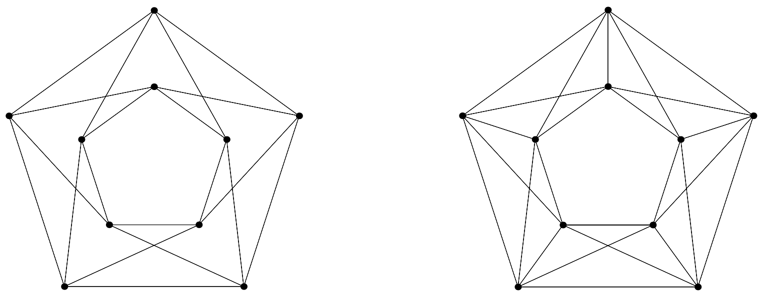

Figure 3.3.1. The graphs $C_5 \otimes J_2$ and $C_5 \circledast J_2$

Proposition 3.3.1. *Let G be a connected regular graph with four distinct eigenvalues and adjacency matrix A. If* $\mathrm{rank}(A) \leq 5$ *and G has no triangles* ($\Delta = 0$), *then G is isomorphic to $C_5 \otimes J_n$ for some n.*

Proof. Let G have v vertices and degree k. First we shall prove that G has diameter 2. Suppose G has diameter 3 and take two vertices x, y at distance 3. Let A be partitioned into two parts, where one part contains y and the neighbours of x. Then

$$A = \begin{pmatrix} O_{k+1,k+1} & N \\ N^T & B \end{pmatrix}.$$

Since $\mathrm{rank}(A) \leq 5$, it follows that $\mathrm{rank}(N) \leq 2$. Now write

$$N = \begin{pmatrix} \underline{1}_k & N_1 \\ 0 & N_2 \end{pmatrix}, \text{ and } N' = \begin{pmatrix} \underline{0}_k & N_1 \\ 1 & N_2 \end{pmatrix}.$$

Since the all-one vector is in the column space of N (N has constant row sums k), $\mathrm{rank}(N') \le \mathrm{rank}(N)$, so $\mathrm{rank}(N_1) \le 1$. But then $N_1 = (J_{k,k-1}\ O)$, and we have a subgraph $K_{k,k}$, so it follows that G is disconnected, which is a contradiction. So G has diameter 2. Next, let A be partitioned into two parts where one part contains the neighbours of x. Then

$$A = \begin{pmatrix} O_{k,k} & N \\ N^T & B \end{pmatrix},$$

with $\mathrm{rank}(N) \le 2$. If $\mathrm{rank}(N) = 1$ then $N = J_{k,k}$, and so G is a bipartite complete graph $K_{k,k}$, but then G only has three distinct eigenvalues. So $\mathrm{rank}(N) = 2$. Now write

$$N = \begin{pmatrix} J_{n,3k-v} & J_{n,v-2k} & O_{n,v-2k} \\ J_{k-n,3k-v} & O_{k-n,v-2k} & J_{k-n,v-2k} \end{pmatrix},$$

for some n. Note that since $\mathrm{rank}(N) = 2$, we have that all parts in N are nonempty. Since G has no triangles, it follows from Lemma 3.2.1 that the number of quadrangles ξ through an edge is constant. If we count the number of quadrangles through x (which corresponds to one of the first $3k - v$ columns of N) and a vertex y which corresponds to one of the first n rows of N (x and y are adjacent), then we see that

$$\xi = (n-1)(k-1) + (k-n)(3k-v-1) = (k-1)^2 + (k-n)(2k-v).$$

On the other hand, if we count the number of quadrangles through x and a vertex z which corresponds to one of the last $k - n$ rows of N, then we see that

$$\xi = (k-n-1)(k-1) + n(3k-v-1) = (k-1)^2 + n(2k-v).$$

So $n = \frac{1}{2}k$ and since A has rank at most 5 and zero diagonal it follows that A is the adjacency matrix of $C_5 \otimes J_n$. □

COROLLARY 3.3.2. *For any n, $C_5 \otimes J_n$ and $C_5 \circledast J_n$ are uniquely determined by their spectra.*

Next consider $IG(l, l-1, l-2)$, the incidence graph of the unique (trivial) 2-$(l, l-1, l-2)$ design. It can be obtained by removing a complete matching from the complete bipartite graph $K_{l,l}$, and is the complement of the $l \times 2$ grid.

PROPOSITION 3.3.3. *For each l and n, the graph $IG(l, l-1, l-2) \circledast J_n$ is uniquely determined by its spectrum.*

Proof. Note that for $l = 1$ or 2, the statement is trivial. So suppose $l > 2$. Let G be a graph with adjacency matrix A and spectrum

$$\{[nl-1]^1, [2n-1]^{l-1}, [-1]^{2nl-l-1}, [-n(l-2)-1]^1\}.$$

Now let $B = (A - (2n-1)I)(A + I)$, then we can partition A and B according to Theorem 3.2.3 such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} n(l-2)J_{nl} & O_{nl} \\ O_{nl} & n(l-2)J_{nl} \end{pmatrix},$$

where A_{11} and A_{22} have row sums $n-1$ and A_{12} has row sums $nl-n$. If two vertices x and y from the same part of the partition are adjacent, then it follows that $A^2_{xy} = n(l-2) + 2n - 2 = k - 1$, so x and y have the same neighbours, different from x and y themselves. Since this holds for any the $n-1$ neighbours of x, which are in the same part as x, it follows that $G = H \circledast J_n$, for some graph H. Since H must have the same spectrum as $IG(l, l-1, l-2)$, and this graph is uniquely determined by its spectrum, G is isomorphic to $IG(l, l-1, l-2) \circledast J_n$. □

If A is the adjacency matrix of a *conference graph* G, that is, a strongly regular graph which has parameters $(v = 4\mu + 1, k = 2\mu, \mu - 1, \mu)$, and spectrum $\{[k]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{v}]^k, [-\frac{1}{2} - \frac{1}{2}\sqrt{v}]^k\}$, then the graph with adjacency matrix

$$\begin{pmatrix} A & I \\ I & J - I - A \end{pmatrix}$$

has spectrum

$$\{[k+1]^1, [k-1]^1, [-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{v+4}]^{2k}, [-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{v+4}]^{2k}\}.$$

We shall call this graph the *twisted double* of G. We shall prove that this is the only way to construct a graph with this spectrum.

PROPOSITION 3.3.4. *Let $v = 4\mu + 1$ and $k = 2\mu$. Then G is a graph with spectrum $\{[k+1]^1, [k-1]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{v+4}]^{2k}, [-\frac{1}{2} - \frac{1}{2}\sqrt{v+4}]^{2k}\}$ if and only if G is the twisted double of a conference graph on v vertices.*

Proof. Let A be the adjacency matrix of G and let B be as in the proof of Theorem 3.2.3, then we find that

$$B = A^2 + A - (\mu + 1)I = \begin{pmatrix} \mu J & J \\ J & \mu J \end{pmatrix},$$

and that we can write A (A_{12} has row and column sums 1) as

$$A = \begin{pmatrix} A_{11} & I \\ I & A_{22} \end{pmatrix}, \text{ and so } B = \begin{pmatrix} A_{11}{}^2 + A_{11} - \mu I & A_{11} + A_{22} + I \\ A_{11} + A_{22} + I & A_{22}{}^2 + A_{22} - \mu I \end{pmatrix}.$$

This implies that $A_{11}{}^2 + A_{11} - \mu I = \mu J$ and $A_{11} + A_{22} + I = J$, so A_{11} is the adjacency matrix of a strongly regular graph with parameters ($v = 4\mu + 1$, $k = 2\mu$, $\mu - 1$, μ), and A_{22} is the adjacency matrix of its complement. □

Since the conference graphs on 9, 13 and 17 vertices are unique, also their twisted doubles are uniquely determined by their spectra. Since there is no conference graph on 21 vertices, there is also no graph on 42 vertices with spectrum $\{[11]^1, [9]^1, [2]^{20}, [-3]^{20}\}$. The twisted double of the 5-cycle is the Petersen graph.

There are 15 conference graphs on 25 vertices, of which only one is isomorphic to its complement (cf. [87]). Since complementary graphs give rise to the same twisted double, it follows that there are precisely 8 graphs on 50 vertices with spectrum $\{[13]^1, [11]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{29}]^{24}, [-\frac{1}{2} - \frac{1}{2}\sqrt{29}]^{24}\}$.

Let G and G' be graphs with adjacency matrices A and A', and eigenvalues λ_i, $i = 0, 1,..., v - 1$, and λ_i', $i = 0, 1,..., v' - 1$, respectively. Then the graph with adjacency matrix $A \otimes I_{v'} + I_v \otimes A'$ has eigenvalues $\lambda_i + \lambda_j'$, $i = 0, 1,..., v - 1$, $j = 0, 1,..., v' - 1$. We shall denote this graph, which is sometimes called the sum [33] or the Cartesian product of G and G' by $G \oplus G'$. If G is a strongly regular graph with spectrum $\{[k]^1, [r]^f, [s]^g\}$, and G' is the complete graph on m vertices, then $G \oplus G'$ is a graph with spectrum

$$\{[k + m - 1]^1, [k - 1]^{m-1}, [r + m - 1]^f, [r - 1]^{f(m-1)}, [s + m - 1]^g, [s - 1]^{g(m-1)}\}.$$

So we get a graph with four distinct eigenvalues if $m = k - r = r - s$. Examples are $G \oplus K_m$, where G is the complete bipartite graph $K_{m,m}$ or the lattice graph $L_2(m)$ and $G \oplus K_4$, where G is the Clebsch or the Shrikhande graph. Also $K_m \oplus K_n$ ($m > n \geq 2$) is a graph with four distinct eigenvalues: it is the same graph as the line graph of the complete bipartite graph $K_{m,n}$.

PROPOSITION 3.3.5. *The graph* $K_{3,3}{\oplus}K_3$ *is uniquely determined by its spectrum.*

Proof. Let G be a graph with spectrum $\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$ and adjacency matrix A. Then G is a 5-regular graph on 18 vertices with one triangle through each vertex. The matrix $C = C(2, -1) = A^2 - A - 2I - J$, as defined in Section 3.2.3 is a positive semidefinite integral matrix of rank two with diagonal 2. Thus C is the Gram matrix of a set of vectors in $\mathbb{R}^2$ of length $\sqrt{2}$ with mutual inner products ± 2, ± 1 or 0.

If two vertices are adjacent and the vectors representing these vertices have inner product -1, then they are in a triangle. This implies that any vertex is adjacent to precisely two vertices such that their inner product is -1, and that the inner product between those two vertices is also -1. If two vertices have inner product -2 then they are adjacent, and if they have inner product 1 or 2 then they are not adjacent. Without loss of generality we assume that there is a vertex represented by vector $\sqrt{2}(1, 0)^T$. This vertex must be in a triangle with vertices represented by vectors $\sqrt{2}(-\frac{1}{2}, \frac{1}{2}\sqrt{3})^T$ and $\sqrt{2}(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})^T$. Furthermore it is adjacent to three vertices represented by $\sqrt{2}(-1, 0)^T$. In turn, such a vertex is in a triangle with vertices represented by vectors $\sqrt{2}(\frac{1}{2}, \frac{1}{2}\sqrt{3})^T$ and $\sqrt{2}(\frac{1}{2}, -\frac{1}{2}\sqrt{3})^T$, and is adjacent to three vertices represented by $\sqrt{2}(1, 0)^T$.

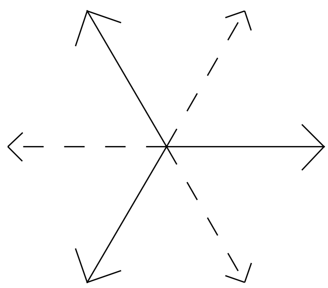

Figure 3.3.2. Vectors representing the vertices of $K_{3,3}{\oplus}K_3$

In this way we find 18 vertices: each of the 6 mentioned vectors represents 3 vertices. Now, up to isomorphism, all adjacencies follow from the inner products and the fact that every vertex is in one triangle. The graph we obtain is $K_{3,3}{\oplus}K_3$. □

3.3.5. Line graphs and other graphs with least eigenvalue −2

If G is a strongly regular graph ($k \neq 2$) or a bipartite regular graph with four distinct eigenvalues (the incidence graph of a symmetric design, cf. Section 3.3.2), then its line graph $L(G)$ has four distinct eigenvalues. If G is strongly regular with v vertices and spectrum $\{[k]^1, [r]^f, [s]^g\}$, then it follows that $L(G)$ has $\frac{1}{2}vk$ vertices and spectrum

$$\{[2k-2]^1, [r+k-2]^f, [s+k-2]^g, [-2]^{\frac{1}{2}vk - v}\}.$$

If G is the incidence graph of a symmetric design, with v vertices and spectrum $\{[k]^1, [r]^f, [-r]^f, [-k]^1\}$, then $L(G)$ has $\frac{1}{2}vk$ vertices and spectrum

$$\{[2k-2]^1, [r+k-2]^f, [-r+k-2]^f, [-2]^{\frac{1}{2}vk-v+1}\}.$$

Also the line graph of the complete bipartite graph $K_{m,n}$ has four distinct eigenvalues (if $m > n \geq 2$): its spectrum is

$$\{[m+n-2]^1, [m-2]^{n-1}, [n-2]^{m-1}, [-2]^{mn-m-n+1}\}.$$

Now these graphs provide almost all connected regular graphs with four distinct eigenvalues and least eigenvalue at least −2. It was proven by Doob and Cvetković [47] that a regular connected graph with least eigenvalue greater than −2 is K_n or C_{2n+1} for some $n \geq 1$. So the only one with four distinct eigenvalues is C_7. Bussemaker, Cvetković and Seidel [17] found all connected regular graphs with least eigenvalue −2, which are neither line graphs, nor cocktail party graphs. Among them are 12 graphs with four distinct eigenvalues.

BCS_9	: one graph on 12 vertices with spectrum	$\{[4]^1, [2]^3, [0]^3, [-2]^5\}$,
BCS_{70}	: one graph on 18 vertices with spectrum	$\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$,
BCS_{153}-BCS_{160}	: eight graphs on 24 vertices with spectrum	$\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$,
BCS_{179}	: one graph on 18 vertices with spectrum	$\{[10]^1, [4]^2, [1]^4, [-2]^{11}\}$,
BCS_{183}	: one graph on 24 vertices with spectrum	$\{[14]^1, [4]^4, [2]^2, [-2]^{17}\}$.

Cocktail party graphs are strongly regular, so we are left with the line graphs. Now Doob [46] showed that if G has four distinct eigenvalues, least eigenvalue −2, and is the line graph of, say H, then H is a strongly regular graph, or the incidence graph of a symmetric design, or a complete bipartite graph $K_{m,n}$, with $m > n \geq 2$.

Furthermore it is known (cf. [33, p. 175]) that $L(K_{m,n})$ is not characterized by its spectrum if and only if $\{m, n\} = \{6, 3\}$ or $\{m, n\} = \{2t^2+t, 2t^2-t\}$ and there exists a symmetric Hadamard matrix with constant diagonal of order $4t^2$. In the first case there is one graph with the same spectrum: BCS_{70}. If G is the line graph of the incidence graph of a symmetric 2-(v, k, λ) design, then the only possible graphs with the same spectrum are the line graphs of the incidence graphs of other symmetric 2-(v, k, λ) designs, unless $(v, k, \lambda) = (4, 3, 2)$ (then the incidence graph of the design is the Cube). In that case there is one exception: BCS_9.

Note that the complement of a connected regular graph with least eigenvalue −2, is a graph with second largest eigenvalue 1.

3.3.6. Other graphs from strongly regular graphs

In the previous sections we already used strongly regular graphs to construct other graphs. In this section we shall construct graphs from strongly regular graphs having certain properties, like having large cliques or cocliques, having a spread, or a regular partition into halves.

3.3.6.1. Hoffman cocliques and cliques

If G is a nonbipartite strongly regular graph on v vertices, with spectrum $\{[k]^1, [r]^f, [s]^g\}$, and C is a coclique of size c meeting the Hoffman (Delsarte) bound, i.e. $c = -vs/(k-s)$, then the induced subgraph $G \setminus C$ on the vertices not in C is a regular, connected graph with spectrum

$$\{[k+s]^1,\ [r]^{f-c+1},\ [r+s]^{c-1},\ [s]^{g-c}\},$$

so it has four distinct eigenvalues if $c < g$. This is an easy consequence of a theorem by Haemers and Higman [60] on strongly regular decompositions of strongly regular graphs. By looking at the complement of the graph, a similar construction works for cliques instead of cocliques. For example, by removing a 3-clique (a line) in the generalized quadrangle $GQ(2,2)$ (the complement of $T(6)$) we obtain a graph with spectrum $\{[5]^1, [1]^6, [-1]^2, [-3]^3\}$. If we remove a 6-coclique from a strongly regular graph with parameters (26, 10, 3, 4) (these exist), then we get a graph with spectrum $\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$.

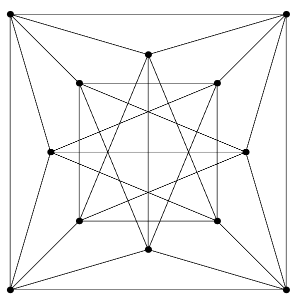

Figure 3.3.3. The graph $GQ(2,2) \setminus$3-clique

3.3.6.2. Spreads

If G admits a *spread*, that is, a partition of the vertices into cliques of size $1 - k/s$ (i.e., meeting the Hoffman bound), then by removing the spread, that is, the edges in these cliques, we obtain a graph with spectrum

$$\{[k + \frac{k}{s}]^1,\ [r + 1]^{k(-s-1)/\mu},\ [r + \frac{k}{s}]^{f - k(-s-1)/\mu},\ [s + 1]^g\}.$$

Here the graphs come from 3-class association schemes. For example, if we remove a spread from the generalized quadrangle $GQ(2, 4)$, we get a distance-regular graph with spectrum $\{[8]^1, [2]^{12}, [-1]^8, [-4]^6\}$. For more on spreads in strongly regular graphs we refer to the paper by Haemers and Tonchev [62].

3.3.6.3. Seidel switching

Let G be a strongly regular graph on v vertices admitting a regular partition into halves with degrees $(\frac{1}{2}(k + s), \frac{1}{2}(k - s))$, so its adjacency matrix A can be written as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

where all parts have the same size and A_{11}, A_{22} have row sums $\frac{1}{2}(k + s)$. When we switch with respect to this partition we obtain a graph with spectrum

$$\{[s + \tfrac{1}{2}v]^1, [r]^f, [s]^{g-1}, [k - \tfrac{1}{2}v]^1\}.$$

Note that we can interchange the role of r and s. It follows from Theorem 3.2.3 that this is the only way to construct a graph with this spectrum.

THEOREM 3.3.6. *If G is an $(s + \frac{1}{2}v)$-regular graph with four distinct eigenvalues on v vertices and with spectrum $\{[s + \frac{1}{2}v]^1, [r]^f, [s]^{g-1}, [k - \frac{1}{2}v]^1\}$, for some k, r and s, then G can be obtained by Seidel switching in a strongly regular graph with spectrum $\{[k]^1, [r]^f, [s]^g\}$, admitting a regular partition into halves with degrees $(\frac{1}{2}(k + s), \frac{1}{2}(k - s))$.*

This theorem may be useful in case we want to prove uniqueness or nonexistence of certain graphs, such as in some of the following examples, where we find some infinite families of graphs with four distinct eigenvalues. The first two families are obtained from the lattice graphs $L_2(n)$ for even n. Recall that the lattice graph is the graph on the n^2

ordered pairs (i, j), with $i, j = 1, 2, \ldots, n$, where two vertices are adjacent if they agree in one of the coordinates. Its spectrum is $\{[2n-2]^1, [n-2]^{2n-2}, [-2]^{(n-1)^2}\}$. If we take for one part of the partition the set $\{(i, j) \mid i, j = 1, \ldots, \frac{1}{2}n\} \cup \{(i, j) \mid i, j = \frac{1}{2}n + 1, \ldots, n\}$, then we have a regular partition into halves with degrees $(n-2, n)$. Thus by Seidel switching we obtain a graph with spectrum

$$\{[\tfrac{1}{2}n^2 - 2]^1, [n-2]^{2n-2}, [-2]^{(n-1)^2-1}, [2n - \tfrac{1}{2}n^2 - 2]^1\}.$$

Note that (in general) there are different ways to obtain regular partitions into halves with these degrees, and so possibly different graphs with this spectrum.

If we take for one part of the partition the set $\{(i, j) \mid i = 1, \ldots, n, j = 1, \ldots, \frac{1}{2}n\}$, then we have a regular partition into halves with degrees $(\frac{3}{2}n - 2, \frac{1}{2}n)$. Thus we obtain a graph with spectrum

$$\{[\tfrac{1}{2}n^2 + n - 2]^1, [n-2]^{2n-3}, [-2]^{(n-1)^2}, [2n - \tfrac{1}{2}n^2 - 2]^1\},$$

so for $n \geq 6$ it has four distinct eigenvalues. The following proposition proves that this graph is uniquely determined by its spectrum.

PROPOSITION 3.3.7. *For each even $n \geq 6$, there is exactly one graph on n^2 vertices with spectrum* $\{[\frac{1}{2}n^2 + n - 2]^1, [n-2]^{2n-3}, [-2]^{(n-1)^2}, [2n - \frac{1}{2}n^2 - 2]^1\}$.

Proof. According to the previous theorem, a graph having the required spectrum must be obtained by Seidel switching in a strongly regular graph with spectrum $\{[2n-2]^1, [n-2]^{2n-2}, [-2]^{(n-1)^2}\}$. For $n \neq 4$ the only graph with this spectrum is the lattice graph $L_2(n)$. Furthermore, we must have a regular partition into halves with degrees $(\frac{3}{2}n - 2, \frac{1}{2}n)$. Now there is (up to isomorphism) exactly one way to do this. □

This partition can also be used for the graphs $L_m(n)$ for "arbitrary" m. Recall that this graph is obtained from an orthogonal array, that is, an $m \times n^2$ matrix M such that for any two rows a, b we have that $\{(M_{ai}, M_{bi}) \mid i = 1, \ldots, n^2\} = \{(i, j) \mid i, j = 1, \ldots, n\}$. The graph has vertices $1, 2, \ldots, n^2$, and two vertices x, y are adjacent if $M_{ix} = M_{iy}$ for some i. This graph is strongly regular with spectrum $\{[mn-m]^1, [n-m]^{m(n-1)}, [-m]^{(n-1)(n-m+1)}\}$. If we now take for one part of the partition the set $\{i \mid M_{1i} = 1, \ldots, \frac{1}{2}n\}$, then we have a regular partition into halves with degrees $(n - 1 + (m-1)(\frac{1}{2}n - 1), \frac{1}{2}(m-1)n)$. Thus we obtain a graph with spectrum

$$\{[\tfrac{1}{2}n^2 + n - m]^1, [n-m]^{m(n-1)-1}, [-m]^{(n-1)(n-m+1)}, [mn - \tfrac{1}{2}n^2 - m]^1\}.$$

Another family of graphs can be obtained from the triangular graphs $T(n)$, for $n \equiv 1 \pmod 4$. Recall that the triangular graph $T(n)$ is the graph on the $\frac{1}{2}n(n-1)$ unordered pairs taken from the n symbols $1, 2, \ldots, n$, where two pairs are adjacent if they have a symbol in common. Its spectrum is $\{[2n-4]^1, [n-4]^{n-1}, [-2]^{\frac{1}{2}n(n-3)}\}$. For each $n \equiv 1 \pmod 4$, we now get a regular partition into halves with degrees $(n-3, n-1)$ by taking for one part the pairs $\{i, j\}$, $i \neq j$ with

$$
\begin{aligned}
&i = 1,\dots, \tfrac{1}{4}(n-1),\ j = 2,\dots, \tfrac{1}{2}(n-1)+1, \text{ or}\\
&i = \tfrac{1}{4}(n-1)+1,\dots, \tfrac{1}{2}(n-1),\ j = \tfrac{1}{2}(n-1)+2,\dots, \tfrac{3}{4}(n-1)+1, \text{ or}\\
&i = \tfrac{1}{2}(n-1)+1,\dots, n-1,\ j = \tfrac{3}{4}(n-1)+2,\dots, n.
\end{aligned}
$$

For $n \equiv 1 \pmod 4$ we thus obtain a graph with spectrum

$$\{[\tfrac{1}{4}n(n-1)-2]^1, [n-4]^{n-1}, [-2]^{\frac{1}{2}n(n-3)-1}, [2n-\tfrac{1}{4}n(n-1)-4]^1\}.$$

Note that (in general) there are more ways to obtain such partitions, and so possibly different graphs with this spectrum. The following lemma shows that we need the restriction $n \equiv 1 \pmod 4$, and gives a property of the partitions.

LEMMA 3.3.8. *If the triangular graph* $T(n)$ *admits a regular partition into halves* V_1 *and* V_2, *with degrees* $(n-3, n-1)$, *then* $n \equiv 1 \pmod 4$ *and for each* $i = 1,\dots, n$, *we have that* $|\{j \neq i \mid \{i, j\} \in V_1\}| = \tfrac{1}{2}(n-1)$.

Proof. First, note that the number of vertices $\tfrac{1}{2}n(n-1)$ should be even, so that $n \equiv 0$ or $1 \pmod 4$. Now fix i, and let $m = |\{j \neq i \mid \{i, j\} \in V_1\}|$. If $\{i, j\} \in V_1$, then we have

$$|\{h \neq i, j \mid \{h, j\} \in V_1\}| + |\{h \neq i, j \mid \{i, h\} \in V_1\}| = n-3,$$

so $|\{h \neq j \mid \{h, j\} \in V_1\}| = n-1-m$. If $\{i, j\} \in V_2$, then we must have that

$$|\{h \neq i, j \mid \{h, j\} \in V_1\}| + |\{h \neq i, j \mid \{i, h\} \in V_1\}| = n-1,$$

and then also $|\{h \neq j \mid \{h, j\} \in V_1\}| = n-1-m$. Now it follows that

$$m + (n-1)(n-1-m) = \sum_{j=1}^{n} |\{h \neq j \mid \{h, j\} \in V_1\}| = 2|V_1| = \tfrac{1}{2}n(n-1),$$

which implies that $m = \tfrac{1}{2}(n-1)$, hence $n \equiv 1 \pmod 4$. □

Since the triangular graph $T(n)$ is uniquely determined by its spectrum unless $n = 8$, Theorem 3.3.6 and Lemma 3.3.8 imply the following result.

PROPOSITION 3.3.9. *For* $n \equiv 0 \pmod 4$, $n \neq 8$, *there is no graph with spectrum* $\{[\tfrac{1}{4}n(n-1)-2]^1, [n-4]^{n-1}, [-2]^{\frac{1}{2}n(n-3)-1}, [2n-\tfrac{1}{4}n(n-1)-4]^1\}$.

In the case $n = 8$ the considered spectrum has only three eigenvalues, and it is the spectrum of the triangular graph $T(8)$ and the Chang graphs. The next lemma shows that the "other" regular partition into halves is not possible, which together with Theorem 3.3.6 proves Proposition 3.3.11.

LEMMA 3.3.10. *For $n \neq 4$, the triangular graph $T(n)$ does not admit a regular partition into halves with degrees* $(\frac{3}{2}n - 4, \frac{1}{2}n)$.

Proof. Suppose we have such a partition with halves V_1 and V_2. Both n and $\frac{1}{2}n(n-1)$ are even, so $n \equiv 0 \pmod 4$. So we may suppose that $n \geq 8$. Now fix i and let $m = |\{j \neq i \mid \{i, j\} \in V_1\}|$. Without loss of generality we may assume that $m > 0$. Then we find that if $\{i, j\} \in V_1$, then

$$|\{h \neq j \mid \{h, j\} \in V_1\}| = \tfrac{3}{2}n - 2 - m.$$

If $\{i, j\} \in V_2$, then we must have that

$$|\{h \neq j \mid \{h, j\} \in V_1\}| = \tfrac{1}{2}n - m.$$

This implies that $m \leq \frac{1}{2}n$ unless there is no j with $\{i, j\} \in V_2$. So $m \leq \frac{1}{2}n$ or $m = n - 1$. Now let j be such that $\{i, j\} \in V_1$, and $m' = |\{h \neq j \mid \{h, j\} \in V_1\}|$, then also $m' \leq \frac{1}{2}n$ or $m' = n - 1$. Without loss of generality we may assume that $m \geq m'$, and since $m + m' = \frac{3}{2}n - 2$, we must have $m = n - 1$ and $m' = \frac{1}{2}n - 1$. Since $m' \geq 3$, there is an $h \neq i, j$ such that $\{i, h\} \in V_1$ and $\{j, h\} \in V_1$. Now let $m'' = |\{g \neq h \mid \{h, g\} \in V_1\}|$, then $m + m'' = \frac{3}{2}n - 2 = m' + m''$, so $m = m'$, which is a contradiction. □

PROPOSITION 3.3.11. *For $n \neq 4$, there is no graph with spectrum* $\{[\frac{1}{4}n(n-1) + n - 4]^1, [n-4]^{n-2}, [-2]^{\frac{1}{2}n(n-3)}, [2n - \frac{1}{4}n(n-1) - 4]^1\}$.

For all parameter sets of strongly regular graphs on at most 63 vertices, except for $T(9)$ and $L_2(6)$, we shall now give an example of how we can obtain a graph with four distinct eigenvalues, using Seidel switching. The only graphs we have to consider are the strongly regular graphs on 40 vertices with spectrum $\{[12]^1, [2]^{24}, [-4]^{15}\}$, the Hoffman-Singleton graph, which is the unique graph on 50 vertices with spectrum $\{[7]^1, [2]^{28}, [-3]^{21}\}$ and the Gewirtz graph, which is the unique graph on 56 vertices with spectrum $\{[10]^1, [2]^{35}, [-4]^{20}\}$.

Now there is one generalized quadrangle $GQ(3, 3)$ (the collinearity graph of which is a strongly regular graph on 40 vertices) with a spread (cf. [89]), and by splitting it into two equal parts, we have a regular partition into halves with degrees (7, 5). Thus we obtain a graph with spectrum $\{[22]^1, [2]^{23}, [-4]^{15}, [-8]^1\}$. Haemers [57, ex. 6.2.2] constructed a strongly regular graph on 40 vertices admitting a regular partition into halves with degrees (4, 8). This yields a graph with spectrum $\{[16]^1, [2]^{24}, [-4]^{14}, [-8]^1\}$.

Since it is possible to partition the vertices of the Hoffman-Singleton graph into two halves such that the induced subgraphs on each of the halves is the union of five pentagons (cf. [16]), we have a regular partition into two halves with degrees (2, 5), and so we can construct a graph with spectrum $\{[22]^1, [2]^{28}, [-3]^{20}, [-18]^1\}$.

Since it is possible to split the Gewirtz graph into two Coxeter graphs (cf. [14]), we have a regular partition into two halves with degrees (3, 7), and so we obtain a graph with spectrum $\{[24]^1, [2]^{35}, [-4]^{19}, [-18]^1\}$. The Gewirtz graph also contains a regular graph on 28 vertices of degree 6 (cf. [14]), and so we have a regular partition into two halves with degrees (6, 4). Thus we obtain a graph with spectrum $\{[30]^1, [2]^{34}, [-4]^{20}, [-18]^1\}$.

3.3.6.4. Subconstituents

Let G be a strongly regular graph with parameters (v, k, λ, μ) and spectrum $\{[k]^1, [r]^f, [s]^g\}$. For any vertex x, we denote by $G(x)$ the induced subgraph on the set of neighbours of x. By $G_2(x)$ we denote the induced subgraph on the vertices distinct from x which are not adjacent to x. These (regular) graphs are called the *subconstituents* of G with respect to x. Cameron, Goethals and Seidel [24] proved that there is a one-one correspondence between the restricted eigenvalues $\notin \{r, s\}$ of the subconstituents of G, such that corresponding eigenvalues have the same restricted multiplicity, and add up to $r + s$. Here we call an eigenvalue restricted if it has an eigenvector orthogonal to the all-one vector. Its restricted multiplicity is the dimension of its eigenspace which is orthogonal to the all-one vector. If $\lambda = 0$, then $G(x)$ is a graph without edges, and $G_2(x)$ is a $(k - \mu)$-regular graph with restricted eigenvalues $r + s$, and possibly r and s, with multiplicities $k - 1$, and say m_r and m_s, respectively. Since $\mu = -(r + s)$, we find that $m_r = f - k$ and $m_s = g - k$, so $G_2(x)$ has spectrum $\{[k+r+s]^1, [r]^{f-k}, [r+s]^{k-1}, [s]^{g-k}\}$. For example, the Gewirtz graph is a strongly regular graph with $\lambda = 0$ and spectrum $\{[10]^1, [2]^{35}, [-4]^{20}\}$, so $\text{Gewirtz}_2(x)$ is a graph with spectrum $\{[8]^1, [2]^{25}, [-2]^9, [-4]^{10}\}$. Also the Hoffman-Singleton graph Ho-Si is a strongly regular graph with $\lambda = 0$, and its spectrum is $\{[7]^1, [2]^{28}, [-3]^{21}\}$, so $\text{Ho-Si}_2(x)$ is a graph with spectrum $\{[6]^1, [2]^{21}, [-1]^6, [-3]^{14}\}$.

If $\lambda = r$ and $G(x)$ is the union of $(r + 1)$-cliques, so it has spectrum $\{[r]^{k/(r+1)}, [-1]^{rk/(r+1)}\}$, then $G_2(x)$ is a $(k - \mu)$-regular graph with restricted eigenvalues $r + s + 1$, and possibly r and s, with multiplicities $rk/(r + 1)$, and say m_r and m_s, respectively. Since $\mu = -s$, we find that $m_r = f - k$ and $m_s = g - rk/(r + 1) - 1$, so $G_2(x)$ has spectrum $\{[k+s]^1, [r]^{f-k}, [r+s+1]^{rk/(r+1)}, [s]^{g-rk/(r+1)-1}\}$. Examples of such graphs can be found when G is the graph of a generalized quadrangle.

3.3.7. Covers

In this section we shall construct n-covers of $C_3 \otimes J_n$, $C_3 \circledast J_n$ (which is isomorphic to K_{3n}),

$C_5 \circledast J_n$, $C_6 \circledast J_n$ and Cube$\circledast J_n$, having four distinct eigenvalues. Let C be the $n \times n$ circulant matrix defined by $C_{ij} = 1$ if $j = i + 1 \pmod n$, and $C_{ij} = 0$ otherwise. Then let A and B be the $n^2 \times n^2$ matrices defined by

$$A = \begin{pmatrix} I & I & \cdots & I \\ C & C & \cdots & C \\ \vdots & \vdots & & \vdots \\ C^{n-1} & C^{n-1} & \cdots & C^{n-1} \end{pmatrix}, \text{ and } B = \begin{pmatrix} I & C & \cdots & C^{n-1} \\ C^{n-1} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & C \\ C & \cdots & C^{n-1} & I \end{pmatrix}.$$

Furthermore, let $D = (J_n - I_n) \otimes I_n$. Then the graphs with adjacency matrices

$$A_3 = \begin{pmatrix} O & A & A^T \\ A^T & O & A \\ A & A^T & O \end{pmatrix}, \; B_3 = \begin{pmatrix} D & A & A^T \\ A^T & D & A \\ A & A^T & D \end{pmatrix}, \; B_5 = \begin{pmatrix} D & A & O & O & A^T \\ A^T & D & A & O & O \\ O & A^T & D & A & O \\ O & O & A^T & D & A \\ A & O & O & A^T & D \end{pmatrix}$$

are n-covers of $C_3 \otimes J_n$, $C_3 \circledast J_n$ and $C_5 \circledast J_n$, respectively. The graphs with adjacency matrices

$$B_6 = \begin{pmatrix} D & O & O & O & A^T & A^T \\ O & D & O & A & O & D+I \\ O & O & D & A & D+I & O \\ O & A^T & A^T & D & O & O \\ A & O & D+I & O & D & O \\ A & D+I & O & O & O & D \end{pmatrix}, B_8 = \begin{pmatrix} D & O & O & O & O & D+I & B & B \\ O & D & O & O & D+I & O & B & B \\ O & O & D & O & B & B & O & D+I \\ O & O & O & D & B & B & D+I & O \\ O & D+I & B & B & D & O & O & O \\ D+I & O & B & B & O & D & O & O \\ B & B & O & D+I & O & O & D & O \\ B & B & D+I & O & O & O & O & D \end{pmatrix}$$

are n-covers of $C_6 \circledast J_n$ and Cube$\circledast J_n$, respectively.

The matrix A_3 has spectrum $\{[2n]^1, [n]^{3n-3}, [0]^{3(n-1)^2}, [-n]^{3n-1}\}$. The crucial step to show this is that $A_3(A_3^2 - n^2 I) = 2nJ$ (The multiplicities follow from the eigenvalues). For $n = 2$ we get the line graph of the Cube, and for $n = 3$ we get a graph, which has the same spectrum as (but is not isomorphic to) the cubic graph $H(3, 3)$.

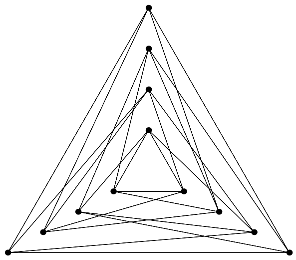

Figure 3.3.4. The line graph of the Cube, a 2-cover of $C_3 \otimes J_2$: three different views

The spectrum of B_3 is $\{[3n-1]^1, [-1]^{3n^2-6n+5}, [-1+\frac{1}{2}n(1\pm\sqrt{5})]^{3n-3}\}$. The crucial step here is that $(B_3+I)((B_3+I)^2 - n(B_3+I) - n^2 I) = 5nJ$. For $n=2$ we get the Icosahedron.

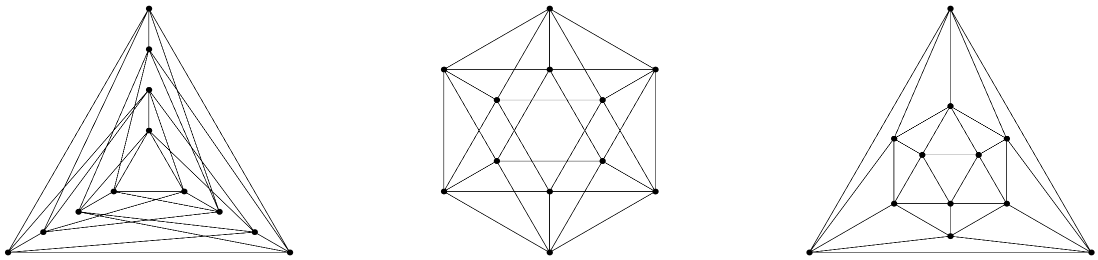

Figure 3.3.5. The Icosahedron, a 2-cover of $C_3 \circledast J_2$: three different views

Similarly we find that B_5 has spectrum $\{[3n-1]^1, [-1]^{5n^2-10n+5}, [-1+\frac{1}{2}n(1\pm\sqrt{5})]^{5n-3}\}$, B_6 has spectrum $\{[3n-1]^1, [2n-1]^{4n-2}, [-1]^{6n^2-6n+2}, [-n-1]^{2n-1}\}$, and B_8 has spectrum $\{[4n-1]^1, [2n-1]^{6n-3}, [-1]^{8n^2-8n+3}, [-2n-1]^{2n-1}\}$.

