Line-transitive linear spaces

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Contents

Abstract

A *linear space* is an incidence structure consisting of a set of points Π and a set of lines Λ in the power set of Π such that any two points are incident with exactly one line. We study those finite linear spaces which admit an automorphism group G which is transitive upon the set of lines of the space.

Within the set of all linear spaces lies a particularly important subset: the *projective planes*. Results exist in the literature [Cam04, CP93] classifying the possible minimal normal subgroups of a group G acting line-transitively on a finite projective plane. We rewrite some of these results to deal with *components* rather than with minimal normal subgroups. We then prove that, if a group G acts on a projective plane which is not Desarguesian, then G does not contain any components. In order to do this we make use of the classification of finite simple groups; our proof consists of examining the different quasisimple groups given in the classification as possible components of G.

We also examine the situation where an almost simple group G with socle $PSL(3,q)$ acts line-transitively on a linear space. This fits into the wider program of examining those almost simple groups which can act line-transitively on linear spaces, a program motivated by the result in [CP01]. We are able to give strong information about the line-transitive actions of G.

Declaration

This dissertation is not substantially the same as any I have submitted for a degree or a diploma or any other qualification at any other university. It is the result of my own work and includes nothing which is the outcome of work done in collaboration, except as specified in the text and acknowledgements.

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"So now I'm goin' back again, I got to get to her somehow.
All the people we used to know, they're an illusion to me now.
Some are mathematicians, some are carpenters' wives.
Don't know how it all got started, I don't know what they're doin' with their lives."

Bob Dylan, *"Tangled Up in Blue"*

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This thesis is in memory of Uncle John.

Chapter 1

Introduction

"I don't know what I would give to do something irrevocable."
Jean-Paul Sartre, *"The Age of Reason"*

1.1 Linear spaces

A *linear space* $\mathcal{S}$ is an incidence structure consisting of a set of points Π and a set of lines Λ in the power set of Π such that any two points are incident with exactly one line. The linear space is called *non-trivial* if every line contains at least three points and there are at least two lines. The space is called *finite* if Π is finite; throughout this thesis all spaces will be assumed to be finite. We write $|\Pi| = v, |\Lambda| = b$.

An automorphism of $\mathcal{S}$ is a permutation of Π which leaves Λ invariant. Our interest is in pairs $(\mathcal{S}, G)$ where $\mathcal{S}$ is a non-trivial finite linear space admitting G, a group of automorphisms that is transitive on the set of lines.

Such pairs have been studied for some time under progressively weaker assumptions about transitivity. In 1959, Ostrom and Wagner[OW59] proved that, in the particular case where $\mathcal{S}$ is a projective plane and G is 2-transitive on points, $\mathcal{S}$ is Desarguesian of order x a prime power and G contains $PSL(3, x)$. In fact we can assert the same conclusion for $\mathcal{S}$ any linear space under the stronger assumption that G is 2-transitive on the set of lines. In this case $\mathcal{S}$ is necessarily a projective plane and we can apply the Ostrom-Wagner theorem to the dual plane to achieve our result.

Early results following the Ostrom-Wagner theorem included that of D.G. Higman and McLaughlin[HM61] who proved, among other things, that if G acts flag-transitively on $\mathcal{S}$ then it acts point-primitively. Here a *flag* is an incident point-line pair. Block[Blo67a] proved that line-transitivity implies point-transitivity.

Buekenhout, Delandtsheer and Doyen[BDD88] have summarised the implications that hold between different transitivity properties of linear spaces as follows:

$$
\begin{tikzcd}
 & \text{L Pr} \arrow[rr, Rightarrow] & & \text{L T} \arrow[dr, Rightarrow] & \\
\text{L 2−T} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{F T} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{P T} \\
 & \text{P 2−T} \arrow[ur, Rightarrow] & & \text{P Pr} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Here L, P and F mean ‘line’, ‘point’ and ‘flag’ respectively, while T and Pr mean ‘transitive’ and ‘primitive’ respectively. In fact the classification of flag-transitive linear spaces is now complete except where G is a one-dimensional affine group. The result was first announced in [BDD$^+$90] with the proof appearing in [Lie98, Del86, Del01, Kle90, Sax02].

Our attention now turns to those linear spaces which are line-transitive but not flag-transitive. Observe that if a linear space $\mathcal{S}$ is line-transitive then every line has the same number, k (where $2 < k < v$), of points; in this case we say that $\mathcal{S}$ is *regular*. It is not hard to see that in a regular space $\mathcal{S}$ every point lies on the same number, r, of lines. In general, when we refer to a linear space $\mathcal{S}$ from here on, we mean that $\mathcal{S}$ is regular and has parameters b, v, k, r as described; on the few occasions when we wish to discuss linear spaces which are not necessarily regular we will refer specifically to *general* linear spaces.

The classification of flag-transitive linear spaces makes heavy use of the fact that a point-stabilizer acts on the set of remaining points with orbits of size a multiple of r. Analysis of the subgroups of potential point-stabilizers then yields much information. This avenue of attack is not open in the more general line-transitive case.

Regular linear spaces have another manifestation in the literature, as $2-(v, k, 1)$ *designs*. We have the following definition:

Definition 1.1. *For $t, v, k, \lambda \in \mathbb{Z}^+$ with $k > t, v > k + t$, a $t - (v, k, \lambda)$* design Λ *is a set Π of points and a set Λ of k-subsets of Π, called the* lines *or* blocks *of Λ, such that every t-subset of Π is contained in exactly λ blocks.*

A $t - (v, k, \lambda)$ design with $\lambda = 1$ is also referred to as a *Steiner system*. The literature on finite linear spaces is sometimes couched in the terminology of $2 - (v, k, 1)$ designs; in particular referring to lines as blocks. Thus, in investigating line-transitive finite linear spaces, we are investigating block-transitive $2 - (v, k, 1)$ designs.

1.2 Projective planes

A particularly important type of linear spaces has already been mentioned, namely the *projective planes*. These can be characterised in many different ways[Dem97, 3.2.3]; we define them as linear spaces for which any two lines intersect at exactly one point. This is equivalent to saying that the number of lines, b, is equal to the number of points, v.

The standard example of a projective plane is the *Desarguesian* plane constructed from a 3-dimensional vector space over a finite field q. These are said to be projective planes *of order* q and it turns out that $v = q^2 + q + 1$ while $k = q + 1$. In general for any projective plane an integer x exists such that $v = x^2 + x + 1$ and $k = x + 1$ and we refer to this integer x as the *order* of the plane.

There is a rich literature on the subject of projective planes. Some very famous conjectures remain unproved in this area, most notably that all projective planes have order x a prime power. For us a motivating conjecture of some fifty years standing is that all point-transitive projective planes are Desarguesian.

Some progress has been made towards a proof of this conjecture. The most significant result is one of Kantor[Kan87] who proved that a projective plane $\mathcal{P}$ of order x admitting a point-primitive collineation group G is Desarguesian and $G \geq PSL(3, x)$, or else $x^2 + x + 1$ is a prime and G is a regular or Frobenius group of order dividing $(x^2 + x + 1)(x + 1)$ or $(x^2 + x + 1)x$.

Kantor's result depends upon the classification of finite simple groups. A corollary of the result is that a group acts primitively on the points of a projective plane $\mathcal{P}$ if and only if it acts primitively on the lines of $\mathcal{P}$. A direct proof of this equivalence is not known and would be of great interest. We also know that a group acts transitively on the points of a projective plane $\mathcal{P}$ if and only if it acts transitively on the lines of $\mathcal{P}$[Dem97, 2.3.1], hence we will refer to a projective plane simply as *primitive* or *transitive* in each case.

Under the stronger assumption that a plane $\mathcal{P}$ is flag-transitive Ott[Ott04] has made some progress towards ruling out the non-Desarguesian cases left open by Kantor. For a good survey article on flag-transitive projective planes see K. Thas[Tha03].

1.3 The classification programme

Spaces which are not projective planes

The framework for investigating line-transitive finite linear spaces which are not projective planes is given by the following theorem. This theorem also applies to the projective plane case but, as will be seen, stronger statements can be made in this case and we will use those as a framework for investigation instead.

Theorem 1.2. *[CP01] Let G be a group acting transitively on the lines of a finite linear space. Then either G,*

1. *is affine (i.e. has an elementary abelian point-transitive normal subgroup),*
2. *is almost simple, OR*
3. *has a normal subgroup which is not transitive on points.*

The first two possibilities given in the theorem are the same as the cases examined in the flag-transitive case. They together cover the line-transitive, *point quasi-primitive* possibilities; here G is said to act quasi-primitively on a set Γ if all normal subgroups of G are transitive on Γ. The proof of Theorem 1.2 makes use of a classification of all quasi-primitive actions given by Praeger[Pra93]. The majority of the proof of Theorem 1.2 consists of eliminating the possibility that the action of G on Π can be a 'product action.'

The investigation of the quasi-primitive possibilities is likely to focus on well-known properties of the particular groups specified. Already results have appeared for the almost simple case: In [CNP03] and [CS00], the cases where G has socle an alternating group and a sporadic group, respectively, are fully classified.

Weijun Liu and others have produced several papers examining the low rank almost simple cases (particularly focusing on the conjecture that line-primitivity implies point-primitivity): in [ZLL00], it is shown that if $^2G_2(q) \trianglelefteq G \leq Aut(^2G_2(q))$ then any line-primitive action of G is indeed point-primitive; in [Liu01, Liu03a, Liu03d, LLM01] all line-primitive actions of $G = G_2(q), G = {}^3D_4(q)$ and G with socle $^2B_2(q)$ are classified; in [Liu03b, LZLF04] Liu examines the line-transitive actions of $G = PSU(3, 2^a)$ and $G = {}^2G_2(q)$ on linear spaces; finally in [Liu03c] Liu gives possibilities for line-transitive actions of $G = PSL(2, q)$ on a linear space. A stronger result for the case $G = PSL(2, q)$ is known to have been proved by Camina, Neumann and Praeger although it is unpublished; we will state this in full in Chapter 3.

Finally Zalesskiĭ and Camina are investigating actions of almost simple groups with socle $PSL(d, q)$ or $PSU(d, q)$ for d large. Their methods are unlikely to work for linear groups of small dimension.

The third possibility given in Theorem 1.2 is being examined using parameters defined in a paper by Delandtsheer and Doyen[DD89]. They suppose that G acts transitively on the lines of a $t-(v, k, \lambda)$ design while leaving invariant a non-trivial partition $\mathcal{C}$ of the point set Π. Now let $d = |\mathcal{C}| \geq 2$ and $c = |C|$, for $C \in \mathcal{C}$. Then Delandtsheer and Doyen prove that there exist positive integers x, y such that

$$c = \frac{\binom{k}{2} - x}{y} \quad , \quad d = \frac{\binom{k}{2} - y}{x}.$$

An immediate corollary of this result is that if $v > (\binom{k}{2} - 1)^2$ then any line-transitive automorphism group G is necessarily point-primitive (and the first two cases given in Theorem 1.2 apply.) Thus the value $(\binom{k}{2} - 1)^2$, for fixed k, gives an upper bound for the value of v in a point-imprimitive line-transitive linear space. By a set of results in [CP96, KMM89, OPP93, NNO$^+$92], all point-imprimitive line-transitive linear spaces for which v is equal to this upper bound are known.

Other results on the third possibility given in Theorem 1.2 are surveyed by Praeger[Pra01]. Further results are given in [PT03, DNP03].

Projective planes

When we consider line-transitive projective planes our framework for investigation is slightly different. Whilst the theorem given in the previous section also applies to projective planes, by making use of results in [CP93] and [Cam04] we can make somewhat more general statements. The key theorem is the following:

Theorem 1.3. *[Cam04, Theorem 2] Let G act transitively on a projective plane $\mathcal{P}$ and let M be a minimal normal subgroup of G. Then M is either abelian or simple.*

In fact we are able to state our results more strongly by rewriting this result in terms of *components*; here a *component* of a group G is defined to be a subnormal quasisimple subgroup of G. Hence the theorem which will provide the framework for our analysis is the following:

Theorem 1.4. *Suppose that G acts transitively on a projective plane $\mathcal{P}$. Then G contains at most one component.*

The proof of this theorem, which involves rewriting proofs of similar theorems from [CP93] and [Cam04], is given in Section 2.1.

We use this theorem to examine those groups G acting transitively on a projective plane; our interest is in examining the possible unique components of G. Existing results in the literature are generally limited to the case where the component is simple and G is almost simple, as described in the previous section.

1.4 Overview of results

Our primary result concerns groups acting transitively on projective planes:

Theorem A. *Suppose that G acts transitively on a projective plane $\mathcal{P}$ of order x. Then one of the following cases holds:*

- *$\mathcal{P}$ is Desarguesian, $G \geq PSL(3, x)$ and the action is 2-transitive on points;*
- *G does not contain a component.*

In particular this theorem implies that if an almost simple, or almost quasi-simple, group G acts on a projective plane $\mathcal{P}$ of order x then $\mathcal{P}$ is Desarguesian and G has socle $PSL(3, x)$.

The theorem is proved in Chapter 2 by an exhaustive case by case analysis of the different possible unique components given by the classification of finite simple groups. We record the following corollary to the theorem:

Corollary 1.5. *Suppose that G acts transitively on a non-Desarguesian projective plane $\mathcal{P}$. Then $F(G) = F^*(G)$, i.e. the generalized Fitting group of G is equal to the Fitting group of G. Let t be a prime dividing into $|F(G)|$ and N_t the Sylow*

t-subgroup of $F(G)$. Then N_t acts semi-regularly on the points of $\mathcal{P}$ for all t except possibly one. Furthermore one of the following holds:

- *x^2+x+1 is a prime and G is a Frobenius group of odd order dividing $(x^2+x+1)x$ or $(x^2+x+1)(x+1)$;*
- *All minimal normal subgroups of G are elementary abelian, semi-regular and intransitive.*

Proof. The statement $F(G) = F^*(G)$ is a direct consequence of the fact that G has no components. We may then apply [CP93, Theorem 3] to give the result about the Sylow t-subgroups of $F(G)$ (see also Proposition 2.2 in this thesis.)

We know that all minimal normal subgroups are elementary abelian. By [CP93, Corollary 1], any elementary abelian group N is semi-regular.

If N is intransitive then we have the second case above. If N is transitive then it is regular and [Kan87, Lemma 6.5] implies that the first case covers all possibilities. $\square$

Our other major result concerns line-transitive actions upon linear spaces by a group with socle $PSL(3,q)$.

Theorem B. *Suppose that $PSL(3,q) \trianglelefteq G \leq AutPSL(3,q)$ and that G acts line-transitively on a finite linear space $\mathcal{S}$. Then one of the following holds:*

- *$S = PG(2,q)$, the Desarguesian projective plane, and G acts 2-transitively on points;*
- *$PSL(3,q)$ is point-transitive but not line-transitive on $\mathcal{S}$. Furthermore, if G_α is a point-stabilizer in G then $G_\alpha \cap PSL(3,q) \cong PSL(3,q_0)$ where $q = q_0^a$ for some integer a.*

The theorem is proved in Chapter 3. The proof involves examining different possible *significant primes* and exploiting some general lemmas about line-transitive linear spaces, as described in Chapter 3. It is hoped that one can extend the results in this chapter to other almost simple groups acting line-transitively on a linear space. Indeed it is possible that $PSL(3,q)$ is one of the hardest cases (as it acts on a Desarguesian projective plane.)

It is worth observing that the lemmas in Section 3.2 are, to our knowledge, new. Some of these lemmas are of interest in their own right; in particular, Lemma 3.11 is a generalization of Fisher's inequality to non-regular linear spaces. In addition the reduction outlined in Section 3.4 can be directly applied to the study of other almost simple groups acting on finite linear spaces, in particular strengthening some of the results of Weijun Liu given earlier.

1.5 General Notation

We record the general notation which will hold throughout the rest of the thesis.

Upper-case letters $A, \ldots, Q$ will be used to denote groups; upper-case letters $R, \ldots, Z$ will denote matrices. There will be some (standard) exceptions to this, e.g. $Z(G)$ for the centre of a group G, U the unipotent radical of a parabolic subgroup, I the identity matrix.

Lower-case letters are reserved for elements of a group and for integers. In particular q will normally be a prime power, p a prime; if we are working with a particular group of Lie type then p will typically be the characteristic prime.

Lower-case Greek letters, $\alpha, \beta, \ldots$, will generally be used to denote points of a space or to denote superscripts $\pm, \pm 1$ as explained in Section 2.8 and Section 2.10. Other uses for lower-case Greek letters will be explained at the relevant places. For a line of a space we use $\mathfrak{L}$. Upper-case Greek letters, $\Omega, \Gamma, \ldots$, are used to denote sets.

Spaces will be denoted by capital calligraphic letters. So $\mathcal{S}$ will be a general linear space with points Π and lines Λ. A projective plane will be denoted by $\mathcal{P}$.

All of the notation described so far may also apply with subscripts. Thus, for instance, d_g will denote an integer, the number of fixed points of a group element g.

Our group theoretic notation will be as follows: We write $H.G$ for an extension of a group H by a group G and $H : G$ for a split extension. An integer n denotes a cyclic group of order n, while $[n]$ denotes an arbitrary soluble group of order n and p^n denotes an elementary abelian group of order p^n where p is a prime. We write $|H|_p$ for the highest divisor of $|H|$ which is a power of a prime p. $G \circ H$ denotes a central product of groups G and H.

We will write $(a_1, \ldots, a_n)$ to mean the greatest common divisor of the integers $a_1, \ldots, a_n$.

Finally note that, unless stated otherwise, all objects studied in this thesis are finite.

Chapter 2

Projective Planes

"Why don't they make the whole plane out of that black box stuff?"
Steven Wright

In this chapter we prove Theorem A. We begin, in Section 2.1, with a proof of Theorem 1.4 which provides a framework for our investigation. In Section 2.2 we give the basic lemmas and notation which will be used throughout the remainder of the chapter. The remaining sections consider possible actions for different possible components of G, as given by the classification of finite simple groups.

2.1 Framework results

We prove Theorem 1.4 which states that if a group G acts transitively upon a projective plane then G contains at most one component. Our proof of Theorem 1.4 starts with some preliminary results.

Note first that if C is a component of G then $C^\circ := < C^g : g \in G > \cong C \circ C^{g_1} \circ \cdots \circ C^{g_m}$ is a normal subgroup of G where $g_1, \ldots, g_m \in G$; furthermore, if C and D are components of G with C not G-conjugate to D then $[C, D] = 1$ and so $[C^\circ, D^\circ] = 1$.

We need some information about the fixed points of automorphisms of a projective plane $\mathcal{P}$ of order x: If a collineation g fixes at least x points then g is called *quasicentral* and g fixes $x+1$, $x+2$ or $x+\sqrt{x}+1$ points[Dem97, 4.1.7]. In the first two cases g fixes a *fan*, namely a line $\mathfrak{L}$ and a point α and all the points on $\mathfrak{L}$ and all the lines incident with α. The distinction between the two cases depends on whether or not α lies on $\mathfrak{L}$. In the third case g fixes a subplane of $\mathcal{P}$ of order $\sqrt{x}$.

In addition we have the following lemma:

Lemma 2.1. *[Dem97, 3.1.2 and 4.1.6] Let $\mathcal{P}$ be a projective plane of order x. If H is a group of collineations of $\mathcal{P}$ which does not fix a subplane of $\mathcal{P}$ then the fixed set of H lies inside a fan. If, on the other hand, H fixes a subplane of $\mathcal{P}$ then that subplane has order at most $\sqrt{x}$.*

We are now ready to prove our first result which is very similar to [CP93, Theorem 3]:

Proposition 2.2. *Let G be a transitive automorphism group of a projective plane $\mathcal{P}$ of order greater than 4. Let G have normal subgroups M and N such that $M_\alpha \neq 1$ and $N_\alpha \neq 1$ for some point α. Then $[N, M] \neq 1$.*

Proof. Let M and N be two normal subgroups of G such that there is a point α so that $M_\alpha \neq 1$ and $N_\alpha \neq 1$ and $[M, N] = 1$.

Consider the point $\beta \in \alpha N$ and let $n \in N$ be such that $\beta = \alpha n$. If $m \in M_\alpha$, then $\beta m = \alpha n m = \alpha m n = \beta$. Thus αN is fixed point-wise by M_α. If $\beta \in \alpha N \backslash \{\alpha\}$ and $\mathfrak{L}$ is the line through α and β, then M_α fixes $\mathfrak{L}$ set-wise. Thus there is a line $\mathfrak{L}$ through α which is fixed by M_α and M_α fixes at least two points. A similar result applies with N replacing M.

Next we show that every line through α is fixed either by M_α or N_α. Assume that this is false and let $\mathfrak{L}$ be a line through α which is fixed by neither. Since G is line-transitive, there is some point β such that M_β fixes $\mathfrak{L}$. Now, since $[M, N] = 1$, N_α acts on the set of fixed lines of M_β. Thus each image of $\mathfrak{L}$ under the action of N_α is a line through α fixed by M_β. Since N_α does not fix $\mathfrak{L}$, it follows that M_β fixes α. However, this means that $M_\beta = M_\alpha$ and hence M_α fixes $\mathfrak{L}$ which is a contradiction to our assumption.

Thus for one of M_α and N_α, the number of lines through α which are fixed must be at least $k/2$. Without loss of generality, this is true for N_α. We now show that the fixed set of N_α is a subplane of $\mathcal{P}$. By the lemma above it is sufficient to prove that $N_G(N_\alpha)$ acts transitively on the set of lines fixed by N_α; to show this we demonstrate that $N_\mathfrak{L} = N_\alpha$ for any line $\mathfrak{L}$ fixed by N_α.

Let $\mathfrak{L}$ be any line through α which is fixed by N_α. Let $m \in M$ such that $\mathfrak{L}m \neq \mathfrak{L}$. Then, since $[M, N] = 1$, it follows that $\mathfrak{L}mN_\mathfrak{L} = \mathfrak{L}N_\mathfrak{L}m = \mathfrak{L}m$, that is $N_\mathfrak{L}$ fixes $\mathfrak{L}m$ and so $N_\mathfrak{L}$ fixes $\mathfrak{L}m \cap \mathfrak{L} = \{\beta\}$, say. Then $N_\alpha \subseteq N_\mathfrak{L} \subseteq N_\beta$, and since N_α is conjugate to N_β, we obtain $N_\alpha = N_\mathfrak{L}$.

Since N is normal in G, $N_G(N_{\mathfrak{L}})$ is transitive on the lines fixed by $N_{\mathfrak{L}} = N_\alpha$. Thus the fixed set of N_α is a subplane of $\mathcal{P}$ with line size at least $k/2$. This is a contradiction of the lemma above. □

Corollary 2.3. *Suppose that G acts transitively on a projective plane $\mathcal{P}$. Then all components of G are conjugate in G.*

Proof. If $\mathcal{P}$ is Desarguesian then G contains at most one component and the statement holds.

By [Dem97, 3.2.15] a non-Desarguesian projective plane has order at least 9. Thus by the previous theorem any two normal subgroups M and N of G with $M_\alpha \neq 1$ and $N_\alpha \neq 1$ for some point α satisfy $[N, M] \neq 1$.

Now suppose that C and D are components of G which are not conjugate in G. Then C° and D° are distinct normal subgroups of G. Note that any component contains an involution and, since the number of points in $\mathcal{P}$ is odd, each involution must fix a point. The theorem implies that $[C^\circ, D^\circ] \neq 1$. This is a contradiction. □

We can now prove Theorem 1.4. Our method of proof is very similar to that of Camina [Cam04, Theorem 1]. First we state some preliminary results:

Lemma 2.4. *[CP93, Theorem 1] Let $\mathcal{P}$ be a finite linear space and let G be a line-transitive automorphism group of $\mathcal{P}$. Let N be a normal subgroup of G. Then N acts faithfully on each of its point orbits.*

Lemma 2.5. *[HP73, XIII.13.1] Let A be an abelian collineation group of a projective plane of order x then $|A| \leq x^2 + x + 1$.*

Lemma 2.6. *[Dem97, 4.1.6] Let A be a collineation group of a projective plane of order x. If A fixes a subplane of order y then either $y^2 = x$ or $y(y+1) \leq x - 2$.*

Theorem 1.4. *Suppose that G acts transitively on a projective plane $\mathcal{P}$. Then G contains at most one component.*

Proof. We may assume that $\mathcal{P}$ is non-Desarguesian of order x and that all components are conjugate in G. Let C be a component of G and write $C^\circ = C_1 \circ \cdots \circ C_m, m \geq 2$, normal in $\mathcal{P}$ with each C_i isomorphic to C.

Let D be a Sylow 2-subgroup of C°. Since $\mathcal{P}$ has an odd number of points there is a point α so that D fixes α. Thus $(C_i)_\alpha \neq 1$ for $1 \leq i \leq m$. Since G acts transitively on $\mathcal{P}$ this is true for all points α. Choose α so that $(C_1)_\alpha$ has maximal order. Observe that $[C_2, (C_1)_\alpha] = 1$ so αC_2 consists of points fixed by $(C_1)_\alpha$.

