UNIVERSITEIT GENT
Faculteit Wetenschappen
Vakgroep Zuivere Wiskunde en Computeralgebra

On the classification of dense near polygons with lines of size 3

Pieter Vandecasteele

Promotoren:
Dr. B. De Bruyn
Prof. Dr. F. De Clerck

Proefschrift voorgelegd aan
de Faculteit Wetenschappen
tot het behalen van de graad van
Doctor in de Wetenschappen: Wiskunde.

Geometry is the science of correct reasoning on incorrect figures.
– George Polya –

Acknowledgements

I solved Rubik's cube for the first time when I was six years old. Although I have to admit it took me another fifteen years before I actually found a real solution — which did not depend on the use of a screwdriver — it was the first time a puzzle of a mathematical kind had stirred my mind. (In fact it was a group theoretical puzzle, which of course I could not have guessed at that time.) The stunning beauty of mathematical structures has fascinated me ever since. From Rubik's cube to the myriads of mind tickling mathematical riddles, from high school mathematics to the many highly interesting topics like Group Theory and Finite Geometry, I feel I have made a compelling journey through the mathematical universe.

Now that I have reached a turning point on my trip, I want to express my gratitude to a number of people. First of all, I want to thank the *Flemish Institute for the Promotion of Scientific and Technological Research in Industry* (IWT) for their financial support. Concerning my research, I am extremely grateful to Dr. Bart De Bruyn, who supervised me in a truly excellent way. During the past four years he has patiently read, reread – and sometimes reread again – the fruits of my research as careful as possible. The countless times I lost the road, he steered me back on track. It is not an understatement to say that my thesis would not look the same without the many hints, corrections, and advises he made. In the same context I would also like to thank Prof. Dr. Frank De Clerck for his many useful remarks and his support, and my colleagues Stefaan, Jan, and Deirdre for their help regarding my questions on Polar Spaces. Concerning the many 'LaTeXnical' difficulties I have experienced, I am in great debt to Geert, again Jan, and especially Tom, for their helpful assistance.

On a more personal level, I want to thank An, Stefaan and David for being great opponents during our badminton games and Jan, Tom, Bart, Geert, Stefaan and Patrick for the many thrilling games of risk and colour-whist. But above all, I want to thank them — as well as many other members of our department — for not only being great colleagues, but also great friends.

Last but not least, I would like to thank my parents for believing in me and always being there for me, and Tiny, for showing me time after time that mathematics in not the only beautiful thing in the world.

Pieter Vandecasteele
October 2004

Preface

In their paper *"Near n-gons and line systems"* of 1980 ([40]), E. Shult and A. Yanushka discussed certain systems of lines in the Euclidean space and showed they were related to a class of geometries which they called *near polygons*. During the past 25 years, near polygons have been thoroughly studied and have earned their own place in the field of incidence geometry. Important examples of near polygons are the dual polar spaces ([8]) and the generalized polygons. Generalized polygons were introduced by Tits in his celebrated paper on triality ([43]) and have already been widely studied ([36], [46]). The class of generalized quadrangles actually coincides with the class of near quadrangles. Generalized quadrangles play a crucial role in the theory of near polygons. Under some mild conditions ([38],[40]), near polygons contain *quads*, i.e. certain substructures isomorphic to generalized quadrangles. Quads can be considered as the 'building blocks' of near polygons. The various results on generalized quadrangles (see e.g. [36]) are therefore very useful in studying near polygons.

In this thesis, we present some new classification results concerning near polygons. We mainly focus our attention on the so-called *slim dense near polygons*. (We assume the existence of quads and we suppose that all lines have size 3.)

In the first chapter, we introduce some basic concepts and define a number of classes of near polygons which we will need in the following chapters.

Chapter 2 is devoted to the characterization of near polygons in terms of the existence of certain *big geodetically closed sub near polygons*. We also prove a number of theorems characterizing *product* near polygons and *spreads of symmetry* of *glued* near polygons. We conclude the chapter with the proof of a nice theorem characterizing slim dense near polygons having a so-called *chain* of big sub near polygons.

In [4], all slim dense near hexagons were classified. Up to an isomorphism, there are eleven examples. The possible relations (or 'positions') between a point and a quad in these hexagons were of great importance for the classification. While trying to do a similar classification for the slim dense near octagons, the necessity arose to study the possible relations between a point and a geodetically closed sub near hexagon (a hex) of a near polygon. In Chapter 3, we tackle this problem in a more general context and investigate the possible relations between a point and any geodetically closed sub near polygon. It turns out that each such relation can be described by a *valuation*, which is a map from the pointset of the considered sub near polygon to the set of natural numbers, satisfying a number of nice properties. We define several standard types of valuations and study in particular the valuations of dense near hexagons and classical near polygons (dual polar spaces).

In Chapter 4, we first classify all possible valuations of the 11 slim dense near hexagons, using the tools provided in Chapter 3. Using this classification and the results of Chapter 2, we then determine in a second step all slim dense near octagons having a big hex. There are 24 examples, up to isomorphism. Finally, we prove that all slim dense near octagons have a big hex. This result then completes the classification.

Table of contents

List of Tables

List of Tables

Chapter 1

Introduction

1.1 Incidence structures

1.1.1 Definition

An *incidence structure* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a triple consisting of

- a nonempty set $\mathcal{P}$, whose elements are called *points*;
- a (possibly empty) set $\mathcal{L}$, disjoint from $\mathcal{P}$, whose elements are called *lines*;
- an *incidence relation* $\mathbf{I} \subseteq (\mathcal{P} \times \mathcal{L})$.

We will always assume that $\mathcal{S}$ is a *finite* incidence structure, i.e. both $\mathcal{P}$ and $\mathcal{L}$ are finite sets. If $(p, L) \in \mathbf{I}$, then we write also $p \ \mathbf{I} \ L$ and we say that the point p is incident with the line L, or that p lies on L, or that L is a line through p, etc. Two points of $\mathcal{S}$ are said to be *collinear* if there is a line through those points. An incidence structure has *order* (s, t) if there are $t + 1$ lines through every point and $s + 1$ points on every line. An incidence structure $\mathcal{S}$ is called a *partial linear space* if every line of $\mathcal{S}$ contains at least two points and if every two different points of $\mathcal{S}$ are contained in at most one line. If every two different points are contained in exactly one line, then a partial linear space is called a *linear space*. Important examples of linear spaces are the *affine spaces* and *projective spaces*. We will assume the reader is familiar with these objects.

1.1.2 Isomorphisms and subgeometries

Two incidence structures $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ are called *isomorphic* if there exists a bijection $\alpha : \mathcal{P} \to \mathcal{P}'$ and a bijection $\beta : \mathcal{L} \to \mathcal{L}'$ such that $p \mathbf{I} L \Leftrightarrow p^\alpha \mathbf{I}' L^\beta$ for every $p \in \mathcal{P}$ and $L \in \mathcal{L}$. We write $\mathcal{S} \cong \mathcal{S}'$ and we say that (α, β) is an *isomorphism* from $\mathcal{S}$ to $\mathcal{S}'$. An isomorphism from $\mathcal{S}$ to itself is called an *automorphism*.

An incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ is called a *subgeometry* of $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and if $\mathbf{I}'$ is the restriction of $\mathbf{I}$ to $\mathcal{P}' \times \mathcal{L}'$.

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a partial linear space, then $\mathcal{S} \cong \mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$, where $\mathcal{P}' := \mathcal{P}$, $\mathcal{L}' := \bigcup_{L \in \mathcal{L}} \{\{x | x \mathbf{I} L\}\}$ and $p' \mathbf{I}' L' \Leftrightarrow p' \in L'$ for every $p' \in \mathcal{P}'$ and $L' \in \mathcal{L}'$. This is the reason why we will often consider lines as sets of points and write $p \in L$ instead of $p \mathbf{I} L$.

1.1.3 Graphs and distances

With every incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$, we associate a graph $\Gamma(\mathcal{S})$, called the *point graph* or *collinearity graph* of $\mathcal{S}$. The vertices of $\Gamma(\mathcal{S})$ are the elements of $\mathcal{P}$ and two different vertices are adjacent if they are collinear. Distances in $\mathcal{S}$ will always be measured in its point graph $\Gamma(\mathcal{S})$. We denote these distances by $\mathrm{d}_{\mathcal{S}}(\cdot, \cdot)$ or also by $\mathrm{d}(\cdot, \cdot)$ if no confusion is possible. If the points x and y belong to different connected components of $\Gamma(\mathcal{S})$, then we will write $\mathrm{d}(x, y) = +\infty$. An incidence structure $\mathcal{S}$ is called *connected* if its point graph is connected. The *diameter* of an incidence structure $\mathcal{S}$ is the diameter of its point graph (i.e. the maximal distance between two points of $\Gamma(\mathcal{S})$) and will often be denoted as $\mathrm{diam}(\mathcal{S})$. If $A, B \subseteq \mathcal{P}$, then $\mathrm{d}(A, B) := \min\{\mathrm{d}(x, y) | x \in A, y \in B\}$. If A consists of only one point x, then we will also write $\mathrm{d}(x, B)$ instead of $\mathrm{d}(\{x\}, B)$. For every $i \in \mathbb{N}$, we define $\Gamma_i(A)$ as the set of points of $\mathcal{S}$ at distance i from A. Again, if A consists of only one point x, we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$. In particular, we write $\Gamma(x) := \Gamma_1(x)$.

1.1.4 Direct product of two incidence structures

Given two incidence structures $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathbf{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathbf{I}_2)$, a new incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows:

- $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$;
- $\mathcal{L} := (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$;
- the point (x, y) of $\mathcal{S}$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y \mathbf{I}_2 L$; the point (x, y) of $\mathcal{S}$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x \mathbf{I}_1 M$ and $y = u$.

The incidence structure $\mathcal{S}$ is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and is denoted by $\mathcal{S}_1 \times \mathcal{S}_2$. Since $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $\mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3) \cong (\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3$, also the direct product $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_k$ of $k \geq 3$ incidence structures $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ is well-defined.

A $(k_1 \times k_2)$-*grid* is the direct product of a line of size k_1 and a line of size k_2. A *dual grid* is the point-line dual of a grid. If $k_1 = k_2$, then the (dual) grid is called *symmetrical*.

1.1.5 Designs

An important class of incidence structures are the $t-(v, k, \lambda)$ *designs*. These structures satisfy the following properties (the elements of $\mathcal{L}$ are here called *blocks* instead of lines):

- there are exactly v points;
- every block contains exactly $k \geq 2$ points;
- every t different points are contained in exactly λ blocks.

If $\lambda = 1$, then the design is called a *Steiner system*, and we denote it as $S(t, k, v)$.

1.1.6 Ovoids and spreads

An *ovoid* of a partial linear space $\mathcal{S}$ is a set O of points such that every line of $\mathcal{S}$ contains a unique point of O. A *spread* of $\mathcal{S}$ is a set of lines partitioning the point set.

1.2 Near Polygons

1.2.1 Roots

In [40], E. Shult and A. Yanushka discussed 'tetrahedrally closed line systems of type $(0, \frac{1}{3})$' in Euclidean spaces and showed that they were related to a class of geometries which they called near polygons. Consider the n-dimensional Euclidean space with origin O. A *line system of type* $(a_1, a_2, \ldots, a_k)$ is a set $\mathcal{A}$ of lines through O with the property that $\cos\alpha \in \{\pm a_1, \pm a_2, \ldots, \pm a_k\}$ for any two different lines of $\mathcal{A}$ at angle α.

A *system of vectors of type* $(a_1, a_2, \ldots, a_k)$ is a set Σ of vectors satisfying:

- $\Sigma = -\Sigma := \{-\vec{x} | \vec{x} \in \Sigma\}$;
- the norm $(\vec{x}, \vec{x})$ of $\vec{x}$ is a nonzero constant c for all $\vec{x} \in \Sigma$;
- if $\vec{x}, \vec{y} \in \Sigma$ and $\vec{x} \neq \pm\vec{y}$, then $\frac{(\vec{x},\vec{y})}{c} \in \{\pm a_1, \pm a_2, \ldots, \pm a_k\}$.

Clearly line systems of type $(a_1, \ldots, a_k)$ are equivalent to systems of vectors of type $(a_1, \ldots, a_k)$. The angle α for which $\cos\alpha = \pm\frac{1}{3}$ is the angle subtended by any two chords drawn from the barycenter of a regular tetrahedron to two of the corner vertices. We say that a system of lines of type $(0, \frac{1}{3})$ is *tetrahedrally closed* when as soon as three of the lines passing through the vertices of a regular tetrahedron centered at the origin are present in the line system, also the line through the fourth vertex is present. A system of vectors of type $(0, \frac{1}{3})$ is called *tetrahedrally closed* if the corresponding line system is tetrahedrally closed.

Let Σ be a tetrahedrally closed system of vectors of type $(0, \frac{1}{3})$, all of them having norm 3. Now fix a vector $\vec{u} \in \Sigma$ and let $\Sigma_{-1}(\vec{u})$ be the set of vectors of Σ having inner product -1 with $\vec{u}$. If $\Sigma_{-1}(\vec{u}) \neq \emptyset$, we can define an incidence structure $\mathcal{S}$ as follows:

- the point set is equal to $\Sigma_{-1}(\vec{u})$;
- the lines are the triplets $\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} \subseteq \Sigma_{-1}(\vec{u})$ such that $\vec{y}_1, \vec{y}_2, \vec{y}_3$ and $\vec{u}$ point to the four vertices of a tetrahedron centered at O;
- incidence is containment.

The following theorem now shows the relation between tetrahedrally closed line systems and near polygons.

Theorem 1.2.1 (Proposition 3.10 of [40]**)**
If for every two vectors of $\Sigma_{-1}(\vec{u})$ having inner product $+1$, there exists at least one vector of $\Sigma_{-1}(\vec{u})$ having inner product -1 with both vectors, then $\mathcal{S}$ is a connected partial linear space satisfying the following properties:

- *the distance between two points is at most 3;*
- *for every point p and every line L, there exists a unique point on L nearest to p (with respect to the distance in the point graph).*

Two different points have distance 1, 2, respectively 3, if their inner product equals -1, $+1$, respectively 0.

1.2.2 Definition

A *near polygon* is a connected partial linear space of finite diameter satisfying the following property.

> *For each point p and every line L, there exists a unique point q on L nearest to p (with respect to the distance in the point graph).*

This axiom will play a very important role throughout this thesis, and we will often call it the *near polygon property*.

In graphtheory, a near polygon is a connected graph Γ satisfying the following properties:

- Γ is connected and has finite diameter;
- for every vertex p and every maximal clique M, there is a unique vertex in M nearest to p.

Here a *maximal clique* is a maximal set of mutually adjacent vertices. Suppose that an incidence structure $\mathcal{S}$ is a near polygon. It follows from the near polygon property that lines of $\mathcal{S}$ correspond with maximal cliques in $\Gamma(\mathcal{S})$. So $\Gamma(\mathcal{S})$ is a near polygon in graphtheoretical sense. Conversely, if a graph Γ is a near polygon, then the incidence structure whose points (respectively lines) are the vertices (respectively maximal cliques) of Γ (natural incidence), is a near polygon in the geometrical sense. So, we have a bijective correspondence between a class of graphs and a class of partial linear spaces.

If d is the diameter of the point graph, then the near polygon is also called a *near $2d$-gon*. The unique near 0-gon $\mathbb{O}$ consists of one point and no lines. Every near 2-gon is a line. In the sequel, we will denote the unique line of size i by $\mathbb{L}_i$.

Theorem 1.2.2 (Line-Line relations, Lemma 1 of [6]**)**
Let L and M be two lines of a near polygon $\mathcal{S}$. Then one of the following possibilities occurs.

- *There exists a unique point p on L and a unique point q on M such that $\mathrm{d}(l,m) = \mathrm{d}(l,p) + \mathrm{d}(p,q) + \mathrm{d}(q,m)$ for all points l on L and m on M.*
- *There exists an $i \in \mathbb{N}$ such that $\mathrm{d}(l,M) = \mathrm{d}(m,L) = i$ for all points l on L and m on M. In this case, L and M are called* parallel *and we write $L \parallel M$.*

Two lines L and M of $\mathcal{S}$ are called *opposite* if $\mathrm{d}(L,M) = \mathrm{diam}(\mathcal{S}) - 1$. Two opposite lines are parallel. A point x and a line L of $\mathcal{S}$ are called *opposite* if $\mathrm{d}(x,L) = \mathrm{diam}(\mathcal{S}) - 1$. Two points x and y of $\mathcal{S}$ are called *opposite* if $\mathrm{d}(x,y) = \mathrm{diam}(\mathcal{S})$.

An important class of near polygons are the so-called *generalized $2d$-gons*. A near $2d$-gon $\mathcal{S}$ ($d \geq 2$) is called a *generalized $2d$-gon* if for every $i \in \{1,\ldots,d-1\}$ and every two points x and y at distance i from one another, $|\Gamma_{i-1}(x) \cap \Gamma(y)| = 1$. By definition, the class of near quadrangles coincides with the class of generalized quadrangles. A generalized $2d$-gon is called *degenerate* if it contains no ordinary $2d$-gon as a subgeometry.

Generalized $2d$-gons and generalized quadrangles in particular have already been widely studied. In Section 1.5, we will have a closer look at these objects and state some important properties.

1.2.3 Substructures

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ denote a near polygon. A nonempty set X of points of $\mathcal{P}$ is called a *subspace* if for every two different collinear points x and y of X, all points of the unique line through x and y are contained in X. If X is a subspace of $\mathcal{S}$, then X induces a partial linear space $\mathcal{S}_X = (X, \mathcal{L}', \mathbf{I}')$, with $\mathcal{L}'$ the set of lines of $\mathcal{S}$ having all their points in X and $\mathbf{I}'$ inherited from $\mathcal{S}$. A subset X of $\mathcal{P}$ is called *geodetically closed* if X is a subspace and if all the points of a *geodesic* (i.e. a shortest path) between two points of X are again contained in X. If X is a geodetically closed subspace, then clearly $\mathcal{S}_X$ is a sub near polygon of $\mathcal{S}$. Every nonempty set X of points is contained in a unique minimal geodetically closed sub near polygon $\mathcal{C}(X)$, namely the intersection of all geodetically closed sub near polygons through X. We call $\mathcal{C}(X)$ the *geodetic closure* of X. We define $\mathcal{C}(\emptyset) := \emptyset$. If $X_1, \ldots, X_k$ are sets of points, then $\mathcal{C}(X_1 \cup \cdots \cup X_k)$ is also denoted by $\mathcal{C}(X_1, \ldots, X_k)$. If one of the arguments of $\mathcal{C}$ is a singleton $\{x\}$, we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \cdots)$. Let x be a point of $\mathcal{S}$ and let X be a subset of $\mathcal{S}$. Then the pair (x, X) is called *classical* if and only if there exists a (necessarily unique) point $\pi_X(x) \in X$ such that $\mathrm{d}(x,y) = \mathrm{d}(x, \pi_X(x)) + \mathrm{d}(\pi_X(x), y)$ for each point $y \in X$. If (x, X) is classical, then $\pi_X(x)$ is called the *projection* of x onto X. A subset X is called *classical* if for every point x of $\mathcal{S}$, the pair (x, X) is classical. It is easy to see that every classical set X is geodetically closed. A subspace X of $\mathcal{S}$ is called *big* if every point of $\mathcal{S}$ has distance at most one to X.

Theorem 1.2.3 (Proposition 2.5 of [40]**)**
Let x and y be two points of a near polygon $\mathcal{S}$ at mutual distance 2. If x and y have two common neighbours c and d such that the line xc contains at least three points, then $\mathcal{C}(x, y)$ is a nondegenerate generalized quadrangle and contains precisely those points of $\mathcal{S}$ which have distance at most 2 from x, y, c and d.

A geodetically closed subset X which induces a nondegenerate generalized quadrangle is called a *quad*. Often, the associated sub generalized quadrangle $\mathcal{S}_X$ will also be called a quad.

Theorem 1.2.4 (Lemma 1.2 of [38]**)**
If there is a quad through two points at distance 2, then this quad is unique. As a consequence, every two different quads are either disjoint, or meet in a point or a line of the near polygon.

Let x be a point of a near polygon $\mathcal{S}$. Let $\mathcal{L}(x, \mathcal{S})$ denote the following incidence structure:

- the point set of $\mathcal{L}(x, \mathcal{S})$ is the set of lines of $\mathcal{S}$ through x;
- the line set of $\mathcal{L}(x, \mathcal{S})$ is the set of quads of $\mathcal{S}$ through x;
- incidence is containment.

Then $\mathcal{L}(x, \mathcal{S})$ is called the *local space of $\mathcal{S}$ at x*. As a corollary of Theorem 1.2.4, the local space of a near polygon in a point is a partial linear space.

All point-point pairs are classical and by the near polygon property also all point-line pairs are classical. The possible point-quad relations are characterized by the following theorem.

Theorem 1.2.5 (Point-Quad relations, Lemma 1.3 of [38], Proposition 2.6 of [40]**)**
Let p be a point of a near polygon $\mathcal{S}$ and let $\mathcal{Q}$ be a quad of $\mathcal{S}$. Then one of the following cases occurs.

- *The pair $(p, \mathcal{Q})$ is classical, i.e.* $\mathrm{d}(p, q) = \mathrm{d}(p, \pi_{\mathcal{Q}}(p)) + \mathrm{d}(\pi_{\mathcal{Q}}(p), q)$ *for every point $q \in \mathcal{Q}$.*
- *The points of $\mathcal{Q}$ which are nearest to p form an ovoid of $\mathcal{Q}$. In this case, $(p, \mathcal{Q})$ is called* ovoidal.
- *The quad $\mathcal{Q}$ is a dual grid and the set of all points of $\mathcal{Q}$ nearest to x is a proper subset of size at least 2 of one of the two cocliques of $\Gamma(\mathcal{Q})$.*

In the next sections, we will define some important classes of near polygons.

1.3 Thin near polygons

A near polygon is called *thin* if every line is incident with precisely two points. A graph Γ is called *bipartite* if its vertex set can be partitioned in two subsets of mutually nonadjacent vertices.

Proposition 1.3.1
The class of thin near polygons coincides with the class of connected bipartite graphs.

Proof
Suppose first that $\mathcal{S}$ is a thin near polygon. Because every line has exactly two points, we can consider $\mathcal{S}$ as a (connected) graph Γ. Suppose now that there exists a closed path of odd length and let (p, L) be an opposite vertex-edge pair in a minimal closed path of odd length in Γ. Since the closed path is minimal, p has equal distance to both vertices of L. Hence (p, L) regarded as point-line pair in $\mathcal{S}$ contradicts the near polygon property. Hence Γ has no closed paths of odd length. So, Γ is a (connected) bipartite graph.

Conversely, suppose that Γ is a connected bipartite graph and let $\mathcal{S}$ be the incidence structure whose points are the vertices of Γ and whose lines are the edges of Γ (with natural incidence). Clearly every line contains two points and every two points are contained in at most one line. Hence $\mathcal{S}$ is a connected partial linear space. Let (p, L) be a point-line pair of $\mathcal{S}$, and let p_1 and p_2 be the points of L. Because Γ contains no closed paths of odd length, it is now easy to see that $\mathrm{d}(p, p_1) = \mathrm{d}(p, p_2) \pm 1$, implying that $\mathcal{S}$ satisfies the near polygon property. $\square$

1.4 Regular near polygons

A near $2d$-gon $\mathcal{S}$ is called *regular* if the following two conditions are satisfied.

RE$_1$ $\mathcal{S}$ has an order (s, t).

RE$_2$ There are constants t_i $(0 \leq i \leq d)$ such that for every two points x and y at distance i, there are exactly $t_i + 1$ points in $\Gamma(x) \cap \Gamma_{i-1}(y)$.

Clearly $t_0 = -1$, $t_1 = 0$ and $t_d = t$. The constants s, t, t_i $(i \in \{2, \ldots, d-1\})$ are called the *parameters* of the regular near polygon. If x is a point of a regular near polygon then by an easy counting argument,

$$\begin{aligned} |\Gamma_0(x)| &= 1; \\ |\Gamma_k(x)| &= s^k \frac{\prod_{i=0}^{k-1}(t - t_i)}{\prod_{i=1}^{k}(1 + t_i)} \text{ for all } k \in \{1, \ldots, d\}. \end{aligned}$$

1.5 Generalized $2d$-gons

Recall that a generalized $2d$-gon is a near $2d$-gon such that for every $i \in \{1, \ldots, d-1\}$ and every two points x and y at distance i from one another, $|\Gamma_{i-1}(x) \cap \Gamma(y)| = 1$. A generalized $2d$-gon is degenerate if it contains no ordinary $2d$-gon as a subgeometry. A generalized $2d$-gon ($d \geq 2$) is called *thick* if there are at least three lines through every point and if there are at least three points on every line. Generalized polygons were introduced by Tits in his celebrated paper on triality ([43]). For a detailed discussion of these structures, we refer to [46]. The generalized quadrangles in particular will be very important throughout this thesis.

1.5.1 Properties

The definition of nondegenerate generalized $2d$-gons given here is equivalent with the definition given in [46], as shown by the following proposition.

Proposition 1.5.1
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized $2d$-gon, then it has no subgeometry which is an ordinary δ-gon, $3 \leq \delta < 2d$. If $\mathcal{S}$ is nondegenerate, then every two elements of $\mathcal{P} \cup \mathcal{L}$ are contained in an ordinary $2d$-gon.

Proof
By the near polygon property, ordinary 3-gons cannot occur. Suppose now that no ordinary i-gons occur for every $i \in \{3, \ldots, k-1\}$ and suppose there exists an ordinary k-gon ($4 \leq k < 2d$). Suppose first that k is odd and let x and L be a point and a line of the ordinary k-gon such that $\mathrm{d}(x, L) = \frac{k-1}{2}$. Then there are two points x_1 and x_2 on L at distance $\frac{k-1}{2}$ from x. Hence there must exist a point x' on L at distance $\frac{k-3}{2}$ from x. It is now easy to see that x and L are contained in an ordinary $(k-1)$-gon, a contradiction. Suppose now that k is even and let x_1 and x_2 be two points of the ordinary k-gon at distance $\frac{k}{2}$. If L_1 and L_2 are two lines through x_2 in the ordinary k-gon, then they both contain a point at distance $\frac{k}{2} - 1$ from x_1, again a contradiction. The proposition now follows by induction.

Suppose now that T is an ordinary $2d$-gon contained in $\mathcal{S}$ and consider the following sets:

- $\mathcal{P}_T^1$: the set of points of T;
- $\mathcal{L}_T$: the set of lines of $\mathcal{S}$ containing two points of T;
- $\mathcal{P}_T^2$: the set of points on lines of $\mathcal{L}_T$ not contained in $\mathcal{P}_T^1$.

For every $x \in \mathcal{P}_T^1$, let $O_T(x)$ be the unique point of $\mathcal{P}_T^1$ opposite to x. If $x \in \mathcal{P}_T^2$ and if L_x is the unique line of $\mathcal{L}_T$ opposite to x, let $O_T(x)$ be the unique point of $\mathcal{P}_T^2 \cap L_x$ nearest to x.

(i) Consider a point $x \in \mathcal{P}_T^2$ and an element E of $\mathcal{P}_T^1 \cup \mathcal{L}_T$. The path of length $d-1$ from x to $O_T(x)$ together with one of the two paths of length $d+1$ from x to $O_T(x)$ defined by T determines a new ordinary $2d$-gon containing x and E.

(ii) Consider a line $L \in \mathcal{L}$ such that $L \cap \mathcal{P}_T^1$ is a point x, and an element E of $\mathcal{P}_T^1 \cup \mathcal{L}_T$. Clearly L contains a unique point x' at distance $d-1$ from $O_T(x)$. The path of length $d-1$ from x' to $O_T(x)$ together with one of the two paths of length $d+1$ from x' to $O_T(x)$ defined by T determines a new ordinary $2d$-gon containing L and E.

Since $\mathcal{S}$ is nondegenerate, at least one ordinary $2d$-gon T exists. Since $\mathcal{S}$ is connected, one can now easily verify that we can construct an ordinary $2d$-gon through any two elements of $\mathcal{P} \cup \mathcal{L}$ starting from T, by a certain sequence of transformations of type (i) and (ii). □

Theorem 1.5.2 (Theorem 1.3.2 of [46]**)**
A generalized $2d$-gon is thick if and only if it has an ordinary $2(d+1)$-gon as a subgeometry.

Theorem 1.5.3 (Theorem 1.5.3 of [46]**)**
Every thick generalized $2d$-gon has an order (s,t) with $s,t \geq 2$ and hence is regular.

Notice that every generalized $2d$-gon which is not thick can be obtained from some ordinary $2d$-gon or some thick generalized 2δ-gon (see [45] and [47] for more information).

1.5.2 Generalized quadrangles

Generalized quadrangles will be very important throughout this thesis, because they will often arise as substructures in other near polygons.

Proposition 1.5.4
Every degenerate generalized quadrangle $\mathcal{Q}$ consists of a number of lines (of possibly different sizes) through a fixed point.

Proof
Suppose there exist two disjoint lines L_1 and L_2. Let x_1 and y_1 be two different points of L_1 and let x_2 (respectively y_2) be the unique point of L_2 collinear with x_1 (respectively y_1). Clearly $x_2 \neq y_2$, but this implies that x_1, x_2, y_2 and y_1 determine an ordinary quadrangle in $\mathcal{S}$, a contradiction. Hence any two lines intersect in a point. Because $\mathcal{S}$ has no triangles as subgeometries, the proposition now follows immediately. $\square$

Remark
The point-line dual of a nondegenerate generalized quadrangle $\mathcal{Q}$ is again a generalized quadrangle, denoted as $\mathcal{Q}^D$. If $\mathcal{Q} \cong \mathcal{Q}^D$, then $\mathcal{Q}$ is called *self-dual*.

Every (dual) grid is a generalized quadrangle. The following result is well known in the theory of generalized quadrangles. For a proof, see e.g. [11].

Theorem 1.5.5
Every nondegenerate generalized quadrangle which is not a grid or a dual grid has an order (s,t) with $s, t \geq 2$.

A generalized quadrangle of order (s,t) will sometimes be denoted as $\mathsf{GQ}(s,t)$. If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a $\mathsf{GQ}(s,t)$, then $|\mathcal{P}| = (1+s)(1+st)$ and $|\mathcal{L}| = (1+t)(1+st)$. If $s = t$ then we say that the generalized quadrangle has order s and we denote it as $\mathsf{GQ}(s)$. A $\mathsf{GQ}(s,1)$ is a symmetrical grid. Dually, a $\mathsf{GQ}(1,t)$ is a symmetrical dual grid.

Standard examples

The following geometries were first recognized as generalized quadrangles by Tits in [43].

(a) Consider in $\mathsf{PG}(d,q)$, $d \in \{3,4,5\}$, a nonsingular quadric Q of projective index 1. The points and lines of Q then form a generalized quadrangle, denoted as $Q(d,q)$. It has the following parameters:

$Q(3,q)$: $(s,t) = (q,1)$;
$Q(4,q)$: $(s,t) = (q,q)$;
$Q(5,q)$: $(s,t) = (q,q^2)$.

(b) Consider in $\mathsf{PG}(d,q^2)$, $d \in \{3,4\}$, a nonsingular hermitian variety H. The points and lines of H then form a generalized quadrangle, denoted as $H(d,q^2)$. It has the following parameters:

$H(3, q^2)$: $(s, t) = (q^2, q)$;
$H(4, q^2)$: $(s, t) = (q^2, q^3)$.

(c) Consider in $\mathsf{PG}(3, q)$ a symplectic polarity ζ. The incidence structure with points the points of $\mathsf{PG}(3, q)$ and lines the totally isotropic lines of $\mathsf{PG}(3, q)$ with respect to ζ is then a generalized quadrangle of order q, denoted as $W(q)$.

The following theorem gives isomorphisms between some of these generalized quadrangles.

Theorem 1.5.6 ([36]**)**

- *$Q(3, q)$ is a grid.*
- *$Q(4, q) \cong W^D(q)$. Moreover, $Q(4, q) \cong W(q)$ if and only if q is even.*
- *$Q(5, q) \cong H^D(3, q^2)$.*

Spreads and ovoids

Let $\mathcal{Q}$ be a generalized quadrangle. Recall that an ovoid of $\mathcal{Q}$ is a set O of points such that every line of $\mathcal{Q}$ contains a unique point of O. A spread of $\mathcal{Q}$ is a set of lines partitioning the point set of $\mathcal{Q}$. A *rosette of ovoids* is a set of ovoids through a given point p of $\mathcal{Q}$ partitioning the set of points at distance 2 from p. A *fan of ovoids* is a set of ovoids partitioning the point set of $\mathcal{Q}$. If $\mathcal{Q}$ had order (s, t), then every ovoid and spread contains $1 + st$ elements. Every rosette of ovoids contains s ovoids and every fan of ovoids contains $s + 1$ ovoids. A lot is known about the existence of spreads and ovoids in generalized quadrangles. We mention the following theorem, which we will need later.

Theorem 1.5.7

- *$W(q)$, q odd, has no ovoids* ([42]).
- *A $\mathsf{GQ}(s, s^2)$, $s \neq 1$, has no ovoids* ([38], [42]).

1.5.3 Parameter restrictions

A lot of restrictions are known concerning the parameters d, s and t of a generalized $2d$-gon of order (s, t).

Theorem 1.5.8 ([32]**)**
Let $\mathcal{S}$ be a generalized $2d$-gon of order (s, t) with $d \geq 2$, then at least one of the following holds:

- *$\mathcal{S}$ is an ordinary $2d$-gon;*
- *$\mathcal{S}$ is a generalized quadrangle;*
- *$d = 3$ and if $\mathcal{S}$ is thick, then $\sqrt{st} \in \mathbb{N}$;*
- *$d = 4$ and if $\mathcal{S}$ is thick, then $\sqrt{2st} \in \mathbb{N}$;*
- *$d = 6$ and $\mathcal{S}$ is not thick.*

Theorem 1.5.9 ([36])
If $\mathcal{Q}$ is a $\mathsf{GQ}(s, t)$ then $s + t$ divides $st(1 + s)(1 + t)$.

Theorem 1.5.10
Let $\mathcal{S}$ be a thick generalized $2d$-gon of order (s, t).

- ([36]) $s \leq t^2$ *and* $t \leq s^2$ *if* $d = 2$;
- ([33]) $s \leq t^3$ *and* $t \leq s^3$ *if* $d = 3$;
- ([34]) $s \leq t^2$ *and* $t \leq s^2$ *if* $d = 4$.

Every known thick generalized quadrangle has order $(q-1, q+1)$, $(q+1, q-1)$, (q, q), (q^2, q^3), (q^3, q^2), (q, q^2) or (q^2, q) for a certain prime power q. All known thick generalized hexagons have order $(q, q), (q, q^3)$ or (q^3, q) for a certain prime power q. All known thick generalized octagons have order $(2^u, 2^{2u})$ or $(2^{2u}, 2^u)$, u odd.

1.6 Dense near polygons

1.6.1 Definition and properties

Let $\mathcal{S}$ be a near polygon such that

DE_1 every line has at least three points;

DE_2 every two points at distance 2 have at least two common neighbours.

Then $\mathcal{S}$ is called a *dense near polygon*.

By Theorem 1.2.3, every two points are contained in a unique quad. As a corollary, the local space of a dense near polygon in a point is a linear space. The following theorem is a generalization of Theorem 1.2.3 in the case of dense near polygons.

Theorem 1.6.1 (Theorem 4 of [6]**)**
If x and y are two points of a dense near $2d$-gon $\mathcal{S}$ at distance i from each other ($i \in \{0, \ldots, d\}$), then $\mathcal{C}(x, y)$ is the unique geodetically closed sub near $2i$-gon through x and y.

A geodetically closed sub near hexagon of a dense near polygon is called a *hex*.

Let x be a fixed point of a dense near polygon $\mathcal{S}$. For every point y of $\mathcal{S}$, let $\mathcal{L}(x, y)$ denote the set of lines through x containing a point at distance $\mathrm{d}(x, y) - 1$ from y.

Theorem 1.6.2 (Lemma 16 of [6], Theorem 4 of [6]**)**
The set $\mathcal{L}(x, y)$ is a subspace of $\mathcal{L}(x, \mathcal{S})$ for all points x and y of a dense near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$. Moreover, $\mathcal{C}(x, y) = \{u \in \mathcal{P} | \mathcal{L}(x, u) \subseteq \mathcal{L}(x, y)\}$.

Dense near polygons satisfy many other nice properties. We refer to [6] for an overview. We mention some properties which we will need later.

Theorem 1.6.3
Let $\mathcal{S}$ be a dense near $2d$-gon, $d \geq 1$, and let $\mathcal{F}$ be a geodetically closed sub near $2i$-gon of $\mathcal{S}$, $0 \leq i \leq d$.

(a) (Lemma 19 of [6]) *Every point of $\mathcal{S}$ is incident with the same number of lines. We denote this number by $t_{\mathcal{S}} + 1$, or $t + 1$ when no confusion is possible.*

(b) (Corollary, page 156 of [6]) *If x is a point of $\mathcal{S}$, then the subgraph of Γ induced by $\Gamma_d(x)$ is connected.*

(c) ([6]) *Let L be a line of $\mathcal{S}$ which intersects $\mathcal{F}$ in a point. Then $\mathcal{C}(\mathcal{F}, L)$ is a geodetically closed sub near $2(i + 1)$-gon.*

(d) ([6]) *There exists a chain $\mathcal{F} = \mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \cdots \subset \mathcal{F}_d = \mathcal{S}$ of geodetically closed sub near polygons of $\mathcal{S}$ such that $\mathrm{diam}(\mathcal{F}_j) = j$ for every $j \in \{i, i + 1, \ldots, d\}$.*

1.6.2 Line-Quad relations

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a dense near polygon. As a corollary of $\mathbf{DE_1}$ and Theorem 1.2.5, every point-quad pair is either classical or ovoidal. Let $\mathcal{Q}$ be a quad of $\mathcal{S}$. Define now the following sets ($i \in \mathbb{N}$):

$$\begin{array}{rcl} N_i(\mathcal{Q}) & := & \{x \in \mathcal{P} |\ \mathrm{d}(x, \mathcal{Q}) = i\}; \\ C_i(\mathcal{Q}) & := & \{x \in N_i(\mathcal{Q}) |\ (x, \mathcal{Q}) \text{ is classical}\}; \\ O_i(\mathcal{Q}) & := & \{x \in N_i(\mathcal{Q}) |\ (x, \mathcal{Q}) \text{ is ovoidal}\}. \end{array}$$

If no confusion is possible, we will write N_i, C_i and O_i. Clearly $N_0 = \mathcal{Q}$ and $O_1 = C_{d-1} = N_d = \emptyset$. Furthermore, for any $x \in O_i$ and $y \in C_i$, $\mathrm{d}(x, y) \neq 1$ (Lemma 4 of [6]).

Theorem 1.6.4 (Line-Quad relations, [6], page 148**)**
Let $(L, \mathcal{Q})$ be a line-quad pair of a dense near polygon $\mathcal{S}$.

- *If $L \subseteq N_i \cup N_{i+1}$ for some i, then $|L \cap N_i| = 1$.*
- *If $L \subseteq C_i$ for some i, then $\{\pi_{\mathcal{Q}}(x) | x \in L\}$ is a line of $\mathcal{Q}$ parallel to L.*
- *If $L \subseteq O_i$ for some i, then the points of L determine a fan of ovoids in $\mathcal{Q}$.*
- *If $L \subseteq O_i \cup N_{i+1}$ for some i, then $L \subseteq O_i \cup O_{i+1}$ and all points of L determine the same ovoid of $\mathcal{Q}$.*
- *If $L \subseteq C_i \cup C_{i+1}$ for some i, then all points of L determine the same point in $\mathcal{Q}$.*
- *If $L \subseteq C_i \cup O_{i+1}$ for some i, then the points of $L \cap O_{i+1}$ determine a rosette of ovoids of $\mathcal{Q}$. The common point of these ovoids is the point of $\mathcal{Q}$ determined by $L \cap C_i$.*

1.6.3 Big sub near polygons

Let $\mathcal{S}$ be a dense near polygon and let $\mathcal{F} \neq \mathcal{S}$ be a geodetically closed sub near polygon of $\mathcal{S}$. Let $v_{\mathcal{S}}$ and $v_{\mathcal{F}}$, respectively, denote the number of points in $\mathcal{S}$ and $\mathcal{F}$. Recall that $\mathcal{F}$ is *big* in $\mathcal{S}$ if every point outside $\mathcal{F}$ has distance one to $\mathcal{F}$. Let x be a point outside $\mathcal{F}$. Clearly $(x, \mathcal{F})$ is a classical pair and the unique point of $\mathcal{F}$ at distance one from x is clearly the point $\pi_{\mathcal{F}}(x)$.

Theorem 1.6.5 ([4]**)**
Let $\mathcal{S}$ be a dense near polygon and let $\mathcal{F}$ be a geodetically closed sub near polygon of $\mathcal{S}$.

- *If $\mathcal{F}$ is big in $\mathcal{S}$, then $\mathcal{F}$ is a classical subset of $\mathcal{S}$.*
- *$\mathcal{F}$ is big in $\mathcal{S}$ if and only if every quad meeting $\mathcal{F}$ either is contained in $\mathcal{F}$ or intersects $\mathcal{F}$ in a line.*
- *If $\mathcal{F}$ is big in $\mathcal{S}$, then every geodetically closed sub near polygon $\mathcal{F}'$ which meets $\mathcal{F}$ either is contained in $\mathcal{F}$ or intersects $\mathcal{F}$ in a big geodetically closed sub near polygon of $\mathcal{F}'$.*

- *Suppose that every line of $\mathcal{S}$ is incident with $s+1$ points. Then $\mathcal{F}$ is big in $\mathcal{S}$ if and only if $v_{\mathcal{S}} = [1 + s(t_{\mathcal{S}} - t_{\mathcal{F}})] \cdot v_{\mathcal{F}}$. Hence, if $\mathcal{F}$ is big, then every geodetically closed sub near polygon $\mathcal{F}'$ isomorphic to $\mathcal{F}$ is also big in $\mathcal{S}$.*

1.7 Decomposable near polygons

For any two near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$, the direct product $\mathcal{S}_1 \times \mathcal{S}_2$ is again a near polygon. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a near $2d_i$-gon then $\mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(d_1 + d_2)$-gon. If $d_1, d_2 \geq 1$, then $\mathcal{S}_1 \times \mathcal{S}_2$ is called a *product near polygon*.

Two near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$ do not only give rise to a product near polygon $\mathcal{S}_1 \times \mathcal{S}_2$, but might also give rise to so-called *glued near polygons*. Glued near $2d$-gons were introduced in [12] for $d = 3$ and in [18] for any $d \geq 3$. Product near polygons and glued near polygons belong to a broader class of near polygons which we will call *decomposable*. The decomposable near polygons are the result of applying the *glueing construction* to two near polygons. For more information, we refer to the work of B. De Bruyn on this subject ([12], [14], [16], [18] and [23]).

1.7.1 Spreads

Let $\mathcal{S}$ be a near $2d$-gon. Two geodetically closed sub near polygons $\mathcal{F}_1$ and $\mathcal{F}_2$ are called *parallel* (notated as $\mathcal{F}_1 \| \mathcal{F}_2$) if for every $i \in \{1, 2\}$ and every point x of $\mathcal{F}_i$ the pair $(x, \mathcal{F}_{3-i})$ is classical and $\mathrm{d}(x, \pi_{\mathcal{F}_{3-i}}(x)) = \mathrm{d}(\mathcal{F}_1, \mathcal{F}_2)$. For every two parallel geodetically closed sub near polygons $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{S}$, the map $\pi_{i,3-i} : \mathcal{F}_i \to \mathcal{F}_{3-i};\ x \mapsto \pi_{\mathcal{F}_{3-i}}(x)$, $i \in \{1, 2\}$, is an isomorphism, which we call the *projection from $\mathcal{F}_i$ to $\mathcal{F}_{3-i}$*. We have that $\pi_{2,1} = \pi_{1,2}^{-1}$.

A *δ-spread* of $\mathcal{S}$ ($\delta \in \{0, \ldots, d\}$) is a partition of $\mathcal{S}$ in geodetically closed sub near 2δ-gons. Hence a 0-spread is just the point set of $\mathcal{S}$ and a 1-spread is just an ordinary spread. A δ-spread is called *admissible* if every two elements of it are parallel. Any two elements of an admissible δ-spread are isomorphic.

Theorem 1.7.1 (Lemma 3 of [23]**)**
Let T be an admissible δ-spread of $\mathcal{S}$ and let $\mathcal{F}$ denote a geodetically closed sub near polygon of $\mathcal{S}$. Let T' denote the set of all elements of T meeting $\mathcal{F}$ and put $T_{\mathcal{F}} := \{\mathcal{G} \cap \mathcal{F} | \mathcal{G} \in T'\}$. Then $T_{\mathcal{F}}$ is an admissible δ'-spread of $\mathcal{F}$ for some $\delta' \in \{0, \ldots, \delta\}$. Hence, if an element of T is completely contained in $\mathcal{F}$, then every element of T which meets $\mathcal{F}$ is completely contained in $\mathcal{F}$.

If $\delta = 1$, we distinguish between the following types of spreads.

- For any two parallel lines K and L of a near polygon $\mathcal{S}$ such that $\mathrm{d}(K, L) = 1$, we define $\{K, L\}^\perp$ as the set of all lines intersecting K and L, and $\{K, L\}^{\perp\perp}$ as the set of all lines meeting every line of $\{K, L\}^\perp$. If $\{K, L\}^\perp$ and $\{K, L\}^{\perp\perp}$ cover the same set of points, then the pair $\{K, L\}$ is called *regular*. An admissible spread S of $\mathcal{S}$ is called *regular* if $\{K, L\}$ is regular and $\{K, L\}^{\perp\perp} \subseteq S$ for all $K, L \in S$ with $\mathrm{d}(K, L) = 1$.

- A spread S of $\mathcal{S}$ is called *a spread of symmetry* if for every line $K \in S$ and for every two points k_1 and k_2 on K there exists an automorphism of $\mathcal{S}$ fixing each line of S and mapping k_1 to k_2. Every spread of symmetry is regular.

- If $\mathcal{S}$ is the direct product of a near polygon $\mathcal{F}$ and a line L, then the set $S := \{L_x | x \text{ is a point of } \mathcal{F}\}$ with $L_x := \{(x, y) | y \in L\}$ is a spread of $\mathcal{S}$. We call any such spread a *trivial spread*. Every trivial spread is a spread of symmetry.

Theorem 1.7.2 (Special case of Theorem 1.7.1, [18]**)**
Let S be a spread of symmetry of a near polygon $\mathcal{S}$ and let $\mathcal{F}$ be a geodetically closed sub near polygon of $\mathcal{S}$. Then the set $S_{\mathcal{F}}$ of all lines of S which are contained in $\mathcal{F}$ is either empty or a spread of symmetry of $\mathcal{F}$.

1.7.2 Glueing construction

Let $\mathcal{B}$ denote a near $2\delta_{\mathcal{B}}$-gon, $\delta_{\mathcal{B}} \geq 0$. For every $i \in \{1, 2\}$, consider the following objects:

- a near polygon $\mathcal{A}_i$ with $\delta_i := \mathrm{diam}(\mathcal{A}_i) > \delta_{\mathcal{B}}$;

- an admissible $\delta_{\mathcal{B}}$-spread $S_i = \{\mathcal{F}_1^{(i)}, \ldots, \mathcal{F}_{n_i}^{(i)}\}$ of $\mathcal{A}_i$;

- an isomorphism $\theta_i : \mathcal{B} \to \mathcal{F}_1^{(i)}$.

We will use the notations $\mathrm{d}_1(\cdot, \cdot)$, $\mathrm{d}_2(\cdot, \cdot)$ and $\mathrm{d}_\beta(\cdot, \cdot)$ to denote the distances in $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$, respectively. Clearly, $\mathcal{F}_j^{(i)} \cong \mathcal{F}_1^{(i)} \cong \mathcal{B}$ for every $i \in \{1, 2\}$ and every $j \in \{1, \ldots, n_i\}$. The geodetically closed sub near polygon $\mathcal{F}_1^{(i)}$ of $\mathcal{A}_i$ is called the *base element* of S_i.

For all $i \in \{1, 2\}$ and all $j, k \in \{1, \ldots, n_i\}$, let $\pi_{j,k}^{(i)}$ denote the projection from $\mathcal{F}_j^{(i)}$ to $\mathcal{F}_k^{(i)}$. We put $\Phi_{j,k}^{(i)} := \theta_i^{-1} \circ \pi_{k,1}^{(i)} \circ \pi_{j,k}^{(i)} \circ \pi_{1,j}^{(i)} \circ \theta_i$ and $\Pi_i := \langle \Phi_{j,k}^{(i)} | 1 \leq j, k \leq$

$n_i\rangle$. Since $\Phi^{(i)}_{j,k}$ is a composition of isomorphisms it is itself an isomorphism and so $\Pi_i \leq \mathrm{Aut}(\mathcal{B})$.

Consider now the following graph Γ with vertex set $\mathcal{B} \times S_1 \times S_2$. Two different vertices $\alpha = (x, \mathcal{F}^{(1)}_{i_1}, \mathcal{F}^{(2)}_{j_1})$ and $\beta = (y, \mathcal{F}^{(1)}_{i_2}, \mathcal{F}^{(2)}_{j_2})$ are adjacent if and only if exactly one of the following three conditions is satisfied:

(a) $\mathcal{F}^{(1)}_{i_1} = \mathcal{F}^{(1)}_{i_2}$, $\mathcal{F}^{(2)}_{j_1} = \mathcal{F}^{(2)}_{j_2}$ and $\mathrm{d}_{\mathcal{B}}(x, y) = 1$;

(b) $\mathcal{F}^{(2)}_{j_1} = \mathcal{F}^{(2)}_{j_2}$, $\mathrm{d}_1(\mathcal{F}^{(1)}_{i_1}, \mathcal{F}^{(1)}_{i_2}) = 1$ and $y = \Phi^{(1)}_{i_1,i_2}(x)$;

(c) $\mathcal{F}^{(1)}_{i_1} = \mathcal{F}^{(1)}_{i_2}$, $\mathrm{d}_2(\mathcal{F}^{(2)}_{j_1}, \mathcal{F}^{(2)}_{j_2}) = 1$ and $y = \Phi^{(2)}_{j_1,j_2}(x)$.

For every element $\mathcal{F}_1 \in S_1$ and $\mathcal{F}_2 \in S_2$, let $\mathcal{A}_1(\mathcal{F}_2)$, respectively $\mathcal{A}_2(\mathcal{F}_1)$, denote the set of all vertices of Γ whose third, respectively middle coordinate is equal to $\mathcal{F}_2$, respectively $\mathcal{F}_1$. We put $T_i := \{\mathcal{A}_i(\mathcal{F}) | \mathcal{F} \in S_{3-i}\}$, $i \in \{1, 2\}$, and $S := \{\mathcal{A}_1(\mathcal{F}_2) \cap \mathcal{A}_2(\mathcal{F}_1) | \mathcal{F}_1 \in S_1$ and $\mathcal{F}_2 \in S_2\}$.

By [23], $\mathrm{diam}(\Gamma) = \delta_1 + \delta_2 - \delta_{\mathcal{B}}$ and every two adjacent vertices are contained in a unique maximal clique. Let $\mathcal{S}$ denote the partial linear space whose points are the vertices of Γ and whose lines are the maximal cliques of Γ (natural incidence).

Theorem 1.7.3 (Theorem 1 of [23]**)**
The following statements are equivalent:

(i) *For every point $x \in \mathcal{S}$ and every element $\mathcal{F} \in T_1 \cup T_2$, the pair $(x, \mathcal{F})$ is classical.*

(ii) *The groups Π_1 and Π_2 commute.*

(iii) *$\mathcal{S}$ is a near polygon and every element of $T_1 \cup T_2$ is geodetically closed.*

If $\delta_{\mathcal{B}} = 1$, then Theorem 1.7.3 can be strengthened.

Theorem 1.7.4 (Theorem 14 of [18]**)**
If $\delta_{\mathcal{B}} = 1$, then $\mathcal{S}$ is a near polygon if and only if Π_1 and Π_2 commute.

Every near polygon $\mathcal{S}$ which can be derived in the above described way from a tuple $(\mathcal{B}, \mathcal{A}_1, \mathcal{A}_2, S_1, S_2, \mathcal{F}^{(1)}_1, \mathcal{F}^{(2)}_1, \theta_1, \theta_2)$, such that the groups Π_1 and Π_2 commute, will be called *decomposable*. We will then say that $\mathcal{S}$ is of type $\mathcal{A}_1 \otimes_{\mathcal{B}} \mathcal{A}_2$ or of type $\mathcal{A}_1 \otimes \mathcal{A}_2$. We call $\mathcal{S}$ a *decomposable near polygon of type* $\delta_{\mathcal{B}}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are dense near polygons, then every decomposable near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$ is also dense.

In the sequel we will suppose that Π_1 and Π_2 commute. Then $\mathcal{S}$ is a near polygon.

Theorem 1.7.5 (Theorems 4 and 5 of [23]**)**

- *The near polygon $\mathcal{S}$ can be obtained starting from any base element $\mathcal{F}^{(1)}_\lambda$ in S_1 and any base element $\mathcal{F}^{(2)}_\mu$ in S_2.*
- *The spreads T_1, T_2 and S are admissible.*

If $\delta_{\mathcal{B}} = 1$, then we can say more.

Theorem 1.7.6 (Theorems 11, 14 and 15 of [18]**)**

- *If none of the spreads S_1 and S_2 is trivial, then S_1 and S_2 are spreads of symmetry.*
- *If each line of $\mathcal{A}_1$ and $\mathcal{A}_2$ is incident with exactly three points and if none of the spreads S_1 and S_2 is trivial, then $\mathcal{S}$ is a near polygon if and only if S_1 and S_2 are spreads of symmetry.*

The following theorem deals with decomposable near polygons of type 0.

Theorem 1.7.7 (Lemma 10 of [23]**)**
Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote two near polygons with diameter at least 1. Then, up to an isomorphism, $\mathcal{A}_1 \times \mathcal{A}_2$ is the unique near polygon of type $\mathcal{A}_1 \otimes_0 \mathcal{A}_2$.

Remark
It is possible that more than one decomposable near polygon of type δ ($\delta \geq 1$) is derived from two near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$ (see e.g. [14]).

1.7.3 Characterization

If we consider a tuple $(\mathcal{B}, \mathcal{A}_1, \mathcal{A}_2, S_1, S_2, \mathcal{F}^{(1)}_1, \mathcal{F}^{(2)}_1, \theta_1, \theta_2)$ which gives rise to a decomposable near polygon $\mathcal{A}_1 \otimes \mathcal{A}_2$ and if we define

$$T_1 := \{\{(x, \mathcal{F}_1, \mathcal{F}_2) | x \in \mathcal{B}, \mathcal{F}_1 \in S_1\} | \mathcal{F}_2 \in S_2\}\,;$$

$$T_2 := \{\{(x, \mathcal{F}_1, \mathcal{F}_2) | x \in \mathcal{B}, \mathcal{F}_2 \in S_2\} | \mathcal{F}_1 \in S_1\}\,;$$

then T_1 and T_2 are two partitions of $\mathcal{A}_1 \otimes \mathcal{A}_2$ in geodetically closed sub near polygons isomorphic to $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively.

Definition
Let $\delta \in \mathbb{N}$. If we have an arbitrary near polygon $\mathcal{S}$, then we can consider all tuples $(\mathcal{B}, \mathcal{A}_1, \mathcal{A}_2, S_1, S_2, \mathcal{F}^{(1)}_1, \mathcal{F}^{(2)}_1, \theta_1, \theta_2)$, $\mathrm{diam}(\mathcal{B}) = \delta$, which give rise to a decomposable near polygon isomorphic to $\mathcal{S}$. With every such tuple there corresponds a pair $\{T_1, T_2\}$ of partitions in $\mathcal{S}$ in geodetically closed sub near polygons. We denote the set of all pairs $\{T_1, T_2\}$ arising in this way by $\Delta_\delta(\mathcal{S})$. If $\{T_1, T_2\} \in \Delta_\delta(\mathcal{S})$, then the following properties hold:

P_1 all elements of T_i, $i \in \{1,2\}$, are isomorphic;

P_2 $\mathrm{diam}(\mathcal{F}_1 \cap \mathcal{F}_2) = \delta$ for every $\mathcal{F}_1 \in T_1$ and $\mathcal{F}_2 \in T_2$;

P_3 for all $\mathcal{F}_1 \in T_1$, $\mathcal{F}_2 \in T_2$ and $x \in \mathcal{F}_1 \cap \mathcal{F}_2$, $\Gamma_1(x) \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$.

(For a discussion on geodetically closed sub near polygons of decomposable near polygons, see Section 3 of [23]).

Theorem 1.7.8 (Theorem 10 of [23]**)**
Let $\mathcal{S}$ be a dense near polygon with diameter at least 2 and let T_1 and T_2 be two partitions of $\mathcal{S}$ in geodetically closed sub near polygons. For every point x of $\mathcal{S}$, let $\mathcal{F}_i(x)$, $i \in \{1,2\}$, denote the unique element of T_i through x and put $\mathcal{I}(x) := \mathcal{F}_1(x) \cap \mathcal{F}_2(x)$. Suppose that $\Gamma_1(x) \subseteq \mathcal{F}_1(x) \cup \mathcal{F}_2(x)$ for every point x of $\mathcal{S}$. If for every point x of $\mathcal{S}$ and every $i \in \{1,2\}$

- *$\mathcal{I}(x)$ is classical in $\mathcal{F}_i(x)$;*
- *$\mathcal{I}(x) \subseteq \mathcal{C}(\Gamma_1(x) \setminus \mathcal{F}_i(x))$;*

then there exist near polygons $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$ such that

- *$\mathcal{F}_1(x) \cong \mathcal{A}_1$, $\mathcal{F}_2(x) \cong \mathcal{A}_2$ and $\mathcal{I}(x) \cong \mathcal{B}$ for every point x of $\mathcal{S}$;*
- *$\mathcal{S}$ is decomposable of type $\mathcal{A}_1 \otimes_{\mathcal{B}} \mathcal{A}_2$.*

Corollary 1.7.9 (Corollary 1 of [23]**)**
If $T_1, T_2 \geq 2$ and $\mathrm{diam}(\mathcal{I}(x)) \leq 1$ for every point $x \in \mathcal{S}$, then there exists sub near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{S}$ is decomposable of type $\mathcal{A}_1 \otimes_k \mathcal{A}_2$ for a certain $k \in \{0,1\}$.

Remark
In the sequel, we will always use the notion 'glued near polygon' when referring to a decomposable near polygon of type 1.

1.8 Dual polar spaces

1.8.1 Polar spaces

In this section, we mean with a *projective space* an incidence structure with the property that any three noncollinear points generate a (possibly degenerate) projective plane. It is called *irreducible* when there are no lines of size 2; otherwise it is called *reducible*. If $(P_i)_{i \in I}$, is a family of irreducible projective spaces whose point sets are pairwise disjoint, then the union P of their point sets carries the structure of a projective space if we take as lines the lines on each P_i on the one hand and all pairs $\{x_i, y_j\}$ with $x_i \in P_i$, $y_j \in P_j$, $i \neq j$, on the other hand. The projective space P is called the *direct sum* of $(P_i)_{i \in I}$. It is known that a reducible projective space is the direct sum of irreducible projective spaces.

A *polar space* $\mathcal{P}$ *of rank* $n \geq 1$ is a set P together with a collection $\mathcal{A}$ of subsets of P, called *subspaces*, satisfying the following axioms.

PS$_1$ Any proper subspace, together with the subspaces it contains, is a projective space of dimension at most $n-1$. This projective space may be reducible.

PS$_2$ The intersection of two subspaces is a subspace.

PS$_3$ Given a subspace V of dimension $n-1$ and a point $p \in P \setminus V$, there is a unique subspace W such that $p \in W$ and $V \cap W$ has dimension $n-2$; W contains all points of V that are collinear with p, i.e. that are contained in a line (i.e. a onedimensional subspace) together with p.

PS$_4$ There exist two disjoint subspaces of dimension $n-1$.

Let $\mathcal{P}_i$ $(i \in I)$ be a set of polar spaces with point sets P_i $(i \in I)$ and with $\mathcal{A}_i$ $(i \in I)$ as collection of subspaces. Suppose that all P_i are mutually disjoint. A new polar space called the *direct sum* can then be constructed on the set $\cup_{i \in I} P_i$. Subspaces are of the form $\cup_{i \in I} a_i$, where $a_i \in \mathcal{A}_i$ $(i \in I)$. A polar space is called *irreducible* if it is not isomorphic to a direct sum of at least two polar spaces of rank ≥ 1.

The polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles. By Tits' classification of polar spaces ([44]), every finite irreducible polar space of rank $n \geq 3$ without lines of size 2 is isomorphic to one of the following examples:

- $W(2n-1, q)$: the polar space associated with a symplectic polarity in $\mathsf{PG}(2n-1, q)$;
- $Q(2n, q)$: the polar space associated with a nonsingular quadric in $\mathsf{PG}(2n, q)$;
- $Q^-(2n+1, q)$: the polar space associated with a nonsingular elliptic quadric in $\mathsf{PG}(2n+1, q)$;

- $Q^+(2n-1, q)$: the polar space associated with a nonsingular hyperbolic quadric in $\mathsf{PG}(2n-1, q)$;
- $H(2n, q^2)$: the polar space associated with a nonsingular hermitian variety in $\mathsf{PG}(2n, q^2)$;
- $H(2n-1, q^2)$: the polar space associated with a nonsingular hermitian variety in $\mathsf{PG}(2n-1, q^2)$.

1.8.2 Dual polar spaces and classical near polygons

Let $\mathcal{P}$ be a polar space of rank $n \geq 1$. Consider the incidence structure $\mathcal{P}^D$ which is constructed as follows:

- the points are the maximal subspaces of $\mathcal{P}$ (i.e. the subspaces of dimension $n-1$);
- the lines are the next-to-maximal subspaces of $\mathcal{P}$ (i.e. the subspaces of dimension $n-2$);
- incidence is reverse containment.

Then $\mathcal{P}^D$ is called a *dual polar space (of rank n)*. Every dual polar space of rank n is a near $2n$-gon ([8]).

Definition
A near polygon is called *classical* if the following conditions are satisfied:

- every two points at distance 2 are contained in a quad;
- every point-quad pair is classical.

According to this definition, all nondegenerate generalized quadrangles are classical.

