Ovoids of Parabolic and Hyperbolic Spaces

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Abstract

This thesis is concerned with ovoids of orthogonal spaces, in particular, of parabolic and hyperbolic spaces, where ovoids of a non–trivial nature occur. The most important results are:

- A new ovoid in a 5–dimensional parabolic space.
- A new infinite family of partitions of (the sets of singular points of) certain 6–dimensional hyperbolic spaces by ovoids, corresponding to a new infinite family of regular packings of certain 3–dimensional projective spaces.

These results are presented within the context of a survey – of the objects above, as well as of ovoids of 7–dimensional parabolic and 8–dimensional hyperbolic spaces.

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Structure of thesis

The material developed in each chapter is as follows.

Chapter 1. We set out all the theory we require concerning polar spaces; particularly, their groups, properties of their subspaces, and how they are classified into isometry and similarity types. Ovoids are introduced, with fundamental points such as ovoid slicing and embedding treated. Constructions involving ovoids and spreads of polar spaces and spreads of projective spaces are given. Some basic theory of translation planes is presented, along with how they are related to ovoids. We describe the current state of knowledge about ovoid existence in the polar spaces of concern to us, and conclude with some background material that will be of use.

Chapter 2. The known $\mathrm{O}(5,q)$ ovoids are given explicitly, and the stabilisers of the Kantor and Ree–Tits slice ovoids are calculated (if these were known previously, then it was as folklore). A new $\mathrm{O}(5,3^5)$ ovoid is presented (and its stabiliser calculated), and we describe how ovoids, spreads, translation planes, flocks, translation generalised quadrangles and eggs arise from the new ovoid (some at least of these objects are new).

Chapter 3. The construction of the two known families of $\mathrm{O}(7,q)$ ovoid via the classical generalised hexagon of order q is given, and it is proven that the Thas and Kantor families are indeed equivalent. We then obtain an explicit description of these Thas–Kantor ovoids, one from which it is clear that their slices are Kantor $\mathrm{O}(5,q)$ ovoids.

Chapter 4. We describe the known $\mathrm{O}^+(8,q)$ ovoids in the models in which they were constructed, and consider the slices of the q not prime ovoids. An interesting alternative description of the Dye ovoid is displayed, and we briefly discuss restraints on the stabilisers of new $\mathrm{O}^+(8,q)$ ovoids. A table of some unsuccessful $\mathrm{O}^+(8,q)$ ovoid searches is given.

Chapter 5. The context of the problem of finding regular packings of $\mathrm{PG}(3,q)$ is given. The new family of regular packings of $\mathrm{PG}(3,q)$ for $q \equiv 2 \pmod 3$ is established, the stabilisers of these packings are determined, and the resulting translation planes are considered.

Chapter 6. We briefly ilustrate how computers can be employed to search for ovoids, describing basic efficiencies that can be implemented. Techniques for distinguishing inequivalent ovoids are included.

Preface

This thesis was originally intended to focus mainly on ovoids of $\mathrm{O}^+(8,q)$; particularly, the construction of new ovoids. However, even after much effort, no success here was forthcoming (note that the table of computer searches that we present in the $\mathrm{O}^+(8,q)$ chapter represents a tiny fraction of what was tried!). But there were some interesting results that arose in the course of this work (such as the alternative description of the Dye ovoid), which we present.

With the exception of the work described above, the research that this thesis is based on was collaborative with Tim Penttila; in particular, all results so obtained were shaped by him.

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Introduction

1.1 Polar spaces

1.1.1 Projective spaces. A **projective space** $\mathrm{P}V$ is the set of all subspaces of a vector space V, partially ordered by inclusion. If V is of dimension $d+1$ over $\mathrm{GF}(q)$, $\mathrm{P}V$ is denoted $\mathrm{PG}(d,q)$.

The group of all invertible semilinear maps from a vector space V to itself is denoted $\Gamma\mathrm{L}(V)$, and contains the subgroup $\mathrm{GL}(V)$ of all invertible linear maps from V to itself (the **general linear group** of V). The groups induced by the action of $\Gamma\mathrm{L}(V)$ and $\mathrm{GL}(V)$ on $\mathrm{P}V$ are denoted $\mathrm{P}\Gamma\mathrm{L}(V)$ and $\mathrm{PGL}(V)$, and called the **projective semilinear group** of V and the **projective general linear group** of V, respectively. Letting $Z(V)$ be the subgroup of $\Gamma\mathrm{L}(V)$ consisting of all scalar maps, we have $\mathrm{P}\Gamma\mathrm{L}(V) \cong \Gamma\mathrm{L}(V)/Z(V)$ and $\mathrm{PGL}(V) \cong \mathrm{GL}(V)/Z(V)$.

For any projective space $\mathrm{P}V$, a **collineation** of $\mathrm{P}V$ is an inclusion–preserving bijection on $\mathrm{P}V$, and a **correlation** of $\mathrm{P}V$ is an inclusion–reversing bijection on $\mathrm{P}V$ (while a correlation of order 2 is a **polarity**). Let $\mathrm{Cor}(\mathrm{P}V)$ denote the group of all collineations and correlations of $\mathrm{P}V$; because the composition of two collineations is a collineation, $\mathrm{Cor}(\mathrm{P}V)$ has as a subgroup $\mathrm{Col}(\mathrm{P}V)$, the group of all collineations of $\mathrm{P}V$ (by the fundamental theorem of projective geometry (see [69, Theorem 3.1]), $\mathrm{Col}(\mathrm{P}V) \cong \mathrm{P}\Gamma\mathrm{L}(V)$ when V has dimension at least 3). Assuming $\dim V$ is finite, we can show that $\mathrm{Col}(\mathrm{P}V)$ has index 2 in $\mathrm{Cor}(\mathrm{P}V)$ (see [69]). For, let $\triangle$, $\triangle' \in \mathrm{Cor}(\mathrm{P}V) - \mathrm{Col}(\mathrm{P}V)$, and note that $g = \triangle^{-1}\triangle' \in \mathrm{Col}(\mathrm{P}V)$ (since the composition of two correlations is a collineation). Then $\triangle' = \triangle g \in \triangle\,\mathrm{Col}(\mathrm{P}V)$, so that the two cosets of $\mathrm{Col}(\mathrm{P}V)$ in $\mathrm{Cor}(\mathrm{P}V)$ are $\mathrm{Col}(\mathrm{P}V)$ and $\triangle\,\mathrm{Col}(\mathrm{P}V)$.

If V is a vector space of dimension d over $\mathrm{GF}(q)$, the groups $\Gamma\mathrm{L}(V)$, $\mathrm{GL}(V)$, $\mathrm{P}\Gamma\mathrm{L}(V)$, $\mathrm{PGL}(V)$ are denoted $\Gamma\mathrm{L}(d,q)$, $\mathrm{GL}(d,q)$, $\mathrm{P}\Gamma\mathrm{L}(d,q)$, $\mathrm{PGL}(d,q)$, respectively.

The 1–dimensional subspaces of a vector space V are **points** of V (and of $\mathrm{P}V$), the 2–dimensional subspaces are **lines** of V (and of $\mathrm{P}V$), the 3–dimensional subspaces of V are **planes** of V (and of $\mathrm{P}V$), and the codimension 1 subspaces are **hyperplanes** of $\mathrm{P}V$ (and of V). Subspaces of V will be said to **meet** if their intersection is non–trivial, and **incident** if one contains the other. The **projective dimension** of a subspace of $\mathrm{P}V$ is one less than its dimension as a subspace of V; inside projective spaces all dimensions will be projective.

For a vector space V of dimension $d+1$ over $\mathrm{GF}(q)$, $V \cong \mathrm{GF}(q)^{d+1}$, so that

$$\#\text{ points of }\mathrm{PG}(d,q) \;=\frac{|V-\{0\}|}{|\,\mathrm{GF}(q)^*|} = q^d + q^{d-1} + \cdots + 1$$

In particular, $\mathrm{PG}(1,q)$ has $q+1$ points, and so this is the number of points on a line of $\mathrm{PG}(d,q)$.

Suppose $d>1$. Let $m>n$, and $\mathcal{W}$ consist of all $\mathrm{PG}(d,q)$ subspaces of dimension at least m that contain a given n–dimensional subspace N. Regard the m–dimensional subspaces of $\mathrm{PG}(d,q)$ on N as the points of $\mathcal{W}$, the $m+1$–dimensional subspaces of $\mathrm{PG}(d,q)$ on N as the lines of $\mathcal{W}$, and so on. Then $\mathcal{W}$ is a $\mathrm{PG}(d-(m-n),q)$ space, and so

$$\begin{aligned}&\text{In } \mathrm{PG}(d,q):\\ &\qquad \#\, m\text{–dimensional subspaces on an } n\text{–dimensional subspace}\\ &\qquad = q^{d-(m-n)} + q^{d-(m-n)-1} + \cdots + 1\end{aligned}$$

By applying a correlation of $\mathrm{PG}(d,q)$, we see that the number of m–dimensional subspaces on an n–dimensional subspace for $m<n$ is the number of $d-m$–dimensional subspaces on a $d-n$–dimensional subspace, and then the result above applies.

Counting the set

$$\{(P,N) : P \text{ a point of } \mathrm{PG}(d,q),\ N \text{ an } n\text{–dimensional subspace of } \mathrm{PG}(d,q)\}$$

in two ways, we obtain

$$\begin{aligned}&\text{In } \mathrm{PG}(d,q):\\ &\qquad \#\, n\text{–dimensional subspaces}\\ &\qquad = \frac{\#\text{ points . }\#\, n\text{–dimensional subspaces on a point}}{\#\text{ points of an } n\text{–dimensional subspace}}\end{aligned}$$

The following hold in any projective space.

(P1) Every two points lie on a unique line.

(P2) Let L and L' be distinct lines meeting in the point P, such that Q and R are points of L distinct from P, and S and T are points of L' distinct from P. Then the line QS through Q and S meets the line RT through R and T.

That (P1) holds is trivial, while to show (P2), first note that L and L' lie in a plane π (as L and L' meet). Now QS and RT are contained in π (as they don't contain P and they intersect with both lines), and therefore QS and RT meet.

1.1.2 Incidence structures. The following material is drawn from [16].

Let $\mathcal{P}$, $\mathcal{L} \neq \emptyset$ be disjoint sets, with $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. Then $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an **incidence structure**, with the elements of $\mathcal{P}$ called **points**, and the elements of $\mathcal{L}$ called **lines**. If $(P, L) \in \mathcal{I}$, P and L are **incident**, in which case we also say that P **lies on** L (and that L **contains** P), and also write $P \subseteq L$. Given $L_1, L_2 \in \mathcal{L}$, we say L_1 and L_2 **meet** if they intersect non–trivially. Given $P \in \mathcal{P}$, we define

$$P^\perp = \{R \in \mathcal{P} : \ P \text{ and } R \text{ lie on a common line}\}$$

and for $L \in \mathcal{L}$,

$$L^\perp = \{N \in \mathcal{L} : \ L \text{ and } N \text{ meet}\}$$

If $|\mathcal{P} \cup \mathcal{L}| < \infty$, Γ is **finite**.

Suppose $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ are incidence structures. If $g : \Gamma \to \Gamma'$ such that

$$(P, L) \in \mathcal{I} \Longleftrightarrow (g(P), g(L)) \in \mathcal{I}'$$

$\forall\ P \in \mathcal{P}, \forall\ L \in \mathcal{L}$, then g is **incidence–preserving**. If g bijectively maps $\mathcal{P}$ to $\mathcal{P}'$ and $\mathcal{L}$ to $\mathcal{L}'$, where g and g^{-1} are incidence–preserving, then g is an **isomorphism** between Γ and Γ', in which case Γ and Γ' are **isomorphic** (denoted $\Gamma \cong \Gamma'$). An isomorphism from Γ to itself is called an **automorphism**.

Suppose $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ are incidence structures. A **duality** from Γ to Γ' is a map θ bijectively taking $\mathcal{P}$ to $\mathcal{L}'$ and $\mathcal{L}$ to $\mathcal{P}'$, where θ and θ^{-1} are incidence–preserving, and a **polarity** is a duality of order 2. If there exists a duality between Γ and Γ', we say that Γ and Γ' are **dual**. If there exists a duality g from Γ to itself, we say Γ is **self–dual**; if g is a polarity, we say Γ is **self-polar**. If a point P of Γ lies on its image under a polarity θ, then P is an **absolute point** of Γ, while if a line L of Γ contains its image under θ, L is an **absolute line** of Γ.

The group G of all automorphisms and dualities of an incidence structure Γ contains the subgroup $\mathrm{Aut}\,\Gamma$ of all automorphisms of Γ. If Γ is self–dual, $\mathrm{Aut}\,\Gamma$ has index 2 in G, otherwise $\mathrm{Aut}\,\Gamma = G$. Another name for $\mathrm{Aut}\,\Gamma$ is the **collineation group** of Γ; its elements are called **collineations**.

Given an incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, the **dual** of Γ is the incidence structure $\Gamma^* = (\mathcal{L}, \mathcal{P}, \mathcal{I}^*)$, where $\mathcal{I}^* = \{(L, P) : (P, L) \in \mathcal{I}\}$ (when we say **dually** concerning a statement about Γ, we mean apply it to Γ^*). Note that Γ need not be isomorphic to Γ^*; if Γ is isomorphic to Γ^*, then Γ is certainly self–dual.

From incidence structures satisfying (P1) and (P2) (plus two extra conditions) can be obtained synthetic projective spaces (see [69, p16]).

1.1.3 Sesquilinear and quadratic forms. Our treatment of polar spaces will for the most part follow that of ([69]).

Let V be a vector space over a field F. A function $f : V \times V \to F$ is a **sesquilinear form** if it is linear in the first variable and semilinear in the second. A sesquilinear form (having companion automorphism σ) is called

- **symmetric** if $f(v_1, v_2) = f(v_2, v_1) \ \ \forall\, v_1, v_2 \in V$
- **skew–symmetric** if $f(v_1, v_2) = -f(v_2, v_1) \ \ \forall\, v_1, v_2 \in V$
- **alternating** if $f(v, v) = 0 \ \ \forall\, v \in V$
- **hermitian** if $f(v_1, v_2) = (f(v_2, v_1))^\sigma \ \ \forall\, v_1, v_2 \in V \ \ (\sigma \neq 1)$

Observe that only hermitian forms have non–trivial companion automorphism. Also, if f is a hermitian form (with companion automorphism σ) on a vector space V over $\mathrm{GF}(q)$ ($q = p^h$, p a prime, $h \geq 1$), then $\sigma^2 = 1$, so $\mathrm{Aut}(\mathrm{GF}(q)) \cong \mathrm{C}_h$ implies that $2 \,\big|\, h$. Thus, q must be a square.

Given a vector space V over a field F, a **quadratic form** is a function $Q : V \to F$ such that

(i) $Q(\lambda v) = \lambda^2 Q(v) \ \ \forall\, \lambda \in F, \forall\, v \in V$

(ii) the function $f_Q : V \times V \to F$ defined by

$$f_Q(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2)$$

is bilinear (f_Q is the **polar form** of Q)

A **polar space** is a vector space equipped with a symmetric, alternating,hermitean or quadratic form. Polar spaces arising from an alternating, hermitian or quadratic form are called **symplectic**, **unitary** or **orthogonal**, respectively.

Note. Let V be a vector space over a field F. A polar form on V is symmetric, while if f is a symmetric form on V (and $\mathrm{char}\, F \neq 2$) then $Q : V \to F$ defined by $Q(v) = \frac{1}{2} f(v, v)$ is a quadratic form with polar form f. So, when $\mathrm{char}\, F \neq 2$, symmetric forms and polar forms correspond; we won't consider the symmetric forms that don't stem from a quadratic form. Also, an alternating form is skew–symmetric, while for $\mathrm{char}\, F \neq 2$ a skew–symmetric form is alternating: skew–symmetric forms that aren't alternating won't be dealt with.

Let V_1 and V_2 be vector spaces over a field F, equipped with sesquilinear forms f_1 and f_2 respectively. A **semisimilarity** from (V_1, f_1) to (V_2, f_2) is an invertible semilinear map $T : V_1 \to V_2$ (with companion automorphism σ) such that $\exists\ c \in F^*$ with $f_2(T(v), T(v')) = c(f_1(v, v'))^\sigma \ \ \forall\ v, v' \in V_1$. If T has $\sigma = 1$ then T is a **similarity**, and if T has $\sigma = 1$ and $c = 1$ then T is an **isometry**. Analogously, for vector spaces V_1 and V_2 (over a field F) equipped with quadratic forms Q_1 and Q_2 respectively, we define a **semisimilarity** between (V_1, Q_1) and (V_2, Q_2) to be an invertible semilinear map $T : V_1 \to V_2$ (with companion automorphism σ) such that $\exists\ c \in F^*$ with $Q_2(T(v)) = c(Q_1(v))^\sigma \ \ \forall\ v \in V_1$. If T has $\sigma = 1$ then T is a **similarity**, and if T has $\sigma = 1$ and $c = 1$ then T is an **isometry**.

We call polar spaces (V_1, β_1) and (V_2, β_2) **semisimilar/similar/isometric** if there exists a semisimilarity/similarity/isometry g between them. The map $\bar{g}$ that g induces between $\mathrm{P}V_1$ and $\mathrm{P}V_2$ is a **projective semisimilarity/similarity/isometry** (note that if g is a scalar map, then $\bar{g}$ is the identity map).

The group of all semisimilarities/similarities/isometries from a polar space $\mathcal{S} = (V, \beta)$ to itself is the **semisimilarity/similarity/isometry group** of $\mathcal{S}$, and is denoted $\Gamma\mathcal{S}/\mathrm{G}\mathcal{S}/\mathcal{S}$. The **projective semisimilarity/similarity/isometry group** of $\mathcal{S}$ is the group induced by the action of $\Gamma\mathcal{S}/\mathrm{G}\mathcal{S}/\mathcal{S}$ on $\mathrm{P}V$, and is denoted $\mathrm{P}\Gamma\mathcal{S}/\mathrm{PG}\mathcal{S}/\mathrm{P}\mathcal{S}$. If $Z(\mathcal{S})$ denotes the group of form–preserving scalar maps on $\mathcal{S}$, then $\mathrm{P}\Gamma\mathcal{S} \cong \Gamma\mathcal{S}\,/Z(\mathcal{S})$, $\mathrm{PG}\mathcal{S} \cong \mathrm{G}\mathcal{S}\,/Z(\mathcal{S})$ and $\mathrm{P}\mathcal{S} \cong \mathcal{S}/Z(\mathcal{S})$.

When a vector space V has a sesquilinear form f on it, we call $v \in V$ **isotropic** if $f(v, v) = 0$, and a subspace W of V **totally isotropic** if $f|_{W \times W} = 0$ (note that in a symplectic space, every vector is isotropic). For Q a quadratic form on a vector space V, $v \in V$ is **singular** if $Q(v) = 0$, while a subspace W of V is **totally singular** if $Q|_W = 0$.

A totally isotropic/singular subspace of a polar space is **maximal** if it is properly contained in no totally isotropic/singular subspace. All maximals of a polar space $\mathcal{S}$ have the same dimension (see Corollary 1.1.5.3), called the **Witt index** of $\mathcal{S}$. We will refer to maximal totally isotropic/singular subspaces as maximals, and totally isotropic/singular points as isotropic/singular points (or just points, when it is implicit that they are isotropic/singular).

Remark. The set of isotropic points of a unitary space is sometimes called a **hermitian variety** of the underlying projective space, with the set of singular points of an orthogonal space a **quadric** of the underlying projective space.

If two isotropic/singular points of a polar space span a totally isotropic/singular line, they are **collinear**. The following (basic) lemma provides an easy means of determining when isotropic/singular points are collinear.

Lemma 1.1.3.1. *Let (V, β) be a polar space over a field F, containing isotropic*

/singular points $P_1 = \langle v_1 \rangle$ and $P_2 = \langle v_2 \rangle$.

(a) For β an alternating or hermitian form, P_1 and P_2 are collinear if and only if $\beta(v_1, v_2) = 0$.

(b) For β a quadratic form (with polar form f_β), P_1 and P_2 are collinear if and only if $f_\beta(v_1, v_2) = 0$.

Proof. (a) Let σ be the companion automorphism of β, and $L = \langle v_1, v_2 \rangle$. Any element of $L \times L$ may be written as $(av_1 + bv_2, cv_1 + dv_2)$ for some $a, b, c, d \in F$, and

$$\begin{aligned} \beta(av_1 + bv_2, cv_1 + dv_2) &= a\sigma(c)\beta(v_1, v_1) + a\sigma(d)\beta(v_1, v_2) + b\sigma(c)\beta(v_2, v_1) \\ &\quad + b\sigma(d)\beta(v_2, v_2) \qquad (1.1.3.1) \\ &= a\sigma(d)\beta(v_1, v_2) + b\sigma(c)\beta(v_2, v_1) \end{aligned}$$

So, if L is totally isotropic, we have $a\sigma(d)\beta(v_1, v_2) + b\sigma(c)\beta(v_2, v_1) = 0 \ \forall\, a, b, c, d \in F$. In particular, setting $a, d \neq 0$ and $c = 0$ yields $\beta(v_1, v_2) = 0$. Conversely, putting $\beta(v_1, v_2) = 0$ and hence $\beta(v_2, v_1) = 0$ (using that an alternating form is skew–symmetric, and that $\beta(v_1, v_2) = \beta(v_2, v_1)^\sigma$ if β is hermitian) in (1.1.3.1) implies that L is totally isotropic.

(b) Let $L = \langle v_1, v_2 \rangle$. Vectors on L are of the form $av_1 + bv_2$ for $a, b \in F$, and

$$\begin{aligned} \beta(av_1 + bv_2) &= a^2\beta(v_1) + b^2\beta(v_2) + abf_\beta(v_1, v_2) \\ &= abf_\beta(v_1, v_2) \end{aligned}$$

from which the result follows. □

As an obvious consequence of Lemma 1.1.3.1, we see that every two points of a totally isotropic/singular subspace of a polar space are collinear.

1.1.4 Perps and radicals. Let f be an alternating, hermitian or polar form on the vector space V. For U a subspace of V, define $U^\perp$ (the **perp** of U) to be $\{v \in V : f(u, v) = 0 \ \forall\, u \in U\}$. The **radical** of f (denoted $\operatorname{rad} f$) is $V^\perp$, while (if f isn't a polar form) the **radical** of V (denoted $\operatorname{rad} V$) is $\operatorname{rad} f$. The form (and space, if f isn't a polar form) is **non–degenerate** if $\operatorname{rad} f = \{0\}$, and **degenerate** otherwise. If V is a vector space coupled with a quadratic form Q (with f_Q the polar form of Q), the **singular radical** of Q and of V (denoted $\operatorname{rad} Q$ and $\operatorname{rad} V$ respectively) is $\{v \in \operatorname{rad} f_Q : Q(v) = 0\}$, with the form (and space) **non–degenerate** if $\operatorname{rad} Q = \{0\}$, and **degenerate** otherwise. For U a subspace of a polar space (V, β), the radical/singular radical of $\beta|_U$ and of U is given by using $\beta|_U$ and U in the definitions above. Clearly, if β is an alternating or hermitian form, $\operatorname{rad} U = U \cap U^\perp$; if β is a quadratic form, $\operatorname{rad} U = \{x \in U \cap U^\perp : \beta|_U(x) = 0\}$ (where perp is defined via the polar form f_β of β).

Given any polar space, it is always possible to obtain a non–degenerate polar space from it, as the following elementary result shows.

Theorem 1.1.4.1. *Let V be a vector space.*

(a) Suppose f is an alternating, hermitian or polar form on V, and let $U = \operatorname{rad} f$. Define f' on V/U via $f'(v_1 + U, v_2 + U) = f(v_1, v_2)$. Then f' is a (well–defined) non–degenerate form of the same type as f.

(b) Suppose Q is a quadratic form on V (with polar form f_Q), and let $U = \operatorname{rad} Q$. Define Q' on V/U by $Q'(v + U) = Q(v)$. Then Q' is a (well–defined) non–degenerate quadratic form.

Proof. (a) Let $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Now

$$\begin{aligned} f'(v_1 + u_1 + U, v_2 + u_2 + U) &= f(v_1 + u_1, v_2 + u_2) \\ &= f(v_1, v_2) + f(v_1, u_2) + f(u_1, v_2) + f(u_1, u_2) \\ &= f(v_1, v_2) \end{aligned}$$

as $u_1, u_2 \in \operatorname{rad} f$, so that f' is determined up to choice of coset representatives. Note that

$$\begin{aligned} \operatorname{rad} f' &= \{w + U \in V/U : f'(v + U, w + U) = 0 \ \ \forall\, v + U \in V/U\} \\ &= \{w + U \in V/U : f(v, w) = 0 \ \ \forall\, v \in V\} \\ &= \{w + U \in V/U : w \in U\} = \{U\} \end{aligned}$$

and U is the zero of V/U.
(b) Let $u \in U$, $v \in V$, and $f_{Q'}$ be the polar form of Q'. Now Q' is well–defined, as

$$Q'(v + u + U) = Q(v + u) = Q(v) + Q(u) + f_Q(v, u) = Q(v)$$

Also

$$\begin{aligned} \operatorname{rad} Q' &= \{w + U \in V/U \mid f_{Q'}(w + U, v + U) = 0 \ \ \forall\, v + U \in V/U : Q'(w + U) = 0\} \\ &= \{w + U \in V/U \mid f_Q(w, v) = 0 \ \ \forall\, v \in V : Q(w) = 0\} \\ &= \{w + U \in V/U \mid w \in U\} = \{U\} \end{aligned}$$

□

In a non–degenerate polar space the following hold.

(PS1) Every two isotropic/singular points lie on at most one totally isotropic/singular line.

(PS2) If an isotropic/singular point P has $P \not\subseteq L$ for L a totally isotropic/singular line, then P is collinear with either one point of L, or all points of L.

(PS3) No isotropic/singular point is collinear with all isotropic/singular points.

Note that (PS1) follows from (P1), while (PS3) is due to the space being non–degenerate. To see (PS2), let the polar space $\mathcal{S}$ (over the field F) admit the alternating or hermitian form β having companion automorphism σ, with $P = \langle u \rangle$ and $L = \langle v, w \rangle$. To show that at least one point of L must be collinear with P, suppose $R_1 = \langle v \rangle$ and $R_2 = \langle w \rangle$ aren't collinear to P, and observe that

$$\beta(u, -\frac{\beta(u,w)}{\beta(u,v)}v + w) = 0$$

Now suppose R_1 and R_2 are collinear to P, in which case $0 = k_1\beta(u,v) + k_2\beta(u,w) = \beta(u, \sigma^{-1}(k_1)v + \sigma^{-1}(k_2)w) \;\; \forall\, k_1, k_2 \in F$, and then we have every point of L being collinear to P (the quadratic form case is similar).

From incidence structures satisfying (PS1), (PS2) and (PS3) (plus an extra condition) can be obtained synthetic polar spaces (see [69, p108]).

A fact we will require shortly is that it is possible to represent by a matrix a sesquilinear form on an n–dimensional vector space. Let V be such a space over a field F, admitting a sesquilinear form f (having companion automorphism σ). Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for V over F, and define the **matrix of f with respect to $\mathcal{B}$** to be the $n \times n$ matrix D having $D_{ij} = f(b_i, b_j)$. Because $\phi(a_1, \ldots, a_n) = a_1b_1 + \ldots a_nb_n$ defines an isomorphism between F^n and V, we obtain a sesquilinear form $f' : F^n \times F^n \to F$ by letting

$$f'(x,y) = x^T D\sigma(y)$$

and now f' on $F^n \times F^n$ corresponds to f on $V \times V$. It is possible to characterise the assorted sesquilinear forms in terms of what type of matrix D is; for us, the use of the above identification will be through the next basic result.

Lemma 1.1.4.2. *Let f be a sesquilinear form on an n–dimensional vector space V over a field F, with $\mathcal{B} = \{b_1, \ldots, b_n\}$ a basis for V over F and D the matrix of f with respect to $\mathcal{B}$. Then f is non–degenerate if and only if* $\det(D) \neq 0$.

Proof. Assume the identification established above, and define $T : F^n \to F$ via $T(y) = D\sigma(y)$. Now

$$y \in \operatorname{rad} f \iff x^T D\sigma(y) = 0 \;\; \forall\, x \in F^n \iff D\sigma(y) = 0 \iff y \in \ker T$$

where we have used the fact that if $z \in F^n$, then $z = 0 \iff w^T z = 0 \;\; \forall\, w \in F^n$. Now h defined by $h : y \to \sigma(y)$ is a bijection on F^n, so $\ker T = \{0\} \iff \det(D) \neq 0$. $\square$

The following hold for an alternating, hermitian or polar form f (having companion automorphism σ) on an n–dimensional vector space V over a field F, where U and W are subspaces of V.

(a) $U \subseteq W \implies W^\perp \subseteq U^\perp$, while the map g on PV defined by $g(U) = U^\perp$ is a polarity.

(b) If $\operatorname{rad} f = \{0\}$, we have $\dim U + \dim U^\perp = \dim V + \dim V^\perp$

The first part of (a) is clear. For the second part, assume the identification established before Lemma 1.1.4.2, and let $\mathcal{U}' = \{u_1', \ldots, u_m'\}$ be a basis for the subspace U' of F^n, where U' corresponds to the subspace U of V. Now

$$\begin{aligned} U'^\perp &= \{v \in F^n : f'(v, k_1 u_1' + \ldots + k_m u_m') = 0 \;\; \forall\, k_1, \ldots, k_m \in F, \;\forall\, u_i' \in \mathcal{U}'\} \\ &= \{v \in F^n : k_1 f'(v, u_1') + \ldots + k_m f'(v, u_m') = 0 \;\; \forall\, k_1, \ldots, k_m \in F, \;\forall\, u_i' \in \mathcal{U}'\} \\ &= \{v \in F^n : f'(v, u_i') = 0 \;\; \forall\, u_i' \in \mathcal{U}'\} \end{aligned}$$

The equations $f'(v, u_i') = 0$ $(i \in \{1, \ldots, n\})$ are linearly independent, since $\det(D) \neq 0$. By the rank–nullity theorem, $\dim U'^\perp = n - m$, so $\dim U^\perp = n - m$. Hence, $\dim(U^{\perp\perp}) = \dim V - \dim U^\perp = \dim V - (\dim V - \dim U) = \dim U$, while $U \subseteq U^{\perp\perp}$, so $U = U^{\perp\perp}$ and thus g is of order 2. Hence, $g = g^{-1}$, so that g is 1–1 and onto.

For (b), note that by Theorem 1.1.4.1 we can define a non–degenerate form f' on $V/V^\perp$ which has the same type as f. By the argument above, we have

$$\dim(U/V^\perp) + \dim(U^\perp/V^\perp) = \dim(V/V^\perp)$$

and so

$$\dim U - \dim V^\perp + \dim U^\perp - \dim V^\perp = \dim V - \dim V^\perp$$

We now require some material to describe the polar spaces that we will work in.

1.1.5 Witt's theorem.

Theorem 1.1.5.1 (Witt). *Let f be an alternating, hermitian or polar form on the vector space V, with U and W subspaces of V and $h : U \to W$ an isometry. There exists an isometry $g : V \to V$ such that $g|_U = h$ if and only if $g(U \cap \operatorname{rad} f) = W \cap \operatorname{rad} f$.*

Proof. See [69, pp57–58]. $\square$

The following elementary corollaries will be frequently required.

Corollary 1.1.5.2. *Let f be an alternating, hermitian or polar form on the vector space V, with $\operatorname{rad} f = \{0\}$. Then every isometry between subspaces of V extends to an isometry on all of V. Consequently, if subspaces U and W of V are isometric, then $U^\perp$ and $W^\perp$ are isometric.*

Proof. The first part is trivial, and for the second, we know by Witt's theorem that an isometry g on V having $g(U) = W$ exists. Then

$$\begin{aligned} g(U^\perp) &= g(\{v \in V : f(u,v) = 0 \ \ \forall\, u \in U\}) \\ &= \{gv \in V : f(gu, gv) = 0 \ \ \forall\, gu \in W\} \\ &= W^\perp \end{aligned}$$

$\square$

Corollary 1.1.5.3. *All maximals of a finite–dimensional polar space have the same dimension.*

Proof. Firstly, note that any two totally isotropic/singular subspaces (of a finite–dimensional polar space $\mathcal{S}$) having the same dimension are isometric, so by Witt's theorem an element of $\mathrm{P\Gamma}\mathcal{S}$ takes one to the other. Now suppose that U_1, U_2 are maximals, and that $\dim U_1 < \dim U_2$. Take a subspace U_1' of U_2 for which $\dim U_1 = \dim U_1'$, and then $\exists\, g \in \mathrm{P\Gamma}\mathcal{S}$ for which $g(U_1) = U_1'$, a contradiction to U_1' not being a maximal. $\square$

Suppose U is a totally isotropic/singular subspace of a polar space $\mathcal{S} = (V, \beta)$. Now $U \subseteq U^\perp$, so $\dim U \leq \dim U^\perp$, and hence $2 \dim U \leq \dim V + \dim V^\perp$. Thus, the Witt index of $\mathcal{S}$ is at most $\frac{1}{2}(\dim V + \dim V^\perp)$.

Let f be an alternating, hermitian or polar form on the vector space V. A pair of isotropic vectors v_1 and v_2 of V that have $f(v_1, v_2) = 1$ is a **hyperbolic pair**, and the line they span is a **hyperbolic line**. If Q is a quadratic form on V (with polar form f_Q), a pair of singular vectors v_1 and v_2 of V that have $f_Q(v_1, v_2) = 1$ is a **hyperbolic pair**, and the line they span is a **hyperbolic line**.

The next result is a basic consequence of these definitions.

Lemma 1.1.5.4. *(a) Any two symplectic hyperbolic lines are isometric.*

(b) Any two unitary hyperbolic lines are isometric.

(c) Any two orthogonal hyperbolic lines are isometric.

Proof. We first consider the case of symplectic/unitary hyperbolic lines; let (V, f) be such a line, where V is over the field F. Then $V = \langle u, v\rangle$ for some $u, v \in V$, where $f(u, u) = f(v, v) = 0$ and $f(u, v) = 1$, and so the matrix of f with respect to $\{u, v\}$ is

$$\begin{pmatrix} 0 & 1 \\ f(v,u) & 0 \end{pmatrix}$$

(note that $f(v, u) = -1$ if f is alternating, while $f(v, u) = 1$ if f is hermitian). Assume the set up before Lemma 1.1.4.2. Given two symplectic/unitary hyperbolic lines, we thus see that the matrix taking one hyperbolic pair to the other is an isometry.

For the orthogonal case, let (V, Q) be an orthogonal hyperbolic line (where V is over the field F), so that $V = \langle u, v\rangle$ for some $u, v \in V$ for which $Q(u) = Q(v) = 0$ and $f_Q(u, v) = 1$ (f_Q the polar form of Q). Identifying V with $\mathrm{GF}(q)^2$, Q corresponds to the quadratic form Q' on $\mathrm{GF}(q)^2$, defined by $Q'(x) = x^T Ax$. Now given $a, b \in F$,

$$\begin{aligned} Q(au + bv) &= Q(au) + Q(bv) + f_Q(au, bv) \\ &= a^2 Q(u) + b^2 Q(v) + ab f_Q(u, v) \\ &= ab \end{aligned}$$

Hence, $Q'(au' + bv') = ab$ (where u' and v' correspond to u and v respectively), so w.l.o.g.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and again we see that the matrix taking one hyperbolic pair to another must be an isometry. $\square$

A subspace of a polar space is **anisotropic** if it contains no non–zero isotropic /singular vectors. A line L of a polar space is **tangent** if it contains exactly one isotropic/singular point P, and then we say that L is a **tangent line at** P.

Lemma 1.1.5.5. *A line of a polar space is either anisotropic, tangent, hyperbolic or totally isotropic/singular.*

Proof. Suppose L is a line of a polar space (V, β) over a field F, where L contains at least one isotropic/singular point $P = \langle v_1 \rangle$. Note that $\operatorname{rad} L$ is either $\{0\}$, a point of L or all of L; we show that this characterises L as being (respectively) hyperbolic, tangent or totally isotropic/singular.

In the former case (following [69, p56]), first suppose β is an alternating form. Now $\exists\ w \in L$ such that $\beta(v_1, w) = a \neq 0$: then $v_2 = a^{-1}w$ is an isotropic vector with $\beta(v_1, v_2) = 1$. Now let β be a quadratic form with polar form f_β. We have $w \in L$ with $\beta(v_1, w) = a \neq 0$: then $v_2 = -\beta(w)a^{-2}v_1 + a^{-1}w$ is a singular vector with $f_\beta(v_1, v_2) = 1$. For β an hermitian form with companion automorphism σ, again we have $w \in L$ such that $\beta(v_1, w) = a \neq 0$. Select $d \in F$ with $d + \sigma(d) \neq 0$, let $c = (d + \sigma(d))^{-1}\sigma(d)\beta(w, w)$, and then $v_2 = -cu(a\sigma(a))^{-1} + w(\sigma(a))^{-1}$ is an isotropic vector with $f(v_1, v_2) = 1$. Thus, in all cases we see that L is a hyperbolic line. Conversely, let β be an alternating or hermitian form and suppose L contains a hyperbolic pair v_1, v_2, with w a non–zero vector of $\operatorname{rad} L$. Writing $v_2 = aw + bv_1$ for some $a, b \in F$, we have $\beta(v_1, v_2) = a\beta(v_1, w) + b\beta(v_1, v_1) = 0$, a contradiction (and similarly for the quadratic form case).

If $\operatorname{rad} L = L$, then $L^\perp = L$, that is, L is a totally isotropic/singular line (and conversely).

If $\operatorname{rad} L$ is a point of L (P say), then L is tangent. For, if β is an alternating or hermitian form and $\langle u \rangle$ is an isotropic point of L distinct from P, then $\beta(v_1, u) = 0$ (as $\operatorname{rad} L = L \cap L^\perp$), so L is a totally isotropic line, in which case $\operatorname{rad} L = L$ and we have a contradiction (the quadratic form case is similar).

$\square$

A vector space V is the **direct sum** of subspaces U and W if $V = \langle U, W \rangle$ and $U \cap W = \{0\}$. If f is an alternating, hermitian or polar form on V with V the direct sum of subspaces U and W, then V is the **orthogonal direct sum** of U and W if $f(u, w) = 0\ \ \forall\ u \in U, \forall\ w \in W$, in which case we write $V = U \perp W$.

1.1.6 Classification results. The following simple lemma means that when classifying polar spaces up to similarity/isometry, we need only consider the non–degenerate ones (recall Theorem 1.1.4.1).

Lemma 1.1.6.1. *Polar spaces (V_1, β_1) and (V_2, β_2) are similar/isometric if and only if $\dim \operatorname{rad} V_1 = \dim \operatorname{rad} V_2$ and $V_1 / \operatorname{rad} V_1$ is similar/isometric to $V_2 / \operatorname{rad} V_2$.*

Proof. Suppose that (V_1, f_1) and (V_2, f_2) are both symplectic or both hermitian spaces (we omit the orthogonal space case) that are similar/isometric via the map T. Now T takes $\operatorname{rad} V_1$ to $\operatorname{rad} V_2$, so certainly $\dim \operatorname{rad} V_1 = \dim \operatorname{rad} V_2$. Define a map $\overline{T} : V_1/\operatorname{rad} V_1 \to V_2/\operatorname{rad} V_2$ via $\overline{T}(v_1 + \operatorname{rad} V_1) = T(v_1) + \operatorname{rad} V_2$. To show that $\overline{T}$ is well–defined, first suppose that $v_1 + \operatorname{rad} V_1 = v_1' + \operatorname{rad} V_1$. Then $v_1 - v_1' \in \operatorname{rad} V_1$, so $T(v_1) - T(v_1') = T(v_1 - v_1') \in T(\operatorname{rad} V_1) = \operatorname{rad} V_2$, which implies that $T(v_1) + \operatorname{rad} V_2 = T(v_1') + \operatorname{rad} V_2$, and hence $\overline{T}(v_1 + \operatorname{rad} V_1) = \overline{T}(v_1' + \operatorname{rad} V_1)$. That $\overline{T}$ is invertible and linear is because T is, while to see that $\overline{T}$ is form–preserving (where the forms $\bar{f}_1$ and $\bar{f}_2$ on the spaces $V_1/\operatorname{rad} V_1$ and $V_2/\operatorname{rad} V_2$ (respectively) are defined as in Theorem 1.1.4.1), note that

$$\begin{aligned}
&\bar{f}_2(\overline{T}(v_1 + \operatorname{rad} V_1), \overline{T}(w_1 + \operatorname{rad} V_1)) \\
&= \bar{f}_2(T(v_1) + \operatorname{rad} V_2, T(w_1) + \operatorname{rad} V_2) \\
&= f_2(T(v_1), T(w_1)) \\
&= f_1(v_1, w_1) \\
&= \bar{f}_1(v_1 + \operatorname{rad} V_1, w_1 + \operatorname{rad} V_1)
\end{aligned}$$

For the other direction, first write $V_1 = \operatorname{rad} V_1 \perp W_1$ and $V_2 = \operatorname{rad} V_2 \perp W_2$, where $W_2 = T(W_1)$ for T an isometry from $V_1/\operatorname{rad} V_1$ to $V_2/\operatorname{rad} V_2$ (the similarity case is similar). Then I defined by $I((r_1, w_1)) = (\phi(r_1), T(w_1))$ is an isometry between (V_1, f_1) and (V_2, f_2), where ϕ is any invertible linear map between $\operatorname{rad} V_1$ and $\operatorname{rad} V_1$ (such a map is guaranteed to exist, since the two radicals were assumed to have the same dimension). □

Of the non–degenerate polar spaces, only those of Witt index n have been classified (see the theorem below), while anisotropic unitary and orthogonal spaces have only been completely classified for finite fields. Given that we are only interested in working in the spaces that have been classified, our focus in this thesis will be on the non–degenerate polar spaces of Witt index n over finite fields.

The next theorem is the main ingredient in the classification theorems we will state.