Now C° is faithful on all its point orbits by Lemma 2.4. This implies that αC_2 contains at least 5 points as C_2 is quasisimple and normal in C°. The fixed set of $(C_1)_\alpha$ is either a subplane or lies inside a fan. But, since C_2 does not fix any point, we conclude that $(C_1)_\alpha$ fixes a subplane whose order is at most $\sqrt{x}$.

We now show that for any line $\mathfrak{L}$ incident with α there is a j so that $(C_j)_\alpha$ fixes $\mathfrak{L}$. Choose a line $\mathfrak{L}$ incident with α. If $(C_1)_\alpha$ fixes $\mathfrak{L}$ there is nothing to prove. We know that there exists a line, $\mathfrak{L}_1$, which is incident with α and is fixed by $(C_1)_\alpha$. But G is transitive on lines so there is $g \in G$ with $\mathfrak{L}_1 g = \mathfrak{L}$. Then $\beta = \alpha g$ is incident with $\mathfrak{L}$ and $((C_1)_\alpha)^g$ fixes $\mathfrak{L}$. But there exists j so that $((C_1)_\alpha)^g = (C_j)_\beta$ since g permutes the factors C_i. Let $i \neq j$. Then $(C_i)_\alpha$ commutes with $(C_j)_\beta$ and so acts on the set of lines fixed by $(C_j)_\beta$. If $(C_i)_\alpha$ fixes $\mathfrak{L}$ then we have proved our claim. If not we see that $(C_j)_\beta$ fixes at least two lines through α and so fixes α. However $((C_1)_\alpha)^g = (C_j)_\beta$ so by the maximality of $(C_1)_\alpha$ we have $(C_j)_\alpha = (C_j)_\beta$ and the claim is proved.

Let y be the order of the subplane fixed by $(C_i)_\alpha$. Then $m(y+1) \geq x+1$. If $y = \sqrt{x}$ then this implies that $m \geq \sqrt{x}$. If $y \neq \sqrt{x}$ then Lemma 2.6 implies that $y(y+1) \leq x-2$. Thus $m \geq y+1$ and so $m \geq \sqrt{x+1} > \sqrt{x}$.

Since C° has an abelian subgroup of order at least 5^m it follows from Lemma 2.5 that $x^2 + x + 1 \geq 5^m \geq 5^{\sqrt{x}}$. This has no solutions. □

For the remainder of this chapter we will consider G acting transitively on a projective plane $\mathcal{P}$ where G has a unique component. We go through the classification of finite simple groups and verify that if $\mathcal{P}$ is not Desarguesian then no such action exists.

2.2 Basic Results and Notation

The notation outlined in this section will hold throughout the rest of the chapter. We also state here a number of basic results which will be used repeatedly throughout the chapter.

Projective Plane Results

Consider a group G acting transitively on a projective plane $\mathcal{P}$ of order x with $v = x^2 + x + 1$ points and lines. We have noted already that v is an odd number. In fact we know more than this:

Lemma 2.7. *[Kan87, p.33] Let G act transitively on a projective plane with G_α a point-stabilizer. Then if p_1 is a prime $\equiv 2(3)$ then G_α contains some Sylow p_1-subgroup of G. Moreover, G_α contains a subgroup of index at most 3 in a Sylow 3-subgroup of G.*

For g an element of G we write n_g for the size of the G-conjugacy class of g in G and r_g for the number of these conjugates lying in a point-stabilizer G_α, for some fixed point α in $\mathcal{P}$. Furthermore, d_g is the number of fixed points of g. We will sometimes also write $r_g(B)$ for the number of G-conjugates of g lying in a subgroup B of G, so $r_g = r_g(G_\alpha)$.

We know already that if a collineation g fixes at least x points then g is called *quasicentral* and g fixes $x+1$, $x+2$ or $x+\sqrt{x}+1$ points[Dem97, 4.1.7]. Furthermore, if a collineation has $x+1$ or $x+2$ fixed points then it is known as a *perspectivity* and Wagner has proved that if G contains a nontrivial perspectivity and G acts transitively on $\mathcal{P}$ then $\mathcal{P}$ is Desarguesian and $G \geq PSL_3(x)$[Wag59].

Now any involution is quasicentral ([Dem97, 3.1.6]) and so all the groups that we consider contain quasicentral collineations. Thus we assume that x is a square, say $x = u^2$, and that all quasicentral collineations, in particular all involutions, have $u^2 + u + 1$ fixed points.

We will be particularly interested in properties of integers of the form u^2+u+1 where u is an integer.

Lemma 2.8. *If $x = u^2$ then $x^2 + x + 1 = (u^2 + u + 1)(u^2 - u + 1)$, where $(u^2 + u + 1, u^2 - u + 1) = 1$.*

Lemma 2.9. *[Lju43, p.11] If $u^2 + u + 1 = p_1^a$ where p_1 is a prime, then either $p_1^a = p_1$ or $p_1^a = 7^3$.*

Lemma 2.10. *[Kan87, p.33] If $x = u^2$ and $x^2 + x + 1 = p^a m$ for a prime p with $a > 1$, then either $m > 8p^a$ or $p^a = u^2 \pm u + 1 = 7^3$.*

Lemma 2.11. *Let $x = u^2$ and let g be an involution acting on projective plane $\mathcal{P}$. Then*

- $\frac{n_g}{r_g} = u^2 - u + 1$;
- $d_g = u^2 + u + 1$;
- $v = \frac{n_g}{r_g} d_g$ *and* $(\frac{n_g}{r_g}, d_g) = 1$.

Proof. Count pairs of the form (α, g), where α is a point and g is an involution fixing α, in two different ways. Then

$$|\{(\alpha, g) : \alpha g = \alpha\}| = v r_g = n_g d_g$$

We know already that $d_g = u^2 + u + 1$ thus we must have $\frac{n_g}{r_g} = u^2 - u + 1$ and the result follows. □

Lemma 2.12. *Suppose that there exists an involution g in G such that $n_g = 2^c p^a m$ where $(m, 2p) = 1$. Then the largest power of p in v is less than or equal to $max(p^a, m + 2\sqrt{m} + 2)$.*

Proof. If $p | \frac{n_g}{r_g}$ then clearly the highest power of p dividing v divides p^a. If not, then $u^2 - u + 1 = \frac{n_g}{r_g}$ divides into m. Then the highest power of p dividing v divides into $d_g = u^2 + u + 1 < (u^2 - u + 1) + 2\sqrt{u^2 - u + 1} + 2$. □

It is in our exploitation of the last two results that our treatment will differ substantially from that of Kantor in the primitive case. We will make use of the equalities outlined in Lemma 2.11, taking g to be a member of a small conjugacy class of involutions.

Group Theory Results and Notation

We begin with a general lemma which will be useful throughout the chapter.

Lemma 2.13. *Let $C < A \times B$. Suppose $|A| < |B : N|$ where N is the largest proper normal subgroup of B. Then either:*

- $C \leq A \times B_1$ *for* $B_1 < B$; *or*
- $C = A_1 \times B$ *for* $A_1 \leq A$.

Proof. Suppose $C \not\leq A \times B_1$ for $B_1 < B$. Then define $B_1 = \{(1, b) : (a, b) \in C\} \cong B$ and observe that the projection $C \to A, (a, b) \mapsto a$ has kernel $K = \{(1, b) \in C\} \lhd B_1$. But $|B_1 : K| \leq |A| < |B : N|$ where N is the largest proper normal subgroup of B. Thus $K = B_1$ and $C = A_1 \times B$ for some $A_1 \leq A$ as required. □

Now we want to show that a group G with unique component L cannot act transitively on a projective plane $\mathcal{P}$ unless it contains a non-trivial perspectivity. Put $L_\alpha = G_\alpha \cap L$, the stabilizer of a point α in the action of L on $\mathcal{P}$. In general, we will set M to be a maximal subgroup of the component L which contains L_α. Define $L^\dagger := L/Z(L)$ and $M^\dagger := M/(Z(L) \cap M)$.

Write $G = (L \circ C_G(L)).N$ where N is a subgroup of $OutL$. Then $G/C_G(L)$ is an almost simple group and we use results about the maximal subgroups of such groups:

When $L^\dagger$ is a classical simple group we use the results of Aschbacher[Asc84] as described in Kleidman and Liebeck [KL90]. These results give information about the maximal subgroups of a group $L^\dagger.N$ with simple socle $L^\dagger$ a classical group. In small dimensions we will refer to the results given by Kleidman[Kle87] who uses identical notation.

We will sometimes precede the structure of a subgroup of a projective group with $\hat{}$ which means that we are giving the structure of the pre-image in the corresponding universal group. For a given element $g \in L$ we will often write g^* for an element in the corresponding universal group which projects onto g. The symbol * will also be used in a different way, with groups, e.g. P_1^*, to signal that a group is a subgroup of a section of L or $L^\dagger$.

We can exclude several atypical situations by observing that, except when $L = P\Omega^+(8,q)$, we may assume that $G/C_G(L) \leq \Gamma L$, the full semilinear classical group associated with L. The cases we have excluded here are when $L^\dagger = PSL(n,q)$ while $G/C_G(L)$ contains an inverse-transpose automorphism of L and when $L = Sp(4,2^f)$ while $G/C_G(L)$ contains a graph automorphism of L. In both cases G contains a normal subgroup H of index 2 such that $H/C_H(L) \leq \Gamma L$. Since we are acting on a set of odd order, any transitive action of G induces a transitive action of H. Thus, except when $L^\dagger = P\Omega^+(8,q)$, we assume that $G/C_G(L) \leq \Gamma L$.

We will write $M \in \mathcal{C}_i$ to mean that $M^\dagger$ is in the i-th family of natural maximal subgroups of $L^\dagger$ given by Kleidman and Liebeck[KL90]. When M is parabolic we will write $M = P_m$ to mean that M is a maximal parabolic subgroup fixing a totally singular subspace W of dimension m inside the natural classical geometry V of dimension n.

When $L^\dagger$ is an exceptional simple group we use different sources to find information about maximal subgroups M of L. When M is parabolic we refer to [Car89, GLS94, GL83]. In some other cases, the maximal subgroups are completely enumerated; in particular for $L^\dagger = {}^2B_2(q)$[Suz62], for $L^\dagger = {}^2G_2(q)$[Kle88a, War66], for $L^\dagger = G_2(q)$[Kle88a, Coo81], for $L^\dagger = {}^2F_4'(q)$[Mal91, CCN$^+$85] and for $L^\dagger = {}^3D_4(q)$[Kle88b].

In both classical and exceptional cases, we appeal to a result of Liebeck and Saxl [LS85] and Kantor[Kan87] which gives the maximal subgroups of odd index in an almost simple group. In particular, when the socle is a finite simple classical group acting on a classical geometry V, such a maximal subgroup either lies in $\mathcal{C}_1$ (stabilizers of totally singular or non-singular subspaces) for characteristic 2 or, when the characteristic is odd, lies in $\mathcal{C}_1$, $\mathcal{C}_2$ (stabilizers of decompositions into subspaces of fixed dimension, $V = \oplus_{i=1}^t V_i$) or $\mathcal{C}_5$ (stabilizers of subfields) or is in a small set of listed exceptions.

Finally, when $L^\dagger$ is a sporadic simple group we refer to [Asc86] which, amongst many other things, lists the maximal subgroups of odd index.

Our analysis becomes slightly simpler by using the following result of Camina and Praeger which is a corollary of Lemma 2.4:

Lemma 2.14. *[CP93, Corollary 1] Let N be an abelian normal subgroup of a group G. Suppose that G acts line-transitively on a finite linear space $\mathcal{P}$. Then N acts semiregularly on the points of $\mathcal{P}$.*

In the case where $\mathcal{P}$ is a projective plane we can apply Lemma 2.7. Thus if L is a unique component of G then $Z(L)$ is normal in G and must have order only divisible by primes congruent to 1(3) or by 3 to the first power. In the case where L is a group of Lie type, for instance, this implies that L is simple unless it is isomorphic to $E_6(q)$, ${}^2E_6(q)$, $U(n,q)$ or $PSL(n,q)$ for certain n.

Hypothesis

Finally, we may state our hypothesis for the rest of the chapter:

Hypothesis. *Assume that G is a group with a unique component L. Assume that G acts transitively on a projective plane $\mathcal{P}$ of order $x = u^2$ such that all involutions fix u^2+u+1 points of $\mathcal{P}$. Assume that $L_\alpha \leq M$ where M is a maximal subgroup of L of odd index. Furthermore assume that $v > |L : M|$.*

Throughout the rest of the chapter we will set $L^\dagger$ to be in a particular family of simple groups of Lie type and will demonstrate that our hypothesis leads to a contradiction.

2.3 $L^\dagger$ is alternating or sporadic

In this section we prove the following proposition:

Proposition 2.15. *Suppose G has a unique component L such that $L^\dagger$ is isomorphic to an alternating group, A_n with $n \geq 5$, or a sporadic simple group. Then G does not act transitively on a projective plane.*

When $L^\dagger$ is a sporadic simple group, the maximal subgroups of $L^\dagger$ of odd index are given by Aschbacher[Asc86]. Aschbacher's list implies that any maximal subgroup M of odd index in L has index divisible by 9 or by a prime congruent to 2(3). Since L_α must lie in such a maximal subgroup this contradicts Lemma 2.7.

Suppose that $L^\dagger \cong A_n$, the alternating group on n letters. If $n \neq 6, 7$ then $Z(L) \leq 2$ [Sch11]; thus, by Lemma 2.14, $L = L^\dagger = A_n$. If $n = 6, 7$ then $Z(L) \leq 6$ and so, by Lemma 2.14, $L = A_n$ or $L = 3.A_n$.

Assume for the moment that $n > 7$ and so $L = A_n$. Let $g \in L = A_n$ be a double transposition. Then $n_g = \frac{n(n-1)(n-2)(n-3)}{8}$. Now A_n contains an abelian subgroup, H, of size $2^{\lfloor \frac{n}{2} \rfloor - 1}$ which contains at least $\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor - 1)$ L-conjugates of g.

Since H lies inside a Sylow 2-subgroup of L, we know that H lies in L_α for some point α. We conclude that

$$\frac{n_g}{r_g} \leq \frac{n(n-1)(n-2)(n-3)}{8\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor - 1)}.$$

Next we refer to Lemma 2.5 and observe that $|H| \leq v$. Furthermore, for $u > 2$, $v < 2(\frac{n_g}{r_g})^2$. Hence

$$\begin{aligned} & 2^{\lfloor \frac{n}{2} \rfloor - 1} \leq 2\frac{n^2(n-1)^2(n-2)^2(n-3)^2}{2^6\lfloor \frac{n}{2} \rfloor^2 (\lfloor \frac{n}{2} \rfloor - 1)^2} \\ \implies \quad & 2^{\lfloor \frac{n}{2} \rfloor} < n^4 \\ \implies \quad & n \leq 43. \end{aligned}$$

If $u = 2$ then $v = 21$ and again we can conclude that $n \leq 43$. Now to examine the cases where $7 < n \leq 43$ we use a method similar to that in [CNP03, Section 5].

Consider the usual permutation action of $L = A_n$ as $Alt(\Omega)$, acting on a set Ω of size n. Then L_α contains a Sylow p-subgroup of L for every prime $p \equiv 2(3)$ and a subgroup of index 3 in a Sylow 3-subgroup of L.

Let Γ be the longest orbit of L_α in Ω. If $8 \leq n \leq 10$ then, since L_α contains a Sylow 2-group and a Sylow 5-group of L, L_α^Γ must be primitive; if $11 \leq n \leq 21$

then the same conclusion comes from the primes 2 and 11; if $22 \leq n \leq 33$ then the same conclusion comes from the primes 2 and 17; and if $34 \leq n \leq 43$ then the same conclusion comes from the primes 2 and 29. Now L_α^Γ has odd index in $Alt(\Gamma)$ and 5 does not divide the index. By [LS85] this means that L_α^Γ contains $Alt(\Gamma)$.

For $n \geq 11, n \neq 39$, we claim that $|\Gamma| \geq n-2$. This is proved using Lemma 2.7 for each individual value of n. We do not reproduce this here but consider, for instance, when $n = 16$: Then L_α contains elements with cycle type (11) and $(8,8)$ and so $|\Gamma| = 16 \geq n-2$.

Let us examine this case, where $n \geq 11, n \neq 39$. Consider again, g, a double transposition with $n_g = \frac{n(n-1)(n-2)(n-3)}{8}$. Then $r_g \geq \frac{(n-2)(n-3)(n-4)(n-5)}{8}$ and so $\frac{n_g}{r_g} \leq \frac{n(n-1)}{(n-4)(n-5)} < 3$ for $n \geq 11$. This is impossible.

For $n = 39$ it turns out, using Lemma 2.7, that $|\Gamma| \geq 34$. Then $\frac{n_g}{r_g} < 3$ and this case is excluded.

For $n = 8$ or 10, the same argument gives $|\Gamma| = n$ and no action exists. For $n = 9$, $|\Gamma| \geq 5$ and, referring to [LS85], L_α lies in an intransitive subgroup of L and this contradicts Lemma 2.7.

Now suppose $n \leq 7$. If $n = 5$ or 6 then Lemma 2.7 implies that $|L : L_\alpha| \leq 3$. This is impossible since no subgroup of such small index exists in L. We are left with $n = 7$.

When $n = 7$ we know that L_α contains an element of order 5. Examining [CCN$^+$85] this means that $M^\dagger = S_5$ or A_6. In fact we must have $L_\alpha = S_5$ or A_6. In both cases $\frac{n_g}{r_g}$ is not an integer. Thus all cases are excluded.

Remark. *It is worth noting that we could immediately conclude, from the transitivity of G and by appealing to [GH00, Theorem 1], that $n \leq 21$. However this is a large result and so we have given a more elementary and direct proof above.*

2.4 $L^\dagger = PSL(n,q)$

In this section we prove the following proposition:

Proposition 2.16. *Suppose G has a unique component such that $L^\dagger$ is isomorphic to $PSL(n,q)$ with $n > 3$. Then G does not act transitively on a projective plane.*

Note that, by the result of Wagner[Wag59] cited above, it is sufficient to prove that our hypothesis, with $L^\dagger = PSL(n,q)$, leads to a contradiction. Recall that, for

$n \neq 8$, we assume that $G/C_G(L) \leq \Gamma L(n,q)$. We also suppose that $n > 3$ for the remainder of this section.

Now consider $SL(n,q)$ acting naturally on a vector space V. Then recall that a *transvection*, g^* say, in $SL(n,q)$ is a collineation of V such that $g^* - I$ has rank 1 and square 0. We now state the following preliminary result:

Lemma 2.17. *Let C be a conjugacy classes of involutions in L corresponding to either,*

- *diagonalizable involutions in the natural modular representation of $SL(n,q)$ with q odd; or to*
- *the projective image of transvections in $SL(n,q)$, where $q = 2^a$ for some integer a.*

Then C is invariant under ΓL.

Proof. Consider the diagonalizable case first. We need to consider the actions by conjugation of automorphisms of $SL(n,q)$ on a diagonal matrix,

$$g^* = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Clearly a field automorphism will preserve g^*. Similarly an automorphism lying in $GL(n,q)$ of form,

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a \end{pmatrix}$$

where $a \in GF(q)^*$, also preserves g^*. These generate the full outer automorphism group of $SL(n,q)$ in $\Gamma L(n,q)$ and we are done. In the case where we have a transvection then we consider the actions by conjugation of automorphisms of $SL(n,q)$ on a matrix,

$$g^* = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ & 1 & & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 1 \end{pmatrix}.$$

Clearly both field automorphisms and the automorphism in $GL(n,q)$ exhibited above preserve g^* and we are done. □

Much of the ensuing treatment will involve counting involutions g. We will take care to ensure that g is always of one of the two types in this lemma thus ensuring that $n_g = r_g(L) = |L : C_L(g)|$ and $r_g = r_g(L_\alpha)$. Also, observe that we may exclude $PSL(4,2) \cong A_8$. We begin by restricting the family within which M, a maximal subgroup of L containing L_α, may lie:

2.4.1 L_α must lie in a parabolic subgroup

By Liebeck and Saxl [LS85], we know that L_α lies inside a maximal subgroup M where

- for q odd, $M \in \mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$; or $n = 4$;
- for q even, $M \in \mathcal{C}_1$.

Lemma 2.18. *L_α cannot lie inside a maximal subgroup from families $\mathcal{C}_i, i > 1$.*

Proof. We may assume that q is odd. In $SL(n,q)$, define

$$g^* = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then g^* is centralized in $SL(n,q)$ by $(SL(2,q) \times SL(n-2,q)).(q-1)$ Then the projective image,g , of g^* is an involution in L and n_g divides into

$$\frac{q^{2(n-2)}(q^{n-1} + \cdots + q + 1)(q^{n-2} + \cdots + q + 1)}{q+1}.$$

Examining the order of subgroups M in $\mathcal{C}_2$ or $\mathcal{C}_5$ we find that $|M|_p \leq q^{\frac{1}{4}(n-1)n}$ and hence $|L : M|_p \geq q^{\frac{1}{4}(n-1)n}$. Since $n > 3$, we know that q^2 divides the index of any maximal subgroup in $\mathcal{C}_2$ or $\mathcal{C}_5$. In the case where $n = 4$, the only maximal subgroups of odd index which do not lie in families $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_5$ also have index divisible by q^2. Hence $p \geq 7$ by Lemma 2.7. Then, by Lemma 2.12, the largest power of p in v is $q^{2(n-2)}$.

Thus, for $n > 4$, $q^{\frac{1}{2}n(n-1)-2(n-2)} = q^{\frac{1}{2}(n^2-5n+8)}$ divides the order of L_α. We therefore need to have $\frac{1}{2}(n^2-5n+8) \leq \frac{1}{4}(n-1)n$ and so $n < 7$.

If n is 5 or 6 then the only possibility that fits this inequality is when $M = N_L(L(n,q_0))$ for $q = q_0^2$. But then $|L : M|$ is even and so this case can be excluded. This possibility can also be excluded when $n = 4$. However when $n = 4$ (and so $L = PSL(4,q)$) we also need to consider the following further possibilities:

- $M = \hat{\ }(SL(2,q) \times SL(2,q)).(q-1).2$. In this case $|L : M| = n_g = \frac{1}{2}q^4(q^2 + 1)(q^2+q+1)$. Then we know that the maximum power of p in v is q^4 hence L_α contains Sylow p-subgroups of M. However the index of a parabolic subgroup in $SL(2,q)$ is even, hence we must have $\hat{\ }(SL(2,q) \times SL(2,q)).2 < L_\alpha$. Then we know that for some α, $L_\alpha > \hat{\ }\begin{pmatrix} SL(2,q) & \\ & SL(2,q) \end{pmatrix}$. Since L_α also contains a Sylow 2-subgroup of $PSL(4,q)$, this implies that L_α must contain the projective image of $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ which is L-conjugate to g and so $r_g \geq q^2(q+1)^2$. Thus $\frac{n_g}{r_g} \leq \frac{1}{2}q^2(q^2+1)$ and $v \leq q^4(q^2+1)(q^2+q+1)$ and so $v = \frac{1}{2}q^4(q^2+1)(q^2+q+1)$ contradicting Lemma 2.10.

- $M = L(4,q_0).[\frac{c}{(q-1,4)}(q_0-1,4)]$ where $c = (q-1)/(q_0-1, \frac{q-1}{(q-1,4)})$ and $q = q_0^3$. Then $|L : M| = (q_0^{12}(q_0^8+q_0^4+1)(q_0^6+q_0^3+1)(q_0^4+q_0^2+1))/(\frac{c}{(q-1,4)}(q_0-1,4))$. Now we know that $p \equiv 1(3)$ and so the highest power of 3 in c is 3. Then we have $9 \big| |L : M|$ which is impossible.

- M is of odd index but does not lie in families $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. Examining the tables of Kleidman[Kle87], we find that there are two possibilities: Either $M \in \mathcal{C}_6$ and $M \cong 2^4.A_6$ or $M \in \mathcal{C}_8$ and $M \cong PGSp(4,q)$. In the former case, q^6 divides $|L : M|$ which is a contradiction. In the latter case, since $p \equiv 1(3)$, we find that 9 divides $|L : M|$ which, again, is a contradiction.

$\square$

Thus we assume from here on that L_α lies inside $M \in \mathcal{C}_1$. This means that L_α must always lie inside a parabolic subgroup, P_m, which stabilizes a subspace W of dimension m in the natural vector space for G. We now seek to bound m.

2.4.2 L_α lies in P_m, m small

We begin by noting some preliminary facts which we will use to establish which parabolic groups P_m are possible candidates to contain L_α. In particular we will show that m is small.