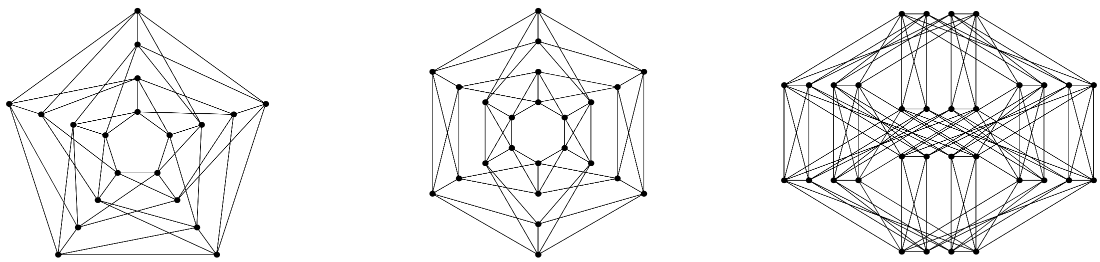

Figure 3.3.6. 2-covers of $C_5 \circledast J_2$, $C_6 \circledast J_2$ and Cube$\circledast J_2$

3.4. Nonexistence results

Let G be a k-regular graph on v vertices with Δ triangles and Ξ quadrangles through every vertex. Fix a vertex x, and let σ_i be the number of vertices y adjacent to x, such that $A^2_{xy} = i$, and let τ_i be the number of vertices y not adjacent to x, such that $A^2_{xy} = i$. Then counting arguments show that

$$\sum_i \sigma_i = k,\ \sum_i i\sigma_i = 2\Delta,\ \sum_i \tau_i = v-k-1,\ \sum_i i\tau_i = k(k-1)-2\Delta, \text{ and } \sum_i \binom{i}{2}(\sigma_i+\tau_i) = \Xi.$$

We shall call this system of equations the (σ, τ)-system.

In the following we examine several feasible spectra and prove the nonexistence of a graph in each case. In each of the proofs of the following propositions we assume the existence of a graph G with the given spectrum and A will denote its adjacency matrix.

PROPOSITION 3.4.1. *There are no graphs with spectrum* $\{[7]^1, [2]^{15}, [-2]^5, [-3]^9\}$, $\{[6]^1, [2]^9, [1]^9, [-3]^{11}\}$, $\{[7]^1, [2]^{12}, [1]^5, [-3]^{12}\}$, $\{[6]^1, [1+\sqrt{10}]^2, [-1]^{10}, [1-\sqrt{10}]^2\}$, $\{[7]^1, [1+2\sqrt{3}]^2, [-1]^{11}, [1-2\sqrt{3}]^2\}$ *or* $\{[8]^1, [-1+\sqrt{6}]^7, [1]^6, [-1-\sqrt{6}]^7\}$.

Proof. A graph with the first spectrum would be 7-regular on 30 vertices with $\Delta = 3$ triangles and $\Xi = 12$ quadrangles through every vertex. Using the idea of Section 3.2.3, let $C = C(2, -3) = A^2 + A - 6I - \frac{5}{3}J$, then $-C$ is a positive semidefinite matrix with diagonal $\frac{2}{3}$. It follows that C can only have entries $-\frac{2}{3}$ and $\frac{1}{3}$, and so if x and y are adjacent then $A^2_{xy} = 0$ or 1, and if x and y are not adjacent then $A^2_{xy} = 1$ or 2. But now the (σ, τ)-system does not have a solution, so we have a contradiction. The other cases go similarly. □

PROPOSITION 3.4.2. *There are no graphs with spectrum*
$\{[8]^1, [2+3\sqrt{2}]^3, [-1]^{20}, [2-3\sqrt{2}]^3\}$ *or* $\{[9]^1, [7]^3, [-1]^{24}, [-3]^2\}$.

Proof. The first spectrum would give an 8-regular graph on 27 vertices with $\Delta = 22$ triangles and $\Xi = 102$ quadrangles through every vertex. The matrix C as defined in Section 3.2.3 by $C = C(2+3\sqrt{2}, -1) = A^2 - (1+3\sqrt{2})A - (2+3\sqrt{2})I - (2-\sqrt{2})J$, is a positive semidefinite matrix with diagonal $4 - 2\sqrt{2}$. It follows that if x and y are adjacent then $A^2_{xy} = 5$, 6 or 7, and if x and y are not adjacent then $A^2_{xy} = 0$ or 1. Now the (σ, τ)-system has one solution $\sigma_7 = 2$, $\sigma_6 = 0$, $\sigma_5 = 6$, $\tau_1 = 12$, $\tau_0 = 6$. But then $G = H \circledast J_3$, for some graph H. It follows that H must have spectrum $\{[2]^1, [\sqrt{2}]^3, [-1]^2, [-\sqrt{2}]^3\}$, but since such a graph does not exist, we have a contradiction. Similarly, a graph with the second spectrum must be of the form $H \circledast J_2$, where H has spectrum $\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$, which is impossible by the results of Section 3.3.5. □

The next proposition uses the fact that the number of quadrangles through an edge is constant (cf. Lemma 3.2.1).

PROPOSITION 3.4.3. *There are no graphs with spectrum*
$\{[8]^1, [-1+\sqrt{21}]^4, [0]^{21}, [-1-\sqrt{21}]^4\}$ *or* $\{[4]^1, [-\frac{1}{2}+\frac{1}{2}\sqrt{21}]^4, [0]^6, [-\frac{1}{2}-\frac{1}{2}\sqrt{21}]^4\}$.

Proof. Note that if H is a graph with the second spectrum, then $H \otimes J_2$ is a graph with the first spectrum. Thus it suffices to show that there is no graph with the first spectrum. Suppose G is such a graph, then G is 8-regular on 30 vertices without triangles, such that every vertex is in $\Xi = 84$ quadrangles and every edge is in $\xi = 21$ quadrangles.

Suppose first of all that G has diameter 2. Suppose x and z are two nonadjacent vertices such that $A^2_{xz} = 1$ and let y be their common neighbour. Now the 21 quadrangles through $\{x, y\}$ and the 21 quadrangles through $\{z, y\}$ are distinct, and since there are 42 edges between $G(y)\setminus\{x, z\}$ and $G_2(y)$, all these edges contain a neighbour of x or z. Then it follows that the number of vertices at distance 2 from y is 14, and so G has diameter 3, which is a contradiction. Thus for any two nonadjacent vertices x and z we must have $A^2_{xz} \geq 2$. But then the (σ, τ)-system has no nonnegative integral solution. Thus G has diameter 3.

Take a vertex x and let y be a vertex at distance 3 from x. Let A be partitioned into two parts, where one part contains y and the neighbours of x. Then

$$A = \begin{pmatrix} O_{9,9} & N \\ N^T & B \end{pmatrix}.$$

Since rank$(A) = 9$, it follows that rank$(N) \leq 4$. Now write

$$N = \begin{pmatrix} \underline{1}_8 & N_1 \\ 0 & N_2 \end{pmatrix}, \text{ and } N' = \begin{pmatrix} \underline{0}_8 & N_1 \\ 1 & N_2 \end{pmatrix}.$$

Since the all-one vector is in the column space of N (N has constant row sums 8), rank$(N') \leq$ rank(N), so rank$(N_1) \leq 3$. Moreover, N_1 has constant row sums 7, and so it follows that N_1 is of the form

$$N_1 = \begin{pmatrix} J_{m_1,7-t_1-t_2} & J_{m_1,t_1} & O_{m_1,t_1} & J_{m_1,t_2} & O_{m_1,t_2} & O_{m_1,13-t_1-t_2} \\ J_{m_2,7-t_1-t_2} & J_{m_2,t_1} & O_{m_2,t_1} & O_{m_2,t_2} & J_{m_2,t_2} & O_{m_2,13-t_1-t_2} \\ J_{m_3,7-t_1-t_2} & O_{m_3,t_1} & J_{m_3,t_1} & J_{m_3,t_2} & O_{m_3,t_2} & O_{m_3,13-t_1-t_2} \\ J_{m_4,7-t_1-t_2} & O_{m_4,t_1} & J_{m_4,t_1} & O_{m_4,t_2} & J_{m_4,t_2} & O_{m_4,13-t_1-t_2} \end{pmatrix},$$

with $m_1 + m_2 + m_3 + m_4 = 8$, and $t_1, t_2 \neq 0$, or that N_1 has at most 3 distinct rows. Suppose we are in the first case. If we count the number of quadrangles through x and a vertex z which corresponds to one of the first m_1 rows, then we see that

$$\xi = 7(m_1 - 1) + (7 - t_2)m_2 + (7 - t_1)m_3 + (7 - t_1 - t_2)m_4.$$

If we count the number of quadrangles through x and a vertex corresponding to one of the m_2 rows of the second block, then

$$\xi = 7(m_2 - 1) + (7 - t_2)m_1 + (7 - t_1)m_4 + (7 - t_1 - t_2)m_3.$$

From this it follows that $m_1 + m_3 = m_2 + m_4 = 4$ and $t_1 + t_2 = 7$. Similarly it follows that $m_1 + m_2 = m_3 + m_4 = 4$, and so that $m_1 = m_4$ and $m_2 = m_3$. This implies that $G_3(x)$ has 7 vertices and that every vertex in $G_2(x)$ has 4 neighbours in $G(x)$. From the Hoffman polynomial it follows that if y is a vertex at distance 3 from x, then $A^3_{xy} = 16$, so in turn every vertex in $G_3(x)$ has 4 neighbours in $G_2(x)$. But then the induced subgraph on $G_3(x)$ is 4-regular on 7 vertices, and this is not possible without triangles.

Thus we are in the second case. Suppose N_1 has 4 identical rows. By counting the number of quadrangles through x and a vertex corresponding to one of these 4 rows it follows that the other 4 rows are disjoint from the first 4. Further counting gives that the other 4 rows must also be the same, and again we have that $G_3(x)$ has 7 vertices and that every vertex in $G_2(x)$ has 4 neighbours in $G(x)$, which leads to a contradiction. It follows that we have one row occuring twice and two rows occuring three times. By counting quadrangles through x and a vertex corresponding to one of the rows occuring twice, we see that

$$\xi = 7 + 3t_1 + 3t_2,$$

for some t_1, t_2, and so 14 should be divisible by 3, which is a contradiction. □

Next we shall prove the nonexistence of some graphs, assuming that they have an eigenvalue with multiplicity 2.

PROPOSITION 3.4.4. *There are no graphs with spectrum* $\{[7]^1, [3]^6, [-1]^{15}, [-5]^2\}$, $\{[10]^1, [2]^3, [0]^{18}, [-8]^2\}$, $\{[10]^1, [4]^2, [0]^{18}, [-6]^3\}$ *or* $\{[9]^1, [4]^6, [-1]^{21}, [-6]^2\}$.

Proof. A graph with the first spectrum is 7-regular on 24 vertices with 5 triangles through each vertex. Let $C = C(3, -1) = A^2 - 2A - 3I - \frac{4}{3}J$, then C is a positive semidefinite matrix of rank two with row sums zero and diagonal $\frac{8}{3}$. Thus C can only have entries $-\frac{7}{3}$, $-\frac{4}{3}$, $-\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{3}$ and $\frac{8}{3}$.

Now suppose that $C_{xy} = -\frac{1}{3}$ for some vertices x and y. Let z be another arbitrary vertex. Since C has rank two it follows that the principal submatrix of C on vertices x, y and z has zero determinant, and so either $C_{xz} = \frac{8}{3}$ and $C_{yz} = -\frac{1}{3}$ or $C_{xz} = -\frac{1}{3}$ and $C_{yz} = \frac{8}{3}$. But then x

and y cannot both have row sums zero, and it follows that C has no entries $-\frac{1}{3}$. Similarly it follows that C cannot have entries $\frac{5}{3}$, $-\frac{7}{3}$ and $\frac{2}{3}$. Thus C can only have entries $\frac{8}{3}$ and $-\frac{4}{3}$.

Now fix x. For all vertices y adjacent to x, we must have $A^2_{xy} = 2$ or 6. But x has 7 neighbours, giving that x is in at least 7 triangles, which is a contradiction. The other cases go similarly. □

PROPOSITION 3.4.5. *There is no graph with spectrum* $\{[12]^1, [3]^2, [0]^{22}, [-9]^2\}$.

Proof. Here we would have a 12-regular graph on 27 vertices with $\Delta = 6$ triangles and $\Xi = 492$ quadrangles through every vertex. The matrix $C(0, -9)$ is positive semidefinite of rank two with row sums zero and diagonal $\frac{8}{3}$. Thus C can only have entries $-\frac{7}{3}$, $-\frac{4}{3}$, $-\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{3}$ and $\frac{8}{3}$.

Now suppose that $C_{xy} = -\frac{1}{3}$ for some vertices x and y. Let z be another arbitrary vertex. Since C has rank two it follows that the principal submatrix of C on vertices x, y and z has zero determinant, and so either $C_{xz} = \frac{8}{3}$ and $C_{yz} = -\frac{1}{3}$ or $C_{xz} = -\frac{1}{3}$ and $C_{yz} = \frac{8}{3}$. But then x and y cannot both have row sums zero. Thus C has no entries $-\frac{1}{3}$. This implies that if x and y are adjacent then $A^2_{xy} \neq 0$, and since there are only 6 triangles through every vertex, it follows that $A^2_{xy} = 1$ ($\sigma_1 = 12$), and so $C_{xy} = \frac{2}{3}$. Again, let z be another vertex, then it follows that $C_{xz} = \frac{2}{3}$, $\frac{8}{3}$ or $-\frac{7}{3}$. Now it follows that if x and z are not adjacent, then $A^2_{xz} = 7$, 10 or 12. But then the (σ, τ)-system has no integral solution, giving a contradiction. □

PROPOSITION 3.4.6. *There are no graphs with spectrum* $\{[9]^1, [3]^8, [-1]^{19}, [-7]^2\}$ *or* $\{[10]^1, [5]^2, [0]^{18}, [-5]^4\}$.

Proof. A graph with the first spectrum would be 9-regular on 30 vertices with $\Delta = 4$ triangles and $\Xi = 124$ quadrangles through every vertex. Take $C(3, -1)$, which is a positive semidefinite integral matrix of rank two with diagonal 4. Thus C can only have entries $-4, -3, \ldots, 3$ and 4. Note that since there are 4 triangles through a vertex, it follows that if x and y are adjacent then $A^2_{xy} \leq 4$.

Now suppose that $C_{xy} = 0$ for some vertices x and y. Let z be another arbitrary vertex. Since C has rank two it follows that the principal submatrix of C on vertices x, y and z has zero determinant, and so $C_{xz} = 0$ or ± 4. This implies that if x and z are adjacent then $A^2_{xz} = 0$ or 4, and if x and z are not adjacent then $A^2_{xz} = 2$ or 6. But then the (σ, τ)-system has no solution, so C has no entries 0. Similarly we can show that C has no entries ± 1 and ± 3. Thus C only has entries ± 2 and ± 4. This implies that if x and y are adjacent then $A^2_{xy} = 0$ or 2, and if x and y are not adjacent, then $A^2_{xy} = 0$, 4 or 6. The (σ, τ)-system now has one solution $\sigma_0 = 5$, $\sigma_2 = 4$, $\tau_0 = 6$, $\tau_4 = 10$, $\tau_6 = 4$. Now it is not hard to show that a graph with these parameters does not exist. The other spectrum is even easier, since here none of the (σ, τ)-systems has a solution. □

PROPOSITION 3.4.7. *There is no graph with spectrum*
$\{[13]^1, [3 + 2\sqrt{10}]^2, [-1]^{25}, [3 - 2\sqrt{10}]^2\}$.

Proof. Such a graph is 13-regular on 30 vertices with $\Delta = 62$ triangles and $\Xi = 570$ quadrangles through every vertex. Take the matrix $C(3 + 2\sqrt{10}, -1)$, so $C = A^2 - (2 + 2\sqrt{10})A - (3 + 2\sqrt{10})I - \frac{7}{15}(10 - 2\sqrt{10})J$, which is a positive semidefinite matrix of rank two with diagonal $\frac{8}{15}(10 - 2\sqrt{10})$. From this it follows that if $A_{xy} = 1$ then $A^2_{xy} = 9$, 10, 11 or 12, and if $A_{xy} = 0$ then $A^2_{xy} = 0$, 1, 2 or 3. For a nonnegative integral solution of the (σ, τ)-system we have $\sigma_9 \geq 6$ and $\sigma_{12} \leq 2$. Now fix a vertex x, and let y and z be two vertices with $A^2_{xy} = A^2_{xz} = 9$, then $C_{xy} = C_{xz} = \frac{8}{15}(10 - 2\sqrt{10}) - 3$. Since the principal submatrix on the vertices x, y and z has zero determinant, it follows that $C_{yz} = \frac{8}{15}(10 - 2\sqrt{10})$, so $A^2_{yz} = 12$. For fixed y we have at least 5 choices for z left ($\sigma_9 \geq 6$), so for y we have $\sigma_{12} \geq 5$, which is a contradiction. □

We finish by giving a case where we use the same technique as in the uniqueness proof of the graph $K_{3,3} \oplus K_3$.

PROPOSITION 3.4.8. *There is no graph with spectrum* $\{[6]^1, [3]^5, [-1]^{13}, [-4]^2\}$.

Proof. Here we have a 6-regular graph on 21 vertices with $\Delta = 5$ triangles and $\Xi = 20$ quadrangles through every vertex. Here we take the matrix $C(3, -1)$, then C is a positive semidefinite integral matrix of rank two with row sums zero and diagonal 2. Thus C is the Gram matrix of a set of vectors in $\mathbb{R}^2$ of length $\sqrt{2}$ with mutual inner products ± 2, ± 1 or 0. Note that not both 0 and ± 1 can occur as inner product, since then also inner products that are not allowed occur.

Suppose that inner product 0 occurs. Without loss of generality we assume that there is a vertex represented by vector $\sqrt{2}(1, 0)^T$. The only vectors that can occur now are $\pm\sqrt{2}(1, 0)^T$ and $\pm\sqrt{2}(0, 1)^T$. Since C has row sums zero, it follows that the number of vertices represented by $\sqrt{2}(1, 0)^T$ equals the number of vertices represented by $-\sqrt{2}(1, 0)^T$, and the number of vertices represented by $\sqrt{2}(0, 1)^T$ equals the number of vertices represented by $-\sqrt{2}(0, 1)^T$. But the number of vertices is odd, which is a contradiction.

It follows that if x and y are adjacent then $A^2_{xy} = 1$, 2, 4 or 5 and if x and y are not adjacent then $A^2_{xy} = 0$, 2 or 3. Now we have the (σ, τ)-system

$$\begin{array}{lclclclclclclcl}
\sigma_1 &+& \sigma_2 &+& \sigma_4 &+& \sigma_5 & & & & & = & k & = & 6, \\
\sigma_1 &+& 2\sigma_2 &+& 4\sigma_4 &+& 5\sigma_5 & & & & & = & 2\Delta & = & 10, \\
 & & & & & & & & \tau_0 + & \tau_2 &+ \tau_3 & = & v-k-1 & = & 14, \\
 & & & & & & & & & 2\tau_2 &+ 3\tau_3 & = & k(k-1)-2\Delta & = & 20, \\
 & & \sigma_2 &+& 6\sigma_4 &+& 10\sigma_5 &+& & \tau_2 &+ 3\tau_3 & = & \Xi & = & 20,
\end{array}$$

which has three solutions:

i. $\sigma_5 = 1, \sigma_4 = 0, \sigma_2 = 0, \sigma_1 = 5, \tau_3 = 0, \tau_2 = 10, \tau_0 = 4$.

ii. $\sigma_5 = 0, \sigma_4 = 1, \sigma_2 = 1, \sigma_1 = 4, \tau_3 = 2, \tau_2 = 7, \tau_0 = 5$.

iii. $\sigma_5 = 0, \sigma_4 = 0, \sigma_2 = 4, \sigma_1 = 2, \tau_3 = 4, \tau_2 = 4, \tau_0 = 6$.

By looking at our vector representation we see that if there is a vertex for which we are in case *ii*, then there are vertices (those represented by vectors opposite to the vector representing our original vertex) for which the (σ, τ)-system does not hold. Similarly, if there is a vertex for which we are in case *iii*, then there must be vertices for which we are in case *i*.

Thus we may assume that there is a vertex x for which we are in case *i*. Let y be the vertex adjacent to x with $A^2_{xy} = 5$, then the other neighbours of x and y are the same, say 1, 2, 3, 4 and 5. Now $A^2_{xi} = 1$ for $i = 1,..., 5$, so $C_{xi} = -2$, and i and j are not adjacent, for all $i, j = 1,..., 5$. From the principal submatrix of C on vertices x, i and j it follows that $C_{ij} = 2$, but then $A^2_{ij} = 3$, so besides x and y, i and j only have one common neighbour. This implies that we can identify the 10 vertices z not adjacent to x such that $A^2_{xz} = 2$ with the pairs $\{i, j\}$, $i, j = 1,..., 5$, $i \neq j$, in such a way that i and j are adjacent to $\{i, j\}$. From the principal submatrix of C on vertices $x, i, \{j, k\}$, with $i \neq j, k$, it follows that $C_{i\,\{j,k\}} = -1$, and so $A^2_{i\,\{j,k\}} = 0$. This implies that the subgraph on the pairs $\{i, j\}$, $i, j = 1,..., 5$ is empty, so that all 10 pairs must be adjacent to the remaining four vertices, which is a contradiction. Thus we may conclude that there is no graph with the given spectrum. □

Chapter 4

Tegenslag is nodig, je wordt er sterker van
Gebruik het als een voordeel, geloof me dat het kan
Want er komen nieuwe kansen, grijp ernaar als je ze ziet
En hou je ogen open, en vergeet het niet
We hebben altijd nog elkaar

(Tröckener Kecks, *Met hart en ziel*)

Three-class association schemes

A large class of graphs with few eigenvalues comes from association schemes with few classes (see Section 1.3.6). The special case of two-class association schemes is widely investigated (cf. [16, 95]), as these are equivalent to strongly regular graphs. Also the case of three-class association schemes is very special: there is more than just applying the general theory. However, there are not many papers about three-class association schemes in general. There is the early paper by Mathon [79], who gives many examples, and the recent thesis of Chang [26], who restricts to the imprimitive case. The special case of distance-regular graphs with diameter three has been paid more attention, and for more results on such graphs we refer to [12].

We shall discuss three-class association schemes, mainly starting from regular graphs with four eigenvalues, since for most of the (interesting) schemes indeed at least one of the relations is such a graph. However, most graphs with four eigenvalues cannot be a relation in a three-class association scheme, and we shall characterize the ones that are, in two different ways. We shall give several constructions, and obtain necessary conditions for existence. At the end of this chapter we shall classify the three-class association schemes into three classes, one which may be considered as degenerate, one in which all three relations are strongly regular, and one in which at least one of the relations is a graph with four distinct eigenvalues. This classification is used to generate all feasible parameter sets of (nondegenerate) three-class association schemes on at most 100 vertices, which are listed in Appendix A.4.

4.1. Examples

The d-class *Hamming scheme* $H(d, q)$ is defined on the ordered d-tuples on q symbols (words of length d over an alphabet with q letters), where two tuples are in relation R_i if they differ in i coordinates. The 2-class Hamming scheme $H(2, q)$ consists of the Lattice graph $L_2(q)$ and its complement. The 3-class Hamming scheme is also known as the *cubic scheme*, as it was introduced by Raghavarao and Chandrasekhararao [92]. The Hamming scheme is characterized by its parameters unless $q = 4$, and then we also have the *Doob schemes*. For $d = 3$ there is one Doob scheme (cf. [12]).

The d-class *Johnson scheme* $J(n, d)$ is defined on the d-subsets of an n-set. Two d-subsets are in relation R_i if they intersect in $d - i$ elements. The 2-class Johnson scheme $J(n, 2)$ consists of the triangular graph $T(n)$ and its complement. The 3-class version is also known as the *tetrahedral scheme*, and was first found as a generalization of the triangular graph by John [73]. The Johnson scheme is characterized by its parameters unless $d = 2$ and $n = 8$ (cf. [12]).

The *rectangular scheme* $R(m, n)$, introduced by Vartak [103], has as vertices the ordered pairs (i, j), with $i = 1,..., m$, and $j = 1,..., n$. For two distinct pairs we can have the following three situations. They agree in the first coordinate, or in the second coordinate, or in neither coordinate, and the relations are defined accordingly. Note that the graph of the third relation is the complement of the line graph of the complete bipartite graph $K_{m,n}$. The scheme is characterized by its parameters.

The Hamming scheme, the Johnson scheme and the rectangular scheme are all in the group case. Only the rectangular scheme does not define a distance-regular graph (unless m or n equals 2). There are many more examples of distance-regular graphs with diameter three. Here we shall mainly focus on 3-class association schemes that are not such graphs, although, of course, the general results do apply. For more examples and specific results on distance-regular graphs we refer to [12]. The antipodal distance-regular graphs with diameter three form a special class, as they are antipodal covers of the complete graph. For more on such graphs, see [13, 21, 53, 74].

4.1.1. The disjoint union of strongly regular graphs

Take the disjoint union of, say n, strongly regular graphs, all with the same parameters $(v^*, k^*, \lambda^*, \mu^*)$ and spectrum $\{[k^*]^1, [r]^f, [s]^g\}$. Then this graph generates a 3-class association scheme (the other relations are given by the disjoint union of the complements of the strongly regular graphs, and the complete n-partite graph). It has eigenmatrix

$$P = \begin{pmatrix} 1 & k^* & v^* - 1 - k^* & (n-1)v^* \\ 1 & k^* & v^* - 1 - k^* & -v^* \\ 1 & r & -1 - r & 0 \\ 1 & s & -1 - s & 0 \end{pmatrix},$$

with multiplicities 1, $n - 1$, nf and ng, respectively, and *reduced* intersection matrices (that is, we delete the first row and column, as they can be considered trivial)

$$\begin{pmatrix} \lambda^* & k^*-1-\lambda^* & 0 \\ \mu^* & k^*-\mu^* & 0 \\ 0 & 0 & k^* \end{pmatrix}, \begin{pmatrix} k^*-1-\lambda^* & v^*-2k^*+\lambda^* & 0 \\ k^*-\mu^* & v^*-2k^*-2+\mu^* & 0 \\ 0 & 0 & v^*-1-k^* \end{pmatrix}, \begin{pmatrix} 0 & 0 & (n-1)v^* \\ 0 & 0 & (n-1)v^* \\ k^* & v^*-1-k^* & (n-2)v^* \end{pmatrix}.$$

Conversely, any association scheme with such parameters must be obtained in the described way. Therefore we may consider this case as degenerate, and it suffices to refer to the extensive literature (for example [16, 95]) on strongly regular graphs. The same remarks hold for the next construction.

4.1.2. A product construction from strongly regular graphs

If G is a strongly regular $(v^*, k^*, \lambda^*, \mu^*)$ graph with spectrum $\{[k^*]^1, [r]^f, [s]^g\}$, then, for any natural number n, the graph $G \otimes J_n$ (see Chapter 3) generates an imprimitive 3-class association scheme (here the other relations are $\overline{G} \otimes J_n$ and a disjoint union of n-cliques). The scheme has eigenmatrix

$$P = \begin{pmatrix} 1 & nk^* & n-1 & n(v^*-1-k^*) \\ 1 & nr & n-1 & n(-1-r) \\ 1 & 0 & -1 & 0 \\ 1 & ns & n-1 & n(-1-s) \end{pmatrix},$$

with multiplicities 1, f, $(n-1)v^*$ and g, respectively, and reduced intersection matrices

$$\begin{pmatrix} n\lambda^* & n-1 & n(k^*-1-\lambda^*) \\ nk^* & 0 & 0 \\ n\mu^* & 0 & n(k^*-\mu^*) \end{pmatrix}, \begin{pmatrix} n-1 & 0 & 0 \\ 0 & n-2 & 0 \\ 0 & 0 & n-1 \end{pmatrix}, \begin{pmatrix} n(k^*-1-\lambda^*) & 0 & n(v^*-2k^*+\lambda^*) \\ 0 & 0 & n(v^*-1-k^*) \\ n(k^*-\mu^*) & n-1 & n(v^*-2k^*-2+\mu^*) \end{pmatrix}.$$

It is easy to show that any 3-class association scheme with $p_{11}^2 = n_1$ must be of this form.

4.1.3. Pseudocyclic schemes

A d-class association scheme is called *pseudocyclic* if all the nontrivial eigenvalues have the same multiplicities m. In this case we also have all degrees equal to m.

If v is a prime power, and $v \equiv 1 \pmod 3$, we can define the 3-class *cyclotomic* association scheme Cycl(v) as follows. Let α be a primitive element of the finite field $GF(v)$. As vertices we take the elements of $GF(v)$. Two vertices will be i-th associates if their difference equals α^{3t+i} for some t (or, if the discrete logarithm (base α) of their difference is congruent to i modulo 3), for $i = 1, 2, 3$.

A similar construction gives pseudocyclic d-class association schemes. Such schemes are used by Mathon [79] to construct antipodal distance-regular graphs with diameter three. The resulting graph has $d(v + 1)$ vertices and we shall denote it by $d(P + 1)$ if P is the original scheme. For $d = 2$, we get the so-called Taylor graphs (cf. [12]).

If v is not a prime power, then only three pseudocyclic 3-class association schemes are known. On 28 vertices Mathon [79] found one, and Hollmann [72] proved that there are precisely two. Furthermore Hollmann [71] found one on 496 vertices.

4.1.4. The block scheme of designs

A *quasi-symmetric* design is a design in which the intersections of two blocks take two sizes x and y. The graph on the blocks of such a design with edges between blocks that intersect in x points is strongly regular, i.e. we have a two-class association scheme.

Now consider a block design with the property that the intersections of two blocks take three sizes. Then possibly the structure on the blocks with relations according to the intersection numbers is a 3-class association scheme. Delsarte [42] proved that if the design is a 4-design then we have a 3-class association scheme. Hobart [67] found several examples in her search for the more general coherent configurations of type (2, 2; 4). She mentions the Witt designs 4-(11, 5, 1) and 5-(24, 8, 1) and their residuals, and the inversive planes of even order, that is, the 3-($2^{2i} + 1$, $2^i + 1$, 1) designs. Of course, in any 3-design with $\lambda = 1$ the blocks can intersect only in 0, 1 or 2 points, but the corresponding relations do not always form a 3-class association scheme.

Hobart and Bridges [68] also constructed a unique 2-(15, 5, 4) design with block intersections 0, 1 and 2, and it defines the distance-regular graph that is also obtained as the second subconstituent in the Hoffman-Singleton graph (see Section 4.3.1).

Beker and Haemers [5] proved that if one of the intersection numbers of a 2-(v, k, λ) design equals $k - r + \lambda$, where $r = \lambda(v - 1)/(k - 1)$ is the replication number of the design, and there are two other intersection numbers, then we have an imprimitive 3-class association scheme, that is generated by $G \otimes J_n$ for some strongly regular graph G (see Section 4.1.2).

4.1.5. Distance schemes and coset schemes of codes

Let C be a linear code with $e + 1$ nonzero weights w_i. Take as vertices the codewords and let a pair of codewords be in relation R_i if their distance is w_i. It is a consequence of a result by Delsarte [42] (cf. [22]) that if the dual code $C^\perp$ is e-error-correcting, then these relations form an $(e + 1)$-class association scheme. This scheme is called the *distance scheme* of the code. Moreover, it has a dual scheme, called the *coset scheme* which is defined on the cosets of $C^{\perp}$. Two cosets $x + C^{\perp}$ and $y + C^{\perp}$ are in relation R_i^* if the minimum weight in the coset $(x - y) + C^{\perp}$ equals i. Relation R_1^* is the *coset graph* of $C^{\perp}$, and is distance-regular.

A small example of a code with three nonzero weights is the binary zero-sum code of length 6, consisting of all 32 words of even weight. Its dual code consist of the zero word and the all-one word and certainly can correct 2 errors. Therefore we have two dual 3-class association schemes on 32 vertices. The graph (in the distance scheme) defined by distance two in the code is a Taylor graph. The coset graph is the incidence graph of a symmetric 2-(16, 6, 2) design. Larger examples are given by the (duals of the) binary Golay code [23, 12, 7] and its punctured [22, 12, 6] code and doubly punctured [21, 12, 5] code. For all three codes the dual codes have nonzero weights 8, 12 and 16, so these define 3-class association schemes on 2^{11}, 2^{10} and 2^9 vertices, respectively. Also the Kasami codes (which are binary BCH codes with minimum distance 5) give rise to 3-class association schemes (cf. [22]).

4.1.6. Quadrics in projective geometries

Let Q be a nondegenerate quadric in $PG(3, q)$ with q odd (i.e. the set of isotropic points of the corresponding quadratic form Q). Let V be the set of projective points x such that $Q(x)$ is a nonzero square. Two distinct vertices are related according as the line through these points is a hyperbolic line (a secant, i.e. intersecting Q in two points), an elliptic line (a passant, i.e. disjoint from Q) or a tangent (i.e. intersecting Q in one point). These relations form a 3-class association scheme (cf. [12]). The number of vertices equals $q(q^2 - \varepsilon)/2$, where $\varepsilon = 1$ if Q is hyperbolic and $\varepsilon = -1$ if Q is elliptic.

For q even, and $n \geq 3$, let Q be a nondegenerate quadric in $PG(n, q)$. Now let V be the set of nonisotropic points (i.e. the points not on Q) distinct from the nucleus (for n odd there is no nucleus, for n even this is the unique point u such that $Q(u + v) = Q(u) + Q(v)$ for all v). The relations as defined above now form a 3-class association scheme (cf. [12]).

4.1.7. Merging classes

Sometimes we obtain a new association scheme by *merging* classes in a given association scheme. Merging means that a new relation is obtained as the union of some original relations, and then we say that the corresponding classes are merged. For example, take the 3-class association scheme with vertex set

$$V = \{(x_1,\{\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}\}) \mid \{x_i, i = 1,\ldots, 7\} = \{1,\ldots, 7\}\}.$$

Two vertices $(x_1,\{\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}\})$ and $(y_1,\{\{y_2, y_3, y_4\}, \{y_5, y_6, y_7\}\})$ are first associates if $x_1 = y_1$. If $x_1 \neq y_1$, then without loss of generality we may assume that $x_1 \in \{y_2, y_3, y_4\}$ and $y_1 \in \{x_2, x_3, x_4\}$. Now the vertices are second associates if $\{x_2, x_3, x_4\} \cap \{y_2, y_3, y_4\} = \emptyset$, otherwise they are third associates. This scheme was obtained by merging two classes in the 4-class association scheme that arose while letting the symmetric group S_7 act on V^2.

On the other hand, it can occur that merging two classes in a 3-class association scheme gives a 2-class association scheme. Of course, this occurs precisely if the remaining relation defines a strongly regular graph. If all three relations of a 3-class association scheme define strongly regular graphs, then we are in a very special situation. It means that by any merging we always get a new association scheme. After [56] we call schemes with this property *amorphic*. The amorphic 3-class association schemes are precisely the 3-class association schemes that are not generated by one of their relations.

4.2. Amorphic three-class association schemes

In the special case that all three relations are strongly regular graphs, we show that the parameters of the graphs are either all of Latin square type, or all of negative Latin square type. The proof is due to D.G. Higman [66]. The same results can be found in [56], where also all such schemes on at most 25 vertices can be found.

THEOREM 4.2.1. *If all three relations of a* 3-*class association scheme are strongly regular graphs, then they either have parameters* $(n^2, l_i(n-1), n-2+(l_i-1)(l_i-2), l_i(l_i-1))$, $i = 1, 2, 3$ *or* $(n^2, l_i(n+1), -n-2+(l_i+1)(l_i+2), l_i(l_i+1))$, $i = 1, 2, 3$.

Proof. Suppose R_i is a strongly regular graph with degree n_i and eigenvalues n_i, r_i and s_i (we do not assume $r_i > s_i$). Without loss of generality we may take

$$P = \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ 1 & r_1 & s_2 & s_3 \\ 1 & s_1 & r_2 & s_3 \\ 1 & s_1 & s_2 & r_3 \end{pmatrix}.$$

Since $PQ = vI$, we see that $1 + r_1 + s_2 + s_3 = 1 + s_1 + r_2 + s_3 = 1 + s_1 + s_2 + r_3 = 0$, and so

$$r_1 - s_1 = r_2 - s_2 = r_3 - s_3.$$

Furthermore, from the orthogonality relations we derive that

$$\frac{s_1}{n_1} = \frac{s_2}{n_2} = \frac{s_3}{n_3} \ ,$$

and we find that $P^2 = vI$, so $P = Q$, and so the scheme is self-dual. Now set $u = r_i - s_i$, then we find from the orthogonality relation

$$0 = 1 + \frac{r_1 s_1}{n_1} + \frac{r_2 s_2}{n_2} + \frac{s_3{}^2}{n_3} = 1 + \frac{s_1}{n_1}(u-1) \ , \text{ so } \frac{n_1}{s_1} = 1 - u \ .$$

Furthermore, we have that

$$\det P = \det\begin{pmatrix} v & n_1 & n_2 & n_3 \\ 0 & r_1 & s_2 & s_3 \\ 0 & s_1 & r_2 & s_3 \\ 0 & s_1 & s_2 & r_3 \end{pmatrix} = \det\begin{pmatrix} v & n_1 & n_2 & n_3 \\ 0 & u & -u & 0 \\ 0 & 0 & u & -u \\ 0 & s_1 & s_2 & r_3 \end{pmatrix} = vu^2(s_1 + s_2 + r_3) = -vu^2 \ ,$$

but on the other hand, $P^2 = vI$, so $(\det P)^2 = v^4$, and we find that $v = u^2$. This proves that the parameters of the relations are either all of Latin square type $(n^2,\ l_i(n-1),\ n-2+(l_i-1)(l_i-2),\ l_i(l_i-1))$ if $n = u > 0$ or all of negative Latin square type $(n^2,\ l_i(n+1),\ -n-2+(l_i+1)(l_i+2),\ l_i(l_i+1))$ if $n = -u > 0$. □

A large family of examples is given by the *Latin square schemes* $L_{i,j}(n)$. Suppose we have $m - 2$ mutually orthogonal Latin squares, or equivalently an orthogonal array OA(n, m). Recall that this is an $m \times n^2$ matrix M such that for any two rows a, b we have that $\{(M_{ai}, M_{bi}) \mid i = 1,\ldots, n^2\} = \{(i, j) \mid i, j = 1,\ldots, n\}$. Now take as vertices $1,\ldots, n^2$. Let I_1 and I_2 be two disjoint nonempty subsets of $\{1,\ldots, m\}$ of sizes i and j, respectively. Now two distinct vertices v and w are l-th associates if $M_{rv} = M_{rw}$ for some $r \in I_l$, for $l = 1, 2$, otherwise they are third associates.

Many constructions for OA(n, m) are known (cf. [11]). For n a prime power, there are constructions of OA(n, m) for every $m \le n + 1$, its maximal value. For $n = 6$, we have $m \le 3$ (Euler's famous 36 officers problem), and for $n = 10$, currently only constructions for $m \le 4$ are known. For $n \ne 4$, a Latin square scheme $L_{1,2}(n)$ is equivalent to the algebraic structure called a *loop* (cf. [90]). Two Latin square schemes are isomorphic if and only if the corresponding loops are *isotopic* (cf. [26]). From [90] we find that there are 22 nonisomorphic $L_{1,2}(6)$ and 563 nonisomorphic $L_{1,2}(7)$.