Theorem 1.1.6.2 (see [69], p69, p116, pp138–139). *Any non–degenerate polar space of Witt index n is an orthogonal direct sum of n hyperbolic lines and an anisotropic space A, where A is determined up to isometry (A is called the* **germ** *of the space).*

Proof. Let $\mathcal{S} = (V, \beta)$ be a non–degenerate polar space of Witt index n. We first show existence of the required decomposition of $\mathcal{S}$, using induction on n. If $n = 0$, the whole of V is anisotropic and so the result follows. Now suppose $n > 0$, so that isotropic/singular vectors of V exist; let $u \neq 0$ be such a vector. Take a line L with $u \in L$ such that $L \nsubseteq u^{\perp}$, so that $L = \langle u, w \rangle$ for some $w \in V - u^{\perp}$. Using Lemma 1.1.5.5, L is a hyperbolic line, so by the proof of Lemma 1.1.5.5, $\operatorname{rad} L = \{0\}$. Because $\operatorname{rad} L^{\perp} = L^{\perp} \cap L^{\perp\perp} = L^{\perp} \cap L = \operatorname{rad} L$, $L^{\perp}$ is a non–degenerate polar space. Also, if W is a totally isotropic/singular subspace with $W \subseteq L^{\perp}$, then $\langle u, W \rangle$ is totally isotropic/singular, so W has dimension at most $n-1$. And if W is a maximal of $L^{\perp}$, $\dim W = n - 1$, since for any maximal M of $\mathcal{S}$ with $u \in M$, $M \cap L^{\perp}$ has dimension $n - 1$. By the induction hypothesis, $L^{\perp} = L_1 \perp \ldots \perp L_{n-1} \perp A$ for some hyperbolic lines $L_1, \ldots, L_{n-1}$ and some anisotropic space A, while $V = L \perp$ Ł$^{\perp}$.

To show that A is determined up to isometry, suppose $V = L_1 \perp \ldots \perp L_n \perp A = L'_1 \perp \ldots \perp L'_n \perp A'$. Since (by Lemma 1.1.5.4) any two hyperbolic lines are isometric, $L_1 \perp \ldots \perp L_n$ and $L'_1 \perp \ldots \perp L'_n$ are isometric. By Witt's theorem, there exists an isometry g of V with $g(L_1 \perp \ldots \perp L_n) = L'_1 \perp \ldots \perp L'_n$. But $A = (L_1 \perp \ldots \perp L_n)^{\perp}$ and $A' = (L'_1 \perp \ldots \perp L'_n)^{\perp}$, so by Corollary 1.1.5.2, $g(A) = A'$.

□

Let (V, β) and (V', β') be non–degenerate polar spaces of the same type, each with Witt index n and having isometric germs. Let $V = L_1 \perp \ldots \perp L_n \perp A$ and $V' = L'_1 \perp \ldots \perp L'_n \perp A'$ be the decompositions guaranteed by Theorem 1.1.6.2, with g an isometry between A and A'. Since any two hyperbolic lines of the same type are isometric (Lemma 1.1.5.4), we have an isometry T_i between L_i and L'_i for all $i \in \{1, \ldots, n\}$. Since we have an isometry defined between each basis vector of V and V', we obtain an isometry between (V, β) and (V', β'); thus, any two spaces satisfying the conditions of Theorem 1.1.6.2 that have isometric germs are isometric. The case where the germs are similar is analogous.

In light of Theorem 1.1.6.2, we need to classify the anisotropic symplectic, unitary and orthogonal spaces; the first of these results is trivial.

Lemma 1.1.6.3. *The dimension of an anisotropic symplectic space is* 0.

Proof. Let A be an anisotropic symplectic space. The form f on A is alternating, so $f(a, a) = 0 \;\; \forall\, a \in A$, and then A being anisotropic forces $A = \{0\}$. □

Corollary 1.1.6.4 (see [69], p69). *Any non–degenerate symplectic space of Witt index n over the field F is isometric to the orthogonal direct sum of n hyperbolic lines, and is denoted* $\mathrm{Sp}(2n, F)$.

Proof. Apply Lemma 1.1.6.3 in Theorem 1.1.6.2. □

Lemma 1.1.6.5 (see [69], p116). *(a) The dimension of an anisotropic unitary space over* $\mathrm{GF}(q)$ *is at most* 1.

(b) The dimension 1 *anisotropic unitary spaces over* $\mathrm{GF}(q)$ *are all isometric.*

Proof. Let f be a hermitian form (with companion automorphism σ defined by $\sigma(x) = x^{q^{\frac{1}{2}}}$) on a vector space V of dimension at least 2 over $\mathrm{GF}(q)$. Suppose v is a non–isotropic vector of V, and set $b = f(v, v)$. Then $b = \sigma(f(v, v))$, so $b \in \mathrm{GF}(q^{\frac{1}{2}})$ (since $\mathrm{GF}(q^{\frac{1}{2}}) = \{x \in \mathrm{GF}(q) : \sigma(x) = x\}$). Now if $c \in \mathrm{GF}(q)$ and $u \in v^{\perp} - \{0\}$, we have

$$f(u + cv, u + cv) = f(u, u) + c\sigma(c)b$$

Because $\mathrm{GF}(q^{\frac{1}{2}}) = \{x\sigma(x) : x \in \mathrm{GF}(q)\}$ and $-f(u, u) \in \mathrm{GF}(q^{\frac{1}{2}})$, there exists $a \in \mathrm{GF}(q)$ such that $a\sigma(a) = \frac{-f(u,u)}{b}$, and then $f(u + av, u + av) = 0$. Now u is not in $\langle v\rangle$ (as otherwise $f(u, v)$ would be some multiple of b, and hence non–zero), so $u + av \neq 0$ and thus V is not anisotropic.

(b) Let f be a hermitian form (with companion automorphism as in (a)) on a vector space V of dimension 1 over $\mathrm{GF}(q)$, such that V is anistropic. Suppose $u \in V - \{0\}$; as above, $f(u, u) \in \mathrm{GF}(q^{\frac{1}{2}})^*$. Now $\exists\ a \in \mathrm{GF}(q)$ with $a\sigma(a) = \frac{1}{f(u,u)}$, so letting $v = au$, we have $f(v, v) = a\sigma(u)f(u, u) = 1$. Hence, given another 1–dimensional vector space W over $\mathrm{GF}(q)$ admitting a hermitian form g such that W is anisotropic, we know that W contains a vector w with $g(w, w) = 1$. Now define an isometry $T : V \to W$ via $T(cv) = cw$. □

Corollary 1.1.6.6 (see [69], p116). *Any non–degenerate unitary space of Witt index* n *over* $\mathrm{GF}(q)$ *is isometric to*

(a) the orthogonal direct sum of n *hyperbolic lines (denoted* $\mathrm{U}(2n, q)$*),*

or

(b) the orthogonal direct sum of n *hyperbolic lines and a* 1*–dimensional anisotropic space (denoted* $\mathrm{U}(2n + 1, q)$*).*

Proof. Apply Lemma 1.1.6.5 in Theorem 1.1.6.2. □

Lemma 1.1.6.7. *(a) The dimension of an anisotropic orthogonal space over* $\mathrm{GF}(q)$ *is at most* 2.

(b) *The dimension* 2 *anisotropic orthogonal spaces over* $\mathrm{GF}(q)$ *are all isometric.*

(c) *The dimension* 1, q *even anisotropic orthogonal spaces over* $\mathrm{GF}(q)$ *are all isometric.*

(d) *The dimension* 1, q *odd anisotropic orthogonal spaces over* $\mathrm{GF}(q)$ *are all similar, and fall into two isometry classes.*

Proof. See [69, pp 138–139]. □

Suppose Q is a non–degenerate quadratic form on a vector space V of dimension d over $\mathrm{GF}(q)$ (q odd), with polar form f_Q. Let $\mathcal{B}$ be a basis for $\mathrm{GF}(q)^d$ over $\mathrm{GF}(q)$, and D be the matrix of f_Q with respect to $\mathcal{B}$; we obtain a quadratic form Q' on $\mathrm{GF}(q)^d$ via $Q'(x) = x^T Dx$, where Q' corresponds to Q. Let $\Box$ denote the set of squares of $\mathrm{GF}(q)^*$. The **discriminant** of Q (denoted $\operatorname{disc} Q$) is $\det D \pmod{\Box}$, and as we show now, gives us an isometry invariant. Suppose $T : (V_1, Q_1) \to (V_2, Q_2)$ is an isometry between non–degenerate orthogonal spaces of dimension d over $\mathrm{GF}(q)$ (q odd), and identify these spaces with $(\mathrm{GF}(q)^d, Q'_1)$ and $(\mathrm{GF}(q)^d, Q'_2)$ respectively, where $Q'_1(x) = x^T A_1 x$ and $Q'_2(x) = x^T A_2 x$ (A_1 and A_2 the matrices of the respective polar forms). If B denotes the matrix corresponding to T, then $A_2 = B^T A_1 B$, so that $\operatorname{disc} Q_2 = \det B^T A_1 B \pmod{\Box} = \det A_1 (\det B)^2 \pmod{\Box} = \operatorname{disc} Q_1$.

Corollary 1.1.6.8 (see [69], pp138–139). *Any non–degenerate orthogonal space of Witt index* n *over* $\mathrm{GF}(q)$ *is isometric to*

(a) *the orthogonal direct sum of* n *hyperbolic lines (denoted* $\mathrm{O}^+(2n, q)$*),*

or

(b) *the orthogonal direct sum of* n *hyperbolic lines and a* 1*–dimensional anisotropic space of discriminant a square/non–square (denoted* $\mathrm{O}(2n+1, q)$*; these spaces falling into two isometry classes and one similarity class for* q *odd, and one isometry class for* q *even),*

or

(c) *the orthogonal direct sum of* n *hyperbolic lines and a* 2*–dimensional anisotropic space (denoted* $\mathrm{O}^-(2n+2, q)$*).*

Proof. Apply Lemma 1.1.6.7 in Theorem 1.1.6.2. □

Note. (a) Given a non–degenerate even–dimensional orthogonal space over $\mathrm{GF}(q)$ with q odd, the discriminant of the quadratic form tells us whether the space is of type (a) or (c) in the above corollary; that is, the discriminant is a complete invariant (see [40, p32]). Since any $\mathrm{O}^+(2n,q)$ space can be written as the orthogonal direct sum of n hyperbolic lines, the matrix of the polar form (with respect to a basis of hyperbolic pairs from these lines) has determinant $(-1)^n$. Thus, a non–degenerate orthogonal space $(\mathrm{GF}(q)^{2n}, Q')$ with q odd is an $\mathrm{O}^+(2n,q)$ space if and only if $\det A = (-1)^n \pmod{\square}$ (A being the matrix of the polar form associated to Q').

(b) The O^+, O and O^- spaces will also be referred to as **hyperbolic**, **parabolic** and **elliptic** spaces, respectively.

(c) The set of singular points of an $\mathrm{O}(3,q)$ space will be called a **conic**, and the set of singular points of an $\mathrm{O}^-(4,q)$ space an **elliptic quadric**.

(d) $\mathrm{W}_n(q)$, $\mathrm{H}(n,q^2)$, $\mathrm{H}(n+1,q^2)$, $\mathrm{Q}^+(2n+1,q)$, $\mathrm{Q}(2n,q)$, $\mathrm{Q}^-(2n+1,q)$ is alternative notation denoting an $\mathrm{Sp}(n+1,q)$, $\mathrm{U}(n+1,q)$, $\mathrm{U}(n+2,q)$, $\mathrm{O}^+(2n+2,q)$, $\mathrm{O}(2n+1,q)$, $\mathrm{O}^-(2n+2,q)$ space, respectively.

(e) We will often regard a given isometry/similarity type of polar space as being a single entity (usually when we are giving some property of that class of space). For example, we will say "$\mathrm{O}^+(8,q)$ has s singular points" instead of "an $\mathrm{O}^+(8,q)$ space has s singular points".

Given vector spaces V_1 and V_2 admitting sesquilinear forms f_1 and f_2 respectively, we form the **external direct sum** $V_1 \perp V_2$ of them by defining a form h on $V_1 \bigoplus V_2$, via

$$h((u,w),(u',w')) = f_1(u,u') + f_2(w,w')$$

where in the case that f_1 and f_2 are unitary forms, we require that they have the same companion automorphism. When taking external direct sums of orthogonal spaces, the following tell us what kind of orthogonal space gets produced.

(a) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are hyperbolic spaces, then $\mathcal{S}_1 \perp \mathcal{S}_2$ is a hyperbolic space.

(b) If $\mathcal{S}_1$ is a hyperbolic space and $\mathcal{S}_2$ an elliptic space, then $\mathcal{S}_1 \perp \mathcal{S}_2$ is an elliptic space.

(c) If $\mathcal{S}_1$ is a parabolic space and $\mathcal{S}_2$ is a hyperbolic or elliptic space, then $\mathcal{S}_1 \perp \mathcal{S}_2$ is a parabolic space.

(d) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are elliptic spaces, then $\mathcal{S}_1 \perp \mathcal{S}_2$ is a hyperbolic space.

Note that (a), (b) and (c) are clear by definition. To prove (d), we show that $\mathcal{S}_1 \perp \mathcal{S}_2$ is an $\mathrm{O}^+(4,q)$ space for $\mathrm{O}^-(2,q)$ spaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Suppose not; then $\mathcal{S}_1 \perp \mathcal{S}_2$ is an elliptic space, and so $\mathcal{S}_1^\perp$ is an $\mathrm{O}^+(2,q)$ space. But $S_1^\perp = S_2$ and so we have a contradiction.

Let $\mathcal{S}$ be a non–degenerate polar space of Witt index n over $\mathrm{GF}(q)$, and $t+1$ be the number of maximals of $\mathcal{S}$ on a totally isotropic/singular subspace of $\mathcal{S}$ of dimension $n-1$. To determine the number of isotropic/singular points of $\mathcal{S}$, we will require the value of t. To this end, let U be a totally isotropic/singular subspace of $\mathcal{S}$ of dimension $n-1$. Then $U^\perp/U$ is a polar space (of the same type as $\mathcal{S}$) of Witt index 1, and is non–degenerate (as $\operatorname{rad} U^\perp = U$). Note that $t+1$ is the number of isotropic/singular points of $U^\perp/U$, and we now have the following basic lemma.

Lemma 1.1.6.9. *Let $t+1$ be the number of isotropic/singular points of a non–degenerate polar space of Witt index 1 over $\mathrm{GF}(q)$.*

(a) In $\mathrm{Sp}(2,q)$, $t+1 = q+1$.

(b) In $\mathrm{U}(2,q)$, $t+1 = q^{\frac{1}{2}}+1$.

(c) In $\mathrm{U}(3,q)$, $t+1 = q^{\frac{3}{2}}+1$.

(d) In $\mathrm{O}^+(2,q)$, $t+1 = 2$.

(e) In $\mathrm{O}(3,q)$, $t+1 = q+1$.

(f) In $\mathrm{O}^-(4,q)$, $t+1 = q^2+1$.

Proof. (a) Each point of $\mathrm{Sp}(2,q)$ is isotropic, while the underlying projective space is $\mathrm{PG}(1,q)$.
(b) Equip $V = \mathrm{GF}(q)^2$ with the hermitian form f defined by

$$f(x,y) = x^{q^{\frac{1}{2}}+1} - y^{q^{\frac{1}{2}}+1}$$

Then (V,f) is a $\mathrm{U}(2,q)$ space. Any isotropic point of (V,f) is spanned by one vector of $\{(t,1) : t^{q^{\frac{1}{2}}+1} = 1\}$, and this set has size $q^{\frac{1}{2}}+1$.
(c) Let the hermitian form f on the vector space V give a $\mathrm{U}(3,q)$ space, containing an isotropic point $P = \langle v\rangle$. Firstly, $P^\perp$ is the unique tangent line at P. For, if L is a tangent line at P, then $L^\perp = \langle w\rangle$ is an isotropic point of L having $f(v,w) = 0$, so that $L^\perp = P$ (otherwise L would be totally isotropic, by Lemma 1.1.3.1) and hence $L = L^{\perp\perp} = P^\perp$ (using that perp defines a polarity). By Lemma 1.1.5.5, any line on

P is either tangent or hyperbolic (noting that no line of (V, f) is totally isotropic). Now every isotropic point is on a line through P (by (P1)), while every hyperbolic line of (V, f) contains $q^{\frac{1}{2}} + 1$ isotropic points (using (b), as a hyperbolic line is a $\mathrm{U}(2, q)$ space). Since there are $q + 1$ lines through P, we see that (V, f) has $q^{\frac{3}{2}} + 1$ isotropic points.
(d) Equip $V = \mathrm{GF}(q)^2$ with the quadratic form Q defined by $Q(x, y) = xy$. Then (V, Q) is an $\mathrm{O}^+(2, q)$ space, and its singular points are $\langle(1, 0)\rangle$ and $\langle(0, 1)\rangle$.
(e) Let P be a singular point of $\mathrm{O}(3, q)$. Argue as in (c) (again, $P^\perp$ is the unique tangent line at P), except now the q hyperbolic lines through P each contain 2 singular points (using that a hyperbolic line is an $\mathrm{O}^+(2, q)$ space), so there are $q+1$ singular points in total.
(f) Let P be a singular point of an $\mathrm{O}^-(4, q)$ space (V, Q). Now $\operatorname{rad} P^\perp = P$, and if L is a tangent line at P, then $\operatorname{rad} L = P$. So, $P = L^\perp \cap L$, which implies that $L \subseteq P^\perp$. Since the number of lines contained in a plane that pass through a point of that plane is $q+1$, there are $q+1$ tangent lines at P. By Lemma 1.1.5.5 (and the fact that $\mathrm{O}^-(4, q)$ has no totally singular lines), the other q^2 lines through P are all hyperbolic; as any singular point of $\mathrm{O}^-(4, q)$ is on one of these lines, there are q^2+1 singular points in total.

□

Incidentally, note that by Lemma 1.1.5.5 and Lemma 1.1.6.9 we now know how many isotropic/singular points that a line of a polar space may have.

1.1.7 Ovoids and spreads. Let $\mathcal{S}$ be a polar space. A **cap** of $\mathcal{S}$ is a set C of isotropic/singular points of $\mathcal{S}$ such that each maximal of $\mathcal{S}$ meets C in at most one point. If each maximal of $\mathcal{S}$ meets C in exactly one point, C is an **ovoid**; clearly, the size of a cap is bounded above by the size of an ovoid. A **partial spread** of $\mathcal{S}$ is a set S' of disjoint maximals of $\mathcal{S}$, while S' is a **spread** if it partitions the set of isotropic/singular points of $\mathcal{S}$ (and the size of a partial spread is bounded above by the size of a spread).

Let Ω be a set of isotropic/singular points of a polar space $\mathcal{S}$ of Witt index at least 2, with $|\Omega| > 1$. If M is a maximal of $\mathcal{S}$ such that $|M \cap \Omega| > 1$, then Ω contains a pair of collinear points. Conversely, given a pair of collinear points of Ω, the line they span is contained in some maximal of $\mathcal{S}$. Thus, no two points of a set of isotropic/singular points of a polar space are collinear if and only if the set is a cap. Because of Lemma 1.1.3.1, this is a convenient way of thinking of caps (and in particular ovoids, once we have Lemma 1.1.7.4).

Let $\mathcal{S}_1 = (V_1, \beta_1)$ and $\mathcal{S}_2 = (V_2, \beta_2)$ be semisimilar polar spaces containing ovoids O_1 and O_2 respectively, with $T : \mathrm{P}V_1 \to \mathrm{P}V_2$ a projective semisimilarity. Then O_1 and O_2 are **equivalent** if $T(O_1) = O_2$. When in the spaces that were classified in Section 1.1.6, we will usually work up to equivalence with ovoids. For example, we will say that "$\mathrm{O}(5, 2)$ contains a unique ovoid" instead of "$\mathrm{O}(5, 2)$ contains a unique equivalence class of ovoids".

The next (basic) result provides a means of showing that two given ovoids of a polar space are inequivalent (it also holds for spreads).

Lemma 1.1.7.1. *Let $\mathcal{S}$ be a polar space containing ovoids O_1 and O_2. If O_1 and O_2 are equivalent, then the stabilisers of O_1 and O_2 are conjugate in $\mathrm{P}\Gamma\mathcal{S}$. Conversely, if H_1 and H_2 are conjugate subgroups of $\mathrm{P}\Gamma\mathcal{S}$, then (up to equivalence) they fix the same set of ovoids.*

Proof. If G is a group acting on a set X, and for $x, y \in X$ $\exists\, g \in G$ with $gx = y$, then $gG_xg^{-1} = G_y$; let X be the set of ovoids of $\mathcal{S}$ and G be $\mathrm{P}\Gamma\mathcal{S}$ to yield the result. For the other direction, suppose $gH_1g^{-1} = H_2$ for $g \in \mathrm{P}\Gamma\mathcal{S}$, and that O is an ovoid of $\mathcal{S}$ fixed by H_1. Then $gGg^{-1}(g(O)) = g(G(O)) = g(O)$, so H fixes an ovoid equivalent to O. □

The next lemma (which can be found in [73]) is fundamental. First, note that if P is an isotropic/singular point of a polar space $\mathcal{S}$, the radical/singular radical of $P^\perp$ is P, so $P^\perp/P$ is a non–degenerate polar space of the same type as $\mathcal{S}$ and with dimension $\dim \mathcal{S} - 2$. Points of $P^\perp/P$ are lines of $\mathcal{S}$ on P, lines of $P^\perp/P$ are planes of $\mathcal{S}$ on P, and so on.

Lemma 1.1.7.2. *Let O be an ovoid of a polar space $\mathcal{S}$, with P an isotropic/singular point of $\mathcal{S}$ not on O. Then $O_P = \{\langle R, P\rangle : R \in P^\perp \cap O\}$ is an ovoid of $P^\perp/P$.*

Proof. Each maximal M' of $P^\perp/P$ corresponds to a maximal M of $\mathcal{S}$ on P. Now $|M \cap O| = 1$, and since each pair of points of a maximal are collinear, we see that $|M' \cap O_P| = 1$. □

Obtaining O_P from O as in Lemma 1.1.7.2 will be referred to as "slicing" by P, with O_P "a slice" of O (the technique is called "slicing"). The next result (which is folklore) means that the number of inequivalent slices that O gives is at most the number of orbits on the set of singular points of $\mathcal{S}$ of the stabiliser of O.

Lemma 1.1.7.3. *Let O be an ovoid of a polar space S, P_1 and P_2 be isotropic/singular points of S with P_1,$P_2 \notin O$, and G be the stabiliser of O. If $P_1 \in [P_2]_G$, then O_{P_1} is equivalent to O_{P_2}.*

Proof. Let $g \in G$ satisfy $g(P_1) = P_2$. Then g maps $P_1^\perp$ to $P_2^\perp$, and so induces a projective semisimilarity $\overline{g} : P_1^\perp/P_1 \to P_2^\perp/P_2$ in the natural way. Now $O_{P_1} = \{\langle R, P_1\rangle : R \in P_1^\perp \cap O\}$, and $\overline{g}O_{P_1} = \{\overline{g}\langle R, P_1\rangle : R \in P_1^\perp \cap O\} = \{\langle \overline{g}R, P_2\rangle : \overline{g}R \in P_2^\perp \cap O\} = O_{P_2}$. □

The following lemma (which can be found in [75]) characterises ovoids and spreads in the spaces of study.

Lemma 1.1.7.4. *Let $\mathcal{S}$ be a polar space of Witt index n over $\mathrm{GF}(q)$, with $t+1$ the number of maximals of $\mathcal{S}$ on a totally isotropic/singular subspace of $\mathcal{S}$ of dimension $n-1$. Then*

(a) An ovoid of $\mathcal{S}$ is a cap of size $q^{n-1}t+1$.

(b) A spread of $\mathcal{S}$ is a partial spread of size $q^{n-1}t+1$.

Proof. To prove (a), we need a formula for the number $M(n)$ of maximals of $\mathcal{S} = (V, \beta)$. Fix an isotropic/singular point P of $\mathcal{S}$. Given a maximal M of $\mathcal{S}$ with $P \nsubseteq M$, $P^\perp$ intersects M in a unique hyperplane $H = P^\perp \cap M$ of M. Furthermore, $M' = \langle P, H\rangle$ is the unique maximal on P that contains H. Thus,

$$\begin{aligned} M(n) - \#\text{ maximals on } P = {} & (\#\text{ maximals } M' \text{ on } P) \\ & .(\#\text{ hyperplanes } H \text{ of } M',\text{ not on } P)(\#\text{ maximals } M \text{ on } H,\ M \neq M') \end{aligned}$$

Now # maximals M' on P = # maximals of $P^\perp/P$, and $P^\perp/P$ is a non–degenerate polar space of Witt index $n-1$. Thus, # maximals M' on P is $M(n-1)$. Because a hyperplane of a maximal is a totally isotropic/singular subspace of $\mathcal{S}$ of dimension $n-1$, # maximals M on H, $M \neq M'$ is t. Also, # hyperplanes of M', not on P = # points of $\mathrm{PG}(n-1, q)$, not on $P^\perp$ (using that perp is dimension reversing), which equals q^{n-1}. Therefore, $M(n) = M(n-1)(q^{n-1}t+1)$.

If C is a cap of $\mathcal{S}$, then

$$|\{(R, M) : R \in C, M \text{ a maximal with } R \subseteq M\}| = |C|M(n-1) \leq M(n)$$

so $|C| \leq q^{n-1}t+1$. The upper bound for $|C|$ is attained if and only if every maximal meets C in one point, that is, if and only if C is an ovoid.

For (b), we need a formula for the number $P(n)$ of isotropic/singular points of $\mathcal{S}$. Fix a maximal M of $\mathcal{S}$. Given an isotropic/singular point P of $\mathcal{S}$ with $P \nsubseteq M$,

$P^\perp$ intersects M in a unique hyperplane $H = P^\perp \cap M$ of M, while $M' = \langle P, H\rangle$ is the unique maximal on P that contains H. Hence,

$$\begin{aligned} P(n) - \#\text{ points on } M &= (\#\text{ hyperplanes } H \text{ of } M)\\ &\quad .(\#\text{ maximals } M' \text{ on } H,\ M \neq M')(\#\text{ points of } M' \text{ not on } H)\end{aligned}$$

and the above equals $(q^{n-1} + \ldots + 1)tq^{n-1}$. Therefore, since # points on M = # points of $\mathrm{PG}(n-1, q)$, we have $P(n) = (q^{n-1} + \cdots + 1)(q^{n-1}t + 1)$.

If S' is a partial spread of (V, β), then

$$\begin{aligned} &|\{(N, P) : N \in S', P \text{ an isotropic/singular point with } P \subseteq N\}|\\ &= |S'|(q^{n-1} + \cdots + 1) \leq P(n)\end{aligned}$$

so that $|S'| \leq q^{n-1}t + 1$. The upper bound for S' is attained if and only if each isotropic/singular point lies on one maximal, that is, if and only if S' is a spread. $\square$

And now we easily obtain

Corollary 1.1.7.5. *Let s denote the number of isotropic/singular points of a non–degenerate polar space of Witt index n over* $\mathrm{GF}(q)$.

(a) In $\mathrm{Sp}(2n, q)$, $s = (q^n + 1)(q^{n-1} + q^{n-2} + \cdots + 1)$.

(b) In $\mathrm{U}(2n, q)$, $s = (q^{n-\frac{1}{2}} + 1)(q^{n-1} + q^{n-2} + \cdots + 1)$.

(c) In $\mathrm{U}(2n + 1, q)$, $s = (q^{n+\frac{1}{2}} + 1)(q^{n-1} + q^{n-2} + \cdots + 1)$.

(d) In $\mathrm{O}^+(2n, q)$, $s = (q^{n-1} + 1)(q^{n-1} + q^{n-2} + \cdots + 1)$.

(e) In $\mathrm{O}(2n + 1, q)$, $s = (q^n + 1)(q^{n-1} + q^{n-2} + \cdots + 1)$.

(f) In $\mathrm{O}^-(2n + 2, q)$, $s = (q^{n+1} + 1)(q^{n-1} + q^{n-2} + \cdots + 1)$.

Proof. In each case above, s is given by the size of a spread of the polar space multiplied by the number of points of a maximal of the polar space; from Lemma 1.1.6.9 and Lemma 1.1.7.4, we have the spread sizes. $\square$

Note. To determine the number of totally isotropic/singular subspaces of dimension k in a non–degenerate polar space $\mathcal{S}$ of Witt index n over $\mathrm{GF}(q)$, count the set

$\{(P, K) : P$ an isotropic/singular point of $\mathcal{S}$, K a k–dimensional totally isotropic /singular subspace of $\mathcal{S}\}$

in two ways to obtain

$$\begin{aligned}&\#\ k\text{–dimensional totally isotropic/singular subspaces of } \mathcal{S}\\ &= \frac{s.\#\ k\text{–dimensional totally isotropic/singular subspaces on an iso./sing. point}}{\#\text{ points of a } k\text{–dimensional totally isotropic/singular subspace}}\end{aligned}$$

where s is the number of totally isotropic/singular points of $\mathcal{S}$. Now Corollary 1.1.7.5 gives us s, while the denominator is just $\frac{q^k-1}{q-1}$. To determine the number ζ of k–dimensional totally isotropic/singular subspaces on an isotropic/singular point P, use induction; ζ is the number of $k-1$–dimensional totally isotropic/singular subspaces of $P^\perp/P$.

For the next lemma (which can be found in [75]), note that if P is a non–isotropic/singular point of a polar space $\mathcal{S}$, then the radical/singular radical of $P^\perp$ is $\{0\}$. Accordingly, $P^\perp$ is a non–degenerate polar space of the same type of $\mathcal{S}$, and of dimension $\dim \mathcal{S} - 1$.

Theorem 1.1.7.6. *(a) A* $\mathrm{U}(2n+1,q)$ *ovoid is a* $\mathrm{U}(2n+2,q)$ *ovoid* $(n \geq 1)$.

(b) An $\mathrm{O}^-(2n,q)$ *ovoid is an* $\mathrm{O}(2n+1,q)$ *ovoid* $(n \geq 1)$.

(c) An $\mathrm{O}(2n+1,q)$ *ovoid is an* $\mathrm{O}^+(2n+2,q)$ *ovoid* $(n \geq 1)$.

Proof. (a) Take a non–isotropic point P of $\mathrm{U}(2n+2,q)$; then $P^\perp$ is a $\mathrm{U}(2n+1,q)$ space. By Lemma 1.1.7.4 and Lemma 1.1.6.9, an ovoid of $P^\perp$ is a set O of $q^{(n+\frac{1}{2})}+1$ isotropic points, no two collinear; O is still such in the ambient $\mathrm{U}(2n+2,q)$ (the proof for (c) is similar).
(b) Take a hyperbolic line L of $\mathrm{O}(2n+1,q)$ (where $\mathrm{O}(2n+1,q)$ is defined via the quadratic form Q); by the proof of Lemma 1.1.5.5, we know that L is non–degenerate. But $\operatorname{rad} L = \{x \in L \cap L^\perp : Q(x) = 0\} = \{x \in L^\perp \cap L^{\perp\perp} : Q(x) = 0\} = \operatorname{rad} L^\perp$, so $L^\perp$ is an $\mathrm{O}(2n-1,q)$ space. Letting P be a singular point not in $(L^\perp)^\perp = L$, we see that the $2n$–dimensional space $\langle L^\perp, P\rangle$ still has Witt index n, that is, $\langle L^\perp, P\rangle$ is an $\mathrm{O}^-(2n,q)$ space. Analogously to (a), ovoids of $\langle L^\perp, P\rangle$ are still ovoids of the ambient $\mathrm{O}(2n+1,q)$. □

1.2 Constructions for ovoids and spreads

1.2.1 Maximals of hyperbolic spaces. Defining a relation $\sim$ on the set $\mathcal{M}$ of maximals of $\mathrm{O}^+(2n,q)$ by letting

$$M_1 \sim M_2 \Longleftrightarrow \dim\ M_1 \cap M_2 \equiv n \pmod 2$$

induces two equivalence classes of $\mathcal{M}$, a fact we shall require a few times in this section (clearly, transitivity is the only difficulty in proving that $\sim$ is an equivalence relation; for a proof, see [69, Theorem 11.60] or [28, Theorem 22.4.12]).

1.2.2 The Klein correspondence. A **partial t–spread** of $\mathrm{PG}(2n-1,q)$ is a set of t–dimensional subspaces of $\mathrm{PG}(2n-1,q)$ that are pairwise disjoint. An $n-1$–**spread** (or just **spread**) of $\mathrm{PG}(2n-1,q)$ is a partial $n-1$–spread of $\mathrm{PG}(2n-1,q)$ that partitions the set of points of $\mathrm{PG}(2n-1,q)$; clearly, such a set has size q^n+1. We say that spreads S_1 and S_2 are **equivalent** if $\exists\ g \in \mathrm{Col}(\mathrm{PG}(2n-1,q))$ such that $g(S_1)=S_2$.

Our chief interest in the following construction is that it relates ovoids of $\mathrm{O}^+(6,q)$ to spreads of $\mathrm{PG}(3,q)$.

Represent $\mathrm{PG}(3,q)$ via $\mathrm{GF}(q)^4$, and put the quadratic form Q defined by

$$Q((x_1,x_2,x_3,x_4,x_5,x_6)) = x_1x_6 + x_2x_5 + x_3x_4$$

on $V=\mathrm{GF}(q)^6$ (that (V,Q) defines an $\mathrm{O}^+(6,q)$ space can be seen since it is explicitly represented as an orthogonal direct sum of 3 hyperbolic lines). Let $l=\langle x,y\rangle$ be a line of $\mathrm{PG}(3,q)$ ($x=(x_1,x_2,x_3,x_4)$, $y=(y_1,y_2,y_3,y_4)$), and define

$$\kappa(l) = \langle (p_{12},p_{13},p_{14},p_{32},p_{24},p_{43})\rangle$$

where

$$p_{ij} = p_{ij}(x,y) = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

Firstly, κ is well–defined, for if $x' = Ax$ and $y' = Ay$ ($A \in \mathrm{GL}(2,q)$), then

$$p_{ij}(x',y') = \left| \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} A^T \right| = |A| \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = |A| p_{ij}(x,y)$$

so that $\kappa(l)$ is still $\langle (p_{12},p_{13},p_{14},p_{32},p_{24},p_{43})\rangle$.

Compute $p_{12}p_{43}+p_{13}p_{24}+p_{14}p_{32}$ to see that it is zero for any choice of x and y, so that κ maps into the set S of singular points of $\mathrm{O}^+(6,q)$. For each possible type of point of S we give below a line mapping to it under κ, thus showing that κ is onto S.

$$\begin{aligned}
\kappa(\langle(1,0,c,-d),(0,1,a,b)\rangle) &= \langle(1,a,b,c,d,-ad-bc)\rangle\\
\kappa(\langle(0,0,1,b),(1,-c,0,d)\rangle) &= \langle(0,1,b,c,-bc,d)\rangle\\
\kappa(\langle(0,0,0,1),(1,b,-c,0)\rangle) &= \langle(0,0,1,0,b,c)\rangle\\
\kappa(\langle(0,0,1,-b),(0,1,0,-c)\rangle) &= \langle(0,0,0,1,b,c)\rangle\\
\kappa(\langle(0,1,0,b),(0,0,1,1)\rangle) &= \langle(0,0,0,0,1,b)\rangle\\
\kappa(\langle(0,0,0,1),(0,0,1,0)\rangle) &= \langle(0,0,0,0,0,1)\rangle
\end{aligned}$$

Also,

$$\#\text{ lines of }\mathrm{PG}(3,q) = (q^2+1)(q^2+q+1) = \#\text{ singular points of }\mathrm{O}^+(6,q)$$

and thus κ is a bijection.

Let $l = \langle(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)\rangle$ and $l' = \langle(x_1',x_2',x_3',x_4'),(y_1',y_2',y_3',y_4')\rangle$ be lines of $\mathrm{PG}(3,q)$. Now l and l' meet if and only if they don't span the whole of $\mathrm{PG}(3,q)$, which is if and only if

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ x_1' & x_2' & x_3' & x_4'\\ y_1' & y_2' & y_3' & y_4' \end{pmatrix}$$

has $\det(A)=0$. Applying a cofactor expansion down the first column of A, we see that $\det(A)$ equals

$$\begin{aligned}
&x_1(y_2x_3'y_4' + y_3x_4'y_2' + y_4x_2'y_3' - y_4x_3'y_2' - y_2x_4'y_3' - y_3x_2'y_4')\\
&-y_1(x_2x_3'y_4' + x_3x_4'y_2' + x_4x_2'y_3' - x_4x_3'y_2' - x_2x_4'y_3' - x_3x_2'y_4')\\
&+x_1'(x_2y_3y_4' + x_3y_4y_2' + x_4y_2y_3' - x_4y_3y_2' - x_2y_4y_3' - x_3y_2y_4')\\
&-y_1'(x_2y_3x_4' + x_3y_4x_2' + x_4y_2x_3' - x_4y_3x_2' - x_2y_4x_3' - x_3y_2x_4')
\end{aligned}$$

Let $\langle(p_{12}',p_{13}',p_{14}',p_{32}',p_{24}',p_{43}')\rangle$ denote $\kappa(l')$; a computation shows that

$$f_Q((p_{12},p_{13},p_{14},p_{32},p_{24},p_{43}),(p_{12}',p_{13}',p_{14}',p_{32}',p_{24}',p_{43}'))$$

is equal to $-\det(A)$. We have shown

Theorem 1.2.2.1 (Klein correspondence). (**see [69], pp188–189**) *There is a bijection κ between the set of lines of $\mathrm{PG}(3,q)$ and the set of singular points of $\mathrm{O}^+(6,q)$, such that lines l and l' of $\mathrm{PG}(3,q)$ meet if and only if $\kappa(l)$ and $\kappa(l')$ are collinear.*

In what follows, let Γ_1 be the graph whose vertices are the lines of $\mathrm{PG}(3,q)$, where two lines are adjacent if they meet. Let Γ_2 be the graph whose vertices are the singular points of $\mathrm{O}^+(6,q)$, where two singular points are adjacent if they are collinear. The Klein correspondence induces an isomorphism between $\mathrm{Aut}\,\Gamma_1$ and $\mathrm{Aut}\,\Gamma_2$, and these groups turn out to be $\mathrm{Cor}(\mathrm{PG}(3,q))$ and $\mathrm{P\Gamma O}^+(6,q)$ respectively (see [69, p190]). Thus we have

Corollary 1.2.2.2. $\mathrm{Cor}(\mathrm{PG}(3,q)) \cong \mathrm{P\Gamma O}^+(6,q)$.

We also have

Corollary 1.2.2.3. *Let κ denote the bijection of the Klein correspondence.*

(a) If S is a spread of $\mathrm{PG}(3,q)$, then $\kappa(S)$ is an ovoid of $\mathrm{O}^+(6,q)$.

(b) If O is an ovoid of $\mathrm{O}^+(6,q)$, then $\kappa^{-1}(O)$ is a spread of $\mathrm{PG}(3,q)$.

Proof. By Theorem 1.2.2.1, two singular points of $\mathrm{O}^+(6,q)$ aren't collinear if and only if the corresponding lines of $\mathrm{PG}(3,q)$ don't meet, while ovoids of $\mathrm{O}^+(6,q)$ and spreads of $\mathrm{PG}(3,q)$ have the same size (using Lemma 1.1.7.4 and Lemma 1.1.6.9). $\square$

Remark. (a) Observe that if S is a spread of $\mathrm{PG}(3,q)$ and $\triangle$ a correlation of $\mathrm{PG}(3,q)$, then $\triangle S$ is a spread of $\mathrm{PG}(3,q)$. For, $\triangle S$ still consists of q^2+1 lines, while if l_1 and l_2 are distinct elements of S and $\triangle l_1$ and $\triangle l_2$ meet in a point p, then l_1 and l_2 lie in the plane $\triangle p$, which implies that they meet, a contradiction.

(b) Let O be an ovoid of $\mathrm{O}^+(6,q)$ and $\triangle$ be a correlation of $\mathrm{PG}(3,q)$.

(i) If $\mathrm{Col}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)} = \mathrm{Cor}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)}$, then $\kappa^{-1}(O)$ and $\triangle\kappa^{-1}(O)$ are inequivalent spreads.

(ii) If $\mathrm{Col}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)} < \mathrm{Cor}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)}$, then $\mathrm{Col}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)}$ is a subgroup of index 2 in $\mathrm{Cor}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)}$.

To see (i), suppose that $g(\kappa^{-1}(O)) = \triangle\kappa^{-1}(O)$ for $g \in \mathrm{Col}(\mathrm{PG}(3,q))$. Then $g^{-1}\triangle \in \mathrm{Cor}(\mathrm{PG}(3,q))_{\kappa^{-1}(O)}$ and $g^{-1}\triangle$ is a correlation. For (ii), recall that $\mathrm{Col}(\mathrm{PG}(3,q))$ has index 2 in $\mathrm{Cor}(\mathrm{PG}(3,q))$.

(c) Note that (ii) applies to any configuration of $\mathrm{PG}(3,q)$, as does (i) (as long as that configuration is of a type preserved by correlations, such as a spread, or a packing of spreads (see Chapter 6)).

We now determine which objects of $\mathrm{PG}(3,q)$ that maximals and totally singular lines of $\mathrm{O}^+(6,q)$ relate to under the Klein correspondence (see [69, pp188-189]).

First, let l_1, l_2, l_3 be three lines of a clique $\mathcal{C}$ of Γ_1. Then we have two cases; either they all meet in a point p, or $l_1 \cap l_2 = p_1$, $l_1 \cap l_3 = p_2$, $l_2 \cap l_3 = p_3$ for distinct points p_1, p_2, p_3. In the former case, a possibility for $\mathcal{C}$ being a maximal clique is that it consists of all lines on p. In the latter case, l_1, l_2, l_3 span a plane π, and any line l of $\mathcal{C}$ is contained in π. For, l doesn't contain at least one of p_1, p_2, p_3, and so is contained in the span of either l_1, l_2, or l_1, l_3, or l_2, l_3 (respectively). Thus, the two types of maximal clique occuring in Γ_1 consist either of all lines on a point, or all lines in a plane.

The Klein correspondence is an isomorphism between Γ_1 and Γ_2, mapping maximal cliques of Γ_1 to maximal cliques of Γ_2. Thus, defining (for p a point and π a plane of $\mathrm{PG}(3,q)$)

$$\kappa(p) = \{\kappa(l) : p \subseteq l,\ l \text{ a line of } \mathrm{PG}(3,q)\}$$

and

$$\kappa(\pi) = \{\kappa(l) : l \subseteq \pi,\ l \text{ a line of } \mathrm{PG}(3,q)\}$$

we have that $\kappa(p)$ and $\kappa(l)$ are maximals of $\mathrm{O}^+(6,q)$.

Suppose p is a point and π is a plane of $\mathrm{PG}(3,q)$, with $p \subseteq \pi$. A **pencil of lines** Ψ of $\mathrm{PG}(3,q)$ is the set of lines on p and in π. Letting $\kappa(\Psi) = \{\kappa(l) : l \in \Psi\}$, we have $\kappa(\Psi) = \kappa(p) \cap \kappa(\pi)$, so that $\kappa(\Psi)$ is a totally singular subspace of $\mathrm{O}^+(6,q)$. Since $|\Psi| = q+1$, $\kappa(\Psi)$ must be a totally singular line of $\mathrm{O}^+(6,q)$. Thus, totally singular lines of $\mathrm{O}^+(6,q)$ correspond to incident point–plane pairs of $\mathrm{PG}(3,q)$.