Theorem 1.8.1 (Cameron, Theorem 1 of [8]**)**
The class of dual polar spaces of rank $n \geq 1$ coincides with the class of classical near polygons of diameter at least 1.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two polar spaces and let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the corresponding dual polar spaces. If $\mathcal{S}$ is the dual polar space corresponding to the direct sum $\mathcal{P}$ of $\mathcal{P}_1$ and $\mathcal{P}_2$, then $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2$. Hence the direct product of dual polar spaces is again a dual polar space, whose corresponding polar space is reducible. In the following table, we list all finite dual polar spaces which are dense and not a product near polygon. The pair (s, t_2) denotes the order of the quads. In the table, we have made use of the following well-known isomorphisms: $Q^D(4,q) \cong W(q)$ and $[Q^-(5,q)]^D = Q^D(5,q) \cong H(3,q^2)$. Every near $2d$-gon mentioned in the table is regular with parameters s and $t_i := \frac{t_2^i - t_2}{t_2 - 1}$, $i \in \{0,\ldots,d\}$. Remark that in the case where q is even, $W(2n-1,q) \cong Q(2n,q)$ and hence $W^D(2n-1,q) \cong Q^D(2n,q)$.

polar space	dual polar space	quads	(s, t_2)
$Q(2n,q)$	$Q^D(2n,q)$	$W(q)$	(q,q)
$Q^-(2n+1,q)$	$[Q^-(2n+1,q)]^D$	$H(3,q^2)$	(q^2,q)
$H(2n-1,q^2)$	$H^D(2n-1,q^2)$	$Q(5,q)$	(q,q^2)
$H(2n,q^2)$	$H^D(2n,q^2)$	$H^D(4,q^2)$	(q^3,q^2)
$W(2n-1,q)$	$W^D(2n-1,q)$	$Q(4,q)$	(q,q)

Table 1.1: Dual polar spaces

Theorem 1.8.2 ([8]**)**
If π is a subspace of dimension k of a polar space $\mathcal{P}$ of rank n ($n \geq 1$, $k \leq n-1$), then the set U_π of all maximal subspaces through π defines a geodetically closed sub near $2(n-1-k)$-gon of $\mathcal{P}^D$. Conversely, every geodetically closed subspace of $\mathcal{P}^D$ is obtained in this way.

1.9 Near polygons with a linear representation

1.9.1 Definition and properties

Let Π_∞ be a projective space of dimension $n \geq 0$ embedded as a hyperplane in a projective space Π and let $\mathcal{K}$ be a nonempty set of points of Π_∞. With every point $p \in \Pi_\infty$, we associate an element $i_\mathcal{K}(p) \in \mathbb{N} \cup \{+\infty\}$, called the *$\mathcal{K}$-index of p*:

- if $p \notin \langle \mathcal{K} \rangle$, then $i_\mathcal{K}(p) = +\infty$;
- if $p \in \langle \mathcal{K} \rangle$, then $i_\mathcal{K}(p) = m$, where m is the smallest integer with the property that there are m points of $\mathcal{K}$ generating a subspace containing p.

The *linear representation* $T_n^*(\mathcal{K})$ is the geometry with points the affine points of Π (i.e. the points not contained in Π_∞), with lines the lines of Π not contained in Π_∞ which intersect $\mathcal{K}$ in a point and with incidence derived from Π.

Theorem 1.9.1 (Lemma 4.2 and Theorem 4.4 of [24]**)**

- *If x and y are two different points of $T_n^*(\mathcal{K})$ and if z is the intersection of xy with Π_∞, then $\mathrm{d}(x,y) = i_{\mathcal{K}}(z)$, where $\mathrm{d}(\cdot,\cdot)$ denotes the distance in the collinearity graph of $T_n^*(\mathcal{K})$.*

- *$T_n^*(\mathcal{K})$ is a near polygon if and only if for every point $x \in \mathcal{K}$ and for every line L of Π_∞ through x, there is a unique point $y \in L \setminus \{x\}$ with smallest $\mathcal{K}$-index.*

Theorem 1.9.2
Consider in Π_∞ two subspaces π_1 and π_2 of dimensions $n_1 \geq 0$ and $n_2 \geq 0$ respectively, such that $\Pi_\infty = \langle \pi_1, \pi_2 \rangle$ and let $\mathcal{K}_i$, $i \in \{1,2\}$, be a set of points in π_i such that $T_{n_i}^*(\mathcal{K}_i)$, $i \in \{1,2\}$, is a near $2d_i$-gon.

- (Lemma 4.7 of [24]) *If $\pi_1 \cap \pi_2 = \emptyset$ then $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$ is a near $2(d_1+d_2)$-gon isomorphic to the direct product $T_{n_1}^*(\mathcal{K}_1) \times T_{n_2}^*(\mathcal{K}_2)$. Every product near polygon with a linear representation is obtained in this way.*

- (Theorems 2, 17 and 18 of [18]) *If $\mathcal{K}_1 \cap \mathcal{K}_2 = \pi_1 \cap \pi_2 = \{p\}$ then $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$ is a glued near $2(d_1+d_2-1)$-gon of type $T_{n_1}^*(\mathcal{K}_1) \otimes T_{n_2}^*(\mathcal{K}_2)$. Every glued near polygon with a linear representation is obtained in this way.*

1.9.2 Known examples of dense near polygons with a linear representation

By Theorem 1.9.2, we may restrict ourselves to those dense near polygons $\mathcal{S}$ which are not a product near polygon nor a glued near polygon.

Generalized quadrangles arising from hyperovals

Suppose that Π_∞ is isomorphic to $\mathsf{PG}(2,q)$, q even, and let $\mathcal{K}$ be a hyperoval in Π_∞. Then $T_2^*(\mathcal{K})$ is a generalized quadrangle of order $(q-1, q+1)$.

A near hexagon arising from the Coxeter cap

Consider the following matrix:

$$\mathsf{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

The columns of A define a set $\mathcal{K}$ of 12 points in $\mathsf{PG}(5,3)$. Any set of points projectively equivalent with $\mathcal{K}$ is called a *Coxeter cap* ([9]). It has the following nice properties.

- No five points of $\mathcal{K}$ are contained in a three-dimensional space.
- Every hyperplane of $\mathsf{PG}(5,3)$ which contains at least five points of $\mathcal{K}$ contains exactly 6 points of $\mathcal{K}$.

It was shown in [24] that $T_5^*(\mathcal{K})$ is a dense near hexagon. We will give an alternative definition of this near polygon in Section 1.10.3.

1.10 Slim dense near polygons

A near polygon $\mathcal{S}$ is called *slim* if every line of $\mathcal{S}$ contains exactly three points. Suppose that $\mathcal{S}$ is a slim near polygon and let $\mathcal{F}$ be a big geodetically closed sub near polygon of $\mathcal{S}$. Then for every point x outside $\mathcal{F}$, the line $x\pi_{\mathcal{F}}(x)$ contains a unique third point which we denote by $r(x)$. If $x \in \mathcal{F}$, then we define $r(x) := x$. The map r is called the *reflection about* $\mathcal{F}$ and is an automorphism of $\mathcal{S}$.

1.10.1 The slim near quadrangles

Consider a generalized quadrangle of order $(2,t)$. By Theorem 1.5.9, $t=1$, $t=2$ or $t=4$. It is an easy exercise to show that every generalized quadrangle of order $(2,1)$ is isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3$. There is also a unique generalized quadrangle of order 2 (see e.g. Theorem 5.2.3 of [36]) and a unique generalized quadrangle of order $(2,4)$ (see e.g. Theorem 5.3.2 of [36]). Summarizing, we have the following theorem.

Theorem 1.10.1
Every generalized quadrangle of order $(2,t)$ is isomorphic to one of the following examples:

- *the grid* $\mathbb{L}_3 \times \mathbb{L}_3 \cong Q(3,2)$;
- $W(2) \cong Q(4,2)$;
- $Q(5,2)$.

In the sequel, the generalized quadrangles $\mathbb{L}_3 \times \mathbb{L}_3$, $W(2)$ and $Q(5,2)$ will often occur as quads in other slim near polygons. We will refer to them as *grid-quads*, $W(2)$-*quads* and $Q(5,2)$-*quads*, respectively.

Remark
Throughout this thesis, we will often refer to a useful model for the generalized quadrangle $W(2)$, due to Sylvester ([41]). A *duad* is an unordered pair ij $(= ji)$ of distinct elements of the set $\{1,2,3,4,5,6\}$. A *syntheme* is a set $\{ij, kl, mn\}$ of three mutually disjoint duads. Let $\mathcal{P}$ be the set of duads, let $\mathcal{L}$ be the set of synthemes and let $\mathbf{I}$ be containment. Then $W(2) \cong (\mathcal{P}, \mathcal{L}, \mathbf{I})$. We will refer to this model as the *Sylvester model* for $W(2)$.

Spreads and ovoids

A $(u \times u)$-grid has precisely $u!$ ovoids. In particular, $\mathbb{L}_3 \times \mathbb{L}_3$ has precisely six ovoids. Now consider the Sylvester model for $W(2)$. For each $i \in \{1, \dots, 6\}$, the set $\{ij | j \neq i\}$ is an ovoid and all ovoids arise in this way. So there are exactly six ovoids, every two distinct ovoids meet in a point and every point is contained in exactly two ovoids. As a corollary of Theorem 1.5.7, $Q(5,2)$ has no ovoids.

Clearly, the grid $\mathbb{L}_3 \times \mathbb{L}_3$ has two spreads. Every line of one spread intersects every line of the other spread in a point. As mentioned before, $W(2)$ is self-dual. So $W(2)$ has exactly 6 spreads. Now consider the generalized quadrangle $H(3,q^2)$. The *classical* ovoids of $H(3,q^2)$ are just the nonsingular plane intersections with $H(3,q^2)$ in $\mathsf{PG}(3,q^2)$. Given such an ovoid O, for distinct points $x, y \in O$, it is always true that $\{x,y\}^{\perp\perp} \subseteq O$. Hence $(O \setminus \{x,y\}^{\perp\perp}) \cup \{x,y\}^{\perp}$ is again an ovoid. The case $n = 2$, i.e. $H(3,4)$, was studied by Brouwer and Wilbrink in [7]. The generalized quadrangle $H(3,4)$ has 200 ovoids, 40 of which are classical. As a matter of fact, the remaining 160 ovoids are obtained in the above described manner from a classical ovoid. As mentioned before, $Q(5,2) \cong H^D(3,4)$. Hence also all spreads of $Q(5,2)$ are known.

1.10.2 Three infinite classes of slim dense near polygons

I. The near polygons $\mathbb{H}_n$, $n \geq 1$

Consider a set V of size $2n+2$, $n \geq 1$, and let $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be the following incidence structure:

- $\mathcal{P}$ is the set of all partitions of V in $n+1$ sets of size 2;

- $\mathcal{L}$ is the set of all partitions of V in $n-1$ sets of size 2 and one set of size 4;
- a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition determined by p is a refinement of the partition determined by L.

By [4], $\mathbb{H}_n$ is a near $2n$-gon. The near polygon $\mathbb{H}_n$ has order $(2, \frac{(n-1)(n+2)}{2})$. Clearly $\mathbb{H}_1 \cong \mathbb{L}_3$ and $\mathbb{H}_2 \cong W(2)$. (For $n=2$, we obtain Sylvester's model, as $W(2)^D \cong W(2)$. All quads of $\mathbb{H}_n$, $n \geq 2$ are isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3$ or $W(2)$.

Theorem 1.10.2 ([19])

- *The geodetically closed sub near $2(n+1-k)$-gons, $k \in \{1, \ldots, n+1\}$, of $\mathbb{H}_n$ are of the form $\mathbb{H}_{n_1-1} \times \cdots \times \mathbb{H}_{n_k-1}$ with $n_1, \ldots, n_k \geq 1$ and $n_1 + \cdots + n_k = n+1$.*
- *Let $\mathcal{M}_n$ denote the partial linear space whose points, respectively lines, are the subsets of size 2, respectively 3 of $\{1, 2, \ldots, n+1\}$ with containment as incidence relation. Then every local space of $\mathbb{H}_n$ is isomorphic to $\mathcal{L}_{\mathbb{H}_n}$, the unique linear space obtained from $\mathcal{M}_n$ by adding lines of size 2.*

II. The near polygons $\mathbb{G}_n$, $n \geq 1$

Let the vector space $V(2n, 4)$, $n \geq 1$, with base $\{\vec{e}_0, \ldots, \vec{e}_{2n-1}\}$, be equipped with the nonsingular hermitian form $(\sum x_i \vec{e}_i, \sum y_i \vec{e}_i) = x_0 y_0^2 + x_1 y_1^2 + \ldots + x_{2n-1} y_{2n-1}^2$, and let $H = H(2n-1, 4)$ denote the corresponding hermitian variety in $\mathsf{PG}(2n-1, 4)$. For every vector $\vec{x}$ of $V(2n, 4)$, we have $\vec{x} = \sum (\vec{x}, \vec{e}_i) \vec{e}_i$. The *support* of a point $p = \langle \vec{x} \rangle$ of $\mathsf{PG}(2n-1, 4)$ is the set of all $i \in \{0, \ldots, 2n-1\}$ for which $(\vec{x}, \vec{e}_i) \neq 0$. The number $|S_p|$ is called the *weight* of p and is equal to the number of nonzero coordinates. A point of $\mathsf{PG}(2n-1, 4)$ belongs to H if and only if its weight is even. A subspace π on H is said to be *good* if it is generated by a (possibly empty) set $\mathcal{G}_\pi \subseteq H$ of points whose supports are two by two disjoint. If π is good, then $\mathcal{G}_\pi$ is uniquely determined. If $\mathcal{G}_\pi$ contains k_{2i} points of weight $2i$, $i \in \mathbb{N} \setminus \{0\}$, then π is said to be of *type* $(2^{k_2}, 4^{k_4}, \ldots)$. Let Y, respectively Y', denote the set of all good subspaces of dimension $n-1$, respectively $n-2$. By [21], the incidence structure $\mathbb{G}_n = (Y, Y', \mathbf{I})$ such that $\mathbf{I}$ is reverse containment is a near $2n$-gon. Moreover, $\mathbb{G}_n$ is a dense near polygon of order $(2, \frac{3n^2-n-2}{2})$. The number of points of $\mathbb{G}_n$ equals $\frac{3^n \cdot (2n)!}{2^n \cdot n!}$. Clearly, $\mathbb{G}_1 \cong \mathbb{L}_3$ and $\mathbb{G}_2 \cong Q(5, 2)$. If $n \geq 3$, then the three types of quads occur in $\mathbb{G}_n$.

Theorem 1.10.3 (Lemma 6 of [21]**)**
There is a one-to-one correspondence between the geodetically closed sub near polygons of $\mathbb{G}_n$ and the good subspaces on H such that a point π of $\mathbb{G}_n$ belongs to a geodetically closed sub near polygon $\mathcal{F}$ of $\mathbb{G}_n$ if and only if $\pi_{\mathcal{F}} \subset \pi$, where $\pi_{\mathcal{F}}$ denotes the good subspace of H corresponding to $\mathcal{F}$.

Theorem 1.10.4 (Theorem 2 of [21]**)**
The geodetically closed sub near $2(n-k)$-gons, $k \in \{0, \dots, n\}$, of $\mathbb{G}_n$ are of the form $\mathbb{H}_{n_1-1} \times \cdots \times \mathbb{H}_{n_k-1} \times \mathbb{G}_{n_{k+1}}$ with $n_1, \dots, n_k \geq 1$, $n_{k+1} \geq 0$ and $n_1 + \cdots + n_{k+1} = n$.

Lines and quads in $\mathbb{G}_n$

Let $n \geq 3$. If L is a line of $\mathbb{G}_n$, then there are two possibilities for π_L:

- π_L has type (2^{n-1}), and we call L a *special line*;
- π_L has type $(2^{n-2}, 4^1)$ and we call L an *ordinary line*.

As mentioned before, all types of quads arise in $\mathbb{G}_n$, $n \geq 3$. If $\mathcal{Q}$ is a $Q(5,2)$-quad, then $\pi_{\mathcal{Q}}$ is of type (2^{n-2}). If $\mathcal{Q}$ is a $W(2)$-quad, then $\pi_{\mathcal{Q}}$ is of type $(2^{n-3}, 6^1)$. There are two kinds of grid-quads:

- grid-quads of type $(2^{n-3}, 4^1)$; such a grid-quad contains a spread of special lines and a spread of ordinary lines;
- grid-quads of type $(2^{n-4}, 4^2)$; such a grid-quad contains no special lines.

Theorem 1.10.5 (Lemmas 8 and 11 of [21]**)**
Consider the near polygon $\mathbb{G}_n$, $n \geq 3$. Then

- *each point is contained in n special lines and $\frac{3}{2}n(n-1)$ ordinary lines;*
- *each special line is contained in $n-1$ quads isomorphic to $Q(5,2)$, no $W(2)$-quads and $\frac{3}{2}(n-1)(n-2)$ grid-quads;*
- *each ordinary line is contained in a unique $Q(5,2)$-quad, $3(n-2)$ $W(2)$-quads and $\frac{(n-2)(3n-7)}{2}$ grid-quads;*
- *each line is contained in exactly $\frac{(n-1)(3n-4)}{2}$ quads;*
- *if $L_1, \dots, L_k$ are different special lines of $\mathbb{G}_n$, $n \geq 3$, through a fixed point x, then $\mathcal{C}(L_1, \dots, L_k) \cong \mathbb{G}_k$.*

Local spaces

Let P_n be an arbitrary base in $\mathsf{PG}(n-1,4)$, let P_n' be the set of all points of $\mathsf{PG}(n-1,4)$ lying on the lines through two points of P_n. Now define $\mathcal{L}_{\mathbb{G}_n}$ as the linear space whose points are the points of P_n' and whose lines are the lines of $\mathsf{PG}(n-1,4)$ containing at least two points of P_n'. Then every local space of $\mathbb{G}_n$ is isomorphic to $\mathcal{L}_{\mathbb{G}_n}$.

III. The near polygons $\mathbb{I}_n$, $n \geq 2$

Consider a nonsingular quadric $Q(2n,2)$, $n \geq 2$, in $\mathsf{PG}(2n,2)$ and a hyperplane Π of $\mathsf{PG}(2n,2)$ intersecting $Q(2n,2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1,2)$. Let $\mathbb{I}_n$ be the following incidence structure:

- the points of $\mathbb{I}_n$ are the maximal subspaces of $Q(2n,2)$ which are not contained in $Q^+(2n-1,2)$;
- the lines of $\mathbb{I}_n$ are the next-to-maximal subspaces of $Q(2n,2)$ which are not contained in $Q^+(2n-1,2)$;
- the incidence relation is reverse containment.

By [4], $\mathbb{I}_n$ is a dense near $2n$-gon of order $(2, 2^n-3)$. The distances in $\mathbb{I}_n$ are inherited from the distances in $Q^D(2n,2)$.

Let π be a subspace of dimension $n-1-i$, $i \in \{0,\ldots,n\}$, on $Q(2n,2)$ which is not contained in $Q^+(2n-1,2)$ if $i \in \{0,1\}$. If X is the set of maximal subspaces of $Q(2n,2)$ through π, then $\mathcal{S}_X$ is a geodetically closed sub near polygon of $\mathbb{I}_n$. Conversely, every geodetically closed subspace of $\mathbb{I}_n$ is obtained this way. If $i \geq 2$ and if π is not contained in Π then $(\mathbb{I}_n)_X$ is isomorphic to $Q^D(2i,2)$. If $i \geq 2$ and if $\pi \subset \Pi$ then $(\mathbb{I}_n)_X$ is isomorphic to $\mathbb{I}_i$. If X is a point outside Π, i.e. $i = n-1$, then $(\mathbb{I}_n)_X$ is big. Since $\mathbb{I}_2 \cong \mathbb{L}_3 \times \mathbb{L}_3$ and $Q^D(4,2) \cong W(2)$, every quad is isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3$ or $W(2)$. Every $(n-2)$-dimensional subspace of $Q(2n,2)$ not contained in $Q^+(2n-1,2)$ contains a unique $(n-3)$-dimensional subspace which is entirely contained in $Q^+(2n-1,2)$. Hence, every line of $\mathbb{I}_n$ is contained in a unique grid-quad. As mentioned before, $\mathbb{I}_n$ has order $(2, 2^n-3)$. So, every line of $\mathbb{I}_n$ is contained in a unique grid-quad and $2^{n-1}-2$ quads isomorphic to $W(2)$. The near hexagon $\mathbb{I}_3$ is isomorphic to $\mathbb{H}_3$.

Every local space of $\mathbb{I}_n$ is isomorphic to $\mathsf{PG}(n-1,2)'$, the incidence structure obtained from $\mathsf{PG}(n-1,2)$ by removing one point.

1.10.3 Three 'exceptional' slim dense near hexagons

I. The near hexagon $\mathbb{E}_1$ associated with the extended ternary Golay code

Consider again the matrix A defined in Section 1.9.2. Let C be the six-dimensional subspace of the vector space $\mathbb{F}_3^{12}$ generated by the rows of A. The subspace C of $\mathbb{F}_3^{12}$ is called the *extended ternary Golay code*. As shown in [40], a near hexagon $\mathbb{E}_1$ is associated with this code as follows:

- the points are the cosets $\vec{v} + \mathsf{C}$ of the code, $\vec{v} \in \mathbb{F}_3^{12}$;
- the lines are all the triplets of the form $\{\vec{v} + \mathsf{C}, \vec{v} + \vec{e}_i + \mathsf{C}, \vec{v} - \vec{e}_i + \mathsf{C}\}$, $\vec{v} \in \mathbb{F}_3^{12}$, $i \in \{1, \ldots, 12\}$, and $\vec{e}_i$ the unique vector of length 12 whose i-th entry equals 1 and all other entries equal 0;
- incidence is containment.

This near hexagon is isomorphic to the near hexagon $T_5^*(\mathcal{K})$ defined in Section 1.9.2. It is a dense near hexagon which has 729 points and is regular with parameters $s = 2$, $t_2 = 1$ and $t = 11$. It was shown in [2] that there is up to an isomorphism only one dense near hexagon with these parameters. All quads are grids and none of them is big in $\mathbb{E}_1$. All local spaces of $\mathbb{E}_1$ are isomorphic to K_{12}, the complete graph on 12 vertices, regarded as linear space.

II. The near hexagon $\mathbb{E}_2$ associated with the Steiner system $S(5, 8, 24)$

The projective plane $\mathsf{PG}(2, 4) = (X, B, \mathbf{I})$ has the following properties:

- The set of 168 hyperovals (i.e. sets of 6 points no three of which are collinear) can be divided into three classes $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ of equal size. Two hyperovals are in the same class if and only if they intersect in an even number of points.
- The set of 360 Baer subplanes (i.e. subplanes of order 2) can be divided into three classes $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ of equal size. Two Baer subplanes are in the same class if and only if they intersect in an odd number of points
- The indices i and j can be chosen in such a way that for $O \in \mathcal{O}_i$ and $S \in \mathcal{B}_j$, $|O \cap S|$ is even if and only if $i = j$.

We now construct the following design. The point set is equal to $X \cup \{\infty_1, \infty_2, \infty_3\}$, with $\infty_1, \infty_2, \infty_3 \notin X$. There are four types of blocks:

- $L \cup \{\infty_1, \infty_2, \infty_3\}$ where $L \in B$;
- $(O \cup \{\infty_1, \infty_2, \infty_3\}) \setminus \{\infty_i\}$ for each $O \in \mathcal{O}_i$;
- $S \cup \{\infty_i\}$ for each $S \in \mathcal{B}_i$;
- the symmetric difference $L \triangle L'$ of L and L' for all $L, L' \in B$, $L \neq L'$.

Taking the natural incidence, we find the unique Steiner system $S(5, 8, 24)$. As shown in [40], a near hexagon $\mathbb{E}_2$ is associated with $S(5, 8, 24)$ as follows:

- the points are the blocks of the Steiner system;
- the lines are the sets of three mutually disjoint blocks;
- incidence is the natural one.

The near hexagon $\mathbb{E}_2$ is dense, has 759 points, is regular with parameters $(s, t_2, t) = (2, 2, 14)$ and is completely determined by its parameters, see [40]. All quads are isomorphic to $W(2)$ and none of them is big in $\mathbb{E}_2$. All local spaces of $\mathbb{E}_2$ are isomorphic to $\mathsf{PG}(3, 2)$.

Theorem 1.10.6 (Theorem 5 of [6]**)**
A dense regular near hexagon with parameters s, t_2 and t is isomorphic to $\mathbb{E}_2$ if and only if $1 + t = (1 + t_2)(1 + st_2)$.

III. The Aschbacher near hexagon $\mathbb{E}_3$

The near hexagon $\mathbb{E}_3$ was first constructed by Aschbacher in [1], but we take the construction from [4]. Consider in $\mathsf{PG}(6, 3)$ a nonsingular quadric $Q(6, 3)$ and a nontangent hyperplane Π intersecting $Q(6, 3)$ in a nonsingular elliptic quadric $Q^-(5, 3)$. There is a polarity associated with $Q(6, 3)$ and we call two points *orthogonal* when one of them is contained in the polar hyperplane of the other. Let N denote the set of 126 internal points of $Q(6, 3)$ which are contained in Π, i.e. the set of all 126 points in Π for which the polar hyperplane intersects $Q(6, 3)$ in a nonsingular elliptic quadric. Let $\mathbb{E}_3$ be the following incidence geometry:

- the points are the 6-tuples of mutually orthogonal points of N;
- the lines are the pairs of orthogonal points of N;
- incidence is reverse inclusion.

Then $\mathbb{E}_3$ is a dense near hexagon of order $(2, 14)$. It has 567 points and contains $W(2)$-quads and $Q(5, 2)$-quads. All $Q(5, 2)$-quads are big. All local spaces of $\mathbb{E}_3$ are isomorphic to $\overline{W(2)}$, the linear space obtained from $W(2)$ by adding its ovoids as extra lines of size 5.

1.10.4 The known slim dense near polygons

Every known slim dense near $2n$-gon, $n \geq 2$, which is not a product near polygon nor a glued near polygon is isomorphic to precisely one of the examples of Table 1.2.

near polygon	local spaces
$Q^D(2n,2) \cong W^D(2n-1,2)$, $n \geq 2$	$\mathsf{PG}(n-1,2)$
$H^D(2n-1,4)$, $n \geq 2$	$\mathsf{PG}(n-1,4)$
$\mathbb{G}_n$, $n \geq 3$	$\mathcal{L}_{\mathbb{G}_n}$
$\mathbb{H}_n$, $n \geq 3$	$\mathcal{L}_{\mathbb{H}_n}$
$\mathbb{I}_n$, $n \geq 4$	$\mathsf{PG}(n-1,2)'$
$\mathbb{E}_1$ $(n=3)$	K_{12}
$\mathbb{E}_2$ $(n=3)$	$\mathsf{PG}(3,2)$
$\mathbb{E}_3$ $(n=3)$	$\overline{W(2)}$

Table 1.2: Known slim dense near polygons

All slim dense near hexagons have been classified (see Section 1.10.5). In Chapter 4, we will classify all slim dense near octagons.

1.10.5 The slim dense near hexagons

The slim dense near hexagons have been classified in [4]. There are eleven examples and we have already encountered all of them (either explicitly or implicitly). They are listed in Table 1.3 together with their number of points $v_{\mathcal{S}}$, their constant number of lines $t_{\mathcal{S}}+1$ through a point, their quads and their local spaces. Grid-quads are denoted as $\boxplus$. Remark that there is up to an isomorphism a unique dense near hexagon of type $Q(5,2) \otimes Q(5,2)$ (see [4]).

As we can see in the table, all local spaces of a given near hexagon $\mathcal{S}$ are isomorphic. The local spaces of $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$ and $Q(5,2) \otimes Q(5,2)$ are so-called *crosses*. An (i,j)-cross $C_{i,j}$ $(i,j \geq 2)$ is the unique linear space on $i+j-1$ points containing a line of size i, a line of size j, and $(i-1)(j-1)$ additional lines of size 2.

Proposition 1.10.7
Let x and y denote two points of a slim dense near polygon $\mathcal{S}$. If x is contained in a quad of order $(2,i)$, $i \in \{1,2,4\}$, then y is also contained in a quad of order $(2,i)$.

near hexagon	$v_{\mathcal{S}}$	$t_{\mathcal{S}}$	big quads	other quads	local spaces
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	27	2	$\boxplus$	–	$C_{2,2}$
$W(2) \times \mathbb{L}_3$	45	3	$\boxplus, W(2)$	–	$C_{3,2}$
$Q(5,2) \times \mathbb{L}_3$	81	5	$\boxplus, Q(5,2)$	–	$C_{5,2}$
$\mathbb{H}_3 \cong \mathbb{I}_3$	105	5	$W(2)$	$\boxplus$	$\mathsf{PG}(2,2)'$
$Q^D(6,2)$	135	6	$W(2)$	–	$\mathsf{PG}(2,2)$
$Q(5,2) \otimes Q(5,2)$	243	8	$Q(5,2)$	$\boxplus$	$C_{5,5}$
$\mathbb{G}_3$	405	11	$Q(5,2)$	$\boxplus, W(2)$	$\mathcal{L}_{\mathbb{G}_3}$
$\mathbb{E}_1$	729	11	–	$\boxplus$	K_{12}
$\mathbb{E}_2$	759	14	–	$W(2)$	$\mathsf{PG}(3,2)$
$\mathbb{E}_3$	567	14	$Q(5,2)$	$W(2)$	$\overline{W(2)}$
$H^D(5,4)$	891	20	$Q(5,2)$	–	$\mathsf{PG}(2,4)$

Table 1.3: Slim dense near hexagons

Proof
By connectedness of $\mathcal{S}$, it is sufficient to prove the result for collinear points x and y. If $\mathcal{Q}$ denotes an arbitrary quad through x, then $\mathcal{C}(y, \mathcal{Q})$ is either the quad $\mathcal{Q}$ itself or a hex $\mathcal{H}$. In any case, there exists a quad through y of the same order as $\mathcal{Q}$ (taking into account that all local spaces in a given hex are isomorphic). $\square$

Proposition 1.10.8
Let $\mathcal{H}$ be a slim dense near hexagon. There exist constants $a_{\mathcal{H}}$, $b_{\mathcal{H}}$ and $c_{\mathcal{H}}$ such that every point of $\mathcal{H}$ is contained in $a_{\mathcal{H}}$ grid-quads, $b_{\mathcal{H}}$ $W(2)$-quads and $c_{\mathcal{H}}$ $Q(5,2)$-quads. Furthermore $t_{\mathcal{H}}(t_{\mathcal{H}}+1) = 2a_{\mathcal{H}} + 6b_{\mathcal{H}} + 20c_{\mathcal{H}}$, and for every point x of $\mathcal{H}$,

- $|\Gamma_0(x)| = 1$;
- $|\Gamma_1(x)| = 2(t_{\mathcal{H}}+1)$;
- $|\Gamma_2(x)| = 4a_{\mathcal{H}} + 8b_{\mathcal{H}} + 16c_{\mathcal{H}}$;
- $|\Gamma_3(x)| = v_{\mathcal{S}} - 1 - [2(t_{\mathcal{H}}+1) + 4a_{\mathcal{H}} + 8b_{\mathcal{H}} + 16c_{\mathcal{H}}]$.

Hence for every $i \in \{0,1,2,3,4\}$, $|\Gamma_i(x)|$ is independent of the chosen point x.

Proof
Since for a given slim dense near hexagon $\mathcal{H}$ all local spaces are isomorphic, there exist constants $a_{\mathcal{H}}, b_{\mathcal{H}}$ and $c_{\mathcal{H}}$ such that every point of $\mathcal{H}$ is contained

in $a_{\mathcal{H}}$ grid-quads, $b_{\mathcal{H}}$ $W(2)$-quads and $c_{\mathcal{H}}$ $Q(5,2)$-quads. Since any two lines are contained in a unique quad, $2a_{\mathcal{H}} + 6b_{\mathcal{H}} + 20c_{\mathcal{H}} = t_{\mathcal{S}}(t_{\mathcal{S}} + 1)$. Since any two points at distance 2 are contained in a unique quad, $|\Gamma_2(x)| = 4a_{\mathcal{H}} + 8b_{\mathcal{H}} + 16c_{\mathcal{H}}$ for every point x of $\mathcal{H}$. The property now follows. $\square$

Chapter 2

Near polygons with a nice chain of sub near polygons

By Theorem 1.6.3, every dense near $2n$-gon $\mathcal{S}$ has a chain $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = \mathcal{S}$ of geodetically closed sub near polygons such that $\mathrm{diam}(\mathcal{F}_i) = i$, $i \in \{0, \ldots, n\}$. A lot of dense near polygons have the property that they contain a big geodetically closed sub near polygon. In this chapter, we will characterize those slim dense near polygons $\mathcal{S}$ which have a chain $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = \mathcal{S}$ of geodetically closed sub near polygons such that $\mathcal{F}_i$ is big in $\mathcal{F}_{i+1}$ for every $i \in \{0, \ldots, n-1\}$.

2.1 Glued near polygons of type $H^D(2n_1-1, q^2) \otimes H^D(2n_2-1, q^2)$

In Chapter 1 we defined the dual polar space $H^D(2n-1, q^2)$. Here, we will determine all spreads of symmetry of $H^D(2n-1, q^2)$ and use this result to prove the existence of glued near polygons of type $H^D(2n_1-1, q^2) \otimes H^D(2n_2-1, q^2)$, $n_1, n_2 \geq 2$. We also show that there exists a unique glued near polygon of type $H^D(2n_1-1, 4) \otimes H^D(2n_2-1, 4)$ for all $n_1, n_2 \geq 2$.

2.1.1 Spreads of symmetry of $H^D(2n-1, q^2)$

Consider the hermitian variety $H := H(2n-1, q^2)$ in $\Pi := \mathsf{PG}(2n-1, q^2)$, $n \geq 2$, and let $H^D := H^D(2n-1, q^2)$ denote its associated dual polar space. We will now determine all spreads of symmetry of the near polygon H^D.

Lemma 2.1.1
Let $\mathcal{L}$ be the linear space whose points are the points of H and whose lines are the lines of Π which contain at least two points of H. Then the subspaces of $\mathcal{L}$ are precisely the intersections of H with subspaces of Π.

Proof
For every set X of points on H, let $\langle X \rangle$ denote the subspace of Π generated by all points of X and let $\overline{X}$ denote the smallest subspace of $\mathcal{L}$ through X. Since $\langle X \rangle \cap H$ is a subspace of $\mathcal{L}$, $\overline{X} \subseteq \langle X \rangle \cap H$. We will now prove that $\overline{X} = \langle X \rangle \cap H$ $(*)$. This property obviously holds if $n := \dim(\langle X \rangle) \leq 1$, and since every intersection of a hermitian variety with a plane is either the plane itself, a line in this plane, a unital or a cone pB with p a point and B a Baer subline, it also holds for $n = 2$. So, suppose that $n \geq 3$ and consider a subset Y of X such that $\dim(\langle Y \rangle) = n-1$. We may suppose that $\langle Y \rangle \cap H = \overline{Y} \subset \overline{X}$ (otherwise use induction). Now consider a fixed point x in $X \setminus \langle Y \rangle$. For every point x' of $\langle X \rangle \cap H$ different from x, the line xx' intersects $\langle Y \rangle$ in a point x'' and one of the following possibilities occurs.

- There exists a line L in $\langle Y \rangle$ through the point x'' intersecting H in a Baer subline. Since property $(*)$ holds for $n = 2$, $x' \in \langle x, L \rangle \cap H = \overline{L \cup \{x\}} \subseteq \overline{X}$.

- Every line of $\langle Y \rangle$ through the point x'' is a tangent line. Then x'' is a singular point of $\langle Y \rangle \cap H$. Hence $x'' \in H$ and the point $x' \in xx''$ belongs to $\overline{X}$.

In any case, $x' \in \overline{X}$. Since x' was an arbitrary point of $\langle X \rangle \cap H$ different from x and since $x \in \overline{X}$, $\langle X \rangle \cap H \subseteq \overline{X}$ and hence $\overline{X} = \langle X \rangle \cap H$. As a consequence the subspaces of $\mathcal{L}$ are precisely the intersections of H with subspaces of Π. This proves the lemma. $\square$

As mentioned in Chapter 1, there exists a bijective correspondence between the geodetically closed sub near polygons of H^D and the subspaces of H. If π is a subspace on H, then the set of all generators of H through π determines a geodetically closed sub near polygon of H^D which we will denote by π^ϕ. There holds $\dim(\pi) + \mathrm{diam}(\pi^\phi) = n - 1$. If π_1 and π_2 are two subspaces on H, then $\pi_1^\phi \subseteq \pi_2^\phi$ if and only if $\pi_2 \subseteq \pi_1$.

Theorem 2.1.2
If V is the set of all $(n-2)$-dimensional subspaces of H which lie in a nontangent hyperplane Π_∞, then $V^\phi := \{\alpha^\phi | \alpha \in V\}$ is a spread of symmetry of H^D. Conversely, every spread of symmetry is obtained in this way.

Proof

(a) Every generator of H contains a unique element of V, or equivalently, every point of H^D is incident with a unique line of V^ϕ. So, V^ϕ is a spread. We can choose our reference system in such a way that H has equation $X_0^{q+1} + \cdots + X_{2n-1}^{q+1} = 0$ and that Π_∞ has equation $X_{2n-1} = 0$. The group

$$G := \{\theta_\lambda : (x_0, \ldots, x_{2n-2}, x_{2n-1}) \mapsto (x_0, \ldots, x_{2n-2}, \lambda x_{2n-1}) \mid \lambda^{q+1} = 1\}$$

of automorphisms of Π fixes H setwise and Π_∞ pointwise. So, G determines a group $\tilde{G}$ of automorphisms of H^D which fixes every element of V^ϕ. This group $\tilde{G}$ acts regularly on each line of V^ϕ, proving that V^ϕ is a spread of symmetry.

(b) Consider a spread of symmetry S of H^D and let X denote the set of those points x of H for which the geodetically closed sub near $2(n-1)$-gon x^ϕ contains a line of S. Take two different points x_1 and x_2 in X, then one of the following possibilities occurs:

- $|x_1x_2 \cap H| = q^2+1$. Let y denote an arbitrary point of $x_1^\phi \cap x_2^\phi = (x_1x_2)^\phi$. By Theorem 1.7.2, the unique line of S through y is contained in x_1^ϕ and x_2^ϕ and hence in $x_1^\phi \cap x_2^\phi$. As a consequence, each of the q^2+1 geodetically closed sub near $2(n-1)$-gons through $(x_1x_2)^\phi$ belongs to X^ϕ, or equivalently, each of the q^2+1 points of x_1x_2 belongs to X.

- $|x_1x_2 \cap H| = q+1$. In this case x_1^ϕ and x_2^ϕ are two disjoint geodetically closed sub near $2(n-1)$-gons of H^D. Put $x_1x_2 \cap H = \{x_1, x_2, \ldots, x_{q+1}\}$. Let L denote an arbitrary line of S contained in x_1^ϕ and let $\mathcal{Q}$ denote the unique quad through L which intersects each of the sub near polygons x_i^ϕ, $i \in \{2, \ldots, q+1\}$, in a line. Now, let y denote an arbitrary point of $\mathcal{Q} \cap x_2^\phi$. The unique line of S through y is contained in $\mathcal{Q}$ and in x_2^ϕ and hence coincides with the line $\mathcal{Q} \cap x_2^\phi$. Since $\mathcal{Q} \cap (x_1^\phi \cup x_2^\phi \cup \cdots \cup x_{q+1}^\phi)$ is a subgrid of $\mathcal{Q}$ and since S is a regular spread, we now see that each line $\mathcal{Q} \cap x_i^\phi$, $i \in \{1, 2, \ldots, q+1\}$, belongs to S. Hence $x_1, x_2, \ldots, x_{q+1} \in X$.

As a consequence, the set X is a subspace of $\mathcal{L}$ and hence the intersection of H with a subspace π. The elements of S are precisely the elements α^ϕ, where α is an $(n-2)$-dimensional subspace contained in $\pi \cap H$. [If $L \in S$, then by Theorem 1.7.2, every point of $L^{\phi^{-1}}$ belongs to X and hence $L^{\phi^{-1}}$ is contained in $\pi \cap H$. Conversely, let α be an $(n-2)$-dimensional subspace contained in $\pi \cap H$, let $x_1, \ldots, x_{n-1}$ denote $n-1$ points of X generating α and let u denote an arbitrary point of α^ϕ. By Theorem 1.7.2 the unique line K of S through u is contained in each geodetically closed sub near polygon

x_i^ϕ and hence coincides with the line $\alpha^\phi = x_1^\phi \cap \cdots \cap x_{n-1}^\phi$.] If $\pi \cap H$ contains a subspace β of dimension $n-1$, then every line through the point β^ϕ would belong to S, which is impossible. As a consequence, $n-2$ is the maximal dimension of the subspaces contained in $\pi \cap H$. If x is a singular point of $\pi \cap H$, then x is contained in all $(n-2)$-dimensional subspaces of $\pi \cap H$ and so all lines of S would be contained in the geodetically closed sub near $2(n-1)$-gon x^ϕ, a contradiction. So, $\pi \cap H$ is a nonsingular hermitian variety of type $H(2n-2, q^2)$ or $H(2n-3, q^2)$, but since S must have the right amount of lines, i.e. $\frac{|H^D|}{q+1}$, we know that $\pi \cap H$ is of type $H(2n-2, q^2)$ and that π is a nontangent hyperplane. $\square$

2.1.2 Near polygons of type $H^D(2n_1-1, q^2) \otimes H^D(2n_1-1, q^2)$

In this section, we will prove that there exist glued near polygons of type $H^D(2n_1-1, q^2) \otimes H^D(2n_2-1, q^2)$ for all $n_1, n_2 \geq 2$ and for every prime power q. We also show that there exists a unique glued near polygon of type $H^D(2n_1-1, 4) \otimes H^D(2n_2-1, 4)$ for all $n_1, n_2 \geq 2$.

Lemma 2.1.3
Let S be a spread of symmetry of $H^D(2n-1, 4)$, $n \geq 2$, and let K denote an arbitrary line of S. Then the group of automorphisms of $H^D(2n-1, 4)$ which fix K and S induces the full group of permutations of the line K.

Proof
We choose our reference system in $\mathsf{PG}(2n-1, 4)$ in such a way that

- $H(2n-1, 4)$ has equation $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$;
- S is the spread determined by the hyperplane $X_{2n-1} = 0$;
- the line K corresponds with the subspace on $H(2n-1, 4)$ generated by the $n-1$ points $\langle \bar{e}_{2i} + \bar{e}_{2i+1} \rangle$, $0 \leq i \leq n-2$.

The elements

$$(x_0, \ldots, x_{2n-2}, x_{2n-1}) \mapsto (x_0^\theta, \ldots, x_{2n-2}^\theta, \lambda x_{2n-1}^\theta),$$

of $\mathsf{P\Gamma L}(2n, 4)$ [$\lambda \in \mathsf{GF}(4)^*$ and $\theta \in \mathsf{Aut}(\mathsf{GF}(4))$], determine a group G of six automorphisms of $H^D(2n-1, 4)$. The group G fixes K and S and induces the full group of permutations of the line K. $\square$

Theorem 2.1.4
For every prime power q and all $n_1, n_2 \in \mathbb{N} \setminus \{0, 1\}$, there exists a glued near polygon of type $H^D(2n_1-1, q^2) \otimes H^D(2n_2-1, q^2)$.

Proof
We will use the notations of Section 1.7.2. Let S_i, $i \in \{1,2\}$ be a spread of symmetry of $H^D(2n_i-1,q^2)$, and let $L_1^{(i)}$ be an arbitrary base element in S_i. By the proof of Theorem 2.1.2, we know that there exists a group $G_i \cong C_{q+1}$ of automorphisms of $H^D(2n_i-1,q^2)$ fixing each element of S_i. In Theorem 10 of [18], the relationship between the groups G_i and Π_i is explained. It follows that $\Pi_1 \cong \Pi_2 \cong C_{q+1}$. So, there exists a bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$ such that $\Pi_1 = \theta^{-1}\Pi_2\theta$ (see Section 5 of [10]). With these choices of S_1, S_2, $L_1^{(1)}$, $L_1^{(2)}$ and θ, we obtain a glued near polygon of type $H^D(2n_1-1,q^2) \otimes H^D(2n_2-1,q^2)$ by Theorem 1.7.4. □

Theorem 2.1.5
For all $n_1, n_2 \in \mathbb{N} \setminus \{0,1\}$, there exists a unique glued near polygon of type $H^D(2n_1-1,4) \otimes H^D(2n_2-1,4)$. Hence, for every $n \in \mathbb{N} \setminus \{0,1\}$, there exists a unique glued near polygon of type $H^D(2n-1,4) \otimes Q(5,2)$.

Proof
By Theorem 2.1.2, all spreads of symmetry of $H^D(2n_i-1,4)$, $i \in \{1,2\}$, are isomorphic. We may therefore fix arbitrary spreads of symmetry S_1 and S_2 in $H^D(2n_1-1,4)$ and $H^D(2n_2-1,4)$, respectively. By Theorem 1.7.5, every near polygon which can be obtained for a certain choice of the base elements can always be obtained for any other choice of the base elements (by changing the map θ accordingly). Hence we may also fix arbitrary base elements $L_1^{(1)} \in S_1$ and $L_1^{(2)} \in S_2$. By Theorem 1.7.6, every bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$, gives rise to a glued near polygon of type $H^D(2n_1-1,4) \otimes_\theta H^D(2n_2-1,4)$. By reasons of symmetry, all these near polygons are isomorphic if the group of automorphisms of $H^D(2n_1-1,4)$ which fix S_1 and the base element $L_1^{(1)} \in S_1$ induces the full group of permutations on this base element. But this is precisely what we have shown in Lemma 2.1.3. □

The near polygon $H^D(2n-1,4) \otimes Q(5,2)$ has $H^D(2n-1,4)$ as a big geodetically closed sub near polygon. In Section 2.4, we will determine all dense near polygons which have $H^D(2n-1,4)$ as a big geodetically closed sub near polygon.

2.2 Characterization of product near polygons

Lemma 2.2.1 (Lemma 4.5 of [4]**)**
Let $\mathcal{F}$ be a big geodetically closed sub near $(2n-2)$-gon of a dense near $2n$-gon $\mathcal{S}$, $n \geq 2$. Then the following are equivalent:

- $\mathcal{S} \cong \mathcal{F} \times \mathbb{L}_i$ *for some line* $\mathbb{L}_i$ *of size* $i \geq 2$;
- $t_{\mathcal{S}} = t_{\mathcal{F}} + 1$;
- *every quad intersecting* $\mathcal{F}$ *in a line is a grid.*

Theorem 2.2.2
Let $\mathcal{S}$ *be a dense near polygon and let* T_1 *and* T_2 *be two sets of geodetically closed sub near polygons of* $\mathcal{S}$ *for which the following holds:*

- *every point* x *of* $\mathcal{S}$ *is contained in a unique element* $\mathcal{F}_1(x)$ *of* T_1 *and a unique element* $\mathcal{F}_2(x)$ *of* T_2;
- $\mathcal{F}_1(x) \cap \mathcal{F}_2(x) = \{x\}$ *for every point* x *of* $\mathcal{S}$;
- *every line* L *through a point* x *is contained in either* $\mathcal{F}_1(x)$ *or* $\mathcal{F}_2(x)$.

Then

- *all elements of* T_1 *are isomorphic;*
- *all elements of* T_2 *are isomorphic;*
- $\mathcal{S} \cong \mathcal{F}_1 \times \mathcal{F}_2$ *for any* $\mathcal{F}_1 \in T_1$ *and any* $\mathcal{F}_2 \in T_2$.

Proof
The proof consists of several steps.
(a) *If* x *and* y *are two points of* $\mathcal{S}$, *then* $|\mathcal{F}_1(x) \cap \mathcal{F}_2(y)| = 1$.
Let y' denote a point of $\mathcal{F}_1(x)$ at minimal distance from y. Since $y' \in \mathcal{F}_1(x)$, $\mathcal{F}_1(y') = \mathcal{F}_1(x)$. If L is a line of $\mathcal{F}_1(y') \cap \mathcal{C}(y, y')$ through y', then the unique point on L nearest to y belongs to $\mathcal{F}_1(x)$ and has distance $\mathrm{d}(y, y') - 1$ from y, a contradiction. Hence every line of $\mathcal{C}(y, y')$ through y' is contained in $\mathcal{F}_2(y')$. This implies that $\mathcal{C}(y, y') \subseteq \mathcal{F}_2(y')$. Hence $y \in \mathcal{F}_2(y')$, $\mathcal{F}_2(y) = \mathcal{F}_2(y')$ and $\mathcal{F}_1(x) \cap \mathcal{F}_2(y) = \mathcal{F}_1(y') \cap \mathcal{F}_2(y') = \{y'\}$.
(b) *If* x *and* y *are two collinear points such that* $xy \subseteq \mathcal{F}_2(x)$, *then* $\mathcal{F}_1(x) \cap \mathcal{F}_2(z)$ *and* $\mathcal{F}_1(y) \cap \mathcal{F}_2(z)$ *are collinear for every point* z *of* $\mathcal{S}$.
We will first prove (by induction on the distance $\mathrm{d}(x, z)$) that the result holds for every point z of $\mathcal{F}_1(x)$. Clearly the result holds if $z = x$. Suppose therefore that $\mathrm{d}(x, z) \geq 1$ and let z' denote a point collinear with z at distance $\mathrm{d}(x, z) - 1$ from x. If u' denotes the unique point in $\mathcal{F}_1(y) \cap \mathcal{F}_2(z')$, then z' and u' are collinear by the induction hypothesis. The quad $\mathcal{C}(z, z', u')$ is a grid. Hence z and u' have a unique common neighbour u different from z'. Since the line zu is not contained in $\mathcal{F}_1(x) = \mathcal{F}_1(z)$, it is contained in

$\mathcal{F}_2(z)$. Since the line $u'u$ is not contained in $\mathcal{F}_2(z') = \mathcal{F}_2(u')$, it is contained in $\mathcal{F}_1(u') = \mathcal{F}_1(y)$. Hence $\mathcal{F}_1(y) \cap \mathcal{F}_2(z) = \{u\}$. Since $\mathcal{F}_1(x) \cap \mathcal{F}_2(z) = \{z\}$, $\mathcal{F}_1(x) \cap \mathcal{F}_2(z)$ and $\mathcal{F}_1(y) \cap \mathcal{F}_2(z)$ are collinear. We will now prove that the result holds for every point z of $\mathcal{S}$. By (a), $\mathcal{F}_2(z)$ intersects $\mathcal{F}_1(x)$ in a point z'. Since $\mathcal{F}_2(z') = \mathcal{F}_2(z)$, $\mathcal{F}_1(x)\cap\mathcal{F}_2(z) = \mathcal{F}_1(x)\cap\mathcal{F}_2(z')$ and $\mathcal{F}_1(y)\cap\mathcal{F}_2(z) = \mathcal{F}_1(y) \cap \mathcal{F}_2(z')$ are collinear.

(c) *All elements of T_1 are isomorphic. All elements of T_2 are isomorphic.*
Take two elements $\mathcal{F}_1(x_1)$ and $\mathcal{F}_1(x_2)$ in T_1. For every point y of $\mathcal{F}_1(x_1)$, let $\theta(y)$ be the unique point in $\mathcal{F}_1(x_2) \cap \mathcal{F}_2(y)$. Clearly, θ is a bijection between the point sets of $\mathcal{F}_1(x_1)$ and $\mathcal{F}_1(x_2)$. By (b), the collinearity graphs of $\mathcal{F}_1(x_1)$ and $\mathcal{F}_1(x_2)$ are isomorphic. Hence also $\mathcal{F}_1(x_1)$ and $\mathcal{F}_1(x_2)$ are isomorphic. (Recall that there is a bijective correspondence between the lines of a near polygon and the maximal cliques in its collinearity graph.) In a similar way one proves that all elements of T_2 are isomorphic.

(d) *$\mathcal{S} \cong \mathcal{F}_1 \times \mathcal{F}_2$ for any $\mathcal{F}_1 \in T_1$ and any $\mathcal{F}_2 \in T_2$.*
For every point x in $\mathcal{S}$, put $\theta(x) := (\mathcal{F}_2(x) \cap \mathcal{F}_1, \mathcal{F}_1(x) \cap \mathcal{F}_2)$. Clearly, θ is a bijection between the point sets of $\mathcal{S}$ and $\mathcal{F}_1 \times \mathcal{F}_2$. If x and y are collinear points such that $xy \subseteq \mathcal{F}_2(x)$, then $\mathcal{F}_2(x) \cap \mathcal{F}_1 = \mathcal{F}_2(y) \cap \mathcal{F}_1$ and $\mathrm{d}(\mathcal{F}_1(x)\cap\mathcal{F}_2, \mathcal{F}_1(y)\cap\mathcal{F}_2) = 1$. If x and y are collinear points such that $xy \subseteq \mathcal{F}_1(x)$, then $\mathcal{F}_1(x)\cap\mathcal{F}_2 = \mathcal{F}_1(y)\cap\mathcal{F}_2$ and $\mathrm{d}(\mathcal{F}_2(x)\cap\mathcal{F}_1, \mathcal{F}_2(y)\cap\mathcal{F}_1) = 1$. One now easily sees that the collinearity graphs of $\mathcal{S}$ and $\mathcal{F}_1 \times \mathcal{F}_2$ are isomorphic. Hence also $\mathcal{S}$ and $\mathcal{F}_1 \times \mathcal{F}_2$ are isomorphic. □

Theorem 1.7.8 in Chapter 1 is actually a generalization of Theorem 2.2.2. In fact, the proof of Theorem 1.7.8 relies on Theorem 2.2.2.

Theorem 2.2.3
Let $\mathcal{S}$ be a dense near $2(n_1 + n_2)$-gon and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two geodetically closed sub near polygons for which the following holds:

- $\mathrm{diam}(\mathcal{F}_i) = n_i \geq 1$ $(i \in \{1, 2\})$;
- *$\mathcal{F}_1$ intersects $\mathcal{F}_2$ in a point x;*
- *every line through x is contained in either $\mathcal{F}_1$ or $\mathcal{F}_2$.*

Then $\mathcal{S} \cong \mathcal{F}_1 \times \mathcal{F}_2$.

Proof
The proof consists of several steps.

(a) *If L is a line of $\mathcal{F}_2$ through x, then $\mathcal{C}(\mathcal{F}_1, L) \cong \mathcal{F}_1 \times L$.*
Let $\mathcal{A}_i$, $i \in \{n_1, \ldots, n_1 + n_2\}$, be geodetically closed sub near polygons through x satisfying:

- $\mathrm{diam}(\mathcal{A}_i) = i$;
- $\mathcal{A}_{n_1} = \mathcal{F}_1$ and $\mathcal{A}_{n_1+1} = \mathcal{C}(\mathcal{F}_1, L)$;
- $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ for every $i \in \{n_1, \ldots, n_1 + n_2 - 1\}$.

Let $\mathcal{B}_i := \mathcal{A}_i \cap \mathcal{F}_2$, $i \in \{n_1, \ldots, n_1+n_2\}$. Since $\mathcal{A}_i \neq \mathcal{A}_{i+1}$, the set of lines of $\mathcal{A}_i$ through x is different from the set of lines of $\mathcal{A}_{i+1}$ through x. Since every line through x is contained in $\mathcal{F}_1 \cup \mathcal{F}_2$ and since $\mathcal{F}_1 \subseteq \mathcal{A}_i$ and $\mathcal{F}_1 \subseteq \mathcal{A}_{i+1}$, $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are different. It follows that $\mathcal{B}_i$, $i \in \{n_1, \ldots, n_1 + n_2\}$, are geodetically closed sub near polygons of $\mathcal{F}_2$ through x such that

- $\mathcal{B}_{n_1} = \{x\}$ and $\mathcal{B}_{n_1+n_2} = \mathcal{F}_2$;
- $\mathcal{B}_i \neq \mathcal{B}_{i+1}$ for every $i \in \{n_1, \ldots, n_1 + n_2 - 1\}$.

Hence $\mathrm{diam}(\mathcal{B}_i) = i - n_1$ for every $i \in \{n_1, \ldots, n_1 + n_2\}$. In particular $\mathcal{B}_{n_1+1} = \mathcal{C}(\mathcal{F}_1, L) \cap \mathcal{F}_2$ is a line. From Theorem 1.6.5 and Lemma 2.2.1, it then follows that $\mathcal{C}(\mathcal{F}_1, L) \cong \mathcal{F}_1 \times L$.

For every point y of $\mathcal{F}_i$, $i \in \{1, 2\}$, let $\mathcal{F}_{3-i}(y)$ denote the geodetically closed sub near polygon generated by all $t_{\mathcal{F}_{3-i}} + 1$ lines through y not contained in $\mathcal{F}_i$. Clearly $\mathcal{F}_1(x) = \mathcal{F}_1$ and $\mathcal{F}_2(x) = \mathcal{F}_2$.

(b) *For every point y of $\mathcal{S}$, there exists a point $y_1 \in \mathcal{F}_1$ and a point $y_2 \in \mathcal{F}_2$ such that $y \in \mathcal{F}_2(y_1) \cap \mathcal{F}_1(y_2)$.*
Let y_i, $i \in \{1, 2\}$, denote a point of $\mathcal{F}_i$ nearest to y. If $\mathcal{C}(y, y_1) \cap \mathcal{F}_1$ contains a line through y_1, then this line would contain a point at distance $\mathrm{d}(y, y_1) - 1$ from y, a contradiction. Hence $\mathcal{C}(y, y_1) \subseteq \mathcal{F}_2(y_1)$ and $y \in \mathcal{F}_2(y_1)$. Similarly, one proves that $y \in \mathcal{F}_1(y_2)$.

(c) *For every point y of $\mathcal{F}_i$, $i \in \{1, 2\}$, $\mathrm{diam}(\mathcal{F}_{3-i}(y)) = n_{3-i}$ and $\mathcal{F}_{3-i}(y) \cap \mathcal{F}_i = \{y\}$.*
Suppose that y_1 and y_2 are collinear points in $\mathcal{F}_1$ such that $\mathcal{F}_2(y_1) \cap \mathcal{F}_1 = \{y_1\}$ and $\mathrm{diam}(\mathcal{F}_2(y_1)) = n_2$. By (a) applied to y_1 instead of x, $\mathcal{C}(y_1y_2, \mathcal{F}_2(y_1)) \cong y_1y_2 \times \mathcal{F}_2(y_1)$ and hence $\mathrm{diam}(\mathcal{F}_2(y_2)) = n_2$ and $\mathcal{F}_2(y_2) \cap \mathcal{F}_1 = \{y_2\}$. Since $\mathrm{diam}(\mathcal{F}_2(x)) = n_2$ and $\mathcal{F}_2(x) \cap \mathcal{F}_1 = \{x\}$, the result follows by connectedness of $\mathcal{F}_1$. A similar reasoning can be used for $i = 2$.

(d) *For all points $y \in \mathcal{F}_1$ and $z \in \mathcal{F}_2$, $\mathcal{F}_2(y)$ intersects $\mathcal{F}_1(z)$ in a unique point.*
By (c), we may suppose that $y \neq x \neq z$. If y_1 and y_2 are collinear points of $\mathcal{F}_1$ such that $|\mathcal{F}_2(y_1) \cap \mathcal{F}_1(z)| = 1$, then by (a) applied to y_1 instead of x, $\mathcal{C}(y_1y_2, \mathcal{F}_2(y_1)) \cong y_1y_2 \times \mathcal{F}_2(y_1)$. As a consequence $|\mathcal{F}_2(y_2) \cap \mathcal{F}_1(z)| = 1$. Since $|\mathcal{F}_2(x) \cap \mathcal{F}_1(z)| = 1$, the result again follows by connectedness of $\mathcal{F}_1$.

(e) *If y_1 and y_2 are different elements of $\mathcal{F}_i$, $i \in \{1,2\}$, then $\mathcal{F}_{3-i}(y_1)$ is disjoint from $\mathcal{F}_{3-i}(y_2)$.*
Let y_1 and y_2 be two different elements of $\mathcal{F}_1$ and suppose that z is a common point of $\mathcal{F}_2(y_1)$ and $\mathcal{F}_2(y_2)$. Let z' be an element of $\mathcal{F}_2$ such that $z \in \mathcal{F}_1(z')$. Since $\mathcal{F}_1(z') \cap \mathcal{F}_2(y_1) = \{z\}$, the lines of $\mathcal{F}_2(y_1)$ through z are precisely those lines through z which are not contained in $\mathcal{F}_1(z')$. Since this property also holds for $\mathcal{F}_2(y_2)$, we necessarily have $\mathcal{F}_2(y_1) = \mathcal{F}_2(y_2)$. As a consequence $\{y_1\} = \mathcal{F}_1 \cap \mathcal{F}_2(y_1) = \mathcal{F}_1 \cap \mathcal{F}_2(y_2) = \{y_2\}$, contradicting $y_1 \neq y_2$. Hence $\mathcal{F}_2(y_1)$ and $\mathcal{F}_2(y_2)$ are disjoint. A similar reasoning can be used for $i = 2$.
(f) $\mathcal{S} \cong \mathcal{F}_1 \times \mathcal{F}_2$.
Since $\mathcal{S}$ is dense, every point of $\mathcal{S}$ is incident with $(t_{\mathcal{F}_1} + 1) + (t_{\mathcal{F}_2} + 1)$ lines. If u is a point of $\mathcal{S}$, then by (b) and (d), there exist points $y \in \mathcal{F}_1$ and $z \in \mathcal{F}_2$ such that $\mathcal{F}_2(y) \cap \mathcal{F}_1(z) = \{u\}$. Since $t_{\mathcal{F}_1} = t_{\mathcal{F}_1(z)}$ and $t_{\mathcal{F}_2} = t_{\mathcal{F}_2(y)}$, every line through u either is contained in $\mathcal{F}_1(z)$ or $\mathcal{F}_2(y)$. All conditions of Theorem 2.2.2 are now satisfied; hence $\mathcal{S} \cong \mathcal{F}_1 \times \mathcal{F}_2$. □

2.3 Substructures of dense glued near polygons

2.3.1 Geodetically closed sub near polygons and spreads of symmetry in glued near polygons

Let $\mathcal{S}$ denote a dense glued near polygon and let $\{T_1, T_2\} \in \Delta_1(\mathcal{S})$. For every point x of $\mathcal{S}$ and every $i \in \{1,2\}$, let $\mathcal{F}_i(x)$ denote the unique element of T_i through x. The relation between geodetically closed sub near polygons of $\mathcal{S}$ through x and the geodetically closed sub near polygons of $\mathcal{F}_i(x)$, $i \in \{1,2\}$, through x was studied in [23]. The following lemma is a corollary of Theorem 6 of [23].

