Faculteit Wetenschappen
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Constructions, characterizations and classifications of SPG-reguli, SPG-systems and the related semipartial geometries

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Wiskunde is.

Wiskunde leeft. Gebeurt. Beweegt. Wiskunde neemt. Wiskunde geeft. Wiskunde weet. Wiskunde spreekt. Wiskunde doet. Wiskunde laat. Wiskunde komt. Wiskunde gaat. Uniek. Muziek. Van vlees en bloed. Wiskunde waagt. Wiskunde wint. Breekt baan. Wiskunde bonkt. Wiskunde staat.

Wiskunde valt. Is overal. Ontroert. Verwarmt. Grijpt bij de keel. Wiskunde knettert. Knalt. Ontketent. Heerst. Wiskunde heelt. Wiskunde zuivert. Lichaam. Geest. Wiskunde swingt. Wiskunde vecht. Is waar. Is echt. Geen loze kreet. Geen leeg gebaar. Wiskunde werkt. Versterkt. Ontwapent. Toont. Wiskunde laaft. Wiskunde loont. Is water. Brood. Wiskunde lacht. Wiskunde huilt. Wiskunde in. Wiskunde uit. Legt bloot. Daagt uit. Wiskunde kookt. Wiskunde bruist. Wiskunde troost. Wiskunde bijt. Wiskunde bloedt. Heeft schijt. Is zwart. Is wit. Is rood. Niet grijs. Wiskunde vloekt. Wiskunde moet. Verbroedert. Zoekt. Wiskunde vindt. Wiskunde wijst. Wiskunde schokt. Wiskunde eist. Wiskunde hoog. Wiskunde laag. Wiskunde voor. Wiskunde na. Wiskunde rookt. Wiskunde jaagt. Is eigen baas. Vereent. Verzoent. Begeestert. Woedt. Bevrijdt. Bewijst. Begrijpt. Vervoert. Wiskunde spreidt. Wiskunde sluit. Bezielt. Verrijkt. Geeft hoop. Verblijdt. Wiskunde shittert. Glanst. Wiskunde flitst. Wiskunde danst. Verhit. Zweept op. Bemint. Verleidt. Wiskunde roept. Wiskunde voelt. Wiskunde groeit. Wiskunde bloeit. Wiskunde blaakt. Wiskunde blijkt. Betovert. Geilt. Wiskunde ademt. Zweet. Wiskunde fluistert. Schreeuwt. Ontmaskert. Snijdt. Wiskunde glijdt. Wiskunde sluipt. Wiskunde slijpt. Wiskunde spuit. Wiskunde klinkt. Wiskunde dwingt. Wiskunde lonkt. Wiskunde blinkt. Wiskunde vraagt. Wiskunde raakt. Verlost. Verbaast. Viert feest. Verklaart. Is bitter. Zoet. Is hot. Is cool. Wiskunde ijlt. Vooruit. Voorbij. Ver weg. Dichtbij. Paraat. Bereid. Op weg. Altijd. Wiskunde was. Wiskunde is.

Wiskunde blijft.

Naar “Jazz” van J. A. Deelder

Acknowledgment

From early kindergarten school mathematics - at that stage mostly numbers - have puzzled and attracted me. I remember - and so do my parents - becoming extremely angry and frustrated because, at the age of four or five, I didn't manage to completely understand the so-called rule of three. At the end it only encouraged me, and from then on I have wanted to become a mathematician.

At this young age I was lucky to make friends with Axelle and Joëlle, two sisters, respectively two and three years older than I, who explained to me their mathematical homework. In the first years of primary school some very good teachers stimulated me by giving me exercises for higher classes. So, when I arrived in the last year of primary school and stumbled on the worst teacher of my life I was so in love with mathematics even he couldn't temper this love. (I still suspect him of hating mathematics and trying to discourage me.) Luckily, once in secondary school, I again came across some very good teachers. Especially during the last three years of secondary school our mathematics course was given by a marvelous man, Fernand Surmont, who prepared me and my companion students in a perfect way for university life. His life came to a tragic end earlier this year, but I will always remember him as a brilliant mathematician (since brilliant mathematicians are not only those able to prove difficult and beautiful theorems, but also those able to transmit to others the love and passion for the "ars mathematicae"). Being well prepared, the change to university life went smoothly and it was here, at Ghent University, when getting this very nice course "projective geometry" that it became clear to me that I especially wanted to become a (finite) geometrician. After my master thesis, the subject of which was inspired by a lecture of Francis Buekenhout, who I want to thank for the interest in my work he has shown ever since, it was thanks to Frank De Clerck and Jef Thas (and my not so bad study results) that I obtained this very nice FWO-grant which allowed me to do the research that has led to this PhD thesis. So of course I want to express my gratitude to Frank De Clerck and Jef Thas, who were my supervisors, for helping me to get this grant, for encouraging me while doing research, for their interesting and helpful remarks, for their co-authorship, for teaching me how to write down mathematics, for letting me present my results at some very nice conferences,... I am also very grateful for the financial support of the Fund for Scientific Research Flanders (FWO-Vlaanderen) which made it possible to work as a full time researcher.

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Stefaan De Winter
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Preface

This thesis is the result of three years of research I did as a Research Assistant of the Fund of Scientific Research Flanders (FWO-Vlaanderen) at the Department of Pure Mathematics and Computer Algebra at Ghent University. The research is situated in the field of finite incidence geometry. More specifically SPG-reguli, SPG-systems and the related (semi)partial geometries are studied.

In Chapter 1 we recall some basic notions, introduce some notations and provide some less elementary results from group theory, the theory of generalized polygons, ... which are used throughout this dissertation.

In Chapter 2 a more detailed introduction is given to partial and semipartial geometries and their constructions via SPG-reguli and SPG-systems, the objects that form the center of this dissertation.

The next three chapters deal with characterizations and classification of the semipartial geometries that arise as a linear representation.

Chapter 3 provides a characterization of the semipartial geometries $T_2^*(\mathcal{B})$ and $T_2^*(\mathcal{U})$ which generalizes a result that was known for the partial geometry $T_2^*(\mathcal{K})$. The axiom introduced here allows to easily obtain that the semipartial geometry constructed in Chapter 8 is not isomorphic to $T_2^*(\mathcal{U})$.

Chapter 4 is the result of our attempt to classify all linear representations of semipartial geometries. However, this turned out to be beyond our forces, but resulted in several parameter restrictions and the classification of semipartial geometries with a linear representation in $\mathrm{AG}(4,q)$, the smallest case that was not solved prior to our work. Using results of De Clerck and Delanote, and De Feyter we obtain then the classification of semipartial geometries (that are no partial quadrangles) embedded in $\mathrm{AG}(4,q)$.

In the fifth chapter it is shown that, under a very mild restriction, every SPG-regulus with $t-\theta=\alpha$ is the image of the points of a Baer-subgeometry under field reduction. This result is used again in Chapter 7.

In Chapter 6 sets of subspaces of a projective space satisfying the polar property are studied and their link with SPG-reguli is explained. Several well known objects are encountered here: unitals, ovoids, m-systems,... and some theorems that were known to be true for all of these, as well as some new results are proved here in a general and uniform setting.

In Chapter 7 a theory of translation semipartial geometries is introduced. It turns out that, with possibly a finite number of exceptions, these geometries provide an alternative description for the semipartial geometries constructed from an SPG-regulus. In this new setting several results, especially about the automorphism group, are obtained. Also, generalized quadrangles admitting a regular abelian group are shown to arise necessarily from a generalized hyperoval.

In the next chapter we construct a new SPG-regulus and show that the semipartial geometry arising from it is new. The results obtained in Chapters 6 and 7 turn out to be very useful in this context.

Finally, in the last chapter, SPG-systems are studied. In the singular case a complete classification is obtained, while in the non-singular case we provide a classification of all SPG-systems of index 2.

Contents

Chapter 1

Introduction

This short introductory chapter is meant to recall some basic notions, to introduce some notation and to provide some less elementary results from group theory, the theory of generalized polygons, ... which will be used throughout this dissertation. A larger introduction to partial and semipartial geometries, the main subject of this thesis, will be given in the next chapter.

1.1 Graphs

Although we expect the reader to be familiar with elementary graph theory, we will fix here the notations for the rest of this work. Note that throughout this work we will only use undirected graphs without multiple edges.

Notation

For a graph G with vertex set C we will use the following notation: $G = (C, \sim)$, the adjacency of two vertices $c_1, c_2 \in C$ being denoted by $c_1 \sim c_2$.

If v is a vertex of a graph G, then we will denote by $\Gamma(v)$ the set of all vertices adjacent to v.

If the graph G is strongly regular, that is, every vertex is adjacent to k vertices (k is called the *valency*), every two adjacent vertices are adjacent to exactly λ common vertices and every two non-adjacent vertices are adjacent to μ common vertices, then we will use the notation $\mathrm{srg}(v, k, \lambda, \mu)$, v being the number of vertices of G.

Recall that the maximal occurring distance between two distinct vertices of a graph is called the *diameter* of that graph, and that the length of a minimal circuit of a graph is called the *girth* of that graph (if there are no circuits the graph is said to have girth ∞).

The following graphs will appear occasionally in this thesis.

The Moore graphs

Consider the following class of graphs G. The valency of G equals $k > 1$ and G is of girth 5. Then one can easily see that $v \geq k^2 + 1$, with v the number of vertices of G. The *Moore graphs* $M(k)$ are exactly the graphs in this class with $v = k^2+1$. It is known that necessarily $k \in \{2, 3, 7, 57\}$ (see for instance Biggs [4]). However for $k = 57$ it is not known whether or not such a graph exists.

1.2 Designs

We only recall the definition and standard notation.

Definition 1.2.1 A $t-(v, k, \lambda)$ *design* $\mathcal{D}$ is a set S of v elements together with a set of k-subsets of S, called *blocks*, such that any t elements of S are contained in exactly λ blocks.

1.3 Incidence geometries

Definition 1.3.1 *An incidence geometry* (or short *geometry*) $\mathcal{S}$ is a triple $\mathcal{S} = (P, B, \mathrm{I})$, with P and B non-empty disjoint sets and with I a symmetric relation between P and B called the *incidence relation*. The elements of P are called the *points* of $\mathcal{S}$, and the elements of B are called the *lines* of $\mathcal{S}$. If for a pair $(p, b) \in P \times B \cup B \times P$ there holds that $p \,\mathrm{I}\, b$, then we say that p is incident with b.

If every line of $\mathcal{S}$ is uniquely determined by the points incident with it, then the lines can be identified with the subsets of P determining them. In such a case the incidence relation simply becomes symmetrized containment and set theoretical notations will also be used in these cases.

Finally we say that two points are collinear if they are contained in a line. Dually we say that two lines are concurrent if and only if they have non-empty intersection.

The dual geometry $\mathcal{S}^D$ of $\mathcal{S}$ is defined as the geometry $\mathcal{S}^D = (B, P, \mathrm{I})$.

Notice that designs provide a class of examples of incidence geometries, the elements of the design being the points, the blocks being the lines and containment (symmetrized) being the incidence relation.

1.3.1 Automorphisms

Definition 1.3.2 An *automorphism* of a geometry $\mathcal{S} = (P, B, \mathrm{I})$ is any $\theta = (\theta_1, \theta_2)$, with θ_1 a bijection from P to P and θ_2 a bijection from B to B, such that for any $(p, b) \in P \times B$, $p \mathrm{~I~} b \iff \theta_1(p) \mathrm{~I~} \theta_2(b)$. However, in the rest of this work we will not use such a strict notation, and just say that θ is an automorphism without referring to θ_1 and θ_2.

The set of all automorphisms of $\mathcal{S}$ forms a group, *the automorphism group*, and will be denoted by $\mathrm{Aut}(\mathcal{S})$. Any subgroup of $\mathrm{Aut}(\mathcal{S})$ will be called *an* automorphism group.

Knowledge of an automorphism group of a geometry is important as this may be very useful in characterizing and recognizing the geometry in an abstract context.

1.3.2 The incidence and collinearity graph

Definition 1.3.3

- The *incidence graph* of a geometry $\mathcal{S}$ is the graph $G = (C, \sim)$ with as vertex set C the set of points and lines of $\mathcal{S}$ and in which two distinct vertices c_1 and c_2 are adjacent if and only if $c_1 \mathrm{~I~} c_2$. (Notice that here one better uses "I" instead of "$\in$" as "I" is symmetric.)
- The *collinearity graph* of a geometry $\mathcal{S}$ is the graph $G = (C, \sim)$ with as vertex set the set of points of $\mathcal{S}$ and in which two distinct vertices are adjacent if and only if they are (seen as points of $\mathcal{S}$) collinear in $\mathcal{S}$.
- We say that a geometry is *connected* if the collinearity graph is a connected graph.

1.3.3 Partial linear spaces

The complete class of incidence geometries being much too broad we will be merely interested in the class of partial linear spaces. We stress the fact that since every geometry in this thesis will be finite we slightly modify the classical definition of a partial linear space in such a way that for us a partial linear space is always finite.

Definition 1.3.4 A finite connected incidence geometry $\mathcal{S}$ is called a *partial linear space* iff every line of $\mathcal{S}$ contains exactly $s+1>1$ points, every point of $\mathcal{S}$ is contained in exactly $t+1>1$ lines, and most importantly, two distinct points are contained in at most one line. If every two distinct points are contained in exactly one line, then $\mathcal{S}$ is called a *linear space*. In either case $\mathcal{S}$ is said to be of *order* (s,t), or of order s if $s=t$. The partial linear space $\mathcal{S}$ is called *thick* if $s,t\geq 2$.

A *spread* of $\mathcal{S}$ is a set of lines of $\mathcal{S}$ partitioning the point set of $\mathcal{S}$. A *packing* of $\mathcal{S}$ is a set of spreads partitioning the line set of $\mathcal{S}$.

Notation

Let (p,L) be a non-incident point-line pair of a partial linear space $\mathcal{S}$. Then we will denote by $\alpha(p,L)$ the number of points incident with L that are collinear with p.

If x and y are distinct collinear points of $\mathcal{S}$ then $\langle x,y\rangle_{\mathcal{S}}$, or $\langle x,y\rangle$ if no confusion is possible, will denote the line determined by x and y.

If x is a point of $\mathcal{S}$, then we denote by $x^{\perp}$ the set of all points of $\mathcal{S}$ collinear with x. If U is a set of points of $\mathcal{S}$, then we denote by $U^{\perp}$ the set $\bigcap_{x\in U} x^{\perp}$.

1.4 Projective spaces

We suppose the reader to be familiar with basic projective geometry, especially Desargues' Theorem, and with the notion of affine space.

Recall that an *axiomatic (finite) projective plane* is just a $2-(q^2+q+1,q+1,1)$ design, $q>1$, and that any (finite) affine plane is a $2-(q^2,q,1)$ design $(q>1)$ and conversely.

Notation

We will use the classical notation $\mathrm{PG}(n, q)$ for a Desarguesian projective space of dimension n over the field $\mathrm{GF}(q)$. We will also use the classical notation for the groups of automorphisms of $\mathrm{PG}(n, q)$.

Finally we will use the following notation.

Suppose that $\mathcal{R}$ is a set of subspaces of the projective space $\mathrm{PG}(n, q)$. Then $\widetilde{\mathcal{R}}$ denotes the set of all points covered by all elements of $\mathcal{R}$.

1.4.1 Perspectivities

We shortly describe this class of automorphisms as it will play an important role in the theory of translation semipartial geometries.

Definition 1.4.1 Let $\mathcal{P} = \mathrm{PG}(n, q)$ be a Desarguesian projective space of dimension at least 2. Let Π be a fixed hyperplane of $\mathcal{P}$ and let p be a fixed point of $\mathcal{P}$. Then, $\mathrm{Persp}(\Pi, p)$ is the set of all automorphisms of $\mathcal{P}$ fixing Π point wise and fixing all hyperplanes containing p. It follows that $\mathrm{Persp}(\Pi, p)$ constitutes a group. The elements of $\mathrm{Persp}(\Pi, p)$ are called *perspectivities* with *axis* Π and *center* p. More specifically the elements of $\mathrm{Persp}(\Pi, p)$ are called *elations* if $p \in \Pi$, and *homologies* if $p \notin \Pi$. Further it is clear that the set of all perspectivities of $\mathcal{P}$ with a given axis or with a given center forms a group. Finally notice that the set of all elations with a given axis Π constitutes a group acting regularly on the point set of $\mathcal{P} \setminus \Pi$.

The following observation is important in the theory of Desarguesian projective spaces. We formulate it as a lemma.

Lemma 1.4.2 ([49]) *Let* $\mathrm{PG}(n, q)$ *be a Desarguesian projective space of dimension at least* 2. *Then the group of all homologies with axis a fixed hyperplane* Π *and with center a fixed point* $p \notin \Pi$ *is isomorphic to the multiplicative group of* $\mathrm{GF}(q)$.

1.4.2 The cross ratio

We recall the definition and a basic property of the cross ratio of 4 collinear points on a projective line $\mathrm{PG}(1, q)$.

Definition 1.4.3 Suppose that p_1, p_2 and p_3 are two by two distinct points on a projective line $\mathrm{PG}(1,q)$, and that p_4 is a fourth point on that line. If one chooses homogeneous coordinates in such a way that the coordinates of p_1, respectively p_2 and p_3, are $(0,1)$, respectively $(1,0)$ and $(1,1)$, then the *cross ratio* $(p_1, p_2; p_3, p_4)$ of the ordered 4-tuple (p_1, p_2, p_3, p_4) is the affine coordinate of the point p_4 with respect to the chosen coordinates if $p_4 \neq p_1$, and equals ∞ if $p_4 = p_1$.

We have the following interesting property which we state here as a lemma.

Lemma 1.4.4 ([49]) *Every linear automorphism of* $\mathrm{PG}(1,q)$ *preserves the cross ratio.*

1.4.3 Field reduction

Consider the Desarguesian projective space $\mathrm{PG}(n,q^l)$. This space corresponds to the $(n+1)$-dimensional vector space $V(n+1,q^l)$ over $\mathrm{GF}(q^l)$, which can be seen as the vector space $V(l(n+1),q)$. This latter vector space corresponds to the projective space $\mathrm{PG}(l(n+1)-1,q)$. If one goes from $\mathrm{PG}(n,q^l)$ to $\mathrm{PG}(l(n+1)-1,q)$ following the above scheme we will say that under the *field reduction* $\mathrm{GF}(q^l) \to \mathrm{GF}(q)$, the projective space $\mathrm{PG}(n,q^l)$ is mapped onto $\mathrm{PG}(l(n+1)-1,q)$. Moreover every $\mathrm{PG}(m,q^l) \subset \mathrm{PG}(n,q^l)$, $0 \leq m \leq n$, is mapped onto some $\mathrm{PG}(l(m+1)-1,q) \subset \mathrm{PG}(l(n+1)-1,q)$ in this way.

Notation

Suppose that X is a (set of) subspace(s) of $\mathrm{PG}(n,q^l)$ which is mapped onto the (set of) subspace(s) $\overline{X}$ under the field reduction $GF(q^l) \to \mathrm{GF}(q)$. Then we write $X \overset{q^l \mapsto q}{\rightarrow} \overline{X}$.

1.4.4 Embeddings

For the purpose of this work we will not use strict definitions of embeddings. Here we only notice that throughout this thesis the terminology "embedded in" will stand for "contained in", where it will always be clear from the context in which sense this is meant.

1.5 Finite classical polar spaces

We suppose the reader to be familiar with the classical approach to polar spaces, that is, via polarities and quadrics in $\mathrm{PG}(n, q)$. We will only overview notations and terminology, and recall the axiomatic approach.

A finite classical non-singular polar space is one of the following:

- a hyperbolic quadric, denoted as $Q^+(2n + 1, q)$;
- a parabolic quadric, denoted as $Q(2n, q)$;
- an elliptic quadric, denoted as $Q^-(2n + 1, q)$;
- a Hermitian variety, denoted as $H(n, q^2)$;
- a symplectic polar space, denoted as $W(n, q)$.

A singular polar space P in $\mathrm{PG}(n, q)$ is a cone $\Pi P'$, with Π a d-dimensional subspace of $\mathrm{PG}(n, q)$, and with P' a non-sigular classical polar space in some $\mathrm{PG}(n - d - 1, q) \subset \mathrm{PG}(n, q)$ disjoint from Π.

The *projective index* of a polar space P is the dimension of a largest totally singular subspace of P; a totally singular subspace of this dimension is called a *generator* of P.

1.5.1 Axiomatic approach

The following very nice theorem is the result of work done by Veldkamp [86], Tits [82], Buekenhout and Shult [11].

Theorem 1.5.1 *Suppose that P is a thick partial linear space of order (s, t), with the property that for every non-incident point-line pair (p, L), $\alpha(p, L) \in \{1, s + 1\}$ and such that no point of P is collinear with all other points of P. If for all non-incident point-line pairs (p, L), $\alpha(p, L) = 1$ then P is a generalized quadrangle (see the following section). If there exists at least one non-incident point-line pair (p, L) for which $\alpha(p, L) = s + 1$, then P is the geometry of points and lines of a finite classical (non-singular) polar space.*

1.5.2 m-Systems

Definition 1.5.2 An *m-system* $\mathcal{M}$ of a finite (non-singular) classical polar space P is a maximum set of mutually disjoint totally singular m-dimensional subspaces of P with the property that no generator of P that contains an element of $\mathcal{M}$ intersects another element of $\mathcal{M}$. These m-systems were introduced by Shult and Thas in [63]. If $m = 0$ then $\mathcal{M}$ is called an *ovoid* of P. If P has projective index n and $m = n$, then $\mathcal{M}$ is a *spread* of P; in this case the elements of $\mathcal{M}$ partition the point set of P.

It is interesting to notice that the number of elements of an m-system $\mathcal{M}$ only depends on the polar space P and not on the dimension m of the elements of $\mathcal{M}$. The number of elements of a (hypothetical) m-system of P will be denoted by $o(P)$. We list these numbers:

P	$o(P)$
$Q^+(2n+1,q)$	q^n+1
$Q(2n,q)$	q^n+1
$Q^-(2n+1,q)$	$q^{n+1}+1$
$H(2n,q^2)$	$q^{2n+1}+1$
$H(2n+1,q^2)$	$q^{2n+1}+1$

1.6 Generalized polygons

Generalized polygons were introduced by Tits in 1959 in the appendix of his celebrated paper “Sur la trialité et certains groupes qui s'en déduisent”[80]. Ever since, they have been extensively studied and have played a key role in incidence geometry. The theory of polygons being much too rich to give even an overview of the most important basic properties, we restrict ourselves to the definition and those examples and properties that we will encounter in this dissertation. For a very good and deep treatise we refer to Van Maldeghem [85].

Definition 1.6.1 A *generalized n-gon* Γ, $n > 2$, is an incidence geometry $\Gamma = (P, B, \mathrm{I})$ such that the following conditions are satisfied.

(i) Γ contains no ordinary k-gon as a subgeometry, $1 < k < n$.

(ii) Any two elements, $x, y \in P \cup B$ are contained in some ordinary n-gon in Γ.

(iii) There exists an ordinary $(n+1)$-gon in Γ.

An incidence geometry which only satisfies (*i*) and (*ii*) from the above definition is called a *weak generalized polygon*.

A (weak) generalized 4-gon, respectively 6-gon and 8-gon, will be called a (weak) *generalized quadrangle*, respectively *generalized hexagon* and *generalized octagon*.

Lemma 1.6.2 ([85]) *Every finite generalized polygon is a thick partial linear space, and hence has an order (s,t), $s,t \geq 2$.*

In the next theorem we collect some restrictions on the parameters of a finite generalized polygon.

Theorem 1.6.3 *Suppose Γ is a finite generalized n-gon of order (s,t), $n \geq 4$. Then one of the following holds.*

- (**Higman [46]**) *$n = 4$ and $s \leq t^2$; dually $t \leq s^2$;*
- (**Haemers and Roos [43]**) *$n = 6$ and $s \leq t^3$; dually $t \leq s^3$;*
- (**Higman [46]**) *$n = 8$, $s \neq t$ and $s \leq t^2$; dually $t \leq s^2$.*

Theorem 1.6.4 *Let $\Gamma = (P, B, \mathrm{I})$ be a generalized n-gon of order (s,t), with $n \in \{3,4,6,8\}$. Then*

$$|P| = \begin{cases} s^2+s+1 & \text{if } n=3 \\ (1+s)(st+1) & \text{if } n=4 \\ (1+s)(1+st+s^2t^2) & \text{if } n=6 \\ (1+s)(1+st)(1+s^2t^2) & \text{if } n=8. \end{cases}$$

Dually

$$|B| = \begin{cases} s^2+s+1 & \text{if } n=3 \\ (1+t)(st+1) & \text{if } n=4 \\ (1+t)(1+st+s^2t^2) & \text{if } n=6 \\ (1+t)(1+st)(1+s^2t^2) & \text{if } n=8. \end{cases}$$

The following theorem provides a nice graph theoretical description of generalized polygons.

Theorem 1.6.5 ([85]) *A geometry Γ is a (weak) generalized n-gon if and only if the incidence graph of Γ is a connected graph of diameter n and girth $2n$, such that each vertex lies on at least three (at least two) edges.*

We will now provide some examples.

Generalized 3-gons

The generalized 3-gons are exactly the axiomatic projective planes.

Generalized quadrangles

The finite classical (weak) generalized quadrangles are the finite classical (non-singular) polar spaces $Q^+(3,q)$, $Q(4,q)$, $Q^-(5,q)$, $H(3,q)$, $H(4,q)$ and $W(3,q)$.

Define in $\mathrm{PG}(2n+m-1,q)$ the set $O = O(n,m,q)$ as follows: O is a set of q^m+1 $(n-1)$-dimensional subspaces, denoted by π_i, such that

(i) every three generate a $\mathrm{PG}(3n-1,q)$;

(ii) for every $i = 0, 1, \ldots, q^m$, there is a $(n+m-1)$-dimensional subspace, denoted by τ_i and called the *tangent* of O at π_i, containing π_i and disjoint from any π_j, $j \neq i$.

For $n = m$ the set O is called a *pseudo-oval* or a *generalized oval*, or an $[n-1]$-*oval*; if $n \neq m$ then the set O is called a *pseudo-ovoid*, or a *generalized ovoid*, or an *egg*, or an $[n-1]$-*ovoid*.

Pseudo-ovals were introduced by Thas [67] and generalized ovoids by Payne and Thas [59]. They show that from any pseudo-oval and any pseudo-ovoid a generalized quadrangle can be constructed. The generalized quadrangles arising in this way are exactly the so-called *translation generalized quadrangles*. We do not give the explicit construction as we will not need it in this thesis.

In Payne and Thas [59] it is shown that if q is even all tangents of a pseudo-oval $O(n,n,q)$ have an $(n-1)$-dimensional space ν in common. In such case the set $\mathcal{H} := O(n,n,q) \cup \{\nu\}$ is called a *generalized hyperoval* of $\mathrm{PG}(3n-1,q)$. This terminology will become clear with the definition of a hyperoval in the next chapter. From any generalized hyperoval a generalized quadrangle $\mathcal{Q}$ can be constructed in the following way: embed $\mathrm{PG}(3n-1,q)$ as a hyperplane Π at infinity in $\mathrm{PG}(3n,q)$. The points of $\mathcal{Q}$ are the points of $\mathrm{PG}(3n,q)\setminus\Pi$, the lines of $\mathcal{Q}$ are the n-dimensional subspaces of $\mathrm{PG}(3n,q)$ that intersect Π exactly in an element of $\mathcal{H}$, the incidence relation of $\mathcal{Q}$ is simply (symmetrized) containment. Then $\mathcal{Q}$ is a generalized quadrangle of order (q^m-1, q^m+1).

Generalized hexagons

The classical generalized hexagon $H(q)$ of order q has the following nice description. The points of $H(q)$ are all the points of the quadric $Q(6, q)$, while the lines of $H(q)$ are some of the lines of $Q(6, q)$. For our purpose it is important to notice that given any point of $H(q)$, the $q + 1$ lines of $H(q)$ through that point all lie in the same totally singular plane of $Q(6, q)$. This observation is at the base of Lemma 9.3.7 in the chapter on SPG-systems.

Finally we give an example of a weak generalized hexagon. The *flag geometry* of the projective plane $\mathrm{PG}(2, q)$ is the geometry with as point set the flags, that are, the incident point-line pairs, of $\mathrm{PG}(2, q)$, and in which the lines are the sets of flags having a point or a line in common. This geometry is a weak generalized hexagon of order $(q, 1)$. We will encounter it in Chapter 9.

1.7 A theorem from group theory

We end this introductory chapter with an important theorem from group theory, the so-called Frobenius Theorem.

Suppose that (X, G) is a permutation group satisfying the following properties:

- G acts transitively but not regularly on X;
- every non-trivial element of G has at most one fixed point in X.

Then (X, G) is said to be a *Frobenius group*. Define $N \subset G$ to be the subset of G consisting of the identity and all those elements of G which act fixed point free on X. Then N is called the *Frobenius kernel* of G and we have the following theorem.

Theorem 1.7.1 (Frobenius' Theorem) *The Frobenius kernel N of a Frobenius group (X, G) is a normal subgroup of G acting regularly on X.*

Chapter 2

Partial and semipartial geometries

In this chapter we discuss the examples and constructions of (semi)partial geometries that will be important for the rest of this thesis. After having given the necessary definitions and having mentioned some generalities in the first section, we focuss in the next two sections on constructions from SPG-reguli and SPG-systems, which form the starting point for the research contained in this dissertation. In the final section those (semi)partial geometries, not arising from SPG-reguli or SPG-systems, that we will encounter in the following chapters are discussed.

2.1 Generalities on (semi)partial geometries

Definition 2.1.1 A *partial geometry* S, denoted $\mathrm{pg}(s,t,\alpha)$, is a partial linear space of order (s,t), such that

- for every non-incident point-line pair (p,L) of S, there are exactly $\alpha > 0$ lines of S incident with p and concurrent with L.

Partial geometries were introduced by Bose in 1963 [6]. Notice that partial geometries generalize in a natural way the generalized quadrangles, since every partial geometry with $\alpha = 1$ is clearly a generalized quadrangle, and conversely. Further it is clear that partial geometries $\mathrm{pg}(s,t,\alpha)$ have a strongly regular point graph:

$$\mathrm{srg}\left((s+1)\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

A partial geometry pg(s, t, t) is called a *Bruck-net* or *net* for short.

Definition 2.1.2 A *semipartial geometry*, denoted spg(s, t, α, μ), is a partial linear space of order (s, t) such that

- (α**-condition**) for every non-incident point-line pair (p, L) of S, there are either 0 or $\alpha > 0$ lines of S incident with p and concurrent with L;
- (μ**-condition**) for every pair (p, l) of non-collinear points of S, there are exactly $\mu > 0$ points of S collinear with both p and l.

Remark 2.1.3

- The two conditions from the definition of semipartial geometry will be referred to as the α- and μ-condition for semipartial geometries, in order to avoid confusion with other α-conditions that we will encounter later on.
- A partial linear space of order (s, t) satisfying the α-condition, but not necessarily the μ-condition, for semipartial geometries will be called a $(0, \alpha)$*-geometry*.

Semipartial geometries were introduced by Debroey and Thas in 1978 [18] as generalizations of partial geometries. It is readily checked that a semipartial geometry spg(s, t, α, μ) has a strongly regular point graph

$$\mathrm{srg}\left(1 + \frac{(t+1)s(\mu + t(s - \alpha + 1))}{\mu}, (t+1)s, s - 1 + t(\alpha - 1), \mu\right).$$

It is clear that every partial geometry is a semipartial geometry with $\mu = \alpha(t + 1)$, and conversely. A semipartial geometry that is not a partial geometry will be called a *proper semipartial geometry.* A semipartial geometry with $\alpha = 1$ is called a *partial quadrangle* $PQ(s, t, \mu)$, and was already introduced in 1975 by Cameron [14] as a generalization of generalized quadrangles. We have the following scheme, in which an arrow means "*generalizes to*":

$$\begin{tikzcd}
 & \boxed{\text{partial geometry}} \arrow[dr] & \\
\boxed{\text{generalized quadrangle}} \arrow[ur] \arrow[dr] & & \boxed{\text{semipartial geometry}} \\
 & \boxed{\text{partial quadrangle}} \arrow[ur] &
\end{tikzcd}$$

Unless stated explicitely we will always assume for partial geometries that $\alpha < \min(s,t)$. This is not really a great restriction, since a partial geometry with $\alpha = s+1$ is a 2-design with $\lambda = 1$, also known as a *balanced incomplete block design* (BIBD); a partial geometry with $\alpha = t+1$ is the dual of a BIBD. A partial geometry with $\alpha = t$ is better known as a *Bruck net* of order $s+1$ and degree $t+1$; a partial geometry with $\alpha = s$ is a *dual net*. As 2-designs, nets and their duals are studied in their own right, they are usually omitted in the study of partial geometries. Finally recall that a partial geometry with $\alpha = 1$ is a generalized quadrangle. Partial geometries for which $1 < \alpha < \min(s,t)$ will be called *proper partial geometries*.

Semipartial geometries with $\alpha = s$ belong to the class of the so-called *copolar spaces*, and we will come back on these geometries in Section 2.4. Finally it is not difficult to see that a semipartial geometry with $\alpha = s+1$ is a BIBD, so we will exclude such a semipartial geometry in most cases.

In the next theorem we collect some interesting basic observations that will be used throughout this work without further notice.

Theorem 2.1.4

- *The dual of a partial geometry* $\mathrm{pg}(s,t,\alpha)$ *is a partial geometry with parameters* $\mathrm{pg}(t,s,\alpha)$.
- **([18])** *For proper semipartial geometries there holds that* $t \geq s$.
- **([18])** *The dual of a semipartial geometry* S *is a semipartial geometry if and only if either* $s = t$ *or* S *is a partial geometry.*
- **([18])** *For semipartial geometries there holds that* $\alpha^2 \leq \mu \leq (t+1)\alpha$.

2.2 Semipartial geometries from SPG-reguli

2.2.1 Definition and construction

In this section SPG-reguli will be defined. The known examples and the related semipartial geometries will be discussed. SPG-reguli and the semipartial geometries constructed from them are the main topics in Chapters 3, 4, 5, 6, 7 and 8.

Definition 2.2.1 Let $\mathcal{R} = \{\mathrm{PG}^{(0)}(m,q), \mathrm{PG}^{(1)}(m,q), \ldots, \mathrm{PG}^{(t)}(m,q)\}$, $t \geq 1$, be a set of mutually disjoint $\mathrm{PG}(m,q)$ in $\mathrm{PG}(n,q)$. We say that $\mathcal{R}$ is an SPG-regulus if and only if the following two conditions are satisfied.

- (**α-condition**) If $\mathrm{PG}(m+1,q)$ contains $\mathrm{PG}^{(i)}(m,q)$, $i = 0, 1, \ldots, t$, then it has a point in common with either 0 or $\alpha > 0$ elements of $\mathcal{R} \setminus \{\mathrm{PG}^{(i)}(m,q)\}$. If it has no point in common with $\mathrm{PG}^{(j)}(m,q)$, for all $j \neq i$, then it is called an *$(m+1)$-dimensional tangent space of $\mathcal{R}$ at* $\mathrm{PG}^{(i)}(m,q)$.

- (**θ-condition**) Every point x of $\mathrm{PG}(n,q)$ that is not contained in an element of $\mathcal{R}$ is contained in a constant number θ of $(m+1)$-dimensional tangent spaces of $\mathcal{R}$.

An SPG-regulus will be called *α-geometric* if the $\mathrm{PG}(2m+1,q)$ generated by any two of its elements contains exactly $\alpha+1$ of its elements.

SPG-reguli were introduced in 1983 by Thas [74].

Remark 2.2.2 The two conditions in the above definition will be referred to as the α- and θ-condition for SPG-reguli.

Construction

Let $\mathcal{R} = \{\mathrm{PG}^{(0)}(m,q), \mathrm{PG}^{(1)}(m,q), \ldots, \mathrm{PG}^{(t)}(m,q)\}$, $t \geq 1$, be a set of disjoint $\mathrm{PG}(m,q)$ in $\mathrm{PG}(n,q)$ generating $\mathrm{PG}(n,q)$. Now embed $\mathrm{PG}(n,q)$ as a hyperplane Π at infinity in $\mathrm{PG}(n+1,q)$. We define the following incidence structure $S(\mathcal{R}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. The point set $\mathcal{P}$ consists of all points of $\mathrm{PG}(n+1,q) \setminus \Pi$. The line set $\mathcal{L}$ is the set of all $(m+1)$-dimensional spaces of $\mathrm{PG}(n+1,q)$, not contained in Π and intersecting Π in an element of $\mathcal{R}$. The incidence relation I is the incidence relation inherited from $\mathrm{PG}(n+1,q)$. The geometry $S(\mathcal{R})$ is called the *generalized linear representation of* $\mathcal{R}$. In some cases we will also use the terminology "generalized linear representation of $S(\mathcal{R})$". In the case where $m = 0$ the geometry is called the *linear representation of* $\mathcal{R}$.

The following theorem is proved in Thas [74]. As the proof is short and this construction is at the base of the following chapters we include the proof.

Theorem 2.2.3 ([74]) *The generalized linear representation $S(\mathcal{R})$ of an SPG-regulus $\mathcal{R}$ is a semipartial geometry* $\mathrm{spg}(q^{m+1}-1, t, \alpha, (t+1-\theta)\alpha)$.

Proof. It is trivial to see that $S(\mathcal{R})$ is a partial linear space of order $(q^{m+1}-1,t)$. Next let L be a line and let p be a point of $S(\mathcal{R})$, such that $p \not\mathrm{I}\, L$ and such that there is a point $r \mathrm{~I~} L$ collinear with p in S. The $(m+1)$-dimensional space $\overline{L}$ determined by L intersects Π in an element of $\mathcal{R}$, say $\mathrm{PG}^{(i)}(m,q)$. The projection of $\overline{L}$ from p onto Π is an $(m+1)$-dimensional space containing $\mathrm{PG}^{(i)}(m,q)$ which intersects at least one element of $\mathcal{R} \setminus \{\mathrm{PG}^{(i)}(m,q)\}$ (since p is collinear with r). Hence it intersects exactly α elements of $\mathcal{R} \setminus \{\mathrm{PG}^{(i)}(m,q)\}$. It now easily follows that p is collinear with exactly α points on the line L, that is, $S(\mathcal{R})$ satisfies the α-condition for semipartial geometries. Finally consider two non-collinear points p and r of $S(\mathcal{R})$. The projective line $\langle p,r\rangle$ defined by p and r intersects Π in a point x that is not contained in an element of $\mathcal{R}$. If $\mathrm{PG}^{(j)}(m,q)$ is such that $\langle x, \mathrm{PG}^{(j)}(m,q)\rangle$ is an $(m+1)$-dimensional tangent space of $\mathcal{R}$, then it is obvious that the line of $S(\mathcal{R})$ determined by $\langle p, \mathrm{PG}^{(j)}(m,q)\rangle$ contains no point collinear with r. On the other hand, if $\mathrm{PG}^{(j)}(m,q)$ is such that $\langle x, \mathrm{PG}^{(j)}(m,q)\rangle$ is not an $(m+1)$-dimensional tangent space, then it is clear that the line of $S(\mathcal{R})$ determined by $\langle p, \mathrm{PG}^{(j)}(m,q)\rangle$ contains exactly α points collinear with r. By the θ-condition for SPG-reguli there are exactly $t+1-\theta$ such lines through p. We conclude that $S(\mathcal{R})$ satisfies the μ-condition for semipartial geometries with $\mu = (t+1-\theta)\alpha$. $\square$

Remark 2.2.4

- The semipartial geometry $S(\mathcal{R})$ is a partial geometry if and only if $\mu = (t+1)\alpha$, that is, if and only if $\theta = 0$.
- From the definition of SPG-regulus it was already clear that $n \geq 2m+1$. If $n = 2m+1$ then we see that $\theta = 0$ and $\alpha = t$, that is, $S(\mathcal{R})$ is a net.
- It is easily seen that the image of an SPG-regulus $\mathcal{R}$ under field reduction is again an SPG-regulus $\mathcal{R}'$, and that $S(\mathcal{R}) \cong S(\mathcal{R}')$.

2.2.2 Examples

Linear representations

Suppose that $\mathcal{M}$ is a point set in $\mathrm{PG}(n,q)$. Then, the linear representation of $\mathcal{M}$ is usually denoted by $T_n^*(\mathcal{M})$ instead of $S(\mathcal{M})$. We now describe the known examples of SPG-reguli consisting of points.

- $T_2^*(\mathcal{K})$

  Let $\mathcal{K}$ be a *maximal arc of degree* d in $\mathrm{PG}(2,q)$, that is, $\mathcal{K}$ is a non-empty set of points of $\mathrm{PG}(2,q)$ such that every line of $\mathrm{PG}(2,q)$ intersects $\mathcal{K}$ in either 0 or d points. It follows readily that $|\mathcal{K}| = qd + d - q$ and that either $d \mid q$ or $d = q+1$. If $d = 1$ then $\mathcal{K}$ is necessarily a single point; if $d = q$ then $\mathcal{K}$ is the complement of a line; if $d = q+1$ then $\mathcal{K}$ must be the complete point set of $\mathrm{PG}(2,q)$. A maximal arc of degree d is called *non-trivial* if $1 < d < q$. From Ball, Blokhuis and Mazzocca [3] we know that non-trivial maximal arcs in $\mathrm{PG}(2,q)$ exist only if q is a power of 2. For examples and constructions we refer to Denniston [34], Thas [71], Mathon [56] and Hamilton and Mathon [45]. The geometry $T_2^*(\mathcal{K})$, with $\mathcal{K}$ a non-trivial maximal arc, is a partial geometry $\mathrm{pg}(q-1,(q+1)(d-1),d-1)$ and was first constructed by Thas [71]. If $\mathcal{K}$ is a maximal arc of degree 2 then it is called a *hyperoval*, usually denoted by $\mathcal{O}$. The geometry $T_2^*(\mathcal{O})$ clearly is a generalized quadrangle of order $(q-1,q+1)$.

- $T_2^*(\mathcal{U})$

  Let $\mathcal{U}$ be a *unital* in $\mathrm{PG}(2,q^2)$, that is, $\mathcal{U}$ is a set of q^3+1 points of $\mathrm{PG}(2,q^2)$ such that every line of $\mathrm{PG}(2,q^2)$ intersects $\mathcal{U}$ in either 1 or $q+1$ points. The classical example is given by a non-singular Hermitian curve $H(2,q^2)$ of $\mathrm{PG}(2,q^2)$. Several non-classical examples are known (see Chapter 8). The geometry $T_2^*(\mathcal{U})$ is a semipartial geometry $\mathrm{spg}(q^2-1,q^3,q,q^2(q^2-1))$.

- $T_3^*(\mathcal{O})$

  Let $\mathcal{O}$ be an *ovoid* of $\mathrm{PG}(3,q)$, that is, $\mathcal{O}$ is a set of q^2+1 points no three of which are collinear if $q \neq 2$, and $\mathcal{O}$ is a set of 5 points no 4 of which are coplanar if $q = 2$. The classical example is given by a non-singular elliptic quadric $Q^-(3,q)$ of $\mathrm{PG}(3,q)$. A non-classical example is given by the so-called Tits-ovoid, see Tits [81]. The geometry $T_3^*(\mathcal{O})$ is a partial quadrangle $\mathrm{spg}(q-1,q^2,1,q(q-1))$, and was first constructed by Cameron [14].

- $T_n^*(\mathcal{B})$

  Consider $\Pi := \mathrm{PG}(n,q^2)$. Let $\mathcal{B} = (\mathcal{X},\mathcal{L},\mathrm{I})$ be a geometry isomorphic to $\mathrm{PG}(n,q)$, whose point set $\mathcal{X}$ is a subset of the point set of Π, whose line set is a subset of the line set of Π and whose incidence is inherited from Π. The geometry $\mathcal{B}$ is called a *Baer-subgeometry* of Π. The geometry $T_n^*(\mathcal{B})$ is a semipartial geometry spg$(q^2-1, (q^{n+1}-1)/(q-1)-1, q, q(q+1))$.

- $T_n^*(\mathcal{C})$

  A *k-cap* $\mathcal{C}$ in PG(n,q) is a non-empty set of k points with the property that every line of PG(n,q) intersects $\mathcal{C}$ in either 0, 1 or 2, points. Hence, if a k-cap $\mathcal{C}$ is such that through every point of PG$(n,q) \setminus \mathcal{C}$ there is a constant number θ of tangent lines, where tangent lines are defined in the natural way, then $\mathcal{C}$ is an SPG-regulus of PG(n,q) and $T_n^*(\mathcal{C})$ is a partial quadrangle. Notice that $T_2^*(\mathcal{O})$ and $T_3^*(\mathcal{O})$ belong to this class. The following other examples of k-caps that are SPG-reguli are known:

  – a projectively unique 11-cap in PG$(4,3)$, see Coxeter [16] and Pellegrino [60];

  – a projectively unique 56-cap in PG$(5,3)$, first constructed by Hill [47].

  – a 78-cap in PG$(5,4)$, an example of which was first constructed by Hill [48].

We mention some important characterization and classification results; but first we introduce the following important property, the so-called *diagonal property*, see for instance [17], for partial linear spaces S.

> [**Diagonal Property**] Let x, y, u, v be points of S and let L be a line of S. Let $x \mathrm{I} L \mathrm{I} y$, $x \neq y$, $u \not\mathrm{I} L \not\mathrm{I} v$, $u \neq v$. If u and v are both collinear with x as well as y, then u is collinear with v.

Debroey [17] proved the following characterization of $T_n^*(\mathcal{B})$.

Theorem 2.2.5 ([17]) *Let S be a semipartial geometry with $\mu = \alpha(\alpha+1)$ and $\alpha > 1$. Then $S \cong T_n^*(\mathcal{B})$ if and only if S satisfies the diagonal property.*

This geometry is also characterized by the following theorem, which is due to Brouwer and Wilbrink [87].

Theorem 2.2.6 ([87]) *Let S be a semipartial geometry* spg(s,t,α,μ) *with $\alpha > 1$ and $\mu = \alpha(\alpha+1)$. Suppose $\alpha \neq 3$. If $\alpha = 2$ and $(s,t) \neq (28,28)$ or if $\alpha \geq 4$ and $s \geq f(\alpha)$, with $f(4) = 12$, $f(5) = 16$, $f(6) = f(7) = 17$, $f(8) = 18$, $f(9) = 19$, $f(10) = 21$, $f(11) = 23$, $f(\alpha) = 2\alpha$ if $\alpha \geq 12$, then $S \cong T_n^*(\mathcal{B})$.*

The following theorem of Thas classifies all partial geometries embedded in affine spaces.

Theorem 2.2.7 ([72]) *If S is a proper partial geometry embedded in* $\mathrm{AG}(n,q)$*, then* $S \cong T_2^*(\mathcal{K})$.

Debroey and Thas [19] obtained the following result on proper semipartial geometries embedded in $\mathrm{AG}(n,q)$.

Theorem 2.2.8 ([19]) *If S is a proper semipartial geometry embedded in* $\mathrm{AG}(n,q)$*, then $n \geq 3$. If $n = 3$ then S is either the pentagon embedded in* $\mathrm{AG}(3,2)$ *(trivial case), or $S \cong T_2^*(\mathcal{U})$, or $S \cong T_2^*(\mathcal{B})$.*

We will improve this result in Chapter 4.

Examples from polar spaces

In [74], in which Thas introduced SPG-reguli, he also provided the following class of examples.

Theorem 2.2.9 ([74]) *If $\mathcal{R}$ is a spread of respectively $Q^-(2m+3,q)$ ($m \geq 0$), $Q^+(2m+1,q)$ ($m \geq 0$), or $H(m,q^2)$ ($m \geq 1$), then $\mathcal{R}$ is an SPG-regulus of respectively* $\mathrm{PG}(2m+3,q)$, $\mathrm{PG}(2m+1,q)$, $\mathrm{PG}(m,q^2)$*. However if $\mathcal{R}$ is a spread of either $Q^+(2m+1,q)$ or $H(m,q^2)$ with m odd, then the related semipartial geometry $S(\mathcal{R})$ is a net.*

Remark 2.2.10 The polar space $Q^-(2m+3,q)$ does admit a spread if either $m \in \{0,1\}$ (Thas [74]) or if q is even (Dye [41]). It is not known whether $Q^-(2m+3,q)$, $m \geq 2$ and q odd, admits a spread.