Let

$$\mathcal{M}_1 = \{\kappa(p) :\ p \text{ a point of } \mathrm{PG}(3,q)\}$$

and

$$\mathcal{M}_2 = \{\kappa(\pi) :\ \pi \text{ a plane of } \mathrm{PG}(3,q)\}$$

We now show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are the two classes of maximal of $\mathrm{O}^+(6,q)$. Recall that maximals M and M' of $\mathrm{O}^+(6,q)$ are equivalent (denoted $M \sim M'$) if dim $(M \cap M') \equiv 1 \pmod 2$.

Let p_1 and p_2 be points of $\mathrm{PG}(3,q)$. If $p_1 = p_2$, then $\kappa(p_1) = \kappa(p_2)$ and so $\kappa(p_1) \sim \kappa(p_2)$. Or, if $p_1 \neq p_2$, let l be the unique line on p_1 and p_2. Then $\kappa(p_1)$ and $\kappa(p_2)$ intersect in the 1–dimensional subspace $\kappa(l)$, so that $\kappa(p_1) \sim \kappa(p_2)$. Now let π_1 and π_2 be planes of $\mathrm{PG}(3,q)$. If $\pi_1 = \pi_2$, then $\kappa(\pi_1) = \kappa(\pi_2)$, and so $\kappa(\pi_1) \sim \kappa(\pi_2)$. If $\pi_1 \neq \pi_2$, let l denote the unique line that π_1 and π_2 intersect in. Then $\kappa(\pi_1)$ and $\kappa(\pi_2)$ intersect in the 1–dimensional subspace $\kappa(l)$, and so $\kappa(\pi_1) \sim \kappa(\pi_2)$.

To show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are distinct, let p be a point and π a plane of $\mathrm{PG}(3,q)$. First suppose $p \subseteq \pi$. Then $\kappa(p) \cap \kappa(\pi) = \kappa(\Psi)$ for Ψ the pencil arising from the pair p, π, and $\kappa(\Psi)$ is a 2–dimensional subspace, so $\kappa(p) \nsim \kappa(\pi)$. Or, if $p \nsubseteq \pi$, then $\kappa(p) \cap \kappa(\pi)$ is the 0–dimensional subspace $\emptyset$, so $\kappa(p) \nsim \kappa(\pi)$. That every maximal of $\mathrm{O}^+(6,q)$ must be in $\mathcal{M}_1$ or $\mathcal{M}_2$ follows from the fact that maximal cliques of Γ_1 correspond to maximal cliques of Γ_2, and that if $\mathcal{C}$ is a maximal clique of Γ_1, $\mathcal{C}$ is either the set of all lines on a point or the set of all lines in a plane.

1.2.3 Generalised quadrangles. A **generalised quadrangle** of **order** (s,t) is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ satisfying

(GQ1) Every point lies on $t+1$ lines, and every two points lie on at most one line.

(GQ2) Every line contains $s+1$ points, and every two lines meet in at most one point.

(GQ3) Given a point P and a line L with $P \nsubseteq L$, there exists a unique line through P that meets L.

When $s = t$, we just say that Γ has order s.

We are interested in GQ's insomuch as they arise from polar spaces. Given a non–degenerate polar space of Witt index 2 over $\mathrm{GF}(q)$, let $\mathcal{P}$ be its set of singular points, $\mathcal{L}$ its set of totally singular lines and $\mathcal{I}$ be subspace incidence; then $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a GQ. From $\mathrm{Sp}(4,q)$, $\mathrm{U}(4,q)$, $\mathrm{U}(5,q)$, $\mathrm{O}^+(4,q)$, $\mathrm{O}(5,q)$ and $\mathrm{O}^-(6,q)$ arise GQ's of orders q, $(q^{\frac{1}{2}}, q)$, $(q^{\frac{3}{2}}, q)$, $(q,1)$, q, and (q^2, q), respectively. If $\mathcal{S}$ is one of these polar spaces, we shall refer to the associated GQ as the "$\mathcal{S}$ GQ".

Ovoids and spreads of GQ's are defined as for ovoids and spreads of the above-mentioned polar spaces. Thus, any ovoid/spread results we prove for GQ's carry over to the associated polar spaces. Note that if GQ's Γ and Γ' are dual, then ovoids/spreads of Γ correspond to spreads/ovoids of Γ'.

Corollary 1.2.3.1. *The* $\mathrm{U}(4,q)$ *and* $\mathrm{O}^-(6,q)$ *GQ's are dual, so ovoids of* $\mathrm{U}(4,q)$ *correspond to spreads of* $\mathrm{O}^-(6,q)$*, and spreads of* $\mathrm{U}(4,q)$ *correspond to ovoids of* $\mathrm{O}^-(6,q)$*.*

Proof. This follows from the Klein correspondence, but the result isn't relevant enough to us to give the non–trivial proof (which can be found in [9]). □

The next result is of much more interest to us.

Corollary 1.2.3.2 (see [55], 3.2.1). *The* $\mathrm{Sp}(4,q)$ *and* $\mathrm{O}(5,q)$ *GQ's are dual, so ovoids of* $\mathrm{Sp}(4,q)$ *correspond to spreads of* $\mathrm{O}(5,q)$*, and spreads of* $\mathrm{Sp}(4,q)$ *correspond to ovoids of* $\mathrm{O}(5,q)$*.*

Proof. The form f defined by

$$f((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$

gives an $\mathrm{Sp}(4,q)$ space $(\mathrm{GF}(q)^4, f)$. Recall that we have a bijection κ between the set of lines of $\mathrm{PG}(3,q)$ and the set of singular points of $\mathrm{O}^+(6,q)$; we will show that κ is the required duality. Suppose $l = \langle x, y\rangle$ $(x = (x_1,x_2,x_3,x_4), y = (y_1,y_2,y_3,y_4))$ is a totally isotropic line of $\mathrm{Sp}(4,q)$. Then $f(x,y) = 0$, so that the (singular) point

$$\kappa(l) = \langle (p_{12}, p_{13}, p_{14}, p_{32}, p_{24}, p_{43})\rangle$$

of the $\mathrm{O}^+(6,q)$ space (V,Q) (where (V,Q) is as before) has $p_{12} + p_{34} = 0$, that is, $p_{12} = p_{43}$. So $\kappa(l)$ lies in the hyperplane $\langle(1,0,0,0,0,1)\rangle^\perp$ of $\mathrm{O}^+(6,q)$; since $\langle(1,0,0,0,0,1)\rangle$ is a non–singular point, $(\langle(1,0,0,0,0,1)\rangle^\perp, Q)$ is an $\mathrm{O}(5,q)$ space. Let $\mathcal{L}$ be the set of totally isotropic lines of $\mathrm{Sp}(4,q)$ and $\mathcal{P}'$ be the set of singular points of $\mathrm{O}(5,q)$. Because the number of singular points of $\mathrm{O}(5,q)$ equals the number of totally isotropic lines of $\mathrm{Sp}(4,q)$, κ is a bijection between $\mathcal{L}$ and $\mathcal{P}'$.

Let $\mathcal{P}$ denote the set of points of $\mathrm{Sp}(4,q)$ (recall that any point of $\mathrm{Sp}(4,q)$ is isotropic), with $\mathcal{L}'$ denoting the set of totally singular lines of $\mathrm{O}(5,q)$. Given $p \in \mathcal{P}$, $\kappa(p) = \{\kappa(l) : p \subseteq l,\ l \text{ a line of } \mathrm{Sp}(4,q)\}$ is a maximal of $\mathrm{O}^+(6,q)$, and $\kappa(p)$ must intersect $\mathrm{O}(5,q)$ in a totally singular line of $\mathrm{O}(5,q)$. To show that κ is 1–1 on $\mathcal{P}$, let $p_1, p_2 \in \mathcal{P}$ be distinct, and p_1p_2 be the unique line joining them. Letting $\tilde{p_1}$ be the set of lines of $\mathrm{Sp}(4,q)$ on p_1 and $\tilde{p_2}$ be the set of lines of $\mathrm{Sp}(4,q)$ on p_2, $\tilde{p_1}$ and $\tilde{p_2}$ have only p_1p_2 in common, so $\kappa(p_1) \neq \kappa(p_2)$. Since the number of points of $\mathrm{Sp}(4,q)$ equals the number of totally singular lines of $\mathrm{O}(5,q)$, κ is a bijection between $\mathcal{P}$ and $\mathcal{L}'$.

That κ and κ^{-1} is incidence–preserving follows from the way that κ is defined on $\mathcal{P}$. Thus, $(\mathcal{P},\mathcal{L},\mathcal{I})$ and $(\mathcal{P}',\mathcal{L}',\mathcal{I}')$ are dual (where $\mathcal{I}$ and $\mathcal{I}'$ denote incidence in $\mathrm{Sp}(4,q)$ and $\mathrm{O}(5,q)$ respectively).

□

Lemma 1.2.3.3. *The* $\mathrm{Sp}(4,q)$ *GQ is self–dual for* q *even, so ovoids and spreads of* $\mathrm{Sp}(4,q)$ *correspond.*

Proof. See [55, 3.2.1]. □

1.2.4 Regular spreads. A **regulus** of $\mathrm{PG}(2n-1,q)$ is a partial $n-1$–spread $\mathcal{R}$ of $\mathrm{PG}(2n-1,q)$ having $|\mathcal{R}|=q+1$, for which any line that meets three of its elements meets all of its elements. If U_1,U_2,U_3 are three mutually disjoint $n-1$–dimensional subspaces of $\mathrm{PG}(2n-1,q)$, then there is a unique regulus $\mathcal{R}(U_1,U_2,U_3)$ containing them (for a proof, see [28, Corollary to Theorem 25.6.1]). We say that a spread S of $\mathrm{PG}(2n-1,q)$ is **regular** if

$$U_1,U_2,U_3\in S \implies \mathcal{R}(U_1,U_2,U_3)\in S \quad \forall\ U_1,U_2,U_3\in S$$

Regular spreads are canonical examples of spreads: they exist for all q. To construct them, let f be a non–degenerate alternating form on a 2–dimensional vector space V over $\mathrm{GF}(q)$, $q=q_0^h$. Because $\mathrm{GF}(q)$ has dimension h over $\mathrm{GF}(q_0)$, we have a $2h$–dimensional vector space V' over $\mathrm{GF}(q_0)$, which we can equip with the non–degenerate alternating form $\mathrm{Tr}\circ f$, where $\mathrm{Tr}:\mathrm{GF}(q)\to\mathrm{GF}(q_0)$ via $\mathrm{Tr}(x)=x+x^{q_0}+\ldots+x^{q_0^{h-1}}$. Let $\{b_1,b_2\}$ be a basis for V over $\mathrm{GF}(q)$ and $\{c_1,\ldots,c_h\}$ be a basis for $\mathrm{GF}(q)$ over $\mathrm{GF}(q_0)$; then $\{b_1c_1,b_1c_2,\ldots,b_1c_h,b_2c_1,b_2c_2,\ldots,b_2c_h\}$ is a basis for V over $\mathrm{GF}(q_0)$. Now ϕ defined via $\phi(\langle b_i\rangle)=\langle b_ic_1,\ldots,b_ic_h\rangle$ $(i\in\{1,2\})$ gives a map taking points of (V,f) to totally isotropic h–dimensional subspaces of $(V',\mathrm{Tr}\circ f)$. Letting S be the set of points of (V,f), ϕ applied to S gives a spread S' of $(V',\mathrm{Tr}\circ f)$. As with any spread of $\mathrm{Sp}(2n,q)$, S' is a spread of $\mathrm{PG}(2n-1,q)$, and S' turns out to be a regular spread (using the characterisation of regular spreads as having kernel isomorphic to $\mathrm{GF}(q^n)$).

For the rest of this section we focus on $\mathrm{PG}(3,q)$ spreads, because of the link to ovoids.

Theorem 1.2.4.1 (see [27], p30). *Regular spreads of* $\mathrm{PG}(3,q)$ *correspond to elliptic quadrics of* $\mathrm{O}^+(6,q)$.

Proof. Let l and m be disjoint lines of $\mathrm{PG}(3,q)$ (and p a point not on l or m). By a **transversal to** l **and** m we mean a line meeting both l and m. First, we show that p lies on a unique transversal to l and m. For existence, $\pi=\langle p,l\rangle$ is a plane which can't wholly contain m (as then l and m would intersect), so π meets m in just one point p'. Then $\langle p,p'\rangle$ is a line on p that meets l and m. Now let n be a transversal to l and m that contains p, and r be the point of l that n meets. Then $n=\langle p,r\rangle$, so $n\subseteq\pi$. Because n meets m and $n\subseteq\pi$, we have $n\cap m\subseteq\pi\cap m=p'$, so $n=\langle p,p'\rangle$.

Hence, given three mutually disjoint lines l,m,n of $\mathrm{PG}(3,q)$, each point of n lies on a unique transversal to l and m, so the resulting set of transversals gives us a set $\mathcal{R}$ of $q+1$ lines. If two elements of $\mathcal{R}$ met, then the plane that they would span would contain n and either l or m, a contradiction. Also, if a line meets three lines of $\mathcal{R}$, it must meet all lines of $\mathcal{R}$. Thus, $\mathcal{R}$ is a regulus of $\mathrm{PG}(3,q)$.

Now $\mathrm{PGL}(4,q)$ is transitive on triples of mutually disjoint lines of $\mathrm{PG}(3,q)$ (this fact is true more generally; $\mathrm{PGL}(2n,q)$ is transitive on triples of mutually disjoint n–dimensional subspaces of $\mathrm{PG}(2n-1,q)$), so is transitive on reguli of $\mathrm{PG}(3,q)$ (since any regulus of $\mathrm{PG}(3,q)$ is a set of transversals to some triple of mutually disjoint lines). Hence, to prove the lemma we may take any regulus of $\mathrm{PG}(3,q)$, say

$$\mathcal{R} = \{\langle (0,0,1,0),(0,0,0,1)\rangle\} \cup \{\langle (1,1,t,0),(0,1,t,0)\rangle : t \in \mathrm{GF}(q)\}$$

Applying the map κ of the Klein correspondence, we obtain a set

$$\kappa(\mathcal{R}) = \{\langle (0,0,0,0,0,1)\rangle\} \cup \{\langle (1,t,0,0,0,0)\rangle : t \in \mathrm{GF}(q)\}$$

of $q+1$ singular points of $\mathrm{O}^+(6,q)$ ($\mathrm{O}^+(6,q)$ represented as usual), where no two points of $\kappa(\mathcal{R})$ are collinear. Note that $\langle \kappa(\mathcal{R})\rangle$ is a 3–dimensional subspace of $\mathrm{O}^+(6,q)$ that is non–degenerate, so $\kappa(\mathcal{R})$ is a conic of $\mathrm{O}^+(6,q)$. Conversely, we know that any two $\mathrm{O}(3,q)$ spaces are isometric, so by Corollary 1.1.5.2, $\mathrm{P\Gamma O}^+(6,q)$ is transitive on $\mathrm{O}(3,q)$ subspaces of $\mathrm{O}^+(6,q)$. Hence, we may choose any conic $\mathcal{C}$ of $\mathrm{O}^+(6,q)$; taking $\mathcal{C} = \{\langle (0,0,0,0,0,1)\rangle\} \cup \{\langle (1,t,0,0,0,0)\rangle : t \in \mathrm{GF}(q)\}$, we have a regulus $\kappa^{-1}(\mathcal{C})$ of $\mathrm{PG}(3,q)$.

Now consider a regular spread S of $\mathrm{PG}(3,q)$. Given three lines l_1, l_2, l_3 of S, they lie on a regulus $\mathcal{R}(l_1,l_2,l_3)$ contained in S, so $\kappa(l_1), \kappa(l_2), \kappa(l_3)$ lie on a conic $\mathcal{C}$, $\mathcal{C} \subseteq \kappa(S)$. Taking $m \in S$ having $m \notin \mathcal{R}(l_1,l_2,l_3)$, we have that $\kappa(m)$ is a point of $\kappa(S)$ with $\kappa(m) \notin \mathcal{C}$, and $\kappa(m)$ is not collinear with any point of $\mathcal{C}$. Thus, $\langle \mathcal{C}, \kappa(m)\rangle = \kappa(S)$ is an $\mathrm{O}^-(4,q)$ space. Conversely, because $\mathrm{P\Gamma O}^+(6,q)$ is transitive on $\mathrm{O}^-(4,q)$ subspaces of $\mathrm{O}^+(6,q)$ (using Corollary 1.1.5.2), we may choose an elliptic quadric $\mathcal{E}$ of the form $\langle \mathcal{C}, \kappa(m)\rangle$, where $\mathcal{C}$ is some conic and $\kappa(m)$ is some singular point not collinear with any point of $\mathcal{C}$ (m a line of $\mathrm{PG}(3,q)$). The result then follows.

□

When in $\mathrm{PG}(3,q)$, the set of transversals to a regulus is again a regulus, which we call the **opposite regulus** (denoted R^{opp}). This fact gives us a means of obtaining new spreads from old, and hence new $\mathrm{O}^+(6,q)$ ovoids from old (and this construction gives evidence that $\mathrm{O}^+(6,q)$ ovoids are common).

Theorem 1.2.4.2 (see [27], p62). *Let S be a spread of $\mathrm{PG}(3,q)$, with $\mathcal{R}$ a regulus contained in S. Then $S' = (S \cup \mathcal{R}^{\mathrm{opp}}) - \mathcal{R}$ is a spread of $\mathrm{PG}(3,q)$.*

Proof. No line L of $\mathcal{R}^{\mathrm{opp}}$ can meet S, since the $q+1$ points of L all lie on lines of $\mathcal{R}$. $\square$

The process of Theorem 1.2.4.2 is known as "switching reguli".

1.2.5 Triality. The following is due to Tits ([79]); a description can be found in [10, p391].

Let $\mathcal{P}$ be the set of singular points of $\mathrm{O}^+(8,q)$, $\mathcal{L}$ the set of totally singular lines, and $\mathcal{M}_1$, $\mathcal{M}_2$ the two classes of maximal. Suppose τ is a map that fixes $\mathcal{L}$, with $\tau(\mathcal{P}) = \mathcal{M}_1$, $\tau(\mathcal{M}_1) = \mathcal{M}_2$ and $\tau(\mathcal{M}_2) = \mathcal{P}$, such that τ is of order 3 and preserves incidence on $\mathcal{P} \cup \mathcal{M}_1 \cup \mathcal{M}_2$, where

- $P \in \mathcal{P}$ is incident with a maximal M if $P \subseteq M$
- $P_1, P_2 \in \mathcal{P}$ are incident if they are collinear
- $M, M' \in \mathcal{M}_1$ or $M, M' \in \mathcal{M}_2$ are incident if $M \cap M'$ is a line
- $M \in \mathcal{M}_1$ and $M' \in \mathcal{M}_2$ are incident if $M \cap M'$ is a plane.

Then τ is called a **triality map**.

The key consequence of triality for us is

Theorem 1.2.5.1 (see [36]). *Let τ be a triality map.*

(a) If O is an ovoid of $\mathrm{O}^+(8,q)$, then $\tau(O)$ is a spread of $\mathrm{O}^+(8,q)$.

(b) If S is a spread of $\mathrm{O}^+(8,q)$ with $S \subset \mathcal{M}_1$, then $\tau^2(S)$ is an ovoid of $\mathcal{S}$, as is $\tau(S)$ if S is a spread of $\mathrm{O}^+(8,q)$ with $S \subset \mathcal{M}_2$.

Proof. First note that all elements of a spread of $\mathrm{O}^+(8,q)$ must belong to one class of maximal, because two maximals are equivalent if and only if the dimension of their intersection is even. Now if distinct elements M and M' of $\tau(O)$ intersect then they do so in a line, and then $\tau^2(M)$ and $\tau^2(M')$ are collinear points of O, contradicting O being an ovoid. Because τ is of order 3, τ is 1–1, so that $\tau(O)$ has size $|O|$. The proof for (b) is similar. $\square$

Let τ be a triality map, $\overline{\tau}$ be the element of $\mathrm{PO}^+(8,q)$ induced by conjugating by τ, and O and S be an ovoid and spread (respectively) of $\mathrm{O}^+(8,q)$ that correspond via τ. We note that the stabiliser of O is conjugate to the stabiliser of S under $\overline{\tau}$.

1.2.6 Spread constructions. While our focus in this thesis is on ovoids, the following constructions show (at least where triality applies) how ovoids can yield spreads, and conversely.

The first of these is by Dillon and Dye (independently).

Theorem 1.2.6.1 (see [35]). *Let q be even. Any $\mathrm{O}^+(4n,q)$ spread yields an $\mathrm{Sp}(4n-2,q)$ spread, and conversely.*

Proof. Define $\mathrm{O}^+(4n,q)$ via a quadratic form Q (with f_Q the polar form of Q), and let P be a non–singular point of $\mathrm{O}^+(4n,q)$; then $P^\perp$ is an $\mathrm{O}(4n-1,q)$ space. Since q is even, f_Q is alternating; the radical of f_Q is P, making $P^\perp/P$ (with the alternating form naturally induced by f_Q) an $\mathrm{Sp}(4n-2,q)$ space.

Let S be a spread of $\mathrm{O}^+(4n,q)$. We claim that

$$S' = \{\langle P, P^\perp \cap M\rangle/P : M \in S\}$$

is a spread of $\mathrm{Sp}(4n-2,q)$. With P being collinear to each point of $P^\perp \cap M$, the elements of S' are totally isotropic. Since $P^\perp$ meets each maximal M in a hyperplane of M, and the corresponding element of S' is obtained by taking the span of that hyperplane with a point and then factoring out by a point, each element of S' has dimension $2n-1$. Given an isotropic point $\langle v\rangle/P$ of $\mathrm{Sp}(4n-2,q)$, $\exists\,!\ M \in S$ such that $\langle v\rangle \subseteq M$, so $\langle P, P^\perp \cap M\rangle/P$ is the unique element of S' containing $\langle v\rangle/P$ and hence S' is a partial spread. Also, each element of S' arises from a unique element of S; otherwise, if $\langle P, P^\perp \cap M\rangle/P = \langle P, P^\perp \cap M'\rangle/P$ for distinct elements M and M' of S, then $P^\perp \cap M = (P^\perp \cap M) \cap (P^\perp \cap M') = P^\perp \cap (M \cap M') = \{0\}$, a contradiction as M must meet $P^\perp$. Thus we have $|S'| = |S|$ (while $|S| = q^{2n-1}+1$), so S' is a spread of $\mathrm{Sp}(4n-2,q)$.

Conversely, suppose S' is a spread of $\mathrm{Sp}(4n-2,q)$, fix a class $\mathcal{M}_1$ of maximal of $\mathrm{O}^+(4n,q)$, and let

$$\mathcal{J}(S') = \{M \in \mathcal{M}_1 : \langle P, P^\perp \cap M\rangle/P \in S'\}$$

where are using that each element of S' has the form $\langle P, P^\perp \cap M\rangle/P$ for some unique maximal M of $\mathrm{O}^+(4n,q)$. Now if M and M' are distinct elements of $\mathcal{M}_1$, then

$$(P^\perp \cap M) \cap (P^\perp \cap M') = P^\perp \cap (M \cap M') = \{0\}$$

as S' is a spread. Also, dim $(M \cap M') \equiv 2n \pmod 2$, so if M and M' intersect non–trivially, they do so in at least a line. But then $M \cap M'$ meets $P^\perp$, a contradiction. Since there are $q^{2n-1}+1$ maximals of $\mathrm{O}^+(4n,q)$, $\mathcal{J}(S')$ is a spread of $\mathrm{O}^+(4n,q)$.

Finally, if S' is a spread of $\mathrm{Sp}(4n-2,q)$ and S is a spread of $\mathrm{O}^+(4n,q)$, it can be seen that $S' = \{\langle P^\perp \cap M\rangle/P : M \in \mathcal{J}(S')\}$ and $S = \mathcal{J}(\{\langle P^\perp \cap M)/P\rangle : M \in S\})$.
$\square$

Note that if S_1 and S_2 are equivalent $\mathrm{Sp}(4n-2,q)$ spreads, then $\mathcal{J}(S_1)$ and $\mathcal{J}(S_2)$ are equivalent $\mathrm{O}^+(4n,q)$ spreads.

The next three theorems are due to Thas.

Theorem 1.2.6.2 ([75]). *(a) Any spread of* $\mathrm{U}(2n+2,q)$ *yields a spread of* $\mathrm{U}(2n+1,q)$ *($n \geq 1$).*

(b) Any spread of $\mathrm{O}^+(4n,q)$ *yields a spread of* $\mathrm{O}(4n-1,q)$*, and conversely ($n \geq 1$).*

(c) Any spread of $\mathrm{O}(2n+1,q)$ *yields a spread of* $\mathrm{O}^-(2n,q)$ *($n \geq 2$).*

Proof. (a) Take a non–isotropic point P of $\mathrm{U}(2n+2,q)$; then $P^\perp$ is a $\mathrm{U}(2n+1,q)$ space. If S is a spread of $\mathrm{U}(2n+2,q)$, then $S' = \{P^\perp \cap M : M \in S\}$ consists of $|S|$ elements (each of dimension n), which is the right size to be a spread of $\mathrm{U}(2n+1,q)$. Also, if $P^\perp \cap M_1$ and $P^\perp \cap M_2$ are distinct elements of S' that intersect non–trivially, then $M_1 \cap M_2 \neq \{0\}$, a contradiction.
(b) First, note that the restriction on the dimension of the space is necessary, as the converse doesn't hold for spreads of $\mathrm{O}(4n+1,q)$ spaces. To see this, recall that regular spreads exist (in particular) in all $\mathrm{Sp}(4n,q)$ spaces for q even, thus giving spreads in all $\mathrm{O}(4n+1,q)$ spaces having q even (for in the set up of the proof of Theorem 1.2.6.1, mapping $P^\perp/P$ to $P^\perp$ in the natural way takes the set of isotropic points of $P^\perp/P$ bijectively to the set of singular points of $P^\perp$, thus taking spreads to spreads). However, spreads don't occur in $\mathrm{O}^+(4n+2,q)$ spaces ($n \geq 1$) for any q, as two maximals of $\mathrm{O}^+(4n+2,q)$ that intersect trivially are in different classes of maximal, but there are only two classes of maximal.

Given a spread of $\mathrm{O}^+(4n,q)$, we obtain a spread of $\mathrm{O}(4n-1,q)$ in an analogous manner to (a). Now suppose S' is a spread of $\mathrm{O}(4n-1,q)$, where $\mathrm{O}(4n-1,q)$ is represented as $P^\perp$ for P a non–singular point of $\mathrm{O}^+(4n,q)$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the two classes of maximal of $\mathrm{O}^+(4n,q)$. Then

$$S_1 = \{M \in \mathcal{M}_1 : M' \subseteq M \text{ for some } M' \in S'\}$$

is a partial spread of $\mathrm{O}^+(4n,q)$, for if distinct maximals of $\mathcal{M}_1$ intersect non–trivially, then they do so in at least a line, in which case the elements of S' that they each contain must intersect, a contradiction. Also, given an element M' of S', there is a unique element M of $\mathcal{M}_1$ containing M' (as if $\overline{M} \in \mathcal{M}_1$ contained M', then M

and $\overline{M}$ would intersect in at least the $2n-1$–dimensional subspace M'. But M and $\overline{M}$ must intersect in an even–dimensional subspace, so $M = \overline{M}$). Thus, $|S_1| = |S|$, and so S_1 has the right size to be a spread. In the same way we obtain a spread $S_2 = \{M \in \mathcal{M}_2 : M' \subseteq M \text{ for some } M' \in S'\}$ of $\mathrm{O}^+(4n, q)$.

(c) Represent $\mathrm{O}^-(2n, q)$ inside $\mathrm{O}(2n+1, q)$ as in the proof of Theorem 1.1.7.6; the result follows in the usual way.

□

Theorem 1.2.6.3 ([75]). *(a) Any* $\mathrm{U}(2n+1, q)$ *spread yields an* $\mathrm{O}^-(4n+2, q)$ *spread* $(n \geq 1)$.

(b) Any $\mathrm{U}(2n+2, q)$ *spread yields an* $\mathrm{O}^+(4n+4, q)$ *spread* $(n \geq 1)$.

Proof. For both (a) and (b), some effort is required to link the unitary and orthogonal spaces; see [75] for a proof that omits detail. □

1.2.7 Unitary ovoids Part (a) of the following lemma is vacously true (as we shall see in Section 1.4), while (b) means that non–existence results about ovoids in hyperbolic spaces yield non–existence results about ovoids in unitary spaces.

Theorem 1.2.7.1 ([75]). *(a) Any* $\mathrm{O}^-(4n+2, q)$ *ovoid yields a* $\mathrm{U}(2n+1, q)$ *ovoid* $(n \geq 1)$.

(b) Any $\mathrm{O}^+(4n, q)$ *ovoid yields a* $\mathrm{U}(2n, q)$ *ovoid* $(n \geq 1)$.

Proof. Same comment as for Theorem 1.2.6.3. □

The following construction (kindly brought to my attention by O'Keefe and Penttila) is a way of obtaining new $\mathrm{U}(4, q)$ ovoids from (certain) old ones, and illustrates how $\mathrm{U}(4, q)$ ovoids are common.

Theorem 1.2.7.2 ([56]). *Let O be a* $\mathrm{U}(4, q)$ *ovoid, with L the (hyperbolic) line spanned by two points P_1, P_2 of O. Suppose that all isotropic points of L are in O. Then $(O \cup L^\perp) - L$ is a* $\mathrm{U}(4, q)$ *ovoid.*

Proof. First, to see that L is a hyperbolic line, use Lemma 1.1.5.5. Also, $\mathrm{rad}\, L = \mathrm{rad}\, L^\perp$, while hyperbolic lines are characterised by being non–degenerate (see the proof of Lemma 1.1.5.5), so $L^\perp$ is a hyperbolic line. Now suppose a point R_1 of $L^\perp$ is collinear to a point R_2 of $\mathrm{U}(4, q)$. Then $R_2 \subseteq R_1^\perp$, so $R_1 = R_1^{\perp\perp} \subseteq R_2^\perp$. Thus, $L^\perp \subseteq R_2^\perp$, so $R_2 \subseteq L$. Hence, no point of $L^\perp$ is collinear with a point of $(O \cup L^\perp) - L$. □

To obtain an example of a $\mathrm{U}(4,q)$ ovoid O satisfying the hypotheses of Theorem 1.2.7.2, take O to be the set of isotropic points of an embedded $\mathrm{U}(3,q)$ (and then in fact every two points of O span a hyperbolic line whose singular points are all in O).

1.2.8 The Hiramine et al construction. An n–**spread set** over $\mathrm{GF}(q)$ is a set S' of q^n $n \times n$ matrices over $\mathrm{GF}(q)$ such that $\det(X-Y) \neq 0$ for all distinct $X, Y \in S'$. A convenient fact is that it is possible to represent spreads of $\mathrm{PG}(2n-1,q)$ via n–spread sets, as shown in the basic lemma below.

Lemma 1.2.8.1. *Any* $\mathrm{PG}(2n-1,q)$ *spread can be represented by an* n*–spread set, and conversely.*

Proof. Without loss of generality, we can represent any element A of a $\mathrm{PG}(2n-1,q)$ spread S as

$$\langle (1,0,\ldots,0,a_1,\ldots,a_n),(0,1,0,\ldots,0,a_{n+1},\ldots,a_{2n}), \\ \ldots,(0,\ldots,0,1,a_{n^2-n},\ldots,a_{n^2})\rangle$$

Suppose

$$B=\langle (1,0,\ldots,0,b_1,\ldots,b_n),(0,1,0,\ldots,0,b_{n+1},\ldots,b_{2n}), \\ \ldots,(0,\ldots,0,1,b_{n^2-n},\ldots,b_{n^2})\rangle$$

is an element of S distinct from A. For the system of equations arising from writing a vector of A as a linear combination of the displayed basis for B, the $2n \times 2n$ matrix D of coefficients must be

$$\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0\\
0 & 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
0 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & -1\\
a_1 & \cdots & \cdots & \cdots & a_n & -b_1 & \cdots & \cdots & \cdots & -b_n\\
a_{n+1} & \cdots & \cdots & \cdots & a_{2n} & -b_{n+1} & \cdots & \cdots & \cdots & -b_{2n}\\
\vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots\\
a_{n^2-n} & \cdots & \cdots & \cdots & a_{n^2} & -b_{n^2-n} & \cdots & \cdots & \cdots & -b_{n^2}
\end{pmatrix}$$

In D, if we add $-b_i$ multiplied by the i–th row to the $i+n$–th row for all $i \in \{1,\ldots,n\}$ and then add the $j+n$–th column to the j–th column for all $j \in \{1,\ldots,n\}$, we see that $\det(D)$ equals

$$\begin{vmatrix} 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \cdots & & \cdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & -1 \\ a_1-b_1 & \cdots & \cdots & \cdots & a_n-b_n & 0 & \cdots & \cdots & \cdots & 0 \\ a_{n+1}-b_{n+1} & \cdots & \cdots & \cdots & a_{2n}-b_{2n} & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{n^2-n}-b_{n^2-n} & \cdots & \cdots & \cdots & a_{n^2}-b_{n^2} & 0 & \cdots & \cdots & \cdots & 0 \end{vmatrix}$$

Taking a cofactor expansion along the first, second, $\ldots$, n–th row in the above determinant, it equals

$$\begin{vmatrix} a_1-b_1 & \cdots & a_n-b_n \\ a_{n+1}-b_{n+1} & \cdots & a_{2n}-b_{2n} \\ \vdots & \cdots & \vdots \\ a_{n^2-n}-b_{n^2-n} & \cdots & a_{n^2}-b_{n^2} \end{vmatrix}$$

Thus, letting S' be a set in which each element

$$\begin{pmatrix} a_1 & \cdots & a_n \\ a_{n+1} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n^2-n} & \cdots & a_{n^2} \end{pmatrix}$$

of S' arises from the element

$$\langle (1,0,\ldots,0,a_1,\ldots,a_n),(0,1,0,\ldots,0,a_{n+1},\ldots,a_{2n}),\\ \ldots,(0,\ldots,0,1,a_{n^2-n},\ldots,a_{n^2})\rangle$$

of S, we see that $\det(X-Y)\neq 0$ for all distinct $X,Y\in S'$, while $|S'|=|S|=q^n$ (for the converse, go in reverse).

□

Note that any 2–spread set S' over $\mathrm{GF}(q)$ can be represented as

$$S'=\{\begin{pmatrix} f(x,y) & g(x,y) \\ x & y \end{pmatrix} : x,y\in \mathrm{GF}(q)\}$$

where f and g are functions on $\mathrm{GF}(q)$ (otherwise, if two elements X and Y of S' had the same bottom coordinates but differed in one of the top coordinates, then $\det(X - Y) = 0$).

In the following trivial lemma, we use the model of $\mathrm{O}^+(6, q)$ that we used in the Klein correspondence. Because $\mathrm{P\Gamma O}^+(6, q)$ is transitive on singular points, note that any $\mathrm{O}^+(6, q)$ ovoid is equivalent to one containing $\langle(0, 0, 0, 0, 0, 1)\rangle$, and such an ovoid may be written as

$$\begin{aligned} O(f, g) = &\{\langle(0, 0, 0, 0, 0, 1)\rangle\} \\ &\cup \{\langle(1, x, y, f(x, y), g(x, y), -yf(x, y) - xg(x, y))\rangle : x, y \in \mathrm{GF}(q)\} \end{aligned}$$

where $f, g : \mathrm{GF}(q)^2 \to \mathrm{GF}(q)$.

Lemma 1.2.8.2. *The 2–spread set*

$$S' = \{\begin{pmatrix} f(x, y) & g(x, y) \\ x & y \end{pmatrix} : \ x, y \in \mathrm{GF}(q)\}$$

over $\mathrm{GF}(q)$ *corresponds to the* $\mathrm{O}^+(6, q)$ *ovoid*

$$\begin{aligned} O(f, -g) = &\{\langle(0, 0, 0, 0, 0, 1)\rangle\} \\ &\cup \{\langle(1, x, y, f(x, y), -g(x, y), -yf(x, y) + xg(x, y))\rangle : x, y \in \mathrm{GF}(q)\} \end{aligned}$$

Proof. By Lemma 1.2.8.1, S' corresponds to the spread

$$S = \{\langle(1, 0, f(x, y), g(x, y)), (0, 1, x, y)\rangle : x, y \in \mathrm{GF}(q)\}$$

of $\mathrm{PG}(3, q)$, and then $\kappa(S) = O(f, -g)$ (κ being the map of the Klein correspondence). □

The following is due to Hiramine, Matsumoto and Oyama.

Theorem 1.2.8.3 ([26]). *For q odd, any 2–spread set over* $\mathrm{GF}(q)$ *yields a 2–spread set over* $\mathrm{GF}(q^2)$.

Proof. Let f, g be functions on $\mathrm{GF}(q)$, $\omega \in \mathrm{GF}(q^2)$ have $\omega^2 = n$ for some non–square n of $\mathrm{GF}(q)$, and

$$S' = \{\begin{pmatrix} f(x, y) & g(x, y) \\ x & y \end{pmatrix} : \ x, y \in \mathrm{GF}(q)\}$$

be a 2–spread set over $\mathrm{GF}(q)$. We claim that

$$H(S') = \{\begin{pmatrix} u & f(x,y) - g(x,y)\omega \\ x + \omega y & u^q \end{pmatrix} : u \in \mathrm{GF}(q^2),\ x, y \in \mathrm{GF}(q)\}$$

is a 2–spread set over $\mathrm{GF}(q^2)$. To see this, let

$$A_1 = \begin{pmatrix} u_1 & f(x_1,y_1) - g(x_1,y_1)\omega \\ x_1 + \omega y_1 & u_1^q \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} u_2 & f(x_2,y_2) - g(x_2,y_2)\omega \\ x_2 + \omega y_2 & u_2^q \end{pmatrix}$$

be elements of $H(S')$. Now

$$\begin{aligned} &\det(A_1 - A_2) \\ &= (u_1 - u_2)^{q+1} - [(x_1 - x_2) + \omega(y_1 - y_2)][f(x_1,y_1) - f(x_2,y_2) - \omega(g(x_1,y_1) - g(x_2,y_2))] \\ &= (u_1 - u_2)^{q+1} - (x_1 - x_2)(f(x_1,y_1) - f(x_2,y_2)) + \omega^2(y_1 - y_2)(g(x_1,y_1) - g(x_2,y_2)) \\ &\qquad + \omega[(x_1 - x_2)(g(x_1,y_1) - g(x_2,y_2)) - (y_1 - y_2)(f(x_1,y_1) - f(x_2,y_2))] \end{aligned}$$

Now, if $a + b\omega = 0$ for $a, b \in \mathrm{GF}(q)$ with at least one of a, b not zero, then $\omega \in \mathrm{GF}(q)$. But $\omega^2 = n$ and n is a non–square of $\mathrm{GF}(q)$, so $\{1, w\}$ is a linearly independent set and hence a basis for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$. Because the expression above is in the form $a + b\omega$ for $a, b \in \mathrm{GF}(q)$, if it is zero we must have $b = 0$, which means (because S' is a 2–spread set) that $x_1 = x_2$ and $y_1 = y_2$. □

Corollary 1.2.8.4. *For q odd, any $\mathrm{O}^+(6,q)$ ovoid yields an $\mathrm{O}^+(6,q^2)$ ovoid.*

Proof. By the Klein correspondence, Lemma 1.2.8.1 and Theorem 1.2.8.3. □

1.3 Translation planes

1.3.1 Projective and affine planes. A **projective plane** is an incidence structure π in which

(PP1) Every two points lie on a unique line.

(PP2) Every two lines meet in a unique point.

(PP3) There exists a set of four points, no three collinear (a **quadrangle**).

Only finite projective planes will concern us. For such a projective plane π, $\exists$ $n \in \mathbb{Z}^+$ (called the **order** of π) such that each point is incident with $n+1$ lines, and each line is incident with $n+1$ points (see [29, Theorem 3.5]). It is trivial that

(a) π has n^2+n+1 points

(b) π has n^2+n+1 lines.

The canonical example of a projective plane π of order q has as its points and lines the points and lines (respectively) of $\mathrm{PG}(2,q)$, with incidence defined via subspace containment (and π is denoted $\mathrm{PG}(2,q)$). If a projective plane is isomorphic to $\mathrm{PG}(2,q)$ for some q, it is called **Desarguesian**, and otherwise **non–Desarguesian**. Note that $\mathrm{Aut}\,\mathrm{PG}(2,q) = \mathrm{P\Gamma L}(3,q)$, by the fundamental theorem of projective geometry (see [69, Theorem 3.1]).

An **affine plane** is an incidence structure $\mathcal{A}$ in which

(AF1) Every two points lie on a unique line.

(AF2) For every point P and line L, there exists a unique line L' on P such that $L' = L$ or $L' \cap L = \emptyset$.

(AF3) There exists a set of three points, not all collinear (a **triangle**).

Lines L and L' such as in (AF2) are called **parallel**. An equivalence relation can be defined on the lines of an affine plane via

$$L_1 \sim L_2 \iff L_1 \text{ and } L_2 \text{ are parallel}$$

The equivalence classes so formed will be called **parallel classes**; denote by L_∞ the set of all parallel classes of an affine plane.

It is possible to pass between projective planes and affine planes, as follows. If $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a projective plane and $L \in \mathcal{L}$, form $\pi_L = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ by letting

$$\mathcal{P}' = \mathcal{P} - \{P \in \mathcal{P} : P \subseteq L\}$$

$$\mathcal{L}' = \mathcal{L} - \{L\}$$

$$\mathcal{I}' = I \cap (\mathcal{P}' \times \mathcal{L}')$$

Note that (PP1) is still satisfied in π_L, while if D is a quadrangle of π, take three points of D to obtain a triangle of π_L. To prove (AF2), let $L' \in \mathcal{L}'$ and $P' \in \mathcal{P}'$ have $P' \not\subseteq L'$. In π, L' meets L in a point R, and then the line of π through P

and R gives the required line of π_L. Thus, π_L is an affine plane. Given $L_1, L_2 \in \mathcal{L}$, $\pi_{L_1} \cong \pi_{L_2}$ if and only if L_1 and L_2 are in the same orbit of $\operatorname{Aut}\pi$ (see [29, Lemma 3.11]).