Lemma 2.3.1
Let $\mathcal{A}_i$, $i \in \{1,2\}$, be a geodetically closed sub near polygon of $\mathcal{F}_i(x)$ through x, such that $\mathcal{A}_1 \cap \mathcal{F}_2(x) = \mathcal{A}_2 \cap \mathcal{F}_1(x)$. Then $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2) \cap \mathcal{F}_i(x) = \mathcal{A}_i$ for every $i \in \{1,2\}$. Moreover,

- *if $\mathcal{A}_1 \cap \mathcal{F}_2(x) = \mathcal{A}_2 \cap \mathcal{F}_1(x) = \{x\}$, then $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2) \cong \mathcal{A}_1 \times \mathcal{A}_2$;*
- *if $\mathcal{A}_1 \cap \mathcal{F}_2(x) = \mathcal{A}_2 \cap \mathcal{F}_1(x) = \mathcal{F}_1(x) \cap \mathcal{F}_2(x)$ (i.e. a line of $\mathcal{S}$) and if both $\mathcal{A}_1$ and $\mathcal{A}_2$ have diameter at least 2, then $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ is a glued near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$.*

Let S^* denote the spread of $\mathcal{S}$ obtained by considering all lines $\mathcal{F}_1 \cap \mathcal{F}_2$ where $\mathcal{F}_1 \in T_1$ and $\mathcal{F}_2 \in T_2$. For every spread S of $\mathcal{S}$ and every geodetically closed sub near polygon $\mathcal{F}$ of $\mathcal{S}$, let $S_{\mathcal{F}}$ denote the set of all lines of S contained in $\mathcal{F}$. For every element $\mathcal{F}$ of $T_1 \cup T_2$ and every point x of $\mathcal{S}$, the pair $(x, \mathcal{F})$ is classical in $\mathcal{S}$. If $\mathcal{F} \in T_i$, $i \in \{1, 2\}$, then $\pi_{\mathcal{F}}(x)$ is the unique point of the line $\mathcal{F}_{3-i}(x) \cap \mathcal{F}$ nearest to x.

Theorem 2.3.2
If S is a spread of symmetry of $\mathcal{S}$ different from S^*, then the following holds for exactly one $i \in \{1, 2\}$:

- (a) *for every $\mathcal{F} \in T_i$, $S_{\mathcal{F}}$ is a spread of symmetry of $\mathcal{F}$;*
- (b) *for all $\mathcal{F}, \mathcal{F}' \in T_i$, $S_{\mathcal{F}'} = \pi_{\mathcal{F}'}(S_{\mathcal{F}}) := \{\pi_{\mathcal{F}'}(L) | L \in S_{\mathcal{F}}\}$.*

Proof

- (a) Let K denote a line of S not contained in S^* and let x be an arbitrary point of K. Choose $i \in \{1, 2\}$ such that K is contained in $\mathcal{F}_i(x)$ but not in $\mathcal{F}_{3-i}(x)$. By Theorem 1.7.2, $S_{\mathcal{F}_{3-i}(x)}$ is empty. Now, let $\mathcal{F}$ denote an arbitrary element of T_i and let y denote an arbitrary point of $\mathcal{F} \cap \mathcal{F}_{3-i}(x)$. Since $S_{\mathcal{F}_{3-i}(x)}$ is empty, the unique line of S through y is contained in $\mathcal{F}$. By Theorem 1.7.2, it then follows that $S_{\mathcal{F}}$ is a spread of symmetry of $\mathcal{F}$.
- (b) Now, suppose that $\mathcal{F}$ and $\mathcal{F}'$ are two different elements of T_i and let L denote an arbitrary line of $S_{\mathcal{F}}$. Let u denote an arbitrary point of L. If $L \in S^*$, then the unique line of S through $\pi_{\mathcal{F}'}(u)$ is contained in $\mathcal{F}_{3-i}(u)$ and $\mathcal{F}'$ and hence coincides with $\pi_{\mathcal{F}'}(L) = \mathcal{F}_{3-i}(u) \cap \mathcal{F}'$. If $L \notin S^*$, then by Lemma 2.3.1, $\mathcal{C}(u, \pi_{\mathcal{F}'}(u), L)$ is isomorphic to $L \times \mathcal{C}(u, \pi_{\mathcal{F}'}(u))$ and intersects $\mathcal{F}'$ in the line $\pi_{\mathcal{F}'}(L)$. By Theorem 1.7.2, the unique line of S through $\pi_{\mathcal{F}'}(u)$ is contained in $\mathcal{C}(u, \pi_{\mathcal{F}'}(u), L)$ and $\mathcal{F}'$ and hence coincides with $\pi_{\mathcal{F}'}(L)$. As a consequence $S_{\mathcal{F}'} = \pi_{\mathcal{F}'}(S_{\mathcal{F}})$. □

2.3.2 Near polygons of type $(\mathcal{F}_1 * \mathcal{F}_2) \circ \mathcal{F}_3$

Theorem 2.3.3
Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be dense near polygons of diameter at least 2. Then every near polygon $\mathcal{S}$ of type $(\mathcal{F}_1 \times \mathcal{F}_2) \otimes \mathcal{F}_3$ is also of type $\mathcal{F}_i \times (\mathcal{F}_{3-i} \otimes \mathcal{F}_3)$ for a certain $i \in \{1, 2\}$.

Proof
Let x denote an arbitrary point of $\mathcal{S}$. Let $\{T, T_3\} \in \Delta_1(\mathcal{S})$ such that $\mathcal{F} \cong \mathcal{F}_1 \times \mathcal{F}_2$ for every $\mathcal{F} \in T$ and $\mathcal{F} \cong \mathcal{F}_3$ for every $\mathcal{F} \in T_3$. Let $\mathcal{F}(x)$ denote

the unique element of T through x and let $\mathcal{F}_3(x)$ denote the unique element of T_3 through x. Let $\{T_1, T_2\} \in \Delta_0(\mathcal{F}(x))$ such that $\mathcal{F} \cong \mathcal{F}_1$ for every $\mathcal{F} \in T_1$ and $\mathcal{F} \cong \mathcal{F}_2$ for every $\mathcal{F} \in T_2$, and let $\mathcal{F}_i(x)$, $i \in \{1,2\}$, denote the unique element of T_i through x. Without loss of generality we may assume that the line $\mathcal{F}(x) \cap \mathcal{F}_3(x)$ is contained in $\mathcal{F}_2(x)$. By Lemma 2.3.1, $\mathcal{F}_4(x) := \mathcal{C}(\mathcal{F}_2(x), \mathcal{F}_3(x))$ is a glued near polygon of type $\mathcal{F}_2 \otimes \mathcal{F}_3$. Now, $\mathrm{diam}(\mathcal{S}) = (\mathrm{diam}(\mathcal{F}_1) + \mathrm{diam}(\mathcal{F}_2)) + \mathrm{diam}(\mathcal{F}_3) - 1 = \mathrm{diam}(\mathcal{F}_1) + (\mathrm{diam}(\mathcal{F}_2) + \mathrm{diam}(\mathcal{F}_3) - 1) = \mathrm{diam}(\mathcal{F}_1) + \mathrm{diam}(\mathcal{F}_4)$ and every line through x is contained in precisely one of the sub near polygons $\mathcal{F}_1(x)$ and $\mathcal{F}_4(x)$. So, $\mathcal{S}$ satisfies the conditions of Theorem 2.2.3. Hence, $\mathcal{S} \cong \mathcal{F}_1(x) \times \mathcal{F}_4(x)$ and $\mathcal{S}$ is of type $\mathcal{F}_1 \times (\mathcal{F}_2 \otimes \mathcal{F}_3)$. This proves the theorem. □

Theorem 2.3.4
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote two dense near polygons of diameter at least 2, and let $\mathcal{S}$ be a near polygon of type $\mathcal{F}_1 \otimes \mathcal{F}_2$. If $\mathcal{S}$ is a product near polygon, then at least one of the near polygons $\mathcal{F}_1$ and $\mathcal{F}_2$ is also a product near polygon.

Proof
Let $\{T_1, T_2\} \in \Delta_1(\mathcal{S})$ such that every element of T_i, $i \in \{1,2\}$, is isomorphic to $\mathcal{F}_i$, and let $\{T_3, T_4\}$ be an arbitrary element of $\Delta_0(\mathcal{S})$. Let x denote an arbitrary point of $\mathcal{S}$ and let $\mathcal{G}_i$, $1 \le i \le 4$, be the unique element of T_i through x. For every $i, j \in \{1,2,3,4\}$ we define $\mathcal{G}_{i,j} = \mathcal{G}_i \cap \mathcal{G}_j$, $d_{i,j} := \mathrm{diam}(\mathcal{G}_{i,j})$ and $d_i := d_{i,i}$. We then have $d := \mathrm{diam}(\mathcal{S}) = d_1 + d_2 - 1 = d_3 + d_4$, $d_{1,2} = 1$ and $d_{3,4} = 0$. Without loss of generality, we may suppose that the line $\mathcal{G}_{1,2}$ is contained in $\mathcal{G}_3$. By Lemma 2.3.1, it follows that $d_3 = d_{1,3} + d_{2,3} - 1$ and $d_4 = d_{1,4} + d_{2,4}$. Hence, $d = (d_{1,3} + d_{1,4}) + (d_{2,3} + d_{2,4}) - 1$. Now, $\mathcal{C}(\mathcal{G}_{1,3}, \mathcal{G}_{1,4}) = \mathcal{G}_1$ and $\mathcal{C}(\mathcal{G}_{2,3}, \mathcal{G}_{2,4}) = \mathcal{G}_2$; so $d_1 \le d_{1,3} + d_{1,4}$ and $d_2 \le d_{2,3} + d_{2,4}$. From $d_1 + d_2 - 1 = d = (d_{1,3} + d_{1,4}) + (d_{2,3} + d_{2,4}) - 1 \ge d_1 + d_2 - 1$, it then follows that $d_1 = d_{1,3} + d_{1,4}$ and $d_2 = d_{2,3} + d_{2,4}$. Since $d_4 = d_{1,4} + d_{2,4} \ge 1$, we have $d_{j,4} \ge 1$ for a $j \in \{1,2\}$. Now, since $\mathcal{G}_3$ contains the line $\mathcal{G}_{1,2}$, we also have $d_{j,3} \ge 1$. Now, the pair $\{\mathcal{G}_{j,3}, \mathcal{G}_{j,4}\}$ satisfies all conditions of Theorem 2.2.3 and hence $\mathcal{F}_j \cong \mathcal{G}_j \cong \mathcal{G}_{j,3} \times \mathcal{G}_{j,4}$. This proves the theorem. □

Theorem 2.3.5
Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ denote three dense near polygons with diameter at least 2 and suppose that none of these near polygons is a product near polygon. If a near polygon $\mathcal{S}$ is of type $(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3$ and if $\{T, T_3\} \in \Delta_1(\mathcal{S})$ such that every element of T is of type $\mathcal{F}_1 \otimes \mathcal{F}_2$ and every element of T_3 is isomorphic to $\mathcal{F}_3$, then there exists an element $\{\tilde{T}_1, \tilde{T}_2\} \in \Delta_1(\mathcal{S})$ and an $i \in \{1,2\}$ such that

- *every element of $\tilde{T}_1$ is isomorphic to $\mathcal{F}_i$;*

- *every element of $\tilde{T}_2$ is of type $\mathcal{F}_{3-i} \otimes \mathcal{F}_3$;*
- *the partition $\tilde{T}_1$ is a refinement of the partition T (i.e. each element of T is the union of some members of $\tilde{T}_1$);*
- *the partition T_3 is a refinement of the partition $\tilde{T}_2$.*

Hence every dense near polygon $\mathcal{S}$ of type $(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3$ is also of type $\mathcal{F}_i \otimes (\mathcal{F}_{3-i} \otimes \mathcal{F}_3)$ for a certain $i \in \{1, 2\}$.

Proof
By Lemma 2.3.4, $\mathcal{S}$ is not a product near polygon. Let $\mathcal{F}^*$ denote an arbitrary element of T, and let $\{T_1', T_2'\} \in \Delta_1(\mathcal{F}^*)$ such that every element of T_i', $i \in \{1, 2\}$, is isomorphic to $\mathcal{F}_i$. Each element $\mathcal{F}$ of T is classical in $\mathcal{S}$ and hence projections can be defined on $\mathcal{F}$. The projection $\pi_{\mathcal{F}}$ from $\mathcal{F}^*$ on $\mathcal{F}$ is an isomorphism between $\mathcal{F}^*$ and $\mathcal{F}$. Put $T_i := \{\pi_{\mathcal{F}}(\mathcal{G}) | \mathcal{F} \in T \text{ and } \mathcal{G} \in T_i'\}$, $i \in \{1, 2\}$. Then T_i is a partition of $\mathcal{S}$ in geodetically closed sub near polygons isomorphic to $\mathcal{F}_i$. For every point x of $\mathcal{S}$ and every $i \in \{1, 2, 3\}$, let $\mathcal{F}_i(x)$ denote the unique element of T_i through x, put $K(x) := \mathcal{F}_1(x) \cap \mathcal{F}_2(x)$ and put $L(x) := \mathcal{C}(\mathcal{F}_1(x), \mathcal{F}_2(x)) \cap \mathcal{F}_3(x)$. Since $\mathcal{F}_1(x)$ and $\mathcal{F}_2(x)$ are projections of elements of $T_1' \cup T_2'$ on the unique element $\mathcal{F}(x)$ of T through x, we find that $K(x)$ is a line, that $\mathcal{C}(\mathcal{F}_1(x), \mathcal{F}_2(x)) = \mathcal{F}(x)$ and that also $L(x)$ is a line. By Theorem 1.7.6 and the fact that $\mathcal{S}$ is not a product near polygon, it follows that $\{L(x) | x \in \mathcal{F}^*\}$ is a spread of symmetry of $\mathcal{F}^*$ and so Theorem 2.3.2 applies. Without loss of generality, we may suppose that every line $L(x)$, $x \in \mathcal{F}^*$, is contained in $\mathcal{F}_2(x)$. Now, choose a point x outside $\mathcal{F}^*$ and let x' denote its projection on $\mathcal{F}^*$. Since $L(x')$ is contained in $\mathcal{F}_2(x')$, the projection $L(x)$ of $L(x')$ on $\mathcal{F}(x)$ is contained in the projection $\mathcal{F}_2(x)$ of $\mathcal{F}_2(x')$ on $\mathcal{F}(x)$. Hence, for every point x of $\mathcal{S}$, $\mathcal{F}_3(x)$ intersects $\mathcal{F}_2(x)$ in a line. By Lemma 2.3.1, it then follows that $\tilde{\mathcal{F}}_2(x) := \mathcal{C}(\mathcal{F}_3(x), \mathcal{F}_2(x))$ is a glued near polygon of type $\mathcal{F}_2 \otimes \mathcal{F}_3$. The set $\tilde{T}_2 := \{\tilde{\mathcal{F}}_2(x) | x \in \mathcal{S}\}$ clearly determines a partition of $\mathcal{S}$ in geodetically closed sub near polygons. Define $\tilde{T}_1 := T_1$. For every point $x \in \mathcal{S}$, $\mathcal{F}_1(x) \cap \tilde{\mathcal{F}}_2(x) = (\mathcal{F}_1(x) \cap \mathcal{F}(x)) \cap \tilde{\mathcal{F}}_2(x) = \mathcal{F}_1(x) \cap (\mathcal{F}(x) \cap \tilde{\mathcal{F}}_2(x)) = \mathcal{F}_1(x) \cap \mathcal{F}_2(x) = K(x)$. By Theorem 1.7.8, it now follows that $\{\tilde{T}_1, \tilde{T}_2\} \in \Delta_1(\mathcal{S})$. So, $\mathcal{S}$ is of type $\mathcal{F}_1 \otimes (\mathcal{F}_2 \otimes \mathcal{F}_3)$. This proves the theorem. $\square$

2.4 Slim dense near polygons with a given big sub near polygon $\mathcal{F}$

In [19] and [22], B. De Bruyn proved the following theorems.

Theorem 2.4.1 (Main result of [19]**)**
Let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \geq 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to $\mathbb{H}_{n-1}$. Then $\mathcal{S}$ is isomorphic to either $\mathbb{H}_n$ or $\mathbb{H}_{n-1} \times \mathbb{L}_3$.

Theorem 2.4.2 (Main theorem of [22]**)**
Let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \geq 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to $\mathbb{G}_{n-1}$. Then $\mathcal{S}$ is isomorphic to either $\mathbb{G}_n$, $\mathbb{G}_{n-1} \times \mathbb{L}_3$, or $\mathbb{G}_{n-1} \otimes Q(5,2)$. Here, $\mathbb{G}_{n-1} \otimes Q(5,2)$ denotes the unique near polygon which can be obtained by glueing $\mathbb{G}_{n-1}$ and $Q(5,2)$.

In this section, we will prove similar results for the other infinite classes of slim dense near polygons.

2.4.1 The case $\mathcal{F} \cong H^D(2n-1,4), n \geq 3$

Since $H^D(3,4)$ is isomorphic to $Q(5,2)$, all slim dense near hexagons containing a big quad isomorphic to $H^D(3,4)$ are already known (see Table 1.3).

Theorem 2.4.3
Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to $H^D(2n-3,4)$. Then $\mathcal{S}$ is isomorphic to one of the following near polygons:

- *$H^D(2n-1,4)$;*
- *$H^D(2n-3,4) \otimes Q(5,2)$;*
- *$H^D(2n-3,4) \times \mathbb{L}_i$ for some line $\mathbb{L}_i$ of size $i \geq 3$.*

We will prove this theorem in several steps.

Lemma 2.4.4
If not all lines are incident with 3 points, then $\mathcal{S} \cong H^D(2n-3,4) \times \mathbb{L}_i$ for some line $\mathbb{L}_i$ of size at least 4.

Proof
Let K be a line with more than three points. Since every line disjoint with $\mathcal{F}$ contains as many points as its projection on $\mathcal{F}$, the line K intersects $\mathcal{F}$ in a point x. Suppose that $t_{\mathcal{S}} > t_{\mathcal{F}} + 1$, then there exists a line $L \neq K$ through x which is not contained in $\mathcal{F}$. By Theorem 1.6.5, the quad $\mathcal{Q} := \mathcal{C}(K, L)$ intersects $\mathcal{F}$ in a line M, so $t_{\mathcal{Q}} \geq 2$. Since $t_{\mathcal{Q}} \geq 2$, every line of $\mathcal{Q}$ is incident with the same number of points, a contradiction, since K has more points

than M. Hence, $t_\mathcal{S} = t_\mathcal{F} + 1$. The lemma now follows from Lemma 2.2.1. $\square$

From now on we suppose that $\mathcal{S}$ is a slim near polygon.

Lemma 2.4.5
The slim near polygon $\mathcal{S}$ does not contain quads isomorphic to $W(2)$.

Proof
Suppose the contrary, then by Proposition 1.10.7, there exists a $W(2)$-quad $\mathcal{Q}$ through a point x of $\mathcal{F}$. By Theorem 1.6.5, this quad $\mathcal{Q}$ intersects $\mathcal{F}$ in a line K. Let L be a line of $\mathcal{Q}$ through x different from K and let $\mathcal{H} \cong H^D(5,4)$ be a hex of $\mathcal{F}$ through K. Let X denote the set of points of $\mathcal{L}(x,\mathcal{H})$ (i.e. the set of lines of $\mathcal{H}$ through x) which are contained in a $W(2)$-quad together with L. We will now show that $|U \cap X| \in \{0,3\}$ for every line U of $\mathcal{L}(x,\mathcal{H})$. Suppose that $|U \cap X| \geq 1$ and let $\mathcal{Q}_U$ denote the $Q(5,2)$-quad corresponding with U. Since $|U \cap X| \geq 1$, there exists a $W(2)$-quad $\mathcal{R}$ through L which intersects $\mathcal{Q}_U$ in a line. By Table 1.3, $\mathcal{C}(\mathcal{Q}_U,\mathcal{R})$ either is isomorphic to $\mathbb{G}_3$ or $\mathbb{E}_3$. In any case, exactly three lines of $\mathcal{Q}_U$ through x are contained in a $W(2)$-quad together with L, or equivalently, $|U \cap X| = 3$. Since $\mathcal{H}$ contains exactly five $Q(5,2)$-quads through K, we have $|X| = 1 + 5 \cdot 2 = 11$. Hence X is a set of 11 points in $\mathsf{PG}(2,4)$ such that every line meets the set in either 0 or 3 points. Such a set does not exist, otherwise, every point of $\mathsf{PG}(2,4)$ outside X would be contained in $\frac{11}{3}$ lines meeting X, which is clearly impossible. $\square$

For every line L intersecting $\mathcal{F}$ in a point x, let A_L be the set of lines of $\mathcal{F}$ through x such that $\mathcal{C}(L,M) \cong Q(5,2)$. By Theorem 1.6.5 and Lemma 2.4.5, $\mathcal{C}(L,L') \cong Q(5,2)$ for every line $L' \neq L$ intersecting $\mathcal{F}$ in x. Hence $t_\mathcal{S} - t_\mathcal{F} = 3|A_L| + 1$. As a consequence, $|A_L|$ is independent from the choice of L and equal to $\alpha := \frac{t_\mathcal{S} - t_\mathcal{F} - 1}{3}$.

Lemma 2.4.6
A_L is a subspace of $\mathcal{L}(x,\mathcal{F})$.

Proof
Let K_1 and K_2 be two different lines of A_L and let K_3 be an arbitrary line through x contained in $\mathcal{C}(K_1,K_2)$. Since the hex $\mathcal{C}(L,K_1,K_2)$ has three $Q(5,2)$-quads through the same point x and no $W(2)$-quads, it must be isomorphic to $H^D(5,4)$. Hence, $\mathcal{C}(L,K_3) \cong Q(5,2)$ and $K_3 \in A_L$. This proves the lemma. $\square$

Lemma 2.4.7
$\alpha \in \{0, 1, t_{\mathcal{F}} + 1\}$.

Proof
We suppose that $2 \leq \alpha \leq t_{\mathcal{F}}$ and derive a contradiction.

(a) Suppose first that $n = 4$, so $\mathcal{F} \cong H^D(5,4)$. Let x be an arbitrary point of $\mathcal{F}$ and let L be an arbitrary line through x not contained in $\mathcal{F}$. Since A_L is a subspace and $2 \leq |A_L| \leq t_{\mathcal{F}} = 20$, there exists a $Q(5,2)$-quad $\mathcal{Q}_x$ through x such that A_L is the set of five lines of $\mathcal{Q}_x$ through x. Also, $\alpha = 5$, $t_{\mathcal{S}} = t_{\mathcal{F}} + 3\alpha + 1 = 36$ and the hex $\mathcal{H}_x := \mathcal{C}(L, \mathcal{Q}_x)$ is isomorphic to $H^D(5,4)$. If L' is a line through x different from L and not contained in $\mathcal{F}$, then $\mathcal{C}(L, L')$ is isomorphic to $Q(5,2)$ and hence intersects $\mathcal{F}$ in a line of $\mathcal{Q}_x$. It follows that every line through x not contained in $\mathcal{F}$ is contained in $\mathcal{H}_x$. Since $\mathcal{H}_x \cong H^D(5,4)$, we then have that $\mathcal{H}_x = \mathcal{C}(\Gamma_1(x) \setminus \mathcal{F})$. So, $\mathcal{H}_x$ and $\mathcal{Q}_x = \mathcal{H}_x \cap \mathcal{F}$ only depend on x and not on the line L. If $y \in \mathcal{Q}_x$, then $\mathcal{H}_y = \mathcal{H}_x$ and $\mathcal{Q}_y = \mathcal{Q}_x$. Hence, the quads $\mathcal{Q}_x$, $x \in \mathcal{F}$, partition the point set of $\mathcal{F}$, and the hexes $\mathcal{H}_x$, $x \in \mathcal{F}$, partition the point set of $\mathcal{S}$. Since $\mathcal{F} \cong H^D(5,4)$ is big in $\mathcal{S}$, every hex isomorphic to $H^D(5,4)$ is big. [The number of points at distance 0 or 1 from $\mathcal{F}$ equals the number of points at distance 0 or 1 from any hex isomorphic to $\mathcal{F}$.] In particular, each of the hexes $\mathcal{H}_x$ is big. The total number of quads $\mathcal{Q}_x$, $x \in \mathcal{F}$, equals $\frac{|H^D(5,4)|}{|Q(5,2)|} = 33$. Let X denote the set of 33 points of $H(5,4)$ corresponding to the quads $\mathcal{Q}_x$, $x \in \mathcal{S}$. If u is a point of X, then we denote the $Q(5,2)$-quad of $\mathcal{F}$ corresponding with it as u^ϕ and the unique hex of $\mathcal{S}$ intersecting $\mathcal{F}$ in u^ϕ by $\mathcal{H}(u)$. We will now show that X is a subspace of the linear space $\mathcal{L}$ defined in Lemma 2.1.1. Consider two different points u_1 and u_2 in X. Since u_1^ϕ and u_2^ϕ are disjoint, $u_1u_2 \cap H(5,4)$ is a Baer subline $\{u_1, u_2, u_3\}$. Since every quad intersecting u_1^ϕ and u_2^ϕ also intersects u_3^ϕ, u_3^ϕ is the reflection of u_1^ϕ about u_2^ϕ (in $\mathcal{F}$). The reflection of $\mathcal{H}(u_1)$ about $\mathcal{H}(u_2)$ (in $\mathcal{S}$) is a hex which meets $\mathcal{F}$ in the quad u_3^ϕ and hence coincides with $\mathcal{H}(u_3)$. As a consequence $u_3 \in X$. This proves that X is a subspace of the linear space $\mathcal{L}$. Hence, there exists a subspace π such that $X = \pi \cap H(5,4)$. Since no two points of X are collinear on $H(5,4)$, we have $|X| \leq |H(2,4)| = 9$, contradicting $|X| = 33$.

(b) Suppose now that $n \geq 5$. Let x denote an arbitrary point of $\mathcal{F}$ and let L denote an arbitrary line through x not contained in $\mathcal{F}$. Since $2 \leq \alpha \leq t_{\mathcal{F}}$, there exist lines K_1, K_2 and K_3 through x such that $K_1, K_2 \in A_L$, $K_1 \neq K_2$ and $K_3 \notin A_L$. Now, the near octagon $\mathcal{C}(L, K_1, K_2, K_3)$ contradicts (a). □

Lemma 2.4.8

- *If $\alpha = 0$, then $\mathcal{S} \cong H^D(2n-3, 4) \times \mathbb{L}_3$;*

- *if* $\alpha = 1$, *then* $\mathcal{S} \cong H^D(2n-3,4) \otimes Q(5,2)$*;*
- *if* $\alpha = t_{\mathcal{F}} + 1$, *then* $\mathcal{S} \cong H^D(2n-1,4)$.

Proof
If $\alpha = 0$, then $t_{\mathcal{S}} - t_{\mathcal{F}} = 1$ and hence $\mathcal{S} \cong H^D(2n-3,4) \times \mathbb{L}_3$ by Lemma 2.2.1. If $\alpha = t_{\mathcal{F}} + 1$, then no grid-quad intersects $\mathcal{F}$ and hence all quads are isomorphic to $Q(5,2)$ by Proposition 1.10.7. Since the generalized quadrangle $Q(5,2)$ has no ovoids (Theorem 1.5.7), all point-quad relations must be classical and $\mathcal{S}$ is a classical near polygon. Now, $H^D(2n-1,4)$ is the only classical near $2n$-gon in which all quads are isomorphic to $Q(5,2)$.

Suppose now that $\alpha = 1$, then $t_{\mathcal{S}} = t_{\mathcal{F}} + 4$. If x is an arbitrary point of $\mathcal{F}$ and if L and M denote two lines through x not contained in $\mathcal{F}$, then $\mathcal{C}(K, L)$ is a $Q(5,2)$-quad. Hence, every point $x \in \mathcal{F}$ is contained in a unique $Q(5,2)$-quad $\mathcal{Q}_x$ which intersects $\mathcal{F}$ in a line L_x. The lines L_x, $x \in \mathcal{F}$, determine a spread of $\mathcal{F}$ and the set $T_1 := \{\mathcal{Q}_x | x \in \mathcal{F}\}$ determines a partition of $\mathcal{S}$ in quads.

Now, consider a point x of $\mathcal{S}$ not contained in $\mathcal{F}$, let y be the unique point of $\mathcal{F}$ collinear with x, and let $A := xy$, B, C, D and E be the lines of $\mathcal{Q} := \mathcal{Q}_y$ through x. Let L be a line intersecting $\mathcal{Q}$ in x and consider the hex $\mathcal{H} := \mathcal{C}(L, \mathcal{Q})$. The hex $\mathcal{H}$ contains a grid-quad $\mathcal{C}(L, A)$, no $W(2)$-quads, and at least two $Q(5,2)$-quads through the line $L_y = \mathcal{Q} \cap \mathcal{F}$, namely $\mathcal{Q}$ and $\mathcal{H} \cap \mathcal{F}$. It follows from Table 1.3 that $\mathcal{H} \cong Q(5,2) \otimes Q(5,2)$. Hence, exactly one line of $\mathcal{Q}$ through x, say B, is contained in a $Q(5,2)$-quad with L.

Now, consider a geodetically closed sub near $2(n-1)$-gon $\mathcal{F}'$ of $\mathcal{S}$ containing $\mathcal{Q}' := \mathcal{C}(B, L)$ and intersecting $\mathcal{Q}$ in B. We will show that every quad of $\mathcal{F}'$ is isomorphic to $Q(5,2)$. Let L_1 and L_2 be two lines of $\mathcal{F}'$ through x. If $L_1 \neq B \neq L_2$, then the hex $\mathcal{H}' := \mathcal{C}(A, L_1, L_2)$ contains grid-quads $\mathcal{C}(A, L_1)$ and $\mathcal{C}(A, L_2)$, a $Q(5,2)$-quad $\mathcal{C}(A, L_1, L_2) \cap \mathcal{F}$, and no $W(2)$-quads. Hence $\mathcal{H}'$ must be isomorphic to either $Q(5,2) \times \mathbb{L}_3$ or $Q(5,2) \otimes Q(5,2)$, see Table 1.3. In any case $\mathcal{C}(L_1, L_2)$ is isomorphic to $Q(5,2)$. If $L_1 = B$ and if L_2 is not contained in $\mathcal{Q}'$, then $\mathcal{C}(L_2, M) \cong Q(5,2)$ for every line $M \neq B$ through x contained in $\mathcal{Q}'$. This is only possible if $\mathcal{C}(L_2, Q')$ is isomorphic to $H^D(5,4)$, see Table 1.3. Hence, also $\mathcal{C}(L_1, L_2) = \mathcal{C}(B, L_2)$ is isomorphic to $Q(5,2)$. It now follows that every quad of $\mathcal{F}'$ through x is isomorphic to $Q(5,2)$. By Proposition 1.10.7 it then follows that every quad of $\mathcal{F}'$ is isomorphic to $Q(5,2)$. As before, we then know that $\mathcal{F}'$ is classical and isomorphic to $H^D(2n-3,4)$. Obviously, $\mathcal{F}'$ is the only geodetically closed sub near polygon through x isomorphic to $H^D(2n-3,4)$.

Repeating the above construction for every point x outside $\mathcal{F}$, we obtain a partition T_2 of $\mathcal{S}$ in geodetically closed sub near polygons isomorphic to

$H^D(2n-3,4)$. We can now apply Theorem 1.7.8 and conclude that $\mathcal{S}$ is a glued near polygon of type $H^D(2n-3,4) \otimes Q(5,2)$. In Section 2.1 we have shown that there exists a unique glued near polygon of such type. $\square$

2.4.2 The case $\mathcal{F} \cong Q^D(2n,2), n \geq 3$

Theorem 2.4.9

Let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \geq 3$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to $Q^D(2n-2,2)$. Then $\mathcal{S}$ is isomorphic to one of the following near polygons:

- $Q^D(2n,2)$;
- $Q^D(2n-2,2) \times \mathbb{L}_3$;
- $\mathbb{I}_n$.

Proof

We will use induction on n. By Table 1.3 the theorem holds if n is equal to 3. Suppose therefore that $n \geq 4$ and that the theorem holds for any near $2n'$-gon with $n' \in \{3,\ldots,n-1\}$. Every geodetically closed sub near 2δ-gon, $\delta \in \{3,\ldots,n-1\}$, intersecting $\mathcal{F}$, intersects $\mathcal{F}$ in a $Q^D(2\delta-2,2)$ and hence is isomorphic to either $Q^D(2\delta,2)$, $Q^D(2\delta-2,2)\times\mathbb{L}_3$ or $\mathbb{I}_\delta$. As a consequence no $Q(5,2)$-quad meets $\mathcal{F}$. If a line of $\mathcal{F}$ is contained in two grid-quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$, then the hex $\mathcal{C}(\mathcal{Q}_1,\mathcal{Q}_2)$ intersects $\mathcal{F}$ in a $W(2)$-quad and hence has a line which is incident with at least two grid-quads and at least one big $W(2)$-quad, contradicting Table 1.3. As a consequence there are at most $t_{\mathcal{F}}+1$ grid-quads through every point of $\mathcal{F}$. For every line K intersecting $\mathcal{F}$ in a point x, let α_K denote the number of grid-quads through K. Since every quad through K meets $\mathcal{F}$ in a line, the number of $W(2)$-quads through K equals $t_{\mathcal{F}}+1-\alpha_K$. Counting the number of lines through x, we have $t_{\mathcal{S}} = \alpha_K + 2(t_{\mathcal{F}}+1-\alpha_K)$. Hence α_K is independent from the line K and equal to $\alpha := 2t_{\mathcal{F}}+2-t_{\mathcal{S}}$. Since there are $t_{\mathcal{S}}-t_{\mathcal{F}} = t_{\mathcal{F}}+2-\alpha$ lines through x not contained in $\mathcal{F}$, there are precisely $\alpha(t_{\mathcal{F}}+2-\alpha)$ grid-quads through x. Since this number is at most $t_{\mathcal{F}}+1$, we necessarily have $\alpha \in \{0,1,t_{\mathcal{F}}+1\}$.

If $\alpha = t_{\mathcal{F}}+1$, then every quad meeting $\mathcal{F}$ but not contained in $\mathcal{F}$ is a grid and hence $\mathcal{S} \cong Q^D(2n-2,2)\times\mathbb{L}_3$ by Lemma 2.2.1.

If $\alpha = 0$, then every quad meeting $\mathcal{F}$ is isomorphic to $W(2)$. Hence, every geodetically closed sub near 2δ-gon, $\delta \in \{2,\ldots,n-1\}$, intersecting $\mathcal{F}$ is isomorphic to $Q^D(2\delta,2)$. Now, consider an arbitrary point-quad pair $(y,\mathcal{Q})$ and let $\mathcal{F}'$ denote an arbitrary geodetically closed sub near $2(n-1)$-gon through $\mathcal{Q}$ intersecting $\mathcal{F}$. Since $\mathcal{F}' \cong Q^D(2n-2,2)$, $\mathcal{F}'$ is big in $\mathcal{S}$ and hence $\mathrm{d}(y,z) = \mathrm{d}(y,\pi_{\mathcal{F}'}(y)) + \mathrm{d}(\pi_{\mathcal{F}'}(y),z)$ for every point $z \in \mathcal{Q}$. Since $(\pi_{\mathcal{F}'}(y),\mathcal{Q})$ is classical, also $(y,\mathcal{Q})$ is classical. Since every point-quad relation is classical, $\mathcal{S}$ itself is classical. Since $\mathcal{S}$ has only $W(2)$-quads, it is necessarily isomorphic to $Q^D(2n,2)$.

We still need to consider the case $\alpha = 1$. Let $\Omega(\mathcal{S},\mathcal{F})$ be the incidence structure whose points are the lines of $\mathcal{S}$ intersecting $\mathcal{F}$ in a point, whose lines are the (not necessarily geodetically closed) subgrids of $\mathcal{S}$ intersecting $\mathcal{F}$ in a line and whose incidence relation is the natural one. For every line L of $\mathcal{S}$ intersecting $\mathcal{F}$ in a point x, let L' denote the unique line of $\mathcal{F}$ through x for which $\mathcal{C}(L,L')$ is a grid. Since $t_{\mathcal{F}} + 1 = \alpha(t_{\mathcal{F}} + 2 - \alpha)$, every line of $\mathcal{F}$ is also contained in a unique grid-quad and hence there is a bijective correspondence between the points L of $\Omega(\mathcal{S},\mathcal{F})$ and the flags (x,L') of $\mathcal{F}$. The lines of $\Omega(\mathcal{S},\mathcal{F})$ then correspond with certain triples of flags. We will now prove that such a triple is either of the form $\{(x,K),(y,K),(z,K)\}$ with $K = \{x,y,z\}$ a line of $\mathcal{F}$, or of the form $\{(L\cap K,L),(M\cap K,M),(N\cap K,N)\}$ with K, L, M, and N four distinct lines of $\mathcal{F}$ satisfying

- $K = (L\cap K) \cup (M \cap K) \cup (N \cap K)$;
- $\mathcal{C}(L,M,N) \cong W(2)$;
- L, M and N are not contained in a subgrid of $\mathcal{C}(L,M,N)$.

Let G be a subgrid of $\mathcal{S}$ intersecting $\mathcal{F}$ in a line K, and let A, B and C be the three points of $\Omega(\mathcal{S},\mathcal{F})$ incident with G. If G is a quad, then $A' = B' = C' = K$ and we obtain a triple of the first kind. If G is not a quad, then the hex $\mathcal{H} := \mathcal{C}(K,A,A')$ contains a grid-quad $\mathcal{C}(A,A')$ and two $W(2)$-quads $\mathcal{C}(A,K)$ and $\mathcal{C}(A',K)$ through the line K and hence is isomorphic to $\mathbb{I}_3$. As a consequence the unique grid-quads through B and C are contained in $\mathcal{H}$. This implies that the three lines A', B' and C' are contained in the $W(2)$-quad $\mathcal{C}(A',K)$. By Application (I) of [17], A', B' and C' are not contained in a grid. [One can also easily verify this property in the near hexagon $\mathbb{H}_3$ which is isomorphic to $\mathbb{I}_3$.] Since there are as many lines in $\Omega(\mathcal{S},\mathcal{F})$ as there are triples of the first or second type, we have found a description of $\Omega(\mathcal{S},\mathcal{F})$ without any reference to $\mathcal{S}$. By the main result of [17], the structure of a dense near polygon $\mathcal{S}'$ of order $(2,t_{\mathcal{S}'})$ with a big geodetically closed sub near polygon $\mathcal{F}'$ is completely determined by the structure of $\Omega(\mathcal{S}',\mathcal{F}')$. Therefore $\mathcal{S}$ necessarily is isomorphic to $\mathbb{I}_n$. $\square$

2.4.3 The case $\mathcal{F} \cong \mathbb{I}_n, n \geq 4$

Theorem 2.4.10
Let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \geq 5$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to $\mathbb{I}_{n-1}$. Then $\mathcal{S}$ is isomorphic to $\mathbb{I}_{n-1} \times \mathbb{L}_3$.

Proof
Suppose that $\mathcal{Q}$ is a $Q(5,2)$-quad intersecting $\mathcal{F}$ in a line and let $\mathcal{Q}'$ denote a $W(2)$-quad of $\mathcal{F}$ through $\mathcal{Q} \cap \mathcal{F}$. The hex $\mathcal{C}(\mathcal{Q}, \mathcal{Q}')$ then contains a $Q(5,2)$-quad $\mathcal{Q}$ and a big $W(2)$-quad $\mathcal{Q}'$, contradicting Table 1.3. Hence, no $Q(5,2)$-quad meets $\mathcal{F}$. Suppose now that $\mathcal{R}$ is a $W(2)$-quad intersecting $\mathcal{F}$ in a line L, let x denote an arbitrary point of L and let L' denote the unique line of $\mathcal{F}$ through x for which $\mathcal{C}(L, L')$ is a grid. The number of $W(2)$-quads of $\mathcal{F}$ through L equals $2^{n-2} - 2$. If $\mathcal{R}'$ is one of these quads, then by Table 1.3, the hex $\mathcal{C}(\mathcal{R}, \mathcal{R}')$ has a line through x not contained in $\mathcal{R} \cup \mathcal{R}'$. As a consequence $t_{\mathcal{S}} - t_{\mathcal{F}} \geq 2^{n-2}$. Also by Table 1.3, $\mathcal{C}(\mathcal{R}, L') \cong W(2) \times \mathbb{L}_3$ and hence $\mathcal{C}(\mathcal{R}, L')$ has two grids through L' which intersect $\mathcal{F}$ in the line L'. As a consequence, the number of grid-quads intersecting $\mathcal{F}$ in L' is at least twice the number of $W(2)$-quads intersecting $\mathcal{F}$ in L. Suppose now that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two grid-quads through L different from $\mathcal{C}(L, L')$. By Table 1.3, the hex $\mathcal{C}(\mathcal{Q}_1, \mathcal{Q}_2)$ cannot intersect $\mathcal{F}$ in a big $W(2)$-quad. Hence $\mathcal{C}(\mathcal{Q}_1, \mathcal{Q}_2) \cap \mathcal{F} = \mathcal{C}(L, L')$ and $\mathcal{Q}_2 \subset \mathcal{C}(\mathcal{Q}_1, L')$. As a consequence, there are at most two grid-quads intersecting $\mathcal{F}$ in the line L. By symmetry, there are also at most two grid-quads intersecting $\mathcal{F}$ in the line L' and hence there is at most one $W(2)$-quad intersecting $\mathcal{F}$ in L. Since every line K through x not contained in $\mathcal{F}$ is contained in a quad together with L, we have $1 + 1 + 2 \geq t_{\mathcal{S}} - t_{\mathcal{F}} \geq 2^{n-2}$ or $n \leq 4$, contradicting our assumption $n \geq 5$. As a consequence, every quad intersecting $\mathcal{F}$ in a line is a grid. The theorem now immediately follows from Lemma 2.2.1. □

2.4.4 The case $\mathcal{F} \cong \mathbb{E}_1$

We will prove the following lemma in Section 4.3 of Chapter 4.

Lemma 2.4.11
Let $\mathcal{S}$ be a slim dense near octagon. There exist constants $a_{\mathcal{S}}, b_{\mathcal{S}}$ and $c_{\mathcal{S}}$ such that every point of $\mathcal{S}$ is contained in $a_{\mathcal{S}}$ grid-quads, $b_{\mathcal{S}}$ $W(2)$-quads and $c_{\mathcal{S}}$ $Q(5,2)$-quads. Furthermore $t_{\mathcal{S}}(t_{\mathcal{S}} + 1) = 2a_{\mathcal{S}} + 6b_{\mathcal{S}} + 20c_{\mathcal{S}}$, and for every point x of $\mathcal{S}$,

- $|\Gamma_0(x)| = 1$;
- $|\Gamma_1(x)| = 2(t_{\mathcal{S}} + 1)$;

- $|\Gamma_2(x)| = 4a_{\mathcal{S}} + 8b_{\mathcal{S}} + 16c_{\mathcal{S}}$;
- $|\Gamma_3(x)| = \frac{v_{\mathcal{S}}}{3} - 1 - 6t_{\mathcal{S}} + 4a_{\mathcal{S}} + 8b_{\mathcal{S}} + 16c_{\mathcal{S}}$;
- $|\Gamma_4(x)| = 2(\frac{v_{\mathcal{S}}}{3} - 1 + 2t_{\mathcal{S}} - 4a_{\mathcal{S}} - 8b_{\mathcal{S}} - 16c_{\mathcal{S}})$.

Hence for every $i \in \{0, 1, 2, 3, 4\}$, $|\Gamma_i(x)|$ is independent of the chosen point x.

Lemma 2.4.12
The near hexagon $\mathbb{E}_1$ has up to an isomorphism exactly one spread of symmetry.

Proof
As explained in Section 1.9.2, the near hexagon can be considered as the linear representation of the Coxeter cap $\mathcal{K}$. Let x be a point of $\mathcal{K}$ and let S be the set of all lines of $\mathcal{S}$ corresponding to the affine parts of the lines of $\mathsf{PG}(6, 3)$ intersecting Π_∞ in x. It was shown in [18] that S is a spread of symmetry of $\mathcal{S}$ and that every spread of symmetry is obtained in this way. Now, by [9], the group of automorphisms of Π_∞ fixing $\mathcal{K}$ acts transitively on the points of $\mathcal{K}$. This proves the lemma. $\square$

Lemma 2.4.13
There exists a unique glued near octagon of type $\mathbb{E}_1 \otimes Q(5, 2)$.

Proof
All spreads of symmetry of $\mathbb{E}_1$ are isomorphic and all spreads of symmetry of $Q(5, 2)$ are isomorphic. We may therefore fix arbitrary spreads of symmetry S_1 and S_2 in $\mathbb{E}_1$ and $Q(5, 2)$ respectively. By Theorem 1.7.5, every near polygon which can be obtained for a certain choice of the base elements can always be obtained for any other choice of the base elements (by changing the map θ accordingly). Hence we may also fix arbitrary base elements $L_1^{(1)} \in S_1$ and $L_1^{(2)} \in S_2$. By Theorem 1.7.6 every bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$ gives rise to a glued near polygon of type $\mathbb{E}_1 \otimes Q(5, 2)$. By reasons of symmetry, all these near polygons are isomorphic if the group of automorphisms of $Q(5, 2)$ which fix S_2 and the base element $L_1^{(2)} \in S_2$ induces the full group of permutations of this base element. This property holds, see e.g. Lemma 2.1.3. $\square$

Theorem 2.4.14
If $\mathcal{S}$ is a dense near octagon containing a big hex isomorphic to $\mathbb{E}_1$, then $\mathcal{S}$ is either isomorphic to $\mathbb{E}_1 \times \mathbb{L}_3$ or $\mathbb{E}_1 \otimes Q(5, 2)$.

Proof

Suppose that $\mathcal{S}$ is not isomorphic to $\mathbb{E}_1 \times \mathbb{L}_3$. Let $\mathcal{H} \cong \mathbb{E}_1$ be a hex of $\mathcal{S}$ and let L be a line of $\mathcal{S}$ intersecting $\mathcal{H}$ in a point x. Because $\mathcal{S}$ is not isomorphic to $\mathbb{E}_1 \times \mathbb{L}_3$, L is contained in at least one quad $\mathcal{Q}_L$ not isomorphic to a grid-quad. Put $L' := \mathcal{Q}_L \cap \mathcal{H}$ and let M be a line of $\mathcal{H}$ through x different from L'. Clearly $\mathcal{H}' := \mathcal{C}(L, L', M)$ is a hex intersecting $\mathcal{H}$ in a grid-quad which is necessarily big in $\mathcal{H}'$. It follows from Table 1.3 that $\mathcal{H}' \cong W(2) \times \mathbb{L}_3$ or $\mathcal{H}' \cong Q(5,2) \times \mathbb{L}_3$. This implies that every quad through L different from $\mathcal{Q}_L$ is isomorphic to a grid-quad. From Lemma 2.4.11, every point y of $\mathcal{S}$ is contained in exactly one quad isomorphic to $\mathcal{Q}_L$. All other quads through y are grid-quads.

Suppose that $\mathcal{Q}_L \cong W(2)$. Then $t_{\mathcal{S}} + 1 = 14$, $b_{\mathcal{S}} = 1$ and $c_{\mathcal{S}} = 0$. From Lemma 2.4.11, $a_{\mathcal{S}} = 88$. It follows from Table 1.3 that only hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3, W(2) \times \mathbb{L}_3$ or $\mathbb{E}_1$ can occur. Consider a point z of $\mathcal{S}$ and let m_1, m_2 and m_3 denote the number of hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, respectively $\mathbb{E}_1$, through z. Because $b_{\mathcal{S}} = 1$, $m_2 = 11$. By considering the local spaces, we can see that every point of $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, respectively $\mathbb{E}_1$, is contained in respectively 3, 3, and 66 grid-quads. Counting in two ways the number of line-grid-quad pairs intersecting each other in z yields that $3m_1 + 6m_2 + 660m_3 = 88 \cdot 12 = 1056$. Since $\mathbb{E}_1$ is big in $\mathcal{S}$, we can easily count that $v_{\mathcal{S}} = v_{\mathbb{E}_1}(1 + 2(t_{\mathcal{S}} - t_{\mathbb{E}_1})) = 3645$. By Lemma 2.4.11, the number of points at distance three from a given point in $\mathcal{S}$ is equal to 1496. Counting the number of points at distance three from a given point in $\mathcal{S}$ then yields (using Proposition 1.10.8) that $8m_1 + 16m_2 + 440m_3 = 1496$. Together with $3m_1 + 6m_2 + 660m_3 = 1056$ and $m_2 = 12$, this implies that $m_3 = 1$. Hence every point of $\mathcal{S}$ is contained in a unique hex isomorphic to $\mathbb{E}_1$. It is now easy to see that the set of $W(2)$-quads of $\mathcal{S}$ and the set of $\mathbb{E}_1$-hexes of $\mathcal{S}$ determine two partitions of $\mathcal{S}$ satisfying the conditions of Theorem 1.7.8. Hence $\mathcal{S}$ is a glued near polygon of type $\mathbb{E}_1 \otimes W(2)$. This implies that $W(2)$ contains a spread of symmetry, contradicting e.g. Corollary 4.5 of [10].

Suppose now that $\mathcal{Q}_L \cong Q(5,2)$. Then $t_{\mathcal{S}} + 1 = 16$, $b_{\mathcal{S}} = 0$ and $c_{\mathcal{S}} = 1$. From Lemma 2.4.11, $a_{\mathcal{S}} = 110$. It follows from Table 1.3 that only hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$ or $\mathbb{E}_1$ can occur. With a similar argument as before, one can show that every point of $\mathcal{S}$ is contained in a unique hex isomorphic to $\mathbb{E}_1$. It follows again that $\mathcal{S}$ is a glued near polygon, this time of type $\mathbb{E}_1 \otimes Q(5,2)$. By the previous lemma, there exists a unique glued near hexagon of type $\mathbb{E}_1 \otimes Q(5,2)$. $\square$

2.4.5 The case $\mathcal{F} \cong \mathbb{E}_2$

Theorem 2.4.15
If $\mathcal{S}$ is a dense near octagon containing a big hex isomorphic to $\mathbb{E}_2$, then $\mathcal{S}$ is isomorphic to $\mathbb{E}_2 \times \mathbb{L}_3$.

Proof
Let $\mathcal{H}$ be a big hex of $\mathcal{S}$ isomorphic to $\mathbb{E}_2$ and suppose that every hex disjoint from $\mathcal{H}$ is isomorphic to $\mathbb{E}_2$. Since two $\mathbb{E}_2$-hexes cannot intersect (if so, their intersection would be a big quad in both hexes), every point of $\mathcal{S}$ is contained in a unique $\mathbb{E}_2$-hex. Let x be a point outside $\mathcal{H}$, let $\mathcal{H}_x$ be the unique $\mathbb{E}_2$-hex through x and suppose that L is a line through x not contained in $\mathcal{H}_x$ and different from the line $x\pi_{\mathcal{H}}(x)$. Then every hex through L not containing $x\pi_{\mathcal{H}}(x)$ leads to a contradiction. Hence $t_{\mathcal{S}} = t_{\mathbb{E}_2} + 1$ and $\mathcal{S} \cong \mathbb{E}_2 \times \mathbb{L}_3$.

We will show that every hex $\mathcal{H}'$ disjoint from $\mathcal{H}$ is isomorphic to $\mathbb{E}_2$. Suppose that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two disjoint isomorphic big quads of $\mathcal{H}'$. Then every point of $\mathcal{Q}_1$ has distance one to $\mathcal{Q}_2$. Clearly, the projection from $\mathcal{H}'$ to $\mathcal{H}$ is distance-preserving. It follows that the projections of the points of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on $\mathcal{H}$ must be contained in two disjoint $W(2)$-quads $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$ such that $|\{x \in \mathcal{Q}_1' | d(x, \mathcal{Q}_2') = 1\}| \geq 9$. But, from Section 3 of [3] and the fact that $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$ are disjoint, $|\{x \in \mathcal{Q}_1' | d(x, \mathcal{Q}_2') = 1\}| = 7$, a contradiction. Hence $\mathcal{H}'$ contains no big quads. By Table 1.3, $\mathcal{H}' \in \{\mathbb{E}_1, \mathbb{E}_2\}$. Suppose that $\mathcal{H}' \cong \mathbb{E}_1$. Then every two lines of $\mathcal{H}'$ through a fixed point x determine a unique grid-quad of $\mathcal{H}'$ whose projection on $\mathcal{H}$ is contained in a unique $W(2)$-quad through $\pi_{\mathcal{H}}(x)$. Hence $\pi_{\mathcal{H}}(x)$ is contained in at least 66 $W(2)$-quads in $\mathcal{H}$, contradicting Table 1.3. Hence $\mathcal{H}'$ is isomorphic to $\mathbb{E}_2$. $\square$

2.4.6 The case $\mathcal{F} \cong \mathbb{E}_3$

Theorem 2.4.16
Let $\mathcal{S}$ be a slim dense near octagon containing a hex $\mathcal{H}$ isomorphic to $\mathbb{E}_3$, then $\mathcal{S} \cong \mathbb{E}_3 \times \mathbb{L}_3$.

Proof
Let $\mathcal{Q}$ denote an arbitrary quad intersecting $\mathcal{H}$ in a line L. Since $\mathcal{H} \cong \mathbb{E}_3$, $\mathcal{H}$ has a $W(2)$-quad $\mathcal{Q}_1$ and a $Q(5,2)$-quad $\mathcal{Q}_2$ through L. If $\mathcal{Q}$ is a $Q(5,2)$-quad, then the hex $\mathcal{C}(\mathcal{Q}, \mathcal{Q}_1)$ contains a $Q(5,2)$-quad $\mathcal{Q}$ and a big $W(2)$-quad $\mathcal{Q}_1$, contradicting Table 1.3. If $\mathcal{Q}$ is a $W(2)$-quad, then by Table 1.3, the hex $\mathcal{C}(\mathcal{Q}, \mathcal{Q}_2)$ is isomorphic to either $\mathbb{G}_3$ or $\mathbb{E}_3$. In any case, $\mathcal{C}(\mathcal{Q}, \mathcal{Q}_2)$ contains a $Q(5,2)$-quad which intersects $\mathcal{H}$ in a line, a contradiction. Hence $\mathcal{Q}$ is a grid. The theorem now follows from Lemma 2.2.1.

2.4.7 The case $\Delta_0(\mathcal{F}) \neq \emptyset$

Theorem 2.4.17
Let $\mathcal{S}$ be a slim dense near polygon containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to the direct product $\mathcal{S}_1 \times \mathcal{S}_2$ of two near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$ of diameter at least 1. Then there exists an $i \in \{1, 2\}$ and a dense near polygon $\mathcal{S}'_i$ such that the following holds:

- *$\mathcal{S}'_i$ has a big geodetically closed sub near polygon isomorphic to $\mathcal{S}_i$;*
- *$\mathcal{S} \cong \mathcal{S}'_i \times \mathcal{S}_{3-i}$.*

Proof
Let x denote an arbitrary point of $\mathcal{F}$. Since $\mathcal{F} \cong \mathcal{S}_1 \times \mathcal{S}_2$, there exist geodetically closed sub near polygons $\mathcal{F}_1$ and $\mathcal{F}_2$ through x such that

- $\mathcal{F}_1 \cong \mathcal{S}_1$ and $\mathcal{F}_2 \cong \mathcal{S}_2$;
- $\mathcal{F}_1 \cap \mathcal{F}_2 = \{x\}$;
- every line through x either is contained in $\mathcal{F}_1$ or in $\mathcal{F}_2$.

If $t_{\mathcal{S}} = t_{\mathcal{F}}+1$, then $\mathcal{S} \cong \mathcal{F}\times L \cong \mathcal{S}_1\times(\mathcal{S}_2\times L)$ by Lemma 2.2.1. Hence, we may suppose that there exist two lines K and L through x which are not contained in $\mathcal{F}$. Since $\mathcal{F}$ is big in $\mathcal{S}$, $\mathcal{C}(K, L)$ intersects $\mathcal{F}$ in a line L'. By reasons of symmetry, we may suppose that $L' \subseteq \mathcal{F}_1$. Put now $\mathcal{F}_3 := \mathcal{C}(\mathcal{F}_1, K)$. Then $\mathcal{F}_1 = \mathcal{F}_3\cap\mathcal{F}$ is big in $\mathcal{F}_3$, $\mathrm{diam}(\mathcal{S}) = \mathrm{diam}(\mathcal{F}_3)+\mathrm{diam}(\mathcal{F}_2)$ and $\mathcal{F}_3\cap\mathcal{F}_2 = \{x\}$. We will show that every line M through x not contained in $\mathcal{F}$ is contained in $\mathcal{F}_3$. The theorem then follows from Theorem 2.2.3. Since $\mathcal{C}(K, L) \subseteq \mathcal{F}_3$, we may suppose that M is not contained in $\mathcal{C}(K, L)$. The quad $\mathcal{C}(K, M)$ then intersects $\mathcal{F}$ in a line $M' \neq L'$. If M' belongs to $\mathcal{F}_2$, then the grid-quad $\mathcal{C}(L', M')$ is big in the hex $\mathcal{C}(K, L, M)$. From Table 1.3 it then follows that $\mathcal{C}(K, L, M) \in \{\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3, W(2) \times \mathbb{L}_3, Q(5, 2) \times \mathbb{L}_3\}$. This contradicts however the fact that none of the quads $\mathcal{C}(K, L)$ and $\mathcal{C}(K, M)$ is a grid. Hence $M' \subseteq \mathcal{F}_1$ and $M \subset \mathcal{C}(K, M') \subseteq \mathcal{F}_3$. □

2.4.8 The case $\mathcal{F} \in \mathcal{C}$ or $\mathcal{F} \in \mathcal{D}$

If Z_1 and Z_2 are two sets of near polygons, then $Z_1 \otimes Z_2$ denotes the (possibly empty) set of all near polygons obtained by glueing an element of Z_1 with one of Z_2. We introduce the following sets.

$$\begin{aligned} C_2 &= \{Q(5,2)\}; \\ C_3 &= \{\mathbb{G}_3, H^D(5,4), \mathbb{E}_1, Q(5,2)\otimes Q(5,2)\}; \\ C_n &= \{\mathbb{G}_n, H^D(2n-1,4)\} \cup \left(\bigcup_{2\le i\le n-1} C_i \otimes C_{n+1-i}\right) \text{ for every } n \ge 4; \\ \mathcal{C} &= C_2 \cup C_3 \cup \cdots \end{aligned}$$

In the next section, we will prove the following theorem.

Theorem 2.4.18
Let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \ge 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ isomorphic to an element of $\mathcal{C}$. Then one of the following possibilities occurs:

- *$\mathcal{S} \cong \mathcal{F} \times \mathbb{L}_3$;*
- *$\mathcal{S}$ is isomorphic to an element of $\mathcal{C}$.*

We now define a subclass $\mathcal{D}$ of $\mathcal{C}$ as follows:

$$\begin{aligned} D_2 &= \{Q(5,2)\}; \\ D_n &= \{\mathbb{G}_n, H^D(2n-1,4)\} \cup \left(\bigcup_{2\le i\le n-1} D_i \otimes D_{n+1-i}\right) \text{ for every } n \ge 3; \\ \mathcal{D} &= D_2 \cup D_3 \cup \cdots \end{aligned}$$

Remark that $\mathcal{D}$ consists of those near polygons of the class $\mathcal{C}$ which do not contain $\mathbb{E}_1$ as a hex. In the next section, we will also prove the following theorem.

Theorem 2.4.19
Let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \ge 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ which is isomorphic to an element of $\mathcal{D}$. Then one of the following possibilities occurs:

- *$\mathcal{S} \cong \mathcal{F} \times \mathbb{L}_3$;*
- *$\mathcal{S}$ is isomorphic to an element of $\mathcal{D}$.*

2.5 Proof of Theorems 2.4.18 and 2.4.19

We first prove Theorem 2.4.18. The proof of 2.4.19 will then follow easily at the end. The proof consists of several steps.

Let $\mathcal{L}(x, \mathcal{S})$ be the local space of a near polygon $\mathcal{S}$ at a point x. The *modified local space* of $\mathcal{S}$ at x is the partial linear space $\mathcal{ML}(x, \mathcal{S})$ obtained from $\mathcal{L}(x, \mathcal{S})$ by removing all lines of size 2. Clearly, no modified local spaces of a product near polygon are connected.

Lemma 2.5.1
The class $\mathcal{C}$ (and hence also $\mathcal{D}$) does not contain product near polygons.

Proof
By Theorem 2.3.4, it follows that it is sufficient to check that no element of the set $\{H^D(2n-1,4)|n \geq 2\} \cup \{\mathbb{G}_n|n \geq 3\} \cup \{\mathbb{E}_1\}$ is a product near polygon.

All quads of $H^D(2n-1,4)$ are isomorphic to $Q(5,2)$. So, all modified local spaces coincide with their corresponding local spaces and hence are linear and connected.

By Theorem 1.10.5, every point x of $\mathbb{G}_n$, $n \geq 3$, is incident with n special lines and $\frac{3n(n-1)}{2}$ ordinary lines. These lines satisfy the following properties:

- $\mathcal{C}(K, L) \cong Q(5,2)$ for every two different special lines K and L through x;

- if M is an ordinary line through x, then there exists a unique pair $\{K, L\}$ of special lines through x such that $M \subseteq \mathcal{C}(K, L)$.

From these properties it is easily seen that $\mathcal{ML}(x, \mathbb{G}_n)$ is connected.

Suppose now that $\mathbb{E}_1$ is a product near hexagon. Then $\mathbb{E}_1$ is the direct product of a slim dense generalized quadrangle $\mathcal{Q}$ and the line $\mathbb{L}_3$. Since $t_{\mathbb{E}_1} = 11$, it follows that $t_{\mathcal{Q}} = 10$, contradicting Theorem 1.10.1. $\square$

Definition
If $\mathcal{S}$ is an element of $\mathcal{C}$ (respectively $\mathcal{D}$), then we define $\Omega_{\mathcal{C}}(\mathcal{S})$ (respectively $\Omega_{\mathcal{D}}(\mathcal{S})$) as the set of all pairs $\{T_1, T_2\} \in \Delta_1(\mathcal{S})$ with the property that every element of $T_1 \cup T_2$ belongs to $\mathcal{C}$ (respectively $\mathcal{D}$). In Section 2.6.2 we will show that $\Omega_{\mathcal{C}}(\mathcal{S}) = \Delta_1(\mathcal{S})$ for every $\mathcal{S} \in \mathcal{C}$ and $\Omega_{\mathcal{D}}(\mathcal{S}) = \Delta_1(\mathcal{S})$ for every $\mathcal{S} \in \mathcal{D}$.

Now let $\mathcal{S}$ be a slim dense near $2n$-gon, $n \geq 4$, containing a big geodetically closed sub near polygon $\mathcal{F}$ belonging to $\mathcal{C}$. There are four possibilities.

(1) $\mathcal{F} \cong H^D(2n-3,4)$ for some $n \geq 3$; by Theorem 2.4.3 we then know that $\mathcal{S}$ is isomorphic to either $H^D(2n-1,4)$, $H^D(2n-3,4) \times \mathbb{L}_3$ or $H^D(2n-3,4) \otimes Q(5,2)$.

(2) $\mathcal{F} \cong \mathbb{G}_{n-1}$ for some $n \geq 3$; by Theorem 2.4.2 we then know that $\mathcal{S}$ is isomorphic to either $\mathbb{G}_n$, $\mathbb{G}_{n-1} \times \mathbb{L}_3$ or $\mathbb{G}_{n-1} \otimes Q(5,2)$.

(3) $\mathcal{F} \cong \mathbb{E}_1$; by Theorem 2.4.14 we then know that $\mathcal{S}$ is isomorphic to either $\mathbb{E}_1 \times \mathbb{L}_3$ or $\mathbb{E}_1 \otimes Q(5,2)$.