The smallest examples of "schemes of negative Latin square type" are given by the cyclotomic scheme Cycl(16) on 16 vertices (see Section 4.1.3 for a definition), and another scheme with the same parameters (cf. [56]). Here all three relations are Clebsch graphs. The second feasible parameter set of negative Latin square type is on 49 vertices.

Here all relations are strongly regular (49, 16, 3, 6) graphs, but such a graph does not exist, according to Bussemaker, Haemers, Mathon and Wilbrink [18]. Van Lint and Schrijver [76] found several strongly regular graphs of negative Latin square type by merging classes in the 8-class cyclotomic scheme on 81 vertices. In fact, in this way you can get amorphic 3-class association schemes of negative Latin square type. We find a scheme with degrees 30, 30 and 20, and at least two nonisomorphic schemes with degrees 40, 20 and 20.

4.3. Regular graphs with four eigenvalues

A graph G which is one of the relations, say R_1, of a 3-class association scheme is regular with at most four distinct eigenvalues. Any two adjacent vertices have a constant number $\lambda = p_{11}^1$ of common neighbours, and any two nonadjacent vertices have $\mu = p_{11}^3$ or $\mu' = p_{11}^2$ common neighbours. If $\mu = \mu'$, then G is strongly regular, so it has at most three distinct eigenvalues (possibly it is disconnected). If $\mu \neq \mu'$, then R_1 generates the scheme, as the other two relations are determined by the number of common neighbours. Then G must have four distinct eigenvalues or be the disjoint union of some strongly regular graphs. If G is a connected regular graph with four distinct eigenvalues, then the following theorem provides us with a handy tool to check whether it is one of the relations of a 3-class association scheme.

THEOREM 4.3.1. *Let G be a connected regular graph with four distinct eigenvalues. Then G is one of the relations of a* 3-*class association scheme if and only if any two adjacent vertices have a constant number of common neighbours, and the number of common neighbours of any two nonadjacent vertices takes precisely two values.*

Proof. Suppose that G is regular of degree k, any two adjacent vertices in G have λ common neighbours, and that any two nonadjacent vertices have either μ or μ' common neighbours. Note that these requirements must necessarily hold in order for G to be one of the relations of a 3-class association scheme, and that $\mu \neq \mu'$, otherwise G is strongly regular, and so it has only three distinct eigenvalues.

Now let G have adjacency matrix A. To prove sufficiency we shall show that the adjacency algebra $\mathbf{A} = \langle A^2, A, I, J \rangle$, which is closed under ordinary matrix multiplication is also closed under entrywise multiplication $\circ$. Since $M \circ J = M$ for any matrix M, and any matrix $M \in \mathbf{A}$ has constant diagonal, so that $M \circ I \in \mathbf{A}$, we only need to show that $A \circ A$, $A^2 \circ A$ and $A^2 \circ A^2$ are in $\mathbf{A}$. Now $A \circ A = A$, $A^2 \circ A = \lambda A$, and

$$A^2 \circ A^2 = k^2 I + \lambda^2 A + ((\mu + \mu')A^2 - \mu\mu' J) \circ (J - I - A)$$

$$= (\mu + \mu')A^2 + (\lambda^2 - \lambda(\mu + \mu') + \mu\mu')A + (k^2 - k(\mu + \mu') + \mu\mu')I - \mu\mu' J.$$

So **A** is also closed under entrywise multiplication, and so G is one of the relations of a 3-class association scheme. □

If μ or μ' equals 0, then it follows that G is distance-regular with diameter three. We shall use the characterization of Theorem 4.3.1 in the following examples.

4.3.1. The second subconstituent of a strongly regular graph

The second subconstituent of a graph with respect to some vertex x is the induced graph on the vertices distinct from x, and that are not adjacent to x. For some strongly regular graphs the second subconstituent generates a 3-class association scheme.

Suppose G is a strongly regular graph without triangles ($\lambda = 0$), with spectrum $\{[k]^1, [r]^f, [s]^g\}$. Then the second subconstituent $G_2(x)$ of G is a regular graph with spectrum $\{[k+r+s]^1, [r]^{f-k}, [r+s]^{k-1}, [s]^{g-k}\}$ (see Chapter 3), so in general it is a connected regular graph with four distinct eigenvalues without triangles. So if the number of common neighbours of two nonadjacent vertices can take at most two values, then we have a 3-class association scheme. This is certainly the case if G is a strongly regular $(v, k, 0, \mu)$ graph with $\mu = 1$ or 2, as we shall see.

If $\mu = 1$ then it follows that in $G_2(x)$ two nonadjacent vertices can have either 0 or 1 common neighbours. For $k > 2$ the graph $G_2(x)$ has four distinct eigenvalues, so then it follows that this graph is distance-regular with diameter three. The distance three relation R_3 is the disjoint union of k cliques of size $k-1$, which easily follows by computing the eigenvalues of $A_3 = J + (k-2)I - A - A^2$, where A is the adjacency matrix of $G_2(x)$. On the other hand, it follows that any distance-regular graph with such parameters can be constructed in this way, that is, given such a distance-regular graph, we can, using the structure of R_3, construct a strongly regular $(v, k, 0, 1)$ graph that has the distance-regular graph as second subconstituent (Take such a distance-regular graph, and order the cliques of the distance three relation. Extend the distance-regular graph with vertices ∞ and $i = 1,..., k$, and with edges $\{\infty, i\}$ and $\{i, y\}$, y is a vertex of the i-th clique, $i = 1,..., k$, then we get a strongly regular $(1 + k^2, k, 0, 1)$ graph). In fact, it now follows from a result by Haemers [59, Cor. 5.4] that any graph with the same spectrum must be constructed in this way. The result by Haemers can also be shown using Corollary 4.3.7, which we shall prove later (see also [39]).

It is well known (cf. [95]) that strongly regular graphs with parameters $(v, k, 0, 1)$ can only exist for $k = 2$, 3, 7 or 57. For the first three cases there are unique graphs: the 5-cycle C_5, the Petersen graph and the Hoffman-Singleton graph. The case $k = 57$ is still undecided. The second subconstituent of the Petersen graph is the 6-cycle C_6. The more interesting case is the second subconstituent Ho-Si$_2(x)$ of the Hoffman-Singleton graph. It is uniquely determined by its spectrum, which now follows from the uniqueness of the Hoffman-Singleton graph and the fact that its automorphism group acts transitively on its vertices.

If $\mu = 2$, then in $G_2(x)$ two nonadjacent vertices can have either 1 or 2 common neighbours (They have at least one common neighbour, since in G they cannot have two common neighbours that are both neighbours of x, as these two vertices then would have three common neighbours). For $k > 5$ the graph $G_2(x)$ has four distinct eigenvalues, so then we have a 3-class association scheme. Here we find for relation R_3 (two vertices are third associates if they have one common neighbour in $G_2(x)$) that $A_3 = 2J + (k-4)I - A - A^2$ with spectrum $\{[2k-4]^1, [k-4]^{k-1}, [-2]^{\frac{1}{2}k(k-3)}\}$, which is the spectrum of the triangular graph $T(k)$. Using this it is also possible to prove that any association scheme with these parameters must be constructed as we did. Consider the graph of the first relation of an association scheme with such parameters. It has degree $k-2$, no triangles, and any two nonadjacent vertices have either 1 or 2 common neighbours (corresponding to relations R_3 and R_2, respectively). Now the third relation has the spectrum of the triangular graph $T(k)$, and since this graph is uniquely determined by its spectrum (unless $k = 8$, but then there is no feasible parameter set: from the integrality of the multiplicities it follows that $k-1$ is a square), it follows that we can rename the vertices by the pairs $\{i, j\}$, $i, j = 1,..., k$, such that two vertices are not adjacent and have one common neighbour if and only if the corresponding pairs intersect. Now we extend the graph with vertices ∞ and $i = 1,..., k$, and with edges $\{\infty, i\}$ and $\{i, \{i, j\}\}$, $i, j = 1,..., k$. Then it follows that this graph is strongly regular with parameters $(1 + \frac{1}{2}k(k+1), k, 0, 2)$. To show this, we have to check that i and $\{j, h\}$ with $i \neq j, h$ have two common neighbours. By considering the original association scheme, we see that the number of vertices that are third associates with $\{i, j\}$ and first associates with $\{j, h\}$ equals $p_{31}^3 = 2$. But such vertices are of the form $\{i, g\}$, which proves that $\mu = 2$. Thus we have proven the following proposition.

PROPOSITION 4.3.2. *Let G be a strongly regular graph without triangles, and with $\mu = 1$ or 2, and degree k, with $k > 2$ if $\mu = 1$, and $k > 5$ if $\mu = 2$. Then the second subconstituent of G with respect to any vertex generates a 3-class association scheme. Conversely, any scheme with the same parameters can be constructed in this way from a strongly regular graph.*

If $\mu = 2$, then the only known example for G with $k > 5$ is the Gewirtz graph, and since this graph is uniquely determined by its parameters, and it has a transitive automorphism group, the association scheme generated by its second subconstituent $\text{Gewirtz}_2(x)$ is uniquely determined by its parameters.

Payne [88] found that the second subconstituent of the collinearity graph of a generalized quadrangle with respect to a quasiregular point generates a 3-class association scheme (or a strongly regular graph). Together with Hobart [69] he found conditions to embed the association scheme back in a generalized quadrangle. Note that the second subconstituent of a generalized quadrangle with respect to a point p is a regular graph with at most four distinct eigenvalues (see Chapter 3). Furthermore any two adjacent vertices have a constant number of common neighbours. The quasiregularity of the point p now implies that the number of common neighbours of two nonadjacent vertices can take only two values.

4.3.2. Hoffman cocliques in strongly regular graphs

Let G be a k-regular graph on v vertices with smallest eigenvalue $\lambda_{\min}$. Recall that a Hoffman coclique in G is a coclique whose size meets the Hoffman (upper) bound $c = v\lambda_{\min}/(\lambda_{\min} - k)$. If C is a Hoffman coclique then every vertex not in C is adjacent to $-\lambda_{\min}$ vertices of C. If G is a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue s, then the adjacencies between C and its complement forms the incidence relation of a 2-$(c, -s, \mu)$ design D (which may be degenerate). Furthermore, the induced graph on the complement of C is a regular graph with at most four distinct eigenvalues (see Chapter 3). A necessary condition for this graph to be one of the relations of a 3-class association scheme is that the design D has at most three distinct block intersection numbers. If it forms a 3-class association scheme then it is the block scheme of D (see Section 4.1.4).

An example is given by an ovoid in the generalized quadrangle $GQ(4, 4)$. An ovoid is a Hoffman coclique in the collinearity graph of the generalized quadrangle. Here the corresponding design is an inversive plane, and the induced graph on the complement of the ovoid is the distance three graph of the distance-regular Doro graph.

4.3.3. A characterization in terms of the spectrum

Now suppose that G is a connected regular graph with spectrum $\{[k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$ that is one of the relations of a 3-class association scheme. The degree $k = n_1$ is its largest eigenvalue, and also λ can be expressed in terms of the spectrum of the graph, since for a connected regular graph with four distinct eigenvalues the number of triangles through a vertex equals $\Delta = Trace(A^3)/2v$, and so

$$\lambda = \frac{2\Delta}{k} = \frac{Trace(A^3)}{vk} = \frac{1}{vk}\sum_{i=0}^{3} m_i\lambda_i^3 \ .$$

In general, μ and μ' do not follow from the spectrum of G. For example, $GQ(2, 4)\backslash$spread and $H(3, 3)_3$ have the same spectrum, and are both graphs from association schemes, but they have distinct parameters (in fact, the first one is a distance-regular graph and the other is not). But in many cases the parameters of the scheme do follow from the spectrum, as they form the only nonnegative integral solution of the following system of equations.

If for every vertex x, the number of nonadjacent vertices that have μ' common neighbours with x equals n_2, and n_3 is the number of nonadjacent vertices that have μ common neighbours with x, then by easy counting arguments we have that the parameters satisfy the following equations.

$$\begin{aligned} n_2 \quad &+ n_3 &&= v - 1 - k \ , \\ n_2\mu' \ &+ n_3\mu &&= k(k - 1 - \lambda) \ , \\ n_2\tbinom{\mu'}{2} &+ n_3\tbinom{\mu}{2} &&= \Xi - k\tbinom{\lambda}{2} \ , \end{aligned}$$

where

$$\Xi = \tfrac{1}{2}\left(\tfrac{1}{v}\sum_{i=0}^{3} m_i\lambda_i^4 - 2k^2 + k\right)$$

is the number of quadrangles through a vertex. Since the number of triangles through an edge is constant, also the number of quadrangles through an edge is constant and equals $\xi = 2\Xi/k$. It follows that given the spectrum Σ of the graph and one extra parameter (for example μ), we can compute all other parameters of the association scheme. For n_3 this gives

$$n_3 = h(\Sigma, \mu) = v - 1 - k - \frac{((v-1-k)\mu - k(k-1-\lambda))^2}{k\xi - k\lambda^2 + k(k-1) + (v-1-k)\mu^2 - 2\mu k(k-1-\lambda)} \ .$$

The next theorem characterizes the regular graphs with four eigenvalues that generate a 3-class association scheme, as those graphs for which this number n_3 is precisely what it should be. We shall use the following lemma, but first we define some vertex partitions. The *neighbourhood partition* of a vertex x is the partition of the vertices into three sets. The first set contains x, the second the neighbours of x, and the third the remaining vertices. The *μ-partition* of a vertex x is a refinement of the neighbourhood partition. The third part (the nonneighbours of x) is further partitioned into the ones that do not have μ common neighbours with x, and the ones that do have μ common neighbours with x.

LEMMA 4.3.3. *Let G be a connected regular graph on v vertices with eigenvalues $k > \lambda_1 > \ldots > \lambda_r$. Let B be the quotient matrix with respect to the neighbourhood partition of an arbitrary vertex x. Suppose B has eigenvalues $k \geq \mu_1 \geq \mu_2$. If for every vertex x one of the equalities $\lambda_1 = \mu_1$ and $\lambda_r = \mu_2$ holds, then G is strongly regular.*

Proof. Let G have adjacency matrix A. Fix an arbitrary vertex x and suppose one of the equalities holds, say μ_i is also an eigenvalue of A. Let V be the 3-dimensional subspace of $\mathbb{R}^v$ of vectors that are constant over the parts of the neighbourhood partition of x. Then A has an eigenvector $u = (u_0, u_1, \ldots, u_1, u_2, \ldots, u_2)^T$ with eigenvalue μ_i in V (cf. [58, Thm. 2.1.ii] or [12, Thm. 3.3.1.ii]). Also the all-one vector $\underline{1}$ is an eigenvector (with eigenvalue k) of A in V. Furthermore $A(1, 0,..., 0)^T = (0, 1,..., 1, 0,..., 0)^T \in V$. Since u, $\underline{1}$ and $(1, 0,..., 0)^T$ are linearly independent vectors in V (otherwise $u_1 = u_2$, and applying A gives $u_0 = u_2$, which gives a contradiction), we have that $AV \subseteq V$. So we have three linearly independent eigenvectors of A in V, and it follows that the neighbourhood partition of x is regular. So the number of common neighbours of x and a vertex y adjacent to x is independent of y. This holds for every x and since G is connected, it follows that this number is also independent of x, and so for every vertex the neighbourhood partition is regular with the same quotient matrix, proving that G is strongly regular. □

THEOREM 4.3.4. *Let G be a connected regular graph on v vertices with four distinct eigenvalues, say with spectrum $\Sigma = \{[k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$. Let p be the polynomial given by $p(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = x^3 + p_2x^2 + p_1x + p_0$ and let λ be given by $\lambda = (k^3 + m_1\lambda_1{}^3 + m_2\lambda_2{}^3 + m_3\lambda_3{}^3)/vk$. Then G is one of the relations of a 3-class association scheme if and only if there is an integer μ such that for every vertex x the number of nonadjacent vertices n_3, that have μ common neighbours with x equals*

$$g(\Sigma, \mu) = v - 1 - k - \frac{k(k - 1 - \lambda - \frac{v-1-k}{k}\mu)^2}{(k - \lambda)(\lambda + p_2) - k - p_1 + p_0 - 2\mu(k - 1 - \lambda) + \frac{v-1-k}{k}\mu^2} \ .$$

Proof. Suppose that G is one of the relations of a 3-class association scheme. Consider the quotient matrix C with respect to the μ-partition of some arbitrary vertex x. Then C is an intersection matrix, and

$$C = \begin{pmatrix} 0 & k & 0 & 0 \\ 1 & \lambda & k - 1 - \lambda - d & d \\ 0 & c & k - c - b & b \\ 0 & \mu & k - \mu - a & a \end{pmatrix},$$

for some a, b, c and d. The number of common neighbours of two adjacent vertices equals $\lambda = Trace(A^3)/vk$. Note that $c \neq \mu$, otherwise G would be strongly regular and hence have only three distinct eigenvalues. Since C has eigenvalues k, λ_1, λ_2 and λ_3, it follows that the characteristic polynomial of C equals

$$(x-k)p(x) = \det(xI-C) = \det\begin{pmatrix} x & x-k & 0 & 0 \\ -1 & x-k & -(k-1-\lambda-d) & -d \\ 0 & x-k & x-(k-c-b) & -b \\ 0 & x-k & -(k-\mu-a) & x-a \end{pmatrix} =$$

$$(x-k)(x^3+(b+c-\lambda-a)x^2+(\lambda a-ca-b\lambda-k+c+\mu b-\mu d+dc)x+ka-ca-bk+\mu b).$$

From this it follows that

$$(k-\lambda)(\lambda+p_2)-k-p_1+p_0=(k-1-\lambda)c+d(\mu-c).$$

Since $(v-1-k-n_3)c = k(k-1-\lambda-d)$ and $n_3\mu = kd$, we derive that

$$(v-1-k)c+n_3(\mu-c) = k(k-1-\lambda) \text{ and } \tfrac{v-1-k}{k}\mu c+d(\mu-c) = \mu(k-1-\lambda).$$

Combining the last equation with the equation that we derived from the characteristic polynomial, we obtain

$$c = \frac{(k-\lambda)(\lambda+p_2)-k-p_1+p_0-\mu(k-1-\lambda)}{k-1-\lambda-\frac{v-1-k}{k}\mu},$$

and so

$$n_3 = \frac{k(k-1-\lambda)-(v-1-k)c}{\mu-c} = v-1-k-\frac{k(k-1-\lambda)-(v-1-k)\mu}{c-\mu} = g(\Sigma,\mu).$$

Suppose now that $n_3 = g(\Sigma, \mu)$ for every vertex x. Note that $g(\Sigma, \mu) < v-1-k$, otherwise G would be strongly regular. Let c be as given above (so that, by the definition of $g(\Sigma, \mu)$ it implicitly follows that $\mu \neq c$), and let a, b and d be given by

$a = ((\lambda - c + p_2)(k-\mu)+p_0)/(\mu-c)$,
$b = a+\lambda-c+p_2$ and
$d = ((k-\lambda)(\lambda+p_2)-k-p_1+p_0-(k-1-\lambda)c)/(\mu-c)$.

Then the matrix C as given above has eigenvalues k, λ_1, λ_2 and λ_3 (again, this follows by inspecting the characteristic polynomial).

First suppose that $g(\Sigma, \mu) > 0$. We shall prove that the quotient matrix B with respect to the μ-partition of x equals C, thus proving that the partition is regular. Without loss of generality we assume that $k > \lambda_1 > \lambda_2 > \lambda_3$. Suppose that B has eigenvalues $k \geq \mu_1 \geq \mu_2 \geq \mu_3$. Since the eigenvalues of B interlace the eigenvalues of the adjacency matrix A of G, it follows that $\lambda_1 \geq \mu_1$ and $\mu_3 \geq \lambda_3$. Since G is a connected regular graph with four distinct eigenvalues, the number of triangles through x equals $\Delta = \frac{1}{2}k\lambda$, so that $B_{22} = \lambda$. Furthermore

$$B_{24} = \frac{g(\Sigma,\mu)\mu}{k} = \frac{\mu}{k}\,\frac{k(k-1-\lambda)-(v-1-k)c}{\mu-c} = \frac{\mu}{k}\,d\,\frac{(k-1-\lambda)-(v-1-k)c}{(k-\lambda)(\lambda+p_2)-k-p_1+p_0-(k-1-\lambda)c} = d,$$

and consequently $B_{23} = k - 1 - \lambda - d$, and

$$B_{32} = \frac{k(k-1-\lambda-d)}{v-1-k-g(\Sigma,\mu)} = \frac{k(k-1-\lambda-\frac{g(\Sigma,\mu)\mu}{k})}{v-1-k-g(\Sigma,\mu)} = c.$$

So $B = C + E$, where E equals

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -\varepsilon \\ 0 & 0 & -\delta & \delta \end{pmatrix}$$

for some ε and δ. To use inequalities for eigenvalues we want symmetric matrices. Therefore we multiply B, C and E from the left by $K^{\frac{1}{2}}$ and from the right by $K^{-\frac{1}{2}}$, where $K = \mathrm{diag}(1, k, v-1-k-g(\Sigma,\mu), g(\Sigma,\mu))$, to get $\tilde{B}$, $\tilde{C}$ and $\tilde{E}$, respectively. Now the eigenvalues have not changed and $\tilde{B}$ is symmetric, but to show that $\tilde{C}$ (and consequently $\tilde{E}$) is symmetric, we have to prove that $g(\Sigma,\mu)(k-a-\mu) = (v-1-k-g(\Sigma,\mu))b$. This follows since

$$(g(\Sigma,\mu)(k-a-\mu) - (v-1-k-g(\Sigma,\mu))b)(c-\mu) =$$

$$(v-1-k)(k-a-\mu)(c-\mu) - (v-1-k-g(\Sigma,\mu))(k-a-\mu+b)(c-\mu) =$$

$$(v-1-k)((\lambda-\mu+p_2)(k-\mu)+p_0) - (k-\mu+\lambda-c+p_2)(k(k-1-\lambda)-(v-1-k)\mu) =$$

$$(v-1-k)(\lambda k+p_2k+p_0) - k(k-1-\lambda)(k+\lambda+p_2) - c\mu(v-1-k) + k(k-1-\lambda)(\mu+c) =$$

$$(v-1-k)(\lambda k+p_2k+p_0) - k(k-1-\lambda)(k+\lambda+p_2) + k((k-\lambda)(\lambda+p_2)-k-p_1+p_0) =$$

$$v(\lambda k+p_2k+p_0) - (k^3+p_2k^2+p_1k+p_0) = 0.$$

The last equation follows by taking the trace of the equation $p(A) = p(k)/vJ$, where A is the adjacency matrix of G, and J is the all-one matrix.

Let $w_0 = K^{\frac{1}{2}}(1, 1, 1, 1)^T$, then it is an eigenvector of both $\tilde{B}$ and $\tilde{C}$ with eigenvalue k. Let w_i be an eigenvector of $\tilde{C}$ with eigenvalue λ_i, $i = 1, 2, 3$, such that $\{w_0, w_1, w_2, w_3\}$ is orthogonal. Let $v_i = K^{-\frac{1}{2}}w_i$, then v_i is eigenvector of C with eigenvalue λ_i, $i = 0, 1, 2, 3$. Now we shall prove that $\tilde{E} = O$ or, equivalently, that $\varepsilon = 0$. Suppose that $\varepsilon > 0$. Now $\tilde{E}$ is positive semidefinite, and so

$$\mu_1 \geq \frac{w_1^T \tilde{B} w_1}{w_1^T w_1} = \lambda_1 + \frac{w_1^T \tilde{E} w_1}{w_1^T w_1} \geq \lambda_1 ,$$

and since $\lambda_1 \geq \mu_1$ it follows that $\mu_1 = \lambda_1$ and $\tilde{E}w_1 = \underline{0}$. Then also $Ev_1 = \underline{0}$, and so $v_{13} = v_{14}$. Using that v_1 is eigenvector of C with eigenvalue λ_1, we find that $v_1 = \underline{0}$, which is a contradiction. Similarly we find that $v_3 = \underline{0}$ by assuming that $\varepsilon < 0$. So $\varepsilon = 0$ and $B = C$. Thus the μ-partition of x is regular, and since this holds for every vertex, we find that G is one of the relations of a 3-class association scheme.

Next suppose that $g(\Sigma, \mu) = 0$. We shall show that this case cannot occur, which finishes the proof. Again, consider the matrix C as given above. From the equation $g(\Sigma, \mu)(k - a - \mu) = (v - 1 - k - g(\Sigma, \mu))b$ it follows that $b = 0$. Furthermore, also $d = g(\Sigma, \mu)\mu/k = 0$. Let B be the quotient matrix with respect to the neighbourhood partition of an arbitrary vertex x, then

$$B = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k-1-\lambda \\ 0 & c & k-c \end{pmatrix}.$$

Let B have eigenvalues k, μ_1 and μ_2, then on one hand the eigenvalues of C are k, μ_1, μ_2 and a, and on the other hand they are k, λ_1, λ_2 and λ_3. Now it follows by Lemma 4.3.3 that G is strongly regular, which is a contradiction with the fact that G has four distinct eigenvalues. □

Obviously, for regular graphs with four eigenvalues that generate a 3-class association scheme, we have that $h(\Sigma, \mu) = g(\Sigma, \mu)$, since they both equal n_3. However, the equality holds for any feasible spectrum Σ of a regular graph with four eigenvalues and any μ. This can be proven in a straightforward way from the equations

$$\lambda k + p_2 k + p_0 = (k^3 + p_2 k^2 + p_1 k + p_0)/v, \text{ and}$$

$$\frac{1}{v}\sum_{i=0}^{3} m_i\lambda_i^4 + p_2\lambda k + p_1 k \;=\; (k^4 + p_2k^3 + p_1k^2 + p_0k)/v \;,$$

which follow by taking traces of the equations $p(A) = p(k)/vJ$ and $Ap(A) = kp(k)/vJ$, respectively.

For $\mu = 0$, in which case we have a distance-regular graph, the characterization was already obtained by Haemers and the author [39]. Together with the previous remarks this gives the following.

COROLLARY 4.3.5. *Let G be a connected regular graph with four distinct eigenvalues, with k, λ and ξ (as functions of the spectrum) as before. Then G is a distance-regular graph (with diameter three) if and only if for every vertex the number of vertices k_2 at distance two equals*

$$k_2 \;=\; \frac{k(k-1-\lambda)^2}{\xi - \lambda^2 + k - 1} \;.$$

This settles a question by Haemers [59] on the characterization of distance-regular graphs with diameter three.

If we have a 3-class association scheme, then $g(\Sigma, \mu)$ must be a nonnegative integer. On the other hand, if we have any graph with spectrum Σ and a μ such that $g(\Sigma, \mu)$ is a nonnegative integer, then for any vertex, we can bound the number of nonadjacent vertices that have μ common neighbours with this vertex.

PROPOSITION 4.3.6. *With the hypothesis of the previous theorem, if $g(\Sigma, \mu)$ is a nonnegative integer, then $n_3 \leq g(\Sigma, \mu)$.*

Proof. Suppose for some vertex x we have $n_3 > g(\Sigma, \mu)$. Consider the μ-partition of x and change this partition by moving $n_3 - g(\Sigma, \mu)$ vertices from the set of vertices not adjacent to x and having μ common neighbours with x to the set of vertices not adjacent to x and not having μ common neighbours with x. By repeating the second part of the proof of Theorem 4.3.4, we find that the partition is regular, which is a contradiction. □

In the special case that H has the same spectrum as one of the relations of a 3-class association scheme, this gives the following.

COROLLARY 4.3.7. *Let G be a connected regular graph with four distinct eigenvalues that is one of the relations of a* 3-*class association scheme. Suppose that for some integer μ the number of vertices nonadjacent to some vertex x, having μ common neighbours with x equals* $n_3 > 0$. *If H is a graph with the same spectrum as G, then for any vertex x in H, the number of vertices that are not adjacent to x and have μ common neighbours with x is at most n_3, with equality for every vertex if and only if H is one of the relations of a* 3-*class association scheme with the same parameters as the scheme of G.*

4.3.4. Hoffman colorings and systems of linked symmetric designs

Let G be a k-regular graph on v vertices with smallest eigenvalue $\lambda_{\min}$. A *Hoffman coloring* in G is a partition of the vertices into Hoffman cocliques, that is, cocliques meeting the Hoffman (upper) bound $c = v\lambda_{\min}/(\lambda_{\min} - k)$. Recall that if C is a Hoffman coclique, then every vertex not in C is adjacent to $-\lambda_{\min}$ vertices of C. A *spread* in G is a partition of the vertices into Hoffman cliques, which is equivalent to a Hoffman coloring in the complement of G. A *regular coloring* of a graph is a partition of the vertices into cocliques of equal size, say c, such that for some l, every vertex outside a coclique C of the coloring is adjacent to precisely l vertices of C. So regular colorings are generalizations of Hoffman colorings. A graph with a regular coloring is regular, with degree $k = l(v/c - 1)$, and it also follows that it has an eigenvalue $\lambda = -l$. Now we find that $c = v\lambda/(\lambda - k)$, similar to the size of a coclique in a Hoffman coloring. In the following we shall say that the regular coloring corresponds to eigenvalue λ.

Suppose G has a regular coloring. Then we define relations R_1 by adjacency in G, R_2 by nonadjacency in G and being in distinct cocliques of the coloring, and R_3 by being distinct, nonadjacent in G and being in the same coclique of the coloring. It is easy to see that these relations form a 3-class association scheme if G is strongly regular (cf. [62]). A lot of Hoffman colorings exist in the triangular graphs $T(n)$, for even n, as these (the schemes) are equivalent to *one-factorizations* of K_n. For $n = 4$ and 6, the one-factorizations of K_n are unique, there are 6 nonisomorphic ones for $n = 8$, and 396 for $n = 10$ (cf. [83]). Dinitz, Garnick and McKay [44] found that there are 526,915,620 nonisomorphic one-factorizations of K_{12}, and they estimated these numbers for $n = 14$, 16 and 18.

If the relations as defined above form an association scheme, then G can have at most four distinct eigenvalues. However, this is not sufficient, as the graph $L_2(3)\otimes J_2$ with spectrum $\{[8]^1, [2]^4, [0]^9, [-4]^4\}$ has a Hoffman coloring, i.e., 3 disjoint cocliques of size 6, but the corresponding relations do not form an association scheme. It turns out that here the multiplicity of the eigenvalue $\lambda_3 = -4$ is too large. In fact, if the relations do form an association scheme, and we assume that the regular coloring corresponds to the eigenvalue λ_3, then it has eigenmatrix

$$P = \begin{pmatrix} 1 & k & v-k-c & c-1 \\ 1 & \lambda_1 & -\lambda_1 & -1 \\ 1 & \lambda_2 & -\lambda_2 & -1 \\ 1 & \lambda_3 & -\lambda_3 - c & c-1 \end{pmatrix},$$

with multiplicities 1, m_1, m_2 and m_3, respectively. Now it easily follows that $c(m_3 + 1) = v$, so that $m_3 = -k/\lambda_3$. On the other hand, this additional condition on the spectrum is sufficient.

THEOREM 4.3.8. *Let G be a connected k-regular graph on v vertices with four distinct eigenvalues. If G has a regular coloring corresponding to eigenvalue, say,* λ_3, *which has multiplicity* $m_3 \le -k/\lambda_3$, *then the corresponding relations form an association scheme.*

Proof. Let A_1 be the adjacency matrix of G (and R_1), and A_3 the adjacency matrix corresponding to the regular coloring (R_3), so $A_3 = I_c \otimes J_{v/c} - I$, where c is the size of the cocliques. Since any vertex outside a coclique C of the coloring is adjacent to $-\lambda_3$ vertices of C, it follows that $A_1(A_3 + I) = -\lambda_3(J - (A_3 + I))$, and so $A_1A_3 \in \langle I, J, A_1, A_3 \rangle$.

Let λ_1 and λ_2 be the remaining two eigenvalues of G, and let $B = (A_1 - \lambda_1 I)(A_1 - \lambda_2 I)$, then the nonzero eigenvalues of B are $(k - \lambda_1)(k - \lambda_2)$ with multiplicity 1, and $(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ with multiplicity m_3. If we let $E_0 = v^{-1}J$, and $E_3 = c^{-1}(A_3 + I) - v^{-1}J$, then

$$BE_0 = (k - \lambda_1)(k - \lambda_2)E_0 \text{ and } BE_3 = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)E_3.$$

By use of $\operatorname{rank}(E_0) = 1$, $\operatorname{rank}(E_3) = v/c - 1 \ge m_3$, $E_0^2 = E_0$, $E_3^2 = E_3$, and $E_0E_3 = O$, it follows that $B - (k - \lambda_1)(k - \lambda_2)E_0 - (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)E_3 = O$, as all its eigenvalues are zero. So $A_1^2 \in \langle I, J, A_1, A_3 \rangle$, and it follows that this algebra is closed under multiplication. Hence we have an association scheme. □

A *system of l linked symmetric* 2-(v, k, λ) *designs* is a collection of sets V_i, $i = 1,..., l + 1$ and an incidence relation between each pair of sets forming a symmetric 2-(v, k, λ) design, such that for any i, j, h the number of $x \in V_i$ incident with both $y \in V_j$ and $z \in V_h$ depends only on whether y and z are incident or not.

Now take as vertex set the union of all V_i, and define relations by being in the same subset V_i, being incident in the system of designs or being not incident in the system of designs. This defines a 3-class association scheme. The association scheme of $l - 1$ linked designs (note that such a system is contained in a system of l linked designs) can also be considered as the block scheme of the 2-(v, k, $l\lambda$) design that is obtained by taking as points the elements of the set V_1 and as blocks the elements of the remaining V_i, with the obvious incidence relation.

The only known nontrivial systems of linked designs have parameters $v = 2^{2m}$, $k = 2^{2m-1} - 2^{m-1}$, $\lambda = 2^{2m-2} - 2^{m-1}$, $l \le 2^{2m-1} - 1$, $m > 1$ (and their complements) (see [23]). Mathon [80] determined all systems of linked 2-(16, 6, 2) designs.

The incidence graph of a system of linked designs is defined on the union of the sets V_i, where adjacency is defined by incidence. If G is a graph with four distinct eigenvalues, that is the incidence graph of a system of linked designs, then G has a regular coloring. The following theorem characterizes these graphs.

THEOREM 4.3.9. *Let G be a connected k-regular graph on v vertices with four distinct eigenvalues. Suppose G has a regular coloring corresponding to, say, λ_3, with cocliques of size c such that the corresponding relations form an association scheme. Let m_1 and m_2 be the multiplicities of the remaining two eigenvalues λ_1 and λ_2, respectively, then* $c - 1 \le \min\{m_1, m_2\}$, *with equality if and only if G is the incidence graph of a system of linked symmetric designs.*

Proof. Let $h = 1, 2$, and take

$$E = \frac{v(v-k-c)}{m_h} E_h + \lambda_h J = (v-k-c+\lambda_h)\, I + \lambda_h \frac{v-c}{k} A_1 + (\lambda_h - \frac{v-k-c}{c-1})\, A_3 \ ,$$

then $\operatorname{rank}(E) \le m_h + 1$. Now partition E and A_1 according to the regular coloring, say $E = (E_{ij})$, $A_1 = (A_{ij})$, $i, j = 1,..., m_3$. Then it follows that if $i \ne j$, then

$$E_{ij} = \lambda_h \frac{v-c}{k} A_{ij} \ , \text{ and } E_{ii} = \frac{c(v-k-c)}{c-1} I + (\lambda_h - \frac{v-k-c}{c-1})\, J \ .$$

Observe that it follows from $m_3 = -k/\lambda_3$ that $m_1\lambda_1 + m_2\lambda_2 = 0$, so $\lambda_h \ne 0$. So E_{ii} is nonsingular, so $c = \operatorname{rank}(E_{ii}) \le \operatorname{rank}(E)$, which proves the inequality. In case of equality we have $\operatorname{rank}(E_{00}) = \operatorname{rank}(E)$, and then it follows that $E_{ij} = E_{i0}E_{00}^{-1}E_{0j}$. From this we derive that $A_{i0}A_{i0}^T = A_{i0}A_{0i} \in \langle I, J \rangle$, and since A_{i0} has constant row and column sums, we find that A_{i0} is the incidence matrix of a symmetric design. Furthermore we find that $A_{i0}A_{0j} \in \langle A_{ij}, J \rangle$ for $i \ne j$, which proves that the designs are linked (cf. [23, Thm. 2]). □

For $l = 1$, a system of linked designs is just one design, and we get the incidence graph and corresponding incidence scheme of a symmetric 2-(v, k, λ) design. It is bipartite distance-regular. In fact, it is well known that any bipartite regular graph with four distinct eigenvalues is the incidence graph of a symmetric design (see Chapter 3). This result now also follows from Theorem 4.3.9. In order to determine all nonisomorphic schemes given a certain parameter set of this form, we should mention that two dual (as well as complementary) designs generate the same association scheme.

Theorem 4.3.9 is the analogue of the following theorem by Haemers and Tonchev

[62, Thm. 5.1] (here g is the multiplicity of the smallest eigenvalue).

THEOREM 4.3.10. *If G is a primitive strongly regular graph with a Hoffman coloring, then $c - 1 \leq g - v/c + 1$, with equality if and only if G is the incidence graph of a system of linked symmetric designs.*

4.4. Number theoretic conditions

Using the Hasse-Minkowski invariant of rational symmetric matrices, Bose and Connor [7] derived number theoretic conditions for the existence of so-called regular group divisible designs, which can be seen as extensions of the well-known Bruck-Ryser conditions for symmetric designs. Godsil and Hensel [53] applied the results of Bose and Connor to imprimitive distance-regular graphs with diameter three. In fact, we find that after slight adjustments of the results of Bose and Connor, they are also applicable to imprimitive 3-class association schemes. Also in the primitive case, Hasse-Minkowski theory can be useful, under the condition that one of the relations is a strongly regular graph, preferably one that is determined by its spectrum. If one of the relations is a lattice graph or a triangular graph, we can use results of Coster [30] or Coster and Haemers [31], respectively. These results are obtained by using the Grothendieck group, a technique similar to Hasse-Minkowski theory. The results are in a sense generalizations of [96] and [86], respectively, which are only applicable to designs. A general reference for applications of Hasse-Minkowski theory to designs is [91].

4.4.1 The Hilbert norm residue symbol and the Hasse-Minkowski invariant

If a and b are nonzero rational numbers, and p a prime, then the *Hilbert norm residue symbol* $(a, b)_p$ is defined to be 1 if the equation

$$ax^2 + by^2 \equiv 1 \pmod{p^r}$$

has a rational solution x, y, for every r, otherwise it is defined to be -1. Here p may also be infinite.

Let A be a rational symmetric nonsingular matrix, with leading principal minor determinant D_r of order r, and take $D_0 = 1$. Now suppose that $D_r \neq 0$ for all r. Then the *Hasse-Minkowski invariant* of A is defined by

$$C_p(A) = (-1,-1)_p \prod_{i=0}^{v-1} (D_{i+1}, -D_i)_p$$

for every prime p.