Now suppose $\mathcal{A} = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is an affine plane. Let

$$\mathcal{P} = \mathcal{P}' \cup \{R : R \in L_\infty\}$$

$$\mathcal{L} = \mathcal{L}' \cup \{L_\infty\}$$

$$\mathcal{I} = \mathcal{I}' \cup \{(R, L) : R \text{ a parallel class with } R \ni L\}$$

$$\cup \{(R, L_\infty) : R \text{ a parallel class}\}$$

Then $\bar{\mathcal{A}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a projective plane having $\bar{\mathcal{A}}_{L_\infty} = \mathcal{A}$, while any projective plane π having $\pi_L = \mathcal{A}$ (for some line L of π) has $\pi \cong \bar{\mathcal{A}}$ (see [29, Theorem 3.10]), where (by construction) $\bar{\mathcal{A}}_{L_\infty} = \mathcal{A}$. Any projective plane π having $\pi_L = \mathcal{A}$ (for some line L of π) has $\pi \cong \bar{\mathcal{A}}$ (see [29]).

Note. The line L_∞ of $\bar{\mathcal{A}}$ is called the **line at infinity** of $\mathcal{A}$.

We define the **order** of an affine plane $\mathcal{A}$ to be the order of $\bar{\mathcal{A}}$. If $\mathcal{A}$ has order n, it follows directly that

(a) every point is incident with $n + 1$ lines

(b) every line is incident with n points

(c) $\mathcal{A}$ has n^2 points and $n^2 + n$ lines

(d) there are $n + 1$ parallel classes (each of size n).

Let L be a line of $\mathrm{PG}(2, q)$. The canonical example of an affine plane of order q is $\mathrm{PG}(2, q)_L$, and is denoted $\mathrm{AG}(2, q)$ (since $\operatorname{Aut}\mathrm{PG}(2, q)$ is 2–transitive on the set of lines of $\mathrm{PG}(2, q)$, this definition of $\mathrm{AG}(2, q)$ is valid).

Suppose $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a projective plane, and $g \in \operatorname{Aut}\pi$. If $P \in \mathcal{P}$ has $g(L) = L \;\; \forall\, L \in \mathcal{L}$ for which $P \subseteq L$, then P is a **centre** of g. If $L \in \mathcal{L}$ has $g(P) = P \;\; \forall\, P \in \mathcal{P}$ for which $P \subseteq L$, then L is an **axis** of g.

In any projective plane π it is the case (see [29, Theorem 4.9]) that

(a) a collineation has a centre if and only if it has an axis

(b) a collineation has at most one centre and at most one axis

(c) if a collineation g has a centre P (and hence an axis L), then P and the points of L are the only fixed points of g, while $\langle g \rangle$ acts regularly on its point orbits that aren't fixed points.

Suppose a collineation g of a projective plane has centre P and axis L. If $P \subseteq L$, then g is an **elation**, and otherwise g is an **homology**.

A line L of a projective plane π is a **translation line** if $\exists\ H \leqslant \operatorname{Aut} \pi$ such that

(a) H consists solely of elations with axis L

(b) H is transitive on the points of π not on L.

We call H the **translation group** of π.

If a projective plane has a translation line, it is called a **translation plane**. If $\mathcal{A}$ is an affine plane such that $\bar{\mathcal{A}}$ is a translation plane with translation line L_∞, then $\mathcal{A}$ is an **affine translation plane**.

Theorem 1.3.1.1. *If π is a finite non–Desarguesian translation plane, then π has a unique translation line L (and so $\operatorname{Aut} \pi = \operatorname{Aut} \pi_L$).*

Proof. See [44]. □

The dual of a projective plane is again a projective plane, but the dual of a translation plane needn't be a translation plane. When the dual of a translation plane π is again such, the point of π which is the translation line of π^* is called the **shears point** of π, while π^* is called a **shears plane**.

1.3.2 Coordinatisation. To be able to describe affine (and hence projective) planes explicitly, we now present a way of coordinatising affine planes ([16, pp127–128] and [29, pp110–112] contain coordinatisation methods for projective planes). Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an affine plane, and $\mathcal{T}$ be a set containing elements called 0 and 1, such that $|\mathcal{T}|$ equals the order of $\mathcal{A}$. Take $L_1, L_2 \in \mathcal{L}$ that intersect (in a point called the origin), and call them the x–axis and y–axis respectively. Pick a point not on the x–axis or y–axis, and call it the unit point. Denote the line through the unit point and parallel to the x–axis by L_x^{unit}, and the line through the unit point and parallel to the y–axis by L_y^{unit}. We now set up a correspondence between $\mathcal{A}$ and $\mathcal{T}^2$.

The set of points of the x–axis is identified with $\{(x, 0) : x \in \mathcal{T}\}$ such that the origin corresponds to $(0, 0)$, while the point which is the intersection of L_y^{unit} with the x-axis is identified with $(1, 0)$. The set of points of the y–axis corresponds to

$\{(0,y) : y \in \mathcal{T}\}$, such that the point which is the intersection of L_x^{unit} with the y–axis is identified with $(0,1)$. The set of points on the line through the origin and the unit point corresponds to $\{(x,x) : x \in \mathcal{T}\}$. Now let R be a point not on the x–axis or y–axis (where R is not the unit point), let L_x^R be the line through R and parallel to the x–axis, and L_y^R be the line through R and parallel to the y–axis. Let $(a,0)$ correspond to the point that is the intersection of L_y^R with the x–axis, and $(0,b)$ correspond to the point that is the intersection of L_x^R with the y–axis. Then R is identified with (a,b).

Given such a coordinatisation of $\mathcal{A}$, let $x,a,b \in \mathcal{T}$, with l the line through $(0,0)$ and $(1,a)$, m the line through $(0,b)$ and parallel to l, and $L_y^{(x,0)}$ the line through $(x,0)$ and parallel to the y–axis. Then m intersects $L_y^{(x,0)}$ in a point which we call $(x,\gamma((x,a,b)))$, and we now have a function $\gamma : \mathcal{T}^3 \to \mathcal{T}$, having the following properties.

(a) $\gamma((x,0,b)) = \gamma((0,x,b)) = b \quad \forall\ x,b \in \mathcal{T}$

(b) $\gamma((x,1,0)) = \gamma((1,x,0)) = x \quad \forall\ x \in \mathcal{T}$

(c) $\forall\ x,y,a \in \mathcal{T}\ \ \exists\,!\ b \in \mathcal{T}$ such that $y = \gamma((x,a,b))$

(d) $\forall\ x_1,y_1,x_2,y_2 \in \mathcal{T}$ (with $x_1 \neq x_2$) $\exists\,!\ (a,b) \in \mathcal{T}^2$ such that $y_1 = \gamma((x_1,a,b))$ and $y_2 = \gamma((x_2,a,b))$

(e) $\forall\ a_1,b_1,a_2,b_2 \in \mathcal{T}$ (with $a_1 \neq a_2$) $\exists\,!\ x \in \mathcal{T}$ such that $\gamma((x,a_1,b_1)) = \gamma((x,a_2,b_2))$

Note that (a) and (b) hold from the definition of γ, (c) is by (AF2), (d) is by (AF1), while (e) is since any two non–parallel lines of $\mathcal{A}$ meet in a unique point.

If a set $\mathcal{T}$ together with a function $\gamma : \mathcal{T}^3 \to \mathcal{T}$ satisfies (a)–(e) above, then $(\mathcal{T},\gamma)$ is called a **planar ternary ring**. From any planar ternary ring $(\mathcal{T},\gamma)$, an affine plane $\mathcal{A} = (\mathcal{P},\mathcal{L},\mathcal{I})$ can be constructed, by letting

$$\mathcal{P} = \mathcal{T}^2$$

$$\mathcal{L} = \{\{(x,y) \mid y \in \mathcal{T}\} : x \in \mathcal{T}\}$$

$$\mathcal{I}\ :\ \text{defined via setwise inclusion}$$

In $\mathcal{A}$, (AF1) holds from (d) and the functionality of γ, (AF2) holds from (c) and (e), while a triangle of $\mathcal{A}$ is $\{(0,0),(1,0),(0,1)\}$. Thus, affine/projective planes and planar ternary rings correspond.

In an planar ternary ring $R = (\mathcal{T}, \gamma)$, we can define binary operations $+$ and $\circ$ (called **addition** and **multiplication** respectively) via

$$x + b = \gamma((x, 1, b))$$

and

$$x \circ b = \gamma((x, b, 0))$$

If

$$\gamma((x, a, b)) = x \circ a + b \quad \forall\, x, a, b \in \mathcal{T}$$

R is **linear**. If R is linear with $(R, +)$ a group, R is a **cartesian group**. A cartesian group R with

$$(x + y) \circ z = x \circ z + y \circ z \quad \forall\, x, y, z \in \mathcal{T}$$

is a **quasifield**. A quasifield R with

$$x \circ (y + z) = x \circ y + x \circ z \quad \forall\, x, y, z \in \mathcal{T}$$

is a **semifield**.

It is the case that quasifields coordinatise translation planes (see [29, Theorem 6.3]). Translation planes coordinatised by semifields are called **semifield planes**, and

Lemma 1.3.2.1. *The dual of a semifield plane is a translation plane.*

Proof. See [29, Corollary 1 to Theorem 6.9]. □

Finally, given a translation plane π in which the point P corresponds to the origin of the quasifield coordinatising π, we call $(\mathrm{Aut}\,\pi)_P$ the **translation complement** of π.

1.3.3 The Bruck–Bose construction.

Theorem 1.3.3.1 (Bruck–Bose construction). *Let S be a spread of a hyperplane H of $\mathrm{PG}(2n, q)$. Define $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ via*

$$\mathcal{P} = \{\text{points } P \text{ of } \mathrm{PG}(2n, q) : P \nsubseteq H\}$$

$$\mathcal{L} = \{n\text{–spaces } \Sigma \text{ of } \mathrm{PG}(2n, q) : \Sigma \cap H \in S\}$$

$$\mathcal{I} = \{(P, L) \in \mathcal{P} \times \mathcal{L} : P \subseteq L\}$$

Then $\mathcal{A}$ is an affine translation plane of order q^2.

Proof. See ([7]). □

As a converse, we have

Theorem 1.3.3.2. *All finite translation planes arise via the Bruck–Bose construction.*

Proof. See [44, Theorem 1.4 and Theorem 1.5]. □

The **kernel** of a quasifield $(\mathcal{T}, \gamma)$ is

$$\mathcal{K} = \{k \in \mathcal{T} : k \circ (x \circ y) = (k \circ x) \circ y \text{ and } k \circ (x + y) = k \circ x + k \circ y \;\forall\; x, y \in \mathcal{T}\}$$

Let R be a quasifield; by definition, R is an algebra over its kernel $\mathcal{K}$. Since multiplication in R is linear over K, if R is n–dimensional over K we have $R \cong K^n$. Given $e \in K^n$ with $e \neq 0$, note that each $v \in K^n$ may be expressed uniquely as $v = eA_v$ for some $n \times n$ matrix A_v. Hence, we obtain an n–spread set $S' = \{A_v : v \in V\}$, which can be used to represent multiplication in R via

$$x \circ y = xA_y$$

Conversely, S' corresponds to a spread S of $\mathrm{PG}(2n-1, q)$ (by Lemma 1.2.8.1), and then the Bruck–Bose construction produces a translation plane from S, coordinatised by R ([7]).

Lemma 1.3.3.3. *(a) Semifields correspond to spreads sets described by additive functions.*

(b) Let H denote the functor of the Hiramine et al construction. If a 2–spread set S' corresponds to a semifield, so does $H(S')$.

Proof. (a) In a semifield, multiplication is left associative. Hence, when representing multiplication as above, the spread set must be closed under addition, that is, the functions describing the spread set must be additive.
(b) By (a), S' is described by additive functions, while H preserves additivity (see Theorem 1.2.8.3). □

1.3.4 Derivation. Let $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane. A **subplane** π_0 of π is a projective plane with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{I}' = \mathcal{I} \cap (\mathcal{P}' \times \mathcal{L}')$. If π is a projective plane of (finite) order n, then the order of a proper subplane π_0 of π is at most $\sqrt{n}$ (see [29, Theorem 3.7]); if π_0 has maximal order it is called a **Baer subplane**.

Let $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane of order n^2, and D be a set of $n+1$ points lying on a common line l such that for all pairs of distinct points P_1, P_2 that are not on l and have $P_1P_2 \cap l \subseteq D$ (where P_1P_2 is the line through P_1 and P_2), there exists a Baer subplane π_0 of π that contains P_1, P_2 and D. Then D is called a **derivation set**. Now define an incidence structure $\mathcal{A}_D = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ via

$$\mathcal{P}' = \{P \in \mathcal{P} : P \nsubseteq l\}$$

$$\mathcal{L}' = \{L \in \mathcal{L} : L \cap l \nsubseteq D\}$$

$$\cup \{\text{Baer subplanes of } \pi \text{ containing } D\}$$

$$\mathcal{I}' = I \cap (\mathcal{P}' \cap \mathcal{L}')$$

Then $\mathcal{A}_D$ is an affine plane (see [29, Theorem 10.2]), and so we obtain a projective plane $\pi_D = \bar{\mathcal{A}}$. We call π_D the projective plane by **deriving** π **with respect to** D, with a projective plane containing a derivation set called **derivable**.

Given any projective plane containing a derivation set, the Bose–Barlotti construction creates a derivation set in its dual (see [6]). By dualising and deriving, we can potentially obtain numbers of new projective planes from a given one.

1.3.5 Translation planes from ovoids of orthogonal spaces. Given an ovoid of a parabolic space of dimension at least 5, or an ovoid of an hyperbolic space of dimension at least 6, that ovoid either is or slices to an $\mathrm{O}^+(6, q)$ ovoid O. Then, by the Klein correspondence and the Bruck–Bose construction, we obtain a translation plane $\pi(O)$ of order q^2. In the case that O is an elliptic quadric, the following theorem (coupled with Theorem 1.2.4.1) implies that $\pi(O)$ is $\mathrm{PG}(2, q^2)$.

Theorem 1.3.5.1. *Regular spreads of* $\mathrm{PG}(2n-1, q)$ *correspond to desarguesian translation planes of order* q^n.

Proof. See [8]. □

In the case of an $\mathrm{O}^+(8, q)$ ovoid O, there are two further ways that translation planes can be obtained. If q is even, then by Theorem 1.2.5.1 and Theorem 1.2.6.1, O yields a spread S of $\mathrm{Sp}(6, q)$, and S is a spread of $\mathrm{PG}(5, q)$; thus, a translation plane of order q^3 is obtained. Alternatively (and for any q), given a singular point P with $P \notin O$,

$$S = \{P^\perp \cap M : M \in \tau(O) \text{ and } \tau(P) \cap M \neq \{0\}\}$$

is a spread of the maximal $\tau(P)$ (where τ is a triality map). Since $\tau(P)$ is 4–dimensional, S yields a translation plane of order q^2 (this construction is due to Kantor in [36]).

1.4 Existence results for ovoids

In Table 1.4.1 we give the current status of ovoid existence in the polar spaces of interest to us (for a survey of spread existence in these spaces, see [75]), a question mark indicating where there are no existence or non–existence results.

space	n	ovoids occur for
$\mathrm{Sp}(2n,q)$	1	all q
	2	q even (none for q odd ([70]))
	3	no q
$\mathrm{U}(2n+1,q)$	1	all q
	2	no q
$\mathrm{U}(2n,q)$	1	all q
	2	all q
	3	?
$\mathrm{O}^{-}(2n+2,q)$	1	all q
	2	no q
$\mathrm{O}(2n+1,q)$	1	all q
	2	all q
	3	$q=3^h$ $(h>0)$ (none for q even ([73]), 5, 7 ([50]), 11 ([57])), ? o/wise
	4	no q ([24])
$\mathrm{O}^{+}(2n,q)$	1	all q
	2	all q
	3	all q
	4	q prime or when $q \not\equiv 1 \pmod 6$ (see [14]), ? o/wise
	5	none for $q=2^h$, 3^h $(h>0)$ ([5]), ? o/wise

Table 1.4.1: Ovoid existence in finite–dimensional polar spaces over $\mathrm{GF}(q)$

1.4.1 Trivial existence results. Let $\mathcal{S}$ be a non–degenerate polar space of Witt index 1 over $\mathrm{GF}(q)$. No totally isotropic/singular lines occur in $\mathcal{S}$, while the set of isotropic/singular points of $\mathcal{S}$ has the right size to be an ovoid, by Corollary 1.1.7.5. Applying Theorem 1.1.7.6, ovoids also exist in $\mathrm{U}(4,q)$, $\mathrm{O}(5,q)$ and $\mathrm{O}^+(6,q)$ for all q, while for q even, ovoids of $\mathrm{O}(5,q)$ and $\mathrm{Sp}(4,q)$ correspond (by Corollary 1.2.3.2 and Lemma 1.2.3.3). To show that ovoids of $\mathrm{O}^+(4,q)$ exist for all q, first note that $\mathrm{O}^+(4,q)$ contains $2(q+1)$ totally singular lines, where each singular point is the intersection of two totally singular lines, one from each class of totally singular line. So, repeatedly taking a singular point and deleting the two totally singular lines on it, we obtain an ovoid of $\mathrm{O}^+(4,q)$.

1.4.2 Non–existence results. The next result enables us to show non–existence of $\mathrm{Sp}(2n,q)$ $(n>2)$, $\mathrm{U}(2n+1,q)$ $(n>1)$ and $\mathrm{O}^-(2n+2,q)$ $(n>1)$ ovoids. In it, we need to know that an **ovoid** of $\mathrm{PG}(3,q)$ is a set of q^2+1 points, no three on a line.

Theorem 1.4.2.1. *([71]) Suppose K is a set of points of $\mathrm{PG}(d,q)$ $(d>2)$, such that there exists an integer $k>1$ for which each hyperplane of $\mathrm{PG}(d,q)$ meets K in 1 or k points, and that there exists a hyperplane of $\mathrm{PG}(d,q)$ meeting K in 1 point. Then either K is a line of $\mathrm{PG}(d,q)$, or $d=3$ and K is an ovoid of $\mathrm{PG}(3,q)$.*

Proof. See ([71]). $\square$

Corollary 1.4.2.2 ([73]). $\mathrm{Sp}(2n,q)$ *has no ovoids for* $n>2$.

Proof. By Lemma 1.1.7.2, it is sufficient to prove that $\mathrm{Sp}(6,q)$ has no ovoids. Let O be an ovoid of $\mathrm{Sp}(6,q)$, H a hyperplane of the underlying $\mathrm{PG}(5,q)$ and $P=H^\perp$. Suppose that $P\in O$. Then $H\cap O=\{P\}$, that is, we have a hyperplane of $\mathrm{PG}(5,q)$ meeting O in 1 point. Now suppose $P\notin O$. By Lemma 1.1.7.2, $O_P=\{\langle P,R\rangle : R\in H\cap O\}$ is an ovoid of H/P (an $\mathrm{Sp}(4,q)$ space), so $|O_P|=q^2+1$. But $|H\cap O|=|O_P|$, so by Theorem 1.4.2.1, O is an ovoid of $\mathrm{PG}(3,q)$ and a contradiction results. $\square$

Corollary 1.4.2.3 ([73]). $\mathrm{U}(2n+1,q)$ *has no ovoids for* $n>1$.

Proof. Let O be an ovoid of $\mathrm{U}(5,q)$ (without loss of generality), H a hyperplane of the underlying $\mathrm{PG}(4,q)$ and $P=H^\perp$. If $P\in O$, then $|H\cap O|=1$. If P is singular and $P\notin O$, then O_P is an ovoid of $\mathrm{U}(3,q)$ and so $|H\cap O|=q^{\frac{3}{2}}+1$. If P isn't singular, then H is a $\mathrm{U}(4,q)$ space and $H\cap O$ is an ovoid of it (as maximals of $\mathrm{U}(4,q)$ are maximals of $\mathrm{U}(5,q)$), implying that $|H\cap O|=q^{\frac{3}{2}}+1$. Thus, by Theorem 1.4.2.1, O is a $\mathrm{PG}(3,q)$ ovoid and we have a contradiction. $\square$

Corollary 1.4.2.4 ([73]). $\mathrm{O}^-(2n+2,q)$ *has no ovoids for* $n>1$.

Proof. Similar to that of Corollary 1.4.2.3. $\square$

In [5] Blokhuis and Moorhouse constructed a bound on the size of caps in non–degenerate m–dimensional unitary and orthogonal spaces over $\mathrm{GF}(q)$ (an improved version of the bound in unitary spaces was established in ([47]). As the characteristic of GF(q) tends to infinity, the bound rules out ovoids once m is large enough. The resulting non–existence results for ovoids of unitary spaces start in U$(8, q)$ (in Table 1.4.1 we chose to stop at the first U$(2n, q)$ and O$^+(2n, q)$ spaces where no ovoids are known).

Ovoids of unitary spaces are not well connected to other geometric objects and structures. This, together with the abundance of U$(4, q)$ ovoids (see Theorem 1.2.7.2) and no known U(m, q) ovoids for $m \geq 5$ (with none existing for odd m) means that we will not be studying ovoids of these spaces; we will be concerned solely with ovoids of parabolic and hyperbolic spaces. With no O$(2n+1, q)$ ovoids for $n > 3$, and no known O$^+(2n, q)$ ovoids for $n > 4$, our focus will be on O$(5, q)$, O$(7, q)$ and O$^+(8, q)$ ovoids. Ovoids of O$^+(6, q)$ are structurally interesting (being linked to spreads and translation planes), but the abundance of them (as evidenced by Theorem 1.2.4.2 and Corollary 1.2.8.4) makes the finding of one of little interest. Ovoids of O$(5, q)$, O$(7, q)$ and O$^+(8, q)$ are rare, making new ovoids, classification and characterisation results in those spaces valuable.

1.5 Miscellaneous background

1.5.1 Group theoretic results. We will assume a knowledge of group theory that includes the orbit–stabiliser theorem, Cauchy's theorem, conjugacy and irreducibility. Some specific results that will be required are as follows; they are all elementary.

Theorem 1.5.1.1 (see [84], p47). *Let G be a group acting transitively on a finite set X, and $H \leqslant G$. If $(|G : H|, |X|) = 1$, then H is transitive on X.*

Proof. Let $x \in X$. Now $|G : H_x| = |G : G_x||G_x : H_x| = |G : H||H : H_x|$, while $|X| = |G : G_x|$ (using that G is transitive on X and the orbit–stabiliser theorem), so $(|G : G_x|, |G : H|) = 1$. Since $|G : G_x|$ divides $|G : H_x|$, $|X| = |G : G_x|$ must divide $|H : H_x|$, and hence H is transitive on X. □

Theorem 1.5.1.2. *Let G be a group acting on a set X, with $H \leqslant G$. Then $\mathrm{N}_G(H)$ permutes the orbits of H, preserving sizes of orbits.*

Proof. Given a permutation representation $\phi : G \to \mathrm{Sym}(X)$, $n \in \mathrm{Sym}(X)$ and an orbit O of G, it is easy to show that nO is an orbit of $n\phi(G)n^{-1}$ on X. In particular, if G acts by conjugation on G, for $n \in \mathrm{N}_G(H)$ we have that nO is an orbit of $nHn^{-1} = H$ (while the second part of the theorem holds since all elements of $\mathrm{N}_G(H)$ are permutations of G). □

Lemma 1.5.1.3. *Suppose f is a polynomial in x over* $\mathrm{GF}(q)$ *with* $f(a) = 0 \ \forall$ $a \in \mathrm{GF}(q)$. *Then* $x^q - x \,\big|\, f(x)$.

Proof. Whenever $f(a) = 0$, we have $x - a \,\big|\, f(x)$. Also,

$$\prod_{a \in \mathrm{GF}(q)} x - a = x^q - x$$

and the result follows. □

1.5.2 Notation for automorphisms. We will frequently be working with functions on $\mathrm{GF}(q)$, (for example, when providing explicit descriptions of ovoids). As a convenient shorthand (the following explanation for which we take from [61]), we will denote by n any function f defined by $f(x) = x^n$, where $n \in E_{q-1} = \{1, 2, \ldots, q-1\}$ (E_{q-1} is a subfield of $\mathbb{Z}_q$). That this convention covers all functions on $\mathrm{GF}(q)$ follows from the fact that for $m, n \in \mathbb{Z}$, the polynomials x^m and x^n map (under the natural homomorphism ϕ from $\mathrm{GF}(q)[x]$ onto to the ring of all functions on $\mathrm{GF}(q)$) to the same function on $\mathrm{GF}(q)$ if and only if $m = n = 0$ or $m \equiv n \pmod{q-1}$ (here we are using that $\ker\phi$ is the principal ideal generated by $x^q - x$).

With the above convention and with $m, n \in E_{q-1}$, $m+n$ will denote the function f defined by $f(x) = x^{m+n}$, while mn will denote the function f defined by $f(x) = x^{mn}$. If f defined by $f(x) = x^n$ is an invertible function, then n must be a unit of E_{q-1}, and we denote f^{-1} by $\frac{1}{n}$. We specify that $\frac{q}{2} = \frac{1}{2}$ and $q - 2 = -1$, as functions.

Ovoids of $\mathrm{O}(5,q)$

2.1 The known ovoids

In Table 3.1.1 we give the stabilisers in $\mathrm{P\Gamma O}(5,q)$ of the known $\mathrm{O}(5,q)$ ovoids, and the values of q for which the ovoids exist (the stabilisers of the Kantor and Ree–Tits slice ovoids are calculated in Theorem 2.2.3 and Theorem 2.2.6 respectively).

name		stabiliser	q
elliptic quadric (see [27, p17])		$\mathrm{P\Gamma O}^{-}(4,q)\times \mathrm{C}_2$	all
Kantor $K(\alpha)$	$\alpha^2\neq 1$	$\mathrm{E}_{q^2}\rtimes((\mathrm{C}_2\times \mathrm{C}_{q-1})\rtimes \mathrm{C}_h)$	p^h $(h>1)$, p
([36])	$\alpha^2=1$	$\mathrm{E}_{q^2}\rtimes((\mathrm{C}_2\times(\mathrm{C}_{q-1}\rtimes \mathrm{C}_2))\rtimes \mathrm{C}_h)$	an odd prime
Ree–Tits slice ([36])		$E_q\rtimes(\mathrm{C}_{q-1}\rtimes \mathrm{C}_h)$	3^{2h+1} $(h>0)$
Thas–Payne ([78])		$\mathrm{E}_{q^2}\rtimes(\mathrm{C}_2\times \mathrm{C}_h)$	3^h $(h>2)$
Tits ([81])		$\mathrm{Sz(q)}\rtimes \mathrm{C}_h$	2^{2h+1} $(h>0)$

Table 2.1.1: Stabilisers of the known $\mathrm{O}(5,q)$ ovoids.

(Here E_n denotes an elementary abelian group of order n.)

2.1.1 Classification results. For $q=3,5,7$ (see [50]) and 11 ([57]) elliptic quadrics are the only $\mathrm{O}(5,q)$ ovoids, while for $q=9$ ovoids are either elliptic quadrics or Kantor ([57]). The other spaces in which ovoids have been classified are those having q even and $q\leq 32$: for $q=2$, 4, 16 only elliptic quadrics occur, while for $q=8$, 32 just elliptic quadric and Tits ovoids arise (see [50]). Currently, $\mathrm{O}(5,q)$ ovoids are rare.

2.1.2 A model of $\mathrm{O}(5,q)$. The model for $\mathrm{O}(5,q)$ that we shall use has $V=\mathrm{GF}(q)^5$, with

$$Q((x_1,x_2,x_3,x_4,x_5))=x_1x_5+x_2x_4+x_3^2$$

defining a quadratic form Q on V. That (V,Q) has Witt index 2 is seen by noting that $\langle(1,0,0,0,0),(0,1,0,0,0)\rangle$ is a totally singular subspace of V, while if $w=(w_1,w_2,w_3,w_4,w_5)\in \mathrm{rad}(f_Q)$ (f_Q the polar form of Q), then $f_Q(w,e_i)=0$ for $0\leq i\leq 5$ (where $\{e_i:1\leq i\leq 5\}$ is the standard basis for V) and so w is the zero vector.

Analogously to the model of $\mathrm{O}^+(6,q)$ that we used in Chapter 1, any $\mathrm{O}(5,q)$ ovoid is equivalent to one of the form

$$O(f)=\{\langle(0,0,0,0,1)\rangle\}\cup\{\langle(1,x,y,f(x,y),-y^2-xf(x,y))\rangle : x,y\in \mathrm{GF}(q)\}$$

where $f : \mathrm{GF}(q)^2 \to \mathrm{GF}(q)$.

2.1.3 The Kantor and elliptic quadric ovoids. Let $q = p^h$ $(h > 1)$ for p an odd prime, n be a non–square of $\mathrm{GF}(q)$ and $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ have $\alpha \neq 1$. Each Kantor ovoid $K(\alpha)$ has $f(x,y) = -nx^\alpha$ for $x, y \in \mathrm{GF}(q)$. It is trivial to check that $K(\alpha)$ is an ovoid; given $x_1, y_1 \in \mathrm{GF}(q)$ we have

$$\begin{aligned}&f_Q((0,0,0,0,1),(1,x_1,y_1,-nx_1^\alpha,-y_1^2+nx_1^{\alpha+1}))\\&= Q((1,x_1,y_1,-nx_1^\alpha,-y_1^2+nx_1^{\alpha+1}+1)) = 1\end{aligned}$$

while for $x_2, y_2 \in \mathrm{GF}(q)$ with $x_1 \neq x_2$ or $y_1 \neq y_2$, we find

$$\begin{aligned}&f_Q((1,x_1,y_1,-nx_1^\alpha,-y_1^2+nx_1^{\alpha+1}),(1,x_2,y_2,-nx_2^\alpha,-y_2^2+nx_2^{\alpha+1}))\\&= Q((2,x_1+x_2,y_1+y_2,-n(x_1+x_2)^\alpha,-y_1^2-y_2^2+n(x_1^{\alpha+1}+x_2^{\alpha+1}))\\&= 2(-y_1^2-y_2^2+n(x_1^{\alpha+1}+x_2^{\alpha+1}))-n(x_1+x_2)^{\alpha+1}+(y_1+y_2)^2\\&= -(y_1-y_2)^2+n(x_1-x_2)^{\alpha+1}\\&= 0 \Longleftrightarrow n = \frac{(y_1-y_2)^2}{(x_1-x_2)^{\alpha+1}}\end{aligned}$$

contradicting n being a non–square of $\mathrm{GF}(q)$.

For any choice of non–square, there is up to equivalence only one ovoid $K(\alpha)$ for each $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$, as we now show. Let $K(n,\alpha)$ denote a Kantor ovoid defined via a non–square n and $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$. For $a \in \mathrm{GF}(q)^*$ define a projective isometry μ_a via

$$\mu_a(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(x_1,x_2,ax_3,a^2x_4,a^2x_5)\rangle$$

Then for $x, y \in \mathrm{GF}(q)$,

$$\begin{aligned}&\mu_a(\langle(1,x,y,-nx^\alpha,-y^2+nx^{\alpha+1})\rangle)\\&= \langle(1,x,ay,-a^2nx^\alpha,a^2(-y^2+nx^{\alpha+1}))\rangle\end{aligned}$$

that is, μ_a takes $K(n,\alpha)$ to $K(a^2n,\alpha)$, and any non–square of $\mathrm{GF}(q)$ may be expressed as a^2n for some $a \in \mathrm{GF}(q)^*$.

In an $\mathrm{O}(5,p^h)$ space where Kantor ovoids occur, there are $[\frac{h}{2}]$ of them, as we will show in Corollary 2.2.4.

When $\alpha = 1$, $K(\alpha)$ is an elliptic quadric, and clearly $f(x,y) = -nx$ describes an elliptic quadric for any odd q. When q is even, there always exists an $a \in \mathrm{GF}(q)$

such that $f(x,y) = ax + y$ describes an elliptic quadric. To see this, first apply f_Q to the pair

$$(1, x_1, y_1, ax_1 + y_1, ax_1^2 + x_1y_1 + y_1^2), \quad (1, x_2, y_2, ax_2 + y_2, ax_2^2 + x_2y_2 + y_2^2)$$

(where $a, x_1, y_1, x_2, y_2 \in \mathrm{GF}(q)$) and let $X = x_1 + x_2$, $Y = y_1 + y_2$; we must show that some value of a has $X^2 + aXY + Y^2$ irreducible. But $X^2 + aXY + Y^2$ is irreducible in characteristic 2 if and only if $a \neq 0$ and $\mathrm{trace}(\frac{1}{a}) = 1$ (see [27]).

2.1.4 The Ree–Tits slice ovoids. The Ree–Tits slice ovoids occur for $q = 3^{2h+1}$ $(h > 0)$, and were obtained by Kantor by slicing the Ree–Tits ovoids of $\mathrm{O}(7,q)$ (see Chapter 4). Given $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ with $a^{\alpha^2} = \alpha^3 \ \ \forall\, a \in \mathrm{GF}(q)$, these ovoids have $f(x,y) = -x^{2\alpha+3} - y^{\alpha}$ for $x, y \in \mathrm{GF}(q)$.

2.1.5 The Thas–Payne ovoids. The Thas–Payne ovoids arise for $q = 3^h$ $(h > 2)$, and were constructed via the Roman generalised quadrangle of order (q, q^2); these ovoids have $f(x,y) = -nx - (n^{-1}x)^{1/9} - y^{1/3}$ for $x, y \in \mathrm{GF}(q)$ and n a non–square of $\mathrm{GF}(q)$. The choice of non–square is irrelevant – there is only one Thas–Payne ovoid for each $q = 3^h$ $(h > 2)$ ([78]).

2.1.6 The Tits ovoids. The set of absolute points of a polarity of $\mathrm{Sp}(4,q)$ is an ovoid of $\mathrm{Sp}(4,q)$ (see [55, 1.8.2]). Since polarities of $\mathrm{Sp}(4,q)$ exist for all $q = 2^{2h+1}$ $(h \geq 0)$, the Tits ovoids of $\mathrm{Sp}(4,q)$ arise for $q = 2^{2h+1}$ $(h > 0)$ (in [53] it was proved that polarities of $\mathrm{Sp}(4,q)$ exist only for these values of q). Defining $\mathrm{Sp}(4,q)$ from $\mathrm{GF}(q)^4$ via the alternating form g, where

$$g((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2$$

and choosing $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ such that $a^{\alpha^2} = a^2 \ \ \forall\, a \in \mathrm{GF}(q)$, these ovoids can be written ([81]) as

$$T = \{\langle(0,0,0,1)\rangle\} \cup \{\langle(1, s, t, s^{\alpha} + st + t^{\alpha+2})\rangle : s, t \in \mathrm{GF}(q)\}$$

By Lemma 1.2.3.3 and Corollary 1.2.3.2 there is a corresponding family of $\mathrm{O}(5,q)$ ovoids. To write these ovoids explicitly, first define an alternating form g' on $V = \mathrm{GF}(q)^5$ via

$$g'((x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3, y_4, y_5)) = x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3$$

Then $P = \langle(0,0,0,0,1)\rangle$ is the radical of g', and by factoring out P we have (with the naturally induced form from g') an $\mathrm{Sp}(4,q)$ space. An isometry I between the two $\mathrm{Sp}(4,q)$ spaces is defined via $I((x_1,x_2,x_3,x_4)) = (x_1,x_4,x_3,x_2,0)$, and applying I (projectively) to T we obtain

$$T' = \{\langle(0,1,0,0,0)\rangle\} \cup \{\langle(1,s^{\alpha+2}+st+t^{\alpha},t,s,0)\rangle : s,t \in \mathrm{GF}(q)\}$$

A quadratic form Q' on V that has g' as its polar form is defined by

$$Q'((x_1,x_2,x_3,x_4,x_5)) = x_1x_2 + x_3x_4 + x_5^2$$

Let $s^{\frac{\alpha+2}{2}} + t^{\frac{\alpha}{2}}$ be the fifth coordinate of each point of $T' - \{\langle(0,1,0,0,0)\rangle\}$ (so that all those points are singular). Define an isometry $\mathcal{J}$ via $\mathcal{J}(x_1,x_2,x_3,x_4,x_5) = (x_1,x_3,x_5,x_4,x_2)$ and apply $\mathcal{J}$ (projectively) to T' to obtain a copy of T' in our usual model for $\mathrm{O}(5,q)$, then switch the second and fourth coordinates of that copy and put $x = s$ and $y = s^{\frac{\alpha+2}{2}} + t^{\frac{\alpha}{2}}$. The resulting ovoid is described by $f(x,y) = x^{\alpha+1} + y^{\alpha}$ for $x,y \in \mathrm{GF}(q)$.

Remark: Let $p = 2$ or 3, $q = p^{2h+1}$ $(h > 0)$ and $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ with $a^{\alpha^2} = a^p$ $\forall\, a \in \mathrm{GF}(q)$. The Tits ovoids and Ree–Tits slices may be described uniformly via $f(x,y) = -x^{(\frac{p-1}{2})(2\alpha+\alpha^2)} - y^{\alpha}$ for $x,y \in \mathrm{GF}(q)$.

2.2 Stabilisers of the known ovoids

In this section we justify the stabilisers given in Table 3.1.1 in the cases where the literature lacks a proof.

To establish the stabilisers of the elliptic quadric ovoids, first write $\mathrm{O}^+(6,q) = \mathrm{O}^-(4,q) \perp \mathrm{O}^-(2,q)$, where we express $\mathrm{O}^-(2,q)$ as $\mathrm{GF}(q^2)$ equipped with the quadratic form Q, defined by $Q(x) = x^{q+1}$. Define an isometry g on $\mathrm{O}^-(2,q)$ via $g(x) = x^q$; it fixes $\mathrm{GF}(q)$ inside $\mathrm{GF}(q^2)$, and then the orthogonal direct sum of that $\mathrm{GF}(q)$ together with $\mathrm{O}^-(4,q)$ gives an $\mathrm{O}(5,q)$ space. Letting ι denote the identity of $\mathrm{P\Gamma O}^-(4,q)$, the group $\langle \iota \times g\rangle$ of order 2 stabilises O and commutes with $\mathrm{P\Gamma O}^-(4,q)$, which also stabilises O. To see that there is no more stabiliser of O inside $\mathrm{P\Gamma O}(5,q)$, first note that each element of $\mathrm{P\Gamma O}^+(6,q)_O$ induces a projective semisimilarity on $\langle O\rangle$, that is, we have a homomorphism from $\mathrm{P\Gamma O}^+(6,q)_O$ onto $\mathrm{P\Gamma O}^-(4,q)$. Now the kernel of this homomorphism is

$$K = \{g \in \mathrm{P\Gamma O}^+(6,q)_O : g(\langle v\rangle) = \langle v\rangle \;\forall\, v \in \langle O\rangle\}$$

If an element of K fixes $\langle O\rangle$ then it fixes $\langle O\rangle^\perp = \mathrm{O}^-(2,q)$, and so

$$\begin{aligned} K &\leqslant \mathrm{PGO}^+(6,q)_{(\mathrm{O}^-(4,q))\perp\mathrm{O}^-(2,q)} \\ &\leqslant \mathrm{P\,O}^+(6,q)_{(\mathrm{O}^-(4,q))\perp\mathrm{O}^-(2,q)} \\ &= \iota \times \mathrm{O}^-(2,q) \end{aligned}$$

To determine $\mathrm{P\Gamma O}(5,q)_O$, note that choosing an $\mathrm{O}(5,q)$ space containing $\langle O\rangle$ is (taking perps) equivalent to choosing a point of $\langle O\rangle^\perp = \mathrm{O}^-(2,q)$, and $\mathrm{P\Gamma O}^+(6,q)_O$ is transitive on the $q+1$ points of $\mathrm{O}^-(2,q)$. Thus, $|\,\mathrm{P\Gamma O}^+(6,q)_O : \mathrm{P\Gamma O}(5,q)_O| = q+1$. Since $\mathrm{O}^-(2,q)$ is $\mathrm{D}_{2(q+1)}$ (see [69, p139]), only the subgroup $\langle \iota \times g\rangle$ of K is in $\mathrm{P\Gamma O}(5,q)_O$.

The next result (which was implicitly used in [78]) will be applied in all the stabiliser calculations that follow.

Lemma 2.2.1. *Let*

$$\tau_{abc}(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle \begin{pmatrix} b^2 & 0 & 0 & 0 & 0 \\ 0 & c^{-1} & -2a & -a^2c & 0 \\ 0 & 0 & b & abc & 0 \\ 0 & 0 & 0 & b^2c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^\sigma \\ x_2^\sigma \\ x_3^\sigma \\ x_4^\sigma \\ x_5^\sigma \end{pmatrix} \rangle$$

and

$$\mu_{rstu}(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) =$$
$$\langle \begin{pmatrix} t^2 & 0 & 0 & 0 & 0 \\ 0 & -s^2u^{-1} & 2s(rs-t) & u(r^2s^2-2rst+t^2) & 0 \\ 0 & su^{-1} & -2rs+t & ru(-rs+t) & 0 \\ 0 & u^{-1} & -2r & -r^2u & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^\sigma \\ x_2^\sigma \\ x_3^\sigma \\ x_4^\sigma \\ x_5^\sigma \end{pmatrix} \rangle$$

where σ is the companion automorphism of τ_{abc} and μ_{rstu}. Then

$$\begin{aligned}&\mathrm{P\Gamma O}(5,q)_{\langle(1,0,0,0,0)\rangle,\langle(0,0,0,0,1)\rangle} \\ &\qquad = \{\tau_{abc} : a,b,c \in \mathrm{GF}(q); bc \neq 0\} \cup \{\mu_{rstu} : r,s,t,u \in \mathrm{GF}(q); tu \neq 0\}.\end{aligned}$$

Proof. Any projective semisimilarity θ fixing $\langle(1,0,0,0,0)\rangle$ and $\langle(0,0,0,0,1)\rangle$ can be written as τ_{abc} or μ_{rstu}, according (respectively) to whether or not θ fixes $\langle(0,0,0,1,0)\rangle$. □

If the translation plane resulting from an $\mathrm{O}^+(6,q)$ ovoid O is a semifield plane, O is called a **translation ovoid**. By Lemma 1.3.3.3 and Lemma 1.2.8.2, an ovoid is such precisely when the functions f and g describing it are additive. Note that any 5–dimensional ovoid of $\mathrm{O}^+(6,q)$ (expressed as in Lemma 1.2.8.2) can be written with $f(x,y) = y$, and so an $\mathrm{O}(5,q)$ ovoid (in the model of $\mathrm{O}(5,q)$ that we are using) is a translation ovoid precisely when the describing function is additive. Thus, the elliptic quadric, Kantor and Thas–Payne ovoids are all translation ovoids.

When calculating the stabiliser of a translation ovoid, the following lemma is convenient.