Lemma 2.19. *Suppose L_α lies inside P_m. For $r|\binom{n}{m}$, r prime, there exists an integer a such that $(1+q^a+\cdots+q^{a(r-1)})$ divides into $|L:P_m|$ which, in turn, divides into v.*

Proof. Let $r^a = |n|_r$. Clearly $1+q^{r^i}+\cdots+q^{r^i(r-1)}$ divides q^n-1 for $i<a$. Then to prove Lemma 2.19 it is sufficient to prove that the polynomial $1+x^{r^i}+\cdots+x^{r^i(r-1)}$ is irreducible in $\mathbb{Q}[x]$.

Now observe that $1+x^{r^i}+\cdots+x^{r^i(r-1)}$ is the product of those roots μ of 1 for which $\mu^{r^{i+1}}=1$ and $\mu^{r^j}\neq 1$ where $j\leq i$. In other words $1+x^{r^i}+\cdots+x^{r^i(r-1)}$ is the r^{i+1}-th cyclotomic polynomial and so is irreducible. □

Corollary 2.20. *Suppose L_α lies inside P_m.*

- *If $p\equiv 1(3)$ then for all primes r dividing $\binom{n}{m}$, we must have $r\equiv 1(3)$ or $r=3$ and $9 \not|\binom{n}{m}$.*
- *If p is odd then $\binom{n}{m}$ must be odd and so either*
    - *n is odd; or*
    - *n is even and m is even.*
- *If $p=2$ then $\binom{n}{m}\not\equiv 0(4)$.*

Proof. We need only prove the final statement. Suppose $4|\binom{n}{m}$. Then either $(q^2+1)|v$ or $(q+1)^2|v$. This means that either v is divisible by a prime equivalent to $2(3)$ or that $9\big|v$. Both of these are impossible. □

Note that, since $(n,q)\neq(4,2)$, the smallest index of a parabolic subgroup in $PSL(n,q)$, $n\geq 4$ is 31 ([KL90, table 5.2A]). Since x is a square we know that $v\geq 91$ and so $d_g<2\frac{n_g}{r_g}$.

Case: n odd, p odd

In this case L contains the projective image, g, of

$$g^* = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Then $n_g = q^{n-1}(q^{n-1} + \cdots + q + 1)$. Furthermore, since $n \geq 4$, g is conjugate in G to the projective image, h, of at least one other diagonal matrix. Then g and h commute and lie in an elementary abelian 2-group. Since L_α contains a Sylow 2-subgroup of L, we must have $r_g \geq 2$.

Thus $\frac{n_g}{r_g} \leq \frac{1}{2}q^{n-1}(q^{n-1} + \cdots + q + 1)$, $d_g \leq q^{n-1}(q^{n-1} + \cdots + q + 1)$ and $v \leq \frac{1}{2}q^{2n-2}(q^{n-1} + \cdots + q + 1)^2$. Now observe that,

$$\begin{aligned} \frac{1}{2}(q^{n-1} + \cdots + q + 1)^2 \geq q^{2n-1} \quad &\Longrightarrow \quad (q^n - 1)^2 \geq 2q^{2n-1}(q-1)^2 \\ &\Longrightarrow \quad q^{2n} \geq 2q^{2n-1}(q-1)^2 \\ &\Longrightarrow \quad q \geq 2(q-1)^2 \\ &\Longrightarrow \quad q < 3. \end{aligned}$$

We know that $q \geq 3$ hence $\frac{1}{2}(q^{n-1} + \cdots + q + 1)^2 < q^{2n-1}$ and $v < q^{4n-3}$. But $|L : P_m| > q^{m(n-m)}$ hence, for $n \geq 23$, we have $m \leq 4$. We use Corollary 2.20 to narrow down the possibilities:

1. For $p \equiv 1(3)$ we find, by explicit calculation using Corollary 2.20, that $m \leq 4$ for all n. In fact, checking small n we find that if $m = 1, 2$ then $n \geq 7$; if $m = 3$ then $n \geq 39$; if $m = 4$ then $n > 70$.

2. For $p \not\equiv 1(3)$ then $\frac{n_g}{r_g} | 3(q^{n-1} + \cdots + q + 1)$. Hence $d_g < 3.q^n$ and so $v < 9q^{2n}$. For $n \geq 11$ this implies that $m \leq 2$.

   Checking the cases where $n < 11$ we find that $m \leq 2$ or $(n, m) = (7, 3)$. This final case will be dealt with along with other exceptional cases at the end of Section 2.4.3.

Case: n even, p odd

Note that in this case we must have m even and L contains the projective image, g, of

$$g^* = \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

Now $n_g = q^{2(n-2)}(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1)$. Again $r_g \geq 2$ and so $\frac{n_g}{r_g} \leq \frac{1}{2}q^{2(n-2)}(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1)$. This gives $d_g \leq q^{2(n-2)}(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1)$ and so $v \leq \frac{1}{2}q^{4(n-2)}(q^{n-2}+\cdots+q^2+1)^2(q^{n-2}+\cdots+q+1)^2$.

In a similar fashion to before we know that, for $q \geq 3$ and $n \geq 4$,

$$\frac{1}{2}(q^{n-2}+\cdots+q^2+1)^2(q^{n-2}+\cdots+q+1)^2 < q^{4n-7}$$

and so $v < q^{8n-15}$. But $|PSL(n,q) : P_m| > q^{m(n-m)}$ hence, for $n \geq 70$, we have $m \leq 8$. Once again we use Corollary 2.20 to narrow down the possibilities:

1. For $p \equiv 1(3)$, we find that $n < 70$ implies that $m = 2$. In fact $(n,m) = (14,2), (38,2)$ or $(62,2)$.

2. For $p \not\equiv 1(3)$, $\frac{n_g}{r_g} | 3(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1) < 3q^{2n-3}$. Thus $v < 9q^{4n-5}$. But $|G : P_m| > q^{m(n-m)}$. Thus for $n \geq 18$ we must have $m \leq 4$. For $n < 18$, $m \leq 4$ or $(n,m) = (14,6)$. This final case will be dealt with along with other exceptional cases in Section 2.4.3.

Case: $p = 2$

In this case G contains the projective image, g, of

$$g^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}.$$

Here g^* is a transvection and $n_g = (q^{n-1}-1)(q^{n-1}+\cdots+q+1)$. Examining a Sylow-2 subgroup of $PSL(n,q)$ we see that it contains at least $2(q^{n-1}-1)$ L-conjugates of g. Since L_α must contain one such Sylow 2-subgroup, we conclude that $r_g \geq 2(q^{n-1}-1)$ and so $\frac{n_g}{r_g} < \frac{1}{2}(q^{n-1}+\cdots+q+1)$. Since $d_g < 2\frac{n_g}{r_g}$, $v < \frac{1}{2}(q^{n-1}+\cdots+q+1)^2$. Also, since $L_\alpha < P_m$ and $|PSL(n,q):P_m| > q^{m(n-m)}$, we conclude that, for $n \geq 10$, $m \leq 2$.

For $n < 10$, the fact that $4 \not| \binom{n}{m}$ implies that $(n,m) = (7,3), (8,4)$ or $(9,4)$ if $m > 2$. We rule these three possibilities out in turn:

- $(9,4)$: This gives $q^{4(9-4)} > q^{2n}$ which is a contradiction.
- $(8,4)$: In this case, $(q^4+1)\big||G:P_4|$ which is impossible.
- $(7,3)$: In this case, $|G:P_3| = (q^2-q+1)(q^4+\cdots+q+1)(q^6+\cdots+q+1) > \frac{1}{2}(q^6+\cdots+q+1)^2 > v$ which is a contradiction.

Note that if $m=2$ and $n \equiv 0,1(4)$ then $(q^2+1)\big|v$ which is impossible. Hence when $m=2$ we assume that $n \equiv 2,3(4)$.

Cases to be examined

We now state those values of m for which $L_\alpha < P_m$ gives a potential transitive action of G on $\mathcal{P}$:

1. $p=2$: $m=1$ $(n \geq 5)$ or 2 $(n \geq 6)$;
2. $p \not\equiv 1(3)$, p odd:
   - n odd: $m=1$ $(n \geq 5)$, $m=2$ $(n \geq 7)$ or $(n,m)=(7,3)$;
   - n even: $m=2$ $(n \geq 6)$, $m=4(n \geq 12)$ or $(n,m)=(14,6)$;
3. $p \equiv 1(3)$:
   - n even: $m=2$ $(n=14$ or $n \geq 38)$, $m=4,6,8$ $(n>70)$;
   - n odd: $m=1,2$ $(n \geq 7)$, $m=3$ $(n \geq 39)$, $m=4$ $(n>70)$.

Remark. *Note that $n=4$ is now done. We will assume that $n \geq 5$ from now on.*

All that remains is to go through the listed cases one at a time assuming that L_α lies inside the given P_m and so $|L:P_m|$ divides v. We seek a contradiction. We begin with a preliminary lemma and corollary which will be useful for counting the number of involutions in L_α:

Lemma 2.21. *Suppose that q is an odd prime power. Assume that the following two matrices are involutions in $SL(n, q)$, then they are conjugate in $SL(n, q)$:*

$$\begin{pmatrix} V & X_1 \\ 0 & W \end{pmatrix}, \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}$$

where $V \in GL(m, q)$, $W \in GL(n-m, q)$ and $X_1 \in M(m \times (n-m), q)$, the set of m by $n-m$ matrices over the field of q elements.

Proof. Since these matrices are involutions we must have

$$VX_1 + X_1W = 0.$$

Take X such that $2X = -X_1W$. Then $AX = X_1 + XW$ and we find that:

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} V & X_1 \\ 0 & W \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

$\square$

Corollary 2.22. *Let q be odd and suppose that L_α lies inside a parabolic subgroup, P_m, of L where $L_\alpha = \hat{}A : (B : C)$ with $C \leq q-1$ and*

$$A \leq \begin{pmatrix} I & M(m \times (n-m), q) \\ & I \end{pmatrix}, \quad B \leq \begin{pmatrix} SL(m, q) & \\ & SL(n-m, q) \end{pmatrix}.$$

Define $\pi(L_\alpha)$ to be equal to the following set:

$$\left\{ \begin{pmatrix} Y_1 & \\ & Y_2 \end{pmatrix} \,\middle|\, \begin{pmatrix} Y_1 & Z \\ & Y_2 \end{pmatrix} \in A : (B : C), \text{ for some } Z \in M(m \times (n-m), q) \right\},$$

the projection of P_m onto the Levi quotient restricted to L_α. Now assume that L_α contains an involution g which is the projective image of an involution in $SL(n, q)$, $g^* = \begin{pmatrix} X_1 & Y \\ & X_2 \end{pmatrix}$.

Then r_g is greater than or equal to the number of $\pi(L_\alpha)$-conjugates of the block diagonal matrix $\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix}$ in $\pi(L_\alpha)$.

Note that in what follows we will assume that L_α lies in a parabolic subgroup which is L-conjugate to one of the above form. In fact, in $PSL(n, q)$ where $n \geq 3$, there are two conjugacy classes of parabolic subgroups. However, since these two classes are fused by a graph automorphism, our method extends trivially to cover the other class.

2.4.3 Remaining Cases

Case: $p = 2, m = 1$

Take g^* a transvection as before, with $n_g = (q^{n-1}-1)(q^{n-1}+\cdots+q+1)$. Recall that $r_g \geq 2(q^{n-1}-1)$ and so $\frac{n_g}{r_g} \leq \frac{1}{2}(q^{n-1}+\cdots+q+1)$ and so $v < \frac{1}{2}(q^{n-1}+\cdots+q+1)^2$.

Then we suppose that $L_\alpha = \hat{}A.B.C \leq P_1 = \hat{}[q^{n-1}] : (SL(n-1,q).(q-1))$. Since L_α contains a Sylow 2-subgroup of L, $A = [q^{n-1}]$ with $B \leq SL(n-1,q)$, $C \leq (q-1)$. Now $|L : P_1| = q^{n-1}+\cdots+q+1$ and thus $|SL(n-1,q) : B| < \frac{1}{2}(q^{n-1}+\cdots+q+1)$. We know that B contains a Sylow 2-subgroup of $SL(n-1,q)$ and so we are in one of the following situations:

- $B \leq P^*_{m_1}$, a parabolic subgroup of $SL(n-1,q)$. For $n \geq 5$ and $m_1 \geq 2$ observe that $|SL(n-1,q) : P^*_{m_1}| > q^{2(n-3)} > \frac{1}{2}(q^{n-1}+\cdots+q+1)$ which is impossible. Thus $m_1 = 1$ and $B < [q^{n-2}] : GL(n-2,q)$. In this case $(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)\big|v$ and $B = [q^{n-2}] : B^*_1$ where $|GL(n-2,q) : B^*_1| < q$. Thus $B > B^*_1 > SL(n-2,q)$.

- $B = SL(n-1,q)$.

Consider the second situation first. We know that, for some α, $\pi(L_\alpha)$ contains $\begin{pmatrix} 1 & \\ & SL(n-1,q) \end{pmatrix}$. We also know that projective images of the following matrices are conjugate in L:

$$g^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}, \quad h^* = \begin{pmatrix} 1 & & & & & \\ & 1 & 0 & \cdots & 0 & 1 \\ & & 1 & & & 0 \\ & & & \ddots & & \vdots \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Thus, by Corollary 2.22, $r_g \geq r_g(\hat{}SL(n-1,q)) \geq (q^{n-2}-1)(q^{n-2}+\cdots+q+1)$. This implies that $\frac{n_g}{r_g} < q(q+1)$ and $v \leq q^4+q^2+1$. This is a contradiction for $n \geq 5$.

Thus we assume that we are in the first situation. The same argument though implies that $r_g \geq r_g(\hat{}SL(n-2,q)) \geq (q^{n-3}-1)(q^{n-3}+\cdots+q+1)$. This implies that $\frac{n_g}{r_g} < (q^2+1)^2$ and so $\frac{n_g}{r_g} \leq q^4+q^2+1$. This means that $v \leq q^8+4q^6+7q^4+6q^2+3$. We know that $(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)|v$ which gives a contradiction for $n \geq 6$.

For $n = 5$ we find that $(q^3+q^2+q+1)|v$ hence $(q^2+1)|v$ which implies that a prime $p_1 \equiv 2(3)$ divides into v which is a contradiction.

Case: $p = 2, m = 2$

We assume here that $n \geq 6$ and $L_\alpha \leq P_2 \cong \hat{}[q^{2(n-2)}] : (SL(2,q) \times SL(n-2,q)).(q-1)$. Now P_2 has index $(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)/(q+1)$. We know, as before, that $v < \frac{1}{2}(q^{n-1}+\cdots+q+1)^2$ hence $|P_2 : L_\alpha| < q(q+1)$. Now observe that $SL(n-2,q)$ does not have a subgroup of index less than $q(q+1)$ hence $L_\alpha > SL(n-2,q)$. As for $m = 1$, this implies that $v \leq q^8+4q^6+7q^4+6q^2+3$. This must be greater than the index of P_2 and so we must have $n = 6$.

In fact when we examine $n = 6$ we find that, to satisfy the bound, we must have $q = 2$. Explicit calculation of n_g, r_g and $|L : P_2|$ excludes this possibility.

Remark. *From here on we assume that p is odd and $n \geq 5$.*

Case: p **odd,** $p \not\equiv 1(3)$**,** n **odd,** m**=1**

For the next two cases take g as before for p odd and n odd with $n_g = q^{n-1}(q^{n-1}+\cdots+q+1)$. We suppose that $L_\alpha = \hat{}A.B.C < P_1 = \hat{}[q^{n-1}] : (SL(n-1,q).(q-1))$. Here $A \leq [q^{n-1}]$, $B \leq SL(n-1,q)$ and $C \leq q-1$. Note that $|L : P_1| = q^{n-1}+\cdots+q+1$.

Suppose first that $p \neq 3$. Then $\frac{n_g}{r_g}|q^{n-1}+\cdots+q+1$ and so $v < 2(q^{n-1}+\cdots+q+1)^2$. Then $|P_1 : L_\alpha| < 2(q^{n-1}+\cdots+q+1)$. Now L_α contains a Sylow-p subgroup of L since $p \equiv 2(3)$. Hence B either lies in a parabolic subgroup, $P^*_{m_1}$, of $SL(n-1,q)$ or $B = SL(n-1,q)$.

Observe that if m_1 is odd then $|SL(n-1,q) : P^*_{m_1}|$ is even. Thus we must assume that m_1 is even, in which case $|SL(n-1,q) : P^*_{m_1}| > q^{2(n-3)} > 2(q^{n-1}+\cdots+q+1)$ for $n \geq 6$. This is a contradiction. For $n = 5$, P^*_2 also has even index in $SL(4,q)$ so can be excluded. Hence we assume that $B = SL(n-1,q)$ and $|C|$ is even. We know that, for some α, $\pi(L_\alpha)$ contains $\begin{pmatrix} \pm 1 & \\ & SL(n-1,q).2 \end{pmatrix}$. Thus, appealing to Corollary 2.22, we conclude that $r_g \geq r_g(\hat{}SL(n-1,q).2) \geq q^{n-2}(q^{n-2}+\cdots+q+1)$ and so $\frac{n_g}{r_g} < q(q+1)$. This means that $v \leq q^4+q^2+1$ which is a contradiction for $n \geq 5$.

We are left with the case where $p = 3$. Now L_α contains a group of index 3 in a Sylow-3 subgroup of L and $|L : L_\alpha|$ is odd. Hence B either lies in a parabolic subgroup, $P^*_{m_1}$ of $SL(n-1,q)$ or $B = SL(n-1,q)$. The case where $B = SL(n-1,q)$ is ruled out exactly as for $p \neq 3$.

Consider $B \leq P^*_{m_1} < SL(n-1,q)$ and suppose that $n \geq 8$. Then $v > q^7 + \cdots + q + 1 > 1333$ and $\frac{n_g}{r_g} > 31$. This, combined with the fact that $\frac{n_g}{r_g} \leq 3(q^{n-1} + \cdots + q + 1)$, means that $v < 12(q^{n-1} + \cdots + q + 1)^2$.

Now B lies in $P^*_{m_1}$ and so m_1 must be even. Then $|SL(n-1,q) : P^*_{m_1}| > q^{2(n-3)} > 12(q^{n-1} + \cdots + q + 1)$ for $n \geq 8$ which is a contradiction. We are left with $n = 5$ or 7. If $n = 5$ then we exclude it as for $p \neq 3$.

For $n = 7$, we know that $d_g < 2\frac{n_g}{r_g} \leq 6(q^6 + \cdots + q + 1)$ and so $v < 18(q^6 + \cdots + q + 1)^2$. Thus we require that $q^{2(7-3)} < |SL(n-1,q) : P^*_{m_1}| < 18(q^6 + \cdots + q + 1)$. This is impossible for $q \geq 9$.

When $q = 3$ we find that $\frac{n_g}{r_g} | 3(q^6 + \cdots + q + 1) = 3279$. Now $\frac{n_g}{r_g} = u^2 - u + 1$ for some integer u and so $\frac{n_g}{r_g} \leq q^6 + \cdots + q + 1$ and we refer to the case where $p \neq 3$.

Remark. *Note that we have now covered all possible cases where $n = 5$ and we assume that $n \geq 6$ from here on.*

Case: p **odd,** $p \not\equiv 1(3)$**,** n **odd,** $m = 2$

In this case $L_\alpha = \hat{\ }A.B.C \leq P_2 \cong \hat{\ }[q^{2(n-2)}] : (SL(2,q) \times SL(n-2,q)).(q-1)$ where $A \leq [q^{n-1}]$, $B \leq SL(2,q) \times SL(n-2,q)$ and $C \leq q-1$. Now $|L : P_2| = (q^{n-3} + \cdots + q^2 + 1)(q^{n-1} + \cdots + q + 1)$.

Now we know that $\frac{n_g}{r_g} | 3(q^{n-1} + \cdots + q + 1)$. Thus $v < 12(q^{n-1} + \cdots + q + 1)^2$ and hence $|P_2 : L_\alpha| < 12(q+1)^2$. If $(n,q) \neq (7,3)$ then no subgroup of $SL(n-2,q)$ has index less than $12(q+1)^2$ unless $(n,q) = (7,3)$. If $(n,q) = (7,3)$ then the only subgroups of $SL(5,q)$ with indices less than $12(3+1)^2$ are the parabolic subgroups. These have indices in $SL(5,q)$ divisible by 11 and so can be excluded. This implies that in all cases $B = B^* \times SL(n-2,q)$ for B^* some subgroup of $SL(2,q)$.

Now $B = B^* \times SL(n-2,q)$ implies that $\pi(L_\alpha) \geq SL(n-2,q).2$ and so, by Corollary 2.22, $r_g > r_g(\hat{\ }SL(n-2,q)) > q^{n-3}(q^{n-3} + \cdots + q + 1)$ and $\frac{n_g}{r_g} < q^2(q^2+1)$ and so $v < q^8 + q^4 + 1$. This gives a contradiction for $n \geq 7$.

Case: p **odd,** $p \not\equiv 1(3)$**,** n **even,** $m = 2$

For the next two cases, take g as earlier for p odd and n even. Then $n_g = q^{2(n-2)}(q^{n-2} + \cdots + q + 1)(q^{n-2} + \cdots + q^2 + 1)$. As in the previous case, $L_\alpha = \hat{\ }A.B.C \leq P_2 \cong \hat{\ }[q^{2(n-2)}] : (SL(2,q) \times SL(n-2,q)).(q-1)$ where $A \leq [q^{2(n-2)}]$,

$B \leq (SL(2,q) \times SL(n-2,q))$, $C \leq q-1$ and $\pi(L_\alpha) = \hat{}B.C$. Now P_2 has index in L, $(q^{n-2} + \cdots + q^2 + 1)(q^{n-2} + \cdots + q + 1)$.

We know, by Lemma 2.13, that one of the following must hold:

- $B \leq (SL(2,q) \times B_1)$ for some $B_1 < SL(n-2,q)$;
- $B = (B_2 \times SL(n-2,q))$ for some $B_2 \leq SL(2,q)$.

Consider the second possibility. As before Corollary 2.22 implies that $r_g \geq r_g(\hat{}SL(n-2,q)) \geq q^{2(n-4)}(q^{n-4}+\cdots+q+1)(q^{n-4}+\cdots+q^2+1)$. Then $\frac{n_g}{r_g} \leq q^4(q^2+1)^2$ and $v \leq q^{18}$ which is a contradiction for $n > 11$. We will need to consider $n = 6, 8, 10$.

We turn to the first possibility above. We know that $\frac{n_g}{r_g} | 3(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$. This implies that $v < 9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1)$ and so $|P_2 : L_\alpha| < 9(q^{n-2}+\cdots+q+1)^2$. Thus we must have B_1 lying inside a parabolic subgroup, $P^*_{m_1}$, in $SL(n-2,q)$ with $|SL(n-2,q) : P^*_{m_1}| < 9(q^{n-2}+\cdots+q+1)^2$. We know that m_1 must be even. If $m_1 \geq 4$ then we know that $|SL(n-2,q) : P^*_{m_1}| > q^{4(n-2-4)}$ which is a contradiction for $n \geq 12$. Thus $n-2 \leq 8$ in which case $m_1 = 4$ is not allowed and so this can also be excluded. Thus we must have $m_1 = 2$. However we know that $\binom{n}{2}$ is odd and so $n \equiv 2(4)$, hence $n-2 \equiv 0(4)$, hence $\binom{n-2}{2}$ is even and $|SL(n-2,q) : P^*_2|$ is even by Lemma 2.19. We may exclude this possibility.

We are left with the possibility that $n = 6, 8$ or 10 and $B = B_2 \times SL(n-2,q)$ for some $B_2 \leq SL(2,q)$.

Observe first that $A.B.C/A$ acts on the non-identity elements of A by conjugation. Since $B = B_2 \times SL(n-2,q)$, this action has orbits of size divisible by $q^{n-2}-1$. When $p = 3$, $q^{n-2}-1$ does not divide into $\frac{q^{2(n-2)}}{3} - 1$ hence in all cases we may assume that $A = [q^{2(n-2)}]$.

Then, for some α, $A : B$ (or its transpose) has the following form and contains the following conjugate of g^*:

$$h^* = \begin{pmatrix} I_{2\times 2} & & & & \\ & -I_{2\times 2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in \begin{pmatrix} B_2 & A \\ & SL(n-2,q) \end{pmatrix}.$$

Observe that $|A : C_A(h^*)| = q^4$. Thus $r_g \geq q^4 r_g(\hat{}SL(n-2,q)) \geq q^{2n-4}(q^{n-4} + \cdots + q + 1)(q^{n-4} + \cdots + q^2 + 1)$. Thus $\frac{n_g}{r_g} \leq (q^2+1)^2$. In fact we may assume that $\frac{n_g}{r_g} \leq q^4 + q^2 + 1$ and so $d_g \leq q^4 + 3q^2 + 3$ and $v \leq (q^4+q^2+1)(q^4+3q^2+3)$.

Now $|L : P_2| = (q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1) > (q^4+q^2+1)(q^4+3q^2+3)$ for $n \geq 6, q \geq 3$. This is a contradiction.