(4) $\mathcal{F}$ is glued. In this case $\Omega := \Omega_{\mathcal{C}}(\mathcal{F}) \neq \emptyset$.

In view of what we need to prove, we may suppose that

- $\mathcal{F}$ is glued, so $\Omega \neq \emptyset$;
- $\mathcal{S} \ncong \mathcal{F} \times \mathbb{L}_3$;
- no proper geodetically closed sub near polygon of $\mathcal{S}$ violates Theorem 2.4.18. (If this were not so, then the reasoning described here would still be applicable to each minimal violating geodetically closed sub near polygon in $\mathcal{S}$, yielding an obvious contradiction.)

Let V denote the set of all $W(2)$-quads and $Q(5,2)$-quads which intersect $\mathcal{F}$ in a line.

Lemma 2.5.2
The set V is not empty.

Proof
Take an arbitrary point x in $\mathcal{F}$. Since $\mathcal{S}$ is not isomorphic to $\mathcal{F} \times \mathbb{L}_3$, there exist two lines K and L through x which are not contained in $\mathcal{F}$. By Theorem 1.6.5, the quad $\mathcal{Q} := \mathcal{C}(K,L)$ intersects $\mathcal{F}$ in a line. Hence $t_{\mathcal{Q}} + 1 \geq 3$ and $\mathcal{Q}$ is isomorphic to either $W(2)$ or $Q(5,2)$. $\square$

Definition
If T is a partition of $\mathcal{F}$ in isomorphic geodetically closed sub near polygons, then $d(T)$ denotes the diameter of an arbitrary element of T. If T_1 and T_2 are two partitions of $\mathcal{F}$ in geodetically closed sub near polygons, then we say that $(T_1, T_2) \in \Omega'$ if $\{T_1, T_2\} \in \Omega$ and if exactly one of the following holds:

(a) $\mathcal{Q} \cap \mathcal{F} \in \mathcal{L}_1 \cap \mathcal{L}_2$ for every $\mathcal{Q} \in V$;

(b) there exists a quad $\mathcal{Q} \in V$ such that $\mathcal{Q} \cap \mathcal{F} \in \mathcal{L}_2 \setminus \mathcal{L}_1$.

Here $\mathcal{L}_i$, $i \in \{1, 2\}$, denotes the set of lines of $\mathcal{F}$ which are contained in a sub near polygon of T_i. If $\{T_1, T_2\} \in \Omega$, then Ω' contains at least one of the elements (T_1, T_2) and (T_2, T_1). Since $\Omega \neq \emptyset$, also $\Omega' \neq \emptyset$.

Lemma 2.5.3
Suppose that $(T_1, T_2) \in \Omega'$.

(i) *If (T_1, T_2) is of type (a), then for every element $\mathcal{A}$ of T_2, there exists a quad $\mathcal{Q}_\mathcal{A} \in V$ such that $\mathcal{Q}_\mathcal{A} \cap \mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{Q}_\mathcal{A} \cap \mathcal{F} \in \mathcal{L}_1 \cap \mathcal{L}_2$.*

(ii) *If (T_1, T_2) is of type (b), then for every element $\mathcal{A}$ of T_2, there exists a quad $\mathcal{Q}_\mathcal{A} \in V$ such that $\mathcal{Q}_\mathcal{A} \cap \mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{Q}_\mathcal{A} \cap \mathcal{F} \in \mathcal{L}_2 \setminus \mathcal{L}_1$.*

(iii) *For every element $\mathcal{A}$ of T_2, there exists a unique geodetically closed sub near $2(d(T_2) + 1)$-gon $\hat{\mathcal{A}}$ through $\mathcal{A}$ such that $\hat{\mathcal{A}} \cap \mathcal{F} = \mathcal{A}$. If $\mathcal{A} \in T_2$ and $\mathcal{B} \in T_1$, then $\hat{\mathcal{A}} \cap \mathcal{B}$ is a line, and every line which intersects $\hat{\mathcal{A}} \cap \mathcal{B}$ in a point either is contained in $\hat{\mathcal{A}}$ or $\mathcal{B}$.*

(iv) *The set $\hat{T}_2 := \{\hat{\mathcal{A}} | \mathcal{A} \in T_2\}$ is a partition of $\mathcal{S}$ in geodetically closed sub near polygons.*

Proof

(i) Let $\mathcal{A}$ denote an arbitrary element of T_2 and x an arbitrary point of $\mathcal{A}$. Since $\mathcal{S} \ncong \mathcal{F} \times \mathbb{L}_3$, there exist two lines K and L through x not contained in $\mathcal{F}$. Using Theorem 1.6.5, we see that the quad $\mathcal{C}(K, L)$ satisfies all required properties.

(ii) We will prove that if this property holds for a certain $\mathcal{A} \in T_2$ (with corresponding quad $\mathcal{Q}_\mathcal{A}$), then it also holds for any $\mathcal{B} \in T_2$ at distance 1 from $\mathcal{A}$. Property (ii) then follows from the connectedness of $\mathcal{F}$. Put $L_\mathcal{A} := \mathcal{Q}_\mathcal{A} \cap \mathcal{F}$, let a denote an arbitrary point of $L_\mathcal{A}$ and let b denote the unique point of $\mathcal{B}$ collinear with a. Since $L_\mathcal{A} \in \mathcal{L}_2 \setminus \mathcal{L}_1$, the quad $\mathcal{C}(L_\mathcal{A}, b)$ is a grid and hence contains a unique line $L_\mathcal{B}$ through b disjoint with $L_\mathcal{A}$. Clearly, the line $L_\mathcal{B}$ belongs to $\mathcal{L}_2 \setminus \mathcal{L}_1$ and is contained in $\mathcal{B}$. We will now construct an element $\mathcal{Q}_\mathcal{B}$ of V through $L_\mathcal{B}$. By Theorem 1.6.5, the grid-quad $\mathcal{C}(L_\mathcal{A}, b)$ is big in the hex $\mathcal{C}(\mathcal{Q}_\mathcal{A}, b)$. By Table 1.3, it then follows that $\mathcal{C}(\mathcal{Q}_\mathcal{A}, b) \cong \mathcal{Q}_\mathcal{A} \times \mathbb{L}_3$. Hence the hex $\mathcal{C}(\mathcal{Q}_\mathcal{A}, b)$ contains a unique quad $\mathcal{Q}_\mathcal{B} \cong \mathcal{Q}_\mathcal{A}$ through $L_\mathcal{B}$. Clearly $\mathcal{Q}_\mathcal{B} \in V$.

(iii) Let $\mathcal{A}$ denote an arbitrary element of T_2. Then there exists a quad $\mathcal{Q} \in V$ intersecting $\mathcal{A}$ in a line K with $K \in \mathcal{L}_1 \cap \mathcal{L}_2$ if (T_1, T_2) is of type (a) or $K \in \mathcal{L}_2 \setminus \mathcal{L}_1$ if (T_1, T_2) is of type (b). Let x denote an arbitrary point of K. Clearly, $\mathcal{C}(\mathcal{A}, Q)$ intersects $\mathcal{F}$ in $\mathcal{A}$. In order to prove that $\mathcal{C}(\mathcal{A}, Q)$ is the sub near polygon which satisfies all required properties, it suffices to show that $\mathcal{C}(\mathcal{A}, L) = \mathcal{C}(\mathcal{A}, Q)$ for any line L through x not contained in $\mathcal{F}$. If (T_1, T_2) is of type (a), consider then in $\mathcal{Q}$ a line L' through x different from K and L. The quad $\mathcal{C}(L, L')$ intersects $\mathcal{F}$ in the line K. Hence $L \subseteq \mathcal{C}(L', K) = \mathcal{Q}$ and $\mathcal{C}(\mathcal{A}, L) = \mathcal{C}(\mathcal{A}, \mathcal{Q})$. Suppose now that (T_1, T_2) is of type (b) and that L is not contained in $\mathcal{Q}$. By Theorem 1.6.5 the hex $\mathcal{H} := \mathcal{C}(\mathcal{Q}, L)$ intersects $\mathcal{F}$ in a big quad $\mathcal{Q}'$ through K. In $\mathcal{H}$ there are at least three quads through K (namely $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{C}(K, L)$) and not all these quads are grids. So, $\mathcal{H}$ cannot be isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$ or $Q(5, 2) \times \mathbb{L}_3$. By Table 1.3, it then follows that $\mathcal{H}$ contains no big grid-quad. So, $\mathcal{Q}'$ is isomorphic to either $W(2)$ or $Q(5, 2)$. Since $K \subseteq \mathcal{L}_2 \setminus \mathcal{L}_1$, $\mathcal{Q}'$ must be contained in $\mathcal{A}$. Hence, $\mathcal{C}(\mathcal{A}, L) = \mathcal{C}(\mathcal{A}, \mathcal{H}) = \mathcal{C}(\mathcal{A}, \mathcal{Q})$.

(iv) It is sufficient to prove that every point x outside $\mathcal{F}$ is contained in a unique element of $\hat{T}_2$. If x' denotes the unique point of $\mathcal{F}$ nearest to x and $\mathcal{A}$ the unique element of T_2 through x', then $x \in \mathcal{C}(\mathcal{A}, x'x) = \hat{\mathcal{A}}$. If there were two different elements $\mathcal{A}_1$ and $\mathcal{A}_2$ in T_2 such that $x \in \hat{\mathcal{A}}_1 \cap \hat{\mathcal{A}}_2$, then x would be collinear with two different elements of $\mathcal{F}$ (notice that $\mathcal{A}_i$, $i \in \{1, 2\}$, is big in $\hat{\mathcal{A}}_i$ by Theorem 1.6.5), contradicting the fact that $\mathcal{F}$ is geodetically closed. $\square$

Let (T_1, T_2) be an element of Ω' with $d(T_1)$ as small as possible. Let $\mathcal{G}$ denote an arbitrary near $2\,d(T_1)$-gon of T_1. If $\Omega(\mathcal{G}) \neq \emptyset$, then we can apply Theorem 2.3.5 and Lemma 2.5.1 and we find that there exists an element $\{\tilde{T}_1, \tilde{T}_2\} \in \Omega(\mathcal{F})$ with $d(\tilde{T}_1) < d(T_1)$ such that T_2 is a refinement of $\tilde{T}_2$. This latter property implies that $(\tilde{T}_1, \tilde{T}_2) \in \Omega'$. But this contradicts our assumption on the minimality of $d(T_1)$. So, $\Omega(\mathcal{G}) = \emptyset$ and $\mathcal{G}$ is isomorphic to an element of the set $\{H^D(2m-1, 4) | m \geq 2\} \cup \{\mathbb{G}_m | m \geq 3\} \cup \{\mathbb{E}_1\}$. We have shown earlier that there exists a unique partition $\hat{T}_2$ of $\mathcal{S}$ in geodetically closed sub near $2(d(T_2)+1)$-gons, such that $\hat{\mathcal{A}} \cap \mathcal{F} \in T_2$ for every $\hat{\mathcal{A}} \in \hat{T}_2$. We will now extend the partition T_1 of $\mathcal{F}$ to a partition $\hat{T}_1$ of $\mathcal{S}$.

Lemma 2.5.4

The partition T_1 of $\mathcal{F}$ can be extended to a partition $\hat{T}_1$ of $\mathcal{S}$ such that every element of $\hat{T}_1$ intersects every element of $\hat{T}_2$ in a line.

Proof

For every point x of $\mathcal{F}$, we define $\hat{\mathcal{F}}_1(x)$ as the unique element of T_1 through x. Suppose that x is a point outside $\mathcal{F}$. Let $\hat{\mathcal{F}}_2(x)$ denote the unique element of $\hat{T}_2$ through x and let x' denote the unique point of $\mathcal{F}$ collinear with x. By Theorem 1.6.5, $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x)$ contains $\hat{\mathcal{F}}_1(x')$ as a big geodetically closed sub near polygon. We distinguish the following cases.

- If $d(T_1) = 2$, then $\hat{\mathcal{F}}_1(x') \cong Q(5,2)$ and $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x)$ is isomorphic to either $Q(5,2) \times \mathbb{L}_3$, $Q(5,2) \otimes Q(5,2)$, $\mathbb{G}_3$, $\mathbb{E}_3$ or $H^D(5,4)$. By property (iii) of Lemma 2.5.3, every line of $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x)$ through x' is contained in one of the quads $\hat{\mathcal{F}}_1(x')$ or $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x) \cap \hat{\mathcal{F}}_2(x)$. So, $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x)$ must be isomorphic to either $Q(5,2) \times \mathbb{L}_3$ or $Q(5,2) \otimes Q(5,2)$.
- If $d(T_1) \geq 3$, then $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x)$ is isomorphic to either $\mathcal{G} \times \mathbb{L}_3$ or $\mathcal{G} \otimes Q(5,2)$ by Theorems 2.4.2, 2.4.3 and 2.4.14.

In any case, there exists a unique geodetically closed sub near polygon $\hat{\mathcal{F}}_1(x)$ in $\mathcal{C}(\hat{\mathcal{F}}_1(x'), x)$ satisfying: (i) $x \in \hat{\mathcal{F}}_1(x)$, (ii) $\hat{\mathcal{F}}_1(x)$ is isomorphic to $\mathcal{G}$ and (iii) $\hat{\mathcal{F}}_1(x)$ is disjoint with $\hat{\mathcal{F}}_1(x')$. Clearly, the set $\hat{T}_1$ of all sub near polygons $\hat{\mathcal{F}}_1(x)$, $x \in \mathcal{S}$, is a partition of $\mathcal{S}$ in sub near polygons isomorphic to $\mathcal{G}$. It is also clear that every element of $\hat{T}_1$ intersects every element of $\hat{T}_2$ in a line. $\square$

Lemma 2.5.5
If $\hat{\mathcal{A}} \in \hat{T}_1 \cup \hat{T}_2$, then either $\hat{\mathcal{A}}$ belongs to $\mathcal{C}$ or $\hat{\mathcal{A}} \cong \mathbb{E}_3$.

Proof
Every element of $\hat{T}_1$ is isomorphic to $\mathcal{G}$ and hence belongs to $\mathcal{C}$. Suppose therefore that $\hat{\mathcal{A}} \in \hat{T}_2$. We know that $\mathcal{A} := \hat{\mathcal{A}} \cap \mathcal{F}$ is big in $\hat{\mathcal{A}}$, that $t_{\hat{\mathcal{A}}} \geq t_{\mathcal{A}}+2$ and that $\mathcal{A}$ belongs to $\mathcal{C}$. If $d(T_2) \geq 3$, then $\hat{\mathcal{A}}$ belongs to $\mathcal{C}$ since we assumed that no proper geodetically closed sub near polygon violates Theorem 2.4.19. If $d(T_2) = 2$, then $\mathcal{A} \cong Q(5,2)$ and by Table 1.3 and the fact that $t_{\hat{\mathcal{A}}} \geq t_{\mathcal{A}}+2$, $\hat{\mathcal{A}}$ is isomorphic to $Q(5,2) \otimes Q(5,2)$, $\mathbb{G}_3$, $\mathbb{E}_3$ or $H^D(5,4)$. Again the lemma holds. $\square$

Lemma 2.5.6
$\mathcal{S}$ is not a product near polygon.

Proof
Since $\mathbb{E}_3$ does not contain grid-quads, all its modified local spaces are linear and connected, implying that $\mathbb{E}_3$ is not a product near polygon. Lemmas 2.5.1 and 2.5.5 now show that $\hat{\mathcal{A}}$ is not a product near polygon for every $\hat{\mathcal{A}} \in \hat{T}_1 \cup \hat{T}_2$.

Suppose now that $\mathcal{S}$ is a product near polygon and let $\{U_1, U_2\} \in \Delta_0(\mathcal{S})$. Consider a point x of $\mathcal{S}$ and let $\hat{\mathcal{A}}_i(x)$ (respectively $\mathcal{U}_i(x)$) be the unique element of $\hat{T}_i$ (respectively U_i) through x, $i \in \{1,2\}$. Clearly for at least one $i \in \{1,2\}$, $\mathrm{diam}(\mathcal{U}_1(x) \cap \hat{\mathcal{A}}_i(x))$ and $\mathrm{diam}(\mathcal{U}_2(x) \cap \hat{\mathcal{A}}_i(x))$ are both at least 1. Without loss of generality, we may assume that this property holds for $i = 1$. Put $\mathcal{B}_i := \mathcal{U}_i(x) \cap \hat{\mathcal{A}}_1(x)$, $i \in \{1,2\}$. Clearly $\mathcal{B}_1 \cap \mathcal{B}_2 = \{x\}$ and every line of $\mathcal{A}_1(x)$ through x is contained in either $\mathcal{B}_1$ or $\mathcal{B}_2$. Since every point of $\mathcal{B}_i$ is classical with respect to $\mathcal{B}_{3-i}$, $i \in \{1,2\}$, it follows that $\mathrm{diam}(\mathcal{B}_1) + \mathrm{diam}(\mathcal{B}_2) = \mathrm{diam}(\hat{\mathcal{A}}_1)$. It now follows from Theorem 2.2.3 that $\hat{\mathcal{A}}_1 \cong \mathcal{B}_1 \times \mathcal{B}_2$, a contradiction. □

Lemma 2.5.7
None of the near polygons $\mathbb{E}_2$, $\mathbb{E}_3$, $Q^D(2n,2)$, $\mathbb{H}_n$, $\mathbb{I}_n$ $(n \geq 2)$ has a spread of symmetry.

Proof
Let $\mathcal{S}$ be one of these near polygons, let S be a spread of symmetry of $\mathcal{S}$ and let K denote an arbitrary line of S. Every line of $\mathcal{S}$ is contained in a $W(2)$-quad. In particular, there exists a $W(2)$-quad $\mathcal{Q}$ through K. By Theorem 1.7.2, $S_{\mathcal{Q}}$ is a spread of symmetry of $\mathcal{Q}$. But by Corollary 4.5 of [10], $W(2)$ has no spread of symmetry. So, we have a contradiction. □

The following lemma finishes the proof of Theorem 2.4.18.

Lemma 2.5.8
- (i) *If $\mathcal{G}_1 \in \hat{T}_1$ and $\mathcal{G}_2 \in \hat{T}_2$, then $\mathcal{S}$ is of type $\mathcal{G}_1 \otimes \mathcal{G}_2$.*
- (ii) *The near polygons $\mathcal{G}_1$ and $\mathcal{G}_2$ belong to $\mathcal{C}$. As a consequence also $\mathcal{S}$ belongs to $\mathcal{C}$.*

Proof

- (i) This follows from Theorem 1.7.8 and Lemmas 2.5.4 and 2.5.6.
- (ii) By Theorem 1.7.6, each element of $\hat{T}_1 \cup \hat{T}_2$ must have a spread of symmetry. Since $\mathbb{E}_3$ has no spread of symmetry (Lemma 2.5.7), each element of $\hat{T}_1 \cup \hat{T}_2$ must belong to $\mathcal{C}$, see Lemma 2.5.5. Theorem 2.4.18 now follows from (i). □

As mentioned before, $\mathcal{D}$ consists of those elements of $\mathcal{C}$ which do not have hexes isomorphic to $\mathbb{E}_1$. The following lemma finishes the proof of Theorem 2.4.19.

Lemma 2.5.9
If $\mathcal{F}$ is a big geodetically closed sub near polygon of a near polygon $\mathcal{S}$ which is isomorphic to an element of $\mathcal{C} \setminus \mathcal{D}$, then $\mathcal{F}$ is not isomorphic to an element of $\mathcal{D}$.

Proof
Let $\mathcal{H}$ be a hex of $\mathcal{S}$ such that $\mathcal{H} \cong \mathbb{E}_1$. Suppose that $\mathcal{F}$ is isomorphic to an element of $\mathcal{D}$. Then $\mathcal{H} \not\subseteq \mathcal{F}$. Since $\mathcal{H}$ has no big quads (see Table 1.3), it follows from Theorem 1.6.5 that $\mathcal{H}$ is disjoint from $\mathcal{F}$. Since $\mathcal{F}$ is big, $\pi_{\mathcal{F}}(\mathcal{H}) := \{\pi_{\mathcal{F}}(x)|x \in \mathcal{H}\}$ is a set of points which is contained in a hex $\mathcal{H}'$ of $\mathcal{F}$. Since $\mathcal{H}'$ has at least as many points as $\mathcal{H}$, it follows from Table 1.3 that $\mathcal{H}' \in \{\mathbb{E}_2, H^D(5,4)\}$. Clearly, $\mathcal{C}(\mathcal{H},\mathcal{H}')$ is a slim dense near octagon which has $\mathcal{H}'$ as a big geodetically closed sub near polygon. A contradiction now follows from Theorems 2.4.3 and 2.4.15. $\square$

2.6 A characterization theorem

Consider a chain $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ of geodetically closed sub near polygons of a dense near $2n$-gon $\mathcal{S}$. Such a chain is called *nice* if it satisfies the following properties:

- $\mathrm{diam}(\mathcal{F}_i) = i$ for every $i \in \{0,\dots,n\}$;
- $\mathcal{F}_i$, $i \in \{0,\dots,n-1\}$, is big in $\mathcal{F}_{i+1}$.

Clearly $\mathcal{F}_n = \mathcal{S}$.

Define now the following sets:

$$\mathcal{M} = \{\mathbb{O},\mathbb{L}_3,\mathbb{E}_2,\mathbb{E}_3\} \cup \mathcal{C} \cup \{Q^D(2n,2)|n \geq 2\} \cup \{\mathbb{H}_n|n \geq 3\} \cup \{\mathbb{I}_n|n \geq 4\};$$

$$\mathcal{N} = \{\mathbb{O},\mathbb{L}_3,\mathbb{E}_3\} \cup \mathcal{D} \cup \{Q^D(2n,2)|n \geq 2\} \cup \{\mathbb{H}_n|n \geq 3\} \cup \{\mathbb{I}_n|n \geq 4\}.$$

Let $\mathcal{M}^\times$, respectively $\mathcal{N}^\times$, denote the set of all near polygons obtained by taking the direct product of some (i.e. at least 1) members of $\mathcal{M}$, respectively $\mathcal{N}$. All members of $\mathcal{M}^\times$ are dense and $\mathcal{N}^\times \subseteq \mathcal{M}^\times$. Every known slim dense near polygon is isomorphic to one of the elements of $\mathcal{M}^\times$.

Theorem 2.6.1
A slim dense near $2n$-gon $\mathcal{S}$ has a nice chain of sub near polygons if and only if it is isomorphic to an element of $\mathcal{N}^\times$.

2.6.1 Proof of Theorem 2.6.1

We will first prove that every element of $\mathcal{N}^\times$ has a nice chain of geodetically closed sub near polygons.

Lemma 2.6.2
If $\mathcal{S}$ is an element of $\mathcal{D} \setminus D_2$, then every line of $\mathcal{S}$ is contained in a big geodetically closed sub near polygon which is isomorphic to an element of $\mathcal{D}$.

Proof
We will use induction on the diameter of $\mathcal{S}$. Obviously, the lemma holds if $\mathcal{S} \cong H^D(2n-1,4)$ for some $n \geq 3$. By considering the local spaces of $\mathbb{G}_n$, one sees that the lemma also holds if $\mathcal{S} \cong \mathbb{G}_n$ for some $n \geq 3$. So, suppose that $\mathcal{S}$ is of the form $\mathcal{A}_1 \otimes \mathcal{A}_2$ with $\mathcal{A}_1$ and $\mathcal{A}_2$ two elements of $\mathcal{D}$. There exist partitions T_1 and T_2 of $\mathcal{S}$ in geodetically closed sub near polygons isomorphic to $\mathcal{A}_1$, respectively $\mathcal{A}_2$, such that every element of T_1 intersects every element of T_2 in a line. Now, let L be an arbitrary line of $\mathcal{S}$, then L is contained in an element of $T_1 \cup T_2$. By reasons of symmetry, we may suppose that L is contained in the element $\mathcal{F}_1$ of T_1. If $\mathcal{A}_2$ is a generalized quadrangle, then $\mathcal{F}_1$ is big in $\mathcal{S}$ and we are done. Suppose therefore that $\mathcal{A}_2$ is not a generalized quadrangle. Let $\mathcal{F}_2$ denote an arbitrary element of T_2. By the induction hypothesis we know that the line $\mathcal{F}_1 \cap \mathcal{F}_2$ of $\mathcal{S}_2$ is contained in a big geodetically closed sub near polygon $\mathcal{F}'$ of $\mathcal{F}_2$ which is isomorphic to an element of $\mathcal{D}$. By Lemma 2.3.1, $\mathcal{C}(\mathcal{F}', \mathcal{F}_1)$ is a glued near polygon of type $\mathcal{F}_1 \otimes \mathcal{F}'$. Since $\mathcal{F}_1, \mathcal{F}' \in \mathcal{D}$, also $\mathcal{C}(\mathcal{F}_1, \mathcal{F}') \in \mathcal{D}$. Clearly $\mathcal{C}(\mathcal{F}_1, \mathcal{F}')$ is big in $\mathcal{S}$ and contains the line L. $\square$

Lemma 2.6.3
Every near polygon $\mathcal{S}$ of $\mathcal{N}^\times \setminus \{\mathbb{O}\}$ has a big geodetically closed sub near polygon which is isomorphic to an element of $\mathcal{N}^\times$.

Proof
Clearly the lemma holds if $\mathcal{S}$ is isomorphic to an element of $\{\mathbb{L}_3, \mathbb{E}_3\} \cup \{Q^D(2n,2) | n \geq 2\} \cup \{\mathbb{H}_n | n \geq 3\} \cup \{\mathbb{I}_n | n \geq 4\}$. Now by Lemma 2.6.2 the result holds for every $\mathcal{S} \in \mathcal{N}$. Now, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two dense near polygons and if $\mathcal{F}_1$ is a big geodetically closed sub near polygon of $\mathcal{S}_1$, then the set of all pairs (x, y) with x a point of $\mathcal{F}_1$ and y a point of $\mathcal{S}_2$ determines a big geodetically closed sub near polygon of $\mathcal{S}_1 \times \mathcal{S}_2$ isomorphic to $\mathcal{F}_1 \times \mathcal{S}_2$. Hence the lemma holds for every $\mathcal{S} \in \mathcal{N}^\times$. $\square$

Corollary 2.6.4
If $\mathcal{S} \in \mathcal{N}^\times$, then $\mathcal{S}$ has a nice chain of geodetically closed sub near polygons.

We still need to prove the theorem in the other direction.

Theorem 2.6.5
Every slim dense near $2n$*-gon,* $n \geq 1$*, containing a nice chain* $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n$ *of geodetically closed sub near polygons is isomorphic to an element of* $\mathcal{N}^\times$*.*

Proof
By Table 1.3, the theorem holds if $n \leq 3$. Hence the theorem will hold if we can prove that every slim dense near $2n$-gon, $n \geq 4$, containing a big geodetically closed sub near $2(n-1)$-gon $\mathcal{F}$ which is isomorphic to an element of $\mathcal{N}^\times$, is itself isomorphic to an element of $\mathcal{N}^\times$. By Theorem 2.4.17, this is true for every $\mathcal{F} \in \mathcal{N}^\times$ as soon as it is true for every $\mathcal{F} \in \mathcal{N}$. By Theorems 2.4.16, 2.4.1, 2.4.9 and 2.4.10, it follows that we only need to consider the case $\mathcal{F} \in \mathcal{D}$. This case is settled by Theorem 2.4.19. □

It is now easy to see that Theorem 2.6.1 follows from Corollary 2.6.4 and Theorem 2.6.5.

2.6.2 A corollary

Lemma 2.6.6

(i) *Every geodetically closed sub near polygon of an element* $\mathcal{F}$ *of* $\mathcal{N}^\times$ *also belongs to the class* $\mathcal{N}^\times$*.*

(ii) *Every geodetically closed sub near polygon of an element* $\mathcal{F}$ *of* $\mathcal{M}^\times$ *also belongs to the class* $\mathcal{M}^\times$*.*

Proof
Clearly (i) and (ii) hold for each of the near polygons $\mathbb{L}_3$, $\mathbb{E}_1$, $\mathbb{E}_3$, $Q^D(2n, 2)$, $H^D(2n-1, 4)$, $\mathbb{G}_n$, $\mathbb{H}_n$ and $\mathbb{I}_n$ ($n \geq 2$), and (ii) holds for each of the near hexagons $\mathbb{E}_1$ and $\mathbb{E}_2$. If the lemma holds for the near polygons $\mathcal{A}_1$ and $\mathcal{A}_2$, then it also holds for the near polygon $\mathcal{A}_1 \times \mathcal{A}_2$ and any glued near polygon of type $\mathcal{A}_1 \otimes \mathcal{A}_2$, by Lemma 2.3.1. □

Theorem 2.6.7
If $\mathcal{S}$ *is an element of* $\mathcal{D}$*, then* $\Omega_{\mathcal{D}}(\mathcal{S}) = \Delta_1(\mathcal{S})$*.*

Proof
Let $\{T_1, T_2\}$ denote an arbitrary element of $\Delta_1(\mathcal{S})$ and let $\mathcal{F}$ denote an arbitrary element of $T_1 \cup T_2$. By Lemma 2.6.6, $\mathcal{F}$ belongs to $\mathcal{N}^\times$. By Theorem 2.3.3 and Lemma 2.5.1, $\mathcal{F}$ is not a product near polygon. By Theorem 1.7.6 and Lemma 2.5.7, it then follows that $\mathcal{F}$ belongs to the class $\mathcal{D}$. This proves that $\Delta_1(\mathcal{S}) \subseteq \Omega_{\mathcal{D}}(\mathcal{S})$. The inclusion $\Omega_{\mathcal{D}}(\mathcal{S}) \subseteq \Delta_1(\mathcal{S})$ is trivial. □

Theorem 2.6.8
If $\mathcal{S}$ is an element of $\mathcal{C}$, then $\Omega_{\mathcal{C}}(\mathcal{S}) = \Delta_1(\mathcal{S})$.

Proof
The proof is similar to the proof of the previous Theorem. Let $\{T_1, T_2\}$ denote an arbitrary element of $\Delta_1(\mathcal{S})$ and let $\mathcal{F}$ denote an arbitrary element of $T_1 \cup T_2$. By Lemma 2.6.6, $\mathcal{F}$ belongs to $\mathcal{M}^{\times}$. By Theorem 2.3.3 and Lemma 2.5.1, $\mathcal{F}$ is not a product near polygon. By Theorem 1.7.6 and Lemma 2.5.7, it then follows that $\mathcal{F}$ belongs to the class $\mathcal{C}$. This proves that $\Delta_1(\mathcal{S}) \subseteq \Omega_{\mathcal{C}}(\mathcal{S})$. The inclusion $\Omega_{\mathcal{C}}(\mathcal{S}) \subseteq \Delta_1(\mathcal{S})$ is trivial. $\square$

2.6.3 Conjecture

As mentioned before, every known slim dense near polygon is isomorphic to an element of $\mathcal{M}^{\times}$. In [25], we introduced the following conjecture.

Conjecture. *Every slim dense near polygon is isomorphic to one of the elements of $\mathcal{M}^{\times}$.*

Chapter 3

Valuations of near polygons

The possible point-quad and line-quad relations were a very important tool in the classification of certain dense near polygons, see e.g. [4] and [15]. In this chapter we will study the possible relations between a point x and a geodetically closed sub near 2δ-gon $\mathcal{F}$, $\delta \geq 3$. The possible relations are described by the so-called *valuations* of $\mathcal{F}$. Valuations are an important tool in the classification of near polygons. We will use them in Chapter 4 to classify all slim dense near octagons.

3.1 Valuations

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote two geodetically closed sub near polygons of a dense near polygon $\mathcal{S}$ and put $d_i := \mathrm{diam}(\mathcal{F}_i)$, $i \in \{1, 2\}$. Depending on how the distances $\mathrm{d}(x_1, x_2)$ behave when x_1 and x_2 range over all elements of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, we will be able to say that $\mathcal{F}_1$ has a 'certain position' with respect to $\mathcal{F}_2$. Some cases have already been studied. For instance, in the case $(d_1, d_2) = (1, 1)$, we can distinguish two possible line-line relations (parallel lines and nonparallel lines, see Theorem 1.2.2), in the case $(d_1, d_2) = (0, 2)$, we can distinguish two possible point-quad relations (classical and ovoidal, see Theorem 1.2.5) and in the case $(d_1, d_2) = (1, 2)$ we can distinguish five possible line-quad relations (see Theorem 1.6.4). We will now have a closer look at the case $(d_1, d_2) = (0, \delta)$ where $\delta \geq 3$.

3.1.1 Definition and elementary properties

Definition
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near $2n$-gon. A function f from $\mathcal{P}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties (we call $f(x)$ the *value*

of x):

$\mathbf{V_1}$ there exists at least one point with value 0;

$\mathbf{V_2}$ every line L of $\mathcal{S}$ contains a unique point x_L with smallest value and $f(x) = f(x_L) + 1$ for every point x of L different from x_L;

$\mathbf{V_3}$ every point x of $\mathcal{S}$ is contained in a geodetically closed sub near polygon $\mathcal{F}_x$ that satisfies the following properties:

- $f(y) \leq f(x)$ for every point y of $\mathcal{F}_x$,
- every point z of $\mathcal{S}$ which is collinear with a point y of $\mathcal{F}_x$ and which satisfies $f(z) = f(y) - 1$ also belongs to $\mathcal{F}_x$.

Proposition 3.1.1
Let f be a valuation of a dense near $2n$-gon $\mathcal{S}$. Then the following holds:

(*i*) *for every two points x and y of $\mathcal{S}$, $|f(x) - f(y)| \leq d(x, y)$;*

(*ii*) *for every point x of $\mathcal{S}$, $f(x) \in \{0, \ldots, n\}$;*

(*iii*) *if x is a point with value 0 and if y is collinear with x, then $f(y) = 1$.*

Proof

(i) This follows from property $\mathbf{V_2}$.

(ii) This follows from (i) and property $\mathbf{V_1}$.

(iii) If $f(y)$ were equal to 0, then the line xy cannot contain a unique point with smallest value. $\square$

Proposition 3.1.2
Let f be a valuation of a dense near polygon $\mathcal{S}$. Then through every point x of $\mathcal{S}$, there exists exactly one geodetically closed sub near polygon $\mathcal{F}_x$ satisfying property $\mathbf{V_3}$.

Proof
By [6], a geodetically closed sub near polygon $\mathcal{F}$ through x is completely determined by the set of lines through x contained in $\mathcal{F}$. Now, by properties $\mathbf{V_2}$ and $\mathbf{V_3}$, a line through x belongs to $\mathcal{F}_x$ if and only if it contains a point with value $f(x) - 1$. This proves that there exists exactly one geodetically closed sub near polygon $\mathcal{F}_x$ satisfying property $\mathbf{V_3}$. $\square$

The following proposition says that the valuations of a dense near polygon $\mathcal{F}$ describe the possible relations between a point of a near polygon $\mathcal{S}$ and any geodetically closed sub near polygon of $\mathcal{S}$ isomorphic to $\mathcal{F}$. The valuations of $\mathcal{F}$ give information on how $\mathcal{F}$ can be embedded in a larger dense near polygon. That is the reason why these objects are important for classifying near polygons.

Proposition 3.1.3
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near $2n$-gon and let $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ be a geodetically closed sub near 2δ-gon of $\mathcal{S}$. For every point x of $\mathcal{S}$ and for every point y of $\mathcal{F}$, we define $f_x(y) := \mathrm{d}(x, y) - \mathrm{d}(x, \mathcal{F})$. Then $f_x : \mathcal{P}' \to \mathbb{N}$ is a valuation of $\mathcal{F}$ for every point x of $\mathcal{S}$.

Proof
Let y be a point of $\mathcal{F}$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \mathcal{F})$. Then $f_x(y) = 0$. Because every line of $\mathcal{F}$ contains a unique point nearest to x, also $\mathbf{V_2}$ is satisfied. For every $y \in \mathcal{F}$, we define $\mathcal{F}_y := \mathcal{C}(x, y) \cap \mathcal{F}$. If $z \in \mathcal{F}_y$ then $f_x(z) = \mathrm{d}(x, z) - \mathrm{d}(x, \mathcal{F}) \leq \mathrm{d}(x, y) - \mathrm{d}(x, \mathcal{F}) = f_x(y)$. If u is a point of $\mathcal{F}_y$ and if u' is a neighbour of u in $\mathcal{F}$ with value $f_x(u) - 1$, then $\mathrm{d}(x, u') = \mathrm{d}(x, u) - 1$, implying that $u' \in \mathcal{C}(x, u) \cap \mathcal{F} \subseteq \mathcal{C}(x, y) \cap \mathcal{F} = \mathcal{F}_y$. This shows that also $\mathbf{V_3}$ is satisfied. $\square$

We will now generalize Proposition 3.1.3, but first we need the following lemma.

Lemma 3.1.4
Let $\mathcal{S}$ be a dense near polygon and let $\mathcal{F}$ be a sub near polygon of $\mathcal{S}$ satisfying the following properties:

- *$\mathcal{F}$ is a subspace of $\mathcal{S}$;*
- *$\mathrm{d}_{\mathcal{F}}(x, y) = \mathrm{d}_{\mathcal{S}}(x, y)$ for all points x and y of $\mathcal{F}$.*

Then for every geodetically closed subspace $\mathcal{G}$ of $\mathcal{S}$, either $\mathcal{G} \cap \mathcal{F} = \emptyset$ or $\mathcal{G} \cap \mathcal{F}$ is a geodetically closed sub near polygon of $\mathcal{F}$.

Proof
Suppose that $\mathcal{G} \cap \mathcal{F} \neq \emptyset$. As intersection of two subspaces, $\mathcal{G} \cap \mathcal{F}$ is again a subspace. Let $a, b \in \mathcal{G} \cap \mathcal{F}$ and let c be a point of $\mathcal{F}$ collinear with b such that $\mathrm{d}_{\mathcal{F}}(a, c) = \mathrm{d}_{\mathcal{F}}(a, b) - 1$. Then $\mathrm{d}_{\mathcal{S}}(a, c) = \mathrm{d}_{\mathcal{S}}(a, b) - 1$ and so $c \in \mathcal{C}(a, b) \subseteq \mathcal{G}$. Hence, $c \in \mathcal{G} \cap \mathcal{F}$. This proves that $\mathcal{G} \cap \mathcal{F}$ is geodetically closed. $\square$

Proposition 3.1.5
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near $2n$-gon and let $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ be a sub near 2δ-gon of $\mathcal{S}$ which satisfies the following properties:

- *$\mathcal{F}$ is a dense near polygon;*
- *$\mathcal{F}$ is a subspace of $\mathcal{S}$;*
- *if x and y are two points of $\mathcal{F}$, then $d_{\mathcal{F}}(x,y) = d_{\mathcal{S}}(x,y)$.*

For every point x of $\mathcal{S}$ and for every point y of $\mathcal{F}$, we define $f_x(y) := d_{\mathcal{S}}(x,y) - d_{\mathcal{S}}(x,\mathcal{F})$. Then $f_x : \mathcal{P}' \to \mathbb{N}$ is a valuation of $\mathcal{F}$ for every point x of $\mathcal{S}$.

Proof
By Lemma 3.1.4, $\mathcal{C}(x,y) \cap \mathcal{F}$ is a geodetically closed subspace of $\mathcal{F}$ for every point x of $\mathcal{S}$ and every point y of $\mathcal{F}$. The proof is now completely similar to the proof of Proposition 3.1.3. □

Valuations of dense near 0-gons and dense near 2-gons are trivial objects. There is a unique point with value 0 and all other points (in the case of near 2-gons) have value 1. In the following section we will show that there are two possible types of valuations in dense generalized quadrangles, corresponding with the two possible point-quad relations (see Theorem 1.2.5).

3.1.2 Classical and ovoidal valuations

Proposition 3.1.6
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near $2n$-gon.

(a) *If y is a point of $\mathcal{S}$, then $f_y : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x,y)$ is a valuation of $\mathcal{S}$.*

(b) *If $\mathcal{O}$ is an ovoid of $\mathcal{S}$, then $f_{\mathcal{O}} : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x,\mathcal{O})$ is a valuation of $\mathcal{S}$.*

Proof
In both cases, $\mathbf{V_1}$ and $\mathbf{V_2}$ are satisfied. In case (a), we put $\mathcal{F}_x := \mathcal{C}(x,y)$. In case ($b$), we put $\mathcal{F}_x := \{x\}$ if $x \in \mathcal{O}$ and $\mathcal{F}_x := \mathcal{S}$ otherwise. For these choices of $\mathcal{F}_x$, also $\mathbf{V_3}$ holds. □

Definition
A valuation of $\mathcal{S}$ is called *classical*, respectively *ovoidal*, if it is obtained like in (a), respectively (b), of the previous proposition. Classical and ovoidal valuations can be characterized as follows.

Proposition 3.1.7
Let f be a valuation of a dense near $2n$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ with $n \geq 1$. Then

- $\max\{f(u)|u \in \mathcal{P}\} \leq n$ *with equality if and only if f is classical;*
- $\max\{f(u)|u \in \mathcal{P}\} \geq 1$ *with equality if and only if f is ovoidal.*

Proof
Obviously, the above inequalities hold and become equalities if f is classical, respectively ovoidal.

(a) Suppose that $\max\{f(u)|u \in \mathcal{P}\} = n$. Let x be a point of $\mathcal{S}$ with value 0 and let y be a point with value n. By Proposition 3.1.1, $\mathrm{d}(x, y) = n$. Let y' be an arbitrary point of $\Gamma_n(x) \cap \Gamma_1(y)$ and let y'' denote the unique point of the line yy' at distance $n-1$ from x. By Proposition 3.1.1, it follows that $f(y'') = f(y'') - f(x) \leq n-1$ and that $f(y'') = f(y) + f(y'') - f(y) \geq n-1$. Hence, $f(y'') = n-1$ and by property $\mathbf{V_2}$ it then follows that $f(y') = n$. So, every point of $\Gamma_n(x) \cap \Gamma_1(y)$ has value n. By the connectedness of $\Gamma_n(x)$ (Theorem 1.6.3), it then follows that every point of $\Gamma_n(x)$ has value n. Now, let z be an arbitrary point of $\mathcal{S}$. Then, by [6], there exists a path of length $n - \mathrm{d}(x, z)$ between z and a point z' of $\Gamma_n(x)$. From $\mathrm{d}(x, z) \geq |f(z) - f(x)| = f(z)$ and $n - f(z) = |f(z') - f(z)| \leq \mathrm{d}(z, z') = n - \mathrm{d}(x, z)$, it follows that $f(z) = \mathrm{d}(x, z)$. This proves that f is classical.

(b) Suppose now that $\max\{f(x)|x \in \mathcal{P}\} = 1$. By property $\mathbf{V_2}$, every line of $\mathcal{S}$ contains a unique point with value 0. So, the points with value 0 determine an ovoid of $\mathcal{S}$ and f is ovoidal. □

Corollary 3.1.8
Every valuation of a dense generalized quadrangle is classical or ovoidal.

Any valuation of a dense near polygon $\mathcal{S}$ induces a valuation in every geodetically closed sub near polygon of $\mathcal{S}$.

Proposition 3.1.9
Let $\mathcal{S}$ be a dense near polygon and let $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ be a sub near polygon of $\mathcal{S}$ which satisfies the following properties:

- *$\mathcal{F}$ is a dense near polygon;*
- *$\mathcal{F}$ is a subspace of $\mathcal{S}$;*
- *if x and y are two points of $\mathcal{F}$ in $\mathcal{S}$, then $d_{\mathcal{F}}(x, y) = d_{\mathcal{S}}(x, y)$.*

Let f denote a valuation of $\mathcal{S}$ and put $m := \min\{f(x)|x \in \mathcal{P}'\}$. Then the map $f_{\mathcal{F}} : \mathcal{P}' \to \mathbb{N};\ x \mapsto f(x) - m$ is a valuation of $\mathcal{F}$.

Proof

For every point x of $\mathcal{S}$, let $\mathcal{F}_x$ denote the unique geodetically closed sub near polygon of $\mathcal{S}$ for which $\mathbf{V_3}$ holds (with respect to the valuation f). By Lemma 3.1.4, $\mathcal{F}_x \cap \mathcal{F}$ is a geodetically closed sub near polygon of $\mathcal{F}$ for every point x of $\mathcal{F}$. Clearly, $f_{\mathcal{F}}$ satisfies properties $\mathbf{V_1}$ and $\mathbf{V_2}$. The map $f_{\mathcal{F}}$ also satisfies $\mathbf{V_3}$ if for every point x of $\mathcal{F}$, one takes $\mathcal{F}'_x := \mathcal{F}_x \cap \mathcal{F}$ as geodetically closed sub near polygon through x. □

Definition

We call $f_{\mathcal{F}}$ an *induced valuation.*

Proposition 3.1.10

Let f be a valuation of a dense near polygon $\mathcal{S}$.

(a) *If every induced quad valuation is classical, then the valuation f itself is classical.*

(b) *If every induced quad valuation is ovoidal, then the valuation f itself is ovoidal.*

Proof

(a) Suppose that f is a nonclassical valuation of $\mathcal{S}$. Let x denote an arbitrary point with value 0 and let i be the smallest nonnegative integer for which there exists a point y satisfying $i = \mathrm{d}(x, y) \neq f(y)$. Obviously, $i \geq 2$. Choose points $y' \in \Gamma_1(y) \cap \Gamma_{i-1}(x)$ and $y'' \in \Gamma_1(y') \cap \Gamma_{i-2}(x)$. Then $f(y'') = i - 2$, $f(y') = i - 1$ and $f(y) \in \{i - 1, i - 2\}$. Every point of $\mathcal{Q} := \mathcal{C}(y, y'')$ collinear with y'' has distance $i - 1$ from x and hence has value $i - 1$. Since the valuation induced in $\mathcal{Q}$ is classical, y'' is the unique point of $\mathcal{Q}$ with smallest value and $f(y) = f(y'') + \mathrm{d}(y'', y) = i - 2 + 2 = i$, a contradiction.

(b) Suppose that f is a valuation of $\mathcal{S}$ which is not ovoidal. Let x denote an arbitrary point with value 0 and let i be the smallest nonnegative integer for which there exists a point y satisfying $i = \mathrm{d}(x, y)$ and $f(y) \geq 2$. Obviously, $i \geq 2$. Choose points $y' \in \Gamma_1(y) \cap \Gamma_{i-1}(x)$ and $y'' \in \Gamma_1(y') \cap \Gamma_{i-2}(x)$. Clearly every point of the line through y' and y'' has value 0 or 1. But then the valuation induced in the quad $\mathcal{C}(y, y'')$ cannot be ovoidal, a contradiction. □

3.1.3 The partial linear space G_f

For every valuation f of $\mathcal{S}$, put $O_f := \{x \in \mathcal{S} | f(x) = 0\}$. If $x, y \in O_f$, then by Proposition 3.1.1, $\mathrm{d}(x, y) \geq 2$. Every quad of $\mathcal{S}$ containing at least two points of O_f is called a *special quad*. Let G_f be the partial linear space whose points are the points of O_f, whose lines are the special quads of $\mathcal{S}$ and with natural incidence. If x and y are two collinear points of G_f, then the line of G_f through x and y corresponds with an ovoid in the quad of $\mathcal{S}$ through x and y. As a corollary, every line of G_f contains at least 3 points.

Proposition 3.1.11
Let f be a valuation of a dense near polygon $\mathcal{S}$ and let x be a point of $\mathcal{S}$. If $\mathrm{d}(x, O_f) \leq 2$, then $f(x) = \mathrm{d}(x, O_f)$.

Proof
Obviously, this holds if $\mathrm{d}(x, O_f) \leq 1$. So, suppose that $\mathrm{d}(x, O_f) = 2$ and let x' denote a point of O_f at distance 2 from x. If the valuation induced in the quad $\mathcal{C}(x, x')$ is ovoidal, then x is collinear with a point of $O_f \cap \mathcal{C}(x, x')$, a contradiction. So, the valuation induced in $\mathcal{C}(x, x')$ is classical and $f(x) = f(x') + \mathrm{d}(x, x') = 2$. $\square$

3.1.4 A property of valuations

Proposition 3.1.12 (Theorem 1 of [6]**)**
Suppose $\mathcal{S}$ is a near polygon with the property that every two points at distance 2 have at least two common neighbours. If $k \geq 2$ different line sizes occur in $\mathcal{S}$, then $\mathcal{S}$ is isomorphic to a direct product of k near polygons each of which has constant line size.

Corollary 3.1.13
If a dense near polygon $\mathcal{S}$ has lines of size $s + 1$, then $\mathcal{S}$ has a partition in isomorphic geodetically closed sub near polygons of order (s, t') for some $t' \geq 0$.

Let $\mathcal{S}$ be a dense near $2n$-gon and let f be a valuation of $\mathcal{S}$. For every $i \in \mathbb{N}$, we define m_i as the number of points of $\mathcal{S}$ with value i. Obviously, $m_i = 0$ if $i \geq n + 1$.

Proposition 3.1.14
If $\mathcal{S}$ contains lines of size $s + 1$, then $\sum_{i=0}^{\infty} \frac{m_i}{(-s)^i} = 0$.

Proof
(a) Suppose first that $\mathcal{S}$ has order (s, t). For every line L of $\mathcal{S}$,

$$\sum_{x\in L}\frac{1}{(-s)^{f(x)}}=\frac{1}{(-s)^{f(x_L)}}+s\frac{1}{(-s)^{f(x_L)+1}}=0.$$

Hence

$$\begin{aligned} 0 &= \sum_{L\in\mathcal{L}}\sum_{x\in L}\frac{1}{(-s)^{f(x)}} \\ &= \sum_{x\in\mathcal{P}}\sum_{L\mathbf{I}x}\frac{1}{(-s)^{f(x)}} \\ &= (t+1)\sum_{x\in\mathcal{P}}\frac{1}{(-s)^{f(x)}} \\ &= (t+1)\sum_{i=0}^{\infty}\frac{m_i}{(-s)^i}. \end{aligned}$$

This shows that the proposition holds if $\mathcal{S}$ has an order.

(b) Suppose next that not every line of $\mathcal{S}$ is incident with the same number of points, then by Corollary 3.1.13, $\mathcal{S}$ has a partition in isomorphic geodetically closed sub near polygons of order (s,t') for some $t'\geq 0$. By (a), the proposition holds for each valuation induced in one of the sub near polygons of the partition. If we add all obtained equations (after multiplying with a suitable power of $-s$), then we obtain the required equation. □

Corollary 3.1.15
Let f be a valuation of a dense near polygon $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathbf{I})$. If k different line sizes $s_1+1,\ldots,s_k+1$ occur in $\mathcal{S}$, then $\max\{f(x)\,|\,x\in\mathcal{P}\}\geq k$.

Proof
Put $m:=\max\{f(x)\,|\,x\in\mathcal{P}\}$. By Proposition 3.1.14, the polynomial $p(s):=\sum_{i=0}^{m}m_i(-s)^{m-i}=0$ has at least k different roots. Hence, $k\leq\deg(f(s))=m$. □

3.2 Some classes of valuations

In Section 3.1.2, we discussed the classical and ovoidal valuations. We will now define several other types of valuations.

3.2.1 Product valuations

Proposition 3.2.1
Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathbf{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathbf{I}_2)$ be two dense near polygons. If f_i, $i \in \{1,2\}$, is a valuation of $\mathcal{S}_i$, then the map $f : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{N}$; $(x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$ is a valuation of $\mathcal{S}_1 \times \mathcal{S}_2$.

Proof
If x_i, $i \in \{1,2\}$, is a point of $\mathcal{S}_i$ for which $f_i(x_i) = 0$, then $f[(x_1, x_2)] = 0$. This proves property $\mathbf{V_1}$. If L is a line of $\mathcal{S}_1 \times \mathcal{S}_2$, then without loss of generality, we may suppose that L is of the form $K \times \{y\}$, with K a line of $\mathcal{S}_1$ and y a point of $\mathcal{S}_2$. Now, $f[(k,y)] = f_1(k) + f_2(y)$ for every point k of K. Property $\mathbf{V_2}$ now immediately follows: the unique point of L with smallest f-value is the point (x_K, y), where x_K denotes the unique point of K with smallest f_1-value. It remains to check property $\mathbf{V_3}$. For every point x_i, $i \in \{1,2\}$, of $\mathcal{S}_i$, let $\mathcal{F}_{x_i}$ denote the sub near polygon of $\mathcal{S}_i$ satisfying $\mathbf{V_3}$. For every point (x_1, x_2) of $\mathcal{S}_1 \times \mathcal{S}_2$, we define $\mathcal{F}_{(x_1,x_2)} := \{(a_1, a_2) \,|\, a_1 \in \mathcal{F}_{x_1} \text{ and } a_2 \in \mathcal{F}_{x_2}\}$. If (a_1, a_2) is a point of $\mathcal{F}_{(x_1,x_2)}$, then $f[(a_1, a_2)] = f_1(a_1) + f_2(a_2) \leq f_1(x_1) + f_2(x_2) = f[(x_1, x_2)]$. If (a_1, a_2) is a point of $\mathcal{F}_{(x_1,x_2)}$ and if (b_1, b_2) is a point of $\mathcal{S}_1 \times \mathcal{S}_2$ collinear with (a_1, a_2) and satisfying $f[(b_1, b_2)] = f(a_1, a_2) - 1$, then without loss of generality, we may suppose that $a_2 = b_2$ and $a_1 \sim b_1$ (in $\mathcal{S}_1$). Then $f_1(b_1) = f[(b_1, b_2)] - f_2(b_2) = f[(a_1, a_2)] - 1 - f_2(a_2) = f_1(a_1) - 1$. Since $a_1 \in \mathcal{F}_{x_1}$, also the point b_1 belongs to $\mathcal{F}_{x_1}$. Hence, the point (b_1, b_2) belongs to $\mathcal{F}_{(x_1,x_2)}$. This proves property $\mathbf{V_3}$. $\square$

Definition
Any valuation of a product near polygon which is obtained in the above-described way is called a *product valuation*.

3.2.2 Semi-classical valuations

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near $2n$-gon, $n \geq 2$, let x be a point of $\mathcal{S}$ and let $\mathcal{A}$ be the distance n-geometry determined by x (i.e. the points of $\mathcal{A}$ are the points of $\mathcal{S}$ at distance n from x, the lines of $\mathcal{A}$ are the lines of $\mathcal{S}$ at distance $n-1$ from x and incidence is the one derived from $\mathcal{S}$). By Theorem 1.6.3, $\mathcal{A}$ is connected. Suppose now that $\mathcal{A}$ has an ovoid O. Then we can define the following function $f_{x,O} : \mathcal{P} \to \mathbb{N}$: if y is a point of $\mathcal{S}$ at distance at most $n-1$ from x, then we define $f_{x,O}(y) := \mathrm{d}(x,y)$; if y is a point of $\mathcal{S}$ at distance n from x, then we define $f_{x,O}(y) := n-2$ if $y \in O$ and $f_{x,O}(y) := n-1$ otherwise.

Proposition 3.2.2
The map $f_{x,O}$ is a valuation of $\mathcal{S}$.

Proof

Since $f_{x,O}(x) = 0$, property $\mathbf{V_1}$ holds. Now, let L be an arbitrary line of $\mathcal{S}$. If $\mathrm{d}(x, L) \leq n-2$, then the unique point on L nearest to x is also the unique point on L with smallest value. If $\mathrm{d}(x, L) = n-1$, then the unique point of O on L is the unique point of L with smallest value. This proves property $\mathbf{V_2}$. Now, property $\mathbf{V_3}$ also holds if we make the following choices for $\mathcal{F}_y$, $y \in \mathcal{P}$: we put $\mathcal{F}_y := \mathcal{C}(x, y)$ if $\mathrm{d}(x, y) \leq n-2$; $\mathcal{F}_y := \{y\}$ if $y \in O$ and $\mathcal{F}_y := \mathcal{S}$ otherwise. □

Definition

Any valuation which can be obtained in the above-mentioned way is called a *semi-classical valuation*. A semi-classical valuation of a generalized quadrangle is just an ovoidal valuation.

Proposition 3.2.3

If f is a valuation of a dense near $2n$-gon and if x is a point of $\mathcal{S}$ such that $f(y) = \mathrm{d}(x, y)$ for every point y at distance at most $n-1$ from y, then f is either classical or semi-classical.

Proof

Suppose that f is not classical and consider a point $z \in \Gamma_n(x)$. Every point of $\Gamma_1(z) \cap \Gamma_{n-1}(x)$ has value $n-1$. Hence by property $\mathbf{V_2}$ and Proposition 3.1.7, $f(z) \in \{n-2, n-1\}$. By property $\mathbf{V_2}$, it now follows that the points of $\Gamma_n(x)$ with value $n-2$ form an ovoid in the distance n-geometry determined by the point x. This proves that f is semi-classical. □

Proposition 3.2.4

Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 2$, of order $(2, t)$ and let x be a point of $\mathcal{S}$. Then there exists a semi-classical valuation f with $f(x) = 0$ if and only if $\Gamma_n(x)$ is bipartite. In that case, there are precisely two semi-classical valuations f such that $f(x) = 0$.

Proof

Let $\mathcal{A}$ denote the distance n-geometry determined by the point x. Then every line of $\mathcal{A}$ contains two points. Clearly, $\mathcal{A}$ has ovoids if and only if the graph induced by $\Gamma_n(x)$ is bipartite. □

3.2.3 Extended valuations

Lemma 3.2.5

If x_1 and x_2 are collinear points of $\mathcal{S}$ such that $\mathrm{d}(x_1, \mathcal{F}) = \mathrm{d}(x_2, \mathcal{F}) - 1$, then $\pi_{\mathcal{F}}(x_1) = \pi_{\mathcal{F}}(x_2)$.

Proof
The point $\pi_{\mathcal{F}}(x_1)$ has distance at most $\mathrm{d}(x_1, \pi_{\mathcal{F}}(x_1)) + \mathrm{d}(x_1, x_2) = \mathrm{d}(x_1, \mathcal{F}) + 1 = \mathrm{d}(x_2, \mathcal{F})$ from x_2 and hence coincides with $\pi_{\mathcal{F}}(x_2)$. $\square$

Lemma 3.2.6
Let $\mathcal{S}$ be a dense near polygon, let K be a line of $\mathcal{S}$ and let $\mathcal{F}$ denote a classical geodetically closed sub near polygon of $\mathcal{S}$. Then one of the following holds.

- *Every point of K has the same distance from $\mathcal{F}$. In this case $\pi_{\mathcal{F}}(K) := \{\pi_{\mathcal{F}}(x) \mid x \in K\}$ is a line of $\mathcal{F}$.*
- *There exists a unique point on K nearest to $\mathcal{F}$. In this case all points $\pi_{\mathcal{F}}(x)$, $x \in K$, are equal.*

Proof
Suppose $\{\pi_{\mathcal{F}}(x) | x \in K\} = u$. Then there exists a unique point on K nearest to $\mathcal{F}$, namely the unique point of K nearest to u. Suppose therefore that there exist points $x_1, x_2 \in K$ such that $\pi_{\mathcal{F}}(x_1) \neq \pi_{\mathcal{F}}(x_2)$. By Lemma 3.2.5, $\mathrm{d}(x_1, \mathcal{F}) = \mathrm{d}(x_2, \mathcal{F})$. Put $i := \mathrm{d}(x_1, \mathcal{F})$. Since

$$\begin{aligned}
\mathrm{d}(\pi_{\mathcal{F}}(x_1), \pi_{\mathcal{F}}(x_2)) &= \mathrm{d}(x_1, \pi_{\mathcal{F}}(x_2)) - \mathrm{d}(x_1, \pi_{\mathcal{F}}(x_1)) \\
&\leq \mathrm{d}(x_1, x_2) + \mathrm{d}(x_2, \pi_{\mathcal{F}}(x_2)) - \mathrm{d}(x_1, \pi_{\mathcal{F}}(x_1)) \\
&= 1,
\end{aligned}$$

$\pi_{\mathcal{F}}(x_1)$ and $\pi_{\mathcal{F}}(x_2)$ are contained in a line K'. If u is a point of K different from x_1 and x_2, then u has distance at most $i+1$ from the points $\pi_{\mathcal{F}}(x_1)$ and $\pi_{\mathcal{F}}(x_2)$ of K'. Hence there exists a point u' on K' at distance at most i from u. By Lemma 3.2.5, it follows that $\mathrm{d}(u, \mathcal{F}) = i$ and $\pi_{\mathcal{F}}(u) = u'$. This proves that $\pi_{\mathcal{F}}(K) \subseteq K'$ and that every point of K has the same distance i from $\mathcal{F}$. Suppose now that there exists a point u' in $K' \setminus \pi_{\mathcal{F}}(K)$. Then u' has distance at most $i+1$ from at least two points of K. Hence u' has distance at most i to a point u of K, showing that $u' = \pi_{\mathcal{F}}(u)$, a contradiction. $\square$

Proposition 3.2.7
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near $2n$-gon, let $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ be a classical geodetically closed sub near polygon of $\mathcal{S}$ and let f' denote a valuation of $\mathcal{F}$. Then the map $f : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x, \pi_{\mathcal{F}}(x)) + f'(\pi_{\mathcal{F}}(x))$, is a valuation of $\mathcal{S}$. If f' is a classical valuation, then also f is classical.

Proof
Obviously, property $\mathbf{V_1}$ is satisfied. By Lemma 3.2.6, it easily follows that also property $\mathbf{V_2}$ is satisfied. For every point x of $\mathcal{S}$, we define $\mathcal{F}_x := \mathcal{C}(x, \mathcal{G}_x)$,

where $\mathcal{G}_x$ denotes the unique geodetically closed sub near polygon of $\mathcal{F}$ through $\pi_{\mathcal{F}}(x)$ satisfying property $\mathbf{V_3}$ with respect to the valuation f' of $\mathcal{F}$. Then $\mathcal{F}_x$ has the following properties.

- $\mathcal{F}_x \cap \mathcal{F} = \mathcal{G}_x$.

  Obviously, $\mathcal{G}_x \subseteq \mathcal{F}_x \cap \mathcal{F}$. If y is a point of $\mathcal{G}_x$ at distance $\mathrm{diam}(\mathcal{G}_x)$ from $\pi_{\mathcal{F}}(x)$, then since $\pi_{\mathcal{F}}(x)$ is contained in a shortest path between x and y, $\mathcal{G}_x = \mathcal{C}(\pi_{\mathcal{F}}(x), y)$ is contained in $\mathcal{C}(x, y)$. Hence, $\mathcal{F}_x$ is equal to $\mathcal{C}(x, y)$ and has diameter $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{diam}(\mathcal{G}_x)$. Suppose that there exists a point z in $\mathcal{F}_x \cap \mathcal{F}$ not contained in $\mathcal{G}_x$. Then $\mathcal{C}(z, \mathcal{G}_x)$ has diameter at least $\mathrm{diam}(\mathcal{G}_x)+1$. As before we have that $\mathcal{C}(x, \mathcal{C}(z, \mathcal{G}_x))$ has diameter $\mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{diam}(\mathcal{C}(\mathcal{G}_x, z)) \geq \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{diam}(\mathcal{G}_x) + 1 = \mathrm{diam}(\mathcal{F}_x)+1$, a contradiction, since $\mathcal{F}_x = \mathcal{C}(x, z, \mathcal{G}_x)$. As a consequence, $\mathcal{F}_x \cap \mathcal{F} = \mathcal{G}_x$.