Lemma 2.2.2. *If O is a translation ovoid, the stabiliser of O fixes some point of O.*

Proof. Let π denote the translation plane arising from O; by Lemma 1.3.2.1, π^* is a translation plane. Let L be the translation line and P the shears point of π. Because $\mathrm{Aut}\,\pi$ is transitive on the points of π not on L, we have $P \subseteq L$ and P is fixed by $\mathrm{Aut}\,\pi$. Then P corresponds to a line l of the associated $\mathrm{PG}(3,q)$ spread, where l is fixed by the group of the spread, so by the Klein correspondence $\mathrm{P\Gamma O}^+(6,q)_O$ fixes some point of O. $\square$

Theorem 2.2.3. *The stabiliser of a Kantor ovoid $K(\alpha)$ with $\alpha^2 = 1$ is $E_{q^2} \rtimes ((C_2 \times (C_{q-1} \rtimes C_2)) \rtimes C_h)$, and $E_{q^2} \rtimes ((C_2 \times C_{q-1}) \rtimes C_h)$ otherwise.*

Proof. The group $G = \{\phi_{d,e} : d, e \in \mathrm{GF}(q)\} < \mathrm{P\Gamma O}(5,q)$ is transitive on $K(\alpha) - \{\langle(0,0,0,0,1)\rangle\}$, where

$$\phi_{d,e}(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \\ e & 0 & 1 & 0 & 0 \\ -nd^\alpha & 0 & 0 & 1 & 0 \\ -e^2 + nd^{\alpha+1} & nd^\alpha & -2e & -d & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \rangle$$

Hence, without loss of generality we may assume that $\langle(1,0,0,0,0)\rangle$ is fixed by $\mathrm{P\Gamma O}(5,q)_{K(\alpha)}$. Also, $\mathrm{P\Gamma O}(5,q)_{K(\alpha)}$ fixes $\langle(0,0,0,0,1)\rangle$ by Lemma 2.2.2, so we need only compute $\mathrm{P\Gamma O}(5,q)_{K(\alpha),\langle(1,0,0,0,0)\rangle,\langle(0,0,0,0,1)\rangle}$. So, let θ be a projective semisimilarity (with companion automorphism σ) fixing $\langle(1,0,0,0,0)\rangle$ and $\langle(0,0,0,0,1)\rangle$. Note that θ stabilises $K(\alpha)$

$$\begin{aligned} &\iff -n(x_2'/x_1')^\alpha = (x_4'/x_1') \\ &\iff -n(x_2')^\alpha (x_1')^{1-\alpha} = x_4' \end{aligned}$$

$\forall\ \langle(x_1', x_2', x_3', x_4', x_5')\rangle \in \theta(K(\alpha) - \{\langle(0,0,0,0,1)\rangle\})$.

We now apply Lemma 2.2.1. If θ is of the form τ_{abc}, then

$$\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(b^2x_1^\sigma, c^{-1}x_2^\sigma - 2ax_3^\sigma - a^2cx_4^\sigma, bx_3^\sigma + abcx_4^\sigma, b^2cx_4^\sigma, x_5^\sigma)\rangle$$

so that $K(\alpha)$ is stabilised by θ if and only if

$$-n(c^{-1}x^{\sigma} - 2ay^{\sigma} - a^2c(-nx^{\alpha})^{\sigma})^{\alpha}(b^2)^{1-\alpha} = b^2c(-nx^{\alpha})^{\sigma} \qquad (2.2.1)$$

$\forall\, x, y \in \mathrm{GF}(q)$. Setting $x = 0$ forces $a = 0$, so when $x = 1$ we have $-nc^{-\alpha}b^{2(1-\alpha)} = -b^2cn^{\sigma}$, that is, $b = \pm(c^{\alpha+1}n^{\sigma-1})^{-1/(2\alpha)}$. With these values for a and b, (2.2.1) holds for all $x, y \in \mathrm{GF}(q)$, and so we have a group $(C_2 \times C_{q-1}) \rtimes C_h$ fixing $K(\alpha)$, $\langle(1,0,0,0,0)\rangle$, $\langle(0,0,0,0,1)\rangle$ and $\langle(0,1,0,0,0)\rangle$.

If θ is of the form μ_{rstu}, then

$$\begin{aligned}\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(&t^2x_1^{\sigma}, -s^2u^{-1}x_2^{\sigma} + 2s(rs-t)x_3^{\sigma} + u(r^2s^2 - 2rst + t^2)x_4^{\sigma},\\ &su^{-1}x_2^{\sigma} + (-2rs+t)x_3^{\sigma} + ru(-rs+t)x_4^{\sigma}, u^{-1}x_2^{\sigma} - 2rx_3^{\sigma} - r^2ux_4^{\sigma}, x_5^{\sigma})\rangle\end{aligned}$$

so that θ stabilises $K(\alpha)$ if and only if

$$\begin{aligned}&-n(-s^2u^{-1}x^{\sigma} + 2s(rs-t)y^{\sigma} + u(r^2s^2 - 2rst + t^2)(-nx^{\alpha})^{\sigma})^{\alpha}(t^2)^{1-\alpha}\\ &\qquad = u^{-1}x^{\sigma} - 2ry^{\sigma} - r^2u(-nx^{\alpha})^{\sigma} \qquad (2.2.2)\end{aligned}$$

$\forall\, x, y \in \mathrm{GF}(q)$. Putting $x = 0$, we obtain $-n(2s(rs-t)y^{\sigma})^{\alpha}(t^2)^{1-\alpha} = -2ry^{\sigma}$ $\forall\, y \in \mathrm{GF}(q)$. Since $\alpha \neq 1$, we have $r = s = 0$, so (2.2.2) becomes $t^2(n^{\frac{\sigma\alpha+1}{\alpha+1}}u)^{\alpha+1}x^{\sigma\alpha^2} = x^{\sigma}$ $\forall\, x \in \mathrm{GF}(q)$. Therefore, $\alpha^2 = 1$. Conversely, with $r = s = 0$, $t^2(n^{\frac{\sigma\alpha+1}{\alpha+1}}u)^{\alpha+1} = 1$ and $\alpha^2 = 1$, (2.2.2) holds for all $x, y \in \mathrm{GF}(q)$. When $\alpha^2 = 1$, $t^2(n^{\frac{\sigma\alpha+1}{\alpha+1}}u)^{\alpha+1} = 1 \iff (t^2(n^{\frac{\sigma\alpha+1}{\alpha+1}}u)^{\alpha+1})^{\alpha} = 1 \iff t^{2\alpha}(n^{\frac{\sigma\alpha+1}{\alpha+1}}u)^{\alpha+1} = 1 \iff t^{2\alpha} = t^2 \iff t = \pm 1$.

So, when $\alpha^2 = 1$ an extra C_2 occurs in $\mathrm{P\Gamma O}(5,q)_O$. Together with the group from the first case, we have the stated stabiliser. □

Corollary 2.2.4. *Let $\alpha, \beta \in \mathrm{Aut}(\mathrm{GF}(q))$. The ovoids $K(\alpha)$ and $K(\beta)$ are equivalent if and only if $\alpha = \beta$ or $\alpha = \beta^{-1}$.*

Proof. Suppose there exists a projective semisimilarity θ between $K(\alpha)$ and $K(\beta)$. By the first part of the proof of Theorem 2.2.3, we can assume that θ fixes $\langle(0,0,0,0,1)\rangle$ and $\langle(1,0,0,0,0)\rangle$. Firstly, suppose that θ is of the form τ_{abc}. Applying θ to $K(\alpha)$, for $x, y \in \mathrm{GF}(q)$ we have

$$\begin{aligned}&\theta(\langle(1, x, y, -nx^{\alpha}, -y^2 + nx^{\alpha+1})\rangle)\\ &= \langle(b^2, c^{-1}x^{\sigma} - 2ay^{\sigma} - a^2c(-nx^{\alpha})^{\sigma}, by^{\sigma} + abc(-nx^{\alpha})^{\sigma}, b^2c(-nx^{\alpha})^{\sigma}, -y^2 + nx^{\alpha+1})\rangle\end{aligned}$$

and since $\theta(K(\alpha)) = K(\beta)$, we have

$$-n(c^{-1}x^{\sigma} - 2ay^{\sigma} - a^2c(-nx^{\alpha})^{\sigma})^{\beta}(b^{-2})^{\beta} = c(-nx^{\alpha})^{\sigma}$$

$\forall\ x,y \in \mathrm{GF}(q)$. Setting $x=0$, we have $a=0$, and then setting $x=1$ implies that the coefficients of $x^{\sigma\beta}$ and $x^{\sigma\alpha}$ are the same. Thus, $\beta=\alpha$.

Now suppose that θ is of the form μ_{rstu}. Applying θ to $K(\alpha)$, for $x,y \in \mathrm{GF}(q)$ we have

$$\begin{aligned}
&\theta(\langle(1,x,y,-nx^\alpha,-y^2+nx^{\alpha+1})\rangle)\\
&= \langle(t^2, -s^2u^{-1}x^\sigma + 2s(rs-t)y^\sigma + u(r^2s^2-2rst+t^2)(-nx^\alpha)^\sigma, su^{-1}x^\sigma +\\
&(-2rs+t)y^\sigma + ru(-rs+t)(-nx^\alpha)^\sigma, u^{-1}x^\sigma - 2ry^\sigma - r^2u(-nx^\alpha), (-y^2+nx^{\alpha+1})^\sigma)\rangle
\end{aligned}$$

and since $\theta(K(\alpha)) = K(\beta)$, we have

$$\begin{aligned}
-n(-s^2u^{-1}x^\sigma + 2s(rs-t)y^\sigma + u(r^2s^2-2rst+t^2)(-nx^\alpha)^\sigma)^\beta(t^{-2})^\beta&\\
= u^{-1}x^\sigma - 2ry^\sigma - r^2u(-nx^\alpha)^\sigma&
\end{aligned}$$

$\forall\ x,y \in \mathrm{GF}(q)$. As in the proof of Theorem 2.2.3, putting $x=0$ we find that $r=s=0$, and then putting $x=1$ implies that the coefficients of $x^{\alpha\sigma\beta}$ and x^σ are the same, so $\alpha=\beta$ and $\alpha^2=1$, or $\alpha=\beta^{-1}$.

For the converse direction, first define the projective similarity τ via

$$\tau(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(x_1,x_4,x_3,x_2,x_5)\rangle$$

Given $x,y \in \mathrm{GF}(q)$, we have

$$\begin{aligned}
&\tau(\langle(1,x,y,-nx^\alpha,-y^2+nx^{\alpha+1})\rangle)\\
&= \langle(1,-nx^\alpha,y,x,-y^2+nx^{\alpha+1})\rangle\\
&= \langle(1,z,y,-n^{\frac{-1}{\alpha}}z^{\frac{1}{\alpha}},-y^2+n^{\frac{-1}{\alpha}}z^{1+\frac{1}{\alpha}})\rangle
\end{aligned}$$

where $z=-nx^\alpha$. Thus, τ takes $K(n,\alpha)$ to $K(n^{-\frac{1}{\alpha}},\frac{1}{\alpha})$. Define the projective isometry μ_a (for $a \in \mathrm{GF}(q)^*$) via

$$\mu_a(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(x_1,x_2,ax_3,a^2x_4,a^2x_5)\rangle$$

Clearly, μ_a maps $K(n,\alpha)$ to $K(na^2,\alpha)$, and then

$$\mu_{n^{\frac{\alpha+1}{2\alpha}}}\tau(K(n,\alpha)) \quad = \quad \mu_{n^{\frac{\alpha+1}{2\alpha}}}(K(n^{-\frac{1}{\alpha}},\frac{1}{\alpha})) \quad = \quad K(n,\frac{1}{\alpha})$$

□

We require the following lemma for the next stabiliser calculation.

Lemma 2.2.5. *The following matrices A,B,C,D represent elements of $\mathrm{P\Gamma O}(5,q)$, where*

- *A takes $\langle(1,0,0,0,0)\rangle$ to $\langle(0,0,0,0,1)\rangle$ and $\langle(0,0,0,0,1)\rangle$ to $\langle(1,1,1,1,1)\rangle$*
- *B fixes $\langle(0,0,0,0,1)\rangle$ and maps $\langle(1,0,0,0,0)\rangle$ to $\langle(1,1,1,1,1)\rangle$*
- *C fixes $\langle(1,0,0,0,0)\rangle$ and maps to $\langle(0,0,0,0,1)\rangle$ to $\langle(1,1,1,1,1)\rangle$*
- *D interchanges $\langle(0,0,0,0,1)\rangle$ and $\langle(1,0,0,0,0)\rangle$*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem 2.2.6. *The stabiliser of a Ree–Tits slice O is $E_q \rtimes (\mathrm{C}_{q-1} \rtimes \mathrm{C}_h)$.*

Proof. The group $G = \langle \eta_d, \psi_e \rangle$ fixes O and has order $q(q-1)$ ([36]), where

$$\eta_d(\langle(x_1, x_2, x_3, x_4, x_5)\rangle) = \langle(x_1, dx_2, d^{\alpha+2}x_3, d^{2\alpha+3}x_4, d^{2\alpha+4}x_5)\rangle$$

and

$$\psi_e(\langle(x_1, x_2, x_3, x_4, x_5)\rangle) = \langle(x_1, x_2, x_3 + ex_1, x_4 - e^{\alpha}x_1, x_5 + e^{\alpha}x_2 + ex_3 + e^2x_4)\rangle$$

($d \in \mathrm{GF}(q)^*$, $e \in \mathrm{GF}(q)$). The orbits of G on O are of length 1, q and $q(q-1)$, with orbit representatives $\langle(0,0,0,0,1)\rangle$, $\langle(1,0,0,0,0)\rangle$ and $\langle(1,1,1,1,1)\rangle$ respectively. Suppose (for a contradiction) that $\mathrm{P\Gamma O}(5,q)_O$ has fewer than three orbits on O. Then O consists either of one orbit (Case (A)), or two orbits:

Case (B): one of which is $O_1 = \{\langle(0,0,0,0,1)\rangle\}$
Case (C): one of which is $O_q = \{\langle(1,0,y,-y^{\alpha},-y^2)\rangle : y \in \mathrm{GF}(q)\}$
Case (D): one of which is $O_{q(q-1)} = \{\langle(1,x,y,-x^{2\alpha+3}-y^{\alpha},-y^2+x^{2\alpha+4}+xy^{\alpha})\rangle : x \neq 0,\ y \in \mathrm{GF}(q)\}$

In Case (D), $\exists\ g \in \mathrm{P\Gamma O}(5,q)_O$ interchanging $\langle(0,0,0,0,1)\rangle$ and $\langle(1,0,0,0,0)\rangle$. In Case (C), $\exists\ g \in \mathrm{P\Gamma O}(5,q)_O$ mapping $\langle(0,0,0,0,1)\rangle$ to $\langle(1,1,1,1,1)\rangle$. As a result, $\mathrm{P\Gamma O}(5,q)_{O,\langle(1,0,0,0,0)\rangle}$ has index q in $\mathrm{P\Gamma O}(5,q)_O$, so that $|O_1 \cup O_{q(q-1)}|$ is coprime to $|\,\mathrm{P\Gamma O}(5,q)_O : \mathrm{P\Gamma O}(5,q)_{O,\langle(1,0,0,0,0)\rangle}|$. Thus, by Theorem 1.5.1.1, $\mathrm{P\Gamma O}(5,q)_{O,\langle(1,0,0,0,0)\rangle}$

is transitive on $O_1 \cup O_{q(q-1)}$, so $\exists\, g \in \mathrm{P\Gamma O}(5,q)_O$ with $g(\langle(1,0,0,0,0)\rangle) = \langle(1,0,0,0,0)\rangle$ and $g(\langle(0,0,0,0,1)\rangle) = \langle(1,1,1,1,1)\rangle$.

In Case (B), $\exists\, g \in \mathrm{P\Gamma O}(5,q)_O$ fixing $\langle(0,0,0,0,1)\rangle$ and mapping $\langle(1,0,0,0,0)\rangle$ to $\langle(1,1,1,1,1)\rangle$. In Case (A), either $\exists\, g \in \mathrm{P\Gamma O}(5,q)_O$ mapping $\langle(1,0,0,0,0)\rangle$ to $\langle(0,0,0,0,1)\rangle$ and $\langle(0,0,0,0,1)\rangle$ to $\langle(1,1,1,1,1)\rangle$, or $\exists\, g \in \mathrm{P\Gamma O}(5,q)_O$ interchanging $\langle(1,0,0,0,0)\rangle$ and $\langle(0,0,0,0,1)\rangle$ (so that we are back in Case (D)).

As usual, let θ denote a projective semisimilarity with companion automorphism σ. In Cases (A)–(D), $M^{-1}\theta \in \mathrm{P\Gamma O}(5,q)_{\langle(1,0,0,0,0)\rangle,\langle(0,0,0,0,1)\rangle}$, where M is respectively A, B, C or D (using Lemma 2.2.5). Thus, by Lemma 2.2.1, $\theta = M\nu$, where ν is of the form τ_{abc} or μ_{rstu}. Note that θ preserves O if and only if

$$\begin{aligned}
&\iff -(x_2'/x_1')^{2\alpha+3} - (x_3'/x_1')^{\alpha} = (x_4'/x_1') \\
&\iff -(x_2')^{2\alpha+3} - (x_1')^{\alpha+3}(x_3')^{\alpha} = (x_1')^{2\alpha+2}x_4'
\end{aligned}$$

$\forall\ \langle(x_1', x_2', x_3', x_4', x_5')\rangle \in \theta(O - \{\langle(0,0,0,0,1)\rangle\})$.

In the four cases below, the strategy will be to obtain a polynomial equation in one variable in which all powers of that variable are different, and then to apply Lemma 1.5.1.3 to deduce that the coefficients of the polynomial are all 0, obtaining a contradiction.

Case (A): If θ is of the form $A\tau_{abc}$, then

$$\begin{aligned}
\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(&x_5^\sigma, b^2cx_4^\sigma + x_5^\sigma, bx_3^\sigma + abcx_4^\sigma + x_5^\sigma, c^{-1}x_2^\sigma - 2ax_3^\sigma, \\
&- a^2cx_4^\sigma + x_5^\sigma, b^2x_1^\sigma - c^{-1}x_2^\sigma + (2a+b)x_3^\sigma + c(a^2+ab-b^2)x_4^\sigma + x_5^\sigma)\rangle
\end{aligned}$$

and if θ fixes O, then

$$\begin{aligned}
&-(b^2c(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^{2\alpha+3} - ((x^{2\alpha+4})^\sigma)^{\alpha+3}(abc(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^\alpha \\
&\qquad = ((x^{2\alpha+4})^\sigma)^{2\alpha+2}(c^{-1}x^\sigma - a^2c(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)
\end{aligned}$$

$\forall\ x \in \mathrm{GF}(q)$. From the first term we obtain $(b^2c)^{2\alpha}x^{\sigma((2\alpha+3)2\alpha+(2\alpha+4)3)}$, which must have coefficient 0, and so a contradiction results.

If θ is of the form $A\mu_{rstu}$, then

$$\begin{aligned}
\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) &= \langle(x_5^\sigma, u^{-1}x_2^\sigma - 2rx_3^\sigma - r^2ux_4^\sigma + x_5^\sigma, su^{-1}x_2^\sigma + (-2rs+t)x_3^\sigma \\
&+ ru(-rs+t)x_4^\sigma + x_5^\sigma, -s^2u^{-1}x_2^\sigma + 2s(rs-t)x_3^\sigma + u(r^2s^2 - 2rst + t^2)x_4^\sigma + x_5^\sigma, \\
t^2x_1^\sigma + u^{-1}(s^2+s-1)x_2^\sigma &- (2s(rs-t) + (-2rs+t) + 2r)x_3^\sigma - (u(r^2s^2 - 2rst + t^2) \\
&+ ru(-rs+t) + r^2u)x_4^\sigma + x_5^\sigma)\rangle
\end{aligned}$$

and if θ stabilises O, then

$$\begin{aligned}
&-(-2ry^\sigma - r^2u(-y^\alpha)^\sigma + (-y^2)^\sigma)^{2\alpha+3} \\
&\qquad - ((-y^2)^\sigma)^{\alpha+3}((-2rs+t)y^\sigma + ru(-rs+t)(-y^\alpha)^\sigma + (-y^2)^\sigma)^\alpha \\
&\qquad\qquad = ((-y^2)^\sigma)^{2\alpha+2}(2s(rs-t)y^\sigma + u(r^2s^2 - 2rst + t^2)(-y^\alpha)^\sigma + (-y^2)^\sigma)
\end{aligned}$$

$\forall\ y \in \mathrm{GF}(q)$. The coefficient of $y^{\sigma(2\alpha+3)}$ must be 0, so $r = 0$, while $(-2rs+t)^\alpha$ (the coefficient of $y^{\sigma(2(\alpha+3)+\alpha)}$) is 0, so $t^\alpha = 0$ and we have a contradiction.

Case (B): If θ is of the form $B\tau_{abc}$, then

$$\begin{aligned}
\theta(\langle(x_1, x_2, x_3, x_4, x_5)\rangle) = \langle(&b^2x_1^\sigma, b^2x_1^\sigma + c^{-1}x_2^\sigma - 2ax_3^\sigma - a^2cx_4^\sigma, b^2x_1^\sigma + bx_3^\sigma + abcx_4^\sigma, \\
&b^2x_1^\sigma + b^2cx_4^\sigma, b^2x_1^\sigma - c^{-1}x_2^\sigma + (2a+b)x_3^\sigma + c(a^2+ab-b^2)x_4^\sigma + x_5^\sigma)\rangle
\end{aligned}$$

and if θ preserves O, then

$$\begin{aligned}
-(b^2 + c^{-1}x^\sigma - a^2c(-x^{2\alpha+3})^\sigma)^{2\alpha+3} - (b^2)^{\alpha+3}(b^2 + abc(-x^{2\alpha+3})^\sigma)^\alpha \\
= (b^2)^{2\alpha+2}(b^2 + b^2c(-x^{2\alpha+3})^\sigma)
\end{aligned}$$

$\forall\ x \in \mathrm{GF}(q)$. From the first term arises $c^{-2\alpha}b^6x^{2\sigma\alpha}$, which must have coefficient 0, a contradiction.

If θ is of the form $B\mu_{rstu}$, then

$$\begin{aligned}
\theta(\langle(x_1, x_2, x_3, x_4, x_5)\rangle) = \langle(&t^2x_1^\sigma, t^2x_1^\sigma - s^2ux_2^\sigma + 2s(rs-t)x_3^\sigma + u(r^2s^2 - 2rst + t^2)x_4^\sigma, \\
&t^2x_1^\sigma + stu^{-1}x_2^\sigma + (-2rs+t)x_3^\sigma + ru(-rs+t)x_4^\sigma, t^2x_1^\sigma + u^{-1}x_2^\sigma - 2rx_3^\sigma - r^2ux_4^\sigma, \\
&t^2x_1^\sigma + u^{-1}(s^2+s-1)x_2^\sigma - (2s(rs-t) + (-2rs+t) + 2r)x_3^\sigma - (u(r^2s^2 - 2rst + t^2) + \\
&ru(-rs+t) + r^2u)x_4^\sigma + x_5^\sigma)\rangle
\end{aligned}$$

and if θ fixes O, then

$$\begin{aligned}
&-(t^2 - s^2u^{-1}x^\sigma + u(r^2s^2 - 2rst + t^2)(-x^{2\alpha+3})^\sigma)^{2\alpha+3} \\
&\qquad - (t^2)^{\alpha+3}(t^2 + stu^{-1}x^\sigma + ru(-rs+t)(-x^{2\alpha+3})^\sigma)^\alpha \\
&\qquad\qquad = (t^2)^{2\alpha+2}(t^2 + u^{-1}x^\sigma - r^2u(-x^{2\alpha+3})^\sigma)
\end{aligned}$$

$\forall\ x \in \mathrm{GF}(q)$. Note that the coefficient of x^σ must be 0, so $t^{2(2\alpha+2)}u^{-1} = 0$, a contradiction.

Case (C): If θ is of the form $C\tau_{abc}$, then

$$\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(b^2x_1^\sigma - c^{-1}x_2^\sigma + (2a+b)x_3^\sigma + c(a^2+bc-b^2)x_4^\sigma + x_5^\sigma, c^{-1}x_2^\sigma - 2ax_3^\sigma - a^2cx_4^\sigma + x_5^\sigma, bx_3^\sigma + abcx_4^\sigma + x_5^\sigma, b^2cx_4^\sigma + x_5^\sigma, x_5^\sigma)\rangle$$

and if O is stabilised by θ, then

$$-(c^{-1}x^\sigma - a^2c(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^{2\alpha+3} - (b^2 - c^{-1}x^\sigma + c(a^2+bc-b^2)(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^{\alpha+3}(abc(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^\alpha =$$
$$(b^2 - c^{-1}x^\sigma + c(a^2+bc-b^2)(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^{2\alpha+2}(b^2c(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)$$

$\forall\ x \in \mathrm{GF}(q)$. Now $-c^{-2\alpha+3}x^{\sigma(2\alpha+3)}$ arises in the first term, and its coefficient must be 0, a contradiction.

If θ is of the form $C\mu_{rstu}$, then

$$\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(t^2x_1^\sigma + u^{-1}(s^2+s-1)x_2^\sigma + (-2s(rs-t)+(-2rs+t)+2r) x_3^\sigma + u(-(r^2s^2-2rst+t^2)+r(-rs+t)+r^2)x_4^\sigma + x_5^\sigma, -s^2u^{-1}x_2^\sigma + 2s(rs-t)x_3^\sigma + u(r^2s^2-2rst+t^2)x_4^\sigma + x_5^\sigma, su^{-1}x_2^\sigma + (-2rs+t)x_3^\sigma + ru(-rs+t)x_4^\sigma + x_5^\sigma, u^{-1}x_2^\sigma - 2rx_3^\sigma - r^2ux_4^\sigma + x_5^\sigma, x_5^\sigma)\rangle$$

and if θ fixes O, then

$$\begin{aligned}
&-(-s^2u^{-1}x^\sigma + u(r^2s^2-2rst+t^2)(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^{2\alpha+3}\\
&-(t^2 + u^{-1}(s^2+s-1)x^\sigma + u(-(r^2s^2-2rst+t^2)+r(-rs+t)+r^2)(-x^{2\alpha+3})^\sigma\\
&\quad + (x^{2\alpha+4})^\sigma)^{\alpha+3}(su^{-1}x^\sigma + ru(-rs+t)(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)^\alpha\\
&= (t^2 + u^{-1}(s^2+s-1)x^\sigma + u(-(r^2s^2-2rst+t^2)+r(-rs+t)+r^2)(-x^{2\alpha+3})^\sigma\\
&\quad + ((x^{2\alpha+4})^\sigma)^{2\alpha+2}(u^{-1}x^\sigma - r^2u(-x^{2\alpha+3})^\sigma + (x^{2\alpha+4})^\sigma)
\end{aligned}$$

$\forall\ x \in \mathrm{GF}(q)$. Note that the coefficient of x^σ is $t^{2(2\alpha+2)}u^{-1} = 0$, a contradiction.

Case (D): If θ is of the form $D\tau_{abc}$, then

$$\theta(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle(x_5^\sigma, b^2cx_4^\sigma, bx_3^\sigma + abcx_4^\sigma, c^{-1}x_2^\sigma - 2ax_3^\sigma - a^2cx_4^\sigma, b^2x_1^\sigma)\rangle$$

so θ preserves O if and only if

$$\begin{aligned}
&-(b^2c(-x^{2\alpha+3}-y^\alpha)^\sigma)^{2\alpha+3} - ((-y^2+x^{2\alpha+4}+xy^\alpha)^\sigma)^{\alpha+3}(by^\sigma + abc(-x^{2\alpha+3}-y^\alpha)^\sigma)^\alpha\\
&= ((-y^2+x^{2\alpha+4}+xy^\alpha)^\sigma)^{2\alpha+2}(c^{-1}x^\sigma - 2ay^\sigma - a^2c(-x^{2\alpha+3}-y^\alpha)^\sigma) \qquad (2.2.3)
\end{aligned}$$

$\forall\ x, y \in \mathrm{GF}(q)$. Set $x = 0$ to obtain $a = 0$, so that (with $x = 0$) (2.2.3) becomes

$$-(b^2c(-y^\alpha)^\sigma)^{2\alpha+3} - ((-y^2)^\sigma)^{\alpha+3}(by^\sigma)^\alpha = 0$$

$\forall\ y \in \mathrm{GF}(q)$, and for $y = 1$ we find that $b^\alpha = -(b^2c)^{2\alpha+3}$. Now set $y = 0$ in (2.2.3), so that

$$-(b^2c(-x^{2\alpha+3})^\sigma)^{2\alpha+3} = ((x^{2\alpha+4})^\sigma)^{2\alpha+2}(c^{-1}x^\sigma)$$

$\forall\ x \in \mathrm{GF}(q)$, and when $x = 1$ we have $c^{-1} = (b^2c)^{2\alpha+3}$. Therefore, $c = -b^{-\alpha}$. Substitute $a = 0$ and $c = -b^{-\alpha}$ into (2.2.3) with $x = y = 1$ to obtain $(b^{2-\alpha}(-2)^\sigma)^{2\alpha+3} - b^\alpha = -b^\alpha$, so $(b^{2-\alpha}(-2)^\sigma)^{2\alpha+3} = 0$ and we have a contradiction.

If θ is of the form $D\mu_{rstu}$, then

$$\begin{aligned}\theta(\langle(x_1, x_2, x_3, x_4, x_5)\rangle) = \langle(&x_5^\sigma, u^{-1}x_2^\sigma - 2rx_3^\sigma - r^2ux_4^\sigma, su^{-1}x_2^\sigma + (-2rs + t)x_3^\sigma + \\ &ru(-rs + t)x_4^\sigma, -s^2u^{-1}x_2^\sigma + 2s(rs - t)x_3^\sigma + u(r^2s^2 - 2rst + t^2)x_4^\sigma, t^2x_1^\sigma)\rangle\end{aligned}$$

so that if θ fixes O, then

$$\begin{aligned}&-(u^{-1}x^\sigma - r^2u(-x^{2\alpha+3})^\sigma)^{2\alpha+3} - ((x^{2\alpha+4})^\sigma)^{\alpha+3}(su^{-1}x^\sigma + ru(-rs+t)(-x^{2\alpha+3})^\sigma)^\alpha \\ &\qquad = ((x^{2\alpha+4})^\sigma)^{2\alpha+2}(-s^2u^{-1}x^\sigma + u(r^2s^2 - 2rst + t^2)(-x^{2\alpha+3})^\sigma)\end{aligned}$$

$\forall\ x \in \mathrm{GF}(q)$. The coefficient of $x^{\sigma(2\alpha+3)}$ is 0, so $-(u^{-1})^{2\alpha+3} = 0$, a contradiction.

We have shown that any projective semisimilarity fixing O has orbits O_1, O_q and $O_{q(q-1)}$ on O. In particular, it will fix $\langle(0, 0, 0, 0, 1)\rangle$ and (without loss of generality) $\langle(1, 0, 0, 0, 0)\rangle$. So, let $\theta \in \mathrm{P\Gamma O}(5, q)$ fix $\langle(0, 0, 0, 0, 1)\rangle$ and $\langle(1, 0, 0, 0, 0)\rangle$. We show that if $\theta \in \mathrm{P\Gamma O}(5, q)_O$, then $\theta \in G \rtimes \mathrm{Aut}(\mathrm{GF}(q))$.

If θ is of the form τ_{abc}, then θ fixes O if and only if

$$\begin{aligned}&-(c^{-1}x^\sigma - 2ay^\sigma - a^2c(-x^{2\alpha+3} - y^\alpha)^\sigma)^{2\alpha+3} \\ &\qquad - (b^2)^{\alpha+3}(by^\sigma + abc(-x^{2\alpha+3} - y^\alpha)^\sigma)^\alpha = (b^2)^{2\alpha+2}b^2c(-x^{2\alpha+3} - y^\alpha)^\sigma\end{aligned}$$

$\forall\ x, y \in \mathrm{GF}(q)$. The coefficient of $y^{\sigma(2\alpha+3)}$ must be 0, so $a = 0$, and when $x = 0$ we have $-b^{3(\alpha+2)}y^{\sigma\alpha} = -b^{2(2\alpha+3)}cy^{\sigma\alpha}\ \ \forall\ y \in \mathrm{GF}(q)$. Hence, $c = b^{-\alpha}$, and note that θ preserves O if and only if $a = 0$ and $c = b^{-\alpha}$. In particular, if $\theta \in \mathrm{P\Gamma O}(5, q)_O$, then θ is of the form η_d and so $\theta \in G \rtimes \mathrm{Aut}(\mathrm{GF}(q))$.

If θ is of the form μ_{rstu} and stabilises O, then

$$\begin{aligned}&-(-s^2u^{-1}x^\sigma + u(r^2s^2 - 2rst + t^2)(-x^{2\alpha+3})^\sigma)^{2\alpha+3}\\ &-(t^2)^{\alpha+3}(su^{-1}x^\sigma + ru(-rs+t)(-x^{2\alpha+3})^\sigma)^\alpha = (t^2)^{2\alpha+2}(u^{-1}x^\sigma - r^2u(-x^{2\alpha+3})^\sigma)\end{aligned}$$

$\forall\, x \in \mathrm{GF}(q)$. The coefficient of x^σ (which is $t^{2(2\alpha+2)}u^{-1}$) is 0, giving a contradiction.

Thus, $G \rtimes \mathrm{Aut}(\mathrm{GF}(q))$ is the stabiliser of O. □

In [78] it is a private communication that the Thas–Payne ovoids are distinct from the Kantor and Ree–Tits slice ovoids. Because of Theorem 2.2.3 and Theorem 2.2.6 we can see this simply by comparing the orders of the corresponding stabilisers.

2.3 The new ovoid

For the proof of the next theorem, we need the following

Lemma 2.3.1. *The equation $x^6 - x = \epsilon$ has no solutions in $\mathrm{GF}(3^5)$ for ϵ an 11th root of unity.*

Proof. By computer. □

Theorem 2.3.2.

$$O = \{\langle(0,0,0,0,1)\rangle\} \cup \{\langle(1,x,y,-x^9-y^{81},-y^2+x^{10}+xy^{81})\rangle : x,y \in \mathrm{GF}(3^5)\}$$

is an ovoid of $\mathrm{O}(5,3^5)$.

Proof. First, note that $f_Q(v,(0,0,0,0,1)) = 1$ for any $v = (1,x_1,y_1,-x_1^9-y_1^{81},-y_1^2+x_1^{10}+x_1y_1^{81})$ $(x_1,y_1 \in \mathrm{GF}(q))$. Letting $w = (1,x_2,y_2,-x_2^9-y_2^{81},-y_2^2+x_2^{10}+x_2y_2^{81})$ $(x_2,y_2 \in \mathrm{GF}(q))$ be distinct from v, we have

$$\begin{aligned}f_Q(v,w) &= Q(v+w)\\ &= Q((2,x_1+x_2,y_1+y_2,-x_1^9-x_2^9-y_1^{81}-y_2^{81},-y_1^2-y_2^2+x_1^{10}+x_2^{10}\\ &\quad +x_1y_1^{81}+x_2y_2^{81}))\\ &= (y_1+y_2)^2+(x_1+x_2)(-x_1^9-x_2^9-y_1^{81}-y_2^{81})\\ &\quad +2(-y_1^2-y_2^2+x_1^{10}+x_2^{10}+x_1y_1^{81}+x_2y_2^{81})\\ &= -y_1^2-y_2^2+2y_1y_2+x_1y_1^{81}+x_2y_2^{81}-x_1y_2^{81}-x_2y_1^{81}+x_1^{10}+x_2^{10}\\ &\quad -x_1x_2^9-x_1^9x_2\\ &= -(y_1-y_2)^2+(x_1-x_2)(y_1-y_2)^{81}+(x_1-x_2)(x_1-x_2)^9\\ &= -Y^2+XY^{81}+X^{10}\end{aligned}$$

where $X = x_1 - x_2$ and $Y = y_1 - y_2$. If one of X and Y is 0, then $f_Q(v,w) \neq 0$, so suppose both are non–zero. Put $Y = X^{49}Z$ to make

$$f_Q(v,w) = X^{98}(-Z^2 + Z^{81}) + X^{10}$$

which equals 0 if and only if $-Z^2 + Z^{81} = -X^{154}$. Taking $Z = W^3$ and noting that X^{154} is an 11th root of unity, we see that $W^6 - W = X^{154}$ has no solutions (by Lemma 2.3.1), so $f_Q(v,w) \neq 0$. □

2.3.1 Calculation of the new ovoid's stabiliser.

Theorem 2.3.1.1. *The stabiliser of O is $\mathrm{E}_{3^{10}} \rtimes (\mathrm{C}_{22} \rtimes \mathrm{C}_5)$, so O is new.*

Proof. Let $G = \{\phi_{d,e} : d, e \in \mathrm{GF}(q)\}$, where

$$\phi_{d,e}(\langle(x_1,x_2,x_3,x_4,x_5)\rangle) = \langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \\ e & 0 & 1 & 0 & 0 \\ -d^9 - e^{81} & 0 & 0 & 1 & 0 \\ -e^2 + d^{10} + de^{81} & d^9 + e^{81} & -2e & -d & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \rangle$$

so G is transitive on $O - \{\langle(0,0,0,0,1)\rangle\}$. Hence, (as in Theorem 2.2.3) we calculate $\mathrm{P\Gamma O}(5,3^5)_{O,\langle(1,0,0,0,0)\rangle,\langle(0,0,0,0,1)\rangle}$. So, let θ (with companion automorphism σ) be an element of $\mathrm{P\Gamma O}(5,3^5)_O$, and note that θ fixes O

$$\begin{aligned} &\iff -(x_2'/x_1')^9 - (x_3'/x_1')^{81} = x_4'/x_1' \\ &\iff -(x_2')^9 - (x_1')^{-72}(x_3')^{81} = (x_1')^8 x_4' \end{aligned}$$

$\forall\ \langle(x_1', x_2', x_3', x_4', x_5')\rangle \in \theta(O - \{\langle(0,0,0,0,1)\rangle\})$.

If θ is of the form τ_{abc}, then θ preserves O if and only if

$$\begin{aligned} -(c^{-1}x^\sigma - 2ay^\sigma - a^2c(-x^9 - y^{81})^\sigma)^9 - (b^2)^{-72}(by^\sigma + abc(-x^9 - y^{81})^\sigma)^{81} \\ = (b^2)^8 b^2 c(-x^9 - y^{81})^\sigma \end{aligned} \tag{2.3.1.1}$$

$\forall\ x, y \in \mathrm{GF}(q)$. When $x = 0$, this condition is

$$(-b^{-63} + b^{18}c)y^{81\sigma} + ((ac)^{81}b^{-63})y^{27\sigma} + (2a)^9 y^{9\sigma} - (a^2c)^9 y^{3\sigma} = 0$$

$\forall\ y \in \mathrm{GF}(q)$, so $a = 0$. Then from (2.3.1.1) we have

$$-c^{-9}x^{9\sigma} - b^{-63}y^{81\sigma} = b^{18}c(-x^9 - y^{81})^\sigma \tag{2.3.1.2}$$

$\forall\, x, y \in \mathrm{GF}(q)$. Putting $y = 0$ in (2.3.1.2), we find $-c^{-9} = -b^{18}c$, that is, $b^{-18} = c^{10}$. Also, letting $x = 0$ in (2.3.1.2) yields $b^{-81} = c$. Therefore, $b^{-18} = b^{-810}$, so that $b^{66} = 1$ and hence $b^{22} = 1$. Whenever $a = 0$, $b^{22} = 1$, and $c = b^{-81}$ we see that (2.3.1.1) holds, so we have shown that a C_{22} occurs in $\mathrm{P\Gamma O}(5, 3^5)_O$.

If θ is of the form μ_{rstu} and θ preserves O, then

$$\begin{aligned}- (2s(rs - t)y^\sigma + u(r^2s^2 - 2rst + t^2)(-y^{81})^\sigma)^9 \\ - (t^2)^{-72}((-2rs + t)y^\sigma + ru(-rs + t)(-y^{81})^\sigma)^{81} \\ = (t^2)^8(-2ry^\sigma - r^2u(-y^{81})^\sigma)\end{aligned}$$

$\forall\, y \in \mathrm{GF}(q)$. As all coefficients are 0, we have $r = 0$ and $(u(r^2s^2 - 2rst + t^2))^9 = 0$, so $(ut^2)^9 = 0$, a contradiction.

Thus, the C_{22} occuring in the first case, G and $\mathrm{Aut}(\mathrm{GF}(3^5)) = C_5$ together comprise the stabiliser of O. Because $|\,\mathrm{P\Gamma O}(5, 3^5)_O|$ is different to the orders of the groups of the known $\mathrm{O}(5, 3^5)$ ovoids, O is new. □

Remark. In our searches for $\mathrm{O}(5, q)$ ovoids, we ran over $f(x, y) = f_1(x) + f_2(y)$ with f_1 and f_2 additive and/or multiplicative. Most searches were run in characteristic 3, and the only new ovoid found was the one in $\mathrm{O}(5, 3^5)$.

2.4 Stabilisers of $\mathrm{Sp}(4, q)$ spreads

It is a theorem of Kantor's ([35]) that if the translation planes corresponding to two $\mathrm{Sp}(2n, q)$ spreads are isomorphic (q even), then there is a semisimilarity taking one spread to the other. When $n = 2$, the Klein correspondence enables the extension of this result to q odd. The resulting corollary enables us (in particular) to determine the stabiliser of the $\mathrm{Sp}(4, q)$ spread arising from the new ovoid.

Theorem 2.4.1. *Let (V_1, f_1) and (V_2, f_2) be $\mathrm{Sp}(4, q)$ spaces, containing spreads S_1 and S_2 respectively. If the associated translation planes $A(S_1)$ and $A(S_2)$ are isomorphic, then there exists a semilinear map $g : V_1 \to V_2$ such that*

(i) $g(S_1) = S_2$

(ii) $f_2(gu, gv) = c(f_1(u, v))^\kappa$ *for some* $c \in \mathrm{GF}(q)$ *and all* $u, v \in \mathrm{GF}(q)$ *(where* κ *is the companion automorphism of* g*).*

Proof. Embed each PV_i as a $\mathrm{PG}(3,q)$ inside $\mathrm{PG}(4,q)$, so that S_1 and S_2 give rise (via the Bruck–Bose construction) to the planes $A(S_1)$ and $A(S_2)$ respectively. If $A(S_1)$ and $A(S_2)$ are desarguesian, then S_1 and S_2 are regular spreads (Theorem 1.3.5.1), in which case the required map exists. So suppose $A(S_1)$ and $A(S_2)$ are non–desarguesian. Because finite non–desarguesian translation planes have a unique translation line (Theorem 1.3.1.1), there exists a semilinear map $g : V_1 \to V_2$ such that $g(V_1) = V_2$ and $g(S_1) = S_2$. By the Klein correspondence, we can associate $\mathrm{O}^+(6,q)$ spaces (W_i, Q_i), $i = 1, 2$ to (V_i, f_i), $i = 1, 2$, so that we have a semisimilarity $\overline{g} : W_1 \to W_2$.