The following theorem is the basic theorem that will supply us with necessary conditions for existence of certain 3-class association schemes. It deals with rationally congruent matrices. Two matrices A and B are *rationally congruent* if there is a nonsingular rational matrix P such that $P^TAP = B$.

THEOREM 4.4.1 (Hasse [63]). *Two rational symmetric positive definite matrices A and B of the same size are rationally congruent if and only if the square free parts of their determinants are the same and their Hasse-Minkowski invariants are equal for all primes p, including the infinite prime.*

Now consider an impritive 3-class association scheme, where one of the relations, say R_3, forms the disjoint union of m cliques of size n. Let A be the adjacency matrix of one of the other (nontrivial) relations, say R_1. Suppose that the graph defined by R_1 has degree k, any two adjacent vertices have λ common neighbours, any two nonadjacent vertices that are in the same clique of relation R_3 have μ common neighbours, and any two nonadjacent vertices from distinct cliques have μ' common neighbours. If $\delta = \frac{1}{2}(\mu' - \lambda)$, then A satisfies the equation

$$(A + \delta I)^2 = (k + \delta^2 - \mu)I + \mu' J + (\mu - \mu')I_m \otimes J_n$$

Since $A + \delta I$ is a symmetric rational matrix, it follows that the right hand side of the equation is rationally congruent to the identity matrix. Note that the matrix has spectrum

$$\{[(k + \delta)^2]^1, [(k + \delta)^2 - mn\mu']^{m-1}, [k + \delta^2 - \mu]^{m(n-1)}\}$$

Now the results of Bose and Connor generalize in an obvious way, and we obtain the following conditions.

PROPOSITION 4.4.2. *If an impritive* 3-*class association scheme as given above exists, then*

a. *if m is even, then $(k + \delta)^2 - mn\mu'$ is a rational square, and if $m \equiv 2 \pmod 4$ and n is even then $(k + \delta^2 - \mu, -1)_p = 1$ for all odd primes p.*

b. *if m is odd, and n is even, then $k + \delta^2 - \mu$ is a rational square, and $((k + \delta)^2 - mn\mu', (-1)^{\frac{1}{2}(m-1)}n\mu')_p = 1$ for all odd primes p.*

c. *if m and n are both odd, then $(k + \delta^2 - \mu, (-1)^{\frac{1}{2}(n-1)}n)_p((k + \delta)^2 - mn\mu', (-1)^{\frac{1}{2}(m-1)}n\mu')_p = 1$ for all odd primes p.*

Actually, we know a little bit more, if $\mu \neq \mu'$, since then A has four distinct eigenvalues, and then it follows that at least one of $k + \delta^2 - \mu$ and $(k + \delta)^2 - mn\mu'$ is a rational square. Examples of parameter sets with $\mu \neq 0$ that are ruled out by these conditions are $(m, n, k, \lambda, \mu', \mu) = (10, 4, 18, 8, 8, 6), (17, 5, 32, 12, 12, 8), (22, 4, 42, 20, 20, 14)$.

4.5. Lists of small feasible parameter sets

In order to generate feasible parameter sets for 3-class association schemes we shall classify them into three sets:

1. At least one of the relations is a graph with four distinct eigenvalues;
2. At least one of the relations is the disjoint union of some (connected) strongly regular graphs having the same parameters;
3. All three relations are strongly regular graphs - The amorphic schemes.

These three cases cover all possibilities. Case 2 is degenerate (see Section 4.1.1). For the remaining two cases we generated all feasible parameter sets on at most 100 vertices. For Case 3 we used Theorem 4.2.1. For Case 1 we started from the algorithms of the previous chapter to generate feasible spectra of graphs with four distinct eigenvalues, added the parameter μ and computed all other parameters, and checked them for necessary conditions (integrality conditions, Krein conditions, and the absolute bound).

Chapter 5

Just concentrate your whole energy into this mu, and do not allow any discontinuation. When you enter this mu and there is no discontinuation, your attainment will be as a candle burning and illuminating the whole universe.

(Mumon)

Bounds on the diameter and special subsets

In Chapter 1 we mentioned that the diameter of a connected graph is smaller than the number of distinct eigenvalues. Chung [28] and Delorme and Solé [41] derived bounds for the diameter of regular graphs in terms of the actual value of (some of) the eigenvalues. In this chapter we shall derive a tool which bounds the sizes of two subsets of vertices, which are at a given distance. The tool uses polynomials, and the bounds depend on the values of the polynomial evaluated at the Laplace eigenvalues. We apply this tool in Section 5.2 to derive upper bounds for the diameter of graphs. By suitable choices of the polynomial we find all diameter bounds mentioned above, but we obtain a better bound by using Chebyshev polynomials. The same bound was found independently by Chung, Faber and Manteuffel [29]. We also improve the bounds of Delorme and Solé [41] for the diameter of bipartite biregular graphs. Also Mohar [84] found diameter bounds in terms of the Laplace eigenvalues of the graph, but his results don't seem to fit in our framework. However, our bound seems to be better. We apply our bounds to the coset graphs of linear codes to obtain an upper bound for the covering radius of a linear code.

In Section 5.3 we shall have a closer look at the polynomials that optimize the bounds. These polynomials were also considered by Fiol, Garriga and Yebra [49] to bound the diameter in terms of the adjacency eigenvalues. Here we shall use them to obtain an upper bound on the number of vertices at maximal distance, and a lower bound on the number of vertices at distance two from a given vertex, in terms of the Laplace spectrum. For graphs with four eigenvalues we prove a more general result. Here we shall bound the number of vertices n_3 that are not adjacent to a given vertex and have a fixed number μ of common neighbours with that vertex, in terms of the spectrum and μ, and we characterize the case of equality. As we have seen in Chapter 4, this particular number n_3 plays an important role in a characterization of the graphs in a three-class association scheme. Our bound is evidence for a conjecture (see Section 5.3.2) on this number.

Another application gives bounds on the size of two equally large sets of vertices at maximal distance, or distance at least two (i.e., with no edges in between). The latter has applications for the bandwidth of a graph. We also give examples of graphs with few eigenvalues for which the bound is tight.

5.1. The tool

THEOREM 5.1.1. *Let G be a connected graph on v vertices with $r + 1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \ldots < \theta_r$. Let m be a nonnegative integer and let X and Y be sets of vertices, such that the distance between any vertex of X and any vertex of Y is at least $m + 1$. If p is a polynomial of degree m such that $p(0) = 1$, then*

$$\frac{|X|\,|Y|}{(v - |X|)(v - |Y|)} \le \max_{i \neq 0} p^2(\theta_i)\,.$$

Proof. Let G have Laplace matrix Q, then $p(Q)_{xy} = 0$ for all vertices $x \in X$ and $y \in Y$. Without loss of generality we assume that the first $|X|$ rows of Q correspond to the vertices in X and the last $|Y|$ rows correspond to the vertices in Y. Now consider the matrix

$$M = \begin{pmatrix} O & p(Q) \\ p(Q) & O \end{pmatrix}.$$

Note that M is symmetric, has row and column sums equal to 1, and its spectrum is $\{\pm\, p(\theta_i) \mid i = 0, 1, \ldots, r\}$, multiplicities included. Let M be partitioned symmetrically in the following way.

$$M = \begin{array}{c} \left(\begin{array}{c|c|c|c} O & O & & O \\ \hline O & O & & \\ \hline & & O & O \\ \hline O & & O & O \end{array}\right) \end{array} \begin{array}{l} \}\ \ |X| \\ \}\ \ v - |X| \\ \}\ \ v - |Y| \\ \}\ \ |Y| \end{array}.$$

Let B be its quotient matrix, then

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 - \dfrac{|Y|}{v - |X|} & \dfrac{|Y|}{v - |X|} \\ \dfrac{|X|}{v - |Y|} & 1 - \dfrac{|X|}{v - |Y|} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_0(B) = -\lambda_3(B) = 1$, $\lambda_1(B) = -\lambda_2(B) = \sqrt{\dfrac{|X|\,|Y|}{(v - |X|)(v - |Y|)}}$. Since the eigenvalues of B interlace those of M, we have that

$$\lambda_1(B) \leq \lambda_1(M) \leq \max_{i \neq 0} |p(\theta_i)| \ ,$$

and the statement follows. □

5.2. The diameter

The first application of Theorem 5.1.1 is to prove that the diameter $d(G)$ of G is smaller than the number of distinct Laplace eigenvalues of G. Suppose G has $r + 1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \ldots < \theta_r$. Now let p be the polynomial given by

$$p(z) = \prod_{i \neq 0} \frac{\theta_i - z}{\theta_i} \ ,$$

then p has degree r, $p(0) = 1$ and $p(\theta_i) = 0$ for $i \neq 0$. So if X and Y are sets of vertices, such that the distance between X and Y is at least $r + 1$, then

$$\frac{|X|\,|Y|}{(v - |X|)(v - |Y|)} \leq 0 \, .$$

which implies that any two vertices have distance smaller than $r + 1$, so $d(G) \leq r$.

Next, suppose G is regular with degree k and adjacency eigenvalues $\lambda_i = k - \theta_i$. Take two vertices x and y at distance $d(G) = m + 1$, and let $X = \{x\}$ and $Y = \{y\}$. By applying Theorem 5.1.1 to the polynomial p given by $p(z) = \left(\frac{k - z}{k}\right)^m$ we find that

$$\frac{1}{v-1} \le \left(\frac{\lambda}{k}\right)^m, \text{ so } d(G) \le \frac{\log(v-1)}{\log\left(\frac{k}{\lambda}\right)} + 1, \text{ where } \lambda = \max_{i \ne 0} |\lambda_i|.$$

This bound (which is only applicable to nonbipartite graphs) was found by Chung [28]. Similarly we find the bound of Delorme and Solé [41] by taking $p(z) = \frac{k-z+t}{k+t}\left(\frac{k-z}{k}\right)^{m-1}$ for arbitrary (real) t. However, we can do better, and we do not require regularity. For example, by taking the polynomial $p(z) = \left(\frac{\theta_r + \theta_1 - 2z}{\theta_r + \theta_1}\right)^m$ we find the bound

$$d(G) \le \frac{\log(v-1)}{\log\left(\dfrac{\theta_r + \theta_1}{\theta_r - \theta_1}\right)} + 1 .$$

Theorem 5.1.1 allows us to use any polynomial p of degree m such that $p(0) = 1$. To get the sharpest bound possible in this way, however, we must choose p such that the right hand side of the inequality in Theorem 5.1.1 is minimized. For $m = 1$, the polynomial we just used is the best possible, but in general we can still do better by looking at the following "relaxation" of the minimization problem, that is, minimize $\max \{ |p(z)| \mid \theta_1 \le z \le \theta_r \}$ over all polynomials p of degree m such that $p(0) = 1$. The solution of this problem can be given in terms of Chebyshev polynomials (cf. [93, Thm. 2.1, Ex. 2.5.12]). We have to take

$$\check{C}_m(z) = \frac{T_m\left(\dfrac{\theta_r + \theta_1 - 2z}{\theta_r - \theta_1}\right)}{T_m\left(\dfrac{\theta_r + \theta_1}{\theta_r - \theta_1}\right)} ,$$

where $T_m(z) = \cosh(m \cosh^{-1}(z)) = \frac{1}{2}(z + \sqrt{z^2-1})^m + \frac{1}{2}(z - \sqrt{z^2-1})^m$ is the m-th *Chebyshev polynomial* (indeed, it is a polynomial). These polynomials have the property that $\max_{z \in [-1,1]} |T_m(z)| = 1$, and this leads to the following bound.

THEOREM 5.2.1. *Let G be a connected noncomplete graph with smallest nonzero Laplace eigenvalue θ_1 and largest Laplace eigenvalue θ_r, then*

$$d(G) \le \frac{\cosh^{-1}(v-1)}{\cosh^{-1}\left(\dfrac{\theta_r + \theta_1}{\theta_r - \theta_1}\right)} + 1 .$$

This bound was found independently by Chung, Faber and Manteuffel [29]. In [37] we approximated the bound by

$$d(G) < \frac{\log 2(v-1)}{\log\left(\dfrac{\sqrt{\theta_r}+\sqrt{\theta_1}}{\sqrt{\theta_r}-\sqrt{\theta_1}}\right)} + 1 \ ,$$

which follows from $2T_m\left(\frac{\theta_r+\theta_1}{\theta_r-\theta_1}\right) = \left(\frac{\sqrt{\theta_r}+\sqrt{\theta_1}}{\sqrt{\theta_r}-\sqrt{\theta_1}}\right)^m + \left(\frac{\sqrt{\theta_r}-\sqrt{\theta_1}}{\sqrt{\theta_r}+\sqrt{\theta_1}}\right)^m > \left(\frac{\sqrt{\theta_r}+\sqrt{\theta_1}}{\sqrt{\theta_r}-\sqrt{\theta_1}}\right)^m$.

As we have seen, a graph G with $r+1$ distinct eigenvalues has diameter at most r. Now what happens if the diameter is precisely r? The polynomial p of degree $r-1$ which minimizes our upper bound is determined by $p(\theta_j) = (-1)^{j-1}c$, $j = 1,\ldots, r$, where c is chosen such that $p(0) = 1$. (It is easily shown that this is indeed the best polynomial we can choose.)

PROPOSITION 5.2.2. *Let G be a connected graph with $r+1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \ldots < \theta_r$. If $d(G) = r$, then*

$$v \geq 1 + \sum_{j\neq 0} \prod_{i\neq 0,j} \frac{\theta_i}{|\theta_j-\theta_i|} \ .$$

Proof. As suggested, we have to take

$$p(z) = \sum_{j\neq 0} (-1)^{j-1} c\, \mathfrak{L}_j(z), \text{ where } \mathfrak{L}_j(z) = \prod_{i\neq 0,j} \frac{\theta_i - z}{\theta_i-\theta_j},\ j \neq 0\,,$$

and where c is such that $p(0) = 1$. Since the degree of p is $r-1$, it follows from Theorem 5.1.1 that $\frac{1}{v-1} \leq |c|$. Since $p(0) = 1$ we have that

$$\frac{1}{c} = \sum_{j\neq 0} (-1)^{j-1} \prod_{i\neq 0,j} \frac{\theta_i}{\theta_i-\theta_j} = \sum_{j\neq 0} \prod_{i\neq 0,j} \frac{\theta_i}{|\theta_j-\theta_i|}\ ,$$

from which the result follows. □

5.2.1. Bipartite biregular graphs

Let G be a bipartite graph with parts V_1 and V_2. We shall call G *biregular* if every vertex in V_1 has vertex degree k_1 and every vertex in V_2 has vertex degree k_2, for some k_1 and k_2. In this case we can improve the diameter bound we just found. The results will be stated in terms of the adjacency eigenvalues, as the proofs are built on the special structure of the adjacency matrix. However, the Laplace eigenvalues θ_i and the adjacency eigenvalues λ_i are related by $\lambda_i^2 = (k_1 - \theta_i)(k_2 - \theta_i)$ (more specifically, for all θ_i, both solutions of the equation for λ_i are eigenvalues, and the other way around). Delorme and Solé [41] found a bound for the diameter of bipartite biregular graphs. Just like them we shall distinguish between the distance of vertices in the same part, and the distance of vertices in distinct parts. However, these maximum distances we distinguish only differ by one. From the following arguments, which are basically the same as those in the proof of Theorem 5.1.1, it is easy to derive and improve the bounds of Delorme and Solé.

In this section, we let G be connected and bipartite biregular, with n_1 vertices of degree k_1, and n_2 vertices of degree k_2, and let A be the adjacency matrix of G, with eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_r = -\lambda_0$. Now let m be odd, X_1 a subset of V_1, and Y_2 a subset of V_2, such that the distance between any vertex in X_1 and any vertex in Y_2 is at least $m + 2$. Let p be an odd polynomial of degree m, such that $p(\lambda_0) = 1$. Then we can partition $p(A)$ symmetrically as follows.

$$p(A) = \left(\begin{array}{c:c:c:c} O & O & & O \\ \hdashline O & O & & \\ \hdashline & & O & O \\ \hdashline O & & O & O \end{array}\right) \begin{array}{ll} \} & |X_1| \\ \} & n_1 - |X_1| \\ \} & n_2 - |Y_2| \\ \} & |Y_2| \end{array} .$$

Let B be its quotient matrix, then

$$B = \begin{pmatrix} 0 & 0 & \kappa & 0 \\ 0 & 0 & \kappa - \dfrac{|Y_2|}{n_1 - |X_1|}\dfrac{1}{\kappa} & \dfrac{|Y_2|}{n_1 - |X_1|}\dfrac{1}{\kappa} \\ \dfrac{|X_1|}{n_2 - |Y_2|}\kappa & \dfrac{1}{\kappa} - \dfrac{|X_1|}{n_2 - |Y_2|}\kappa & 0 & 0 \\ 0 & \dfrac{1}{\kappa} & 0 & 0 \end{pmatrix}, \text{ where } \kappa = \sqrt{\frac{k_1}{k_2}} .$$

The eigenvalues of B are ± 1, $\pm\sqrt{\dfrac{|X_1|\,|Y_2|}{(n_1-|X_1|)(n_2-|Y_2|)}}$, and since they interlace those of $p(A)$, we find that

$$\frac{|X_1|\,|Y_2|}{(n_1-|X_1|)(n_2-|Y_2|)} \le \max_{i \ne 0, r} p^2(\lambda_i) \ .$$

Now denote by $d_{ij}(G)$ the maximum distance between any vertex in V_i and any vertex in V_j, for i, $j = 1$ or 2. If we let $m = d_{12}(G) - 2$, so m is odd, and take two vertices at distance $d_{12}(G)$, and let the polynomial p be given by

$$p(z) = \frac{T_m\left(\frac{z}{\lambda_1}\right)}{T_m\left(\frac{\lambda_0}{\lambda_1}\right)} \ ,$$

then p is odd ($T_m(z)$ is an even, respectively odd polynomial, if m is even, respectively odd), and we find the following bound.

PROPOSITION 5.2.3. *Let G be connected and bipartite biregular, with n_1 vertices of degree k_1, and n_2 vertices of degree k_2, then*

$$d_{12}(G) \le \frac{\cosh^{-1}\left(\sqrt{(n_1-1)(n_2-1)}\right)}{\cosh^{-1}\left(\frac{\lambda_0}{\lambda_1}\right)} + 2 \ .$$

Next, let m be even, $j = 1$ or 2, and let X_j and Y_j be subsets of V_j, such that the distance between any vertex in X_j and any vertex in Y_j is at least $m + 2$. Let p be an even polynomial of degree m, such that $p(\lambda_0) = 1$, then there is a polynomial q of degree $\tfrac{1}{2}m$ such that $p(z) = q(z^2)$. Now we consider the matrix A^2, which has row and column sums equal to $\lambda_0^2 = k_1k_2$, and eigenvalues λ_i^2. Furthermore, we can write A^2 as

$$A^2 = \begin{array}{cl} \begin{pmatrix} M_1 & O \\ O & M_2 \end{pmatrix} & \begin{array}{l} \}\ n_1 \\ \}\ n_2 \end{array} \end{array} .$$

Note that the spectrum of A^2 is the union of the spectra of M_1 and M_2, and both M_1 and M_2

have one eigenvalue k_1k_2. Now consider the (multi)graph on V_j with adjacency matrix M_j, which has Laplace eigenvalues $k_1k_2 - \lambda_i^2$ for some of the i-s, $i < r$. By applying Theorem 5.1.1 to this graph and the polynomial $q'(z) = q(k_1k_2 - z)$, we show that

$$\frac{|X_j|\,|Y_j|}{(n_j - |X_j|)(n_j - |Y_j|)} \leq \max_{i \neq 0, r} p^2(\lambda_i) \ ,$$

since the distance in this new graph (which is one of the so-called halved (multi)graphs) is half of the original distance (in G). Now the maximal distance follows.

PROPOSITION 5.2.4. *Let G be connected and bipartite biregular, with n_1 vertices of degree k_1, and n_2 vertices of degree k_2. Let j = 1 or 2, then*

$$d_{jj}(G) \leq \frac{\cosh^{-1}(n_j - 1)}{\cosh^{-1}\left(\dfrac{\lambda_0}{\lambda_1}\right)} + 2 \ .$$

When G is bipartite and regular (so $k = k_1 = k_2$), we can combine Propositions 5.2.3 and 5.2.4.

COROLLARY 5.2.5. *Let G be a connected graph on v vertices, which is bipartite and regular with degree k, and has second largest eigenvalue λ_1. Then*

$$d(G) \leq \frac{\cosh^{-1}(\tfrac{1}{2}v - 1)}{\cosh^{-1}\left(\dfrac{k}{\lambda_1}\right)} + 2 \ .$$

For bipartite regular graphs this bound improves the general bound of Theorem 5.2.1. For example, suppose we have a bipartite regular graph with $k = 6$, and $\lambda_1 = 2$ (for example, the incidence graph of a 2-(16, 6, 2) design, which has diameter three). After rounding, the general bound gives $d(G) \leq 4$, while the bound of Corollary 5.2.5 gives $d(G) \leq 3$.

5.2.2. The covering radius of error-correcting codes

In this section we give the applications of the diameter bounds to the covering radius of error-correcting codes. If C is a code of length n over an alphabet Q of q elements, then we can apply Theorem 5.1.1 to the Hamming graph $H(n, q)$ to derive an upper bound for the covering radius of C in terms of n, q and the size of C, since the code C is a subset of vertices in the Hamming graph, and the covering radius of C is the maximum distance of any of the vertices to C. Unfortunately, we only derive a useless bound. For linear codes, however, we can follow the approach of Delorme and Solé [41], and use its coset graph, and this will turn out to be more successful. The coset graph of a code has diameter equal to the covering radius of the code, and the eigenvalues of the graph can be expressed in terms of the weights of the dual code (recall the duality of the distance scheme and coset scheme of a code, cf. Section 4.1.5). In this way we can derive bounds for the covering radius of a code, using the bounds for the diameter of the graph.

Let C be a linear code over $GF(q)$ of length n and dimension K. Formally speaking, the coset graph of C is a multigraph G with vertex set $GF(q)^n/C$ (the cosets of C) of size $v = q^{n-K}$. The number of edges between two cosets $x + C$ and $y + C$ is the number of words of weight one in $x - y + C$. In this way we get a regular graph of degree $k = n(q-1)$, possibly with loops (which occur whenever there is a codeword of weight one) or multiple edges (which occur whenever there is a coset containing more than one word of weight one; this is the case if and only if there is a codeword of weight two or there are at least two codewords of weight one (in the latter case we have multiple loops)). The coset graph is an ordinary graph if the minimum distance of the code is at least three. The nice thing is that the covering radius of C is equal to the diameter of G. Furthermore the adjacency spectrum of G is (the multiset) $\{n(q-1) - qw(x) \mid x \in C^{\perp}\}$ (cf. [22]), and so G has Laplace spectrum $\{qw(x) \mid x \in C^{\perp}\}$, where $w(x)$ is the weight of x. The first observation we make is that the covering radius is not larger than the external distance, since the latter equals the number of nonzero weights in the dual code (note that in case of equality, we can apply Proposition 5.2.2). Application of our bounds for the diameter of G gives the following results.

THEOREM 5.2.6. *Let C be a linear code over $GF(q)$ of length n and dimension K, and covering radius ρ. Let $\Delta^{\perp}$ and $\delta^{\perp}$ be the maximum, respectively minimum nonzero weight (distance) of the dual code $C^{\perp}$. Then*

$$\rho \le \frac{\cosh^{-1}(q^{n-K}-1)}{\cosh^{-1}\left(\dfrac{\Delta^{\perp}+\delta^{\perp}}{\Delta^{\perp}-\delta^{\perp}}\right)} + 1 \ .$$

If the coset graph is bipartite then we can apply the improved bound of Corollary 5.2.5. This happens precisely when $q = 2$ and $C^{\perp}$ contains the all-one word, or equivalently, when C is a binary even weight code. To show this, let n be the length of the code, and λ_r be the smallest adjacency eigenvalue of G. Suppose $q = 2$ and $C^{\perp}$ contains the all-one word, so $\Delta^{\perp} = n$, then $\lambda_r = -n = -k$, from which we may conclude that G is bipartite. Conversely, if G is bipartite, then $\lambda_r = -n(q-1)$, so there is an $x \in C^{\perp}$ such that $n(q-1) - qw(x) = -n(q-1)$, which can only be the case if $q = 2$ and $w(x) = n$, so x is the all-one word.

THEOREM 5.2.7. *Let C be a binary linear even weight code of length n and dimension K, and covering radius ρ. Let $\delta^{\perp}$ be the minimum nonzero weight (distance) of the dual code $C^{\perp}$. Then*

$$\rho \le \frac{\cosh^{-1}(2^{n-K-1}-1)}{\cosh^{-1}\left(\frac{n}{n-2\delta^{\perp}}\right)} + 2 .$$

5.3. Bounds on special subsets

Let G be a graph. Now let X be an arbitrary set of vertices, and Y be the set of vertices that are at distance at least d from every vertex in X. When we use the polynomial $\check{C}_{d-1}(z)$ in Theorem 5.1.1, we derive a bound on the size of Y, in terms of the extreme Laplace eigenvalues, d, and the size of X.

PROPOSITION 5.3.1. *Let G a connected noncomplete graph with smallest nonzero Laplace eigenvalue θ_1 and largest Laplace eigenvalue θ_r. Let X be an arbitrary set of vertices, and Y the set of vertices at distance at least d from X, where d is a positive integer, then*

$$|Y| \le \frac{v}{1 + \frac{|X|}{v-|X|} T_{d-1}\left(\frac{\theta_r+\theta_1}{\theta_r-\theta_1}\right)^2} .$$

Recall that to obtain the sharpest bound we have to minimize $\max\{|p(\theta_i)| \mid i \ne 0\}$ over all polynomials p of degree m such that $p(0) = 1$. Chebyshev polynomials are certainly good, but not optimal. In the paper by Fiol, Garriga and Yebra [49] the optimal polynomials were investigated to bound the diameter of a graph in terms of its adjacency eigenvalues. The problem of finding the optimal polynomials in fact is one from the theory of uniform approximations of continuous functions (cf. [27, 93]).

Let S be a compact set of real numbers and let $C(S)$ be the set of continuous functions on S to the reals. Let $f \in C(S)$, with uniform norm

$$\| f \|_{\infty} = \max_{z \in S} | f(z) | .$$

Let W be a subspace of $C(S)$ of dimension n, then w^* is called a *best approximation* of f in W if

$$\min_{w \in W} \| f - w \|_\infty = \| f - w^* \|_\infty .$$

The set of critical points of a function is the set $E(f, S) = \{z \in S \mid \|f\|_\infty = |f(z)|\}$. The sign of $z \neq 0$ is defined by $\mathrm{sgn}(z) = z|z|^{-1}$ ($\mathrm{sgn}(0) = 0$). Now we have the following characterization of best approximations (cf. [93]).

LEMMA 5.3.2. *The function w^* is a best approximation of f if and only if there are distinct points $z_1, \ldots, z_t \in E(f - w^*, S)$, and positive numbers $\alpha_1, \ldots, \alpha_t$ such that for all $w \in W$*

$$\sum_{i=1}^{t} \alpha_i \,\mathrm{sgn}(f(z_i) - w^*(z_i))\, w(z_i) = 0 ,$$

where $t \leq n + 1$.

After substitution of $p(0) = 1$ our problem is to find

$$\min_{p_m, \ldots, p_1} \max_{i \neq 0} |p_m \theta_i^{\,m} + \ldots + p_1 \theta_i + 1| ,$$

so we want a best approximation of the function -1 on $S = \{\theta_1, \ldots, \theta_r\}$ from $W = \{w \mid w(z) = p_m z^m + \ldots + p_1 z\}$, which is an m-dimensional subspace of $C(S)$. It follows that p is the unique optimal polynomial if and only if there are $z_j \in \{\theta_i \mid i = 1, \ldots, r\}$, $j = 1, \ldots, m + 1$, such that $z_1 < z_2 < \ldots < z_{m+1}$, and $p(z_j)$ is alternating $\pm \max\{|p(\theta_i)| \mid i \neq 0\}$ (cf. [93, Thm. 2.8 and 2.10]). It also follows that we must have $z_1 = \theta_1$ and $z_{m+1} = \theta_r$. For $m = 2$, where we have to find the optimal quadratic polynomial, it is easily verified that we have to take $z_2 = \theta_h$, the Laplace eigenvalue closest to $\frac{1}{2}(\theta_1 + \theta_r)$. In the general case it follows (cf. [27, Thm. 7.1.6]) that there is a subset T of $\{1, \ldots, r\}$ of size $m + 1$ such that the polynomial p given by

$$p(z) = c_T \sum_{j \in T} \prod_{i \in T \setminus \{j\}} \frac{z - \theta_i}{|\theta_j - \theta_i|} ,$$

where c_T is such that $p(0) = 1$, is the unique optimal polynomial. Now let P_m be the set of polynomials of degree m such that $p(0) = 1$, then it follows that $|c_T| = \min_{p \in P_m} \max_{i \neq 0} |p(\theta_i)|$.

If T' is an arbitrary subset of $\{1, \ldots, r\}$ of size $m + 1$, then

$$|c_{T'}| = \min_{p \in P_m} \max_{i \in T'} |p(\theta_i)| = \left(\sum_{j \in T'} \prod_{i \in T' \setminus \{j\}} \frac{\theta_i}{|\theta_j - \theta_i|} \right)^{-1} .$$

Now it follows that $|c_{T'}| \le |c_T|$, and so $|c_T| \le \max_{T' \subset \{1,\dots,r\},\, |T'|=m+1} |c_{T'}| \le |c_T|$. Thus we find that the required minimum equals

$$|c_T| = \max_{T' \subset \{1,\dots,r\},\, |T'|=m+1} \left(\sum_{j \in T'} \prod_{i \in T' \setminus \{j\}} \frac{\theta_i}{|\theta_j - \theta_i|} \right)^{-1} .$$

5.3.1. The number of vertices at maximal distance and distance two

In Section 5.2 we started by proving that if a graph has $r + 1$ distinct Laplace eigenvalues, then it has diameter at most r. Using the results of the previous section we find a bound on the number of vertices that are at maximal distance r from a fixed vertex. By G_i we denote the distance i graph of G.

THEOREM 5.3.3. *Let G be a connected graph on v vertices with $r + 1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \dots < \theta_r$. Let x be an arbitrary vertex, and let k_r be the number of vertices at distance r from x. Then*

$$k_r \le \frac{v}{1 + \dfrac{\gamma^2}{v-1}}, \text{ where } \gamma = \sum_{j \neq 0} \prod_{i \neq 0, j} \frac{\theta_i}{|\theta_j - \theta_i|} .$$

If equality holds for every vertex, then G_r is a strongly regular $(v, k_r, \lambda, \lambda)$ graph. If G is a distance-regular graph with diameter r such that G_r is a strongly regular $(v, k_r, \lambda, \lambda)$ graph then the bound is tight for every vertex.

Proof. Take $X = \{x\}$, and let Y be the set of vertices at distance r from x. Now take the optimal polynomial of degree $r - 1$ given in the previous section, with $\gamma = |c_T|^{-1}$ and apply Theorem 5.1.1, then the bound follows. If the bound is tight, then it follows that in the proof of Theorem 5.1.1 we have tight interlacing, and so the partition of M is regular. Therefore

$$p(Q) = \begin{pmatrix} a & \vdots & a\underline{1}^T & \vdots & \underline{0}^T \\ \cdots & & \cdots & & \cdots \\ a\underline{1} & \vdots & S_{11} & \vdots & S_{12} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & S_{12}^T & \vdots & S_{22} \end{pmatrix} \begin{array}{l} \} \quad 1 \\ \\ \} \quad v-1-k_r \\ \\ \} \quad k_r \end{array} ,$$

where $a = 1/(v - k_r)$, is regularly partitioned with S_{12} and S_{22} having the same row sums. If the bound is tight for every vertex, then it follows that $J - (v - k_r)p(Q)$ is the adjacency matrix of G_r, and that this graph is a strongly regular $(v, k_r, \lambda, \lambda)$ graph.

On the other hand, if G is a distance-regular graph with diameter r such that G_r is a strongly regular $(v, k_r, \lambda, \lambda)$ graph then we shall show that

$$k_r = \frac{v}{1 + \frac{\gamma^2}{v-1}}, \text{ where } \gamma^{-1} = \max_{i \neq 0} |p(\theta_i)|,$$

for some polynomial p of degree $r - 1$ such that $p(0) = 1$. Because of the optimality of the bound this suffices to prove that the bound is tight for every vertex. Assume that G has degree k, then its Laplace eigenvalues θ_i and its adjacency eigenvalues λ_i are related by $\lambda_i = k - \theta_i$. Furthermore, let A be the adjacency matrix of G, and let A_i be the adjacency matrix of the distance i graph G_i of G. Since G is distance-regular, there is a polynomial q of degree $r - 1$ such that

$$q(A) = (J - A_r)/(v - k_r) = (A_{r-1} + ... + A + I)/(v - k_r),$$

and then $q(k) = 1$. Now let $p(z) = q(k - z)$. We have that G_r is a strongly regular $(v, k_r, \lambda, \lambda)$ graph, and such a graph has adjacency eigenvalues k_r and $\pm\sqrt{k_r(v - k_r)/(v-1)}$. From this it follows that

$$\max_{i \neq 0} |p(\theta_i)| = \max_{i \neq 0} |q(\lambda_i)| = \sqrt{\frac{k_r}{(v-1)(v-k_r)}},$$

which proves the claim. □

A side result of Theorem 5.3.3 is that if $v < 1 + \gamma$, so that $k_r < 1$, then the diameter of G is at most $r - 1$, a result we already found in Section 5.2.

Examples of graphs for which the bound is tight for every vertex are given by the 2-antipodal distance-regular graphs, with $k_r = 1$ (G_r being a disjoint union of edges). Other examples are given by the Odd graph on 7 points ($k_3 = 18$) and the generalized hexagons $GH(q, q)$ ($k_3 = q^5$). If G is a connected regular graph with four eigenvalues then we can prove that a tight bound for every vertex implies distance-regularity, but we shall prove this in more generality in the next section.

The reader may have noticed that we have omitted examples of strongly regular graphs for which the bound is tight for every vertex. By taking $r = 2$ in Theorem 5.3.3, we see that the bound is tight for strongly regular $(v, k, \lambda', \lambda' + 2)$ graphs. Using Theorem 2.2.1, it is not hard to show that for any connected graph with three Laplace eigenvalues the bound also follows from the parameter restrictions of such a graph. However, it may be surprising that the bound turns out to be tight for *some* vertex if and only if G comes from a polarity in a symmetric design with at least one absolute point (see Section 2.2.4). The absolute points correspond to the vertices for which the bound is tight.

For graphs with four eigenvalues, the upper bound for k_3 gives a lower bound for k_2, the number of vertices at distance 2 from x, since $k_2 = v - 1 - d_x - k_3$, where d_x is the vertex degree of x. This lower bound generalizes to graphs with more than four eigenvalues, since we can bound the number of vertices k_{3+} at distance at least three, using the optimal quadratic polynomial. By $G_{1,2}$ we denote the graph on the same vertices as G, where two vertices are adjacent if they have distance 1 or 2 in G.

THEOREM 5.3.4. *Let G be a connected graph on v vertices with $r + 1 \geq 4$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \ldots < \theta_r$, and let θ_h be an eigenvalue unequal to θ_1 and θ_r, which is closest to $\frac{1}{2}(\theta_1 + \theta_r)$. Let x be an arbitrary vertex with vertex degree d_x, and let $k_{2,x}$ be the number of vertices at distance 2 from x. Then*

$$k_{2,x} \geq v - 1 - d_x - \frac{v}{1 + \dfrac{\gamma^2}{v-1}} \;, \; \textit{where } \gamma = \sum_{j=1,h,r} \prod_{\substack{i=1,h,r \\ i \neq j}} \frac{\theta_i}{|\theta_j - \theta_i|} \;.$$

If equality holds for every vertex, then $G_{1,2}$ is a strongly regular $(v, d_x + k_{2,x}, \lambda', \lambda' + 2)$ graph. If G is a distance-regular graph such that $G_{1,2}$ is a strongly regular $(v, k + k_2, \lambda', \lambda' + 2)$ graph then the bound is tight for every vertex.

Proof. The proof is similar to the proof of Theorem 5.3.3. Here equality for every vertex implies that "the distance at least 3 graph" G_{3+} is a strongly regular $(v, k_{3+}, \lambda, \lambda)$ graph, and so $G_{1,2}$ is a strongly regular $(v, d_x + k_{2,x}, \lambda', \lambda' + 2)$ graph. Note that in that case G must have diameter 3 or 4. □

Examples for $r = 3$ for which this bound is tight were already given above. We do not know of any graph with more than four eigenvalues for which the bound is tight.

5.3.2. Special subsets in graphs with four eigenvalues

In a graph with four eigenvalues two vertices are at distance three if and only if they are not adjacent and have no common neighbours. The purpose of this section is to generalize the bound on the number of vertices k_3 at distance 3 from a vertex x to a bound on the number of vertices n_3 that are not adjacent to x and have μ common neighbours with x. Here the reader should keep in mind the analogue of the generalization of distance-regular graphs with diameter three to three-class association schemes. The question of bounding n_3 was raised after we characterized the graphs in a three-class association scheme as those regular graphs with four eigenvalues for which n_3 equals $g(\Sigma, \mu)$, for every vertex, for some μ (Theorem 4.3.4). Recall that $g(\Sigma, \mu)$ is a (rather complicated) function of the spectrum Σ of the graph and μ. Furthermore, we know that if $g(\Sigma, \mu)$ is a nonnegative integer then n_3 is at most $g(\Sigma, \mu)$. We strongly believe that the integrality condition can be dropped, but are (so far) unable to prove so. Still, the bound we obtain in this section is close, giving some evidence for the conjecture.

Let us define G_μ as the graph on the same vertices as G, where two vertices are adjacent if in G they are not adjacent, and have μ common neighbours. Let $G_{\neg\mu}$ be the graph with two vertices being adjacent if in G they are not adjacent, and do not have μ common neighbours.