The polar space $Q^+(2m+1,q)$ with m even does not admit a spread (Hirschfeld and Thas [50]). The following polar spaces admit a spread: $Q^+(3,q)$; $Q^+(7,q)$ with either q an odd prime or q odd and such that $q \equiv 0$ or $2 \pmod 3$; $Q^+(2m+1,q)$ with m odd and q even (Dye [41]). In all other cases the existence of a spread is an open problem.

The polar spaces $H(4,4)$ (unpublished computer result by Brouwer, 1981) and $H(m,q^2)$ with m odd (Thas [75]) do not admit a spread. In all other cases the existence of a spread is an open problem.

For an interesting overview on the existence of spreads, ovoids and m-systems of finite classical polar spaces and their applications we refer to Thas [77].

Theorem 2.2.9 has been generalized by Luyckx [54].

Theorem 2.2.11 ([54]) *Let $\mathcal{R}$ be an m-system of the polar space $P \in \{Q^-(2n+1,q), W_{2n+1}(q), H(2n,q^2)\}$. Then $\mathcal{R}$ is an SPG-regulus of the ambient space of P.*

We will encounter this result again in Chapter 6 as a corollary of a more general theorem.

All the SPG-reguli $\mathcal{R}$ arising from the previous two theorems, as well as the ovoids of $\mathrm{PG}(3,q)$ and the unitals of $\mathrm{PG}(2,q^2)$, satisfy the following property (see also Chapter 6).

> [**Polar Property**] Let $\mathcal{R} = \{\mathrm{PG}^{(0)}(m,q), \mathrm{PG}^{(1)}(m,q), \dots, \mathrm{PG}^{(t)}(m,q)\}$ be an SPG-regulus in $\mathrm{PG}(n,q)$, with $n \neq 2m+1$. The union of all $(m+1)$-dimensional tangent spaces of $\mathcal{R}$ at $\mathrm{PG}^{(i)}(m,q)$ form an $(n-m-1)$-dimensional space, for all $i \in \{0,1,\dots,t\}$.

In Thas [74] the following is shown.

Theorem 2.2.12 ([74]) *Let $\mathcal{R} = \{\mathrm{PG}^{(0)}(m,q), \mathrm{PG}^{(1)}(m,q), ..., \mathrm{PG}^{(t)}(m,q)\}$ be a set consisting of disjoint $\mathrm{PG}(m,q)$ in $\mathrm{PG}(n,q)$, for which the α-condition for SPG-reguli holds. Suppose $\mathcal{R}$ satisfies the polar property. Then $\mathcal{R}$ is an SPG-regulus if and only if $t = q^{\frac{n+1}{2}}$.*

The next theorem gives an overview of the basic restrictions on the parameters of SPG-reguli satisfying the polar property. These restrictions can all be found in [74].

Theorem 2.2.13 ([74]) *If $\mathcal{R} = \{\mathrm{PG}^{(0)}(m,q), \mathrm{PG}^{(1)}(m,q), ..., \mathrm{PG}^{(t)}(m,q)\}$ is an SPG-regulus consisting of $\mathrm{PG}(m,q)$'s in $\mathrm{PG}(n,q)$ and if $\mathcal{R}$ satisfies the polar property, then*

- $t = q^{\frac{n+1}{2}}$;
- $\alpha = q^{\frac{4m-n+3}{2}}$;
- $\theta = 1 + q^{\frac{n-2m-1}{2}}$;
- $\mu = q^{m+1}(q^{m+1}-1)$;
- $\frac{n-3}{4} \leq m < \frac{n-1}{2}$.

In Chapter 6 (Theorem 6.5.1) a considerably stronger version of Theorem 2.2.12 will be proved.

2.2.3 Parallelism

We end this section by introducing a natural parallelism on the lines of $S(\mathcal{R})$.

We say that two lines of $S(\mathcal{R})$ are *$\mathcal{R}$-parallel* if and only if they determine the same element of $\mathcal{R}$, where $S(\mathcal{R})$ is considered in its natural representation in $\mathrm{PG}(n+1,q)$.

It is obvious that $\mathcal{R}$-parallelism determines an equivalence relation on the lines of $S(\mathcal{R})$ and hence naturally gives rise to a packing of $S(\mathcal{R})$. It is not clear at all whether or not $\mathcal{R}$-parallelism is independent of the representation of $S(\mathcal{R})$, that is whether every automorphism of $S(\mathcal{R})$ maps $\mathcal{R}$-parallel lines to $\mathcal{R}$-parallel lines; but we will come back later on this, see Chapter 7

2.3 Semipartial geometries from SPG-systems

2.3.1 Definition and construction

Definition 2.3.1 Let $Q(2n+2,q)$, $n \geq 1$, be a non-singular parabolic quadric of $\mathrm{PG}(2n+2,q)$. An *SPG-system of* $Q(2n+2,q)$ is a set τ of $(n-1)$-dimensional totally singular subspaces of $Q(2n+2,q)$ such that the elements of τ on any non-singular elliptic quadric $Q^-(2n+1,q) \subset Q(2n+2,q)$ form a spread of $Q^-(2n+1,q)$.

Let $Q^+(2n+1,q)$, $n \geq 1$, be a non-singular hyperbolic quadric of $\mathrm{PG}(2n+1,q)$. An *SPG-system of* $Q^+(2n+1,q)$ is a set τ of $(n-1)$-dimensional totally singular subspaces of $Q^+(2n+1,q)$ such that the elements of τ on any non-singular parabolic quadric $Q(2n,q) \subset Q^+(2n+1,q)$ constitute a spread of $Q(2n,q)$.

Let $H(2n+1,q^2)$, $n \geq 1$, be a non-singular Hermitian variety of $\mathrm{PG}(2n+1,q^2)$. An *SPG-system of* $H(2n+1,q^2)$ is a set τ of $(n-1)$-dimensional totally singular subspaces of $H(2n+1,q^2)$ such that the elements of τ on any non-singular Hermitian variety $H(2n,q^2) \subset H(2n+1,q^2)$ constitute a spread of $H(2n,q^2)$.

Let P be a singular polar space with ambient space $\mathrm{PG}(d,q)$, having as radical the point x. Assume that the projective index of P is n, with $n \geq 1$, that is, n is the dimension of the maximal totally singular subspaces on P. An *SPG-system of* P is a set τ of $(n-1)$-dimensional totally singular subspaces of P, not containing x, such that the elements of τ which are (maximal) totally singular for the polar subspace P' of P induced by any $\mathrm{PG}(d-1,q) \subset \mathrm{PG}(d,q)$ not containing x constitute a spread of P'.

In all cases we will call the dimension $n-1$ of the elements of τ, *the index* of the SPG-system. Note that an SPG-system τ of a polar space $P \in \{Q(4,q), Q^+(3,q), H(3,q^2)\}$, hence τ is of index 0, is the set of points on P. If P is a singular polar space with radical $\{x\}$, then an SPG-system of index 0 is the point set of $P \setminus \{x\}$.

SPG-systems were introduced by Thas [76] in 2001.

Construction

Let P be either a non-singular polar space, with $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q^2)\}$, $n \geq 1$, or a singular polar space with projective index n, $n \geq 1$, and having as radical the point x. Let $\mathrm{PG}(d,q')$, with $q' = q$ if $P \neq H(2n+1,q^2)$ and $q' = q^2$ if $P = H(2n+1,q^2)$, be the ambient space of P. Further, let τ be an SPG-system of P and let P be embedded in a non-singular polar space $\overline{P}$ with ambient space $\mathrm{PG}(d+1,q')$, of the same type as P and with projective index n (so, P and $\overline{P}$ are both orthogonal, Hermitian, or symplectic). Hence for $P = Q(2n+2,q)$ we have $\overline{P} = Q^-(2n+3,q)$, for $P = Q^+(2n+1,q)$ we have $\overline{P} = Q(2n+2,q)$, and for $P = H(2n+1,q^2)$ we have $\overline{P} = H(2n+2,q^2)$.

If $\overline{P}$ is not symplectic and $y \in \overline{P}$, let T_y be the tangent hyperplane of $\overline{P}$ at y. If $\overline{P}$ is symplectic and θ is the corresponding symplectic polarity of $\mathrm{PG}(d+1,q')$, let $T_y = y^\theta$ for any $y \in \mathrm{PG}(d+1,q')$.

For $y \in \overline{P} \setminus P$, let $\overline{y}$ be the set of all points z of $\overline{P} \setminus P$ for which $T_z \cap P = T_y \cap P$. As $T_u \neq T_z$ for distinct points u, z of $\overline{P}$, no two distinct points of $\overline{y}$ are collinear in $\overline{P}$. For P non-singular and orthogonal we have $|\overline{y}| = 2$, except when $P = Q^+(2n+1,q)$ and q even where $|\overline{y}| = 1$. For P non-singular and Hermitian we have $|\overline{y}| = q+1$. For P singular and orthogonal we have $|\overline{y}| = 1$, except when P is parabolic and q is even where $|\overline{y}| = q$, for P singular and Hermitian we have $|\overline{y}| = q$. For P singular and symplectic we have $|\overline{y}| = q$. Notice also that all points of $\overline{y}$ belong to some common line of $\mathrm{PG}(d+1,q')$ except when P is singular parabolic and q is even where they belong to a common non-singular conic.

Let ξ be any maximal totally singular subspace of $\overline{P}$, not contained in P for which $\xi \cap P \in \tau$ and let $y \in (\overline{P} \setminus P) \cap \xi$. Further, let $\overline{\xi}$ be the set of all maximal totally singular subspaces η of $\overline{P}$, not contained in P for which $\xi \cap P = \eta \cap P$ and $\eta \cap \overline{y} \neq \emptyset$. From $\xi \cap P = \eta \cap P$ follows that no point of $\xi \setminus P$ is collinear on $\overline{P}$ with a point of $\eta \setminus P$. Let z be a point of $\xi \setminus P$, $z \neq y$, and let $\langle y, z\rangle \cap P = \{u\}$. Further let y' be the element of $\overline{y}$ in η. The hyperplanes $T_l \cap \mathrm{PG}(d,q')$ of $\mathrm{PG}(d,q')$, $l \in \langle u, y\rangle$, form a bundle of hyperplanes which coincides with the bundle of hyperplanes with elements $T_{l'} \cap \mathrm{PG}(d,q')$ of $\mathrm{PG}(d,q')$, $l' \in \langle u, y' \rangle$. Hence on $\langle u, y' \rangle$ there is a point z' with $T_z \cap \mathrm{PG}(d,q') = T_{z'} \cap \mathrm{PG}(d,q')$, and hence $T_z \cap P = T_{z'} \cap P$. So $\overline{z}$ has also a point in common with η. With the above definitions and notations we define the following incidence structure $\mathcal{S}$. It has point set $\mathcal{P} = \{\overline{y} \mid y \in \overline{P} \setminus P\}$. The set $\mathcal{L}$ of lines consists of all sets $\overline{\xi}$. Incidence in $\mathcal{S}$ is defined as follows. If $\overline{y} \in \mathcal{P}$ and $\overline{\xi} \in \mathcal{L}$, then $\overline{y} \,\mathrm{I}\, \overline{\xi}$ if and only if for some $z \in \overline{y}$ and some $\eta \in \overline{\xi}$, we have $z \in \eta$.

Theorem 2.3.2 ([76]) *If the polar space P is either non-singular with $P \in \{Q(2n+2,q), Q^+(2n+1,q)$ with $q \in \{2,3\}, H(2n+1,q^2)\}$ or singular with $P/\{x\} \in \{Q(2n,q)$ with q even, $Q^+(2n-1,q), Q^-(2n+1,q), H(2n-1,q^2)$, $H(2n,q^2)$, $W_{2n-1}(q)\}$, then $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry.*

2.3.2 Examples, characterization and classification

We will now state the characterization and classification results for SPG-systems obtained in [76].

Theorem 2.3.3 ([76]) *If τ is an SPG-system of the non-singular polar space $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q^2)\}$, then $|\tau| = |P|$.*

Let P be a singular polar space having as radical the point x, with projective index n, and for which the quotient $P/\{x\}$ is the non-singular polar space P'. If τ is an SPG-system of P then $|\tau| = o(P')q^n$.

Theorem 2.3.4 ([76]) *Let τ be a set of $(n-1)$-dimensional totally singular subspaces of the non-singular polar space $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q^2)\}$. Then τ is an SPG-system of P if and only if the following conditions are satisfied:*

(i) $|\tau| = |P|$,

(ii) *if $\pi, \pi' \in \tau$, with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator of P.*

Let τ be a set of $(n-1)$-dimensional totally singular subspaces of the singular polar space P with projective index n and having as radical the point x. Assume also that no element of τ contains x. The non-singular quotient $P/\{x\}$ will be denoted by P'. Then τ is an SPG-system of P if and only if the following conditions are satisfied:

(i) $|\tau| = o(P')q^n$,

(ii) *if* $\pi, \pi' \in \tau$, *with* $\pi \neq \pi'$ *and* $\pi \cap \pi' \neq \emptyset$, *then* $\langle \pi, \pi' \rangle$ *contains a generator of* P.

Theorem 2.3.5 ([76]) *If* $\mathcal{R}$ *is a spread of* $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q^2)\}$ *and* τ *is the set of all* $(n-1)$*-dimensional subspaces contained in the elements of* $\mathcal{R}$*, then* τ *is an SPG-system of* P.

Similarly if P *is a singular polar space of projective index* n*, with ambient space* $\mathrm{PG}(d,q)$*, having as radical the point* x*, and if* $\mathcal{R}$ *is a set of* n*-dimensional totally singular subspaces of* P *(hence containing* x*), with the property that* $\mathcal{R}$ *induces a spread of the polar subspace* P' *induced by any* $\mathrm{PG}(d-1,q)$ *not containing* x *(that is, the intersections of the elements of* $\mathcal{R}$ *with* P' *constitute a spread of* P'*), then the set* τ *of all* $(n-1)$*-dimensional subspaces of the ambient space of* P *contained in the elements of* $\mathcal{R}$*, but not containing* x*, is an SPG-system of* P.

An SPG-system constructed as in Theorem 2.3.5 will be called a *spread-SPG-system*.

Let $H(q)$ be the classical generalized hexagon of order q embedded in the quadric $Q(6,q)$. Two lines of $H(q)$ intersect if and only if they lie in a common plane of $Q(6,q)$, see Section 1.6. As the number of lines of $H(q)$ is equal to $(q^6-1)/(q-1) = |(Q(6,q)|$, it follows from Theorem 2.3.4 that the line set τ of $H(q)$ is an SPG-system of $Q(6,q)$.

All other known SPG-systems are spread-SPG-systems. In Thas [76] the following classification result was proved.

Theorem 2.3.6 ([76]) *There are exactly two classes of SPG-systems of index 1 on a non-singular polar space and both are SPG-systems of* $Q(6,q)$*. One of them is a spread-SPG-system, the other one consists of the lines of the classical generalized hexagon* $H(q)$ *embedded in* $Q(6,q)$.

Note that $Q(6,q)$ has at least one spread for all values of q, except possibly for the case where $q = p^h$, $h > 1$, $p \neq 2,3$ and $q \equiv 1 \pmod 3$ in which case the existence of a spread is still open.

We will improve this classification in Chapter 9.

We will not describe in detail the classes of semipartial geometries that arise from the SPG-system construction as our results on these objects are concerned with the characterization and classification of the SPG-systems themselves. We only define the semipartial geometry $\mathrm{TQ}(4,q)$, and mention two interesting isomorphisms, providing a link with the SPG-reguli from the previous section.

The semipartial geometry arising from an SPG-system of $Q(4,q)$, that is the point set of $Q(4,q)$, is denoted by $\mathrm{TQ}(4,q)$. It was first constructed by Metz in 1976 (unpublished) using a construction which is essentially the SPG-system construction, see [27]. We will encounter this geometry again in Chapter 4.

Let S be the semipartial geometry constructed from a spread-SPG-system τ of the singular polar space P with radical the point x. By considering the projection of τ from x onto a hyperplane of the ambient space of P not containing x one can deduce the following isomorphisms, see Thas [76].

- If $P/\{x\} = Q^-(2n+1,q)$, then $S \cong S(\mathcal{R})$, where $S(\mathcal{R})$ is the semipartial geometry constructed from an SPG-regulus $\mathcal{R}$ in $\mathrm{PG}(2n+1,q)$ being a spread of a non-singular elliptic quadric $Q^-(2n+1,q)$.
- If $P/\{x\} = H(2n,q^2)$, then $S \cong S(\mathcal{R})$, where $S(\mathcal{R})$ is the semipartial geometry constructed from an SPG-regulus $\mathcal{R}$ in $\mathrm{PG}(2n,q^2)$ being a spread of a non-singular Hermitian variety $H(2n,q^2)$.

2.4 Other examples

In the final section of this chapter we will briefly discuss some semipartial geometries that we will encounter in this thesis, but that are not arising from SPG-reguli, nor from SPG-systems.

2.4.1 The geometries $\mathrm{LP}(n,q)$, $U_{2,3}(n)$ and $\overline{M(t+1)}$.

The semipartial geometries $\mathrm{LP}(n,q)$

Let $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the geometry with as point set $\mathcal{P}$ the set of lines of $\mathrm{PG}(n,q)$, $n \geq 4$, with as line set $\mathcal{L}$ the set of planes of $\mathrm{PG}(n,q)$, and with as incidence relation I the incidence inherited from $\mathrm{PG}(n,q)$. Then S is a semipartial geometry $\mathrm{spg}(q(q+1), (q^{n-1}-1)/(q-1)-1, q+1, (q+1)^2)$, see Debroey and Thas [18]. It is usually denoted by $\mathrm{LP}(n,q)$.

The semipartial geometries $U_{2,3}(n)$

Let U be a set of finite cardinality n, $n \geq 4$. Define the following geometry $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. The point set $\mathcal{P}$ of S is the set of pairs $\{a, b\}$ for which $\{a, b\} \subset U$. The line set $\mathcal{L}$ is the set of triples $\{a, b, c\}$ for which $\{a, b, c\} \subset U$. The incidence relation I is the inclusion relation. Then S is a semipartial geometry spg$(2, n-3, 2, 4)$, see Debroey and Thas [18]. It is usually denoted by $U_{2,3}(n)$.

The semipartial geometry $\overline{M(t+1)}$

Let $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the geometry with as point set $\mathcal{P}$ the set of vertices of the Moore graph $M(t+1)$, with as line set $\mathcal{L}$ the set $\{\Gamma(x) \mid x \in \mathcal{P}\}$, and with I the natural incidence. Then S is a semipartial geometry spg(t, t, t, t^2), see Debroey and Thas [18]. It is usually denoted by $\overline{M(t+1)}$. Notice that necessarily $t+1 \in \{2, 3, 7, 57\}$ (see [4]). However it is not known whether the Moore graph $M(57)$ exists.

Characterizations

Theorem 2.4.1 ([18]) *If S is a proper semipartial geometry with $\alpha = t$, then $S \cong \overline{M(t+1)}$.*

Theorem 2.4.2 ([17], [87]) *Let S be a proper semipartial geometry with $\mu = \alpha^2$.*

- *If $\alpha = 2$, then $S \cong U_{2,3}(n)$.*
- *If $2 < \alpha = s$, then $S \cong \overline{M(t+1)}$.*
- *If $2 < \alpha < s$, then $S \cong \mathrm{LP}(n, q)$.*

2.4.2 Copolar spaces

Definition 2.4.3 A *copolar space* S is a partial linear space of order (s, t), such that for every non-incident point-line pair (p, L) there are either 0 or s points incident with L and collinear with p, that is, S satisfies the α-condition for semipartial geometries with $\alpha = s$. A copolar space of order $(2, t)$ is also called a *cotriangular space*.

A copolar space S is called *indecomposable* if and only if S is not the union of two or more copolar spaces on disjoint point sets.

A copolar space S is called *reduced* if it is indecomposable and such that for all vertices x and y in the point graph of S there holds that $\Gamma(x) = \Gamma(y)$ implies $x = y$.

Examples

If the copolar space S is a partial geometry, then clearly S is a dual net.

The semipartial geometries $U_{2,3}(n)$ and $\overline{M(t+1)}$ are easily seen to be reduced copolar spaces.

Let σ be a symplectic polarity of $\mathrm{PG}(2n+1, q)$, $n \geq 1$. Define S to be the geometry $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with as point set $\mathcal{P}$ the set of all points of $\mathrm{PG}(2n+1, q)$, with as line set $\mathcal{L}$ the set of lines of $\mathrm{PG}(2n+1, q)$ that are not totally isotropic with respect to σ, and with as incidence relation I the incidence inherited from $\mathrm{PG}(2n+1, q)$. Then S is a semipartial geometry $\mathrm{spg}(q, q^{2n}-1, q, q^{2n}(q-1))$, see Debroey and Thas [18]. It is usually denoted by $\overline{W(2n+1, q)}$. It is readily checked that $\overline{W(2n+1, q)}$ is a reduced copolar space.

Let Q be a non-singular quadric in $\mathrm{PG}(2n-1, 2)$, with $n \geq 2$ if Q is elliptic and with $n \geq 3$ if Q is hyperbolic. Define S to be the geometry $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with as point set $\mathcal{P}$ the set of all points of $\mathrm{PG}(2n-1, 2)$ not on Q, with as line set $\mathcal{L}$ the set of lines of $\mathrm{PG}(2n-1, 2)$ not intersecting Q, and with as incidence relation I the incidence inherited from $\mathrm{PG}(2n-1, 2)$. Then S is a semipartial geometry $\mathrm{spg}(2, 2^{2n-3} - \epsilon 2^{n-2} - 1, 2, 2^{2n-3} - \epsilon 2^{n-1})$, where $\epsilon = -1$ if Q is an elliptic quadric, and where $\epsilon = 1$ if Q is a hyperbolic quadric. In the respective cases S is usually denoted by $\mathrm{NQ}^-(2n-1, 2)$ and $\mathrm{NQ}^+(2n-1, 2)$. These geometries are due to Wilbrink (unpublished). Here as well it is clear that $\mathrm{NQ}^\pm(2n-1, 2)$ is a reduced copolar space (in fact it is a cotriangular space).

Classification

The following classification result is due to Hall [44].

Theorem 2.4.4 ([44]) *If S is a finite reduced copolar space of order (s, t), $s \geq 2$, then S is isomorphic to one of the following semipartial geometries:*

- $\overline{M(t+1)}$,

- $U_{2,3}(n)$,
- $\overline{W(2n-1,q)}$,
- $\mathrm{NQ}^{\pm}(2n-1,2)$.

Chapter 3

Net-inducible (semi)partial geometries

In this chapter, as well as in the next two chapters, we will be merely concerned with characterizations of the semipartial geometries having a linear representation. This chapter, which is based on De Clerck, De Winter and Thas [25], is a generalization of a characterization of the partial geometry $T_2^*(\mathcal{K})$ due to De Clerck, De Soete and Gevaert [24].

3.1 Introduction

Let us consider any linear representation in $\mathrm{AG}(n,q)$ of a (semi)partial geometry S. Clearly there is a natural parallelism on the lines of S induced by the parallelism on the subspaces of $\mathrm{AG}(n,q)$. (Note that this is the same parallelism as mentioned in Subsection 2.2.3.) Furthermore it is clear that if we consider any plane $\beta \subset \mathrm{AG}(n,q)$ containing at least two intersecting lines of S, then the geometry induced by S in β is a Bruck net of order q and degree $\alpha+1$. So S contains "a lot of" these nets and one might wonder whether the (semi)partial geometry S can be characterized by these nets.

A first result in this sense was obtained by De Clerck, De Soete and Gevaert [24], where this problem was solved if S is a partial geometry. They introduced axiom (A) (see below) and showed that if a proper partial geometry S satisfies axiom (A), then it contains "a lot of" subnets. Two natural extra conditions were sufficient to force $S \cong T_2^*(\mathcal{K})$.

We will generalize this for semipartial geometries and obtain a characterization of the geometries $T_2^*(\mathcal{B})$ and $T_2^*(\mathcal{U})$. We will also explain the geometric importance of one of the extra conditions that is imposed on the subnets, and its link with an interesting open problem.

3.2 Parallelism

The following lemma shows that the abstract definition of parallelism in semipartial geometries arises in a natural way.

Lemma 3.2.1 *Suppose that S is a semipartial geometry , with $\alpha \neq 1$, being a linear representation in* $\mathrm{AG}(3,q)$ *of a point set $\mathcal{R}$ in the plane at infinity. Define two lines of S to be parallel iff they are parallel in* $\mathrm{AG}(3,q)$. *Then two lines L and L' of S are parallel if and only if either $L = L'$, or L and L' are two different non-intersecting lines and either there is no point of L collinear with a point of L' or there exist two transversal lines B and B' of L and L' intersecting in a point $x \notin L \cup L'$.*

Proof. Clearly if the lines L and L' are parallel then we are in one of these three situations. Conversely if L coincides with L' then of course L is parallel with L'. So next suppose that L and L' are two different non-intersecting lines such that there exist two transversal lines B and B' (of S) intersecting in a point $x \notin L \cup L'$. Then the lines L and L' are contained in the plane $\langle B, B' \rangle$ (here lines in S are identified with the lines in $\mathrm{AG}(3,q)$ representing them). Since L and L' do not intersect they must be parallel in the affine plane $\langle B, B' \rangle$, hence in $\mathrm{AG}(3,q)$ and hence in S. Finally suppose that L and L' are two different non-intersecting lines and there is no point of L collinear with a point of L'. If L and L' are contained in a plane then they are clearly parallel. If they would not be contained in a plane, then they would generate $\mathrm{AG}(3,q)$ and in such case it is easily seen that there must exist a transversal of L and L'. $\square$

Notice that the previous lemma is not valid anymore in $\mathrm{AG}(n,q)$, $n \geq 4$. For example in the geometry $T_3^*(\mathcal{B})$ it is possible to find two different non-intersecting lines L and L' with the property that there is no point of L collinear with a point of L', but such that L and L' are not parallel. In a way this explains why we will obtain characterizations of the semipartial geometries $T_2^*(\mathcal{B})$ and $T_2^*(\mathcal{U})$, but not of the geometry $T_n^*(\mathcal{B})$, $n \geq 3$.

The previous lemma is at the base of the following definition of parallelism in abstract semipartial geometries .

Definition 3.2.2 Let S be an spg(s,t,α,μ) with $\alpha > 1$. We call two lines L and L' of S *parallel* (notation: $L \parallel L'$) if either $L = L'$, or if L and L' are two different non-intersecting lines and either there is no point of L collinear with a point of L' or there exist two transversal lines B and B' intersecting in a point $x \notin L \cup L'$. If L and L' are parallel and there is no point on L collinear with a point on L' we will write $L \widehat{\parallel} L'$; in such a case we say that L and L' are *ultraparallel*.

Let L, M be different lines, $L \cap M = \{x\}$ or $L \parallel M$. We define for any point p, $p \notin L \cup M$, the integer $\beta(p)$ with respect to L and M, as $|\{K \mid p \in K, K \in \{L,M\}^{\perp}\}|$.

The following axiom (A) was introduced in [24] and will be important for the rest of the chapter.

> AXIOM (A). *Let L and M be any two distinct intersecting or parallel lines of S. For any point p on a transversal of L and M, $p \notin L \cup M$, there holds $\beta(p) = \alpha - |L \cap M|$ with respect to L and M.*

Remark 3.2.3 Assume that axiom (A) is satisfied. Let p be a point on a transversal of two different intersecting lines L and M, $p \notin L \cup M$. Then it is easy to check that the unique line L', respectively M', containing p, for which $L' \cap L \neq \emptyset$ but $L' \cap M = \emptyset$, respectively $M \cap M' \neq \emptyset$ but $M' \cap L = \emptyset$, satisfies $L' \parallel M$, respectively $M' \parallel L$. Notice that L' and M' exist and are unique because $\beta(p) = \alpha - 1$ with respect to L and M.

In [24] any partial geometry with $\alpha > 1$ satisfying axiom (A) was called a *net-inducible partial geometry*. In the same spirit we will call a semipartial geometry with $\alpha > 1$ satisfying axiom (A) a *net-inducible semipartial geometry*. The reason for this name will become clear soon.

Lemma 3.2.4 *Let S be a net-inducible semipartial geometry, then for any two non-concurrent lines L and M, with $\{L,M\}^{\perp} \neq \emptyset$, the set $\{L,M\}^{\perp\perp}$ is a set of pairwise non-concurrent lines.*

Proof. Suppose there exists a line $K \in \{L,M\}^{\perp}$ and let $k \in K$. We will prove that k is on at most one line of $\{L,M\}^{\perp\perp}$. Assume that $N, N' \in \{L,M\}^{\perp\perp}$ with $\{k\} = N \cap N'$ and $N \neq N'$. Assume without loss of generality that $k \notin L$ and put $l := K \cap L$. Then l is contained in a transversal of N

and N' ($\alpha > 1$ implies a second line through l intersecting M and hence N and N'). So axiom (A) implies that $\beta(l) = \alpha - 1$ with respect to N and N' while it is clear that l is contained in α lines of $\{N, N'\}^\perp$, a contradiction. Hence the lemma. $\square$

Definition 3.2.5 From Lemma 3.2.4 follows that if S is net-inducible, then for any two lines L and M as in Lemma 3.2.4 we have $\left|\{L, M\}^{\perp\perp}\right| \leq s+1$. As in [24] we call two non-concurrent lines L and M of a net-inducible semipartial geometry *α-regular* if and only if $\left|\{L, M\}^{\perp\perp}\right| = s + 1$.

The proof of the following lemma is basically the same as the proof of Theorem 3.4 of [24]; we include it here in order to be complete and because in [24] the case $s = 2$ was not treated.

Lemma 3.2.6 (Theorem 3.4 of [24]) *Let S be a net-inducible semipartial geometry. If L and M are two different parallel lines for which there exists a transversal, then the pair $\{L, M\}$ is an α-regular pair.*

Proof. We need to prove that $\left|\{L, M\}^{\perp\perp}\right| = s + 1$. Let B and B' be two transversals of L and M intersecting in $x \notin L \cup M$. First suppose $s > 2$. By axiom (A) and Remark 3.2.3 we know that we can construct a unique line N_y, parallel to L and intersecting B', through every point y of $B \setminus \{x, L\cap B, M\cap B\}$ (since $s > 2$ the point y exists). Since L is parallel to M and hence, because of axiom (A), $\beta(y) = \alpha$ with respect to L and M, we see that N_y does not intersect M and so is parallel to M as well. It is easily seen that each of the points $N_y \cap B$ and $N_y \cap B'$ are contained in α transversals of L and M. Now consider any point $z \in N_y \setminus \{N_y \cap B, N_y \cap B'\}$. There is a unique line B_2 through z parallel to B and intersecting B' in a point z'. The point z' is on the line B' which is a transversal of B and L; hence axiom (A) implies that B_2 must intersect L. Analogously we see that B_2 intersects M. Consequently z is on one, and hence on α transversals of L and M. We conclude that every point of N_y is on α transversals, from which we deduce that $N_y \in \{L, M\}^{\perp\perp}$. Repeating the above construction with B and another transversal of L and M, intersecting B in a point distinct from x, $L \cap B$ and $M \cap B$, proves that also x is contained in a unique line parallel to L and M that is contained in $\{L, M\}^{\perp\perp}$. We have obtained that $\left|\{L, M\}^{\perp\perp}\right| = s + 1$, that is, $\{L, M\}$ is an α-regular pair.

Finally suppose that $s = 2$, that is, every line contains exactly three points. Introduce the following notation: $l := B \cap L$, $l' := B' \cap L$, $l'' := L \setminus \{l, l'\}$, $m := B \cap M$, $m' := B' \cap M$ and $m'' := M \setminus \{m, m'\}$. The point l'' is on a transversal of B and B', and so axiom (A) implies that there is a unique line N parallel to B through l''. Furthermore this line must intersect B' and clearly the only possibility is that it intersects B' in m'. Let n be the third point of N. Since N is a transversal of B' and L, axiom (A) implies the existence of a unique line K through n parallel to L and intersecting B', necessarily in the point x. It is now easy to show that $K \in \{L, M\}^{\perp\perp}$, that is, $\left|\{L, M\}^{\perp\perp}\right| = 3 = s + 1$. $\square$

From this it follows (see De Clerck, Gevaert and Thas [26]) that two parallel lines L, M for which there exists a transversal, generate a unique net of order $s+1$ and degree $\alpha+1$; this net will be denoted by $\mathcal{N}(L, M)$. As in [26] and [24] one easily sees that any two distinct intersecting lines L' and M' of S generate a net $\mathcal{N}(L', M')$ of order $s+1$ and degree $\alpha+1$, and that also y and L'', with y a point not on the line L'' but collinear with α points of L'', generate a net $\mathcal{N}(y, L'') = \mathcal{N}(L'', y)$ of order $s + 1$ and degree $\alpha + 1$. If L', M' are lines of $\mathcal{N}(L, M)$, with $L \parallel M$, then clearly $\mathcal{N}(L, M) = \mathcal{N}(L', M')$; if L'' is a line of $\mathcal{N}(L, M)$ and y is a point of $\mathcal{N}(L, M)$ then again $\mathcal{N}(L, M) = \mathcal{N}(y, L'')$ (recall that $y \notin L''$). Finally three points a, b and c, such that a and b determine a line of S and c is not on this line but is collinear with α points of this line, define a unique net $\mathcal{N}(a, b, c) = \mathcal{N}(\langle a, b\rangle, c)$ and if a, b and c are points of $\mathcal{N}(L, M)$, then $\mathcal{N}(a, b, c) = \mathcal{N}(L, M)$.

In the rest of this chapter, whenever we are talking about a net, we mean a (Bruck) net of order $s + 1$ and degree $\alpha + 1$ which is a subgeometry of S.

Remark 3.2.7 It is now clear why we call a semipartial geometry satisfying axiom (A), a net-inducible semipartial geometry.

Lemma 3.2.8 *Let S be a net-inducible semipartial geometry. If x is a point and L a line, with x not on L but collinear with α points of L, then x is contained in exactly one line M parallel to L.*

Proof. Assume that M as well as N would contain x and would be parallel to L. From the above it follows that $\mathcal{N}(L, M) = \mathcal{N}(x, L) = \mathcal{N}(L, N)$. This implies that in this net x would be contained in two lines parallel to L, a contradiction. Furthermore there exists a line M through x parallel to L, namely the line through x not intersecting L in $\mathcal{N}(x, L)$. $\square$

Lemma 3.2.9 *Let S be a net-inducible semipartial geometry. If L, M, N are distinct lines with $L \cap M \neq \emptyset$, $L \cap N \neq \emptyset$, $\mathcal{N}(L, M) \neq \mathcal{N}(L, N)$, then M is not parallel to N.*

Proof. If $N \cap M \neq \emptyset$ then this is trivial. So suppose $M \parallel N$, $M \cap L = m$ and $N \cap L = n$ with $n \neq m$. Then $\mathcal{N}(L, N) = \mathcal{N}(m, N) = \mathcal{N}(M, N) = \mathcal{N}(n, M) = \mathcal{N}(L, M)$, a contradiction. □

Corollary 3.2.10 *Let $\mathcal{N}$ be a net in a net-inducible semipartial geometry, and let p be a point of $S \setminus \mathcal{N}$ that is collinear with at least one point x of $\mathcal{N}$. Then the $\alpha + 1$ lines through p parallel to the $\alpha + 1$ lines in $\mathcal{N}$ through x do not intersect $\mathcal{N}$.*

Lemma 3.2.11 *If S is a net-inducible semipartial geometry, then the following properties hold.*

- *Any two non-collinear points are contained in exactly $\frac{\mu}{\alpha(\alpha+1)}$ nets.*
- *Every line is contained in $\frac{t}{\alpha}$ nets.*

Moreover,

$$\mu \leq \frac{t^2 - \alpha t - \alpha(\alpha+1)(t-s-1)}{s+1-\alpha}, \tag{3.1}$$

$$\mu \leq \frac{t(t-\alpha)}{s+1}, \tag{3.2}$$

where the second inequality is only valid for proper net-inducible semipartial geometries .

Proof. The proof of the first two statements is basically the proof of the analogue statements for partial geometries as given in [24]. The first one follows noticing that every point collinear with two non-collinear points determines a unique net containing these two points, and that conversely every net containing two non-collinear points contains exactly $\alpha(\alpha + 1)$ points collinear with both. The second one follows by noticing that given a line L, every line $M \neq L$ intersecting L determines a unique net containing L, and that conversely every net containing L contains $(s + 1)\alpha$ lines, distinct from L, intersecting L.

We now prove the two inequalities.

Consider a line L and a point p not on L such that p is collinear with α points on L. We count the number of points, distinct from p and the points on L, collinear with p and with at least one point on L. First of all the α lines through p intersecting L, each contain $s - 1$ such points. Secondly if

x is a point on L collinear with p, then each of the $t-1$ lines through x, distinct from L and distinct from the line through x and p, contains $\alpha - 1$ such points. This yields $\alpha(t-1)(\alpha-1)$ points. Finally if x is a point on L not collinear with p, then there are $\mu - \alpha$ points collinear with both p and x that are not incident with L. This yields another $(s+1-\alpha)(\mu-\alpha)$ points. Clearly all these points are counted α times, as they are all collinear with α points on L. Hence we find that the number of points, distinct from p and the points on L, collinear with p and with at least one point on L equals

$$X = \frac{\alpha(s-1+(t-1)(\alpha-1)) + (s+1-\alpha)(\mu-\alpha)}{\alpha}.$$

The net $\mathcal{N}(p, L)$ contains $\alpha(s-1)+s$ such points. Let $\mathcal{N}$ be a net through L different from $\mathcal{N}(p, L)$. Let p be incident with k lines intersecting $\mathcal{N} \setminus L$. Note that $k \geq 1$ and that p is incident with α lines intersecting L. If x is a point of $\mathcal{N} \setminus L$ collinear with p, then p is incident with $\alpha + 1$ lines parallel to the $\alpha + 1$ lines in $\mathcal{N}$ through x. Hence $k \leq t - 2\alpha$. Since there are $\frac{t}{\alpha} - 1$ such nets $\mathcal{N}$, we obtain $(\frac{t}{\alpha} - 1)(t - 2\alpha) + \alpha(s-1) + s \geq X$. Inequality (3.1) follows.

To prove inequality (3.2) we consider a line L and a point p with p collinear with no point on L. Here a net through L contains at most $t - \alpha$ points collinear with p while there are $\frac{t}{\alpha}$ nets through L. On the other hand there are exactly $\frac{\mu(s+1)}{\alpha}$ points collinear with p and with α points on L. So $(t-\alpha)\frac{t}{\alpha} \geq \frac{\mu(s+1)}{\alpha}$, and inequality (3.2) follows. □

Remark 3.2.12

1. From the proof of both inequalities it is clear that equality holds in (3.1) and in (3.2) if and only if for every net $\mathcal{N}$ and every point $p \notin \mathcal{N}$, p is collinear with exactly $t - \alpha$ points of the net.

2. Let S be a partial geometry, which is equivalent to $\mu = \alpha(t+1)$. Then only the proof of the first inequality is valid and becomes $t^2 - (3 + s)\alpha t + \alpha^2(s+2) \geq 0$. Assuming the partial geometry is not a net itself (i.e. $t \neq \alpha$) this inequality is satisfied iff $t \geq \alpha(s+2)$, which is exactly the inequality obtained in [24].

In analogy with [24] we will assume now that equality holds in (3.1) and (3.2). A net-inducible semipartial geometry for which this holds will be called a *maximal net-inducible semipartial geometry*. Hence a *maximal net-inducible partial geometry* is a net-inducible partial geometry for which equality in (3.1) holds. We have the following lemma.

Lemma 3.2.13 *Let S be a maximal net-inducible semipartial geometry. Let $\mathcal{N}$ be a net and suppose that p is a point such that $p \notin \mathcal{N}$. If $x \in \mathcal{N}$ is a point collinear with p, then every line through x in $\mathcal{N}$ is parallel to exactly one line through p (not intersecting $\mathcal{N}$) and the remaining $t-\alpha$ lines through p intersect $\mathcal{N}$.*

Proof. This follows from the previous remark and the corollary after Lemma 3.2.9. □

Lemma 3.2.14 *If S is a proper maximal net-inducible semipartial geometry* $\mathrm{spg}(s,t,\alpha,\mu)$*, then $\mu = (\alpha+1)(t-s-1)$. Furthermore α must divide $s+1$.*

Proof. Using the second equation to eliminate t^2 from (3.1) and then solving (3.1) for μ yields the desired form for μ. Since by Lemma 3.2.11 α must divide μ and t and obviously cannot divide $\alpha+1$ it follows that $\alpha \mid s+1$. □

Lemma 3.2.15 *Let S be a maximal net-inducible semipartial geometry and let $\mathcal{N}_1$, $\mathcal{N}_2$ be two distinct nets through a line L. Suppose $M \in \mathcal{N}_1$, $M \parallel L$, $M \neq L$, and $N \in \mathcal{N}_2$, $N \parallel L$, $N \neq L$. Then $M \parallel N$.*

Proof. If no point of N is collinear with a point of M, then clearly $M \parallel N$. So suppose that there exists a point n on N that is collinear with α points on M. Consider a point $m \in M$ with $n \sim m$. From the above we know that N must be parallel to a line K of $\mathcal{N}_1$ through m. Assume that this line intersects L in a point x. Then x would be contained in two lines parallel to N, namely L and K, a contradiction by Lemma 3.2.8. We conclude that $K = M$. □

Notation

We denote by $\bigcup \mathcal{N}_L$ the set of all points that are contained in a net through the line L.

Lemma 3.2.16 *Let S be a proper maximal net-inducible semipartial geometry and let (m, L) be an anti-flag of S, such that m is not collinear with any point of L. Then there are t lines through m intersecting $\bigcup \mathcal{N}_L$ in $\frac{t}{\alpha} - 1$ points and there is a unique line through m whose points are not collinear with any point of L.*

Proof. Let K be a line through m with the property that $K \cap \bigcup \mathcal{N}_L \neq \emptyset$, and such that there are at least two nets, say $\mathcal{N}_1$ and $\mathcal{N}_2$, through L having empty intersection with K. It is clear that K is not parallel to L. Let $k \in K \cap \bigcup \mathcal{N}_L$, and let l be a point on L collinear with k. The net $\mathcal{N}_i$ contains a line M_i through l parallel to K, with $i = 1, 2$. By Lemma 3.2.8 we have $M_1 = M_2$. So $\mathcal{N}_1 = \mathcal{N}_2$, a contradiction.
Let R be a line through m with the property that $R \cap \bigcup \mathcal{N}_L \neq \emptyset$, say $r \in R \cap \bigcup \mathcal{N}_L$. Let l be a point on L collinear with r. Let R' be the unique line through l parallel to R. Then R has no point in common with the net $\mathcal{N}(L, R')$.
Hence, if T is a line through m having a nonempty intersection with $\bigcup \mathcal{N}_L$, then T intersects exactly $\frac{t}{\alpha} - 1$ nets through L in a point, while it has no point in common with a unique net through L. Counting the number of triples $\{l, d, T\}$ with $l \in L$, $d \in T$, $m \in T$, $l \sim d$, in two different ways we obtain $\gamma(\frac{t}{\alpha} - 1)\alpha = (s + 1)\mu$, with γ the number of lines through m intersecting $\bigcup \mathcal{N}_L$. As we have assumed equality in formula (3.2) of Lemma 3.2.11 it follows that $\gamma = t$. Now the lemma follows. $\square$

Theorem 3.2.17 *Let S be a maximal net-inducible semipartial geometry . Then parallelism between lines is an equivalence relation and each point of S belongs to exactly one line of each parallel class.*

Proof. Let $L \parallel M$ and $M \parallel N$, with L, M, N distinct lines. There are three possibilities, the first two of which only occur when S is a proper semipartial geometry .

- Suppose $L \mathrel{\widehat{\parallel}} M$ and $M \mathrel{\widehat{\parallel}} N$. Hence no point of $L \cup N$ is collinear with a point of M. If there is no point on L collinear with a point on N, then obviously $L \mathrel{\widehat{\parallel}} N$. So suppose there is a line K intersecting L and N. Clearly the line L has no point in common with $\bigcup \mathcal{N}_M$. From the previous lemma we know that K has $\frac{t}{\alpha} - 1$ points in $\bigcup \mathcal{N}_M$. If x is such a point, then we know that since $L \cap \mathcal{N}(x, M) = \emptyset$, L is parallel to a line M' through x in $\mathcal{N}(x, M)$. From the fact that $L \parallel M$ it follows that M' must be the unique line of $\mathcal{N}(x, M)$ through x parallel to M. Similarly we see that N is parallel to M'. Hence if we consider the net $\mathcal{N}(L, M')$ we see that this net must contain K and hence N, so we may conclude that $L \parallel N$.
- Suppose M is ultraparallel to L but not to N. If there is no point on L collinear with a point on N we have that $L \mathrel{\widehat{\parallel}} N$. So suppose there

is a point $l \in L$ collinear with a point $n \in N$. Since $L \cap \mathcal{N}(N, M) = \emptyset$ it follows that L must be parallel to a line N' of $\mathcal{N}(N, M)$ containing n. Since L is parallel to M it follows that $N' \cap M = \emptyset$, so we may conclude that $N' = N$, i.e. $N \parallel L$.

- Suppose that M is neither ultraparallel to L nor to N. In this case Lemma 3.2.15 applies and we may conclude $L \parallel N$.

Let x be a point and let L be a line not containing x. If x is collinear with α points of L, then by Lemma 3.2.8 the point x is on exactly one line parallel to L. Now assume that x is collinear with no point of L. By Lemma 3.2.16 there is a unique line M through x having no point in common with $\bigcup \mathcal{N}_L$. Then no point of M is collinear with a point of L, so $L \parallel M$. As parallelism is an equivalence relation this line M is the unique line through x parallel to L. □

It is now possible to define parallelism between nets. We call two nets $\mathcal{N}_1, \mathcal{N}_2$ parallel whenever $\mathcal{N}_1 = \mathcal{N}_2$ or $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ (notation: $\mathcal{N}_1 \parallel \mathcal{N}_2$).

Theorem 3.2.18 *Let S be a maximal net-inducible semipartial geometry , then parallelism between nets is an equivalence relation and each point of S belongs to exactly one net of each parallel class.*

Proof. Let $\mathcal{N}_1 \parallel \mathcal{N}_2$ and $\mathcal{N}_1 \parallel \mathcal{N}_3$, with $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ distinct nets. Let x_i be a point of $\mathcal{N}_i$, with $i = 1, 2, 3$. By Theorem 3.2.17 the lines of $\mathcal{N}_1$ through x_1 are parallel to the respective lines of $\mathcal{N}_2$ through x_2, and also to the respective lines of $\mathcal{N}_3$ through x_3. Hence, again by Theorem 3.2.17, the lines of $\mathcal{N}_2$ through x_2 are parallel to the respective lines of $\mathcal{N}_3$ through x_3. If y would be a common point of $\mathcal{N}_2$ and $\mathcal{N}_3$, then the $\alpha + 1$ lines through y parallel to the $\alpha + 1$ lines of $\mathcal{N}_2$ through x_2 would belong to $\mathcal{N}_2$ and also to $\mathcal{N}_3$. Hence $\mathcal{N}_2 = \mathcal{N}_3$, a contradiction. So $\mathcal{N}_2$ and $\mathcal{N}_3$ are parallel.
Let $\mathcal{N}$ be a net and let x be a point not in $\mathcal{N}$. We have to show that x is contained in a net $\mathcal{N}'$ having no point in common with $\mathcal{N}$. Let M, M' be lines through x parallel to distinct intersecting lines of $\mathcal{N}$. Put $\mathcal{N}' = \mathcal{N}(M, M')$. If $\mathcal{N}$ and $\mathcal{N}'$ would have a point in common, then, as before, we would have $\mathcal{N} = \mathcal{N}'$, a contradiction. Hence $\mathcal{N} \parallel \mathcal{N}'$. □

Lemma 3.2.19 *A maximal net-inducible semipartial geometry S has $(s+1)^3$ points.*

Proof. Let L be a line and $\mathcal{N}$ a net such that L is parallel to no line of $\mathcal{N}$. It follows that L intersects every net parallel to $\mathcal{N}$ (including $\mathcal{N}$). The previous theorem now implies that S can be covered by $s+1$ disjoint nets. $\square$

3.3 The characterization of $T_2^*(\mathcal{K})$

In this section we only state the characterization of $T_2^*(\mathcal{K})$ as obtained in [24]. For the proof we refer to that paper.