The spreads S_i of (V_i, f_i), $i = 1, 2$ correspond to ovoids O_i of (W_i, Q_i), $i = 1, 2$, and that neither spread corresponds to a desarguesian plane implies that $\dim\langle O_i\rangle = 5$, $i = 1, 2$. Now let $U_i = \langle O_i\rangle$, $i = 1, 2$ and $\overline{m} = \overline{g}\,|_{U_1}$. Applying the Klein correspondence, each U_i corresponds to a linear complex L_i of PV_i (see [27, pp5–6, p30]), and $\overline{m}$ is induced by a semilinear map $m : V_1 \to V_2$ having $m(L_1) = L_2$. Because m is an isomorphism from the (V_1, f_1) GQ to the (V_2, f_2) GQ, m is a projective semisimilarity (see for example [27]). □

In general, the stabiliser G of an $\mathrm{Sp}(2n,q)$ spread has $G \le \Gamma\mathrm{L}(2n,q)$. When the Kantor result or Theorem 2.4.1 holds, we have that G is a subgroup of $\Gamma\mathrm{Sp}(2n,q)Z$ (where $Z = Z(\mathrm{GL}(2n,q))$), resulting in

Corollary 2.4.2. *Let S be an $Sp(2n,q)$ spread and $A(S)$ its translation plane, for q even or $n = 2$. Then every collineation of $A(S)$ fixing 0 (0 being the origin of the quasifield coordinatising $A(S)$) can be written in the form g_1g_2, where $g_1 \in \Gamma\mathrm{Sp}(2n,q)$ and g_2 is an homology of $A(S)$.*

Proof. From [44, p5], $\mathrm{Aut}\,A(S)_0 = T \times \Gamma\mathrm{L}(2n,q)_S$, where T is the translation group of $A(S)$. Apply Kantor's theorem and Theorem 2.4.1 to yield the result. □

2.5 Objects arising from the new ovoid

2.5.1 Ovoids, spreads and planes. To the new $\mathrm{O}(5,3^5)$ ovoid corresponds a new $\mathrm{Sp}(4,3^5)$ spread, which yields a translation plane π of order 3^{10}, the automorphism group of which we calculate in the next theorem (let $A : B$ denote an extension of A by B which may or may not split).

Theorem 2.5.1.1. $\mathrm{Aut}\,\pi = E_{3^{20}} \rtimes (C_{3^5-1} : E_{3^{10}} \rtimes (C_{22} \rtimes C_5))$.

Proof. $\mathrm{Aut}\,\pi$ is a semidirect product of the translation group of π (which is $E_{3^{20}}$) with the translation complement of π (see [44, p5]). To calculate the latter group, apply Corollary 2.4.2; here the homology group of π is C_{3^5-1}, while $\mathrm{PSp}(4,q)_S \cong \mathrm{P\Gamma O}(5,q)_O$. $\square$

Applying Corollary 1.2.8.4 to the new ovoid, we obtain an infinite family of $\mathrm{O}^+(6,q)$ ovoids. By Lemma 1.2.8.2 and Lemma 1.3.3.3 this family corresponds to an infinite family of semifield planes of order $(3^5)^{2^h}$ $(h \geq 1)$. Aside from when $h = 1$, it is not clear that these planes (and the associated spreads and ovoids) are new.

By ([21]), any spread set having the form

$$\{\begin{pmatrix} v + a(t) & b(t) \\ t & v^\gamma \end{pmatrix} : v, t \in \mathrm{GF}(q), \gamma \in \mathrm{Aut}(\mathrm{GF}(q)), a, b : \mathrm{GF}(q) \to \mathrm{GF}(q)\} \quad (2.5.1.1)$$

corresponds to a derivable translation plane. The spread set corresponding to the new ovoid is (using Lemma 1.2.8.2)

$$N = \{\begin{pmatrix} y & x^9 + y^{81} \\ x & y \end{pmatrix} : \; x, y \in \mathrm{GF}(3^5)\}$$

and we can rewrite N as

$$\{\begin{pmatrix} x^9 + y^{81} & y \\ y & x \end{pmatrix} : \; x, y \in \mathrm{GF}(3^5)\}$$

which is in the form (2.5.1.1). Furthermore, given an arbitrary spread set S', note that $H(S')$ (where S' and $H(S')$ are written as in Theorem 1.2.8.3) has the form (2.5.1.1), so that all of the semifields arising from the new ovoid are derivable. By the Bose–Barlotti construction, the duals of these planes may be derived.

2.5.2 Flocks. An **oval** of a projective plane of order n is a set of $n+1$ points, no three lying on a common line. A **quadratic cone** K of $\mathrm{PG}(3,q)$ comprises the $q+1$ lines joining a point P (where $P \in \mathrm{PG}(3,q) - \mathrm{PG}(2,q)$) to the points of an oval of $\mathrm{PG}(2,q)$ (P is the **vertex** of K). When q is odd, every oval of $\mathrm{PG}(2,q)$ is a conic ([64]), and then we can write

$$K = \{\langle(x_1, x_2, x_3, x_4)\rangle : x_1 x_2 = x_3^2\}$$

with $\langle(0,0,0,1)\rangle$ the vertex of K.

A **flock** of a quadratic cone K of $\mathrm{PG}(3,q)$ is a partition of $K - \{P\}$ by q conics. Any flock F of K can be explicitly described as

$$F = F(f,g) = \{K \cap \pi_t : t \in \mathrm{GF}(q)\}$$

for functions $f, g : \mathrm{GF}(q) \to \mathrm{GF}(q)$ with $f(0) = g(0) = 0$, where each element of F is contained in some plane π_t having equation $tx_0 - f(t)x_1 + g(t)x_2 + x_3 = 0$ ([21]). We call flocks $F = \{K \cap \pi_t : t \in \mathrm{GF}(q)\}$ and $F' = \{K \cap \pi'_t : t \in \mathrm{GF}(q)\}$ **equivalent** if $g(\{\pi_t : t \in \mathrm{GF}(q)\}) = \{\pi'_t : t \in \mathrm{GF}(q)\}$ for some $g \in \mathrm{P\Gamma L}(4, q)$ that fixes K.

Let $F = F(f, g)$ be a flock of a quadratic cone K of $\mathrm{PG}(3, q)$ and embed K in $\mathrm{O}^+(6, q)$, where Q is the quadratic form on the underlying vector space. Let c_s, c_t be two elements of F, with π_s,π_t the corresponding planes. Now $\pi_s \cap \pi_t$ is a line external to K, and so is an $\mathrm{O}^-(2, q)$ space. Hence, $(\pi_s \cap \pi_t)^\perp$ is an $\mathrm{O}^-(4, q)$ space, containing the conics $c_s^\perp, c_t^\perp$. Thus,

$$O = \{\langle v \rangle \subseteq c_t^\perp : Q(v) = 0; t \in \mathrm{GF}(q)\}$$

is a cap of $\mathrm{O}^+(6, q)$. Also, $|O| = q^2 + 1$ (with all the conics $c_t^\perp$ intersecting in P), so that O is an ovoid (this construction is due independently to Walker in [83] and Thas in [20]). If the resulting translation plane is a semifield plane, F is called a **semifield flock**. If f and g are additive functions, F is a semifield flock, and conversely ([21]).

Suppose K is a quadratic cone of $\mathrm{PG}(3, q)$. The canonical examples of flocks of K are the **linear** flocks, existing for all q; a flock is such if the planes corresponding to its elements all meet in a common line. While flocks are not uncommon (see [58] for a survey of the known examples), semifield flocks are still rare. To see that linear flocks are semifield ones, first note that in the construction above, the ovoid resulting from a linear flock is an elliptic quadric (for if all the planes π_t meet in $\mathrm{O}^-(2, q)$, then all the conics $c_t^\perp$ lie in $\mathrm{O}^-(4, q)$), and such ovoids are translation ovoids. For q even, all semifield flocks are linear ([33]), while for q odd there are only two known classes of non–linear semifield flocks. The Kantor flocks occur for $q = p^h$, p an odd prime and $h > 1$, and have $f(t) = -nt^\alpha$ and $g(t) = 0$ for n a non–square of $\mathrm{GF}(q)$ and $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ with $\alpha \neq 1$. The Ganley flocks exist for $q = 3^h$ $(h > 2)$, and have $f(t) = n^{-1}t^9 + nt$ and $g(t) = -t^3$ for n a non–square of $\mathrm{GF}(q)$.

For q odd, there is a construction of Thas ([76] and [77]) which, given a semifield flock F, yields an ovoid $O(F)$ of $\mathrm{O}(5, q)$; in [43] it was shown that $O(F)$ is a translation ovoid. In the other direction, given any translation ovoid O of $\mathrm{O}(5, q)$, there exists a semifield flock F such that O is equivalent to $O(F)$ ([43]). The Kantor $\mathrm{O}(5, q)$ ovoids give rise in this way to the Kantor semifield flocks, while the Thas–Payne ovoids yield the Ganley flocks ([76] and [77]). Also, since flocks F_1 and F_2 are equivalent if and only if $O(F_1)$ and $O(F_2)$ are equivalent ([43]), the flock corresponding to the new $\mathrm{O}(5, 3^5)$ ovoid is new ([1]); explicitly, this flock has $f(t) = 2t^9$

and $g(t) = t^{27}$ ([1]). The semifield plane that this flock yields (via the construction described previously) is new, being distinct from the one arising directly from the new ovoid ([1]).

2.5.3 Translation GQs and eggs. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a GQ of order (s,t), with $s,t \neq 1$. If a collineation g of Γ fixes each line incident with a point P, then g is a **whorl about** P. If a whorl g about P is the identity or fixes no point of $\mathcal{P} - P^\perp$, g is called an **elation about** P. A whorl about a line is defined dually to a whorl about a point; let g be a whorl about a line L. If g fixes each point of $L^\perp$, g is a **symmetry** about L. We call Γ an **elation generalised quadrangle** (EGQ for short) with **elation group** G and **base point** P if there is a group G of elations about a point P that is regular on $\mathcal{P} - P^\perp$. If Γ is an EGQ with elation group G and base point P for which G has a subgroup of s symmetries about each line through P, then Γ is a **translation generalised quadrangle** (TGQ for short) with **translation group** G and **base point** P.

There is a construction that yields a generalised quadrangle $Q(F)$ of order (q^2, q) from a flock F of a quadratic cone of $\mathrm{PG}(3,q)$, for any q (Kantor did q odd in [34] and Payne q even in [54]). If F is linear, then $Q(F)$ is classical, and conversely ([74]). Also, F is a semifield flock if and only if $Q(F)$ is the dual of a translation generalised quadrangle ([33]). In [1] it was shown that the TGQ arising in this way from the flock of the new $\mathrm{O}(5,3^5)$ ovoid is new.

An **egg** $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,s)$ is a partial $n-1$–spread of $\mathrm{PG}(2n+m-1,s)$ satisfying

(i) $|\mathcal{E}| = s^m + 1$

(ii) Each triple of $\mathcal{E}$ spans a $3n-1$–dimensional subspace of $\mathrm{PG}(2n+m-1,s)$.

(iii) For each $U \in \mathcal{E}$, there exists an $n+m-1$–dimensional subspace T_U of $\mathrm{PG}(2n+m-1,s)$ such that T_U meets no point of an element of $\mathcal{E} - \{U\}$.

For any egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,s)$, there is a construction of a TGQ $T(\mathcal{E})$ of order (s^n, s^m), while each TGQ of order (r,t) is isomorphic to a TGQ $T(\mathcal{E})$ for some egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,s)$ ([55, 8.7.1]). So, to the new TGQ corresponds an egg of $\mathrm{PG}(19,3)$, new by [1].

Let $\mathrm{PG}(4n-1,s)^*$ denote the dual space of $\mathrm{PG}(4n-1,s)$, that is, the space whose points are the hyperplanes of $\mathrm{PG}(4n-1,s)$, whose lines are the codimension 2 subspaces of $\mathrm{PG}(4n-1,s)$, etc. If $\mathcal{E}$ is an egg of $\mathrm{PG}(4n-1,s)$, the spaces T_U associated to $\mathcal{E}$ comprise an egg $\mathcal{E}^*$ of $\mathrm{PG}(4n-1,s)^*$, where $\mathcal{E}$ and $\mathcal{E}^*$ have the same parameters ([55]) (and $\mathcal{E}^*$ is also new). The resulting TGQ $T(\mathcal{E}^*)$ is called the **translation dual** of $T(\mathcal{E})$. Let $\mathcal{E}$ be the egg of $\mathrm{PG}(19,3)$ corresponding to the new TGQ; it was shown in [1] that $T(\mathcal{E}^*)$ is also a new TGQ.

Ovoids of $\mathrm{O}(7, q)$

3.1 The known ovoids

In Table 4.1.1 we give the stabilisers in $\mathrm{P\Gamma O}(7, q)$ of the known $\mathrm{O}(7, q)$ ovoids, and the values of q for which the ovoids exist.

name	stabiliser	q
Thas–Kantor ([11] and [36])	$\mathrm{PGU}(3, q) \rtimes \mathrm{C}_h$	3^h $(h > 0)$
Ree–Tits ([80])	${}^2\,\mathrm{G}_2(q) \rtimes \mathrm{C}_{2h+1}$	3^{2h+1} $(h > 0)$

Table 3.1.1: Stabilisers of the known $\mathrm{O}(7, q)$ ovoids.

3.1.1 Non–existence and classification results. There is a unique ovoid in $\mathrm{O}(7, 3)$ (see [35]), so that the Thas–Kantor and Ree–Tits families coincide there. In [50] it was proved that no $\mathrm{O}(7, q)$ ovoid with $q \not\equiv 0 \pmod 3$ can have only elliptic quadrics as its slices. Since elliptic quadrics are the only ovoids occuring in $\mathrm{O}(5, 5)$, $\mathrm{O}(5, 7)$ (see [50]) and $\mathrm{O}(5, 11)$ ([57]), there are no $\mathrm{O}(7, q)$ ovoids for $q = 5, 7, 11$. It is a result of [73] that $\mathrm{O}(7, q)$ ovoids don't exist for q even. In [50] it was conjectured that $\mathrm{O}(7, q)$ ovoids don't occur for $q \not\equiv 0 \pmod 3$. Even more so than in $\mathrm{O}(5, q)$, ovoids are currently rare.

3.1.2 A model of $\mathrm{O}(7, q)$. The model for $\mathrm{O}(7, q)$ that we shall use is analogous to the one we used for $\mathrm{O}(5, q)$, namely, $\mathrm{GF}(q)^7$ equipped with the quadratic form Q defined by

$$Q((x_1, x_2, x_3, x_4, x_5, x_6, x_7)) = x_1x_7 + x_2x_6 + x_3x_5 + x_4^2$$

Analogously to $\mathrm{O}(5, q)$, any ovoid of $\mathrm{O}(7, q)$ is equivalent to one of the form

$$\begin{aligned} O(f_1, f_2) = \{&\langle(0, 0, 0, 0, 0, 0, 1)\rangle\} \cup \\ &\{\langle(1, x, y, z, f_1(x, y, z), f_2(x, y, z), -z^2 - yf_1(x, y, z) - xf_2(x, y, z))\rangle : x, y, z\} \end{aligned}$$

where $f_1, f_2 : \mathrm{GF}(q)^3 \to \mathrm{GF}(q)$.

Before giving the known ovoids explicitly, we will describe how they were constructed, first introducing the incidence structures in which this was done.

3.1.3 Generalised hexagons. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure. Define an associated graph G_Γ whose vertex set is $\mathcal{P} \cup \mathcal{L}$, and where adjacency is defined via $\mathcal{I}$. Note that no two points of G_Γ are adjacent and no two lines of G_Γ are adjacent, so that the distance $d(P_1, P_2)$ between any two points P_1,P_2 is even (and the same for lines).

We call an incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ a **generalised hexagon of order** q if it satisfies

(H1) each point lies on $q+1$ lines, and each line contains $q+1$ points.

(H2) $|\mathcal{P}| = |\mathcal{L}| = 1 + q + q^2 + q^3 + q^4 + q^5$

(H3) G_Γ has diameter 6 and girth 12.

Given a generalised hexagon Γ of order q, an **ovoid** of Γ is a set of q^3+1 points such that any two are at maximal distance in G_Γ. A **spread** of Γ is a set of q^3+1 lines such that any two are at maximal distance in G_Γ.

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure having the set of totally singular points of $\mathrm{O}(7,q)$ as $\mathcal{P}$, with $\mathcal{L}$ consisting of those lines

$$\langle (x_1, x_2, x_3, x_4, x_5, x_6, x_7), (y_1, y_2, y_3, y_4, y_5, y_6, y_7) \rangle$$

of $\mathrm{PG}(6,q)$ whose Grassman coordinates (a tuple whose coordinates are of the form

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

as i,j range over $\mathrm{GF}(q)$) satisfy

$$\begin{aligned} p_{34} &= p_{12}, \quad & p_{35} &= p_{20}, \quad & p_{36} &= p_{01} \\ p_{03} &= p_{56}, \quad & p_{13} &= p_{64}, \quad & p_{23} &= p_{45} \end{aligned}$$

(all lines of $\mathcal{L}$ are in fact totally singular) and $\mathcal{I}$ the incidence inherited from $\mathrm{O}(7,q)$. Then Γ is called the **classical generalised hexagon of order** q, and denoted $\mathrm{H}(q)$. It is the case that $\mathrm{Aut}\,\mathrm{H}(q) \cap \mathrm{PGL}(7,q) = \mathrm{G}_2(q)$.

Lemma 3.1.3.1. *(a) Two points of $\mathrm{H}(q)$ are at distance at most 4 in $G_{\mathrm{H}(q)}$ if and only if they are collinear.*

(b) If two lines of $\mathrm{H}(q)$ meet, then the plane they span is totally singular.

Proof. (a) See [85].
(b) Let R be a point of $H(q)$ lying on the lines L_1 and L_2 of $\mathrm{H}(q)$, where P_1 and P_2 are points (distinct from R) lying on L_1 and L_2 respectively. Since $d(P_1, P_2) = 4$, we see (by (a)) that P_1 and P_2 are collinear, so the plane π spanned by L_1 and L_2 is totally singular. $\square$

Remark. In [72] it is shown (with notation as above) that the $q+1$ lines of $\mathrm{H}(q)$ through R are the $q+1$ totally singular lines of π that pass through R. It follows that given an ovoid O of $\mathrm{H}(q)$, the q^3+1 totally singular planes that arise in this way from O comprise a spread of $\mathrm{O}(7,q)$ (for if two of these planes met, then points P_1, P_2 of O would be collinear to the same point, implying (by Lemma 3.1.3.1) that $d(P_1, P_2) = 4$).

Theorem 3.1.3.2 ([73]). *An ovoid of* $\mathrm{H}(q)$ *is an ovoid of* $\mathrm{O}(7,q)$, *and conversely.*

Proof. Apply Lemma 3.1.3.1 and the fact that ovoids of $\mathrm{H}(q)$ and $\mathrm{O}(7,q)$ have the same size. $\square$

3.1.4 The Thas–Kantor ovoids. The Thas–Kantor ovoids of $\mathrm{O}(7,q)$ are due originally to Thas (see for example [72]), and his construction is as follows. Take an $\mathrm{O}^-(6,q)$ subspace of $\mathrm{O}(7,q)$, and let S consist of all lines of $\mathrm{H}(q)$ that lie in $\mathrm{O}^-(6,q)$. If two lines of S intersect, the plane they span is totally singular (by Lemma 3.1.3.1), which is not possible in $\mathrm{O}^-(6,q)$. Now (noting that a line of $\mathrm{H}(q)$ not contained in $\mathrm{O}^-(6,q)$ meets $\mathrm{O}^-(6,q)$ in a point)

$$\begin{aligned}&(\#\text{ points of }\mathrm{O}^-(6,q)\text{ on a line of }S\ .\ |S|) + (1\ .\ \#\text{ lines of }\mathrm{H}(q)\text{ not in }S)\\ &= \#\text{ singular points of }\mathrm{O}^-(6,q)\ .\ \#\text{ lines of }\mathrm{H}(q)\text{ on a singular point of }\mathrm{O}^-(6,q)\end{aligned}$$

and so

$$(q+1)|S| + (q^5+q^4+q^3+q^2+q+1-|S|) = (q^3+1)(q+1)^2$$

Thus $|S| = q^3+1$, implying that S is a spread of $\mathrm{O}^-(6,q)$. No two lines of S can be at distance 2 in $G_{\mathrm{H}(q)}$, while if two lines of S were at distance 4 in $G_{\mathrm{H}(q)}$, then there would be a line L of $\mathrm{H}(q)$ meeting both of them. But since any line of $\mathrm{H}(q)$ not in $\mathrm{O}^-(6,q)$ meets $\mathrm{O}^-(6,q)$ in a point, and L meets two points of $\mathrm{O}^-(6,q)$, then L is in $\mathrm{O}^-(6,q)$ and hence S, a contradiction. Thus, S is a spread of $\mathrm{H}(q)$.

It is result of [80] that $\mathrm{H}(q)$ is self–dual for $q = 3^h$ $(h > 0)$, and so for such q a family of $\mathrm{H}(q)$ ovoids is obtained, which (by Theorem 3.1.3.2) yields a family of $\mathrm{O}(7,q)$ ovoids. These ovoids are actually equivalent to a family of ovoids that Kantor subsequently found; to show this, we need only know about Kantor's construction that the stabiliser in $\mathrm{PGO}(7,q)$ of each ovoid is $\mathrm{PGU}(3,q)$.

Theorem 3.1.4.1 ([59]). *The Thas and Kantor ovoids of* $\mathrm{O}(7,q)$ *are equivalent.*

Proof. Let $q=3^h$ $(h>0)$. We show that the stabiliser $K\cong\mathrm{PGU}(3,q)$ of the Kantor ovoid O_K in $\mathrm{PGO}(7,q)$ ([36]) is conjugate in $\mathrm{P\Gamma O}(7,q)$ to the stabiliser of the Thas ovoid O_T in $\mathrm{PGO}(7,q)$, so that (by Lemma 1.1.7.1) the sets of ovoids they fix are setwise equivalent (and because K has two orbits on singular points of $\mathrm{O}(7,q)$ ([36]), it only stabilises one ovoid). First, let S denote a Thas spread of $\mathrm{H}(q)$ constructed via an $\mathrm{O}^-(6,q)$ hyperplane H, and note that $\mathrm{G}_2(q)_H=\mathrm{G}_2(q)_S$. Choose a duality $\triangle$ of $\mathrm{H}(q)$ such that O_T is $S^{\triangle}$. Then

$$\mathrm{G}_2(q)_{O_T}=\triangle(\mathrm{G}_2(q)_S)\triangle^{-1}=\triangle(\mathrm{G}_2(q)_H)\triangle^{-1}$$

From [38], there is a unique $\mathrm{Aut}(\mathrm{G}_2(q))$–conjugacy class of maximal subgroups isomorphic to $\mathrm{PSU}(3,q)$ (which equals $\mathrm{PGU}(3,q)$ for characteristic 3). There are two $\mathrm{G}_2(q)$ conjugacy classes of maximal subgroups isomorphic to $\mathrm{PGU}(3,q)$, according to the subgroups being irreducible or reducible. Each of the reducible subgroups arises as the stabiliser in $\mathrm{G}_2(q)$ of an $\mathrm{O}^-(6,q)$ hyperplane, so $\mathrm{G}_2(q)_H\cong\mathrm{PGU}(3,q)$, while $\triangle\in\mathrm{Aut}(\mathrm{G}_2(q))\setminus\mathrm{G}_2(q)$ conjugates $\mathrm{G}_2(q)_H$ to the irreducible copy $\mathrm{G}_2(q)_{O_T}$ of $\mathrm{PGU}(3,q)$. Now K acts irreducibly on $\mathrm{O}(7,q)$, and there is a unique conjugacy class of irreducible $\mathrm{PGU}(3,q)$ subgroups of $\mathrm{P\Gamma O}(7,q)$ (using [41, Theorem 2.2] and [67, 13.1 and 13.3]). Therefore, $\mathrm{G}_2(q)_{O_T}=\mathrm{P\Gamma O}(7,q)_{O_T}$ is conjugate to $\mathrm{P\Gamma O}(7,q)_{O_K}$. $\square$

To express the Thas–Kantor ovoids in the form $O(f_1,f_2)$, we start with the Kantor construction of these ovoids ([36]). (In what follows we will continually recycle notation, as we will be changing model a number of times.) Let

$$W=\{\begin{pmatrix}\alpha&\beta&c\\ \gamma&a&\overline{\beta}\\ b&\overline{\gamma}&\overline{\alpha}\end{pmatrix} : \alpha,\beta,\gamma\in\mathrm{GF}(q^2),a,b,c\in\mathrm{GF}(q);a+\mathrm{Tr}(\alpha)=0\}$$

with Q' defined by

$$Q'(\begin{pmatrix}\alpha&\beta&c\\ \gamma&a&\overline{\beta}\\ b&\overline{\gamma}&\overline{\alpha}\end{pmatrix})=\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2+\mathrm{Tr}(\beta\gamma)+bc$$

(here $\overline{\beta}=\beta^q$, and $\mathrm{Tr}(\alpha)=\alpha+\alpha^q$). The singular radical of W is $\langle I\rangle$, making $W/\langle I\rangle$ an $\mathrm{O}(7,q)$ space. For $q=3^h$ $(h\geq 1)$, a single orbit of $\mathrm{PGU}(3,q)$ on $W/\langle I\rangle$ comprises an ovoid

$$O = \{\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rangle\} \bigcup \{\langle \begin{pmatrix} \overline{\rho} & \overline{\rho}\sigma & \mathrm{N}(\rho) \\ \overline{\sigma} & \mathrm{N}(\sigma) & \rho\overline{\sigma} \\ 1 & \sigma & \rho \end{pmatrix} \rangle : \mathrm{Tr}(\rho) + \mathrm{N}(\sigma) = 0\}$$

where $\mathrm{N}(\sigma) = \sigma\overline{\sigma}$.

Now let $W = \{(\alpha, \beta, \gamma, b, c) : \alpha, \beta, \gamma \in \mathrm{GF}(q^2), b, c \in \mathrm{GF}(q)\}$, with Q' defined by

$$Q'((\alpha, \beta, \gamma, b, c)) = \alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2 + \mathrm{Tr}(\beta\gamma) + bc$$

Note that the singular radical of W is $\langle(1, 0, 0, 0, 0)\rangle$, so that $W/\langle(1, 0, 0, 0, 0)\rangle$ is the space where we will work (though we will write $v + \langle(1, 0, 0, 0, 0)\rangle$ as v, for $v \in W$). Now

$$O = \{\langle(0, 0, 0, 0, 1)\rangle\} \cup \{\langle(\overline{\rho}, \overline{\rho}\sigma, \overline{\sigma}, 1, \mathrm{N}(\rho))\rangle : \rho, \sigma \in \mathrm{GF}(q^2); \mathrm{Tr}(\rho) + \mathrm{N}(\sigma) = 0\}$$

Let $\omega \in \mathrm{GF}(q^2)$ satisfy $\omega^2 = n$ (n a non–square of $\mathrm{GF}(q)$). Then $\{1, \omega\}$ is a basis for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$, implying that $\rho = v + z\omega$ and $\sigma = x + y\omega$ for $v, z, x, y \in \mathrm{GF}(q)$ (note that in the following versions of O, for space reasons it is implicit that x, y, v, z are running over $\mathrm{GF}(q)$ subject to the constraint $\mathrm{Tr}(v+z\omega)+\mathrm{N}(x+y\omega) = 0$). Since $\omega^q = -\omega$, we have

$$O = \{\langle(0, 0, 0, 0, 1)\rangle\} \cup \{\langle(v - z\omega, (v - z\omega)(x + y\omega), x - y\omega, 1, (v + z\omega)(v - z\omega))\rangle\}$$

$$= \{\langle(0, 0, 0, 0, 1)\rangle\} \cup \{\langle(v - z\omega, vx - nyz + (vy - xz)\omega, x - y\omega, 1, v^2 - nz^2)\rangle\}$$

Expand W into 8–tuples by putting the $\mathrm{GF}(q)$ components of each $\mathrm{GF}(q^2)$ element into separate coordinates, so that

$$O = \{\langle(0, 0, 0, 0, 0, 0, 0, 1)\rangle\} \cup \{\langle(v, -z, vx - nyz, vy - xz, x, -y, 1, v^2 - nz^2)\rangle\}$$

on which we have Q' defined by $Q'((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8))$

$$\begin{aligned} &= (x_1 + x_2\omega)^2 + \mathrm{N}((x_1 + x_2\omega)) + (x_1 - x_2\omega)^2 + \mathrm{Tr}((x_3 + x_4\omega)(x_5 + x_6\omega)) \\ &\quad + x_7x_8 \\ &= 3x_1^2 + nx_2^2 + 2x_3x_5 + 2nx_4x_6 + x_7x_8 \end{aligned}$$

If we convert Q' to the form

$$Q'((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)) = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$$

O becomes (upon rearranging)

$$O = \{\langle(0,0,0,0,0,0,0,1)\rangle\} \cup \{\langle(1,x,y,z,nz,2n(-vy+xz),2(vx-nyz),v^2-nz^2)\rangle\}$$

The condition $\mathrm{Tr}(v+z\omega)+\mathrm{N}(x+y\omega)=0$ means that $v=\frac{(-x^2+ny^2)}{2}$, so

$$O = \{\langle(0,0,0,0,0,0,0,1)\rangle\} \cup \{\langle(1,x,y,z,nz,n(x^2y-ny^3+2xz),-x^3+nxy^2-2nyz,x^4+n^2y^4-2nx^2y^2-nz^2)\rangle\}$$

Scaling the last four coordinates of O above by n^{-1}, and switching to our usual model for $\mathrm{O}(7,q)$, we obtain

$$O = \{\langle(0,0,0,0,0,0,1)\rangle\} \cup \{\langle(1,x,y,z,x^2y-ny^3+2xz,-\frac{1}{n}x^3+xy^2-2yz,\frac{1}{n}x^4+ny^4-2x^2y^2-z^2)\rangle\}$$

so that O is now in the form $O(f_1,f_2)$. It is not difficult to prove directly that $O(f_1,f_2)$ is an ovoid; to show that $O-\{\langle(0,0,0,0,0,0,1)\rangle\}$ is a cap, consider $P_1=\langle v_1\rangle=\langle(1,x_1,y_1,z_1,x_1^2y_1-ny_1^3+2x_1z_1,-\frac{1}{n}x_1^3+x_1y_1^2-2y_1z_1,\frac{1}{n}x_1^4+ny_1^4-2x_1^2y_1^2-z_1^2)\rangle$ and $P_2=\langle v_2\rangle=\langle(1,x_2,y_2,z_2,x_2^2y_2-ny_2^3+2x_2z_2,-\frac{1}{n}x_2^3+x_2y_2^2-2y_2z_2,\frac{1}{n}x_2^4+ny_2^4-2x_2^2y_2^2-z_2^2)\rangle$. Now

$$\begin{aligned}
f_Q(v_1,v_2) &= Q(v_1+v_2)\\
&= 2(\frac{1}{n}(x_1^4+x_2^4)+n(y_1^4+y_2^4)-2(x_1^2y_1^2+x_2^2y_2^2)-z_1^2-z_2^2)\\
&\quad+(x_1+x_2)(-\frac{1}{n}(x_1^3+x_2^3)+x_1y_1^2+x_2y_2^2-2(y_1z_1+y_2z_2))\\
&\quad+(y_1+y_2)(x_1^2y_1+x_2^2y_2-n(y_1^3+y_2^3)+2(x_1z_1+x_2z_2))\\
&= -\frac{1}{n}(x_1^2-ny_1^2)^2-z_1^2-\frac{1}{n}(x_2^2-ny_2^2)^2-z_2^2\\
&\quad+x_1(-\frac{1}{n}x_2^3+x_2y_2^2-2y_2z_2)+x_2(-\frac{1}{n}x_1^3+x_1y_1^2-2y_1z_1)\\
&\quad+y_1(x_2^2y_2-ny_2^3+2x_2z_2)+y_2(x_1^2y_1-ny_1^3+2x_1z_1)+2z_1z_2\\
&= -(z_1-z_2)^2+\frac{1}{n}(x_1-x_2)^4+n(y_1-y_2)^4-2(z_1-z_2)(x_2y_1-x_1y_2)\\
&\quad+(x_1-x_2)^2(y_1-y_2)^2-(x_2y_1-x_1y_2)^2
\end{aligned}$$

and this last value for $f_Q(v_1,v_2)$ is a quadratic in (z_1-z_2) with discriminant $\frac{1}{n}(x_1-x_2)^4+(x_1-x_2)^2(y_1-y_2)^2+n(y_1-y_2)^4=\frac{1}{n}((x_1-x_2)^2-n(y_1-y_2)^2)^2$. This is a square (or 0) in $\mathrm{GF}(q)$ if and only if $(x_1-x_2)^2-n(y_1-y_2)^2=0$ (since n is a non–square), which is if and only if $x_1=x_2$ and $y_1=y_2$, in which case $z_1=z_2$.

3.1.5 The Ree–Tits ovoids. In [80] Tits proved that the set of absolute points of a polarity of $\mathrm{H}(q)$ is an ovoid of $\mathrm{H}(q)$, and that a polarity of $\mathrm{H}(q)$ exists if (and only if) $q = 3^{2h+1}$ $(h \geq 0)$. Via Theorem 3.1.3.2, this gave rise to a family of $\mathrm{O}(7, q)$ ovoids. Incidentally, these theorems of [80] extend in the following way to arbitrary generalised hexagons of order q.

Theorem 3.1.5.1. *(a) Given a polarity θ of a generalised hexagon Γ of order q, the set of absolute points of θ is an ovoid of Γ, and the set of absolute lines of Γ is a spread of Γ.*

(b) If a generalised hexagon of order q admits a polarity, then $q = 1$ or $q = 3^{2h+1}$ $(h \geq 0)$.

Proof. See [11] and [51] respectively. □

Let $q = 3^{2h+1}$ $(h > 0)$. Given $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ with $a^\alpha = a^{3^{h+1}}$ $\forall\, a \in \mathrm{GF}(q)$, the Ree–Tits ovoids are described by $f_1(x, y, z) = x^2y - xz + y^\alpha - x^{\alpha+3}$ and $f_2(x, y, z) = x^\alpha y^\alpha - z^\alpha + xy^2 + yz - x^{2\alpha+3}$ ([80]).

3.2 Slices of the known ovoids

Theorem 3.2.1 ([59]). *Slicing a Thas–Kantor ovoid of $\mathrm{O}(7, q)$ gives an elliptic quadric ovoid of $\mathrm{O}(5, q)$ if $q = 3$, and a Kantor $\mathrm{O}(5, q)$ ovoid $K(\alpha)$ if $q > 3$ (where $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ has $a^\alpha = a^3$ $\forall\, a \in \mathrm{GF}(q)$).*

Proof. By Lemma 1.1.7.3, we need only slice by one representative of each orbit of $\mathrm{PGU}(3, q)$ on singular points of $\mathrm{O}(7, q)$. Because there are two such orbits, we just slice by the orbit representative $P = \langle(0, 0, 0, 0, 0, 1, 0)\rangle$. Letting O be a Thas–Kantor ovoid (as written above), we see that $P^\perp \cap O$ is

$$\{\langle(0, 0, 0, 0, 0, 0, 1)\rangle\} \cup \{\langle(1, 0, y, z, -ny^3, -2yz, -z^2 + ny^4)\rangle : y, z \in \mathrm{GF}(q)\}$$

which in our usual $\mathrm{O}(5, q)$ model becomes

$$O' = \{\langle(0, 0, 0, 0, 1)\rangle\} \cup \{\langle(1, y, z, -ny^3, -z^2 + ny^4)\rangle : y, z \in \mathrm{GF}(q)\}$$

so that $O' = K(\alpha)$, where $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ has $a^\alpha = a^3$ $\forall\, a \in \mathrm{GF}(q)$. □

A weaker version of Theorem 3.2.1 has subsequently been proven ([4]); the automorphism α defining the slice was not determined.

Theorem 3.2.2 ([36]). *Slicing a Ree–Tits ovoid of $\mathrm{O}(7, q)$ gives a Kantor or Ree–Tits slice ovoid of $\mathrm{O}(5, q)$.*

Proof. Since ${}^2\,\mathrm{G}_2(q)$ has three orbits on singular points of $\mathrm{O}(7,q)$, we just slice by the orbit representatives $\langle(0,0,0,0,0,1,0)\rangle$ and $\langle(0,0,0,0,1,0,0)\rangle$. The former gives a Kantor ovoid

$$\{\langle(0,0,0,0,1)\rangle\} \cup \{\langle(1,y,z,y^\alpha,-y^{\alpha+1}-z^2)\rangle : y,z \in \mathrm{GF}(q)\}$$

(where $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ has $a^\alpha = a^{3^{h+1}}$ $\forall\, a \in \mathrm{GF}(q)$) while the latter gives an ovoid

$$\{\langle(0,0,0,0,1)\rangle\} \cup \{\langle(1,x,z,-z^\alpha - x^{2\alpha+3}, xz^\alpha - z^2 + x^{2\alpha+4})\rangle : x,z \in \mathrm{GF}(q)\}$$

$\square$

Incidentally, there is a construction that yields $\mathrm{O}(5,q)$ ovoids from $\mathrm{H}(q)$ spreads ([4]). In particular, the Thas–Payne ovoids were shown to arise in this way.

Ovoids of $\mathrm{O}^+(8,q)$

4.1 The known ovoids

In Table 4.1.1 we give the stabilisers in $\mathrm{P\Gamma O}^+(8,q)$ of the known $\mathrm{O}^+(8,q)$ ovoids, and the values of q for which the ovoids exist.

name	stabiliser	q
Thas–Kantor ([11] and [36])	$\mathrm{PGU}(3,q) \rtimes \mathrm{C}_h$	3^h $(h>0)$
Ree–Tits ([80])	${}^2\mathrm{G}_2(q) \rtimes \mathrm{C}_{2h+1}$	3^{2h+1} $(h>0)$
Kantor ([36])	$\mathrm{PGU}(3,q) \rtimes \mathrm{C}_h$	p^h, $p \equiv 2 \pmod 3$ and prime, h odd
Kantor ([36])	$\mathrm{PSL}(2,q^3) \rtimes \mathrm{C}_h$	2^h $(h>1)$
Dye ([19])	$\mathrm{S}_9 \times \mathrm{C}_3$	8
Conway et al/Moorhouse ([14] and [46])	refer to text	prime $(q \geq 5)$

Table 4.1.1: Stabilisers of the known $\mathrm{O}^+(8,q)$ ovoids.

We note in passing that no $\mathrm{O}^+(8,q)$ ovoid O can contain an elliptic quadric ovoid O'. For, suppose the contrary; then given $P \in O - O'$, $P^\perp$ intersects the $\mathrm{O}^-(4,q)$ space spanned by O' in a plane, and so $P^\perp$ must contain a point of O'.

4.1.1 Classification results. There is a unique $\mathrm{O}^+(8,q)$ ovoid for $q=2$ ([35]), 3 ([52]) and 4 ([23]). Note that no ovoids of $\mathrm{O}^+(8,q)$ are known for $q \equiv 1 \pmod 6$, q not prime; of the spaces of concern to us, $\mathrm{O}^+(8,q)$ is the only class for which non–existence results haven't been proven but where there are spaces currently barren of ovoids.

4.1.2 The Kantor $\mathrm{PGU}(3,q)$ $(q \equiv 2 \pmod 3)$ ovoids. Let $q = p^h$ ($p \equiv 2 \pmod 3$ and prime, h odd), and

$$V = \left\{ \begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \overline{\beta} \\ b & \overline{\gamma} & \overline{\alpha} \end{pmatrix} : \alpha,\beta,\gamma \in \mathrm{GF}(q^2), a,b,c \in \mathrm{GF}(q); a + \mathrm{Tr}(\alpha) = 0 \right\}$$

with Q defined by

$$Q\left(\begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \overline{\beta} \\ b & \overline{\gamma} & \overline{\alpha} \end{pmatrix} \right) = \alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2 + \mathrm{Tr}(\beta\gamma) + bc$$

(here $\overline{\beta} = \beta^q$, and $\mathrm{Tr}(\alpha) = \alpha + \alpha^q$). Then (V, Q) is an $\mathrm{O}^+(8,q)$ space, containing

$$O = \quad \{\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rangle\} \bigcup \{\langle \begin{pmatrix} \overline{\rho} & \overline{\rho}\sigma & \mathrm{N}(\rho) \\ \overline{\sigma} & \mathrm{N}(\sigma) & \rho\overline{\sigma} \\ 1 & \sigma & \rho \end{pmatrix} \rangle : \mathrm{Tr}(\rho) + \mathrm{N}(\sigma) = 0\}$$

where $\mathrm{N}(\sigma) = \sigma\overline{\sigma}$. Now $\mathrm{PGU}(3,q)$ is 2–transitive on O ([36]), so to prove that O is an ovoid, just note that the pair of points

$$\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rangle, \langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rangle$$

aren't collinear.

4.1.3 The Kantor $\mathrm{PSL}(2,q^3)$ ovoids. Let $q = 2^h$ $(h \geq 1)$, with $V = \mathrm{GF}(q) \bigoplus \mathrm{GF}(q^3) \bigoplus \mathrm{GF}(q^3) \bigoplus \mathrm{GF}(q)$ equipped with Q, defined by

$$Q((a, \beta, \alpha, d)) = ad + \mathrm{Tr}(\beta\alpha)$$

(here $\mathrm{Tr}(\beta\alpha) = \beta\alpha + (\beta\alpha)^q + (\beta\alpha)^{q^2}$). Let

$$O = \{\langle (0,0,0,1) \rangle\} \cup \{\langle (1, t, t^{q+q^2}, N(t)) \rangle : t \in \mathrm{GF}(q^3)\}$$

where $N(t) = t^{1+q+q^2}$. Note that for $s, t \in \mathrm{GF}(q^3)$ with $s \neq t$ and f_Q the polar form of Q, we have

$$\begin{aligned} f_Q((1, s, s^{q+q^2}, N(s)), (1, t, t^{q+q^2}, N(t))) &= Q((0, s+t, (s+t)^{q+q^2}, N(s) + N(t))) \\ &= \mathrm{Tr}(N(s+t)) = N(s+t) \neq 0 \end{aligned}$$

while $f_Q((0,0,0,1), (1, t, t^{q+q^2}, N(t))) = Q((1, t, t^{q+q^2}, N(t) + 1)) = 1$. Proving just the latter would be sufficient to show that O is an ovoid, given that $\mathrm{PSL}(2,q^3)$ is 3–transitive on O ([36]).