- *For every $y \in \mathcal{F}_x$, $\pi_{\mathcal{F}}(y) \in \mathcal{G}_x$.*

  Clearly every shortest path between y and a point $z \in \mathcal{G}_x$ is contained in $\mathcal{F}_x$. Since the point $\pi_{\mathcal{F}}(y)$ is contained in a shortest path between y and z, the point $\pi_{\mathcal{F}}(y)$ belongs to $\mathcal{F}_x \cap \mathcal{F} = \mathcal{G}_x$.

- *For every point y of $\mathcal{F}_x$, $d(y, \pi_{\mathcal{F}}(y)) \leq d(x, \pi_{\mathcal{F}}(x))$.*

  As before, $\mathcal{C}(y, \mathcal{G}_x)$ has diameter $\mathrm{d}(y, \pi_{\mathcal{F}}(y)) + \mathrm{diam}(\mathcal{G}_x)$. Since $\mathcal{C}(y, \mathcal{G}_x) \subseteq \mathcal{C}(x, \mathcal{G}_x)$, it follows that $\mathrm{d}(y, \pi_{\mathcal{F}}(y)) + \mathrm{diam}(\mathcal{G}_x) \leq \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{diam}(\mathcal{G}_x)$.

Let u be a point of $\mathcal{F}_x$. Since $\pi_{\mathcal{F}}(u) \in \mathcal{G}_x$, $f'(\pi_{\mathcal{F}}(u)) \leq f'(\pi_{\mathcal{F}}(x))$. Hence, $f(u) = \mathrm{d}(u, \pi_{\mathcal{F}}(u)) + f'(\pi_{\mathcal{F}}(u)) \leq \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + f'(\pi_{\mathcal{F}}(x)) = f(x)$. Let v be a neighbour of u with value $f(u) - 1$. In order to prove property $\mathbf{V_3}$, we distinguish two possibilities.

- $\mathrm{d}(v, \pi_{\mathcal{F}}(v)) \neq \mathrm{d}(u, \pi_{\mathcal{F}}(u))$. Then $\pi_{\mathcal{F}}(u) = \pi_{\mathcal{F}}(v)$ by Lemma 3.2.5. In this case we have $\mathrm{d}(v, \pi_{\mathcal{F}}(v)) = \mathrm{d}(u, \pi_{\mathcal{F}}(u)) - 1$. So, v is on a shortest path between u and $\pi_{\mathcal{F}}(u) = \pi_{\mathcal{F}}(v)$. Since $u, \pi_{\mathcal{F}}(u) \in \mathcal{F}_x$, also v belongs to $\mathcal{F}_x$.

- $\mathrm{d}(v, \pi_{\mathcal{F}}(v)) = \mathrm{d}(u, \pi_{\mathcal{F}}(u))$. In this case we have $f'(\pi_{\mathcal{F}}(v)) = f'(\pi_{\mathcal{F}}(u)) - 1$. By Lemma 3.2.6, $\mathrm{d}(\pi_{\mathcal{F}}(u), \pi_{\mathcal{F}}(v)) = 1$. So, $\pi_{\mathcal{F}}(v) \in \mathcal{G}_{\pi_{\mathcal{F}}(u)} \subseteq \mathcal{G}_x$. Now, v lies on a shortest path between $\pi_{\mathcal{F}}(v)$ and u. Since $\pi_{\mathcal{F}}(v) \in \mathcal{F}_x$ and $u \in \mathcal{F}_x$, also v belongs to $\mathcal{F}_x$.

If f' is a classical valuation of $\mathcal{F}$, then $f(x) = \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + f'(\pi_{\mathcal{F}}(x)) = \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{d}(\pi_{\mathcal{F}}(x), x^*) = \mathrm{d}(x, x^*)$, where x^* denotes the unique point of $\mathcal{F}$ for which $f'(x^*) = 0$. Hence f is classical if f' is classical. $\square$

Definition
The valuation f is called an *extension* of f'. If $\mathcal{F} = \mathcal{S}$, then $f = f'$ and we say that f is the *trivial extension* (of f'). We call a valuation *extended* if it is a nontrivial extension of another valuation.

3.2.4 Distance-j-ovoidal valuations

Distance-j-ovoids of generalized $2n$-gons have been considered at several places in the literature (see e.g. [35] and [46]). We will give the definition in the more general setting of near polygons.

Definition
Let $\mathcal{S}$ be a near $2n$-gon, $n \geq 2$. A *distance-j-ovoid* $(2 \leq j \leq n)$ of $\mathcal{S}$ is a set X of points satisfying:

(1) $\mathrm{d}(x, y) \geq j$ for every two different points x and y of X;

(2) for every point a of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(a, x) \leq \frac{j}{2}$;

(3) for every line L of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(L, x) \leq \frac{j-1}{2}$.

A distance-2-ovoid is just an ovoid. From (1), (2) and (3), we immediately have the following.

- If j is odd, then for every point a of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(a, x) \leq \frac{j-1}{2}$.
- If j is even, then for every line L of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(L, x) \leq \frac{j-2}{2}$.

Proposition 3.2.8
If X is a distance-j-ovoid of a dense near $2n$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ with $2 \leq j \leq n$ and j even, then the map $f : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x, X)$ is a valuation of $\mathcal{S}$.

Proof
Since $f(x) = 0$ for every point $x \in X$, property $\mathbf{V_1}$ holds.
Let L be a line of $\mathcal{S}$. Then there exists a unique point $x^* \in X$ such that $\mathrm{d}(x^*, L) \leq \frac{j-2}{2} = \frac{j}{2} - 1$. So, $\mathrm{d}(a, x^*) \leq \frac{j}{2}$ for every point a of L. By property

(1), $\mathrm{d}(a, X) = \mathrm{d}(a, x^*)$ for every point a of L. It is now easy to see that property $\mathbf{V_2}$ holds: the point x_L is the unique point of L nearest to x^*.
Let x denote an arbitrary point of $\mathcal{S}$. If $\mathrm{d}(x, X) = \frac{j}{2}$, then we define $\mathcal{F}_x := \mathcal{S}$. If $\mathrm{d}(x, X) < \frac{j}{2}$, then by property (1), there exists a unique point $x' \in X$ at distance $\mathrm{d}(x, X)$ from x and we define $\mathcal{F}_x := \mathcal{C}(x, x')$. Clearly, property $\mathbf{V_3}$ holds for any point x for which $\mathrm{d}(x, X) = \frac{j}{2}$. Suppose therefore that $\mathrm{d}(x, X) < \frac{j}{2}$ and let x' denote the unique point of X at distance $\mathrm{d}(x, X)$ from x. Then for every point y of $\mathcal{F}_x$, $\mathrm{d}(y, x') \leq \mathrm{d}(x, x') < \frac{j}{2}$. So, $f(y) = \mathrm{d}(y, x') \leq f(x)$. Now, let y be a point of $\mathcal{F}_x$ and let z be a point of $\mathcal{S}$ collinear with y such that $f(z) = f(y) - 1$. Then there exists a point $x'' \in X$ such that $\mathrm{d}(z, x'') = \mathrm{d}(y, x') - 1$. Suppose that $z \notin \mathcal{F}_x$, then $\mathrm{d}(z, x') = 1 + \mathrm{d}(y, x')$ by Theorem 1.6.3. So, $x' \neq x''$. Since $\mathrm{d}(y, x') \leq \frac{j}{2} - 1$, $\mathrm{d}(y, z) = 1$ and $\mathrm{d}(z, x'') = \mathrm{d}(y, x') - 1 \leq \frac{j}{2} - 2$, there exists a path of length $j - 2$ between x' and x'', contradicting property (1). Hence, the point z belongs to $\mathcal{F}_x$. This proves that also $\mathbf{V_3}$ holds. $\square$

Definition
Any valuation f which can be obtained in the above mentioned way is called an *distance-j-ovoidal valuation*. A distance-2-ovoidal valuation is the same as an ovoidal valuation.

3.2.5 SDPS-valuations

Definition
Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be one of the following classical near $4n$-gons:

(a) a point ($n = 0$);

(b) a dense generalized quadrangle ($n = 1$);

(c) $W^D(4n - 1, q)$ with $n \geq 2$;

(d) $[Q^-(4n + 1, q)]^D$ with $n \geq 2$.

A subset X of $\mathcal{P}$ is called an *SDPS-set* (SDPS = sub dual polar space) of $\mathcal{A}$ if it satisfies the following properties.

(1) No two points of X are collinear in $\mathcal{A}$.

(2) If $x, y \in X$ such that $\mathrm{d}(x, y) = 2$, then $X \cap \mathcal{C}(x, y)$ is an ovoid of the quad $\mathcal{C}(x, y)$.

(3) The point-line incidence structure $\tilde{\mathcal{A}}$ whose points are the elements of X and whose lines are quads of $\mathcal{A}$ containing at least two points of X (natural incidence) is isomorphic to one of the following near $2n$-gons:

- case (a): a point;
- case (b): a line of size at least 2;
- case (c): $W^D(2n-1, q^2)$;
- case (d): $H^D(2n, q^2)$.

(4) For all $x, y \in X$, $\mathrm{d}(x, y) = 2 \cdot \delta(x, y)$, where $\delta(\cdot, \cdot)$ denotes the distance in $\tilde{\mathcal{A}}$.

Remark
An SDPS-set of the near 0-gon consists of the unique point of the near 0-gon. An SDPS-set of a generalized quadrangle is just an ovoid of that generalized quadrangle. SDPS-sets of type (c) and (d) also exist, see Section 9 (ii)+(iii) of [39] or Section 1.3 of [37].

Proposition 3.2.9 (Section 3.3**)**
If X is an SDPS-set of the near $4n$-gon $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$, then the map $f : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x, X)$ is a valuation of $\mathcal{A}$.

Definition
Any valuation f which can be obtained in the above mentioned way is called an *SDPS-valuation.*

3.3 Proof of Proposition 3.2.9

Before we prove Proposition 3.2.9, we will first derive some nice properties of SDPS-sets. We will use the same notations as in Section 3.2.5. So, we suppose that X is an SDPS-set of $\mathcal{A}$, where $\mathcal{A}$ is one of the classical near polygons mentioned in the definition. We call a quad of $\mathcal{A}$ *special* if it contains at least two points of X.

Lemma 3.3.1
Suppose $n \geq 2$. If (s, t_2), respectively $(\tilde{s}, \tilde{t}_2)$, denotes the order of the quads of $\mathcal{A}$, respectively $\tilde{\mathcal{A}}$, then $(\tilde{s}, \tilde{t}_2) = (st_2, t_2^2)$.

Proof

In case (c), we have $(s, t_2) = (q, q)$ and $(\tilde{s}, \tilde{t}_2) = (q^2, q^2)$. In case (d), we have $(s, t_2) = (q^2, q)$ and $(\tilde{s}, \tilde{t}_2) = (q^3, q^2)$. So, $(\tilde{s}, \tilde{t}_2) = (st_2, t_2^2)$ in any case. $\square$

Lemma 3.3.2

No two special quads intersect in a line.

Proof

Suppose the contrary. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ denote two special quads which intersect in a line L. Let x denote the unique point of X on L and let x_1 and x_2 denote two other points of L. Let y_i, $i \in \{1, 2\}$, denote a point of $(\mathcal{Q}_i \cap X) \setminus \{x\}$ collinear with x_i. Then $\mathrm{d}(y_1, y_2) = \mathrm{d}(y_1, x_1) + \mathrm{d}(x_1, y_2) = 1 + 2 = 3$, contradicting property (4) in the definition of SDPS-set. $\square$

Lemma 3.3.3

If $x \in X$, then every line of $\mathcal{A}$ through x is contained in a unique special quad.

Proof

Obviously, the lemma holds if $n \leq 1$. Suppose therefore that $n \geq 2$. Put $t = t_2 + t_2^2 + \cdots + t_2^{2n-1}$ and $\tilde{t} = \tilde{t}_2 + \tilde{t}_2^2 + \cdots + \tilde{t}_2^{n-1}$. Then $\mathcal{A}$ has order (s, t) and $\tilde{\mathcal{A}}$ has order $(\tilde{s}, \tilde{t})$. By Lemma 3.3.2, the number of lines through x which are contained in a special quad is equal to $(t_2 + 1)(\tilde{t} + 1) = (t_2 + 1)(1 + \tilde{t}_2 + \tilde{t}_2^2 + \cdots + \tilde{t}_2^{n-1}) = (t_2 + 1)(1 + t_2^2 + \cdots + t_2^{2n-2}) = 1 + t_2 + \ldots + t_2^{2n-1} = t + 1$. This proves the lemma. $\square$

Lemma 3.3.4

Let $\mathcal{F}$ be a geodetically closed sub near polygon of $\mathcal{A}$. Then $\mathcal{F} \cap X$ is either empty or a geodetically closed subspace of $\tilde{\mathcal{A}}$.

Proof

We suppose that $\mathcal{F} \cap X$ is nonempty. Let x_1 and x_2 denote two points of $\mathcal{F} \cap X$ such that $\delta(x_1, x_2) = 1$. Then $\mathrm{d}(x_1, x_2) = 2$. Since $\mathcal{F}$ is geodetically closed, $\mathcal{C}(x_1, x_2) \subseteq \mathcal{F}$ and hence $\mathcal{C}(x_1, x_2) \cap X \subseteq \mathcal{F} \cap X$. This proves that $\mathcal{F} \cap X$ is a subspace of $\tilde{\mathcal{A}}$. Now, let a, b and c denote points of X such that $a, b \in \mathcal{F} \cap X$, $\delta(a, c) = \delta(a, b) - 1$ and $\delta(c, b) = 1$. Then $\mathrm{d}(a, c) = \mathrm{d}(a, b) - 2$ and $\mathrm{d}(c, b) = 2$. Since $\mathcal{F}$ is geodetically closed and $\mathrm{d}(a, c) + \mathrm{d}(c, b) = \mathrm{d}(a, b)$, $c \in \mathcal{F} \cap X$. Hence $\mathcal{F} \cap X$ is also geodetically closed (in $\tilde{\mathcal{A}}$). $\square$

Lemma 3.3.5
Let $\tilde{\mathcal{F}}$ be a geodetically closed sub near polygon of $\tilde{\mathcal{A}}$. Then there exists a unique geodetically closed sub near polygon $\mathcal{F}$ of $\mathcal{A}$ such that $\mathrm{diam}_{\mathcal{A}}(\mathcal{F}) = 2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$ *and $\tilde{\mathcal{F}} = \mathcal{F} \cap X$. Moreover, $\tilde{\mathcal{F}}$ is an SDPS-set of $\mathcal{F}$.*

Proof
Let x and y be two points of $\tilde{\mathcal{F}}$ such that $\delta(x, y) = \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$. If $\mathcal{F}$ is a geodetically closed sub near polygon of $\mathcal{A}$ such that $\mathrm{diam}_{\mathcal{A}}(\mathcal{F}) = 2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$ and $\tilde{\mathcal{F}} = \mathcal{F} \cap X$, then $\mathcal{F}$ necessarily equals $\mathcal{C}(x, y)$ since $x, y \in \mathcal{F}$ and $\mathrm{d}(x, y) = 2 \cdot \delta(x, y) = 2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}}) = \mathrm{diam}_{\mathcal{A}}(\mathcal{F})$. We will now show that $\mathcal{F} := \mathcal{C}(x, y)$ satisfies all required properties. By Lemma 3.3.4, $\mathcal{F} \cap X$ is a geodetically closed sub near polygon of $\tilde{\mathcal{A}}$ containing x and y and hence also $\tilde{\mathcal{F}}$. On the other hand, $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) = \frac{1}{2}\mathrm{diam}_{\mathcal{A}}(\mathcal{F} \cap X) \leq \frac{1}{2}\mathrm{diam}(\mathcal{F}) = \frac{1}{2}\,\mathrm{d}(x, y) = \delta(x, y) = \mathrm{diam}_{\tilde{\mathcal{A}}}(\tilde{\mathcal{F}})$. This proves that $\mathcal{F} \cap X = \tilde{\mathcal{F}}$.
One now easily sees that $(\mathcal{F}, \tilde{\mathcal{F}})$ is an admissible pair of classical near polygons, i.e. $(\mathcal{F}, \tilde{\mathcal{F}})$ is of one of the types (a), (b), (c) or (d) described in the definition of SDPS-set. So, $\tilde{\mathcal{F}}$ is an SDPS-set of $\mathcal{F}$. □

Definition
A geodetically closed sub near polygon $\mathcal{F}$ of $\mathcal{A}$ is called *special* if $\mathcal{F} \cap X$ is an SDPS-set of $\mathcal{F}$. The special sub near polygons of $\mathcal{A}$ are those sub near polygons of the form $\mathcal{C}(x_1, x_2)$ where $x_1, x_2 \in X$.

From Lemmas 3.3.4 and 3.3.5, we have the following corollary.

Corollary 3.3.6
If $\mathcal{F}$ is a geodetically closed sub near polygon of $\mathcal{A}$, then $\mathcal{F} \cap X$ is either empty or an SDPS-set in a geodetically closed sub near polygon of $\mathcal{F}$.

Lemma 3.3.7
Let $n \geq 1$. If $\mathcal{F}$ is a geodetically closed sub near $(4n-2)$-gon of $\mathcal{A}$, then $\mathcal{F} \cap X$ is a geodetically closed sub near $(2n-2)$-gon of $\tilde{\mathcal{A}}$.

Proof
We first show that $\mathcal{F} \cap X$ is nonempty. Let x be an arbitrary point of X. If $x \in \mathcal{F}$, then we are done. If $x \notin \mathcal{F}$, let $\mathcal{Q}$ denote the unique special quad through the line $x\pi_{\mathcal{F}}(x)$. Since $\mathcal{F}$ is big in $\mathcal{A}$, $\mathcal{F} \cap \mathcal{Q}$ is a line. Since this line contains a point of X, also $\mathcal{F}$ contains a point of X. Now, let x^* denote a point of $\mathcal{F} \cap X$ and let y denote a point of X at maximal distance from x^*. Then $\mathrm{d}(x^*, y) = 2n$ and $y \notin \mathcal{F}$. Let $\mathcal{Q}$ denote the unique special quad through the line $y\pi_{\mathcal{F}}(y)$. The quad $\mathcal{Q}$ intersects $\mathcal{F}$ in a line which contains a point y'' of X. Now, $\mathrm{d}(x^*, y'') \geq 2n-2$ and hence $\delta(x^*, y'') \geq n-1$. This proves that $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) \geq n-1$. On the other hand, $2 \cdot \mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) =$

$\mathrm{diam}_{\mathcal{A}}(\mathcal{F} \cap X) \leq \mathrm{diam}_{\mathcal{A}}(\mathcal{F}) = 2n-1$. Hence, $\mathrm{diam}_{\tilde{\mathcal{A}}}(\mathcal{F} \cap X) = n-1$. In Lemma 3.3.4, we have already shown that $\mathcal{F} \cap X$ is geodetically closed. $\square$

Lemma 3.3.8

Let $n \geq 1$. Let x be a point of $\mathcal{A}$ and let $\mathcal{F}$ denote a geodetically closed sub near $(4n-2)$-gon through x. Then $d(x, X) = d(x, \mathcal{F} \cap X)$.

Proof

Let y be a point of $X \setminus \mathcal{F}$, let $\mathcal{Q}$ denote the unique special quad through the line $y\pi_{\mathcal{F}}(y)$ and let y' denote the unique point of X on the line $\mathcal{Q} \cap \mathcal{F}$. Then $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_{\mathcal{F}}(y)) + 1 \geq \mathrm{d}(x, y')$. As a consequence, $\mathrm{d}(x, X \setminus \mathcal{F}) \geq \mathrm{d}(x, \mathcal{F} \cap X)$ and $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F} \cap X)$. $\square$

Lemma 3.3.9

Let x be a point of $\mathcal{A}$ and let $\mathcal{F}$ denote a special sub near polygon through x. Then $d(x, X) = d(x, \mathcal{F} \cap X)$.

Proof

The lemma holds if $\mathcal{F} = \mathcal{A}$. So, suppose that $\mathcal{F} \neq \mathcal{A}$ and that the lemma holds for any special sub near polygon of diameter $\mathrm{diam}(\mathcal{F}) + 2$. Let $\mathcal{F}''$ denote a special sub near polygon of diameter $\mathrm{diam}(\mathcal{F}) + 2$ through $\mathcal{F}$ and let $\mathcal{F}'$ denote a geodetically closed sub near polygon of diameter $\mathrm{diam}(\mathcal{F}) + 1$ such that $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{F}''$. By our assumption, $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F}'' \cap X)$. By Lemma 3.3.8, $\mathrm{d}(x, \mathcal{F}'' \cap X) = \mathrm{d}(x, \mathcal{F}'' \cap X \cap \mathcal{F}')$ and by Lemma 3.3.7, $\mathcal{F}'' \cap X \cap \mathcal{F}' = \mathcal{F} \cap X$. The lemma now follows. $\square$

Lemma 3.3.10

For every point x of $\mathcal{A}$, $d(x, X) \leq n$. Moreover, there exists a point x^* of $\mathcal{A}$ such that $d(x^*, X) = n$.

Proof

We will prove this by induction on n. Obviously, the lemma holds if n is equal to 0 or 1. Suppose therefore that $n \geq 2$. Let $\mathcal{F}$ denote a geodetically closed sub near $(4n-2)$-gon through x and let $\mathcal{F}'$ denote the unique geodetically closed sub near $(4n-4)$-gon of $\mathcal{A}$ containing all points of $\mathcal{F} \cap X$. Since $\mathcal{F}'$ is big in $\mathcal{F}$, $\mathrm{d}(x, \pi_{\mathcal{F}'}(x)) \leq 1$. Since $\mathcal{F} \cap X$ is an SDPS-set in $\mathcal{F}'$, we have $\mathrm{d}(\pi_{\mathcal{F}'}(x), \mathcal{F} \cap X) \leq n-1$ and hence $\mathrm{d}(x, \mathcal{F}) \leq n$. By the induction hypothesis, we know that there exists a point $y \in \mathcal{F}'$ such that $\mathrm{d}(y, X \cap \mathcal{F}) = n-1$. If x^* denotes a point of $\mathcal{F} \setminus \mathcal{F}'$ collinear with y, then $\mathrm{d}(x^*, X) = \mathrm{d}(x^*, X \cap \mathcal{F}) = 1 + \mathrm{d}(y, X \cap \mathcal{F}) = n$. This proves the lemma. $\square$

Lemma 3.3.11
Let x be a point of $\mathcal{A}$. Then there exist two points $x_1, x_2 \in X$ such that $d(x, x_1) = d(x, x_2) = d(x, X)$ and $d(x_1, x_2) = 2 \cdot d(x, X)$. As a consequence, every point is contained in a special geodetically closed sub near $4 \cdot d(x, X)$-gon.

Proof
We will prove this by induction on the distance $\mathrm{d}(x, X)$. Obviously, the property holds if $\mathrm{d}(x, X) = 0$. Suppose therefore that $\mathrm{d}(x, X) \geq 1$ and that the property holds for any point at distance at most $\mathrm{d}(x, X) - 1$ from X. Take a point x' collinear with x at distance $\mathrm{d}(x, X) - 1$ from X and let x'_1 and x'_2 denote two points of X such that $\mathrm{d}(x', x'_1) = \mathrm{d}(x', x'_2) = \mathrm{d}(x, X) - 1$ and $\mathrm{d}(x'_1, x'_2) = 2 \cdot \mathrm{d}(x, X) - 2$. Let $\mathcal{F}'$ denote the unique geodetically closed sub near $4[\mathrm{d}(x, X) - 1]$-gon through x'_1 and x'_2. Then $x' \in \mathcal{F}'$ since x' is on a shortest path between x'_1 and x'_2. If $x \in \mathcal{F}'$, then by Lemma 3.3.10, x would have distance at most $\mathrm{d}(x, X) - 1$ to $X \cap \mathcal{F}'$, a contradiction. So, x is not contained in $\mathcal{F}'$ and $\mathrm{d}(x, x'_1) = \mathrm{d}(x, x'_2) = 1 + \mathrm{d}(x', x'_1) = 1 + \mathrm{d}(x', x'_2) = \mathrm{d}(x, X)$. Let $\mathcal{F}$ denote the unique special geodetically closed sub near $4 \cdot \mathrm{d}(x, X)$-gon through $\mathcal{F}'$ containing the line xx'. Let L_2 denote a line of $\mathcal{C}(x, x'_2)$ through x'_2 not contained in $\mathcal{F}'$ and let $\mathcal{Q}_2$ denote the unique special quad through L_2. Then L_2 contains a unique point y_2 at distance $\mathrm{d}(x, X) - 1$ from x and $\mathcal{Q}_2 \subseteq \mathcal{F}$. Now, put $x_1 := x'_1$ and let x_2 denote a point of $(X \cap \mathcal{Q}_2) \setminus \{x'_2\}$ collinear with y_2. Then $\mathrm{d}(x, x_2) \leq \mathrm{d}(x, y_2) + \mathrm{d}(y_2, x_2) = \mathrm{d}(x, X)$ and hence $\mathrm{d}(x, x_2) = \mathrm{d}(x, X)$. Since the quad $\mathcal{Q}_2$ intersects $\mathcal{F}'$ only in the point x'_2, $\mathrm{d}(x_1, x_2) = \mathrm{d}(x_1, x'_2) + \mathrm{d}(x'_2, x_2) = 2 \cdot \mathrm{d}(x, X)$. We have already noticed that $\mathrm{d}(x, x_1) = \mathrm{d}(x, x'_1) = \mathrm{d}(x, X)$. This proves the lemma. □

Lemma 3.3.12
Every line L of $\mathcal{A}$ contains a unique point at smallest distance from X.

Proof
We will use induction on n. Obviously, the lemma holds if n is equal to 0 or 1. Suppose therefore that $n \geq 2$. Let $\mathcal{F}$ denote a geodetically closed sub near $(4n - 2)$-gon through L and let $\mathcal{F}'$ denote the unique special sub near $(4n - 4)$-gon contained in $\mathcal{F}$. For every point x of L, $\mathrm{d}(x, X) = \mathrm{d}(x, \mathcal{F}' \cap X)$ by Lemma 3.3.9. If L is contained in $\mathcal{F}'$, then the lemma holds for the line L by the induction hypothesis. If L is disjoint from $\mathcal{F}'$, then the lemma holds for the line $\pi_{\mathcal{F}'}(L)$ and hence also for the line L since $\mathrm{d}(x, \mathcal{F}' \cap X) = 1 + \mathrm{d}(\pi_{\mathcal{F}'}(x), \mathcal{F}' \cap X)$. If L intersects $\mathcal{F}'$ in a unique point, then this point is the unique point of L at smallest distance from X. □

Lemma 3.3.13
Let x be a point of $\mathcal{A}$ and let $\mathcal{F}$ denote a special sub near $4 \cdot \mathrm{d}(x, X)$-gon through x. Then a line L through x contains a point at distance $\mathrm{d}(x, X) - 1$ from X if and only if L is contained in $\mathcal{F}$. As a consequence, x is contained in a unique special sub near $4 \cdot \mathrm{d}(x, X)$-gon $\mathcal{F}_x$.

Proof
Suppose that L is contained in $\mathcal{F}$. By Lemma 3.3.12, L contains a unique point nearest to X and by Lemma 3.3.10 every point of $\mathcal{F}$ has distance at most $\mathrm{d}(x, X)$ to X. Hence L contains a unique point at distance $\mathrm{d}(x, X) - 1$ from x.
Suppose that L is not contained in $\mathcal{F}$ and that L contains a point y at distance $\mathrm{d}(x, X) - 1$ from $\mathcal{F}$. Let $\mathcal{F}'$ denote the unique special $4[\mathrm{d}(x, X) + 1]$-gon through $\mathcal{C}(y, \mathcal{F})$. Then $\mathrm{d}(y, X) = \mathrm{d}(y, \mathcal{F}' \cap X)$ by Lemma 3.3.9. Let y' denote a point of $X \cap \mathcal{F}'$ at distance $\mathrm{d}(x, X) - 1$ from y. Then $y' \notin \mathcal{F}$ and hence also $y' \notin \mathcal{C}(y, \mathcal{F})$. Let y'' denote the unique point of $\mathcal{C}(y, \mathcal{F})$ collinear with y' and let $\mathcal{Q}$ denote the unique special quad through the line $y'y''$. Then the line $\mathcal{Q} \cap \mathcal{C}(y, \mathcal{F})$ contains a point z of X which necessarily belongs to $\mathcal{F}$. We have $\mathrm{d}(y, z) \geq \mathrm{d}(x, X) + 1$ and $\mathrm{d}(y, z) \leq \mathrm{d}(y, y') + \mathrm{d}(y', z) = \mathrm{d}(x, X) - 1 + 2 = \mathrm{d}(x, X) + 1$. Hence y' is contained in a shortest path between y and z. But this is impossible since $y' \notin \mathcal{C}(y, \mathcal{F})$. $\square$

Lemma 3.3.14
For every point x of $\mathcal{A}$ and every point $y \in \mathcal{F}_x$, $\mathcal{F}_y \subseteq \mathcal{F}_x$.

Proof
In the near polygon $\mathcal{F}_x$, there exists a unique special sub near $4 \cdot \mathrm{d}(y, \mathcal{F}_x \cap X)$-gon $\mathcal{F}'_y$ through y. By Lemma 3.3.9, $\mathrm{d}(y, \mathcal{F}_x \cap X) = \mathrm{d}(y, X)$. Hence $\mathcal{F}'_y$ must coincide with $\mathcal{F}_y$. This proves the lemma. $\square$

Theorem 3.3.15
The map $f : \mathcal{P} \to \mathbb{N};\ x \mapsto d(x, X)$ is a valuation of $\mathcal{A}$.

Proof
Obviously, there exists a point with value 0. By Lemma 3.3.12, every line contains a unique point with smallest value. We will now show that f also satisfies property $\mathbf{V_3}$. For every point y of $\mathcal{F}_x$, $\mathrm{d}(y, X) \leq \mathrm{d}(x, X)$ by Lemma 3.3.10. If y is a point of $\mathcal{F}_x$ and if z is a point of $\mathcal{A}$ collinear with x for which $\mathrm{d}(z, X) = \mathrm{d}(y, X) - 1$, then $z \in \mathcal{F}_y$ by Lemma 3.3.13. By Lemma 3.3.14, it now follows that $z \in \mathcal{F}_x$. Hence f is a valuation of $\mathcal{A}$. $\square$

3.4 Valuations of dense near hexagons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near hexagon and let f be a valuation of $\mathcal{S}$. There are three possibilities.

- $\max\{f(x)|x \in \mathcal{P}\} = 3$. In this case f is a classical valuation.
- $\max\{f(x)|x \in \mathcal{P}\} = 1$. In this case f is an ovoidal valuation.
- $\max\{f(x)|x \in \mathcal{P}\} = 2$.

Proposition 3.4.1
If $|O_f| = 1$, then f is a classical or a semi-classical valuation.

Proof
This follows directly from Propositions 3.1.11 and 3.2.3. □

Proposition 3.4.2
Suppose that $|O_f| \geq 2$ and that f is not ovoidal. Then every two points of O_f lie at distance 2 from each other. As a consequence, G_f is a linear space.

Proof
Let x and y denote two distinct points of O_f. Then $\mathrm{d}(x, y) \in \{2, 3\}$. Suppose that $\mathrm{d}(x, y) = 3$ and consider a shortest path x, x_1, x_2, y from x to y. By property $\mathbf{V_2}$, the points x_1 and x_2 have value 1, and there exists a point p on x_1x_2 with value 0. Let $\mathcal{F}_{x_1}$ denote the sub near polygon through x_1 satisfying property $\mathbf{V_3}$. Since x and p are points with value 0 collinear with x_1, they are both contained in $\mathcal{F}_{x_1}$. Since x_1 and p belong to $\mathcal{F}_{x_1}$, also the point x_2 belongs to $\mathcal{F}_{x_1}$. Since y is a point with value 0 collinear with x_2, also $y \in \mathcal{F}_{x_1}$. Hence, $x, y \in \mathcal{F}_{x_1}$ and $\mathcal{C}(x, y) \subseteq \mathcal{F}_{x_1}$. Since $\mathrm{d}(x, y) = 3$, $\mathcal{S} = \mathcal{C}(x, y) = \mathcal{F}_{x_1}$, a contradiction, since every point of $\mathcal{F}_{x_1}$ has value at most 1 and $\mathcal{S}$ contains points with value 2. □

Proposition 3.4.3
If not every line of a dense near hexagon $\mathcal{S}$ is incident with the same number of points, then f is classical or an extended valuation arising from an ovoidal valuation in a quad of $\mathcal{S}$.

Proof
Suppose that $\mathcal{S}$ has $k \geq 2$ different line sizes $s_1 + 1, \ldots, s_k + 1$. By Corollary 3.1.15, f is not ovoidal and $k \leq 3$. If $k = 3$, then by Proposition 3.1.12, $\mathcal{S}$ is the direct product of three lines of different sizes. Any quad of $\mathcal{S}$ is a nonsymmetrical grid and hence does not contain ovoids. So, every induced

quad valuation is classical. By Proposition 3.1.10, it then follows that the valuation f itself is also classical. So, we may suppose that $k = 2$. By Proposition 3.1.12, it follows that $\mathcal{S}$ is the direct product of a line L and a generalized quadrangle $\mathcal{Q}$. Without loss of generality, we may suppose that L has size $s_1 + 1$ and that $\mathcal{Q}$ has order (s_2, t_2) for a certain $t_2 \in \mathbb{N} \setminus \{0\}$. By Corollary 3.1.15, $\mathcal{S}$ contains points with value 2. If f contains points with value 3, then f is classical by Proposition 3.1.7. So, we may suppose that there are only points with value 0, 1 or 2. There are $(t_2 + 1)(s_2t_2 + 1)$ quads in $\mathcal{S}$ isomorphic to an $[(s_1+1)\times(s_2+1)]$-grid. The induced valuation in each such quad cannot be ovoidal and hence is classical. As a consequence, each such quad contains a unique point of O_f. Since a point of $\mathcal{S}$ is contained in precisely $t_2 + 1$ of these $[(s_1 + 1) \times (s_2 + 1)]$-grids, $|O_f| = \frac{(t_2+1)(s_2t_2+1)}{t_2+1} = s_2t_2 + 1 \geq 2$. We can now apply Proposition 3.4.2 and we find that any two points of O_f lie at distance 2 from each other. Since f is not classical, there exists a quad $\mathcal{R}$ such that the valuation induced in $\mathcal{R}$ is ovoidal, see Proposition 3.1.10. Obviously, the quad $\mathcal{R}$ is isomorphic to $\mathcal{Q}$. For any point x of $\mathcal{S}$ outside $\mathcal{Q}$, there always exists a point of the ovoid $O_f \cap \mathcal{R}$ at distance 3 from x. So, $f(x) \neq 0$ and $O_f \subset \mathcal{R}$. By Proposition 3.1.11, it now follows that $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{d}(\pi_{\mathcal{F}}(x), O_f)$ for every point x of $\mathcal{S}$. Hence, f is the extension of an ovoidal valuation in $\mathcal{R}$. $\square$

If all lines of $\mathcal{S}$ are incident with $s + 1$ points, then by Proposition 3.1.14, $m_0 - \frac{m_1}{s} + \frac{m_2}{s^2} = 0$, where m_i, $i \in \{0, 1, 2\}$, denotes the total number of points with value i.

3.5 Valuations of classical near polygons

3.5.1 Results

Every dense near $2d$-gon with $d \leq 2$ is a classical near polygon. We have studied the valuations of these near polygons earlier. The following proposition treats the case of classical near hexagons.

Proposition 3.5.1
Let $\mathcal{S}$ be a dense classical near hexagon. Then every valuation f of $\mathcal{S}$ is either a classical valuation, an ovoidal valuation, a semi-classical valuation or the extension of an ovoidal valuation in a quad of $\mathcal{S}$.

Proof
If $|O_f| = 1$, then f is classical or semi-classical by Proposition 3.4.1. So, we may suppose that $|O_f| \geq 2$ and that f is not ovoidal. By Proposition 3.4.2,

every two points of O_f lie at distance 2 from each other. As a consequence, there exists a special quad $\mathcal{Q}$. Suppose that there exists a point x in O_f not contained in $\mathcal{Q}$. Then $\pi_{\mathcal{Q}}(x) \notin O_f$ and there exists a point y of the ovoid $O_f \cap \mathcal{Q}$ at distance 2 from $\pi_{\mathcal{Q}}(x)$. But then $\mathrm{d}(x, y) = 3$, contradicting the fact that any two points of O_f lie at distance 2 from each other. As a consequence, O_f is an ovoid in $\mathcal{Q}$. From Proposition 3.1.11, it now follows that $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_{\mathcal{Q}}(x)) + \mathrm{d}(\pi_{\mathcal{Q}}(x), O_f)$. Hence, f is the extension of an ovoidal valuation of $\mathcal{Q}$. $\square$

Proposition 3.5.2

(a) *Every valuation f of $Q^D(2n, q)$, $n \geq 2$ and q odd, is classical.*

(b) *Every valuation f of $H^D(2n-1, q^2)$, $n \geq 2$, is classical.*

(c) *If the generalized quadrangle $H(4, q^2)$ has no spreads, then every valuation f of $H^D(2n, q^2)$ is classical. As a consequence, every valuation of $H^D(2n, 4)$ is classical.*

Proof

The quads of $Q^D(2n, q)$ are isomorphic to $Q^D(4, q) \cong W(q)$. Since $W(q)$, q odd, has no ovoids (see Theorem 1.5.7), every induced quad valuation is classical. Hence f itself is also classical. The proofs of (b) and (c) are similar. Every quad of $H^D(2n-1, q^2)$ is isomorphic to $H^D(3, q^2) \cong Q(5, q)$ and $Q(5, q)$ has no ovoids (see again Theorem 1.5.7). Finally, every quad of $H^D(2n, q^2)$ is isomorphic to $H^D(4, q^2)$ and if the generalized quadrangle $H(4, q^2)$ has no spreads, then its point-line dual $H^D(4, 4)$ has no ovoids. By a computer result of Brouwer, we know that the generalized quadrangle $H(4, 4)$ has no spreads. $\square$

Proposition 3.5.3 (Section 3.5.2**)**

Let $\mathcal{S}$ be a classical dense near $2n$-gon which is not a product near polygon. Let f be a valuation of $\mathcal{S}$ such that no induced hex-valuation is semi-classical or ovoidal. Then f is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{S}$.

3.5.2 Proof of Proposition 3.5.3

To prove Proposition 3.5.3, we will use induction on n. Obviously, the proposition holds if $n \leq 2$. The proposition also holds if n is equal to 3 by Proposition 3.5.1. So, suppose that $n \geq 4$ and that the proposition holds for any classical near $2m$-gon $\mathcal{S}'$ with $m \leq n-1$ (= Induction Hypothesis). So, we suppose that the proposition holds for any proper geodetically closed sub near polygon of $\mathcal{S}$ (since such a sub near polygon is again classical).

Lemma 3.5.4
No two special quads intersect in a line.

Proof
Suppose that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two special quads intersecting in a line. Then $\mathcal{C}(\mathcal{Q}_1, \mathcal{Q}_2)$ is a hex $\mathcal{H}$. By Proposition 3.5.1, the valuation induced in $\mathcal{H}$ is either classical, semi-classical, ovoidal or extended. It is easily seen that none of these possibilities can occur. □

Lemma 3.5.5
For all $x_1, x_2 \in O_f$, $\mathrm{d}(x_1, x_2)$ is even.

Proof
We distinguish three possibilities.

(1) $\mathrm{d}(x_1, x_2) < n$. If f' denotes the valuation induced in $\mathcal{C}(x_1, x_2)$, then by the Induction Hypothesis, f' is an SDPS-valuation in $\mathcal{C}(x_1, x_2)$. Since $x_1, x_2 \in O_{f'}$, $\mathrm{d}(x_1, x_2)$ is even.

(2) $\mathrm{d}(x_1, x_2) = n$ and n is even.

(3) $\mathrm{d}(x_1, x_2) = n$ and n is odd. Let $\mathcal{F}$ denote a geodetically closed sub near $(2n-2)$-gon through x_1 and let x_2' denote the unique point of $\mathcal{F}$ collinear with x_2. Then $f(x_2') = 1$. By the Induction Hypothesis, the valuation f' induced in $\mathcal{F}$ is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{F}$. Since $f(x_2') = 1$, x_2' is collinear with a point x_2'' of $O_f \cap \mathcal{F}$. Since $\mathrm{d}(x_1, x_2'')$ is even and $\mathrm{d}(x_1, x_2'') \geq n-2$, $\mathrm{d}(x_1, x_2'') = n-1$. So, f' is an SDPS-valuation of $\mathcal{F} = \mathcal{C}(x_1, x_2'')$. Now, the line $x_2'x_2''$ is contained in at least two special quads, $\mathcal{C}(x_2, x_2'')$ and the unique special quad through $x_2'x_2''$ contained in $\mathcal{F}$ (see Lemma 3.3.3). This contradicts Lemma 3.5.4. □

Lemma 3.5.6
If $x_1, x_2 \in O_f$ with $\mathrm{d}(x_1, x_2)$ as big as possible, then $O_f \subseteq \mathcal{C}(x_1, x_2)$.

Proof
Obviously, this holds if $\mathrm{d}(x_1, x_2) = n$. Suppose therefore that $\mathrm{d}(x_1, x_2) < n$ and let $\mathcal{F}$ denote a geodetically closed sub near $(2n-2)$-gon through $\mathcal{C}(x_1, x_2)$. By the Induction Hypothesis applied to $\mathcal{F}$, it follows that every point of $O_f \cap \mathcal{F}$ is contained in $\mathcal{C}(x_1, x_2)$. Suppose now that there exists a point $y \in O_f$ not contained in $\mathcal{F}$ and let y' denote the unique point of $\mathcal{F}$ collinear with y. Since $f(y') = 1$, there exists a point $x_3 \in O_f \cap \mathcal{F}$ collinear with y'. Then $x_3 \in \mathcal{C}(x_1, x_2)$. Since the valuation induced in $\mathcal{C}(x_1, x_2)$ is an SDPS-valuation,

there exists a point $x_4 \in O_f \cap \mathcal{C}(x_1, x_2)$ at distance $\mathrm{d}(x_1, x_2)$ from x_3. We now distinguish two possibilities.

- The quad $\mathcal{C}(y, x_3)$ intersects $\mathcal{C}(x_1, x_2)$ in a line L. Then L is contained in two special quads, $\mathcal{C}(y, x_3)$ and the unique special quad through L contained in $\mathcal{C}(x_1, x_2)$. This contradicts Lemma 3.5.4.

- The quad $\mathcal{C}(y, x_3)$ intersects $\mathcal{C}(x_1, x_2)$ in a point x_3. Then $\mathrm{d}(y, x_4) = 2 + \mathrm{d}(x_1, x_2)$, contradicting the maximality of $\mathrm{d}(x_1, x_2)$. □

Lemma 3.5.7
The valuation f is not semi-classical.

Proof
Suppose the contrary. Let x denote the unique point of $\mathcal{S}$ with value 0 and let $\mathcal{H}$ denote a hex containing a point at maximal distance n from x. Then it is easy to see that the valuation induced in $\mathcal{H}$ is semi-classical, contradicting our assumptions. □

Lemma 3.5.8
If $|O_f| = 1$, then f is a classical valuation.

Proof
Let x denote the unique point of O_f. If y is a point at distance at most $n-1$ from x, then by the Induction Hypothesis, the valuation induced in $\mathcal{C}(x, y)$ is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{C}(x, y)$. Since $|O_f| = 1$, this induced valuation must be classical. Hence, $f(y) = \mathrm{d}(x, y)$ for every point y at distance at most $n-1$ from x. The lemma now follows from Proposition 3.2.3 and Lemma 3.5.7. □

Lemma 3.5.9
If $|O_f| \geq 2$, then every point x of O_f is contained in a special quad.

Proof
Suppose the contrary. Then, by the Induction Hypothesis, the valuation induced in every geodetically closed sub near $(2n-2)$-gon through x must be classical. Hence, $f(y) = \mathrm{d}(x, y)$ for every point y at distance at most $n-1$ from x. By Proposition 3.2.3 and Lemma 3.5.7, it now follows that f is classical, contradicting $|O_f| \geq 2$. □

Lemma 3.5.10
For every point x of $\mathcal{S}$, $f(x) = \mathrm{d}(x, O_f)$.

Proof
By Lemma 3.5.8, this holds if $|O_f| = 1$. Suppose therefore that $|O_f| \geq 2$. Let x^* denote a point of O_f nearest to x and let $\mathcal{Q}$ denote a special quad through x^*. Then either $\pi_{\mathcal{Q}}(x) = x^*$ or $\pi_{\mathcal{Q}}(x) \sim x^*$. Since $\mathrm{d}(x, \pi_{\mathcal{Q}}(x)) \leq n-2$, $\mathrm{d}(x, x^*) \leq n-1$. By the Induction Hypothesis, the valuation induced in $\mathcal{C}(x, x^*)$ is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{C}(x, x^*)$. So, $f(x) = \mathrm{d}(x, O_f \cap \mathcal{C}(x, x^*)) = \mathrm{d}(x, x^*)$. □

Lemma 3.5.11
If the maximal distance between two points of O_f is smaller than n, then f is the extension of an SDPS-valuation in a geodetically closed sub near polygon of $\mathcal{S}$.

Proof
Let $x_1, x_2 \in O_f$ with $\mathrm{d}(x_1, x_2)$ as big as possible. Since $\mathrm{d}(x_1, x_2) < n$, the valuation induced in $\mathcal{F} := \mathcal{C}(x_1, x_2)$ is an SDPS-valuation f' with $O_{f'} = O_f$ (see Lemma 3.5.6). For every point x of $\mathcal{S}$, $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + \mathrm{d}(\pi_{\mathcal{F}}(x), O_{f'})$, proving the lemma. □

In the sequel, we will suppose that the maximal distance between two points of O_f is equal to n. This implies that n is even, see Lemma 3.5.5. Let $\tilde{S}$ denote the following partial linear space:

- the points of $\tilde{S}$ are the elements of O_f;
- the lines of $\tilde{S}$ are the special quads;
- incidence is containment.

If x_1 and x_2 are two points of O_f, then (as before) we denote by $\mathrm{d}(x_1, x_2)$ the distance between x_1 and x_2 in the geometry $\mathcal{S}$ and by $\mathrm{d}'(x_1, x_2)$ the distance between x_1 and x_2 in the geometry $\tilde{\mathcal{S}}$.

Lemma 3.5.12
For all points $x_1, x_2 \in O_f$, $d(x_1, x_2) = 2 \cdot d'(x_1, x_2)$. As a consequence, the diameter of $\tilde{\mathcal{S}}$ is half the diameter of $\mathcal{S}$.

Proof
Every path of $\tilde{\mathcal{S}}$ between x_1 and x_2 can be turned into a path of $\mathcal{S}$ with double length. This proves that $\mathrm{d}(x_1, x_2) \leq 2 \cdot \mathrm{d}'(x_1, x_2)$ for all points $x_1, x_2 \in O_f$. We will prove the lemma by induction on the distance $\mathrm{d}(x_1, x_2)$ which is always even by Lemma 3.5.5. Obviously, the lemma holds if $\mathrm{d}(x_1, x_2)$ is 0 or

2. Suppose therefore that $\mathrm{d}(x_1, x_2) = 2k \geq 4$. Then we already know that $\mathrm{d}'(x_1, x_2) \geq k$. We will now show that there exists a special quad $\mathcal{Q}$ through x_2 containing a point x_3 at distance $2k-2$ from x_1. If $2k < n$, this follows from the fact that the valuation induced in $\mathcal{C}(x_1, x_2)$ is an SDPS-valuation. If $2k = n$, this follows from Lemma 3.5.9. If $x_3 \notin O_f$, then there exists a point in $\mathcal{Q} \cap O_f$ at distance $2k-1$ from x_1, contradicting Lemma 3.5.5. Hence $x_3 \in O_f$. Since $\mathrm{d}(x_1, x_3) = 2k-2$, $\mathrm{d}'(x_1, x_3) = k-1$ and $\mathrm{d}'(x_1, x_2) \leq k$. Together with $\mathrm{d}'(x_1, x_2) \geq k$, this implies that $\mathrm{d}(x_1, x_2) = 2 \cdot \mathrm{d}'(x_1, x_2)$. □

Lemma 3.5.13
$\tilde{\mathcal{S}}$ is a near polygon (and hence a near n-gon).

Proof
Let x denote a point of O_f and let $\mathcal{Q}$ denote a special quad. If $\pi_{\mathcal{Q}}(x) \notin O_f$, then there exists a point in $O_f \cap \mathcal{Q}$ at distance $\mathrm{d}(x, \pi_{\mathcal{Q}}(x)) + 1$ from x and a point in $O_f \cap \mathcal{Q}$ at distance $\mathrm{d}(x, \pi_{\mathcal{Q}}(x)) + 2$ from x, contradicting Lemma 3.5.5. So, $\pi_{\mathcal{Q}}(x) \in O_f$ and $\pi_{\mathcal{Q}}(x)$ is the unique point of $O_f \cap \mathcal{Q}$ nearest to x. The lemma now follows from Lemma 3.5.12. □

Lemma 3.5.14
Let (s, t_2) denote the order of the quads of $\mathcal{S}$. Then every line of $\tilde{\mathcal{S}}$ is incident with $\tilde{s} + 1 := st_2 + 1$ points and every two points at distance 2 in $\tilde{\mathcal{S}}$ have $\tilde{t}_2 + 1 := t_2^2 + 1$ common neighbours. So, $\tilde{\mathcal{S}}$ is a dense near polygon.

Proof
Obviously, every line of $\tilde{\mathcal{S}}$ is incident with $st_2 + 1$ points. Now, choose points $x_1, x_2 \in O_f$ at distance 4 from each other. Let $\mathcal{H}$ be a hex through x_1 contained in $\mathcal{C}(x_1, x_2)$ and let x_2' denote the unique point of $\mathcal{H}$ collinear with x_2. Since $f(x_1) = 0$ and $f(x_2') = 1$, the valuation induced in $\mathcal{H}$ is the extension of an ovoidal valuation of a quad $\mathcal{Q}$ of $\mathcal{H}$. Hence, $\mathcal{H}$ contains a unique special quad. Now, we calculate the number $\tilde{t}_2 + 1$ of common neighbours of x_1 and x_2 (in $\tilde{\mathcal{S}}$). Any such common neighbour is contained in $\mathcal{C}(x_1, x_2)$. The number $\tilde{t}_2 + 1$ is equal to the number of special quads of $\mathcal{C}(x_1, x_2)$ through x_1. Since every hex of $\mathcal{C}(x_1, x_2)$ through x_1 contains a unique such quad, $\tilde{t}_2 + 1 = \frac{t_2^3 + t_2^2 + t_2 + 1}{t_2 + 1} = t_2^2 + 1$ or $\tilde{t}_2 = t_2^2$. □

If x_1 and x_2 are two points of O_f at distance 4 from each other, then $O_f \cap \mathcal{C}(x_1, x_2)$ is a geodetically closed subspace and hence a quad of $\tilde{\mathcal{S}}$. Conversely, every quad is obtained in this way.

Lemma 3.5.15
The near polygon $\tilde{\mathcal{S}}$ is classical.

Proof

The lemma holds trivially if $\tilde{\mathcal{S}}$ is a generalized quadrangle. So, suppose $n > 4$. Let x denote a point of $\tilde{\mathcal{S}}$ and let $\tilde{\mathcal{Q}}$ denote a quad of $\tilde{\mathcal{S}}$. Let $\mathcal{F}$ denote the geodetically closed sub near octagon of $\mathcal{S}$ containing all points of $\tilde{\mathcal{Q}}$. Then, by the Induction Hypothesis, the valuation induced in $\mathcal{F}$ is an SDPS-valuation. We distinguish the following possibilities.

(a) $\pi_{\mathcal{F}}(x) \in O_f \cap \mathcal{F}$. Then $\pi_{\mathcal{F}}(x)$ is indeed the unique point of $\tilde{\mathcal{Q}}$ nearest to x.

(b) $\mathrm{d}(\pi_{\mathcal{F}}(x), O_f \cap \mathcal{F}) = 1$. Then $\pi_{\mathcal{F}}(x)$ is contained in a special quad $\mathcal{Q}$ of $\mathcal{F}$, see Lemma 3.3.3. Then there exists a point in $O_f \cap \mathcal{Q}$ at distance $\mathrm{d}(x, \pi_{\mathcal{F}}(x))+1$ from x and a point in $O_f \cap \mathcal{Q}$ at distance $\mathrm{d}(x, \pi_{\mathcal{F}}(x))+2$ from x, contradicting Lemma 3.5.5.

(c) $\mathrm{d}(\pi_{\mathcal{F}}(x), O_f \cap \mathcal{F}) = 2$. Let x' denote a neighbour of $\pi_{\mathcal{F}}(x)$ collinear with a point of $O_f \cap \mathcal{F}$, then x' is contained in a special quad $\mathcal{Q}$ of $\mathcal{F}$. Then there exists a point in $O_f \cap \mathcal{Q}$ at distance $\mathrm{d}(x, \pi_{\mathcal{F}}(x)) + 2$ from x and a point in $O_f \cap \mathcal{Q}$ at distance $\mathrm{d}(x, \pi_{\mathcal{F}}(x)) + 3$ from x, contradicting Lemma 3.5.5. □

Lemma 3.5.16

The near polygon $\mathcal{S}$ is isomorphic to $W^D(2n-1, q)$ or to $[Q^-(2n+1, q)]^D$ for some prime power q. If $\mathcal{S} \cong W^D(2n-1, q)$ then $\tilde{\mathcal{S}}$ is isomorphic to $W^D(n-1, q^2)$. If $\mathcal{S} \cong [Q^-(2n+1, q)]^D$, then $\tilde{\mathcal{S}}$ is isomorphic to $H^D(n, q^2)$.

Proof

The near n-gon $\tilde{\mathcal{S}}$ is a regular, classical and dense near polygon. So, if $n \geq 6$, then since $\tilde{t}_2 \neq 1$, we know that $\tilde{\mathcal{S}}$ is isomorphic to either $H^D(n-1, r^2)$, $Q^D(n, r)$, $H^D(n, r^2)$, $W^D(n-1, r)$ or $[Q^-(n+1, r)]^D$ for some prime power r. The near $2n$-gon $\mathcal{S}$ is isomorphic to one of the following examples (for some prime power q).

(1) $H^D(2n-1, q^2)$. In this case f is a classical valuation (by Proposition 3.5.2), contradicting our assumption that the maximal distance between two points of O_f is equal to n.

(2) $Q^D(2n, q)$. If q is odd, then f is a classical valuation (by Proposition 3.5.2), contradicting our assumption that the maximal distance between two points of O_f is equal to n. So, q is even. Then $Q^D(2n, q) \cong W^D(2n-1, q)$. We will treat this case in (4).

(3) $H^D(2n, q^2)$. In this case $\tilde{\mathcal{S}}$ is a regular near n-gon with parameters $(\tilde{s}, \tilde{t}_2)$ equal to $(st_2, t_2^2) = (q^5, q^4)$. This case cannot occur, see Theorem 2 of [37].

(4) $W^D(2n-1, q)$. In this case $\tilde{\mathcal{S}}$ is a regular near n-gon with parameters $(\tilde{s}, \tilde{t}_2)$ equal to $(st_2, t_2^2) = (q^2, q^2)$. From Theorem 2 of [37], it follows that $\tilde{\mathcal{S}} \cong W^D(n-1, q^2)$.

(5) $[Q^-(2n+1, q)]^D$. In this case $\tilde{\mathcal{S}}$ is a regular near n-gon with parameters $(\tilde{s}, \tilde{t}_2)$ equal to $(st_2, t_2^2) = (q^3, q^2)$. It follows that $\tilde{\mathcal{S}} \cong H^D(n, q^2)$. $\square$

Chapter 4

The classification of slim dense near octagons

In [4], all slim dense near hexagons were classified. In this chapter, we will classify all slim dense near octagons, using the results of Chapters 2 and 3. It turns out that every slim dense near octagon has big hexes.

4.1 Slim dense near hexagons

Using Proposition 1.10.8, we can construct the following table.

$\mathcal{H}$	$\mathsf{v}_{\mathcal{H}}$	$\mathsf{t}_{\mathcal{H}}$	$\mathsf{a}_{\mathcal{H}}$	$\mathsf{b}_{\mathcal{H}}$	$\mathsf{c}_{\mathcal{H}}$	n_1	n_2	n_3
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	27	2	3	–	–	6	12	8
$W(2) \times \mathbb{L}_3$	45	3	3	1	–	8	20	16
$Q(5,2) \times \mathbb{L}_3$	81	5	5	–	1	12	36	32
$\mathbb{H}_3 \cong \mathbb{I}_3$	105	5	3	4	–	12	44	48
$Q^D(6,2)$	135	6	–	7	–	14	56	64
$Q(5,2) \otimes Q(5,2)$	243	8	16	–	2	18	96	128
$\mathbb{G}_3$	405	11	9	9	3	24	156	224
$\mathbb{E}_1$	729	11	66	–	–	24	264	440
$\mathbb{E}_2$	759	14	–	35	–	30	280	448
$\mathbb{E}_3$	567	14	–	15	6	30	216	320
$H^D(5,4)$	891	20	–	–	21	42	336	512

Table 4.1: Slim dense near hexagons

Here, $a_{\mathcal{H}}$ (resp. $b_{\mathcal{H}}$, $c_{\mathcal{H}}$) denotes again the number of grid-quads (resp. $W(2)$-quads, $Q(5,2)$-quads) through a given point of $\mathcal{H}$ and n_i denotes the number of points at distance i from a given point in $\mathcal{H}$, $i \in \{1,2,3\}$.

4.2 Valuations of slim dense near hexagons

In this section, we assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a slim dense near hexagon and that $f : \mathcal{P} \to \{0, 1, 2, 3\}$ is a valuation of $\mathcal{S}$. If x and y are two points of $\mathcal{S}$ such that $\mathrm{d}(x, y) = 2$, then $O(x, y)$ denotes the unique ovoid of $\mathcal{C}(x, y)$ containing x and y.

4.2.1 Some observations

Lemma 4.2.1
The valuation f is ovoidal if and only if O_f contains two points at distance three.

Proof
Suppose that f is ovoidal. Let $x, y \in O_f$ such that $\mathrm{d}(x, y) = 2$ and consider the quad $\mathcal{Q} := \mathcal{C}(x, y)$. Let z be a point of $\mathcal{Q}$ with value 1 and let L be a line intersecting $\mathcal{Q}$ in z. Since f is ovoidal, L contains a unique point z' of O_f. It is now easy to see that z' has distance three to at least one of the points of $O_f \cap \mathcal{Q}$. We already proved the lemma in the other direction, see Proposition 3.4.2. $\square$

Lemma 4.2.2
If $\mathcal{S}$ has $Q(5, 2)$-quads or big $W(2)$-quads, then it has no ovoids.

Proof
Suppose the contrary and let O be an ovoid of $\mathcal{S}$. If $\mathcal{S}$ contains a big $W(2)$-quad, then $\mathcal{S}$ contains two disjoint $W(2)$-quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Put $O_i := \mathcal{Q}_i \cap O$, $i \in \{1, 2\}$. The ovoids O_2 and $\pi_{\mathcal{Q}_2}(O_1)$ have at least one point in common. So, there exist two collinear points of O, a contradiction. If $\mathcal{S}$ contains a $Q(5, 2)$-quad $\mathcal{Q}$, then $O \cap \mathcal{Q}$ is an ovoid of $\mathcal{Q}$, contradicting the fact that $Q(5, 2)$ has no ovoids. $\square$

Lemma 4.2.3
If $\mathcal{S}$ contains $Q(5, 2)$-quads and f is not classical, then $|O_f| = 2t - 7$ and every $Q(5, 2)$-quad contains a unique point of O_f.

Proof
Since f is not classical, it follows from Proposition 3.1.7 that $0 \leq f(x) \leq 2$ for every point x. Let $\mathcal{Q}$ be a $Q(5, 2)$-quad. Because $\mathcal{Q}$ has no ovoids, the induced valuation is necessarily classical. Hence there exists a point x' in $\mathcal{Q}$ such that $f(x') - \min\{f(y) | y \in \mathcal{Q}\} = 2$. It is now easy to see that $\min\{f(y) | y \in Q\} = 0$ and because the induced valuation on $\mathcal{Q}$ is classical, exactly one point of $\mathcal{Q}$ has value 0. It follows that $|O_f \cap \mathcal{Q}| = 1$. Counting the

number of $Q(5,2)$-quads in two different ways yields that $|O_f| \cdot c_{\mathcal{H}} = \frac{|\mathcal{P}| \cdot c_{\mathcal{H}}}{27}$ or that $|O_f| = \frac{|\mathcal{P}|}{27}$. Counting the number of points of $\mathcal{S}$ around a $Q(5,2)$-quad (which is necessarily big in $\mathcal{S}$), yields that $|\mathcal{P}| = 27(1 + 2(t-4))$. Hence $|O_f| = 2t - 7$. $\square$

Lemma 4.2.4
Suppose that f is not classical. Then the valuation f is extended if and only if O_f is an ovoid in a big grid-quad or a big $W(2)$-quad.

Proof
Suppose that f is an extended valuation. Since the extension of a classical valuation is again classical (Proposition 3.2.7), f is the extension of an ovoidal valuation of a big quad $\mathcal{Q}$ in $\mathcal{S}$. Since $Q(5,2)$-quads have no ovoids, $\mathcal{Q}$ is either a grid-quad or a $W(2)$-quad. Conversely, suppose that $\mathcal{Q}$ is a big quad of $\mathcal{S}$ having an ovoid O such that $O_f = O$. Let g be the unique ovoidal valuation of $\mathcal{Q}$ such that $O_g = O$. By Proposition 3.1.11, $f(x) = 1 + g(\pi_{\mathcal{Q}}(x))$. Hence f is the extension of g. $\square$

Definition
For every set K of points of $\mathcal{S}$ at mutual distance 2, put $[K] := \bigcup \mathcal{C}(a,b)$, where the union ranges over all elements $a, b \in K$, with $a \neq b$. For every point p in $\mathcal{S} \setminus [K]$, put $C_p(K) := \Gamma_1(p) \cap [K]$. If O_f is a set of points at mutual distance 2, then we will also write C_p instead of $C_p(O_f)$.

Lemma 4.2.5
Suppose that f is not classical, semi-classical, ovoidal or extended. Then

(1) $\max\{f(x) | x \in \mathcal{P}\} = 2$;

(2) $|O_f| \geq 2$;

(3) $d(x,y) = 2$ *for every* $x, y \in O_f$, *so,* G_f *is a linear space;*

(4) $m_0 - \frac{m_1}{2} + \frac{m_2}{4} = 0$;

(5) *no big quads are special;*

(6) *if* G_f *is a line, then* $G_f \cong \mathbb{L}_3$ *and* $\mathcal{S} \cong \mathbb{H}_3$;

(7) *if* $Q(5,2)$*-quads occur, then either*

- $|C_p| = |C_p \cap O_f| = 1$, *or*
- $|C_p| \geq 2$ *and* $C_p \cap O_f = \emptyset$.