Theorem 3.3.1 ([24]) *Let S be a maximal net-inducible partial geometry ($s+1 \neq \alpha$ and $\alpha > 1$) for which each triad of points is contained in $0, 1$ or $\frac{t+1}{\alpha+1}$ subnets of S of order $s+1$ and degree $\alpha+1$. Then $S \cong T_2^*(\mathcal{K})$.*

3.4 The characterization of $T_2^*(\mathcal{U})$

From Lemma's 3.2.11 and 3.2.14 follows that, assuming S is a proper maximal net-inducible semipartial geometry, $t \geq s+1+\alpha$.

In this section we will prove, in analogy with [24], that assuming S is a maximal net-inducible semipartial geometry $\mathrm{spg}(s, t, \alpha, \mu)$ with $t > s+1+\alpha$, then under some restriction each of the nets can be embedded in an affine plane, from which will follow that $S \cong T_2^*(\mathcal{U})$.

Hence in this section we will assume that S is a proper maximal net-inducible semipartial geometry and that $t > s+1+\alpha$ (note that by Lemma 3.2.14 this is equivalent to $\mu > \alpha(\alpha+1)$).

Lemma 3.4.1 *If $\mathcal{N}_1, \mathcal{N}_2$ are two distinct nets, then only the following three configurations can occur:*

- $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$*;*
- $\mathcal{N}_1 \cap \mathcal{N}_2$ *is a line;*
- $\mathcal{N}_1 \cap \mathcal{N}_2$ *is a set of $s+1$ pairwise non-collinear points.*

Proof. It is clear that the first two configurations can occur, so suppose that the intersection of the point sets of $\mathcal{N}_1$ and $\mathcal{N}_2$, which we will denote by K, is neither empty nor a line. It is easily seen that two points of K can

never be collinear. Let p be a point of $\mathcal{N}_1$ that is not contained in K and suppose that p is collinear with β points in K (note that $\beta \leq \alpha + 1$). Then there are $t - \alpha - \beta$ lines through p intersecting $\mathcal{N}_2$ not in K and $\alpha + 1$ lines through p belonging to the net $\mathcal{N}_1$. Also the unique net through p parallel to $\mathcal{N}_2$ contains $\alpha + 1$ lines incident with p. If one of these $\alpha + 1$ lines, say M, would be contained in $\mathcal{N}_1$, then the line M' parallel to M through any given point of K would be contained in $\mathcal{N}_1 \cap \mathcal{N}_2$, clearly a contradiction. So we have found at least $t + \alpha + 2 - \beta$ lines through p, which is only possible if $\beta \geq \alpha + 1$. Hence there must hold that $\beta = \alpha + 1$.
Now suppose that $|K| = \kappa$. Each of these κ points is collinear with $(\alpha + 1)s$ points of $\mathcal{N}_1 \setminus K$ while from what precedes we know that each point of $\mathcal{N}_1 \setminus K$ is collinear with $\alpha + 1$ points in K. Hence the following equality must hold:

$$\left[(s+1)^2 - \kappa\right](\alpha + 1) = \kappa(\alpha + 1)s.$$

Solving for κ yields $\kappa = s + 1$, which proves the lemma. □

3.4.1 Reconstructing the affine space

We will denote with

- $\mathcal{E}'(\mathcal{N})$ the set of all sets of $s+1$ pairwise non-collinear points contained in $\mathcal{N}$ that arise as intersection of $\mathcal{N}$ with another net;
- $\mathcal{E}'$ the set of all sets of $s + 1$ pairwise non-collinear points that arise as intersection of two nets;
- $\mathcal{E}(\mathcal{N})$ the set of all lines of S contained in $\mathcal{N}$;
- $\mathcal{E}$ the set of all lines of S.

Lemma 3.4.2 *If $\mathcal{N}$ is a net, then the incidence structure $\mathcal{A} = (\mathcal{P}(\mathcal{N}), \mathcal{E}(\mathcal{N}) \cup \mathcal{E}'(\mathcal{N}), \mathrm{I})$, with $\mathcal{P}(\mathcal{N})$ the point set of $\mathcal{N}$ and I the natural incidence, is an affine plane iff each element of $\mathcal{E}'(\mathcal{N})$ is contained in $\frac{\mu}{\alpha(\alpha+1)}$ nets.*

Proof. Suppose that each element of $\mathcal{E}'(\mathcal{N})$ is contained in $\frac{\mu}{\alpha(\alpha+1)}$ nets. It is then clear that through any two points a and b of $\mathcal{A}$ there passes a unique line of $\mathcal{A}$: if a and b are collinear in S then this line will be the unique element of $\mathcal{E}$ determined by a and b; if a and b are not collinear in S, then, because of our assumption, the $\frac{\mu}{\alpha(\alpha+1)} - 1$ nets distinct from $\mathcal{N}$ and containing a an b will all determine the same element of $\mathcal{E}'(\mathcal{N})$. It is also clear that each line

of $\mathcal{A}$ contains $s+1$ points. Hence $\mathcal{A}$ is a $2-((s+1)^2, s+1, 1)$-design, i.e., $\mathcal{A}$ is an affine plane of order $s+1$.
If there would be an element of $\mathcal{E}'(\mathcal{N})$ not contained in $\frac{\mu}{\alpha(\alpha+1)}$ nets it is easily seen that $\mathcal{A}$ contains two non-collinear points of S contained in more than one element of $\mathcal{E}'(\mathcal{N})$, so $\mathcal{A}$ could never be an affine plane. $\square$

Remark 3.4.3 It is clear that each element of $\mathcal{E}'$ is contained in $\frac{\mu}{\alpha(\alpha+1)}$ nets iff any triad of points of S is contained in $0, 1$ or $\frac{\mu}{\alpha(\alpha+1)}$ nets.

Notation

We will denote by $\mathcal{A}(\mathcal{N})$ the affine plane in which the net $\mathcal{N}$ is embedded.

Definition 3.4.4 A *coclique K of nexus* (n, m) in a semipartial geometry is a set of two by two non-collinear points such that each point not belonging to K is collinear with either n or m points in K.

Lemma 3.4.5 *Let S be a proper maximal net-inducible semipartial geometry with the property that each element of $\mathcal{E}'$ is contained in $\frac{\mu}{\alpha(\alpha+1)}$ nets. Then any element of $\mathcal{E}'$ is a coclique of nexus* $(1, \alpha+1)$ *of S.*

Proof. Let $K \in \mathcal{E}'$ and let p be a point not in K. If there exists a net containing p and K, then from Lemma 3.4.1 it follows that p is collinear with $\alpha+1$ points in K. Furthermore it is obvious that whenever p is collinear with two points of K there exists a net through K containing p. From now on we may assume that p is not collinear with $\alpha+1$ points of K, hence p is collinear with either 0 or 1 point of K. We have to show that 0 cannot occur. So assume that p is collinear with no points in K. Let $\mathcal{N}$ be any net containing K and let K' be a line parallel to K in $\mathcal{A}(\mathcal{N})$ that contains $\alpha+1$ points collinear with p (note that K' exists since $t > s+1+\alpha$ and p is collinear with $t-\alpha$ points of $\mathcal{N}$). Furthermore let $\mathcal{N}'$ be the unique net through p parallel to $\mathcal{N}$. If we let $\mathcal{N}_p$ be the net through p containing K', then it is easily seen that $\mathcal{N}_p$ will intersect $\mathcal{N}'$ in an element K_p of $\mathcal{E}'$. With exception of $\mathcal{N}'$ and $\mathcal{N}_p$, all nets through K_p will intersect $\mathcal{N}$ in an element of $\mathcal{E}'$ disjoint from K', hence parallel to it in $\mathcal{A}(\mathcal{N})$. On the other hand, if $K'' \in \mathcal{E}'$ is contained in $\mathcal{N}$, parallel to K in $\mathcal{A}(\mathcal{N})$ and such that p is collinear with $\alpha+1$ points of K'', then the net parallel to $\mathcal{N}_p$ containing any point of K'' will contain K'' and will intersect $\mathcal{N}'$ in an element of $\mathcal{E}'$ parallel to K_p. Hence the net through p containing K'' will intersect $\mathcal{N}'$ in K_p. This implies that in $\mathcal{N}$ there are exactly $\frac{\mu}{\alpha(\alpha+1)} - 1$ elements of $\mathcal{E}'$ parallel to K in $\mathcal{A}(\mathcal{N})$ that contain $\alpha + 1$ points collinear with p. If we furthermore suppose that there are y (respectively z) elements of $\mathcal{E}'$ contained in $\mathcal{N}$ parallel to K in $\mathcal{A}(\mathcal{N})$ that contain 1 (respectively 0) point(s) collinear with p, we have the following:

$$\begin{cases} \left(\frac{\mu}{\alpha(\alpha+1)} - 1\right)(\alpha+1) + y & = & t - \alpha \\ \frac{\mu}{\alpha(\alpha+1)} - 1 + y + z & = & s + 1. \end{cases}$$

Using $\mu = (\alpha+1)(t-s-1)$ (by Lemma 3.2.14) this yields $z = 0$. This proves the lemma. □

Lemma 3.4.6 *There holds that $t = (s+1)^{3/2}$ and $\alpha = (s+1)^{1/2}$.*

Proof. Let $\mathcal{N}$ be a net and p a point not contained in $\mathcal{N}$. Then there is a set C of $t - \alpha$ points in $\mathcal{N}$ collinear with p. We can extend $\mathcal{A}(\mathcal{N})$ to a projective plane $\mathcal{P}$ by adding as line at infinity a new line L_∞ with as point set the set of parallel classes of $\mathcal{A}(\mathcal{N})$ and where such a parallel class is also incident with each line belonging to it. Let C' be the set of points of L_∞ defined by the lines of $\mathcal{N}$. Then the set $C'' = C \cup C'$ is a set of $t+1$ points in $\mathcal{P}$ with the property that each line of $\mathcal{P}$ contains either 1 or $\alpha + 1$ points of C''. Since C'' clearly is not a line of $\mathcal{P}$ it follows from Tallini Scafati [66] (see also Theorem 4.1.1) that C'' is either a unital or a Baer subplane. Hence we have that $\alpha = (s+1)^{1/2}$. If now C'' would be a Baer subplane, then $t = s + 1 + (s+1)^{1/2}$ and hence it would follow that $\mu = \alpha(\alpha+1)$, a contradiction with the assumptions of this section. It follows that C'' is a unital and hence $t = (s+1)^{3/2}$. □

In order to show that we can embed S in an affine space we will use the following elegant and useful characterization of affine spaces due to F. Buekenhout (see [8]).

Theorem 3.4.7 ([8]) *Let $\mathcal{L}$ be a linear space such that*

(i) every three non-collinear points generate an affine plane;

(ii) $\mathcal{L}$ contains at least three non-collinear points;

(iii) every line of $\mathcal{L}$ contains at least four points.

Then $\mathcal{L}$ is an affine space.

As this theorem only holds whenever lines contain at least four points we should first deal with the cases $s = 1$ and $s = 2$. But since $\alpha \mid s+1$ (by Lemma 3.2.14), $\alpha \neq s+1$ and $\alpha > 1$ it follows that these cases cannot occur.

So from now on we may suppose that $s \geq 3$. If we consider the incidence structure $\mathbb{P} = (\mathcal{P}, \mathcal{E} \cup \mathcal{E}', \mathrm{I})$, with $\mathcal{P}$ the point set of S and with I the natural incidence relation, then $\mathbb{P}$ is clearly a linear space satisfying (ii) and (iii) of Theorem 3.4.7.

Notation

We denote by $\langle a, b\rangle$ the element of $\mathcal{E} \cup \mathcal{E}'$ generated by a and b, where a and b are two distinct points of $\mathbb{P}$.

Lemma 3.4.8 *Any three non-collinear points of $\mathbb{P}$ generate an affine plane.*

Proof. Let a, b and c be three non-collinear points of $\mathbb{P}$. There are a number of different cases to handle.

- If $\langle a, b\rangle \in \mathcal{E}$ and c is collinear (in S) with a point of the line $\langle a, b\rangle$, then a, b, c generate a net $\mathcal{N}$, hence also the affine plane $\mathcal{A}(\mathcal{N})$ in $\mathbb{P}$.
- Assume that $\langle a, b\rangle \in \mathcal{E}$ and that c is collinear (in S) with no point on this line. Consider the lines $\langle a, c\rangle$ and $\langle b, c\rangle$ of $\mathcal{E}'$ and any point $x \in \langle a, c\rangle \setminus \{a, c\}$. Then x has to be collinear (in S) with 1 or $\alpha + 1$ points of $\langle b, c\rangle$. Assume that x would be collinear with $\alpha + 1$ such points. The net containing two of these points and x would clearly contain the line $\langle a, b\rangle$ and the point c, hence c would be collinear with α points on $\langle a, b\rangle$, a contradiction. So x is collinear with a unique point x' of $\langle b, c\rangle$. Notice that in fact x is collinear (in S) with exactly one point p of $\langle c, d\rangle \in \mathcal{E}'$ with $d \in \langle a, b\rangle \setminus \{a\}$ and $p \neq d$. If $x' = b$ the net containing a, b and x would contain c, again yielding a contradiction. Hence $x' \neq b$. We next show that the line $\langle x, x'\rangle$ is in fact parallel to $\langle a, b\rangle$. From Lemma 3.4.6 and the fact that the number of points of S equals $(s+1)^3$ it follows that in S there are exactly $s(s+1)$ points collinear with no point of $\langle a, b\rangle$. Hence this point set must be exactly the set $F = \{u \mid u \in \langle c, d\rangle \setminus \{d\}, d \in \langle a, b\rangle\} \cup \{u \mid u \in \langle c, d\rangle, \langle c, d\rangle \parallel \langle a, b\rangle\}$. Since the line L_x through x parallel to $\langle a, b\rangle$ must be completely contained in F (since x is collinear, in S, with no point of $\langle a, b\rangle$) and obviously can intersect neither the line through c parallel to

$\langle a,b\rangle$ nor $\langle a,b\rangle$, we have that L_x must have a unique point in common with $\langle c,d\rangle \in \mathcal{E}'$, $d \in \langle a,b\rangle$. Hence $L_x = \langle x,x'\rangle$, i.e., $\langle x,x'\rangle \parallel \langle a,b\rangle$. This implies that each line parallel to $\langle a,b\rangle$ that intersects $\langle a,c\rangle \setminus \{c\}$ intersects $\langle b,c\rangle \setminus \{c\}$ and hence belongs to the subspace of $\mathbb{P}$ generated by a,b and c. If one now considers the points a,x and x' it is easily seen that the element of $\mathcal{E}$ through c parallel to $\langle x,x'\rangle$ and hence parallel to $\langle a,b\rangle$ must belong to the subspace generated by a,x and x' and hence to the subspace generated by a,b and c. In this way we have a set of $s+1$ parallel lines that belong to the subspace generated by a,b and c. Let y be any point of $\langle a,b\rangle \setminus \{b\}$. Then by applying the foregoing to the points y,b and c, and using the fact that the line through any point of $\langle b,c\rangle$ parallel to $\langle a,b\rangle$ is unique, we see that $\langle c,y\rangle \in \mathcal{E}'$ must have a point in common with each of the $s+1$ parallel lines. Now consider any point z on any of these $s+1$ parallel lines and any point z' on the line parallel to $\langle a,b\rangle$ containing c, where z,z' and c are distinct. Using the same ideas for the triple of points $\{z,z',c\}$, we see that $\langle z,z'\rangle$ is either the line through c parallel to $\langle a,b\rangle$ or an element of $\mathcal{E}'$ that intersects each of the $s+1$ parallel lines. We obtain that the subspace generated by a,b and c consists exactly of the points on these $s+1$ parallel lines. It is now easily deduced that the subspace generated by a,b and c is a $2-((s+1)^2,s+1,1)$-design, that is, an affine plane of order $s+1$.

- Finally we have the case where no two elements of $\{a,b,c\}$ are collinear in S. Since $\langle a,b\rangle$ is a coclique of nexus $(1,\alpha+1)$ in S, c is collinear (in S) with either 1 or $\alpha+1$ points of $\langle a,b\rangle$. The first possibility clearly is equivalent to one of the previous cases, while the second possibility yields that a,b and c are contained in a net $\mathcal{N}$ and hence generate an affine plane $\mathcal{A}(\mathcal{N})$ in $\mathbb{P}$. This concludes the proof of the lemma. □

3.4.2 The characterization

We can now prove the following theorem.

Theorem 3.4.9 *Let S be a proper maximal net-inducible semipartial geometry* $\mathrm{spg}(s,t,\alpha,\mu)$ *with $\alpha > 1$. If in addition each triad of points of S is contained in* 0, 1 *or* $\frac{\mu}{\alpha(\alpha+1)}$ *nets, with $\mu > \alpha(\alpha+1)$, then $S \cong T_2^*(\mathcal{U})$.*

Proof. From the previous section we know that S can be embedded in an affine space $\mathrm{AG}(n,s+1)$. Since the number of points of S equals $(s+1)^3$ we see that $n=3$. From Theorem 2.2.8 it then follows that S must be isomorphic to $T_2^*(\mathcal{U})$. □

Remark 3.4.10

- Notice that it is not really necessary to use the Debroey-Thas Theorem (Theorem 2.2.8) that classifies the semipartial geometries in $\mathrm{AG}(3,q)$. In order to do so we first prove that two lines are parallel in S if and only if they are parallel in $\mathrm{AG}(3,q)$. Assume the contrary, that is, assume that L and M are two lines parallel in S that are not parallel in $\mathrm{AG}(3,q)$. It is easily seen that L and M must be ultraparallel in S. Choose any point x that is collinear with α points on L and α points on M. The line X through x parallel to L in S clearly is contained in the net $\mathcal{N}(x,L)$, and hence must be parallel to L in $\mathrm{AG}(3,q)$. Analogously the line Y through x parallel to M in S is contained in the net $\mathcal{N}(x,M)$, and hence must be parallel to M in $\mathrm{AG}(3,q)$. Since parallelism in S is an equivalence relation and each point is contained in one element of each equivalence class we see that $X=Y$, a contradiction. Conversely, if L and M are distinct lines of S parallel in $\mathrm{AG}(3,q)$, we should show that L and M are also parallel in S. Choose any point x on M. Then we know there exists a unique line M' through x parallel to L in S. By the above we know that M' has to be parallel to L in $\mathrm{AG}(3,q)$. Hence $M=M'$. From Lemma 3.4.6 it then readily follows that S must be the linear representation of a unital.
- It is worthwhile to notice that the condition on the triads is necessary. It is related to the following interesting open problem. Does there exist in $\mathrm{PG}(6n-1,q)$ an SPG-regulus $\mathcal{R}$, consisting of $q^{3n}+1$ $(2n-1)$-dimensional subspaces, that is (q^n+1)-regular and such that $\mathcal{R}$ does not arise from a unital in $\mathrm{PG}(2,q^{2n})$ under field reduction. The semipartial geometry $S(\mathcal{R})$ constructed from such an SPG-regulus would then be a maximal net-inducible semipartial geometry with the same parameters as $T_2^*(\mathcal{U})$ but not isomorphic to $T_2^*(\mathcal{U})$, and hence not satisfying the condition on the triads.

3.5 The characterization of $T_2^*(\mathcal{B})$

In this section we will prove that a maximal net-inducible semipartial geometry with $t=s+1+\alpha$ (and hence $\mu=\alpha(\alpha+1)$) is isomorphic to $T_2^*(\mathcal{B})$. In [87] Wilbrink and Brouwer showed that under a mild restriction on the parameters (which is satisfied here) a proper $\mathrm{spg}(s,t,\alpha,\mu)$ with $\mu=\alpha(\alpha+1)$ and $\alpha\neq 3$ is isomorphic to $T_n^*(\mathcal{B})$ for some n. Because of the parameters of S we then have that $S\cong T_2^*(\mathcal{B})$ whenever $\alpha\neq 3$.

However the technique that we will use to prove the same result when $\alpha = 3$ will work for general α and so we will formulate and prove the following theorem for general α.

Theorem 3.5.1 *If S is a (proper) maximal net-inducible semipartial geometry* $\mathrm{spg}(s, t, \alpha, \alpha(\alpha + 1))$ *with $\alpha > 1$, then $S \cong T_2^*(\mathcal{B})$.*

Proof. First notice the following. Let $\mathcal{N}$ be any net and let p be a point not in $\mathcal{N}$. Because of the equality in (3.2) of Lemma 3.2.11 and the fact that $t = s + 1 + \alpha$ we have that $t - \alpha = \alpha^2$. By Lemma 3.2.11 there will hold that the $t - \alpha = \alpha^2$ points collinear with p in $\mathcal{N}$ will be two by two collinear.
In order to prove the theorem it suffices to show that the diagonal property holds in S. So let x_1, x_2, x_1' and x_2' be four distinct points such that x_1 is collinear with x_2 and $\{x_1', x_2'\} \in \{x_1, x_2\}^\perp$ while $\{x_1', x_2'\} \cap \langle x_1, x_2\rangle = \emptyset$. We must prove that x_1' and x_2' are collinear. We may suppose that x_1' is not contained in $\langle x_1, x_2'\rangle$ nor in $\langle x_2, x_2'\rangle$. First suppose that x_2 is not contained in $\mathcal{N}(x_1, x_1', x_2')$. It then follows from the above that x_1' is collinear with x_2'. So from now on we may suppose that x_1, x_2, x_1' and x_2' are contained in a net $\mathcal{N}$. An easy counting shows that there are exactly $\alpha^2(\alpha-1)$ points not in $\mathcal{N}$ collinear with both x_1 and x_2. If we consider a net $\mathcal{N}' \neq \mathcal{N}$ through the line $\langle x_1, x_1'\rangle$ we see that this net contains exactly $t - 2\alpha = \alpha^2 - \alpha$ points collinear with x_2 not in $\mathcal{N}$. Notice that all these points are collinear with both x_1 and x_1'. Since there are $\frac{t}{\alpha} - 1 = \alpha$ possible choices for $\mathcal{N}'$ we find $\alpha^2(\alpha-1)$ points not in $\mathcal{N}$ collinear with x_1, x_2 and x_1'. Hence there holds that each point not in $\mathcal{N}$ that is collinear with both x_1 and x_2 is collinear with x_1'. Furthermore there exist such points. Of course we can interchange x_1' and x_2', and so each point not in $\mathcal{N}$ collinear with both x_1 and x_2 is collinear with both x_1' and x_2'. This implies the existence of a point outside $\mathcal{N}$ collinear with x_1' and x_2', hence x_1' is collinear with x_2'. By Theorem 2.2.5 we are done. $\square$

3.6 Main result

We can now reformulate the results of the previous sections in the following theorem.

Theorem 3.6.1 *Let S be a maximal net-inducible semipartial geometry* $\mathrm{spg}(s, t, \alpha, \mu)$, *such that each triad of points is contained in $0, 1$ or $\frac{\mu}{\alpha(\alpha+1)}$ nets. Then either $S \cong T_2^*(\mathcal{K})$, or $S \cong T_2^*(\mathcal{U})$, or $S \cong T_2^*(\mathcal{B})$.*

Remark 3.6.2 Notice that in the $T_2^*(\mathcal{B})$ case the condition on the triads is trivially satisfied with $\frac{\mu}{\alpha(\alpha+1)} = 1$.

Chapter 4

Linear representations of semipartial geometries

The problem of classifying the semipartial geometries that can be embedded in affine space arose as soon as semipartial geometries were introduced, in Debroey and Thas [18] in 1978, since several of the examples that were given in that paper were indeed embedded in affine space. In the same year, 1978, Debroey and Thas published a first classification result in [19], proving that every proper semipartial geometry embedded in $\mathrm{AG}(3,q)$ is either a pentagon, $T_2^*(\mathcal{B})$ or $T_2^*(\mathcal{U})$, see Theorem 2.2.8. In the same year the analogous problem for partial geometries had been settled by Thas, see Theorem 2.2.7. Recently De Clerck and Delanote [23] showed that if a proper semipartial geometry S is embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, then either S is a linear representation or $\alpha = 2$. By the classification of $(0,2)$-geometries embedded in affine space, due to De Feyter [29, 30, 31, 32, 33] in 2004 (see also De Clerck, De Feyter and Thas [21]), the problem of classifying embeddable proper semipartial geometries is reduced to classifying all linear representations of semipartial geometries . Prior to our work on this subject the best result concerning these linear representations was the Debroey-Thas classification from [19]. In the present chapter, which is based on De Winter [37], we will improve their result by classifying the linear representations of proper semipartial geometries in $\mathrm{AG}(4,q)$ and deriving restrictions on the parameters in the higher dimensional case.

In the mean time several authors, notably Cameron [14] and Calderbank [12], also had been working on the embeddings in $\mathrm{AG}(n,q)$ of partial quadrangles, more specifically on linear representations of these geometries. We will mention their results in Subsection 4.1.

4.1 Introduction

In this introduction we will collect some theorems, mostly concerned with special point sets $\mathcal{K}$ in $\mathrm{PG}(n, q)$, which will be of use in the rest of this chapter.

The following theorems deal with characterizations of point sets $\mathcal{K}$ in terms of intersections with respect to lines.

Theorem 4.1.1 (Tallini Scafati [66]) *If $\mathcal{K}$ is a set of points of $\mathrm{PG}(2, q)$, with the property that every line intersects $\mathcal{K}$ in either 1 or $m > 0$ points, and such that both intersections occur, then $\mathcal{K}$ is either a line, a unital, or a Baer-subplane.*

Theorem 4.1.2 (Tallini Scafati [66]) *Let $\mathcal{K}$ be a set of points in $\mathrm{PG}(n, q)$, $n \geq 3$, such that each line intersects $\mathcal{K}$ in either α or β points. If $\mathcal{K}$ nor its complement is empty, a point or a hyperplane, then q is an odd square and if $\alpha \leq \beta$ then*

$$\alpha = \frac{1}{2}(q + 1 - \sqrt{q}(1 - \epsilon)),$$

$$\beta = \frac{1}{2}(q + 1 + \sqrt{q}(1 + \epsilon)) \text{ and}$$

$$|\mathcal{K}| = \frac{1}{2}(1 + \frac{q^{n-1} - 1}{q - 1}(q + \epsilon\sqrt{q}) + \delta\sqrt{q^{n-1}}),$$

where $\epsilon = \pm 1$ and $\delta = \pm 1$.

Theorem 4.1.3 (Ueberberg [84]) *If $\mathcal{K}$ is a point set in $\mathrm{PG}(n, q)$, $n \geq 3$, with the property that $\mathcal{K}$ spans $\mathrm{PG}(n, q)$ and such that each line of $\mathrm{PG}(n, q)$ intersects $\mathcal{K}$ in either 0, 1 or $\alpha \geq \sqrt{q} + 1$ points, then $\mathcal{K}$ is either a Baer-subgeometry, an affine subspace of $\mathrm{PG}(n, q)$, or $\mathcal{K}$ equals the point set of $\mathrm{PG}(n, q)$.*

The next two theorems deal with characterizations of $\mathcal{K}$ in terms of intersections with respect to hyperplanes.

Theorem 4.1.4 (Tallini [65]) *If $\mathcal{K}$ is a non-empty set of points of $\mathrm{PG}(n, q)$, $\mathcal{K} \neq \mathrm{PG}(n, q)$ and $n \geq 3$, with the property that every hyperplane of $\mathrm{PG}(n, q)$ intersects $\mathcal{K}$ in either 0 or $m > 0$ points, then $\mathcal{K}$ is either a unique point or the complement of a hyperplane of $\mathrm{PG}(n, q)$.*

Theorem 4.1.5 (J. A. Thas [68]) *Suppose $\mathcal{K}$ is a point set in $\mathrm{PG}(n,q)$, $\mathcal{K} \neq \mathrm{PG}(n,q)$ and $n \geq 3$, with the property that any hyperplane contains either 1 or $m > 0$ points of $\mathcal{K}$. Then $\mathcal{K}$ is the point set of a line of $\mathrm{PG}(n,q)$ or $\mathcal{K}$ is an ovoid of $\mathrm{PG}(3,q)$.*

Finally we will use the following important results, linking strongly regular graphs and point sets with two intersection numbers with respect to hyperplanes.

Theorem 4.1.6 (Delsarte [28]) *Suppose that $\mathcal{K}$ is a non-empty point set of $\mathrm{PG}(n,q)$, spanning $\mathrm{PG}(n,q)$. Then $T_n^*(\mathcal{K})$ has a strongly regular point graph if and only if $\mathcal{K}$ has two intersection numbers with respect to hyperplanes.*

The proof of this theorem uses characters. However in Chapter 6 we will give a purely combinatorial proof in an interesting special case.

Now suppose that $\mathcal{K}$ is a point set of $\mathrm{PG}(n,q)$ spanning $\mathrm{PG}(n,q)$ and having two intersection numbers, w_1 and w_2, with respect to hyperplanes. It is easy to see that $w_1 \neq w_2$, unless $\mathcal{K}$ is the set of all points of $\mathrm{PG}(n,q)$, a trivial case which we exclude.

Theorem 4.1.7 (Calderbank and Kantor [13]) *If $\mathcal{K}$ is a non-trivial point set in $\mathrm{PG}(n,q)$, that is, $\mathcal{K}$ is not a subspace nor its complement, $q = p^h$, with the property that $T_n^*(\mathcal{K})$ has a strongly regular point graph with parameters $(N = q^{n+1}, K = |\mathcal{K}|\,(q-1), \lambda, \mu)$, then $w_2 - w_1 = p^w$ and*

$$q(w_2 - w_1) = ((\lambda - \mu)^2 + 4(K - \mu))^{1/2},$$

where $w_1 < w_2$ are the two intersection numbers of $\mathcal{K}$ with respect to hyperplanes of $\mathrm{PG}(n,q)$. Further, $\mathcal{K}$ defines a projective linear two weight code.

It is the above equation that will be exploited to prove the main result of this chapter.

We end this introductory section with some of the classification results that were already mentioned in the preface of this chapter.

Theorem 4.1.8 (De Clerck and Delanote [23]) *Suppose that S is a $(0,\alpha)$-geometry embedded in $\mathrm{AG}(n,q)$. If $\alpha \geq 3$ then S is a linear representation.*

Theorem 4.1.9 (De Feyter [21]) *Suppose that S is a proper semipartial geometry embedded in* $\mathrm{AG}(n,q)$*, $n \geq 4$. If $n > 4$, then S is a linear representation. If $n = 4$, then either $S \cong TQ(4,q)$, with $q = 2^h$, or S is a linear representation.*

Concerning the linear representations of partial quadrangles the following is known.

Theorem 4.1.10 ([14], [12], [83]) *If $\mathcal{K}$ is a non-trivial point set in* $\mathrm{PG}(n,q)$ *such that $T_n^*(\mathcal{K})$ is a partial quadrangle, then only the following cases can occur:*

- *$\mathcal{K}$ is a hyperoval in* $\mathrm{PG}(2,2^m)$*;*
- *$\mathcal{K}$ is an ovoid in* $\mathrm{PG}(3,q)$*;*
- *$\mathcal{K}$ is the projectively unique* 11*-cap in* $\mathrm{PG}(4,3)$*;*
- *$\mathcal{K}$ is the projectively unique* 56*-cap in* $\mathrm{PG}(5,3)$*;*
- *$\mathcal{K}$ is a* 78*-cap in* $\mathrm{PG}(5,4)$ *such that each external point is on* 7 *secants;*
- *$\mathcal{K}$ is a* 430*-cap in* $\mathrm{PG}(6,4)$*.*

Basically it is Cameron [14] who derives what the point set in the hyperplane at infinity should look like. Then Calderbank [12] associated an algebraic equation related to the Pell-equation to these sets and solved it for the smallest cases. Finally, Tzanakis and Wolfskill [83] obtain the complete solution of the equation of Calderbank, yielding Theorem 4.1.10.

It is not known whether such a 430-cap in $\mathrm{PG}(6,4)$ exists. However for each of the other possibilities an example is known (see Chapter 2, Subsection 2.2.2).

4.2 General results

From now on let $\mathcal{K}$ be a non-trivial set of points in $\mathrm{PG}(n,q)$, $n \geq 3$, that is, $\mathcal{K}$ is not a subspace nor its complement. Embed $\mathrm{PG}(n,q)$ as a hyperplane Π in $\mathrm{PG}(n+1,q)$. We suppose that the linear representation $T_n^*(\mathcal{K})$ of $\mathcal{K}$ is an $\mathrm{spg}(q-1, |\mathcal{K}|-1, \alpha, \mu)$. In this section we will derive some general results for such a set $\mathcal{K}$, which will enable us in the following sections to obtain a classification when $n = 3$. We always suppose that $\alpha > 1$.

Lemma 4.2.1 *Every line of* Π *intersects* $\mathcal{K}$ *in either* 0, 1 *or* $\alpha+1$ *points, and the set* $\mathcal{K}$ *consists of* $1+x\alpha$, $x \in \mathbb{N}$, *points. There exists a constant* θ *such that each point not belonging to* $\mathcal{K}$ *is incident with* θ *lines intersecting* $\mathcal{K}$ *in* 1 *point, that is,* $\mathcal{K}$ *is an SPG-regulus. Furthermore* $\mathcal{K}$ *has two intersection numbers with respect to hyperplanes.*

Proof. It is readily checked that the α-condition for semipartial geometries implies that a line intersecting $\mathcal{K}$ in 2 points must intersect it in $\alpha+1$ points. Now consider a fixed point of $\mathcal{K}$. Then any line through this point contains either 0 or α other points of $\mathcal{K}$. Hence $|\mathcal{K}| = 1 + x\alpha$, with x the number of lines through a given point of $\mathcal{K}$ intersecting $\mathcal{K}$ in at least 2 points. The existence of the constant θ is a consequence of the μ-condition for semipartial geometries . There holds $\mu = (|\mathcal{K}| - \theta)\alpha$. Finally, since the point graph of $T_n^*(\mathcal{K})$ is strongly regular, the last assertion of the lemma follows from Delsarte [28], see Theorem 4.1.6. □

We will call a line intersecting $\mathcal{K}$ in 0, 1, respectively $\alpha+1$ points, an *exterior line*, a *tangent*, respectively an $(\alpha+1)$*-secant*.

Lemma 4.2.2

(i) There exist exterior lines with respect to $\mathcal{K}$.

(ii) Every hyperplane of Π *has at least one point in common with* $\mathcal{K}$.

Proof.

(i) Suppose by way of contradiction that every line of Π would have at least one point in common with $\mathcal{K}$. Then clearly each line intersects $\mathcal{K}$ in either 1 or $\alpha+1$ points. From Theorem 4.1.2 there follows that $1 = \frac{1}{2}(q+1-\sqrt{q}(1-\epsilon))$, with $\epsilon = \pm 1$, clearly a contradiction.

(ii) If there exist hyperplanes exterior to $\mathcal{K}$, then Lemma 4.2.1 implies that a hyperplane contains either 0 or $m > 0$ points of $\mathcal{K}$. Hence Theorem 4.1.4 yields that $\mathcal{K}$ is either a point or the point set of the complement of a hyperplane, in contradiction with our assumptions. □

Together with Lemma 4.2.1 the previous lemma implies that a hyperplane meets $\mathcal{K}$ in either $1 + y\alpha$ points or $1 + z\alpha$ points, $y, z \in \mathbb{N}$, with $y < z$. We will call a hyperplane of the former type, respectively of the latter type, a *y-hyperplane*, respectively a *z-hyperplane*.

Lemma 4.2.3 *If $\mathcal{K}$ is a set of points in $\Pi := \mathrm{PG}(n,q)$, $n \geq 3$, $q = p^m$, p prime, with the property that $T_n^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, \alpha, \mu)$, then $\alpha = p^i$, $0 \leq i \leq m$.*

Proof. By the previous lemma we can choose a subspace $\gamma := \mathrm{PG}(l,q) \subset \Pi$ exterior to $\mathcal{K}$, $0 < l < n-1$, such that no $(l+1)$-dimensional subspace is exterior to $\mathcal{K}$. Now consider any $\Gamma := \mathrm{PG}(l+2,q) \subset \Pi$ containing γ. Clearly $|\Gamma \cap \mathcal{K}| = 1 + c\alpha$, $c \in \mathbb{N} \setminus \{0\}$. Every $(l+1)$-dimensional subspace of Γ containing γ will contain $1 + c_j\alpha$, $c_j \in \mathbb{N}$, points of $\mathcal{K}$. We obtain

$$\sum_{j=0}^{q}(1 + c_j\alpha) = 1 + c\alpha$$

and hence $q + \alpha \sum_{j=0}^{q} c_j = c\alpha$. This proves the lemma. $\square$

Lemma 4.2.4 *There holds that $z - y$ equals p^k for some $k > 0$.*

Proof. From Theorem 4.1.7 it follows that $w_2 - w_1 = p^w$, $w \in \mathbb{N}$. Hence the previous lemma implies that $w_2 - w_1 = (z-y)p^i = p^w$. Thus $z - y = p^k$ for some $k \in \mathbb{N}$. We will now show that $k > 0$. Consider an exterior line L and let δ be the number of z-hyperplanes containing L. We count the pairs (u, η), where $u \in \mathcal{K}$, $u \in \eta$, η a hyperplane containing L, in two ways:

$$\delta(1 + z\alpha) + (\frac{q^{n-1}-1}{q-1} - \delta)(1 + y\alpha) = (1 + x\alpha)\frac{q^{n-2}-1}{q-1}. \tag{4.1}$$

Now consider an $(\alpha+1)$-secant M and let δ' be the number of z-hyperplanes containing M. Here we count the pairs (v, ξ), where $v \in \mathcal{K} \setminus L$, $v \in \xi$, ξ a hyperplane containing M,

$$\delta'(z-1)\alpha + (\frac{q^{n-1}-1}{q-1} - \delta')(y-1)\alpha = (x-1)\alpha\frac{q^{n-2}-1}{q-1}. \tag{4.2}$$

Subtracting equation (4.2) from equation (4.1) yields

$$(\delta' - \delta)p^k = \frac{\alpha+1}{\alpha}q^{n-2}.$$

Since $\alpha \neq q$ (otherwise $\mathcal{K}$ would be the point set of a subspace) we find that $\frac{\alpha+1}{\alpha}q^{n-2} \geq q^{n-2} + 2q^{n-3}$. As $\delta, \delta' \in \mathbb{N}$, we see that if $k = 0$ it follows that $\delta' \geq q^{n-2} + 2q^{n-3}$, a contradiction since $\delta' \leq (q^{n-1}-1)/(q-1)$. $\square$

Lemma 4.2.5 *If $\mathcal{K}$ is a set of points in $\Pi := \mathrm{PG}(n,q)$, $n \geq 3$, $q = p^m$, p prime, with the property that $T_n^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, p^i, \mu)$, then the strongly regular point graph of $T_n^*(\mathcal{K})$ has parameters*

- $\mu = p^i \frac{x(1+xp^i)(p^m-p^i)}{p^{mn}+\cdots+p^m-xp^i}$,
- $\lambda = p^m - 2 + xp^i(p^i-1)$ *and*
- $K = (xp^i+1)(p^m-1)$, *with K the valency of the graph.*

Proof. From the previous lemmas we know that $\mu = (xp^i + 1 - \theta)p^i$. So we should now determine θ. We count in two ways the pairs (u,v), where $u \notin \mathcal{K}$, $v \in \mathcal{K}$ and $\langle u, v\rangle$ is a tangent. We obtain $(1+xp^i)(\frac{q^n-1}{q-1} - x)q = (\frac{q^{n+1}-1}{q-1} - 1 - xp^i)\theta$ from which θ follows. It now easily follows that $\mu = p^i \frac{x(1+xp^i)(p^m-p^i)}{p^{mn}+\cdots+p^m-xp^i}$.

The values for λ and K follow trivially. □

Theorem 4.2.6 *Let $\mathcal{K}$ be a non-trivial set of points in $\Pi := \mathrm{PG}(n,q)$, $n \geq 3$, $q = p^m$, p prime, with the property that $T_n^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, p^i, \mu)$. If $i \geq m/2$, then $T_n^*(\mathcal{K}) \cong T_n^*(\mathcal{B})$, with $\mathcal{B}$ a Baer-subgeometry of* $\mathrm{PG}(n,q)$.

Proof. Since $T_2^*(\mathcal{K})$ is connected it follows that $\mathcal{K}$ spans $\mathrm{PG}(n,q)$. Now Theorem 4.1.3 immediately implies that $\mathcal{K}$ is a Baer-subgeometry. □

From now on we may suppose that $i < m/2$.

Since the point graph of $T_n^*(\mathcal{K})$ is strongly regular Theorem 4.1.7 now implies that

$$p^{2m}p^{2k+2i} = (\lambda - \mu)^2 + 4(K - \mu), \tag{4.3}$$

with λ, μ and K as in Lemma 4.2.5.

Finally we show that from $\mathcal{K}$ we can construct a point set in $\mathrm{PG}(n-1,q)$ having two intersection numbers with respect to hyperplanes. Let u be any point of $\mathcal{K}$ and consider a hyperplane Δ not containing u. As every hyperplane through u contains either y or z $(\alpha+1)$-secants through u, we see that the projection of $\mathcal{K}$ from u onto Δ yields a point set $\mathcal{L}$ of cardinality x in Δ with the property that every hyperplane of Δ contains either y or z points of $\mathcal{L}$. Notice that both intersection numbers occur.

Lemma 4.2.7 *There holds*

$$x^2(q^{n-2}-1) + x(q^{n-2}(q-1) - (y+z)(q^{n-1}-1)) + yz(q^n-1) = 0. \tag{4.4}$$

Proof. In Hirschfeld [49] this is shown for $n = 3$ (not in the context of projections of a set $\mathcal{K}$). The proof we give for the general case is analogous. Let $\mathcal{L}$ and Δ be as above. Denote by τ_y, respectively τ_z, the number of hyperplanes of Δ containing y, respectively z, points of $\mathcal{L}$. We obtain

$$\tau_y + \tau_z = \frac{q^n - 1}{q - 1}$$

$$\tau_y y + \tau_z z = x\frac{q^{n-1} - 1}{q - 1}$$

$$\tau_y y(y-1) + \tau_z z(z-1) = x(x-1)\frac{q^{n-2} - 1}{q - 1}.$$

Eliminating τ_y and τ_z from these equations yields equation (4.4). $\square$

4.3 The case $n = 3$

In this section we suppose that the setup is as in the previous section with $n = 3$, $\alpha = p^i$ and $0 < i < m/2$. Furthermore we use the same notations. We start by handling some special cases.

Theorem 4.3.1 *The case $y = 0$ cannot occur.*

Proof. If $y = 0$, then $\mathcal{K}$ is a point set satisfying the conditions of Theorem 4.1.5. Hence $\mathcal{K}$ is either a line or an ovoid, in contradiction either with the non-triviality of $\mathcal{K}$ or with the fact that $\alpha > 1$. $\square$

Next we exclude the other end of the spectrum.

Theorem 4.3.2 *The case $z = q + 1$ cannot occur.*

Proof. If $z = q + 1$, then in a z-plane of Π every point of $\mathcal{K}$ is clearly contained in $q + 1$ $(\alpha + 1)$-secants. There follows that a z-plane contains no tangent lines and hence that $\mathcal{K}$ induces a maximal arc in every z-plane. This implies that $\alpha = p^l - 1$ (see Chapter 2, Subsection 2.2.2), in contradiction with Lemma 4.2.3. $\square$

We will now start with an analysis of equation (4.3), but first we introduce a new notation.

Notation

We will denote by $\mathcal{O}(p^f)$ any polynomial in p of degree at least f with coefficients in $\mathbb{N}$.

The calculations in the rest of this section are tedious, and can easily be carried out in MAPLE. That is the reason why in most steps we only mention the terms we need and use shortened expressions.

If $n = 3$ equation (4.3) becomes, after multiplying both sides with $(p^{3m} + p^{2m} + p^m - xp^i)^2$,

$$p^{2m+4i+2k}(p^{3m-i} + p^{2m-i} + p^{m-i} - x)^2 = U, \tag{4.5}$$

with

$$U = (x^2 - 2x^3 + x^4)p^{4i} + \mathcal{O}(p^{4i+1}).$$

Lemma 4.3.3 *Either p divides x, or p divides $x - 1$.*

Proof. Considering the above equation modulo p^{4i+1} we find that p divides $x^2(x-1)^2$. □

Lemma 4.3.4 *There holds that $x \equiv y \equiv z \pmod p$.*

Proof. We first show that every plane contains exterior lines. Assume that a z-plane π would contain no exterior lines. It then follows that $\mathcal{K}$ induces in π either a line, a Baer-subgeometry or a unital (by Theorem 4.1.1), yielding $\alpha \geq p^{m/2}$, a contradiction. Now let L be an exterior line to $\mathcal{K}$, and suppose that there are δ z-planes containing L. We obtain

$$\delta(1 + z\alpha) + (q + 1 - \delta)(1 + y\alpha) = 1 + x\alpha,$$

which yields

$$\delta p^k + y + p^{m-i} + yp^m = x.$$

Hence $x \equiv y \pmod p$. As $z - y = p^k$, with $k > 0$, the result follows. □

Notice that since there are z-planes, and each z-plane must contain at least one exterior line, $\delta \neq 0$. The fact that $\delta \neq 0$ will be of use later on.

4.3.1 The case p divides x

We write $x = ap$ and substitute this in equation (4.5). We obtain

$$\begin{aligned} U &= p^{4m} - 2ap^{2m+2i+1} + a^2p^{4i+2} + \mathcal{O}(p^{4m+1}) + a\mathcal{O}(p^{3m+i+1}) \\ &\quad + a^2\mathcal{O}(p^{m+3i+2}) + a^3\mathcal{O}(p^{4i+3}) + a^4\mathcal{O}(p^{4i+4}). \end{aligned}$$

Lemma 4.3.5 *There holds that $p^{2m-2i-1}$ divides a.*

Proof. Suppose that p^j divides a with $0 \leq j < 2m-2i-1$.We write $a = a'p^j$. As $4i+2+2j < 2m+2i+1+j$ and $4i+2+2j < 4m$ if $j < 2m-2i-1$, clearly $U = a'^2p^{4i+2+2j} + \mathcal{O}(p^{4i+2j+3})$. There are two possibilities. If $j < m-i$, the left hand side of equation (4.5) becomes

$$p^{2m+4i+2k+2j}(p^{3m-i-j} + p^{2m-i-j} + p^{m-i-j} - a'p)^2.$$

If $j \geq m-i$ we obtain

$$p^{4m+2i+2k}(p^{2m} + p^m + 1 - a'p^{j-m+i+1})^2$$

In either case there holds that the left hand side of equation (4.5) is 0 (mod $p^{4i+2j+3}$). This implies that a'^2 is 0 (mod p) and hence that p^{j+1} divides a. The lemma follows. □

As an immediate consequence we can write from now on $x = bp^{2m-2i}$. We will now turn to the analysis of equation (4.4) which will allow us to exclude the case p divides x.

Theorem 4.3.6 *The case p divides x cannot occur.*

Proof. Write y and z in p-ary representation: $y = y_fp^f + \ldots$ and $z = z_lp^l + \ldots$ with $y_f \neq 0$ (because of Theorem 4.3.1) and $z_l \neq 0$. After division by $q-1$ equation (4.4) becomes

$$x^2 - x(q(y+z-1) + y + z) + yz(q^2+q+1) = 0, \tag{4.6}$$

with $q = p^m$. Because of the previous lemma p^{4m-4i} divides x^2, the terms of lowest degree in $x(q(y+z-1)+y+z)$ are

$$\overline{by_f}p^{2m-2i+f} \quad \text{and} \quad \overline{bz_l}p^{2m-2i+l}$$

while the term of lowest degree in $yz(q^2+q+1)$ is

$$\overline{y_fz_l}p^{f+l}$$

where $\overline{uv}$ denotes multiplication of u and v modulo p. Clearly $f+l < 4m-4i$, as f and l are at most m (y and z are numbers of lines through a point in a plane) and $i < m/2$. Furthermore $2m-2i+f \leq f+l$ would imply $l \geq m+1$, a contradiction. In an analogous way $2m-2i+l \leq f+l$ cannot occur. Hence if we consider equation (4.6) modulo p^{f+l+1} we obtain that $\overline{y_f z_l} \equiv 0$, the final contradiction. □

4.3.2 The case p divides $x - 1$

The basic ideas for handling this case are the same as in the previous subsection, but it will turn out that there are more subcases to deal with. We will write $x = ap + 1$.

In the following series of lemmas we will derive a form for x in function of p.