Remark. *Observe that we have now completed the case where $n = 6$. We assume that $n \geq 7$ from now on.*

Case: p **odd,** $p \not\equiv 1(3)$**,** n **even,** $m = 4$

We assume, for this case, that $n \geq 12$. Similarly to the previous case, $L_\alpha = \hat{}A.B.C \leq P_4 \cong \hat{}[q^{4(n-4)}] : (SL(4,q) \times SL(n-4,q)).(q-1)$ where $A \leq [q^{4(n-4)}]$, $B \leq (SL(4,q) \times SL(n-4,q))$, $C \leq q-1$ and $\pi(L_\alpha) = \hat{}B.C$.

As before, $n_g = q^{2(n-2)}(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$ and so $\frac{n_g}{r_g} | 3(q^{n-2}+\cdots+q+1)(q^{n-2}+\cdots+q^2+1)$. This implies that $v < 9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1)$. Then we have

$$|L : P_4||P_4 : L_\alpha| < 9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1)$$

Since $9(q^{n-2}+\cdots+q+1)^3(q^{n-2}+\cdots+q^2+1) < q^{4n-4}$ we must have $|P_4 : L_\alpha| < q^{12}$. We know, by Lemma 2.13, that one of the following must hold:

- $B \leq (SL(2,q) \times B_1)$ for some $B_1 < SL(n-4,q)$. In this case $|SL(n-4,q) : B_1| < q^{12}$. For $n \geq 12$ this implies that B_1 lies in the parabolic subgroup P_1^* of $SL(n-4,q)$. But this has even index and so can be excluded.
- $B = (B_2 \times SL(n-4,q))$ for some $B_2 \leq SL(4,q)$.

Thus the second possibility must hold. As before Corollary 2.22 implies that $r_g \geq r_g(\hat{}SL(n-4,q)) \geq q^{2(n-6)}(q^{n-6}+\cdots+q+1)(q^{n-6}+\cdots+q^2+1)$. Then $\frac{n_g}{r_g} < q^8(q^4+1)^2$ and

$$d_g < \frac{n_g}{r_g} + 2\sqrt{\frac{n_g}{r_g}} + 2 < (q^8+q^4+3)q^4(q^4+1)$$

giving $v \leq q^{12}(q^4+1)^3(q^8+q^4+3)$ which is a contradiction for $n \geq 12$.

Case: p **odd,** $p \equiv 1(3)$**,** n **even,** $m = 2, 4, 6$ **or** 8

We will take g to be the projective image of,

$$g^* = \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

Then $n_g = q^{2(n-2)}(q^{n-2}+\cdots+q^2+1)(q^{n-2}+\cdots+q+1)$ and we know that $v < q^{8n-15}$. Recall that when $m = 2$ we may assume that $n = 14$ or $n \geq 38$, otherwise $n > 70$.

Let $L_\alpha = \hat{}A.B.C \leq P_m \cong \hat{}[q^{2(n-m)}] : (SL(m,q) \times SL(n-m,q)).(q-1)$ where $A \leq [q^{m(n-m)}]$, $B \leq (SL(m,q) \times SL(n-m,q))$, $C \leq q-1$ and $\pi(L_\alpha) = \hat{}B.C$. Note that $|L : P_m| > q^{m(n-m)}$ and so $|P_m : L_\alpha| < q^{8n-15-mn+m^2}$.

There are two possibilities for B, by Lemma 2.13:

- $B = (B_2 \times SL(n-m,q))$ for some $B_2 \leq SL(m,q)$. Then Corollary 2.22 implies that $r_g \geq r_g(\hat{}SL(n-m,q)) \geq q^{2(n-m-2)}(q^{n-m-2}+\cdots+q+1)(q^{n-m-2}+\cdots+q^2+1)$. Then $\frac{n_g}{r_g} \leq q^{2m}(q^m+1)^2$ and $v \leq q^{8m+3}$ Thus we need $m(n-m) < 8m+3$ which implies that $m > \frac{n-8}{2}$ which is a contradiction.

- $B \leq (SL(m,q) \times B_1)$ for some $B_1 < SL(n-m,q)$. By Liebeck and Saxl [LS85], the projective image of B_1 in $PSL(n-m,q)$ must lie in families $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. The latter two possibilities imply that,

$$\begin{aligned} & \frac{1}{4}n(n-1) < 8n - 15 - mn + m^2 \\ \implies \quad & n^2 - (33-m)n + (60-m^2) < 0 \\ \implies \quad & n < 33 - m \\ \implies \quad & n = 14, m = 2. \end{aligned}$$

  We examine the remaining situation with $n = 14, m = 2$. Then one subgroup in $\mathcal{C}_2$ has index less than $q^{8n-15-mn+m^2} = q^{6n-11}$, namely the projective image of $Q_2 \cong (SL(6,q) \times SL(6,q)).(q-1).2$ which has even index in $PSL(12,q)$. Similarly the only subgroup in $\mathcal{C}_5$ with index less than q^{6n-11} is $N_{PSL(12,q)}(PSL(12,q_0))$ where $q = q_0^2$. This also has even index in $PSL(12,q)$ and so can be excluded.

Thus B_1 lies in a parabolic subgroup $P^*_{m_1}$ of $SL(n-m,q)$. Since $n-m$ is even, we must have m_1 even to have $i := |SL(n-m,q) : P^*_{m_1}|$ odd. Observe that $q^{m_1(n-m-m_1)} < i < q^{8n-15-mn+m^2}$. Suppose first that $m+m_1 \geq 10$. The upper and lower bounds for i imply that

$$\begin{aligned} & (10-m)(n-10) < 8n-15-mn+m^2 \\ \implies \quad & 2n < m^2 - 10m + 85 \\ \implies \quad & n < 35, m = 2. \end{aligned}$$

We examine the remaining situation with $n < 35, m = 2$. Referring to Corollary 2.20 the only value of n less than 35 for which P_2 has admissible index is $n = 14$. But in this case $m_1 = 8$ is too large to define a parabolic group in $SL(12,q)$. This case is excluded. Thus we assume that $m + m_1 \leq 8$ and $m \leq 6$. We split into cases:

- Suppose that $m = 6$ and so $m_1 = 2$. Then $|L : P_6|$ odd implies that $\binom{n}{6}$ is odd and hence $n \equiv 2(4)$. However this implies that $\binom{n-6}{2}$ is even and so i is even which is impossible.
- Suppose that $m = 4$ and so $m_1 \leq 4$. Recall that, by Corollary 2.20, 5 does not divide into $\binom{n}{4}$ hence $n \equiv 4(5)$. However this implies that 5 divides into $\binom{n-4}{m_1}$ which implies, by Lemma 2.19, that i is divisible by a prime $p_1 \equiv 2(3)$ which is impossible.
- Suppose that $m = 2$ and so $m_1 \leq 6$. We exclude $m_1 = 2$ or 6 in the same way as we excluded $m_1 = 2$ for $m = 6$. We exclude $m_1 = 4$ in the same way as we excluded $m_1 = 4$ for $m = 4$. Hence we are done.

Case: p odd, $p \equiv 1(3)$, n odd, $m = 1, 2, 3$ or 4

We will take g to be the projective image of,

$$g^* = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

Then $n_g = q^{n-1}(q^{n-1} + \cdots + q + 1)$ and we know that $v < q^{4n-3}$. Furthermore, by Lemma 2.12, we know that $|v|_p \leq q^{n-1}$. Recall that, for $m = 1$ or 2, we have $n = 7$ or $n \geq 13$, for $m = 3$ we have $n \geq 39$ and for $m = 4$ we have $n > 70$.

Then, in this case, $L_\alpha = \hat{}A.B.C \leq P_m = \hat{}[q^{n-m}] : (SL(n-m,q).(q-1))$ where $A \leq [q^{n-m}]$, $B \leq SL(n-m,q)$, $C \leq q-1$ and $\pi(L_\alpha) = \hat{}B.C$. Note that $|L : P_m| > q^{m(n-m)}$ and so $|SL(n-m,q) : B| < q^{4n-3-mn+m^2}$.

There are two possibilities for B, by Lemma 2.13:

- $B = (B_2 \times SL(n-m,q))$ for some $B_2 \leq SL(m,q)$. We know that $2 \leq C$ and so, by Corollary 2.22, $r_g \geq r_g(\hat{}SL(n-m,q).2) \geq q^{n-m-1}(q^{n-m-1}+\cdots+q+1)$. Hence $\frac{n_g}{r_g} < q^m(q^m+1)$ and $v \leq q^{4m}+q^{2m}+1$. Thus we must have

$$\begin{aligned} & m(n-m) < 4m+1 \\ \Longrightarrow\quad & m^2+(4-n)m+1 > 0 \\ \Longrightarrow\quad & m > n-5. \end{aligned}$$

  This is a contradiction.

- $B \leq (SL(m,q) \times B_1)$ for some $B_1 < SL(n-m,q)$. By Liebeck and Saxl [LS85], the projective image of B_1 in $PSL(n-m,q)$ must lie in a subgroup M of $PSL(m,q)$ from families $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. The latter two possibilities imply that,

$$\begin{aligned} & \frac{1}{4}n(n-1) < 4n-3-mn+m^2 \\ \Longrightarrow\quad & n^2-(17-4m)n+(12-4m^2) < 0 \\ \Longrightarrow\quad & n < 17-2m. \end{aligned}$$

  This implies that either $m=2$ and $n=7$ or $m=1$ and $n=7,13$. In fact, when $m=1$ and $n=13$ the initial inequality is not satisfied and this possibility can be excluded. When $m=2$ and $n=7$, the only possibility is if $B_1 \leq M = N_{L_5(q)}(L_5(q_0))$ where $q=q_0^2$. But $|SL(n-2,q) : M|$ is even here and can be excluded. When $m=1$ and $n=7$ we must have M a subgroup of $SL(6,q)$ in $\mathcal{C}_2$ or $\mathcal{C}_5$ and $|SL(6,q) : M| < q^{19}$. The only such subgroups are $M = \hat{}(SL(3,q))^2.(q-1).2$ and $M = N_{L(6,q)}(L(6,q_0))$ where $q=q_0^2$. Both of these subgroups have even index in $SL(6,q)$ and hence B_1 does not lie inside such an M.

  Thus B_1 lies in a parabolic subgroup, $P^*_{m_1}$ of $SL(n-m,q)$. Write $i := |SL(n-m,q) : P^*_{m_1}|$ and observe that $q^{m_1(n-m-m_1)} < i < q^{4n-3-mn+m^2}$. Suppose first that $m+m_1 \geq 5$. The upper and lower bounds for i imply that

$$\begin{aligned} & (5-m)(n-5) < 4n-3-mn+m^2 \\ \Longrightarrow\quad & n < m^2-5m+28. \end{aligned}$$

This implies that $n < 24$ and either $m = 1$ or $m = 2$. These cases imply that $m_1 \geq 3$. Now for i to be divisible only by primes equivalent to $1(3)$ or by 3 but not 9, we must have $\binom{n-m}{m_1}$ divisible only by primes equivalent to $1(3)$ or by 3 but not 9 and hence $n - m \geq 39$ which is a contradiction.

Thus $m + m_1 \leq 4$ and $m \leq 3$. Note that if m is odd then m_1 must be even since i is odd implies that $\binom{n-m}{m_1}$ is odd. This excludes $m = 3$ and ensures that, for $m = 1$, $m_1 = 2$.

Observe some facts about the remaining cases:

– Suppose that $m = 1$ and $m_1 = 2$. We must have $n \geq 39$ to ensure that n and $\binom{n-1}{2}$ are divisible only by primes equivalent to $1(3)$ or by 3 but not 9. Then we have $B_1 \leq P_2^* \cong [q^{2(n-3)}] : (SL(2, q) \times SL(n-3, q)).(q-1)$ and, since $|SL(n-1, q) : P_2^*| > q^{2(n-3)}$, then $|P_2^* : B_1| < q^{n+4}$.

– Suppose that $m = 2$. If $n = 7$ then B_1 lies inside a parabolic subgroup of $SL(5, q)$. But 5 divides into $\binom{5}{j}$ for $j = 1, 2$ which is not allowed. Thus $n \geq 39$ as this is the next smallest number with allowable divisors of $\binom{n}{2}$. Consider $m_1 = 2$. Since $\binom{n}{2}$ is odd we must have $n \equiv 3(4)$ and so $\binom{n-2}{2}$ is even which is a contradiction. Hence $m_1 = 1$ and $B_1 \leq P_1^* \cong [q^{n-3}] : SL(n-3, q).(q-1)$. Now $|SL(n-2, q) : P_1^*| \geq q^{n-3}$ and so $|P_1^* : B_1| < q^{n+4}$.

Now the only subgroup of $SL(n-3, q)$ in $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$ with index less than q^{n+4} is a parabolic subgroup P_1^* which has even index. Thus, for $m = 1$ and $m = 2$, $B_1 \geq SL(n-3, q).2$ and so, by Corollary 2.22, $r_g \geq r_g(\hat{\ }SL(n-3, q).2) \geq q^{n-4}(q^{n-4} + \cdots + q + 1)$. Hence $\frac{n_g}{r_g} < q^3(q^3 + 1)$ and $v \leq q^{12} + q^6 + 1$ which is a contradiction.

Exceptional cases

We have deferred two cases in the process of our proof. Firstly we need to consider the possibility that $n = 7, p \not\equiv 1(3)$ is odd and $L_\alpha \leq P_3$, a parabolic subgroup stabilizing a 3-dimensional subspace in the vector space for G. We exclude this possibility as follows:

Refer to Section 2.4.2 when np is odd and suppose that $L_\alpha < P_3$. In this case $\frac{n_g}{r_g} | 3(q^6 + \cdots + q + 1)$ and $|L : P_3| = (q^6 + \cdots + q + 1)(q^6 + q^4 + q^3 + q^2 + 1)$. Thus $v > q^{12}$ and $\frac{n_g}{r_g} > q^5 \geq 243$.

Suppose first that $\frac{n_g}{r_g} < q^6+\cdots+q+1$. Then $u^2-u+1 = \frac{n_g}{r_g} \leq \frac{3}{5}(q^6+\cdots+q+1)$ and $u^2+u+1 = d_g < q^6+q^4+q^3+q^2+1$ since $\frac{n_g}{r_g} > 243$. Thus $v < |L : P_3|$ which is a contradiction.

Then consider the case where $\frac{n_g}{r_g} \geq q^6+\cdots+q+1$. We must have $v \geq 3(q^6+\cdots+q+1)(q^6+q^4+q^3+q^2+1)$. Suppose that $\frac{n_g}{r_g} = q^6+\cdots+q+1$. Then our lower bound on v implies that $d_g \geq 3(q^6+q^4+q^3+q^2+1) > 2\frac{n_g}{r_g}$ which is impossible. The only other possibility is that $\frac{n_g}{r_g} = 3(q^6+\cdots+q+1) = u^2-u+1$. But then $u^2+u+1 = d_g < 7(q^6+q^4+q^3+q^2+1)$ which again is impossible for $q \geq 7$. For $q = 3, 5$ we find that $3(q^6+\cdots+q+1) \neq u^2-u+1$ for integer u and so these cases can be excluded.

The second possibility that we need to consider is when $n = 14, p \not\equiv 1(3)$ is odd and $L_\alpha \leq P_6$, a parabolic subgroup stabilizing a 6-dimensional subspace in the vector space for G. We exclude this possibility as follows:

Refer to Section 2.4.2 when n is even and p is odd and observe that $v < 9q^{51}$ and $n_g < q^{49}$. Furthermore

$$L_\alpha \leq P_6 = \hat{}[q^{48}] : (SL(6,q) \times SL(8,q)).(q-1)$$

which has index greater than q^{48}. Thus $|P_6 : L_\alpha| < 9q^3$. Now $SL(6,q)$ and $SL(8,q)$ do not have any subgroups with index this small, hence $L_\alpha > \hat{}A.(SL(6,q)\times SL(8,q))$ where $A = [q^{48}] \cap L_\alpha$. Observe that $|[q^{48}] : A| \leq 3$. In fact, $A.(SL(6,q)\times SL(8,q))/A$ acts by conjugation on the non-identity elements of A with orbits of size divisible by $q^5+\cdots+q+1$, hence $A = [q^{48}]$. Then, for some α, $A : (SL(6,q) \times SL(8,q))$ (or its transpose) has the following form and contains the following conjugate of g^*:

$$h^* = \begin{pmatrix} -1 & & & \\ & I_{5\times5} & & \\ & & -1 & \\ & & & I_{7\times7} \end{pmatrix} \in \begin{pmatrix} SL(6,q) & A \\ & SL(8,q) \end{pmatrix}.$$

Let h be the projective image of h^*. Then $r_g > r_h(\hat{}(SL(6,q)\times SL(8,q))) > q^{10}.q^{14} = q^{24}$. Then h is certainly centralized by a subgroup of A of size no more than q^{36}. Hence $r_g > q^{36}$. This implies that $\frac{n_g}{r_g} < q^{13}$ and $v < q^{27}$ which is a contradiction.

Our proof of Proposition 2.16 is complete.

2.5 $L = PSL(2,q)$ or $L^\dagger = PSL(3,q)$

In this section we prove the following proposition:

Proposition 2.23. *Suppose that G contains a minimal normal subgroup L isomorphic to $PSL(2,q)$ with $q \geq 4$ or that G has a unique component L such that $L^\dagger$ is isomorphic to $PSL(3,q)$ with $q \geq 2$. If G acts transitively on a projective plane $\mathcal{P}$ of order x then $\mathcal{P}$ is Desarguesian and $G \geq PSL(3,x)$.*

Again we seek to demonstrate that our hypothesis leads to a contradiction. Note that the result for $n = 2$ is known to have been proven for the case where $G = PSL(2,q)$ by Camina, Neumann and Praeger but has not been published. Note also that the reason that our statement distinguishes between a *component* in the $PSL(3,q)$ case and a *minimal normal subgroup* in the $PSL(2,q)$ case is given by Lemma 2.14.

2.5.1 Preliminary facts

We will need some preliminary facts about $PSL(2,q)$ and $PSL(3,q)$. As before we assume that $(G/C_G(L))/Z(L) \leq P\Gamma L(n,q)$ since $|AutL : P\Gamma L(n,q)| \leq 2,\ n = 2,3$. Observe that both $PSL(2,q)$ and $PSL(3,q)$ have a single conjugacy class of involutions of size, in odd characteristic, $\frac{1}{2}q(q \pm 1)$ and $q^2(q^2+q+1)$ respectively and, in even characteristic, q^2-1 and $(q^2-1)(q^2+q+1)$ respectively. Both also have the property that a Sylow 2-subgroup contains at least 2 such involutions. Since a point-stabilizer must contain such a Sylow 2-subgroup we conclude that $r_g \geq 2$. Note also that $PSL(3,q)$ has a single conjugacy class of transvections and this class does not fuse with any other in $P\Gamma L(n,q)$.

Kleidman [Kle87] lists explicitly the maximal subgroups of $PSL(3,q)$ and Liebeck and Saxl[LS85] assert that the maximal subgroups of odd degree lie, as before, in families $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_5$ for $q > 2$. Note that $PSL(3,2) \cong PSL(2,7)$ and so we will deal with this group in the $PSL(2,q)$ case. We state a result from Suzuki[Suz82] which gives the structure of all the subgroups of $PSL(2,q)$:

Theorem 2.24. *Let q be a power of the prime p. Let $d = (q-1,2)$. Then a subgroup of $PSL(2,q)$ is isomorphic to one of the following groups.*

1. *The dihedral groups of order $2(q \pm 1)/d$ and their subgroups.*

2. *A parabolic group P_1 of order $q(q-1)/d$ and its subgroups. A Sylow p-subgroup P of P_1 is elementary abelian, $P \lhd P_1$ and the factor group P_1/P is a cyclic group of order $(q-1)/d$.*

3. *$PSL(2,r)$ or $PGL(2,r)$, where r is a power of p such that $r^m = q$.*

4. *A_4, S_4 or A_5.*

Proof. See [Suz82, Theorem 6.25]. □

Note that when $p = 2$, the above list is complete without the final entry. Furthermore, referring to [Kle87], we see that there are unique $PSL(2,q)$ conjugacy classes of the maximal dihedral subgroups of size $2(q \pm 1)/d$ as well as a unique $PSL(2,q)$ conjugacy class of parabolic subgroups P_1.

The result of Liebeck and Saxl[LS85] asserts that all of the families of maximal subgroups can, for some q, contain a subgroup of odd index in $PSL(2,q)$ thus we will simply go through the possibilities given by Suzuki in the $PSL(2,q)$ case. We will use the results of Kleidman [Kle87] to examine conjugacy classes of subgroups of $PSL(2,q)$.

In the $PSL(3,q)$ case we will also need the subgroups of $GL(2,q)$ which can be easily obtained from the subgroups of $PSL(2,q)$.

Theorem 2.25. *H, a subgroup of $GL(2,q)$, $q = p^a$, is amongst the following up to conjugacy in $GL(2,q)$. Note that the last two cases may be omitted when $p = 2$.*

1. *H is cyclic;*

2. *$H = AD$ where*
$$A \leq \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} : \lambda \in GF(q) \right\}$$
and $D \leq N(A)$, is a subgroup of the group of diagonal matrices;

3. *$H =< c, S >$ where $c|q^2 - 1$, S^2 is a scalar 2-element in c;*

4. *$H =< D, S >$ where D is a subgroup of the group of diagonal matrices, S is an anti-diagonal 2-element and $|H : D| = 2$;*

5. *$H =< SL(2,p^b), V >$ or contains $< SL(2,p^b), V >$ as a subgroup of index 2 and here $b|a$, V is a scalar matrix. In the second case, $p^b > 3$;*

6. *$H/ < -I >$ is isomorphic to $S_4 \times C$, $A_4 \times C$, or (with $p \neq 5$) $A_5 \times C$, where C is a scalar subgroup of $GL(2,q)/ < -I >$;*

7. *$H/ < -I >$ contains $A_4 \times C$ as a subgroup of index 2 and A_4 as a subgroup with cyclic quotient group, C is a scalar subgroup of $GL(2,q)/ < -I >$.*

Proof. In this proof and subsequently, we will refer to subgroups of $GL(2, q)$ as being *of type* y, where y is a number between 1 and 7 corresponding to the list above.

When the characteristic is odd, the proof of this result is given in [Blo67b, Theorem 3.4]. When the characteristic is even we know that $GL(2, q) \cong PSL(2, q) \times (q - 1)$. Then, for $H < GL(2, q)$ either $H \geq SL(2, q)$ and we are in type 5 above, or we have $H \leq H_1 \times (q - 1)$ where H_1 is maximal in $PSL(2, q)$.

If $H_1 = D_{2(q-1)}$ then H of types 1 and 4 accounts for all $H \leq H_1 \times (q - 1)$. Since there is only one conjugacy class of $D_{2(q-1)}$ in $PSL(2, q)$ all $H \leq H_1 \times (q - 1)$ must be of type 1 or 4 in $GL(2, q)$.

Similarly $H_1 = D_{2(q+1)}$ is accounted for by H of types 1 and 3 in $GL(2, q)$, while $H_1 = P_1$ is accounted for by H of type 2 in $GL(2, q)$.

Finally consider $H \leq PSL(2, q_0) \times (q - 1)$. Any maximal subgroup of $PSL(2, q_0)$ must be an intersection with $D_{2(q\pm1)}$ or P_1 (since these intersections exist and there is only one conjugacy class of such maximal subgroups) or is of type $PSL(2, q_1)$ where $q_o = q_1^m, m$ prime. Thus any subgroup of $PSL(2, q_0)$ lies inside $D_{2(q\pm1)}$ or P_1 and so is already accounted for, or else equals $PSL(2, q_1)$ where $q = q_1^b$.

Thus we must consider $H \leq PSL(2, q_1) \times (q - 1)$ and $H \not\leq B \times (q - 1)$ for $B < PSL(2, q_1)$. Then $\{a : (a, 1) \in H\}$ is normal in $\{a : (a, b) \in H\} \cong PSL(2, q_1)$. Provided $q_1 > 2$ this implies that $\{a : (a, 1) \in H\} = 1$ and H is cyclic or $\{a : (a, 1) \in H\} \cong PSL(2, q_1)$ and H is a subgroup of $GL(2, q)$ of type 5.

If $q_1 = 2$ then $PSL(2, q_1) \leq D_{2(q\pm1)}$ and the case is already accounted for. $\square$

Note that a subgroup of type 1 in $GL(2, q)$ is never maximal in $GL(2, q)$. Furthermore type 5 includes $GL(2, q)$ itself. We now proceed with our analysis.

2.5.2 $L = PSL(2, q)$

Assume that $L = PSL(2, q), q \geq 4$. We exclude the case where $G/C_G(L)$ contains $PGL(2, q)$ since then G has a normal subgroup N of index 2. Then $N/C_N(L)$ contains only field automorphisms and N acts transitively on $\mathcal{P}$ (since the number of points in $\mathcal{P}$ is odd.) Hence we assume that $G/C_G(L)$ contains only field automorphisms and $|G/C_G(L)| \leq |PSL(2, q)|. \log_p q$.