THEOREM 5.3.5. *Let G be a connected graph on v vertices with four distinct Laplace eigenvalues* $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3$. *Let μ be a nonnegative integer, let x be an arbitrary vertex, and let n_3 be the number of vertices that are not adjacent to x and have exactly μ common neighbours with x. Then*

$$n_3 \le \frac{v}{1 + \dfrac{\gamma^2}{v-1}}, \textit{ where } \gamma = \begin{cases} \dfrac{2(\theta_1\theta_3 - v\mu)}{(\theta_3-\theta_2)(\theta_2-\theta_1)} + 1 & \textit{if } \; v\mu \le \theta_1\theta_2 \textit{ or } \theta_2\theta_3 < v\mu, \\ \dfrac{2(\theta_2\theta_3 - v\mu)}{(\theta_3-\theta_1)(\theta_1-\theta_2)} + 1 & \textit{if } \; \theta_1\theta_2 < v\mu \le \theta_1\theta_3, \\ \dfrac{2(\theta_1\theta_2 - v\mu)}{(\theta_2-\theta_3)(\theta_3-\theta_1)} + 1 & \textit{if } \; \theta_1\theta_3 < v\mu \le \theta_2\theta_3. \end{cases}$$

If equality holds for every vertex, then G_μ is a strongly regular $(v, n_3, \lambda, \lambda)$ *graph. If G is regular then equality holds for every vertex if and only if G, G_μ and $G_{\neg\mu}$ form a three-class association scheme and G_μ is a strongly regular* $(v, n_3, \lambda, \lambda)$ *graph.*

Proof. Here we use a slight variation to the technique we used in the proof of Theorem 5.1.1. Let $p(z) = p_2z^2 + p_1z + p_0$ be a quadratic polynomial such that $p(0) = p_0 = 1 + p_2v\mu$. Let Q be the Laplace matrix of G, then $(p_2(Q^2 - \mu J) + p_1Q + p_0I)_{xy} = 0$ for all vertices y that are not adjacent to x and have μ common neighbours with x. If we replace $p(Q)$ by $p_2(Q^2 - \mu J) + p_1Q + p_0I$ in the proof of Theorem 5.1.1, then the matrix M has row and column sums equal to 1, and spectrum $\{\pm 1\} \cup \{\pm p(\theta_i) \mid i = 1, 2, 3\}$ with corresponding multiplicities. Now it follows that

$$n_3 \le \frac{v}{1 + \dfrac{\gamma^2}{v-1}}, \textit{ where } \gamma^{-1} = \max_{i \ne 0} |p(\theta_i)| \,.$$

So here the sharpest bound is obtained by minimizing $\max\{|p(\theta_i)| \mid i \ne 0\}$ over all polynomials $p(z) = p_2z^2 + p_1z + p_0$ such that $p(0) = 1 + p_2\nu\mu$. For $\mu = 0$ we know the solution: there is a unique optimal polynomial p, and $p(\theta_1) = -p(\theta_2) = p(\theta_3)$. In general the situation is more complicated. We shall see that the polynomial is not always unique anymore. However, we can use Lemma 5.3.2 to optimize the bound explicitly. In order to characterize the case of equality, we need to be sure that the bound we find is indeed derived with the best possible polynomial. After substitution of $p(0) = 1 + p_2\nu\mu$ our problem becomes to find

$$\min_{p_2, p_1} \max_{i \neq 0} |p_2(\theta_i^2 + \nu\mu) + p_1\theta_i + 1| \ ,$$

so we are looking for a best approximation of the function -1 on $S = \{\theta_1, \theta_2, \theta_3\}$ from $W = \{w |\ w(z) = p_2(z^2 + \nu\mu) + p_1z\}$, which is a two-dimensional subspace of $C(S)$.

Now suppose we have a best approximation w^* (these always exist), and suppose that it has one critical point ($t = 1$), say θ_i. Then it follows from the lemma that for all $w \in W$, $w(\theta_i) = 0$, which implies that $\theta_i = 0$, a contradiction.

Now suppose that it has two critical points, say θ_i and θ_j with $s_i = \mathrm{sgn}(w^*(\theta_i) + 1)$ and $s_j = \mathrm{sgn}(w^*(\theta_j) + 1)$. Then there are $\alpha_i, \alpha_j > 0$ such that for all p_2 and p_1 we have

$$\alpha_i s_i(p_2(\theta_i^2 + \nu\mu) + p_1\theta_i) + \alpha_j s_j(p_2(\theta_j^2 + \nu\mu) + p_1\theta_j) = 0.$$

Setting $p_2 = 0$ gives $\alpha_i s_i\theta_i + \alpha_j s_j\theta_j = 0$, from which we find that $s_i = -s_j$. By setting $p_1 = 0$ and by use of the derived equation, we have that $(\theta_i^2 + \nu\mu)\theta_j = (\theta_j^2 + \nu\mu)\theta_i$, which is equivalent to $\nu\mu = \theta_i\theta_j$. Using that $w^*(\theta_i) + 1 = -(w^*(\theta_j) + 1)$, we find that in this case the optimal min-max value of the above problem equals

$$\frac{|\theta_i - \theta_j|}{\theta_i + \theta_j} \ .$$

Note that here the optimal polynomial is not unique, in fact there are infinitely many.

Next, consider the case that all three eigenvalues θ_i are critical points with $s_i = \mathrm{sgn}(w^*(\theta_i) + 1)$. Then it follows from Lemma 5.3.2 that there are $\alpha_i > 0$ such that

$$\sum_{i=1}^{3} \alpha_i s_i(\theta_i^2 + \nu\mu) \ = \ 0 \ , \text{ and } \sum_{i=1}^{3} \alpha_i s_i\theta_i \ = \ 0 \ ,$$

which is equivalent to

$$\alpha_1 s_1(\theta_3 - \theta_1)(\theta_1\theta_3 - \nu\mu) + \alpha_2 s_2(\theta_3 - \theta_2)(\theta_2\theta_3 - \nu\mu) = 0,$$
$$\alpha_3 s_3(\theta_3 - \theta_1)(\theta_1\theta_3 - \nu\mu) + \alpha_2 s_2(\theta_2 - \theta_1)(\theta_1\theta_2 - \nu\mu) = 0.$$

So it follows that if $v\mu < \theta_1\theta_2$ or $v\mu > \theta_2\theta_3$, then $s_1 = -s_2 = s_3$, and the optimal polynomial is uniquely determined giving optimal value

$$\frac{(\theta_3 - \theta_2)(\theta_2 - \theta_1)}{|\,2(\theta_1\theta_3 - v\mu) + (\theta_3 - \theta_2)(\theta_2 - \theta_1)\,|}\ .$$

Similarly we find that if $\theta_1\theta_2 < v\mu < \theta_1\theta_3$, then $-s_1 = s_2 = s_3$ and if $\theta_1\theta_3 < v\mu < \theta_2\theta_3$, then $s_1 = s_2 = -s_3$, giving similar expressions as above for the optimal value. We see that the optimal value is a continuous function of μ. Thus we find the "optimal" bound.

If for every vertex the bound is tight, then it follows (similarly as before) that $J - (v - n_3)(p_2(Q^2 - \mu J) + p_1Q + p_0I)$ is the adjacency matrix of G_μ and that this graph is a strongly regular $(v, n_3, \lambda, \lambda)$ graph. Moreover, if G is regular, then we have to prove that we have a 3-class association scheme. To show this, suppose that G is regular with degree k and adjacency matrix A. Furthermore, let A_3 be the adjacency matrix of G_μ, and $A_2 = J - I - A - A_3$ be the adjacency matrix of $G_{\neg\mu}$. As $Q = kI - A$, it follows that $A_3, A_2 \in \langle A^2, A, I, J \rangle$, the adjacency algebra $\mathbf{A}$ of G. Since G is regular with four eigenvalues, it follows that $A^3 \in \mathbf{A}$. This implies that $\langle A_3, A_2, A, I \rangle = \mathbf{A}$, and so G, G_μ and $G_{\neg\mu}$ form a 3-class association scheme.

On the other hand, if G is a graph with four eigenvalues such that G, G_μ and $G_{\neg\mu}$ form a 3-class association scheme and G_μ is a strongly regular $(v, n_3, \lambda, \lambda)$ graph then the bound is tight for every vertex. The proof is similar to the situation in the previous section. Here we have to show that the bound is tight for some polynomial $p(z) = p_2z^2 + p_1z + p_0$ such that $p(0) = 1 + p_2v\mu$. Now there are q_2, q_1 and q_0 such that $(J - A_3)/(v - n_3) = q_2(A^2 - \mu J) + q_1A + q_0I$. If we now take $q(z) = q_2z^2 + q_1z + q_0$, then it follows by taking row sums in the matrix equation that $q(k) = 1 + q_2v\mu$, and by taking $p(z) = q(k - z)$, we find the required polynomial (note that $p_2 = q_2$). It gives a tight bound, which is proven similarly as in the proof of Theorem 5.3.3. □

Examples of graphs for which the bound is tight, and $\mu \neq 0$, are given by the line graph of the Petersen graph ($\mu = 1$, $n_3 = 8$), the Johnson graph $J(7, 3)$ ($\mu = 4$, $n_3 = 18$), the distance two graph of the generalized hexagon $GH(q, q)$ ($\mu = q^3 + q^2 - q - 1$, $n_3 = q^5$) and several graphs in the association schemes that are obtained by Hoffman colorings in strongly regular $(v, n_3, \lambda, \lambda)$ graphs.

The bound does in general not prove the conjecture. For example, suppose we have a regular graph with spectrum $\{[5]^1, [\sqrt{5}]^7, [-1]^5, [-\sqrt{5}]^7\}$. After rounding the numbers, the bound gives $n_3 \leq 2, 15, 3, 1, 0, 0$ for $\mu = 0, 1, 2, 3, 4, 5$, respectively. The conjectured bounds, however, are 2, 14, 2, 0, 0, 0, respectively. There is precisely one graph with the given spectrum, a 2-cover of $C_5 \circledast J_2$ (cf. Section 3.3.7), for which every vertex has $n_3 = 1, 12, 1, 0, 0, 0$, respectively.

5.3.3. Equally large sets at maximal distance

As a last illustration of Theorem 5.1.1 we derive bounds on the sizes of two equally large sets at maximal distance, and distance at least two.

PROPOSITION 5.3.6. *Let G be a connected graph on v vertices with $r+1$ distinct Laplace eigenvalues* $0=\theta_0<\theta_1<\ldots<\theta_r$. *Let X_1 and X_2 be sets of vertices of size κ, such that the distance between any vertex of X_1 and any vertex of X_2 is r, then*

$$\kappa \le \frac{v}{1+\gamma}\ , \ \textit{where}\ \gamma = \sum_{j\neq 0}\prod_{i\neq 0,j}\frac{\theta_i}{|\theta_j-\theta_i|}\ .$$

If the bound is tight then again we must have tight interlacing in Theorem 5.1.1, and so the partition of M is regular. It now follows that the partition of $p(Q)$ induced by the partition of the vertices into X_1, X_2 and the set of remaining vertices is regular with quotient matrix

$$\begin{pmatrix} \frac{\kappa}{v-\kappa} & 1-\frac{\kappa}{v-\kappa} & 0 \\ \frac{\kappa}{v-\kappa} & 1-\frac{2\kappa}{v-\kappa} & \frac{\kappa}{v-\kappa} \\ 0 & 1-\frac{\kappa}{v-\kappa} & \frac{\kappa}{v-\kappa} \end{pmatrix}.$$

Consider the connected regular graphs with four eigenvalues. Let G be a 2-antipodal distance-regular graph with diameter three, so that it has eigenvalues $k>\lambda_1>\lambda_2=-1>\lambda_3$, with $\lambda_1\lambda_3=-k$, then $G\circledast J_n$ is a connected regular graph with four eigenvalues. For such graphs Proposition 5.3.6 gives $\kappa\le n$, and it is easy to find vertex sets for which this bound is tight. Checking the list of feasible parameter sets in Appendix A.3, it follows that the only other examples of regular graphs with four eigenvalues on at most 30 vertices, for which the bound is tight, are given by the four incidence graphs of 2-(15, 8, 4) designs, which all have a tight bound $\kappa\le 3$. The problem of finding two sets of size three at distance three is equivalent to finding three points all of which are incident with three blocks in the corresponding complementary 2-(15, 7, 3) designs.

Another example is given by the Hamming graph $H(d,q)$, which has Laplace eigenvalues jq, $j=0,\ldots,d$. Here we find that $1+\gamma=2^d$, so $\kappa\le(\frac{1}{2}q)^d$. For q even, the bound is tight: split the alphabet into two equally large parts Q_1 and Q_2, and take as vertex sets the set of words with letters in Q_1, and the set of words with letters in Q_2.

If we have only three Laplace eigenvalues then Proposition 5.3.6 provides a bound on the size of two equally large disconnected vertex sets, that is, two sets with no edges in between. We also find such a bound in case we have more Laplace eigenvalues.

PROPOSITION 5.3.7. *Let G be a connected graph on v vertices with $r + 1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < ... < \theta_r$. Let X_1 and X_2 be two disconnected vertex sets of size κ', then $\kappa' \le \frac{1}{2}v(1 - \theta_1/\theta_r)$.*

Proof. Use the first degree polynomial $p(z) = 1 - 2z/(\theta_1 + \theta_r)$. □

This method was used by Haemers [58] to find a bound due to Helmberg, Mohar, Poljak and Rendl [64] on the bandwidth of a graph.

Next, we consider the case that the bound on κ' is tight. Then the Laplace matrix Q is regularly partitioned with quotient matrix

$$\begin{pmatrix} \theta_1 & -\theta_1 & 0 \\ \frac{1}{2}(\theta_1 - \theta_r) & \theta_r - \theta_1 & \frac{1}{2}(\theta_1 - \theta_r) \\ 0 & -\theta_1 & \theta_1 \end{pmatrix}.$$

Thus a necessary condition for tightness is that $\theta_r - \theta_1$ is an even integer.

Families of (strongly regular) graphs for which we have a tight bound are given by the complete multipartite graphs $K_{m \times n}$ for even n, with $\kappa \le \frac{1}{2}n$, the triangular graphs $T(n)$ for even n, with $\kappa \le \binom{\frac{1}{2}n}{2}$, and the lattice graphs $L_2(n)$ for even n, with $\kappa \le (\frac{1}{2}n)^2$. Checking the list of feasible parameter sets in Appendix A.2, it follows that besides the mentioned graphs, the only connected graphs with three Laplace eigenvalues on at most 27 vertices for which the bound can be tight are the graphs obtained from polarities in 2-(15, 8, 4), 2-(16, 6, 2) and 2-(21, 5, 1) designs. For example, the matrices given by

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{D_1}} & I & I & I & P & O & O \\ & & I & I & P & I & O & O \\ I & I & \multicolumn{2}{c}{\multirow{2}{*}{D_2}} & O & O & I & P \\ I & I & & & O & O & P & I \\ I & P & O & O & \multicolumn{2}{c}{\multirow{2}{*}{D_3}} & I & I \\ P & I & O & O & & & I & I \\ O & O & I & P & I & I & \multicolumn{2}{c}{\multirow{2}{*}{D_4}} \\ O & O & P & I & I & I & & \end{pmatrix}, \text{ with } D_i \in \left\{ \begin{pmatrix} O & J \\ J & O \end{pmatrix}, \begin{pmatrix} J & O \\ O & J \end{pmatrix} \right\},$$

where

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

are incidence matrices of 2-(16, 6, 2) designs with a polarity, and we obtain graphs with Laplace spectrum $\{[8]^m, [4]^{15-m}, [0]^1\}$ for $m = 5, 6, 7, 8$, and 9. For these graphs we have $\kappa \leq 4$, and the bound is tight, as we can see from the matrices. The regular graphs in this example are the Clebsch graph and the lattice graph $L_2(4)$. The only other regular graph obtained from a 2-(16, 6, 2) design with a polarity is the Shrikhande graph, and also here the bound is tight. The triangular graph $T(6)$ is an (the only regular) example obtained from a 2-(15, 8, 4) design with a polarity, and it has tight bound $\kappa \leq 3$. Furthermore, there are precisely two graphs that can be obtained from a polarity in the 2-(21, 5, 1) design (the projective plane of order 4), and for both graphs the bound $\kappa \leq 6$ is tight.

Besides the graphs we already mentioned, there are only two other strongly regular graphs on at most 35 vertices for which the bound is tight: these are two of the three Chang graphs. These graphs have the same spectrum as and are obtained from switching in the triangular graph $T(8)$. The one that is obtained from switching with respect to a 4-coclique and the one that is obtained from switching with respect to an 8-cycle have a tight bound, the one that is obtained from switching with respect to the union of a 3-cycle and a 5-cycle not.

Appendices

A.2. Graphs with three Laplace eigenvalues

By computer we generated all feasible parameter sets for graphs on v vertices with constant μ and $\overline{\mu}$, having restricted Laplace eigenvalues $\theta_1 > \theta_2$ and vertex degrees $k_1 \geq k_2$, for $v \leq 40$, satisfying $0 < \mu \leq \overline{\mu}$. If $\lambda_{22} < 0$, then the condition $n_2 \leq v(\theta_1 - k_2)/\theta_1$ is satisfied. By # we denote the number of nonregular graphs. If there are any strongly regular graphs, then their number is denoted in between brackets. By `Bruck-Ryser(p)` we denote that the Bruck-Ryser condition is not satisfied modulo p.

v	μ	$\overline{\mu}$	θ_1	θ_2	k_1	k_2	n_1	n_2	λ_{22}	#	Notes	Subsection 2.2.*
5	1	1	3.6180	1.3820	2	2			0	× (1)	C_5, $G(3,1,0)$	5
7	1	2	4.4142	1.5858	3	2	4	3	-1	1	$G(4,1,0)$, $P(7,3,1)$	3.*a*,*d*, 4, 5
9	1	3	5.3028	1.6972	4	2	5	4	-2	1	$G(5,1,0)$	3.*a*, 5
9	2	2	6	3	4	4			1	× (1)	$L_2(3)$	
10	1	4	5	2	3	3			0	× (1)	Petersen	
11	1	4	6.2361	1.7639	5	2	6	5	-3	1	$G(6,1,0)$	3.*a*, 5
11	2	3	6.7321	3.2679	5	4	6	5	0	1	$P(11,5,2)$	4
13	1	5	7.1926	1.8074	6	2	7	6	-4	1	$G(7,1,0)$	3.*a*, 5
13	1	6	5.7321	2.2679	4	3	9	4	-1	1	$P(13,4,1)$	3.*c*,*d*, 4
13	2	4	7.5616	3.4384	6	4	7	6	-1	1	$G(7,3,1)$	5
13	3	3	8.3028	4.6972	6	6			2	× (1)	$P(13)$	
15	1	6	8.1623	1.8377	7	2	8	7	-5	1	$G(8,1,0)$	3.*a*, 5
15	2	5	8.4495	3.5505	7	4	8	7	-2	0	$G(D)$	5
15	3	4	9	5	7	6			1	≥ 3 (1)	$P(15,7,3)$ ($\overline{T(6)}$)	4
16	2	6	8	4	6	5			0	≥ 4 (3)	$P(16,6,2)$ (Clebsch, $L_2(4)$, Shrikhande)	4
17	1	7	9.1401	1.8599	8	2	9	8	-6	1	$G(9,1,0)$	3.*a*, 5
17	2	6	9.3723	3.6277	8	4	9	8	-3	0	Bruck-Ryser(3), $G(D)$	1, 5
17	3	5	9.7913	5.2087	8	6	9	8	0	0	Bruck-Ryser(7)	1
17	4	4	10.5616	6.4384	8	8			3	× (1)	$P(17)$	
19	1	8	10.1231	1.8769	9	2	10	9	-7	1	$G(10,1,0)$	3.*a*, 5
19	1	10	7.4495	2.5505	6	3	11	8	-3	0	Bruck-Ryser(3)	1, 3
19	2	7	10.3166	3.6834	9	4	10	9	-4	0	$G(D)$	5
19	4	5	11.2361	6.7639	9	8	10	9	2	≥ 1	$P(19,9,4)$	4
21	1	9	11.1098	1.8902	10	2	11	10	-8	1	$G(11,1,0)$	3.*a*, 5
21	1	12	7	3	5	4			-1	2	$P(21,5,1)$	3.*b*,*d*, 4
21	2	8	11.2749	3.7251	10	4	11	10	-5	0	Bruck-Ryser(3), $G(D)$	1, 5
21	3	7	11.5414	5.4586	10	6	11	10	-2	1	$G(11,5,2)$	5
21	4	6	12	7	10	8			1	≥ 1 (1)	$T(7)$, switched $T(7)$	6
21	5	5	12.7913	8.2087	10	10			4	× (0)	Bruck-Ryser(3)	1
22	3	8	11	6	9	7			0	≥ 2	switched $T(7)$	6
23	1	10	12.0990	1.9010	11	2	12	11	-9	1	$G(12,1,0)$	3.*a*, 5
23	2	9	12.2426	3.7574	11	4	12	11	-6	0	$G(D)$	5
23	3	8	12.4641	5.5359	11	6	12	11	-3	0	$G(D)$	5
23	4	7	12.8284	7.1716	11	8	12	11	0	?		
23	5	6	13.4495	8.5505	11	10	12	11	3	≥ 1	$P(23,11,5)$	4
25	1	11	13.0902	1.9098	12	2	13	12	-10	1	$G(13,1,0)$	3.*a*, 5
25	1	15	7.7913	3.2087	6	4	16	9	-2	1		3.*c*
25	2	10	13.2170	3.7830	12	4	13	12	-7	0	$G(D)$	5
25	2	12	10	5	8	6			-1	? (1)	$L_2(5)$	2
25	3	9	13.4051	5.5949	12	6	13	12	-4	1	$G(13,4,1)$	5

v	μ	$\overline{\mu}$	θ_1	θ_2	k_1	k_2	n_1	n_2	λ_{22}	#	Notes	Subsection 2.2.*
25	3	10	11.4495	6.5505	9	8	16	9	1	≥ 1	$P(25,9,3)$	4
25	5	7	14.1926	8.8074	12	10	13	12	2	?		
25	6	6	15	10	12	12			5	× (15)	$L_3(5)$	
26	4	9	13	8	10	10			3	× (10)		
27	1	12	14.0828	1.9172	13	2	14	13	-11	1	$G(14,1,0)$	3.*a*, 5
27	2	11	14.1962	3.8038	13	4	14	13	-8	0	$G(D)$	5
27	3	10	14.3589	5.6411	13	6	14	13	-5	0	$G(D)$	5
27	5	8	15	9	13	10			1	? (1)	Schläfli	
27	6	7	15.6458	10.3542	13	12	14	13	4	≥ 1	$P(27,13,6)$	4
28	4	10	14	8	12	9			0	? (4)	$T(8)$, Chang	
29	1	13	15.0765	1.9235	14	2	15	14	-12	1	$G(15,1,0)$	3.*a*, 5
29	2	12	15.1789	3.8211	14	4	15	14	-9	0	Bruck-Ryser(3), $G(D)$	1, 5
29	2	15	10.4495	5.5505	8	7	21	8	0	0	Bruck-Ryser(3), $P(D)$	1, 4
29	3	11	15.3218	5.6782	14	6	15	14	-6	0	Bruck-Ryser(31), $G(D)$	1, 5
29	4	10	15.5311	7.4689	14	8	15	14	-3	10	$G(15,7,3)$	5
29	5	9	15.8541	9.1459	14	10	15	14	0	?		
29	6	8	16.3723	10.6277	14	12	15	14	3	0	Bruck-Ryser(11)	1
29	7	7	17.1926	11.8074	14	14			6	× (41)	$P(29)$	
31	1	14	16.0711	1.9289	15	2	16	15	-13	1	$G(16,1,0)$	3.*a*, 5
31	1	20	8.2361	3.7639	6	5	25	6	-1	1	$P(31,6,1)$	3.*d*, 4
31	2	13	16.1644	3.8356	15	4	16	15	-10	0	$G(D)$	5
31	3	12	16.2915	5.7085	15	6	16	15	-7	0	$G(D)$	5
31	3	14	12.6458	7.3542	10	9	21	10	1	≥ 1	$P(31,10,3)$	4
31	4	11	16.4721	7.5279	15	8	16	15	-4	3	$G(16,6,2)$	5
31	6	9	17.1623	10.8377	15	12	16	15	2	?		
31	7	8	17.8284	12.1716	15	14	16	15	5	≥ 1	$P(31,15,7)$	4
33	1	15	17.0664	1.9336	16	2	17	16	-14	1	$G(17,1,0)$	3.*a*, 5
33	1	21	9.5414	3.4586	8	4	19	14	-4	0		3
33	2	14	17.1521	3.8479	16	4	17	16	-11	0	Bruck-Ryser(3), $G(D)$	1, 5
33	3	13	17.2663	5.7337	16	6	17	16	-8	0	Bruck-Ryser(7), $G(D)$	1, 5
33	4	12	17.4244	7.5756	16	8	17	16	-5	0	$G(D)$	5
33	6	10	18	11	16	12			1	?		
33	7	9	18.5414	12.4586	16	14	17	16	4	?		
33	8	8	19.3723	13.6277	16	16			7	× (0)	Bruck-Ryser(3)	1
34	5	12	17	10	15	11			0	?		
35	1	16	18.0623	1.9377	17	2	18	17	-15	1	$G(18,1,0)$	3.*a*, 5
35	2	15	18.1414	3.8586	17	4	18	17	-12	0	$G(D)$	5
35	3	14	18.2450	5.7550	17	6	18	17	-9	0	$G(D)$	5
35	4	13	18.3852	7.6148	17	8	18	17	-6	0	$G(D)$	5
35	6	11	18.8730	11.1270	17	12	18	17	0	?		
35	7	10	19.3166	12.6834	17	14	18	17	3	?		
35	8	9	20	14	17	16			6	≥ 5 (≥ 3854)	$P(35,17,8)$	4
36	1	24	9	4	7	5			-2	1		2, 3.*b*
36	2	20	12	6	10	7			-2	? (1)	$L_2(6)$	
36	4	15	16	9	14	10			-1	? (1)	$T(9)$	
36	6	12	18	12	15	14			4	≥ 5 (≥ 32728)	$P(36,15,6)$ ($L_3(6)$)	4
37	1	17	19.0586	1.9414	18	2	19	18	-16	1	$G(19,1,0)$	3.*a*, 5
37	2	16	19.1322	3.8678	18	4	19	18	-13	0	$G(D)$	5
37	2	20	13.5311	5.4689	12	6	20	17	-5	0	Bruck-Ryser(5)	1
37	2	21	11.6458	6.3542	9	8	28	9	0	≥ 1	$P(37,9,2)$	4
37	3	15	19.2268	5.7732	18	6	19	18	-10	0	$G(D)$	5
37	4	14	19.3523	7.6477	18	8	19	18	-7	0	$G(D)$	5
37	5	13	19.5249	9.4751	18	10	19	18	-4	22	$G(19,9,4)$	5
37	5	14	17.3166	10.6834	15	12	20	17	1	?		
37	7	11	20.1401	12.8599	18	14	19	18	2	?		
37	8	10	20.7016	14.2984	18	16	19	18	5	?		
37	9	9	21.5414	15.4586	18	18			8	× (≥ 82)	$P(37)$	
39	1	18	20.0554	1.9446	19	2	20	19	-17	1	$G(20,1,0)$	3.*a*, 5
39	2	17	20.1240	3.8760	19	4	20	19	-14	0	$G(D)$	5
39	3	16	20.2111	5.7889	19	6	20	19	-11	0	$G(D)$	5
39	4	15	20.3246	7.6754	19	8	20	19	-8	0	$G(D)$	5
39	5	14	20.4772	9.5228	19	10	20	19	-5	0	$G(D)$	5
39	7	12	21	13	19	14			1	?		
39	8	11	21.4641	14.5359	19	16	20	19	4	?		
39	9	10	22.1623	15.8377	19	18	20	19	7	≥ 1	$P(39,19,9)$	4
40	3	20	15	8	13	9			-2	?		
40	4	18	16	10	13	12			2	≥ 5 (≥ 27)	$P(40,13,4)$	4
40	6	14	20	12	18	13			0	?		

A.3. Regular graphs with four eigenvalues

In this Appendix we list all feasible spectra for connected regular graphs with four eigenvalues and at most 30 vertices. If both the spectrum and its complementary spectrum correspond to connected graphs then only the one with least degree is mentioned. # denotes the number of graphs. In between brackets the number of such graphs or their complements that are a relation in a three-class association scheme is denoted (if positive). These numbers are obtained from Appendix A.4. The references refer to the subsections of Chapter 3 or the literature. For more on the computer results, see [40]. McKay and Royle [82] determined all vertex-transitive graphs with at most 26 vertices. Godsil [private communication] ran a program to extract the ones with four eigenvalues and found five more graphs: two with spectrum $\{[9]^1, [3]^7, [-1]^9, [-3]^7\}$ and three with spectrum $\{[11]^1, [3]^7, [-1]^8, [-3]^8\}$.

A.3.1. Four integral eigenvalues

Nr	v	spectrum	Δ	Ξ	#	Notes	References
1	6	$\{[2]^1, [1]^2, [-1]^2, [-2]^1\}$	0	0	1 (1)	C_6	3.2
2	8	$\{[5]^1, [1]^2, [-1]^4, [-3]^1\}$	6	22	1 (1)	$\overline{G} = 2C_4$	3.3
3	8	$\{[3]^1, [1]^3, [-1]^3, [-3]^1\}$	0	3	1 (1)	Cube	3.2
4	10	$\{[4]^1, [1]^4, [-1]^4, [-4]^1\}$	0	12	1 (1)	$IG(5,4,3)$	3.2
5	12	$\{[9]^1, [1]^3, [-1]^6, [-3]^2\}$	28	204	1 (1)	$\overline{G} = 3C_4$	3.3
6	12	$\{[8]^1, [2]^2, [-1]^8, [-4]^1\}$	19	123	1 (1)	$\overline{G} = 2K_{3,3}$	3.3
7	12	$\{[4]^1, [2]^3, [0]^3, [-2]^5\}$	2	2	2	L(Cube), BCS_9	3.5
8	12	$\{[7]^1, [1]^4, [-1]^6, [-5]^1\}$	9	81	1 (1)	$\overline{G} = 2CP(3)$	3.3
9	12	$\{[5]^1, [1]^5, [-1]^5, [-5]^1\}$	0	30	1 (1)	$IG(6,5,4)$	3.2
10	12	$\{[5]^1, [1]^3, [0]^6, [-4]^2\}$	0	25	0	$\lambda_1 = 1$	3.5
11	12	$\{[5]^1, [1]^6, [-1]^2, [-3]^3\}$	2	14	1	$GQ(2,2)$\3-cl, $\overline{L(CP(3))}$	3.5, 3.6
12	12	$\{[5]^1, [3]^2, [-1]^8, [-3]^1\}$	6	14	1	$C_6 \circledast J_2$	3.4
13	12	$\{[5]^1, [2]^2, [1]^3, [-2]^6\}$	4	9	1 (1)	$L(K_{3,4})$	3.5
14	14	$\{[6]^1, [1]^6, [-1]^6, [-6]^1\}$	0	60	1 (1)	$IG(7,6,5)$	3.2
15	15	$\{[4]^1, [2]^5, [-1]^4, [-2]^5\}$	2	0	1 (1)	L(Petersen)	3.1, 3.5
16	15	$\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$	4	4	0	$\lambda_3 = -2$	3.5
17	15	$\{[6]^1, [1]^6, [0]^5, [-4]^3\}$	1	36	0	$\lambda_1 = 1$	3.5
18	15	$\{[6]^1, [3]^2, [1]^4, [-2]^8\}$	7	20	1 (1)	$L(K_{3,5})$	3.5
19	16	$\{[13]^1, [1]^4, [-1]^8, [-3]^3\}$	66	738	1 (1)	$\overline{G} = 4C_4$	3.3
20	16	$\{[11]^1, [3]^2, [-1]^{12}, [-5]^1\}$	39	367	1 (1)	$\overline{G} = 2K_{4,4}$	3.3
21	16	$\{[6]^1, [4]^2, [0]^6, [-2]^7\}$	9	27	0	$\lambda_3 = -2$	3.5
22	16	$\{[9]^1, [1]^6, [-1]^8, [-7]^1\}$	12	204	1 (1)	$\overline{G} = 2CP(4)$	3.3
23	16	$\{[7]^1, [1]^7, [-1]^7, [-7]^1\}$	0	105	1 (1)	$IG(8,7,6)$	3.2
24	16	$\{[7]^1, [1]^8, [-1]^5, [-5]^2\}$	3	69	0	$\lambda_1 = 1$	3.5
25	16	$\{[7]^1, [3]^3, [-1]^{11}, [-5]^1\}$	9	57	1	Cube$\circledast J_2$	3.4
26	18	$\{[14]^1, [2]^3, [-1]^{12}, [-4]^2\}$	73	894	1 (1)	$\overline{G} = 3K_{3,3}$	3.3
27	18	$\{[13]^1, [1]^6, [-1]^9, [-5]^2\}$	54	666	1 (1)	$\overline{G} = 3CP(3)$	3.3
28	18	$\{[13]^1, [1]^8, [-2]^8, [-5]^1\}$	56	652	1 (1)	$\overline{G} = 2L_2(3)$	3.3
29	18	$\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$	1	12	1	$K_{3,3} \oplus K_3$	3.4
30	18	$\{[5]^1, [2]^7, [-1]^1, [-2]^9\}$	3	2	0	$\lambda_3 = -2$	3.5
31	18	$\{[11]^1, [2]^4, [-1]^{12}, [-7]^1\}$	28	360	1 (1)	$\overline{G} = 2K_{3,3,3}$	3.3
32	18	$\{[6]^1, [3]^4, [0]^4, [-2]^9\}$	7	16	1	$L(L_2(3))$	3.5
33	18	$\{[7]^1, [1]^{11}, [-2]^4, [-5]^2\}$	2	58	1	$\overline{BCS_{179}}$	3.5
34	18	$\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$	11	40	2 (1)	$L(K_{3,6})$, BCS_{70}	3.5
35	18	$\{[8]^1, [1]^8, [-1]^8, [-8]^1\}$	0	168	1 (1)	$IG(9,8,7)$	3.2
36	18	$\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$	12	68	0		2.2
37	18	$\{[8]^1, [2]^6, [-1]^8, [-4]^3\}$	10	78	2		computer
38	18	$\{[8]^1, [5]^2, [-1]^{14}, [-4]^1\}$	19	96	1	$C_6 \circledast J_3$	3.4
39	18	$\{[8]^1, [2]^4, [0]^9, [-4]^4\}$	8	84	3 (1)	$L_2(3) \otimes J_2$	3.4, computer

40	18	$\{[8]^1, [4]^3, [-1]^8, [-2]^6\}$	18	78	0		$\lambda_3 = -2$	3.5
41	20	$\{[17]^1, [1]^5, [-1]^{10}, [-3]^4\}$	120	1816	1	(1)	$\overline{G} = 5C_4$	3.3
42	20	$\{[16]^1, [1]^8, [-2]^{10}, [-4]^1\}$	99	1401	1	(1)	$\overline{G} = 2\text{Petersen}$	3.3
43	20	$\{[14]^1, [4]^2, [-1]^{16}, [-6]^1\}$	66	817	1	(1)	$\overline{G} = 2K_{5,5}$	3.3
44	20	$\{[6]^1, [2]^5, [0]^{10}, [-4]^4\}$	0	27	2	(1)	$\text{Petersen}\otimes J_2$	3.4, computer
45	20	$\{[6]^1, [3]^4, [1]^4, [-2]^{11}\}$	6	12	1		$L(IG(5,4,3))$	3.5
46	20	$\{[13]^1, [1]^{10}, [-2]^8, [-7]^1\}$	45	615	1	(1)	$\overline{G} = 2\overline{\text{Petersen}}$	3.3
47	20	$\{[7]^1, [2]^4, [0]^{12}, [-5]^3\}$	0	63	0			computer
48	20	$\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$	6	30	9		SR(26,10,3,4)\6-cocl. Dodecahedron$_{3,5}$	3.1, 3.6, computer
49	20	$\{[7]^1, [3]^5, [-1]^{10}, [-3]^4\}$	9	33	4	(1)	$\text{Petersen}\circledast J_2$	3.4, computer
50	20	$\{[7]^1, [3]^3, [2]^4, [-2]^{12}\}$	9	27	1	(1)	$L(K_{4,5})$	3.5
51	20	$\{[11]^1, [1]^8, [-1]^{10}, [-9]^1\}$	15	415	1	(1)	$\overline{G} = 2\text{CP}(5)$	3.3
52	20	$\{[9]^1, [1]^9, [-1]^9, [-9]^1\}$	0	252	1	(1)	$IG(10,9,8)$	3.2
53	20	$\{[9]^1, [3]^4, [-1]^{14}, [-7]^1\}$	12	156	1		$IG(5,4,3)\circledast J_2$	3.4
54	20	$\{[9]^1, [2]^8, [-1]^4, [-3]^7\}$	15	105	26		$L_3(5)$\5-coclique	3.6, computer
55	20	$\{[9]^1, [3]^5, [-1]^9, [-3]^5\}$	18	108	9	(1)	$J(6,3)$	3.1, [61]
56	21	$\{[6]^1, [3]^5, [-1]^{13}, [-4]^2\}$	5	20	0			4
57	21	$\{[8]^1, [5]^2, [1]^6, [-2]^{12}\}$	16	72	1	(1)	$L(K_{3,7})$	3.5
58	22	$\{[10]^1, [1]^{10}, [-1]^{10}, [-10]^1\}$	0	360	1	(1)	$IG(11,10,9)$	3.2
59	24	$\{[21]^1, [1]^6, [-1]^{12}, [-3]^5\}$	190	3630	1	(1)	$\overline{G} = 6C_4$	3.3
60	24	$\{[20]^1, [2]^4, [-1]^{16}, [-4]^3\}$	163	2961	1	(1)	$\overline{G} = 4K_{3,3}$	3.3
61	24	$\{[19]^1, [3]^3, [-1]^{18}, [-5]^2\}$	139	2395	1	(1)	$\overline{G} = 3K_{4,4}$	3.3
62	24	$\{[19]^1, [1]^8, [-1]^{12}, [-5]^3\}$	135	2403	1	(1)	$\overline{G} = 4\text{CP}(3)$	3.3
63	24	$\{[5]^1, [3]^6, [-1]^{14}, [-3]^3\}$	4	6	1		2-cover $C_6\circledast J_2$	3.7, computer
64	24	$\{[5]^1, [2]^8, [0]^8, [-3]^7\}$	0	5	0			computer
65	24	$\{[17]^1, [5]^2, [-1]^{20}, [-7]^1\}$	100	1536	1	(1)	$\overline{G} = 2K_{6,6}$	3.3
66	24	$\{[6]^1, [4]^4, [0]^8, [-2]^{11}\}$	8	19	0		$\lambda_3 = -2$	3.5
67	24	$\{[17]^1, [1]^9, [-1]^{12}, [-7]^2\}$	88	1560	1	(1)	$\overline{G} = 3\text{CP}(4)$	3.3
68	24	$\{[7]^1, [3]^6, [-1]^{15}, [-5]^2\}$	5	41	0			computer
69	24	$\{[7]^1, [3]^3, [1]^{11}, [-3]^9\}$	4	25	5			computer
70	24	$\{[8]^1, [2]^{11}, [-2]^9, [-4]^3\}$	7	48	5			computer
71	24	$\{[8]^1, [2]^8, [0]^9, [-4]^6\}$	4	60	4			computer
72	24	$\{[8]^1, [4]^3, [0]^{15}, [-4]^5\}$	8	68	5		$L(\text{Cube})\otimes J_2$, $\text{BCS}_9\otimes J_2$	3.4, computer
73	24	$\{[15]^1, [3]^4, [-1]^{18}, [-9]^1\}$	57	981	1	(1)	$\overline{G} = 2K_{4,4,4}$	3.3
74	24	$\{[8]^1, [4]^3, [2]^5, [-2]^{15}\}$	13	48	1	(1)	$L(K_{4,6})$	3.5
75	24	$\{[9]^1, [1]^{17}, [-3]^2, [-5]^4\}$	4	116	1		$GQ(2,4)$\3-cl, $\overline{\text{BCS}_{183}}$	3.5, 3.6
76	24	$\{[9]^1, [3]^7, [-1]^9, [-3]^7\}$	15	84	≥ 5			computer
77	24	$\{[14]^1, [2]^6, [-1]^{16}, [-10]^1\}$	37	822	1	(1)	$\overline{G} = 2K_{3,3,3,3}$	3.3
78	24	$\{[9]^1, [3]^4, [1]^9, [-3]^{10}\}$	12	84	87			computer
79	24	$\{[9]^1, [6]^2, [1]^7, [-2]^{14}\}$	22	119	1	(1)	$L(K_{3,8})$	3.5
80	24	$\{[10]^1, [2]^3, [0]^{18}, [-8]^2\}$	0	285	0			4
81	24	$\{[10]^1, [1]^{16}, [-2]^4, [-6]^3\}$	7	196	0		$\lambda_1 = 1$	3.5
82	24	$\{[10]^1, [4]^2, [0]^{18}, [-6]^3\}$	10	205	0			4
83	24	$\{[10]^1, [2]^{11}, [-2]^8, [-4]^4\}$	16	141	183			computer
84	24	$\{[10]^1, [4]^5, [0]^3, [-2]^{15}\}$	25	145	0		$\lambda_3 = -2$	3.5
85	24	$\{[13]^1, [1]^{10}, [-1]^{12}, [-11]^1\}$	18	738	1	(1)	$\overline{G} = 2\text{CP}(6)$	3.3
86	24	$\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$	24	141	9		$L(\text{CP}(4))$, $\text{BCS}_{153\text{-}160}$	3.5
87	24	$\{[11]^1, [1]^{11}, [-1]^{11}, [-11]^1\}$	0	495	1	(1)	$IG(12,11,10)$	3.2
88	24	$\{[11]^1, [3]^5, [-1]^{17}, [-9]^1\}$	15	335	1		$IG(6,5,4)\circledast J_2$	3.4
89	24	$\{[11]^1, [5]^3, [-1]^{19}, [-7]^1\}$	28	279	1		$\text{Cube}\circledast J_3$	3.4
90	24	$\{[11]^1, [1]^{16}, [-1]^2, [-5]^5\}$	15	255	0		$\lambda_1 = 1$	3.5
91	24	$\{[11]^1, [3]^6, [-1]^{14}, [-5]^3\}$	23	239	28		$(GQ(2,2)\text{\3-clique})\circledast J_2$	3.4, computer
92	24	$\{[11]^1, [7]^2, [-1]^{20}, [-5]^1\}$	39	303	1		$C_6\circledast J_4$	3.4
93	24	$\{[11]^1, [3]^7, [-1]^8, [-3]^8\}$	27	215	≥ 16			computer
94	25	$\{[10]^1, [5]^2, [0]^{18}, [-5]^4\}$	15	180	0			4
95	26	$\{[12]^1, [1]^{12}, [-1]^{12}, [-12]^1\}$	0	660	1	(1)	$IG(13,12,11)$	3.2
96	27	$\{[22]^1, [1]^{12}, [-2]^{12}, [-5]^2\}$	191	3892	1	(1)	$\overline{G} = 3L_2(3)$	3.3
97	27	$\{[6]^1, [3]^6, [0]^{12}, [-3]^8\}$	3	12	4	(1)	$H(3,3)$, 3-cover $C_3\otimes J_3$	3.1, 3.7, [61]
98	27	$\{[20]^1, [2]^6, [-1]^{18}, [-7]^2\}$	136	2664	1	(1)	$\overline{G} = 3K_{3,3,3}$	3.3
99	27	$\{[8]^1, [2]^{12}, [-1]^8, [-4]^6\}$	4	48	13	(3)	$GQ(2,4)$\spread (2×) $H(3,3)_3$, $GQ(3,3)_2(x)$	3.1, 3.6, [61]
100	27	$\{[8]^1, [5]^4, [-1]^{20}, [-4]^2\}$	16	72	1			computer
101	27	$\{[10]^1, [4]^6, [1]^2, [-2]^{18}\}$	23	124	1		$L(K_{3,3,3})$	3.5
102	27	$\{[10]^1, [7]^2, [1]^8, [-2]^{16}\}$	29	184	1	(1)	$L(K_{3,9})$	3.5
103	27	$\{[12]^1, [3]^2, [0]^{22}, [-9]^2\}$	6	492	0			4
104	27	$\{[12]^1, [3]^4, [0]^{18}, [-6]^4\}$	18	348	5	(1)	$L_2(3)\otimes J_3$	3.4, computer
105	27	$\{[12]^1, [3]^8, [0]^6, [-3]^{12}\}$	30	276	≥ 1	(1)	$H(3,3)_2$	3.1
106	28	$\{[25]^1, [1]^7, [-1]^{14}, [-3]^6\}$	276	6372	1	(1)	$\overline{G} = 7C_4$	3.3
107	28	$\{[6]^1, [5]^4, [-1]^{20}, [-2]^3\}$	12	36	0		$\lambda_3 = -2$	3.5
108	28	$\{[20]^1, [6]^2, [-1]^{24}, [-8]^1\}$	141	2587	1	(1)	$\overline{G} = 2K_{7,7}$	3.3