Lemma 4.3.7 *There holds that p^{m-i-1} divides a.*

Proof. Substituting $x = ap + 1$ in U we obtain

$$\begin{aligned} U &= a^2p^{4i+2} - 2ap^{m+3i+1} + p^{2m+2i} + \mathcal{O}(p^{2m+3i}) + a\mathcal{O}(p^{m+4i+1}) \\ &\quad + a^2\mathcal{O}(p^{m+3i+2}) + a^3\mathcal{O}(p^{4i+3}) + a^4\mathcal{O}(p^{4+4i}). \end{aligned}$$

Suppose that p^j divides a with $j < m - i - 1$. We write $a = a'p^j$. As $4i+2+2j < m+3i+1+j$ and $4i+2+2j < 2m+2i$ whenever $j < m-i-1$, clearly we obtain $U = a'^2p^{4i+2+2j} + \mathcal{O}(p^{4i+3+2j})$. The left hand side of equation (4.5) becomes

$$p^{2m+4i+2k}(p^{3m-i} + p^{2m-i} + p^{m-i} - 1 - a'p^j)^2$$

and is clearly 0 (mod $p^{4i+3+2j}$). We find that p must divide a'^2, that is, p^{j+1} divides a. The lemma follows. □

From now on we write $x = bp^{m-i} + 1$.

Lemma 4.3.8 *There holds that $b = cp + 1$ with $c \in \mathbb{N} \setminus \{0\}$.*

Proof. Substituting $x = 1 + bp^{m-i}$ in U we obtain

$$U = p^{2m+2i} - 2bp^{2m+2i} + b^2p^{2m+2i} + \mathcal{O}(p^{2m+2i+1}).$$

As the left hand side of equation (4.5) is 0 (mod $p^{2m+2i+1}$) we see that p must divide $(b-1)^2$. Hence $b = cp + 1$ with $c \in \mathbb{N}$. If $c = 0$ we find $x < 1 + p^m$, a contradiction as any non-trivial two weight set in Δ must contain at least $2 + p^m$ points, i.e. $|\mathcal{L}| = x \geq 2 + p^m$. □

We obtain $x = 1 + p^{m-i} + cp^{m-i+1}$.

Lemma 4.3.9 *There holds that p^{i-1} divides c.*

Proof. The lemma becomes of course trivial if $i = 1$, so we suppose that $i > 1$. Again in the same spirit we obtain

$$\begin{aligned} U &= 4p^{2m+4i} - 4cp^{2m+3i+1} + c^2p^{2m+2i+2} + \mathcal{O}(p^{3m+2i}) + c\mathcal{O}(p^{3m+i+1}) + \\ & \quad c^2\mathcal{O}(p^{3m+i+2}) + c^3\mathcal{O}(p^{3m+3+i}) + c^4\mathcal{O}(p^{4m+4}). \end{aligned}$$

We suppose that $c = c'p^j$ with $j < i-1$. There holds that $2m+2i+2+2j < 2m+3i+1+j$ and $2m+2i+2+2j < 2m+4i$ if $j < i-1$. Furthermore the left hand side of equation (4.5) is 0 (mod $p^{2m+2i+3+2j}$), implying that p divides c' and hence that p^{j+1} divides c. The lemma follows. □

There follows that $x = 1 + p^{m-i} + dp^m$ with $d \in \mathbb{N} \setminus \{0\}$.

Recall that $z - y = p^k$.

Lemma 4.3.10 *If $p \neq 2$, then $d \equiv 2 \pmod p$. If $p = 2$, then d is even; furthermore in this case, if 2^f, with $f > 1$, divides d, and $k > 1$, then $i = (m-1)/2$.*

Proof. There holds that

$$U = (d^2 - 4d + 4)p^{2m+4i} + (-4 + 2d)p^{3m+2i} + p^{4m} + \mathcal{O}(p^{3m+3i}).$$

As the left hand side of equation (4.5) is 0 (mod $p^{2m+4i+1}$), we see that $(d-2)^2$ is divisible by p. The first two assertions follow. Suppose that $p = 2$, 2^f, with $f > 1$, divides d, and $k > 1$. Then

$$U = 2^{2m+4i+2} + 2^{4m} + \mathcal{O}(2^{2m+4i+3})$$

(notice that $2m+4i+3 \leq \min(3m+2i+2, 3m+3i)$, unless $i = 1$ and $m = 3$, in which case we have the desired property $i = (m-1)/2$). Since $k > 1$ the left hand side of equation (4.5) is 0 (mod $2^{2m+4i+3}$). This implies that $2m + 4i + 2 = 4m$, that is, $i = (m-1)/2$. □

In the following series of lemmas we will now exclude step by step the possibility that p divides $x - 1$.

Lemma 4.3.11 *The case $y = 1$ cannot occur.*

Proof. Suppose that $y = 1$. Then, by Theorem 4.1.1, the set $\mathcal{L}$ in Δ must either be a line, a Baer-subplane or a unital. As $z = p^m + 1$ is impossible, by Theorem 4.3.2, $\mathcal{L}$ cannot be a line. If $\mathcal{L}$ is a Baer-subplane, respectively a unital, there follows that $x = 1 + p^{m/2} + p^m$, respectively $1 + p^{3m/2}$, both in contradiction with the derived form of x. □

Lemma 4.3.12 *p^k divides $y - 1$.*

Proof. First notice that, since $y, z \leq p^m$, $k < m$. Using the equation obtained in the proof of Lemma 4.3.4 and the form of x we find

$$\delta p^k + (y-1) + p^m + (y-1)p^m = dp^m,$$

with δ the number of z-planes through an exterior line of $\mathcal{K}$. The lemma follows immediately since $\delta \neq 0$, see Lemma 4.3.4. □

There are three possibilities for y and z:

- **(I)** $y = 1 + up^l$ and $z = 1 + p^k + up^l$, with $u \in \mathbb{N}$, u not divisible by p and $l > k$ (notice that $l < m$);
- **(II)** $y = 1 + y_k p^k + up^{k+1}$ and $z = 1 + (1 + y_k)p^k + up^{k+1}$, with $0 < y_k < p - 1$ and $u \in \mathbb{N}$ (notice that $p \neq 2$ in this case);
- **(III)** $y = 1 - p^k + up^l$ and $z = 1 + up^l$, with $u \in \mathbb{N}$, u not divisible by p and $l > k$.

Lemma 4.3.13 *$k \leq m - i$.*

Proof. Consider a tangent line L to $\mathcal{K}$ and let β be the number of y-planes through L. We find

$$\beta y + (p^m + 1 - \beta)z = x,$$

which yields

$$-\beta p^k + zp^m + (z-1) = p^{m-i} + dp^m.$$

Since p^k divides $z - 1$, we see that $k \leq m - i$. □

Lemma 4.3.14 *The following cases cannot occur: $l < m-i$ in* (**I**)*, $k < m-i$ in* (**II**) *and $l < m - i$ in* (**III**)*.*

Proof. Assume by way of contradiction that $l < m-i$, respectively $k < m-i$. Substituting the obtained values for x, y and z in equation (4.6) we find

$$up^{l+k} + \mathcal{O}(p^{l+k+1}) = 0 \text{ in case } (\mathbf{I}),$$

$$y_k(y_k+1)p^{2k} + \mathcal{O}(p^{2k+1}) = 0 \text{ in case } (\mathbf{II}) \text{ and}$$
$$up^{k+l} + \mathcal{O}(p^{k+l+1}) = 0 \text{ in case } (\mathbf{III}).$$

In case (**I**) and (**III**) we obtain a contradiction modulo p^{k+l+1}, since p does not divide u. In case (**II**) we see that $y_k \in \{0, p-1\}$, also a contradiction.□

Lemma 4.3.15 *In case* (**I**) $l > m-i$ *cannot occur.*

Proof. Assume by way of contradiction that $l > m-i$. First suppose that $k < m-i$. Then equation (4.6) becomes

$$p^{m-i+k} + \mathcal{O}(p^{m-i+k+1}) = 0,$$

yielding a contradiction modulo $p^{m-i+k+1}$.

If $k = m-i$ we obtain

$$up^{m-i+l} + \mathcal{O}(p^{m-i+l+1}) = 0,$$

yielding a contradiction modulo $p^{m-i+l+1}$ since p does not divide u. □

Lemma 4.3.16 *Case* (**I**) *cannot occur.*

Proof. We are left with showing that also $l = m-i$ is impossible in this case, so assume the contrary. Substituting the appropriate values for x, y and z in equation (4.6), we obtain

$$up^{m-i+k} - p^{m-i+k} + \mathcal{O}(p^{m-i+k+1}) = 0,$$

implying $u = 1 + vp$ with $v \in \mathbb{N}$. If $v \neq 0$ equation (4.6) becomes

$$-dp^{m+k} + vp^{m-i+k+1} + \mathcal{O}(p^{2m-i}) + v\mathcal{O}(p^{2m-i+1}) + v^2\mathcal{O}(p^{2m-2i+2}) = 0,$$

from which easily follows that p^{i-1} divides v. We find $y \geq 1 + p^{m-i}(1+p^i) = 1 + p^{m-i} + p^m$, a contradiction. Hence $v = 0$. Equation (4.6) becomes, modulo p^{2m}:

$$-dp^{m+k} - p^{2m-i} = 0.$$

This implies that $p^{m-i-k} > 1$ divides d, which in view of Lemma 4.3.10 yields that $p = 2$. Now first suppose that $k = 1$. It follows that the number of $(\alpha+1)$-secants (with respect to $\mathcal{K}$) in a y-plane equals

$$\frac{(1+p^i+p^m)(1+p^{m-i})}{1+p^i},$$

from which we deduce that $1+p^i$ divides $1+p^{m-i}$. If we now count the number of $(\alpha+1)$-secants in a z-plane we find

$$\frac{(1+p^i+p^{i+1}+p^m)(1+p+p^{m-i})}{1+p^i},$$

which implies (using the fact that $1+p^i$ divides $1+p^{m-i}$ and rewriting $p+p^{m-i}$ as $p-1+1+p^{m-i}$) that $1+p^i$ divides $p-1$, clearly a contradiction. Now suppose that $k > 1$ with $m-i-k \neq 1$. From Lemma 4.3.10 it follows that $i = (m-1)/2$, and hence the number of $(\alpha+1)$-secants (with respect to $\mathcal{K}$) in a y-plane can never be an integer, a contradiction. Finally suppose that $k > 1$ with $m-i-k = 1$. Here as well, $1+p^i$ divides $1+p^{m-i}$ and since the number of $(\alpha+1)$-secants in a z-plane must be an integer we find that $1+p^i$ divides $p^{i+k}+p^m = p^{i+k}(1+p)$ and hence $1+p^i$ divides $1+p$, that is, $i = 1$ and hence $k = m-2$. Equation (4.6) becomes $2^{2m}d^2 - 5.2^{2m-2}(2^m+1)d + 2^{2m-1}(3.2^{2m-2}+2^{m+1}+1) = 0$. As this equation must have at least one integer solution in d it follows that the square root of its discriminant D must be an integer. We obtain $D = 2^{4m-4}(2^{2m}-7.2^{m+1}-7)$ and hence $2^{2m}-7.2^{m+1}-7$ has an integer square root. Rewriting this we see that $(2^m-7)^2-56$ is a^2 for some $a \in \mathbb{N}$ with $a+\beta = 2^m-7$. There follows that $a = \frac{56-\beta^2}{2\beta}$. We find that β must divide 56 and must be even, so $\beta \in \{2,4,8,14,28,-2,-4,-8,-14,-28\}$. We now easily see that $\beta = 4$, $a = 5$ and $m = 4$ is the only solution. The unique solution for equation (4.6) then is given by $x = 169$, $y = 9$, $z = 13$ and $q = 16$. As we supposed that $\mathcal{K}$ yields a semipartial geometry these values should also satisfy equation (4.3). Plugging in these values in this equation we obtain $266090905600 = 338923945984$, the final contradiction (another way to obtain a contradiction here is to check that $\mu \notin \mathbb{N}$). $\square$

Lemma 4.3.17 *Case* (**II**) *cannot occur.*

Proof. We are left with showing that also $k = m-i$ is impossible in this case, so assume the contrary. Equation (4.6) becomes

$$y_k(y_k-1)p^{2m-2i} + \mathcal{O}(p^{2m-2i+1}) = 0,$$

from which we see that $y_k = 1$. Assume that $u \neq 0$. We find

$$up^{2m-2i+1} - (d+1)p^{2m-i} + \mathcal{O}(p^{2m}) + u\mathcal{O}(p^{2m-i+1}) + u^2\mathcal{O}(p^{2m-2i+2}) = 0,$$

which enables us to deduce that p^{i-1} divides u. Hence $y \geq 1 + p^{m-i}(1+p^i) > 1 + p^m$, a contradiction.

Now suppose that $u = 0$. We count the number of $(\alpha + 1)$-secants (with respect to $\mathcal{K}$) in a y-plane, and obtain:

$$\frac{(1 + p^i + p^m)(1 + p^{m-i})}{1 + p^i}.$$

This implies that $1 + p^i$ divides $1 + p^{m-i}$. The number of $(\alpha + 1)$-secants in a z-plane equals

$$\frac{(1 + p^i + 2p^m)(1 + 2p^{m-i})}{1 + p^i}.$$

From this we deduce (using the fact that $1 + p^i$ divides $1 + p^{m-i}$) that $1 + p^i$ divides $2p^{2m-i}$ and hence that $2/(1 + p^i) \in \mathbb{N}$, a contradiction. □

Lemma 4.3.18 *In case* (**III**) $l > m - i$ *cannot occur.*

Proof. Suppose $l > m - i$. Since equation (4.6) becomes

$$p^{m-i+k} + p^{2m-2i} + \mathcal{O}(p^{2m-2i+1}) = 0,$$

we see that $k = m - i$ and $p = 2$. The equation now is of the form

$$2^{2m-2i+1} + (3d-1)2^{2m-i} - 3u2^{m-i+l} + u^2 2^{2l} + \mathcal{O}(2^{2m-i+1}) = 0.$$

Recall that $u \neq 0$. If $2m - 2i + 1 = 2m - i$, then $i = 1$ and $z \geq 1 + p^m$, a contradiction. Hence $l = m - i + 1$ and $u = 1 + 2v$ (since $l < m - i + 1$ or $l > m - i + 1$ would yield a contradiction modulo $p^{m-i+l+1}$, respectively modulo $p^{2m-2i+2}$). If $v \neq 0$, we obtain

$$-v2^{2m-2i+2} - v2^{2m-2i+3} + v(v+1)2^{2m-2i+4} + (3d-1)2^{2m-i} + \mathcal{O}(2^{2m-i+1}) = 0$$

and hence 2^{i-2} must divide v, yielding $z \geq 1 + 2^{m-i+1} + 2^m$, a contradiction. Consequently $v = 0$, but then, as d is even, we find a contradiction modulo p^{2m-i+1}. □

Lemma 4.3.19 *Case* (**III**) *cannot occur.*

We may suppose that $l = m - i$. Just as in the previous lemmas we find that $u = 1 + vp$, and see that $v = 0$ since otherwise $z > 1 + p^m$. Equation (4.6) becomes

$$dp^{m+k} - p^{2m-i} + \mathcal{O}(p^{2m-i+1}) = 0,$$

from which we deduce that $p^{m-i-k} > 1$ divides d. In view of Lemma 4.3.10 this implies that $p = 2$. This case now is handled analogously as in Lemma 4.3.16. If $k > 1$ and $m - i - k > 1$, then $i = (m-1)/2$ and the number of $(\alpha+1)$-secants in a z-plane can never be an integer. If $k > 1$ and $m-i-k = 1$, we find that $1 + p^i$ divides $p - 1$, a contradiction. Finally, if $k = 1$, we see that $1 + p^i$ divides $1 + p$ and hence $i = 1$. Like in Lemma 4.3.16 we obtain a quadratic equation in d which should have at least one integer solution. But as its discriminant equals $-2^{5m} + 2^{4m+1} + 2^{4m} + 2^{3m+2} + 2^{2m+2}$, which is always negative (unless $m = 1$ which cannot occur), we obtain the final contradiction. □

4.3.3 Summary

Theorem 4.3.20 *If $\mathcal{K}$ is a non-trivial set of points in $\mathrm{PG}(3, q)$ such that $T_3^*(\mathcal{K})$ is an spg$(q-1, |\mathcal{K}|-1, \alpha, \mu)$, then either $\alpha = 1$ and $\mathcal{K}$ is an ovoid or q is a square, $\alpha = \sqrt{q}$ and $\mathcal{K}$ is the point set of a Baer-subgeometry.*

Proof. Let $q = p^m$. If $\alpha = 1$ or $\alpha \geq p^{m/2}$ Theorems 4.1.5 and 4.2.6 imply the result, so suppose that $1 < \alpha < p^{m/2}$. In Lemma 4.3.3 it was shown that if x is the number of $(\alpha+1)$-secants through a point of $\mathcal{K}$, then either p divides x or $x - 1$. If p would divide x, then Theorem 4.3.6 implies that such $\mathcal{K}$ cannot exist. If p would divide $x - 1$ then Lemmas 4.3.11, 4.3.16, 4.3.17 and 4.3.19 yield that such $\mathcal{K}$ cannot exist. Hence there follows that necessarily $\alpha = 1$ or $\alpha \geq p^{m/2}$. The theorem is proved. □

Theorem 4.3.21 *If S is a semipartial geometry with $\alpha > 1$, embedded in $\mathrm{AG}(4, q)$, then either $S \cong \mathrm{TQ}(4, q)$, with $q = 2^h$, or $S \cong T_3^*(\mathcal{B})$.*

Proof. By Theorem 4.1.8 and Theorem 4.1.9 we know that such a semipartial geometry is either $\mathrm{TQ}(4, 2^h)$ or a linear representation. The result now follows immediately from the previous theorem. □

Hence if we omit the restriction that $\alpha > 1$, we obtain the following possibilities for semipartial geometries embedded in $\mathrm{AG}(4, q)$:

- $\alpha = 1$ and S is the linear representation of an ovoid of $\mathrm{PG}(3, q)$ (see Theorem 4.1.10);
- $\alpha = 1$ and S is no linear representation (not much is known about this case);
- $S \cong \mathrm{TQ}(4, q)$, with $q = 2^h$;
- $S \cong T_3^*(\mathcal{B})$.

4.4 Some remarks on the case $n > 3$

The objective of this final section is to prove that the conclusions of Lemmas 4.3.3, 4.3.5, 4.3.7, 4.3.8, 4.3.9 and 4.3.10 remain valid in the higher dimensional case. We use the same notations as before and we suppose that $n \geq 4$.

Theorem 4.4.1 *There holds that either p^{2m-2i} divides x or that $x = 1 + p^{m-i} + dp^m$ with $d \in \mathbb{N} \setminus \{0\}$. In the latter case $d \equiv 2 \pmod{p}$ if $p \neq 2$ and d is even if $p = 2$; furthermore if $p = 2$, 4 divides d and $k > 1$, then $i = (m-1)/2$.*

Proof. Denote by U the right hand side of equation (4.5), set $D = p^{3m} + p^{2m} + p^m - xp^i$, $D(n) = p^{4m} + \ldots + p^{nm}$ and $N = p^i x(1 + xp^i)(p^m - p^i)$; furthermore let λ and μ be as in Lemma 4.2.5. We calculate the right hand side of equation (4.3):

$$\begin{array}{cl} & (\lambda - \frac{N}{D+D(n)})^2 + 4(K - \frac{N}{D+D(n)}) \\ = & \frac{1}{(D+D(n))^2}\left[\lambda^2(D+D(n))^2 - 2\lambda(D+D(n))N\right. \\ & \left. +N^2 + 4K(D+D(n)) - 4N(D+D(n))\right] \\ = & \frac{1}{(D+D(n))^2}\left[U + D(n)\left(2D\lambda^2 + D(n)\lambda^2 - 2\lambda N + 4K - 4N\right)\right]. \end{array}$$

Hence, after multiplication of both sides of equation (4.3) with $(D + D(n))^2$, we obtain

$$p^{2m+2k+2i}(p^{nm} + p^{(n-1)m} + \ldots + p^m - xp^i)^2 = U_n.$$

with

$$U_n = U + D(n)\left(2D\lambda^2 + D(n)\lambda^2 - 2\lambda N + 4K - 4N\right).$$

Since $U_n - U = 0 \pmod{p^{4m}}$ we immediately see that the conclusions and proofs of Lemmas 4.3.3, 4.3.5, 4.3.7, 4.3.8, 4.3.9 remain valid in the higher dimensional case. In order to see that Lemma 4.3.10 remains valid it suffices to notice that if $p = 2$ there holds that $U_n - U \equiv 0 \pmod{p^{4m+1}}$, and so also that proof can be copied. $\square$

Conjecture. If S is a semipartial geometry with $\alpha > 1$, with the property that S is the linear representation of a non-trivial point set in $\mathrm{PG}(n, q)$, $n \geq 4$, then $S \cong T_n^*(\mathcal{B})$.

Although the techniques applied to prove this conjecture for $n = 3$ seem suitable to attack the general case, the main problems when trying to do so arise from the fact that k can be larger than m if $n \geq 4$.

Chapter 5

Interlude: SPG-reguli and Baer-subgeometries

As mentioned in Chapter 2, a Baer-subgeometry $\mathcal{B}$ in $\mathrm{PG}(n, p^{2m})$ constitutes an SPG-regulus in $\mathrm{PG}(n, p^{2m})$ consisting of points. However, applying a field reduction $p^{2m} \mapsto p^l$, with $2m = lk$, $\mathcal{B}$ will be mapped onto a set $\overline{\mathcal{B}}$ of $(k-1)$-dimensional subspaces in $\mathrm{PG}(k(n+1)-1, p^l)$, which is again an SPG-regulus; see Remark 2.2.4. Furthermore $T_n^*(\mathcal{B}) \cong S(\overline{\mathcal{B}})$. In this chapter we will provide a positive answer to the following natural question:

> *Given an SPG-regulus $\mathcal{R}$ in $\mathrm{PG}(n, q)$, with $\mu = \alpha(\alpha+1)$ and $\alpha > 1$ (which is the case for the linear representation of a Baer-subgeometry), does there exist some n', q' and a Baer-subgeometry $\mathcal{B}$ in $\mathrm{PG}(n', q')$ such that $\mathrm{PG}(n', q') \overset{q' \mapsto q}{\rightarrow} \mathrm{PG}(n, q)$ and $\mathcal{B} \overset{q' \mapsto q}{\rightarrow} \mathcal{R}$?*

Of course, for "most" parameter sets $(s, t, \alpha, \alpha(\alpha+1))$ Theorem 2.2.6 already provides a positive answer. Nevertheless an infinite number of parameter sets is not covered by that theorem, so from the point of view of classifying SPG-reguli, the main theorem of this chapter is still useful. It will turn out to be helpful in the theory of translation semipartial geometries as well (see Chapter 7).

Finally we could say that this chapter, which is based on a part of [36], is a natural transition from the SPG-reguli (and the related geometries) consisting of points that were studied in the previous chapters, to those consisting of higher dimensional spaces that will be considered in the next chapters.

Throughout this chapter s, t, α, μ and θ will denote the usual parameters of SPG-reguli and the related semipartial geometries.

5.1 The Main Theorem

Theorem 5.1.1 *If $\mathcal{R} = \{\pi_0, \pi_1, \ldots, \pi_t\}$ is an SPG-regulus in* $\mathrm{PG}(n, q)$*, each π_i being a* $\mathrm{PG}(l, q)$*, for which $t - \theta = \alpha \notin \{1, q^{l+1} - 1, q^{l+1}\}$ and with $\theta \neq 0$, then $S(\mathcal{R}) \cong T^*_m(\mathcal{B})$ for certain m.*

Proof. First notice that if a space $\langle p, \pi_i\rangle$, $p \notin \pi_i$, p a point, intersects α elements $\pi_1, \ldots, \pi_\alpha$ of $\mathcal{R}$ distinct from π_i, then for every point $p' \in \langle p, \pi_i\rangle \setminus \widetilde{\mathcal{R}}$ there holds that $\langle p', \pi_j\rangle$ is not a tangent space if $j \in \{1, \ldots, \alpha, i\}$, while $\langle p', \pi_j\rangle$ is a tangent space for each $j \notin \{1, \ldots, \alpha, i\}$.

STEP 1. $\mathcal{R}$ *is α-geometric*

If the π_i are points then this is trivial. So suppose that the π_i are of dimension $l > 0$.

- We suppose that $\alpha < q^{l+1} - q^l + 2$. Let p be any point not belonging to $\widetilde{\mathcal{R}}$ with the property that $\langle p, \pi_0\rangle$ is not a tangent space. We may assume that $\pi_1, \ldots, \pi_\alpha$ are the α elements of $\mathcal{R}$ intersecting $\langle p, \pi_0\rangle$. We must prove that $\pi = \pi_2$ is completely contained in $\langle \pi_0, \pi_1\rangle$. Define X to be the set $X = \langle p, \pi_0\rangle \setminus \widetilde{\mathcal{R}}$. Furthermore let $m := \pi_1 \cap \langle p, \pi_0\rangle$ and $m_1 := \pi \cap \langle p, \pi_0\rangle$. Choose $x_1 \in X \setminus \langle m, m_1\rangle$. There holds that $\pi \cap \langle x_1, \pi_1\rangle \neq \emptyset$, say $m_2 := \pi \cap \langle x_1, \pi_1\rangle$. If $l = 1$ then we are done as $m_1, m_2 \in \langle \pi_0, \pi_1\rangle$ implies that $\pi \subset \langle \pi_0, \pi_1\rangle$. So suppose that $l > 1$. Now choose some $x_2 \in X \setminus \langle m, m_1, x_1\rangle$. There holds that $\langle x_2, \pi_1\rangle \cap \langle x_1, m_1, \pi_1\rangle = \pi_1$. Furthermore there must hold that $\pi \cap \langle x_2, \pi_1\rangle \neq \emptyset$, so we can put $m_3 := \pi \cap \langle x_2, \pi_1\rangle$. Because $\alpha < q^{l+1} - q^l + 2$ we can continue this process, that is, choose $x_i \in X \setminus \langle m, m_1, x_1, \ldots, x_{i-1}\rangle$ and put $m_{i+1} := \pi \cap \langle x_i, \pi_1\rangle$, for $i = 3, \ldots, l$. As $\pi = \langle m_1, \ldots, m_{l+1}\rangle$ and $m_1, \ldots, m_{l+1} \in \langle \pi_0, \pi_1\rangle$ it follows that $\pi \subset \langle \pi_0, \pi_1\rangle$. Hence $\mathcal{R}$ is α-geometric.

- We suppose that $\alpha \geq q^{l+1} - q^l + 2$ and $q \geq 3$. Let $p, \pi_0, \pi_1, \ldots, \pi_\alpha$ be as before. Choose $p_1 \in \langle p, \pi_0\rangle \setminus \widetilde{\mathcal{R}}$, with $p_1 \neq p$. We may assume that $\pi_1 \cap \langle p, p_1\rangle = \emptyset$. If $l = 1$ then every line in $\{\pi_0, \pi_2, \ldots, \pi_\alpha\}$ must have a point in common with $\langle p, \pi_1\rangle$ as well as with $\langle p_1, \pi_1\rangle$. This implies that all these lines belong to the 3-space $\langle p, p_1, \pi_1\rangle$, i.e. $\mathcal{R}$ is α-geometric. So suppose that $l > 1$. Choose a point $p_2 \in \langle p, \pi_1\rangle \setminus \widetilde{\mathcal{R}}$, with $p_2 \neq p$. We may assume that $\pi_2 \cap \langle p, p_1, p_2\rangle = \emptyset$ (as $\alpha > q^2 + q - 2$). If $l = 2$ we are done. If $l > 2$ we can choose $p_3 \in \langle p, \pi_2\rangle \setminus (\widetilde{\mathcal{R}} \cup \{p\})$. As $q^l + q^{l-1} + \cdots + 1 - (l + 1) < q^{l+1} - q^l + 2 \leq \alpha$ we can continue this process till we have chosen a point p_l. The same inequality then implies that there must exist some $\pi_j \in \{\pi_0, \ldots, \pi_\alpha\}$ with π_j disjoint from the l-space $\langle p, p_1, \ldots, p_l\rangle$. Every element of $\{\pi_0, \ldots, \pi_\alpha\} \setminus \{\pi_j\}$ must have a point in common with each of the spaces $\langle p, \pi_j\rangle$, $\langle p_1, \pi_j\rangle, \ldots, \langle p_l, \pi_j\rangle$ implying $\pi_i \subset \langle p, p_1, \ldots, p_l, \pi_j\rangle$ for every $i \in \{0, \ldots, \alpha\}$, i.e. $\mathcal{R}$ is α-geometric. Notice that this proof does not work whenever $q = 2$ as the inequality we used fails to be true in that case.

- Finally we suppose that $\alpha \geq q^{l+1} - q^l + 2$ with $q = 2$. Again let $p, \pi_0, \ldots, \pi_\alpha$ be as before. Define x to be such that $\alpha = 2^{l+1} - x$. Define z to be such that $2^z \geq x > 2^{z-1}$. Finally define k to be such that $kz \geq l > (k-1)z$. Then the points of $\langle p, \pi_0\rangle \setminus \widetilde{\mathcal{R}}$ generate at least a z-space (with $z \geq 1$). Let γ_1 be a z-space generated by such points containing p. If not every element of $\{\pi_0, \ldots, \pi_\alpha\}$ intersects γ_1, say $\gamma_1 \cap \pi_1 = \emptyset$, then we can choose in an analogue way a z-space $\beta \subset \langle p, \pi_1\rangle$ containing p. In this way there will hold that $\gamma_2 := \langle \gamma_1, \beta\rangle$ is a $2z$-space generated by points not belonging to the points covered by the spaces in $\{\pi_0, \ldots, \pi_\alpha\}$. If not every element of $\{\pi_0, \ldots, \pi_\alpha\}$ has a point in common with γ_2 we can continue the process and construct a $3z$-space γ_3. Suppose that finally we find some az-space γ_a such that every element of $\{\pi_0, \ldots, \pi_\alpha\}$ has a point in common with γ_a, but not with γ_{a-1} (notice that $a \geq 2$ since $z < l+1$). As we know that a of these spaces intersect γ_a in a $(z-1)$-space this implies

  $$(2^{az+1} - 1) - a(2^z - 1) - 1 - az \geq \alpha + 1 - a.$$

  If we replace α by $2^{l+1} - 2^z$ this yields

  $$2^{az+1} - (a-1)2^z - 2 - a(z-2) > 2^{l+1}$$

  from which we can deduce that $a \geq k$.

  First suppose that $a > k$. Then $\dim(\gamma_{a-1}) = (a-1)z \geq l$. Consequently we can find some l-space generated by points $p, p_1, \ldots, p_l$ not belonging to $\widetilde{\mathcal{R}}$, with the property that there exist some $\pi_i \in \{\pi_0, \ldots, \pi_\alpha\}$ with $\pi_i \cap \langle p, p_1, \ldots, p_l\rangle = \emptyset$ and such that π_j has non-empty intersection with each of the spaces $\langle p, \pi_i\rangle$, $\langle p_1, \pi_i\rangle, \ldots \langle p_l, \pi_i\rangle$, for every $j \in \{0, \ldots, \alpha\} \setminus \{i\}$. There follows that $\mathcal{R}$ is α-geometric.

  Now suppose that $a = k$. Then there holds that $\dim(\gamma_{a-1}) = (k-1)z < l \leq kz = \dim(\gamma_a)$. This implies that we can construct an l-space γ with $\gamma_{a-1} \subset \gamma \subseteq \gamma_a$ where γ is generated by $l+1$ points, among which the point p, not belonging to $\widetilde{\mathcal{R}}$. Define $u := l - \dim(\gamma_{a-1}) > 0$. Assume that every element of $\{\pi_0, \ldots \pi_\alpha\}$ intersects γ. As we know that $a-1$ of these spaces intersect γ in a $(z-1)$-space and one of these spaces intersects γ in a $(u-1)$-space, there must hold that

$$(2^{l+1}-1)-(a-1)(2^z-1)-(2^u-1)-(l+1) \geq \alpha+1-a.$$

Replacing $\alpha+1$ by $2^{l+1}-2^z$ yields

$$-2-a2^z+2.2^z+2a-2^u-l>0.$$

If we now write a as $2+\delta$ we find (notice that $\delta \geq 0$)

$$2+2\delta-\delta 2^z-2^u-l>0,$$

a contradiction as $z \geq 1$ and $u \geq 1$. Hence there exists an element of $\{\pi_0, \ldots, \pi_\alpha\}$ not intersecting γ. As before there follows that $\mathcal{R}$ is α-geometric. This concludes the proof of STEP 1.

STEP 2. *$S(\mathcal{R})$ satisfies the diagonal property.*

Let x, y, u and v be four distinct points of $S(\mathcal{R})$ with $x \sim y$, with $u, v \in \{x, y\}^\perp$ and such that u, v do not belong to the line L incident with x and y. We must show that u and v are collinear. Suppose that $v \notin U$ and $v \notin V$, with U the line incident with x and u, and with V the line incident with y and u, as otherwise $u \sim v$ is trivially satisfied. We consider $S(\mathcal{R})$ in its representation in $\mathrm{PG}(n+1,q)$ where $\mathcal{R}$ consists of $\mathrm{PG}(l,q)$ in $\Pi := \mathrm{PG}(n,q)$. Define $z := \langle u, v\rangle \cap \Pi$. Let π_0 be the element of $\mathcal{R}$ determined by $\langle x, y\rangle$ and suppose that $\pi_1, \ldots, \pi_\alpha$ are the α elements of $\mathcal{R}$ determined by the lines of $S(\mathcal{R})$ through u intersecting L. As $\mathcal{R}$ is α-geometric by STEP 1, there holds that $\pi_0, \pi_1, \ldots, \pi_\alpha$ are completely contained in some $(2l+1)$-space. Define Σ to be the $(2l+2)$-space $\langle u, \pi_0, \pi_1\rangle$. First suppose that $v \notin \Sigma$. Let π_1 be the element of $\mathcal{R}$ determined by $\langle u, x\rangle$ and π_2 the element determined by $\langle u, y\rangle$. The projection of $\langle u, \pi_1\rangle$ from v on Π yields the $(l+1)$-space $\langle z, \pi_1\rangle$. This space intersects $\alpha+1$ elements of $\mathcal{R}$, but not π_2 as $z \notin \langle \pi_0, \pi_1\rangle$. Similarly the projection of $\langle u, \pi_2\rangle$ from v on Π will be $\langle z, \pi_2\rangle$ and this space will intersect $\alpha+1$ elements of $\mathcal{R}$, but not π_1. If u would not be collinear with v the point z would be a point not belonging to $\widetilde{\mathcal{R}}$ with the property that there are at least $\alpha+2$ elements π_i of $\mathcal{R}$ such that $\langle z, \pi_i\rangle$ is not a tangent space of $\mathcal{R}$, a contradiction as $t-\theta=\alpha$. Hence u and v are collinear. Next suppose that v belongs to Σ. In $S(\mathcal{R})$ there are $t(\alpha-1)$ points not on L collinear with both x and y. The space Σ contains exactly $\alpha(\alpha-1)$ of these points. As $t>\alpha$ we can choose a point $w \notin \Sigma$ with the property that w is collinear with x and y in $S(\mathcal{R})$. Because of the foregoing, there holds that u and w as well as v and w are collinear in $S(\mathcal{R})$. Let π be the element of $\mathcal{R}$ determined by $\langle u, w\rangle$. As $\pi \notin \{\pi_0, \pi_1, \ldots, \pi_\alpha\}$ it is easily seen that $v \notin \langle u, \pi, \pi_1\rangle$. The above (where we let w play the role of y) implies that u and v are collinear in $S(\mathcal{R})$. Hence $S(\mathcal{R})$ satisfies the diagonal property.

From $t-\theta = \alpha$ it follows that $S(\mathcal{R})$ is a semipartial geometry with parameters $(s, t, \alpha, \alpha(\alpha+1))$. As $S(\mathcal{R})$ satisfies the diagonal property Theorem 2.2.5 implies that $S(\mathcal{R}) \cong T_m^*(\mathcal{B})$ for certain m. $\square$

Remark 5.1.2 Notice that the restriction $\alpha \notin \{1, q^{l+1}-1, q^{l+1}\}$ is in fact a weak restriction as in each of these cases $S(\mathcal{R})$ turns out to be a special type of semipartial geometry. In the first case $S(\mathcal{R})$ is a partial quadrangle, and examples of partial quadrangles with $\mu = 2$, so not isomorphic to a $T_m^*(\mathcal{B})$, constructed from an SPG-regulus, are known. An example is given by $T_4^*(\mathcal{C})$, with $\mathcal{C}$ the 11-cap in $\mathrm{PG}(4,3)$. In the second case $S(\mathcal{R})$ is a finite reduced copolar space, and these were classified in Hall [44], see Theorem 2.4.4 (it is reduced as we supposed that $S(\mathcal{R})$ is not a partial geometry). From this classification it is clear that no finite reduced copolar space has $\mu = \alpha(\alpha+1)$. In the final case $S(\mathcal{R})$ is a 2-design, which we do not consider.

Theorem 5.1.1 now yields the following characterization of "blown up" Baer-subgeometries.

Corollary 5.1.3 *Let $\mathcal{R}$ be an SPG-regulus in* $\mathrm{PG}(n,q)$ *with the property that* $|\mathcal{R}| - \theta = \alpha + 1$*, with* $\theta \neq 0$*. Then there exist natural numbers l and m and a Baer-subgeometry $\mathcal{B}$ in* $\mathrm{PG}(l, q^m)$ *such that* $n = m(l+1) - 1$ *and such that* $\mathcal{B} \overset{q^m \mapsto q}{\longrightarrow} \mathcal{R}$.

Chapter 6

The polar property

This chapter is devoted to the study of a special class of sets of subspaces of a projective space: sets of subspaces satisfying the so called polar property. The results of this chapter, although of interest in their own right, will turn out to be useful for the theory of SPG-reguli.

As terminology the polar property was introduced in De Winter and Thas [38] in 2003, but as a concept it is much older and one could say that it dates from the beginning of the theory of eggs [67], where the generalized polar property is a requirement in the definition. Later on it is encountered again in [74] where Thas introduced SPG-reguli. The study of sets of $(n-1)$-dimensional spaces in $\mathrm{PG}(h, q)$ satisfying the polar property turns out to contain the study of (semi-)ovals, (semi-)ovoids, pseudo-ovals and pseudo-ovoids, explaining the terminology of semi-pseudo-ovoid that will be introduced in this chapter. It will be shown that SPG-reguli satisfying the polar property have an elegant and useful characterization as semi-pseudo-ovoids. Further the Main Theorem 6.2.1 restricting the parameters of semi-pseudo-ovoids shall allow us to show that the axioms defining a pseudo-oval and pseudo-ovoid can be (considerably) relaxed. This chapter is based on De Winter and Thas [39]. However Corollary 6.3.3 and Theorem 6.4.2 appeared as the two main theorems (formulated in a slightly different way) in the first part of another paper of De Winter and Thas [38].

Throughout this chapter we will consider $(n-1)$-dimensional subspaces of $\mathrm{PG}(h, q)$, while, following the notations introduced so far, it would be more natural to consider m- or l-dimensional subspaces in $\mathrm{PG}(n, q)$. The reason for using different notation in this chapter is that the objects studied are closely related to pseudo-ovals and pseudo-ovoids, the elements of which are classically said to have dimension $n-1$. Further, with this notation, several dimensions we will encounter can be expressed in a nicer way.

6.1 Semi-pseudo-ovoids

Definition 6.1.1 Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a set of mutually disjoint $(n-1)$-dimensional spaces in $\mathrm{PG}(h,q)$, with $h > 2n-1$. Then we say that $\mathcal{O}$ satisfies the *generalized polar property* if and only if there exists an $(h-n)$-dimensional space τ_i containing π_i and disjoint from all elements of $\mathcal{O} \setminus \{\pi_i\}$, $i = 1, 2, \ldots, \eta$. If the union of all n-dimensional spaces containing π_i and disjoint from all elements of $\mathcal{O} \setminus \{\pi_i\}$ is an $(h-n)$-dimensional space τ_i, for all $i = 1, 2, \ldots, \eta$, then $\mathcal{O}$ is said to satisfy the *polar property* (see also Chapter 2 on page 21). In the latter case the space τ_i will be called the *tangent space* of $\mathcal{O}$ at π_i. In general, for any set $\mathcal{O}$ as above, a *k-dimensional tangent space* at π_i, will be a k-dimensional space containing π_i, but skew to all elements of $\mathcal{O} \setminus \{\pi_i\}$.

A *(generalized) semi-pseudo-ovoid* or a *(generalized) semi-egg* of $\mathrm{PG}(h,q)$ is just a non-empty set $\mathcal{O}$ of η mutually skew $(n-1)$-dimensional subspaces satisfying the (generalized) polar property.

For $n = 1$ semi-pseudo-ovoids are just semi-ovals ($h = 2$) and semi-ovoids ($h > 2$); these objects were introduced by Buekenhout [9]. The question of Buekenhout (posed at the conference "Colloquio Internazionale sulle Theorie Combinatorie" in 1973 and published in [9]) whether every semi-ovoid is necessarily an ovoid in $\mathrm{PG}(3,q)$ was answered affirmatively by Thas [70].

6.1.1 Examples of semi-pseudo-ovoids

(a) Pseudo-ovals and pseudo-ovoids are semi-pseudo-ovoids.

(b) Several types of SPG-reguli provide examples, see Chapter 2, Subsection 2.2.2.

(c) Each m-system $\mathcal{M}$ of the polar space P, for which any $(m+1)$-dimensional subspace containing any $\pi \in \mathcal{M}$ (π has dimension m) and not contained in $\pi^\perp$ if P is defined by a polarity, or not contained in the tangent space of P at π if P is a quadric in even dimension over a field with characteristic two, has a point in common with at least one element of $\mathcal{M} \setminus \{\pi\}$; see Shult and Thas [64].

(d) Let $\mathcal{O}$ be a semi-pseudo-ovoid consisting of $(n-1)$-dimensional subspaces of $\mathrm{PG}(h,q)$. Let $\pi \in \mathcal{O}$ and assume that any n-dimensional subspace containing any element γ of $\mathcal{O} \setminus \{\pi\}$ and any point of π, has a point in common with at least one element of $\mathcal{O} \setminus \{\pi, \gamma\}$. Then $\mathcal{O} \setminus \{\pi\}$ is still a semi-pseudo-ovoid of $\mathrm{PG}(h,q)$. If the semi-pseudo-ovoid $\mathcal{O}$ is an SPG-regulus, then this condition is satisfied for any $\pi \in \mathcal{O}$ if the parameter α of the SPG-regulus is at least two.

6.1.2 α-regular semi-pseudo-ovoids

A semi-pseudo-ovoid $\mathcal{O}$ in $\mathrm{PG}(h,q)$ is called *α-regular* if any n-dimensional subspace containing any element $\pi \in \mathcal{O}$ but not contained in the tangent space of $\mathcal{O}$ at π, has a point in common with exactly α elements of $\mathcal{O} \setminus \{\pi\}$. Analogously one can define *α-regular generalized semi-pseudo-ovoids.*

Any pseudo-oval and pseudo-ovoid is 1-regular. If the semi-pseudo-ovoid $\mathcal{O}$ is an SPG-regulus, then it is also α-regular.

Suppose that $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ is an α-regular semi-pseudo-ovoid consisting of $(n-1)$-dimensional spaces in $\mathrm{PG}(h,q)$, and let τ_i be the tangent space of $\mathcal{O}$ at π_i. Further let γ be any hyperplane containing τ_i. The elements of $\mathcal{O} \setminus \{\pi_i\}$ all intersect γ in an $(n-2)$-dimensional space. Counting in two ways the pairs (p, ξ), with p a point of γ contained in an element of $\mathcal{O} \setminus \{\pi_i\}$, $\pi_i \subset \xi \subset \gamma$, with ξ an n-dimensional space, and with $p \in \xi$, we obtain

$$(|\mathcal{O}| - 1)\frac{q^{n-1}-1}{q-1} = \alpha\left(\frac{q^{h-n}-1}{q-1} - \frac{q^{h-2n+1}-1}{q-1}\right).$$

We conclude that $|\mathcal{O}| - 1 = \alpha q^{h-2n+1}$.

Blokhuis and Szőnyi [5] studied α-regular semi-ovals. They obtained the following results.

Theorem 6.1.2 ([5]) *Let $\mathcal{O}$ be an α-regular semi-oval in a projective plane Π of order q.*

- *Then $\mathcal{O}$ is an oval or $\alpha + 1$ divides $q - 1$.*
- *If Π is Desarguesian, then $\mathcal{O}$ is a unital or α and q are relatively prime.*
- *If Π is Desarguesian, then $\mathcal{O}$ is a unital or the tangents at collinear points of $\mathcal{O}$ are concurrent.*

6.2 The main inequalities

6.2.1 Main theorem

Theorem 6.2.1 *If $\mathcal{O}$ is a semi-pseudo-ovoid consisting of η $(n-1)$-dimensional subspaces of* $\mathrm{PG}(h,q)$*, then*

$$1 + q^{h-2n+1} \leq \eta \leq 1 + q^{\frac{h+1}{2}}.$$

It follows that $h \leq 4n-1$.

Proof. Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a semi-pseudo-ovoid in $\mathrm{PG}(h,q)$ consisting of η $(n-1)$-dimensional subspaces. The tangent space of $\mathcal{O}$ at π_i will be denoted by τ_i, with $i = 1, 2, \ldots, \eta$. Further, recall that $\widetilde{O} = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_\eta$. Consider an n-dimensional subspace β with $\pi_1 \subset \beta \subset \tau_1$ and let γ be an $(n+1)$-dimensional subspace with $\beta \subset \gamma$, $\gamma \not\subset \tau_1$. Each n-dimensional subspace δ of γ containing π_1, with $\delta \neq \beta$, contains a point of $\widetilde{\mathcal{O}} \setminus \pi_1$. Hence

$$\left|(\widetilde{\mathcal{O}} \cap \gamma) \setminus \pi_1\right| \geq q.$$

There are exactly $\frac{q^{h-n}-1}{q-1} - \frac{q^{h-2n}-1}{q-1}$ spaces γ containing a fixed β. It follows that

$$\left|\widetilde{\mathcal{O}}\right| \geq \frac{q^n-1}{q-1} + q\frac{q^{h-n}-q^{h-2n}}{q-1},$$

that is,

$$\left|\widetilde{\mathcal{O}}\right| \geq \frac{q^n-1}{q-1}(1+q^{h-2n+1}).$$

Consequently,

$$|\mathcal{O}| \geq 1 + q^{h-2n+1}. \tag{6.1}$$

Next, let x_i be any point of $\mathrm{PG}(h,q) \setminus \widetilde{\mathcal{O}}$, and let t_i be the number of n-dimensional subspaces ξ on x_i with $\pi_j \subset \xi \subset \tau_j$ for some j. First we count the number of pairs (x_i, ξ), with $x_i \in \xi$, ξ n-dimensional, and $\pi_j \subset \xi \subset \tau_j$ for some j. We obtain

$$\sum_i t_i = \eta q^n \frac{q^{h-2n+1}-1}{q-1}. \tag{6.2}$$

Next, we count the number of ordered triples (x_i, ξ, ξ'), with $x_i \in \xi$, $x_i \in \xi'$, $\xi \neq \xi'$, ξ and ξ' n-dimensional, $\pi_j \subset \xi \subset \tau_j$ for some j, and $\pi_{j'} \subset \xi' \subset \tau_{j'}$ for some j'. We obtain

$$\sum_i t_i(t_i - 1) = \eta(\eta - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{6.3}$$

Hence

$$\sum_i t_i^2 = \eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{6.4}$$

The number of points x_i is equal to

$$d = \left|\mathrm{PG}(h, q) - \widetilde{\mathcal{O}}\right| = \frac{q^{h+1} - 1}{q - 1} - \eta\frac{q^n - 1}{q - 1}. \tag{6.5}$$

Now $d\sum_i t_i^2 - \left(\sum_i t_i\right)^2 \geq 0$, and so, by (6.2), (6.4) and (6.5)

$$\left(\frac{q^{h+1} - 1}{q - 1} - \eta\frac{q^n - 1}{q - 1}\right)\eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1} - \left(\eta q^n \frac{q^{h-2n+1} - 1}{q - 1}\right)^2 \geq 0,$$

that is, $\eta^2 - 2\eta - (q^{h+1} - 1) \leq 0$, and so,

$$\eta \leq 1 + q^{\frac{h+1}{2}}. \tag{6.6}$$

Finally, from (6.1) and (6.6) follows that $1 + q^{h-2n+1} \leq 1 + q^{\frac{h+1}{2}}$, and so

$$h \leq 4n - 1.$$

$\square$

Corollary 6.2.2 *If $\mathcal{O}$ is an α-regular semi-pseudo-ovoid consisting of $\mathrm{PG}(n-1, q)$ in $\mathrm{PG}(h, q)$, then $\alpha \leq q^{2n-\frac{h+1}{2}}$.*

Proof. Since $|\mathcal{O}| - 1 = \alpha q^{h-2n+1}$, and $|\mathcal{O}| - 1 \leq q^{\frac{h+1}{2}}$ by the previous theorem, we obtain $\alpha \leq q^{2n-\frac{h+1}{2}}$. $\square$

Notice that this bound is sharp (unitals provide examples).