Proof

By the results of Section 3.4, (1), (2), (3) and (4) hold.

(5) Suppose the contrary and let $\mathcal{Q}$ be a big special quad. Suppose there exists a point $x \in O_f \setminus \mathcal{Q}$. Let x' be the unique point of $\mathcal{Q}$ at distance one from x. Because $\mathrm{d}(x,y) = 2$ for every point y of $O_f \cap \mathcal{Q}$, x' must be collinear with every point of the induced ovoid $O_f \cap \mathcal{Q}$ in $\mathcal{Q}$, a contradiction. Hence O_f is an ovoid in $\mathcal{Q}$, contradicting Lemma 4.2.4.

(6) Suppose that G_f is a line and let $\mathcal{Q}$ be the unique quad of $\mathcal{S}$ containing all points of O_f. Suppose first that $\mathcal{Q} \cong W(2)$. By (5), there exists a point x at distance 2 from $\mathcal{Q}$. Because every two ovoids of $\mathcal{Q}$ intersect in a point, x has distance two to at least one point y of O_f. Because $f(y) = 0$ and from Proposition 3.1.11, it follows that $f(x) = 2$. Without loss of generality, we may suppose that x induces an ovoid in $\mathcal{Q}$ different from O_f. Let y' be a point of that ovoid, different from y. Clearly $f(y') = 1$. By Proposition 3.1.11, any common neighbour z of x and y' has value 2. But then the third point of the line through x and z must have value 1 and has also distance 2 to $\mathcal{Q}$, contradicting the fact that $f(p) = 2$ for every point p at distance 2 from $\mathcal{Q}$. Hence $G_f \cong \mathbb{L}_3$ and $\mathcal{Q} \cong \mathbb{L}_3 \times \mathbb{L}_3$. By (5) and Table 1.3, $\mathcal{S}$ is isomorphic to $\mathbb{H}_3$, $Q(5,2) \otimes Q(5,2)$, $\mathbb{G}_3$ or $\mathbb{E}_1$. Suppose there exists a line at distance two from $\mathcal{Q}$. Then there exists a line $L = \{x_1, x_2, x_3\} \subseteq \Gamma_2(\mathcal{Q})$ such that the set O_{x_i} of points of $\mathcal{Q}$ closest to x_i is not equal to O_f for every $i \in \{1,2,3\}$. It is then clear that $O_f \cap O_{x_i}$ is a singleton $\{x_i'\}$ and $f(x_i') = 0$, $i = 1,2,3$. It now follows that $f(x_1) = f(x_2) = f(x_3) = 2$, contradicting $\mathbf{V_2}$. Hence no lines at distance two from $\mathcal{Q}$ exist. Only the near hexagon $\mathbb{H}_3$ satisfies this property.

(7) Let p be a point of $\mathcal{S} \setminus [O_f]$. Consider a $Q(5,2)$-quad $\mathcal{Q}$ through p and let q be the unique point of $O_f \cap \mathcal{Q}$ (see Lemma 4.2.3). We distinguish two possibilities.

- $\mathrm{d}(p,q) = 1$.

  Then $q \in C_p$. Suppose that C_p contains another point r. Since $p \notin [O_f]$, $r \notin O_f$. Let $\mathcal{Q}'$ denote a special quad through r and let q' denote a point of $\mathcal{Q}' \cap O_f$ collinear with r. If $q = q'$, then p, r and q are on the same line. So $p \in \mathcal{Q}'$, contradicting $p \notin [O_f]$. So $q \neq q'$ and $\mathrm{d}(q,q') = 2$. If $r \notin \mathcal{C}(q,q')$ then $\mathrm{d}(r,q) = \mathrm{d}(r,q') + \mathrm{d}(q',q) = 3$, contradicting $\mathrm{d}(r,q) \leq \mathrm{d}(r,p) + \mathrm{d}(p,q) = 2$. So, $r \in \mathcal{C}(q,q')$. But then also $p \in \mathcal{C}(q,q')$ since p is a common neighbour of r and q. This however contradicts the fact that $p \notin [O_f]$.

- $\mathrm{d}(p,q) = 2$.

Every special quad through q intersects $\mathcal{Q}$ in a line and this line has a point at distance 1 from p. By (6), there are at least two special quads through q and since no two special quads intersect in a line, we find that $|C_p| \geq 2$. If $q' \in C_p \cap O_f$, then $\mathrm{d}(q', q) = \mathrm{d}(q', p) + \mathrm{d}(p, q) = 3$, a contradiction. Hence $C_p \cap O_f = \emptyset$. □

4.2.2 The classical slim dense near hexagons

Suppose that $\mathcal{S}$ is a classical near hexagon. So $\mathcal{S}$ is isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$, $Q^D(6,2)$ or $H^D(5,4)$. Clearly all quads are big and by Proposition 3.5.1, all valuations are either classical, semi-classical, ovoidal or extended. By Proposition 3.5.2, all valuations of $H^D(5,4)$ are classical.

Theorem 4.2.6
The near hexagon $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ has 12 ovoidal valuations. The other classical slim dense near hexagons have no ovoidal valuations.

Proof
It is straightforward to calculate that $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ has 12 ovoids and hence 12 ovoidal valuations. By Lemma 4.2.2, the other classical slim dense near hexagons have no ovoids and hence no ovoidal valuations. □

Theorem 4.2.7
The near hexagon $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ has 54 semi-classical valuations. The near hexagon $W(2) \times \mathbb{L}_3$ has 90 semi-classical valuations. The other classical slim dense near hexagons have no semi-classical valuations.

Proof
By Proposition 3.2.4, $\mathcal{S}$ has a semi-classical valuation f with $O_f = \{x\}$ if and only if $\Gamma_3(x)$ is bipartite. This is clearly the case if $\mathcal{S}$ is isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ or $W(2) \times \mathbb{L}_3$. Again by Proposition 3.2.4, $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ has 54 semi-classical valuations and $W(2) \times \mathbb{L}_3$ has 90 semi-classical valuations. If $\mathcal{S}$ is isomorphic to $H^D(5,4)$ or $Q(5,2) \times \mathbb{L}_3$, then $\mathcal{S}$ has no semi-classical valuations by Lemma 4.2.3. Suppose now that $\mathcal{S} \cong Q^D(6,2)$ and that x is a point of $\mathcal{S}$ such that $\Gamma_3(x)$ is bipartite. Let G be the distance 3-geometry determined by x. As shown in Proposition 7.1 of [4], $\mathrm{diam}(G) \leq 4$. Consider a point y of G and let G_i, $i \in \{0, \ldots, 4\}$, denote the set of points at distance i from y in the geometry G.

(a) Because every line through y contains exactly one other point of $\Gamma_3(x)$, it follows that $|G_1| = 7$. Clearly, if $z \in G_2$ then $\mathrm{d}(y, z) = 2$ and if $z \in G_3$ then $\mathrm{d}(y, z) \in \{2, 3\}$. A quad $\mathcal{Q}$ through y contains a unique point x' of $\Gamma_1(x)$.

Clearly, $\mathcal{Q} \cap \Gamma_3(x)$ consists of all points of $\mathcal{Q}$ at distance two from x'. This set containing 8 points including y is bipartite. It follows that $|\mathcal{Q} \cap G_2| = 3$ and $|\mathcal{Q} \cap G_3| = 1$. Because there are 7 quads through y, $|G_2| = 21$ and $|G_3 \cap \Gamma_2(y)| = 7$. By Table 4.1, $|G| = |\Gamma_3(x)| = 64$. Since $\Gamma_3(x)$ is bipartite, $|G_0 \cup G_2 \cup G_4| = |G_1 \cup G_3| = 32$. Hence $|G_4| = 32 - |G_0| - |G_2| = 10$, $|G_3| = 32 - |G_1| = 25$ and $|G_3 \cap \Gamma_3(y)| = 18$.

(b) Consider $z \in G_4$. Then $z \in \Gamma_3(y)$. Put $S_z^1 := \Gamma_1(z) \cap \Gamma_2(y) \cap G_3$ and $S_z^2 := \Gamma_1(z) \cap \Gamma_2(y) \cap \Gamma_2(x)$. Clearly $S_z^1 \cup S_z^2 = \Gamma_1(z) \cap \Gamma_2(y)$ and hence $|S_z^1| + |S_z^2| = 7$. Let $z' \in S_z^2$, put $M := \mathcal{C}(x, z') \cap \mathcal{C}(y, z')$ and $\mathcal{Q} := \mathcal{C}(z, M)$. Put $y' := \pi_{\mathcal{Q}}(y)$ and $x' := \pi_{\mathcal{Q}}(x)$. Because $\mathcal{Q}$ is classical, it is clear that $M = \{x', y', z'\}$. Suppose that z'' is another point of S_z^2 in $\mathcal{Q}$. Again because $\mathcal{Q}$ is classical, $\mathrm{d}(x', z'') = \mathrm{d}(y', z'') = 1$ and hence $z'' = z'$, a contradiction. Because $|(S_z^1 \cup S_z^2) \cap \mathcal{Q}| = 3$, $\mathcal{Q}$ contains two elements of S_z^1. Repeating this argument for another point of $S_z^1 \cup S_z^2$ not contained in $\mathcal{Q}$, we obtain at least one other element of S_z^1. Hence $|S_z^1| \geq 3$ for every $z \in G_4$.

(c) Every point $u \in G_3 \cap \Gamma_2(y)$ is collinear with exactly four points of G_4, namely the unique points of $\Gamma_3(x)$ on every line through u not contained in $\mathcal{C}(y, u)$. Put $N := |\{(a, b) | a \in G_3 \cap \Gamma_2(y) \text{ and } b \in G_4\}|$. By (a) and (b), $28 = 4|G_3 \cap \Gamma_2(y)| \geq N \geq 3|G_4| = 30$, a contradiction. □

Theorem 4.2.8

Let F_1, respectively F_2, denote the number of extended valuations f of a classical slim dense near hexagon $\mathcal{H}$, for which $O_f \cong \mathbb{L}_3$, respectively $O_f \cong \mathbb{L}_5$. We have the following possibilities.

$\mathcal{H}$	F$_1$	F$_2$
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	54	0
$W(2) \times \mathbb{L}_3$	90	18
$Q(5,2) \times \mathbb{L}_3$	270	0
$Q^D(6,2)$	0	378
$H^D(5,4)$	0	0

Proof

Every grid-quad and every $W(2)$-quad has 6 ovoids. Hence F_1 is equal to 6 times the number of grid-quads of $\mathcal{S}$. So, $F_1 = 6\frac{v_{\mathcal{S}} \cdot a_{\mathcal{S}}}{9} = \frac{2}{3} v_{\mathcal{S}} \cdot a_{\mathcal{S}}$. Similarly, $F_2 = \frac{2}{5} v_{\mathcal{S}} \cdot b_{\mathcal{S}}$. The theorem now follows from Table 4.1. □

Remark

Since $Q^D(6,2)$ has no semi-classical and ovoidal valuations, every valuation of $Q^D(2n,2)$, $n \geq 2$, is the (possibly trivial) extension of an SDPS-valuation in a geodetically closed sub near polygon of $Q^D(2n,2)$, see Proposition 3.5.3.

4.2.3 The near hexagon $\mathbb{I}_3 \cong \mathbb{H}_3$

Projective sets

We will use the same notations as in Section 1.10.2 (III). Let α be a given $(n-1)$-dimensional subspace contained in $Q^+(2n-1,2)$. Every $(n-2)$-dimensional subspace β contained in α is contained in a unique $(n-1)$-dimensional subspace $\bar{\beta}$ which is not contained in $Q^+(2n-1,2)$. Define $V_\alpha := \{\bar{\beta} | \beta \subseteq \alpha \text{ and } \dim(\beta) = n-2\}$. If x is a point of $Q(2n,2)$ not contained in $Q^+(2n-1,2)$, then the tangent hyperplane in x intersects α in an $(n-2)$-dimensional subspace β. Obviously, $\bar{\beta}$ is the unique element of V_α through x. If $\bar{\beta}_1$ and $\bar{\beta}_2$ are two different elements of V_α, then $\dim(\bar{\beta}_1 \cap \bar{\beta}_2) = \dim(\beta_1 \cap \beta_2) = n-3$ and $\mathrm{d}(\bar{\beta}_1, \bar{\beta}_2) = 2$. Hence, V_α is a set of $2^n - 1$ points at mutual distance 2. If $\bar{\beta}_1$ and $\bar{\beta}_2$ are two different elements of V_α, then $\bar{\beta}_1 \cap \bar{\beta}_2 = \beta_1 \cap \beta_2 \subset Q^+(2n-1,2)$. Hence, β_1 and β_2 determine a grid-quad. If $\mathcal{Q}$ is a grid-quad of $\mathbb{I}_n$ containing at least one element of V_α, then the $(n-3)$-dimensional subspace γ of $Q^+(2n-1,2)$ corresponding with $\mathcal{Q}$ is contained in α. There are precisely three maximal subspaces through γ which intersect α in a subspace of dimension $n-2$. These three points of V_α form an ovoid in $\mathcal{Q}$. The linear space whose points are the elements of V_α and whose lines are the nonempty intersections of V_α with grid-quads of $\mathbb{I}_n$ (natural incidence) is isomorphic to the linear space of the points and lines of $\mathsf{PG}(n-1,2)$. We therefore call V_α a *projective set* of $\mathbb{I}_n$. We have the following properties.

- Every point of $\mathbb{I}_n$ is contained in precisely two projective sets.
- Every two points of $\mathbb{I}_n$ at distance 2 not contained in a $W(2)$-quad are contained in a unique projective set.
- Every point of $Q^D(2n,2)$ not contained in $\mathbb{I}_n$ is collinear with precisely $2^n - 1$ points of $\mathbb{I}_n$ (one on each line through x) and these $2^n - 1$ points determine a projective set of $\mathbb{I}_n$.
- Consider the following incidence structure $\mathcal{S}$ derived from $\mathbb{I}_n$. The points of $\mathcal{S}$ are of two types: (a) the points of $\mathbb{I}_n$, (b) the projective sets of $\mathbb{I}_n$. The lines of $\mathcal{S}$ are of two types: (i) the lines of $\mathbb{I}_n$, (ii) the points of $\mathbb{I}_n$. Incidence is as follows: a point of type (a) is incident with a line of type (i) if and only if they are incident as elements of $\mathbb{I}_n$, a

point of type (a) is incident with a line of type (ii) if and only if they coincide, a point of type (b) is never incident with a line of type (i), a point of type (b) is incident with every line of type (ii) contained in it. Obviously, $\mathcal{S} \cong Q^D(2n, 2)$. So, the near polygon $Q^D(2n, 2)$ can easily be reconstructed from the near polygon $\mathbb{I}_n$.

Induced valuations

Again we will use the same notations as in Section 1.10.2 (III). Recall that the embedding of $\mathbb{I}_3$ in $Q^D(6, 2)$ is distance-preserving. This implies that every valuation f of $Q^D(6, 2)$ induces a valuation $\tilde{f}$ in $\mathbb{I}_3$, see Proposition 3.1.5.

Suppose first that f is a classical valuation of $Q^D(6, 2)$ and that x is the unique point of $Q^D(6, 2)$ with value 0. So, x is a 2-dimensional subspace of $Q(6, 2)$. There are 2 possibilities.

- If x is a point of $\mathbb{I}_3$, then $\tilde{f}$ is a classical valuation of $\mathbb{I}_3$ with $O_{\tilde{f}} = \{x\}$.
- If x is not a point of $\mathbb{I}_3$, then x is a generator of $Q^+(5, 2)$. In this case $O_{\tilde{f}}$ is the projective set determined by the point x. In this case, $\tilde{f}$ is a valuation of type $\mathsf{PG}(2, 2)$.

Suppose next that f is an extended valuation arising from an ovoid O in a quad $\mathcal{Q}$ of $Q^D(6, 2)$. The quad $\mathcal{Q}$ corresponds with a point $\alpha_{\mathcal{Q}}$ on $Q(6, 2)$. There are two possibilities.

- If $\alpha_{\mathcal{Q}}$ is not contained in Π, then $\alpha_{\mathcal{Q}}$ also determines a $W(2)$-quad of $\mathbb{I}_3$. In this case, $\tilde{f}$ is a valuation of type $\mathbb{L}_5$. The valuation $\tilde{f}$ is the extension of an ovoidal valuation in a $W(2)$-quad.
- If $\alpha_{\mathcal{Q}}$ is contained in Π, then $\mathcal{Q} \cap \mathbb{I}_3$ is a grid-quad $\mathcal{Q}'$ of $\mathbb{I}_3$ and $O_{\tilde{f}}$ is an ovoid of $\mathcal{Q}'$. So, $\tilde{f}$ is a valuation of type $\mathbb{L}_3$.

Remark
The near hexagon $\mathbb{I}_3$ has 28 big $W(2)$-quads and each such quad contains 6 ovoids. Hence there are 168 (extended) valuations of type $\mathbb{L}_5$ in $\mathbb{I}_3$.

Lemma 4.2.9
The near hexagon $\mathbb{I}_3$ has no ovoidal or semi-classical valuations.

Proof
Since $\mathbb{I}_3$ has big $W(2)$-quads, it follows from Lemma 4.2.2 that $\mathbb{I}_3$ has no ovoids. Suppose that x is the point (12)(34)(56)(78) (in the model for $\mathbb{H}_3 \cong \mathbb{I}_3$ described in Chapter 1) and consider the following points of $\Gamma_3(x)$:

$$\begin{aligned}
x_1 &:= (13)(25)(47)(68)\\
x_2 &:= (15)(23)(47)(68)\\
x_3 &:= (15)(24)(37)(68)\\
x_4 &:= (15)(24)(38)(67)\\
x_5 &:= (15)(27)(38)(46)\\
x_6 &:= (13)(27)(46)(58)\\
x_7 &:= (13)(27)(45)(68).
\end{aligned}$$

Then $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_1$ is a closed path of length 7, yielding that $\Gamma_3(x)$ cannot be bipartite. Hence $\mathbb{I}_3$ has no semi-classical valuations. $\square$

Theorem 4.2.10
Every valuation of $\mathbb{I}_3$ which is not classical nor extended is an induced valuation of type $\mathsf{PG}(2,2)$ or of type $\mathbb{L}_3$. Furthermore, there are 30 valuations of type $\mathsf{PG}(2,2)$ and 210 valuations of type $\mathbb{L}_3$.

Proof
By Lemma 4.2.5, there are two possibilities.

(a) *G_f is a line of size 3.*
Let $\mathcal{Q}$ denote the unique special grid-quad. By Lemma 4.2.5, $m_0 - \frac{m_1}{2} + \frac{m_2}{4} = 0$. It follows from $m_0 = 3$ and $m_0 + m_1 + m_2 = 105$ that $m_0 = 3$, $m_1 = 38$ and $m_2 = 64$. From Proposition 3.1.11, f is now completely determined by O_f: the 64 points with value 2 are on the one hand the 48 points in $\Gamma_1(\mathcal{Q})$ which are not collinear with a point of O_f and the 16 points of $\Gamma_2(\mathcal{Q})$ which lie at distance 2 from at least one point of O_f. So, f necessarily is an induced valuation of type $\mathbb{L}_3$. Since $\mathbb{I}_3$ has 35 grid-quads and every grid-quad has six ovoids, there are 210 valuations of type $\mathbb{L}_3$ in $\mathbb{I}_3$.

(b) *G_f contains at least 2 lines and all lines have size 3.*
Since $\mathrm{d}(x,y) = 2$ and $\mathcal{C}(x,y) \cong \mathbb{L}_3 \times \mathbb{L}_3$ for every two different points x and y of O_f, O_f is contained in a unique projective set V. So, $|O_f| \leq 7$. Let $\mathcal{Q}$ denote a special quad and suppose that $z \in O_f$ is not contained in $\mathcal{Q}$. If there exists a point $z' \in \mathcal{Q}$ collinear with z, then $\Gamma_3(z) \cap O_f \cap \mathcal{Q} \neq \emptyset$, contradicting the fact that $\mathrm{d}(x,y) = 2$ for every $x, y \in O_f$. Hence $\mathrm{d}(z,\mathcal{Q}) = 2$ and $\mathrm{d}(z,u) = 2$ for each point of $u \in O_f \cap \mathcal{Q}$. So, there are three special grid-quads through

z and $|O_f| \geq 7$. So, $|O_f| = 7$ and O_f is a projective set. The valuation is now completely determined by O_f. Every $W(2)$-quad of $\mathbb{I}_3$ contains a unique point with value 0. Because every point of $\mathbb{I}_3$ is contained in such a quad, it follows that $f(v) = \mathrm{d}(v, O_f)$ for every point v of $\mathbb{I}_3$, see Proposition 3.1.11. In this case f is an induced valuation of type $\mathsf{PG}(2,2)$. Since every point outside $\mathbb{I}_3$ in $Q^D(6,2)$ determines a unique induced valuation of type $\mathsf{PG}(2,2)$ in $\mathbb{I}_3$, there are $v_{Q^D(6,2)} - v_{\mathbb{I}_3} = 30$ valuations of type $\mathsf{PG}(2,2)$ in $\mathbb{I}_3$. □

4.2.4 The near hexagon $Q(5,2) \otimes Q(5,2)$

Lemma 4.2.11
If f is a nonclassical valuation of $Q(5,2) \otimes Q(5,2)$, then $G_f \cong \mathsf{AG}(2,3)$.

Proof
Suppose that f is a nonclassical valuation of $Q(5,2) \otimes Q(5,2)$. By Lemmas 4.2.3 and 4.2.5, G_f is a linear space on 9 points containing only lines of size 3. Clearly $G_f \cong \mathsf{AG}(2,3)$. □

Theorem 4.2.12
Let x be a point and $\mathcal{Q}$ a grid-quad of $\mathcal{S}$ such that $\mathrm{d}(x, \mathcal{Q}) = 2$. Then there exists a unique valuation such that $\{x\} \cup (\Gamma_2(x) \cap \mathcal{Q}) \subset O_f$.

Proof
Put $\Gamma_2(x) \cap \mathcal{Q} = \{a_1, b_1, c_1\}$, $\mathcal{A} := \mathcal{C}(x, a_1)$, $\mathcal{B} := \mathcal{C}(x, b_1)$ and $\mathcal{C} := \mathcal{C}(x, c_1)$. Clearly, $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{C} = \mathcal{B} \cap \mathcal{C} = \{x\}$ and since $\mathcal{L}(x, \mathcal{S}) \cong C_{5,5}$ (see Table 1.3), there exists a unique grid-quad $\mathcal{D}$ through x such that $\mathcal{A} \cap \mathcal{D} = \mathcal{B} \cap \mathcal{D} = \mathcal{C} \cap \mathcal{D} = \{x\}$. Let a_2, b_2 respectively c_2 be the remaining point of $O(x, a_1)$, $O(x, b_1)$ respectively $O(x, c_1)$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ denote the two $Q(5,2)$-quads through x and let $\sigma_{\mathcal{Q}_i}$, $i \in \{1,2\}$, denote the reflection about $\mathcal{Q}_i$. Exactly one line of $\mathcal{A}$ (respectively $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$) through x is contained in $\mathcal{Q}_1$ and the other line is contained in $\mathcal{Q}_2$. The automorphism $\sigma := \sigma_{\mathcal{Q}_2} \circ \sigma_{\mathcal{Q}_1}$ fixes the quads $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ and maps the ovoid $\{a_1, b_1, c_1\}$ of $\mathcal{Q}$ to the ovoid $\{a_2, b_2, c_2\}$ of the quad $\mathcal{Q}' := \sigma(\mathcal{Q})$. The quad $\mathcal{Q}'$ intersects $\mathcal{A}, \mathcal{B}$ respectively $\mathcal{C}$ in the point a_2, b_2 respectively c_2.

Now consider the quad $\mathcal{T} := \mathcal{C}(a_1, b_2)$. Because $\mathrm{d}(a_1, x) = \mathrm{d}(a_1, b_1) = 2$ and $\mathrm{d}(a_1, \mathcal{B}) = 2$, the point a_1 is ovoidal with respect to $\mathcal{B}$ and $\mathcal{T}$ is a grid-quad such that $\mathcal{T} \cap \mathcal{A}, \mathcal{T} \cap \mathcal{B}$ and $\mathcal{T} \cap \mathcal{Q}$ are points. Let e be the third point of $O(a_1, b_2)$. Clearly $\mathrm{d}(x, e) = 2$. Let $\mathcal{E} := \mathcal{C}(x, e)$. Clearly $\mathcal{A} \cap \mathcal{E} = \mathcal{B} \cap \mathcal{E} = \{x\}$. If $\mathcal{C} = \mathcal{E}$, then the point e of $\mathcal{C}$ has distance 2 from a_1 and hence coincides with c_2. But then $\{a_1\} = \mathcal{C}(e, b_2) \cap \mathcal{A} = \mathcal{C}(c_2, b_2) \cap \mathcal{A} = \{a_2\}$, a contradiction. Suppose now that $\mathcal{C} \cap \mathcal{E}$ is a line. Then e is contained in a $Q(5,2)$-quad $\mathcal{Q}''$

with a point $y \in \{c_1, c_2\}$. But $\mathcal{T} \cap \mathcal{Q}''$ is a line through e, implying that $\mathrm{d}(y, \mathcal{T}) = 1$. But then $\mathrm{d}(y, a_1) = 3$ or $\mathrm{d}(y, b_2) = 3$, a contradiction. It follows that $\mathcal{E} = \mathcal{D}$. Put $d_1 := e$ and let d_2 be the third point $O(x, d_1)$. We can now repeat the previous argument for other pairs of points in the set $K := \{x, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$. It follows that for every $a, b \in K$ with $a \neq b$, $\mathrm{d}(a, b) = 2$ and $O(a, b) \subset K$. Hence, the incidence structure whose points are the points of K, whose lines are of the form $O(a, b)$, $a, b \in K$ and $a \neq b$ (natural incidence) is isomorphic to $\mathsf{AG}(2, 3)$.

One easily counts that there are 18 $Q(5, 2)$-quads intersecting K in a (necessarily unique) point. Since this is the total number of $Q(5, 2)$-quads of $\mathcal{S}$, every $Q(5, 2)$-quad of $\mathcal{S}$ intersects K in a unique point. Since every point of $\mathcal{S}$ lies in a $Q(5, 2)$-quad, every point of $\mathcal{S}$ lies at distance at most 2 from K.

If f is a valuation of $\mathcal{S}$ such that $\{x\} \cup (\Gamma_2(x) \cap \mathcal{Q}) \subset O_f$, then by Lemma 4.2.11 and the previous reasoning it follows that $O_f = K$. By Proposition 3.1.11, we then obtain that $f(y) = \mathrm{d}(y, K)$ for every point y of $\mathcal{S}$. So, there is at most one valuation satisfying the required properties. We will now show that the map $f : y \mapsto \mathrm{d}(y, K)$ is a valuation. Clearly condition $\mathbf{V_1}$ is satisfied. Now consider a line L of $\mathcal{S}$. Let $\mathcal{Q}$ be a $Q(5, 2)$-quad through L and let p be the unique point of $\mathcal{Q} \cap K$. If $p \in L$ then condition $\mathbf{V_2}$ is satisfied. If $p \notin L$, then it is clear to see that exactly one point of L has value 1 (namely the unique point of L nearest to p) and the other points have value 2. Again condition $\mathbf{V_2}$ is satisfied. It remains to show that f also satisfies property $\mathbf{V_3}$. If $f(y) = 2$, we define $\mathcal{F}_y := \mathcal{S}$. If $f(y) = 0$, we define $\mathcal{F}_y := \{y\}$. If $f(y) = 1$ and y is contained in a quad $\mathcal{Q}_y$ with $|\mathcal{Q}_y \cap K| \geq 2$, we define $\mathcal{F}_y := \mathcal{Q}_y$. If $f(y) = 1$ and y is not contained in a quad $\mathcal{R}$ satisfying $|\mathcal{R} \cap K| \geq 2$, let $\mathcal{F}_y$ be the unique line through y containing a point of K. With these choices of $\mathcal{F}_y$, we easily see that f satisfies property $\mathbf{V_3}$. □

Theorem 4.2.13
$Q(5, 2) \otimes Q(5, 2)$ has 648 valuations of type $\mathsf{AG}(2, 3)$.

Proof
From Theorem 4.2.12, every ovoidal point-quad pair $(x, \mathcal{Q})$ in $\mathcal{S}$ determines a unique valuation f of type $\mathsf{AG}(2, 3)$. The pair $(x, \mathcal{Q})$ corresponds with an antiflag in G_f. There are 72 antiflags in $\mathsf{AG}(2, 3)$. The number of grid-quads in $\mathcal{S}$ is equal to $\frac{16 v_{\mathcal{S}}}{9} = 432$ and the number of points at distance two from a grid-quad is equal to $v_{\mathcal{S}} - 9 - 9 \cdot 2(t-1) = 243 - 135 = 108$. Hence, the total number of valuations of type $\mathsf{AG}(2, 3)$ is equal to $\frac{432 \cdot 108}{72} = 648$. □

4.2.5 The near hexagon $\mathbb{G}_3$.

Lemma 4.2.14
A linear space $\mathcal{L}$ is isomorphic to $\overline{W(2)}$ if and only if each point of $\mathcal{L}$ is incident with exactly three lines of size 3 and two lines of size 5.

Proof
One calculates that $\mathcal{L}$ has fifteen points, fifteen lines of size 3 and six lines of size 5. If L is a line of size 5 then precisely 20 other lines meet L in a point. Hence no line is disjoint from L. Let $\mathcal{L}'$ be the partial linear space obtained from $\mathcal{L}$ by removing all lines of size 5. We will show that $\mathcal{L}' \cong W(2)$. Obviously, $\mathcal{L}'$ has order $(2,2)$. Let (y,L) be a non-incident point-line pair in $\mathcal{L}'$. Because both lines of size five through x intersect L in a point, exactly one line of size three through x intersects L in a point. It follows that $\mathcal{L}'$ is the generalized quadrangle of order two. Since every line of size 5 in $\mathcal{L}$ intersects every line of size three in exactly one point, every line of size 5 determines an ovoid in $\mathcal{L}$. This proves the lemma. □

Lemma 4.2.15
If f is a nonclassical valuation of $\mathcal{S}$, then $G_f \cong \overline{W(2)}$.

Proof
By Lemmas 4.2.3 and 4.2.5, G_f is a linear space on 15 points containing lines of size 3 and 5. Let x be a point of O_f and suppose that x is contained in α_x special grid-quads and β_x special $W(2)$-quads. Since $|O_f| = 15$ and G_f is a linear space, $2\alpha_x + 4\beta_x = 14$. Since no two special quads intersect in a line, $2\alpha_x + 3\beta_x \leq 12$. Hence $\beta_x \geq 2$. From the structure of the local spaces of $\mathbb{G}_3$, it follows that $\beta_x \neq 3$. Hence $\alpha_x = 3$ and $\beta_x = 2$. The lemma now follows from Lemma 4.2.14. □

Lemma 4.2.16
There exist at most 486 valuations of type $\overline{W(2)}$ in $\mathbb{G}_3$.

Proof
Let f be a valuation of type $\overline{W(2)}$, let $\mathcal{Q}$ be a special $W(2)$-quad and let x be a point of $O_f \setminus \mathcal{Q}$. Put $\Gamma_2(x) \cap \mathcal{Q} = \{x_1, x_2, \ldots, x_5\}$. Then $O_f = \cup_{i \in \{1,\ldots,5\}} O(x, x_i)$. Every $Q(5,2)$-quad of $\mathbb{G}_3$ has a classical induced valuation and hence contains a unique point of O_f. Since every point y of $\mathbb{G}_3$ is contained in a $Q(5,2)$-quad, $\mathrm{d}(y, O_f) \leq 2$ and $f(y) = \mathrm{d}(y, O_f)$. Hence, for every $W(2)$-quad $\mathcal{Q}$ and every point $x \in \Gamma_2(\mathcal{Q})$, there is at most one valuation f such that $\{x\} \cup (\Gamma_2(x) \cap \mathcal{Q}) \subseteq O_f$. Now, one counts that there are 243 $W(2)$-quads in $\mathbb{G}_3$ and 120 points at distance 2 from any such quad.

Hence there are at most $\frac{243 \cdot 120}{60} = 486$ nonclassical valuations in $\mathbb{G}_3$. (In $\overline{W(2)}$ there are 60 antiflags (p, L) where L is a line of size 5.) □

As seen in Chapter 1, there exists a distance-preserving embedding of $\mathbb{G}_3$ in the classical near hexagon $H^D(5, 4)$.

Lemma 4.2.17
Let x be a point of $H^D(5, 4)$ not contained in $\mathbb{G}_3$, let f_x denote the classical valuation of $H^D(5, 4)$ determined by x and let $\tilde{f}_x$ denote the valuation of $\mathbb{G}_3$ induced by f_x. Then $\tilde{f}_x$ is a nonclassical valuation of $\mathbb{G}_3$. Moreover, if x_1 and x_2 are two different points of $H^D(5, 4) \setminus \mathbb{G}_3$, then $O_{\tilde{f}_{x_1}} \neq O_{\tilde{f}_{x_2}}$.

Proof
Suppose that $\tilde{f}_x$ is a classical valuation of $\mathbb{G}_3$. Let y_1 and y_2 be two points of $\mathbb{G}_3$ such that $\tilde{f}_x(y_1) = 0$ and $\tilde{f}_x(y_2) = 3$. Then $\mathrm{d}(x, y_2) = \mathrm{d}(x, y_1) + \tilde{f}_x(y_2) > 3$, a contradiction. Suppose now that $O := O_{\tilde{f}_{x_1}} = O_{\tilde{f}_{x_2}}$ for two different points x_1, x_2 of $H^D(5, 4) \setminus \mathbb{G}_3$. Let $y \in O$ and let $\tilde{\mathcal{Q}}_1$ and $\tilde{\mathcal{Q}}_2$ denote the two special $W(2)$-quads through y. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ denote the unique $Q(5, 2)$-quads of $H^D(5, 4)$ through $\tilde{\mathcal{Q}}_1$ and $\tilde{\mathcal{Q}}_2$ respectively. If $yx_1 = yx_2$, then every point of $O \cap \tilde{\mathcal{Q}}_1 \setminus \{y\}$ has distance 1 from x_1 and x_2 and hence coincides with y, a contradiction. Hence $yx_1 \neq yx_2$. Since x_1 is collinear with the five points of $O \cap \tilde{\mathcal{Q}}_1$, the line yx_1 is contained in $\mathcal{Q}_1$. Similarly, also yx_2 is contained in $\mathcal{Q}_1$ and $\mathcal{Q}_1 = \mathcal{C}(yx_1, yx_2)$ (in $H^D(5, 4)$). In a similar way one proves that $\mathcal{Q}_2 = \mathcal{C}(yx_1, yx_2)$ (in $H^D(5, 4)$). Hence $\mathcal{Q}_1 = \mathcal{Q}_2$ and $\tilde{\mathcal{Q}}_1 = \tilde{\mathcal{Q}}_2$, a contradiction. □

Corollary 4.2.18

- *There are precisely 486 valuations of type $\overline{W(2)}$ in $\mathbb{G}_3$.*
- *Every valuation of $\mathbb{G}_3$ is induced by a valuation in $H^D(5, 4)$.*

Proof
There are 486 points of $H^D(5, 4)$ not contained in $\mathbb{G}_3$. The lemma now follows from Lemmas and 4.2.16 and 4.2.17. □

4.2.6 The near hexagon $\mathbb{E}_1$

The near hexagon $\mathbb{E}_1$ has 36 ovoids, see [13]. Hence $\mathbb{E}_1$ has 36 ovoidal valuations. In Section 5 (C) of [20], it was shown that for every point $x \in \mathbb{E}_1$, there exists a closed path of length 7 in $\Gamma_3(x)$. Hence $\Gamma_3(x)$ cannot be bipartite and $\mathbb{E}_1$ has no semi-classical valuations. Since no quads of $\mathbb{E}_1$ are big, $\mathbb{E}_1$ has no extended valuations. We will now show that all valuations of $\mathbb{E}_1$ are either classical or ovoidal. Suppose the contrary and let f be a valuation of $\mathbb{E}_1$ which is not classical or ovoidal. By the above remarks we know that f is not semi-classical nor extended. We will now derive some properties of the valuation f. By Lemma 4.2.5 (3), the following lemma holds

Lemma 4.2.19
$d(x, y) = 2$ for all $x, y \in O_f$ with $x \neq y$.

Let θ denote an isomorphism between $\mathbb{E}_1$ and $T_5^*(\mathcal{K})$ as defined in Chapter 1. So θ is a bijection between the point sets of $\mathbb{E}_1$ and $\mathsf{AG}(6, 3) := \Pi \setminus \Pi_\infty$ such that $\mathrm{d}(x, y) = i_{\mathcal{K}}[\theta(x)\theta(y) \cap \Pi_\infty]$ for all points x and y in $\mathbb{E}_1$ with $x \neq y$.

Lemma 4.2.20
The set $\theta(O_f)$ is a proper subspace of $\mathsf{AG}(6, 3)$. As a consequence, $|O_f| \in \{3, 9, 27, 81, 243\}$.

Proof
Let $\theta(x_1)$ and $\theta(x_2)$ denote two arbitrary different points of $\theta(O_f)$. Then the quad $\mathcal{C}(\theta(x_1), \theta(x_2))$ of $T_5^*(\mathcal{K})$ consists of all affine points in a plane α of Π and the intersection line $\alpha \cap \Pi_\infty$ contains precisely two points of $\mathcal{K}$. The third affine point of the line $\theta(x_1)\theta(x_2)$ is not $T_5^*(\mathcal{K})$-collinear with $\theta(x_1)$ and $\theta(x_2)$ and hence equals $\theta(x_3)$, where x_3 is the unique point of $\mathcal{C}(x_1, x_2)$ not collinear with x_1 and x_2. Since $|O_f \cap \mathcal{C}(x_1, x_2)| \geq 2$, the induced valuation in $\mathcal{C}(x_1, x_2)$ must necessarily be ovoidal and hence $x_3 \in O_f$ and $\theta(x_3) \in \theta(O_f)$. This proves that $\theta(O_f)$ is a subspace of $\mathsf{AG}(6, 3)$. Since no two points of O_f are collinear, $\theta(O_f) \neq \mathsf{AG}(6, 3)$. By Lemma 4.2.5 (2), $|O_f| \geq 2$ and hence $|O_f| \geq 3$. It follows that $|O_f| \in \{3, 9, 27, 81, 243\}$. □

Lemma 4.2.21
$|O_f| \leq 13$.

Proof
Let $x \in O_f$ and let V denote the set of all quads of the form $\mathcal{C}(x, z)$ where $z \in O_f \setminus \{x\}$. Suppose that $|O_f| \geq 14$. Then $|V| \geq \frac{14-1}{2}$ or $|V| \geq 7$. Since $t_{\mathbb{E}_1} + 1 = 12$, there exists a line L through x which is contained in two quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of V. Let u denote an arbitrary point of $L \setminus \{x\}$, let u_1 denote the unique point of $(\mathcal{Q}_1 \cap O_f) \setminus \{x\}$ collinear with u and let u_2 denote the unique point of $\mathcal{Q}_2 \cap O_f$ not collinear with u. Then $\mathrm{d}(u_1, u_2) = \mathrm{d}(u_1, u) + \mathrm{d}(u, u_2) = 3$, contradicting Lemma 4.2.19. □

By Properties 4.2.20 and 4.2.21, $|O_f| \in \{3, 9\}$. If $|O_f| = 3$, then clearly G_f is a line of size 3. This is impossible by Lemma 4.2.5 (6). Hence $|O_f| = 9$. Let T denote the set of all quads of the form $\mathcal{C}(x_1, x_2)$ where x_1 and x_2 are two points of O_f. Since $\theta(O_f)$ is a subplane of $\mathsf{AG}(6,3)$, the incidence structure defined by O_f and T is an affine plane of order 3. So, $|T| = 12$.

Lemma 4.2.22
$\Gamma_3(O_f) = \emptyset$.

Proof
Suppose that u is a point of $\Gamma_3(O_f)$, then u has distance 3 to every point of O_f and distance 2 to every quad of T. There are precisely $6 \cdot |T| = 72$ pairs $(\mathcal{Q}, L)$ with $\mathcal{Q}$ a quad of T and L a line through u contained in $\Gamma_2(\mathcal{Q})$. For every line L through u, let $T(L)$ denote the set of all quads $\mathcal{Q} \in T$ for which $L \subset \Gamma_2(\mathcal{Q})$. Since u is contained in precisely 12 lines, there exists a line $L = \{u, u_1, u_2\}$ through u for which $|T(L)| \geq 6$. For every $x \in O_f$, L contains a unique point nearest to x of O_f and hence $\{\mathrm{d}(x, u_1), \mathrm{d}(x, u_2)\} = \{2, 3\}$.

Let $\mathcal{Q}$ denote a quad of T and put $\mathcal{Q} \cap O_f = \{x_1, x_2, x_3\}$. If $\mathcal{Q} \in T(L)$, then the points u, u_1 and u_2 determine three mutually disjoint ovoids O_u, O_{u_1} and O_{u_2} of $\mathcal{Q}$. Since O_u is disjoint from the ovoid $\{x_1, x_2, x_3\}$, either O_{u_1} or O_{u_2} coincides with $\{x_1, x_2, x_3\}$. In any case, we have $\mathrm{d}(u_1, x_1) = \mathrm{d}(u_1, x_2) = \mathrm{d}(u_1, x_3)$ and $\mathrm{d}(u_2, x_1) = \mathrm{d}(u_2, x_2) = \mathrm{d}(u_2, x_3)$. If $\mathcal{Q} \notin T(L)$, then $u_i \in \Gamma_1(\mathcal{Q})$ for a certain $i \in \{1, 2\}$. Since $(u_i, \mathcal{Q})$ is classical, $|\{\mathrm{d}(u_i, x_1), \mathrm{d}(u_i, x_2), \mathrm{d}(u_i, x_3)\}| = 2$ and hence also $|\{\mathrm{d}(u_{3-i}, x_1), \mathrm{d}(u_{3-i}, x_2), \mathrm{d}(u_{3-i}, x_3)\}| = 2$.

Now, consider the partial linear space with point set O_f and line set $T(L)$ (natural incidence). Since $|T(L)| \geq 6$, we have two possibilities.

- The partial linear space is connected. Then $\mathrm{d}(x, u_i)$, $i \in \{1, 2\}$, is independent of the chosen point $x \in O_f$. So, $T = T(L)$. Suppose that $\mathrm{d}(u_1, x) = 2$ for every point of $x \in O_f$. If x_1 and x_2 are two different points of O_f, then $\mathcal{C}(u_1, x_1) \cap \mathcal{C}(u_1, x_2) = \{u_1\}$ and so the quads $\mathcal{C}(u_1, x_1)$ and $\mathcal{C}(u_1, x_2)$ determine four different lines through u_1. Now, the nine quads $\mathcal{C}(u_1, x)$, $x \in O_f$, will determine 18 different lines of $\mathbb{E}_1$ through u_1. This is a contradiction, since there are only twelve such lines.

- The partial linear space is disconnected: there is one connected component O'_f of size 8 and one single point x^*. Then $\mathrm{d}(x, u_i)$, $i \in \{1, 2\}$, is independent of the chosen point $x \in O'_f$ and different from $\mathrm{d}(x^*, u_i)$. So, $|T(L)| = 8$ and $T \setminus T(L) = \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4\}$, where $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$ are the four quads of T through x^*. By symmetry, we may suppose that $\mathrm{d}(u_1, x) = 2$ for every point of $x \in O'_f$. For every $i \in \{1, 2, 3, 4\}$, $\mathrm{d}(u_1, \mathcal{Q}_i) = 1$ and hence there exists a unique point v_i in $\mathcal{Q}_i$ collinear with u_1. Let w_{i1} and w_{i2} denote the two points of $\mathcal{Q}_i \cap O_f$ collinear with v_i. For every $i \in \{1,2,3,4\}$ and every $j \in \{1,2\}$, let A_{ij} denote the set of two lines through u_1 contained in the quad $\mathcal{C}(u_1, w_{ij})$. Since $A_{i1} \cap A_{i2} = u_1 v_i$, $|A_{i1} \cup A_{i2}| = 3$ for every $i \in \{1,2,3,4\}$. Now, for all $i, i' \in \{1,2,3,4\}$ and all $j, j' \in \{1,2\}$ with $i \neq i'$, $\mathcal{C}(w_{ij}, w_{i'j'}) \in T(L)$ and hence $A_{ij} \cap A_{i'j'} = \emptyset$. It now easily follows that $|\bigcup A_{ij}| = 12$. So, $L \in A_{i^*j^*}$ for a certain $i^* \in \{1,2,3,4\}$ and a certain $j^* \in \{1,2\}$. Now, $u \in L$ and $\mathrm{d}(w_{i^*j^*}, L) = 1$ and so $\mathrm{d}(u, O_f) \leq 2$, a contradiction. $\square$

Put $Y := \Gamma_1(O_f)$ and $Z := \Gamma_2(O_f)$. Let Y_i, $i \in \mathbb{N} \setminus \{0\}$, denote the set of points of Y which are collinear with precisely i points of O_f. Then $\sum |Y_i| \cdot i = |O_f| \cdot 24 = 216$.

Lemma 4.2.23
$|Y_1| = |Y_2| = 72$ and $|Y_i| = 0$ for all $i \geq 3$. So, $|Y| = 144$ and $|Z| = 576$.

Proof
Suppose that the point u of $\Gamma_1(O_f)$ is collinear with three different points x_1, x_2 and x_3 of O_f. Then u is contained in the quad $\mathcal{C}(x_1, x_2)$. Since x_1 and x_2 are the only points of $\mathcal{C}(x_1, x_2) \cap O_f$ collinear with u, we have $x_3 \notin \mathcal{C}(x_1, x_2)$. Now, u is also contained in $\mathcal{C}(x_1, x_3)$ and so the quads $\mathcal{C}(x_1, x_3)$ and $\mathcal{C}(x_1, x_2)$ intersect in a line ux_1. If x_4 denotes the unique point of $\mathcal{C}(x_1, x_3) \cap O_f$ not collinear with u, then $\mathrm{d}(x_2, x_4) = \mathrm{d}(x_2, u) + \mathrm{d}(u, x_4) = 1 + 2 = 3$, contradicting Lemma 4.2.19. So, $|Y_i| = 0$ if $i \geq 3$. Since every two points of O_f have precisely 2 common neighbours, $|Y_2| = 2 \cdot \binom{|O_f|}{2} = 72$, $|Y_1| = 216 - 2|Y_2| = 72$, $|Y| = |Y_1| + |Y_2| = 144$ and $|Z| = v_{\mathbb{E}_1} - |O_f| - |Y| = 576$. $\square$

Using the notations of Section 3.1.4, $m_0 = |O_f| = 9$. By Proposition 3.1.11, $m_1 = |Y| = 144$ and $m_2 = |Z| = 576$. By Lemmas 4.2.22 and 4.2.5 (1), $m_i = 0$ if $i \geq 3$. Hence $\sum_{i=0}^{\infty} \frac{m_i}{(-2)^i} = 81$. This contradicts Proposition 3.1.14. This proves the following theorem.

Theorem 4.2.24
All valuations of $\mathbb{E}_1$ are either classical or ovoidal.

4.2.7 The near hexagon $\mathbb{E}_2$

In [5], all ovoids of $\mathbb{E}_2$ were classified. There are 24 ovoids. (For an alternative proof, see the appendix of [11].) So, $\mathbb{E}_2$ has 24 ovoidal valuations. Consider a $W(2)$-quad $\mathcal{Q}$ which is ovoidal with respect to a point x. Suppose the ovoid $\Gamma_2(x) \cap \mathcal{Q}$ of $\mathcal{Q}$ is given by $\{(12), (13), (14), (15), (16)\}$ in the Sylvester model of $W(2)$. Then $(25), (46), (35), (24), (36), (25)$ is a closed path of odd length in $\Gamma_3(x)$, implying that $\Gamma_3(x)$ cannot be bipartite. Hence $\mathbb{E}_2$ has no semi-classical valuations. Since no quad of $\mathbb{E}_2$ is big, $\mathbb{E}_2$ has also no extended valuations.

Theorem 4.2.25
All valuations of $\mathbb{E}_2$ are classical or ovoidal.

Proof
We may suppose that f is not classical nor ovoidal. Hence by Lemma 4.2.5, G_f is a linear space with at least two lines and since all lines have size 5, $|O_f| \geq 21$. If $|O_f| \geq 21$, then through every point $x \in O_f$ there exists a line L which is contained in two special quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then there exists a point $u_1 \in O_f \cap \mathcal{Q}_1$ and a point $u_2 \in O_f \cap \mathcal{Q}_2$ such that $\mathrm{d}(u_1, u_2) = 3$, a contradiction. As a consequence, $|O_f| = 21$ and $G_f \cong \mathsf{PG}(2,4)$.

Clearly, every point of O_f has value 0 and every neighbour of a point of O_f has value 1. Suppose that x is a point with value 1 and suppose that x is not collinear with a point of value 0. Clearly, every neighbour of x has value 2. It follows that x has distance 2 to every special quad. Hence x is ovoidal with respect to every special quad. So there exists a point $x' \in O_f$ at distance 2 from x (any two ovoids of $W(2)$ intersect in at least one point), contradicting the fact that every neighbour of x has value 2. Hence every point with value 1 is collinear with a point of value 0 and it is now easy to see that these points are the points of $[O_f] \setminus O_f$. Hence $m_0 = 21$ and $m_1 = 210$ (every two special quads intersect in a point of O_f). Since $m_0 - \frac{m_1}{2} + \frac{m_2}{4} = 0$, $m_2 = 336$, contradicting $m_0 + m_1 + m_2 = v_{\mathbb{E}_2} = 759$. □

4.2.8 The near hexagon $\mathbb{E}_3$

Lemma 4.2.26
If f is a nonclassical valuation of $\mathbb{E}_3$, then $G_f \cong \mathsf{PG}(2,4)$.

Proof
By Lemmas 4.2.3 and 4.2.5, G_f is a linear space on 21 points containing only lines of size 5. Hence $G_f \cong \mathsf{PG}(2,4)$. □

Lemma 4.2.27
Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quads of $\mathbb{E}_3$ intersecting each other in a point x. Then there is a unique set of quads through x containing $\mathcal{Q}_1$ and $\mathcal{Q}_2$ and partitioning the set of lines through x.

Proof
Because $Q(5,2)$-quads are big, $\mathcal{Q}_1 \cong \mathcal{Q}_2 \cong W(2)$. Since $\mathcal{L}(x, \mathbb{E}_3) \cong \overline{W(2)}$, every two disjoint lines (of size three) of $\mathcal{L}_x$ are contained in a unique spread of $\mathcal{L}_x$. □

Definition
Let $(x, \mathcal{Q})$ be a ovoidal point-quad pair in $\mathcal{S}$ and suppose $\Gamma_2(x) \cap \mathcal{Q} = \{x_1, \dots, x_5\}$. Then

- $P(x, \mathcal{Q}) := \{\mathcal{C}(x, x_i) | 1 \leq i \leq 5\}$,
- $S(x, \mathcal{Q}) := \{y | y \in O(x, x_i), 1 \leq i \leq 5\}$.

Theorem 4.2.28
Let $(x, \mathcal{Q})$ be a point-quad pair such that $\mathrm{d}(x, \mathcal{Q}) = 2$. Then there exists a unique valuation f such that $\{x\} \cup (\Gamma_2(x) \cap \mathcal{Q}) \subset O_f$.

Proof
Clearly $\mathcal{Q} \cong W(2)$. Let $\Gamma_2(x) \cap \mathcal{Q} = \{x_1, \dots, x_5\}$ and put $\mathcal{Q}_i := \mathcal{C}(x, x_i)$, $1 \leq i \leq 5$. Clearly $|S(x, \mathcal{Q})| = 21$. Consider a point x_i, $i \in \{1, \dots, 5\}$, and let P be the set of quads containing $\mathcal{Q}$ and $\mathcal{Q}_i$ partitioning the set of lines through x_i (see Lemma 4.2.27). One easily sees that $P(x_i, \mathcal{Q}_j) = P$ for every $j \in \{1, \dots, 5\} \setminus \{i\}$. Hence $S(x_i, \mathcal{Q}_j) = S(x, \mathcal{Q})$ for all $i, j \in \{1, \dots, 5\}$ with $i \neq j$. It follows that

- for all $a, b \in S(x, \mathcal{Q})$, $\mathrm{d}(a, b) = 2$ and $O(a, b) \subseteq S(x, \mathcal{Q})$;
- the incidence structure whose points are the elements of $S(x, \mathcal{Q})$ and whose lines are the sets $O(a, b)$, $a, b \in S(x, \mathcal{Q})$ and $a \neq b$ (natural incidence), is isomorphic to $\mathsf{PG}(2, 4)$.

The number of $Q(5, 2)$-quads which intersect $K := S(x, \mathcal{Q})$ in a (necessarily unique) point is equal to $6 \cdot |K| = 126$. Since this is the total number of $Q(5, 2)$-quads, every $Q(5, 2)$-quad intersects K in a unique point.

If f is a valuation such that $\{x\} \cup (\Gamma_2(x) \cap \mathcal{Q}) \subseteq O_f$, then O_f necessarily coincides with K. Since every point y of $\mathbb{E}_3$ is contained in a $Q(5, 2)$-quad, $\mathrm{d}(y, K) \leq 2$ and $f(y) = \mathrm{d}(y, K)$. We will now show that the map $f : y \mapsto \mathrm{d}(y, K)$ is a valuation. Obviously, property $\mathbf{V_1}$ is satisfied. Let L denote an arbitrary line of $\mathbb{E}_3$, let $\mathcal{R}$ denote a $Q(5, 2)$-quad through L and let p denote the unique point of K in $\mathcal{R}$. If $p \in L$, then property $\mathbf{V_2}$ is satisfied. If $p \notin L$, then the unique point p' of L nearest to p has value 1 and every other point of L has value 2. Again property $\mathbf{V_2}$ is satisfied. If y is a point such that $f(y) = 0$, then we define $\mathcal{F}_y := \{y\}$. If $f(y) = 1$, let $\mathcal{F}_y$ denote the unique quad through y intersecting K in 5 points. If $f(y) = 2$, put $\mathcal{F}_y := \mathbb{E}_3$. For these choices of $\mathcal{F}_y$, also property $\mathbf{V_3}$ is satisfied. □

Theorem 4.2.29
The near hexagon $\mathbb{E}_3$ has 324 valuations of type $\mathsf{PG}(2, 4)$.

Proof
From Theorem 4.2.28, every ovoidal point-quad pair $(x, \mathcal{Q})$ in $\mathbb{E}_3$ determines a unique valuation of type $\mathsf{PG}(2,4)$. In $\mathbb{E}_3$ there are 567 $W(2)$-quads and 192 points at distance 2 from any such quad. Since there are 336 antiflags in $\mathsf{PG}(2,4)$, $\mathbb{E}_3$ has $\frac{567 \cdot 192}{336} = 324$ nonclassical valuations. $\square$

4.2.9 Overview

We now summarize our results in Table 4.2. In the last two columns, we mention the types of the valuations.

near hexagon	class.	ovoid.	semi-cl.	extended	other
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	$\surd$	$\surd$	$\surd$	$\mathbb{L}_3$	–
$W(2) \times \mathbb{L}_3$	$\surd$	–	$\surd$	$\mathbb{L}_3$, $\mathbb{L}_5$	–
$Q(5,2) \times \mathbb{L}_3$	$\surd$	–	–	$\mathbb{L}_3$	–
$\mathbb{H}_3$	$\surd$	–	–	$\mathbb{L}_5$	$\mathbb{L}_3$, $\mathsf{PG}(2,2)$
$Q^D(6,2)$	$\surd$	–	–	$\mathbb{L}_5$	–
$Q(5,2) \otimes Q(5,2)$	$\surd$	–	–	–	$\underline{\mathsf{AG}(2,3)}$
$\mathbb{G}_3$	$\surd$	–	–	–	$\overline{W(2)}$
$\mathbb{E}_1$	$\surd$	$\surd$	–	–	–
$\mathbb{E}_2$	$\surd$	$\surd$	–	–	–
$\mathbb{E}_3$	$\surd$	–	–	–	$\mathsf{PG}(2,4)$
$H^D(5,4)$	$\surd$	–	–	–	–

Table 4.2: Valuations in slim dense near hexagons

4.3 Properties of slim dense near octagons

In this section, we assume that $\mathcal{S}$ is a slim dense near octagon. As before, $v_\mathcal{S}$ denotes the number of points of $\mathcal{S}$ and $t_\mathcal{S} + 1$ denotes the constant number of lines through a point of $\mathcal{S}$.

Lemma 4.3.1
If $\mathcal{H}$ is a hex of $\mathcal{S}$ containing a $Q(5,2)$-quad, then every hex intersecting $\mathcal{H}$ intersects $\mathcal{H}$ in at least a line.

Proof
Let $\mathcal{H}'$ be a hex which meets $\mathcal{H}$. Let $u \in \mathcal{H} \cap \mathcal{H}'$, let $\mathcal{Q}$ denote a $Q(5,2)$-quad of $\mathcal{H}$ through u and let v denote a point of $\mathcal{H}'$ at distance 3 from

u. Since $Q(5,2)$ has no ovoids, v is classical with respect to $\mathcal{Q}$. Let v' denote the point of $\mathcal{Q}$ nearest to v. Then $\mathrm{d}(v,v') \leq 2$, so $v' \neq u$. Since $\mathrm{d}(v,u) = \mathrm{d}(v,v') + \mathrm{d}(v',u)$, the point v' is contained in a shortest path between the points v and u of $\mathcal{H}'$. Hence $v' \in \mathcal{H}'$ and $\mathcal{C}(u,v') \subseteq \mathcal{H} \cap \mathcal{H}'$. So $\mathcal{H}$ and $\mathcal{H}'$ intersect in at least a line. $\square$

Lemma 4.3.2

There exist constants $a_{\mathcal{S}}$, $b_{\mathcal{S}}$ and $c_{\mathcal{S}}$ such that every point of $\mathcal{S}$ is contained in $a_{\mathcal{S}}$ grid-quads, $b_{\mathcal{S}}$ $W(2)$-quads and $c_{\mathcal{S}}$ $Q(5,2)$-quads. Furthermore $t_{\mathcal{S}}(t_{\mathcal{S}}+1) = 2a_{\mathcal{S}} + 6b_{\mathcal{S}} + 20c_{\mathcal{S}}$, and

- $|\Gamma_0(x)| = 1$;
- $|\Gamma_1(x)| = 2(t_{\mathcal{S}}+1)$;
- $|\Gamma_2(x)| = 4a_{\mathcal{S}} + 8b_{\mathcal{S}} + 16c_{\mathcal{S}}$;
- $|\Gamma_3(x)| = \frac{v_{\mathcal{S}}}{3} - 1 - 6t_{\mathcal{S}} + 4a_{\mathcal{S}} + 8b_{\mathcal{S}} + 16c_{\mathcal{S}}$;
- $|\Gamma_4(x)| = 2(\frac{v_{\mathcal{S}}}{3} - 1 + 2t_{\mathcal{S}} - 4a_{\mathcal{S}} - 8b_{\mathcal{S}} - 16c_{\mathcal{S}})$;

for every point x of $\mathcal{S}$. Hence $|\Gamma_i(x)|$, $i \in \{0,1,2,3,4\}$, is independent of the chosen point x.

Proof

(a) Consider two collinear points x and y and let μ denote the number of grid-quads through xy. For every hex $\mathcal{H}$ through xy, let $\lambda_{\mathcal{H}}$ denote the number of grid-quads through xy. Then the total number of grid-quads through x is equal to $\mu + \sum(a_{\mathcal{H}} - \lambda_{\mathcal{H}})$ where the summation ranges over all hexes $\mathcal{H}$ through the line xy. By symmetry, the number of grid-quads through y is also equal to $\mu + \sum(a_{\mathcal{H}} - \lambda_{\mathcal{H}})$. Hence every two collinear points are contained in the same number of grid-quads. By connectedness of $\mathcal{S}$, it follows that every point of $\mathcal{S}$ is contained in the same number of grid-quads. In a similar way, one shows that this property also holds for the $W(2)$-quads and the $Q(5,2)$-quads. Because every two intersecting lines of $\mathcal{S}$ are contained in a unique quad, $t_{\mathcal{S}}(t_{\mathcal{S}}+1) = 2a_{\mathcal{S}} + 6b_{\mathcal{S}} + 20c_{\mathcal{S}}$.

(b) Let x be an arbitrary point of $\mathcal{S}$. Obviously, $|\Gamma_0(x)| = 1$ and $|\Gamma_1(x)| = 2(t_{\mathcal{S}}+1)$. Since every point at distance two from x is contained in a unique quad together with x, $|\Gamma_2(x)| = 4a_{\mathcal{S}} + 8b_{\mathcal{S}} + 16c_{\mathcal{S}}$.

Let N_i, $i \in \{0,1,2,3\}$, be the total number of lines of $\mathcal{S}$ at distance i from x. An easy counting yields

$$\begin{aligned}
N_2 &= |\Gamma_2(x)|(t_\mathcal{S}+1) - 2N_1; \\
N_3 &= |\Gamma_3(x)|(t_\mathcal{S}+1) - 2N_2; \\
2N_3 &= |\Gamma_4(x)|(t_\mathcal{S}+1); \\
|\Gamma_3(x)| + |\Gamma_4(x)| &= v_\mathcal{S} - 1 - 2(t_\mathcal{S}+1) - (4a_\mathcal{S} + 8b_\mathcal{S} + 16c_\mathcal{S}).
\end{aligned}$$

These equations allow us to calculate $|\Gamma_3(x)|$ and $|\Gamma_4(x)|$ in terms of $v_\mathcal{S}$, $t_\mathcal{S}$, $a_\mathcal{S}$, $b_\mathcal{S}$ and $c_\mathcal{S}$. We obtain the required equations. □

Lemma 4.3.3
Let $(x, \mathcal{H})$ be a point-hex pair of $\mathcal{S}$. Let $f_x : \mathcal{H} \to \mathbb{N};\ y \mapsto \mathrm{d}(x,y) - \mathrm{d}(x,\mathcal{H})$. Then

- *f_x is a classical valuation of $\mathcal{H}$ if and only if $\mathrm{d}(x,\mathcal{H}) \leq 1$;*
- *f_x is an ovoidal valuation of $\mathcal{H}$ if and only if $\mathrm{d}(x,\mathcal{H}) = 3$.*

Proof
By Proposition 3.1.3, f_x is a valuation of $\mathcal{H}$.

(a) If f_x is classical, then $\max\{f_x(u)|u \in \mathcal{H}\} = 3$. Since $f_x(u) = \mathrm{d}(x,u) - \mathrm{d}(x,\mathcal{H})$, and $\mathrm{d}(x,u) \leq 4$, it follows that $\mathrm{d}(x,\mathcal{H}) \leq 1$. Conversely, if $\mathrm{d}(x,\mathcal{H}) \leq 1$, then $(x,\mathcal{H})$ is a classical pair and one easily sees that f_x must be classical.