For $q = 4, 5$ or 9, L is isomorphic to an alternating group. This case has already been examined [CNP03] and so these values of q can be excluded. Observe that P_1, a parabolic subgroup of $PSL(2, q)$, has odd index if and only if $p = 2$. Furthermore if $p = 2$ then $L_\alpha \leq P_1$ since L_α must contain a Sylow 2-subgroup of $PSL(2, q)$. This implies that $n_g = q^2-1$, $r_g = q-1$ and $u^2-u+1 = \frac{n_g}{r_g} = q+1$. But then $u^2-u = q$ which is impossible. Hence we assume L_α does not lie in a parabolic subgroup of $PSL(2, q)$ and that p is odd.

Now the only maximal subgroups of $PSL(2, q)$ which contain a Sylow p-subgroup of $PSL(2, q)$ are the parabolic subgroups. Also, for $q = 3^a$ with $a \geq 3$, the only maximal subgroups containing a subgroup of index p in a Sylow p-subgroup of $PSL(2, q)$ are the parabolic subgroups. Thus Lemma 2.7 implies that $p \equiv 1(3)$ and we assume this from here on. Note that, for an involution $g \in PSL(2, q)$, $n_g = \frac{1}{2}q(q \pm 1)$.

We examine the non-parabolic subgroups of L as candidates to be L_α, using Theorem 2.24.

If $L_\alpha = A_4$ then $r_g = 3$ and, since $r_g \big| n_g$ and $p \equiv 1(3)$, we must have $n_g = \frac{1}{2}q(q-1)$ and $q \equiv 3(4)$. Similarly if $L_\alpha = A_5$ then $r_g = 15$ and $q \equiv 3(4)$. But then $\frac{q+1}{4}$ divides into $|L : L_\alpha|$. Since $\frac{q+1}{4} \equiv 2(3)$ this contradicts Lemma 2.7.

If $L_\alpha = S_4$ then $r_g = 9$ and once more $q \equiv 3(4)$. In fact $\frac{n_g}{r_g} = \frac{q(q-1)}{18}$. Then in $PSL(2, q)$ there is a unique conjugacy class of elements of order 4. Let h be such an element and observe that $r_h = 6$. Now the fixed set of h lies inside the fixed set of $g = h^2$ and $d_h = \frac{2}{3}d_g = \frac{2}{3}(u^2+u+1)$. But g fixes a Baer subplane and so $Fixh$ is the fixed set of a quasicentral collineation of $Fixg$ (see Section 2.2) and $|Fixh|$ divides $|Fixg|$. Thus $\frac{1}{3}(u^2 + u + 1) = u + \sqrt{u} + 1$ and $u = 4$. But then $\frac{q(q-1)}{18} = \frac{n_g}{r_g} = 13$ which is impossible.

Now suppose that $L_\alpha \leq D_{q\pm1}$ so $q \pm 1 \equiv 0(4)$. Then $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q\mp1)}{\frac{1}{2}|L_\alpha|+1}$. Now $|\frac{n_g}{r_g}|_p \neq 1$ and so $|\frac{n_g}{r_g}|_p = |v|_p = q$. Thus $|L_\alpha| + 2$ divides into $q \mp 1$.

Define $m := \frac{q\pm1}{|L_\alpha|}$ and assume first that $m > 1$. Observe that $v = q\frac{q\pm1}{|L_\alpha|}\frac{q\mp1}{2}a$ for some integer a and $d_g = \frac{|L_\alpha|+2}{2}\frac{q\pm1}{|L_\alpha|}a$. If $|L_\alpha| = 4$ then $\frac{n_g}{r_g} = \frac{q(q\mp1)}{6}$ and, in fact, since $q \equiv 1(3)$, $\frac{n_g}{r_g} = \frac{q(q-1)}{6}$. But then $d_g = \frac{3(q+1)}{4}$ and, since $\frac{q+1}{4} \equiv 2(3)$, this is a contradiction. Thus $|L_\alpha| > 4$.

Now observe that $m(|L_\alpha| + 2) > q \mp 1$; furthermore if $(m-1)(|L_\alpha| + 2) = q \mp 1$ then $q \pm 1 - |L_\alpha| + 2m - 2 = q \mp 1$. Reducing modulo 4, this equation gives $2m \equiv 0(4)$ which is a contradiction since $m \big| v$. Thus $(m-2)(|L_\alpha| + 2) \geq q \mp 1$. This implies that $m \geq |L_\alpha| + 1$ and so $|L_\alpha|^2 + |L_\alpha| \leq q \pm 1$.

Since $\frac{n_g}{r_g} < d_g$ we have

$$\begin{aligned} & \frac{q(q\mp 1)}{|L_\alpha|+2} < \frac{|L_\alpha|+2}{2}\frac{q\pm 1}{|L_\alpha|}a \\ \implies \quad & 2|L_\alpha|q(q\mp 1) < (|L_\alpha|^2+4|L_\alpha|+4)(q\pm 1)a \\ \implies \quad & |L_\alpha| < \frac{q+1}{q}a. \end{aligned}$$

The final inequality follows by using the fact that $|L_\alpha| > 4$ and $|L_\alpha|^2+|L_\alpha| \leq q\pm 1$. It then implies that $a > 3$.

Take h of maximal order in L_α. Since $|L_\alpha| > 4$ we know that h is not an involution and $n_h = q(q\mp 1)$ and so $\frac{n_h}{r_h} = \frac{q(q\mp 1)}{2}$. Thus $d_h = \frac{q\pm 1}{|L_\alpha|}a$ which means that $d_h < d_g$ and h acts as an automorphism on the Baer subplane, $Fixg$. This implies that $d_h^2 < 3d_g$ and so $\frac{(q\pm1)^2}{|L_\alpha|^2}a^2 < 3\frac{|L_\alpha|+2}{2}\frac{q\pm 1}{|L_\alpha|}a$. This implies that $q\pm 1 < \frac{1}{2}|L_\alpha|^2+|L_\alpha|$ which is a contradiction.

Hence $m = 1$ and $|L_\alpha| = q\pm 1$. We have two situations. If $q \equiv 3(4)$ then $n_g = \frac{1}{2}q(q-1)$ and $r_g = \frac{1}{2}(q+1)+1$. This means that $\frac{n_g}{r_g}$ is a not an integer, which is impossible. If $q \equiv 1(4)$ then $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q+1)}{\frac{1}{2}(q-1)+1} = q$. Since $|L : L_\alpha| = \frac{1}{2}q(q+1)$ we must have d_g a multiple of $\frac{q+1}{2}$. The only possibility is that $d_g = \frac{3(q+1)}{2}$ which means that $q = 13$ and $v = 273$.

In this case an involution fixes a Baer subplane with 21 points. Within this Baer subplane a Sylow 2-subgroup of $PSL(2,q)$ fixes 9 points. But the fixed set of a collineation group is a *closed set* and so can have at most 7 points [Dem97, 3.1.2 and 3.2.18].

Now suppose that $L_\alpha = PGL(2,r)$ and $q = r^a$ where $a \equiv 2(4)$. Thus $q \equiv 1(4)$ and $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q+1)}{r^2}$. Now $\frac{q}{r^2} = |\frac{n_g}{r_g}|_p \neq |v|_p \geq \frac{q}{r}$ and so $|\frac{n_g}{r_g}|_p = 1$ and $r = \sqrt{q}$. Then $u^2-u+1 = \frac{n_g}{r_g} = \frac{1}{2}(q+1)$. Then $u = \frac{c+1}{2}$ where $c = \sqrt{2q-1}$. This implies that $u^2+u+1 = \frac{q+3+2c}{2}$. Now $|L : L_\alpha| = \frac{1}{2}(q+1)\sqrt{q}$ and so $\sqrt{q}$ divides into u^2+u+1. Now observe that $\sqrt{q}(\frac{\sqrt{q}+5}{2}) > \frac{q+3+2c}{2}$. Furthermore $\sqrt{q}(\frac{\sqrt{q}-1}{2}) < \frac{n_g}{r_g}$. Thus $d_g = \sqrt{q}(\frac{\sqrt{q}+e}{2})$ where $e = 1$ or 3.

Now $2u = d_g - \frac{n_g}{r_g} = \frac{e\sqrt{q}-1}{2}$. We also know that $u = \frac{c+1}{2}$ and so we must have $e\sqrt{q}-3 = 2\sqrt{2q-1}$. Since $e = 1$ or 3 we must have $e = 3$. Then

$$\begin{aligned} 2\sqrt{2q-1} = 3\sqrt{q}-3 \quad \implies \quad & 2\sqrt{2q} > 3\sqrt{q}-3 \\ \implies \quad & q < (\frac{3}{3-2\sqrt{2}})^2 < 18^2. \end{aligned}$$

This implies that $q = 7^2$ or 13^2. But neither of these satisfy the equality $2\sqrt{2q-1} = 3\sqrt{q}-3$ and so can be excluded.

Now suppose that $L_\alpha = PSL(2, r)$ and $q = r^a$ where a is odd. Then $\frac{n_g}{r_g} = \frac{\frac{1}{2}q(q\pm1)}{\frac{1}{2}r(r\pm1)}$ where $q \mp 1 \equiv 0(4)$. Now let h be an element of order $\frac{r\pm1}{2}$. Then $\frac{n_h}{r_h} = \frac{q(q\mp1)}{r(r\mp1)}$. If $r \equiv 3(4)$ then

$$\frac{n_g}{r_g} = r^{a-1}(r^{a-1} + r^{a-1} + \cdots + r + 1) > r^{a-1}(r^{a-1} - r^{a-1} + \cdots - r + 1) = \frac{n_h}{r_h}.$$

Hence $d_g < d_h$ which is impossible.

Now if $r \equiv 1(4)$ then $u^2 - u + 1 = \frac{n_g}{r_g} = r^{a-1}(r^{a-1} - r^{a-2} + \cdots - r + 1)$ and so $r^{a-1} - r^{a-2} < u < r^{a-1}$. This means that

$$\begin{aligned} r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1} - 2r^{a-2} \quad &< \quad d_g = \frac{n_g}{r_g} + 2u; \\ d_g = \frac{n_g}{r_g} + 2u \quad &< \quad r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1}. \end{aligned}$$

Now $r^{a-1} + r^{a-2} + \cdots + r + 1$ divides into d_g. But observe that

$$\begin{aligned} & (r^{a-1} + r^{a-2} + \cdots + r + 1)(r^{a-1} - 2r^{a-2} + 2r^{a-3} \cdots - 2r + 3) \\ < \quad & r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1} - 2r^{a-2}; \end{aligned}$$

$$\begin{aligned} & (r^{a-1} + r^{a-2} + \cdots + r + 1)(r^{a-1} - 2r^{a-2} + 2r^{a-3} \cdots - 2r + 4) \\ > \quad & r^{2a-2} - r^{2a-3} + \cdots - r^a + 3r^{a-1}. \end{aligned}$$

This gives a contradiction and all possibilities are excluded.

2.5.3 $L^\dagger = PSL(3, q)$

Once again we seek to show that our hypothesis leads to a contradiction; the usual action of $PSL(3, q)$ on a Desarguesian projective plane $PG(2, q)$ will not arise due to our restriction that all involutions fix a Baer subplane.

Recall that, for g an involution, $n_g = q^2(q^2 + q + 1)$ for q odd and $n_g = (q^2 - 1)(q^2 + q + 1)$ for q even. We assume here that $q > 2$ and we know that $L_\alpha \leq M$ where M is a member of $\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{C}_5$. We consider the latter two possibilities first. Observe that, in both cases, $p \equiv 1(3)$ since p^2 divides $|PSL(3, q) : M|$.

Suppose that $M \in \mathcal{C}_2$. Then v is divisible by $\frac{q^3(q+1)(q^2+q+1)}{6}$. Now the highest power of q in $\frac{n_g}{r_g}$ is q^2. Since $v = \frac{n_g}{r_g}d_g$ and $(\frac{n_g}{r_g}, d_g) = 1$ we must have q^3 dividing into d_g and q^2 dividing into r_g. But then $u^2 - u + 1 = \frac{n_g}{r_g} \leq q^2 + q + 1$. This means that $v \leq (q^2 + q + 1)(q^2 + 3q + 3)$ which is a contradiction.

Suppose that $M = N_{PSL(3,q)}(PSL(3, r)) \in \mathfrak{C}_5$ where $q = r^a$ and $a \geq 3$ is an odd integer. Then $|v|_p = \frac{q^3}{r^3}$. Suppose first that $|v|_p = |\frac{n_g}{r_g}|_p \leq q^2$ and so $q \leq r^3$. Then we must have $a = 3$, $r_g|(q^2 + q + 1)$ and r^3 dividing $|L_\alpha|$. Since $r_g|(q^2 + q + 1)$ we cannot have $L_\alpha = PSL(3, r)$ or $PSL(3, r).3$. But since r^3 divides into $|L_\alpha|$ we must have L_α inside a parabolic subgroup P of $PSL(3, r).3$. But observe that then v is divisible by

$$|PSL(3, q) : P| = \frac{q^3(q^3 - 1)(q^2 - 1)}{3r^3(r - 1)(r^2 - 1)}$$

which is divisible by 9, a contradiction. The only other possibility is that $p \not| \frac{n_g}{r_g}$ and $\frac{n_g}{r_g} \leq q^2 + q + 1$. But then $q^2 \leq r_g \leq r^2(r^2 + r + 1)$. This is impossible.

Hence we conclude that $M \in \mathfrak{C}_1$. Thus $L_\alpha = \hat{}A.B$ where A is a subgroup of an elementary abelian unipotent subgroup, U, of order q^2 and B is a subgroup of odd index in $GL(2, q)$. We will write $B \cap SL(2, q) = (2, q-1).B_1$ where $B_1 \leq PSL(2, q)$.

We will take α to be such that $L_\alpha \leq P_1$ where

$$P_1 = \hat{} \left\{ \begin{pmatrix} \frac{1}{detY} & a\ b \\ 0 & Y \end{pmatrix} : Y \in GL_2(q), a, b \in GF(q) \right\}.$$

Case: $p \not\equiv 1(3)$

In this case $|U : A| \leq 3$ and $|P : B_1 \cap P| \leq 3$ for some $P \in Syl_p PSL(2, q)$. If B_1 is a subgroup of P_1^*, a parabolic subgroup of $PSL(2, q)$, then $q + 1$ divides the index of B in $GL(2, q)$ and $p = 2$. Then L_α is a subgroup of the Borel subgroup of $PSL(3, q)$ and contains a normal Sylow 2-subgroup P. Thus $r_g = r_g(P) = 2q^2 - q - 1$ and so $r_g \not| n_g$ which is a contradiction.

If $B_1 = PSL(2, q)$ then $B \geq SL(2, q)$. In fact, in odd characteristic, B must contain all matrices of determinant ± 1 since $|GL(2, q) : B|$ is odd. Furthermore in its action by conjugation on the non-identity elements of U, $SL(2, q)$ is transitive. Hence $A = U$. Thus, in both odd and even characteristic, L_α contains all involutions of the parabolic group: $q^2(q + 2)$ of them in the odd case, $(q^2 - 1)(q + 1)$ of them in the even case. In both cases $r_g \not| n_g$ which is a contradiction.

For the remaining cases $p|v$ and so $p = 3$. If $B_1 \leq D_{q\pm1}$ then $q|v$ and we must have $q = 3$. In this case $n_g = 3^2 13$ and so $u^2 - u + 1 = \frac{n_g}{r_g} = 3$ or 13. If $\frac{n_g}{r_g} = 3$ then $v = 21$. This contradicts the fact that $|L : M| = 13$ and this divides into v. So $\frac{n_g}{r_g} = 13, r_g = 9, d_g = 21$ and, since $B_1 \leq D_{q\pm1}$ we must have $L_\alpha = [3^2] : (8.2)$. But then L_α contains more than 9 involutions and this case is excluded.

If B_1 is a proper subgroup of $PSL(2,q)$ isomorphic to A_4, S_4 or A_5 then $q = 3$ or 9. Now $PSL(2,3) \cong A_4$ and so $q = 3$ is already excluded. If $q = 9$ then 5 divides $PSL(2,q)$ and so $B_1 \cong A_5$, but $|PSL(2,9) : A_5|$ is even which is impossible.

If $B_1 \cong PSL(2,r)$ or $B_1 \cong PGL(2,r)$ for $q = r^a, a > 1$ then $\frac{q}{r}|v$. Hence $q = 9$ and $r = 3$. but then 5 divides $|PSL(2,9) : B_1|$ which is a contradiction.

Case: $p \equiv 1(3)$

In this case 3 divides $|PSL(3,q) : M|$ and thus we assume that B contains both the Sylow 2 and Sylow 3-subgroups of $GL(2,q)$. In fact $L = PSL(3,q)$ since $Z(L)$ is semiregular (see Lemma 2.14.) Then B is a subgroup of $GL(2,q)$ of type 4, 5, 6 or 7 in the list given earlier. Note that B contains the scalar subgroup of order 3 and so $|GL(2,q) : B| = |\hat{}GL(2,q) : \hat{}B|$.

Observe first that there are two P_1-conjugacy classes of involutions in P_1. Only one of these is centralized by a whole Sylow 2-subgroup, P, of P_1. Call this conjugacy class $\mathcal{A}$.

In the case where $L_\alpha = A : B$, that is we have a split extension, we know that B contains a Sylow 2-subgroup of P_1 and so the involution in the centre of B must lie in $\mathcal{A}$. This implies that we can conjugate by elements of P_1 (i.e. choose α) such that this involution g is the projective image of

$$g^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We conclude that

$$B \leq \left\{ \begin{pmatrix} \frac{1}{detY} & \\ & Y \end{pmatrix} : Y \in GL(2,q) \right\}.$$

We begin with two preliminary lemmas:

Lemma 2.26. *Let p be odd and $L_\alpha = \hat{}A : B \leq P_1$. Suppose that $|A| = q^2$ and that $(|B|, p) = 1$. Then $|B| > \frac{|GL(2,q)|}{q^2+q+1}$.*

Proof. Let h be an element of order p. Then

$$v = \frac{n_h}{r_h} d_h = \frac{(q^2-1)(q^2+q+1)}{q^2-1} d_h = (q^2+q+1)d_h.$$

We have two possibilities:

1. Suppose that h is quasi-central. We must have $d_h = u^2 + u + 1$ where $v = u^4 + u^2 + 1$. Then $u^2 - u + 1 = \frac{n_h}{r_h} = q^2 + q + 1$ and so $d_h = q^2 + 3q + 3$. Thus $|B| = \frac{|GL(2,q)|}{q^2+3q+3}a$ for some integer a. If $a = 1$ then $|B|$ is not an integer for $q > 1$. If $a \geq 2$ then $|B| > \frac{|GL(2,q)|}{q^2+q+1}$ as required.

2. Suppose that h is not quasi-central. Then $d_h^2 < v$ and so,

$$\left(\frac{v}{q^2+q+1}\right)^2 < v \implies v < (q^2+q+1)^2.$$

This implies that $|B| > \frac{|GL(2,q)|}{q^2+q+1}$ as required.

□

Lemma 2.27. *Let p be odd and $L_\alpha = A : B \leq P_1$. Suppose that $(|B|, p) = 1$. Then $|A| \neq q$.*

Proof. Let h be an element of order p and suppose that $|A| = q$. Then

$$v = \frac{n_h}{r_h} d_h = \frac{(q^2-1)(q^2+q+1)}{q-1} d_h = (q+1)(q^2+q+1)d_h.$$

But, since v is odd and $q + 1$ is even, this implies that d_h is not an integer. This is a contradiction. □

We now begin our analysis of the different possibilities for B. In the case where $B < GL(2,q)$ is of type 4, 6 or 7 then Schur-Zassenhaus implies that $A.B$ is a split extension.

Suppose first that B is a subgroup of type 4 in $GL(2,q)$. Let α be such that $B \leq < D, S >$ where D is the subgroup of diagonal matrices and S is an anti-diagonal 2-element. Note that we must have q dividing into $|A|$.

Now observe that, since B contains a Sylow 2-subgroup of D, we can choose α such that

$$\begin{aligned}
\begin{pmatrix} 1 & e & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A \quad &\implies \quad \begin{pmatrix} -1 & e & f \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \in A \\
&\implies \begin{pmatrix} 1 & -2e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A \\
&\implies \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A.
\end{aligned}$$

We conclude that $A = A_1 \times A_2$ where

$$A_1 \leq \left\{ \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : e \in GF(q) \right\} , \; A_2 \leq \left\{ \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in GF(q) \right\}.$$

Now consider an element, as given, of A_1. Then,

$$\begin{aligned} X = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & a \\ 0 & a^{-1} & 0 \end{pmatrix} \in B \quad &\Longrightarrow \quad \hat{} \begin{pmatrix} -1 & e & 0 \\ 0 & 0 & a \\ 0 & a^{-1} & 0 \end{pmatrix}^2 \in L_\alpha \\ &\Longrightarrow \quad \hat{} \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -e & -ae \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in L_\alpha \\ &\Longrightarrow \quad \begin{pmatrix} 1 & 0 & ae \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A_2. \end{aligned}$$

Thus, for fixed X, we have an injection from A_1 into A_2. There is a similar injection from A_2 into A_1 and so $|A_1| = |A_2| = \sqrt{|A|}$. Now let

$$E = B \cap \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & a \\ 0 & a^{-1} & 0 \end{pmatrix} : a \in GF(q) \right\}$$

and observe that

$$\begin{aligned} \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A_1, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & a \\ 0 & a^{-1} & 0 \end{pmatrix} \in E, \quad &\Longrightarrow \quad \hat{} \begin{pmatrix} 1 & e & 0 \\ 0 & 0 & a \\ 0 & a^{-1} & 0 \end{pmatrix}^2 \in L_\alpha \\ &\Longrightarrow \quad \hat{} \begin{pmatrix} -1 & e & ae \\ 0 & 0 & a \\ 0 & a^{-1} & 0 \end{pmatrix} \in L_\alpha \end{aligned}$$

and this last element is an involution. We now count all the involutions in L_α as follows:

Pre-image of involution g in $SL(3,q)$	Number of such involutions in L_α
$\begin{pmatrix} 1 & c & d \\ & -1 & \\ & & -1 \end{pmatrix}$	$\lvert A\rvert$
$\begin{pmatrix} -1 & 0 & d \\ & -1 & \\ & & 1 \end{pmatrix}$	$\sqrt{\lvert A\rvert}$
$\begin{pmatrix} -1 & c & 0 \\ & 1 & \\ & & -1 \end{pmatrix}$	$\sqrt{\lvert A\rvert}$
$\begin{pmatrix} -1 & c & d \\ & & a \\ & a^{-1} & \end{pmatrix}$	$\lvert E\rvert\sqrt{\lvert A\rvert}$

Thus $r_g = \sqrt{|A|}(\sqrt{|A|}+|E|+2)$ and note that $r_g \leq q(2q+1)$ since $|E| \leq q-1$. Suppose that $(\frac{n_g}{r_g}, p) = 1$. Then $r_g \geq q^2$ and we must have $|A| = q^2$. Alternatively suppose that $(\frac{n_g}{r_g}, p) \neq 1$. Then

$$\begin{aligned} |\frac{n_g}{r_g}|_p = |v|_p \geq \frac{q^3}{|A|} \quad &\Longrightarrow \quad \frac{q^2}{\sqrt{|A|}} \geq |\frac{n_g}{r_g}|_p \geq \frac{q^3}{|A|} \\ &\Longrightarrow \quad |A| \geq q^2. \end{aligned}$$

Thus, in either case, $|A| = q^2$. Then, by Lemma 2.26, $|B| > \frac{|GL(2,q)|}{q^2+q+1}$. But $\frac{2(q-1)^2}{7} < \frac{|GL(2,q)|}{q^2+q+1} = \frac{q(q-1)^2(q+1)}{q^2+q+1}$ for $q > 1$. Hence $|B| = 2(q-1)^2$ and $|E| = q-1$. Then $r_g = q(2q+1)$ which makes $\frac{n_g}{r_g}$ a non-integer unless $q = 1$. This is a contradiction.

Next assume that B is of type 6 or 7. To ensure that B has odd index in $GL(2,q)$ we assume that $B \cong 2.(S_4 \times C)$ or $B \cong 2.(A_4 \times C).2$ where $C \leq Z(GL(2,q))/ < -I >$.

Then we must have q dividing into $|A|$ since $|v|_p \leq q^2$. We write $|A| = qp^a$ where

$a \geq 1$ by Lemma 2.27. Since $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \in B$ this means that $r_g > |A|$.

Suppose first that $q = p^a$ and $|A| = q^2$. By Lemma 2.26,

$$\begin{aligned} & \frac{|GL(2,q)|}{q^2+q+1} < |B| \leq 24(q-1) \\ \Longrightarrow \quad & 24(q^2+q+1) > q^3 - q \\ \Longrightarrow \quad & q < 30. \end{aligned}$$

Then $q = 7, 13$ or 19. Note that in $GL(2,7)$ subgroups of type 6 or 7 have even index and in $GL(2,19)$ subgroups of type 6 and 7 have index divisible by 3. Hence we are left with $q = 13$. In this case $n_g = 3^2.13.61$ and v is divisible by $|L : M| = 3.7.13.61$. Now since $u^2 - u + 1 = \frac{n_g}{r_g}$ divides into n_g we must have $u = 2, 4, 14$ or 23. But in all of these case $u^2 + u + 1$ is not divisible by both 7 and 61. Thus v is not divisible by both 7 and 61 which is a contradiction.