6.2.2 Pseudo-ovoids

In this section we will show that by Theorem 6.2.1 the original definition of pseudo-ovoid (see Chapter 1 on page 10) $O(n, 2n, q)$ can be considerably weakened.

Theorem 6.2.3 *Let $\mathcal{O}$ be a semi-pseudo-ovoid consisting of η $(n-1)$-dimensional subspaces of* $\mathrm{PG}(h, q)$*; let $h = 4n - 1$. Then $\eta = 1 + q^{2n}$ and so $\mathcal{O}$ is a pseudo-ovoid.*

Proof. Assume that $h = 4n - 1$ for the semi-pseudo-ovoid $\mathcal{O}$. Then $h - 2n + 1 = (h + 1)/2$, and so by Theorem 6.2.1 $\eta = 1 + q^{h-2n+1} = 1 + q^{2n}$. As equality holds in equation (6.6), $d \sum_i t_i^2 - (\sum_i t_i)^2 = 0$, with the notations of the proof of Theorem 6.2.1. So t_i is a constant. Hence

$$t_i = \frac{\sum_i t_i}{d} = q^n + 1$$

for all i. As $\eta = 1 + q^{h-2n+1}$, each n-dimensional subspace containing $\pi_i \in \mathcal{O}$, but not contained in the tangent space of $\mathcal{O}$ at π_i, contains exactly one point of $\widetilde{\mathcal{O}} \setminus \pi$, with $i = 1, 2, \ldots, q^{2n} + 1$. It follows that any three distinct elements of $\mathcal{O}$ generate a $(3n - 1)$-dimensional subspace of $\mathrm{PG}(h, q)$. Consequently $\mathcal{O}$ is a pseudo-ovoid of $\mathrm{PG}(4n - 1, q)$. $\square$

Remark 6.2.4

(a) It follows that an egg $O(n, 2n, q)$ is nothing else than a set of $(n-1)$-dimensional subspaces of $\mathrm{PG}(4n - 1, q)$, such that for each $\pi_i \in O(n, 2n, q)$ the union of all n-dimensional subspaces containing π_i but skew to all elements of $O(n, 2n, q) \setminus \{\pi_i\}$ is a $(3n - 1)$-dimensional subspace of $\mathrm{PG}(4n - 1, q)$.

(b) A *weak egg* of $\mathrm{PG}(4n - 1, q)$ is a set of $1 + q^{2n}$ $(n - 1)$-dimensional subspaces of $\mathrm{PG}(4n - 1, q)$, every three of which generate a $(3n - 1)$-dimensional subspace. It is an open question whether or not each weak egg is an egg; see Lavrauw [52].

6.3 Interpretation of the equalities

We will use the notations introduced in the previous sections.

Theorem 6.3.1 *For a semi-pseudo-ovoid $\mathcal{O}$, $\eta = 1 + q^{\frac{h+1}{2}}$ if and only if each point not in an element of $\mathcal{O}$ is on a constant number of tangent spaces. This constant equals $1 + q^{\frac{h-2n+1}{2}}$.*

Proof. Each point not in an element of $\mathcal{O}$ is on a constant number of tangent spaces if and only if t_i in the proof of Theorem 6.2.1 is a constant, if and only if $d\sum_i t_i^2 - (\sum_i t_i)^2 = 0$, if and only if $\eta = 1 + q^{\frac{h+1}{2}}$. In such a case

$$t_i = \frac{\sum_i t_i}{d} = 1 + q^{\frac{h-2n+1}{2}}.$$

□

Theorem 6.3.2 *For a semi-pseudo-ovoid $\mathcal{O}$, $\eta = 1 + q^{\frac{h+1}{2}}$ if and only if each hyperplane not containing a tangent space of $\mathcal{O}$, contains a constant numbers of elements of O. This constant equals $1 + q^{\frac{h-2n+1}{2}}$.*

Proof. Let γ_i be any hyperplane not containing a tangent space of the semi-pseudo-ovoid $\mathcal{O}$. The number of elements of $\mathcal{O}$ in γ_i will be denoted by u_i. Now we count the number of pairs (γ_i, π), with $\pi \in \mathcal{O}$ in γ_i. We obtain

$$\sum_i u_i = \eta q^n \frac{q^{h-2n+1} - 1}{q - 1}. \tag{6.7}$$

Next we count the number of ordered triples (γ_i, π, π'), with $\pi \neq \pi'$, $\pi \in \mathcal{O}$, $\pi' \in \mathcal{O}$ and π, π' in γ_i. We obtain

$$\sum_i u_i(u_i - 1) = \eta(\eta - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{6.8}$$

From (6.7) and (6.8) it follows that

$$\sum_i u_i^2 = \eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{6.9}$$

The number of hyperplanes of $\mathrm{PG}(h, q)$ not containing a space τ_j equals $\frac{q^{h+1}-1}{q-1} - \eta\frac{q^n-1}{q-1} = g$. As $g\sum_i u_i^2 - (\sum_i u_i)^2 \geq 0$, we obtain

$$\left(\frac{q^{h+1} - 1}{q - 1} - \eta\frac{q^n - 1}{q - 1}\right)\eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1} - \left(\eta q^n \frac{q^{h-2n+1} - 1}{q - 1}\right)^2 \geq 0,$$

that is, $\eta^2 - 2\eta - (q^{h+1} - 1) \leq 0$, or equivalently,

$$\eta \leq 1 + q^{\frac{h+1}{2}}. \tag{6.10}$$

Equality holds in (6.10) if and only if u_i is a constant. In such a case this constant equals

$$u_i = \frac{\sum_i u_i}{g} = 1 + q^{\frac{h-2n+1}{2}}.$$

The theorem is proved. $\square$

Corollary 6.3.3 *If $\eta = 1 + q^{\frac{h+1}{2}}$ for a semi-pseudo-ovoid $\mathcal{O}$, then $\widetilde{\mathcal{O}}$ has two intersection numbers with respect to hyperplanes. Hence $\widetilde{\mathcal{O}}$ defines a projective linear two-weight code and a strongly regular graph.*

Proof. If the hyperplane γ contains a tangent space of $\mathcal{O}$, then $\gamma \cap \widetilde{\mathcal{O}}$ is the disjoint union of one element of $\mathcal{O}$ and $q^{\frac{h+1}{2}}$ $(n-2)$-dimensional subspaces. If γ does not contain a tangent space of $\mathcal{O}$, then $\gamma \cap \widetilde{\mathcal{O}}$ is the disjoint union of $1 + q^{\frac{h-2n+1}{2}}$ elements of $\mathcal{O}$ and $q^{\frac{h+1}{2}} - q^{\frac{h-2n+1}{2}}$ $(n-2)$-dimensional subspaces. The fact that $\widetilde{\mathcal{O}}$ defines a projective linear two-weight code and a strongly regular graph now follows from Theorem 4.1.6 and Theorem 4.1.7. $\square$

Theorem 6.3.4 *A semi-pseudo-ovoid $\mathcal{O}$ is either a pseudo-oval or a pseudo-ovoid if and only if $\eta = 1 + q^{h-2n+1}$.*

Proof. Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a semi-pseudo-ovoid. Then, by the proof of Theorem 6.2.1, $\eta = 1 + q^{h-2n+1}$ if and only if any n-dimensional subspace containing π_i, but not contained in the tangent space of $\mathcal{O}$ at π_i, has exactly one point in common with $\widetilde{\mathcal{O}} \setminus \pi_i$, for all $i = 1, 2, \ldots, \eta$, that is, if and only if any three distinct elements of $\mathcal{O}$ generate a $(3n-1)$-dimensional subspace, that is, if and only if $\mathcal{O}$ is either a pseudo-oval or a pseudo-ovoid. $\square$

6.4 Translation duals

If $\mathcal{O}$ is a pseudo-ovoid consisting of $q^{2n} + 1$ $(n-1)$-dimensional subspaces of $\mathrm{PG}(4n-1, q)$, then the tangent spaces of $\mathcal{O}$ form a pseudo-ovoid $\mathcal{O}^*$ in the dual space of $\mathrm{PG}(4n-1, q)$; see Payne and Thas [59]. The pseudo-ovoid $\mathcal{O}^*$ is called the *translation dual* of $\mathcal{O}$. If q is even, then for every known pseudo-ovoid $\mathcal{O}$, $\mathcal{O} \cong \mathcal{O}^*$; for q odd, there are examples with $\mathcal{O} \not\cong \mathcal{O}^*$, see e.g. Payne [58]. Now we extend the notion of translation dual to semi-pseudo-ovoids.

Lemma 6.4.1 *Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$, with $\eta = 1 + q^{\frac{h+1}{2}}$, be a semi-pseudo-ovoid in $\mathrm{PG}(h, q)$ and let τ_i be the tangent space of $\mathcal{O}$ at π_i, $i =$*

$1, 2, \ldots, \eta$. If γ is a hyperplane of τ_i not containing π_i, with $i = 1, 2, \ldots, \eta$, then there is at least one τ_j, with $j \neq i$, for which $\tau_j \cap \gamma$ is $(h-2n)$-dimensional, that is, for which $\tau_j \cap \gamma = \tau_i \cap \tau_j$.

Proof. Assume, by way of contradiction, that $\tau_j \cap \gamma$ is $(h-2n-1)$-dimensional for any $j \neq i$. Now we count in two ways the number of pairs (z, τ_j), with $z \in \gamma \setminus \pi_i$, $z \in \tau_j$, and $j \neq i$. We obtain

$$\frac{q^{h-n} - q^{n-1}}{q-1} q^{\frac{h-2n+1}{2}} = q^{\frac{h+1}{2}} \frac{q^{h-2n} - 1}{q-1},$$

clearly a contradiction. □

Theorem 6.4.2 *Let $\mathcal{O}$ be a semi-pseudo-ovoid in* PG(h, q)*, with $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$. Then the tangent spaces of $\mathcal{O}$ form a semi-pseudo-ovoid $\mathcal{O}^*$ in the dual space of* PG(h, q)*.*

Proof. Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$, with $\eta = 1 + q^{\frac{h+1}{2}}$, and let τ_i be the tangent space of $\mathcal{O}$ at π_i, $i = 1, 2, \ldots, \eta$. By Lemma 6.4.1 the space π_i is the intersection of all hyperplanes γ of τ_i, for which the space $\langle \gamma, \tau_j \rangle$ generated by γ and τ_j is PG(h, q), for all $j \neq i$. It follows that the tangent spaces of $\mathcal{O}$ form a semi-pseudo-ovoid $\mathcal{O}^*$ in the dual space of PG(h, q). □

The semi-pseudo-ovoid $\mathcal{O}^*$ will be called the *translation dual* of the semi-pseudo-ovoid $\mathcal{O}$.

6.5 SPG-reguli satisfying the polar property

In this section we will have a closer look at the link between semi-pseudo-ovoids and SPG-reguli satisfying the polar property. As mentioned already in Chapter 2 we will obtain a considerably stronger version of Theorem 2.2.12.

Theorem 6.5.1 *A semi-pseudo-ovoid $\mathcal{O}$ is an SPG-regulus satisfying the polar property if and only if $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$.*

Proof. First, suppose that $\mathcal{O}$ is an SPG-regulus satisfying the polar property. Then $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$ by Theorem 2.2.13.

Next, suppose that $\mathcal{O}$ is a semi-pseudo-ovoid satisfying $|\mathcal{O}| = \eta = 1 + q^{\frac{h+1}{2}}$. Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ and let τ_i be the tangent space of $\mathcal{O}$ at π_i, $i =$

$1, 2, \ldots, \eta$. Further, let $\mathrm{PG}(n, q)$ contain π_i, with $\mathrm{PG}(n, q) \not\subset \tau_i$. Let α be the number of elements of $\mathcal{O} \setminus \{\pi_i\}$ intersecting $\mathrm{PG}(n, q)$. Now we count in two ways the number of pairs (π_j, ϕ), with $\pi_j \subset \phi$, $j \neq i$, ϕ a hyperplane containing $\mathrm{PG}(n, q)$. We obtain

$$\alpha \frac{q^{h-2n+1} - 1}{q - 1} + (q^{\frac{h+1}{2}} - \alpha) \frac{q^{h-2n} - 1}{q - 1} = \frac{q^{h-n} - q^{n-1}}{q - 1} \cdot q^{\frac{h-2n+1}{2}}.$$

Hence $\alpha = q^{2n - \frac{h+1}{2}}$. As α is independent from i and the choice of $\mathrm{PG}(n, q)$, it follows that $\mathcal{O}$ is an SPG-regulus. $\square$

The problem on weak eggs in $\mathrm{PG}(4n-1, q)$ mentioned in Section 6.2.2 now generalizes in a natural way to the following problem. Suppose $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ is a set of $\eta = 1 + q^{\frac{h+1}{2}}$ mutually disjoint $(n-1)$-dimensional spaces in $\mathrm{PG}(h, q)$. Suppose that every n-dimensional space containing π_i, with $i = 1, 2, \ldots, \eta$, intersects either 0 or $\alpha = q^{2n - \frac{h+1}{2}}$ elements of $\mathcal{O} \setminus \{\pi_i\}$. Then one can ask whether or not $\mathcal{O}$ is a semi-pseudo-ovoid (and hence an SPG-regulus). Notice that for $h = 4n-1$ this problem is exactly the problem for weak eggs mentioned before.

Theorem 6.5.2 *If $\mathcal{O}$ is a semi-pseudo-ovoid with $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$, then the translation dual $\mathcal{O}^*$ of $\mathcal{O}$ is also an SPG-regulus satisfying the polar property.*

Proof. Immediate from Theorems 6.4.2 and 6.5.1. $\square$

There is a variant of Theorem 6.5.1 for generalized semi-pseudo-ovoids that can be useful, as it is sometimes easy to check that on π_i there is an $(h-n)$-dimensional space τ_i disjoint from π_j, $j \neq i$, but difficult to check that every n-dimensional space containing π_i, but not contained in τ_i, has non-empty intersection with $\widetilde{O} \setminus \pi_i$.

Theorem 6.5.3 *Let $\mathcal{O}$ be a generalized semi-pseudo-ovoid in $\mathrm{PG}(h, q)$, with $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$. Then $\mathcal{O}$ is an SPG-regulus satisfying the polar property.*

Proof. Let $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ with $\eta = 1 + q^{\frac{h+1}{2}}$, and let τ_i be a fixed $(h-n)$-dimensional space containing π_i and disjoint from π_j, $j \neq i$, for $i = 1, 2, \ldots, \eta$. Following the proof of Theorem 6.3.2 we see that every hyperplane containing no τ_i, $i = 1, 2, \ldots, \eta$, contains exactly $1 + q^{\frac{h-2n+1}{2}}$ elements of $\mathcal{O}$. Now let $\mathrm{PG}(n, q)$ be any n-dimensional space containing π_i and having non-empty intersection with $\widetilde{\mathcal{O}} \setminus \pi_i$. As in Theorem 6.5.1 we find that $\mathrm{PG}(n,q)$ intersects exactly $\alpha = q^{2n-\frac{h+1}{2}}$ elements of $\mathcal{O} \setminus \{\pi_i\}$. We now easily obtain that there are exactly $\frac{q^{h-2n+1}-1}{q-1}$ n-dimensional spaces containing π_i and having empty intersection with $\widetilde{\mathcal{O}} \setminus \pi_i$. We conclude that τ_i is the union of all n-dimensional spaces containing π_i and having empty intersection with $\widetilde{\mathcal{O}} \setminus \pi_i$, $i = 1, 2, \ldots, \eta$, that is, $\mathcal{O}$ is a semi-pseudo-ovoid. Applying Theorem 6.5.1 finishes the proof. $\square$

As an application we give a very short proof of a theorem of Luyckx [54] (Theorem 2.2.11) and provide an alternative proof for Theorem 6.2.3.

Corollary 6.5.4 ([54]) *Let $\mathcal{O}$ be an m-system of the polar space $P \in \{Q^-(2n+1,q), W_{2n+1}(q), H(2n,q^2)\}$, but not a spread of $W_{2n+1}(q)$. Then $\mathcal{O}$ is an SPG-regulus satisfying the polar property of the ambient space of P.*

Proof. If we denote the ambient space of P as $\mathrm{PG}(h,q)$, then in each case there holds $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$. Furthermore, the definition of an m-system implies immediately that $\mathcal{O}$ is a generalized semi-pseudo-ovoid. The result now follows from Theorem 6.5.3. $\square$

The following variant of Theorem 6.2.3 shows that in the original definition of pseudo-ovoid $O(n,2n,q)$ the restriction that every three distinct elements of $O(n,2n,q)$ should generate a $\mathrm{PG}(3n-1,q)$ is superfluous.

Corollary 6.5.5 *Let $\mathcal{O}$ be a generalized semi-pseudo-ovoid consisting of* $1+q^{2n}$ *mutually disjoint $(n-1)$-dimensional spaces in* $\mathrm{PG}(4n-1,q)$. *Then $\mathcal{O}$ is a pseudo-ovoid.*

Proof. Theorem 6.5.3 implies that $\mathcal{O}$ is an SPG-regulus with $\alpha = 1$. Hence every three distinct elements of $\mathcal{O}$ generate a $(3n-1)$-dimensional space, that is, $\mathcal{O}$ is a pseudo-ovoid. $\square$

Chapter 7

Translation semipartial geometries

In this chapter we introduce a theory of elation and translation semipartial geometries. The translation semipartial geometries will, except for possibly a finite number of exceptions, provide a group theoretical characterization of semipartial geometries constructed from SPG-reguli. In De Clerck, Gevaert and Thas [26] a theory of translation partial geometries was introduced and thoroughly studied. However, because of the rather strong restrictions on the groups and the geometries, not every partial geometry constructed from an SPG-regulus was a translation partial geometry, which is of course a pity if one wants to obtain a general group theoretical setting for the geometries arising from SPG-reguli. In this chapter we will overcome this problem.

Starting from an SPG-family (G, J), that is, a not necessarily abelian group G and a collection of subgroups $J = \{S_0, ..., S_t\}$ satisfying some extra conditions, we shall construct a semipartial geometry S as a coset geometry. It will be clear that there are strong relations between the theory of these geometries and that of elation and translation generalized quadrangles. We show for example that the theory of translation semipartial geometries is in almost all cases equivalent to the study of SPG-reguli in $\mathrm{PG}(n, q)$. We then introduce a special class of automorphisms for these geometries, called parallelisms, and examine the structure of fixed points and lines under these automorphisms. In the case where G is abelian we show that in almost all cases $\mathrm{Aut}(\mathcal{S}) \leq \mathrm{A\Gamma L}(n{+}2, q)$ for certain n and q. Most results of this chapter are taken from De Winter [36]; the application on generalized quadrangles with an abelian regular group is taken from S. De Winter and K. Thas [40].

For the rest of this chapter let I denote the set $\{0, 1, \ldots, t\}$.

7.1 SPG-families

Definition 7.1.1 Let G be a finite not necessarily abelian group. Except in a few cases mentioned we will use a multiplicative notation for G. Suppose that $J = \{S_0, \ldots, S_t\}$ is a set of $t+1$ ($t \geq 1$) subgroups of order $s+1$ of G ($s \geq 1$) with $S_i \cap S_j = \{id\}$ whenever $i \neq j$, for $i, j \in I$. Then the pair (G, J) is called a *geometric family*. Define for each $i \in I$ the set S_i^* as follows: $S_i^* := S_i \cup \{g \in G \mid S_i g \cap S_j = \emptyset, \forall j \in I\}$. We say that the pair (G, J) is an *SPG-family* with parameters (s, t, α, θ) if the following two conditions are satisfied.

1. (α-**condition**) There exists an α with $0 < \alpha < s+1$ such that for each S_i and each $g \in S_j \setminus \{id\}$, $j \neq i$, there exists a unique set $\{j_1 = j, j_2, \ldots, j_\alpha\} \subset I$ with the property that $S_{j_k} g \cap S_i \neq \emptyset$, $\forall k \in \{1, \ldots, \alpha\}$.

2. (θ-**condition**) There exists a θ such that for each $g \in G \setminus \bigcup_{i \in I} S_i$ there exists a unique set $\{j_1, \ldots, j_\theta\}$ with the property that $g \in S_{j_k}^*, \forall k \in \{1, \ldots, \theta\}$.

This definition generalizes the concept of a 4-gonal family, see Kantor [51], which was used to study elation and translation generalized quadrangles, see chapter 8 and 10 of Payne and Thas [59].
It is our aim to develop a theory for semipartial geometries analogously to the theory of elation and translation generalized quadrangles. We will start by showing that each SPG-family gives rise to a semipartial geometry. In order to do so we first introduce the concept of a *(right) coset geometry* [51].

Definition 7.1.2 Let G be a group and let $J = \{S_i \mid i \in I\}$ be a set of subgroups of G. Then the *right coset geometry* $S(G, J)$ is the incidence geometry with as points, the elements of G, as lines the right cosets $S_i g$, $i \in I$ and $g \in G$, and for which the incidence relation is containment.

Lemma 7.1.3 *If (G, J) is an SPG-family, then*

(a) $|S_i g \cap S_j h| \in \{0, 1\}, \forall i \neq j, \forall g, h \in G$;

(b) $G = \langle S_0, \ldots, S_t \rangle$.

Proof. (a) Suppose that $y, z \in S_i g \cap S_j h$. This implies the existence of elements $s_{i_y}, s_{i_z} \in S_i$ and $s_{j_y}, s_{j_z} \in S_j$ for which $y = s_{i_y} g = s_{j_y} h$ and $z =$

$s_{i_z}g = s_{j_z}h$. Hence $yz^{-1} = s_{i_y}s_{i_z}^{-1} = s_{j_y}s_{j_z}^{-1}$, yielding $yz^{-1} \in S_i \cap S_j$, that is, $y = z$.

(b) Consider $s_0 \in S_0 \setminus \{id\}$ and the coset S_1s_0. If each element of S_1s_0 would belong to a certain S_j then we would find for each $s_1 \in S_1$ a $j \in I$ such that $s_1s_0 = s_j$, with $s_j \in S_j$. From (a) it then follows easily that distinct elements of S_1 determine distinct indices j, yielding $\alpha \geq s+1$, a contradiction. Hence there exists a $g \in S_1s_0$ such that $g \in G \setminus \bigcup_i S_i$. Clearly $g \notin S_1^*$, implying $\theta < t+1$. Thus we may conclude that for each $g \in G \setminus \bigcup_i S_i$ there are indices $i \neq j$ such that $g = s_is_j$, with $s_i \in S_i$ and $s_j \in S_j$, that is, $G = \langle S_0, \ldots, S_t \rangle$.□

Theorem 7.1.4 *The right coset geometry $S(G, J)$ is a semipartial geometry* spg$(s, t, \alpha, (t+1-\theta)\alpha)$.

Proof. From Lemma 7.1.3 it follows that $S(G, J)$ is a partial linear space of order (s, t). Now let S_ih be any line of $S(G, J)$ and let g be any point of $S(G, J)$ not contained in S_ih. Suppose that g is collinear with a point $s_ih \in S_ih$. This implies the existence of some $j \in I \setminus \{i\}$ such that $gh^{-1}s_i^{-1} \in S_j$. Because of the α-condition there exists a unique set $\{j_1 = j, \ldots, j_\alpha\}$ such that $S_{j_k}gh^{-1}s_i^{-1} \cap S_i \neq \emptyset$, for $k = 1, \ldots, \alpha$. Consequently there exist elements $s_{i_k} \in S_i$ and $s_{j_k} \in S_{j_k}$, for $k = 1, \ldots, \alpha$, such that $s_{j_k}g = s_{i_k}s_ih$. Hence we have constructed α points of S_ih collinear with g in $S(G, J)$. On the other hand it is clear that each point of S_ih collinear with g must be as constructed above. This implies that $S(G, J)$ is a $(0, \alpha)$-geometry. Finally suppose that g and h are two non-collinear points of $S(G, J)$. This is equivalent with $hg^{-1} \in G \setminus \bigcup_i S_i$. Because of the θ-condition and Lemma 7.1.3 there exist exactly $t+1-\theta > 0$ indices i such that $hg^{-1} \notin S_i^*$. For each of these indices i there are exactly α indices j such that $S_ihg^{-1} \cap S_j \neq \emptyset$. This yields the existence of exactly $(t+1-\theta)\alpha$ points collinear with g as well as with h. This proves the theorem. □

Corollary 7.1.5 *If (G, J) is an SPG-family, then $t+1-\theta \geq \alpha$.*

Proof. This follows immediately from the fact that for a semipartial geometry there must hold that $\mu \geq \alpha^2$, see Theorem 2.1.4. □

Definition 7.1.6 A semipartial geometry constructed as above will be called an *elation semipartial geometry* (ESPG); whenever G is abelian it will be called a *translation semipartial geometry* (TSPG).

Example 7.1.7 Let $S(\mathcal{R})$ be the spg(s,t,α,μ) constructed from an SPG-regulus $\mathcal{R}$ in $\Pi := \mathrm{PG}(n,q)$. Let G be the group of all elations of $\mathrm{PG}(n+1,q)$ with axis Π. Then G is a regular group of automorphisms of $S(\mathcal{R})$. Choose a point x of $S(\mathcal{R})$ and denote by $L_0,\ldots,L_t$ the lines of $S(\mathcal{R})$ through x. Define $S_i := \{g \in G \mid L_i^g = L_i\}$, which are clearly subgroups of G of order $s+1$. Clearly $S_i \cap S_j = \{id\}$ for all $i \neq j$, with $i,j \in I$. Furthermore there holds that $S_i^* = S_i \cup \{g \in G \mid \alpha(x^g, L_i) = 0\}$. Now consider S_i and suppose that $g \in S_j \setminus \{id\}$, with $i \neq j$. The point x^g is collinear with exactly α distinct points $x^{h_1}, x^{h_2}, \ldots, x^{h_\alpha}$ on the line L_i, with $\{h_1,h_2,\ldots,h_\alpha\} \subset S_i$. This implies that $h_k g^{-1}$, $k = 1,2,\ldots,\alpha$, is an elation of $\mathrm{PG}(n+1,q)$ with axis Π and center a point c of some element of $\mathcal{R}$. Hence $x^{h_k g^{-1}}$ will be collinear with x in $S(\mathcal{R})$, that is, there exists a $j_k \in I$ such that $h_k g^{-1} \in S_{j_k}$, that is, there exists a $j_k \in I$ such that $S_{j_k} g \cap S_i \neq \emptyset$, with $k = 1,2,\ldots,\alpha$. Conversely, it is now obvious that every S_l for which $S_l g \cap S_i \neq \emptyset$ gives rise to a point of L_i collinear with x^g. We conclude that (G,J) satisfies the α-condition for SPG-families. Next suppose that $g \in G \setminus \bigcup_{i\in I} S_i$. Then x is not collinear with x^g. Since there are μ points collinear with both x and x^g, that is, there are μ/α lines incident with x containing points collinear with x^g, there are exactly $t+1-\mu/\alpha$ lines L_i such that $\alpha(x^g, L_i) = 0$, that is, such that $g \in S_i^*$. There follows that (G,J) satisfies the θ-condition for SPG-families with $\theta = t+1-\mu/\alpha$. We have obtained that (G,J) is an SPG-family. It is now easily seen that $\phi : G \to S(\mathcal{R}) : g \mapsto x^g$ determines an isomorphism between $S(G,J)$ and $S(\mathcal{R})$.

7.2 Translation semipartial geometries

For the rest of this section suppose that (G,J) is an SPG-family with parameters (s,t,α,θ) and with G abelian. Let K be the set of all endomorphisms β of G satisfying $S_i^\beta \subset S_i, \forall i \in I$. From the fact that G is abelian it follows immediately that $K,+,.$, with the usual addition and multiplication of endomorphisms, is a ring. This ring will be called the *kernel* of the TSPG.

Theorem 7.2.1 *If $t-\theta > \alpha$, then $K,+,.$ is a field.*

Proof. It is clearly sufficient to show that the kernel of $\beta : G \to G$, $\beta \in K \setminus \{0\}$, is trivial, that is, it is sufficient to show that $g^\beta = id$, with $\beta \in K$, $g \in S_i \setminus \{id\}$, implies $\beta = 0$. So suppose without loss of generality that $g^\beta = \{id\}$, with $g \in S_0 \setminus \{id\}$ and $\beta \in K$. Furthermore consider any $h \in S_i \setminus \{id\}$, $i > 0$. There are two possibilities. If there exists a $j \in I$ such that $hg \in S_j$, then we see that applying β yields $(hg)^\beta = h^\beta \in S_j \cap S_i$ (notice that $i \neq j$), that is, $h^\beta = id$. So from now on we suppose that $hg \in G \setminus \bigcup_i S_i$. Since $S(G,J)$ is a semipartial geometry and $hg = gh$ is collinear with $h \in S_i$ in $S(G,J)$, we know there exists a unique set $\{j_1,\ldots,j_\alpha\} \subset I$ with the property that $S_{j_k}hg \cap S_i \neq \emptyset$, with $k = 1,\ldots,\alpha$. By our assumption that $t-\theta > \alpha$ we can always find some $l \in I$ such that $l \notin \{i, j_1 \ldots, j_\alpha\}$ and $hg \notin S_l^*$. Consequently there is a (unique) set of α indices $\{m_1,\ldots,m_\alpha\}$ with the property that $S_l hg \cap S_{m_k} \neq \emptyset$, $k = 1,\ldots,\alpha$. Notice that $i \notin \{m_1,\ldots,m_\alpha\}$, since otherwise l would belong to $\{j_1,\ldots,j_\alpha\}$. This means that we can put $hg = s_1 s_{m_1} = s_2 s_{m_2} = \cdots = s_\alpha s_{m_\alpha}$, with $s_1,\ldots,s_\alpha \in S_l$ and $s_{m_k} \in S_{m_k}$ ($k = 1,\ldots,\alpha$). If we now let β act, we find: $h^\beta = s_1^\beta s_{m_1}^\beta = s_2^\beta s_{m_2}^\beta = \cdots = s_\alpha^\beta s_{m_\alpha}^\beta$. Assume that $h^\beta \neq id$. If $s_{m_k}^\beta \neq id$ for all $k \in \{1,\ldots,\alpha\}$, then we see that also $s_k^\beta \neq id$ for all k (since $h^\beta \in S_i \setminus \{id\}$ and $i \notin \{m_1,\ldots,m_\alpha\}$). Furthermore it is clear that all s_k^β are distinct, as $s_a^\beta = s_b^\beta$, $a \neq b$, for $a,b \in \{1,\ldots,\alpha\}$, would imply $s_{m_a}^\beta = s_{m_b}^\beta$. Clearly a contradiction as $S_{m_a} \cap S_{m_b} = \{id\}$. Hence h^β would be collinear with at least $\alpha+1$ points of S_l in $S(G,J)$, namely $s_1^\beta,\ldots,s_\alpha^\beta$ and id, a contradiction. So assume without loss of generality that $s_{m_1}^\beta = id$. This would imply that $h^\beta = s_1^\beta$, that is, $l = i$, again a contradiction. Hence we have proved that $h^\beta = id$. It easily follows that $S_i^\beta = id$, $\forall i \in I$. Since, by Lemma 7.1.3, $G = \langle S_0,\ldots,S_t\rangle$, we have that $h^\beta = id$, $\forall h \in G$, that is, $\beta = 0$. This concludes the proof. □

Theorem 7.2.2 *Each TSPG with $t-\theta > \alpha$ is isomorphic to a semipartial geometry constructed from an SPG-regulus $\mathcal{R}$ in* $\mathrm{PG}(n,q)$, *where* $\mathrm{GF}(q)$ *is a subfield of the kernel. Conversely every semipartial geometry constructed from an SPG-regulus $\mathcal{R}$ in* $\mathrm{PG}(n,q)$ *is isomorphic to a TSPG $S(G,J)$, with G the group of all elations of* $\mathrm{PG}(n+1,q)$ *with axis* $\mathrm{PG}(n,q)$, *and with* $\mathrm{GF}(q)$ *a subfield of the kernel.*

Proof. Let $\mathrm{GF}(q)$ be any subfield of the kernel K of the TSPG $S(G,J)$. Then clearly G as well as the sets S_i $i = 0,\ldots,t$, can be seen as vector spaces over $\mathrm{GF}(q)$. Let $\mathrm{PG}(n,q)$ denote the projective space arising from the vector space G. Then the S_i $(i = 0,\ldots t)$ induce $t+1$ mutually disjoint m-dimensional subspaces $\mathrm{PG}^{(0)}(m,q),\ldots,\mathrm{PG}^{(t)}(m,q)$ of $\mathrm{PG}(n,q)$. Hence $s+1 = q^{m+1}$. Clearly the α-condition, respectively the θ-condition, of SPG-families implies the α-condition, respectively the θ-condition, of SPG-reguli. Hence we have proved the first part of the theorem.

Now suppose that $S(\mathcal{R})$ is a semipartial geometry constructed from an SPG-regulus $\mathcal{R}$ in $\Pi := \mathrm{PG}(n,q)$. From the example in Section 7.1 we already know that $S(\mathcal{R})$ is a TSPG $S(G,J)$. Let x be the point of $\mathrm{PG}(n+1,q) \setminus \Pi$

corresponding to $id \in G$. It is well known that the multiplicative group of GF(q) is isomorphic to the group of homologies of PG$(n+1,q)$ with axis Π and center x, see Lemma 1.4.2. We will now show in detail that this group can be seen as a group of automorphisms of G. In order to simplify notation we will identify the elements of G with the corresponding points of PG$(n+1,q) \setminus \Pi$. So let $\beta \neq 1$ be a homology of PG$(n+1,q)$ with axis Π and center x and let $g \neq h$ be elements of $G \setminus \{x\}$. First suppose that x, g and h are not collinear in PG$(n+1,q)$. Define $g_\infty := \langle x, g\rangle \cap \Pi$, $h_\infty := \langle x, h\rangle \cap \Pi$ and $l_\infty := \langle g, h\rangle \cap \Pi$. Then the triangle with vertices g, g^β, h_∞ and the triangle with vertices h, h^β, g_∞ are perspective triangles with center l_∞. The Desargues Theorem now implies that x, gh and $g^\beta h^\beta$ are collinear, which in turn implies that $g^\beta h^\beta$ and $(gh)^\beta$ must coincide, that is, $(gh)^\beta = g^\beta h^\beta$. The cross ratio $(x, g_\infty, g, g^{-1}) = -1$ and as β preserves the cross ratio (see Lemma 1.4.4) there holds that $(x^\beta, g_\infty^\beta, g^\beta, (g^{-1})^\beta) = (x, g_\infty, g^\beta, (g^{-1})^\beta) = -1$, implying $(g^{-1})^\beta = (g^\beta)^{-1}$. Finally suppose that x, g and h are collinear and that $h \neq g^{-1}$. Choose any l not on $\langle x, g\rangle$. Then x, gl and $l^{-1}h$ are not collinear in PG$(n+1,q)$ since otherwise $gll^{-1}h = gh$ would be a point of $\langle x, gl\rangle$, implying $h = g^{-1}$ as $\langle x, gh\rangle \cap \langle x, gl\rangle = \{x\}$. Hence $(gh)^\beta = (gll^{-1}h)^\beta = (gl)^\beta (l^{-1}h)^\beta = g^\beta l^\beta (l^{-1})^\beta h^\beta = g^\beta h^\beta$. It is now clear that the group of homologies with axis Π and center x can indeed be seen as a group of automorphisms of G and the last part of the theorem follows easily. □

Hence we have shown that whenever $t - \theta > \alpha$ the theory of TSPG is equivalent to the theory of SPG-reguli. Furthermore it follows that in this case K is the largest field such that there exists an SPG-regulus $\mathcal{R}$ in PG(n, K) with the property that $S(G,J) \cong S(\mathcal{R})$. Notice that because of Corollary 7.1.5 this is in fact the general case. The cases $t + 1 - \theta \in \{\alpha, \alpha+1\}$ that were not handled yet will be solved almost completely in the following theorems.

Theorem 7.2.3 *Translation semipartial geometries $S(G,J)$ with $\mu = \alpha^2$, that is, for which (G,J) satisfies $t + 1 - \theta = \alpha$, do not exist.*

Proof. Assume by way of contradiction that (G,J) is an SPG-family with G abelian and $t + 1 - \theta = \alpha$. Let g be any point of $G \setminus \bigcup_{i\in I} S_i$. We may suppose that $g \notin S_i^*$, $i = 0, \ldots, \alpha-1$, and that $g \in S_i^*$, $i = \alpha, \ldots, t$. Hence there exists an index $j \in I$ such that $s_0 g = s_j$ with $s_0 \in S_0$ and $s_j \in S_j$. As G is abelian this implies that $s_j^{-1} g = s_0^{-1}$. Hence g is collinear with α points of S_0 in $S(G,J)$, that is, there exists a unique set $\{j_1, \ldots, j_\alpha\} \subset I \setminus \{0\}$ such that $s_{j_k} g = s_{0_k}$ with $s_{j_k} \in S_{j_k}$ and $s_{0_k} \in S_0$, $k = 1, \ldots, \alpha$. We find that

$g \notin S^*_{j_k}$. As $g \notin S^*_0$, this implies that g belongs to at most $t-\alpha$ sets S^*_i, a contradiction. $\square$

Lemma 7.2.4 *There does not exist an ESPG S, with S an* spg$(28, 28, 2, 6)$.

Proof. Suppose (G, J) would be the SPG-family corresponding with S. Since the number of points of S equals 103125, we have $|G| = 103125$. On the other hand we know that the elements of J have order $s+1 = 29$. Hence 29 should divide 103125, a contradiction. $\square$

Theorem 7.2.5 *If $S(G, J)$ is a TSPG with $t - \theta = \alpha$, $\alpha > 1$, $\alpha \neq 3$, then $S(G, J) \cong S(\mathcal{R})$ for some SPG-regulus $\mathcal{R}$ in* $\mathrm{PG}(n, q)$*, where* $\mathrm{GF}(q)$ *is a subfield of the kernel.*

Proof. First suppose that $\alpha = 2$. By Lemma 7.2.4 we know that $(s, t) \neq (28, 28)$, and hence by Theorem 2.2.6 we obtain $S(G, J) \cong T^*_m(\mathcal{B})$ for a certain m.

So from now on we will suppose that $\alpha \geq 4$. Let the ring K be defined as before and let β be an element of K. Let $g \in G \setminus \{id\}$ be such that $g^\beta = id$ and let $h \in S_j$ be such that $h \in S_i g$ for some $i \neq j$. Then by writing $h = s_i g$, $s_i \in S_i$, and letting act β on this relation we see that $h^\beta = s_i^\beta \in S_i \cap S_j$, implying $h^\beta = id$. Next suppose that $h \in G \setminus \bigcup_i S_i$ and that $h = s_1 g_1 = s_2 g_2$ with $s_1 \in S_i$, $s_2 \in S_j$, $i \neq j$ and $g_1^\beta = g_2^\beta = id$. Letting act β on this relation implies that $h^\beta = s_1^\beta = s_2^\beta \in S_i \cap S_j$, that is, $h^\beta = id$.
Now define an α-line to be any line of $S(G, J)$ intersecting $\bigcup_i S_i$ in exactly α points. Put $C = \bigcup_i S_i \setminus \{id\}$ and consider the graph $(C, \sim)$, in which two distinct vertices are adjacent if and only if there exists an α-line containing both. The above easily implies that if $h \in C$ is such that $h^\beta = id$, then $k^\beta = id$ for every $k \sim h$. Furthermore, if $g \notin C$, with $g^\beta = id$ and $g \neq id$, then again the above shows the existence of some $h \in C$ such that $h^\beta = id$. Hence, if $(C, \sim)$ is connected we obtain that $\beta = 0$ (remember that $G = \langle S_0, S_1, \ldots, S_t \rangle$) and that K is a field. Consequently, whenever $(C, \sim)$ is connected, one shows as in Theorem 7.2.2 that $S(G, J) \cong S(\mathcal{R})$, with $\mathcal{R}$ an SPG-regulus in some $\mathrm{PG}(n, q)$. So from now on assume that $(C, \sim)$ is not connected. Define $A := \{c \in C \mid c^\beta = id\}$ and $B := C \setminus A$. It is easy to deduce that $|S_i \cap A| = x$ for some $x \geq \alpha - 1$, where x is independent of our choice of $i \in I$. If $B = \emptyset$ then clearly K is a field and there is nothing left to prove. So suppose $B \neq \emptyset$ and choose a fixed $y \in B$. We will count in two ways the pairs (p, l), with p collinear with y in $S(G, J)$, $p \notin C$, $p \neq id$,

p collinear with l and $l \in A$. There are $t(s+1-\alpha)$ choices for p. Define x_i to be number of possible choices for l, given the ith choice for p. Notice that no vertex of A is adjacent to a vertex of B. We find

$$\sum_{i=1}^{t(s+1-\alpha)} x_i = xt(\alpha-1) + tx(\alpha^2-\alpha+1)$$

where $x_i \leq \alpha$. Hence

$$t(s+1-\alpha)\alpha \geq t(\alpha-1)^2 + t(\alpha-1)(\alpha^2-\alpha+1).$$

This implies that $s \geq \alpha^2 - 1$. Because of Theorem 2.2.6 we now have that $S(G,J) \cong T_m^*(\mathcal{B})$ for some m. The fact that $\mathrm{GF}(q)$ is a subfield of the kernel follows as in Theorem 7.2.2 $\square$

Remark 7.2.6 Notice that the kernel of $T_m^*(\mathcal{B})$ is not a field. This can easily be seen as follows. Let $\mathcal{B}$ be a $\mathrm{PG}(m,q)$ canonically contained in $\Pi := \mathrm{PG}(m,q^2)$ as a subgeometry (with this we mean that $\mathrm{PG}(m,q^2)$ is coordinatized in such a way that all points of $\mathcal{B}$ have all their coordinates in $\mathrm{GF}(q)$). We consider the geometry $T_m^*(\mathcal{B})$ in its representation in $\mathrm{PG}(m+1,q^2)$, where we suppose that $\Pi \subset \mathrm{PG}(m+1,q^2)$ is the hyperplane with equation $x_0 = 0$. Let us identify the point $(1,0,0,\ldots,0)$ with $id \in G$. It is well known that the map $c : p(x_0,x_1,\ldots,x_{m+1}) \mapsto p^q(x_0^q,x_1^q,\ldots,x_{m+1}^q)$ determines an automorphism of $\mathrm{PG}(m+1,q^2)$ which stabilizes Π and fixes exactly all points of $\mathrm{PG}(m+1,q^2)$ having all their coordinates in $\mathrm{GF}(q)$ (so also $\mathcal{B}$). It follows that c determines a non-trivial element of the kernel with non-trivial fixed points. Hence $c-1 \in K$ cannot have an inverse.

Corollary 7.2.7 *If S is a proper translation semipartial geometry with $\mu = \alpha(\alpha+1) > 2$ and with $\alpha \neq 3$, then $S \cong T_m^*(\mathcal{B})$ for certain m.*

Proof. This is an immediate consequence of the previous theorem and Theorem 5.1.1. $\square$

Remark 7.2.8 In this section we have shown that the theory of TSPG is almost completely equivalent to the theory of SPG-reguli, where the small gap is due to gaps in Theorem 2.2.6. In the case $t-\theta = \alpha$ with $\alpha = 3$ we still can say something, although little. Namely that a same technique as in Theorem 7.2.5 allows us to state that if $s < 8$ the graph $(C,\sim)$ must be connected and hence that the TSPG is isomorphic to a semipartial geometry constructed from an SPG-regulus. Theorem 5.1.1 then implies that such a semipartial geometry cannot exist.

We believe that the foregoing results show that the following conjecture is realistic.

Conjecture. If $\alpha > 1$, then the theory of TSPG is equivalent to the theory of SPG-reguli. Furthermore, each TSPG with $\mu = \alpha(\alpha + 1) > 2$ that is not a Bruck-net is isomorphic to $T_m^*(\mathcal{B})$ for some m.

7.3 Parallelism in ESPG and TSPG

In this section we will examine a special class of automorphisms of ESPG and TSPG. In the case of TSPG this will have important implications for the complete automorphism group of the TSPG.

Let $S(G, J)$ be an ESPG.

Definition 7.3.1

- We say that two lines S_ig and S_jh of $S(G, J)$ are *G-parallel* if and only if $i = j$. For two G-parallel lines L and M we write $L \parallel_G M$ or $L \parallel M$ if no confusion is possible. Clearly parallelism is an equivalence relation.
- A *G-parallelism* is any automorphism β of $S(G, J)$ with the property that $L \parallel_G L^\beta$ for every line L of $S(G, J)$. A G-parallelism is called a *G-elation* whenever it acts fixed point free; by definition the identity mapping will also be called a G-elation. If G is abelian, a G-elation will also be called a *G-translation*. If no confusion is possible we will just call these automorphisms *parallelism*, *elation* and *translation*.

Remark 7.3.2 Let $S(\mathcal{R})$ be a semipartial geometry constructed from an SPG-regulus $\mathcal{R}$ in $\mathrm{PG}(n, q)$ and let $S(G, J)$ be the TSPG, constructed as in the example of Section 7.1, isomorphic to $S(\mathcal{R})$. If ϕ is the natural isomorphism between $S(\mathcal{R})$ and $S(G, J)$, then ϕ maps $\mathcal{R}$-parallel lines (see Chapter 2, Subsection 2.2.3) onto G-parallel lines. This allows us to identify both parallelisms.

Clearly the group G of an ESPG $S(G, J)$ acts as a group of G-elations by right multiplication.

Theorem 7.3.3 *Let β be a G-parallelism of the ESPG $S(G, J)$, with $\alpha > 1$. Then one of the following situations occurs:*

- *β is a G-elation;*
- *β has a unique fixed point;*
- *the incidence structure $S' = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ the set of fixed points of β, with $\mathcal{L}$ the set of lines of $S(G, J)$ that contain a fixed point and with I the natural incidence is a semipartial geometry* $\mathrm{spg}(s', t, \alpha, \mu)$.