(b) If $\mathrm{d}(x,\mathcal{H}) = 3$, then clearly $\max\{f_x(u)|u \in \mathcal{H}\} = 1$. Hence by Proposition 3.1.7, f_x is ovoidal. Conversely, suppose that f_x is ovoidal. Then $\max\{f_x(u)|u \in \mathcal{H}\} = 1$. Suppose that $\mathrm{d}(x,\mathcal{H}) = 2$. By Lemma 4.2.1, there exist two points y and z in O_{f_x} such that $\mathrm{d}(y,z) = 3$. Let $\mathcal{Q} := \mathcal{C}(x,y)$. Because $\mathcal{Q} \cap \mathcal{H} = \{y\}$, $\mathrm{d}(z,\mathcal{Q}) = 2$. If z is classical with respect to $\mathcal{Q}$, then $\mathrm{d}(z,y) = \mathrm{d}(z,x) + \mathrm{d}(x,y) = 4$, a contradiction. So z is ovoidal with respect to $\mathcal{Q}$. But then every line of $\mathcal{Q}$ through y contains a point at distance two from z, implying that this line is contained in $\mathcal{H}$, contradicting $\mathcal{Q} \cap \mathcal{H} = \{y\}$. This proves the lemma. □

4.4 Near octagons of type $(Q(5,2) \otimes Q(5,2)) \otimes Q(5,2)$

In this section, we will prove that there are up to isomorphism exactly two glued near octagons of type $(Q(5,2) \otimes Q(5,2)) \otimes Q(5,2)$.

4.4.1 Spreads of symmetry of $Q(5,2)$

Lemma 4.4.1
If S_1 and S_2 are two spreads of symmetry of $Q(5,2)$, then $|S_1 \cap S_2| \in \{1,3,9\}$.

Proof
Consider two different spreads of symmetry S_1 and S_2 of $Q(5,2)$. As shown in Theorem 2.1.2, spreads of symmetry of $Q(5,2) \cong H^D(3,4)$ correspond to nontangent plane intersections with the hermitian variety $H(3,4)$ in $\mathsf{PG}(3,4)$. Let π_i, $i \in \{1,2\}$, be the nontangent plane of $\mathsf{PG}(3,4)$ corresponding with S_i. Since π_1 and π_2 are both nontangent, $\pi_1 \cap \pi_2 \not\subset H(3,4)$. Hence $|S_1 \cap S_2| = |\pi_1 \cap \pi_2 \cap H(3,4)| \in \{1,3\}$. □

Lemma 4.4.2
Let S be a spread of symmetry of $Q(5,2)$ and let A and B be two reguli of S. Then there exists an element $\theta \in \mathsf{Aut}(Q(5,2))$ stabilizing S, and mapping A to B such that $d(x,x^\theta) = d(y,y^\theta) = d(z,z^\theta)$ for every line $\{x,y,z\}$ of S.

Proof
Define a graph Γ on $\mathsf{GF}(3)^3$ as follows. Two vertices (x_1,y_1,z_1) and (x_2,y_2,z_2) are adjacent if and only if

- $x_1 \neq x_2$, $(y_1,z_1) = (y_2,z_2)$, or
- $x_2 = x_1 + y_1 z_2 - y_2 z_1$, $(y_1,z_1) \neq (y_2,z_2)$.

Then by Section 2.5.2 of [11], the incidence structure $\mathcal{S}$ with points the vertices of Γ, with lines the maximal cliques of Γ and with containment as incidence relation, is isomorphic to $Q(5,2)$. As explained in Section 2.8 of [11], we may assume that S corresponds to the set of lines of the form $\{(\sigma,a,b) | \sigma \in \mathsf{GF}(3)\}$, $a,b \in \mathsf{GF}(3)$. Let $\mathcal{A}_S$ be the set of automorphisms of $Q(5,2)$ stabilizing S. Every element $\theta \in \mathcal{A}_S$ is of the following form (see again Section 2.8 of [11]):

$$\begin{bmatrix} x^\theta \\ y^\theta \\ z^\theta \end{bmatrix} = \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & ba_{21} - ca_{11} & ba_{22} - ca_{12} \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d \\ b \\ c \end{bmatrix},$$

with $a_{11}, a_{12}, a_{21}, a_{22}, b, c, d \in \mathsf{GF}(3)$ such that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. The linear space whose points are the lines of S, whose lines are the reguli of S and with natural incidence, is isomorphic to $\mathsf{AG}(2,3)$. An element of $\mathcal{A}_S$ induces an automorphism of $\mathsf{AG}(2,3)$ of the following form:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix}.$$

Since all automorphisms of $\mathsf{AG}(2,3)$ are of this form, it is easy to see that there exists a $\theta \in \mathcal{A}_S$ such that the induced automorphism of $\mathsf{AG}(2,3)$ maps the line A to the line B. Furthermore, we can choose θ in such a way that $a_{11}a_{22}-a_{12}a_{21} = 1$. One can easily check that each such θ is an automorphism of $Q(5,2)$ satisfying the conditions of the lemma. □

4.4.2 Spreads of symmetry of $Q(5,2) \otimes Q(5,2)$

Consider a near hexagon $\mathcal{H} \cong Q(5,2) \otimes Q(5,2)$. Let T_1 and T_2 be the two partitions of $\mathcal{H}$ in mutually disjoint $Q(5,2)$-quads and let $S^* := \{\mathcal{F}_1 \cap \mathcal{F}_2 | \mathcal{F}_1 \in T_1, \mathcal{F}_2 \in T_2\}$, i.e. the spread of $\mathcal{H}$ arising in a natural way from T_1 and T_2. By Theorem 16 of [14], S^* is a spread of symmetry of $\mathcal{H}$. We will use the following notations.

- The unique element of S^* through a point x of $\mathcal{H}$ is denoted as L_x.
- Let $L = \{x_1, x_2, x_3\}$ be a line of $\mathcal{H}$. If $L \in S^*$, put $[L] := \{L\}$, else put $[L] := \{L_{x_1}, L_{x_2}, L_{x_3}\}^\perp$.
- For every $i \in \{1,2\}$, let K_i be the set of all spreads of symmetry S of $\mathcal{H}$ different from S^*such that for every $\mathcal{Q} \in T_i$, $S_\mathcal{Q}$ is a spread of symmetry of $\mathcal{Q}$, and for all $\mathcal{Q}, \mathcal{Q}' \in T_i$, $S_{\mathcal{Q}'} = \pi_{\mathcal{Q}'}(S_\mathcal{Q}) := \{\pi_{\mathcal{Q}'}(L) | L \in S_\mathcal{Q}\}$. Every spread of symmetry of $\mathcal{H}$ different from S^* is contained in $K_1 \cup K_2$, see Theorem 2.3.2.
- A spread S of a quad $\mathcal{Q} \in T_1 \cup T_2$ is called a *good* spread of $\mathcal{Q}$ if S is a spread of symmetry of $\mathcal{Q}$ different from $S^*_\mathcal{Q}$ such that $[L] \in S$ for every $L \in S$. If A, B and C are three disjoint reguli of $S^*_\mathcal{Q}$, then $A \cup B^\perp \cup C^\perp$ is a good spread of $\mathcal{Q}$.
- Let S be a good spread of a quad $\mathcal{Q} \in T_i$, $i \in \{1,2\}$. Then $\overline{S} := \cup_{\mathcal{Q}' \in T_i} \pi_{\mathcal{Q}'}(S)$.

Lemma 4.4.3
If $S \in K_i$ with $i \in \{1,2\}$, then for every $\mathcal{Q} \in T_i$, $S_\mathcal{Q} \cap S^*_\mathcal{Q}$ is a regulus of $S^*_\mathcal{Q}$ and $S_\mathcal{Q}$ is a good spread of $\mathcal{Q}$.

Proof
Suppose that $S \in K_i$. Let $L = \{x_1, x_2, x_3\}$ be a line of S such that $L \notin S^*$ and let $\mathcal{Q}$ be the unique element of T_i containing L. Let $\mathcal{Q}_j$ be the unique element of T_{3-i} through the line L_{x_j}, $j \in \{1, 2, 3\}$. Since $\mathcal{Q}_1 \cong Q(5, 2)$, $S^*_{\mathcal{Q}_1}$ is not trivial and hence by Theorem 11 of [18], the group $\Pi_{S^*_{\mathcal{Q}_1}}(L_{x_1})$ of projectivities of L_{x_1} with respect to $S^*_{\mathcal{Q}_1}$ acts regularly on the points of L_{x_1}. Clearly, for every $\mathcal{Q}' \in T_i$, the projection of L on $\mathcal{Q}'$ is a line intersecting $\mathcal{Q}' \cap \mathcal{Q}_1$, $\mathcal{Q}' \cap \mathcal{Q}_2$ and $\mathcal{Q}' \cap \mathcal{Q}_3$. It follows that the group $\{(\pi_{\mathcal{Q}} \circ \pi_{\mathcal{Q}'} \circ \pi_{\mathcal{Q}''})_{|\mathcal{Q}} | \mathcal{Q}', \mathcal{Q}'' \in T_i\}$ has a regular action on the lines of $[L]$. Since $S_{\mathcal{Q}'} = \pi_{\mathcal{Q}'}(S_{\mathcal{Q}''})$ for every $\mathcal{Q}', \mathcal{Q}'' \in T_i$, it follows that $[L] \subset S_{\mathcal{Q}}$. One now easily sees that $|S_{\mathcal{Q}} \cap S^*_{\mathcal{Q}}| \in \{0, 3, 6\}$ and hence by Lemma 4.4.1, $|S_{\mathcal{Q}} \cap S^*_{\mathcal{Q}}| = 3$ and $S_{\mathcal{Q}} \cap S^*_{\mathcal{Q}}$ is a regulus. Again since $S_{\mathcal{Q}'} = \pi_{\mathcal{Q}'}(S_{\mathcal{Q}''})$ for all $\mathcal{Q}', \mathcal{Q}'' \in T_i$, $|S_{\mathcal{Q}'} \cap S^*_{\mathcal{Q}'}| = 3$ for every $\mathcal{Q}' \in T_i$. As a corollary, for every $\mathcal{Q}' \in T_i$ and for every line $L \in S_{\mathcal{Q}'}$, $[L] \subset S_{\mathcal{Q}'}$. □

Let S be a good spread of $\mathcal{Q} \in T_i$, $i \in \{1, 2\}$. Then $S = A \cup B^{\perp} \cup C^{\perp}$, with A, B and C three disjoint reguli of $S^*_{\mathcal{Q}}$. If $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are the quads of T_{3-i} containing the lines of A, then we say that S is of type $[\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3]$. Clearly if a good spread of certain type in $\mathcal{Q}$ exists, it is unique. One easily sees that the projection of a good spread S of $\mathcal{Q} \in T_i$ on $\mathcal{Q}' \in T_i$, is a good spread of $\mathcal{Q}'$ of the same type as S. This proves the following lemma.

Lemma 4.4.4
If S_1 (in $\mathcal{Q}_1$) and S_2 (in $\mathcal{Q}_2$) are good spreads of same type, then $\overline{S_1} = \overline{S_2}$.

Lemma 4.4.5
Let θ be an automorphism of $\mathcal{Q} \in T_i$, $i \in \{1, 2\}$, such that θ stabilizes $S^*_{\mathcal{Q}}$ and $\mathrm{d}(x, x^\theta) = \mathrm{d}(y, y^\theta) = \mathrm{d}(z, z^\theta)$ for every line $\{x, y, z\}$ of $S^*_{\mathcal{Q}}$. Then the map $\phi : x \mapsto \pi_{\mathcal{Q}'} \circ \theta \circ \pi_{\mathcal{Q}}(x)$, with $\mathcal{Q}'$ the unique quad of T_i through x, is an automorphism of $\mathcal{H}$.

Proof
We may assume without loss of generality that $i = 1$. By definition, $x^\phi = x^\theta$ if $x \in \mathcal{Q}$. One easily checks that ϕ is a bijection from $\mathcal{H}$ to $\mathcal{H}$ fixing every element of T_1. Now consider a line M of $\mathcal{H}$. Suppose first that M is a line of $\mathcal{Q}' \in T_1$. Then M^ϕ is obtained from M by two projections and an automorphism. It is now easy to see that M^ϕ is again a line of $\mathcal{H}$ and that $M^\phi \in S^*$ if $M \in S^*$. Suppose now that $M = \{x_1, x_2, x_3\}$ is a line of $\mathcal{R} \in T_2$ and $M \notin S^*$. Let $L := \mathcal{R} \cap \mathcal{Q}$ and let $\mathcal{R}'$ be the unique quad of T_2 through L^θ. Let $\mathcal{Q}_i$ be the quad of T_i intersecting M in x_i, put $L_i := \mathcal{Q}_i \cap \mathcal{R}$, and $L'_i := \mathcal{Q}_i \cap \mathcal{R}'$, $i \in \{1, 2, 3\}$. We have two possibilities.

- $M \cap L \neq \emptyset$.

We may assume that $M \cap L = \{x_1\}$. Hence $\mathcal{Q}_1 = \mathcal{Q}$ and $L_1 = L$. We have $\pi_{\mathcal{Q}}(x_i) = x_1$ for every $i \in \{1,2,3\}$ and hence x_i^ϕ is the unique point of $\mathcal{R}' \cap \mathcal{Q}_i$ collinear with x_1^θ. Since $\pi_{\mathcal{Q}_2}(x_1)$ and $\pi_{\mathcal{Q}_3}(x_1)$ are collinear, also $\pi_{\mathcal{Q}_2}(x_1^\theta)$ and $\pi_{\mathcal{Q}_3}(x_1^\theta)$ are collinear. It follows that M^ϕ is a line.

- $M \cap L = \emptyset$.

  Let $\mathcal{G}$ be the unique grid of $\mathcal{R}'$ through L_1', L_2' and L_3' and let $M' \in \mathcal{G}$ be the projection of M on $\mathcal{R}'$. Since M and L are parallel, the points x_1, x_2 and x_3 of M are projected on three different points of L. Since $\mathrm{d}(x, x^\theta) = \mathrm{d}(y, y^\theta) = \mathrm{d}(z, z^\theta)$ for every line $\{x, y, z\}$ of $S^*_{\mathcal{Q}}$, the action of θ on L consists of a projection from L on the line L^θ, possibly followed by a cyclic permutation of the points of L^θ. Suppose first that for one of the points of M, say x_1, $x_1^\phi \in M'$. If $L = L^\theta$, then $M = M'$ and one can easily see that θ must fix L pointwise. It follows that $M^\phi = M$. We may therefore assume that $L \neq L^\theta$. Then $\mathrm{d}(x_1, x_1^\phi) = 1$ and since projections from one element of T_1 to another preserve distances, $\mathrm{d}(\pi_{\mathcal{Q}}(x_1), [\pi_{\mathcal{Q}}(x_1)]^\theta) = 1$. It follows that the action of θ on L consists of the projection from L on L^θ (followed by the trivial permutation). Hence also $\mathrm{d}(\pi_{\mathcal{Q}}(x_2), [\pi_{\mathcal{Q}}(x_2)]^\theta) = 1$ and $\mathrm{d}(\pi_{\mathcal{Q}}(x_3), [\pi_{\mathcal{Q}}(x_3)]^\theta) = 1$. Again since projections preserve distances, $\mathrm{d}(x_2, x_2^\phi) = 1$ and $\mathrm{d}(x_3, x_3^\phi) = 1$. It is now easy to see that x_2^ϕ and x_3^ϕ are the remaining points of M'. Hence M^ϕ is a line. Suppose now that none of the points x_1^ϕ, x_2^ϕ and x_3^ϕ are contained in M'. The projections of the points of L^θ on $\mathcal{G}$ determine a partition of $\mathcal{G}$ in ovoids and M^ϕ must necessarily contain an element of every ovoid. It can now be easily verified that M^ϕ must be one of the two lines of $\mathcal{G}$ disjoint from M'.

It follows that ϕ is a bijection of $\mathcal{H}$ mapping lines onto lines. Hence ϕ is an automorphism of $\mathcal{H}$. □

Lemma 4.4.6
If S is a good spread of $\mathcal{Q} \in T_i$, $i \in \{1,2\}$, then $\overline{S} \in K_i$.

Proof
Let $L \in \overline{S}$ and $u, v \in L$. We will now construct an automorphism ϕ of $\mathcal{H}$ fixing every line of $\overline{S}$ and mapping u on v. By Lemma 4.4.4, we may assume that $L \in \mathcal{Q}$ ($\in T_i$). Let θ be an automorphism of $\mathcal{Q}$ fixing every line of S and mapping u on v. Let $A := S \cap S^*_{\mathcal{Q}}$ and let B and C be the two reguli of S disjoint from A. Hence $S^*_{\mathcal{Q}} = A \cup B^\perp \cup C^\perp$. Since every line of S is fixed by θ, all lines of $A \subset S^*_{\mathcal{Q}}$ are fixed. Since all lines of $B \cup C$ are fixed, one can easily see that $L^\theta \in B^\perp$ if $L \in B^\perp$ and $L^\theta \in C^\perp$ if $L \in C^\perp$. Hence $S^*_{\mathcal{Q}}$

is stabilized by θ and one easily sees that $\mathrm{d}(x, x^\theta) = \mathrm{d}(y, y^\theta) = \mathrm{d}(z, z^\theta)$ for every line $\{x, y, z\}$ of $S^*_\mathcal{Q}$.

Now consider the map $\phi : x \mapsto \pi_{\mathcal{Q}'} \circ \theta \circ \pi_\mathcal{Q}(x)$, with $\mathcal{Q}'$ the unique quad of T_i through x. By the previous lemma, ϕ is an automorphism of $\mathcal{H}$ and it is easy to see that ϕ fixes every line of $\overline{S}$. Hence $\overline{S}$ is a spread of symmetry of $\mathcal{H}$ which is clearly contained in K_i. □

Lemma 4.4.7

Up to an automorphism, $Q(5,2) \otimes Q(5,2)$ has two spreads of symmetry.

Proof

Let S^1 and S^2 be two spreads of symmetry of $Q(5,2) \otimes Q(5,2)$ different from S^*. We have two possibilities.

(i) $S^1, S^2 \in K_i$ *for an* $i \in \{1, 2\}$.

Choose $\mathcal{Q} \in T_i$. By Lemma 4.4.2, there exists an automorphism of $\mathcal{Q}$ stabilizing $S^*_\mathcal{Q}$ and mapping $S^1_\mathcal{Q}$ on $S^2_\mathcal{Q}$ in such a way that $\mathrm{d}(x, x^\theta) = \mathrm{d}(y, y^\theta) = \mathrm{d}(z, z^\theta)$ for every line $\{x, y, z\}$ of $S^*_\mathcal{Q}$. By Lemma 4.4.5, there exists an automorphism of $Q(5,2) \otimes Q(5,2)$ mapping S^1 on S^2.

(ii) $S^1 \in K_i$ *and* $S^2 \in K_{3-i}$, $i \in \{1, 2\}$.

There exists an automorphism of $\mathcal{H}$ switching T_1 and T_2. This observation together with (i) shows that all spreads of symmetry of $\mathcal{H}$ different from S^* are equivalent. □

4.4.3 Near polygons of type $(Q(5,2) \otimes Q(5,2)) \otimes Q(5,2)$

Since there are up to isomorphism two spreads of symmetry in $Q(5,2) \otimes Q(5,2)$, there are exactly two glued near octagons of type $(Q(5,2) \otimes Q(5,2)) \otimes Q(5,2)$. We will denote the unique glued near octagon obtained by glueing $Q(5,2) \otimes Q(5,2)$ and $Q(5,2)$ using the 'natural' spread of symmetry S^* of $Q(5,2) \otimes Q(5,2)$ as $(Q(5,2) \otimes Q(5,2)) \otimes_1 Q(5,2)$, and the other one as $(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$.

4.5 Slim dense near octagons containing a big hex

Theorem 4.5.1

If $\mathcal{S}$ is a slim dense near octagon containing a big hex, then $\mathcal{S}$ is isomorphic to one of the examples given in Table 4.3.

Proof
Let $\mathcal{H}$ be a big hex of $\mathcal{S}$. By Table 1.3, we have three cases.

- $\mathcal{H}$ contains a big quad. Then $\mathcal{S} \in \mathcal{N}^{\times}$ by Theorem 2.6.5;
- $\mathcal{H} \cong \mathbb{E}_1$. Then $\mathcal{S} \cong \mathbb{E}_1 \times \mathbb{L}_3$ or $\mathcal{S} \cong \mathbb{E}_1 \otimes Q(5,2)$ by Theorem 2.4.14;,
- $\mathcal{H} \cong \mathbb{E}_2$. Then $\mathcal{S}$ is isomorphic to $\mathbb{E}_2 \times \mathbb{L}_3$ by Theorem 2.4.15. $\square$

In Table 4.4, we give an overview of the remaining (non big) hexes in the near octagons of Table 4.3.

near octagon	v	$t+1$	big hexes
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	81	4	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$W(2) \times \mathbb{L}_3 \times \mathbb{L}_3$	135	5	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$
$Q(5,2) \times \mathbb{L}_3 \times \mathbb{L}_3$	243	7	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$\mathbb{I}_3 \times \mathbb{L}_3$	315	7	$W(2) \times \mathbb{L}_3$, $\mathbb{I}_3$
$Q^D(6,2) \times \mathbb{L}_3$	405	8	$W(2) \times \mathbb{L}_3$, $Q^D(6,2)$
$(Q(5,2) \otimes Q(5,2)) \times \mathbb{L}_3$	729	10	$Q(5,2) \times \mathbb{L}_3$, $Q(5,2) \otimes Q(5,2)$
$\mathbb{G}_3 \times \mathbb{L}_3$	1215	13	$Q(5,2) \times \mathbb{L}_3$, $\mathbb{G}_3$
$\mathbb{E}_1 \times \mathbb{L}_3$	2187	13	$\mathbb{E}_1$
$\mathbb{E}_2 \times \mathbb{L}_3$	2277	16	$\mathbb{E}_2$
$\mathbb{E}_3 \times \mathbb{L}_3$	1701	16	$Q(5,2) \times \mathbb{L}_3$, $\mathbb{E}_3$
$H^D(5,4) \times \mathbb{L}_3$	2673	22	$Q(5,2) \times \mathbb{L}_3$, $H^D(5,4)$
$W(2) \times W(2)$	225	6	$W(2) \times \mathbb{L}_3$
$Q(5,2) \times W(2)$	405	8	$W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$Q(5,2) \times Q(5,2)$	729	10	$Q(5,2) \times \mathbb{L}_3$
$(Q(5,2) \otimes Q(5,2)) \otimes_1 Q(5,2)$	2187	13	$Q(5,2) \otimes Q(5,2)$
$(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$	2187	13	$Q(5,2) \otimes Q(5,2)$
$\mathbb{G}_3 \otimes Q(5,2)$	3645	16	$Q(5,2) \otimes Q(5,2)$, $\mathbb{G}_3$
$\mathbb{E}_1 \otimes Q(5,2)$	6561	16	$\mathbb{E}_1$
$H^D(5,4) \otimes Q(5,2)$	8019	25	$Q(5,2) \otimes Q(5,2)$, $H^D(5,4)$
$\mathbb{G}_4$	8505	22	$\mathbb{G}_3$
$\mathbb{H}_4$	945	10	$\mathbb{H}_3$
$\mathbb{I}_4$	2025	14	$Q^D(6,2)$
$Q^D(8,2)$	2295	15	$Q^D(6,2)$
$H^D(7,4)$	114939	85	$H^D(5,4)$

Table 4.3: Slim dense near octagons having a big hex

near octagon	non big hexes
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	–
$W(2) \times \mathbb{L}_3 \times \mathbb{L}_3$	–
$Q(5,2) \times \mathbb{L}_3 \times \mathbb{L}_3$	–
$\mathbb{I}_3 \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$Q^D(6,2) \times \mathbb{L}_3$	–
$(Q(5,2) \otimes Q(5,2)) \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$\mathbb{G}_3 \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$
$\mathbb{E}_1 \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$\mathbb{E}_2 \times \mathbb{L}_3$	$W(2) \times \mathbb{L}_3$
$\mathbb{E}_3 \times \mathbb{L}_3$	$W(2) \times \mathbb{L}_3$
$H^D(5,4) \times \mathbb{L}_3$	–
$W(2) \times W(2)$	–
$Q(5,2) \times W(2)$	–
$Q(5,2) \times Q(5,2)$	–
$(Q(5,2) \otimes Q(5,2)) \otimes_1 Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$\mathbb{G}_3 \otimes Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$\mathbb{E}_1 \otimes Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$H^D(5,4) \otimes Q(5,2)$	$Q(5,2) \times \mathbb{L}_3$
$\mathbb{G}_4$	$W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$, $\mathbb{H}_3$
$\mathbb{H}_4$	$W(2) \times \mathbb{L}_3$
$\mathbb{I}_4$	$\mathbb{H}_3$
$Q^D(8,2)$	–
$H^D(7,4)$	–

Table 4.4: Non big hexes of slim dense near octagons having a big hex

4.6 Classification of all slim dense near octagons

Consider the following ordering of the 11 slim dense near hexagons.

$\mathcal{N}_i$	near hexagon
$\mathcal{N}_1$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$\mathcal{N}_2$	$W(2) \times \mathbb{L}_3$
$\mathcal{N}_3$	$Q(5,2) \times \mathbb{L}_3$
$\mathcal{N}_4$	$Q(5,2) \otimes Q(5,2)$
$\mathcal{N}_5$	$\mathbb{H}_3 \cong \mathbb{I}_3$
$\mathcal{N}_6$	$Q^D(6,2)$
$\mathcal{N}_7$	$\mathbb{E}_3$
$\mathcal{N}_8$	$\mathbb{G}_3$
$\mathcal{N}_9$	$\mathbb{E}_1$
$\mathcal{N}_{10}$	$\mathbb{E}_2$
$\mathcal{N}_{11}$	$H^D(5,4)$

In this section, we will prove the following theorem.

Theorem 4.6.1
Let $\mathcal{S}$ be a slim dense near octagon and let i be the biggest integer such that $\mathcal{S}$ contains a hex isomorphic to $\mathcal{N}_i$. Then every hex of $\mathcal{S}$ isomorphic to $\mathcal{N}_i$ is big in $\mathcal{S}$. As a corollary, every slim dense near octagon is isomorphic to one of the examples given in Table 4.3.

Suppose the contrary. Let $\mathcal{H}$ be a hex isomorphic to $\mathcal{N}_i$ and let x denote a point of $\mathcal{S}$ at distance 2 from $\mathcal{H}$. Let f_x denote the valuation of $\mathcal{H}$ induced by x (see Proposition 3.1.3). We call a quad of $\mathcal{H}$ special if it is special with respect to the valuation f_x. We need to consider several cases.

Case 1. $i \in \{9, 10, 11\}$.

Proof
By Table 4.2, all valuations of $\mathbb{E}_1, \mathbb{E}_2$ and $H^D(5,4)$ are either classical or ovoidal. A contradiction now follows from Lemma 4.3.3. □

Case 2. $i = 8$.

Proof
By Table 4.2, $G_{f_x} \cong \overline{W(2)}$. Let $x' \in O_{f_x}$.
(a) Consider a special $W(2)$-quad $\mathcal{Q}$ through x'. Since x is ovoidal with respect to $\mathcal{Q}$ and $\mathrm{d}(x, \mathcal{Q}) = 2$, $\mathcal{H}' := \mathcal{C}(x, \mathcal{Q})$ is a hexagon containing a

$W(2)$-quad which is not big in $\mathcal{H}'$. By Table 1.3, $\mathcal{H}' \cong \mathbb{G}_3$ or $\mathcal{H}' \cong \mathbb{E}_3$. In both cases,

(i) there exists a $W(2)$-quad $\mathcal{Q}'$ through x intersecting $\mathcal{H}$ in a point z of $\mathcal{Q} \cap O_{f_x}$,

(ii) every line of $\mathcal{H}$ through z is contained in a unique special quad of $\mathcal{H}$.

The hexes through $\mathcal{Q}'$ determine a partition of the lines intersecting $\mathcal{Q}'$ in x. By Lemma 4.3.1, every hex through $\mathcal{Q}'$ intersects $\mathcal{H}$ in at least a line through z. By (ii) such a line is contained in a unique special quad through z and since x is ovoidal with respect to this special quad, one easily sees that every hex through $\mathcal{Q}'$ intersects $\mathcal{H}$ in a special quad through z. Since $G_{f_x} \cong \overline{W(2)}$, there are five special quads through z: three grid-quads and two $W(2)$-quads. Every hex through $\mathcal{Q}'$ intersecting $\mathcal{H}$ in a special grid-quad is necessary isomorphic to $\mathbb{G}_3$ and the remaining two hexes through $\mathcal{Q}'$ are isomorphic to $\mathbb{E}_3$ or $\mathbb{G}_3$. Suppose that α of them are isomorphic to $\mathbb{G}_3$. It follows that

$$t + 1 = 3 + (3 + \alpha)(t_{\mathbb{G}_3} + 1 - 3) + (2 - \alpha)(t_{\mathbb{E}_3} + 1 - 3),$$

where $\alpha \in \{0, 1, 2\}$. It follows that $t + 1 \in \{48, 51, 54\}$.

(b) Consider a special grid-quad $\mathcal{Q}$ through x'. Then $\mathcal{H}' := \mathcal{C}(x, \mathcal{Q})$ is a hexagon containing a grid-quad which is not big in $\mathcal{H}'$. Because $i = 8$, it follows that $\mathcal{H}' \cong \mathbb{H}_3$, $\mathcal{H}' \cong Q(5,2) \otimes Q(5,2)$ or $\mathcal{H}' \cong \mathbb{G}_3$. Since x is ovoidal with respect to $\mathcal{Q}$, there exist quads $\mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{Q}_3$ through x intersecting $\mathcal{Q}$ in the points x_1, x_2 and x_3 of $\mathcal{Q} \cap O_{f_x}$. Clearly these quads are not big in $\mathcal{H}'$. If $\mathcal{H}' \cong \mathbb{H}_3$ or $\mathcal{H}' \cong Q(5,2) \otimes Q(5,2)$, then $\mathcal{Q}_1 \cong \mathcal{Q}_2 \cong \mathcal{Q}_3 \cong \mathbb{L}_3 \times \mathbb{L}_3$. Suppose that $\mathcal{H}' \cong \mathbb{G}_3$ and that $\mathcal{Q}_1 \cong W(2)$. Then $(x_2, \mathcal{Q}_1)$ is an ovoidal point-$W(2)$-quad pair in $\mathcal{H}'$. By Section 4.2.5 one can now see that one of the quads $\mathcal{Q}_2, \mathcal{Q}_3$ is a grid. Hence there always exists a grid-quad $\mathcal{Q}'$ through x intersecting $\mathcal{H}$ in a point z of $\mathcal{Q} \cap O_{f_x}$. As in (a), $\mathcal{Q}'$ is contained in five hexes partitioning the lines intersecting $\mathcal{Q}'$ in x, three of them intersecting $\mathcal{H}$ in a special grid-quad and the remaining two intersecting $\mathcal{H}$ in a special $W(2)$-quad. From Table 1.3, the hexes through $\mathcal{Q}'$ intersecting $\mathcal{H}$ in a $W(2)$-quad are isomorphic to $\mathbb{G}_3$, and the remaining three hexes are isomorphic to $\mathbb{G}_3, \mathbb{H}_3$ or $Q(5,2) \otimes Q(5,2)$. Suppose α' of them are isomorphic to $\mathbb{G}_3$ and β' of them are isomorphic to $\mathbb{H}_3$. It follows that the number of lines through a point of $\mathcal{S}$ equals

$$2 + (2 + \alpha')(t_{\mathbb{G}_3} + 1 - 2) + \beta'(t_{\mathbb{H}_3} + 1 - 2) + (3 - \alpha' - \beta')(t_{Q(5,2) \otimes Q(5,2)} + 1 - 2),$$

where $\alpha' + \beta' \in \{0, 1, 2, 3\}$. Hence $t + 1 \in \{34, 37, 40, 43, 46, 49, 52\}$, contradicting (a). $\square$

Case 3. $i = 7$.

Proof
By Table 4.2, $G_{f_x} \cong \mathsf{PG}(2, 4)$. Hence $|O_{f_x}| = 21$. By Lemma 4.2.3, every $Q(5, 2)$-quad of $\mathcal{H}$ contains a unique point of O_{f_x}.

(a) Suppose that L is a line through x such that also the remaining points y and z of L have distance 2 to $\mathcal{H}$. Consider a point x' of O_{f_x} and let $\mathcal{Q}$ be a $Q(5, 2)$-quad of $\mathcal{H}$ through x'. Let y' and z' be the unique points of $\mathcal{Q} \cap O_{f_y}$ respectively $\mathcal{Q} \cap O_{f_z}$. Since $\mathrm{d}(x, y') = \mathrm{d}(x, z') = \mathrm{d}(y, z') = 3$ and all induced $Q(5, 2)$-quad valuations are classical, x', y' and z' are on a line $L_{\mathcal{Q}}$ through x'. Clearly we can repeat this argument for every $Q(5, 2)$-quad through x' in $\mathcal{H}$. Suppose now that for a $Q(5, 2)$-quad $\mathcal{Q}'$ of $\mathcal{H}$ through x', $L_{\mathcal{Q}'} \neq L_{\mathcal{Q}}$. Let y'' and z'' be the remaining points of $L_{\mathcal{Q}'}$ such that $y'' \in O_{f_y}$ and $z'' \in O_{f_z}$. Clearly $\mathcal{Q}'' := \mathcal{C}(L_{\mathcal{Q}}, L_{\mathcal{Q}'}) \cong W(2)$ and $(x, \mathcal{Q}'')$, $(y, \mathcal{Q}'')$ and $(z, \mathcal{Q}'')$ are ovoidal point-quad pairs. Any two ovoids of a $W(2)$ intersect in a point. So, let u be a point of $\mathcal{Q}''$ contained in the ovoids determined by x and y. Then since $\mathrm{d}(x, u) = \mathrm{d}(y, u) = 2$, it follows that $\mathrm{d}(z, u) = 1$, a contradiction. Hence the line $L_{\mathcal{Q}}$ must then be contained in the intersection of all $Q(5, 2)$-quads through x' in $\mathcal{H}$. But no line of $\mathcal{H}$ satisfies this property. It follows that every line through x is contained in a quad through x intersecting $\mathcal{H}$ in a point.

(b) Consider a special $W(2)$-quad $\mathcal{Q}$ of $\mathcal{H}$ and put $\mathcal{H}' := \mathcal{C}(x, \mathcal{Q})$. Since $\mathcal{H}'$ is a hex containing a $W(2)$-quad which is not big, $\mathcal{H}' \cong \mathbb{E}_3$ by Table 1.3 and the fact that $i = 7$. Hence every quad through x intersecting $\mathcal{Q}$ is isomorphic to $W(2)$. Repeating this for every special $W(2)$-quad and taking (a) into account, it follows that $t + 1 = 3|O_{f_x}| = 63$.

(c) Suppose that grid-quads occur. From Lemma 4.3.2, also x is contained in a grid-quad $\mathcal{G}$. The two lines of $\mathcal{G}$ through x are contained in two of the $W(2)$-quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$ through x intersecting $\mathcal{H}$. But from (b), $\mathcal{G} \subset \mathcal{C}(\mathcal{Q}_1, \mathcal{Q}_2) \cong \mathbb{E}_3$, a contradiction. Hence no grid-quads can occur. Because $i = 7$, it follows that only hexes isomorphic to $Q^D(6, 2)$ or $\mathbb{E}_3$ can occur.

(d) Consider a $Q(5, 2)$-quad $\mathcal{Q}$ in $\mathcal{S}$ and a point $x' \in \mathcal{Q}$. The α hexes through $\mathcal{Q}$ determine a partition of the lines intersecting $\mathcal{Q}$ in x. From (c) all these hexes are isomorphic to $\mathbb{E}_3$. It follows that $t+1 = 5+\alpha(15-5)$. From (b), a contradiction follows. $\square$

Case 4. $i = 6$.

Proof
By Table 4.2, O_{f_x} is the set of points of an ovoid in a $W(2)$-quad $\mathcal{Q}$ in $\mathcal{H}$. But then $\mathcal{C}(x, \mathcal{Q})$ is a hex containing a $W(2)$-quad which is not big. From $i = 6$ and Table 1.3, a contradiction follows. $\square$

Case 5. $i = 5$.

Proof
(a) If $\mathcal{Q}$ is a special $W(2)$-quad, then $\mathcal{C}(x, \mathcal{Q})$ is a hex containing a $W(2)$-quad which is not big. Since $i = 5$, a contradiction follows. Hence all special quads of $\mathcal{H}$ are grids. Consider a special grid-quad $\mathcal{Q}$. It is then clear to see that $\mathcal{C}(x, \mathcal{Q}) \cong \mathbb{H}_3$ or $\mathcal{C}(x, \mathcal{Q}) \cong Q(5,2) \otimes Q(5,2)$. As in (c) of Case 2, we see that every quad through x intersecting $\mathcal{H}$ in a point is a grid. As a corollary, $Q(5,2)$-quads cannot occur: such a quad $\mathcal{Q}'$ through a point of $\mathcal{H}$ must then intersect $\mathcal{H}$ in a line L and from Table 1.3, every hex through $\mathcal{Q}'$ and a $W(2)$-quad of $\mathcal{H}$ through L is isomorphic to $\mathbb{G}_3$ or $\mathbb{E}_3$, contradicting $i = 5$. It follows that only hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3, W(2) \times \mathbb{L}_3$ or $\mathbb{H}_3$ can occur.

Consider a line L of $\mathcal{S}$ intersecting $\mathcal{H}$ in a point y and let $z \in L \setminus \{y\}$. Let $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ be the grid-quads of $\mathcal{H}$ through y and let $\mathcal{H}_i := \mathcal{C}(L, \mathcal{G}_i)$.

(b) Let L' be another line through z. If L' contains a point z' at distance two from $\mathcal{H}$, then by (a), z' induces an ovoid in at least one of the grid-quads $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ containing the point y. Hence L' is contained in at least one $\mathcal{H}_i$, $i = 1,2,3$. It is also easy to see that every line through z at distance 1 from $\mathcal{H}$ is contained in exactly one $\mathcal{H}_i$, $i = 1,2,3$. Hence each line through z is contained in one of the hexes $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$.

(c) Suppose now that $\mathcal{Q}$ is a quad through L intersecting $\mathcal{H}$ in y and suppose that $\mathcal{Q}$ is contained in $\mathcal{H}_j$ and $\mathcal{H}_k$ ($1 \leq j < k \leq 3$). By (a), $\mathcal{Q}$ is a grid-quad. Let y' be a point of $\mathcal{Q}$ at distance 2 from y. Clearly, both $\mathcal{G}_j$ and $\mathcal{G}_k$ contain three points of $O_{f_{y'}}$. It follows that $G_{f_{y'}} \cong \mathsf{PG}(2,2)$, implying that also $\mathcal{G}_l$ ($l \in \{1,2,3\} \setminus \{j,k\}$) contains three points of $O_{f_{y'}}$. It follows that $\mathcal{H}_i \cong \mathbb{H}_3$ for every $i \in \{1,2,3\}$ and hence by (b), $t+1 = 2+3(t_{\mathbb{H}_3}-1) = 14$. By (a), $\mathcal{C}(y', v)$ is a grid-quad for every $v \in O_{f_{y'}}$ and any two of these grid-quads are contained in a unique hex isomorphic to $\mathbb{H}_3$. As a corollary, $(a_{\mathcal{S}}, b_{\mathcal{S}}, c_{\mathcal{S}}) = (7, 28, 0)$.

Suppose that y' is contained in α hexes of type $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, β hexes of type $W(2) \times \mathbb{L}_3$ and γ hexes of type $\mathbb{H}_3$. Counting pairs $(\mathcal{G}, L)$ where $\mathcal{G}$ is a grid-quad through y' and L a line through y' not contained in $\mathcal{G}$ yields that $7(t+1-2) = 3\alpha + 6\beta + 12\gamma$ or $28 = \alpha + 2\beta + 4\gamma$. Counting pairs $(\mathcal{W}, L)$ where $\mathcal{W}$ is a $W(2)$-quad through y' and L a line through y' not contained in $\mathcal{W}$ yields that $28(t+1-3) = \beta + 12\gamma$ or $308 = \beta + 12\gamma$. A contradiction follows.

(d) By (b) and (c), we may suppose that $\mathcal{H}_i \cap \mathcal{H}_j = \{L\}$, for every $1 \leq i < j \leq 3$. Because only hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$ or $\mathbb{H}_3$ occur, it follows from (b) that $7 \leq t+1 \leq 14$ or $t+1 = 16$. Now consider again a point y' at distance two from $\mathcal{H}$ such that $y \in O_{f_{y'}}$. From (c), $G_{f_{y'}} \cong \mathbb{L}_3$. Let $\mathcal{Q}$ be the grid-quad of $\mathcal{H}$ containing all points of $O_{f_{y'}}$. Now let $\mathcal{Q}'$ be a $W(2)$-quad of $\mathcal{H}$ disjoint from $\mathcal{Q}$. Clearly, y' is ovoidal with respect to $\mathcal{Q}'$. Let $\mathcal{U}$ and $\mathcal{V}$ be two hexes through y' intersecting $\mathcal{Q}'$ in, say, u and v. Suppose that M is a line through y' contained in $\mathcal{U}$ and $\mathcal{V}$. Clearly, M contains a point m at distance 2 from u. But then m is classical with respect to $\mathcal{Q}'$, implying that $\mathrm{d}(m, v) = 4$, a contradiction. Hence every line through y' is contained in a unique hex intersecting $\mathcal{Q}'$ in a point. It follows that $t+1 \geq 15$, and hence $t+1 = 16$. By (b) each $\mathcal{H}_i$, $i \in \{1,2,3\}$ must contain five more lines through z, implying that $\mathcal{H}_i \cong \mathbb{H}_3$ for every $i \in \{1,2,3\}$. It follows that every quad through L intersecting $\mathcal{H}$ in a line is a $W(2)$-quad. Now let $\mathcal{Q}''$ be a $W(2)$-quad of $\mathcal{H}$ through y and consider $\mathcal{C}(L, \mathcal{Q}'')$. Then the line L is contained in at least three $W(2)$-quads in $\mathcal{C}(L, \mathcal{Q}'')$. A contradiction follows from (a) and Table 1.3. $\square$

Case 6. $i = 4$.

Proof

Clearly $G_{f_x} \cong \mathsf{AG}(2,3)$ and hence every point of O_{f_x} is contained in four special grid-quads. Let $y \in O_{f_x}$, let $\mathcal{Q}$ be the quad through x and y and let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4$ be the four special grid-quads of $\mathcal{H}$ through y. Each of the hexes $\mathcal{C}(x, \mathcal{G}_j)$, $1 \leq j \leq 4$, contains a grid-quad which is not big. Because $i = 4$ and from Table 1.3, it follows that these hexes are isomorphic to $Q(5,2) \otimes Q(5,2)$. Let M be a line through y in $\mathcal{Q}$. Let M_1, M_2, M_3 respectively M_4 be the lines of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ respectively $\mathcal{G}_4$ that are contained in a $Q(5,2)$-quad with M. Then at least two of these lines are contained in the same $Q(5,2)$-quad of $\mathcal{H}$, implying that the hex through M and these lines has three $Q(5,2)$-quads through a point, a contradiction. $\square$

Case 7. $i = 3$.

Proof
Clearly O_{f_x} is an ovoid in a grid-quad of $\mathcal{H}$. It follows that hexes occur containing grid-quads that are not big, a contradiction. $\square$

Case 8. $i = 2$.

Proof
(a) If a point x' has distance two to a hex, then x' induces a semi-classical valuation in that hex: if the induced valuation has two points with value 0, then there exists two points u and v with value 0 such that $\mathrm{d}(u, v) = 2$, implying that $\mathcal{C}(x, u, v)$ is a hex of $\mathcal{S}$ containing quads that are not big, contradicting $i = 2$. Clearly the induced valuation cannot be classical. The property now follows from Table 4.2.

(b) Since there exist points at distance 2 from every hex isomorphic to $W(2)\times\mathbb{L}_3$, there exists a point-$W(2)$-quad pair $(x, \mathcal{Q})$ such that $\mathrm{d}(x, \mathcal{Q}) = 3$ and hence x induces an ovoid $\{x_1, \ldots, x_5\}$ of points at distance three from x in $\mathcal{Q}$. Since every two ovoids of $W(2)$ intersect in a point, every line through x is contained in one of the hexes $\mathcal{H}_i := \mathcal{C}(x, x_i)$, $i \in \{1, 2, 3, 4, 5\}$. It follows that $t + 1 \leq 20$. Suppose now that $\mathcal{H}_i \cap \mathcal{H}_j \neq \{x\}$ for certain $i, j \in \{1, 2, 3, 4, 5\}$ with $i \neq j$. Let $L \in \mathcal{H}_i \cap \mathcal{H}_j$ through x and let y be the unique point of L at distance two from x_i. It follows that $\mathrm{d}(y, x_j) = 4$, contradicting $y, x_j \in \mathcal{H}_j$. Hence $\mathcal{H}_i \cap \mathcal{H}_j = \{x\}$. Since every hex is isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ or $W(2) \times \mathbb{L}_3$, $15 \leq t + 1 \leq 20$ and $b_{\mathcal{S}} \geq t + 1 - 15 = t - 14$.

(c) Let $\mathcal{H}' \cong W(2) \times \mathbb{L}_3$ and let $x' \in \mathcal{H}'$. For every $i \in \{0, 1, 2, 3\}$, let N_i be the set of all points y of $\mathcal{S}$ such that $\mathrm{d}(y, \mathcal{H}') = \mathrm{d}(y, x') = i$. Clearly every point of N_1 has a unique neighbour in $\mathcal{H}'$, and by (a), every point of N_2 has a unique point in $\mathcal{H}'$ at distance 2. One can now verify that

- $|N_0| = 1$;
- $|N_1| = 2(t - t_{\mathcal{H}'}) = 2t - 6$;
- $|N_2| = |\Gamma_2(x')| - |\Gamma_2(x') \cap \mathcal{H}'| - 2(t_{\mathcal{H}'} + 1)|N_1|$.

Here $|\Gamma_2(x')| = 4a_{\mathcal{S}} + 8b_{\mathcal{S}}$ (see Lemma 4.3.2) and $|\Gamma_2(x') \cap \mathcal{H}'| = 20$ (see Section 4.1). Hence $|N_i|$, $i \in \{0, 1, 2\}$ is independent of the point x'. By Table 4.1 $v_{\mathcal{H}'} = 45$. Since $\mathrm{d}(y, \mathcal{H}') \leq 2$ for every $y \in \mathcal{S}$, it follows that

$$v_{\mathcal{S}} = v_{\mathcal{H}'}(|N_0| + |N_1| + |N_2|) = 45(23 - 14t + 4a_{\mathcal{S}} + 8b_{\mathcal{S}}).$$

Hence by Lemma 4.3.2,

$$n_3 \;=\; \frac{v_{\mathcal{S}}}{3} - 1 - 6t + 4a_{\mathcal{S}} + 8b_{\mathcal{S}} \;=\; 344 - 216t + 64a_{\mathcal{S}} + 128b_{\mathcal{S}}. \qquad (1)$$

(d) Consider a point x' and let A (respectively B) be the number of hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ (respectively $W(2) \times \mathbb{L}_3$) through x'. Consider the $a_{\mathcal{S}}$ grid-quads $\mathcal{G}_1, \ldots, \mathcal{G}_{a_{\mathcal{S}}}$ through x and suppose that $\mathcal{G}_i$ is contained in u_i hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ and v_i hexes isomorphic to $W(2) \times \mathbb{L}_3$. Since each line through x' outside $\mathcal{G}_i$ is contained in a unique hex through $\mathcal{G}_i$, $u_i + 2v_i = t - 1$ for every $i \in \{1, \ldots, a_{\mathcal{S}}\}$. Since each hex of $\mathcal{S}$ contains three grid-quads through a point, it easily follows that

$$A + 2 \cdot B \;=\; \frac{1}{3}\sum_{i=1}^{a_{\mathcal{S}}}(u_i + 2v_i) \;=\; \frac{1}{3}\sum_{i=1}^{a_{\mathcal{S}}}(t-1) \;=\; \frac{1}{3}a_{\mathcal{S}}(t-1). \qquad (2)$$

Since each hex isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ (respectively $W(2) \times \mathbb{L}_3$) through x' contains 8 (respectively 16) points at distance 3 from x',

$$n_3 \;=\; 8A + 16B. \qquad (3)$$

Equations (1), (2) and (3) yield

$$a_{\mathcal{S}}(t-1) \;=\; 129 - 81t + 24a_{\mathcal{S}} + 48b_{\mathcal{S}}. \qquad (4)$$

Lemma 2.4.11 yields

$$2a_{\mathcal{S}} + 6b_{\mathcal{S}} \;=\; t(t+1). \qquad (5)$$

From equations (4) and (5), it follows that

$$b_{\mathcal{S}} \;=\; \frac{t^3 - 24t^2 + 137t - 258}{6(t-9)}.$$

Since no two $W(2)$-quads intersect in a line, $3b_{\mathcal{S}} \leq t + 1 \leq 20$. Hence $0 \leq b_{\mathcal{S}} \leq 6$. Since $b_{\mathcal{S}} \in \mathbb{N}$ and $15 \leq t + 1 \leq 20$, it follows that $t = 17$ and $b_{\mathcal{S}} = 1$. This contradicts $b_{\mathcal{S}} \geq t - 14$. □

Case 9. $i = 1$.

Proof

(a) Because only grid-quads and hexes isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$ can occur, $\mathcal{S}$ is a regular near octagon with parameters $(s, t, t_2, t_3) = (2, t, 1, 2)$. Since

$$|\Gamma_k(x)| = 2^k \frac{\Pi_{i=0}^{k-1}(t - t_i)}{\Pi_{i=1}^{k}(1 + t_i)},\ k \in \{0, \dots 4\},$$

$\mathcal{S}$ has $v_{\mathcal{S}} = \sum_{k=0}^{4} \Gamma_k(x) = 4t^3 - 6t^2 + 8t + 3$ points.

For every $0 \leq k \leq 4$, let $A^{(k)}$ be the $v_{\mathcal{S}} \times v_{\mathcal{S}}$ matrix whose rows and columns are indexed by the points of $\mathcal{S}$ and such that $A^{(k)}_{x,y} = 1$ if $\mathrm{d}(x, y) = k$ and $A^{(k)}_{x,y} = 0$ in all other cases. Clearly $A^{(0)} = I$, the $v_{\mathcal{S}} \times v_{\mathcal{S}}$ identity matrix. Put $A := A^{(1)}$. It is now easy to calculate the following equality for every $i \in \{1, 2, 3\}$:

$$A \cdot A^{(i)} = 2(t - t_{i-1})A^{(i-1)} + (t_i + 1)A^{(i)} + (t_{i+1} + 1)A^{(i+1)}.$$

We can now easily write $A^{(2)}$, $A^{(3)}$ and $A^{(4)}$ as functions of A:

$$A^{(2)} = \frac{A^2 - A - 2(t+1)I}{2}; \tag{1}$$

$$A^{(3)} = \frac{A^3 - 3A^2 - 6tA + 4(t+1)I}{6}; \tag{2}$$

$$A^{(4)} = \frac{A^4 - 6A^3 + (15 - 12t)A^2 + (28t - 2)A + 12(t^2 - t - 2)I}{6(t+1)}. \tag{3}$$

Let J be the all-one $v_{\mathcal{S}} \times v_{\mathcal{S}}$ matrix. It follows that

$$\begin{aligned} J &= I + A + A^{(2)} + A^{(3)} + A^{(4)} \\ &= \frac{1}{6(t+1)}(A + (t+1)I)(A^3 - 6A^2 + (21 - 6t)A + (10t - 20)I). \end{aligned}$$

The matrix J has eigenvalues 0 and v, with multiplicities respectively $v_{\mathcal{S}} - 1$ and 1. With every eigenvalue of A, there corresponds an eigenvalue of J. Clearly $u_0 := 2(t+1)$ is the eigenvalue of A (with multiplicity $f_0 := 1$) corresponding to the eigenvalue v of J and every other eigenvalue of A is a root of the polynomial

$$p(x) := (x + (t+1))(x^3 - 6x^2 + (21 - 6t)x + (10t - 20))$$

in x. Put $u_1 := -(t+1)$ and let u_2, u_3 and u_4 be the three roots of $p(x)$. So, $u_2 + u_3 + u_4 = 6$, $u_2u_3 + u_2u_4 + u_3u_4 = 21 - 6t$ and $u_2u_3u_4 = 20 - 10t$. Let

f_i, $i \in \{1,2,3,4\}$, denote the multiplicity of u_i regarded as eigenvalue of A. We can determine f_i, $i \in \{1,2,3,4\}$, by solving the nonsingular system

$$\sum_{i=1}^{4} f_i \cdot u_i^k = tr(A^k),\ k = 0,1,2,3.$$

From (1), (2) and (3), we find

$$\begin{aligned} A^2 &= 2(t+1)I + A^{(1)} + 2A^{(2)}; \\ A^3 &= 2(t+1)I + 3(2t+1)A^{(1)} + 6A^{(2)} + 6A^{(3)}. \end{aligned}$$

Clearly $tr(I) = v_{\mathcal{S}}$ and $tr(A^{(i)}) = 0$ if $i > 0$. Hence

$$tr(A^2) = tr(A^3) = 2(t+1)tr(I) = 2(t+1)v_{\mathcal{S}}.$$

It now easily follows that

$$f_1 = \frac{e(u_2+u_3+u_4) - 2u_1(u_2u_3+u_3u_4+u_2u_4) + (v_{\mathcal{S}}-1)u_2u_3u_4 + f}{(u_2-u_1)(u_3-u_1)(u_4-u_1)},$$

where $e = 2(1+t)[v_{\mathcal{S}} + 2(1+t)]$ and $f = -2(1+t)[v_{\mathcal{S}} + 4(1+t)^2]$. Plugging in all values, we obtain that

$$f_1 = \frac{96 + 256t + 128t^3 - 192t^2}{48 + 20t + 3t^2 + t^3}.$$

From Corollary 2 of [20], $t+1 \leq 54$. Because $t+1 > 3$ and $f_1 \in \mathbb{N}$, it follows that $t+1 \in \{4,5,11\}$.

(b) Suppose now that O_{f_x} contains at least two points y and z, then $\mathcal{C}(x,y,z)$ is a hex containing a grid $\mathcal{C}(y,z)$ which is not big, a contradiction. It follows that there is a unique point $y \in \mathcal{H}$ at distance two from x. Since f_x is a semi-classical valuation, four points x_1, x_2, x_3 and x_4 of $\Gamma_3(y) \cap \mathcal{H}$ lie at distance three from x. Put now $\mathcal{H}_i := \mathcal{C}(x, x_i)$ for every $i \in \{1,2,3,4\}$. Then $\mathcal{H}_i \cap \mathcal{H} = \{x_i\}$ for every $1 \leq i \leq 4$. Suppose that a line L through x is contained in $\mathcal{H}_i$ and $\mathcal{H}_j$ $(1 \leq i < j \leq 4)$. Then a point x' on L lies at distance 2 from x_i. But also $\mathrm{d}(x_i, x_j) = 2$ and $\mathrm{d}(x_i, \mathcal{H}_j) = 2$. [If $\mathrm{d}(x_i, \mathcal{H}_j) = 1$, then because x_j is the unique point of $\mathcal{H}_j \cap \mathcal{H}$, x_i is collinear with a point y' of $\mathcal{H}_j$ outside $\mathcal{H}$, and $\mathrm{d}(y', x_j) = 1$. Since $\mathcal{H}$ is geodetically closed, y' must be contained in $\mathcal{H}$, a contradiction.] This contradicts the fact that $|O_{f'_{x_i}}| = 1$ (where f'_{x_i} is the valuation of $\mathcal{H}_j$ induced by x_i). It follows that $t+1 \geq 12$, contradicting $t+1 \in \{4,5,11\}$. $\square$

Corollary 4.6.2
Let $\mathcal{S}$ be a slim dense near polygon not containing a geodetically closed sub near octagon isomorphic to $(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$ and let $\mathcal{H}$ be one of the 11 slim dense near hexagons. Then there exists a constant $\alpha_{\mathcal{H}}$ such that every point of $\mathcal{S}$ is contained in precisely $\alpha_{\mathcal{H}}$ hexes isomorphic to $\mathcal{H}$.

Proof
For every slim dense near octagon $\mathcal{O}$, let $a_{\mathcal{O}}$ denote the constant number of $\mathcal{H}$-hexes through a point. Now, consider two different collinear points x and y of $\mathcal{S}$. Let μ denote the number of $\mathcal{H}$-hexes through xy. For every geodetically closed sub near octagon $\mathcal{O}$ through xy, let $\lambda_{\mathcal{O}}$ denote the number of $\mathcal{H}$-hexes through xy. Then the total number of $\mathcal{H}$-hexes through x is equal to $\mu + \sum(a_{\mathcal{O}} - \lambda_{\mathcal{O}})$, where the summation ranges over all geodetically closed sub near octagons $\mathcal{O}$ through the line xy. By symmetry, the number of $\mathcal{H}$-hexes through y is also equal to $\mu + \sum(a_{\mathcal{O}} - \lambda_{\mathcal{O}})$. Hence, every two collinear points of $\mathcal{S}$ are contained in the same number of $\mathcal{H}$-hexes. By connectedness of $\mathcal{S}$, it follows that every point of $\mathcal{S}$ is contained in the same number of $\mathcal{H}$-hexes. $\square$

Appendix A

Nederlandstalige samenvatting

In deze appendix vatten we kort samen wat zich in de thesis bevindt. Enkel de belangrijkste resultaten zullen vermeld worden. Voor meer details en bewijzen verwijzen we naar de Engelstalige tekst.

A.1 Inleiding

A.1.1 Incidentiestructuren

De in deze thesis beschouwde incidentiestructuren zijn *punt-rechte-incidentie-structuren*. Afstanden worden steeds gemeten in de *puntgraaf*. De *diameter* van een incidentiestructuur is de maximale afstand tussen twee punten. Een incidentiestructuur wordt een *partieel lineaire ruimte* genoemd als er door elke twee verschillende punten ten hoogste één rechte gaat en als elke rechte minstens twee punten bevat. Een partieel lineaire ruimte wordt een *lineaire ruimte* genoemd als elke twee verschillende punten bevat zijn in een unieke rechte.

A.1.2 Schierveelhoeken

Een *schierveelhoek* is een samenhangende partieel lineaire ruimte met eindige diameter die aan de volgende voorwaarde voldoet.

> *Voor elk punt p en elke rechte L, bestaat er een uniek punt op L dat het dichtst bij p is gelegen.*

Heeft een schierveelhoek diameter d, dan wordt deze ook een schier-$2d$-hoek genoemd. De unieke schier-0-hoek bestaat uit één enkel punt. De schier-2-hoeken zijn rechten. We noteren de unieke rechte met i punten als $\mathbb{L}_i$. De schier-4-hoeken zijn precies de *veralgemeende vierhoeken*.

Heeft een schierveelhoek $\mathcal{S}$ een constant aantal punten $s+1$ op elke rechte en een constant aantal rechten $t+1$ door een punt, dan zeggen we dat $\mathcal{S}$ orde (s,t) heeft. Een schierveelhoek wordt *dun* (respectievelijk *slank*) genoemd als elke rechte precies twee (respectievelijk drie) punten bevat. Een schierveelhoek wordt *dicht* genoemd als elke rechte minstens drie punten bevat en als elke twee punten op afstand 2 minstens twee gemeenschappelijke buren hebben.

A.1.3 Deelstructuren

Veronderstel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ een schierveelhoek is. Een niet-ledige deelverzameling X van $\mathcal{P}$ wordt een *deelruimte* genoemd als voor elke twee verschillende collineaire punten x en y van X, alle punten van de unieke rechte door x en y eveneens bevat zijn in X. Elke deelruimte X van $\mathcal{S}$ induceert een partieel lineaire ruimte $\mathcal{S}_X = (X, \mathcal{L}', \mathbf{I}')$, met $\mathcal{L}'$ de verzameling rechten van $\mathcal{S}$ die volledig in X gelegen zijn en met incidentie $\mathbf{I}'$ geïnduceerd door $\mathcal{S}$. Een deelverzameling X van $\mathcal{P}$ wordt *geodetisch gesloten* genoemd als X een deelruimte is en als alle punten op een geodetisch pad tussen twee punten van X opnieuw in X gelegen zijn. Als X een geodetisch gesloten deelruimte is, dan is $\mathcal{S}_X$ een deelschierveelhoek van $\mathcal{S}$. Een niet-ontaarde geodetisch gesloten deelschiervierhoek wordt een *quad* genoemd. Een geodetisch gesloten deelschierzeshoek in een dichte schierveelhoek wordt een *hex* genoemd. Een *ovoïde* van een schierveelhoek $\mathcal{S}$ is een verzameling O van punten van $\mathcal{S}$ zodanig dat elke rechte van $\mathcal{S}$ een uniek punt van O bevat. Een *spread* van $\mathcal{S}$ is een verzameling rechten van $\mathcal{S}$ die de puntenverzameling partitioneert.

De volgende stelling is van groot belang in de theorie van de dichte schierveelhoeken.

Stelling A.1.1 (Stelling 4 van [6]**)**
Als x en y twee punten zijn van een dichte schierveelhoek $\mathcal{S}$ op afstand i van elkaar, dan zijn ze bevat in een unieke geodetisch gesloten deelschier-$2i$-hoek van $\mathcal{S}$.

A.1.4 Veralgemeende 2d-hoeken

Een schier-$2d$-hoek wordt een *veralgemeende $2d$-hoek* genoemd als voor elke $i \in \{1, \ldots, d-1\}$ en voor elke twee punten x en y op afstand i, $|\Gamma_{i-1}(x) \cap \Gamma(y)| = 1$. De *veralgemeende vierhoeken* zijn precies de schiervierhoeken. Veralgemeende vierhoeken zijn reeds uitvoerig bestudeerd (zie bv. [36]). Alle slanke dichte veralgemeende vierhoeken zijn gekend ([36]).

- Het 3×3-rooster $\mathbb{L}_3 \times \mathbb{L}_3$ heeft orde $(2,1)$.

- De veralgemeende vierhoek $W(2)$ heeft orde $(2,2)$ of kortweg orde 2.
- De veralgemeende vierhoek $Q(5,2)$ heeft orde $(2,4)$.

Deze veralgemeende vierhoeken komen vaak voor als quads in schierveelhoeken. We zullen naar deze quads verwijzen als grid-quads, $W(2)$-quads of $Q(5,2)$-quads.

A.1.5 Productschierveelhoeken en gelijmde schierveelhoeken

In de thesis zijn twee methodes om nieuwe schierveelhoeken te construeren uit gekende schierveelhoeken van groot belang.