Thus assume now that $q > p^a$ and $|A| < q^2$. Then,

$$\begin{aligned} \frac{n_g}{r_g} < \frac{q^2(q^2+q+1)}{|A|} \quad & \Longrightarrow \quad d_g < \frac{q^2(q^2+q+1)}{|A|} + 2\frac{q^2+q+1}{\sqrt{|A|}} + 2 \\ & \Longrightarrow \quad d_g < \frac{(q^2+2q+1)(q^2+q+1)}{|A|} \\ & \Longrightarrow \quad v < \frac{(q+1)^2 q^2 (q^2+q+1)^2}{|A|^2}. \end{aligned}$$

This implies that,

$$\begin{aligned} & \frac{(q^2+q+1)q^3(q-1)^2(q+1)}{|A||B|} \leq v < \frac{q^2(q^2+q+1)^2(q+1)^2}{|A|^2} \\ \Longrightarrow \quad & |A| < \frac{(q+1)(q^2+q+1)}{q(q-1)^2}|B| \end{aligned}$$

which implies that $|A| < 2.|B|$ for $q \geq 7$.

Now elements from $\hat{}2.C$ do not centralize any element of $\hat{}A$. Thus let $m = \frac{(q-1)/2}{|C|}$ and observe that $\frac{q-1}{3m} = |\hat{}2.C|$ divides into $|A| - 1 = qp^a - 1$. This in turn means that $\frac{q-1}{3m}$ divides into $p^a - 1$. Since $q > p^a$ this means that $3m > p$. Then

$$\begin{aligned} |B| > \frac{|A|}{2} \quad & \Longrightarrow \quad 48|C| > \frac{q.p^a}{2} \\ & \Longrightarrow \quad 48\frac{q-1}{m} > q.p^a \\ & \Longrightarrow \quad p^{a+1} < 144. \end{aligned}$$

Since $p \geq 7$, $a \geq 1$ we must have $p = 7$, $a = 1$. But when $p = 7$, $2.(A_4 \times C).2$ and $2.(S_4 \times C)$ have even index in $GL(2,q)$ which is a contradiction.

Thus we are left with the possibility that B is of type 5 in $GL(2,q)$. We want to show that $L_\alpha = \hat{}A.B$ is a split extension and we can choose α such that

$$B \leq \left\{ \begin{pmatrix} \frac{1}{detY} & \\ & Y \end{pmatrix} : Y \in B^* \right\} \cong B^* \leq GL(2,q).$$

Observe first that each Sylow 2-subgroup of L_α contains a unique element of $\mathcal{A}$. thus $\mathcal{A} \cap L_\alpha$ is a L_α conjugacy class. Furthermore there exist at least two non-conjugate maximal subgroups, M_1, M_2, of B which are of order not divisible by p and index in B not divisible by 2. Then, by Schur-Zassenhaus, $A : M_1$ and $A : M_2$ are subgroups of L_α. But M_1, M_2 must both have centres which are conjugate in L_α, in fact must lie in $\mathcal{A}$. This implies that there exist conjugates of M_1, M_2 which both lie in

$$\left\{ \begin{pmatrix} \frac{1}{detY} & \\ & Y \end{pmatrix} : Y \in B^* \right\} \cong B^* \leq GL(2,q).$$

These conjugates must generate a complement to A as required.

Now note first that $SL(2,r) \leq GL(2,q)$ implies that $SL(2,r) \leq SL(2,q)$. In fact if we examine the maximal subgroups of $PSL(2,q)$ given by Suzuki[Suz82] then, for $f = p_1 \ldots p_n$ where p_i is prime,

$$SL(2,r) < SL(2,r^{p_1}) < \cdots < SL(2,r^{p_1 \cdots p_{n-1}}) < SL(2,q).$$

We assume that at most one of these primes is equal to 2 since otherwise B has even index in $GL(2,q)$. If we assume that $p_2, \ldots p_n$ are all odd then the chain of subgroups given here is maximal except for the first inclusion when $p_1 = 2$. Now $SL(2,r)$ maximal in $SL(2,q)$ has a unique conjugacy class hence, stepping down the chain of inclusion, we assume that $SL(2,r^{p_1})$ has a unique conjugacy class in $SL(2,q)$. If $p_1 = 2$ then two conjugacy classes of $SL(2,r)$ exist in $SL(2,r^{p_1})$ and hence in $SL(2,q)$, otherwise $SL(2,r)$ has a unique conjugacy class in $SL(2,q)$.

By examining [KL90, Action Table 3.5G]) we find that, when f is even, the two conjugacy classes are fused in $GL(2,r^2)$ through conjugation by $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ where λ generates the group $GF(r^2)^*$. Thus, in $GL(2,q)$ there is a unique conjugacy class of $SL(2,r)$ and we take α such that B^* contains the copy of $SL(2,r)$ consisting of matrices of determinant 1 with entries in $GF(r)$.

Observe that $B^* \ni \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and so

$$\begin{pmatrix} 1 & e & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A \implies \begin{pmatrix} -1 & e & f \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \in A$$
$$\implies \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in A$$

Once again we conclude that $A = A_1 \times A_2$ where

$$A_1 \leq \left\{ \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : e \in GF(q) \right\},\ A_2 \leq \left\{ \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in GF(q) \right\}.$$

In the same way as earlier we also know that $|A_1| = |A_2| = \sqrt{|A|}$. We count involutions in L_α:

Pre-image of involution g in $SL(3,q)$	Number of such involutions in L_α
$\begin{pmatrix} 1 & c & d \\ & -1 & \\ & & -1 \end{pmatrix}$	$\lvert A\rvert$
$\begin{pmatrix} -1 & c & d \\ & \pm 1 & \\ & & \mp 1 \end{pmatrix}$	$2\sqrt{\lvert A\rvert}$
$\begin{pmatrix} -1 & c & d \\ & \pm 1 & x \\ & & \mp 1 \end{pmatrix},\ x \neq 0$	$2(r-1)\sqrt{\lvert A\rvert}$
$\begin{pmatrix} -1 & c & d \\ & v & w \\ & x & -v \end{pmatrix},\ x \neq 0$	$r(r-1)\sqrt{\lvert A\rvert}$

Thus $r_g = \sqrt{|A|}(\sqrt{|A|}+r^2+r)$. Now $SL(2,r)$ has orbits of size r^2-1 in its action by conjugation on non-identity elements of A. Hence either $|A|=1$ or $\sqrt{|A|} \geq r$. If $|A|=1$ then, since q divides into $|L_\alpha|$, we must have $r=q$ and so $\frac{n_g}{r_g} = q^2$. This contradicts Lemma 2.9. Hence $\sqrt{|A|} \geq r$ and so $|\frac{n_g}{r_g}|_p = \frac{q^2}{\sqrt{|A|}r}$.

Then either $|\frac{n_g}{r_g}|_p = 1$, $r=q$ and $\sqrt{|A|} = q$ or $|\frac{n_g}{r_g}|_p = |v|_p \geq \frac{q^3}{|A|r}p^a$ where $p^a = \frac{|G|/|L|}{|G_\alpha|/|L_\alpha|}$. In the latter case this means that

$$\frac{q^2}{\sqrt{|A|}r} \geq \frac{q^3}{|A|r}p^a$$

and so $|A| \geq q^2.p^{2a}$. This implies that $|A| = q^2$ and $a=0$. In both cases we find that $|A|=q^2$ and so $r_g = qr(\frac{q}{r}+1+r)$. In order for this to divide into n_g we find that we must have r^4+2r^3-r+1 divisible by $\frac{q}{r}+1+r$. For $q \geq r^6$ this is clearly a contradiction. Examining cases individually for $q \leq r^5$ we find only contradictions.

Thus Proposition 2.23 is proved.

2.6 $L^\dagger = U(n,q)$

In this section we prove the following proposition:

Proposition 2.28. *Suppose G contains a unique component L such that $L^\dagger$ is isomorphic to $U(n,q)$. Then G does not act transitively on a projective plane.*

We may assume that $n \geq 3$ and $(n,q) \neq (3,2)$. Once again, we seek to show that our hypothesis leads to a contradiction. We know ([KL90, Proposition 2.3.2]) that our unitary geometry (V,κ) has a hyperbolic basis. Unless stated otherwise, we will write all matrix representations of elements of $SU(n,q)$ according to this basis:

$$\begin{cases} \{e_1, f_1, \ldots, e_m, f_m\}, & \text{if } n = 2m; \\ \{e_1, f_1, \ldots, e_m, f_m, x\}, & \text{if } n = 2m+1. \end{cases}$$

where $\kappa(e_i,e_j) = \kappa(f_i,f_j) = 0$, $\kappa(e_i,f_j) = \delta_{ij}$, $\kappa(e_i,x) = \kappa(f_i,x) = 0$ for all i,j and $\kappa(x,x) = 1$.

We will also need to make use of an orthonormal basis for (V,κ). Let v_i, w_i with $i = 1, \ldots, m$ be orthonormal vectors such that $< v_i, w_i > = < e_i, f_i >$ for all $i = 1, \ldots, m$. Our orthonormal basis $\mathcal{B}$ will consist of these vectors v_i, w_i with $i = 1, \ldots, m$, as well as the vector x in the case where n is odd.

Now the result of Liebeck and Saxl [LS85] implies that L_α lies inside a maximal subgroup M where

- for q odd, $M \in \mathcal{C}_1, M \in \mathcal{C}_2$, $M^\dagger = N_{U(n,q)}(U(n,q_0))$ where $q = q_0^a$ and a is odd, or $M^\dagger = M_{10}$ and $(n,q) = (3,5)$, or $n = 4$;

- for q even, $M \in \mathcal{C}_1$.

We show next that, in all cases, M must lie in $\mathcal{C}_1$:

Lemma 2.29. *L_α lies inside M, where M maximal in L lies inside $\mathcal{C}_1$.*

Proof. We may assume that p is odd. Define g to be the projective image of

$$g^* = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

For $n \neq 4$, g lies in the centre of a maximal subgroup $\hat{}(SU(2,q)\times SU(n-2,q)).(q+1)$. For $n = 4$, g lies in the centre of a maximal subgroup $\hat{}(SU(2,q)\times SU(2,q)).(q+1).2$. Furthermore, g has the same form under our orthonormal basis $\mathcal{B}$ and, under this basis, $P\Gamma U(n,q) = U(n,q).<\delta,\phi>$ where ϕ is a field automorphism and δ is conjugation by

$$\hat{}\begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

for some $a \in GF(q^2)^*$, a primitive $(q+1)$-th root of unity. Then g is centralised by $<\sigma,\phi>$ hence $n_g | q^{2(n-2)}b$ where $(q,b) = 1$ and $b < q^{2(n-2)}$. Then, by Lemma 2.12, $|v|_p \leq q^{2(n-2)}$.

Suppose that $L_\alpha \leq M$ where $M \in \mathcal{C}_2$, or $M^\dagger = N_{U(n,q)}(U(n,q_0))$ where $q = q_0^a$ and a is odd, or $M^\dagger = M_{10}$ and $(n,q) = (3,5)$, or $n = 4$. Observe that $|U(n,q)|_p = q^{\frac{1}{2}n(n-1)}$ while, for $n \neq 4$, $|M|_p \leq q^{\frac{1}{4}n(n-1)}$. Thus we must have $\frac{1}{2}n(n-1) - 2(n-2) = \frac{1}{2}(n^2 - 5n + 8) \leq \frac{1}{4}n(n-1)$. This implies that $n \leq 6$. We assume this from here on.

Note that we may also assume that $p \equiv 1(3)$ since, in all given cases, $|U(n,q) : M^\dagger|$ odd implies that p^2 divides into $|U(n,q) : M^\dagger|$. We may immediately rule out the possibility that $M^\dagger = M_{10}$.

Consider first the case where $n \neq 4$. If $M \in \mathcal{C}_2$ then $|U(n,q) : M^\dagger|_p > q^{2(n-2)}$ for $n = 3, 5$ and 6 which is a contradiction. If $M = N_{U(n,q)}(U(n,q_0))$ then $q = q_0^a$

where a is an odd prime. Then $|M|_p \leq q^{\frac{1}{2a}n(n-1)}$ hence we must have $\frac{1}{2}(n^2 - 5n+8) \leq \frac{1}{2a}n(n-1)$ which implies that $n=3$ and $q=q_0^3$. Now, when $n=3$, $n_g = q^2(q^2-q+1)$ and L_α contains a Sylow p-subgroup of M. If $L_\alpha \geq U(3,q_0)$ then $r_g = q_0^2(q_0^2-q_0+1)$ but then $r_g \not| n_g$ which is a contradiction. The only other possibility is that $L_\alpha \cap U(3,q_0) \leq P_1^*$, where P_1^* is a parabolic subgroup of $U(3,q_0)$. But this has even index in $U(3,q_0)$ which is a contradiction.

Now suppose that $n=4, p \equiv 1(3)$. Note that here $L = U(4,q)$ and that $n_g = \frac{1}{2}q^4(q^2-q+1)(q^2+1)$. We need to consider the cases where M is a maximal subgroup of odd index not lying in $\mathcal{C}_1$. Furthermore we need $|U(4,q):M|_p \leq q^4$. We go through the possibilities in turn.

- Suppose that $M \in \mathcal{C}_2$. There exist two subgroups $M \in \mathcal{C}_2$ such that $|U(4,q): M|_p \leq q^4$ but only one has odd index. We need to rule out this possibility, when $M \cong \hat{}(SU(2,q) \times SU(2,q)).(q+1).2$ and $|U(4,q):M|_p = q^4$. Then L_α must contain a Sylow p-subgroup of M. But the parabolic subgroup of $SU(2,q)$ has even index hence we may conclude that, for some α,

$$L_\alpha > \hat{} \begin{pmatrix} SU(2,q) & \\ & SU(2,q) \end{pmatrix}.$$

  Then L_α contains,

$$h = \hat{} \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

  Now h is a $U(4,q)$-conjugate of g, thus $r_g \geq \frac{1}{2}(q^2-q)^2$. Hence $\frac{n_g}{r_g} < q^2(q+1)(q+2)$. If $q^4 \big| \frac{n_g}{r_g}$ then we must have $\frac{n_g}{r_g} = q^4$ which is a contradiction of Lemma 2.9. The only other possibility is that $\frac{n_g}{r_g} \leq \frac{1}{2}(q^2-q+1)(q^2+1) < \frac{1}{2}q^4$. But then $d_g < q^4$ and so $v < \frac{1}{2}q^4(q^2-q+1)(q^2+1)$ which contradicts $L_\alpha \leq M$.

- Suppose that $M \in \mathcal{C}_6$ or $M \in S$. The only odd index subgroup is $M = 2^4.A_6$ where $q \equiv 3(8)$. But then $|U(4,q):M|_p > q^4$ which is a contradiction.

- Suppose that $M \in \mathcal{C}_5$. If $M = N_{U(4,q)}(U(4,q_0))$ then $q = q_0^a$ where a is an odd prime. Then $|M|_P \leq q^{\frac{6}{a}}$ hence we must have $\frac{1}{2}(n^2-5n+8) = 2 \leq \frac{6}{a}$ which implies that $q = q_0^3$. However this implies that 9 divides into $|U(n,q):M|$ which is a contradiction.

The only other odd index subgroup in $\mathcal{C}_5$ is $M = PGSp(4,q)$ when $q \equiv 1(4)$. Now, given our original basis $\{e_1, f_1, e_2, f_2\}$ and our original hermitian form κ, define the form $\kappa_\sharp = \zeta^{-1}\kappa$ over the $GF(q)$-vector space $V_\sharp$ spanned by $\{\zeta e_1, f_1, \zeta e_2, f_2\}$. Here ζ is an element of $GF(q^2)$ such that $\zeta^q = -\zeta$. Then $\kappa_\sharp$ is a symplectic form over $V_\sharp$.

Clearly if g^* is an isometry for $(\kappa_\sharp, V_\sharp)$ then g^* is an isometry for (κ, V) and we have an embedding $Sp(4,q) < SU(4,q)$. This embedding corresponds to a maximal subgroup $PSp(4,q) < U(4,q)$ when $q \not\equiv 1(4)$ and $PGSp(4,q) < U(4,q)$ when $q \equiv 1(4)$. In the latter case, there are two conjugacy classes of $PGSp(4,q)$ in $U(4,q)$; it is this case which concerns us.

Under the orthonormal basis $\{v_1, w_1, v_2, w_2\}$, the two conjugacy classes of $PGSp(4,q)$ in $U(4,q)$ are fused by x, the projective image of

$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

where $\lambda \in GF(q^2)$ is a $(q+1)$-primitive element. Thus r_g is the same no matter which of the two conjugacy classes we lie in. Assume from here on that $L_\alpha \leq M = PGSp(4,q)$ preserving $(\kappa_\sharp, V_\sharp)$.

Then $|U(4,q) : M|_p = q^2$, thus $|M : L_\alpha|_p \leq q^2$. The only maximal subgroup, M_1, of $PSp(4,q)$ such that $|PSp(4,q) : M_1|$ is odd and $|PSp(4,q) : M_1|_p \leq q^2$ is $(Sp(2,q) \circ Sp(2,q)).2$. Thus either

- $L_\alpha = M$ with v divisible by $\frac{1}{2}q^2(q+1)(q^2-q+1)$; or
- $L_\alpha \cap PSp(4,q) \leq B = (Sp(2,q) \circ Sp(2,q)).2$. Note that $|(U(4,q) : B|_p = q^4$. Since the parabolic subgroups of $Sp(2,q)$ are of even index we must have $L_\alpha \cap PSp(4,q) = B$ and so $L_\alpha = B.2$ with v divisible by $\frac{1}{4}q^4(q+1)(q^2-q+1)(q^2+1)$.

Under our original basis this implies that, for some α,

$$L_\alpha >\hat{} \begin{pmatrix} SU(2,q) & \\ & SU(2,q) \end{pmatrix}.$$

Now $PSp(4,q)$ is normalized in $U(4,q)$ by,

$$h = \hat{}\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

Thus h lies in L_α and, as before, we know that h is a $U(n,q)$-conjugate of g. We may conclude that $r_g \geq \frac{1}{2}(q^2-q)^2$ and so $\frac{n_g}{r_g} < q^2(q+1)(q+2)$. As in the case where $M \in \mathcal{C}_2$ this contradicts $L_\alpha = B.2$. We conclude that $M = PGSp(4,q)$.

Now observe that $C_{PSp(4,q)}(h) \cong \hat{}GL(2,q).2$ thus $r_g \geq \frac{1}{2}q^3(q+1)(q^2+1)$ and $\frac{n_g}{r_g} < q^2$. This implies that $v < q^2(q+1)(q+2)$ which is a contradiction for $q > 4$.

□

Thus L_α lies inside a maximal subgroup $M \in \mathcal{C}_1$. There are two types of $M \in \mathcal{C}_1$ [KL90, Table 3.5B]:

- The parabolic subgroups, $P_m, 1 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Observe that $(q+1)^m$ divides $L : P_m|$. This implies that $p = 2$. If $q \equiv 1(3)$ then $(q+1) \equiv 2(3)$ and $q+1$ divides into v. If $m > 1$ and $q \equiv 2(3)$ then $9|v$. Neither of these situations are allowed. Hence $m = 1$ and we must have $q = 2^a$, a odd.

- The subgroups B_m of type $GU(m,q) \perp GU(n-m,q)$ with $1 \leq m < n/2$. In this case $q^{m(n-m)}$ divides $|L : B_m|$ and we must have $p \equiv 1(3)$. Observe that $q^{m(n-m)} > q^{2(n-2)}$ for $\frac{n}{2} > m > 2$. But we know, by the argument in the previous lemma, that $|v|_p \leq q^{2(n-2)}$ hence $m \leq 2$.

We now examine these two situations in turn and seek a contradiction.

2.6.1 Case: $p = 2$, $q = 2^a$, a odd, $L_\alpha \leq P_1$

Set n_e to be the even element of $\{n, n-1\}$ while n_o is the odd element. Then $i := |U(n,q) : P_1| = \frac{(q^{n_e}-1)(q^{n_o}+1)}{q^2-1}$. We know that $3\big|(q+1)\big|i$. In addition, $q^{n_e-2}+\cdots+q^2+1|i$ and so for all $r|\frac{n_e}{2}, q^{2r-2}+\cdots+q^2+1|i$ which means that for all $r|\frac{n_e}{2}, r \equiv 1(3)$. A similar argument allows us to conclude from the fact that $(q^{n_o-1}-\cdots+q^2-q+1)|i$ that for all $r|n_o, r \equiv 1(3)$. We may conclude from this that n is even and $n \equiv 2(12)$. Thus $n \geq 14$.

Now $L_\alpha = [q^{2n-3}] : B \le P_1$ where $B \le \hat{}((q^2-1) \times SU(n-2,q))$. We consider the two possibilities given by Lemma 2.13:

- $B \le \hat{}((q^2-1) \times B_1)$ for some $B_1 < SU(n-2,q)$. We know that B_1 must lie in a parabolic subgroup of $SU(n-2,q)$ by Liebeck, Saxl [LS85]. However any parabolic subgroup of $SU(n-2,q)$ has index divisible by $q+1$ which would result in $9|v$ which is a contradiction.

- $B = \hat{}(A_1 \times SU(n-2,q))$ for some $A_1 \le (q^2-1)$. For some α

$$L_\alpha \ge \hat{} \begin{pmatrix} SU(n-2,q) & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Now consider transvections in $SU(n,q)$. All transvections are conjugate to

$$g^* : V \to V, v \mapsto v + s\kappa(v, e_1)e_1$$

for some $s \in GF(q^2), s + s^q = 0$[Tay92, p119]. For $W =< e_1 >$, define $X_{W,W^\perp}$ to be the subgroup of $SU(n,q)$ consisting of all transvections of this form. Now suppose that $h \in SU(n,q)$ preserves W. Then, for $v \in V$,

$$\begin{aligned} v(h^{-1}g^*h) &= (vh^{-1} + s\kappa(vh^{-1}, e_1)e_1)h \\ &= v + s\kappa(vh^{-1}, e_1hh^{-1})e_1h \\ &= v + s\kappa(v, e_1h)e_1h \\ &= v + stt^q\kappa(v, e_1)e_1 \end{aligned}$$

where $t \in GF(q)^*$ is defined via $e_1h = te_1$. Then $(stt^q)^q + stt^q = tt^q(s+s^q) = 0$. Thus $X_{W,W^\perp}$ is normal in the parabolic subgroup of $SU(n,q)$ stabilizing W. Since $|X_{W,W^\perp}| = q$[Tay92, p114], we may conclude that, for g the projective image of g^*, $\frac{|P_1|}{q-1}$ divides into $C_L(g)$. Then, since the only maximal subgroup of $U(n,q)$ whose order is divisible by $\frac{|P_1|}{q-1}$ is P_1, we find that $n_g \le \frac{|U(n,q)|(q-1)(n,q+1)2\log_2 q}{|P_1|}$.

Furthermore, $g \in L_\alpha$ and, by the same argument, $r_g \ge \frac{|SU(n-2,q)|}{|P_1^*|}$ where P_1^* is a parabolic subgroup of $SU(n-2,q)$. Thus,

$$\frac{n_g}{r_g} \le \frac{|U(n,q)|(q-1)(n,q+1)2\log_2 q}{|P_1|}\frac{|P_1^*|}{|SU(n-2,q)|} < q^8.$$

Then $v < q^{17}$ which is a contradiction.

2.6.2 Case: $p \equiv 1(3), L_\alpha \leq B_m, m \leq 2$

Observe that $|L : B_m| = q^{m(n-m)}\frac{(q^n-(-1)^n)\ldots(q^{n-m+1}-(-1)^{n-m+1})}{(q+1)\ldots(q^m-(-1)^m)}$. Consider two situations:

- Suppose n is odd. Then L contains the projective image, g, of

$$g^* = \begin{pmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

  Then g is centralized in $U(n,q)$ by $\hat{}GU(n-1,q)$. Furthermore, as in Lemma 2.29, g has the same form, under the basis $\mathcal{B}$, as above and so is centralised by $< \sigma, \phi >$. Hence $n_g | (q^{n-1})(q^{n-1} - \cdots - q + 1)$. Thus $|v|_p \leq q^{n-1}$. Suppose that $m \geq 2$, in which case $|L : B_m|$ is divisible by $q^{2(n-2)}$. Thus we need $2(n-2) \leq n-1$ which gives $n \leq 3$. For $n = 3$ we know that $m = 1$. Thus, in general, $L_\alpha \leq B_1 = \hat{}GU(n-1,q)$. Furthermore L_α contains a Sylow p-subgroup of $\hat{}GU(n-1,q)$.