Proof. Let $\mathcal{F}_L$ denote the set of fixed points of any line L of $S(G, J)$. Suppose that β is neither a G-elation nor has a unique fixed point. First note that any line containing a fixed point is also fixed. Let p be an element of $\mathcal{P}$ and let L be an element of $\mathcal{L}$ not containing p. If p is collinear with a point l of L in $S(G, J)$ then clearly $\langle p, l\rangle$ is fixed under β, implying that $l = L \cap \langle p, l\rangle$ belongs to $\mathcal{P}$. This also yields that if $p \in G$ is collinear in $S(G, J)$ with two non-collinear elements of $\mathcal{P}$, there must hold that $p \in \mathcal{P}$. Now we show that all lines of S' contain the same number of points. Since β has at least two fixed points we can always choose a line L such that $|\mathcal{F}_L| \geq 2$. Now let M be any other line of $S(G, J)$ with $\mathcal{F}_M \neq \emptyset$. If L and M are concurrent in a point z we see that $z \in \mathcal{F}_L \cap \mathcal{F}_M$, since any line that contains a fixed point is a fixed line. Each point of $\mathcal{F}_L \setminus \{z\}$ determines, by collinearity in $S(G, J)$, $\alpha - 1$ points of $\mathcal{F}_M \setminus \{z\}$, and vice versa each point of $\mathcal{F}_M \setminus \{z\}$ determines $\alpha - 1$ points of $\mathcal{F}_L \setminus \{z\}$, implying $|\mathcal{F}_L| = |\mathcal{F}_M|$. (Notice that this count is invalid whenever $\alpha = 1$.) Now suppose that L and M are disjoint. Choose points $l \in \mathcal{F}_L$ and $m \in \mathcal{F}_M$. There are two possibilities. If l and m are collinear in $S(G, J)$ the foregoing implies that $|\mathcal{F}_L| = \left|\mathcal{F}_{\langle l,m\rangle}\right| = |\mathcal{F}_M|$. If l and m are not collinear we choose a point $k \in \{l, m\}^{\perp}$. By the first part of the proof we know that k must be fixed by β. Hence $|\mathcal{F}_L| = \left|\mathcal{F}_{\langle l,k\rangle}\right| = \left|\mathcal{F}_{\langle k,m\rangle}\right| = |\mathcal{F}_M|$. We have proved that all lines of S' contain the same number of points, say $s' + 1$, with $s' \geq 1$. So clearly S' is a partial linear space of order (s', t). Again by the first part of the proof it follows that S' is in fact an $\mathrm{spg}(s', t, \alpha, \mu)$. $\square$

Theorem 7.3.4 *Let $S(G, J)$ be a TSPG, with $\mu \neq \alpha(\alpha + 1)$ and $\alpha > 1$. Then a G-parallelism of $S(G, J)$ that is not the identity has at most one fixed point.*

Proof. Let $\beta \neq 1$ be a G-parallelism of $S(G, J)$ and suppose that β has more than one fixed point. Since $S(G, J)$ is a TSPG Theorem 7.3.3 implies that the incidence structure of fixed points and fixed lines of β is an $\mathrm{spg}(s', t, \alpha, \mu)$. Furthermore Theorem 7.2.2 states that $S(G, J) \cong S(\mathcal{R})$, with $\mathcal{R}$ an SPG-regulus in $\mathrm{PG}(n, q)$. Now choose a point y that is fixed under β and a line L

through y. Consider a point $z \in L$ that is not fixed by β and finally choose a point x collinear with y not on L that is fixed by β. Since $\mu > \alpha(\alpha+1)$, we can always choose a line $K_z \neq L$ through z with the property that x is collinear with α points on K_z, but such that the line L_x through x and G-parallel to L does not intersect K_z (notice that z is not collinear with x). Denote the lines through x intersecting K_z by $M_1, \ldots, M_\alpha$. There holds that $M_i^\beta = M_i$, $i = 1, \ldots, \alpha$. Hence, if we put $K_{z^\beta} = (K_z)^\beta$, we see that (since β preserves incidence) $M_i \cap K_{z^\beta} \neq \emptyset$. For the rest of the proof we shall consider $S(G,J) \cong S(\mathcal{R})$ in its representation in $\mathrm{AG}(n+1,q)$, where $\Pi_\infty := \mathrm{PG}(n,q)$ is the hyperplane at infinity of $\mathrm{AG}(n+1,q)$. Suppose that $\mathcal{R}$ is an SPG-regulus consisting of m-dimensional spaces. Let Π_z be the projection from x onto Π_∞ of the $(m+1)$-dimensional space determined by K_z and define in an analogous way the space Π_{z^β} as the projection of the space determined by K_{z^β}. There holds that Π_z and Π_{z^β} are $(m+1)$-dimensional and both contain the same element K of $\mathcal{R}$. Let m_i denote the point of Π_z determined by $M_i \cap K_z$ under the projection from x. Similarly let m'_i denote the point of Π_{z^β} determined by $M_i \cap K_{z^\beta}$. There are two possibilities. First suppose that $\Pi_z = \Pi_{z^\beta}$. This implies that the point z' of Π_∞ determined by the projective line $\langle z, z^\beta \rangle$ belongs to Π_z. Notice that z' belongs to some element of $\mathcal{R}$ since z and z^β both belong to the line L of $S(\mathcal{R})$. Furthermore there must hold that $m_i = m'_i$ and $z' \neq m_i$ (since $M_i \not\parallel L$, $i = 1, \ldots, \alpha$). It follows that Π_z intersects at least $\alpha+1$ elements of $\mathcal{R}$ distinct from K, in contradiction with the α-condition for SPG-reguli. So we may suppose that $\Pi_z \neq \Pi_{z^\beta}$, that is, $\Pi_z \cap \Pi_{z^\beta} = K$ and $\langle \Pi_z, \Pi_{z^\beta} \rangle$ is an $(m+2)$-dimensional subspace Π of Π_∞. Clearly all points m_i and m'_i, $i = 1, \ldots, \alpha$, belong to Π and all these points are distinct. Hence we have found α projective lines $\langle m_i, m'_i \rangle \subset \Pi$, $i = 1, \ldots, \alpha$, each belonging to some element of $\mathcal{R}$; clearly all these elements of $\mathcal{R}$ are distinct. If $m = 0$, we obtain a contradiction, so we may suppose that $m > 0$. The point $z' = \langle z, z^\beta \rangle \cap \Pi_\infty$ belongs to the intersection of Π with some element of $\mathcal{R}$, but $z' \notin \langle m_i, m'_i \rangle$ since $M_i \not\parallel L$, $i = 1, \ldots, \alpha$. Consider the $(m+1)$-dimensional space $\langle z', K \rangle$. Since it must have a point in common with each line $\langle m_i, m'_i \rangle$ and clearly contains z', we found an $(m+1)$-dimensional space containing K and intersecting at least $\alpha+1$ other elements of $\mathcal{R}$, a contradiction. This concludes the proof. $\square$

Remark 7.3.5 It is interesting to notice that the condition $\mu > \alpha(\alpha+1)$ is necessary in the previous theorem. Consider $T_n^*(\mathcal{B})$ in its representation in $\mathrm{AG}(n+1,q^2)$. Then the automorphism of $\mathrm{AG}(n+1,q^2)$ induced by conjugation in the field $\mathrm{GF}(q^2)$ will determine a parallelism of $T_n^*(\mathcal{B})$ that is not the identity and has more than one fixed point (the fixed point structure will be an $\mathrm{AG}(n+1,q)$).

Corollary 7.3.6 *The set of all translations of a TSPG $S(G,J)$ with $t-\theta > \alpha > 1$, that is, $\mu > \alpha(\alpha+1) > 2$, is an elementary abelian group which acts regularly on the point set of $S(G,J)$; this group, which is isomorphic to G, is isomorphic to the group of elations of* $\mathrm{PG}(n+1,q)$*, for some n, with axis a fixed* $\mathrm{PG}(n,q)$ *and with $|G| = q^{n+1}$.*

Proof. By Theorem 7.3.4 the group of all parallelisms of $S(G,J)$ is a Frobenius group (as permutation group on the points of $S(G,J)$). This implies that the set E of all elations constitutes a group with a regular action on the points of $S(G,J)$. Hence $E \cong G$. Furthermore, since $S(G,J) \cong S(\mathcal{R})$ for an SPG-regulus in some $\mathrm{PG}(n,q)$, there holds that G is isomorphic to the group of all elations of $\mathrm{PG}(n+1,q)$ with axis $\mathrm{PG}(n,q)$, hence G is elementary abelian. $\square$

We can now describe all parallelisms of a TSPG for which $\mu > \alpha(\alpha+1) > 2$.

Theorem 7.3.7 *If $S(\mathcal{R})$ is a semipartial geometry with $\alpha > 1$ constructed from an SPG-regulus $\mathcal{R}$ with $t - \theta > \alpha$ in* $\mathrm{PG}(n,K)$*, with K the kernel of $S(\mathcal{R})$, then the group of all parallelisms of $S(\mathcal{R})$ is isomorphic to the group of all perspectivities of* $\mathrm{PG}(n+1,K)$ *with axis a fixed* $\mathrm{PG}(n,K)$*.*

Proof. By Theorem 7.3.4 we know that a parallelism of $S(\mathcal{R})$ has at most one fixed point and by Corollary 7.3.6 all translations of $S(\mathcal{R})$ are induced by the elations of $\mathrm{PG}(n+1,K)$ with axis $\mathrm{PG}(n,K)$. There only remains to show that every parallelism with a fixed point is induced by a homology of $\mathrm{PG}(n+1,K)$ with axis $\mathrm{PG}(n,K)$; the converse, namely that every homology of $\mathrm{PG}(n+1,K)$ with axis $\mathrm{PG}(n,K)$ is a parallelism of $S(\mathcal{R})$, is trivial. This is essentially the same as the analogue for TGQ, but as it is short and for the sake of completeness, we prove it here. So let β be a parallelism of $S(\mathcal{R})$ having a unique fixed point. Denote the unique translation group of $S(\mathcal{R})$ by G and let x be the unique fixed point of β. With the point z of $S(\mathcal{R})$ there corresponds the element of G which maps x onto z. So with the lines through x there correspond the subgroups $S_0, \ldots, S_t$ of G. We have $S(\mathcal{R}) \cong S(G,J)$, where $S(G,J)$ is defined in the natural way. By the uniqueness of G it follows that $\beta^{-1}G\beta = G$. We will show that β defines an element of the kernel of $S(G,J)$. Of course β defines a permutation γ of the elements of G, with $g^\gamma = \beta^{-1}g\beta$, and as β fixes x, $S_i^\gamma = S_i$ for all $i = 1, \ldots, t$. So we only need to show that γ preserves the operation of G. But this is obvious as $(hg)^\gamma = \beta^{-1}(hg)\beta = (\beta^{-1}h\beta)(\beta^{-1}g\beta) = h^\gamma g^\gamma$. Hence γ belongs to the kernel K of $S(G,J)$. As the multiplicative group of K corresponds to the group of all homologies of $\mathrm{PG}(n+1,K)$ with axis $\mathrm{PG}(n,K)$ and center the point x, the theorem follows. $\square$

Corollary 7.3.8 *Let $\mathcal{R}$ be an SPG-regulus in* $\mathrm{PG}(n,q)$ *consisting of* $\mathrm{PG}(m,q)$*'s, for which $\alpha > 1$. Suppose that $\theta \neq 0$ and $\alpha \notin \{q^{m+1}-1, q^{m+1}\}$ if $t-\theta = \alpha$. Let H be the following group: $H = \{\delta \in \mathrm{P\Gamma L}(n+1,q) \mid \pi^\delta = \pi, \forall \pi \in \mathcal{R}\}$. Then there exists some $l \in \mathbb{N} \setminus \{0\}$ with $l \mid n+1$ such that one of the following occurs:*

- $|H| = 2\frac{q^l-1}{q-1}$ *and there exists a Baer-subgeometry $\mathcal{B}$ in* $\mathrm{PG}(\frac{n+1-l}{l}, q^l)$ *such that $\mathcal{B} \overset{q^l \mapsto q}{\longrightarrow} \mathcal{R}$;*
- $|H| = \frac{q^l-1}{q-1}$ *and there exists an SPG-regulus $\mathcal{R}'$ with $t-\theta > \alpha$ in* $\mathrm{PG}(\frac{n+1-l}{l}, q^l)$ *such that $\mathcal{R}' \overset{q^l \mapsto q}{\longrightarrow} \mathcal{R}$.*

Proof. Immediate from Theorem 7.3.7, Theorem 5.1.1 and the fact that there is a unique non-trivial automorphism in $\mathrm{PG}(\frac{n+1-l}{l}, q^l)$ fixing a Baer-subgeometry pointwise. $\square$

Theorem 7.3.9 *If $S(\mathcal{R})$ is a semipartial geometry with $\alpha > 1$ constructed from an SPG-regulus $\mathcal{R}$ with $t-\theta > \alpha$ in* $\mathrm{PG}(n,K)$*, with K the kernel of $S(\mathcal{R})$, then the subgroup* $\mathrm{Aut}_\parallel(S(\mathcal{R}))$ *of* $\mathrm{Aut}(S(\mathcal{R}))$*, which is the group of automorphisms of $S(\mathcal{R})$ that preserve $\mathcal{R}$-parallelism, is a subgroup of* $\mathrm{A\Gamma L}(n+2,K)$*.*

Proof. Let (G,J) be the SPG-family such that $S(\mathcal{R}) \cong S(G,J)$ and let Ψ be any automorphism of $S(G,J)$ preserving $\mathcal{R}$-parallelism. We will use the additive representation of the group G. Notice that G is a K-vector space. Denote by τ_t the translation $\tau_t : G \to G : g \mapsto g+t$, for $t \in G$. Define τ_t such that $\Phi(0_G) = 0_G$ with $\Phi := \tau_t\Psi$. Let τ_a be any translation. Because of our assumptions $\Phi\tau_a\Phi^{-1}$ is a parallelism. Assume that $\tau_a \neq \tau_{0_G}$ and that $\Phi\tau_a\Phi^{-1}(g) = g$ for some $g \in G$. Then clearly $\tau_a\Phi^{-1}(g) = \Phi^{-1}(g)$, implying τ_a to have a fixed point, a contradiction. Hence $\Phi\tau_a\Phi^{-1}$ is a translation. By the regularity and the uniqueness of G we have that $\tau_a^\Phi = \tau_{\Phi(a)}$. We will now show that Φ can be seen as an endomorphism on G. To do so it is clearly sufficient to prove that Φ is additive. If $x, y \in G$, then $\tau_{\Phi(x+y)} = \Phi\tau_{x+y}\Phi^{-1} = \Phi\tau_x\tau_y\Phi^{-1} = \Phi\tau_x\Phi^{-1}\Phi\tau_y\Phi^{-1} = \tau_{\Phi(x)}\tau_{\Phi(y)} = \tau_{\Phi(x)+\Phi(y)}$. Since G is regular and unique it follows that $\Phi(x+y) = \Phi(x)+\Phi(y)$. Now define $\phi : K \to K, k \mapsto \Phi k \Phi^{-1}$. It is obvious that $\phi(k)$ is a parallelism if $k \neq 0$ as well as an endomorphism on G fixing 0_G. Hence $\phi(k)$ belongs indeed to K. There holds that $\phi(0_K) = 0_K$; $\phi(1) = 1$; $\phi(k+l)(g) = \Phi(k+l)\Phi^{-1}(g) = \Phi(k\Phi^{-1}(g) + l\Phi^{-1}(g)) = \Phi k\Phi^{-1}(g) + \Phi l\Phi^{-1}(g) = \phi(k)(g) + \phi(l)(g) = (\phi(k) +$

$\phi(l))(g)$ for all $g \in G$ and all $k, l \in K$. Hence $\phi(k+l) = \phi(k) + \phi(l)$.
Finally $\phi(kl) = \Phi kl\Phi^{-1} = \Phi k\Phi^{-1}\Phi l\Phi^{-1} = \phi(k)\phi(l)$ for all $k, l \in K$. Henceforth ϕ belongs to $\mathrm{Aut}(K)$. Now consider any $g \in G$ and any $k \in K$. There holds that $\Phi(kg) = \Phi k(g) = \Phi k\Phi^{-1}\Phi(g) = \phi(k)\Phi(g)$. This, together with the additivity of Φ, implies that Φ can be seen as an automorphism of the K-vector space G, which in turn implies that Φ is induced by an element of $\mathrm{A\Gamma L}(n+2, K)$. Finally, as $\Psi = \tau_{-t}\Phi$, it follows that also Ψ is induced by an element of $\mathrm{A\Gamma L}(n+2, K)$. $\square$

This theorem will allow us to determine the full automorphism group of almost all known semipartial geometries constructed from SPG-reguli. We need the following lemmas.

Lemma 7.3.10 *If $\mathcal{R}$ is an α-geometric SPG-regulus, with $\alpha > 1$, consisting of m-dimensional spaces in* $\mathrm{PG}(n, q)$*, then every automorphism of $S(\mathcal{R})$ preserves $\mathcal{R}$-parallelism.*

Proof. We define the following types of parallelism. We say that two distinct lines L and M of $S(\mathcal{R})$ are $\|_1$-parallel ($L \|_1 M$) if and only if they do not intersect and there exist two transversals of L and M that intersect in a point not belonging to $L \cup M$. We say that two distinct lines L and M are $\|_2$-parallel if and only if they do not intersect, there exists no transversal of L and M and there exists a line Q such that $L \|_1 Q \|_1 M$. Finally we say that two lines L and M are $\|$-parallel if and only if either $L = M$, $L \|_1 M$ or $L \|_2 M$. From the definition it is clear that every automorphism of $S(\mathcal{R})$ preserves $\|$-parallelism. We will now show that $\|$-parallelism coincides with $\mathcal{R}$-parallelism. We consider $S(\mathcal{R})$ in its natural representation in the affine space $\mathrm{AG}(n+1, q)$. Suppose that L and M are two distinct lines that are $\|_1$-parallel. Assume by way of contradiction that they would not be $\mathcal{R}$-parallel. Using the fact that $\mathcal{R}$ is α-regular, it readily follows that the $(m+1)$-dimensional spaces $\langle L\rangle$ and $\langle M\rangle$ are contained in a $(2m+2)$-dimensional space. Hence L and M must intersect, a contradiction.We see that L and M are $\mathcal{R}$-parallel. By transitivity it now also follows that $\|_2$-parallel lines are $\mathcal{R}$-parallel. Conversely suppose that L and M are two distinct $\mathcal{R}$-parallel lines. If there exists a transversal of L and M, then we immediately see that $L \|_1 M$ (recall that $\alpha > 1$). Next suppose that no point on L is collinear with a point on M. Choose any point l on L and any point m on M, and let x be a point collinear with l and m. Then clearly the line $\mathcal{R}$-parallel with L and M through x is $\|_1$-parallel with L and M. Hence $L \|_2 M$. This proves the lemma. $\square$

Notice that the previous lemma applies to SPG-reguli consisting of points, as these are always α-geometric.

Lemma 7.3.11 *If $\mathcal{R}$ is an SPG-regulus consisting of m-dimensional spaces in* $\mathrm{PG}(n,q)$ *that satisfies the polar property, then every automorphism of* $S(\mathcal{R})$ *preserves $\mathcal{R}$-parallelism.*

Proof. We consider $S(\mathcal{R})$ in its natural representation in $\mathrm{AG}(n+1,q)$. We will show that two non-intersecting lines L and M of $S(\mathcal{R})$ are $\mathcal{R}$-parallel if and only if either every point on L is collinear with a point on M, or if no point on L is collinear with a point on M. This clearly is sufficient to prove the lemma. If L and M are $\mathcal{R}$-parallel and π is the element of $\mathcal{R}$ determined by L and M, then it is trivial that we are in one of the above situations, since either $\langle L, M\rangle \cap \mathrm{PG}(n,q)$ is or is not contained in the tangent space at π. Conversely suppose that L and M are not $\mathcal{R}$-parallel and that π_L is the element of $\mathcal{R}$ determined by L. The space $\langle L, \tau_L\rangle$, with τ_L the tangent space at π_L, is an $(n-m)$-dimensional space and its affine part consists exactly of all the points of L and those that are collinear with no point on L (in $S(\mathcal{R})$). Further we see that M must have exactly one (affine) point in common with $\langle L, \tau_L\rangle$. This proves the lemma. □

Notice that this lemma shows that if $\mathcal{R}$ satisfies the polar property, then there are four possibilities for two distinct lines L and M in $S(\mathcal{R})$:

- every point of L is collinear with a point of M and vice versa, with $L \cap M = \emptyset$;
- no point of L is collinear with a point of M and vice versa;
- all but one of the points of L are collinear with a point of M and vice versa;
- L and M intersect.

Corollary 7.3.12 *If $S(\mathcal{R})$ is a semipartial geometry with $\alpha > 1$ and $t - \theta > \alpha$ constructed from an SPG-regulus $\mathcal{R}$ in* $\mathrm{PG}(n,q)$ *that is α-geometric or that satisfies the polar property, then* $\mathrm{Aut}(S(\mathcal{R})) \leq \mathrm{A\Gamma L}(n+2,q)$. *More precisely* $\mathrm{Aut}(S(\mathcal{R}))$ *is the subgroup of* $\mathrm{A\Gamma L}(n+2,q)$ *induced by* $\mathrm{Aut}(\mathcal{R}) \leq \mathrm{P\Gamma L}(n+1,q)$.

Proof. As Lemma 7.3.10 and Lemma 7.3.11 imply that every automorphism must map parallel lines onto parallel lines, Theorem 7.3.9 implies that $\mathrm{Aut}_{\|}(S(\mathcal{R})) = \mathrm{Aut}(S(\mathcal{R}))$. The result now easily follows. □

Remark 7.3.13 Notice that $t - \theta > \alpha$ is not really a restriction in the previous corollary, since $t - \theta = \alpha > 1$ implies that $S(\mathcal{R}) \cong T_m^*(\mathcal{B})$ for certain m, and in such a case it is well known that the conclusion of the previous corollary holds.

7.4 Recognition of ESPG and TSPG

In this section we provide the analogue for TSPG of Theorem 8.3.1 of Payne and Thas [59] on the recognition of TGQ.

Theorem 7.4.1 *Let S be an* $\mathrm{spg}(s, t, \alpha, \mu)$ *with $\alpha < s+1$, and suppose that G is a regular automorphism group of S. If there exists a point x such that either $L^g = L$ or $L^g \cap L = \emptyset$ for all lines L incident with x and all $g \in G$, then S is an ESPG.*

Proof. We first show that in fact there holds that either $M^g = M$ or $M^g \cap M = \emptyset$ for any line M of S. Assume by way of contradiction that $M^g \cap M = y$ for some line M and some point y. Let h be the element of G mapping x onto y. Put $L = M^{h^{-1}}$. One sees that $L^{hgh^{-1}} \neq L$, while from $y = M \cap M^g$ it follows that $x \in L^{hgh^{-1}}$, implying $L \cap L^{hgh^{-1}} \neq \emptyset$, a contradiction. Choose any point z and denote by $L_0, \ldots, L_t$ the lines of S through z. Now define $S_i := \{g \in G \mid L_i^g = L_i\}$, $i = 0, \ldots, t$. By the regularity of G and the above it follows that each S_i, $i = 0, \ldots, t$, is a subgroup of G of order $s + 1$. Suppose that $g \in S_i \cap S_j$, with $i \neq j$. This implies $z^g = L_i^g \cap L_j^g = L_i \cap L_j = z$ and hence by the regularity of G that $g = id$. So there holds that $S_i \cap S_j = id$ whenever $i \neq j$. It is now an easy exercise to prove that (G, J) with $J = \{S_0, \ldots, S_t\}$ is an SPG-family. Define the mapping $\iota : S \to S(G, J) : y \mapsto g$ with $z^g = y$. Clearly ι is an isomorphism, implying that S is an ESPG. $\square$

Definition 7.4.2 Let S be an $\mathrm{spg}(s, t, \alpha, \mu)$, with $\alpha > 1$, and suppose that $P = \{\pi_0, \ldots, \pi_t\}$ is a packing of S.

- A *P-parallelism* of S is any automorphism β of S with the property that for every line L, whenever $L \in \pi_i$ then also $L^\beta \in \pi_i$. If a P-parallelism acts fixed point free it is called a *P-elation.* By definition the identity is also a P-elation.
- A *P-symmetry* about π_i is any P-parallelism β such that for every line $L \in \pi_i$ there holds that $L^\beta = L$.

Lemma 7.4.3 *Let S be an* $\mathrm{spg}(s,t,\alpha,\mu)$*, with* $\alpha > 1$*, and suppose that* $P = \{\pi_0, \ldots, \pi_t\}$ *is a packing of S.*

- *The conclusion of Theorem 7.3.3 holds for any P-parallelism.*
- *Any P-symmetry that is not the identity acts fixed point free.*
- *If we denote by* $\mathrm{Sym}_P(\pi_i)$ *the set of all P-symmetries about* π_i *then* $\mathrm{Sym}_P(\pi_i) \leq \mathrm{Aut}(S)$*; furthermore* $|\,\mathrm{Sym}_P(\pi_i)\,| \leq s+1$*.*

Proof. The first assertion is proved as in Theorem 7.3.3, one only has to replace G-parallelism by P-parallelism. Let β be a P-symmetry about π_i, and suppose that β fixes the point x. If $y \neq x$ is any point collinear with x not on the line of π_i through x, then $y = L \cap M$, with L a line through x and $M \in \pi_i$. As β is a P-parallelism it must fix every line through x, hence L. On the other hand it fixes the line M by definition. There follows that β fixes y. Now let z be collinear with x, where $x \neq z$ and $\langle x, z\rangle \in \pi_i$. Consider a line $N \neq \langle x, z\rangle$ through z and let $u \neq z$ be a point on N collinear with x. By the foregoing β fixes u and, as N does not belong to π_i, there follows that β also fixes z. Hence β fixes every point collinear with x. By the connectedness of the geometry it follows that β must be the identity. Hence the second assertion. The third assertion is now an immediate consequence. $\square$

Lemma 7.4.4 *If* $\lambda \in \mathrm{Sym}_P(\pi_i)$ *and* $\mu \in \mathrm{Sym}_P(\pi_j)$*, with* $i \neq j$*, then* $\lambda\mu = \mu\lambda$*.*

Proof. Consider any point x of S and put $L \in \pi_i$ with $x \in L$ and $M \in \pi_j$ with $x \in M$. We clearly have that $x^{\lambda\mu} = L^{\lambda\mu} \cap M^{\lambda\mu} = L^{\mu} \cap M^{\lambda} = L^{\mu\lambda} \cap M^{\mu\lambda} = x^{\mu\lambda}$, proving $\lambda\mu = \mu\lambda$. $\square$

Theorem 7.4.5 *Let S be an* $\mathrm{spg}(s,t,\alpha,\mu)$*, with* $t > 2$ *and* $2 \leq \alpha < s+1$*. Suppose that* $P = \{\pi_0, \ldots, \pi_t\}$ *is a packing of S. If* $|\,\mathrm{Sym}_P(\pi_i)\,| = s+1$ *for all* $i \in I = \{0, \ldots, t\}$*, then S is a TSPG.*

Proof. Put $S_i := \mathrm{Sym}_P(\pi_i)$, $G := \langle S_i\rangle_{i\in I}$, $G_i := \langle S_j\rangle_{j\in I\setminus\{i\}}$, and finally $G_{ij} := \langle S_k\rangle_{k\in I\setminus\{i,j\}}$. We will denote by $\pi_i(x)$ the line of π_i through the point x. Because of the previous lemma we know that $[S_i, G_i] = id$. We will first show that G_{ij} acts transitively for any $\{i,j\} \subset I$. Consider points $x \neq y$ and suppose that x is collinear with y. If $y \in \pi_k(x)$ with $k \notin \{i,j\}$, then clearly there exists an element of $S_k \leq G_{ij}$ mapping x onto y. So suppose without loss of generality that $y \in \pi_i(x)$. Since y is contained in at least four lines and $\alpha \geq 2$ we can always find some $k \in I \setminus \{i, j\}$ such that there exists a point $z \in \pi_k(y)$ with the property that $x \in \pi_l(z)$, with $l \in I \setminus \{i, j\}$. Hence there are elements $s_l \in S_l$ and $s_k \in S_k$ such that $x^{s_l s_k} = z^{s_k} = y$. It is clear that $s_l s_k \in G_{ij}$. If x is not collinear with y, the existence of some $g \in G_{ij}$ such that $x^g = y$ follows from the above together with the connectedness of S. Henceforth G_i and G_j are transitive. We will now prove that G_i is in fact regular. Suppose that $x^g = x$ for some $g \in G_i$. Consider any $y \in \pi_i(x)$. Because of our assumptions there exists an element $s \in S_i$ such that $x^s = y$. We see that $y^g = y^{s^{-1}gs} = x^{gs} = x^s = y$. This implies that g fixes all points of $\pi_i(x)$. Lemma 7.4.3 now implies that the structure of fixed points and fixed lines of g is an spg(s, t, α, μ), that is, all points of S are fixed and $g = id$. Hence also G_{ij} and G_j are regular, yielding $G_i = G_{ij} = G_j$. Since $G = \langle S_i, G_i \rangle = \langle S_i, G_j \rangle = G_j$ it follows that G is an abelian regular group of automorphisms of S. It is an easy exercise to check that (G, J), with $J = \{S_0, \ldots, S_t\}$, is an SPG-family. Hence $S(G, J)$ is a TSPG. Choose some point x of S and consider the map $\iota : S \to S(G, J) : y \mapsto g$, with $x^g = y$. It is not difficult to show that ι is an isomorphism between S and $S(G, J)$. This proves the theorem. $\square$

Remark 7.4.6 The assumption $t > 2$ in the previous theorem is not really a restriction. Suppose namely that $t = 2$. Since for semipartial geometries there holds that $s \leq t$ (if S is a partial geometry with $s > t$ we consider its dual) it follows from the assumption $1 < \alpha < s + 1$ that $s = \alpha = 2$, that is, S is a cotriangular space. Clearly S is indecomposable, since we always assume our geometries to be connected. Suppose S is not reduced. Then there exists distinct points x and y such that $\Gamma(x) = \Gamma(y)$. This implies that S is a partial geometry pg$(2, 2, 2)$, and it is easily seen that this geometry is uniquely determined by its parameters. Next suppose that S is reduced. Then Hall [44], see Theorem 2.4.4, classified the indecomposable and reduced copolar spaces, yielding either $S \cong \overline{M(3)}$ or $S \cong U_{2,3}(5)$. So in the case $t = 2$ the geometry S is always known.

7.5 A special property

In this section we will explain the link between TSPG and the "translation partial geometries" as introduced by De Clerck, Gevaert and Thas [26] (see also the preface of this chapter).

Definition 7.5.1 Let (G, J), with $J = \{S_0, \ldots, S_t\}$, be an SPG-family with parameters (s, t, α, θ). We say that (G, J) is *special* at S_i iff for each $j \in I \setminus \{i\}$, with $I = \{0, \ldots, t\}$, there exists a set $W(i, j) \subset I \setminus \{j\}$, with $|W(i, j)| = \alpha$, and such that $S_i S_j = S_k S_j$, for each $k \in W(i, j)$. Notice that for G abelian and $t - \theta > \alpha$ the fact that (G, J) is special at each element is equivalent with the fact that the corresponding SPG-regulus is α-geometric.

In [26] a theory of translation partial geometries was introduced as follows. Let G be an abelian group of order $(s+1)^3$ $(s \geq 1)$ and let $J = \{A_0, \ldots, A_t\}$ be a set of subgroups of G of order $s + 1$, with $t = \alpha(s + 2)$, and satisfying:

(1) for any pair $\{i, j\} \subset I = \{0, \ldots, t\}$, there exists a subset $V(i, j)$ of I, $|V(i, j)| = \alpha + 1$, such that $A_i A_j = A_k A_l$, for all $\{k, l\} \subset V(i, j)$;

(2) $A_i A_j \cap A_k = \{id\}$ for all $k \in I \setminus V(i, j)$.

Then it is possible to prove, see [26] (but it also follows from the following theorem), that the coset geometry $S(G, J)$ is a partial geometry $\mathrm{pg}(s, t, \alpha)$. In [26] these geometries are called translation partial geometries and it is shown that they are equivalent to α-geometric SPG-reguli in $\mathrm{PG}(3n - 1, q)$ consisting of $\mathrm{PG}(n - 1, q)$. We prefer to call them *special translation partial geometries*, and use the name translation partial geometries for any TSPG that is a partial geometry. The reason for this being the following theorem.

Theorem 7.5.2 *Any special translation partial geometry $S(G, J)$ with $s + 1 \neq \alpha$ is a TSPG $S(G, J)$ such that (G, J) has parameters $(s, \alpha(s + 2), \alpha, 0)$ and such that (G, J) is special at each element; and conversely.*

Proof. Let (G, J) with $J = \{A_0, \ldots, A_t\}$, be such that $S(G, J)$ is a special translation partial geometry. We will show directly that (G, J) is an SPG-family. Lemma 3.3.4 of [26] states that $A_i \cap A_j = \{id\}$ whenever $i \neq j$. The α-condition for SPG-families is an immediate corollary of (1) and (2) in the above definition. Now consider any $i \in I$. As G is abelian, the $A_i A_j$, $i \neq j$, are groups of order $(s + 1)^2$ with the property that $A_i A_j \cap A_i A_k = A_i$, whenever $k \notin V(i, j)$. As $t = \alpha(s+2)$ this implies that for every point $g \in G \setminus \bigcup_{i \in I} A_i$ there exists some $j \in I$ such that $g \in A_i A_j$. Hence (G, J) satisfies the θ-condition with $\theta = 0$. Thus (G, J) is an SPG-family with the desired parameters. The fact that (G, J) is special at each element follows immediately from (1) putting $W(i, j) = V(i, j) \setminus \{j\}$. Conversely the parameters $(s, \alpha(s + 2), \alpha, 0)$ imply that the TSPG is a partial geometry constructed from an SPG-regulus consisting of $\mathrm{PG}(n-1,q)$ in $\mathrm{PG}(3n-1,q)$ for certain n. It can easily be seen that the fact that the TSPG is special at each element implies that the constructed partial geometry is a special partial geometry (by putting $V(i,j) = W(i,j) \cup \{j\}$). $\square$

That translation partial geometries with parameters $(s, \alpha(s+2), \alpha)$ arising from SPG-reguli consisting of $\mathrm{PG}(n-1,q)$ in $\mathrm{PG}(3n-1,q)$ are indeed more general than special translation partial geometries does not only follow from the definitions but also from the fact that there is an example of such a translation partial geometry that is not special. Namely the partial geometry $\mathrm{pg}(8,20,2)$ constructed from the SPG-regulus of Mathon in $\mathrm{PG}(5,3)$, see De Clerck, Delanote, Hamilton and Mathon [22]. The authors list the elements (lines) of the SPG-regulus. It is easily checked that the $\mathrm{PG}(3,3)$ generated by the first and the sixth line has equations $x_2 - x_3 = 0$ and $x_5 - 2x_4 = 0$ and does not contain a third element of the SPG-regulus. Hence the constructed partial geometry is a translation partial geometry that is not special.

7.6 Generalized quadrangles and partial quadrangles with an abelian regular group

In the section on recognition of TSPG we always assumed that $\alpha > 1$. In the final section of this chapter we will have a closer look at the case $\alpha = 1$. The motivation for this section comes from a question of J. C. Fisher at the La Roche Conference on Incidence Geometry (May 23-29, 2004):

> What can one say about generalized quadrangles that admit a regular group G of automorphisms?

In this section we will handle the case where G is abelian. In fact a more general theorem will be proved. This work is based on De Winter and K. Thas [40].

Lemma 7.6.1 *Let $\mathcal{Q}$ be a finite $(0,1)$-geometry of order (s,t), with $s > 2$. Furthermore suppose that $G \leq \mathrm{Aut}(\mathcal{Q})$ is an abelian group of automorphisms acting regularly on the point set of $\mathcal{Q}$. Then the stabilizer, $\mathrm{Stab}_G(L)$, in G of any line L of Q is a group of order $|L| = s+1$.*

Proof. Consider a line L of $\mathcal{Q}$ and choose any point x on L. Let S denote the unique set $S = \{g_0 = id, g_1, \ldots, g_s\} \subset G$ with the property that $x^{g_i} \in L$, $i = 0, 1, \ldots, s$. It is our aim to show that S is a subgroup of G stabilizing L. Consider any two distinct elements, g_i and g_j, of S, with $i \neq 0 \neq j$. From $x \sim x^{g_i}$ it follows that $x^{g_j} \sim x^{g_i g_j}$. Analogously we obtain from $x \sim x^{g_j}$ that $x^{g_i} \sim x^{g_j g_i} = x^{g_i g_j}$. Hence the point $x^{g_i g_j}$ is collinear with two distinct points, x^{g_i} and x^{g_j}, on L and consequently belongs to L. From this it follows that $L^{g_i} = \langle x, x^{g_j} \rangle^{g_i} = \langle x^{g_i}, x^{g_i g_j} \rangle = L$, so that $S \subset \mathrm{Stab}_G(L)$. Since $|\mathrm{Stab}_G(L)| \leq s + 1$ by the regularity of G, it follows that $S = \mathrm{Stab}_G(L)$. $\square$

This lemma allows us to derive a nice divisibility condition, but let us first introduce the concept of a spread of symmetry.

A *spread of symmetry* T of $\mathcal{Q}$ is a spread of $\mathcal{Q}$ for which there is an automorphism group of $\mathcal{Q}$ (called the "associated group") fixing T linewise and acting regularly on the points of any of its lines.

Suppose that $\mathcal{Q}$ is a finite connected $(0, 1)$-geometry of order (s, t), with $s, t \geq 2$. Further let G be a regular abelian automorphism group of $\mathcal{Q}$. Choose any fixed point x of $\mathcal{Q}$ and denote the $t + 1$ lines through x by $L_0, L_1, \ldots, L_t$. By the previous lemma there exist $t + 1$ subgroups of order $s + 1$ of G, denoted by $S_0, S_1, \ldots, S_t$, with the property that S_i stabilizes the line L_i. It is easily seen that the orbit of L_i under G, $i = 0, 1, \ldots, t$, is a spread of symmetry of $\mathcal{Q}$.

Remark 7.6.2 Note that it follows already that no finite generalized n-gons with $n > 4$ admit an abelian group acting regularly on its point set (the existence of a spread of symmetry implies that the geometry has (ordinary) quadrangles).[1]

We will now derive the above mentioned divisibility condition.

Lemma 7.6.3 *Suppose $\mathcal{Q}$ is as in Lemma 7.6.1, and suppose that $t > 1$. Then $(s + 1)^3$ divides $|G|$. If in particular $\mathcal{Q}$ is a thick GQ of order (s, t), then $t = s + 2$.*

Proof. Let L be a line of $\mathcal{Q}$, and suppose $M \sim L \neq M$. Denote by T_L, respectively T_M, the spread of symmetry containing L, respectively M, as described prior to the lemma, and by S_L, respectively S_M, its associated group. Then there is an $(s+1) \times (s+1)$-grid Γ containing L and M which is

[1]The finiteness assumption is not essential here; Lemma 7.6.1 clearly has an infinite analogue with similar proof.

fixed by $S_LS_M = \langle S_L, S_M\rangle$, so that S_LS_M acts regularly on each regulus of Γ. Now consider a line N which meets Γ in a unique point. Then $S_N \cap S_LS_M = \{id\}$, so the subgroup $S_LS_MS_N$ of G has size $(s+1)^3$. If $\mathcal{Q}$ is a GQ of order (s,t), then $(s+1)^3$ divides $|G| = (s+1)(st+1)$, so that $s+1$ divides $t-1$, whence

$$s+1 \quad | \quad \frac{t-1}{s+1} - 1.$$

So either $\frac{t-1}{s+1} - 1 = 0$, or $s^2+3s+3 \leq t$, contradicting Higman's inequality $t \leq s^2$ (see Theorem 1.6.3). We conclude that $t = s+2$. $\square$

Suppose that $\mathcal{Q}$ is a finite $(0,1)$-geometry of order (s,t), with $s,t \geq 2$. Further let G be a regular abelian automorphism group of $\mathcal{Q}$. Choose any fixed point x of $\mathcal{Q}$ and denote the $t+1$ lines through x by $L_0, L_1, \ldots, L_t$. By Lemma 7.6.1 there exists a set J $t+1$ subgroups of order $s+1$ of G, denoted by $S_0, S_1, \ldots, S_t$, with the property that S_i stabilizes the line L_i. It is not difficult to check that (G,J) is a geometric family and further that $S(G,J) \cong \mathcal{Q}$.

Here as well, define K to be the set of all endomorphisms β of G with the property that $S_i^\beta \subset S_i$, for all $i \in \{0,1,\ldots,t\}$. Then, since G is abelian, $K,+,.$, that is, K endowed with the usual addition and multiplication of endomorphisms, is a ring.

It is not difficult to see that the restriction $t-\theta > \alpha$ as well as the proof of Theorem 7.2.1 can be adjusted for general $(0,1)$-geometries. We obtain the following Theorem. (We include the proof as it is short.)

Theorem 7.6.4 *Suppose that the setting and the notations are as in the above paragraph. If for any two points y and z of $\mathcal{Q}$ that are at distance two from each other in the collinearity graph of $\mathcal{Q}$ there holds that $\left|\{y,z\}^\perp\right| > 2$, then $K,+,.$ is a field.*

Proof. It is sufficient to show that every element of $K\backslash\{0\}$ is a bijection from G to G, and hence by the finiteness of G we only need to show injectivity. Suppose that $\beta \in K$ is such that $s_0^\beta = id$ for some $s_0 \in S_0\backslash\{id\}$; then we must show that $\beta = 0$. Assume the contrary. Choose any element $s_i \in S_i \setminus \{id\}$, with $i \neq 0$. Then the point s_0s_i is at distance two from id in the collinearity graph of $S(G)$. Because of our assumption there exist elements $s_l \in S_l \setminus \{id\}$ and $s_k \in S_k \setminus \{id\}$, $l,k \notin \{0,i\}$ and $l \neq k$, such that $s_0s_i = s_ls_k$ (note that $\left|\{y,z\}^\perp\right|$, $y \not\sim z$, is even since G is abelian). Letting act β yields $s_i^\beta = s_l^\beta s_k^\beta$. First suppose that $s_l^\beta = id$; then $s_i^\beta = s_k^\beta$. Since $S_i \cap S_k = \{id\}$ we obtain that $s_i^\beta = id$. Analogously $s_k^\beta = id$ implies that $s_i^\beta = id$. Next suppose that neither s_l^β nor s_k^β equals id. In this case the line $S_l s_k^\beta$ (of $S(G)$) intersects the line S_k in $s_k^\beta \neq id$ and intersects the line S_i in $s_i^\beta \neq id$. Hence we have found a triangle in the $(0,1)$-geometry $S(G)$, a contradiction. We conclude that $s_i^\beta = id$, and henceforth that $S_j^\beta = id$, for all $j \in \{0, 1, \ldots, t\}$. By the connectedness of the geometry $\mathcal{Q} \cong S(G)$ we know that $G = \langle S_0, S_1, \ldots, S_t \rangle$, and hence it follows that $G^\beta = id$, that is, $\beta = 0$. □

Theorem 7.6.5 *Let $\mathcal{Q}$ be a finite connected $(0,1)$-geometry of order (s,t), $s, t \geq 2$, in which any two points at distance two in the collinearity graph of $\mathcal{Q}$ have at least three common neighbours. Then $\mathcal{Q}$ has a generalized linear representation if and only if $\mathcal{Q}$ admits a regular abelian group of automorphisms.*

Proof. It is clear that if $\mathcal{Q}$ has a generalized linear representation, say in $\mathrm{AG}(n,q)$, then the group G of all translations of $\mathrm{AG}(n,q)$ is abelian and acts as a regular group of automorphisms on $\mathcal{Q}$.

Conversely, if every two points at distance two in the collinearity graph of $\mathcal{Q}$ have at least three common neighbours, then the previous theorem implies that $K, +, .$ is a field. Consequently G can be seen as a K-vector space of dimension $\log_{|K|}(|G|)$, and the S_i as vector subspaces of dimension $\log_{|K|}(s+1)$. It is clear that the line set of $\mathcal{Q}$ is exactly the set of all translates of these vector subspaces, and hence $\mathcal{Q}$ has a generalized linear representation in the affine space constructed from the K-vector space G. This generalized linear representation is a linear representation if and only if $\log_{|K|}(s+1) = 1$. □

We can now provide an answer to the question of J. C. Fisher for abelian groups. Recall that generalized hyperovals were defined in Chapter 1, Section 1.6 on page 10.

Corollary 7.6.6 *If $\mathcal{Q}$ is thick finite GQ of order (s,t) admitting a regular abelian automorphism group, then $\mathcal{Q}$ is isomorphic to a GQ arising from a generalized hyperoval $\mathcal{O}$.*

Proof. Since $t = s + 2$, by Lemma 7.6.3, and every two points at distance two have at least three common neighbours in a thick GQ this result is an immediate consequence of Theorem 7.6.5. □

Notice that since $\mathcal{Q}$ arises from a generalized hyperoval, $s - 1$ necessarily is even.

Remark 7.6.7 It should be noticed that the classification of (generalized quadrangles arising from) generalized hyperovals is equivalent to the classification of the so-called translation generalized quadrangles of order $s > 1$, s even, see Thas and Thas [78] for a recent reference.

We can now also apply Theorem 7.6.5 to the class of partial quadrangles.

Corollary 7.6.8 *If $\mathcal{Q}$ is a thick partial quadrangle $PQ(s,t,\mu)$ admitting a regular abelian group G of automorphisms, then $\mathcal{Q}$ has a generalized linear representation.*

Proof. If $\mu > 2$, then Theorem 7.6.5 immediately implies the result.

So suppose that $\mu = 2$. (Notice that since G is abelian $\mu = 1$ cannot occur.) With the same notations as before, we need to show that $K, +, .$ is a field. First notice that for every point g of $S(G)$ there exist i and j such that g can be written as $s_i s_j$ for some $s_i \in S_i$ and some $s_j \in S_j$. Since $\mu = 2$, it is easily seen that the grid $S_i S_j$ (in $S(G) \cong \mathcal{Q}$) determined by S_i and S_j, and the grid $S_l S_k$ determined by S_l and S_k have trivial intersection (that is $S_i S_j \cap S_l S_k = \{id\}$) unless $\{i,j\} \cap \{k,l\} \neq \emptyset$. Suppose that $\beta \in K$ is such that $s_0^\beta = id$, with $s_0 \in S_0 \setminus \{id\}$. Consider any $s_i \in S_i \setminus \{id\}$, with $i \neq 0$. Further choose any $s_j \in S_j \setminus \{j\}$, with $j \notin \{0,i\}$. Then the point $s_0 s_i s_j$ is at distance two from id and is only collinear with the point $s_0 s_i$ in the grid $S_0 S_i$. Hence there must exist k and l, such that $\{k,l\} \cap \{0,i,j\} = \emptyset$ and such that $s_0 s_i s_j = s_l s_k$, for some $s_l \in S_l \setminus \{id\}$ and some $s_k \in S_k \setminus \{id\}$ (if not there would arise triangles or the assumption $\mu = 2$ would be violated). Letting act β yields $s_i^\beta s_j^\beta = s_l^\beta s_k^\beta$. Since $S_i S_j \cap S_l S_k = \{id\}$, we obtain that $s_i^\beta = id$. It now easily follows that $\beta = 0$ and hence that $K, +, .$ is a field. The proof can be finished as before. $\square$

Chapter 8

A new semipartial geometry satisfying the polar property

In this chapter we will construct a new SPG-regulus satisfying the polar property. We will show that the semipartial geometry arising from it is new and we will determine its automorphism group. The construction is a generalization of a construction for m-systems by Luyckx and Thas [55]. Although the results of this chapter are taken from De Winter and Thas [38], the proofs given here are new and much shorter than in that paper. The reason for this is that at the time of [38] we did not have the results of De Winter and Thas [39] and De Winter [36] (see Chapters 6 and 7), which turn out to be very useful in this context.

8.1 The construction

8.1.1 Buekenhout-Metz unitals

We first recall the André-Bruck-Bose construction (see André [1] and Bruck and Bose [7]) for translation planes.

André-Bruck-Bose construction

Consider a line spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$, and embed $\mathrm{PG}(3,q)$ as a hyperplane Π at infinity in $\mathrm{PG}(4,q)$. We consider the following incidence structure $\mathcal{P}$. The point set of $\mathcal{P}$ is the union of $\mathcal{S}$ and the set of points of $\mathrm{PG}(4,q) \setminus \Pi$. The line set of $\mathcal{P}$ consist of the spread $\mathcal{S}$ and all the planes of $\mathrm{PG}(4,q)$

which intersect Π exactly in an element of $\mathcal{S}$. The incidence relation of $\mathcal{P}$ is containment. Then $\mathcal{P}$ is a projective (translation) plane of order q^2. Further, $\mathcal{P}$ is Desarguesian if and only if the spread $\mathcal{S}$ is regular, that is, the regulus of Π determined by any three elements of $\mathcal{S}$ belongs to $\mathcal{S}$.

We can now explain the construction of Buekenhout for unitals. We will restrict ourselves to the Desarguesian case as we will only use this case.

Buekenhout's construction

Let $\mathcal{S}, \Pi \subset \mathrm{PG}(4,q)$ and $\mathcal{P}$ be as above. We suppose that $\mathcal{S}$ is regular. Hence $\mathcal{P} \cong \mathrm{PG}(2,q^2)$. Consider a 3-dimensional subspace $\Omega \neq \Pi$ of $\mathrm{PG}(4,q)$. Then $\Omega \cap \Pi$ is a plane that contains a unique line R of $\mathcal{S}$. Further let $\mathcal{O}$ be an ovoid of Ω such that $\mathcal{O}$ intersects Π in a unique point $o \notin R$. Let L be the line of $\mathcal{S}$ through o and let p be a point of $L \setminus \{o\}$. Finally let $\mathcal{C}$ be the cone $p\mathcal{O}$. Then Buekenhout [10] showed that $\mathcal{U} := (\mathcal{C} \cap (\mathrm{PG}(4,q) \setminus \Pi)) \cup \{L\}$ is a unital in $\mathcal{P} \cong \mathrm{PG}(2,q^2)$.

The unital $\mathcal{U}$ will be called a *Buekenhout-Metz unital.* In fact Buekenhout [10] constructed the classical unital of $\mathrm{PG}(2,q^2)$ using the above construction with a classical ovoid, that is, an elliptic quadric. He further proved that a non-classical unital was constructed in this way, starting from a Tits-ovoid. In [57] Metz showed that it was possible to construct a non-classical example starting from a classical ovoid (this explains why this class of unitals is referred to as Buekenhout-Metz unitals). In fact many non-isomorphic Buekenhout-Metz unitals of $\mathrm{PG}(2,q^2)$ are known; for a complete overview on these unitals we refer to Baker and Ebert [2] and Ebert [42].