109	28	$\{[9]^1, [5]^3, [2]^6, [-2]^{18}\}$	18	81	1	(1)	$L(K_{4,7})$	3.5
110	28	$\{[10]^1, [2]^{14}, [-2]^7, [-4]^6\}$	12	117	≥ 2		SR(36,14,4,6)\8-cocl.	3.6, computer
111	28	$\{[11]^1, [3]^7, [1]^7, [-3]^{13}\}$	21	175	≥ 10350			computer
112	28	$\{[12]^1, [2]^{14}, [-2]^6, [-4]^7\}$	24	270	≥ 8472	(56)	$\overline{T(8)}$\spread SR(35,16,6,8)\7-cocl.	3.6, computer
113	28	$\{[15]^1, [1]^{12}, [-1]^{14}, [-13]^1\}$	21	1197	1	(1)	$\overline{G} = 2CP(7)$	3.3
114	28	$\{[13]^1, [1]^{13}, [-1]^{13}, [-13]^1\}$	0	858	1	(1)	$IG(14,13,12)$	3.2
115	28	$\{[13]^1, [3]^6, [-1]^{20}, [-11]^1\}$	18	618	1		$IG(7,6,5) \circledast J_2$	3.4
116	28	$\{[13]^1, [5]^5, [-1]^6, [-2]^{16}\}$	48	408	0		$\lambda_3 = -2$	3.5
117	30	$\{[26]^1, [2]^5, [-1]^{20}, [-4]^4\}$	289	6972	1	(1)	$\overline{G} = 5K_{3,3}$	3.3
118	30	$\{[26]^1, [1]^{12}, [-2]^{15}, [-4]^2\}$	289	6966	1	(1)	$\overline{G}$ = 3Petersen	3.3
119	30	$\{[25]^1, [1]^{10}, [-1]^{15}, [-5]^4\}$	252	5940	1	(1)	$\overline{G} = 5CP(3)$	3.3
120	30	$\{[24]^1, [4]^3, [-1]^{24}, [-6]^2\}$	226	5022	1	(1)	$\overline{G} = 3K_{5,5}$	3.3
121	30	$\{[6]^1, [2]^9, [1]^9, [-3]^{11}\}$	0	6	0			4
122	30	$\{[23]^1, [2]^{10}, [-2]^{18}, [-7]^1\}$	196	4194	1	(1)	$\overline{G} = 2GQ(2,2)$	3.3
123	30	$\{[6]^1, [3]^8, [1]^4, [-2]^{17}\}$	5	4	0		$\lambda_3 = -2$	3.5
124	30	$\{[23]^1, [1]^{15}, [-2]^{12}, [-7]^2\}$	190	4230	1	(1)	$\overline{G} = 3\overline{\text{Petersen}}$	3.3
125	30	$\{[7]^1, [2]^{14}, [-2]^{14}, [-7]^1\}$	0	42	4	(4)	$IG(15,7,3)$	3.2, [61]
126	30	$\{[7]^1, [2]^{15}, [-2]^5, [-3]^9\}$	3	12	0			4
127	30	$\{[7]^1, [4]^5, [0]^{15}, [-3]^9\}$	7	28	0			computer
128	30	$\{[7]^1, [2]^{12}, [1]^5, [-3]^{12}\}$	2	14	0			4
129	30	$\{[8]^1, [2]^{14}, [-2]^{14}, [-8]^1\}$	0	84	4	(4)	$IG(15,8,4)$	3.2, [61]
130	30	$\{[8]^1, [2]^{15}, [-2]^9, [-4]^5\}$	4	36	11		$GQ(3,3)$\10-coclique	3.6, computer
131	30	$\{[8]^1, [3]^9, [-1]^{15}, [-4]^5\}$	7	42	0			computer
132	30	$\{[8]^1, [4]^7, [-1]^8, [-2]^{14}\}$	14	42	0		$\lambda_3 = -2$	3.5
133	30	$\{[21]^1, [1]^{12}, [-1]^{15}, [-9]^2\}$	130	3030	1	(1)	$\overline{G} = 3CP(5)$	3.3
134	30	$\{[8]^1, [4]^5, [2]^5, [-2]^{19}\}$	12	36	1		$L(\overline{IG(6,5,4)})$	3.5
135	30	$\{[21]^1, [1]^{18}, [-3]^{10}, [-9]^1\}$	138	2934	1	(1)	$\overline{G} = 2\overline{GQ(2,2)}$	3.3
136	30	$\{[9]^1, [3]^8, [-1]^{19}, [-7]^2\}$	4	124	0			4
137	30	$\{[9]^1, [4]^6, [-1]^{21}, [-6]^2\}$	11	102	0			4
138	30	$\{[9]^1, [3]^5, [0]^{20}, [-6]^4\}$	0	126	2	(1)	Petersen$\otimes J_3$	3.4, computer
139	30	$\{[9]^1, [4]^4, [0]^{20}, [-5]^5\}$	6	102	0			computer
140	30	$\{[9]^1, [3]^{10}, [-1]^9, [-3]^{10}\}$	12	60	≥ 17			computer
141	30	$\{[9]^1, [5]^5, [-1]^{19}, [-3]^5\}$	20	92	1		L(Petersen)$\circledast J_2$	3.4, computer
142	30	$\{[9]^1, [7]^3, [-1]^{24}, [-3]^2\}$	28	156	0			4
143	30	$\{[9]^1, [4]^4, [3]^5, [-2]^{20}\}$	16	62	1	(1)	$L(K_{5,6})$	3.5
144	30	$\{[10]^1, [2]^{15}, [-2]^{10}, [-5]^4\}$	9	120	3			computer
145	30	$\{[19]^1, [4]^4, [-1]^{24}, [-11]^1\}$	96	2082	1	(1)	$\overline{G} = 2K_{5,5,5}$	3.3
146	30	$\{[10]^1, [5]^4, [2]^5, [-2]^{20}\}$	23	120	1		$L(\overline{\text{Petersen}})$	3.5
147	30	$\{[11]^1, [2]^{16}, [-3]^9, [-4]^4\}$	16	162	0			computer
148	30	$\{[11]^1, [5]^5, [-1]^{20}, [-4]^4\}$	28	198	8	(1)	Petersen$\circledast J_3$	3.4, computer
149	30	$\{[11]^1, [2]^{10}, [1]^9, [-4]^{10}\}$	13	174	?			
150	30	$\{[11]^1, [6]^4, [-1]^{20}, [-3]^5\}$	34	222	0			computer
151	30	$\{[11]^1, [5]^5, [1]^4, [-2]^{20}\}$	30	186	0		$\lambda_3 = -2$	3.5
152	30	$\{[11]^1, [8]^2, [1]^9, [-2]^{18}\}$	37	270	1	(1)	$L(K_{3,10})$	3.5
153	30	$\{[12]^1, [2]^6, [0]^{20}, [-8]^3\}$	4	414	0			computer
154	30	$\{[12]^1, [2]^9, [0]^{15}, [-6]^5\}$	12	318	2	(1)	$GQ(2,2) \otimes J_2$	3.4, computer
155	30	$\{[12]^1, [2]^{16}, [-3]^8, [-4]^5\}$	22	244	?			
156	30	$\{[12]^1, [2]^{14}, [0]^5, [-4]^{10}\}$	20	254	?			
157	30	$\{[12]^1, [3]^{10}, [0]^5, [-3]^{14}\}$	27	240	≥ 68876		$L_3(6)$\6-coclique	3.6, computer
158	30	$\{[17]^1, [2]^8, [-1]^{20}, [-13]^1\}$	46	1590	1	(1)	$\overline{G} = 2K_{3,3,3,3,3}$	3.3
159	30	$\{[12]^1, [4]^5, [1]^{10}, [-3]^{14}\}$	28	248	≥ 50			computer
160	30	$\{[13]^1, [1]^{20}, [-1]^5, [-7]^4\}$	14	474	0		$\lambda_1 = 1$	3.5
161	30	$\{[13]^1, [2]^{15}, [-2]^9, [-5]^5\}$	27	372	?			
162	30	$\{[13]^1, [3]^9, [-1]^{15}, [-5]^5\}$	30	378	≥ 1487	(1)	$GQ(2,2) \circledast J_2$	3.4, computer
163	30	$\{[13]^1, [3]^{11}, [-2]^8, [-3]^{10}\}$	36	344	?			
164	30	$\{[13]^1, [3]^9, [1]^5, [-3]^{15}\}$	34	346	≥ 82		$L_3(6)$\6-clique	3.6, computer
165	30	$\{[14]^1, [1]^{14}, [-1]^{14}, [-14]^1\}$	0	1092	1	(1)	$IG(15,14,13)$	3.2
166	30	$\{[14]^1, [2]^9, [-1]^{19}, [-13]^1\}$	10	930	0			2.2
167	30	$\{[14]^1, [5]^4, [-1]^{24}, [-10]^1\}$	37	660	1		$IG(5,4,3) \circledast J_3$	3.4
168	30	$\{[14]^1, [4]^6, [-1]^{20}, [-6]^3\}$	41	542	?			
169	30	$\{[14]^1, [9]^2, [-1]^{26}, [-6]^1\}$	66	692	1		$C_6 \circledast J_5$	3.4
170	30	$\{[14]^1, [2]^{15}, [-1]^4, [-4]^{10}\}$	37	498	≥ 24931		SR(35,16,6,8)\5-cl.	3.6, computer

A.3.2. Two integral eigenvalues

Nr	v	spectrum	Δ	Ξ	#		Notes	References
1	10	$\{[7]^1, [-3]^1, [0.618]^4, [-1.618]^4\}$	15	80	1	(1)	$\overline{G} = 2C_5$	3.3
2	10	$\{[4]^1, [0]^5, [1.236]^2, [-3.236]^2\}$	0	10	1	(1)	$C_5 \otimes J_2$	3.4

3	12	$\{[5]^1, [-1]^5, [2.236]^3, [-2.236]^3\}$	5	10	1	(1)	Icosahedron	3.1, [61]
4	12	$\{[5]^1, [1]^3, [0.732]^4, [-2.732]^4\}$	2	13	0		$\lambda_1 = 1$	3.5
5	14	$\{[3]^1, [-3]^1, [1.414]^6, [-1.414]^6\}$	0	0	1	(1)	$IG(7,3,1)$	3.2
6	14	$\{[4]^1, [-4]^1, [1.414]^6, [-1.414]^6\}$	0	6	1	(1)	$IG(7,4,2)$	3.2
7	14	$\{[6]^1, [0]^7, [1.646]^3, [-3.646]^3\}$	3	33	0			computer
8	15	$\{[12]^1, [-3]^2, [0.618]^6, [-1.618]^6\}$	55	560	1	(1)	$\overline{G} = 3C_5$	3.3
9	15	$\{[4]^1, [0]^6, [1.791]^4, [-2.791]^4\}$	0	4	0			4
10	15	$\{[6]^1, [-1]^{10}, [4.162]^2, [-2.162]^2\}$	11	32	0			4
11	15	$\{[6]^1, [-1]^6, [2.449]^4, [-2.449]^4\}$	7	20	0			computer
12	15	$\{[6]^1, [0]^{10}, [1.854]^2, [-4.854]^2\}$	0	48	1	(1)	$C_5 \otimes J_3$	3.4
13	16	$\{[7]^1, [-1]^{11}, [4.464]^2, [-2.464]^2\}$	15	57	0			4
14	18	$\{[5]^1, [3]^1, [1.303]^8, [-2.303]^8\}$	2	4	1		td$L_2(3)$	3.4
15	20	$\{[17]^1, [-3]^3, [0.618]^8, [-1.618]^8\}$	120	1815	1	(1)	$\overline{G} = 4C_5$	3.3
16	20	$\{[5]^1, [-1]^5, [2.236]^7, [-2.236]^7\}$	3	2	1		2-cover $C_5 \circledast J_2$	3.7, computer
17	20	$\{[7]^1, [-1]^{15}, [5.873]^2, [-1.873]^2\}$	18	75	0		$\lambda_3 > -2$	3.5
18	20	$\{[8]^1, [-2]^9, [3.236]^5, [-1.236]^5\}$	15	60	0		$\lambda_3 = -2$	3.5
19	20	$\{[8]^1, [0]^{15}, [2.472]^2, [-6.472]^2\}$	0	132	1	(1)	$C_5 \otimes J_4$	3.4
20	20	$\{[8]^1, [0]^{11}, [2.317]^4, [-4.317]^4\}$	6	80	0			computer
21	21	$\{[4]^1, [-2]^8, [2.414]^6, [-0.414]^6\}$	2	0	1	(1)	$L(IG(7,3,1))$	3.5, 3.1
22	21	$\{[6]^1, [-1]^6, [2.449]^7, [-2.449]^7\}$	5	10	0			computer
23	21	$\{[6]^1, [0]^8, [2.193]^6, [-3.193]^6\}$	2	16	1			computer
24	21	$\{[8]^1, [-1]^{14}, [4.742]^3, [-2.742]^3\}$	18	78	0			computer
25	21	$\{[8]^1, [-1]^8, [2.828]^6, [-2.828]^6\}$	12	56	6	(1)	$L(IG(7,3,1))_3$	3.1, computer
26	21	$\{[8]^1, [1]^{12}, [-0.209]^4, [-4.791]^4\}$	2	88	0		$\lambda_1 = 1$	3.5
27	21	$\{[8]^1, [1]^6, [1.449]^7, [-3.449]^7\}$	6	62	0			4
28	21	$\{[8]^1, [2]^8, [-0.586]^6, [-3.414]^6\}$	8	60	28	(1)	$L(IG(7,3,1))_2$	3.1, computer
29	22	$\{[5]^1, [-5]^1, [1.732]^{10}, [-1.732]^{10}\}$	0	10	1	(1)	$IG(11,5,2)$	3.2
30	22	$\{[5]^1, [0]^{11}, [2.372]^5, [-3.372]^5\}$	0	10	0			computer
31	22	$\{[6]^1, [-6]^1, [1.732]^{10}, [-1.732]^{10}\}$	0	30	1	(1)	$IG(11,6,3)$	3.2
32	22	$\{[10]^1, [0]^{11}, [2.317]^5, [-4.317]^5\}$	15	175	0			computer
33	24	$\{[7]^1, [-1]^{15}, [4.464]^4, [-2.464]^4\}$	13	41	0			computer
34	24	$\{[7]^1, [-1]^7, [2.646]^8, [-2.646]^8\}$	7	21	10	(1)	Klein	3.1, [61]
35	24	$\{[8]^1, [0]^{15}, [2.873]^4, [-4.873]^4\}$	3	78	0			computer
36	24	$\{[9]^1, [1]^{15}, [-0.551]^4, [-5.449]^4\}$	2	134	0		$\lambda_1 = 1$	3.5
37	24	$\{[9]^1, [1]^7, [1.646]^8, [-3.646]^8\}$	8	91	1	(1)	$\text{Klein}_{1,3}$	3.1, computer
38	24	$\{[11]^1, [-1]^{17}, [5.472]^3, [-3.472]^3\}$	35	255	1		Icosahedron$\circledast J_2$	3.4, computer
39	25	$\{[22]^1, [-3]^4, [0.618]^{10}, [-1.618]^{10}\}$	210	4220	1	(1)	$\overline{G} = 5C_5$	3.3
40	25	$\{[10]^1, [0]^{20}, [3.090]^2, [-8.090]^2\}$	0	280	1	(1)	$C_5 \otimes J_5$	3.4
41	26	$\{[4]^1, [-4]^1, [1.732]^{12}, [-1.732]^{12}\}$	0	0	1	(1)	$IG(13,4,1)$	3.2
42	26	$\{[19]^1, [-7]^1, [1.303]^{12}, [-2.303]^{12}\}$	123	2208	1	(1)	$\overline{G} = 2P(13)$	3.3
43	26	$\{[7]^1, [5]^1, [1.562]^{12}, [-2.562]^{12}\}$	6	24	1		td$P(13)$	3.4
44	26	$\{[9]^1, [-9]^1, [1.732]^{12}, [-1.732]^{12}\}$	0	180	1	(1)	$IG(13,9,6)$	3.2
45	26	$\{[12]^1, [0]^{13}, [2.606]^6, [-4.606]^6\}$	24	318	≥ 85	(1)	$P(13) \otimes J_2$	3.4, computer
46	27	$\{[8]^1, [-1]^{20}, [6.243]^3, [-2.243]^3\}$	22	102	0			4
47	27	$\{[8]^1, [-1]^{14}, [3.854]^6, [-2.854]^6\}$	13	48	1		3-cover $C_3 \circledast J_3$	3.7, computer
48	28	$\{[6]^1, [-2]^{15}, [3.414]^6, [0.586]^6\}$	6	9	1		$L(IG(7,4,2))$, Coxeter_4	3.1, 3.5
49	28	$\{[7]^1, [-1]^7, [2.646]^{10}, [-2.646]^{10}\}$	6	15	0			computer
50	28	$\{[9]^1, [-1]^{21}, [6.583]^3, [-2.583]^3\}$	27	144	0			computer
51	28	$\{[9]^1, [0]^{21}, [3.623]^3, [-6.623]^3\}$	0	153	0			computer
52	28	$\{[12]^1, [0]^{21}, [3.292]^3, [-7.292]^3\}$	12	390	0			computer
53	28	$\{[12]^1, [0]^{15}, [2.873]^6, [-4.873]^6\}$	21	300	?			
54	28	$\{[13]^1, [-1]^{13}, [3.606]^7, [-3.606]^7\}$	39	390	≥ 515	(1)	Taylor	3.1, [61]
55	30	$\{[27]^1, [-3]^5, [0.618]^{12}, [-1.618]^{12}\}$	325	8150	1	(1)	$\overline{G} = 6C_5$	3.3
56	30	$\{[7]^1, [-3]^9, [2.732]^{10}, [-0.732]^{10}\}$	5	16	?			
57	30	$\{[8]^1, [0]^{21}, [3.583]^4, [-5.583]^4\}$	0	84	0			4
58	30	$\{[9]^1, [3]^5, [1.236]^{12}, [-3.236]^{12}\}$	8	62	?			
59	30	$\{[10]^1, [0]^{19}, [3.359]^5, [-5.359]^5\}$	7	151	0			computer
60	30	$\{[11]^1, [-1]^{11}, [3.317]^9, [-3.317]^9\}$	22	165	?			
61	30	$\{[11]^1, [1]^{19}, [0.162]^5, [-6.162]^5\}$	3	249	0		$\lambda_1 = 1$	3.5
62	30	$\{[11]^1, [5]^5, [-0.382]^{12}, [-2.618]^{12}\}$	29	190	?			
63	30	$\{[12]^1, [0]^{25}, [3.708]^2, [-9.708]^2\}$	0	510	1	(1)	$C_5 \otimes J_6$	3.4
64	30	$\{[13]^1, [-2]^{19}, [5.372]^5, [-0.372]^5\}$	47	388	0		$\lambda_3 = -2$	3.5
65	30	$\{[13]^1, [-1]^{25}, [9.325]^2, [-3.325]^2\}$	62	570	0			4
66	30	$\{[13]^1, [-1]^{21}, [5.899]^4, [-3.899]^4\}$	46	410	?			
67	30	$\{[13]^1, [3]^9, [-0.268]^{10}, [-3.732]^{10}\}$	32	358	?			
68	30	$\{[14]^1, [-2]^{21}, [5.791]^4, [1.209]^4\}$	56	532	0		$\lambda_3 = -2$	3.5
69	30	$\{[14]^1, [0]^{15}, [2.873]^7, [-4.873]^7\}$	35	525	?			
70	30	$\{[14]^1, [2]^{11}, [0.449]^9, [-4.449]^9\}$	34	513	?			

A.3.3. One integral eigenvalue

Nr	v	spectrum	Δ	Ξ	#	Notes	References
1	7	$\{[2]^1, [1.247]^2, [-0.445]^2, [-1.802]^2\}$	0	0	1 (1)	C_7	3.1, 3.5
2	13	$\{[4]^1, [1.377]^4, [\ 0.274]^4, [-2.651]^4\}$	0	4	1 (1)	Cycl(13)	3.1
3	19	$\{[6]^1, [2.507]^6, [-1.222]^6, [-2.285]^6\}$	6	12	1 (1)	Cycl(19)	3.1
4	28	$\{[9]^1, [2.604]^9, [-0.110]^9, [-3.494]^9\}$	9	72	≥ 2 (2)	Mathon, Hollmann	3.1

A.4. Three-class association schemes

In the following Appendices all possible parameter sets for 3-class association schemes on at most 100 vertices are listed, except for the less interesting schemes generated by the disjoint union of strongly regular graphs, the schemes generated by SRG⊗J_n, and the rectangular schemes $R(m, n)$ for $m \neq n$. For the parameters of the first two kind of schemes, see Sections 4.1.1 and 4.1.2, respectively. The parameters of the rectangular scheme are given below. The number of vertices of the scheme is denoted by v. If the scheme is primitive, then this number is in **bold face**. The "spectrum" is given by the last three rows of P^T, and so the first row represents the spectrum of the first relation, and similarly for the second and third relation. In the first row of the spectrum, the multiplicities of the (eigenvalues of the) scheme are denoted in superscript. In Appendices A.4.1 and A.4.4 the multiplicities are omitted, since there the schemes are self-dual, so the multiplicities are equal to the degrees. L_1, L_2 and L_3 here denote the reduced intersection matrices, that is, the first row and column are omitted. # denotes the number of nonisomorphic schemes of that type. At the end of the line remarks are made. The rectangular scheme $R(m, n)$ would read as follows.

v	spectrum	L_1	L_2	L_3	#	
mn	$\{(m-1)(n-1),\ 1,\ 1-m,\ 1-n\}$	$(m-2)(n-2)$ $n-2$ $m-2$	$n-2$ 0 1	$m-2$ 1 0	1	$R(m,n)$, if m or
	$\{\ n-1,\ -1,\ -1,\ n-1\}$	$(m-1)(n-2)$ 0 $m-1$	0 $n-2$ 0	$m-1$ 0 0		n equals 2 then
	$\{\ m-1,\ -1,\ m-1,\ -1\}$	$(m-2)(n-1)$ $n-1$ 0	$n-1$ 0 0	0 0 $m-2$		DRG and Q-pol.

A.4.1. Amorphic three-class association schemes

v	spectrum	L_1	L_2	L_3	#	
4	{ 1, 1, -1, -1}	0 0 0	0 0 1	0 1 0	1	$L_{1,1}(2) \approx R(2,2)$
	{ 1, -1, 1, -1}	0 0 1	0 0 0	1 0 0		
	{ 1, -1, -1, 1}	0 1 0	1 0 0	0 0 0		
9	{ 4, 1, -2, -2}	1 1 1	1 0 1	1 1 0	1	$L_{1,1}(3) \approx R(3,3)$
	{ 2, -1, 2, -1}	2 0 2	0 1 0	2 0 0		
	{ 2, -1, -1, 2}	2 2 0	2 0 0	0 0 1		
16	{ 9, 1, -3, -3}	4 2 2	2 0 1	2 1 0	1	$L_{1,1}(4) \approx R(4,4)$
	{ 3, -1, 3, -1}	6 0 3	0 2 0	3 0 0		
	{ 3, -1, -1, 3}	6 3 0	3 0 0	0 0 2		
16	{ 6, 2, -2, -2}	2 2 1	2 2 2	1 2 0	4	$L_{1,2}(4)$ [56]
	{ 6, -2, 2, -2}	2 2 2	2 2 1	2 1 0		
	{ 3, -1, -1, 3}	2 4 0	4 2 0	0 0 2		

16	{5, -3, 1, 1}	0	2	2	2	2	1	2	1	2	2	Cycl(16) [56]
	{5, 1, -3, 1}	2	2	1	2	0	2	1	2	2		
	{5, 1, 1, -3}	2	1	2	1	2	2	2	2	0		
25	{16, 1, -4, -4}	9	3	3	3	0	1	3	1	0	1	$L_{1,1}(5) \simeq R(5,5)$
	{4, -1, 4, -1}	12	0	4	0	3	0	4	0	0		
	{4, -1, -1, 4}	12	4	0	4	0	0	0	0	3		
25	{12, 2, -3, -3}	5	4	2	4	2	2	2	2	0	2	$L_{1,2}(5)$
	{8, -2, 3, -2}	6	3	3	3	3	1	3	1	0		
	{4, -1, -1, 4}	6	6	0	6	2	0	0	0	3		
25	{8, 3, -2, -2}	3	2	2	2	2	4	2	4	2	2	$L_{2,2}(5)$, Cycl(25) [56]
	{8, -2, 3, -2}	2	2	4	2	3	2	4	2	2		
	{8, -2, -2, 3}	2	4	2	4	2	2	2	2	3		
36	{25, 1, -5, -5}	16	4	4	4	0	1	4	1	0	1	$L_{1,1}(6) \simeq R(6,6)$
	{5, -1, 5, -1}	20	0	5	0	4	0	5	0	0		
	{5, -1, -1, 5}	20	5	0	5	0	0	0	0	4		
36	{20, 2, -4, -4}	10	6	3	6	2	2	3	2	0	22	$L_{1,2}(6)$ [26, 90]
	{10, -2, 4, -2}	12	4	4	4	4	1	4	1	0		
	{5, -1, -1, 5}	12	8	0	8	2	0	0	0	4		
36	{15, 3, -3, -3}	6	6	2	6	6	3	2	3	0	?	SRG(36,20,10,12)\spread
	{15, -3, 3, -3}	6	6	3	6	6	2	3	2	0		
	{5, -1, -1, 5}	6	9	0	9	6	0	0	0	4		
36	{15, 3, -3, -3}	6	4	4	4	2	4	4	4	2	0	$L_{2,2}(6)$
	{10, -2, 4, -2}	6	3	6	3	4	2	6	2	2		
	{10, -2, -2, 4}	6	6	3	6	2	2	3	2	4		
49	{36, 1, -6, -6}	25	5	5	5	0	1	5	1	0	1	$L_{1,1}(7) \simeq R(7,7)$
	{6, -1, 6, -1}	30	0	6	0	5	0	6	0	0		
	{6, -1, -1, 6}	30	6	0	6	0	0	0	0	5		
49	{30, 2, -5, -5}	17	8	4	8	2	2	4	2	0	563	$L_{1,2}(7)$ [90]
	{12, -2, 5, -2}	20	5	5	5	5	1	5	1	0		
	{6, -1, -1, 6}	20	10	0	10	2	0	0	0	5		
49	{24, 3, -4, -4}	11	9	3	9	6	3	3	3	0	≥ 1	$L_{1,3}(7)$
	{18, -3, 4, -3}	12	8	4	8	7	2	4	2	0		
	{6, -1, -1, 6}	12	12	0	12	6	0	0	0	5		
49	{24, 3, -4, -4}	11	6	6	6	2	4	6	4	2	≥ 1	$L_{2,2}(7)$
	{12, -2, 5, -2}	12	4	8	4	5	2	8	2	2		
	{12, -2, -2, 5}	12	8	4	8	2	2	4	2	5		
49	{18, 4, -3, -3}	7	6	4	6	6	6	4	6	2	≥ 1	$L_{2,3}(7)$
	{18, -3, 4, -3}	6	6	6	6	7	4	6	4	2		
	{12, -2, -2, 5}	6	9	3	9	6	3	3	3	5		
49	{16, -5, 2, 2}	3	6	6	6	6	4	6	4	6	0	SRG does not exist [18]
	{16, 2, -5, 2}	6	6	4	6	3	6	4	6	6		
	{16, 2, 2, -5}	6	4	6	4	6	6	6	6	3		
64	{49, 1, -7, -7}	36	6	6	6	0	1	6	1	0	1	$L_{1,1}(8) \simeq R(8,8)$
	{7, -1, 7, -1}	42	0	7	0	6	0	7	0	0		
	{7, -1, -1, 7}	42	7	0	7	0	0	0	0	6		
64	{42, 2, -6, -6}	26	10	5	10	2	2	5	2	0	≥ 1	$L_{1,2}(8)$
	{14, -2, 6, -2}	30	6	6	6	6	1	6	1	0		
	{7, -1, -1, 7}	30	12	0	12	2	0	0	0	6		
64	{35, 3, -5, -5}	18	12	4	12	6	3	4	3	0	≥ 1	$L_{1,3}(8)$
	{21, -3, 5, -3}	20	10	5	10	8	2	5	2	0		
	{7, -1, -1, 7}	20	15	0	15	6	0	0	0	6		
64	{28, 4, -4, -4}	12	12	3	12	12	4	3	4	0	≥ 1	$L_{1,4}(8)$
	{28, -4, 4, -4}	12	12	4	12	12	3	4	3	0		
	{7, -1, -1, 7}	12	16	0	16	12	0	0	0	6		
64	{35, 3, -5, -5}	18	8	8	8	2	4	8	4	2	≥ 1	$L_{2,2}(8)$
	{14, -2, 6, -2}	20	5	10	5	6	2	10	2	2		
	{14, -2, -2, 6}	20	10	5	10	2	2	5	2	6		
64	{28, 4, -4, -4}	12	9	6	9	6	6	6	6	2	≥ 1	$L_{2,3}(8)$
	{21, -3, 5, -3}	12	8	8	8	8	4	8	4	2		
	{14, -2, -2, 6}	12	12	4	12	6	3	4	3	6		
64	{27, -5, 3, 3}	10	8	8	8	6	4	8	4	6	?	
	{18, 2, -6, 2}	12	9	6	9	2	6	6	6	6		
	{18, 2, 2, -6}	12	6	9	6	6	6	9	6	2		
64	{21, 5, -3, -3}	8	6	6	6	6	9	6	9	6	≥ 1	$L_{3,3}(8)$, Cycl(64)
	{21, -3, 5, -3}	6	6	9	6	8	6	9	6	6		
	{21, -3, -3, 5}	6	9	6	9	6	6	6	6	8		
81	{64, 1, -8, -8}	49	7	7	7	0	1	7	1	0	1	$L_{1,1}(9) \simeq R(9,9)$
	{8, -1, 8, -1}	56	0	8	0	7	0	8	0	0		
	{8, -1, -1, 8}	56	8	0	8	0	0	0	0	7		

81	{56, 2, -7, -7}	37 12 6	12 2 2	6 2 0	≥ 1	$L_{1,2}(9)$
	{16, -2, 7, -2}	42 7 7	7 7 1	7 1 0		
	{ 8, -1, -1, 8}	42 14 0	14 2 0	0 0 7		
81	{48, 3, -6, -6}	27 15 5	15 6 3	5 3 0	≥ 1	$L_{1,3}(9)$
	{24, -3, 6, -3}	30 12 6	12 9 2	6 2 0		
	{ 8, -1, -1, 8}	30 18 0	18 6 0	0 0 7		
81	{40, 4, -5, -5}	19 16 4	16 12 4	4 4 0	≥ 1	$L_{1,4}(9)$
	{32, -4, 5, -4}	20 15 5	15 13 3	5 3 0		
	{ 8, -1, -1, 8}	20 20 0	20 12 0	0 0 7		
81	{48, 3, -6, -6}	27 10 10	10 2 4	10 4 2	≥ 1	$L_{2,2}(9)$
	{16, -2, 7, -2}	30 6 12	6 7 2	12 2 2		
	{16, -2, -2, 7}	30 12 6	12 2 2	6 2 7		
81	{40, 4, -5, -5}	19 12 8	12 6 6	8 6 2	≥ 1	$L_{2,3}(9)$
	{24, -3, 6, -3}	20 10 10	10 9 4	10 4 2		
	{16, -2, -2, 7}	20 15 5	15 6 3	5 3 7		
81	{32, 5, -4, -4}	13 12 6	12 12 8	6 8 2	≥ 1	$L_{2,4}(9)$
	{32, -4, 5, -4}	12 12 8	12 13 6	8 6 2		
	{16, -2, -2, 7}	12 16 4	16 12 4	4 4 7		
81	{40, -5, 4, 4}	19 10 10	10 6 4	10 4 6	≥ 2	Van Lint-Schrijver
	{20, 2, -7, 2}	20 12 8	12 1 6	8 6 6		
	{20, 2, 2, -7}	20 8 12	8 6 6	12 6 1		
81	{30, -6, 3, 3}	9 12 8	12 12 6	8 6 6	≥ 1	Van Lint-Schrijver
	{30, 3, -6, 3}	12 12 6	12 9 8	6 8 6		
	{20, 2, 2, -7}	12 9 9	9 12 9	9 9 1		
81	{32, 5, -4, -4}	13 9 9	9 6 9	9 9 6	≥ 1	$L_{3,3}(9)$
	{24, -3, 6, -3}	12 8 12	8 9 6	12 6 6		
	{24, -3, -3, 6}	12 12 8	12 6 6	8 6 9		
100	{81, 1, -9, -9}	64 8 8	8 0 1	8 1 0	1	$L_{1,1}(10) \simeq R(10,10)$
	{ 9, -1, 9, -1}	72 0 9	0 8 0	9 0 0		
	{ 9, -1, -1, 9}	72 9 0	9 0 0	0 0 8		
100	{72, 2, -8, -8}	50 14 7	14 2 2	7 2 0	≥ 1	$L_{1,2}(10)$
	{18, -2, 8, -2}	56 8 8	8 8 1	8 1 0		
	{ 9, -1, -1, 9}	56 16 0	16 2 0	0 0 8		
100	{63, 3, -7, -7}	38 18 6	18 6 3	6 3 0	≥ 1	$L_{1,3}(10)$
	{27, -3, 7, -3}	42 14 7	14 10 2	7 2 0		
	{ 9, -1, -1, 9}	42 21 0	21 6 0	0 0 8		
100	{54, 4, -6, -6}	28 20 5	20 12 4	5 4 0	?	$L_{1,4}(10)$, SRG(100,63,38,42)$\setminus$spread
	{36, -4, 6, -4}	30 18 6	18 14 3	6 3 0		
	{ 9, -1, -1, 9}	30 24 0	24 12 0	0 0 8		
100	{45, 5, -5, -5}	20 20 4	20 20 5	4 5 0	?	$L_{1,5}(10)$, SRG(100,54,28,30)$\setminus$spread
	{45, -5, 5, -5}	20 20 5	20 20 4	5 4 0		
	{ 9, -1, -1, 9}	20 25 0	25 20 0	0 0 8		
100	{63, 3, -7, -7}	38 12 12	12 2 4	12 4 2	≥ 1	$L_{2,2}(10)$
	{18, -2, 8, -2}	42 7 14	7 8 2	14 2 2		
	{18, -2, -2, 8}	42 14 7	14 2 2	7 2 8		
100	{54, 4, -6, -6}	28 15 10	15 6 6	10 6 2	?	$L_{2,3}(10)$
	{27, -3, 7, -3}	30 12 12	12 10 4	12 4 2		
	{18, -2, -2, 8}	30 18 6	18 6 3	6 3 8		
100	{45, 5, -5, -5}	20 16 8	16 12 8	8 8 2	?	$L_{2,4}(10)$
	{36, -4, 6, -4}	20 15 10	15 14 6	10 6 2		
	{18, -2, -2, 8}	20 20 5	20 12 4	5 4 8		
100	{55, -5, 5, 5}	30 12 12	12 6 4	12 4 6	?	
	{22, 2, -8, 2}	30 15 10	15 0 6	10 6 6		
	{22, 2, 2, -8}	30 10 15	10 6 6	15 6 0		
100	{44, -6, 4, 4}	18 15 10	15 12 6	10 6 6	?	
	{33, 3, -7, 3}	20 16 8	16 8 8	8 8 6		
	{22, 2, 2, -8}	20 12 12	12 12 9	12 9 0		
100	{45, 5, -5, -5}	20 12 12	12 6 9	12 9 6	?	$L_{3,3}(10)$
	{27, -3, 7, -3}	20 10 15	10 10 6	15 6 6		
	{27, -3, -3, 7}	20 15 10	15 6 6	10 6 10		
100	{36, 6, -4, -4}	14 12 9	12 12 12	9 12 6	?	$L_{3,4}(10)$
	{36, -4, 6, -4}	12 12 12	12 14 9	12 9 6		
	{27, -3, -3, 7}	12 16 8	16 12 8	8 8 10		
100	{33, -7, 3, 3}	8 12 12	12 12 9	12 9 12	?	
	{33, 3, -7, 3}	12 12 9	12 8 12	9 12 12		
	{33, 3, 3, -7}	12 9 12	9 12 12	12 12 8		