4.1.4 The Dye ovoid. The only sporadic known ovoid of $\mathrm{O}^+(8,q)$ is a particularly intriguing one. To construct it, first let $V = \mathrm{GF}(q)^9$, with Q defined by

$$Q((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)) = \sum_{1 \leq i < j \leq 9} x_i x_j$$

and f_Q the polar form of Q. The singular radical of V is $\langle (1,1,1,1,1,1,1,1,1) \rangle$, and then $V / \langle (1,1,1,1,1,1,1,1,1) \rangle$ is an $\mathrm{O}^+(8,q)$ space (for convenience, we will write vectors $v + \langle(1,1,1,1,1,1,1,1,1)\rangle$ of $V/\langle(1,1,1,1,1,1,1,1,1)\rangle$ just as v). Let $q = 8$, $a \in \mathrm{GF}(q)$ satisfy $1 + a^2 + a^3 = 0$, and

$$O_1 = \langle(1,0,0,0,0,0,0,0,0)\rangle^{\mathrm{S}_9}, \quad O_2 = \langle(a,a^2,a^4,0,0,0,0,0,0)\rangle^{\mathrm{S}_9}$$

Then $|O_1| = 9$ and $|O_2| = 504$, so $O = O_1 \cup O_2$ has the right size to be an ovoid. With O_1 being the $\mathrm{O}^+(8,2)$ ovoid, we just need to show that O_2 is a cap and that O_1 and O_2 are compatible. By Lemma 6.0.2.1, it is enough to show that some point P of O_2 is not collinear with any other point of O_2 ((A)), and that P is not collinear with any point of O_1 ((B)); let $P = \langle(a,a^2,a^4,0,0,0,0,0,0)\rangle$.

(A): For a point R of $O_2 - \{P\}$, there are four possibilities for its three non–zero coordinates: that they overlap the three non–zero coordinates of P in no, one, two or three coordinates. In each case, we just need consider the different possibilities for the first three coordinates of R, as the contribution of the last six coordinates of R to the value of the polar form will be the same no matter how we permute them. When calculating in GF(8) we will be using suitable multliples of the identity $1 + a^2 + a^3 = 0$.

overlap in no non–zero coordinates:

$$f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,0,0,a,a^2,a^4,0,0,0)) = a^2 + a^5 + a^6 = a^4 + a^6$$

overlap in one non–zero coordinate:

$$\begin{aligned}
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a,0,0,a^2,a^4,0,0,0,0)) &= a^4 + a \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a,0,a^2,a^4,0,0,0,0)) &= a^2 \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,0,a,a^2,a^4,0,0,0,0)) &= a \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,0,0,a,a^4,0,0,0,0)) &= a^2 \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a^2,0,a,a^4,0,0,0,0)) &= a^2 + a \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,0,a^2,a,a^4,0,0,0,0)) &= a^2 + a^4 + a^6 + a \\
&= a^2 + a^3 + a^6 = 1 + a^6 \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,0,0,a,a^2,0,0,0,0)) &= a + a^2 + a^4 + a^5 = a \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a^4,0,a,a^2,0,0,0,0)) &= a^2 + a + a^4 + a^6 \\
&= 1 + a^6 \\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,0,a^4,a,a^2,0,0,0,0)) &= a^3 + a^2 + a^4 = 1 + a^4
\end{aligned}$$

overlap in two non–zero coordinates:

$$
\begin{aligned}
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a,a^2,0,a^4,0,0,0,0,0)) &= a\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a,0,a^2,a^4,0,0,0,0,0)) &= a^4+a^6+a = a^3+a^6\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a,a^2,a^4,0,0,0,0,0)) &= a^2+a^3+a^4+a^6+a\\
&= a^2+a^6\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a^2,a,a^4,0,0,0,0,0)) &= a^2+a^5+a = a+a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,0,a,a^4,0,0,0,0,0)) &= a^2+a^5\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,a,0,a^4,0,0,0,0,0)) &= a^2+a^4+a = a^2+a^3\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,a^4,0,a,0,0,0,0,0)) &= a^6+a^2\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,0,a^4,a,0,0,0,0,0)) &= a^3+a^4+a^2 = 1+a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a^2,a^4,a,0,0,0,0,0)) &= a^2\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a^4,a^2,a,0,0,0,0,0)) &= a^2+a^4+a = a^2+a^3\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,0,a^2,a,0,0,0,0,0)) &= a+a^2\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,a^2,0,a,0,0,0,0,0)) &= a^5+a+a^2 = a+a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a,a^4,0,a^2,0,0,0,0,0)) &= a+a^4+a^6 = a^3+a^6\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a,0,a^4,a^2,0,0,0,0,0)) &= a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a,a^4,a^2,0,0,0,0,0)) &= a^2+a^3+a^4 = 1+a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,a^4,a,a^2,0,0,0,0,0)) &= a^6+a\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,0,a,a^2,0,0,0,0,0)) &= a^2+a+a^4 = a^2+a^3\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,a,0,a^2,0,0,0,0,0)) &= a^2+a^5
\end{aligned}
$$

overlap in three non–zero coordinates:

$$
\begin{aligned}
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a,a^4,a^2,0,0,0,0,0,0)) &= (a^2+a^4)^2\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,a,a^4,0,0,0,0,0,0)) &= (a+a^2)^2\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^2,a^4,a,0,0,0,0,0,0)) &= a^2+a^6+a^5 = a^4+a^6\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,a,a^2,0,0,0,0,0,0)) &= a^2+a^3\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(a^4,a^2,a,0,0,0,0,0,0)) &= (a+a^4)^2
\end{aligned}
$$

(B): It is enough to note that

$$\begin{aligned}
f_Q((a,a^2,a^4,0,0,0,0,0,0),(1,0,0,0,0,0,0,0,0)) &= a^2+a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,1,0,0,0,0,0,0,0)) &= a+a^4\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,0,1,0,0,0,0,0,0)) &= a+a^2\\
f_Q((a,a^2,a^4,0,0,0,0,0,0),(0,0,0,1,0,0,0,0,0)) &= a+a^2+a^4 = a^2+a^3
\end{aligned}$$

Note that a purely geometrical construction of the Dye ovoid can be given; embed the $\mathrm{O}^+(8,2)$ ovoid O inside $\mathrm{O}^+(8,8)$, and take the union of O with all conics spanned by triples of O.

4.1.5 The Conway et al and Moorhouse ovoids. Conway, Kleidman and Wilson constructed four infinite families of $\mathrm{O}^+(8,q)$ ovoid for q prime. Of these, the binary family exists for $q \geq 3$, with each ovoid stabilised by $\mathrm{C}_{2^2} \times \mathrm{Sp}(6,2)$ (here and below, the groups given are the stabilisers in $\mathrm{O}^+(8,q)$ of the ovoids). The first ternary family occurs for $q \equiv 2 \pmod 3$ ($q \geq 5$), with its members also fixed by $\mathrm{C}_{2^2} \times \mathrm{Sp}(6,2)$ (the binary and first ternary ovoids of $\mathrm{O}^+(8,5)$ are equivalent). The second ternary family exists for $q \equiv 1 \pmod 3$, the ovoids stabilised by $\mathrm{C}_{2^8} : \mathrm{S}_7$ for $q = 7$ and $\mathrm{C}_{2^7} : \mathrm{S}_7$ for $q \geq 13$. The third ternary family arises for $q \equiv 2 \pmod 3$, with ovoid stabiliser $\mathrm{C}_2 \times \mathrm{S}_9$ for $p = 2$ and $p \geq 11$ and $\mathrm{C}_2 \times \mathrm{S}_{10}$ for $p = 5$. These families generalise the unique ovoids of $\mathrm{O}^+(8,2)$ and $\mathrm{O}^+(8,3)$ (into the third ternary family and binary family respectively), Cooperstein's ovoid of $\mathrm{O}^+(8,5)$ ([15]) (into the third ternary family) and Shult's ovoid of $\mathrm{O}^+(8,7)$ ([65]) (into the second ternary family).

To represent $\mathrm{O}^+(8,q)$, Conway et al take the E_8 root lattice E (for more on root lattices see [66]), where

$$E = \{\tfrac{1}{2}(a_1,a_2,\ldots,a_8) : a_i \in \mathbb{Z}, a_1 \equiv a_2 \equiv \cdots \equiv a_8 \pmod 2, \sum_{i=1}^{8} a_i \equiv 0 \pmod 4\}$$

and form the quotient space $\overline{E} = E/qE$ over $\mathrm{GF}(q)$ (here $\{\overline{a} = a + qE : a \in \mathbb{Z}\}$ is identified with $\mathrm{GF}(q)$).

A quadratic form Q is defined on $\overline{E}$ via

$$Q(\overline{x}) = \overline{\tfrac{1}{2}x \cdot x}$$

(where $\overline{x} = x + qE$, and $\cdot$ denotes the usual inner product), and then $(\overline{E},Q)$ is an $\mathrm{O}^+(8,q)$ space.

For $m \in \mathbb{Z}^+$ let $E_{2m} = \{v \in E : v \cdot v = 2m\}$, and (for q an odd prime and $x \in E_{2q}$) $\mathcal{L}_2(x) = \{v \in E_{2q} : v \equiv x \pmod{2E}\}$. The binary family has the form

$$\mathcal{O}_2(x) = \{\langle \overline{v} \rangle : v \in \mathcal{L}_2(x)\}$$

Letting $q \geq 5$, $x \in E_{2q}$ and $\mathcal{L}_3(x) = \{v \in E_{4q} : v \equiv x \pmod{3E}\}$, the ternary families have the form

$$\mathcal{O}_3(x) = \{\langle \overline{v} \rangle : v \in \mathcal{L}_3(x)\}$$

Moorhouse realised that a prime parameter r ($\neq q$) lay in the subscript of the $\mathcal{O}_2(x)$ and $\mathcal{O}_3(x)$ ovoids. To describe his construction ([46]), let $i \in \mathbb{Z}^+$ with $i \leq [\frac{r}{2}]$, n_i be the integer $1 \leq n_i \leq [\frac{r}{2}]$ satisfying $i^2 n_i^2 \equiv 1 \pmod{r}$, $nE_{2m} = \{nv : v \in E_{2m}\}$ for $n \in \mathbb{Z}^+$, and

$$\mathcal{L}_{r,q} = \bigcup_{i=1}^{[\frac{r}{2}]} n_i E_{2i(r-i)q}$$

which contains

$$\mathcal{L}'_{r,q} = \bigcup_{i=1}^{[\frac{r}{2}]} n_i q E_{\frac{2i(r-i)}{q}}$$

For $x \in E$ having $\frac{-q(x \cdot x)}{2} \equiv s \pmod{r}$ (s a non–zero square of $\mathrm{GF}(q)$), let n_x be the integer $1 \leq n_x \leq [\frac{r}{2}]$ with $(x \cdot x)n_x^2 \equiv -2q \pmod{r}$. To form Moorhouse's general version of the sets $\mathcal{L}_2(x)$ and $\mathcal{L}_3(x)$, let $[x]_{r,q} = \{v \in \mathcal{L}_{r,q} : v \equiv n_x x \pmod{rE}\}$. Then

$$\mathcal{O}_{r,q}(x) = \{\langle \overline{v} \rangle : v \in [x]_{r,q} - qE\}$$

generalises the $\mathcal{O}_2(x)$ and $\mathcal{O}_3(x)$ ovoids. Moorhouse proves that

$$\mathcal{O}_{r,q}(x) \text{ is an ovoid } \iff (n_x x + rE) \cap \mathcal{L}'_{r,q} = \emptyset$$

and that the right hand side holds whenever $r < q$, and that for $r > q$ $\exists\, x \in E$ for which it holds.

Testing $r \leq 13$ and $p \leq 11$, Moorhouse found only one new $\mathcal{O}_{r,q}$ ovoid via his construction; an $\mathrm{O}^+(8,11)$ ovoid with $\mathrm{PGO}(8,11)$ stabiliser $\mathrm{S}_3 \times (C_{2^5} \rtimes \mathrm{S}_6)$. In general, Moorhouse was unable to determine the stabiliser G of an arbitrary ovoid $\mathcal{O}_{r,q}(x)$. He did prove that G contains $W_{x+rE}\langle -1 \rangle / \langle -1 \rangle$ (W being the Weyl group of E), so that a subgroup of G is established.

4.2 Slices of the known ovoids.

Because their stabilisers don't grow with q, the number of slices that the Conway et al ovoids yield should tend to infinity as q does (if the converse to Lemma 1.1.7.3

holds most of the time). However, it is easy to determine the inequivalent slices of the other known ovoids.

Theorem 4.2.1 ([36]). *Slicing a Kantor* $\mathrm{PGU}(3,q)$ *(*$q \equiv 2 \pmod 3$*) ovoid yields one of two inequivalent* $\mathrm{O}^+(6,q)$ *ovoids.*

Proof. For the singular points not on the ovoid, the points

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix} \right\rangle$$

(where $\omega \in \mathrm{GF}(q^2)$ is arbitrary) are orbit representatives for $\mathrm{PGU}(3,q)$. For information on the translation planes arising from the resulting slices, see [36]; their groups tell apart the slices. □

Theorem 4.2.2 ([36]). *Slicing a Kantor* $\mathrm{PSL}(2,q^3)$ *ovoid of* $\mathrm{O}^+(8,q)$ *yields one* $\mathrm{O}^+(6,q)$ *ovoid.*

Proof. An orbit representative for $\mathrm{PSL}(2,q^3)$ acting on the singular points not on the ovoid is $\langle(0,0,1,0)\rangle$ (an explicit description of $\mathrm{PSL}(2,q^3)$ in terms of its action on the relevant model of $\mathrm{O}^+(8,q)$ is given in [35]). □

Theorem 4.2.3. *Slicing the Dye* $\mathrm{O}^+(8,8)$ *ovoid yields one of* 13 *inequivalent* $\mathrm{O}^+(6,q)$ *ovoids.*

Proof. Below we give orbit representatives for S_9 acting on $\mathrm{O}^+(8,8)$ (excluding the two orbits comprising the Dye ovoid), and the array of the corresponding $\mathrm{O}^+(6,q)$ ovoid (using the invariant of Theorem 6.0.3.1), all obtained by computer. Here, ω was our primitive element for $\mathrm{GF}(8)$.

$$\begin{aligned}
\langle(0,1,w^5,1,w^4,1,w^4,w^5,0)\rangle &: [0,0,33396,8720,1200,0,0,280,84] \\
\langle(0,1,w^3,w^4,w^3,1,1,1,w^2)\rangle &: [0,0,33940,8280,800,520,140,0,0] \\
\langle(0,0,1,1,w^6,w,w^2,w^2,w^6)\rangle &: [0,0,34940,7240,860,640,0,0,0] \\
\langle(0,1,0,w^6,w^3,w^2,w^3,w^5,w^6)\rangle &: [0,0,35936,7004,440,300,0,0,0] \\
\langle(0,1,w,1,0,w^5,w^5,w^5,w^6)\rangle &: [0,0,34412,7808,1020,440,0,0,0] \\
\langle(0,1,1,1,1,1,0,0,0)\rangle &: [0,0,0,0,0,0,0,0,43680] \\
\langle(0,1,1,0,w^2,w^5,0,0,0)\rangle &: [0,0,30740,10240,2100,600,0,0,0] \\
\langle(0,1,0,w^5,w^5,w^2,0,0,w^6)\rangle &: [0,0,32508,8216,700,1920,0,0,336]
\end{aligned}$$

$$\langle(0,1,w,w,w^6,w,w^2,w^2,w^5)\rangle : [0,0,35576,6504,1190,340,70,0,0]$$
$$\langle(0,1,0,w^5,w^5,w^2,w^5,w^6,0)\rangle : [0,0,34023,8268,600,600,105,0,84]$$
$$\langle(0,1,w^6,w^4,w^5,w,w^2,0,w^3)\rangle : [0,0,36365,6720,140,420,35,0,0]$$
$$\langle(0,1,1,1,1,w^3,0,0,0)\rangle : [0,0,28096,13824,480,1280,0,0,0]$$
$$\langle(0,1,w^3,w^4,w^3,1,1,w^3,0)\rangle : [0,0,33526,7356,1680,740,210,0,168]$$

Because the arrays are all different, the ovoids are all inequivalent. Note the array of the ovoid obtained by slicing by $\langle(0,1,1,1,1,1,0,0,0)\rangle$: with the span of every triple on that ovoid meeting the ovoid in a conic, the ovoid is an elliptic quadric.

□

4.3 Normal coordinates.

The analogue for $\mathrm{O}^+(8,q)$ of the models we used for $\mathrm{O}(5,q)$ and $\mathrm{O}(7,q)$ has $V = \mathrm{GF}(q)^8$, with Q defined by

$$Q((x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)) = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$$

and we shall refer to (V,Q) as **normal coordinates**. In normal coordinates, any ovoid is equivalent to one of the form

$$O(f_1,f_2,f_3) = \{\langle(0,0,0,0,0,0,0,1)\rangle\} \cup \{\langle(1,x,y,z,f_1(x,y,z),f_2(x,y,z),$$
$$f_3(x,y,z), -zf_1(x,y,z) - yf_2(x,y,z) - xf_3(x,y,z))\rangle : x,y,z \in \mathrm{GF}(q)\}$$

where $f_1,f_2,f_3 : \mathrm{GF}(q)^3 \to \mathrm{GF}(q)$.

While we would like to give all the known ovoids in the above form (in the same way that we did for our standard models of $\mathrm{O}(5,q)$ and $\mathrm{O}(7,q)$), there are some families for which we do not know of such expressions. For example, we have been unable to convert the Kantor $\mathrm{PGU}(3,q)$ ($q \equiv 2 \pmod 3$) ovoids into normal coordinates, even though they are parametrised in the same way as their 7–dimensional analogues. Additionally, the Conway et al families haven't been parametrised, hindering their conversion into normal coordinates.

However, we do have some of the known $\mathrm{O}^+(8,q)$ ovoids in the form $O(f_1,f_2,f_3)$. In Chapter 3 we saw the two known $\mathrm{O}(7,q)$ families in the form $O(f_1,f_2)$, which (defining f via $f(x,y,z) = z$) in normal coordinates becomes $O(f,f_1,f_2)$. To express the Dye ovoid in normal coordinates, consider the model (V,Q') for $\mathrm{O}^+(8,q)$, where Q' is defined by

$$Q'((x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)) = \sum_{1\le i<j\le 8} x_ix_j$$

Of the orbits of S_8 on singular points of $(\mathrm{GF}(8), Q')$, the following are caps:

$$\begin{array}{rl}
\langle(1,1,1,1,1,1,1,1)\rangle^{\mathrm{S}_8} & \text{(length 1)} \\
\langle(1,0,0,0,0,0,0,0)\rangle^{\mathrm{S}_8} & \text{(length 8)} \\
\langle(1,w,w^4,w^4,w^4,w^4,w^4,w^4)\rangle^{\mathrm{S}_8} & \text{(length 56)} \\
\langle(1,w^3,w^5,w^5,w^5,w^5,w^5,w^5)\rangle^{\mathrm{S}_8} & \text{(length 56)} \\
\langle(1,w^5,w^6,w^6,w^6,w^6,w^6,w^6)\rangle^{\mathrm{S}_8} & \text{(length 56)} \\
\langle(1,w,w^5,0,0,0,0,0)\rangle^{\mathrm{S}_8} & \text{(length 336)}
\end{array}$$

and these orbits comprise an ovoid O. Since S_8 fixes the Dye ovoid (as $\mathrm{S}_8 \leqslant \mathrm{S}_9$), O is the Dye ovoid.

A basis for (V, Q') consisting of hyperbolic pairs (b_1, b_2), (b_3, b_4), (b_5, b_6), (b_7, b_8) is

$$\begin{array}{ll}
b_1 = (0,0,0,0,0,0,0,1) & b_5 = (1,1,1,0,0,1,0,1) \\
b_2 = (1,0,0,1,1,0,0,1) & b_6 = (1,1,1,1,0,0,0,0) \\
b_3 = (0,0,1,1,0,1,1,1) & b_7 = (1,0,1,0,1,0,1,0) \\
b_4 = (0,0,1,1,1,1,0,0) & b_8 = (0,1,1,0,1,0,1,1)
\end{array}$$

By taking each point of O and then the canonical vector spanning that point, finding its coefficients with respect to the above basis and then taking the linear combination with those coefficients of the standard basis vectors for (V, Q) (where the i–th standard basis vector for (V, Q) is identified with b_i), we obtain a copy of O in normal coordinates. Applying the projective isometry T defined by

$$T(\langle(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\rangle) = \langle(x_1+x_2+x_6, x_2, x_3-x_8, x_4, x_5, x_6, x_7-x_8, x_8)\rangle$$

to that copy, we obtain an ovoid containing the point $\langle(0,0,0,0,0,0,0,1)\rangle$, and then interpolating we find that

$$\begin{aligned}
f_1 &= x + y + z + x^2y + x^4y^2 + xy^2 + x^2y^4 + x^4y^4 \\
f_2 &= y + x^2z + x^4z^2 + xz^2 + x^2z^4 + x^4z^4 \\
f_3 &= x + y + y^2z + y^4z^2 + yz^2 + y^2z^4 + y^4z^4
\end{aligned}$$

gives $O(f_1, f_2, f_3) = O$. It is interesting that the symmetry present in this description of the Dye ovoid is not evident in the original presentation of the ovoid.

Finally, for $q = 2, 4, 16$ we have found that

$$f_1 = xy + z^2$$
$$f_2 = xz + y^2 + z^2$$
$$f_3 = yz + x^2 + y^2 + z^2$$

gives an ovoid $O(f_1, f_2, f_3)$ (but not when $q = 8$). If these functions describe an infinite family of ovoids, then it consists of Kantor $\mathrm{PSL}(2, q^3)$ ovoids.

4.4 Restrictions on stabilisers of new ovoids.

In [46] Moorhouse claimed computer evidence of many new r-ary ovoids for large q, and suggests that his construction may produce unboundedly many ovoids as $q \to \infty$. Even if his construction does do this (he couldn't tell when the $\mathcal{O}_{r,q}(x)$ ovoids are equivalent), ovoids are still rare in the vast bulk of $\mathrm{O}^+(8,q)$ spaces. When q is not prime, $\mathrm{O}^+(8,q)$ currently contains at most two ovoids (with the exception of $\mathrm{O}^+(8,8)$), and many of these spaces contain no ovoid or one ovoid.

The $\mathrm{O}^+(8,q)$ (q not prime) spaces get big very quickly, and so (for ovoid–fixing purposes) it is desirable to consider groups that fix ovoids with as few orbits as possible. Unfortunately, Kleidman in [39] classified $\mathrm{O}^+(8,q)$ ovoids fixed by 2–transitive groups as just being the known ones (this result was later extended by Gunarwardena in [22] to primitive groups).

For $q \equiv 1 \pmod 3$ (q odd), we have some evidence that ovoids fixed by transitive groups don't occur, gained as follows. Let $V = \mathrm{GF}(q^6) \bigoplus \mathrm{GF}(q^2)$, and define Q on V via

$$Q((v,x)) = \mathrm{Tr}(v^{q^3+1}) + x^{q+1}$$

where $\mathrm{Tr}(v^{q^3+1}) = v^{q^3+1} + (v^{q^3+1})^q + (v^{q^3+1})^{q^2}$. Let $G = \langle g_\lambda \rangle$, where

$$g_\lambda((v,x)) = (\lambda v, x)$$

with $\lambda \in \mathrm{GF}(q^6)$ having $\lambda^{q^2-q+1} = 1$. Since G has order $q^2 - q + 1$, $q+1$ mutually compatible orbits of G are required to make an ovoid. For $q = 7, 25$ we found by computer that G does not preserve an ovoid.

If a transitive ovoid exists in $\mathrm{O}^+(8,q)$ (for any q), by the orbit–stabiliser theorem $\mathrm{P\Gamma O}^+(8,q)_O$ has to have order divisible by $q^2 - q + 1$. If $q^2 - q + 1$ happens to be a prime, then by Cauchy's theorem $\mathrm{P\Gamma O}^+(8,q)_O$ has to have an element r of that order, and then $\langle r \rangle$ is a Sylow q^2+q-1–subgroup of $\mathrm{P\Gamma O}^+(8,q)_O$ (for, q^3+1 divides $|\,\mathrm{P\Gamma O}^+(8,q)_O|$ only once (see [40, p19]), and $q^3+1 = (q+1)(q^2-q+1)$). Since any two such groups are conjugate in $\mathrm{P\Gamma O}^+(8,q)_O$, the sets of ovoids they fix are setwise equivalent (Lemma 1.1.7.1), and so the computer results above tell us that $\mathrm{O}^+(8,7)$ and $\mathrm{O}^+(8,25)$ don't contain transitive ovoids.

4.5 Results of computer searches.

On the next page we give a very small selection of the group actions tried in our searches for $\mathrm{O}^+(8,q)$ ovoids (the right–hand column denotes the values of q that were tested); none of our searches has been successful (we omit the searches described in the section above). Only searches producing no ovoids have been listed; as mentioned in Chapter 2, when a search turns up numbers of ovoids all having the same array with respect to some invariant, it is difficult to prove that all the ovoids are known.

model (V,Q)	groups tried	q
$V=\{A\in \mathrm{M}(3,q) : \mathrm{trace}(A)=0\}$,	$\mathrm{O}(3,q)$, $\mathrm{Syl}_2(\mathrm{GL}(3,q))$,	7
$Q(A)=\frac{1}{2}\,\mathrm{trace}(A^2)$	$\mathrm{Syl}_3(\mathrm{GL}(3,q))$, $\mathrm{Syl}_7(\mathrm{GL}(3,q))$	
$V=\mathrm{GF}(q^6)\bigoplus \mathrm{GF}(q^2)$,	$\langle g,h\rangle$, with $g((v,x))=(\lambda v,x)$,	
$Q((v,x))=\mathrm{Tr}(v^{q^3+1})+x^{q+1}$,	$h((v,x))=(v^q,x^q)$	
$\mathrm{Tr}(v^{q^3+1})=v^{q^3+1}+(v^{q^3+1})^q+(v^{q^3+1})^{q^2}$	$(\lambda^a=1)$	
	$a=q+1$	$5,7,8$
	$a=\frac{q+1}{2}$	$7,9$
	$a=\frac{q+1}{3}$	5
$V=\mathrm{GF}(q)^8$,	S_8	$9,27$
$Q((x_1,\ldots,x_8))=x_1^2+\cdots+x_8^2$		
$V=\mathrm{GF}(q)^9$,	S_9	32
$Q((x_1,\ldots,x_9))=\sum_{1\le i<j\le 9}x_ix_j$		

Table 4.5.1: Some searches for $\mathrm{O}^+(8,q)$ ovoids.

Regular packings of $\mathrm{PG}(3, q)$

A **packing** of $\mathrm{PG}(3, q)$ is a partition of the set of lines of $\mathrm{PG}(3, q)$ by spreads, and so has size q^2+q+1. If a packing consists entirely of regular spreads, it is **regular**. If a packing has stabiliser in $\mathrm{P\Gamma L}(4, q)$ admitting a cyclic group that acts regularly on the packing, it is **cyclic**. Given packings Υ_1 and Υ_2 of $\mathrm{PG}(3, q)$, we say they are **equivalent** if $\exists\, g \in \mathrm{P\Gamma L}(4, q)$ with $g(\Upsilon_1) = \Upsilon_2$.

5.1 History of the problem

The study of packings of $\mathrm{PG}(3, q)$ dates back to Kirkman's schoolgirl problem, which was set in the *Lady's and Gentleman's Diary* of 1850 (see [3]): "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast." The first published solution to the problem was due to Cayley in [12], while Kirkman's own solution was presented in the *Lady's and Gentleman's Diary* of 1851. There are in fact seven solutions in all, as shown by Woolhouse in 1862/3 (see [27, p91] for references).

By identifying each schoolgirl with one of the fifteen points of $\mathrm{PG}(3, 2)$, each row of schoolgirls with a line of $\mathrm{PG}(3, 2)$ and each day with a spread of $\mathrm{PG}(3, 2)$, the schoolgirl problem is solved by finding a packing of $\mathrm{PG}(3, 2)$. There are two inequivalent packings of $\mathrm{PG}(3, 2)$ (see [27, Theorem 17.5.6]), while the other solutions to the schoolgirl problem are not expressible as packings of $\mathrm{PG}(3, 2)$.

Denniston in [17] (and then independently Beutelspacher in [2]) showed that $\mathrm{PG}(3, q)$ always admits a packing, so the question arose as to when $\mathrm{PG}(3, q)$ admits a regular packing. Certainly $\mathrm{PG}(3, 2)$ admits regular packings, as every spread of $\mathrm{PG}(3, 2)$ is regular. In [18], Denniston found a cyclic regular packing Υ of $\mathrm{PG}(3, 8)$, which (by applying a correlation $\triangle$ of $\mathrm{PG}(3, 8)$) yields a cyclic regular packing $\triangle\Upsilon$ of $\mathrm{PG}(3, 8)$, inequivalent to Υ (recall the remark after Corollary 1.2.2.3). In [62], Prince (actually) discovers two inequivalent cyclic regular packings of $\mathrm{PG}(3, 5)$. (Lunardon had claimed in [42] that $\mathrm{PG}(3, q)$ contains no regular packings for q odd, but his proof was erroneous.)

5.2 The new regular packings

Shortly we will present a construction that yields two cyclic regular packings of $\mathrm{PG}(3, q)$ for $q \equiv 2 \pmod 3$, subsuming the known regular packings for $q = 2, 5, 8$.

With a packing of $\mathrm{PG}(3, q)$ associated (via the bijection κ of the Klein correspondence) to a partition of the set of singular points of $\mathrm{O}^+(6, q)$ by ovoids, we see (applying Theorem 1.2.4.1) that the problem of finding a regular packing of $\mathrm{PG}(3, q)$ is equivalent to the problem of finding a partition of the set of singular points of $\mathrm{O}^+(6, q)$ by elliptic quadric ovoids – the construction is made possible by this fact.

In the below, we denote by $\overline{\kappa}$ the isomorphism from $\mathrm{Cor}(\mathrm{PG}(3, q))$ to $\mathrm{P\Gamma O}^+(6, q)$ that results from κ.

5.2.1 A model of $\mathrm{O}^+(6, q)$. We now give the model of $\mathrm{O}^+(6, q)$ in which we will be working. Let $T : \mathrm{GF}(q^3) \to \mathrm{GF}(q)$ via $T(x) = x + x^q + x^{q^2}$, and V be the $\mathrm{GF}(q)$ vector space whose underlying set is $\mathrm{GF}(q^3)^2$. Define a quadratic form Q on V via $Q((x, y)) = T(xy)$; the polar form f_Q of Q has $f_Q(u, v) = T(u_1v_2)+T(u_2v_1)$ for $u = (u_1, u_2)$, $v = (v_1, v_2)$. To show that f_Q (and hence Q) is non–degenerate, suppose that $(v_1, v_2) \in V$ has $f_Q((v_1, v_2), (x, y)) = 0 \;\; \forall\, (x, y) \in V$. Then $T(v_1y)+T(v_2x) = 0$ $\forall\, (x, y) \in V$, so that

$$v_1y + v_1^q y^q + v_1^{q^2} y^{q^2} + v_2x + v_2^q x^q + v_2^{q^2} x^{q^2} = 0 \;\; \forall\, (x, y) \in V$$

Putting $x = 0$ and then $y = 0$ forces $(v_1, v_2) = (0, 0)$. To see that (V, Q) has maximal Witt index, note that $\{(x, 0) : x \in \mathrm{GF}(q^3)\}$ is a totally singular subspace of V of dimension 3. Thus, (V, Q) is an $\mathrm{O}^+(6, q)$ space.

Theorem 5.2.1.1 ([60]). *Let $q \equiv 2 \pmod 3$ and $\Sigma = \{(y, z) \in V : y^{q^2} + z \in \mathrm{GF}(q)\}$. Define g on V via $g((x, y)) = (\mu x, \frac{1}{\mu} y)$ for $\mu \in \mathrm{GF}(q^3)^*$ with $|\mu| = q^2+q+1$, and denote by $\hat{g}$ the resulting map on $\mathrm{P}V$. Let $I = \{i \in \mathbb{Z} : 0 \le i < q^2 + q + 1\}$. If S_i denotes the set of singular points of $g^i(\Sigma)$ for each $i \in I$, then*

$$\Pi = \{S_i : i \in I\}$$

is a partition of the set of singular points of $\mathrm{O}^+(6, q)$ by elliptic quadric ovoids, and $\langle \hat{g} \rangle \leqslant \mathrm{PGO}^+(6, q)$ acts regularly on Π. Hence,

$$\Upsilon = \{\kappa^{-1}(S_i) : i \in I\}$$

is a packing of $\mathrm{PG}(3, q)$, where $\overline{\kappa}^{-1}(\langle \hat{g} \rangle) \leqslant \mathrm{PGL}(4, q)$ acts regularly on Υ.

Proof. First note that g is an isometry and $\langle \hat{g} \rangle$ indeed acts regularly on Π, since (for $i \in I$) $\hat{g}^i$ can only fix $\langle (x, y) \rangle$ if $\mu^i \in \mathrm{GF}(q)$, while $(q^2 + q + 1, q - 1) = (q - 1, 3)$, which is 1 when $q \equiv 2 \pmod 3$.

We need to show that Σ is an $\mathrm{O}^-(4,q)$ space. Given $x \in \mathrm{GF}(q^3)$ with $T(x)=0$ and $(y,z)\in\Sigma$ (with $b=y^{q^2}+z$), note that

$$\begin{aligned} f_Q((x,x^q),(y,z)) &= T(xz)+T(x^q y) \\ &= T(x(b-y^{q^2}))+T(x^q y) \\ &= bT(x)-T(xy^{q^2})+T(x^q y) \\ &= bT(x)=0 \end{aligned}$$

so that $\{(x,x^q)\in V : T(x)=0\}\subseteq \Sigma^\perp$. Since $\Sigma^\perp$ has dimension 2, we must have equality.

Now we prove that $\Sigma^\perp$ is anisotropic or totally singular. Observe that $\Sigma^\perp = \langle u,u^q\rangle$ for any non–zero vector $u=(x,x^q)$ of $\Sigma^\perp$. For, $\langle u\rangle = \langle u^q\rangle$ implies that $x^{2q-1}=x^{q^2}$, giving $x^{(q-1)^2}=1$. Since $q\equiv 2 \pmod 3$, $(q^2+q+1,q-1)=1$, so $x^{q-1}=1$. But then $x\in \mathrm{GF}(q)$, so that $T(x)=3x=0$, implying $x=0$, a contradiction. So suppose u is singular; then u^q is singular, and now

$$\begin{aligned} f(u,u^q) = f((x,x^q),(x^q,x^{q^2})) &= T(xx^{q^2})+T(x^qx^q)=T(xx^{q^2})+T(xx) \\ &= T(x(x^{q^2}+x)) = T(x(x^q+x^{q^2}+x))-T(xx^q) \\ &= T(xT(x^q))-T(xx^q)=T(x^q)T(x)-T(xx^q)=T(x^q)T(x)=0 \end{aligned}$$

so that $\langle u,u^q\rangle = \Sigma^\perp$ is totally singular.

Now suppose that $\Sigma^\perp$ is totally singular. Then for $x\in \mathrm{GF}(q^3)$, $T(x)=0$ implies that $(x,x^q)\in\Sigma^\perp$, so that $Q((x,x^q))=0$ and therefore $T(x^{q+1})=0$. Note that $T(T(x))=3T(x)$, so $T(x-\frac{T(x)}{3})=0$ and hence $T((x-\frac{T(x)}{3})^{q+1})=0$. But

$$\begin{aligned} &(x-\frac{T(x)}{3})^{q+1} \\ &\qquad = -(\frac{2}{9})x^2+(\frac{5}{9})x^{q+1}-(\frac{2}{9})x^{2q}-(\frac{1}{9})x^{q^2+1}-(\frac{1}{9})x^{q^2+q}+(\frac{1}{9})x^{2q^2} \end{aligned}$$

so that

$$\begin{aligned} &T((x-\frac{T(x)}{3})^{q+1}) \\ &\qquad = -(\frac{1}{3})x^2+(\frac{1}{3})x^{q+1}-(\frac{1}{3})x^{2q}+(\frac{1}{3})x^{q^2+1}+(\frac{1}{3})x^{q^2+q}-(\frac{1}{3})x^{2q^2} \end{aligned}$$

which is not divisible by $x^{q^3}-x$, a contradiction. Thus, $\Sigma^\perp$ is an $\mathrm{O}^-(2,q)$ space. Hence Σ is an $\mathrm{O}^-(4,q)$ space, and so all $g^i(\Sigma)$ spaces are such for $i\in I$.

Suppose $i,j\in I$ with $i<j$. It remains to show that $g^i(\Sigma)$ and $g^j(\Sigma)$ have no singular points in common, which is if $g^i(\Sigma)\cap g^j(\Sigma)$ is anisotropic. Since we have

$\langle g\rangle$ acting regularly on $\{g^i(\Sigma) : i \in I\}$, it is enough to show that each $\Sigma \cap g^j(\Sigma)$ is anisotropic. Since each $\Sigma\cap g^j(\Sigma)$ has dimension at least 2 and anisotropic orthogonal spaces have dimension at most 2, we want to show that each $\Sigma\cap g^j(\Sigma)$ is an $\mathrm{O}^-(2,q)$ space. Letting $\lambda = \mu^j$, we have $|\lambda| \,\big|\, q^2+q+1$, and then

$$g^j(\Sigma) = \{(\lambda y, \frac{z}{\lambda}) : y^{q^2} + z \in \mathrm{GF}(q)\}$$

Now,

$$v_1 = (\frac{\lambda^q}{-\lambda^{q^2+q}+\lambda^q}, \frac{-\lambda}{\lambda-\lambda^{q+1}}+1)$$

and

$$v_2 = (\frac{\lambda^q-1}{-\lambda^{q^2+q}+\lambda^q}, \frac{-\lambda+1}{\lambda-\lambda^{q+1}}+1)$$

are vectors of $\Sigma \cap g^j(\Sigma)$, and (because $\Sigma^\perp \cap (g^j(\Sigma))^\perp = \{(0,0)\}$) v_1 and v_2 span $\Sigma \cap g^j(\Sigma)$. To show that $\Sigma \cap g^j(\Sigma)$ is an $\mathrm{O}^-(2,q)$ space, we will require $f_Q(v_1,v_1)$, $f_Q(v_2,v_2)$, $f_Q(v_1,v_2)$. To follow the algebra in these calculations, note that whenever two bracketed terms are multiplied, the multiplication proceeds from left to right, starting with the first element of the first bracketed term.

Now $f_Q(v_1,v_1) = 2Q(v_1) = 2T(A)$, where

$$\begin{aligned} A &= (\frac{\lambda^q}{-\lambda^{q^2+q}+\lambda^q})(\frac{-\lambda}{\lambda-\lambda^{q+1}}+1) = \frac{-\lambda^{q+1}+\lambda^q(\lambda-\lambda^{q+1})}{(-\lambda^{q^2+q}+\lambda^q)(\lambda-\lambda^{q+1})} \\ &= \frac{-\lambda^{2q+1}}{-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1}} \end{aligned}$$

Writing the three terms of $T(A)$ as one fraction with denominator $(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2}$, we know that $T(A)=1$ if the numerator of that fraction equals the denominator, which is if

$$\begin{aligned} (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2-1}(-\lambda^{2q+1}) + (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2-q}(-\lambda^{2q+1})^q \\ + (-\lambda^{2q+1})^{q^2} = (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2} \end{aligned} \tag{5.2.1.1}$$

or (raising both sides of (5.2.1.1) to the q and multiplying through by $(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+q}$) if

$$\begin{aligned} (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+1}(-\lambda^{2q+1})^q + (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q+1}(-\lambda^{2q+1})^{q^2} \\ + (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+q}(-\lambda^{2q+1}) = (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+q+1} \end{aligned} \tag{5.2.1.2}$$

Denote the terms of (5.2.1.2) as d_1, d_2, d_3, d_4 respectively; we now evaluate them. Firstly,

$$
\begin{aligned}
&(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+1}\\
&=(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})(-1+\lambda+\lambda^{q^2+1}-\lambda^{q^2+2})\\
&=1-\lambda-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q+\lambda^{q+1}+\lambda^{q^2+q+1}-\lambda^{q^2+q+2}-\lambda^{q+1}+\lambda^{q+2}+\lambda^{q^2+q+2}\\
&\quad-\lambda^{q^2+q+3}+\lambda^{2q+1}-\lambda^{2q+2}-\lambda^{q^2+2q+2}+\lambda^{q^2+2q+3}\\
&=1-\lambda-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q+\lambda^{q+1}+1-\lambda-\lambda^{q+1}+\lambda^{q+2}+\lambda\\
&\quad-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2}-\lambda^{q+1}+\lambda^{q+2}\\
&=2-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q-\lambda-\lambda^{q+1}+2\lambda^{q+2}-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2}
\end{aligned}
$$

so

$$
\begin{aligned}
d_1&=(2-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q-\lambda-\lambda^{q+1}+2\lambda^{q+2}-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2})(-\lambda^{2q^2+q})\\
&=-2\lambda^{2q^2+q}+\lambda^{3q^2+q+1}-\lambda^{3q^2+q+2}+\lambda^{2q^2+2q}+\lambda^{2q^2+q+1}+\lambda^{2q^2+2q+1}-2\lambda^{2q^2+2q+2}\\
&\quad+\lambda^{2q^2+q+2}-\lambda^{2q^2+3q+1}+\lambda^{2q^2+3q+2}\\
&=-2-2\lambda^{2q^2+q}+\lambda^{2q^2}-\lambda^{2q^2+1}+\lambda^{2q^2+2q}+\lambda^{q^2}+\lambda^{q^2+q}+\lambda^{q^2+1}-\lambda^{q^2+2q}+\lambda^q
\end{aligned}
$$

and

$$
\begin{aligned}
d_4&=(2-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q-\lambda-\lambda^{q+1}+2\lambda^{q+2}-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2})\\
&\quad.(-1+\lambda^{q^2}+\lambda^{q^2+q}-\lambda^{2q^2+q})\\
&=-2+2\lambda^{q^2}+2\lambda^{q^2+q}-2\lambda^{2q^2+q}+\lambda^{q^2+1}-\lambda^{2q^2+1}-\lambda^{2q^2+q+1}+\lambda^{3q^2+q+1}-\lambda^{q^2+2}\\
&\quad+\lambda^{2q^2+2}+\lambda^{2q^2+q+2}-\lambda^{3q^2+q+2}+\lambda^q-\lambda^{q^2+q}-\lambda^{q^2+2q}+\lambda^{2q^2+2q}+\lambda-\lambda^{q^2+1}\\
&\quad-\lambda^{q^2+q+1}+\lambda^{2q^2+q+1}+\lambda^{q+1}-\lambda^{q^2+q+1}-\lambda^{q^2+2q+1}+\lambda^{2q^2+2q+1}-2\lambda^{q+2}\\
&\quad+2\lambda^{q^2+q+2}+2\lambda^{q^2+2q+2}-2\lambda^{2q^2+2q+2}+\lambda^2-\lambda^{q^2+2}-\lambda^{q^2+q+2}+\lambda^{2q^2+q+2}-\lambda^{2q+1}\\
&\quad+\lambda^{q^2+2q+1}+\lambda^{q^2+3q+1}-\lambda^{2q^2+3q+1}+\lambda^{2q+2}-\lambda^{q^2+2q+2}-\lambda^{q^2+3q+2}+\lambda^{2q^2+3q+2}\\
&=-2+2\lambda^{q^2}+2\lambda^{q^2+q}-2\lambda^{2q^2+q}+\lambda^{q^2+1}-\lambda^{2q^2+1}-\lambda^{q^2}+\lambda^{2q^2}-\lambda^{q^2+2}+\lambda^{2q^2+2}\\
&\quad+\lambda^{q^2+1}-\lambda^{2q^2+1}+\lambda^q-\lambda^{q^2+q}-\lambda^{q^2+2q}+\lambda^{2q^2+2q}+\lambda-\lambda^{q^2+1}-1+\lambda^{q^2}+\lambda^{q+1}\\
&\quad-1-\lambda^q+\lambda^{q^2+q}-2\lambda^{q+2}+2\lambda+2\lambda^{q+1}-2+\lambda^2-\lambda^{q^2+2}-\lambda+\lambda^{q^2+1}-\lambda^{2q+1}\\
&\quad+\lambda^q+\lambda^{2q}-\lambda^{q^2+2q}+\lambda^{2q+2}-\lambda^{q+1}-\lambda^{2q+1}+\lambda^q\\
&=-6+2\lambda^{q^2}+2\lambda^{q^2+q}-2\lambda^{2q^2+q}+2\lambda^{q^2+1}-2\lambda^{2q^2+1}+\lambda^{2q^2}-2\lambda^{q^2+2}+\lambda^{2q^2+2}\\
&\quad+2\lambda^q-2\lambda^{q^2+2q}+\lambda^{2q^2+2q}+2\lambda+2\lambda^{q+1}-2\lambda^{q+2}+\lambda^2-2\lambda^{2q+1}+\lambda^{2q}+\lambda^{2q+2}
\end{aligned}
$$

while

$$\begin{aligned}
d_2 &= d_1^q \\
&= (-2 - 2\lambda^{2q^2+q} + \lambda^{2q^2} - \lambda^{2q^2+1} + \lambda^{2q^2+2q} + \lambda^{q^2} + \lambda^{q^2+q} + \lambda^{q^2+1} - \lambda^{q^2+2q} \\
&\quad + \lambda^q)^q \\
&= -2 - 2\lambda^{q^2+2} + \lambda^2 - \lambda^{q+2} + \lambda^{2q^2+2} + \lambda + \lambda^{q^2+1} + \lambda^{q+1} - \lambda^{2q^2+1} + \lambda^{q^2}
\end{aligned}$$

and

$$\begin{aligned}
d_3 &= d_2^q \\
&= (-2 - 2\lambda^{q^2+2} + \lambda^2 - \lambda^{q+2} + \lambda^{2q^2+2} + \lambda + \lambda^{q^2+1} + \lambda^{q+1} - \lambda^{2q^2+1} + \lambda^{q^2})^q \\
&= -2 - 2\lambda^{2q+1} + \lambda^{2q} - \lambda^{q^2+2q} + \lambda^{2q+2} + \lambda^q + \lambda^{q+1} + \lambda^{q^2+q} - \lambda^{q+2} + \lambda
\end{aligned}$$

It can be checked that $d_1 + d_2 + d_3 = d_4$; thus, $f_Q(v_1, v_1) = 2$.