  Thus either $L_\alpha \geq \hat{}SU(n-1,q)$ or L_α lies in a parabolic subgroup of $\hat{}GU(n-1,q)$. But $(q+1)$ divides $|\hat{}GU(n-1,q) : P|$ for P a parabolic subgroup of $\hat{}GU(n-1,q)$ which is impossible. Thus $L_\alpha \geq \hat{}SU(n-1,q)$ and L_α contains all the involutions of $\hat{}GU(n-1,q)$.

  Now, for $n > 3$, consider a different involution g as in Lemma 2.29. Then $n_g = q^{2(n-2)}\frac{(q^n+1)(q^{n-1}-1)}{(q+1)(q^2-1)}$ and $r_g \geq r_g(\hat{}GU(n-1,q)) = q^{2(n-3)}\frac{(q^{n-1}-1)(q^{n-2}+1)}{(q+1)(q^2-1)}$. This implies that $\frac{n_g}{r_g} \leq q^4$ and so $\frac{n_g}{r_g} \leq q^4 - q^2 + 1$ and $v < q^8 + q^4 + 1$. But $|L : B_1| = q^{n-1}(q^{n-1} - \cdots - q + 1)$ which is greater than $q^8 + q^4 + 1$ for $n \geq 7$. For $n = 5$, $2|U(5,q) : B_1| > q^8 + q^4 + 1$ and so have $L = U(5,q)$, $L_\alpha = B_1$ and $v = q^4(q^4 - q^3 + q^2 - q + 1)$. But, since $q^4 > \sqrt{v}$, this implies that $d_g = q^4$ which contradicts Lemma 2.9.

  For $n = 3$ there is a unique conjugacy class of involutions of size $q^2(q^2 - q + 1)$. Since $\hat{}SU(2,q) \leq L_\alpha \leq \hat{}GU(2,q)$, L_α must contain precisely the involutions lying in $\hat{}GU(2,q)$ of which there are $q^2 - q + 1$. Then $\frac{n_g}{r_g} = q^2$ which contradicts Lemma 2.9.

- Suppose n is even and let g be as in the proof of Lemma 2.29. Now $|U(n,q) : B_1|$ is even and thus $L_\alpha < B_2 \cong \hat{}(SU(n-2,q) \times SU(2,q)).(q+1)$ and, since

$|v|_p \leq q^{2(n-2)}$, L_α contains a Sylow p-subgroup of $\hat{}(SU(n-2,q) \times SU(2,q))$. Note that, since B_2 is non-maximal in $L = U(4,q)$, we may assume that $n \geq 6$.

Now the index of the parabolic subgroups of $SU(n-2,q)$ in $SU(n-2,q)$ is even. Hence we must have $L_\alpha > \hat{}SU(n-2,q)$. For some α, we may assume that

$$L_\alpha \geq \hat{} \begin{pmatrix} SU(n-2,q) & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Now g is centralized in L by some conjugate of B_2. This implies that

$$n_g = q^{2(n-2)} \frac{(q^n-1)(q^{n-1}+1)}{(q+1)(q^2-1)} \text{ and } r_g \geq q^{2(n-4)} \frac{(q^{n-2}-1)(q^{n-3}+1)}{(q+1)(q^2-1)}.$$

Thus $\frac{n_g}{r_g} \leq q^6(q^2+1)$ and $v \leq q^{16}+q^{15}$ and, for $n \geq 8$, this contradicts $L_\alpha \leq B_2$. We are left with the possibility that $n = 6$. But $2|U(6,q) : B_2| > q^{16} + q^{15}$, thus $L_\alpha = B_2$ and $v = q^8(q^4+q^2+1)(q^4-q^3+q^2-q+1)$. But then $q^8 \geq \sqrt{v}$ and so $d_g = q^8$ which contradicts Lemma 2.9.

Thus Proposition 2.28 is proven.

2.7 $L = PSp(n,q)$

In this section we prove the following proposition:

Proposition 2.30. *Suppose G contains a minimal normal subgroup isomorphic to $PSp(n,q)$ with $n \geq 4$. Then G does not act transitively on a projective plane.*

We know [KL90, Proposition 2.4.1] that our symplectic geometry (V,κ) has a symplectic basis. Unless stated otherwise, we will write all matrix representations of $Sp(n,q)$ according to this basis, $\{e_1, f_1, \dots, e_m, f_m\}$, where $n = 2m$. Here $\kappa(e_i,e_j) = \kappa(f_i,f_j) = 0$ and $\kappa(e_i,f_j) = \delta_{ij}$.

By Liebeck and Saxl [LS85], we know that L_α lies inside a maximal subgroup M where

- for q odd, $M \in \mathcal{C}_1, \mathcal{C}_2$ or $M = N_{PSp(n,q)}(PSp(n,q_0))$ or $n = 4$;
- for q even, $M \in \mathcal{C}_1$.

Note that when $n = 4$ we can assume that $q > 3$ since $PSp(4, 3) \cong U(4, 2)$ which has already been covered.

Lemma 2.31. *L_α lies inside a maximal subgroup from family $\mathcal{C}_1$.*

Proof. Assume that q is odd and that $L_\alpha \leq M$ where M is a maximal subgroup of $PSp(n, q)$ that does not lie in $\mathcal{C}_1$. Observe that in $PSp(n, q)$ there exists a subgroup $B \cong Sp(2, q) \circ Sp(n-2, q)$.

For $n \neq 4$, by [KL90, Lemma 3.2.1 and Table 3.5.c], B is normal in a $P\Gamma Sp(n, q)$-maximal subgroup B_Γ such that $|P\Gamma Sp(n, q) : B_\Gamma| = |L : B|$. Thus, for $n \neq 4$, the involution $g \in Z(B)$ has $n_g = |L : B| = q^{n-2}(q^{n-2} + \cdots + q^2 + 1)$.

When $n = 4$ the same argument applies to $B \cong (Sp(2, q) \circ Sp(2, q)).2$ and the involution $g \in Z(B)$ has $n_g = \frac{1}{2}q^2(q^2 + 1)$.

Therefore the highest value of p in v is at most q^{n-2}. The lowest index of p among maximal subgroups $M \in \mathcal{C}_2$ or $M = N_{PSp(n,q)}(PSp(n, q_0))$ is $q^{\frac{1}{8}n^2}$. This implies that $n - 2 \geq \frac{1}{8}n^2$ which is a contradiction for $n > 4$.

Now suppose that M is maximal in $PSp(4, q)$, $M \notin \mathcal{C}_1$, $|PSp(4, q) : M|$ is odd and $|PSp(4, q) : M|_p \leq q^2$. We must have $M = (Sp(2, q) \circ Sp(2, q)).2$. Then $L_\alpha \leq M$ and $L_\alpha \geq P$ for some P a Sylow p-subgroup of M. Since the parabolic subgroups of $Sp(2, q)$ have even index in $Sp(2, q)$ we must have $L_\alpha = (Sp(2, q) \circ Sp(2, q)).2$.

Now we can choose α such that

$$L_\alpha = \hat{}\left\langle \begin{pmatrix} Sp(2,q) & \\ & Sp(2,q) \end{pmatrix}, h := \begin{pmatrix} & I_{2\times 2} \\ I_{2\times 2} & \end{pmatrix} \right\rangle.$$

Observe that h is conjugate to g in PSp(4,q). Now h has at least $\frac{1}{2}q^2(q^2 - 1)$ L_α-conjugates in L_α, thus $\frac{n_g}{r_g} \leq \frac{\frac{1}{2}q^2(q^2+1)}{\frac{1}{2}q(q^2-1)} \leq 2q$. Then $v \leq 8q^2$. But $v > |L : L_\alpha| = \frac{1}{2}q^2(q^2 + 1)$ which is a contradiction for $q > 3$.

Hence in all cases $M \in \mathcal{C}_1$. $\square$

In $\mathcal{C}_1$ we have subgroups of two types:

- Parabolic subgroups, $P_m \cong [q^a].(\frac{q-1}{(q-1,2)}).(PGL(m, q) \times PSp(n-2m, q))$ where $1 \leq m \leq \frac{n}{2}, a = \frac{m}{2} - \frac{3m^2}{2} + mn$. If $L_\alpha \leq P_m$ then $(q+1)\big||PSp(n, q) : P_m|$ divides into v. Hence we must have $p = 2$.

- Subgroups, B_m, of type $Sp_m \perp Sp_{n-m}$ isomorphic to $Sp(m, q) \circ Sp(n-m, q)$ where $2 \leq m < \frac{n}{2}$ and m is even. In this case q^2 divides into $|PSp(n, q) : B_m|$ which in turn divides into v. Hence we must have $p \equiv 1(3)$.

2.7.1 Case: $p = 2, L_\alpha \leq P_m$

The index of P_m in $Sp(n,q)$ is divisible by q^2+1 for all $m > 1$ which is impossible and so $m = 1$. Then $P_1 \cong [q^{n-1}] : ((q-1) \times Sp(n-2,q))$ and $|Sp(n,q) : P_1| = (q+1)(q^{n-2} + \cdots + q^2 + 1)$. We conclude that $q \equiv 2(3)$ and that every prime dividing into $\frac{n}{2}$ is equivalent to $1(3)$. Hence $n \geq 14$ and $n \equiv 2(4)$. This implies that $n-2 \equiv 0(4)$ and every parabolic subgroup of $Sp(n-2,q)$ has index divisible by q^2+1. Thus $L_\alpha = [q^{n-1}] : (A \times Sp(n-2,q))$ for some $A \leq q-1$.

Now consider $Sp(n,q)$ acting on a vector space V preserving a symplectic form κ. For $u \in V, a \in GF(q)$ we have transvections in $Sp(n,q)$ defined by,

$$g_{a,u} : V \to V, v \mapsto v + a\kappa(v,u)u$$

Fix u, set $W =< u >$ and let $X_{W,W^\perp} = \{g_{a,u} : a \in GF(q)\}$. Then $X_{W,W^\perp} < Sp(n,q)$ is of size q. The parabolic subgroup of $Sp(n,q)$ which preserves W normalizes $X_{W,W^\perp}$.

Now let $g = g_{1,u}$. Then, since the only maximal subgroup whose order is divisible by $\frac{|P_1|}{q-1}$ is P_1, we have

$$n_g \leq \frac{|Sp(n,q)|}{|P_1|}(q-1)\log_2 q.$$

Similarly $r_g \geq \frac{|Sp(n-2,q)|}{|P_1^*|}$ where P_1^* is a parabolic subgroup of $Sp(n-2,q)$. Then

$$\frac{n_g}{r_g} \leq \frac{|Sp(n,q)||P_1^*|(q-1)\log_2 q}{|Sp(n-2,q)||P_1|} \leq q^4.$$

Thus $v \leq q^9$ which contradicts $n \geq 14$ and this case is excluded.

2.7.2 Case: $p \equiv 1(3), L_\alpha < B_m$

We know that the maximum power of p in v is at most q^{n-2}. Now $|PSp(n,q) : B_m|_p = \frac{q^{\frac{1}{4}n^2}}{q^{\frac{1}{4}m^2}q^{\frac{1}{4}(n-m)^2}}$. Thus we need,

$$n-2 \geq \frac{1}{4}(n^2 - m^2 - (n-m)^2) = \frac{1}{2}m(n-m).$$

This implies that $m = 2$ and so $L_\alpha \leq Sp(2,q) \circ Sp(n-2,q)$. If $n = 4$ then B_2 is not maximal and so we assume that $n > 4$. Furthermore we know that L_α must contain a Sylow p-subgroup of $Sp(2,q) \circ Sp(n-2,q)$. But the indices of a parabolic subgroup of $Sp(2,q)$ in $Sp(2,q)$ and of a parabolic subgroup of $Sp(n-2,q)$ in

$Sp(n-2, q)$ are both divisible by $q+1$, hence are even. Thus we conclude that $L_\alpha = Sp(2, q) \circ Sp(n-2, q)$.

Now $r_g \geq \frac{1}{2}q^{n-4}(q^{n-4} + \ldots q^2 + 1)$ and so $\frac{n_g}{r_g} \leq 2q^2(q^2+1)$ and $v \leq 8q^4(q^2+1)^2$. But $v > |L : L_\alpha| = q^{n-2}(q^{n-2} + \ldots q^2 + 1)$ which is a contradiction for $n > 6$.

Thus we must assume that $n = 6$ and $|L : L_\alpha| = q^4(q^4 + q^2 + 1)$ and $\frac{n_g}{r_g} \leq 2q^2(q^2+1)$. If $|\frac{n_g}{r_g}|_p = |v|_p \geq q^4$ then $\frac{n_g}{r_g} = q^4$ which contradicts Lemma 2.9. Thus $|\frac{n_g}{r_g}|_p = 1$ and so $\frac{n_g}{r_g} | q^4 + q^2 + 1$. If $\frac{n_g}{r_g} = q^4 + q^2 + 1$ then d_g is not divisible by q^4 which contradicts the fact that $|L : L_\alpha|$ divides into v. If $\frac{n_g}{r_g} < \frac{1}{2}(q^4 + q^2 + 1)$ then $v < |L : L_\alpha|$ which is also a contradiction.

Our proof of Proposition 2.30 is complete.

2.8 $L = \Omega(n, q)$, nq odd

Throughout the next two sections, Greek letters such as ϵ, η and ζ will stand for either $+, -$ or $\circ$. We will write polynomials such as $x - \epsilon$ to mean $x - \epsilon 1$. We write $\Omega^\circ(n, q)$ to mean $\Omega(n, q)$ when n is odd.

In this section we assume that $n \geq 7$ and q is odd and we prove the following proposition:

Proposition 2.32. *Suppose that n is odd, $n \geq 7$ and G has a minimal normal subgroup isomorphic to $\Omega(n, q)$. Then G does not act transitively on a projective plane.*

Observe that L contains $\Omega^\epsilon(n-1, q).2$ for $\epsilon = -$ and $\epsilon = +$. One of these groups contains a central involution and hence L contains an involution g such that $r_g(L) = \frac{1}{2}q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}} + \epsilon)$. Examining [KL90, Table 3.5.D] for fusion of conjugacy classes, we see that $n_g = r_g(L)$ and thus $|v|_p \leq q^{\frac{n-1}{2}}$.

We begin by proving that L_α must lie in a maximal subgroup $M \in \mathcal{C}_1$:

Lemma 2.33. *L_α does not lie inside a subgroup $M \in \mathcal{C}_i, i > 1$.*

Proof. We examine the list of odd index maximal subgroups in G as given by Liebeck and Saxl[LS85]. The following possibilities are available for a maximal subgroup M of odd index. We exclude them in turn.

- $L = \Omega(7, q)$ and $M = \Omega(7, 2)$. We know that $|v|_p \leq q^3$ and so $|L_\alpha|$ must be divisible by q^6. This is impossible for $L_\alpha \leq M$.

- $M \in \mathcal{C}_2$ or $M = N_{\Omega(n,q)}(\Omega(n,q_0))$ where $q = q_0^c$ for c an odd prime. In both cases $|M|_p \leq \sqrt{|\Omega^\epsilon(n,q)|_p}$. Now $|\Omega^\epsilon(n,q)|_p = q^{\frac{1}{4}(n-1)^2}$ and so we must have,

$$\frac{1}{8}(n-1)^2 + \frac{1}{2}(n-1) \geq \frac{1}{4}(n-1)^2.$$

This is impossible for $n \geq 7$.

$\square$

Thus L_α lies inside a parabolic subgroup or a subgroup B_m of type $O(m,q) \perp O^\eta(n-m,q)$ for some odd $m < n$. In fact parabolic subgroups have even index in $P\Omega(n,q)$ hence we may assume that $L_\alpha \leq B_m$ for some m.

Since $|v|_p \leq q^{\frac{n-1}{2}}$ we know that $L_\alpha \leq B_1 = \Omega^\eta(n-1,q).2$ and that L_α contains a Sylow p-subgroup of $\Omega^\eta(n-1,q)$. Now the parabolic subgroups of $\Omega^\eta(n-1,q)$ have even index. Hence we must have $L_\alpha = \Omega^\eta(n-1,q)$ and v is divisible by $|\Omega(n,q) : \Omega^\eta(n-1,q).2| = \frac{1}{2}q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}} + \eta)$.

Now consider the involution h centralized in L by $(\Omega^\zeta(2,q) \times \Omega(n-2,q)).[4]$. Then $n_h = \frac{q^{n-2}(q^{n-1}-1)}{2(q-\zeta)}$. Now $\Omega^\eta(n-1,q)$ contains a conjugate of h centralized by, at most, $(\Omega^\zeta(2,q) \times \Omega^{\zeta\eta}(n-3,q)).[4]$. then $r_h \geq \frac{q^{n-3}(q^{\frac{n-3}{2}}+\eta\zeta)(q^{\frac{n-1}{2}}-\eta)}{2(q-\zeta)}$. This implies that $\frac{n_h}{r_h} \leq q(q+1)$ and so $v \leq 2q^2(q+1)^2$. But then $v < |L : L_\alpha|$ which is a contradiction.

Hence we have proved Proposition 2.32.

2.9 $L = P\Omega^\epsilon(n,q)$, n even

In this section we assume that $n \geq 8$ and we prove the following proposition:

Proposition 2.34. *Suppose that n is even, $n \geq 8$ and G has a minimal normal subgroup isomorphic to $P\Omega^\epsilon(n,q)$. Then G does not act transitively on a projective plane.*

First we examine what happens when $p = 2$:

Lemma 2.35. *Suppose $n \geq 8$ is even and G has a minimal normal subgroup isomorphic to $P\Omega^\epsilon(n,2^a)$. Then G does not act transitively on a projective plane.*

Proof. Write $q = 2^a$. We know that $L_\alpha \leq P_m$ for some integer m. If $m > 1$ then $q^b + 1$ divides $|P\Omega^\epsilon(n,q) : P_m|$ where b is some even integer. Since $q^b + 1 \equiv 2(3)$ this is impossible. Thus L_α lies inside some parabolic subgroup P_1. Now

$$|P\Omega^\epsilon(n,q) : P_1| = \frac{(q^{\frac{n}{2}} - \epsilon)(q^{\frac{n-2}{2}} + \epsilon)}{q-1}.$$

If $q \equiv 2(3)$ then $q^{\frac{n-2}{2}} + 1 \equiv q^{\frac{n}{2}} + 1 \equiv 2(3)$. Since one of these divides $|P\Omega^\epsilon(n,q) : P_m|$, this is impossible. Hence $q \equiv 1(3)$. Now let n_e be the even one of $\frac{n}{2}$ and $\frac{n-2}{2}$, n_o the odd one. Then one of the following holds:

- $|\Omega^\epsilon(n,q) : P_1| = \frac{q^{n_e}-1}{q-1}(q^{n_0}+1)$ and 9 divides $|\Omega^\epsilon(n,q) : P_1|$; or
- $|\Omega^\epsilon(n,q) : P_1| = \frac{q^{n_o}-1}{q-1}(q^{n_e}+1)$ and $q^{n_e} + 1 \equiv 2(3)$.

Both of these cases are impossible. $\square$

Throughout the rest of the section p is odd. Now L contains maximal subgroups in $\mathcal{C}_1$ of type $O^\zeta(2,q) \perp O^\eta(n-2,q)$ for $\zeta\eta = \epsilon$. One of these groups contains a central involution and hence L contains an involution g such that $|L : C_L(g)| = \frac{q^{n-2}(q^{\frac{n-2}{2}}+\eta)(q^{\frac{n}{2}}-\epsilon)}{2(q-\zeta)}$. Examining for fusion of conjugacy classes in [KL90, Tables 3.5.E and 3.5.F] we see that, except when $(n,\epsilon) = (8,+)$, $n_g = |L : C_L(g)|$. When $(n,\epsilon) = (8,+)$, we know that $n_g \leq 3|L : C_L(g)|$ and so, in all cases, $|v|_p \leq q^{n-2}$.

We begin by proving that L_α must lie in a maximal subgroup $M \in \mathcal{C}_1$:

Lemma 2.36. *L_α does not lie inside a subgroup $M \in \mathcal{C}_i, i > 1$.*

Proof. We examine the list of odd index maximal subgroups in G as given by Liebeck and Saxl[LS85]. The following possibilities are available for a maximal subgroup of odd index $M \notin \mathcal{C}_1$. We exclude them in turn.

- $L = P\Omega^+(8,q)$ and either $M = \Omega^+(8,2)$ or $M = 2^3.2^6.PSL(3,2)$. We know that $|v|_p \leq q^6$ and so $|L_\alpha|_p \geq q^6$. This is impossible for $L_\alpha \leq M$ in both cases.
- $M \in \mathcal{C}_2$ or $M = N_{P\Omega^\epsilon(n,q)}(P\Omega^\epsilon(n,q_0))$ where $q = q_0^c$ for c an odd prime. In both cases $|M|_p \leq \sqrt{|P\Omega^\epsilon(n,q)|_p}$. Now $|P\Omega^\epsilon(n,q)|_p = q^{\frac{1}{4}n(n-2)}$ and so we must have
$$\frac{1}{8}n(n-2) + n - 2 \geq \frac{1}{4}n(n-2).$$
This is impossible for $n > 8$. When $n = 8$, no subgroup M of odd index has $|M|_p \geq 6$ so the result stands.

$\square$

Thus L_α lies inside a parabolic subgroup P_m or a subgroup B_m of type $O^{\zeta_1}(m,q) \perp O^{\eta_1}(n-m,q)$ for some $m < \frac{n}{2}$. In fact parabolic subgroups have even index in $P\Omega^{\epsilon}(n,q)$ for p odd. Hence we assume that $L_\alpha \leq B_m$ for some integer m. We know that $|v|_p \leq q^{n-2}$ and so $|P\Omega^{\epsilon}(n,q) : B_m|_p \leq q^{n-2}$. This implies that $m=1$ or $m=2$. Note also that $p \equiv 1(3)$.

Suppose first that $L_\alpha \leq B_2$ where B_2 is of type $O^{\zeta_1}(2,q) \perp O^{\eta_1}(n-2,q)$ for $\zeta_1\eta_1 = \epsilon$. Then $|P\Omega^{\epsilon}(n,q) : B_2| = \frac{q^{n-2}(q^{\frac{n-2}{2}}+\eta_1)(q^{\frac{n}{2}}-\epsilon)}{2(q-\zeta_1)}$ and so L_α must contain a Sylow p-subgroup of B_2. Since the parabolic subgroups of $P\Omega^{\eta_1}(n-2,q)$ have even index we must have $L_\alpha > \Omega^{\eta_1}(n-2,q)$.

In the case where $L_\alpha \leq B_1$ then $L_\alpha \leq \Omega(n-1,q).c_1$ where $c_1 \in \{1,2\}$. Now $|P\Omega^{\epsilon}(n,q) : B_1|_p = q^{\frac{n-2}{2}}$ hence $|B_1 : L_\alpha|_p \leq q^{\frac{n-2}{2}}$. Examining the proof of Lemma 2.33 this means that $L_\alpha \cap \Omega(n-1,q)$ lies inside a maximal subgroup of $\Omega(n-1,q)$ in family $\mathcal{C}_1$.

Since the parabolic subgroups of $\Omega(n-1,q)$ have even index in $\Omega(n-1,q)$ this means that $L_\alpha \cap \Omega(n-1,q) \leq B^*_{m_1}$; here $B^*_{m_1}$ is a maximal subgroup of $\Omega(n-1,q)$ of type $O_{m_1}(q) \perp O^{\gamma}(n-1-m_1,q)$ for some odd $m_1 < n-1$. In fact $|B_1 : L_\alpha|_p \leq q^{\frac{n-2}{2}}$ implies that $m_1 = 1$ and that L_α contains a Sylow p-subgroup of $B^*_1 = \Omega^{\eta_1}(n-2,q).c_2$ where $c_2 \in \{1,2\}$. Once again, since the parabolic subgroups of $\Omega^{\eta_1}(n-2,q)$ have even index we must have $L_\alpha > \Omega^{\eta_1}(n-2,q)$.

Thus in both cases, when $m=1$ and when $m=2$, we see that $L_\alpha > \Omega^{\eta_1}(n-2,q)$ is a subgroup of $P\Omega^{\epsilon}(n,q)$ which preserves a decomposition of the associated vector space V into subspaces, $V = W_2 \perp W_{n-2}$, where $\dim W_i = i$ and the W_i are non-degenerate subspaces of V.

Then $H = \Omega^{\eta_1}(n-2,q)$ contains h a conjugate of g, and $C_H(h)$ is isomorphic to either $(\Omega^{\gamma_1}(2,q) \times \Omega^{\gamma_2}(n-4,q)).2$ or $2.(P\Omega^{\gamma_1}(2,q) \times P\Omega^{\gamma_2}(n-4,q)).[4]$ (see [KL90, Proposition 4.1.6]). In either case $r_g \geq \frac{q^{n-4}(q^{\frac{n-4}{2}}+\gamma_2)(q^{\frac{n-2}{2}}-\eta_1)}{2(q-\gamma_1)}$.