A.4.2. Four integral eigenvalues

Excluded here are association schemes generated by SRG⊗J_n, and the rectangular schemes $R(m, n)$, except the 6-cycle C_6 and the Cube.

v	spectrum	L_1	L_2	L_3	#	
6	$\{2, 1^{2}, -1^{2}, -2^{1}\}$	0 1 0	1 0 1	0 1 0	1	C_6
	$\{2, -1, -1, 2\}$	1 0 1	0 1 0	1 0 0		DRG
	$\{1, -1, 1, -1\}$	0 2 0	2 0 0	0 0 0		Q-123
8	$\{3, 1^{3}, -1^{3}, -3^{1}\}$	0 2 0	2 0 1	0 1 0	1	Cube
	$\{3, -1, -1, 3\}$	2 0 1	0 2 0	1 0 0		DRG
	$\{1, -1, 1, -1\}$	0 3 0	3 0 0	0 0 0		Q-123
15	$\{4, 2^{5}, -1^{4}, -2^{5}\}$	1 2 0	2 4 2	0 2 0	1	L(Petersen)
	$\{8, -2, -2, 2\}$	1 2 1	2 4 1	1 1 0		DRG, R_2 SRG
	$\{2, -1, 2, -1\}$	0 4 0	4 4 0	0 0 1		
20	$\{9, 3^{5}, -1^{9}, -3^{5}\}$	4 4 0	4 4 1	0 1 0	1	$J(6,3)$
	$\{9, -3, -1, 3\}$	4 4 1	4 4 0	1 0 0		$R_1 \simeq R_2$ DRG
	$\{1, -1, 1, -1\}$	0 9 0	9 0 0	0 0 0		Q-123, Q-321
27	$\{6, 3^{6}, 0^{12}, -3^{8}\}$	1 4 0	4 4 4	0 4 4	1	$H(3,3)$
	$\{12, 0, -3, 3\}$	2 2 2	2 5 4	2 4 2		DRG
	$\{8, -4, 2, -1\}$	0 3 3	3 6 3	3 3 1		Q-123
27	$\{8, 2^{12}, -1^{8}, -4^{6}\}$	1 6 0	6 8 2	0 2 0	2	$GQ(2,4)\setminus$spread
	$\{16, -2, -2, 4\}$	3 4 1	4 10 1	1 1 0		R_1 DRG, R_2 SRG
	$\{2, -1, 2, -1\}$	0 8 0	8 8 0	0 0 1		
28	$\{12, 2^{14}, -2^{6}, -4^{7}\}$	4 6 1	6 4 2	1 2 0	56	$\overline{T(8)}\setminus$spread, $\overline{\text{Chang}}\setminus$spread [102]
	$\{12, -2, -2, 4\}$	6 4 2	4 6 1	2 1 0		R_2 SRG
	$\{3, -1, 3, -1\}$	4 8 0	8 4 0	0 0 2		
30	$\{7, 2^{14}, -2^{14}, -7^{1}\}$	0 6 0	6 0 8	0 8 0	4	$IG(15,7,3)$
	$\{14, -1, -1, 14\}$	3 0 4	0 13 0	4 0 4		R_1 and R_3 DRG
	$\{8, -2, 2, -8\}$	0 7 0	7 0 7	0 7 0		Q-123, Q-213
32	$\{6, 2^{15}, -2^{15}, -6^{1}\}$	0 5 0	5 0 10	0 10 0	3	$IG(16,6,2)$
	$\{15, -1, -1, 15\}$	2 0 4	0 14 0	4 0 6		R_1 and R_3 DRG
	$\{10, -2, 2, -10\}$	0 6 0	6 0 9	0 9 0		Q-123, Q-213
32	$\{15, 3^{10}, -1^{15}, -5^{6}\}$	6 8 0	8 6 1	0 1 0	1	$2(GQ(2,2)+1)$
	$\{15, -3, -1, 5\}$	8 6 1	6 8 0	1 0 0		R_1 and R_2 DRG
	$\{1, -1, 1, -1\}$	0 15 0	15 0 0	0 0 0		Q-123, Q-321
35	$\{12, 5^{6}, 0^{14}, -3^{14}\}$	5 6 0	6 9 3	0 3 1	1	$J(7,3)$
	$\{18, -3, -3, 3\}$	4 6 2	6 9 2	2 2 0		R_1 and R_3 DRG, R_2 SRG
	$\{4, -3, 2, -1\}$	0 9 3	9 9 0	3 0 0		Q-123
35	$\{12, 3^{14}, -2^{6}, -3^{14}\}$	4 6 1	6 9 3	1 3 0	≥ 1	SRG(35,16,6,8)$\setminus$spread
	$\{18, -3, -3, 3\}$	4 6 2	6 9 2	2 2 0		R_2 SRG
	$\{4, -1, 4, -1\}$	3 9 0	9 9 0	0 0 3		
35	$\{12, 4^{10}, -2^{20}, -3^{4}\}$	5 2 4	2 0 4	4 4 8	0	SRG(35,18,9,9)$\setminus$spread
	$\{6, -1, -1, 6\}$	4 0 8	0 5 0	8 0 8		R_3 SRG
	$\{16, -4, 2, -4\}$	3 3 6	3 0 3	6 3 6		
36	$\{5, 2^{16}, -1^{10}, -3^{9}\}$	0 4 0	4 8 8	0 8 2	1	Sylvester, residual of 4-(11,5,1)
	$\{20, -1, -4, 4\}$	1 2 2	2 11 6	2 6 2		DRG, $R_3 \simeq L_2(6)$
	$\{10, -2, 4, -2\}$	0 4 1	4 12 4	1 4 4		
40	$\{9, 3^{15}, -1^{9}, -3^{15}\}$	2 6 0	6 18 3	0 3 0	≥ 1	$GQ(3,3)\setminus$spread
	$\{27, -3, -3, 3\}$	2 6 1	6 18 2	1 2 0		R_1 DRG, R_2 SRG
	$\{3, -1, 3, -1\}$	0 9 0	9 18 0	0 0 2		
40	$\{18, 4^{12}, -2^{24}, -6^{3}\}$	8 5 4	5 0 4	4 4 4	0	SRG(40,27,18,18)$\setminus$spread
	$\{9, -1, -1, 9\}$	10 0 8	0 8 0	8 0 4		R_3 SRG
	$\{12, -4, 2, -4\}$	6 6 6	6 0 3	6 3 2		
42	$\{5, 2^{20}, -2^{20}, -5^{1}\}$	0 4 0	4 0 16	0 16 0	1	$IG(21,5,1)$
	$\{20, -1, -1, 20\}$	1 0 4	0 19 0	4 0 12		R_1 and R_3 DRG
	$\{16, -2, 2, -16\}$	0 5 0	5 0 15	0 15 0		Q-123, Q-213
42	$\{6, 2^{21}, -1^{6}, -3^{14}\}$	0 5 0	5 20 5	0 5 0	1	Ho-Si$_2(x)$, 2-(15,5,4)
	$\{30, -2, -5, 3\}$	1 4 1	4 21 4	1 4 0		DRG
	$\{5, -1, 5, -1\}$	0 6 0	6 24 0	0 0 4		
45	$\{8, 2^{25}, -2^{9}, -4^{10}\}$	0 5 2	5 5 10	2 10 4	1	Gewirtz$_2(x)$
	$\{20, -1, -5, 5\}$	2 2 4	2 9 8	4 8 4		R_3 SRG
	$\{16, -2, 6, -2\}$	1 5 2	5 10 5	2 5 8		
45	$\{8, 4^{12}, -1^{8}, -2^{24}\}$	3 4 0	4 24 4	0 4 0	0	SRG(45,12,3,3)$\setminus$spread
	$\{32, -4, -4, 2\}$	1 6 1	6 22 3	1 3 0		R_2 SRG
	$\{4, -1, 4, -1\}$	0 8 0	8 24 0	0 0 3		
45	$\{16, 4^{15}, -2^{20}, -4^{9}\}$	6 6 3	6 4 6	3 6 3	?	
	$\{16, -2, -2, 6\}$	6 4 6	4 8 3	6 3 3		$R_2 \simeq T(10)$, R_3 SRG
	$\{12, -3, 3, -3\}$	4 8 4	8 4 4	4 4 3		

45	$\{24, 3^{20}, -3^{20}, -6^{4}\}$	12 5 6	5 0 3	6 3 3	≥ 2	$\overline{GQ(4,2)}\backslash$spread
	$\{8, -1, -1, 8\}$	15 0 9	0 7 0	9 0 3		R_3 SRG
	$\{12, -3, 3, -3\}$	12 6 6	6 0 2	6 2 3		
45	$\{24, 2^{27}, -3^{8}, -6^{9}\}$	11 10 2	10 4 2	2 2 0	396	$\overline{T(10)}\backslash$spread [83]
	$\{16, -2, -2, 6\}$	15 6 3	6 8 1	3 1 0		R_2 SRG
	$\{4, -1, 4, -1\}$	12 12 0	12 4 0	0 0 3		
48	$\{12, 2^{30}, -4^{15}, -6^{2}\}$	1 5 5	5 0 10	5 10 5	3	2 linked 2-(16,6,2) [80]
	$\{15, -1, -1, 15\}$	4 0 8	0 14 0	8 0 12		
	$\{20, -2, 4, -10\}$	3 6 3	6 0 9	3 9 7		Q-213
48	$\{15, 5^{12}, -1^{15}, -3^{20}\}$	6 8 0	8 20 2	0 2 0	0	
	$\{30, -5, -2, 3\}$	4 10 1	10 18 1	1 1 0		DRG
	$\{2, -1, 2, -1\}$	0 15 0	15 15 0	0 0 1		
51	$\{16, 4^{17}, -1^{16}, -4^{17}\}$	5 10 0	10 20 2	0 2 0	≥ 1	3(Cycl(16)+1)
	$\{32, -4, -2, 4\}$	5 10 1	10 20 1	1 1 0		DRG
	$\{2, -1, 2, -1\}$	0 16 0	16 16 0	0 0 1		
52	$\{25, 5^{13}, -1^{25}, -5^{13}\}$	12 12 0	12 12 1	0 1 0	4	Taylor [19, 101]
	$\{25, -5, -1, 5\}$	12 12 1	12 12 0	1 0 0		R_1 and R_2 DRG
	$\{1, -1, 1, -1\}$	0 25 0	25 0 0	0 0 0		Q-123, Q-321
56	$\{15, 7^{7}, 1^{20}, -3^{28}\}$	6 8 0	8 16 6	0 6 4	1	$J(8,3)$
	$\{30, -2, -5, 3\}$	4 8 3	8 15 6	3 6 1		DRG
	$\{10, -6, 3, -1\}$	0 9 6	9 18 3	6 3 0		Q-123
56	$\{27, 3^{21}, -1^{27}, -9^{7}\}$	10 16 0	16 10 1	0 1 0	1	2(Schläfli+1)
	$\{27, -3, -1, 9\}$	16 10 1	10 16 0	1 0 0		R_1 and R_2 DRG
	$\{1, -1, 1, -1\}$	0 27 0	27 0 0	0 0 0		Q-123, Q-321
60	$\{15, 3^{25}, 0^{16}, -5^{18}\}$	2 8 4	8 8 8	4 8 8	≥ 1	hyperbolic quadric in $PG(3,5)$
	$\{24, 0, -6, 4\}$	5 5 5	5 8 10	5 10 5		
	$\{20, -4, 5, 0\}$	3 6 6	6 12 6	6 6 7		
60	$\{21, 3^{32}, -4^{24}, -7^{3}\}$	6 6 8	6 0 8	8 8 8	?	
	$\{14, -1, -1, 14\}$	9 0 12	0 13 0	12 0 12		
	$\{24, -3, 4, -8\}$	7 7 7	7 0 7	7 7 9		
63	$\{6, 3^{21}, -1^{27}, -3^{14}\}$	1 4 0	4 4 16	0 16 16	2	$GH(2,2)$
	$\{24, 0, -4, 6\}$	1 1 4	1 10 12	4 12 16		DRG, R_3 SRG
	$\{32, -4, 4, -4\}$	0 3 3	3 9 12	3 12 16		
63	$\{24, 4^{27}, -3^{8}, -4^{27}\}$	9 12 2	12 16 4	2 4 0	≥ 1	SRG(63,30,13,15)$\backslash$spread
	$\{32, -4, -4, 4\}$	9 12 3	12 16 3	3 3 0		R_2 SRG
	$\{6, -1, 6, -1\}$	8 16 0	16 16 0	0 0 5		
63	$\{24, 5^{21}, -3^{35}, -4^{6}\}$	10 3 10	3 0 5	10 5 15	?	SRG(63,32,16,16)$\backslash$spread
	$\{8, -1, -1, 8\}$	9 0 15	0 7 0	15 0 15		R_3 SRG
	$\{30, -5, 3, -5\}$	8 4 12	4 0 4	12 4 13		
64	$\{7, 3^{21}, -1^{35}, -5^{7}\}$	0 6 0	6 0 15	0 15 20	1	Folded 7-cube
	$\{21, 1, -3, 9\}$	2 0 5	0 10 10	5 10 20		R_1 and R_2 DRG, R_3 SRG
	$\{35, -5, 3, -5\}$	0 3 4	3 6 12	4 12 18		Q-123, Q-312
64	$\{9, 5^{9}, 1^{27}, -3^{27}\}$	2 6 0	6 12 9	0 9 18	2	$H(3,4)$, Doob
	$\{27, 3, -5, 3\}$	2 4 3	4 10 12	3 12 12		DRG, R_2 SRG
	$\{27, -9, 3, -1\}$	0 3 6	3 12 12	6 12 8		Q-123
64	$\{14, 2^{42}, -2^{7}, -6^{14}\}$	0 12 1	12 24 6	1 6 0	?	
	$\{42, -2, -6, 6\}$	4 8 2	8 28 5	2 5 0		
	$\{7, -1, 7, -1\}$	2 12 0	12 30 0	0 0 6		
64	$\{15, 3^{30}, -1^{15}, -5^{18}\}$	2 12 0	12 30 3	0 3 0	≥ 5	SRG(64,18,2,6)$\backslash$spread
	$\{45, -3, -3, 5\}$	4 10 1	10 32 2	1 2 0		DRG, R_2 SRG
	$\{3, -1, 3, -1\}$	0 15 0	15 30 0	0 0 2		
64	$\{18, 6^{15}, -2^{45}, -6^{3}\}$	7 5 5	5 0 10	5 10 15	0	linked designs
	$\{15, -1, -1, 15\}$	6 0 12	0 14 0	12 0 18		
	$\{30, -6, 2, -10\}$	3 6 9	6 0 9	9 9 11		Q-123
64	$\{30, 6^{15}, -2^{45}, -10^{3}\}$	14 9 6	9 0 6	6 6 6	12	3 linked 2-(16,6,2), SRG$\backslash$spread [80]
	$\{15, -1, -1, 15\}$	18 0 12	0 14 0	12 0 6		R_3 SRG
	$\{18, -6, 2, -6\}$	10 10 10	10 0 5	10 5 2		Q-123
65	$\{10, 5^{13}, 0^{26}, -3^{25}\}$	3 6 0	6 12 12	0 12 12	1	Locally Petersen
	$\{30, 0, -5, 4\}$	2 4 4	4 13 12	4 12 8		DRG
	$\{24, -6, 4, -2\}$	0 5 5	5 15 10	5 10 8		
66	$\{15, 2^{44}, -3^{11}, -7^{10}\}$	0 10 4	10 8 12	4 12 4	≥ 1	block scheme 4-(11,5,1)
	$\{30, -1, -6, 8\}$	5 4 6	4 15 10	6 10 4		$R_3 \simeq T(12)$
	$\{20, -2, 8, -2\}$	3 9 3	9 15 6	3 6 10		Q-312
66	$\{20, 2^{44}, -2^{10}, -8^{11}\}$	2 16 1	16 20 4	1 4 0	?	
	$\{40, -2, -4, 8\}$	8 10 2	10 26 3	2 3 0		
	$\{5, -1, 5, -1\}$	4 16 0	16 24 0	0 0 4		
66	$\{40, 2^{44}, -4^{10}, -8^{11}\}$	22 14 3	14 4 2	3 2 0	526915620	$\overline{T(12)}\backslash$spread [44]
	$\{20, -2, -2, 8\}$	28 8 4	8 10 1	4 1 0		R_2 SRG
	$\{5, -1, 5, -1\}$	24 16 0	16 4 0	0 0 4		

68	{12,	4^{17},	0^{34},	-5^{16}}	1	10	0	10	20	10	0	10	5	1	Doro, 3-(17,5,1)
	{40,	0 ,	-4 ,	6 }	3	6	3	6	24	9	3	9	3		DRG
	{15,	-5 ,	3 ,	-2 }	0	8	4	8	24	8	4	8	2		
70	{17,	3^{34},	-3^{34},	-17^{1}}	0	16	0	16	0	18	0	18	0	≥ 53387	*IG*(35,17,8) [102]
	{34,	-1 ,	-1 ,	34 }	8	0	9	0	33	0	9	0	9		R_1 and R_3 DRG
	{18,	-3 ,	3 ,	-18 }	0	17	0	17	0	17	0	17	0		*Q*-123, *Q*-213
70	{18,	2^{49},	-3^{6},	-7^{14}}	1	14	2	14	21	7	2	7	0	≥ 1	Merging example (Section 4.1.7)
	{42,	-2 ,	-7 ,	7 }	6	9	3	9	26	6	3	6	0		
	{ 9,	-1 ,	9 ,	-1 }	4	14	0	14	28	0	0	0	8		
70	{18,	7^{14},	-2^{49},	-3^{6}}	8	2	7	2	0	7	7	7	28	?	SRG(70,27,12,9)\spread
	{ 9,	-1 ,	-1 ,	9 }	4	0	14	0	8	0	14	0	28		R_3 SRG
	{42,	-7 ,	2 ,	-7 }	3	3	12	3	0	6	12	6	23		
70	{36,	3^{40},	-4^{9},	-6^{20}}	17	15	3	15	9	3	3	3	0	≥ 1	SRG(70,42,23,28)\spread
	{27,	-3 ,	-3 ,	6 }	20	12	4	12	12	2	4	2	0		R_2 SRG
	{ 6,	-1 ,	6 ,	-1 }	18	18	0	18	9	0	0	0	5		
72	{15,	3^{35},	-3^{35},	-15^{1}}	0	14	0	14	0	21	0	21	0	≥ 25634	*IG*(36,15,6) [100, 101]
	{35,	-1 ,	-1 ,	35 }	6	0	9	0	34	0	9	0	12		R_1 and R_3 DRG
	{21,	-3 ,	3 ,	-21 }	0	15	0	15	0	20	0	20	0		*Q*-123, *Q*-213
72	{35,	5^{21},	-1^{35},	-7^{15}}	16	18	0	18	16	1	0	1	0	≥ 227	Taylor [101]
	{35,	-5 ,	-1 ,	7 }	18	16	1	16	18	0	1	0	0		R_1 and R_2 DRG
	{ 1,	-1 ,	1 ,	-1 }	0	35	0	35	0	0	0	0	0		*Q*-123, *Q*-321
75	{24,	6^{20},	-1^{24},	-4^{30}}	9	14	0	14	32	2	0	2	0	?	
	{48,	-6 ,	-2 ,	4 }	7	16	1	16	30	1	1	1	0		DRG
	{ 2,	-1 ,	2 ,	-1 }	0	24	0	24	24	0	0	0	1		
75	{28,	3^{42},	-2^{14},	-7^{18}}	8	18	1	18	21	3	1	3	0	?	SRG(75,32,10,16)\spread
	{42,	-3 ,	-3 ,	7 }	12	14	2	14	25	2	2	2	0		R_2 SRG
	{ 4,	-1 ,	4 ,	-1 }	7	21	0	21	21	0	0	0	3		
75	{28,	8^{14},	-2^{56},	-7^{4}}	13	6	8	6	0	8	8	8	16	0	SRG(75,42,25,21)\spread
	{14,	-1 ,	-1 ,	14 }	12	0	16	0	13	0	16	0	16		R_3 SRG
	{32,	-8 ,	2 ,	-8 }	7	7	14	7	0	7	14	7	10		*Q*-123
76	{18,	3^{38},	-1^{18},	-6^{19}}	2	15	0	15	36	3	0	3	0	0	SRG(76,21,2,7)\spread
	{54,	-3 ,	-3 ,	6 }	5	12	1	12	39	2	1	2	0		DRG, R_2 SRG
	{ 3,	-1 ,	3 ,	-1 }	0	18	0	18	36	0	0	0	2		
78	{25,	5^{26},	-1^{25},	-5^{26}}	8	16	0	16	32	2	0	2	0	≥ 1	3(Cycl(25)+1)
	{50,	-5 ,	-2 ,	5 }	8	16	1	16	32	1	1	1	0		DRG
	{ 2,	-1 ,	2 ,	-1 }	0	25	0	25	25	0	0	0	1		
80	{13,	3^{39},	-3^{39},	-13^{1}}	0	12	0	12	0	27	0	27	0	≥ 930	*IG*(40,13,4) [102]
	{39,	-1 ,	-1 ,	39 }	4	0	9	0	38	0	9	0	18		R_1 and R_3 SRG
	{27,	-3 ,	3 ,	-27 }	0	13	0	13	0	26	0	26	0		*Q*-123, *Q*-213
80	{24,	2^{60},	-6^{4},	-8^{15}}	3	15	5	15	15	10	5	10	0	1	4 linked 2-(16,6,2) [80]
	{40,	-2 ,	-10 ,	8 }	9	9	6	9	21	9	6	9	0		
	{15,	-1 ,	15 ,	-1 }	8	16	0	16	24	0	0	0	14		*Q*-312
81	{10,	7^{10},	1^{20},	-2^{50}}	5	4	0	4	6	10	0	10	40	0	$\lambda_3 = -2$
	{20,	2 ,	-7 ,	2 }	2	3	5	3	1	15	5	15	30		DRG, R_2 SRG
	{50,	-10 ,	5 ,	-1 }	0	2	8	2	6	12	8	12	29		*Q*-123
81	{16,	4^{32},	-2^{32},	-5^{16}}	3	8	4	8	8	16	4	16	12	?	
	{32,	-1 ,	-4 ,	8 }	4	4	8	4	15	12	8	12	12		R_3 SRG
	{32,	-4 ,	5 ,	-4 }	2	8	6	8	12	12	6	12	13		
81	{20,	5^{20},	2^{20},	-4^{40}}	5	6	8	6	6	8	8	8	24	?	
	{20,	2 ,	-7 ,	2 }	6	6	8	6	1	12	8	12	20		R_2 SRG (unique)
	{40,	-8 ,	4 ,	1 }	4	4	12	4	6	10	12	10	17		
81	{24,	3^{48},	-3^{8},	-6^{24}}	5	16	2	16	26	6	2	6	0	?	
	{48,	-3 ,	-6 ,	6 }	8	13	3	13	29	5	3	5	0		
	{ 8,	-1 ,	8 ,	-1 }	6	18	0	18	30	0	0	0	7		
84	{18,	9^{8},	2^{27},	-3^{48}}	7	10	0	10	25	10	0	10	10	1	*J*(9,3)
	{45,	0 ,	-7 ,	3 }	4	10	4	10	22	12	4	12	4		DRG
	{20,	-10 ,	4 ,	-1 }	0	9	9	9	27	9	9	9	1		*Q*-123
84	{20,	4^{35},	-1^{20},	-5^{28}}	4	15	0	15	42	3	0	3	0	?	
	{60,	-4 ,	-3 ,	5 }	5	14	1	14	43	2	1	2	0		DRG
	{ 3,	-1 ,	3 ,	-1 }	0	20	0	20	40	0	0	0	2		
85	{16,	4^{34},	-1^{16},	-4^{34}}	3	12	0	12	48	4	0	4	0	≥ 2	*GQ*(4,4)\spread
	{64,	-4 ,	-4 ,	4 }	3	12	1	12	48	3	1	3	0		DRG, R_2 SRG
	{ 4,	-1 ,	4 ,	-1 }	0	16	0	16	48	0	0	0	3		
85	{48,	5^{30},	-3^{50},	-12^{4}}	26	11	10	11	0	5	10	5	5	?	SRG(85,64,48,48)\spread
	{16,	-1 ,	-1 ,	16 }	33	0	15	0	15	0	15	0	5		R_3 SRG
	{20,	-5 ,	3 ,	-5 }	24	12	12	12	0	4	12	4	3		
88	{12,	4^{22},	1^{32},	-4^{33}}	1	10	0	10	40	10	0	10	5	?	
	{60,	0 ,	-6 ,	4 }	2	8	2	8	40	11	2	11	2		DRG
	{15,	-5 ,	4 ,	-1 }	0	8	4	8	44	8	4	8	2		

90	{12, 3^{44}, -3^{44}, -12^{1}}	0 11 0	11 0 33	0 33 0	≥ 2285	$IG(45,12,3)$ [81]
	{44, -1, -1, 44}	3 0 9	0 43 0	9 0 24		R_1 and R_3 DRG
	{33, -3, 3, -33}	0 12 0	12 0 32	0 32 0		Q-123, Q-213
90	{44, 4^{33}, -1^{44}, -11^{12}}	18 25 0	25 18 1	0 1 0	0	Taylor
	{44, -4, -1, 11}	25 18 1	18 25 0	1 0 0		R_1 and R_2 DRG
	{1, -1, 1, -1}	0 44 0	44 0 0	0 0 0		Q-123, Q-321
91	{20, 7^{12}, 0^{65}, -8^{13}}	3 12 4	12 6 12	4 12 24	?	
	{30, 4, -3, 9}	8 4 8	4 13 12	8 12 20		
	{40, -12, 2, -2}	2 6 12	6 9 15	12 15 12		Q-123
91	{60, 2^{65}, -5^{12}, -10^{13}}	37 18 4	18 4 2	4 2 0	≈$1.13*10^{18}$	$\overline{T(14)}\setminus$spread [44]
	{24, -2, -2, 10}	45 10 5	10 12 1	5 1 0		R_2 SRG
	{6, -1, 6, -1}	40 20 0	20 4 0	0 0 5		
95	{36, 3^{57}, -2^{18}, -9^{19}}	10 24 1	24 27 3	1 3 0	?	SRG(95,40,12,20)$\setminus$spread
	{54, -3, -3, 9}	16 18 2	18 33 2	2 2 0		R_2 SRG
	{4, -1, 4, -1}	9 27 0	27 27 0	0 0 3		
96	{15, 5^{30}, -1^{15}, -3^{50}}	4 10 0	10 60 5	0 5 0	≥ 1	$GQ(5,3)\setminus$spread
	{75, -5, -5, 3}	2 12 1	12 58 4	1 4 0		DRG, R_2 SRG
	{5, -1, 5, -1}	0 15 0	15 60 0	0 0 4		
96	{15, 7^{18}, -1^{45}, -3^{32}}	6 8 0	8 36 16	0 16 4	0	
	{60, -4, -4, 6}	2 9 4	9 38 12	4 12 4		DRG, R_2 and R_3 SRG
	{20, -4, 4, -4}	0 12 3	12 36 12	3 12 4		
96	{19, 7^{19}, -1^{57}, -5^{19}}	6 12 0	12 30 15	0 15 4	?	
	{57, -3, -3, 9}	4 10 5	10 36 10	5 10 4		DRG, R_2 and R_3 SRG
	{19, -5, 3, -5}	0 15 4	15 30 12	4 12 2		Q-123
96	{25, 5^{20}, 1^{50}, -7^{25}}	4 8 12	8 4 8	12 8 30	?	
	{20, 4, -4, 4}	10 5 10	5 4 10	10 10 30		R_2 and R_3 SRG
	{50, -10, 2, 2}	6 4 15	4 4 12	15 12 22		
96	{30, 2^{75}, -6^{5}, -10^{15}}	4 20 5	20 20 10	5 10 0	1	5 linked 2-(16,6,2) [80]
	{50, -2, -10, 10}	12 12 6	12 28 9	6 9 0		
	{15, -1, 15, -1}	10 20 0	20 30 0	0 0 14		Q-312
96	{30, 4^{48}, -2^{15}, -6^{32}}	8 20 1	20 36 4	1 4 0	?	SRG(96,35,10,14)$\setminus$spread
	{60, -4, -4, 6}	10 18 2	18 38 3	2 3 0		R_2 SRG
	{5, -1, 5, -1}	6 24 0	24 36 0	0 0 4		
96	{30, 6^{30}, -2^{45}, -6^{20}}	10 15 4	15 18 12	4 12 4	?	
	{45, -3, -3, 9}	10 12 8	12 24 8	8 8 4		R_2 and R_3 SRG
	{20, -4, 4, -4}	6 18 6	18 18 9	6 9 4		
96	{30, 10^{15}, -2^{75}, -6^{5}}	14 5 10	5 0 10	10 10 30	0	SRG(96,45,24,18)$\setminus$spread
	{15, -1, -1, 15}	10 0 20	0 14 0	20 0 30		R_3 SRG
	{50, -10, 2, -10}	6 6 18	6 0 9	18 9 22		Q-123
96	{60, 4^{45}, -4^{45}, -12^{5}}	36 11 12	11 0 4	12 4 4	≥ 1	$\overline{GQ(5,3)}\setminus$spread
	{15, -1, -1, 15}	44 0 16	0 14 0	16 0 4		R_3 SRG
	{20, -4, 4, -4}	36 12 12	12 0 3	12 3 4		
96	{38, 6^{19}, 2^{38}, -6^{38}}	14 9 14	9 4 6	14 6 18	?	
	{19, 3, -5, 3}	18 8 12	8 2 8	12 8 18		R_2 and R_3 SRG
	{38, -10, 2, 2}	14 6 18	6 4 9	18 9 10		
96	{45, 3^{60}, -3^{15}, -9^{20}}	18 24 2	24 18 3	2 3 0	?	SRG(96,50,22,30)$\setminus$spread
	{45, -3, -3, 9}	24 18 3	18 24 2	3 2 0		R_2 SRG
	{5, -1, 5, -1}	18 27 0	27 18 0	0 0 4		
96	{45, 7^{27}, -3^{63}, -9^{5}}	22 8 14	8 0 7	14 7 14	?	SRG(96,60,38,36)$\setminus$spread
	{15, -1, -1, 15}	24 0 21	0 14 0	21 0 14		R_3 SRG
	{35, -7, 3, -7}	18 9 18	9 0 6	18 6 10		
99	{28, 3^{63}, -5^{21}, -8^{14}}	5 10 12	10 3 15	12 15 15	?	
	{28, -1, -5, 10}	10 3 15	3 12 12	15 12 15		R_3 SRG
	{42, -3, 9, -3}	8 10 10	10 8 10	10 10 21		Q-312
99	{28, 6^{21}, 1^{44}, -6^{33}}	7 6 14	6 2 6	14 6 36	?	
	{14, 3, -4, 3}	12 4 12	4 1 8	12 8 36		R_2 and R_3 SRG
	{56, -10, 2, 2}	7 3 18	3 2 9	18 9 28		
99	{32, 8^{22}, -1^{32}, -4^{44}}	13 18 0	18 44 2	0 2 0	?	
	{64, -8, -2, 4}	9 22 1	22 40 1	1 1 0		DRG
	{2, -1, 2, -1}	0 32 0	32 32 0	0 0 1		
99	{40, 5^{44}, -4^{10}, -5^{44}}	16 20 3	20 25 5	3 5 0	≥ 1	SRG(99,48,22,24)$\setminus$spread
	{50, -5, -5, 5}	16 20 4	20 25 4	4 4 0		R_2 SRG
	{8, -1, 8, -1}	15 25 0	25 25 0	0 0 7		
99	{40, 6^{36}, -4^{54}, -5^{8}}	17 4 18	4 0 6	18 6 24	?	SRG(99,50,25,25)$\setminus$spread
	{10, -1, -1, 10}	16 0 24	0 9 0	24 0 24		R_3 SRG
	{48, -6, 4, -6}	15 5 20	5 0 5	20 5 22		
100	{18, 3^{56}, -2^{18}, -6^{25}}	1 14 2	14 35 14	2 14 2	?	
	{63, -2, -7, 7}	4 10 4	10 40 12	4 12 2		$R_3 \simeq L_2(10)$
	{18, -2, 8, -2}	2 14 2	14 42 7	2 7 8		

100	{22, 7^{16}, 2^{33}, -4^{50}}	6 9 6	9 12 12	6 12 26	?	
	{33, 3 , -7 , 3 }	6 8 8	8 8 16	8 16 20		R_2 SRG
	{44, -11 , 4 , 0 }	3 6 13	6 12 15	13 15 15		
100	{22, 8^{20}, -2^{55}, -3^{24}}	9 6 6	6 9 18	6 18 20	?	
	{33, -3 , -3 , 8 }	4 6 12	6 14 12	12 12 20		R_2 and R_3 SRG
	{44, -6 , 4 , -6 }	3 9 10	9 9 15	10 15 18		
100	{48, 4^{50}, -2^{24}, -8^{25}}	21 25 1	25 21 2	1 2 0	?	
	{48, -4 , -2 , 8 }	25 21 2	21 25 1	2 1 0		
	{ 3, -1 , 3 , -1 }	16 32 0	32 16 0	0 0 2		
100	{49, 7^{25}, -1^{49}, -7^{25}}	24 24 0	24 24 1	0 1 0	≥ 18	2(P(49)+1) [19, 101]
	{49, -7 , -1 , 7 }	24 24 1	24 24 0	1 0 0		R_1 and R_2 DRG
	{ 1, -1 , 1 , -1 }	0 49 0	49 0 0	0 0 0		Q-123, Q-321

A.4.3. Two integral eigenvalues

Excluded here are association schemes generated by SRG⊗J_n

v	spectrum	L_1	L_2	L_3	#	
12	{ 5, -1^{5}, 2.236^{3}, -2.236^{3}}	2 2 0	2 2 1	0 1 0	1	Icosahedron
	{ 5, -1 , -2.236 , 2.236 }	2 2 1	2 2 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1 , -1.000 , -1.000 }	0 5 0	5 0 0	0 0 0		Q-213, Q-312
14	{ 3, -3^{1}, 1.414^{6}, -1.414^{6}}	0 2 0	2 0 4	0 4 0	1	IG(7,3,1)
	{ 6, 6 , -1.000 , -1.000 }	1 0 2	0 5 0	2 0 2		R_1 and R_3 DRG
	{ 4, -4 , -1.414 , 1.414 }	0 3 0	3 0 3	0 3 0		Q-231, Q-321
21	{ 4, -2^{8}, 2.414^{6}, -0.414^{6}}	1 2 0	2 2 4	0 4 4	1	L(IG(7,3,1))
	{ 8, 2 , -0.586 , -3.414 }	1 1 2	1 2 4	2 4 2		DRG
	{ 8, -1 , -2.828 , 2.828 }	0 2 2	2 4 2	2 2 3		
22	{ 5, -5^{1}, 1.732^{10}, -1.732^{10}}	0 4 0	4 0 6	0 6 0	1	IG(11,5,2)
	{10, 10 , -1.000 , -1.000 }	2 0 3	0 9 0	3 0 3		R_1 and R_3 DRG
	{ 6, -6 , -1.732 , 1.732 }	0 5 0	5 0 5	0 5 0		Q-231, Q-321
24	{ 7, -1^{7}, 2.646^{8}, -2.646^{8}}	2 4 0	4 8 2	0 2 0	1	Klein
	{14, -2 , -2.646 , 2.646 }	2 4 1	4 8 1	1 1 0		DRG
	{ 2, 2 , -1.000 , -1.000 }	0 7 0	7 7 0	0 0 1		
26	{ 4, -4^{1}, 1.732^{12}, -1.732^{12}}	0 3 0	3 0 9	0 9 0	1	IG(13,4,1)
	{12, 12 , -1.000 , -1.000 }	1 0 3	0 11 0	3 0 6		R_1 and R_3 DRG
	{ 9, -9 , -1.732 , 1.732 }	0 4 0	4 0 8	0 8 0		Q-231, Q-321
28	{13, -1^{13}, 3.606^{7}, -3.606^{7}}	6 6 0	6 6 1	0 1 0	1	Taylor
	{13, -1 , -3.606 , 3.606 }	6 6 1	6 6 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1 , -1.000 , -1.000 }	0 13 0	13 0 0	0 0 0		Q-213, Q-312
33	{10, -1^{10}, 3.162^{11}, -3.162^{11}}	3 6 0	6 12 2	0 2 0	0	Hasse-Minkowski
	{20, -2 , -3.162 , 3.162 }	3 6 1	6 12 1	1 1 0		DRG
	{ 2, 2 , -1.000 , -1.000 }	0 10 0	10 10 0	0 0 1		
35	{ 6, -1^{6}, 2.449^{14}, -2.449^{14}}	1 4 0	4 16 4	0 4 0	0	Hasse-Minkowski, PG(2,6)
	{24, -4 , -2.449 , 2.449 }	1 4 1	4 16 3	1 3 0		DRG
	{ 4, 4 , -1.000 , -1.000 }	0 6 0	6 18 0	0 0 3		
36	{17, -1^{17}, 4.123^{9}, -4.123^{9}}	8 8 0	8 8 1	0 1 0	1	2(P(17)+1)
	{17, -1 , -4.123 , 4.123 }	8 8 1	8 8 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1 , -1.000 , -1.000 }	0 17 0	17 0 0	0 0 0		Q-213, Q-312
38	{ 9, -9^{1}, 2.236^{18}, -2.236^{18}}	0 8 0	8 0 10	0 10 0	6	IG(19,9,4)
	{18, -18 , -1.000 , -1.000 }	4 0 5	0 17 0	5 0 5		R_1 and R_3 DRG
	{10, -10 , -2.236 , 2.236 }	0 9 0	9 0 9	0 9 0		Q-231, Q-321
40	{ 9, 1^{15}, 2.162^{12}, -4.162^{12}}	0 4 4	4 4 4	4 4 10	?	
	{12, -4 , 2.000 , 2.000 }	3 3 3	3 2 6	3 6 9		R_2 SRG
	{18, 2 , -5.162 , 1.162 }	2 2 5	2 4 6	5 6 6		
40	{18, -2^{9}, 3.464^{15}, -3.464^{15}}	8 8 1	8 8 2	1 2 0	0	Hasse-Minkowski
	{18, -2 , -3.464 , 3.464 }	8 8 2	8 8 1	2 1 0		
	{ 3, 3 , -1.000 , -1.000 }	6 12 0	12 6 0	0 0 2		
42	{13, -1^{13}, 3.606^{14}, -3.606^{14}}	4 8 0	8 16 2	0 2 0	≥ 1	3(Cycl(13)+1)
	{26, -2 , -3.606 , 3.606 }	4 8 1	8 16 1	1 1 0		DRG
	{ 2, 2 , -1.000 , -1.000 }	0 13 0	13 13 0	0 0 1		
44	{ 7, -7^{1}, 2.236^{21}, -2.236^{21}}	0 6 0	6 0 15	0 15 0	0	IG(22,7,2)
	{21, 21 , -1.000 , -1.000 }	2 0 5	0 20 0	5 0 10		R_1 and R_3 DRG
	{15, -15 , -2.236 , 2.236 }	0 7 0	7 0 14	0 14 0		Q-231, Q-321
44	{21, -1^{21}, 4.583^{11}, -4.583^{11}}	10 10 0	10 10 1	0 1 0	0	Hasse-Minkowski
	{21, -1 , -4.583 , 4.583 }	10 10 1	10 10 0	1 0 0		$R_1 \simeq R_2$ DRG
	{ 1, 1 , -1.000 , -1.000 }	0 21 0	21 0 0	0 0 0		Q-213, Q-312