The following interesting characterization of Buekenhout-Metz unitals in $\mathrm{PG}(2,q^2)$ is due to Casse, O' Keefe and Penttila [15] and Quinn and Casse [61].

Theorem 8.1.1 ([61], [15]) *A unital $\mathcal{U}$ in $\mathrm{PG}(2,q^2)$, $q > 2$, is a Buekenhout-Metz unital if and only if there exists a point u of $\mathcal{U}$ such that the points of $\mathcal{U}$ on each of the q^2 secant lines through u form a Baer-subline. The tangents of $\mathcal{U}$ at the points of $\mathcal{U}$ on a secant through u are concurrent.*

The restriction $q > 2$ is in fact not a restriction since $\mathrm{PG}(2,4)$ contains, up to isomorphism, only one unital.

Remark 8.1.2 Notice that if $\mathcal{U}$ is a classical unital of $\mathrm{PG}(2,q^2)$, then every point $u \in \mathcal{U}$ satisfies the property described in Theorem 8.1.1.

8.1.2 A new semipartial geometry

In this subsection it will be shown that starting from a Buekenhout-Metz unital it is possible to construct a new semipartial geometry (with the same parameters as $T_2^*(\mathcal{U})$). The proof given is not the original proof as it appeared in [38]. Here the result will arise from a more general theorem on semi-pseudo-ovoids.

Derivation of semi-pseudo-ovoids

Theorem 8.1.3 *Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a semi-pseudo-ovoid consisting of $(n-1)$-dimensional spaces in $\mathrm{PG}(h, q)$, with $\eta = 1 + q^{\frac{h+1}{2}}$. Let τ_i be the tangent space of O at π_i. Suppose that the tangent spaces $\tau_1, \tau_2, \ldots, \tau_s$ have a $\mathrm{PG}(h-2n, q) =: \Gamma$ in common. If $\{\overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_s}\}$ is a set of mutually disjoint $(n-1)$-dimensional spaces covering exactly the same point set as $\pi_1 \cup \pi_2 \cup \ldots \cup \pi_s$, then $\overline{O} = (O \cup \{\overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_s}\}) \setminus \{\pi_1, \pi_2, \ldots, \pi_s\}$ is also a semi-pseudo-ovoid and hence an SPG-regulus satisfying the polar property.*

Proof. Clearly τ_i has empty intersection with the elements of $\overline{O} \setminus \{\pi_i\}$, if $i \notin \{1, 2, \ldots, s\}$. Furthermore it is obvious that the $(h-n)$-dimensional space $\langle \overline{\pi_j}, \Gamma \rangle$ has empty intersection with the elements of $\overline{O} \setminus \{\overline{\pi_j}\}$, $j = 1, 2, \ldots, s$. We conclude that $\overline{O}$ is a generalized semi-pseudo-ovoid with $\left|\overline{O}\right| = 1 + q^{\frac{h+1}{2}}$. Theorem 6.5.3 finishes the proof. □

The semipartial geometry $S(\overline{\mathcal{U}})$.

Suppose that $\mathcal{U}$ is a Buekenhout-Metz unital in $\mathrm{PG}(2, q^2)$ and suppose that $u \in \mathcal{U}$ is as in Theorem 8.1.1. Under the field reduction $\mathrm{PG}(2, q^2) \stackrel{q^2 \mapsto q}{\longrightarrow} \mathrm{PG}(5, q)$, $\mathcal{U}$ will be mapped onto a set $\mathcal{U}'$ consisting of q^3+1 disjoint lines in $\mathrm{PG}(5, q)$. By Remark 2.2.4 and Theorem 6.5.1 $\mathcal{U}'$ is an SPG-regulus satisfying the polar property, such that $S(\mathcal{U}') \cong T_2^*(\mathcal{U})$. If (in $\mathrm{PG}(2, q^2)$) L is a secant line (with respect to $\mathcal{U}$) through u, then the Baer-subline $B := L \cap \mathcal{U}$ will be mapped onto a regulus $R := \{\pi_0, \pi_1, \ldots, \pi_q\}$ in the 3-dimensional subspace of $\mathrm{PG}(5, q)$ which corresponds to L. Further, by Theorem 8.1.1, the tangent spaces $\tau_0, \tau_1, \ldots, \tau_q$ of $\mathcal{U}'$ at $\pi_0, \pi_1, \ldots, \pi_q$ will intersect in a common line Γ. If we denote by $\overline{R} := \{\overline{\pi_0}, \overline{\pi_1}, \ldots, \overline{\pi_q}\}$ the opposite regulus of R and define $\overline{\mathcal{U}} := (\mathcal{U}' \setminus R) \cup \overline{R}$, then we have the following theorem.

Theorem 8.1.4 *The set $\overline{\mathcal{U}}$ of q^3+1 lines of $\mathrm{PG}(5, q)$ is an SPG-regulus satisfying the polar property.*

Proof. This is an immediate consequence of Theorem 8.1.3. □

Remark 8.1.5

- If $\mathcal{U}$ is the classical unital, then $\mathcal{U}'$ is in fact a spread of the classical polar space $Q^-(5,q)$ and in that case it was already shown in [55] by Luyckx and Thas that a regulus may be replaced by its opposite regulus. In all other cases the set $\overline{\mathcal{U}}$ is a new SPG-regulus.
- The fact that the semipartial geometry $S(\overline{\mathcal{U}})$ is not isomorphic to $T_2^*(\mathcal{U})$ will follow from the determination of the automorphism group in the next section, but can also easily be seen in a geometric way, as $T_2^*(\mathcal{U})$ satisfies axiom (A) (see Chapter 3), while it is not difficult to see that $S(\overline{\mathcal{U}})$ does not satisfy this axiom.

8.2 The automorphism group

In this section we will determine the automorphism group of $S(\overline{\mathcal{U}})$, from which will follow that $S(\overline{\mathcal{U}})$ is indeed a new semipartial geometry. (Notice that the only known semipartial geometries $\mathrm{spg}(q^2-1, q^3, q, q^2(q^2-1))$ were $T_2^*(\mathcal{U})$ and those that arise by replacing a regulus by its opposite regulus in a spread of $Q^-(5,q)$, see Remark 8.1.5.)

Here the results from Chapter 7 will turn out to be very useful, as the determination of $\mathrm{Aut}(S(\overline{\mathcal{U}}))$ will take less than one page, whereas the original proof (De Winter and Thas [38]) was much longer and rather tedious.

We will use the same notation as in the previous section. Since $\overline{\mathcal{U}}$ satisfies the polar property, we should, by Corollary 7.3.12, only determine $\mathrm{Aut}(\overline{\mathcal{U}}) \leq \mathrm{P\Gamma L}(6,q)$ in order to find the complete automorphism group of $S(\overline{\mathcal{U}})$.

We start by noticing that, by Corollary 7.3.8, the group fixing all lines of $\mathcal{U}'$ must have order $q+1$, and hence must be C_{q+1} (this group arises from the group of homologies of $\mathrm{AG}(3,q^2)$ with a fixed center, which is a cyclic group, see Lemma 1.4.2).

Lemma 8.2.1 *Every automorphism of $\overline{\mathcal{U}}$ in $\mathrm{P\Gamma L}(6,q)$ stabilizes $\overline{R}$.*

Proof. Notice that $\overline{R}$ is the unique subset Z of $\overline{\mathcal{U}}$ consisting of $q+1$ lines of $\overline{\mathcal{U}}$ that are contained in a $\mathrm{PG}(3,q)$, and has the additional property that for all $\pi, \pi' \in \overline{\mathcal{U}} \setminus Z$ $(\pi \neq \pi')$, $\langle \pi, \pi' \rangle$ intersects either all or no elements of Z (since $\mathcal{U}'$ is q-geometric). This implies that for all $\theta \in \mathrm{Aut}(\overline{\mathcal{U}})$ $\overline{R}^{\theta} = \overline{R}$. □

It follows that every automorphism of $\overline{\mathcal{U}}$ stabilizes R as well, and hence is an automorphism of $\mathcal{U}'$. Conversely, it is clear that every automorphism of $\mathcal{U}'$ that stabilizes R is an automorphism of $\overline{\mathcal{U}}$. Furthermore, by the foregoing we know that $\mathrm{Stab}_{\mathrm{Aut}(\mathcal{U}')}(R) \cong \mathrm{Stab}_{\mathcal{U}}(B) \rtimes C_{q+1}$ (recall that $B \overset{q^2 \mapsto q}{\rightarrow} R$). We have the following theorem.

Theorem 8.2.2 *It holds that* $\mathrm{Aut}(\overline{\mathcal{U}}) \cong \mathrm{Stab}_{\mathcal{U}}(B) \rtimes C_{q+1}$, *with* $\mathrm{Aut}(\overline{\mathcal{U}}) \leq \mathrm{P\Gamma L}(6, q)$.

And consequently we obtain the following theorem.

Theorem 8.2.3 *The full automorphism group of the semipartial geometry* $S(\overline{\mathcal{U}})$ *is given by* $\mathrm{Aut}(S(\overline{\mathcal{U}})) \cong \mathrm{Aut}(\overline{\mathcal{U}}) \rtimes \mathrm{Persp}(\Pi)$, *where* $\mathrm{Persp}(\Pi)$ *is the group of all perspectivities of* $\mathrm{PG}(6, q)$ *with axis the* $\Pi := \mathrm{PG}(5, q)$ *in which* $\overline{\mathcal{U}}$ *is embedded.*

Corollary 8.2.4 *The semipartial geometry* $S(\overline{\mathcal{U}})$ *is not isomorphic to* $T_2^*(\mathcal{U})$.

Proof. This follows immediately from the fact that the respective automorphism groups are not isomorphic. Notice that by Corollary 7.3.12 the automorphism group of $T_2^*(\mathcal{U})$ is isomorphic to the subgroup of $\mathrm{P\Gamma L}(4, q^2)$ induced by $\mathrm{Aut}(\mathcal{U}) \leq \mathrm{P\Gamma L}(3, q^2)$. □

Corollary 8.2.5 *Suppose that* $\mathcal{U}_1$ *is a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$ *constructed from a classical ovoid, and that* $\mathcal{U}_2$ *is a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^2)$ *constructed from a Tits ovoid. Then* $S(\overline{\mathcal{U}}_1) \ncong S(\overline{\mathcal{U}}_2)$.

Proof. This is again an immediate consequence from the fact that the respective automorphism groups cannot be isomorphic. □

Chapter 9

SPG-systems

In this final chapter we will investigate the existence and classification of SPG-systems (see Chapter 2 for the definition and for a link with SPG-reguli). In Thas [76] where SPG-systems were introduced, it was shown that the non-singular polar space P admits an SPG-system if P admits a spread, and that the singular polar space P with radical x admits an SPG-system if P/x admits a spread. Further he classified all SPG-systems of index 1 (Theorem 2.3.6), the classification of SPG-systems of index 0 being trivial. Here we will obtain the classification of SPG-systems of index 2 in the non-singular case and of all SPG-systems in the singular case. In either case it will turn out that only spread-SPG-systems can exist. This chapter is based on De Winter [35] which appeared in "Advances in Geometry".

9.1 Two helpful propositions

The following two propositions will be used frequently in the following sections. The first one generalizes a result for SPG-systems of index 1 that can be found in Thas [76].

Proposition 9.1.1 *Let τ be an SPG-system of a polar space P.*

1. *If P is non-singular of projective index n then each point of P is contained in exactly $(q^n - 1)/(q - 1)$ elements of τ.*
2. *If P is singular of projective index n then each point of P (except the radical) is contained in exactly q^{n-1} elements of τ.*

Proof.

1. Assume τ is an SPG-system of a non-singular polar space P. We give the proof for $P = Q(2n+2,q)$, the other cases are similar. The polar space $Q(2n+2,q)$ contains $q^{n+1}(q^{n+1}-1)/2$ non-singular polar spaces $Q^-(2n+1,q)$ which implies that each point of $Q(2n+2,q)$ is contained in $q^{n+1}(q^n-1)/2$ of these $Q^-(2n+1,q)$. Furthermore straightforward counting shows that each $\mathrm{PG}(n-1,q) \subset Q(2n+2,q)$ is contained in $q^{n+1}(q-1)/2$ of these polar subspaces. Double counting now proves that each point of $Q(2n+2,q)$ is contained in exactly $(q^n-1)/(q-1)$ elements of τ.

2. Assume τ is an SPG-system of a singular polar space P with radical x and let $\mathrm{PG}(d,q)$ be the ambient space of P. Then each element of τ is contained in $(q^{d-n+1}-1)/(q-1)-(q^{d-n}-1)/(q-1) = q^{d-n}$ hyperplanes of $\mathrm{PG}(d,q)$ not containing x. On the other hand a point of $P \setminus \{x\}$ is contained in q^{d-1} hyperplanes not through x. Again straightforward double counting proves that each point of $P \setminus \{x\}$ is contained in q^{n-1} elements of τ.

□

The next proposition provides a characterization of spread-SPG-systems that will be of great importance for our classification.

Proposition 9.1.2 *Let τ be an SPG-system of index $n-1 \geq 2$ of a polar space P. Then τ is a spread-SPG-system if and only if every two intersecting elements of τ intersect in an $(n-2)$-dimensional space.*

Proof. One direction of the lemma is trivial, so assume that τ is an SPG-system such that every two intersecting elements of τ intersect in an $(n-2)$-dimensional space. Let $\pi \in \tau$ and let $\alpha_1, ..., \alpha_k$ be the generators of P through π. Now each element of τ intersecting π should be completely contained in some α_i. Assume that γ_1 and γ_2 are elements of τ such that $\gamma_1 \in \alpha_i$ and $\gamma_2 \in \alpha_j$ with $i \neq j$ and $\gamma_1 \neq \pi \neq \gamma_2$. Since $\gamma_1 \cap \gamma_2 \neq \emptyset$, it follows that $\gamma_1 \cap \pi = \gamma_2 \cap \pi$. This implies that $\langle \gamma_1, \gamma_2 \rangle$ cannot contain a generator of P, contradicting the fact that τ is an SPG-system (recall Theorem 2.3.4). So all elements of τ intersecting π are contained in a unique generator α_i. It now follows easily that τ is a spread-SPG-system. □

9.2 The cases $Q^+(7,q)$ and $H(7,q^2)$

For both the polar spaces $Q^+(7,q)$ and $H(7,q^2)$ the same short proof allows us to obtain the classification of SPG-systems of these polar spaces.

Theorem 9.2.1 *The only SPG-systems of $Q^+(7,q)$ are spread-SPG-systems. The polar space $H(7,q^2)$ does not admit SPG-systems.*

Proof. Let τ be an SPG-system of $Q^+(7,q)$ and assume that $\alpha, \beta_1 \in \tau$ with α and β_1 intersecting in a unique point. Now let $Q(6,q) \subset Q^+(7,q)$ be a non-singular parabolic quadric containing β_1 and let $\beta_2, ..., \beta_{q+1}$ be the q other elements of τ contained in $Q(6,q)$ and intersecting the line $L := \alpha \cap Q(6,q)$. Suppose π_1 and π_2 are the two generators of $Q^+(7,q)$ containing α. Using the fact that $\langle \alpha, \beta_i \rangle$ must contain a generator, it follows that either $\beta_i \cap \pi_1$ or $\beta_i \cap \pi_2$ is a line, for each $i \in \{1, ..., q+1\}$. Since $\pi_j \cap Q(6,q)$ is a plane $\overline{\pi_j}$, with $j \in \{1,2\}$, the above implies that we have found $q+1$ disjoint lines in the union of the two planes $\overline{\pi_1}$ and $\overline{\pi_2}$, clearly a contradiction. We conclude that any two distinct intersecting elements of τ always intersect in a line or, using Proposition 9.1.2, that each SPG-system of $Q^+(7,q)$ is a spread-SPG-system.

Using the same technique on $H(7,q^2)$ (here one constructs q^2+1 disjoint lines in the union of $q+1$ planes), we find that $H(7,q^2)$ can only admit spread-SPG-systems. But since $H(7,q^2)$ does not admit a spread, see Thas [73], this implies that $H(7,q^2)$ does not admit an SPG-system. □

Remark 9.2.2 Note that $Q^+(7,q)$ contains a spread if q is even as well as in the case that q is odd, except when q is odd, with $q \equiv 1 \pmod 3$ and not prime, in which case the existence of a spread of $Q^+(7,q)$ is still open, see for instance Dye [41]. See also Remark 2.2.10.

9.3 The case $Q(8,q)$

Because the polar space $Q(8,q)$ contains in a way more generators through a singular plane than $Q^+(7,q)$ and $H(7,q^2)$, the elementary technique from the previous section does not work for $Q(8,q)$. A more involved method will be needed. In a first step we will study hypothetical SPG-systems of $Q(8,q)$ locally. In the second step these results will be used to study the global structure of such an SPG-system.

9.3.1 Determination of the local structure

Let p be a point of the polar space P and let τ be an SPG-system of P. Then we define *the local structure of* τ *at* p as the structure in the polar space P/p, induced by the elements of τ containing p. Note that P/p is contained in $T_p(P)/p$, with $T_p(P)$ the tangent space of P at p if θ is not symplectic and with $T_p(P) = p^\theta$ in the symplectic case, where θ is the polarity defining P.

By Proposition 9.1.1 and Theorem 2.3.4 (ii) the local structure of τ at any point of $Q(8,q)$ consists of q^2+q+1 lines $L_0, ..., L_{q^2+q}$ contained in a non-singular parabolic quadric $Q(6,q)$ with the property that $\langle L_i, L_j\rangle$ contains a generator of $Q(6,q)$ for every $i, j \in \{0, ..., q^2+q\}$ with $i \neq j$.

Assume that all lines L_i are two by two disjoint. Then for every pair $\{L_i, L_j\}$ there is a unique line M_{ij} with $M_{ij} \cap L_i \neq \emptyset \neq M_{ij} \cap L_j$ and such that $\langle L_i, M_{ij}\rangle$ and $\langle L_j, M_{ij}\rangle$ are planes of $Q(6,q)$. Furthermore it is clear that if a line M_{ij} intersects the line L_k, with $i \neq j \neq k \neq i$, there holds that $M_{ij} = M_{ik} = M_{jk}$ (the span of L_i, L_j and L_k in $Q(6,q)$ must be a cone $M_{ij}Q(2,q)$). So in fact, if a line L_i meets a line M_{kl} the two span a plane of $Q(6,q)$ and $M_{kl} = M_{ij}$ for precisely q different indices j. We now consider the incidence structure $\mathbb{S} = (\mathbb{P}, \mathbb{L}, \mathrm{I})$, with $\mathbb{P}$ the set of lines L_i, $\mathbb{L}$ the set of lines M_{ij} and with I the incidence relation defined by having nonempty intersection. We will call the elements of $\mathbb{P}$ the $\mathbb{S}$-points, while the elements of $\mathbb{L}$ will be called the $\mathbb{S}$-lines. We will show that $\mathbb{S}$ is a projective plane. It is clear that the number of $\mathbb{S}$-points is q^2+q+1 and that two different $\mathbb{S}$-points define a unique $\mathbb{S}$-line. Consider the line L_0 of $Q(6,q)$ and suppose that the point x of L_0 is contained in both M_{0i} and M_{0j} with $M_{0i} \neq M_{0j}$. This implies that $\langle L_i, L_j\rangle \subset x^\perp$, giving a $\mathrm{PG}(3,q) = \langle L_i, M_{ij}, x\rangle \subset Q(6,q)$, a contradiction. Hence an $\mathbb{S}$-point L_i is incident with at most $q+1$ $\mathbb{S}$-lines. Since an $\mathbb{S}$-line contains at most $q+1$ $\mathbb{S}$-points and each $\mathbb{S}$-point is collinear with each $\mathbb{S}$-point we easily see that each $\mathbb{S}$-point is incident with $q+1$ $\mathbb{S}$-lines and each $\mathbb{S}$-line is incident with $q+1$ $\mathbb{S}$-points. From this it follows trivially that two $\mathbb{S}$-lines must always intersect. Hence we find from the above that $\mathbb{S}$ is a projective plane of order q. The flag geometry of $\mathbb{S}$ (see Chapter 1, Section 1.6) has as point set the set of points on the lines L_i and as line set the set of lines L_i and M_{ij}. This flag-geometry is embedded in $Q(6,q)$. By Thas and Van Maldeghem [79] all such embeddings in $\mathrm{PG}(6,q)$ are isomorphic and moreover q must be a power of 3.

We call a point of $Q(8,q)$ of *type H* if its local structure consists of q^2+q+1 disjoint lines. Hence, we have proved the following lemma.

Lemma 9.3.1 *If τ is an SPG-system of $Q(8,q)$ such that there exists a point of type H, then q is a power of 3 and the incidence structure $\mathbb{S}$ defined as above is a projective plane.*

Remark 9.3.2 A set of q^2+q+1 lines as above can easily be constructed as follows (see Thas and Van Maldeghem [79]). Let q be a power of 3 and consider the embedding of the classical hexagon $H(q)$ in $Q(6,q)$. Now let $Q^+(5,q)$ be a non-singular hyperbolic quadric contained in $Q(6,q)$. It is well known that $Q^+(5,q)$ contains two disjoint planes, called *ideal planes*, each consisting of q^2+q+1 points that are two by two at distance four in $H(q)$. Since q is a power of three, $H(q)$ is self-dual, so there exist in $H(q)$ two sets of q^2+q+1 mutually disjoint lines, where any two distinct lines of such a set are at distance four in $H(q)$. In fact it is even clear that if we call the lines in one set L_i (with $i \in \{0,...,q^2+q\}$) the other set consists of the lines M_{ij}.

Now we shall investigate the possible configurations for the local structure if two of its lines intersect. A first possibility is clearly that the q^2+q+1 lines are the lines of a plane of $Q(6,q)$. We will call a point with such a local structure a point of *type S*. For the rest of this section we assume that the local structure is not of type S.

Lemma 9.3.3 *For every two intersecting lines in the local structure at a point of an SPG-system τ on $Q(8,q)$, the $q+1$ lines in the pencil defined by these two lines belong to the local structure.*

Proof. Let π_1 and π_2 be elements of τ such that $\pi_1 \cap \pi_2$ is a line M and let $\alpha_0 = \langle \pi_1, \pi_2 \rangle, ..., \alpha_q$ be the $q+1$ generators of $Q(8,q)$ through π_1 and let $\omega_0 = \alpha_0, ..., \omega_q$ be the generators through π_2. Furthermore take a third plane π_3 through M in α_0 and consider a non-singular elliptic quadric $Q^-(7,q) \subset Q(8,q)$ containing π_3. Assume by way of contradiction that $\pi_3 \notin \tau$. The $2q+1$ generators $\alpha_0, ..., \alpha_q, \omega_1, ..., \omega_q$ will intersect $Q^-(7,q)$ in $2q+1$ planes $\overline{\alpha_0} = \overline{\omega_0} = \pi_3, ..., \overline{\omega_q}$ containing M. Now let $m \in M$ and call π_m the element of τ through m in $Q^-(7,q)$. The plane π_m must then contain a line N_1 of $\overline{\alpha_0} \cup ... \cup \overline{\alpha_q}$ different from M (since $\langle \pi_1, \pi_m \rangle$ must contain a generator), as well as a line N_2 of $\overline{\omega_0} \cup .. \cup \overline{\omega_q}$ different from M. This is only possible if $N_1 = N_2 \subset \overline{\alpha_0}$ (since otherwise one could construct a $\mathrm{PG}(3,q)$ on $Q^-(7,q)$). Analogously we find for a point $m' \in M \setminus \{m\}$ a line N' in $\overline{\alpha_0}$. Since N_1 and N' must have nonempty intersection, we find a contradiction. This proves that the $q+1$ planes through M in $\langle \pi_1, \pi_2 \rangle$ belong to τ, hence the lemma. $\square$

Suppose the point l is not of type H and not of type S, so without loss of generality we may assume that in its local structure the lines L_0,...,L_q form a pencil of lines in the plane $\langle L_0, L_1\rangle$. Using the previous lemma it is evident that no other L_i (with $i \in \{q+1, ..., q^2+q\}$) can be contained in this plane. There are two cases.

(i) For all $i \in \{q+1, ..., q^2+q\}$, $L_i \cap \langle L_0, L_1\rangle = \emptyset$. If p is the vertex of the pencil, then $L_i \subset p^\perp$ (since each of the lines $L_0, ..., L_q$ must contain a point collinear with all points of L_i) and L_j with $j \notin \{0, ...q, i\}$ intersects $\langle p, L_i\rangle$ in a unique point. The last assertion follows from the fact that $L_j \not\subset \langle p, L_i\rangle$ (since otherwise the previous lemma would imply that there is a line L_k through p not in $\langle L_0, L_1\rangle$, leading to a $\mathrm{PG}(3, q) \subset Q(6, q)$, a contradiction) and the fact that $L_j \cap \langle p, L_i\rangle \neq \emptyset$ (since otherwise $p \notin \langle L_i, L_j\rangle \subset p^\perp$ would lead to a $\mathrm{PG}(3, q) \subset Q(6, q)$). Now if $L_i \cap L_j \neq \emptyset$ we would find a $\mathrm{PG}(3, q) \subset Q(6, q)$, a contradiction. So we see that each point of $\langle p, L_i\rangle \setminus (\{p\} \cup L_i)$ is contained in a unique L_j. Put $s := L_k \cap \langle p, L_i\rangle$ and $t := L_j \cap \langle p, L_i\rangle$ with L_k and L_j chosen in such a way that p, s and t are not collinear. It is clear that $L_k \not\subset t^\perp$ since otherwise we would have $\langle p, t\rangle \subset L_k^\perp$. So there exists a $t' \in L_j \setminus \{t\}$ and an $s' \in L_k \setminus \{s\}$ such that $L_k \subset t'^\perp$ and $L_j \subset s'^\perp$. This implies that $\langle t, t', s'\rangle \subset s^\perp$, a contradiction since $s \notin \langle t, t', s'\rangle$. Hence this case cannot occur.

(ii) Assume $L_{q+1} \cap L_0$ is a point r. This means that the pencil of lines through r in $\langle L_0, L_{q+1}\rangle$ belongs to the local structure. If $L_j \cap \langle L_0, L_1\rangle = \emptyset = L_j \cap \langle L_0, L_{q+1}\rangle$ for a certain line L_j we would have $L_j \subset \langle p, r\rangle^\perp$, with p the vertex of the pencil determined by L_0 and L_q, a contradiction. So without loss of generality we may assume that there is a line L_j intersecting $\langle L_0, L_1\rangle$ in a point r'. Now suppose $r' \notin L_0$, without loss of generality we may assume $r' \in L_1$. Clearly $L_j \cap L_{q+1} = \emptyset$. Then there exists a point $t \in L_j$ with $L_{q+1} \subset t^\perp$. We now see that $t = r'$ (respectively $t \neq r'$) leads to the contradiction $\langle L_0, L_{q+1}\rangle \subset t^\perp$ (respectively $\langle L_0, L_1\rangle \subset t^\perp$). Hence for any line L_j not belonging to $\langle L_0, L_1\rangle$ and $\langle L_0, L_{q+1}\rangle$ we have $r' \in L_0 \setminus \{p, r\}$ and the only possibility that is left for the local structure is the following one. Let $\alpha_0, ..., \alpha_q$ be the $q+1$ planes of $Q(6, q)$ through a line L and let $r_0, ..., r_q$ be the $q+1$ points of L. Then the local structure consists of the pencils of lines through r_i in α_i with $i = 0, ..., q$. A point of $Q(8, q)$ with such a local structure will be said to be of *type O*. In fact we have proved the following lemma.

Lemma 9.3.4 *A point of $Q(8,q)$ is either of type S, type O or type H. If there exists a point of type H then q is a power of 3.*

9.3.2 The global structure

If a point x of $Q(8,q)$ is of type O then from the previous section we know that there is a special element of τ through x that we will denote by ξ_x such that for each generator α_i containing ξ_x there is a unique line M_i $(i = 0, ..., q)$ through x in ξ_x with the property that all planes in α_i containing M_i belong to τ. Furthermore we will denote the union of all points in all generators containing ξ_x by $\widetilde{x}$.

Lemma 9.3.5 *Let τ be an SPG-system of $Q(8,q)$. If there is a point p of type S in $Q(8,q)$, then no point of type O can exist.*

Proof. Let Σ be the $\mathrm{PG}(3,q)$ spanned by the elements of τ through p and assume that no other point in Σ is of type S. Then a point $x \in \Sigma \setminus \{p\}$ is necessarily of type O and since the line $\langle x,p\rangle$ is already contained in $q+1$ elements of τ it follows that one of these planes must be ξ_x. This implies that $p \in \xi_x$ for all $x \in \Sigma \setminus \{p\}$. Consequently there are distinct points $x, y \in \Sigma \setminus \{p\}$ with the property that $\xi_x = \xi_y$. Consider a line L_1 through x in ξ_x not through y or p. Then L_1 determines a 3-dimensional space M containing ξ_x with the property that all planes through L_1 in M belong to τ. Now there must be a line L_2 through y in ξ_x such that all planes through L_2 in M belong to τ. With $z := L_1 \cap L_2$ we see that M contains $2q+1$ elements of τ through z. Hence z is of type S, a contradiction since $z \in \Sigma$.

So we may assume that in Σ there is a second point n of type S. An arbitrary point $z \in \Sigma \setminus \langle p,n\rangle$ will then be contained in $2q+1$ elements of τ that are completely contained in Σ. Hence z is of type S. It now easily follows that all points of Σ are of type S (that is, all planes of Σ belong to τ). Suppose now that $x \notin \Sigma$ is of type O. Then we know that each point of $\widetilde{x}$ is contained in an element of τ through x. Since $\widetilde{x} \cap \Sigma \neq \emptyset$ we find a point in Σ that would be contained in q^2+q+2 elements of τ, a contradiction. □

Lemma 9.3.6 *Let τ be an SPG-system of $Q(8,q)$. If there exists one point of type O, then all points of $Q(8,q)$ will be of type O.*

Proof. Suppose that x and y are two distinct points of type O such that $\xi_x = \xi_y$. Like in the previous lemma this leads to a point of type S, contradicting the conclusion of the previous lemma. So for all x and y of type O we have that $\xi_x \neq \xi_y$.

Using Lemma 9.3.5 it is evident that $p \in \xi_l$ for certain l of type O implies that p is of type O. The line $\langle p, l\rangle$ then determines a solid Σ. Since each point $x \in \langle p, l\rangle$ is contained in the $q+1$ elements of τ through $\langle p, l\rangle$ in Σ it follows that one of these planes must be ξ_x, hence each plane through $\langle p, l\rangle$ in Σ is of the form ξ_x for a certain $x \in \langle p, l\rangle$. This implies that each point of $\widetilde{l}$ is of type O. Let z be a point outside $\widetilde{l}$ and suppose $z \in p^\perp$ with $p \in \xi_l$. Then there will be a line $\langle p, n\rangle \in \xi_p$ such that $\langle p, n\rangle \subset z^\perp$. Let Ω be the $\mathrm{PG}(3, q)$ through ξ_p determined by $\langle p, n\rangle$, then it is clear that if $z \in \Omega$ we can use the foregoing to see that z will be of type O; so assume that $z \notin \Omega$. Now there is a plane $\alpha \supset \langle p, n\rangle$ in Ω such that $\alpha \subset z^\perp$. Since we know that α is of the form ξ_y for certain $y \in \langle p, n\rangle$, we see that $z \in \widetilde{y}$, proving that z is of type O. $\square$

Lemma 9.3.7 *Let τ be an SPG-system of $Q(8, q)$ with the property that each point is of type O. Then the incidence structure $\mathbb{O} = (P, B, \mathrm{I})$ with P the point set of $Q(8, q)$, with $B = \{\text{lines } L \text{ of } Q(8, q) \| (\exists x)(x \in L \subset \xi_x)\}$ and with I the natural incidence is a generalized octagon.*

Proof. First of all notice that $a \in \xi_b$ implies that $b \in \xi_a$. We recall from the previous lemma that for $x \in L \subset \xi_x$ (x a point of type O) the map $y \mapsto \xi_y$ is a bijection from the points of L to the planes on L in the solid determined by x and L.

First we show that $x \notin z^\perp$ implies that $\xi_x \cap \widetilde{z} = \emptyset$. Assume that $u \in \xi_x \cap \widetilde{z}$. If $u \in \xi_z$ we would find $x \in \xi_u \subset \widetilde{z}$, a contradiction. If $u \notin \xi_z$ it follows that $u \in \xi_y$ for certain $y \in \xi_z$. This implies that $\xi_u = \langle u, y, x\rangle$. Since $\xi_u \subset \widetilde{y}$, $\xi_y \subset x^\perp$ and so $z \in x^\perp$, a contradiction.

Let z be an arbitrary point of $Q(8, q)$ and let d be the distance function in the incidence graph of $\mathbb{O}$. It is clear that if $x \in \xi_z$, $x \neq z$, then $d(x, z) = 2$ and for all points u collinear with x in $\mathbb{O}$ but not on the line $\langle x, z\rangle$, $d(u, z) > 2$.

If $x \in \widetilde{z} \setminus \xi_z$ we see that x belongs to a unique plane ξ_y with $y \in \xi_z$, hence $d(x, z) = 4$, the path is unique and for all u collinear with x in $\mathbb{O}$ but not on the line $\langle x, y\rangle$, $d(u, z) > 4$ (the last assertion since $\xi_x \cap \widetilde{z} = \langle x, y\rangle$, i.e. all points collinear with x not on $\langle x, y\rangle$ lay outside $\widetilde{z}$).

In the case $x \notin \widetilde{z}$ and $x \in z^\perp$ there is a unique line $\langle z, y\rangle \subset \xi_z$ with $\langle z, y\rangle \subset x^\perp$ and $\xi_y \subset x^\perp$, i.e. there is a unique $y \in \xi_z$ such that $x \in \widetilde{y}$. Hence $d(x, z) = 6$ and there is a unique path of this length.

Finally assume that $x \notin z^{\perp}$. Then we know that $\xi_x \cap \widetilde{z} = \emptyset$ and that each line through x contains a point collinear with z in $Q(8, q)$, in other words $d(x, z) = 8$. From this it follows that the diameter of the incidence graph of $\mathbb{O}$ is 8.

From the uniqueness of the paths of length $2, 4$ and 6 it is easy to conclude that no circuits of length smaller then 14 can occur. Assume that $z \sim y_1 \sim y_2 \sim x \sim y_3 \sim y_4 \sim y_5 \sim z$ would determine a path of length 14, with $\sim$ denoting collinearity in $\mathbb{O}$. Using the foregoing we see that $d(x, z) = 6$. Furthermore we notice that $\xi_x = \langle x, y_2, y_3 \rangle$ and that $\{x, y_2, y_3\} \subset z^{\perp}$ (since $d(y, z) \leq 6$ implies that $y \in z^{\perp}$). This implies that $z \in \widetilde{x}$ and so $d(x, z) = 4$, a contradiction. Hence circuits have minimal length 16.

So we have proved that the incidence graph of $\mathbb{O}$ has diameter 8 and girth 16, in other words that $\mathbb{O}$ is a generalized octagon (see Theorem 1.6.5). $\square$

Proposition 9.3.8 *If τ is an SPG-system of $Q(8, q)$, then there cannot exist a point of type O.*

Proof. This follows immediately from the Lemmas 9.3.6, 9.3.7 and the fact that no generalized octagon of order $s > 1$ can exist, see Theorem 1.6.3. $\square$

Now we will show that points of type H cannot exist either. We use the same notations as mentioned before.

Lemma 9.3.9 *Let $\mathcal{O}$ be the point set of the local structure $L_0, ..., L_{q^2+q}$ of a point of type H. Then if $|\mathcal{O} \cap \alpha| > q + 1$ for a plane $\alpha \subset Q(6, q)$, then α contains a line L_i.*

Proof. Assume that α does not contain a line L_i. It then follows from Lemma 9.3.1 that in this case α cannot contain a line M_{ij}. Assume that there exists a line K of α containing at least three distinct points x_1, x_2 and x_3 of $\mathcal{O}$. Furthermore let $x_1 \in L_1, x_2 \in L_2$ and $x_3 \in L_3$. Since there is a point on L_1 collinear with all points of L_2 we find that either $\langle L_1, K \rangle$ or $\langle L_2, K \rangle$ is a plane of $Q(6, q)$. So we may assume that $\langle L_1, K \rangle$ is contained in $Q(6, q)$. This plane will intersect exactly q other lines L_i along the line M_{12}. Since K is contained in this plane and intersects L_1, L_2 as well as L_3 it follows that $K = M_{12}$, a contradiction. From this follows that a line of α contains at most two points of $\mathcal{O}$, in other words, $\alpha \cap \mathcal{O}$ is a (partial) oval in α. Since q is a power of 3 (Lemma 9.3.1), so certainly not even, we see that $|\alpha \cap \mathcal{O}| \leq q + 1$. $\square$

Proposition 9.3.10 *The quadric $Q(8,q)$ cannot contain a point of type H with respect to an SPG-system τ.*

Proof. Let x be a point of type H and let α be an element of τ containing x. It is clear that each point of α must be of type H. Now let Σ be a $\mathrm{PG}(3,q) \subset Q(8,q)$ containing α. By looking at the local structure, each point of α is contained in q elements of τ that intersect Σ in a line; also, α and each of these q planes of τ intersect a common plane of $Q(8,q)$ in a line. In this way each point of α determines q^2 points of $\Sigma \setminus \alpha$. Since $q^2(q^2+q+1)/q^3 > q+1$, this means that there is a point n in $\Sigma \setminus \alpha$ with the property that more than $q+1$ elements of τ through n have nonempty intersection with α. The previous lemma (notice that n must be of type H) now implies that there is an element of τ through n that intersects α in a line L. Hence a point of L can never be of type H, a contradiction. $\square$

Bringing together the results of this section, we have proved the following theorem.

Theorem 9.3.11 *The only SPG-systems of $Q(8,q)$ are spread-SPG-systems.*

Proof. Immediate from Propositions 9.3.8 and 9.3.10. $\square$

Remark 9.3.12 Note that $Q(8,q)$ with q odd does not admit a spread (see for example Thas [73] and Thas [75]; see also Remark 2.2.10) and hence this theorem implies that only $Q(8,2^h)$ admits an SPG-system.

9.4 The singular case

In this section we will classify the SPG-systems of the singular polar spaces P (having as radical a unique point) for every $n \in \mathbb{N}_0$. One easily sees that a same technique as used in the cases $Q^+(7,q)$ and $H(7,q^2)$ would provide a classification in the case P has projective index 3, but since here an element of τ is contained in a unique generator of P we can do better.

Theorem 9.4.1 *The only SPG-systems of a singular polar space P of projective index n ($n \in \mathbb{N}_0$) are spread-SPG-systems.*

Proof. Note that the cases $n \in \{1,2\}$ were already handled in Thas [76]; the following proof will work for general n.

Let $\mathrm{PG}(d,q)$ be the ambient space of P, let α be an element of τ and let P' be a polar subspace of P induced by a $\mathrm{PG}(d-1,q)$ not containing α nor the radical of P. If Σ is the generator of P containing α, then all elements γ of τ contained in P' and intersecting $\beta := \alpha \cap P'$ in a $\mathrm{PG}(i-1,q) =: K$ will intersect $\Omega := \Sigma \cap P'$ in a $\mathrm{PG}(i,q) =: L$ (since $\langle \gamma, \alpha \rangle$ must contain a generator of P, see Theorem 2.3.4). So K determines q^i points of $\Omega \setminus \beta$. We will then say that the average number of points of $\Omega \setminus \beta$ determined by a point of K is $\delta_i := q^i(q-1)/(q^i-1)$. If we define δ to be the average number of points of $\Omega \setminus \beta$ determined by a point of β we must have $\delta \leq \delta_{n-1} = q^{n-1}(q-1)/(q^{n-1}-1)$ (since all elements of τ in P' intersecting β have to be disjoint). We want to prove that there is a unique element σ of τ in P' that contains β; so assume the contrary, i.e. for no point of β the average number of points of $\Omega \setminus \beta$ determined by that point equals δ_{n-1}. If we define a_i to be the number of points in β that determine δ_i points in $\Omega \setminus \beta$ we find

$$\delta = \frac{q-1}{q^{n-1}-1} \sum_{i=1}^{n-2} a_i \delta_i > \frac{q-1}{q^{n-1}-1} \delta_{n-1} \sum_{i=1}^{n-2} a_i = \delta_{n-1},$$

clearly a contradiction. It follows that all SPG-systems of P are as constructed in Thas [76], that is, are all spread-SPG-systems. $\square$

Appendix A

Nederlandstalige samenvatting

In deze appendix is het ons doel om in het Nederlands een overzicht te geven van de in deze dissertatie bekomen en behandelde resultaten. Het ligt echter niet in onze bedoeling om hier een Nederlandse vertaling van het voorgestelde doctoraat te presenteren. Zo zullen de belangrijkste stellingen zonder bewijs voorgesteld worden, of hoogstens met een summiere schets ervan. Wel zullen we in deze synopsis trouw de structuur van de Engelstalige tekst volgen. Tenslotte merken we nog op dat we voor bepaalde wiskundige termen, die nu eenmaal natuurlijker klinken in de Engelse taal, geen Nederlandstalige vertaling gebruiken.

A.1 Inleiding

In het eerste hoofdstuk worden een aantal goed gekende zaken herhaald die van verder nut voor dit proefschrift zijn. Zo worden kort designs, partieel lineaire ruimten, projectieve en polaire ruimten en veralgemeende veelhoeken, alsook de belangrijke stelling van Frobenius, behandeld. Voor deze samenvatting veronderstellen we dat de lezer vertrouwd is met deze concepten en de gebruikelijke notaties of verwijzen we naar hoofdstuk 1 (pagina 1 t.e.m. 12 van deze thesis).

A.2 Partiële en semipartiële meetkunden

In het tweede hoofdstuk introduceren en bespreken we die concepten die aan de basis liggen van het eigenlijke onderzoek waarvan deze thesis een neerslag is: (semi)partiële meetkunden en hun constructies via SPG-reguli en SPG-systemen.

A.2.1 (Semi)partiële meetkunden

Definitie A.2.1

- Een *partiële meetkunde* $\mathrm{pg}(s,t,\alpha)$ is een partieel lineaire ruimte van de orde (s,t) met de eigenschap dat voor elk niet-incident punt-rechte paar (p,L) er exact $\alpha > 0$ rechten M bestaan zodanig dat M incident is met p en concurrent is met L.
- Een *semipartiële meetkunde* $\mathrm{spg}(s,t,\alpha,\mu)$ is een partieel lineaire ruimte van de orde (s,t) zodanig dat
  - voor elk niet-incident punt-rechte paar (p,L) er ofwel geen, ofwel exact $\alpha > 0$ rechten M bestaan zodat M incident is met p en concurrent is met L;
  - voor elk niet-collineair puntenpaar $\{p,q\}$ er juist $\mu > 0$ punten r bestaan die collineair zijn met zowel p als q.

Partiële meetkunden werden geïntroduceerd door Bose [6] en semipartiële meetkunden door Debroey en Thas [18]. Een $\mathrm{pg}(s,t,t)$ wordt een *Bruck-net* genoemd, een partiële meetkunde waarvoor $1 < \alpha < \min(s,t)$ een *eigenlijke partiële meetkunde*, een semipartiële meetkunde die geen partiële meetkunde is een *eigenlijke semipartiële meetkunde*, en tenslotte een eigelijke semipartiële meetkunde met $\alpha = 1$ een *partiële vierhoek* (zie Cameron [14]).

A.2.2 SPG-reguli

Voor de definitie van *(veralgemeende) lineaire representatie* verwijzen we de lezer naar de Engelse tekst (hoofdstuk 2, pagina 16).

Definitie A.2.2 Een *SPG-regulus* $\mathcal{R}$ in $\mathrm{PG}(n,q)$ is een verzameling $\{PG^{(0)}(m,q),\ldots,\mathrm{PG}^{(t)}(m,q)\}$ bestaande uit $t+1 \geq 2$ disjuncte m-dimensionale deelruimten van $\mathrm{PG}(n,q)$ die voldoen aan de volgende eigenschappen.

- Elke $\mathrm{PG}(m+1,q)$ die $\mathrm{PG}^{(i)}(m,q)$, $i = 0,\ldots,t$, bevat, heeft een punt gemeen met ofwel geen, ofwel juist $\alpha > 0$ elementen van $\mathcal{R}\backslash\{\mathrm{PG}^{(i)}(m,q)\}$. Indien $\mathrm{PG}(m+1,q)$ geen elementen van $\mathcal{R} \setminus \{\mathrm{PG}^{(i)}(m,q)\}$ snijdt, dan wordt $\mathrm{PG}(m+1,q)$ een $(m+1)$-*dimensionale raakruimte* genoemd.

- Elk punt van $\mathrm{PG}(n, q)$ dat niet bevat is in een element van $\mathcal{R}$ is bevat in een constant aantal θ $(m+1)$-dimensionale raakruimten.

In [74] introduceerde Thas SPG-reguli en bewees hij de volgende stelling.

Stelling A.2.3 ([74]) *De veralgemeende lineaire representatie van een SPG-regulus is een semipartiële meetkunde* $\mathrm{spg}(q^{m+1}-1, t, \alpha, (t+1-\theta)\alpha)$.

In hoofdstuk 2 worden daarna de belangrijkste voorbeelden van SPG-reguli besproken, alsook enkele karakterisatiestellingen voor de bekomen meetkunden.

A.2.3 SPG-systemen

Definitie A.2.4 Zij $Q(2n+2, q)$, $n \geq 1$, een niet-singuliere parabolische kwadriek in $\mathrm{PG}(2n+2, q)$. Een *SPG-systeem van* $Q(2n+2, q)$ is een verzameling τ van $(n-1)$-dimensionale totaal singuliere deelruimten van $Q(2n+2, q)$ met de eigenschap dat voor elke niet-singuliere elliptische kwadriek $Q^-(2n+1, q) \subset Q(2n+2, q)$ de elementen van τ op $Q^-(2n+1, q)$ een spread vormen van deze kwadriek.

Voor de definitie van een SPG-systeem van $Q^+(2n+1, q)$, respectievelijk $H(2n+1, q^2)$, dient men enkel "niet-singuliere elliptische kwadriek $Q^-(2n+1, q) \subset Q(2n+2, q)$" te vervangen door "niet-singuliere parabolische kwadriek $Q(2n, q) \subset Q^+(2n+1, q)$", respectievelijk "niet-singuliere hermitische variëteit $H(2n, q^2) \subset H(2n+1, q^2)$".

Zij P een singuliere polaire ruimte met omgevende ruimte $\mathrm{PG}(d, q)$ en met als radicaal het unieke punt x. Onderstel dat $n > 0$ de projectieve index van P is. Een *SPG-systeem van* P is een verzameling τ van $(n-1)$-dimensionale totaal singuliere deelruimten van P die x niet bevatten, met de eigenschap dat de elementen van τ een spread induceren in elke polaire ruimte $P' \subset P$ bevat in een willekeurige $\mathrm{PG}(d-1, q) \subset \mathrm{PG}(d, q)$ die x niet bevat.

In alle gevallen wordt de dimensie $n-1$ van de elementen van τ de *index* van het SPG-systeem genoemd.

Merk op dat de SPG-systemen van index 0 triviaal zijn, dat is, ze bestaan uit alle punten van de polaire ruimte indien deze niet singulier is, en ze bestaan uit alle punten van de polaire ruimte min het radicaal indien deze singulier is.

In [76] introduceerde Thas SPG-systemen en toonde aan dat deze aanleiding geven tot de constructie van semipartiële meetkunden. Verder bekwam hij in [76] ook volgende constructie en classificatiestelling.

Stelling A.2.5 ([76]) *Zij $\mathcal{S}$ een spread van $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q^2)\}$. Dan is de verzameling τ van alle $(n-1)$-dimensionale deelruimten bevat in de elementen van $\mathcal{S}$ een SPG-systeem van P.*

Zij P een singuliere polaire ruimte met omgevende ruimte $\mathrm{PG}(d,q)$ en met als radicaal het unieke punt x. Zij $n > 0$ de projectieve index van P. Onderstel dat $\mathcal{S}$ een verzameling van n-dimensionale totaal singuliere deelruimten van P is met de eigenschap dat de elementen van $\mathcal{S}$ een spread induceren in elke polaire ruimte $P' \subset P$ bepaald door een $\mathrm{PG}(d-1,q) \subset \mathrm{PG}(d,q)$ die x niet bevat. De verzameling τ van alle $(n-1)$-dimensionale deelruimten van de elementen van $\mathcal{S}$ die x niet bevatten, is een SPG-systeem van P.