Note that the product B of the two components of v_2 is

$$\begin{aligned}
B &= \left(\frac{\lambda^q - 1}{-\lambda^{q^2+q} + \lambda^q}\right)\left(\frac{-\lambda + 1}{\lambda - \lambda^{q+1}} + 1\right) \\
&= \frac{(\lambda^q - 1)(-\lambda + 1) + (\lambda^q - 1)(\lambda - \lambda^{q+1})}{(-\lambda^{q^2+q} + \lambda^q)(\lambda - \lambda^{q+1})} \\
&= \frac{-\lambda^{q+1} + \lambda^q + \lambda - 1 + \lambda^{q+1} - \lambda^{2q+1} - \lambda + \lambda^{q+1}}{-1 + \lambda^q + \lambda^{q+1} - \lambda^{2q+1}} \\
&= 1
\end{aligned}$$

so $Q(v_2) = T(1) = 3$, and hence $f_Q(v_2, v_2) = 6$.

To compute $f_Q(v_1, v_2)$, we determine $Q(v_1 + v_2)$ (it will be seen to be 7). Now

$$Q(v_1 + v_2) = T\left(\left(\frac{2\lambda^q - 1}{-\lambda^{q^2+q} + \lambda^q}\right)\left(\frac{-2\lambda + 1}{\lambda - \lambda^{q+1}} + 2\right)\right) = T(C)$$

where

$$\begin{aligned}
C &= \frac{(2\lambda^q - 1)(-2\lambda + 1) + 2(2\lambda^q - 1)(\lambda - \lambda^{q+1})}{(-\lambda^{q^2+q} + \lambda^q)(\lambda - \lambda^{q+1})} \\
&= \frac{-4\lambda^{q+1} + 2\lambda^q + 2\lambda - 1 + 4\lambda^{q+1} - 4\lambda^{2q+1} - 2\lambda + 2\lambda^{q+1}}{-1 + \lambda^q + \lambda^{q+1} - \lambda^{2q+1}}
\end{aligned}$$

and so we may write $Q(v_1 + v_2)$ as

$$\begin{aligned}
T\left(\frac{2(\lambda^q + \lambda^{q+1})}{-1 + \lambda^q + \lambda^{q+1} - \lambda^{2q+1}}\right) &+ T\left(\frac{-4\lambda^{2q+1}}{-1 + \lambda^q + \lambda^{q+1} - \lambda^{2q+1}}\right) \\
&+ T\left(\frac{-1}{-1 + \lambda^q + \lambda^{q+1} - \lambda^{2q+1}}\right) \quad (5.2.1.3)
\end{aligned}$$

Consider the first term of (5.2.1.3). Pulling the 2 through and then proceeding as we did to obtain the equation (5.2.1.1), we see that it is 2 if

$$(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+1}(\lambda^q+\lambda^{q+1})^q+(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q+1}(\lambda^q+\lambda^{q+1})^{q^2} + (-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+q}(\lambda^q+\lambda^{q+1}) = d_4 \quad (5.2.1.4)$$

Denote the first three terms of (5.2.1.4) as s_1, s_2, s_3 respectively. Using the calculation for d_1, we have

$$\begin{aligned} &s_1 \\ &= (2-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q-\lambda-\lambda^{q+1}+2\lambda^{q+2}-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2})(\lambda^{q^2}+\lambda^{q^2+q}) \\ &= 2\lambda^{q^2}+2\lambda^{q^2+q}-\lambda^{2q^2+1}-\lambda^{2q^2+q+1}+\lambda^{2q^2+2}+\lambda^{2q^2+q+2}-\lambda^{q^2+q}-\lambda^{q^2+2q}-\lambda^{q^2+1} \\ &\quad -\lambda^{q^2+q+1}-\lambda^{q^2+q+1}-\lambda^{q^2+2q+1}+2\lambda^{q^2+q+2}+2\lambda^{q^2+2q+2}-\lambda^{q^2+2}-\lambda^{q^2+q+2} \\ &\quad +\lambda^{q^2+2q+1}+\lambda^{q^2+3q+1}-\lambda^{q^2+2q+2}-\lambda^{q^2+3q+2} \\ &= 2\lambda^{q^2}+2\lambda^{q^2+q}-\lambda^{2q^2+1}-\lambda^{q^2}+\lambda^{2q^2+2}+\lambda^{q^2+1}-\lambda^{q^2+q}-\lambda^{q^2+2q}-\lambda^{q^2+1}-1-1 \\ &\quad -\lambda^q+2\lambda+2\lambda^{q+1}-\lambda^{q^2+2}-\lambda+\lambda^q+\lambda^{2q}-\lambda^{q+1}-\lambda^{2q+1} \\ &= -2+\lambda^{q^2}+\lambda^{q^2+q}-\lambda^{2q^2+1}+\lambda^{2q^2+2}-\lambda^{q^2+2q}+\lambda+\lambda^{q+1}-\lambda^{q^2+2}+\lambda^{2q}-\lambda^{2q+1} \end{aligned}$$

and then

$$\begin{aligned} &s_2 = s_1^q \\ &= (-2+\lambda^{q^2}+\lambda^{q^2+q}-\lambda^{2q^2+1}+\lambda^{2q^2+2}-\lambda^{q^2+2q}+\lambda+\lambda^{q+1}-\lambda^{q^2+2}+\lambda^{2q}-\lambda^{2q+1})^q \\ &= -2+\lambda+\lambda^{q^2+1}-\lambda^{q+2}+\lambda^{2q+2}-\lambda^{2q^2+1}+\lambda^q+\lambda^{q^2+q}-\lambda^{2q+1}+\lambda^{2q^2}-\lambda^{2q^2+q} \end{aligned}$$

while

$$\begin{aligned} &s_3 = s_2^q \\ &= (-2+\lambda+\lambda^{q^2+1}-\lambda^{q+2}+\lambda^{2q+2}-\lambda^{2q^2+1}+\lambda^q+\lambda^{q^2+q}-\lambda^{2q+1}+\lambda^{2q^2}-\lambda^{2q^2+q})^q \\ &= -2+\lambda^q+\lambda^{q+1}-\lambda^{q^2+2q}+\lambda^{2q^2+2q}-\lambda^{q+2}+\lambda^{q^2}+\lambda^{q^2+1}-\lambda^{2q^2+q}+\lambda^2-\lambda^{q^2+2} \end{aligned}$$

and $s_1+s_2+s_3=d_4$. Now the second term of (5.2.1.3) equals $4T(A)$, which is 4 (using the calculation for $f_Q(v_1,v_1)$). The third term of (5.2.1.3) is 1 if

$$-(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+1}-(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q+1} -(-1+\lambda^q+\lambda^{q+1}-\lambda^{2q+1})^{q^2+q}=d_4 \quad (5.2.1.5)$$

Denote the first three terms of (5.2.1.5) by t_1, t_2, t_3. Again using the calculation for d_1, we have

$$t_1 = 2-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q-\lambda-\lambda^{q+1}+2\lambda^{q+2}-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2}$$

so that

$$\begin{aligned}t_2 &= t_1^q\\&= (2-\lambda^{q^2+1}+\lambda^{q^2+2}-\lambda^q-\lambda-\lambda^{q+1}+2\lambda^{q+2}-\lambda^2+\lambda^{2q+1}-\lambda^{2q+2})^q\\&= 2-\lambda^{q+1}+\lambda^{2q+1}-\lambda^{q^2}-\lambda^q-\lambda^{q^2+q}+2\lambda^{q^2+2q}-\lambda^{2q}+\lambda^{2q^2+q}-\lambda^{2q^2+2q}\end{aligned}$$

and

$$\begin{aligned}t_3 &= t_2^q\\&= (2-\lambda^{q+1}+\lambda^{2q+1}-\lambda^{q^2}-\lambda^q-\lambda^{q^2+q}+2\lambda^{q^2+2q}-\lambda^{2q}+\lambda^{2q^2+q}-\lambda^{2q^2+2q})^q\\&= 2-\lambda^{q^2+q}+\lambda^{2q^2+q}-\lambda-\lambda^{q^2}-\lambda^{q^2+1}+2\lambda^{2q^2+1}-\lambda^{2q^2}+\lambda^{q^2+2}-\lambda^{2q^2+2}\end{aligned}$$

and then $t_1+t_2+t_3=d_4$.

Now given $x, y \in \mathrm{GF}(q)$,

$$\begin{aligned}Q(xv_1+yv_2) &= x^2Q(v_1)+xyf_Q(v_1,v_2)+y^2Q(v_2)\\&= x^2+3xy+3y^2\end{aligned}$$

so that $\Sigma \cap g^j\Sigma$ is anisotropic if $x^2+3xy+3y^2$ has only the trivial zero in $\mathrm{GF}(q)^2$. Dividing by y^2, we want to know if $P(z)=z^2+3z+3$ has zeros (where $z=\frac{x}{y}$). When q is odd, $P(z)$ has no zeros since -3 is a non–square of $\mathrm{GF}(q)$. When q is even, $P(z)$ has no zeros since (letting Tr denote the absolute trace function $\mathrm{Tr}: \mathrm{GF}(q)\to\mathrm{GF}(2)$) $\mathrm{Tr}(\frac{1.3}{3^2})=\mathrm{Tr}(1)=1$.

5.3 Stabilisers of the new packings

Theorem 5.3.1 ([60]). *Let $q\equiv 2 \pmod 3$ and Υ be a packing of* $\mathrm{PG}(3,q)$ *constructed as in Theorem 4.1. Then* $\mathrm{P\Gamma L}(4,q)_\Upsilon \cong \mathrm{C}_{q^2+q+1}\rtimes \mathrm{C}_{3h}$ *for $q>2$, while* $\mathrm{P\Gamma L}(4,q)_\Upsilon\cong \mathrm{P\Gamma L}(3,2)$ *for $q=2$.*

Proof. First, note that $|G|=(q^2+q+1)|G_S|$ for any spread S of Υ, while $|\mathrm{PGL}(4,q)_S|=2(q+1)(q^2+1)q^2(q^2-1)$ (see [27]) and $|G_S|\,\big|\,|\mathrm{PGL}(4,q)_S|$. Hence,

$$|G|\,\big|\,(q^2+q+1)2(q+1)(q^2+1)q^2(q^2-1) \tag{5.3.1}$$

while $q^6\,\big|\,|\mathrm{PSL}(4,q)|$, so $\mathrm{PSL}(4,q)\nleq G$. By Aschbacher's theorem (see [40, Theorem 1.2.1]), any subgroup of $\mathrm{GL}(4,q)$ (any q) not containing $\mathrm{SL}(4,q)$ is contained in a group of one of the following categories.

$\mathcal{C}_1$: stabiliser of a subspace of $\mathrm{PG}(3,q)$

$\mathcal{C}_2$: stabiliser of four points of $\mathrm{PG}(3,q)$, no three coplanar, or stabiliser of a pair of non–incident lines of $\mathrm{PG}(3,q)$

$\mathcal{C}_3$: stabiliser of an extension field $\mathrm{GF}(q^2)$ of $\mathrm{GF}(q)$

$\mathcal{C}_5$: stabiliser of a subfield $\mathrm{GF}(q_0)$ of $\mathrm{GF}(q)$ of prime index b

$\mathcal{C}_6$: normaliser of a symplectic group over $\mathrm{GF}(r)$ (r prime) in an absolutely irreducible representation

$\mathcal{C}_8$: similarity group of a non–degenerate 4–dimensional polar space over $\mathrm{GF}(q)$

$\mathcal{C}_9$: refer to [40, p3]

We will be ruling out all but $\mathcal{C}_1$; the groups and their orders for the categories $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_8$ are as follows.

$\mathcal{C}_2$: $\mathrm{GL}(a,q)\,\mathrm{wr}\,\mathrm{S}_t$, $at=4$ $(t\geq 2)$; $\quad (q^{\frac{a(a-1)}{2}}\Pi_{i=1}^{a}(q^i-1))t!$

$\mathcal{C}_3$: $\mathrm{GL}(2,q^2):\mathrm{C}_2$; $\quad q^2(q^2-1)(q^4-1)2$

$\mathcal{C}_5$: $\mathrm{GL}(4,q_0)$; $\quad q_0^6\prod_{i=1}^{4}(q_0^i-1)$

$\mathcal{C}_6$: $(\mathrm{C}_{q-1}\circ 2^{1+4}):\mathrm{Sp}(4,2)$; $\quad (q-1)2^9.3^2.5$

$\mathcal{C}_8$: $\mathrm{Sp}(4,q)$, $\mathrm{O}^+(4,q)$, $\mathrm{O}^-(4,q)$, $\mathrm{U}(4,q^{\frac{1}{2}})$; $\quad q^4\Pi_{i=1}^{2}(q^{2i}-1)$, $2q^2(q^2-1)^2$, $\quad 2q^2(q^2+1)(q^2-1)$, $\quad (q^{\frac{1}{2}})^6\Pi_{i=1}^{4}((q^{\frac{1}{2}})^i-(-1)^i)$ (respectively)

Suppose $n\geq 3$. A q–**primitive prime divisor** of q^n-1 is a prime s with $s\,|\,q^n-1$, such that $s\nmid q^i-1$ for $i<n$. As long as $(q,n)\neq(2,6)$, a q–primitive prime divisor of q^n-1 exists ([87]); we now use these numbers to rule out the categories $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_8$.

Let $G=\mathrm{PGL}(4,q)_\Upsilon$, $r\in G$ have $|r|=q^2+q+1$ and s be a q–primitive prime divisor of q^3-1. If G is to be a subgroup of a group of $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_5$, $\mathcal{C}_6$ or $\mathcal{C}_8$, then s has to divide that group's order. First consider $\mathcal{C}_2$. If $a=1$, $|\,\mathrm{GL}(a,q)\,\mathrm{wr}\,\mathrm{S}_t\,|=(q-1)4!$, and note that $s\nmid q-1$, while $s\neq 2$ since q^2+q+1 is odd (we will use that q^2+q+1 is odd a number of times in the below). If $a=2$, $|\,\mathrm{GL}(a,q)\,\mathrm{wr}\,\mathrm{S}_t\,|=q(q-1)(q^2-1)2$, and s divides none of q, $q-1$, q^2-1, 2. For $\mathcal{C}_3$, if $s\,|\,q^4-1$ then $s\,|\,q^2+1$, which implies that $s\,|\,q$, again a contradiction. To rule out $\mathcal{C}_5$, let $q_0=p^d$ for some prime p, and (except when $q_0=4$) we can take a p–primitive divisor of q^3-1 to see that r can't be an element of $\mathrm{GL}(4,q_0)$ (in the exceptional case, it is enough to note that $\mathrm{GL}(4,2)$ has no elements of order 21). For $\mathcal{C}_6$, recall that $(q^2+q+1,q-1)=1$, while

$(q^2+q+1, 2^9) = 1$. Also, if $s = 3$ then $3 \mid q^3-1$, a contradiction, so $(q^2+q+1, 3^2) = 1$. Hence, $q^2+q+1 \mid 5$, a contradiction. Finally, the groups of $\mathcal{C}_8$ can be ruled out in a similar way to those of $\mathcal{C}_2$ and $\mathcal{C}_3$.

By [49], [86] and [68], the groups of $\mathcal{C}_9$ are $\mathrm{PSL}(2,q_0)$ $(q_0^{2v} = q)$, $\mathrm{Sz}(q_0)$ $(q_0^{2v} = q)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(4,3)$, A_5, A_6, A_7 (some of these groups occur according to a condition on q). For the former two, we have

$$|\,\mathrm{PSL}(2,q_0)| = \frac{q_0(q_0^2-1)}{(2,q_0-1)}$$

and

$$\mathrm{Sz}(q_0) = q_0^2(q_0-1)(q_0^2+1)$$

and these groups are dealt with as for $\mathrm{GL}(4,q_0)$. The condition for $\mathrm{PSL}(2,7)$ to occur is $q^3 \equiv 1 \pmod 7$. Now if $q^2+q+1 \mid |\,\mathrm{PSL}(2,7)| = 2^3.21$, then $q^2+q+1 \leq 21$ and so q is 2 or 4; since $\mathrm{PSL}(2,7)$ contains no elements of order 21, $q = 2$. The condition for $\mathrm{PSL}(3,4)$ to occur is that q is a power of 9, while if $q^2+q+1 \mid |\,\mathrm{PSL}(3,4)| = 2^6.3^2.5.7$, then $q = 9$ is forced, a contradiction. The condition for $\mathrm{PSp}(4,3)$ to appear is that $3 \nmid q$ and $q \geq 5$. If $q^2+q+1 \mid |\,\mathrm{PSp}(4,3)| = 2^6.3^4.5$, we have $q \leq 19$, and then for each possible q we obtain a contradiction. If A_5 or A_6 occurs then $q^2+q+1 \mid 15$ or $q^2+q+1 \mid 45$ respectively, both not possible. The condition for A_7 to appear is that $q^3 \equiv 1 \pmod 7$ or $7 \mid q$. Now if $q^2+q+1 \mid |\,\mathrm{A}_7\,|$ then $q \leq 17$, and the only possibilities are $q = 2$ and 4, while A_7 having no elements of order 21 implies that $q = 2$. Hence, if G is a group of $\mathcal{C}_9$, we have $q = 2$ and the socle of G is isomorphic to A_7 or $\mathrm{PSL}(2,7)$. If $q = 2$, we have $G \cong \mathrm{P\Gamma L}(3,2) \cong \mathrm{PSL}(2,7)$ (see [27, Theorem 17.5.6]), and so A_7 doesn't arise.

We are left with $\mathcal{C}_1$. Because G is acting regularly on a partition of the set of lines of $\mathrm{PG}(3,q)$, G doesn't fix any line; hence, G fixes a point P or a plane π of $\mathrm{PG}(3,q)$. We now show that G acts faithfully on $\mathrm{PG}(3,q)/P$ or π (we prove these two facts concurrently). Let $g \in G$, and note that g is a collineation of the projective plane $\mathrm{PG}(3,q)/P$ or π. If g acts trivially on $\mathrm{PG}(3,q)/P$ or π, then g has centre P or π. Let l be a line on P or in π and $S \in \Upsilon$ with $l \in S$. Since g fixes l, g fixes S, so g fixes every line of S. Now let $k \in \mathrm{GL}(4,q)$ have $kz = g$, for some scalar map z of $\mathrm{GL}(4,q)$. Then k is in the kernel of S, and the kernel of a regular spread of $\mathrm{PG}(3,q)$ is $\mathrm{GF}(q^2)$, while the only elements of $\mathrm{GF}(q^2)$ with fixed points are those in $\mathrm{GF}(q)$. Hence, $k \in \mathrm{GF}(q)^*$, and thus g is the identity. As a result, G induces a group $\overline{G}$ on $\mathrm{PG}(2,q)$, and $G \cong \overline{G} \leqslant \mathrm{PGL}(3,q)$ (let $\overline{r}$ be the resulting image in $\mathrm{PGL}(3,q)$ of r). Note that $\mathrm{PSL}(3,q) \nleqslant \overline{G}$, since $q^3 \mid |\,\mathrm{PSL}(3,q)|$ and $q^3 \nmid |\overline{G}|$ (using (5.3.1)).

Again we apply Aschbacher's theorem, this time to determine which sort of subgroup of $\mathrm{PGL}(3,q)$ that $\overline{G}$ is. The categories that arise are

$\mathcal{C}_2$: stabiliser of three points of $\mathrm{PG}(2,q)$, no two on a line

$\mathcal{C}_3$: stabiliser of an extension field $\mathrm{GF}(q^3)$ of $\mathrm{GF}(q)$

$\mathcal{C}_5$: stabiliser of a subfield $\mathrm{GF}(q_0)$ of $\mathrm{GF}(q)$ of prime index b

$\mathcal{C}_6$: normaliser of a symplectic group over $\mathrm{GF}(r)$ (r prime) in an absolutely irreducible representation

$\mathcal{C}_8$: similarity group of a non–degenerate 3–dimensional polar space over $\mathrm{GF}(q)$

$\mathcal{C}_9$: refer to [40, p3]

The category $\mathcal{C}_1$ consists of groups stabilising a subspace of $\mathrm{PG}(2,q)$; to see that it doesn't occur, first consider $\mathrm{PG}(3,q)/P$, and suppose that a point of $\mathrm{PG}(3,q)/P$ is fixed by $\overline{G}$. Then a line of $\mathrm{PG}(3,q)$ on P is fixed by G, which is impossible. If a line of $\mathrm{PG}(3,q)/P$ is fixed, then a plane U of $\mathrm{PG}(3,q)$ on P is fixed by G. Now there are $q+1$ lines of U on P, and G stabilises these lines setwise. But G has order at least q^2+q+1, so some of these lines must be fixed, a contradiction (these arguments also treat the case where G fixes π).

The groups and their orders for the categories $\mathcal{C}_2$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_8$ are as follows.

$\mathcal{C}_2$: $\mathrm{GL}(1,q)\,\mathrm{wr}\,\mathrm{S}_3$; $(q-1)3!$

$\mathcal{C}_5$: $\mathrm{GL}(3,q_0)$ (where $q=q_0^b$) ; $q_0^3\prod_{i=1}^3(q_0^i-1)$

$\mathcal{C}_6$: $(\mathrm{C}_{q-1}\circ 3^{1+2}) : \mathrm{Sp}(2,3)$; $(q-1)2^3.3^4$

$\mathcal{C}_8$: $\mathrm{GO}(3,q)$, $\mathrm{GU}(3,q^{\frac{1}{2}})$; $(2,q-1)q(q^2-1)$, $(q^{\frac{1}{2}})^3\Pi_{i=1}^3((q^{\frac{1}{2}})^i-(-1)^i)$

Using primitive prime divisors, we can rule out the above cases with the same arguments as we used for the categories of $\mathrm{GL}(4,q)$. By [45] and [25], the groups of $\mathcal{C}_9$ are A_5, A_6, $\mathrm{PSL}(2,7)$, and we can argue as before to rule out A_5 and A_6, and to show that $\mathrm{PSL}(2,7)$ occurs only if $q=2$.

We are left with $\mathcal{C}_3$, so that $\overline{G}\leqslant \mathrm{N}_{\mathrm{PGL}(3,q)}(\langle\overline{r}\rangle)$, that is, $G\leqslant \mathrm{N}_{\mathrm{PGL}(4,q)}(\langle r\rangle)$. By [30, II.7.3],

$$\mathrm{N}_{\mathrm{PGL}(4,q)}(\langle r\rangle)\cong \mathrm{C}_{q^3-1}\rtimes \mathrm{C}_3$$

Define maps $\underline{\hat{g}}$, $\hat{s}$, $\hat{t}_1$, $\hat{t}_2$ on $\mathrm{P\Gamma O}^+(6,q)$ via

$$\begin{aligned}
\underline{\hat{g}}(\langle(x,y)\rangle) &= \langle(\omega x, \frac{1}{\omega}y)\rangle \\
\hat{s}(\langle(x,y)\rangle) &= \langle(x^q, y^q)\rangle \\
\hat{t}_1(\langle(x,y)\rangle) &= \langle(y,x)\rangle \\
\hat{t}_2(\langle(x,y)\rangle) &= \langle(-x,y)\rangle
\end{aligned}$$

(where $\mathrm{GF}(q^3)^* = \langle\omega\rangle$). These maps belong to $N = \mathrm{N}_{\mathrm{PGO}^+(6,q)}(\langle\hat{g}\rangle)$, and

$$|\langle\underline{\hat{g}}, \hat{s}, \hat{t}_1, \hat{t}_2\rangle| = 6(q^3-1)$$

(note that $\underline{\hat{g}}$ has order $\frac{q^3-1}{(2,q-1)}$). Also,

$$\frac{N}{\mathrm{N}_{\mathrm{PGL}(4,q)}(\langle r\rangle)} = 1 \text{ or } 2$$

depending on whether or not $\mathrm{P\Gamma O}^+(6,q)_\Pi$ (Π as in Theorem 5.2.1.1) contains an element corresponding to a correlation of $\mathrm{PG}(3,q)$, respectively (for, as in the remark after Corollary 1.2.2.3, $|\,\mathrm{P\Gamma O}^+(6,q)_\Pi| = 2|\,\mathrm{P\Gamma L}(4,q)_\Upsilon|$ in the latter instance). In any case, $|N|\,\big|\,6(q^3-1)$, and so $N = \langle\underline{\hat{g}}, \hat{s}, \hat{t}_1, \hat{t}_2\rangle$. We now show that $N \cap \langle\underline{\hat{g}}\rangle = \langle\hat{g}\rangle$. Clearly, $\langle\hat{g}\rangle \leqslant \langle\underline{\hat{g}}\rangle$, so that $\langle\hat{g}\rangle \leqslant N \cap \langle\underline{\hat{g}}\rangle$. If $l_1 \in (N \cap \langle\underline{\hat{g}}\rangle) - \langle\hat{g}\rangle$, then $\exists\, l_2 \in N \cap \langle\underline{\hat{g}}\rangle$ defined by $l_2(\langle(x,y)\rangle) = \langle(ax, \frac{1}{a}y)\rangle$, where $a \in \mathrm{GF}(q)^*$. But

$$l_2(\Sigma^\perp) = \{(x, \frac{1}{a^2}x^q) : T(x) = 0\}$$

which equals $\Sigma^\perp$ if and only if $a^2 = 1$, and thus $l_2 = 1$. But $l_2 = l_1\hat{g}^i$ for some $i \in I$ (I as in Theorem 5.2.1.1), so that $l_1 = \hat{g}^{-i}$, a contradiction.

Now

$$\frac{N}{\langle\underline{\hat{g}}\rangle} \cong \mathrm{C}_6 \times \mathrm{C}_{(2,q-1)}$$

and

$$\frac{\langle\hat{s}, \underline{\hat{g}}\rangle}{\langle\underline{\hat{g}}\rangle} \lhd \frac{N}{\langle\underline{\hat{g}}\rangle}$$

Thus,

$$\frac{N}{\langle\hat{s}, \underline{\hat{g}}\rangle} \cong \mathrm{C}_2 \times \mathrm{C}_{(2,q-1)}$$

Suppose that $\hat{t}_1$ stabilises Π; we may assume that $\hat{t}_1$ stabilises Σ, since we have a group acting regularly on Π. So, given $(x,y) \in \Sigma$, we must have $(y,x) \in \Sigma$. Let $x^{q^2} + y = a \in \mathrm{GF}(q)$ and $x + y^{q^2} = b \in \mathrm{GF}(q)$. From these equations we easily obtain (by eliminating the term in y) that $x^q - x = x^{q^2} - x^q = x - x^{q^2} = a - b$, so that $x^q - x + x^{q^2} - x^q + x - x^{q^2} = 3(a-b)$, and hence $a = b$. Thus, $x^q = x$, which implies that $x \in \mathrm{GF}(q)$ and therefore that $y \in \mathrm{GF}(q)$, a contradiction. It can also be checked that $\hat{t}_2$, $\hat{t}_1\hat{t}_2$ don't stabilise Π for q odd, while defining the map $\hat{t}_3$ as for l_2, we see that $\hat{t}_1$, $\hat{t}_1\hat{t}_3$, $\hat{t}_2\hat{t}_3$, $\hat{t}_1\hat{t}_2\hat{t}_3$ do not stabilise Π. Hence, $N_\Pi = \langle \hat{g}, \hat{s}\rangle$.

Let $q = p^h$, p prime. Then $|\,\mathrm{P\Gamma O}^+(6,q) : \mathrm{PGO}^+(6,q)| = h$, and so

$$|\,\mathrm{P\Gamma O}^+(6,q)_\Pi : \mathrm{PGO}^+(6,q)_\Pi| \leq h$$

Defining $\underline{\hat{s}}$ via $\underline{\hat{s}}(\langle (x,y)\rangle) = \langle (x^p, y^p)\rangle$, we have $\underline{\hat{s}} \in \mathrm{P\Gamma O}^+(6,q)_\Pi$, so equality occurs and hence

$$\mathrm{P\Gamma O}^+(6,q)_\Pi = \langle \hat{g}, \underline{\hat{s}}\rangle \cong \mathrm{C}_{q^2+q+1} \rtimes \mathrm{C}_{3h}$$

Thus, $\mathrm{P\Gamma L}(4,q)_\Upsilon$ is of the stated form.

Remark. Because $\hat{t}_1$ corresponds under $\overline{\kappa}^{-1}$ to a correlation of $\mathrm{PG}(3,q)$, two inequivalent regular packings of $\mathrm{PG}(3,q)$ are obtained for each $q \equiv 2 \pmod 3$ (recall the remark after Corollary 1.2.2.3). Explicitly, replacing Σ by $\Sigma' = \{(y,z) : y^q + z \in \mathrm{GF}(q)\}$ gives a partition $\Pi' = \{S_i' : i \in I\}$, with $\Upsilon' = \{\kappa^{-1}(S_i') : i \in I\}$ inequivalent to Υ.

□

5.4 Regular packings and translation planes

Prince found all cyclic packings of $\mathrm{PG}(3,5)$, finding two regular ones; thus his packings are the same as those we construct for $q = 5$. Denniston found all packings of $\mathrm{PG}(3,8)$ admitting $\mathrm{C}_{73} \rtimes \mathrm{C}_9$, finding two regular ones; thus his packings are the same as those we construct for $q = 8$.

There is a construction of Walker ([82], and due independently to Lunardon in [42]) which yields translation planes of order q^4 with kernel $\mathrm{GF}(q)$ from regular packings of $\mathrm{PG}(3,q)$. The construction is non–trivial, but in short it works by obtaining a $\mathrm{PG}(7,q)$ spread from the given packing, to which the Bruck–Bose construction applies. In particular, a translation plane admitting $\mathrm{SL}(2,q) \times \mathrm{C}_{q^2+q+1}$ in the translation complement corresponds to a cyclic regular packing of $\mathrm{PG}(3,q)$ (this can be extracted from [31]). Hence, since regular packings of $\mathrm{PG}(3,q)$ were previously known to exist only for $q = 2,5,8$, the translation planes arising from our construction are presumably new for $q > 8$.

The Lorimer–Rahilly and Johnson–Walker planes of order 16 (given in [32]) each admit $\mathrm{SL}(2,2) \times \mathrm{C}_7$ in the translation complement, and so arise from the two inequivalent packings of $\mathrm{PG}(3,2)$ (while the Prince planes of order 625 and the Denniston–Walker planes of order 4096 arose directly from the cyclic regular packings of $\mathrm{PG}(3, 5)$ and $\mathrm{PG}(3, 8)$, respectively).

$\square$

Searching for ovoids

The techniques we describe in this chapter are often applicable to searches for configurations in any incidence structure, even though we will only discuss them with regards to ovoids of polar spaces.

6.0.1 The standard approach. When conducting a computer search for ovoids in a polar space $\mathcal{S}$, often we make an hypothesis that some group G (where $G \leqslant \mathrm{P}\Gamma\mathcal{S}$) fixes an ovoid. After breaking down the set $\mathcal{P}$ of isotropic/singular points of $\mathcal{S}$ into orbits of G, we determine the set $\mathcal{C}$ of those orbits that are caps. We are then left with a graph–theoretic problem: let Γ be a graph with vertex set $\mathcal{C}$ in which two vertices C_1, C_2 are adjacent if $C_1 \cup C_2$ is a cap (in which case we say C_1 and C_2 are **compatible**), and search for a clique of Γ that has the right size to be an ovoid of $\mathcal{S}$.

The model for $\mathcal{S}$ is often chosen so that G can act naturally. For example, if G is a symmetric group and $\mathcal{S}$ an orthogonal space, an appropriate model would be a vector space consisting of tuples, equipped with the "dot product" quadratic form. If G is a matrix group, an appropriate model would be one such as Kantor used in his construction of the Thas–Kantor ovoids (see Chapter 3).

6.0.2 Computational savings. There are a number of elementary ways of speeding up ovoid searches. To begin with, the set $\mathcal{P}$ can be parametrised, and then representatives for the orbits of G on $\mathcal{P}$ can be determined. To find those orbits that are caps and then which of those caps are compatible, the following result is most useful.

Lemma 6.0.2.1. *Let $G \leqslant \mathrm{P}\Gamma\mathcal{S}$ for $\mathcal{S} = (V, \beta)$ a polar space.*

(a) Suppose Ω is an orbit of G on isotropic/singular points of $\mathcal{S}$. If Ω contains a point not collinear with any other point of Ω, then Ω is a cap.

(b) Suppose Ω_1 and Ω_2 are orbits of G on isotropic/singular points of $\mathcal{S}$, such that Ω_1 and Ω_2 are caps. If Ω_1 contains a point not collinear with any point of Ω_2, then Ω_1 and Ω_2 are compatible.

Proof. (a) Let P be a point of Ω not collinear with any other point of Ω. If points R_1 and R_2 of Ω ($P \notin \{R_1, R_2\}$) are collinear, take $g \in G$ such that $gR_1 = P$. Then P and gR_2 are collinear, a contradiction.

(b) Let P be a point of Ω_1 not collinear with any point of Ω_2. If a point R_1 of Ω_1 is collinear to a point R_2 of Ω_2, take $g \in G$ such that $gR_1 = P$. Then P and gR_2 are collinear, a contradiction. □

At the clique search stage, we have

Lemma 6.0.2.2. *Let $G \leqslant \mathrm{P\Gamma}\mathcal{S}$ for $\mathcal{S} = (V, \beta)$ a polar space, and suppose that there is an ovoid fixed by G. Then $\mathrm{N}_{\mathrm{P\Gamma}\mathcal{S}}(G)$ permutes the ovoids fixed by G.*

Proof. By Theorem 1.5.1.2, $\mathrm{N}_{\mathrm{P\Gamma}\mathcal{S}}(G)$ permutes the orbits of G – being a subgroup of $\mathrm{P\Gamma}\mathcal{S}$, $\mathrm{N}_{\mathrm{P\Gamma}\mathcal{S}}(G)$ takes caps to caps and ovoids to ovoids. □

Knowing any orbits of G which by virtue of their length are or aren't on any ovoid fixed by G also aids a clique search (assuming that there are orbits of different sizes to begin with). Most of all, the bigger G is, the easier the clique search is.

6.0.3 Invariants. If we have found a new ovoid in some orthogonal space, the following invariants can show quite quickly that it is new. The first one is elementary.

Theorem 6.0.3.1. *Let O be an ovoid of an orthogonal space $\mathcal{S}$ over $\mathrm{GF}(q)$, and F be an array of length $q+1$, consisting of 0's. For each triple t_1, t_2, t_3 of O, $\langle t_1, t_2, t_3 \rangle$ is a non–degenerate plane; let the i–th place of F be the number of such planes meeting O in i points (for $i \geq 3$). Then any ovoid of $\mathcal{S}$ equivalent to O has array F.*

Proof. Clear. □

Note that this invariant is quite cheap to apply to ovoids whose stabilisers are 2–transitive or 3–transitive on them, as then just all triples on an arbitrary pair or just one triple (respectively) need be considered to establish the array for that ovoid. The next invariant we will consider is significantly cheaper than the one above when dealing with an ovoid that doesn't admit a transitive group.

Suppose we have determined the arrays of Theorem 6.0.3.1 for all the known ovoids of some orthogonal space $\mathcal{S}$, and that a search in $\mathcal{S}$ turns up an ovoid O. If each r–th place of these arrays of the known ovoids is 0, then as soon as the r–th place of the array of O becomes 1, we know that O is a new ovoid. This principle also applies to the following invariant.

Theorem 6.0.3.2 ([46]). *Let $O = \{\langle v_i \rangle : 1 \leq i \leq q^{n-1}+1\}$ be an ovoid of $\mathrm{O}^+(2n, q)$ (q odd), where the space admits a polar form f_Q. Define a matrix A having $A_{ij} = 0$ for $i = j$, and $A_{ij} = \pm 1$ for (respectively) $f_Q(v_i, v_j)$ a square or non–square of $\mathrm{GF}(q)$.*

Let $|AA^T|$ be the matrix obtained by taking the absolute value of each entry of AA^T. Form an array F of length $|O|$, whose i–th place is the number of entries of $|AA^T|$ equal to i. Then any ovoid of $\mathrm{O}^+(2n,q)$ equivalent to O has array F.

Proof. Moorhouse in [46] states that the invariant is due to Conway when $n = 3$; see [13] for a proof. $\square$

The invariant of Theorem 6.0.3.2 is not complete for $n = 2$ and complete for the known ovoids for $p \leq 11$ when $n = 4$ ([46]), while the invariant of Theorem 6.0.3.1 is not complete.

6.0.4 Some specific strategies. **(1)** There is a method (learnt from Gordon Royle who applied it to BLT sets) that can inform the choice of group hypothesis and model to work in. The idea is as follows: take a known ovoid O of a polar space $\mathcal{S}$, and consider $\mathrm{P\Gamma}\mathcal{S}_O$. Any subgroup of $\mathrm{P\Gamma}\mathcal{S}_O$ is guaranteed to fix O – moving down the subgroup lattice of $\mathrm{P\Gamma}\mathcal{S}_O$, the hope is that some other ovoid inequivalent to O will be fixed. This approach is especially useful when group actions yielding ovoids are hard to come by.

What can make this method time consuming (aside from when a small group is used) is when multiple numbers of ovoids are obtained: they have to be tested for equivalence. In practice, it is often the case that two ovoids have the same array with respect to the invariant being used. If we want to make sure that we don't miss any new ovoids, we must look for a projective semisimilarity between all such ovoids, or else calculate their stabilisers and then test conjugacy.

In any case, when applying this method, the invariant we use can give us more information than just telling us that ovoids are inequivalent. For example, suppose that some group G is conjugate to a subgroup of each stabiliser of m of the known ovoids of a polar space $\mathcal{S}$, and that searching in $\mathcal{S}$ with G we obtain n ovoids ($m \leq n$), with m different arrays occuring with respect to some invariant (this situation is typical when no new ovoids have been found). Now each of the m known ovoids must be represented among the ovoids we have found (by Lemma 1.1.7.1). To tell where they occur, first take the m known ovoids and determine the array of each. Divide the n ovoids we have found into m classes, with ovoids having the same array put in the same class. For each class, if its array matches that of one of the m known ovoids, then that known ovoid must occur somewhere in that class.

(2) Another tack (one not involving a group hypothesis) is to parametrise the form of an arbitrary ovoid, and then determine conditions that the parametrising functions have to satisfy in order to describe an ovoid. For example, in Chapter 2 we saw that any $\mathrm{O}(5,q)$ ovoid can be expressed in the form

$$O(f) = \{\langle(0,0,0,0,1)\rangle\} \cup \{\langle(1,x,y,f(x,y),-y^2-xf(x,y))\rangle : x,y \in \mathrm{GF}(q)\}$$

where $f : \mathrm{GF}(q)^2 \to \mathrm{GF}(q)$. By applying the polar form to two arbitrary points of $O(f)$, the condition for $O(f)$ to be an ovoid is that

$$-(y_1-y_2)^2 - (x_1-x_2)(f(x_1,y_1)-f(x_2,y_2)) \neq 0$$

$\forall\ x_1, y_1, x_2, y_2 \in \mathrm{GF}(q)$ with $x_1 \neq x_2$ or $y_1 \neq y_2$. Although it is usually not possible to loop over all f, searches can be restricted to certain classes of f. Sometimes, doing this results in computational savings; for example, if f is an additive function, we can eliminate two variables in the above condition – it was along these lines that the new $\mathrm{O}(5,3^5)$ ovoid was found.

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