Productschierveelhoeken

Zijn $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathbf{I}_1)$ en $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathbf{I}_2)$ twee schierveelhoeken, dan kan men een nieuwe schierveelhoek $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ als volgt uit $\mathcal{S}_1$ en $\mathcal{S}_2$ construeren:

- $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$;
- $\mathcal{L} := (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$;
- het punt (x,y) van $\mathcal{S}$ is incident met de rechte $(z,L) \in \mathcal{P}_1 \times \mathcal{L}_2$ als en slechts als $x = z$ en $y \ \mathbf{I}_2 \ L$; het punt (x,y) van $\mathcal{S}$ is incident met de rechte $(M,u) \in \mathcal{L}_1 \times \mathcal{P}_2$ als en slechts als $y = u$ en $x \ \mathbf{I}_2 \ M$.

De schierveelhoek wordt *het direct product van $\mathcal{S}_1$ en $\mathcal{S}_2$* genoemd en we noteren $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$. Is $\mathrm{diam}(\mathcal{S}_1) \geq 1$ en $\mathrm{diam}(\mathcal{S}_2) \geq 1$, dan wordt $\mathcal{S}$ een *productschierveelhoek* genoemd.

Gelijmde schierveelhoeken

Een andere methode om nieuwe schierveelhoeken te construeren is het zogenaamd *lijmen* van schierveelhoeken. Zie onder andere [16], [18] en [23]. We kunnen lijmingen van 'type' δ beschouwen voor elke $\delta \in \mathbb{N}$. De gelijmde schierveelhoeken van type 0 zijn precies de productschierveelhoeken. Het resultaat van een lijming van type δ toegepast op twee schierveelhoeken $\mathcal{S}_1$ en $\mathcal{S}_2$ noteren we als $\mathcal{S}_1 \otimes_\delta \mathcal{S}_2$. In [23] worden nodige en voldoende voorwaarden opgesomd opdat $\mathcal{S}_1 \otimes_\delta \mathcal{S}_2$ opnieuw een schierveelhoek zou zijn. Uit de precieze definitie van lijming volgt onmiddelijk dat er twee partities T_1 en T_2 van $\mathcal{S}_1 \otimes_\delta \mathcal{S}_2$ bestaan, die aan de volgende voorwaarden voldoen:

P₁ alle elementen van T_i, $i \in \{1,2\}$, zijn isomorf met $\mathcal{S}_i$;

P₂ $\mathrm{diam}(\mathcal{F}_1 \cap \mathcal{F}_2) = \delta$ voor elke $\mathcal{F}_1 \in T_1$ en $\mathcal{F}_2 \in T_2$;

P₃ voor alle $\mathcal{F}_1 \in T_1$, $\mathcal{F}_2 \in T_2$ en $x \in \mathcal{F}_1 \cap \mathcal{F}_2$, $\Gamma_1(x) \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$.

In deze thesis bedoelen we met 'lijming' altijd een lijming van type $\delta = 1$ en schrijven we ook $\mathcal{S}_1 \otimes \mathcal{S}_2$ in plaats van $\mathcal{S}_1 \otimes_1 \mathcal{S}_2$. Zijn $\mathcal{S}_1$ en $\mathcal{S}_2$ twee schierveelhoeken die geen productschierveelhoeken zijn en is $\mathcal{S}_1 \otimes \mathcal{S}_2$ een gelijmde schierveelhoek, dan moeten $\mathcal{S}_1$ en $\mathcal{S}_2$ noodzakelijk een *symmetriespread* bevatten. Een spread S van een schierveelhoek $\mathcal{S}$ wordt een symmetriespread genoemd als er voor elke rechte $K \in S$ en voor elke twee punten k_1 en k_2 van K een automorfisme van $\mathcal{S}$ bestaat die elke rechte van S fixeert en k_1 op k_2 afbeeldt. Een deel van de thesis bestond erin om voor bepaalde schierveelhoeken na te gaan of ze al dan niet een symmetriespread bevatten en bijgevolg kunnen gelijmd worden.

A.1.6 De gekende slanke dichte schierveelhoeken

In deze thesis werd hoofdzakelijk de aandacht gevestigd op de slanke dichte schierveelhoeken. In deze sectie geven we een overzicht van de gekende slanke dichte schierveelhoeken. De slanke dichte veralgemeende vierhoeken hebben we reeds eerder vermeld.

De duale polaire ruimten $Q^D(2n,2)$ en $H^D(2n-1,4)$, $n \geq 2$

Beschouw $Q(2n,2)$, de polaire ruimte van rang n geassocieerd met een niet-singuliere kwadriek in $\mathsf{PG}(2n,2)$, $n \geq 2$. Vertrekkende van $Q(2n,2)$ kan men als volgt de *duale polaire ruimte* $Q^D(2n,2)$ construeren:

- de punten van $Q^D(2n,2)$ zijn de maximale deelruimten van $Q(2n,2)$ (de deelruimten van dimensie $n-1$);
- de rechten van $Q^D(2n,2)$ zijn de deelruimten van $Q(2n,2)$ die dimensie $n-2$ hebben;
- incidentie is omgekeerde inclusie.

Dan is $Q^D(2n,2)$ een slanke dichte schier-$2n$-hoek voor elke $n \geq 2$ ([8]). Op analoge wijze kunnen we een slanke dichte schier-$2n$-hoek $H^D(2n-1,4)$ definiëren, vertrekkend van de polaire ruimte $H(2n-1,4)$ van rang n, geassocieerd met een niet-singuliere hermitische variëteit in $\mathsf{PG}(2n-1,4)$, $n \geq 2$. De punten en de rechten van $H^D(2n-1,4)$ zijn de deelruimten van $H(2n-1,4)$ die respectievelijke dimensies $n-1$ en $n-2$ hebben.

De oneindige klassen $\mathbb{G}_n$, $\mathbb{H}_n$ en $\mathbb{I}_n$, $n \geq 2$

• Beschouw de vectorruimte $V(2n,4)$, $n \geq 1$, met basis $\{\vec{e}_0, \ldots, \vec{e}_{2n-1}\}$, uitgerust met de hermitische vorm $(\sum x_i \vec{e}_i, \sum y_i \vec{e}_i) = x_0 y_0^2 + x_1 y_1^2 + \ldots + x_{2n-1} y_{2n-1}^2$. Stel dat $H = H(2n-1,4)$ de corresponderende hermitische variëteit is in $\mathsf{PG}(2n-1,4)$. Voor elke vector $\vec{x}$ van $V(2n,4)$ hebben we dat $\vec{x} = \sum (\vec{x}, \vec{e}_i)\vec{e}_i$. De *ondersteuning* van een punt $p = \langle \vec{x} \rangle$ van $\mathsf{PG}(2n-1,4)$ is de verzameling van alle $i \in \{0, \ldots, 2n-1\}$ zodanig dat $(\vec{x}, \vec{e}_i) \neq 0$. Een deelruimte π van H wordt *goed* genoemd als ze wordt voortgebracht door een (mogelijks lege) verzameling $\mathcal{G}_\pi \subseteq H$ van punten wiens ondersteuningen onderling disjunct zijn. Als π goed is, dan is $\mathcal{G}_\pi$ uniek bepaald. Veronderstel dat Y, respectievelijk Y' de verzameling is van alle goede deelruimten van dimensie $n-1$, respectievelijk $n-2$. Dan is de incidentiestructuur $\mathbb{G}_n = (Y, Y', \mathbf{I})$, met $\mathbf{I}$ de omgekeerde inclusie, een slanke dichte schier-$2n$-hoek ([21]).

• Beschouw een verzameling V met $2n+2$ elementen, $n \geq 1$, en stel dat $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ de volgende incidentiestructuur is:

- $\mathcal{P}$ is de verzameling van alle partities van V in $n+1$ verzamelingen van 2 elementen;
- $\mathcal{L}$ is de verzameling van alle partities van V in $n-1$ verzamelingen van 2 elementen en één verzameling van 4 elementen;
- een punt $p \in \mathcal{P}$ is incident met een rechte $L \in \mathcal{L}$ als en slechts als de partitie bepaald door p een verfijning is van de partitie bepaald door L.

Dan is $\mathbb{H}_n$ een slanke dichte schier-$2n$-hoek ([4]).

• Beschouw de niet-singuliere kwadriek $Q(2n,2)$, $n \geq 2$, in $\mathsf{PG}(2n,2)$ en een hypervlak Π van $\mathsf{PG}(2n,2)$ die $Q(2n,2)$ snijdt in een niet-singuliere hyperbolische kwadriek $Q^+(2n-1,2)$. Veronderstel dat $\mathbb{I}_n$ de volgende incidentiestructuur is:

- de punten van $\mathbb{I}_n$ zijn de maximale deelruimten van $Q(2n,2)$ die niet bevat zijn in $Q^+(2n-1,2)$;
- de rechten van $\mathbb{I}_n$ zijn de op één na maximale deelruimten van $Q(2n,2)$ die niet bevat zijn in $Q^+(2n-1,2)$;
- incidentie is de omgekeerde inclusie.

Dan is $\mathbb{I}_n$ een slanke dichte schier-$2n$-hoek ([4]).

We vermelden enkele isomorfismes tussen de tot nu toe opgesomde slanke dichte schierveelhoeken.

- $Q(5,2) \cong \mathbb{G}_2 \cong H^D(3,4)$;
- $W(2) \cong \mathbb{H}_2 \cong Q^D(4,2)$;
- $\mathbb{L}_3 \times \mathbb{L}_3 \cong \mathbb{I}_2$;
- $\mathbb{I}_3 \cong \mathbb{H}_3$.

Drie 'uitzonderlijke' slanke dichte schierveelhoeken

Er zijn drie slanke dichte schierzeshoeken gekend die tot geen enkele van de reeds genoemde oneindige klassen behoren.

▶ Beschouw de volgende matrix:

$$\mathsf{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

Stel dat C de zesdimensionale deelruimte van de vectorruimte $\mathbb{F}_3^{12}$ is, voortgebracht door de rijen van A. De deelruimte C wordt de *uitgebreide ternaire Golay code* genoemd.

Veronderstel dat $\mathbb{E}_1$ de volgende incidentiestructuur is:

- de punten zijn de nevenklassen $\vec{v} + \mathsf{C}$, $\vec{v} \in \mathbb{F}_3^{12}$, van de code C;
- de rechten zijn alle tripletten van de vorm $\{\vec{v}+\mathsf{C}, \vec{v}+\vec{e}_i+\mathsf{C}, \vec{v}-\vec{e}_i+\mathsf{C}\}$, $\vec{v} \in \mathbb{F}_3^{12}$, $i \in \{1, \ldots, 12\}$; hierbij is $\vec{e}_i$ de unieke vector van lengte 12 met waarde 1 op positie i en waarde 0 op de andere posities;
- incidentie is inclusie.

Dan is $\mathbb{E}_1$ een slanke dichte schierzeshoek ([40]).

▶ Beschouw het unieke Steiner systeem $S(5,8,24)$ en beschouw de volgende incidentiestructuur $\mathbb{E}_2$:

- de punten zijn de blokken van het Steiner systeem;
- de rechten zijn de verzamelingen van drie onderling disjuncte blokken;
- incidentie is inclusie.

Dan is $\mathbb{E}_2$ een slanke dichte schierzeshoek ([40]).

► Beschouw in $\mathsf{PG}(6,3)$ een niet-singuliere kwadriek $Q(6,3)$ en een niet-rakend hypervlak Π dat $Q(6,3)$ snijdt in een niet-singuliere elliptische kwadriek $Q^-(5,3)$. Er is een polariteit geassocieerd met $Q(6,3)$ en we noemen twee punten *orthogonaal* als één van hen bevat is in het polaire hypervlak van het andere punt. Noem N de verzameling van 126 interne punten van $Q(6,3)$ die bevat zijn in Π, i.e. de verzameling van alle 126 punten van Π waarvoor het polaire hypervlak $Q(6,3)$ snijdt in een niet-singuliere elliptische kwadriek. Stel dat $\mathbb{E}_3$ de volgende incidentiestructuur is:

- de punten zijn de 6-tupels van onderling orthogonale punten van N;
- de rechten zijn de paren van orthogonale punten van N;
- incidentie is omgekeerde inclusie.

Dan is $\mathbb{E}_3$ een slanke dichte schierzeshoek ([4]).

Overzicht

Elke gekende slanke dichte schier-$2n$-hoek, $n \geq 2$, die geen productschierveelhoek of gelijmde schierveelhoek is, is isomorf met juist één van de volgende voorbeelden.

$Q^D(2n,2)$, $n \geq 2$
$H^D(2n-1,4)$, $n \geq 2$
$\mathbb{G}_n$, $n \geq 3$
$\mathbb{H}_n$, $n \geq 3$
$\mathbb{I}_n$, $n \geq 4$
$\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$ $(n = 3)$

A.2 Goede ketens van grote deelschierveelhoeken

In Hoofdstuk 2 staan schierveelhoeken centraal die een zogeheten *grote* geodetisch gesloten deelschierveelhoek bevatten. Een geodetisch gesloten deelschierveelhoek $\mathcal{F}$ van een schierveelhoek $\mathcal{S}$ wordt *groot* genoemd als voor elk punt x van $\mathcal{S}$, $\mathrm{d}(x, \mathcal{F}) \leq 1$. Doel van Hoofdstuk 2 is de classificatie van alle slanke dichte schierveelhoeken die een zogeheten *goede keten* van grote geodetisch gesloten deelschierveelhoeken bevatten.

A.2.1 Gelijmde schierveelhoeken van type $H^D(2n_1-1, q^2) \otimes H^D(2n_2 - 1, q^2)$

Zoals reeds opgemerkt in Sectie A.1 kunnen we onder bepaalde voorwaarden een nieuwe schierveelhoek $\mathcal{S}_1 \otimes \mathcal{S}_2$ construeren vertrekkend van twee bestaande schierveelhoeken $\mathcal{S}_1$ en $\mathcal{S}_2$. Een nodige voorwaarde is dat beide schierveelhoeken een symmetriespread bevatten.

Beschouw nu de duale polaire ruimte $H^D(2n - 1, q^2)$. Er bestaat een natuurlijke bijectie θ tussen de deelruimten van dimensie k van de polaire ruimte $H := H(2n-1, q^2)$ en de geodetisch gesloten deelschierveelhoeken van diameter $2(n-1-k)$ in $H^D(2n-1, q^2)$. (De generatoren van $H(2n-1, q^2)$ door α corresponderen met de punten van α^θ.) Wat betreft de symmetriespreads van de duale polaire ruimte $H^D(2n-1, q^2)$ werd het volgende resultaat gevonden.

Stelling A.2.1 (Stelling 2.1.2, pagina 36**)**
Stel dat V de verzameling is van alle $(n-2)$-dimensionale deelruimten van H die in een niet-rakend hypervlak Π_∞ van $\mathsf{PG}(2n-1, q^2)$ liggen. Dan is $V^\theta := \{\alpha^\theta | \alpha \in V\}$ een symmetriespread van $H^D(2n-1, q^2)$. Elke symmetriespread van $H^D(2n-1, q^2)$ wordt op deze manier bekomen.

Deze karakterisatie van de symmetriespreads stelde ons in staat om het volgende resultaat aan te tonen.

Stelling A.2.2 (Stelling 2.1.4, pagina 38**)**
Voor elke priemmacht q en voor elke $n_1, n_2 \in \mathbb{N}\setminus\{0,1\}$ bestaat er een gelijmde schierveelhoek van type $H^D(2n_1 - 1, q^2) \otimes H^D(2n_2 - 1, q^2)$.

In het geval $q = 2$, bewezen we de volgende sterkere stelling.

Stelling A.2.3 (Stelling 2.1.5, pagina 39**)**
Voor elke $n_1, n_2 \in \mathbb{N}\setminus\{0,1\}$ bestaat er een unieke gelijmde schierveelhoek van

type $H^D(2n_1-1,4)\otimes H^D(2n_2-1,4)$. *In het bijzonder bestaat er voor elke* $n \in \mathbb{N}\setminus\{0,1\}$ *een unieke gelijmde schierveelhoek van type* $H^D(2n-1,4)\otimes Q(5,2)$.

A.2.2 Karakterisatie van productschierveelhoeken

In Hoofdstuk 2 werden twee belangrijke karakterisaties voor productschierveelhoeken bewezen. Deze speelden een belangrijke rol bij het herkennen van productschierveelhoeken in de rest van deze thesis.

Stelling A.2.4 (Stelling 2.2.2, pagina 40**)**
Stel dat $\mathcal{S}$ *een dichte schierveelhoek is en dat* T_1 *en* T_2 *twee verzamelingen deelschierveelhoeken zijn van* $\mathcal{S}$ *die aan de volgende eigenschappen voldoen:*

- *elk punt van* $\mathcal{S}$ *is bevat in een uniek element* $\mathcal{F}_1(x)$ *van* T_1 *en een uniek element* $\mathcal{F}_2(x)$ *van* T_2;
- $\mathcal{F}_1(x)\cap\mathcal{F}_2(x)=\{x\}$ *voor elk punt* x *van* $\mathcal{S}$;
- *elke rechte* L *door een punt* x *is bevat in ofwel* $\mathcal{F}_1(x)$, *ofwel* $\mathcal{F}_2(x)$.

Dan geldt:

- *alle elementen van* T_1 *zijn isomorf en alle elementen van* T_2 *zijn isomorf;*
- $\mathcal{S}\cong\mathcal{F}_1\times\mathcal{F}_2$ *voor elke* $\mathcal{F}_1\in T_1$ *en* $\mathcal{F}_2\in T_2$.

Stelling A.2.5 (Stelling 2.2.3, pagina 41**)**
Stel dat $\mathcal{S}$ *een dichte schier-*$2(n_1+n_2)$*-hoek is en stel dat* $\mathcal{F}_1$ *en* $\mathcal{F}_2$ *twee deelschierveelhoeken zijn waarvoor het volgende geldt:*

- $\mathrm{diam}(\mathcal{F}_i)=n_i\geq 1$ $(i\in\{1,2\})$;
- $\mathcal{F}_1$ *snijdt* $\mathcal{F}_2$ *in een punt* x;
- *elke rechte door* x *is bevat in ofwel* $\mathcal{F}_1$, *ofwel* $\mathcal{F}_2$.

Dan is $\mathcal{S}$ *isomorf met* $\mathcal{F}_1\times\mathcal{F}_2$.

A.2.3 Symmetriespreads in gelijmde schierveelhoeken

Veronderstel dat $\mathcal{S}$ een dichte gelijmde schierveelhoek is en veronderstel dat T_1 en T_2 twee partities zijn van $\mathcal{S}$ in geodetisch gesloten deelschierveelhoeken die volgen uit de lijming-constructie, zie Sectie A.1.5. Noem S^* de spread van $\mathcal{S}$ die wordt gevormd door alle rechten $\mathcal{F}_1 \cap \mathcal{F}_2$, waarbij $\mathcal{F}_1 \in T_1$ en $\mathcal{F}_2 \in T_2$. Voor elke spread S van $\mathcal{S}$ en voor elke geodetisch gesloten deelschierveelhoek $\mathcal{F}$ van $\mathcal{S}$ definiëren we $S_{\mathcal{F}}$ als de verzameling van alle rechten van S die bevat zijn in $\mathcal{F}$.

Stelling A.2.6 (Stelling 2.3.2, pagina 44**)**
Als S een symmetriespread van $\mathcal{S}$ is verschillend van S^*, dan geldt het volgende voor precies één $i \in \{1,2\}$.

- *Voor elke $\mathcal{F} \in T_i$ is $S_{\mathcal{F}}$ een symmetriespread van $\mathcal{F}$.*
- *Voor elke $\mathcal{F}, \mathcal{F}' \in T_i$, $S_{\mathcal{F}'} = \pi_{\mathcal{F}'}(S_{\mathcal{F}}) := \{\pi_{\mathcal{F}'}(L) | L \in S_{\mathcal{F}}\}$.*

A.2.4 Schierveelhoeken van type $(\mathcal{F}_1 * \mathcal{F}_2) \circ \mathcal{F}_3$

Een aantal belangrijke eigenschappen aangaande de 'associativiteit' van de productoperator en de lijmoperator werden bewezen.

Stelling A.2.7 (Stelling 2.3.3, pagina 44**)**
Zijn $\mathcal{F}_1, \mathcal{F}_2$ en $\mathcal{F}_3$ dichte schierveelhoeken van diameter ten minste 2, dan is elke schierveelhoek van type $(\mathcal{F}_1 \times \mathcal{F}_2) \otimes \mathcal{F}_3$ eveneens van type $\mathcal{F}_i \times (\mathcal{F}_{3-i} \otimes \mathcal{F}_3)$ voor een zekere $i \in \{1,2\}$.

Stelling A.2.8 (Stelling 2.3.4, pagina 45**)**
Stel dat $\mathcal{F}_1$ en $\mathcal{F}_2$ dichte schierveelhoeken zijn van diameter ten minste 2 en dat $\mathcal{S}$ een schierveelhoek is van type $\mathcal{F}_1 \otimes \mathcal{F}_2$. Als $\mathcal{S}$ een productschierveelhoek is, dan is minstens een van de schierveelhoeken $\mathcal{F}_1$ en $\mathcal{F}_2$ ook een productschierveelhoek.

Stelling A.2.9 (Stelling 2.3.5, pagina 45**)**
Stel dat $\mathcal{F}_1, \mathcal{F}_2$ en $\mathcal{F}_3$ dichte schierveelhoeken zijn met diameter ten minste 2 en dat geen van hen een productschierveelhoek is. Dan is elke schierveelhoek van type $(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3$ eveneens van type $\mathcal{F}_i \otimes (\mathcal{F}_{3-i} \otimes \mathcal{F}_3)$ voor een zekere $i \in \{1,2\}$.

A.2.5 Slanke dichte schierveelhoeken met een gegeven grote geodetisch gesloten deelschierveelhoek $\mathcal{F}$

In [19] en [22] bewees B. De Bruyn de volgende stellingen.

Stelling A.2.10
Stel dat $\mathcal{S}$ een slanke dichte schier-$2n$-hoek is, $n \geq 4$, die een grote geodetisch gesloten deelschierveelhoek $\mathcal{F}$ bevat isomorf met $\mathbb{H}_{n-1}$. Dan is $\mathcal{S}$ isomorf met $\mathbb{H}_n$ of $\mathbb{H}_{n-1} \times \mathbb{L}_3$.

Stelling A.2.11
Stel dat $\mathcal{S}$ een slanke dichte schier-$2n$-hoek is, $n \geq 4$, die een grote geodetisch gesloten deelschierveelhoek $\mathcal{F}$ bevat isomorf met $\mathbb{G}_{n-1}$. Dan is $\mathcal{S}$ isomorf met $\mathbb{G}_n$, $\mathbb{G}_{n-1} \times \mathbb{L}_3$ of met $\mathbb{G}_{n-1} \otimes Q(5,2)$, de unieke schierveelhoek die wordt bekomen door $\mathbb{G}_{n-1}$ en $Q(5,2)$ te lijmen.

We hebben ons toegespitst op het vinden van gelijkaardige resultaten voor de andere gekende (klassen van) slanke dichte schierveelhoeken. We vatten de gevonden resultaten samen in de volgende stelling.

Stelling A.2.12
Veronderstel dat $\mathcal{S}$ een slanke dichte schier-$2n$-hoek is die een grote geodetisch gesloten deelschierveelhoek $\mathcal{F}$ bevat.

- *Is $\mathcal{F} \cong H^D(2n-3,4)$, $n \geq 4$, dan is $\mathcal{S}$ isomorf met $H^D(2n-1,4)$, $H^D(2n-3,4) \otimes Q(5,2)$ of $H^D(2n-3,4) \times \mathbb{L}_3$.* (Stelling 2.4.3, pagina 47)
- *Is $\mathcal{F} \cong Q^D(2n-2,2)$, $n \geq 3$, dan is $\mathcal{S}$ isomorf met $Q^D(2n,2)$, $Q^D(2n-2,2) \times \mathbb{L}_3$ of $\mathbb{I}_n$.* (Stelling 2.4.9, pagina 51)
- *Is $\mathcal{F} \cong \mathbb{I}_{n-1}$, $n \geq 5$, dan is $\mathcal{S}$ isomorf met $\mathbb{I}_{n-1} \times \mathbb{L}_3$.* (Stelling 2.4.10, pagina 53)
- *Is $\mathcal{F} \cong \mathbb{E}_1$, dan is $\mathcal{S}$ isomorf met $\mathbb{E}_1 \times \mathbb{L}_3$ of $\mathbb{E}_1 \otimes Q(5,2)$, de unieke schierveelhoek die bekomen wordt door $\mathbb{E}_1$ en $Q(5,2)$ te lijmen.* (Stelling 2.4.14, pagina 54)
- *Is $\mathcal{F} \cong \mathbb{E}_2$, dan is $\mathcal{S}$ isomorf met $\mathbb{E}_2 \times \mathbb{L}_3$.* (Stelling 2.4.15, pagina 56)
- *Is $\mathcal{F} \cong \mathbb{E}_3$, dan is $\mathcal{S}$ isomorf met $\mathbb{E}_3 \times \mathbb{L}_3$.* (Stelling 2.4.16, pagina 56)

We slaagden er ook in een gelijkaardig resultaat te vinden voor productschierveelhoeken.

Stelling A.2.13 (Stelling 2.4.17, pagina 57**)**
Veronderstel dat $\mathcal{S}$ een slanke dichte schierveelhoek is die een grote geodetisch gesloten deelschierveelhoek $\mathcal{F}$ bevat die isomorf is met het direct product $\mathcal{S}_1 \times \mathcal{S}_2$ van twee schierveelhoeken $\mathcal{S}_1$ en $\mathcal{S}_2$ van diameter minstens 1. Dan bestaat er een $i \in \{1,2\}$ en een dichte schierveelhoek $\mathcal{S}_i'$ zodanig dat het volgende geldt.

- *$\mathcal{S}_i'$ heeft een grote geodetisch gesloten deelschierveelhoek isomorf met $\mathcal{S}_i$;*
- *$\mathcal{S}$ is isomorf met $\mathcal{S}_i' \times \mathcal{S}_{3-i}$.*

De klassen $\mathcal{C}$ en $\mathcal{D}$

Als Z_1 en Z_2 twee verzamelingen van schierveelhoeken zijn, dan definiëren we $Z_1 \otimes Z_2$ als de (mogelijks lege) verzameling van alle schierveelhoeken die bekomen worden door een element van Z_1 te lijmen met een element van Z_2. Definieer nu de volgende verzamelingen.

$$\begin{aligned}
C_2 &= \{Q(5,2)\};\\
C_3 &= \{\mathbb{G}_3, H^D(5,4), \mathbb{E}_1\};\\
C_n &= \{\mathbb{G}_n, H^D(2n-1,4)\} \;\cup\; \left(\bigcup_{2\le i\le n-1} C_i \otimes C_{n+1-i}\right) \text{ voor elke } n \ge 4;\\
\mathcal{C} &= C_2 \cup C_3 \cup \cdots
\end{aligned}$$

We definiëren nu een deelklasse $\mathcal{D}$ van $\mathcal{C}$ als volgt:

$$\begin{aligned}
D_2 &= \{Q(5,2)\};\\
D_n &= \{\mathbb{G}_n, H^D(2n-1,4)\} \;\cup\; \left(\bigcup_{2\le i\le n-1} D_i \otimes D_{n+1-i}\right) \text{ voor elke } n \ge 3;\\
\mathcal{D} &= D_2 \cup D_3 \cup \cdots
\end{aligned}$$

Merk op dat $\mathcal{D}$ bestaat uit die schierveelhoeken uit $\mathcal{C}$ die $\mathbb{E}_1$ niet als hex bevatten. We toonden de volgende stelling aan.

Stelling A.2.14 (Stellingen 2.4.18 en 2.4.19, pagina 58**)**
Veronderstel dat $\mathcal{S}$ een slanke dichte schier-$2n$-hoek is, $n \ge 4$, die een grote geodetisch gesloten deelschierveelhoek $\mathcal{F}$ bevat isomorf met een element van $\mathcal{C}$ (respectievelijk $\mathcal{D}$). Dan hebben we één van de volgende mogelijkheden:

- *$\mathcal{S} \cong \mathcal{F} \times \mathbb{L}_3$;*
- *$\mathcal{S}$ is isomorf met een element van $\mathcal{C}$ (respectievelijk $\mathcal{D}$).*

In de volgende sectie zal het duidelijk worden waarom we de klassen $\mathcal{C}$ en $\mathcal{D}$ hebben geïntroduceerd.

A.2.6 Een karakterisatiestelling

Beschouw een ketting $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n$ van geodetisch gesloten deelschierveelhoeken van een dichte schier-$2n$-hoek $\mathcal{S}$. We noemen zo'n ketting *goed* als ze aan de volgende voorwaarden voldoet.

- $\mathrm{diam}(\mathcal{F}_i) = i$ voor elke $i \in \{0, \ldots, n\}$;
- $\mathcal{F}_i$, $i \in \{0, \ldots, n-1\}$, is groot in $\mathcal{F}_{i+1}$.

Definieer nu de volgende verzamelingen.

$$\mathcal{M} \;=\; \{\mathbb{O}, \mathbb{L}_3, \mathbb{E}_2, \mathbb{E}_3\} \cup \mathcal{C} \cup \{Q^D(2n,2) | n \geq 2\} \cup \{\mathbb{H}_n | n \geq 3\} \cup \{\mathbb{I}_n | n \geq 4\};$$

$$\mathcal{N} \;=\; \{\mathbb{O}, \mathbb{L}_3, \mathbb{E}_3\} \;\cup\; \mathcal{D} \;\cup\; \{Q^D(2n,2) | n \geq 2\} \;\cup\; \{\mathbb{H}_n | n \geq 3\} \;\cup\; \{\mathbb{I}_n | n \geq 4\}.$$

We definiëren $\mathcal{M}^\times$, respectievelijk $\mathcal{N}^\times$, als de verzameling van alle schierveelhoeken die worden bekomen door directe producten te nemen van een aantal (i.e. minstens 1) elementen uit $\mathcal{M}$, respectievelijk $\mathcal{N}$. Alle elementen van $\mathcal{M}^\times$ zijn dicht en $\mathcal{N}^\times \subset \mathcal{M}^\times$. Elke gekende slanke dichte schierveelhoek is isomorf met een element uit $\mathcal{M}^\times$.

We zijn erin geslaagd de verzameling $\mathcal{N}^\times$ als volgt te karakteriseren.

Stelling A.2.15 (Stelling 2.6.1, pagina 65**)**
Een slanke dichte schier-$2n$-hoek $\mathcal{S}$ heeft een goede ketting van deelschierveelhoeken als en slechts als $\mathcal{S}$ isomorf is met een element uit $\mathcal{N}^\times$.

Wat betreft de klasse $\mathcal{M}^\times$ hebben we het volgend vermoeden.

Conjectuur. Elke slanke dichte schierveelhoek is isomorf met een element van $\mathcal{M}^\times$.

A.3 Valuaties van schierveelhoeken

De mogelijke punt-quad relaties en rechte-quad relaties zijn van groot belang bij het classificeren van bepaalde dichte schierveelhoeken (zie bv. [4] en [15]). In Hoofdstuk 3 bestudeerden we de mogelijke relaties tussen een punt x en een geodetisch gesloten deelschier-2δ-hoek $\mathcal{F}$ van een dichte schierveelhoek $\mathcal{S}$, en dit voor algemene δ.

A.3.1 Definitie en elementaire eigenschappen

Veronderstel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ een dichte schier-$2n$-hoek is. Een functie f van $\mathcal{P}$ naar $\mathbb{N}$ wordt een *valuatie* genoemd als ze voldoet aan de volgende eigenschappen (we noemen $f(x)$ de waarde van x).

$\mathbf{V_1}$ Er bestaat minstens één punt met waarde 0.

$\mathbf{V_2}$ Elke rechte L in $\mathcal{S}$ bevat een uniek punt x_L met kleinste waarde en $f(x) = f(x_L) + 1$ voor elk ander punt x van L verschillend van x_L.

$\mathbf{V_3}$ Elk punt x in $\mathcal{S}$ is bevat in een geodetisch gesloten deelschierveelhoek $\mathcal{F}_x$ die aan de volgende eigenschappen voldoet:

- $f(y) \leq f(x)$ voor elk punt $y \in \mathcal{F}_x$,
- elk punt z van $\mathcal{S}$ dat collineair is met een punt y van $\mathcal{F}$ zodanig dat $f(z) = f(y) - 1$, behoort eveneens tot $\mathcal{F}_x$.

De geodetisch gesloten deelschierveelhoek $\mathcal{F}_x$ die aan voorwaarde **$\mathbf{V_3}$** voldoet is uniek.

De valuaties van een dichte schierveelhoek geven informatie over hoe deze ingebed kan worden in een grotere schierveelhoek. Het volgende resultaat werd immers bewezen.

Stelling A.3.1 (Propositie 3.1.5, pagina 72**)**
Stel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ een dichte schier-$2n$-hoek is en dat $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ een deelschier-2δ-hoek is van $\mathcal{S}$ die aan de volgende eigenschappen voldoet.

- *$\mathcal{F}$ is dicht,*
- *$\mathcal{F}$ is een deelruimte van $\mathcal{S}$,*
- *als x en y twee punten zijn van $\mathcal{F}$, dan in $\mathrm{d}_{\mathcal{F}}(x, y) = \mathrm{d}_{\mathcal{S}}(x, y)$.*

Voor elk punt x van $\mathcal{S}$ en elk punt y van $\mathcal{F}$, definiëren we nu $f_x(y) := \mathrm{d}_{\mathcal{S}}(x, y) - \mathrm{d}_{\mathcal{S}}(x, \mathcal{F})$. Dan is $f_x : \mathcal{P}' \to \mathbb{N}$ een valuatie van $\mathcal{F}$ (voor elk punt x in $\mathcal{S}$).

A.3.2 Klassieke en ovoïdale valuaties

Stelling A.3.2 (Propositie 3.1.6, pagina 72**)**
Stel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ een dichte schier-$2n$-hoek is.

(a) *Als y een punt is van $\mathcal{S}$, dan is $f_y : \mathcal{P} \to \mathbb{N}; x \mapsto d(x, y)$ een valuatie van $\mathcal{S}$. We noemen f_y een* klassieke *valuatie $\mathcal{S}$.*

(b) *Als $\mathcal{O}$ een ovoïde is van $\mathcal{S}$, dan is $f_{\mathcal{O}} : \mathcal{P} \to \mathbb{N}; x \mapsto d(x, \mathcal{O})$ een valuatie van $\mathcal{S}$. We noemen $f_{\mathcal{O}}$ een* ovoïdale *valuatie van $\mathcal{S}$.*

Elke valuatie in een dichte veralgemeende vierhoek is klassiek of ovoïdaal.

A.3.3 Geïnduceerde valuaties

Valuaties kunnen ook geïnduceerd worden door een valuatie in een grotere schierveelhoek.

Stelling A.3.3 (Propositie 3.1.9, pagina 73**)**
Beschouw een dichte schierveelhoek met bijhorende valuatie f en stel dat $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ een dichte deelschierveelhoek is van $\mathcal{S}$, zodanig dat $\mathcal{F}$ een deelruimte is van $\mathcal{S}$ en zodanig dat voor elke twee punten x en y van $\mathcal{F}$, $d_{\mathcal{F}}(x, y) = d_{\mathcal{S}}(x, y)$. Is $m := \min\{f(x)|x \in \mathcal{F}\}$, dan is $f_x : \mathcal{P}' \to \mathbb{N}; x \mapsto f(x) - m$ een valuatie van $\mathcal{F}$. We noemen f_x een geïnduceerde valuatie.

A.3.4 De partieel lineaire ruimte G_f

Stel dat f een valuatie is van een dichte schierveelhoek $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$. We definiëren $O_f := \{x \in \mathcal{P}|f(x) = 0\}$. Een quad van $\mathcal{S}$ die minstens 2 punten van O_f bevat, bevat noodzakelijk een ovoïde van punten uit O_f en wordt een *speciale quad* genoemd. We definiëren nu G_f als de incidentiestructuur wiens punten de punten van O_f zijn, wiens rechten de speciale quads zijn en met inclusie als incidentierelatie.

A.3.5 Een aantal types van valuaties

In de thesis worden de volgende types van valuaties beschreven.

Productvaluaties (Sectie 3.2.1, pagina 77)

Stel dat $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathbf{I}_1)$ en $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathbf{I}_2)$ twee dichte schierveelhoeken zijn. Is f_i, $i \in \{1, 2\}$, een valuatie van $\mathcal{S}_i$, dan is de afbeelding $f : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{N} : (x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$ een valuatie van $\mathcal{S}_1 \times \mathcal{S}_2$. We noemen f een *productvaluatie*.

Semiklassieke valuaties (Sectie 3.2.2, pagina 77)

Stel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ een dichte schier-$2n$-hoek is, $n \geq 2$, en beschouw een punt x van $\mathcal{S}$. Stel dat $\mathcal{A}$ de afstands-n-meetkunde is bepaald door x (i.e. de punten zijn de punten van $\Gamma_n(x)$, de rechten zijn de rechten van $\mathcal{S}$ op afstand $n-1$ van x en incidentie is inclusie). Stel dat $\mathcal{A}$ een ovoïde $\mathcal{O}$ heeft. Definieer de functie $f_{x,\mathcal{O}} : \mathcal{P} \to \mathbb{N}$ als volgt.

- $f_{x,\mathcal{O}}(y) = \mathrm{d}(x, y)$ voor elk punt $y \in \mathcal{P}$ op afstand ten hoogste $n-1$ van x;
- $f_{x,\mathcal{O}}(y) = n-2$ voor elk punt $y \in \mathcal{O}$;
- $f_{x,\mathcal{O}}(y) = n-1$ voor elk punt $y \in \Gamma_n(x)$ niet bevat in $\mathcal{O}$.

Dan is $f_{x,\mathcal{O}}$ een valuatie van f. We noemen f een *semiklassieke valuatie*.

Is $\mathcal{S}$ een slanke dichte schier-$2n$-hoek, $n \geq 2$, dan heeft $\mathcal{S}$ een semiklassieke valuatie f zodanig dat $f(x) = 0$ als en slechts als $\Gamma_n(x)$ bipartite is. In dat geval zijn er precies twee dergelijke semiklassieke valuaties.

Extensievaluaties (Sectie 3.2.3, pagina 78)

Stel dat $\mathcal{S}$ een dichte schierveelhoek is en dat $\mathcal{F} = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ een klassieke geodetisch gesloten deelschierveelhoek is van $\mathcal{S}$. Veronderstel dat f' een valuatie is van $\mathcal{F}$. Dan is $f : \mathcal{P} \to \mathbb{N} : \ x \mapsto \mathrm{d}(x, \pi_{\mathcal{F}}(x)) + f'(\pi_{\mathcal{F}}(x))$ een valuatie van $\mathcal{S}$ ($\pi_{\mathcal{F}}(x)$ is het unieke punt van $\mathcal{F}$ dichtst bij x gelegen). We noemen f een *extensie* van f. Is f' klassiek, dan is ook f klassiek. Is $\mathcal{F} = \mathcal{S}$ en $f = f'$, dan wordt f een *triviale extensie* van f' genoemd. Een valuatie is een *extensievaluatie* als het een niet-triviale extensie is van een andere valuatie.

Afstands-j-ovoïdale valuaties (Sectie 3.2.4, pagina 81)

Stel dat $\mathcal{S}$ een schier-$2n$-hoek is, $n \geq 2$. Een *afstands-j-ovoïde* ($2 \leq j \leq n$) van $\mathcal{S}$ is een verzameling X van punten die aan de volgende voorwaarden voldoet:

- $\mathrm{d}(x, y) \geq j$ voor alle $x, y \in X$ met $x \neq y$;
- voor elk punt a van $\mathcal{S}$ bestaat er een punt $x \in X$ zodanig dat $\mathrm{d}(a, x) \leq \frac{j}{2}$;
- voor elke rechte L van $\mathcal{S}$ bestaat er een punt $x \in X$ zodanig dat $\mathrm{d}(L, x) \leq \frac{j-1}{2}$.

Merk op dat een afstands-2-ovoïde gewoon een ovoïde is.

Stel dat X een afstands-j-ovoïde van een dichte schier-$2n$-hoek $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is ($2 \leq j \leq n$ en j even). Dan is de afbeelding $f : \mathcal{P} \to \mathbb{N} : x \mapsto \mathrm{d}(x, X)$ een valuatie van $\mathcal{S}$. We noemen f een *afstands-j-ovoïdale* valuatie.

SDPS-valuaties (Sectie 3.2.5, pagina 82)

Veronderstel dat $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ één van de volgende klassieke schier-$4n$-hoeken is.

(a) een punt ($n = 0$);

(b) een dichte veralgemeende vierhoek ($n = 1$);

(c) $W^D(4n-1, q)$, $n \geq 2$;

(d) $[Q^-(4n+1, q)]^D$, $n \geq 2$.

Een deelverzameling X van $\mathcal{P}$ wordt een *SDPS-verzameling* (SDPS = sub dual polar space) in $\mathcal{A}$ genoemd als het aan de volgende voorwaarden voldoet.

(1) Geen twee punten van X zijn collineair in $\mathcal{A}$.

(2) Als $x, y \in X$ zodanig dat $\mathrm{d}(x, y) = 2$, dan is $X \cap \mathcal{C}(x, y)$ een ovoïde van de quad $\mathcal{C}(x, y)$.

(3) De incidentiestructuur $\tilde{\mathcal{A}}$ met punten de elementen van X, met rechten de quads van $\mathcal{A}$ die minstens twee punten van X bevatten, en met inclusie als incidentierelatie, is één van de volgende schier-$2n$-hoeken.

- geval (a): een punt;
- geval (b): een rechte van grootte minstens 2;
- geval (c): $W^D(2n-1, q^2)$;
- geval (d): $H^D(2n, q^2)$;

(4) Voor elke $x, y \in X$, $\mathrm{d}(x, y) = 2 \cdot \delta(x, y)$, met $\delta(x, y)$ de afstand tussen twee punten x en y in $\tilde{\mathcal{A}}$.

Als X een SDPS-verzameling is van de schier-$4n$-hoek $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$, dan is de afbeelding $f : \mathcal{P} \to \mathbb{N} : x \mapsto \mathrm{d}(x, X)$ een valuatie van $\mathcal{A}$. De valuatie f wordt een *SDPS-valuatie* genoemd.

A.3.6 Valuaties in dichte schierzeshoeken

Veronderstel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ een dichte schierzeshoek is en dat f een valuatie is van $\mathcal{S}$. Er zijn drie mogelijkheden.

- $\max\{f(x)|x \in \mathcal{P}\} = 3$. In dit geval is f een klassieke valuatie.
- $\max\{f(x)|x \in \mathcal{P}\} = 1$. In dit geval is f een ovoïdale valuatie.
- $\max\{f(x)|x \in \mathcal{P}\} = 2$.

Stelling A.3.4 (Propositie 3.4.1, pagina 89**)**
Is $|O_f| = 1$, dan is f klassiek of semiklassiek.

Stelling A.3.5 (Propositie 3.4.2, pagina 89**)**
Veronderstel dat $|O_f| \geq 2$ en dat f niet ovoïdaal is. Dan liggen elke twee punten van O_f op afstand 2 van elkaar. Bijgevolg is G_f een lineaire ruimte.

A.3.7 Valuaties in klassieke schierveelhoeken

Elke dichte schier-$2d$-hoek, $d \leq 2$, is klassiek. We hebben de valuaties in deze schierveelhoeken reeds eerder beschouwd. Wat betreft de dichte klassieke schierzeshoeken, hebben we de volgende stelling.

Stelling A.3.6 (Propositie 3.5.1, pagina 90**)**
Veronderstel dat $\mathcal{S}$ een dichte klassieke schierzeshoek is. Dan is elke valuatie f van $\mathcal{S}$ of klassiek, of ovoïdaal, of semiklassiek of de extensie van een ovoïdale valuatie in een quad van $\mathcal{S}$.

Stelling A.3.7 (Propositie 3.5.2, pagina 91**)**

(a) Elke valuatie f van $Q^D(2n, q)$, $n \geq 2$ en q is oneven, is klassiek.

(b) Elke valuatie f van $H^D(2n-1, q^2)$, $n \geq 2$, is klassiek.

(c) Als de veralgemeende vierhoek $H(4, q^2)$ geen spreads heeft, dan is elke valuatie f van $H^D(2n, q^2)$ klassiek. Elke valuatie van $H^D(2n, 4)$ is bijgevolg klassiek.

Stelling A.3.8 (Propositie 3.5.3, pagina 91**)**
Veronderstel dat $\mathcal{S}$ een klassieke dichte schier-$2n$-hoek is die geen productschierveelhoek is. Veronderstel dat f een valuatie is van $\mathcal{S}$ zodanig dat geen geïnduceerde hex-valuatie semiklassiek of ovoïdaal is. Dan is f de (mogelijks triviale) extensie van een SDPS-valuatie in een geodetisch gesloten deelschierveelhoek van $\mathcal{S}$.

A.4 De classificatie van de slanke dichte schierachthoeken

In Hoofdstuk 4 hebben we alle slanke dichte schierachthoeken geclassificeerd. We maakten hierbij gebruik van de resultaten uit Hoofdstukken 2 en 3.

A.4.1 De classificatie van de slanke dichte schierzeshoeken

In [4] werden alle slanke dichte schierzeshoeken geclassificeerd. Het aantal punten wordt gegeven door $v_{\mathcal{S}}$, het aantal rechten door een punt wordt gegeven door $t_{\mathcal{S}} + 1$.

schierzeshoek	$v_{\mathcal{S}}$	$t_{\mathcal{S}}$	grote quads	andere quads
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	27	2	$\mathbb{L}_3 \times \mathbb{L}_3$	–
$W(2) \times \mathbb{L}_3$	45	3	$\mathbb{L}_3 \times \mathbb{L}_3, W(2)$	–
$Q(5,2) \times \mathbb{L}_3$	81	5	$\mathbb{L}_3 \times \mathbb{L}_3, Q(5,2)$	–
$\mathbb{H}_3 \cong \mathbb{I}_3$	105	5	$W(2)$	$\mathbb{L}_3 \times \mathbb{L}_3$
$Q^D(6,2)$	135	6	$W(2)$	–
$Q(5,2) \otimes Q(5,2)$	243	8	$Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3$
$\mathbb{G}_3$	405	11	$Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3, W(2)$
$\mathbb{E}_1$	729	11	–	$\mathbb{L}_3 \times \mathbb{L}_3$
$\mathbb{E}_2$	759	14	–	$W(2)$
$\mathbb{E}_3$	567	14	$Q(5,2)$	$W(2)$
$H^D(5,4)$	891	20	$Q(5,2)$	–

Tabel A.1: Slanke dichte schierzeshoeken

A.4.2 Valuaties in slanke dichte schierzeshoeken

Om de slanke dichte schierachthoeken te kunnen bepalen, hebben we eerst alle valuaties van de 11 slanke dichte schierzeshoeken moeten classificeren. Deze classificatie nam een groot deel van Hoofdstuk 4 in beslag.

In Tabel A.2 geven we een overzicht van de gevonden valuaties. In de laatste twee kolommen vermelden we eveneens tot welke isomorfieklasse de partieel lineaire ruimte G_f behoort, waarbij f de beschouwde valuatie is. De meetkunde $\overline{W(2)}$ wordt bekomen uit $W(2)$ door de zes ovoïden als extra rechten te beschouwen.

schierzeshoek	klass.	ovoïd.	semikl.	extensie	andere
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	$\surd$	$\surd$	$\surd$	$\mathbb{L}_3$	–
$W(2) \times \mathbb{L}_3$	$\surd$	–	$\surd$	$\mathbb{L}_3$, $\mathbb{L}_5$	–
$Q(5,2) \times \mathbb{L}_3$	$\surd$	–	–	$\mathbb{L}_3$	–
$\mathbb{H}_3$	$\surd$	–	–	$\mathbb{L}_5$	$\mathbb{L}_3$, $\mathsf{PG}(2,2)$
$Q^D(6,2)$	$\surd$	–	–	$\mathbb{L}_5$	–
$Q(5,2) \otimes Q(5,2)$	$\surd$	–	–	–	$\underline{\mathsf{AG}(2,3)}$
$\mathbb{G}_3$	$\surd$	–	–	–	$\overline{W(2)}$
$\mathbb{E}_1$	$\surd$	$\surd$	–	–	–
$\mathbb{E}_2$	$\surd$	$\surd$	–	–	–
$\mathbb{E}_3$	$\surd$	–	–	–	$\mathsf{PG}(2,4)$
$H^D(5,4)$	$\surd$	–	–	–	–

Tabel A.2: Valuaties in slanke dichte schierzeshoeken

A.4.3 Slanke dichte schierveelhoeken met een grote deelschierveelhoek

De eigenlijke classificatie van slanke dichte schierachthoeken hebben we gerealiseerd in verschillende stappen.

▶ In een eerste stap hebben we alle 'gekende' voorbeelden in een lijst verzameld. Naast de productschierveelhoeken en de slanke dichte schierachthoeken $\mathbb{G}_4$, $\mathbb{H}_4$, $\mathbb{I}_4$, $Q^D(8,2)$ en $H^D(7,4)$, hebben we ook nagegaan welke slanke dichte schierachthoeken worden bekomen door het lijmen van een schierzeshoek met een veralgemeende vierhoek. Daartoe moesten we onder andere alle symmetriespreads in de 11 slanke dichte schierzeshoeken bepalen. Op deze manier hebben we 24 slanke dichte schierachthoeken gevonden, zie Tabel A.3 en Tabel A.4.

▶ Steunend op de resultaten uit Hoofdstuk 2 zijn we erin geslaagd de volgende stelling te bewijzen.

Stelling A.4.1 (Stelling 4.5.1, pagina 124**)**
Elke slanke dichte schierachthoek die een grote hex bevat is isomorf met één van de elementen uit Tabel A.3.

schierachthoek	v	t + 1	grote hexen
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	81	4	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$W(2) \times \mathbb{L}_3 \times \mathbb{L}_3$	135	5	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$
$Q(5,2) \times \mathbb{L}_3 \times \mathbb{L}_3$	243	7	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$\mathbb{I}_3 \times \mathbb{L}_3$	315	7	$W(2) \times \mathbb{L}_3$, $\mathbb{I}_3$
$Q^D(6,2) \times \mathbb{L}_3$	405	8	$W(2) \times \mathbb{L}_3$, $Q^D(6,2)$
$(Q(5,2) \otimes Q(5,2)) \times \mathbb{L}_3$	729	10	$Q(5,2) \times \mathbb{L}_3$, $Q(5,2) \otimes Q(5,2)$
$\mathbb{G}_3 \times \mathbb{L}_3$	1215	13	$Q(5,2) \times \mathbb{L}_3$, $\mathbb{G}_3$
$\mathbb{E}_1 \times \mathbb{L}_3$	2187	13	$\mathbb{E}_1$
$\mathbb{E}_2 \times \mathbb{L}_3$	2277	16	$\mathbb{E}_2$
$\mathbb{E}_3 \times \mathbb{L}_3$	1701	16	$Q(5,2) \times \mathbb{L}_3$, $\mathbb{E}_3$
$H^D(5,4) \times \mathbb{L}_3$	2673	22	$Q(5,2) \times \mathbb{L}_3$, $H^D(5,4)$
$W(2) \times W(2)$	225	6	$W(2) \times \mathbb{L}_3$
$Q(5,2) \times W(2)$	405	8	$W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$Q(5,2) \times Q(5,2)$	729	10	$Q(5,2) \times \mathbb{L}_3$
$(Q(5,2) \otimes Q(5,2)) \otimes_1 Q(5,2)$	2187	13	$Q(5,2) \otimes Q(5,2)$
$(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$	2187	13	$Q(5,2) \otimes Q(5,2)$
$\mathbb{G}_3 \otimes Q(5,2)$	3645	16	$Q(5,2) \otimes Q(5,2)$, $\mathbb{G}_3$
$\mathbb{E}_1 \otimes Q(5,2)$	6561	16	$\mathbb{E}_1$
$H^D(5,4) \otimes Q(5,2)$	8019	25	$Q(5,2) \otimes Q(5,2)$, $H^D(5,4)$
$\mathbb{G}_4$	8505	22	$\mathbb{G}_3$
$\mathbb{H}_4$	945	10	$\mathbb{H}_3$
$\mathbb{I}_4$	2025	14	$Q^D(6,2)$
$Q^D(8,2)$	2295	15	$Q^D(6,2)$
$H^D(7,4)$	114939	85	$H^D(5,4)$

Tabel A.3: Slanke dichte schierachthoeken

schierachthoek	niet-grote hexen
$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$	–
$W(2) \times \mathbb{L}_3 \times \mathbb{L}_3$	–
$Q(5,2) \times \mathbb{L}_3 \times \mathbb{L}_3$	–
$\mathbb{I}_3 \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$Q^D(6,2) \times \mathbb{L}_3$	–
$(Q(5,2) \otimes Q(5,2)) \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$\mathbb{G}_3 \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$
$\mathbb{E}_1 \times \mathbb{L}_3$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$\mathbb{E}_2 \times \mathbb{L}_3$	$W(2) \times \mathbb{L}_3$
$\mathbb{E}_3 \times \mathbb{L}_3$	$W(2) \times \mathbb{L}_3$
$H^D(5,4) \times \mathbb{L}_3$	–
$W(2) \times W(2)$	–
$Q(5,2) \times W(2)$	–
$Q(5,2) \times Q(5,2)$	–
$(Q(5,2) \otimes Q(5,2)) \otimes_1 Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$\mathbb{G}_3 \otimes Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$\mathbb{E}_1 \otimes Q(5,2)$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$
$H^D(5,4) \otimes Q(5,2)$	$Q(5,2) \times \mathbb{L}_3$
$\mathbb{G}_4$	$W(2) \times \mathbb{L}_3$, $Q(5,2) \times \mathbb{L}_3$, $\mathbb{H}_3$
$\mathbb{H}_4$	$W(2) \times \mathbb{L}_3$
$\mathbb{I}_4$	$\mathbb{H}_3$
$Q^D(8,2)$	–
$H^D(7,4)$	–

Tabel A.4: Niet-grote hexen in de slanke dichte schierachthoeken

$\mathcal{N}_i$	schierzeshoek
$\mathcal{N}_1$	$\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$
$\mathcal{N}_2$	$W(2) \times \mathbb{L}_3$
$\mathcal{N}_3$	$Q(5,2) \times \mathbb{L}_3$
$\mathcal{N}_4$	$Q(5,2) \otimes Q(5,2)$
$\mathcal{N}_5$	$\mathbb{H}_3 \cong \mathbb{I}_3$
$\mathcal{N}_6$	$Q^D(6,2)$
$\mathcal{N}_7$	$\mathbb{E}_3$
$\mathcal{N}_8$	$\mathbb{G}_3$
$\mathcal{N}_9$	$\mathbb{E}_1$
$\mathcal{N}_{10}$	$\mathbb{E}_2$
$\mathcal{N}_{11}$	$H^D(5,4)$

Tabel A.5: Slanke dichte schierzeshoeken

► We zijn er tenslotte in geslaagd aan te tonen dat elke slanke dichte schierachthoek een grote hex moet bevatten.

Stelling A.4.2 (Stelling 4.6.1, pagina 128**)**
Stel dat $\mathcal{S}$ een slanke dichte schierachthoek is en stel dat i het grootste natuurlijk getal is zodanig dat $\mathcal{S}$ een hex bevat isomorf met $\mathcal{N}_i$ (zie Tabel A.5). Dan is elke hex van $\mathcal{S}$ isomorf met $\mathcal{N}_i$ groot in $\mathcal{S}$. Bijgevolg is elke slanke dichte schierachthoek isomorf met een van de voorbeelden uit Tabel A.3.

Deze stelling vervolledigt de classificatie van alle slanke dichte schierachthoeken.De volgende stelling is een onmiddelijk gevolg van de classificatie.

Stelling A.4.3 (Gevolg 4.6.2, pagina 137**)**
Stel dat $\mathcal{S}$ een slanke dichte schierveelhoek is die geen geodetisch gesloten deelschierveelhoeken bevat isomorf met $(Q(5,2) \otimes Q(5,2)) \otimes_2 Q(5,2)$, en stel dat $\mathcal{H}$ één van de 11 slanke dichte schierzeshoeken is. Dan bestaat er een constante $\alpha_{\mathcal{H}}$ zodanig dat elk punt van $\mathcal{S}$ bevat is in precies $\alpha_{\mathcal{H}}$ hexen isomorf met $\mathcal{H}$.

Bibliography

[1] M. Aschbacher. Flag structures on Tits geometries. *Geom. Dedicata*, 14:21–32, 1983.

[2] A. E. Brouwer. The uniqueness of the near hexagon on 729 points. *Combinatorica*, 2:333–340, 1982.

[3] A. E. Brouwer. The uniqueness of the near hexagon on 759 points. In N. L. Johnson, M. J. Kallaher, and C. T. Long, editors, *Finite Geometries*, volume 82 of *Lecture Notes in Pure and Appl. Math.*, pages 47–60, Marcel Dekker, New York, Basel, 1982.

[4] A. E. Brouwer, A. M. Cohen, J. I. Hall, and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata*, 49:349–368, 1994.

[5] A. E. Brouwer and E. W. Lambeck. An inequality on the parameters of distance regular graphs and the uniqueness of a graph related to M_{23}. *Annals of discrete Mathematics*, 34:101–106, 1987.

[6] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata*, 14:145–176, 1983.

[7] A. E. Brouwer and H. A. Wilbrink. Ovoids and fans in the generalized quadrangle $Q(4,2)$. *Geom. Dedicata*, 36:121–124, 1990.

[8] P. J. Cameron. Dual polar spaces. *Geom. Dedicata*, 12:75–86, 1982.

[9] H. S. M. Coxeter. Twelve points in $\mathsf{PG}(5,3)$ with 95040 self-transformations. *Proc. Roy. Soc. London Ser. A*, 247:279–293, 1958.

[10] B. De Bruyn. Generalized quadrangles with a spread of symmetry. *Europ. J. Comb.*, 20:759–771, 1999.

[11] B. De Bruyn. Constructions and characterizations of near polygons. *Ph.D. Thesis*, Ghent University, 2000.

[12] B. De Bruyn. On near hexagons and spreads of generalized quadrangles. *J. Alg. Comb.*, 11:211–226, 2000.

[13] B. De Bruyn. On near polygons and the Coxeter cap in $\mathsf{PG}(5,3)$. *J. Geom.*, 68:23–33, 2000.

[14] B. De Bruyn. On the number of nonisomorphic glued near hexagons. *Bull. Belg. Math. Soc.*, 7:493–510, 2000.

[15] B. De Bruyn. Near hexagons with four points on a line. *Adv. Geom.*, 1:211–228, 2001.

[16] B. De Bruyn. Glued near polygons. *Europ. J. Comb.*, 22:973–981, 2001.

[17] B. De Bruyn. On the uniqueness of near polygons with three points on every line. *Europ. J. Comb.*, 23:523–528, 2002.

[18] B. De Bruyn. The glueing of near polygons. *Bull. Belg. Math. Soc.*, 9:621-630, 2002.

[19] B. De Bruyn. Near polygons having a big sub near polygon isomorphic to $\mathbb{H}_n$. *Ann. Comb.*, 6:285–294, 2002.

[20] B. De Bruyn. On the finiteness of near polygons with 3 points on every line. *J. Alg. Comb.*, 18:41–46, 2003.

[21] B. De Bruyn. New near polygons from Hermitian varieties. *Bull. Belg. Math. Soc.*, 10:561–577, 2003.

[22] B. De Bruyn. Near polygons having a big sub near polygon isomorphic to $\mathbb{G}_n$. *Bull. Belg. Math. Soc.*, 11:321–341, 2004.

[23] B. De Bruyn. Decomposable near polygons. *Ann. of Comb.*, 8:251–267, 2004.

[24] B. De Bruyn and F. De Clerck. On linear representations of near hexagons. *Europ. J. Comb.*, 20:45–60, 1999.

[25] B. De Bruyn and P. Vandecasteele. Two conjectures regarding dense near polygons with three points on each line. *Europ. J. Comb.*, 24:631–647, 2003.

[26] B. De Bruyn and P. Vandecasteele. Near polygons with a nice chain of sub near polygons. *J. Combin. Theory Ser. A*, 108:297–311, 2004.

[27] B. De Bruyn and P. Vandecasteele. Near polygons having a big sub near polygon isomorphic to $H^D(2n-1,4)$. *Preprint.*

[28] B. De Bruyn and P. Vandecasteele. Valuations of near polygons. *Preprint.*

[29] B. De Bruyn and P. Vandecasteele. The valuations of the near hexagons related to the Witt designs $S(5,6,12)$ and $S(5,8,24)$. *Preprint.*

[30] B. De Bruyn and P. Vandecasteele. The distance-2-sets of the slim dense near hexagons. *Preprint.*

[31] B. De Bruyn and P. Vandecasteele. The classification of the slim dense near octagons *Preprint.*

[32] W. Feit and G. Higman. The non-existence of certain generalized polygons. *J. Algebra*, 1:114–131, 1964.

[33] W. H. Haemers and C. Roos. An inequality for generalized hexagons. *Geom. Dedicata*, 10:219–222, 1981.

[34] D. G. Higman. Partial geometries, generalized quadrangles and strongly regular grahps. In A. Barlotti, editor, *Atti Convegno di Geometria e sue Applicazioni*, 263–293. Academic Press, London, 1971.

[35] A. Offer and H. Van Maldeghem. Distance j-ovoids and related structures in generalized polygons. Proceedings of the First Irsee Conference. To appear in *Designs, Codes and Cryptography.*

[36] S. E. Payne and J. A. Thas. Finite generalized quadrangles. Volume 110 of *Research Notes in Mathematics*, Pitman, Boston, 1984.

[37] H. Pralle and S. Shpectorov. The ovoidal hyperplane of a dual polar space of rank 4. *Preprint*

[38] S. A. Shad and E. E. Shult. The near n-gon geometries. *Unpublished*, 1979.

[39] E. E. Shult and J. A. Thas. Constructions of polygons from buildings. *Proc. London Math. Soc.*, 71:397–440, 1995.

[40] E. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata*, 9:1–72, 1980.

[41] J. J. Sylvester. Collected Mathematical Papers I. Cambridge University Press, Cambridge, 1904.

[42] J. A. Thas. Some results on quadrics and a new class of partial geometries. *Simon Stevin*, 55:129–139, 1981.

[43] J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Etudes Sci. Publ. Math.*, 2:14–60, 1959.

[44] J. Tits. Buildings and BN-pairs of spherical type. *Lecture notes in Math.*, Springer Verlag, 386, 1974.

[45] J. Tits. Classification of buildings of spherical type and Moufang polygons: a survey. *Teorie Combinatorie, Proc. Intern. Colloq. (Roma 1973)*, volume I, pages 229–246. Accad. Naz. Lincei, 1976.

[46] H. Van Maldeghem. Generalized polygons. Volume 93 of *Monographs in Mathematics*. Birkhäuser, Basel, Boston, Berlin, 1998.

[47] A. Yanushka. On order in generalized polygons. *Geom. Dedicata*, 10:451–458, 1981.

Index