SPG-systemen geconstrueerd zoals in bovenstaande stelling worden *spread-SPG-systemen* genoemd.

Stelling A.2.6 ([76]) *Er zijn juist twee klassen van SPG-systemen van index 1, en beide zijn SPG-systemen van $Q(6,q)$. De een is een spread-SPG-systeem, de andere bestaat uit alle rechten van de klassieke veralgemeende zeshoek $H(q)$ ingebed in $Q(6,q)$.*

De meetkunde $\mathrm{TQ}(4,q)$ is de semipartiële meetkunde geconstrueerd uit het unieke SPG-systeem van $Q(4,q)$ en werd eerder geconstrueerd door Metz (niet gepubliceerd).

In de Engelse tekst wordt tenslotte ook het verband tussen SPG-systemen en SPG-reguli besproken.

A.2.4 Andere semipartiële meetkunden

In de laatste sectie van Hoofdstuk 2 worden tenslotte die (semi)partiële meetkunden besproken die niet geconstrueerd worden via een SPG-regulus of een SPG-systeem, maar die toch op zeker punt in deze dissertatie opduiken. De belangrijkste voorbeelden hier zijn de copolaire ruimten en de classificatiestelling voor deze meetkunden door Hall [44].

A.3 Net-induceerbare semipartiële meetkunden

Het doel van het derde hoofdstuk is het veralgemenen van een karakterisatie van de partiële meetkunde $T_2^*(\mathcal{K})$ die bekomen werd door De Clerck, De Soete en Gevaert [24]. De resultaten van dit hoofdstuk werden gepubliceerd in De Clerck, De Winter en Thas [25].

A.3.1 Inleiding

Wanneer men de lineaire representatie van eender welke (semi)partiële meetkunde S in $\mathrm{AG}(n,q)$ beschouwt, dan is het duidelijk dat het parallellisme van $\mathrm{AG}(n,q)$ op natuurlijke wijze een parallellisme induceert op de rechten van S. Dit is de reden waarom in de tweede sectie van hoofdstuk 3 parallellisme in semipartiële meetkunden bestudeerd wordt. Verder ziet men onmiddellijk dat in elk vlak $\beta \subset \mathrm{AG}(n,q)$ dat twee snijdende rechten van S bevat een Bruck-net van orde q en graad $\alpha+1$ geïnduceerd wordt door S. Bijgevolg bevat S "zeer veel" dergelijke netten en kan men zich afvragen of S kan gekarakteriseerd worden door de aanwezigheid van deze netten. Een eerste resultaat dat in deze richting bekomen werd is de hoger vermelde karakterisatie van de partiële meetkunde $T_2^*(\mathcal{K})$. In hoofdstuk 3 bekomen we een veralgemening die ook de semipartiële meetkunden $T_2^*(\mathcal{U})$ en $T_2^*(\mathcal{B})$ in deze geest karakteriseert.

A.3.2 Parallellisme

Definitie A.3.1 Zij S een $\mathrm{spg}(s,t,\alpha,\mu)$ met $\alpha > 1$. We zeggen dat twee rechten L en L' *parallel* zijn als zich één van de volgende situaties voordoet.

- $L = L'$;
- L en L' zijn twee verschillende niet-snijdende rechten en geen enkel punt van L is collineair met een punt van L';
- L en L' zijn twee verschillende niet-snijdende rechten en er bestaan twee verschillende transversalen B en B' van L en L' die elkaar snijden in een punt $b \notin L \cup L'$.

Merk op dat we met transversaal van twee verschillende (mogelijk snijdende) rechten L en M, elke rechte bedoelen die beide rechten snijdt en niet het mogelijke snijpunt van L en M bevat.

Het volgende axioma dat werd geïntroduceerd in [24] is van groot belang in hoofdstuk 3.

> AXIOMA (A). *Zij L en M eender welke twee verschillende snijdende of parallelle rechten van S en zij p eender welk punt op een transversaal van L en M, met $p \notin L \cup M$. Dan is p bevat in juist $\alpha - |L \cap M|$ elementen van $\{L, M\}^{\perp}$.*

Elke semipartiële meetkunde waarvoor $\alpha > 1$ die voldoet aan axioma (A) wordt een *net-induceerbare semipartiële meetkunde* genoemd.

Vervolgens wordt deze terminologie verklaard door aan te tonen dat indien S aan axioma (A) voldoet elke twee snijdende rechten van S op unieke wijze een Bruck-net van orde q en graad $\alpha + 1$ als deelmeetkunde van S induceren. Deze geïnduceerde netten zullen wij hier *A-netten* noemen.

Het volgende lemma ligt aan de basis van de twee voorwaarden die verder in de karakterisatie opduiken.

Lemma A.3.2 *Is S een net-induceerbare semipartiële meetkunde, dan gelden de volgende eigenschappen.*

- *Elke twee niet-collineaire punten zijn bevat in juist $\frac{\mu}{\alpha(\alpha+1)}$ A-netten.*
- *Elke rechte is bevat in juist $\frac{t}{\alpha}$ A-netten.*

Verder geldt

$$\mu \leq \frac{t^2 - \alpha t - \alpha(\alpha+1)(t-s-1)}{s+1-\alpha}, \tag{A.1}$$

$$\mu \leq \frac{t(t-\alpha)}{s+1}, \tag{A.2}$$

waarbij de tweede ongelijkheid enkel geldig is voor eigenlijke net-induceerbare semipartiële meetkunden.

Vanaf hier wordt verondersteld dat gelijkheid geldt in ongelijkheden (A.1) en (A.2) en een semipartiële meetkunde waarvoor dit voldaan is wordt *maximaal net-induceerbaar* genoemd. De volgende restrictie op de parameters kan dan worden aangetoond.

Lemma A.3.3 *Zij S een eigenlijke maximale net-induceerbare semipartiële meetkunde* $\mathrm{spg}(s,t,\alpha,\mu)$. *Dan is $\mu = (\alpha+1)(t-s-1)$ en is α een deler van $s+1$.*

Op het einde van de sectie wordt tenslotte aangetoond dat parallellisme goed gedefinieerd is.

Stelling A.3.4 *Is S een maximale net-induceerbare semipartiële meetkunde, dan is parallellisme tussen rechten een equivalentierelatie en elk punt van S behoort tot juist één rechte uit elke equivalentieklasse.*

Ook tussen de A-netten kan een parallellisme gedefinieerd worden: we zeggen dat twee A-netten $\mathcal{N}_1$ en $\mathcal{N}_2$ parallel zijn als en slechts als $\mathcal{N}_1 = \mathcal{N}_2$ of $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$.

Stelling A.3.5 *Is S een maximale net-induceerbare semipartiële meetkunde, dan is parallellisme tussen A-netten een equivalentierelatie en elk punt van S behoort tot juist één net uit elke equivalentieklasse.*

Gevolg A.3.6 *Is S een maximale net-induceerbare semipartiële meetkunde, dan heeft S juist $(s+1)^3$ punten.*

A.3.3 De karakterisatie van $T_2^*(\mathcal{K})$

Hier vermelden we enkel de karakterisatie zoals bekomen door De Clerck, De Soete en Gevaert [24].

Stelling A.3.7 ([24]) *Zij S een maximale net-induceerbare partiële meetkunde ($s+1 \neq \alpha$ en $\alpha > 1$). Zij elke triade van punten (dit is een verzameling van drie twee aan twee niet-collineaire punten) van S bevat in $0, 1$ of $\frac{t+1}{\alpha+1}$ subnetten van orde $s+1$ en graad $\alpha+1$. Dan is $S \cong T_2^*(\mathcal{K})$.*

A.3.4 De karakterisatie van $T_2^*(\mathcal{U})$

Een gevolg van Lemma A.3.3 is dat $t \geq s+1+\alpha$. In de vierde sectie van Hoofdstuk 3 wordt het geval $t > s+1+\alpha$ behandeld, hetgeen gelijkwaardig is met $\mu > \alpha(\alpha+1)$. Ons eerste doel bestond eruit om de affiene ruimte $\mathrm{AG}(3,q)$ te reconstrueren. Hiervoor werd gesteund op de volgende elegante en practische stelling van F. Buekenhout [8].

Stelling A.3.8 *Zij $\mathcal{L}$ een lineaire ruimte die voldoet aan*

(i) elke drie niet-collineaire punten brengen een affien vlak voort;

(ii) $\mathcal{L}$ bevat minstens drie niet-collineaire punten;

(iii) elke rechte van $\mathcal{L}$ bevat minstens vier punten.

Dan is $\mathcal{L}$ een affiene ruimte.

Hiervan gebruik makend konden we volgende stelling bewijzen.

Stelling A.3.9 *Zij S een maximale net-induceerbare semipartiële meetkunde met $\alpha > 1$. Onderstel dat elke triade van S bevat is in $0, 1$ of $\frac{\mu}{\alpha(\alpha+1)}$ A-netten. Beschouw de incidentiestructuur $\mathbb{P} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, met $\mathcal{P}$ de verzameling van alle punten van S, $\mathcal{L}$ de verzameling van alle puntenverzamelingen die ontstaan als intersectie van twee niet-parallelle A-netten en I de natuurlijke incidentie. Dan is $\mathbb{P}$ een driedimensionale affiene ruimte.*

Het volgende lemma leidt dan tot de uiteindelijke karakterisatie.

Lemma A.3.10 *Er geldt dat $t = \sqrt{(s+1)^3}$ en $\alpha = \sqrt{s+1}$.*

We bekomen

Stelling A.3.11 *Zij S een eigenlijke maximale net-induceerbare semipartiële meetkunde met $\alpha > 1$. Indien alle triaden bevat zijn in $0, 1$ of $\frac{\mu}{\alpha(\alpha+1)} > 1$ A-netten, dan is $S \cong T_2^*(\mathcal{U})$.*

A.3.5 De karakterisatie van $T_2^*(\mathcal{B})$

In deze sectie wordt het geval $t = s + 1 + \alpha$ (of equivalent $\mu = \alpha(\alpha + 1)$) behandeld. Indien $\alpha \neq 3$ volgt, gebruik makend van een stelling van Brouwer en Wilbrink, dat $S \cong T_2^*(\mathcal{B})$. Het bewijs dat in hoofdstuk 3 gegeven wordt om dit resultaat te bewijzen als $\alpha = 3$ werkt echter voor alle $\alpha > 1$. We bekomen de volgende stelling.

Stelling A.3.12 *Is S een eigenlijke maximale net-induceerbare semipartiële meetkunde waarvoor $\mu = \alpha(\alpha + 1) > 2$, dan is $S \cong T_2^*(\mathcal{B})$.*

A.3.6 Hoofdresultaat

De bekomen resultaten van hoofdstuk 3 kunnen worden samengebracht in de volgende stelling.

Stelling A.3.13 *Zij S een maximale net-induceerbare semipartiële meetkunde waarvoor elke triade bevat is in* $0, 1$ *of* $\frac{\mu}{\alpha(\alpha+1)}$ *A-netten. Dan is ofwel* $S \cong T_2^*(\mathcal{K})$, *ofwel* $S \cong T_2^*(\mathcal{U})$, *ofwel* $S \cong T_2^*(\mathcal{B})$. *In het laatste geval is de voorwaarde op de triades triviaal vervuld.*

A.4 Lineaire representaties van semipartiële meetkunden

In [19] werd begonnen met de classificatie van de semipartiële meetkunden die kunnen ingebed worden in een affiene ruimte en toonden Debroey en Thas aan dat elke eigenlijke semipartiële meetkunde ingebed in $\mathrm{AG}(3, q)$ ofwel een pentagon (in $\mathrm{AG}(3, 2)$), ofwel $T_2^*(\mathcal{U})$, ofwel $T_2^*(\mathcal{B})$ is. Eerder dat jaar had Thas reeds het analoge probleem voor partiële meetkunden opgelost. Recent toonden De Clerck en Delanote [23] aan dat indien een eigenlijke semipartiële meetkunde S ingebed is in $\mathrm{AG}(n, q)$, $n \geq 4$, S een lineaire representatie is of dat $\alpha = 2$. De door De Feyter [29, 30, 31, 32, 33] (zie ook De Clerck, De Feyter en Thas [21]) bekomen klassificatie van $(0, 2)$-meetkunden in $\mathrm{AG}(n, q)$ brengt het oorspronkelijke probleem, dus de classificatie van semipartiële meetkunden in $\mathrm{AG}(n, q)$, terug tot het classificeren van semipartiële meetkunden met een lineaire representatie. Het best gekende resultaat in deze richting was het resultaat uit de Debroey-Thas classificatie in [19]. In hoofdstuk 4 verbeteren we hun resultaat en bekomen zo de classificatie van alle eigenlijke semipartiële meetkunden ingebed in $\mathrm{AG}(4, q)$. In het hoger dimensionale geval worden restricties op de parameters bekomen. Gelet op het feit dat de tussenresultaten en hun bewijzen relatief technisch zijn en heel wat berekeningen vergen, beperken we ons wat deze samenvatting betreft tot het formuleren van de hoofdresultaten en het in grote lijnen bespreken van hoe we die bekomen.

A.4.1 Algemene resultaten

We onderstellen dat $\mathcal{K}$ een niet-triviale puntenverzameling in $\mathrm{PG}(n, q)$, $n \geq 3$, is met de eigenschap dat $S := T_n^*(\mathcal{K})$, de lineaire representatie van $\mathcal{K}$, een spg$(q-1, |\mathcal{K}|-1, \alpha, \mu)$ is met $\alpha > 1$. In het volgende lemma brengen we een aantal algemene resultaten uit Hoofdstuk 4, Sectie 4.2, samen.

Lemma A.4.1 *Zij $q = p^m$, met p priem. Er geldt:*

- *$\mathcal{K}$ is een SPG-regulus;*
- *$|\mathcal{K}| = 1 + x\alpha$, $x \in \mathbb{N}$;*
- *$\alpha = p^i$;*
- *$\mathcal{K}$ heeft twee intersectiegetallen, $1 + y\alpha < 1 + z\alpha$, ten opzichte van hypervlakken;*
- *$z - y = p^k$, met $k > 0$;*
- *$\mu = p^i \frac{x(1+xp^i)(p^m-p^i)}{p^{mn}+\cdots+p^m-xp^i}$;*
- *$\lambda = p^m - 2 + xp^i(p^i - 1)$, met λ het aantal punten dat collineair is met twee gegeven collineaire punten, de gegeven twee punten niet meegerekend;*
- *$K = (xp^i + 1)(p^m - 1)$, met K het aantal punten collineair met een gegeven punt, het gegeven punt niet meegerekend, dat is, K is de valentie van de collineariteitsgraaf.*

De volgende stelling, die een onmiddellijk gevolg is van een stelling van Ueberberg [84], laat ons toe reeds de helft van het spectrum voor α uit te sluiten.

Stelling A.4.2 *Indien $i \geq m/2$, dan is $S \cong T_n^*(\mathcal{B})$, met $\mathcal{B}$ een Baer-deelmeetkunde van* PG(n, q).

Tenslotte worden nog twee belangrijke gelijkheden bekomen.

Stelling A.4.3 *Er geldt dat*

$$p^{2m+2k+2i} = (\lambda - \mu)^2 + 4(K - \mu). \tag{A.3}$$

Deze stelling is een onmiddellijk gevolg van een stelling van Calderbank en Kantor [13] omtrent verzamelingen met twee intersectiegetallen.

Stelling A.4.4 *Er geldt dat*

$$x^2(q^{n-2} - 1) + x(q^{n-2}(q - 1) - (y + z)(q^{n-1} - 1)) + yz(q^n - 1) = 0. \tag{A.4}$$

A.4.2 Het geval $n = 3$

Doorheen deze sectie gebruiken we dezelfde notaties als in de voorgaande sectie en wordt ondersteld dat $n = 3$ en dat $i < m/2$.

Stelling A.4.5 *Er geldt dat $y \neq 0$ en dat $z \neq q + 1$.*

Een analyse van vergelijking (A.3), gebaseerd op congruentie-argumenten, levert snel volgende resultaten.

Lemma A.4.6 *Ofwel is p een deler van x ofwel van $x - 1$.*

Lemma A.4.7 *Er geldt dat $x \equiv y \equiv z \pmod{p}$.*

De volgende opsplitsing is nu logisch.

Het geval p deelt x

Een diepgaandere analyse van vergelijking (A.3), nog steeds gebaseerd op congruentie-argumenten, laat toe het volgende aan te tonen.

Lemma A.4.8 *Er geldt dat p^{2m-2i} een deler is van x.*

Hierop steunend laat een analyse van vergelijking (A.4) dan toe te beslissen dat dit geval zich niet kan voordoen.

Stelling A.4.9 *Het geval waarbij p een deler is van x kan zich niet voordoen.*

Het geval p deelt $x - 1$

De ideeën om dit geval aan te pakken zijn in wezen analoog aan die gebruikt in het geval waar p een deler is van x. Het blijkt echter dat de analyses van vergelijkingen (A.3) en (A.4) veel dieper moeten doorgevoerd worden en op sommige punten zeer technisch worden.

De analyse van vergelijking (A.3) leert ons uiteindelijk de volgende resultaten.

Lemma A.4.10 *Er geldt dat $x = 1 + p^{m-i} + dp^m$, met $d \in \mathbb{N} \setminus \{0\}$. Indien $p \neq 2$ is $d \equiv 2 \pmod{p}$. Indien $p = 2$ is d even en zal, indien 4 een deler is van d en $k > 1$, gelden dat $i = \frac{m-1}{2}$.*

Lemma A.4.11 *Er geldt dat p^k een deler is van $y - 1 \neq 0$.*

Dit alles laat ons uiteindelijk toe om een onderscheid te maken tussen de volgende drie mogelijkheden.

(**I**) $y = 1 + up^l$ en $z = 1 + p^k + up^l$, met $u \in \mathbb{N}$, u niet deelbaar door p en $l > k$ (met $l < m$);

(**II**) $y = 1 + y_k p^k + up^{k+1}$ en $z = 1 + (1 + y_k)p^k + up^{k+1}$, met $0 < y_k < p - 1$ en $u \in \mathbb{N}$ (merk op dat $p \neq 2$ in dit geval);

(**III**) $y = 1 - p^k + up^l$ en $z = 1 + up^l$, met $u \in \mathbb{N}$, u niet deelbaar door p en $l > k$.

Een diepgaande analyse van vergelijking (A.4) in elk van deze gevallen, waarbij sterk gesteund wordt op het technisch resultaat uit Lemma A.4.10, laat ons toe de gevallen (**I**), (**II**) en (**III**) uit te sluiten, zodat volgende stelling bewezen werd.

Stelling A.4.12 *Het geval p deelt $x - 1$ kan zich niet voordoen.*

A.4.3 Overzicht

De resultaten uit de vorige sectie leiden tot de volgende stellingen.

Stelling A.4.13 *Zij S een eigenlijke semipartiële meetkunde, waarvoor $\alpha > 1$, die de lineaire representatie is van een puntenverzameling $\mathcal{K}$ in $\mathrm{PG}(3, q)$. Dan is $S = T_3^*(\mathcal{B})$, met $\mathcal{B}$ een Baer-deelmeetkunde van $\mathrm{PG}(3, q)$.*

Bewijs. Uit Stelling A.4.2 volgt dat indien $\alpha \geq \sqrt{q}$ de stelling geldt. We mogen dus onderstellen dat $\alpha < \sqrt{q}$. In dit geval impliceert Lemma A.4.6 dat, met de notaties van de vorige sectie, owel p een deler is van x ofwel van $x - 1$. Deze gevallen worden uitgesloten door Stelling A.4.9 en A.4.12. Bijgevolg is $\alpha \geq \sqrt{q}$. □

Steunend op een resultaat uit De Clerck en Delanote [23] en de classificatie van De Feyter [30] volgt

Stelling A.4.14 *Is S een eigenlijke semipartiële meetkunde, waarvoor $\alpha > 1$, ingebed in $\mathrm{AG}(4, q)$, dan is ofwel $S = T_3^*(\mathcal{B})$, ofwel $S = \mathrm{TQ}(4, 2^m)$.*

A.4.4 Het hogerdimensionaal geval

In Sectie 4.4 van Hoofdstuk 4 wordt aangetoond dat heel wat van de restricties op de parameters, bekomen voor drie dimensies, geldig blijven in hogere dimensies. Desondanks laten deze restricties ons niet toe een analoog resultaat te bewijzen als $n \geq 4$. Wel geloven we dat de bekomen resultaten toelaten om te stellen dat de volgende conjectuur realistisch is.

Conjectuur. *Is S een eigenlijke semipartiële meetkunde, waarvoor $\alpha > 1$, ingebed in $\mathrm{AG}(n, q)$, met $n \geq 5$, dan is $S = T^*_{n-1}(\mathcal{B})$.*

A.5 SPG-reguli en Baer-deelmeetkunden

In het korte vijfde hoofdstuk wordt een karakterisatie bewezen van die SPG-reguli (die niet noodzakelijk uit punten bestaan) die aanleiding geven tot een meetkunde isomorf met $T^*_n(\mathcal{B})$. De stelling die we bewijzen levert een antwoord op de volgende natuurlijke vraag:

> *Zij $\mathcal{R}$ een SPG-regulus in $\mathrm{PG}(n, q)$, waarvoor $\mu = \alpha(\alpha + 1) > 2$ (hetgeen het geval is in het geval van de lineaire representatie van een Baer-deelmeetkunde). Bestaan er dan zekere n', q' en een Baer-deelmeetkunde $\mathcal{B}$ met de eigenschap dat $\mathrm{PG}(n', q') \overset{q' \mapsto q}{\to}$ $\mathrm{PG}(n, q)$ en $\mathcal{B} \overset{q' \mapsto q}{\to} \mathcal{R}$?*

De volgende stelling levert een positief antwoord.

Stelling A.5.1 *Zij $\mathcal{R}$ een SPG-regulus bestaande uit $\mathrm{PG}(l, q)$'s in $\mathrm{PG}(n, q)$, waarvoor $t - \theta = \alpha \notin \{1, q^{l+1} - 1, q^{l+1}\}$ en $\theta \neq 0$. Dan is $S(\mathcal{R}) \cong T^*_m(\mathcal{B})$ voor zekere m.*

Tenslotte wordt in Hoofdstuk 5 nog uitgelegd waarom de restricties uit de opgave van de vorige stelling natuurlijk en niet te streng zijn.

A.6 De polaire eigenschap

In hoofdstuk 6 wordt een speciale klasse van verzamelingen van deelruimten van een projectieve ruimte bestudeerd: verzamelingen van deelruimten die voldoen aan de polaire eigenschap. Het hoofdstuk is gebaseerd op De Winter en Thas [39].

A.6.1 Semi-pseudo-ovoïden

Definitie A.6.1 Zij $\mathcal{O} = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ een verzameling twee aan twee disjuncte $(n-1)$-dimensionale deelruimten van $\mathrm{PG}(h,q)$, met $h > 2n-1$. Dan zeggen we dat $\mathcal{O}$ voldoet aan de *polaire eigenschap* als en slechts als de unie van alle n-dimensionale deelruimten die π_i bevatten en disjunct zijn van elke $\pi_j \neq \pi_i$ een $(h-n)$-dimensionale ruimte τ_i is; we noemen τ_i dan de *raakruimte aan* π_i. Een verzameling $\mathcal{O}$ die voldoet aan de polaire eigenschap wordt een *semi-pseudo-ovoïde* genoemd.

Voorbeelden van semi-pseudo-ovoïden zijn pseudo-ovalen, pseudo-ovoïden, verscheidene types van SPG-reguli, alsook m-systemen van bepaalde polaire ruimten.

A.6.2 Een belangrijke ongelijkheid

Stelling A.6.2 *Zij $\mathcal{O}$ een semi-pseudo-ovoïde bestaande uit η $(n-1)$-dimensionale deelruimten van* $\mathrm{PG}(h,q)$. *Dan is*

$$1 + q^{h-2n+1} \leq \eta \leq 1 + q^{\frac{h+1}{2}}.$$

Er volgt dat $h \leq 4n-1$.

Op deze stelling steunend kan dan worden aangetoond dat de oorspronkelijke definitie van pseudo-ovoïde (zie de Engelse tekst op pagina 10) sterk kan vereenvoudigd worden.

Stelling A.6.3 *Zij $\mathcal{O}$ een semi-pseudo-ovoïde bestaande uit η $(n-1)$-dimensionale deelruimten van* $\mathrm{PG}(h,q)$. *Onderstel dat $h = 4n-1$. Dan is $\eta = 1 + q^{2n}$ en is $\mathcal{O}$ een pseudo-ovoïde.*

A.6.3 Interpretatie van de gelijkheden

Er wordt onderzocht wat gelijkheid betekent in de ongelijkheden die bekomen werden in de voorgaande sectie.

Stelling A.6.4 *Zij $\mathcal{O}$ een semi-pseudo-ovoïde met η elementen. Dan zal $\eta = 1 + q^{\frac{h+1}{2}}$ als en slechts als elk punt dat niet tot een element van $\mathcal{O}$ behoort, bevat is in een constant aantal raakruimten. Deze constante is gelijk aan $1 + q^{\frac{h-2n+1}{2}}$.*

Stelling A.6.5 *Zij $\mathcal{O}$ een semi-pseudo-ovoïde met η elementen. Dan zal $\eta = 1 + q^{\frac{h+1}{2}}$ als en slechts als elk hypervlak dat geen raakruimte bevat, een constant aantal elementen van $\mathcal{O}$ bevat. Deze constante is gelijk aan $1 + q^{\frac{h-2n+1}{2}}$.*

Gevolg A.6.6 *Zij $\mathcal{O}$ een semi-pseudo-ovoïde waarvoor $\eta = 1 + q^{\frac{h+1}{2}}$. Dan heeft de unie van de elementen van $\mathcal{O}$ twee intersectiegetallen ten opzichte van hypervlakken. Bijgevolg bepaalt $\mathcal{O}$ een projectieve lineaire tweegewichtscode en een sterk reguliere graaf.*

Stelling A.6.7 *Een semi-pseudo-ovoïde bestaande uit η elementen is een pseudo-ovaal of een pseudo-ovoïde als en slechts als $\eta = 1 + q^{h-2n+1}$.*

A.6.4 Translatie dualen

In de vierde sectie van Hoofdstuk 6 wordt een resultaat van Payne en Thas [59] omtrent translatie dualen van pseudo-ovoïden veralgemeend voor semi-pseudo-ovoïden.

Stelling A.6.8 *Zij $\mathcal{O}$ een semi-pseudo-ovoïde in* $\mathrm{PG}(h,q)$*, waarvoor $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$. Dan vormen de raakruimten van $\mathcal{O}$ een semi-pseudo-ovoïde in de duale ruimte van* $\mathrm{PG}(h,q)$.

A.6.5 SPG-reguli en de polaire eigenschap

In de laatste sectie van Hoofdstuk 6 wordt het verband tussen de polaire eigenschap en SPG-reguli behandeld. De bekomen resultaten blijken zeer nuttig te zijn in Hoofdstuk 8.

Stelling A.6.9 *Een semi-pseudo-ovoïde is een SPG-regulus (die voldoet aan de polaire eigenschap) als en slechts als $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$.*

De volgende veralgemening van semi-pseudo-ovoïde kan worden ingevoerd.

Definitie A.6.10 Een *veralgemeende semi-pseudo-ovoïde* $\mathcal{O} = \{\pi_0, \ldots, \pi_\eta\}$ is een verzameling van twee aan twee disjuncte $(n-1)$-dimensionale deelruimten van $\mathrm{PG}(h,q)$, $h > 2n-1$, met de eigenschap dat voor elk element $\pi_i \in \mathcal{O}$ er een $(h-n)$-dimensionale ruimte τ_i bestaat zodat $\pi_i \subset \tau_i$ en $\tau_i \cap \pi_j = \emptyset$ voor alle $j \neq i$.

We kunnen dan de volgende veralgemening van de voorgaande stelling bewijzen.

Stelling A.6.11 *Zij $\mathcal{O}$ een veralgemeende semi-pseudo-ovoïde in* PG(h, q), *waarvoor* $|\mathcal{O}| = 1 + q^{\frac{h+1}{2}}$. *Dan is $\mathcal{O}$ een SPG-regulus die voldoet aan de polaire eigenschap.*

A.7 Translatie semipartiële meetkunden

In Hoofdstuk 7 wordt een theorie van elatie en translatie semipartiële meetkunden ingevoerd. Behalve voor een eindig aantal uitzonderingen blijken deze translatie semipartiële meetkunden de semipartiële meetkunden te zijn die geconstrueerd worden uit een SPG-regulus. Eerst wordt een studie gemaakt van speciale automorfismen, parallellismen genaamd, voor deze meetkunden. Dit leidt tot het resultaat dat voor bijna alle gekende semipartiële meetkunden S, geconstrueerd uit een SPG-regulus, geldt dat de automorfismegroep van S geïnduceerd wordt door de automorfismen van de affiene ruimte waarin S is ingebed. Deze resultaten zijn gebaseerd op De Winter [36]. Tenslotte wordt in hoofdstuk 7 ook een gedeeltelijk antwoord gegeven op een vraag van J. C. Fisher omtrent veralgemeende vierhoeken met een reguliere automorfismegroep. Dit laatste gedeelte is gebaseerd op De Winter en K. Thas [40].

A.7.1 SPG-families

Definitie A.7.1 Zij G een eindige niet noodzakelijk abelse groep. Onderstel dat $J = \{S_0, \ldots, S_t\}$ een verzameling is van $t + 1 > 1$ deelgroepen van G van de orde $s + 1 > 1$ met de eigenschap dat $S_i \cap S_j = \{id\}$ voor alle $i \neq j$. Definieer voor elke $i \in I := \{0, 1, \ldots, t\}$ de verzameling $S_i^* := S_i \cup \{g \in G \mid S_i g \cap S_j = \emptyset, \forall j \in I\}$. Het paar (G, J) is een *SPG-familie* met parameters (s, t, α, θ) indien de volgende twee voorwaarden vervuld zijn.

- Er bestaat een α met $0 < \alpha < s + 1$ zodat voor elke S_i en elke $g \in S_j \setminus \{id\}$, $j \neq i$, er een unieke verzameling $\{j_1 = j, j_2, \ldots, j_\alpha\} \subset I$ bestaat met de eigenschap dat $S_{j_k} g \cap S_i \neq \emptyset$, $\forall k \in \{1, \ldots, \alpha\}$.
- Er bestaat een θ zodat voor elke $g \in G \setminus \bigcup_{i \in I} S_i$ er een unieke verzameling $\{j_1, \ldots, j_\theta\} \subset I$ bestaat met de eigenschap dat $g \in S_{j_k}^*$, $\forall k \in \{1, \ldots, \theta\}$.

Deze definitie veralgemeent het concept van 4-gonale families zoals ingevoerd door Kantor [51].

Definitie A.7.2 *Zij G een groep en $J = \{S_i \mid i \in I\}$ een verzameling van deelgroepen van G. De rechtse nevenklasse-meetkunde $S(G, J)$ is de incidentiemeetkunde met als punten de elementen van G, als rechten de rechtse nevenklassen $S_i g$, $i \in I$ en $g \in G$, en met als incidentierelatie "bevat zijn in".*

Stelling A.7.3 *Zij $S(G, J)$ een SPG-familie met parameters (s, t, α, θ). Dan is de rechtse nevenklasse-meetkunde $S(G, J)$ een semipartiële meetkunde $\mathrm{spg}(s, t, \alpha, (t + 1 - \theta)\alpha)$.*

Een semipartiële meetkunde geconstrueerd uit een SPG-familie (G, J) wordt een *elatie semipartiële meetkunde* (ESPG) genoemd. Indien G abels is, spreken van een *translatie semipartiële meetkunde* (TSPG).

Tot slot van deze sectie merken we op dat elke semipartiële meetkunde S geconstrueerd uit een SPG-regulus een voorbeeld is van een TSPG. Stel G de groep van alle translaties van de affiene ruimte waarin S is ingebed. Kies een vast punt x van S en noteer door L_0 tot en met L_t de rechten van S door x. Definieer dan $S_i = \{g \in G \mid L_i^g = L_i\}$. Het is duidelijk dat de S_i deelgroepen zijn van G van de orde $s+1$. Het is niet moeilijk om te zien dat (G, J) een SPG-familie is en dat $S \cong S(G, J)$.

A.7.2 Translatie semipartiële meetkunden

Het hoofddoel van de tweede sectie van Hoofdstuk 7 is om aan te tonen dat in bijna alle gevallen een TSPG isomorf is met een semipartiële meetkunde geconstrueerd uit een SPG-regulus.

Stelling A.7.4 *Elke TSPG waarvoor $t - \theta > \alpha$ (dit is $\mu > \alpha(\alpha + 1)$) is isomorf met een semipartiële meetkunde geconstrueerd uit een SPG-regulus.*

Stelling A.7.5 *Er bestaat geen TSPG met $\mu = \alpha^2$.*

De volgende stelling steunt sterk op het resultaat bekomen in Hoofdstuk 5.

Stelling A.7.6 *Zij S een eigenlijke TSPG waarvoor $\mu = \alpha(\alpha + 1) > 2$ en $\alpha \neq 3$. Dan is $S \cong T_m^*(\mathcal{B})$ voor zekere m.*

We denken dat de voorgaande resultaten toelaten om de volgende conjectuur te formuleren.

Conjectuur. Indien $\alpha > 1$, dan is de theorie der TSPG equivalent met de theorie der SPG-reguli. Verder is elke TSPG waarvoor $\mu = \alpha(\alpha+1) > 2$, die geen Bruck-net is, isomorf met $T^*_m(\mathcal{B})$ voor zekere m.

A.7.3 Parallellisme in ESPG en TSPG

Zij $S(G, J)$ een ESPG.

Definitie A.7.7

- Twee rechten $S_i g$ en $S_j h$ van $S(G, J)$ zijn *G-parallel* of kortweg *parallel* (indien geen verwarring mogelijk is) als en slechts als $i = j$.
- Een *G-parallellisme* of kortweg *parallellisme* is elk automorfisme β van $S(G, J)$ waarvoor elke rechte parallel is met zijn beeld. Een *elatie* is elk fixpuntvrij parallellisme; per definitie is ook de identiteit een elatie. Indien G abels is wordt een elatie ook een *translatie* genoemd.

De studie van parallellismen voor ESPG en TSPG levert volgende stellingen.

Stelling A.7.8 *Zij β een parallellisme van de ESPG $S(G, J)$, met $\alpha > 1$. Dan doet zich één van de volgende situaties voor:*

- *β is een elatie;*
- *β heeft een uniek fixpunt;*
- *de incidentiestructuur $S' = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, met $\mathcal{P}$ de verzameling der fixpunten van β, met $\mathcal{L}$ de verzameling van die rechten van $S(G, J)$ die een fixpunt bevatten, en met I de natuurlijke incidentie, is een semipartiële meetkunde $\mathrm{spg}(s', t, \alpha, \mu)$.*

Stelling A.7.9 *Zij $S(G, J)$ een TSPG, waarvoor $\mu \neq \alpha(\alpha + 1) > 2$. Dan heeft een parallellisme van $S(G, J)$ dat niet de identiteit is hoogstens één fixpunt.*

Hierop steunend kunnen dan alle translaties van een TSPG waarvoor $\mu \neq \alpha(\alpha + 1) > 2$ bepaald worden.

Stelling A.7.10 *De verzameling van alle translaties van een TSPG waarvoor* $\mu \neq \alpha(\alpha+1) > 2$ *vormt een elementair abelse groep die regulier werkt op de puntenverzameling van* $S(G, J)$. *Deze groep is isomorf met de groep der elaties van* $\mathrm{PG}(n+1, q)$ *met gegeven as* $\mathrm{PG}(n, q)$ *voor zekere* n.

Hiervan gebruik makend kan dan uiteindelijk de groep van alle parallellismen voor dergelijke meetkunden beschreven worden. Dit leidt tenslotte tot volgende stelling.

Stelling A.7.11 *Zij* $S(\mathcal{R})$ *een semipartiële meetkunde, met* $\mu \neq \alpha(\alpha+1)$, *geconstrueerd uit een SPG-regulus* $\mathcal{R}$ *in* $\mathrm{PG}(n, q)$ *waarvoor* $\alpha > 1$. *Indien* $\mathcal{R}$ α*-geometrisch is of voldoet aan de polaire eigenschap, dan is* $\mathrm{Aut}(S(\mathcal{R})) \leq \mathrm{A\Gamma L}(n+2, q)$. *In het bijzonder is* $\mathrm{Aut}(S(\mathcal{R}))$ *de deelgroep van* $\mathrm{A\Gamma L}(n+2, q)$ *geïnduceerd door* $\mathrm{Aut}(\mathcal{R}) \leq \mathrm{P\Gamma L}(n+1, q)$.

A.7.4 Karakterisatie van ESPG en TSPG

Het doel van Sectie 7.4 was voor ESPG en TSPG een analogon te bewijzen van een stelling van Payne en Thas [59] omtrent een karakterisatie van translatie veralgemeende vierhoeken.

Stelling A.7.12 *Zij* S *een* $\mathrm{spg}(s, t, \alpha, \mu)$ *waarvoor* $\alpha < s+1$. *Zij* G *een reguliere automorfismegroep van* S. *Indien er een punt* x *bestaat zodat* $L^g = L$ *of* $L^g \cap L = \emptyset$ *voor alle rechten* L *door* x *en alle* $g \in G$, *dan is* S *een ESPG.*

Definitie A.7.13 Zij S een $\mathrm{spg}(s, t, \alpha, \mu)$ waarvoor $\alpha > 0$. Onderstel dat $P = \{\pi_0, \ldots, \pi_t\}$ een packing is van S.

- Een *P-parallellisme* van S is elk automorfisme β van S met de eigenschap dat voor elke rechte L geldt dat indien $L \in \pi_i$, dan $L^\beta \in \pi_i$.
- Een *P-symmetrie om π_i* is elk P-parallellisme β waarvoor $L^\beta = L$ voor elke $L \in \pi_i$. De verzameling van alle P-symmetrieën om π_i vormt een groep $\mathrm{Sym}_P(\pi_i)$.

Stelling A.7.14 *Zij* S *een* $\mathrm{spg}(s, t, \alpha, \mu)$ *waarvoor* $t > 2$ *en* $2 \leq \alpha < s+1$. *Zij* $P = \{\pi_0, \ldots, \pi_t\}$ *een packing van* S. *Indien* $|\mathrm{Sym}_P(\pi_i)| = s+1$ *voor elke* $i \in \{0, \ldots, t\}$, *dan is* S *een TSPG.*

A.7.5 Een speciale eigenschap

In de vijde sectie van Hoofdstuk 7 bespreken we het verband met "translatie partiële meetkunden" zoals ingevoerd door De Clerck, Gevaert en Thas [26]. Wat deze samenvatting betreft gaan we hier niet dieper op in.

A.7.6 Veralgemeende vierhoeken met een reguliere abelse groep

In de laatste sectie van Hoofdstuk 7, die gebaseerd is op De Winter en K. Thas [40], wordt de volgende vraag, gesteld door J. C. Fisher op de La Roche Conference on Incidence Geometry (23-29 mei 2004), beantwoord in het geval G abels is.

> *Wat kan men zeggen over veralgemeende vierhoeken die een reguliere groep G van automorfismen hebben?*

Definitie A.7.15 Een *veralgemeende hyperovaal* $\mathcal{O}$ in $\mathrm{PG}(3n-1,q)$ is een verzameling van q^n+2 twee aan twee disjuncte $\mathrm{PG}(n-1,q)$s in $\mathrm{PG}(3n-1,q)$ met de eigenschap dat elke drie elementen van $\mathcal{O}$ de volledige ruimte $\mathrm{PG}(3n-1,q)$ genereren.

Stelling A.7.16 *Zij $\mathcal{Q}$ een dikke eindige veralgemeende vierhoek. Zij G een abelse reguliere automorfismegroep van $\mathcal{Q}$. Dan is $\mathcal{Q}$ isomorf met de veralgemeende lineaire representatie van een veralgemeende hyperovaal.*

Tenslotte wordt ook nog aangetoond dat dit resultaat te veralgemenen is voor partiële vierhoeken en andere $(0,1)$-meetkunden.

A.8 Een nieuwe semipartiële meetkunde

In Hoofdstuk 8 wordt een nieuwe SPG-regulus $\mathcal{R}$ geconstrueerd en wordt aangetoond dat de semipartiële meetkunde $S(\mathcal{R})$ eveneens nieuw is. Hoewel de resultaten uit Hoofdstuk 8 oorspronkelijk verschenen zijn in De Winter en Thas [38], zijn de bewijzen uit Hoofdstuk 8 nieuw en heel wat korter dan in bovengenoemd artikel. De reden hiervoor is dat de resultaten uit Hoofdstukken 6 en 7, die van latere datum zijn dan [38], uiterst nuttig zijn in deze context.

A.8.1 De constructie

Stelling A.8.1 *Zij $\mathcal{O} = \{\pi_0, \ldots, \pi_\eta\}$ een semi-pseudo-ovoïde bestaande uit $(n-1)$-dimensionale deelruimten in* $\mathrm{PG}(h,q)$*, met $\eta = 1 + q^{\frac{h+1}{2}}$. Zij τ_i de raakruimte van $\mathcal{O}$ aan π_i. Onderstel dat de doorsnede van de raakruimten $\tau_1, \tau_2, \ldots, \tau_s$ een* $\mathrm{PG}(h-2n,q)$ *is. Zij $\{\overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_s}\}$ een verzameling van twee aan twee disjuncte $(n-1)$-dimensionale ruimten die juist dezelfde puntenverzameling als $\pi_1 \cup \pi_2 \cup \ldots \cup \pi_s$ bedekt. Dan is $\overline{O} = (O \cup \{\overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_s}\}) \setminus \{\pi_1, \pi_2, \ldots, \pi_s\}$ eveneens een semi-pseudo-ovoïde en bijgevolg een SPG-regulus die aan de polaire eigenschap voldoet.*

Zij $\mathcal{U}$ een Buekenhout-Metz unitaal in $\mathrm{PG}(2,q^2)$. Dan bestaat er steeds een speciaal punt $u \in \mathcal{U}$ met de eigenschap dat voor elke secant L door u geldt dat $L \cap \mathcal{U}$ een Baer-deelrechte is. Onder de reductie $\mathrm{PG}(2,q^2) \overset{q^2 \mapsto q}{\longrightarrow} \mathrm{PG}(5,q)$ wordt $\mathcal{U}$ afgebeeld op een verzameling $\mathcal{U}'$ van q^3+1 disjuncte rechten van $\mathrm{PG}(5,q)$. Er geldt dat $S(\mathcal{U}') \cong T_2^*(\mathcal{U})$. Zij L een secant (in $\mathrm{PG}(2,q^2)$) door u. Dan wordt $L \cap \mathcal{U}$ afgebeeld op een regulus $R := \{\pi_0, \ldots, \pi_q\}$ in de 3-dimensionale ruimte van $\mathrm{PG}(5,q)$ waarop L wordt afgebeeld. Uit eigenschappen van Buekenhout-Metz unitalen weten we dan dat de raakruimten $\tau_0, \ldots, \tau_q$ van $\mathcal{U}'$ aan $\pi_0, \ldots, \pi_q$ zullen snijden in een gemeenschappelijke rechte Γ. Noem $\overline{R}$ de regulus van transversalen van R en $\overline{\mathcal{U}} := (\mathcal{U}' \setminus R) \cup \overline{R}$. We bekomen de volgende stelling.

Stelling A.8.2 *De verzameling $\overline{\mathcal{U}}$ is een SPG-regulus in* $\mathrm{PG}(5,q)$ *die voldoet aan de polaire eigenschap.*

Bewijs. Dit is een directe toepassing van Stelling A.8.1. □

A.8.2 De automorfismegroep

In de tweede sectie van Hoofdstuk 8 wordt de automorfismegroep van de semipartiële meetkunde $S(\overline{\mathcal{U}})$ bepaald. Hiervoor wordt gesteund op de resultaten bekomen in Hoofdstuk 7. Een gevolg is dat de semipartiële meetkunde $S(\overline{\mathcal{U}})$ nieuw is, dit is, $S(\overline{\mathcal{U}}) \ncong T_2^*(\mathcal{U})$. Dit laatste kan ook meetkundig ingezien worden: $T_2^*(\mathcal{U})$ voldoet aan axioma (A) (zie Hoofdstuk 3), terwijl $S(\overline{\mathcal{U}})$ niet aan dit axioma voldoet.

A.9 Classificatie van SPG-systemen van index 2

In het laatste hoofdstuk van dit proefschrift wordt het bestaan van SPG-systemen onderzocht en bekomen we een classificatie van alle SPG-systemen van index 2 in het niet-singuliere geval en van alle SPG-systemen in het singuliere geval. Hoofdstuk 9 is gebaseerd op De Winter [35] dat verscheen in “Advances in Geometry”.

A.9.1 Twee hulpstellingen

De in Sectie 9.1 bekomen hulpstellingen blijken van groot nut te zijn in de rest van Hoofdstuk 9. De eerste veralgemeent een resultaat van Thas [76] voor SPG-systemen van index 1. De tweede levert een karakterisatie van spread-SPG-systemen en is cruciaal voor de in dit hoofdstuk bekomen resultaten.

Lemma A.9.1 *Zij τ een SPG-systeem van een polaire ruimte P.*

- *Indien P niet-singulier is en projectieve index n heeft, dan is elk punt van P bevat in juist $(q^n - 1)/(q - 1)$ elementen van τ.*
- *Indien P singulier is en projectieve index n heeft, dan is elk punt van P, behalve het radicaal, bevat in juist q^{n-1} elementen van τ.*

Lemma A.9.2 *Zij τ een SPG-systeem van index $n - 1 \geq 2$ van een polaire ruimte P. Dan is τ een spread-SPG-systeem als en slechts als elke twee snijdende elementen van τ snijden in een $(n - 2)$-dimensionale ruimte.*

A.9.2 De gevallen $Q^+(7, q)$ en $H(7, q^2)$

Steunend op de hogervermelde karakterisatie van spread-SPG-systemen bekomen we snel volgende classificatie.

Stelling A.9.3 *De enige SPG-systemen van $Q^+(7, q)$ zijn spread-SPG-systemen. De polaire ruimte $H(7, q^2)$ heeft geen SPG-systemen.*

A.9.3 Het geval $Q(8, q)$

Gelet op het feit dat $Q(8, q)$ relatief meer generatoren door een singulier vlak bevat dan $Q^+(7, q)$ en $H(7, q^2)$, blijkt een ingewikkelder methode nodig te zijn om de SPG-systemen van $Q(8, q)$ te classificeren. In deze samenvatting geven we slechts een ruwe schets van hoe we te werk zijn gegaan.

In een eerste fase hebben we de locale structuur van een hypothetisch SPG-systeem onderzocht. Zij p een punt van $Q(8, q)$, en zij τ een SPG-systeem van $Q(8, q)$. Beschouwen we de raakkegel $pQ(6, q)$ aan p en kiezen we hierin een vaste $Q(6, q)$ als basis. De $q^2 + q + 1$ elementen van τ die p bevatten, induceren een verzameling van q^2+q+1 rechten in $Q(6, q)$, de *locale structuur van τ in p* genaamd. We hebben aangetoond dat er, op isomorfie na, juist drie mogelijke locale structuren, *type S, type O* en *type H*, kunnen optreden.

In de tweede fase hebben we steunend op de bovenstaande resultaten de globale structuur van τ onderzocht. In eerste instantie volgde dat als ten minste één punt een locale structuur van type O had, alle punten een locale structuur van dit type moesten hebben. Dit zou dan aanleiding geven tot het bestaan van een dikke veralgemeende octagon van orde s, een strijdigheid. Vervolgens kon worden aangetoond dat ook een locale structuur van type H zich niet kan voordoen. Bijgevolg heeft elk punt van $Q(8, q)$ een locale structuur van type S. Uit Lemma A.9.2 volgt dan volgende classificatie.

Stelling A.9.4 *De enige SPG-systemen van de polaire ruimte $Q(8, q)$ zijn spread-SPG-systemen.*

A.9.4 Het singulier geval

Tenslotte bekomen we in de laatste sectie van deze thesis de classificatie van alle SPG-systemen van singuliere polaire ruimten. Voor index 1 was reeds een classifictie bekomen door Thas [76].

Stelling A.9.5 *De enige SPG-systemen van een singuliere polaire ruimte van projectieve index n zijn spread-SPG-systemen.*

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