45	$\{16, -2^{20}, 4.873^{12}, -2.873^{12}\}$	7 5 3	5 5 6	3 6 3	?	
	$\{16, -2, -2.873, 4.873\}$	5 5 6	5 7 3	6 3 3		R_3 SRG
	$\{12, 3, -3.000, -3.000\}$	4 8 4	8 4 4	4 4 3		
46	$\{11, -11^{1}, 2.449^{22}, -2.449^{22}\}$	0 10 0	10 0 12	0 12 0	582	*IG*(23,11,5) [102]
	$\{22, 22, -1.000, -1.000\}$	5 0 6	0 21 0	6 0 6		R_1 and R_3 DRG
	$\{12, -12, -2.449, 2.449\}$	0 11 0	11 0 11	0 11 0		*Q*-231, *Q*-231
50	$\{9, -9^{1}, 2.449^{24}, -2.449^{24}\}$	0 8 0	8 0 16	0 16 0	50	*IG*(25,9,3) [43]
	$\{24, 24, -1.000, -1.000\}$	3 0 6	0 23 0	6 0 10		R_1 and R_3 DRG
	$\{16, -16, -2.449, 2.449\}$	0 9 0	9 0 15	0 15 0		*Q*-231, *Q*-321
52	$\{6, -2^{27}, 3.732^{12}, 0.268^{12}\}$	2 3 0	3 6 9	0 9 18	1	*L*(*IG*(13,4,1))
	$\{18, 2, 0.464, -6.464\}$	1 2 3	2 3 12	3 12 12		DRG
	$\{27, -1, -5.196, 5.196\}$	0 2 4	2 8 8	4 8 14		
54	$\{13, -13^{1}, 2.646^{26}, -2.646^{26}\}$	0 12 0	12 0 14	0 14 0	105041	*IG*(27,13,6) [102]
	$\{26, 26, -1.000, -1.000\}$	6 0 7	0 25 0	7 0 7		R_1 and R_3 DRG
	$\{14, -14, -2.646, 2.646\}$	0 13 0	13 0 13	0 13 0		*Q*-231, *Q*-321
55	$\{18, -4^{10}, 3.854^{22}, -2.854^{22}\}$	6 6 5	6 4 8	5 8 5	?	
	$\{18, 7, -2.000, -2.000\}$	6 4 8	4 9 4	8 4 6		$R_2 \simeq T(11)$
	$\{18, -4, -2.854, 3.854\}$	5 8 5	8 4 6	5 6 6		
56	$\{5, -3^{15}, 2.414^{20}, -0.414^{20}\}$	0 4 0	4 4 12	0 12 18	0	Fon-der-Flaass [50]
	$\{20, 4, 0.828, -4.828\}$	1 1 3	1 6 12	3 12 15		DRG
	$\{30, -2, -4.243, 4.243\}$	0 2 3	2 8 10	3 10 16		
57	$\{6, -3^{20}, 2.618^{18}, 0.382^{18}\}$	0 5 0	5 15 10	0 10 10	≥ 1	Perkel
	$\{30, 3, 0.854, -5.854\}$	1 3 2	3 14 12	2 12 6		DRG
	$\{20, -1, -4.472, 4.472\}$	0 3 3	3 18 9	3 9 7		
58	$\{8, -8^{1}, 2.449^{28}, -2.449^{28}\}$	0 7 0	7 0 21	0 21 0	0	*IG*(29,8,2)
	$\{28, 28, -1.000, -1.000\}$	2 0 6	0 27 0	6 0 15		R_1 and R_3 DRG
	$\{21, -21, -2.449, 2.449\}$	0 8 0	8 0 20	0 20 0		*Q*-231, *Q*-321
60	$\{11, -1^{11}, 3.317^{24}, -3.317^{24}\}$	2 8 0	8 32 4	0 4 0	≥ 1	Mathon
	$\{44, -4, -3.317, 3.317\}$	2 8 1	8 32 3	1 3 0		DRG
	$\{4, 4, -1.000, -1.000\}$	0 11 0	11 33 0	0 0 3		
60	$\{19, -1^{19}, 4.359^{20}, -4.359^{20}\}$	6 12 0	12 24 2	0 2 0	≥ 1	3(Cycl(19)+1)
	$\{38, -2, -4.359, 4.359\}$	6 12 1	12 24 1	1 1 0		DRG
	$\{2, 2, -1.000, -1.000\}$	0 19 0	19 19 0	0 0 1		
60	$\{29, -1^{29}, 5.385^{15}, -5.385^{15}\}$	14 14 0	14 14 1	0 1 0	6	2(*P*(29)+1) [101]
	$\{29, -1, -5.385, 5.385\}$	14 14 1	14 14 0	1 0 0		R_1 and R_2 DRG
	$\{1, 1, -1.000, -1.000\}$	0 29 0	29 0 0	0 0 0		*Q*-213, *Q*-312
62	$\{6, -6^{1}, 2.236^{30}, -2.236^{30}\}$	0 5 0	5 0 25	0 25 0	1	*IG*(31,6,1)
	$\{30, 30, -1.000, -1.000\}$	1 0 5	0 29 0	5 0 20		R_1 and R_3 DRG
	$\{25, -25, -2.236, 2.236\}$	0 6 0	6 0 24	0 24 0		*Q*-231, *Q*-321
62	$\{10, -10^{1}, 2.646^{30}, -2.646^{30}\}$	0 9 0	9 0 21	0 21 0	82	*IG*(31,10,3) [98, 99]
	$\{30, 30, -1.000, -1.000\}$	3 0 7	0 29 0	7 0 14		R_1 and R_3 DRG
	$\{21, -21, -2.646, 2.646\}$	0 10 0	10 0 20	0 20 0		*Q*-231, *Q*-321
62	$\{15, -15^{1}, 2.828^{30}, -2.828^{30}\}$	0 14 0	14 0 16	0 16 0	≥ 633446	*IG*(31,15,7) [100]
	$\{30, 30, -1.000, -1.000\}$	7 0 8	0 29 0	8 0 8		R_1 and R_3 DRG
	$\{16, -16, -2.828, 2.828\}$	0 15 0	15 0 15	0 15 0		*Q*-231, *Q*-321
63	$\{8, -1^{8}, 2.828^{27}, -2.828^{27}\}$	1 6 0	6 36 6	0 6 0	1	*PG*(2,8)
	$\{48, -6, -2.828, 2.828\}$	1 6 1	6 36 5	1 5 0		DRG
	$\{6, 6, -1.000, -1.000\}$	0 8 0	8 40 0	0 0 5		
64	$\{14, -2^{7}, 3.464^{28}, -3.464^{28}\}$	3 9 1	9 27 6	1 6 0	?	
	$\{42, -6, -3.464, 3.464\}$	3 9 2	9 27 5	2 5 0		
	$\{7, 7, -1.000, -1.000\}$	2 12 0	12 30 0	0 0 6		
64	$\{30, -2^{15}, 4.472^{24}, -4.472^{24}\}$	14 14 1	14 14 2	1 2 0	?	
	$\{30, -2, -4.472, 4.472\}$	14 14 2	14 14 1	2 1 0		
	$\{3, 3, -1.000, -1.000\}$	10 20 0	20 10 0	0 0 2		
68	$\{12, -12^{1}, 2.828^{33}, -2.828^{33}\}$	0 11 0	11 0 22	0 22 0	0	*IG*(34,12,4)
	$\{33, 33, -1.000, -1.000\}$	4 0 8	0 32 0	8 0 14		R_1 and R_3 DRG
	$\{22, -22, -2.828, 2.828\}$	0 12 0	12 0 21	0 21 0		*Q*-231, *Q*-321
68	$\{33, -1^{33}, 5.745^{17}, -5.745^{17}\}$	16 16 0	16 16 1	0 1 0	0	Taylor, Hasse-Minkowski
	$\{33, -1, -5.745, 5.745\}$	16 16 1	16 16 0	1 0 0		R_1 and R_2 DRG
	$\{1, 1, -1.000, -1.000\}$	0 33 0	33 0 0	0 0 0		*Q*-213, *Q*-312
69	$\{22, -1^{22}, 4.690^{23}, -4.690^{23}\}$	7 14 0	14 28 2	0 2 0	0	Hasse-Minkowski
	$\{44, -2, -4.690, 4.690\}$	7 14 1	14 28 1	1 1 0		DRG
	$\{2, 2, -1.000, -1.000\}$	0 22 0	22 22 0	0 0 1		
72	$\{17, -1^{17}, 4.123^{27}, -4.123^{27}\}$	4 12 0	12 36 3	0 3 0	≥ 1	Mathon
	$\{51, -3, -4.123, 4.123\}$	4 12 1	12 36 2	1 2 0		DRG
	$\{3, 3, -1.000, -1.000\}$	0 17 0	17 34 0	0 0 2		
74	$\{9, -9^{1}, 2.646^{36}, -2.646^{36}\}$	0 8 0	8 0 28	0 28 0	3	*IG*(37,9,2) [2]
	$\{36, 36, -1.000, -1.000\}$	2 0 7	0 35 0	7 0 21		R_1 and R_3 DRG
	$\{28, -28, -2.646, 2.646\}$	0 9 0	9 0 27	0 27 0		*Q*-231, *Q*-321

76	{37, -1^{37}, 6.083^{19}, -6.083^{19}}	18 18 0	18 18 1	0 1 0	≥ 11	Taylor [19, 101]
	{37, -1 , -6.083 , 6.083 }	18 18 1	18 18 0	1 0 0		R_1 and R_2 DRG
	{ 1, 1 , -1.000 , -1.000 }	0 37 0	37 0 0	0 0 0		*Q*-213, *Q*-312
78	{19, -19^{1}, 3.162^{38}, -3.162^{38}}	0 18 0	18 0 20	0 20 0	≥ 19	*IG*(39,19,9) [100]
	{38, 38 , -1.000 , -1.000 }	9 0 10	0 37 0	10 0 10		R_1 and R_3 DRG
	{20,-20 , -3.162 , 3.162 }	0 19 0	19 0 19	0 19 0		*Q*-231, *Q*-321
81	{ 8, -1^{32}, 3.854^{24}, -2.854^{24}}	2 5 0	5 15 20	0 20 12	0	
	{40, -5 , -0.854 , 5.854 }	1 3 4	3 20 16	4 16 12		DRG
	{32, 5 , -4.000 , -4.000 }	0 5 3	5 20 15	3 15 13		
81	{10, 1^{20}, 2.854^{30}, -3.854^{30}}	0 4 5	4 6 10	5 10 35	?	
	{20, -7 , 2.000 , 2.000 }	2 3 5	3 1 15	5 15 30		R_2 SRG (unique)
	{50, 5 , -5.854 , 0.854 }	1 2 7	2 6 12	7 12 30		
81	{16, -2^{32}, 5.243^{24}, -3.243^{24}}	5 6 4	6 10 16	4 16 12	?	
	{32, -4 , -2.243 , 6.243 }	3 5 8	5 14 12	8 12 12		R_3 SRG
	{32, 5 , -4.000 , -4.000 }	2 8 6	8 12 12	6 12 13		
81	{20, 2^{20}, 3.243^{30}, -5.243^{30}}	3 6 10	6 6 8	10 8 22	?	
	{20, -7 , 2.000 , 2.000 }	6 6 8	6 1 12	8 12 20		R_2 SRG (unique)
	{40, 4 , -6.243 , 2.243 }	5 4 11	4 6 10	11 10 18		
81	{24, -3^{8}, 4.243^{36}, -4.243^{36}}	7 14 2	14 28 6	2 6 0	?	
	{48, -6 , -4.243 , 4.243 }	7 14 3	14 28 5	3 5 0		
	{ 8, 8 , -1.000 , -1.000 }	6 18 0	18 30 0	0 0 7		
81	{28, 1^{56}, 3.374^{12}, -10.374^{12}}	4 12 11	12 6 6	11 6 11	?	
	{24, -3 , 6.000 , 6.000 }	14 7 7	7 9 7	7 7 14		R_2 SRG
	{28, 1 ,-10.374 , 3.374 }	11 6 11	6 6 12	11 12 4		
82	{16, -16^{1}, 3.162^{40}, -3.162^{40}}	0 15 0	15 0 25	0 25 0	≥ 56000	*IG*(41,16,6) [100]
	{40, 40 , -1.000 , -1.000 }	6 0 10	0 39 0	10 0 15		R_1 and R_3 DRG
	{25,-25 , -3.162 , 3.162 }	0 16 0	16 0 24	0 24 0		*Q*-231, *Q*-321
84	{13, -1^{13}, 3.606^{35}, -3.606^{35}}	2 10 0	10 50 5	0 5 0	≥ 1	Mathon
	{65, -5 , -3.606 , 3.606 }	2 10 1	10 50 4	1 4 0		DRG
	{ 5, 5 , -1.000 , -1.000 }	0 13 0	13 52 0	0 0 4		
84	{41, -1^{41}, 6.403^{21}, -6.403^{21}}	20 20 0	20 20 1	0 1 0	≥ 18	Taylor [19, 101]
	{41, -1 , -6.403 , 6.403 }	20 20 1	20 20 0	1 0 0		R_1 and R_2 DRG
	{ 1, 1 , -1.000 , -1.000 }	0 41 0	41 0 0	0 0 0		*Q*-213, *Q*-312
85	{12, -5^{16}, 3.449^{34}, -1.449^{34}}	1 6 4	6 2 16	4 16 28	?	
	{24, 7 , 0.449 , -4.449 }	3 1 8	1 8 14	8 14 26		
	{48, -3 , -4.899 , 4.899 }	1 4 7	4 7 13	7 13 27		
85	{32, -2^{50}, 8.325^{17}, -4.325^{17}}	15 11 5	11 11 10	5 10 5	?	
	{32, -2 , -4.325 , 8.325 }	11 11 10	11 15 5	10 5 5		R_3 SRG
	{20, 3 , -5.000 , -5.000 }	8 16 8	16 8 8	8 8 3		
85	{32, -2^{16}, 4.899^{34}, -4.899^{34}}	12 18 1	18 27 3	1 3 0	0	Hasse-Minkowski
	{48, -3 , -4.899 , 4.899 }	12 18 2	18 27 2	2 2 0		
	{ 4, 4 , -1.000 , -1.000 }	8 24 0	24 24 0	0 0 3		
86	{ 7, -7^{1}, 2.449^{42}, -2.449^{42}}	0 6 0	6 0 36	0 36 0	0	*IG*(43,7,1)
	{42, 42 , -1.000 , -1.000 }	1 0 6	0 41 0	6 0 30		R_1 and R_3 DRG
	{36,-36 , -2.449 , 2.449 }	0 7 0	7 0 35	0 35 0		*Q*-231, *Q*-321
86	{15, -15^{1}, 3.162^{42}, -3.162^{42}}	0 14 0	14 0 28	0 28 0	0	*IG*(43,15,5)
	{42, 42 , -1.000 , -1.000 }	5 0 10	0 41 0	10 0 18		R_1 and R_3 DRG
	{28,-28 , -3.162 , 3.162 }	0 15 0	15 0 27	0 27 0		*Q*-231, *Q*-321
86	{21, -21^{1}, 3.317^{42}, -3.317^{42}}	0 20 0	20 0 22	0 22 0	≥ 14	*IG*(43,21,10) [102]
	{42, 42 , -1.000 , -1.000 }	10 0 11	0 41 0	11 0 11		R_1 and R_3 DRG
	{22,-22 , -3.317 , 3.317 }	0 21 0	21 0 21	0 21 0		*Q*-231, *Q*-321
87	{28, -1^{28}, 5.292^{29}, -5.292^{29}}	9 18 0	18 36 2	0 2 0	≥ 1	3(PseudoCycl(28)+1)
	{56, -2 , -5.292 , 5.292 }	9 18 1	18 36 1	1 1 0		DRG
	{ 2, 2 , -1.000 , -1.000 }	0 28 0	28 28 0	0 0 1		
88	{42, -2^{21}, 5.292^{33}, -5.292^{33}}	20 20 1	20 20 2	1 2 0	0	Hasse-Minkowski
	{42, -2 , -5.292 , 5.292 }	20 20 2	20 20 1	2 1 0		
	{ 3, 3 , -1.000 , -1.000 }	14 28 0	28 14 0	0 0 2		
92	{10, -10^{1}, 2.828^{45}, -2.828^{45}}	0 9 0	9 0 36	0 36 0	0	*IG*(46,10,2)
	{45, 45 , -1.000 , -1.000 }	2 0 8	0 44 0	8 0 28		R_1 and R_3 DRG
	{36,-36 , -2.828 , 2.828 }	0 10 0	10 0 35	0 35 0		*Q*-231, *Q*-321
92	{45, -1^{45}, 6.708^{23}, -6.708^{23}}	22 22 0	22 22 1	0 1 0	≥ 80	Taylor [19, 101]
	{45, -1 , -6.708 , 6.708 }	22 22 1	22 22 0	1 0 0		R_1 and R_2 DRG
	{ 1, 1 , -1.000 , -1.000 }	0 45 0	45 0 0	0 0 0		*Q*-213, *Q*-312
93	{20, 5^{32}, -1.586^{30}, -4.414^{30}}	5 8 6	8 8 16	6 16 18	?	
	{32, -1 , -5.657 , 5.657 }	5 5 10	5 11 15	10 15 15		
	{40, -5 , 6.243 , -2.243 }	3 8 9	8 12 12	9 12 18		
94	{23, -23^{1}, 3.464^{46}, -3.464^{46}}	0 22 0	22 0 24	0 24 0	≥ 1	*IG*(47,23,11)
	{46, 46 , -1.000 , -1.000 }	11 0 12	0 45 0	12 0 12		R_1 and R_3 DRG
	{24,-24 , -3.464 , 3.464 }	0 23 0	23 0 23	0 23 0		*Q*-231, *Q*-321

v	spectrum	L_1	L_2	L_3	#	
96	{19, -5^{19}, 4.464^{38}, -2.464^{38}}	4 12 2	12 30 15	2 15 2	?	
	{57, 9, -3.000, -3.000}	4 10 5	10 36 10	5 10 4		R_2 SRG
	{19, -5, -2.464, 4.464}	2 15 2	15 30 12	2 12 4		
96	{30, -6^{5}, 4.472^{45}, -4.472^{45}}	9 5 15	5 0 10	15 10 25	?	
	{15, 15, -1.000, -1.000}	10 0 20	0 14 0	20 0 30		
	{50, -10, -4.472, 4.472}	9 6 15	6 0 9	15 9 25		
96	{30, -2^{63}, 9.708^{16}, -3.708^{16}}	14 8 7	8 8 14	7 14 14	?	
	{30, -2, -3.708, 9.708}	8 8 14	8 14 7	14 7 14		R_3 SRG
	{35, 3, -7.000, -7.000}	6 12 12	12 6 12	12 12 10		
96	{31, -1^{31}, 5.568^{32}, -5.568^{32}}	10 20 0	20 40 2	0 2 0	≥ 1	3(Cycl(31)+1)
	{62, -2, -5.568, 5.568}	10 20 1	20 40 1	1 1 0		DRG
	{2, 2, -1.000, -1.000}	0 31 0	31 31 0	0 0 1		
96	{38, -2^{57}, 8.928^{19}, -4.928^{19}}	18 14 5	14 14 10	5 10 4	?	
	{38, -2, -4.928, 8.928}	14 14 10	14 18 5	10 5 4		R_3 SRG
	{19, 3, -5.000, -5.000}	10 20 8	20 10 8	8 8 2		
98	{16, -16^{1}, 3.317^{48}, -3.317^{48}}	0 15 0	15 0 33	0 33 0	≥ 22	*IG*(49,16,5)
	{48, 48, -1.000, -1.000}	5 0 11	0 47 0	11 0 22		R_1 and R_3 DRG
	{33, -33, -3.317, 3.317}	0 16 0	16 0 32	0 32 0		*Q*-231, *Q*-321
99	{10, -1^{10}, 3.162^{44}, -3.162^{44}}	1 8 0	8 64 8	0 8 0	0	*PG*(2,10)
	{80, -8, -3.162, 3.162}	1 8 1	8 64 7	1 7 0		DRG
	{8, 8, -1.000, -1.000}	0 10 0	10 70 0	0 0 7		
99	{42, -2^{54}, 8.374^{22}, -5.374^{22}}	20 17 4	17 17 8	4 8 2	?	
	{42, -2, -5.374, 8.374}	17 17 8	17 20 4	8 4 2		R_3 SRG
	{14, 3, -4.000, -4.000}	12 24 6	24 12 6	6 6 1		

A.4.4. One integral eigenvalue

v	spectrum	L_1	L_2	L_3	#	
7	{2, 1.247, -0.445, -1.802}	0 1 0	1 0 1	0 1 1	1	C_7
	{2, -0.445, -1.802, 1.247}	1 0 1	0 0 1	1 1 0		$R_1 \simeq R_2 \simeq R_3$ DRG
	{2, -1.802, 1.247, -0.445}	0 1 1	1 1 0	1 0 0		*Q*-123, *Q*-231, *Q*-312
13	{4, 1.377, 0.274, -2.651}	0 2 1	2 1 1	1 1 2	1	Cycl(13)
	{4, 0.274, -2.651, 1.377}	2 1 1	1 0 2	1 2 1		
	{4, -2.651, 1.377, 0.274}	1 1 2	1 2 1	2 1 0		
19	{6, 2.507, -1.222, -2.285}	2 2 1	2 1 3	1 3 2	1	Cycl(19)
	{6, -1.222, -2.285, 2.507}	2 1 3	1 2 2	3 2 1		
	{6, -2.285, 2.507, -1.222}	1 3 2	3 2 1	2 1 2		
28	{9, 2.604, -0.110, -3.494}	2 4 2	4 2 3	2 3 4	2	Mathon, Hollmann
	{9, -0.110, -3.494, 2.604}	4 2 3	2 2 4	3 4 2		
	{9, -3.494, 2.604, -0.110}	2 3 4	3 4 2	4 2 2		
31	{10, 3.084, -0.787, -3.297}	3 4 2	4 2 4	2 4 4	≥ 1	Cycl(31)
	{10, -0.787, -3.297, 3.084}	4 2 4	2 3 4	4 4 2		
	{10, -3.297, 3.084, -0.787}	2 4 4	4 4 2	4 2 3		
37	{12, 2.187, 1.158, -4.345}	2 5 4	5 4 3	4 3 5	≥ 1	Cycl(37)
	{12, 1.158, -4.345, 2.187}	5 4 3	4 2 5	3 5 4		
	{12, -4.345, 2.187, 1.158}	4 3 5	3 5 4	5 4 2		
43	{14, 2.888, 0.615, -4.503}	3 6 4	6 4 4	4 4 6	≥ 1	Cycl(43)
	{14, 0.615, -4.503, 2.888}	6 4 4	4 3 6	4 6 4		
	{14, -4.503, 2.888, 0.615}	4 4 6	4 6 4	6 4 3		
49	{16, 4.296, -2.137, -3.159}	6 5 4	5 4 7	4 7 5	≥ 1	Cycl(49)
	{16, -2.137, -3.159, 4.296}	5 4 7	4 6 5	7 5 4		
	{16, -3.159, 4.296, -2.137}	4 7 5	7 5 4	5 4 6		
52	{17, 4.302, -1.548, -3.754}	6 6 4	6 4 7	4 7 6	?	
	{17, -1.548, -3.754, 4.302}	6 4 7	4 6 6	7 6 4		
	{17, -3.754, 4.302, -1.548}	4 7 6	7 6 4	6 4 6		
61	{20, 4.230, -0.445, -4.786}	6 8 5	8 5 7	5 7 8	≥ 1	Cycl(61)
	{20, -0.445, -4.786, 4.230}	8 5 7	5 6 8	7 8 5		
	{20, -4.786, 4.230, -0.445}	5 7 8	7 8 5	8 5 6		
67	{22, 4.085, 0.230, -5.316}	6 9 6	9 6 7	6 7 9	≥ 1	Cycl(67)
	{22, 0.230, -5.316, 4.085}	9 6 7	6 6 9	7 9 6		
	{22, -5.316, 4.085, 0.230}	6 7 9	7 9 6	9 6 6		
73	{24, 4.950, -1.132, -4.818}	8 9 6	9 6 9	6 9 9	≥ 1	Cycl(73)
	{24, -1.132, -4.818, 4.950}	9 6 9	6 8 9	9 9 6		
	{24, -4.818, 4.950, -1.132}	6 9 9	9 9 6	9 6 8		
76	{25, 3.570, 1.444, -6.014}	6 10 8	10 8 7	8 7 10	?	
	{25, 1.444, -6.014, 3.570}	10 8 7	8 6 10	7 10 8		
	{25, -6.014, 3.570, 1.444}	8 7 10	7 10 8	10 8 6		

79	{26, 3.122, 2.108, -6.230}	6 10 9	10 9 7	9 7 10	≥ 1	Cycl(79)
	{26, 2.108, -6.230, 3.122}	10 9 7	9 6 10	7 10 9		
	{26, -6.230, 3.122, 2.108}	9 7 10	7 10 9	10 9 6		
91	{30, 4.412, 0.960, -6.373}	8 12 9	12 9 9	9 9 12	?	
	{30, 0.960, -6.373, 4.412}	12 9 9	9 8 12	9 12 9		
	{30, -6.373, 4.412, 0.960}	9 9 12	9 12 9	12 9 8		
91	{30, 5.909, -2.404, -4.506}	11 10 8	10 8 12	8 12 10	?	
	{30, -2.404, -4.506, 5.909}	10 8 12	8 11 10	12 10 8		
	{30, -4.506, 5.909, -2.404}	8 12 10	12 10 8	10 8 11		
97	{32, 6.207, -3.098, -4.109}	12 10 9	10 9 13	9 13 10	≥ 1	Cycl(97)
	{32, -3.098, -4.109, 6.207}	10 9 13	9 12 10	13 10 9		
	{32, -4.109, 6.207, -3.098}	9 13 10	13 10 9	10 9 12		

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Index

Samenvatting

Een graaf is in essentie een (eenvoudig) wiskundig model van een netwerk van, bijvoorbeeld, steden, computers, atomen, enz., maar ook van meer abstracte (wiskundige) objecten. Grafen worden toegepast in verscheidene gebieden, zoals scheikunde, besliskunde, elektrotechniek, architectuur en informatica. Grofweg gezegd is een graaf een verzameling punten die de knopen van het netwerk (in de voorbeelden zijn dit de steden, computers en atomen) representeren, en tussen ieder tweetal punten zit een zogenaamde kant, of niet, al naar gelang er een weg is tussen de steden, de computers verbonden zijn, of dat er verbindingen zijn tussen de atomen. Deze kanten kunnen gewichten hebben, waarmee we afstanden, capaciteiten, krachten representeren, en ze kunnen gericht zijn (eenrichtingsverkeer). Alhoewel het model vrij eenvoudig is, dat wil zeggen, in die mate dat we niet aan een graaf kunnen zien wat voor netwerk het representeert, is de onderliggende theorie zeer rijk en divers.

Er is een grote verscheidenheid aan problemen in grafentheorie, bijvoorbeeld het beroemde *handelsreizigersprobleem*, het probleem om een kortste ronde door de graaf te vinden die ieder punt bezoekt. Voor kleine grafen lijkt dit probleem eenvoudig, maar als het aantal punten toeneemt, kan het probleem zeer moeilijk worden. De naam van het probleem geeft een aanwijzing waar het oorspronkelijk vandaan komt, maar het is interessant om te zien dat het probleem toegepast wordt in heel andere gebieden, zoals in het ontwerp van zeer grote I.C.'s (VLSI). Een ander type probleem is het samenhangsprobleem: hoeveel kanten kunnen we uit de graaf verwijderen (door wegversperringen, verbroken lijnen) zodanig dat we toch nog vanuit ieder punt in de graaf naar ieder ander punt kunnen door over kanten door de graaf te lopen.

In dit proefschrift bestuderen we speciale klassen van grafen, die veel structuur hebben. In het oog van de wiskundige aanschouwer zijn grafen met veel structuur en symmetrie de mooiste grafen. Belangrijke klassen van mooie grafen zijn de sterk reguliere grafen, en algemener, afstandsreguliere grafen of grafen in associatieschema's. De grafen van Plato's reguliere lichamen mogen beschouwd worden als antieke voorbeelden: de tetraëder, de octaëder, de kubus, de icosaëder en de dodecaëder. Associatieschema's komen ook voor in andere gebieden van de wiskunde en haar toepassingen, zoals in de coderingstheorie van boodschappen, om fouten tegen te gaan die optreden tijdens verzending of opslag (op een CD bijvoorbeeld), of om geheime boodschappen (zoals PIN-codes) te coderen. Associatieschema's komen oorspronkelijk vanuit het ontwerp van statistische testen, en ze zijn ook belangrijk in de eindige groepentheorie.

Vooral in de theorie van grafen met veel symmetrie, maar zeker ook in andere delen van de grafentheorie, heeft het gebruik van lineaire algebra zijn kracht bewezen. Afhankelijk van de specifieke problemen en persoonlijke voorkeur gebruiken grafentheoretici verschillende matrices om een graaf te representeren. De meest populaire zijn de (0, 1)-verbindingsmatrix en de Laplace matrix. Vaak worden de algebraïsche eigenschappen van de matrix gebruikt als brug tussen verschillende structurele eigenschappen van een graaf. De relatie tussen de structurele (combinatorische, topologische) eigenschappen van een graaf en de algebraïsche van de corresponderende matrix is daarom een zeer interessante. Soms gaat de theorie zelfs verder, bijvoorbeeld, in de theoretische scheikunde, waar de eigenwaarden van de matrix van de graaf die correspondeert met een koolwaterstofmolecuul gebruikt worden om zijn stabiliteit te voorspellen.

Enkele voorbeelden van elementaire vragen in algebraïsche grafentheorie zijn: kunnen we zien aan het spectrum van de matrix of een graaf regulier is (is ieder punt het eindpunt van een constant aantal kanten), of samenhangend (kunnen we vanuit ieder punt in ieder ander punt komen), of bipartiet (is het mogelijk om de verzameling punten in twee delen te splitsen zodanig dat alle kanten van het ene deel naar het andere lopen)? Het antwoord hangt af van de gebruikte matrix. Zowel het verbindingsspectrum als het Laplace spectrum geven aan of een graaf regulier is, echter, het verbindingsspectrum herkent of een graaf bipartiet is, maar niet of het samenhangend is. Voor het Laplace spectrum is het net andersom: het herkent of een graaf samenhangend is, maar niet of het bipartiet is.

Een graaf bepaalt zijn spectrum, maar zeker niet andersom. Derhalve is het zinvol om te onderzoeken welke structurele eigenschappen afgeleid kunnen worden uit de eigenwaarden, of algemener, van sommige eigenschappen van de eigenwaarden.

Bijvoorbeeld, is het mogelijk om een graaf volledig te bepalen uit zijn verbindingsspectrum $\{[6]^1, [2]^6, [-2]^9\}$? Het antwoord is nee, er zijn twee verschillende grafen met dit spectrum, maar ze hebben soortgelijke combinatorische eigenschappen.

Andere vragen gaan over de kleinste verbindingseigenwaarde van een graaf. Er is bijvoorbeeld een grote verzameling grafen met alle verbindingseigenwaarden ten minste -2, de gegeneraliseerde lijngrafen. Er zijn echter meer voorbeelden, en deze zijn gekarakteriseerd met behulp van zogenaamde wortelroosters door Cameron, Goethals, Seidel en Shult [25]. Ander type resultaten zijn grenzen voor speciale deelstructuren in een graaf in termen van (sommige van) de eigenwaarden, zoals Hoffmans cokliekgrens.

Ook als we zoeken naar grafen met speciale structurele eigenschappen, kan het handig zijn om eerst de eigenschappen in spectrale eigenschappen te vertalen, alvorens ons geluk te beproeven. Bijvoorbeeld, stel dat we alle reguliere grafen willen vinden, waarvoor ieder tweetal punten precies één gemeenschappelijke buur heeft (dat wil zeggen, een punt dat op kanten ligt met alle twee die punten). De vriendschapsstelling stelt dat de enige graaf die aan die eigenschap voldoet de driehoek is, en haar bewijs berust op een eenvoudige algebraïsche eigenschap.

Natuurlijk zijn er veel meer (type) resultaten op het gebied van de spectrale grafentheorie, en we verwijzen de geïnteresseerde lezer naar het boek van Cvetković, Doob en Sachs [33], bijvoorbeeld.

In het algemeen zijn de meeste eigenwaarden van een graaf verschillend, maar als er veel eigenwaarden samenvallen, dan blijkt dat we in een zeer speciale situatie zitten. Als alle eigenwaarden hetzelfde zijn, dan moeten we een lege graaf hebben (een graaf zonder kanten). Als we slechts twee eigenwaarden hebben, dan hebben we in essentie een volledige graaf (een graaf met kanten tussen ieder tweetal punten). Hier bestuderen we *grafen met weinig eigenwaarden*, waar *weinig* meestal drie of vier betekent. Zulke grafen kunnen gezien worden als algebraïsche generalisaties van zogenaamde sterk reguliere grafen. Sterk reguliere grafen (cf. [16, 95]) worden gedefinieerd in termen van combinatorische eigenschappen, maar ze hebben een makkelijke algebraïsche karakterisering: grofweg gezegd zijn het de reguliere grafen met drie (verbindings- of Laplace) eigenwaarden. Door de regulariteit te laten varen, en grafen met drie verbindingseigenwaarden, en grafen met drie Laplace eigenwaarden te beschouwen, verkrijgen we twee zeer natuurlijke generalisaties. Seidel (cf. [94]) deed iets soortgelijks voor het Seidel spectrum, en vond grafen die nauw verbonden zijn met de combinatorische structuren genaamd reguliere twee-grafen. Er is weinig bekend over niet-reguliere grafen met drie verbindingseigenwaarden. Er zijn twee artikelen over zulke grafen, van Bridges en Mena [10] en Muzychuk en Klin [85]. Het blijkt dat de zaken hier zeer gecompliceerd kunnen zijn, maar met behulp van de Perron-Frobenius eigenvector kunnen we toch enige combinatorische eigenschappen afleiden, en het aantal verschillende graden beperken. We zullen de grafen met kleinste eigenwaarde -2 nader bekijken, en ze bijna allemaal vinden. Niet-reguliere grafen met drie Laplace eigenwaarden lijken tot op heden niet onderzocht (behalve de geodetische grafen met diameter twee, maar die werden niet als zodanig herkend), wat verrassend genoemd mag worden, want we vinden een betrekkelijk eenvoudige combinatorische karakterisering van zulke grafen.

Het fundamentele probleem van grafen met weinig (verbindings)eigenwaarden is gesteld door Doob [45]. Volgens zijn standpunt is *weinig* ten hoogste vijf, en hij karakteriseerde een familie van reguliere grafen met vijf eigenwaarden afkomstig van Steiner tripelsystemen. Het lijkt echter te ingewikkeld om reguliere grafen met vijf eigenwaarden in het algemeen te bestuderen. Doob [46] bestudeerde ook reguliere grafen met vier eigenwaarden, waarvan de kleinste -2 is. In het algemene geval van vier eigenwaarden zullen we enkele mooie eigenschappen afleiden, zoals wandel-regulariteit, maar er is geen gemakkelijke combinatorische karakterisering, zoals in het geval van drie eigenwaarden. Niettemin vinden we veel constructies.

Associatieschema's (cf. [3, 12, 15, 52]) vormen een combinatorische generalisatie van sterk reguliere grafen, en na deze grafen vormt het volgende punt van onderzoek de drie-klasse associatieschema's. In zulke schema's zijn alle grafen regulier met ten hoogste vier eigenwaarden, dus we kunnen de resultaten over zulke grafen toepassen. Zo bereiken we meer dan door de algemene theorie over associatieschema's toe te passen, en we vinden twee verrassende karakteriseringsstellingen. De literatuur over drie-klasse associatieschema's bestaat voornamelijk uit speciale constructies, en resultaten over het speciale geval van afstandsreguliere grafen met diameter drie. Resultaten over drie-klasse associatieschema's kunnen worden gevonden in het artikel van Mathon [79], waarin vele voorbeelden worden gegeven, en het proefschrift van Chang [26], dat zich beperkt tot het imprimitieve geval.

In het afsluitende hoofdstuk van dit proefschrift leiden we grenzen voor de diameter en de grootte van speciale deelverzamelingen in grafen af. Het geval van scherpe grenzen wordt onderzocht, en hier komen afstandsreguliere grafen en drie-klasse associatieschema's tevoorschijn. Alle grenzen worden afgeleid met behulp van het scheiden van eigenwaarden en het vinden van geschikte polynomen. De diametergrenzen worden toegepast op foutenverbeterende codes.

Aan dit proefschrift zijn lijsten met parameterverzamelingen voor grafen met drie Laplace eigenwaarden, reguliere grafen met vier eigenwaarden, en drie-klasse associatieschema's (op een begrensd aantal punten) toegevoegd. Mede dankzij computerresultaten van Spence zijn alle grafen voor bijna alle parameterverzamelingen voor reguliere grafen met vier eigenwaarden en ten hoogste dertig punten gevonden.

Delen van de resultaten in dit proefschrift zijn elders verschenen. De resultaten over grafen met drie Laplace eigenwaarden in [38], reguliere grafen met vier eigenwaarden in [34] en [40], drie-klasse associatieschema's in [35] en [39], grenzen voor de diameter in [37] en grenzen voor speciale deelverzamelingen in [36].