If $n > 8$ this means that $\frac{n_g}{r_g} \leq \frac{q^2(q+1)^3}{(q-1)^2}$ and so $v \leq 2q^4(q+1)^4$. Since $|L : L_\alpha| < v$ we must have $n = 10, q = 7$ and $L_\alpha = B_1$. But then $|L : B_1|$ is divisible by $\frac{1}{2}7^4(7^5 \pm 1)$. This is impossible since then $|L : B_1|$ is divisible by a prime $s \equiv 2(3)$.

If $n = 8$ then $\frac{n_g}{r_g} < 4q^2(q+1)^2$. Then $v < 28q^4(q+1)^4$ which is less than $|L : B_2|$. Thus $L_\alpha = B_1$. But then $|L : L_\alpha|$ is even which is a contradiction.

Proposition 2.34 is now proven.

2.10 L is an exceptional group of Lie type in odd characteristic

In this section we prove the following proposition:

Proposition 2.37. *Suppose that G has a minimal normal subgroup L where L is an exceptional group of Lie type in odd characteristic or that G has a unique component L such that $L^\dagger$ is isomorphic to a simple group $E_6(q)$ or ${}^2E_6(q)$ where q is odd. Then G does not act transitively on a projective plane.*

We introduce some extra notation for this section and the following one. We will write E_6^- for ${}^2E_6$, E_6^+ for E_6. Similarly SL^- will stand for SU, SL^+ for SL. We will use ϵ to denote either ± 1 or $\pm$ depending on the context. Generally our notation refers to the adjoint version of the exceptional group, any variation on this will be specified. For a group G, we will write $\frac{1}{2}G$ to mean a subgroup in G of index 2. We define $P(G) := \min\{|G : H| : H < G\}$. Finally, for a group H we write $O^{p'}H$ to mean the unique smallest normal subgroup N of H such that $|H/N|_p = 1$.

We have eight possibilities for L which we will examine in turn. As usual we will examine odd-index maximal subgroups of L, treating these as candidates to contain a stabilizer L_α, and seek to show a contradiction.

We immediately exclude the case where $L = {}^2G_2(q)$, $q > 3$, by examining the list of maximal subgroups of ${}^2G_2(q)$ given in [Kle88a, Theorem C] (see also [War66]). We see that any maximal subgroup of odd index must have index divisible by 9 and hence cannot contain a point-stabilizer. Hence this case is excluded. Note that the list given by Kleidman [Kle88a] contains a maximal subgroup of odd index (with structure $(2^2 \times D_{\frac{1}{2}(q+1)}) : 3$) which has been omitted by Liebeck and Saxl[LS85] and by Kantor[Kan87].

For the remaining cases we will refer to the results of Liebeck and Saxl giving the maximal subgroups $M^\dagger$ of odd index in $L^\dagger$.[LS85] These maximal subgroups $M^\dagger$ take one of two forms: Either $M^\dagger = N_{L^\dagger}(L^\dagger(q_0))$, where $q = q_0^a$ for a an odd prime and the subgroup $L^\dagger(q_0)$ of $L^\dagger(q)$ corresponds to the centralizer of a field automorphism of $L^\dagger(q)$ (see [Kan87, Theorem C]), or $M^\dagger$ is enumerated in [LS85, Table 1].

Note that, by [KL90, Table 5.1.B], Out$L$, the outer automorphism group of L, has order strictly less than q provided $L \neq {}^3D_4(3), {}^2E_6(5)$. We also use the following lemma:

Lemma 2.38. *Let ϕ be a field automorphism of $L(q)$ of prime order a. Let $L(q_0) = O^{p'}C_{L(q)}(\phi)$ where $q = q_0^a$. Then $N_{L(q)}(L(q_0)) \lesssim \mathrm{Inndiag}(L(q_0))$ and, furthermore, $\mathrm{Inndiag}(L(q_0)) = L(q_0).d$ where*

$$d = \begin{cases} (3, q_0 - \epsilon) & L = E_6^\epsilon \\ (2, q_0 - 1) & L = E_7 \\ 1 & \text{otherwise} \end{cases}$$

Proof. Our notation is consistent with that in [GLS94]. Write $L(q) = O^{p'}C_{\overline{L}}(\sigma)$ where $\overline{L}$ is a simple adjoint $\overline{\mathbb{F}_p}$-algebraic group, $\overline{\mathbb{F}_p}$ is the algebraic closure of $GF(q)$ and σ is a Steinberg automorphism [GLS94, Definition 2.2.1].

By [GLS94, Proposition 2.5.17], there exists a Steinberg automorphism τ of $\overline{L}$ such that $\tau^a = \sigma$ and τ induces ϕ on L. Then $L(q_o) = O^{p'}C_{\overline{L}}(\tau)$ and, by [GLS94, Proposition 2.5.9], $N_{\overline{L}}(L(q_0)) = C_{\overline{L}}(\tau)$.

Thus $N_{L(q)}(L(q_0)) = C_{L(q)}(\tau) \leq C_{L(q)}(\phi) \lesssim \mathrm{Inndiag}(L(q_0))$ by [GLS94, Proposition 4.9.1]. The structure of $\mathrm{Inndiag}(L(q_0))$ is given in [GLS94, Theorem 2.5.12]. □

2.10.1 Case: $L = E_8(q)$

Referring to [GLS94, Table 4.5.1], we see that $E_8(q)$ contains an involution g such that $C_L(g) \geq 2.(PSL(2, q) \times E_7(q))$. There is one such $E_8(q)$ conjugacy class of involutions in L and so n_g divides into

$$\frac{2q^{56}(q^{10} + 1)(q^{12} + 1)(q^6 + 1)(q^{30} - 1)}{q^2 - 1}.$$

Using Lemma 2.12 this implies that $|v|_p \leq q^{56}$ and hence that $|L_\alpha|_p \geq q^{64}$. The list in [LS85, Table 1] contains no maximal subgroups M such that $|M|_p \geq q^{64}$. Similarly Lemma 2.38 implies that $|N_L(E_8(q_0))|_p = |E_8(q_0)|_p = q_0^{120}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{120} \leq q^{40}$ and so this possibility is excluded.

2.10.2 Case: $L = E_7(q)$

Referring to [GLS94, Table 4.5.1], we see that $E_7(q)$ contains an involution g such that $C_L(g)$ contains $SL^\epsilon(8, q)/(4, q - \epsilon)$ for ϵ either $+$ or $-$. There is one such $\mathrm{Inndiag}(E_7(q))$ conjugacy class of involutions in L and so n_g divides into

$$(4, q - 1)q^{35}(q^7 + \epsilon)(q^5 + \epsilon)(q^3 + \epsilon)(q^8 + q^4 + 1)(q^{12} + q^6 + 1).$$

This implies that $|v|_p \leq q^{35}$ and hence that $|L_\alpha|_p \geq q^{28}$. The list in [LS85, Table 1] contains one maximal subgroup such that $|M|_p \geq q^{28}$, namely $M = N_L(2.(PSL(2,q) \times P\Omega^+(12,q))$. Then $|L : M|_p = q^{32}$ and so $p \equiv 1(3)$. But this implies that 9 divides into $|L : M|$ and so it is not possible that $L_\alpha \leq M$.

Similarly Lemma 2.38 implies that $|N_L(E_7(q_0))|_p \leq |E_7(q_0).2|_p = q_0^{63}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{63} \leq q^{21}$ and so this possibility is excluded.

2.10.3 Case: $L^\dagger = E_6^\epsilon(q)$

Referring to [GLS94, Table 4.5.1], we see that L contains an involution g such that $C_L(g)$ contains $Spin_{10}^\epsilon(q)$. Here $Spin_{10}^\epsilon(q) \cong (4, q-\epsilon).P\Omega^\epsilon(10,q)$. There is only one such Inndiag($E_6^\epsilon(q)$) conjugacy class of involutions in L and so,

$$n_g = q^{16}(q^6 + \epsilon q^3 + 1)(q^2 + \epsilon q + 1)(q^8 + q^4 + 1).$$

This implies that $|v|_p \leq q^{16}$ and hence that $|L_\alpha|_p \geq q^{20}$. Then Lemma 2.38 implies that $|N_{L^\dagger}(L^\dagger(q_0))|_p \leq |L^\dagger(q_0).(3, q-\epsilon)|_p$ which divides into $3q_0^{36}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{36} \leq q^{12}$ and so this possibility is excluded.

Subcase: $\epsilon = +$

In this case the list in [LS85, Table 1] contains two maximal subgroups $M^\dagger$ such that $|M^\dagger|_p \geq q^{20}$: $M^\dagger = N_{L^\dagger}((4, q-1).P\Omega^+(10,q))$ or $M^\dagger$ is parabolic of type D_5. If $p \equiv 1(3)$ in either case then 9 divides $|L : M|$ which is a contradiction. Hence $p \not\equiv 1(3)$, the universal and adjoint versions coincide and L is simple.

In the non-parabolic case, $|L : M|_p > p^2$ which is impossible for $p \not\equiv 1(3)$. Hence M is a parabolic subgroup of $E_6^+(q)$ of type D_5 and $|L : M| = (q^6 + q^3 + 1)(q^2 + q + 1)(q^8 + q^4 + 1)$.

Now $M \cong [q^{16}] : (Spin_{10}^+(q)H)$ where H is a Cartan subgroup of $E_6(q)$ and H normalizes $Spin_{10}^+(q)$. Here $Spin_{10}^+(q) \cong (4, q-1).P\Omega^+(10,q)$ and $P\Omega^+(10,q)$ has parabolic subgroups of even index. This implies that $L_\alpha \geq [q^{16}] : (Spin_{10}^+(q).2)$ for $p \neq 3$.

Furthermore, for $p = 3$, every non-parabolic subgroup of $P\Omega^+(10,q)$ has index divisible by 9[Kle87]. This means that $L_\alpha \geq [\frac{q^{16}}{3}].(Spin_{10}^+(q).2)$. Now E, the commutator subgroup of the Levi complement in M, is isomorphic to $Spin_{10}^+(q)$ and

$|E : L_\alpha \cap E|$ is at most $\frac{3}{2}(q-1)$. But $P(Spin^+_{10}(q)) > \frac{3}{2}(q-1)$ [KL90, Table 5.2.A]. Thus $L_\alpha > E$.

Now if $q = 3^a$ then $|E|$ is divisible by $3^{8a}-1$; in particular, $|E|$ is divisible by the primitive prime divisors of $3^{8a}-1$. This implies that if $\phi : E \to GL(m, 3)$ is a non-trivial representation of E over $GF(3)$ then $m \geq 8a$. Now consider the action of E on the unipotent radical of the full parabolic group, $[q^{16}]$, considered as a module over $GF(3)$. We know that E does not act trivially on any submodule of the unipotent radical (otherwise $Z(E)$ would have too large a centralizer; see [GLS94, Table 4.5.1]). Thus the action must be either irreducible or split into two modules both of size q^8. In either case we must have $L_\alpha \geq [q^{16}] : (Spin^+_{10}(q).2)$.

We return to the general case where $p \not\equiv 1(3)$ and assume that M contains $C_L(g) = Spin^+_{10}(q)H$. Furthermore we know that L acts on the cosets of M as a rank 3 permutation group with subdegrees $1, q(q^3+1)(q^8-1)/(q-1)$ and $q^8(q^4+1)(q^5-1)/(q-1)$([Kan87]). Then we have two possibilities:

- Suppose $C_M(h) \geq Spin^+_{10}(q)$ for all h in L_α where h is L-conjugate to g. Now if $M = [q^{16}] : C_L(g)$ then M contains q^{16} M-conjugates of $C_L(g)$ each containing a unique copy of $Spin^+_{10}(q)$. Any other L-conjugate of $C_L(g)$ lies inside a non-trivial conjugate of M. But these intersect M with non-trivial indices as above. These intersections cannot contain $Spin^+_{10}(q)$. Hence M contains only M-conjugates of g and, in fact, all these must lie in L_α. Thus $r_g = q^{16}$ and $\frac{n_g}{r_g} = (q^8+q^4+1)(q^6+q^3+1)(q^2+q+1)$. Set

$$u = q^8 + \frac{1}{2}q^7 + \frac{3}{8}q^6 + \frac{5}{16}q^5\frac{99}{128}q^4 + \frac{127}{256}q^3 + \frac{423}{1024}q^2 + \frac{749}{2048}q + \frac{39587}{32768}.$$

  Then $u^2 - u + 1 > \frac{n_g}{r_g}$ for $q \geq 47$. If we set $u_1 = u - \frac{1}{32768}$ then $u_1^2 - u_1 + 1 < \frac{n_g}{r_g}$ for $q > 1$. Thus we need to check $q < 47$ but no such q satisfies $u^2 - u + 1 = \frac{n_g}{r_g}$ for integer u.

- Suppose there exists h in L_α which is L-conjugate to g and $C_M(h)$ does not contain a copy of $Spin^+_{10}(q)$. Then $C_L(h)$ lies inside a non-trivial conjugate of M. Hence $|M : C_M(h)|$ is a multiple of $q(q^3+1)(q^8-1)/(q-1)$ or $q^8(q^4+1)(q^5-1)/(q-1)$. Furthermore we know that q^{16} divides $|M : C_M(h)|$ since $|M|_p = q^{16}|C_L(g)|_p$. Hence $|M : C_M(h)| \geq q^{16}(q^4+1)(q^5-1)/(q-1)$.

  Now, if $L_\alpha \geq [q^{16}] : (Spin^+_{10}(q).2)$ then $r_g = r_g(M)$ since $L_\alpha \trianglelefteq M$ and $|M : L_\alpha|$ is odd. Thus $r_g \geq q^{16}(q^4+1)(q^5-1)/(q-1)$ and $\frac{n_g}{r_g} < q^8 + q^4 + 1$. Then

$d_g \leq q^8 + q^4 + 1 < (q^6 + q^3 + 1)(q^2 + q + 1)$. Thus $v < |L : M|$ which is a contradiction.

Subcase: $\epsilon = -$

In this case the list in [LS85, Table 1] contains one maximal subgroup $M^\dagger$ in $L^\dagger$ such that $|M^\dagger|_p \geq q^{20}$, namely $M^\dagger = N_{L^\dagger}((4, q+1).P\Omega^-(10, q))$. In fact $|M|_p = q^{20}$ and so $p \equiv 1(3)$ and the universal and adjoint versions of E_6^- coincide and L is simple. Then $M = N_L(Spin_{10}^-(q)) \cong Spin_{10}^-(q).(q+1)$ ([GLS94, Table 4.5.2]). Furthermore L_α must contain a Sylow p-subgroup of M. But the parabolic subgroups of $P\Omega_{10}^-(q)$ have even index, hence $Spin_{10}^-(q).2 \leq L_\alpha \leq Spin_{10}^-(q).(q+1)$.

Now, using [GLS94, Table 4.5.2], we see that $E_6^-(q)$ contains two conjugacy classes of involutions: those conjugate to g, centralized by $Spin_{10}^-(q)$, and those conjugate to g_1 say, centralized by $SL(2, q) \circ SU(6, q)$. Then $n_g = q^{16}(q^2 - q + 1)(q^6 - q^3 + 1)(q^8 + q^4 + 1)$ and $n_{g_1} = q^{20}(q^4 + 1)(q^2 + 1)(q^6 - q^3 + 1)(q^8 + q^4 + 1)$.

We examine the involutions lying in $Spin_{10}^-(q)$ using [GLS94, Table 4.5.2]. Apart from the central involution, $Spin_{10}^-(q)$ contains two conjugacy classes of involutions. Let h be an involution in $Spin_{10}^-(q)$ centralized by $Spin_4^+(q) \circ Spin_6^-(q)$. Then L_α contains at least $\frac{1}{4}q^{12}(q^4 + q^3 + q^2 + q + 1)(q^2 - q + 1)(q^4 + 1)(q^2 + 1)$ conjugates of h. If h is L-conjugate to g, then $\frac{n_g}{r_g} < 4q^8$ which is a contradiction. Thus assume that h is L-conjugate to g_1.

In this case $\frac{n_g}{r_g} \leq 4q^{16} + 4q^{12} + 4q^8$. Then

$$d_g < \frac{n_g}{r_g} + 2\sqrt{\frac{n_g}{r_g}} + 2 < 4q^{16} + 4q^{12} + 6q^8 + 2q^4 + 2.$$

This implies that $v < 19|L : M|$ for $q \geq 7$.

Now suppose that q^{16} does not divide into $\frac{n_g}{r_g}$. Then $\frac{n_g}{r_g}$ divides into $(q^2 - q + 1)(q^6 - q^3 + 1)(q^8 + q^4 + 1)$ and so $d_g < 3q^{16}$ and $v = |L : M|$. This contradicts Lemma 2.10. Thus $v = 7|L : M|$ or $v = 13|L : M|$ and $q^{16} \Big| \frac{n_g}{r_g}$.

If $\frac{n_g}{r_g} \geq 7q^{16}$ then $v > 49q^{32} > 13|L : M|$ which is a contradiction. Thus, by Lemma 2.9, $\frac{n_g}{r_g} = 3q^{16}$. This implies that $3q^{16} < d_g < 3q^{16} + 2\sqrt{3}q^8 + 2$ and so $9q^{32} < v < 9q^{32} + 12q^{24} + 6q^{16}$. But then $7|L : M| < v < 13|L : M|$ which is a contradiction.

2.10.4 Case: $L = {}^3D_4(q)$

We know that ${}^3D_4(q)$ has a single conjugacy class of involutions[GLS94] which is centralized by a maximal subgroup isomorphic to $(SL(2, q^3) \circ SL(2, q)).2$ [Kle88b]. Hence, for g an involution in L, $n_g = q^8(q^8+q^4+1)$ and so $|v|_p \leq q^8$ and $|L_\alpha|_p \geq q^4$.

If $L_\alpha < M = N_L({}^3D_4(q_0))$ then this condition implies that $q = q_0^3$. No such subfield subgroup exists.

There are two other odd index maximal subgroups M such that $|M|_p \geq q^4$.[LS85] The first possibility is that $M = G_2(q)$ and $|L : M|_p = q^6$. But then odd index subgroups of $G_2(q)$ have p-index strictly greater than q^2.[LS85] Thus $L_\alpha = G_2(q)$. Now $r_g(G_2(q)) = q^4(q^4 + q^2 + 1)$ and so $\frac{n_g}{r_g} = q^4(q^4 - q^2 + 1)$. But this implies that $|v|_p \leq q^4$ which is impossible.

The second possibility is that $L_\alpha \leq M = 2.(PSL(2, q) \times PSL(2, q^3)).2$. Then $|L : M| = q^8(q^8 + q^4 + 1)$ and so $p \equiv 1(3)$ and L_α contains a Sylow p-subgroup of M. But the parabolic subgroups of $PSL(2, q)$ have even index, hence we conclude that $L_\alpha = M$.

Now $r_g(2.(PSL(2, q) \times PSL(2, q^3))) \geq 1 + \frac{1}{2}q^3(q^3 - 1)\frac{1}{2}q(q - 1)$. This implies that $\frac{n_g}{r_g} < 7q^8$. Suppose that $|\frac{n_g}{r_g}|_p = 1$ and hence $\frac{n_g}{r_g} \leq q^8 + q^4 + 1$. Then $d_g < 3q^8$ and so $d_g = q^8$. This contradicts Lemma 2.10.

Thus $|\frac{n_g}{r_g}|_p > 1$ and so we must have either $\frac{n_g}{r_g} = q^8$ (contradicting Lemma 2.9) or $\frac{n_g}{r_g} = 3q^8$. If $\frac{n_g}{r_g} = 3q^8$ then $d_g < \frac{13}{3}(q^8 + q^4 + 1)$ which is the smallest possibility for d_g that is larger than $\frac{n_g}{r_g}$. Thus we have a contradiction.

2.10.5 Case: $L = G_2(q)$

Referring to [GLS94, Table 4.5.1], we see that $G_2(q)$ contains an involution g such that $C_L(g)$ contains $SL(2, q) \circ SL(2, q)$. There is one such conjugacy class of involutions in L and, examining [Kle88a], we see that $C_L(g) \cong (SL(2, q) \circ SL(2, q)).2$. Hence $n_g = q^4(q^4 + q^2 + 1)$. Using Lemma 2.12, we may conclude that $|v|_p \leq q^4$ and hence that $|L_\alpha|_p > q^2$.

Examining the odd-index maximal subgroups [KL90], we find that all have p-index divisible by p^2 and so $p \equiv 1(3)$. We have a number of possibilities for M an odd-index maximal subgroup, $|M|_p \geq q^2$, M containing L_α:

- Suppose $M = N_L(G_2(q_0))$. Then using Lemma 2.38 we find that $q = q_0^3$. But this means that 9 divides $|L : M|$ which is impossible.

- Suppose $M = (SL(2,q) \circ SL(2,q)).2$. Then $L_\alpha \geq 2.P.2$ where P is a Sylow p-subgroup of $PSL(2,q) \times PSL(2,q)$. Since the parabolic subgroup of $PSL(2,q)$ have even index we must have $L_\alpha = M$ and $v = q^4(q^4+q^2+1)a$ for some integer a. Then Lemma 2.10 implies that $a \neq 1$ and so $a \geq 7$.

  Now $PSL(2,q) \times PSL(2,q)$ has at least $\frac{1}{4}q^2(q \pm 1)^2$ involutions and thus so does $SL(2,q) \circ SL(2,q)$. Then

  $$\frac{n_g}{r_g} < 4q^2 \frac{q^4+q^2+1}{q^2-2q+1} < 7q^4$$

  for $q \geq 7$. Thus either $\frac{n_g}{r_g} = q^4$ (contradicting Lemma 2.9) or $\frac{n_g}{r_g} = 3q^4$ or $\frac{n_g}{r_g}$ divides into q^4+q^2+1.

  If $u^2-u+1 = \frac{n_g}{r_g} = 3q^4$ then $u^2+u+1 = d_g < 3q^4+2\sqrt{3q^4}+2 < 4q^4+4q^2+4$. This implies that $v < 12q^4(q^4+q^2+1)$ and so $a = 7$. But then $d_g = \frac{7}{3}(q^4+q^2+1)$ which is less than $\frac{n_g}{r_g}$ for $q \geq 7$. This is impossible.

  If $u^2-u+1 = \frac{n_g}{r_g} = q^4+q^2+1$ then $u = q^2+1$ and $d_g = q^4+3q^2+3$. But then $(v,p) = 1$ which is impossible. If $\frac{n_g}{r_g} < q^4+q^2+1$ then $u \leq q^2$ which implies that $\frac{n_g}{r_g} \leq q^4-q^2+1$ and $d_g \leq q^4+q^2+1$. Then $\frac{n_g}{r_g}d_g < |L:M|$ which is a contradiction.

- Suppose $M = SL^\epsilon(3,q).2$ and so $p \equiv 1(3)$. Consider first the situation where $L_\alpha = M$. When $\epsilon = +$, $M =< SL(3,q), \phi >$ where ϕ is a graph automorphism [Cha68, (2.6)]. When $\epsilon = -$, $M \leq P\Gamma U(3,q)$ [Kle88a, Proposition 2.2]. In both cases M is equal to a universal version of $A_2^\epsilon(q)$ extended by a graph automorphism [GLS94, Definition 2.5.13].

  Examining [GLS94, Table 4.5.2] we see that M has 2 conjugacy classes of involutions. These have size $q^2(q^2+\epsilon q+1)$ and $q^2(q^2+\epsilon q+1)(q-\epsilon)$. When $\epsilon = +$ this gives $r_g = q^3(q^2+q+1)$ and $\frac{n_g}{r_g} = q(q^2-q+1)$. This is impossible since either $|\frac{n_g}{r_g}|_p = 1$ or $|\frac{n_g}{r_g}|_p \geq q^3$. When $\epsilon = -$ we have $r_g = q^2(q^2-q+1)(q+2)$ and $\frac{n_g}{r_g} = \frac{q^2(q^2+q+1)}{q+2}$. This is not an integer for $q > 1$ hence can be excluded.

  Thus we must have $L_\alpha < M$ and we know that $|M : L_\alpha|_p \leq q$. Examining the subgroups of $SL^\epsilon(3,q)$ we find that $L_\alpha \cap SL^\epsilon(3,q) \leq P_1$, a parabolic subgroup of $SL^\epsilon(3,q)$.

  When $\epsilon = -$, $|SL^\epsilon(3,q) : P_1|$ is even hence this possibility can be excluded.

When $\epsilon = +$, $M =< SL(3,q), m >$ where m is a graph automorphism of $SL(3,q)$. Since L_α has odd index in $G_2(q)$, L_α must contain a graph automorphism. Examining [KL90, Table 3.5.A] we find that $L_\alpha \cap SL(3,q)$ lies inside a subgroup M_1 of $SL(3,q)$ of type $GL(2,q) \oplus GL(1,q)$ or of type $P_{1,2}$. In the former case we find that $|v|_p \geq q^5$. Since $|n_g|_p = q^4$ we must have $|\frac{n_g}{r_g}|_p = 1$ which implies that $\frac{n_g}{r_g} \leq q^4 + q^2 + 1$ and $|d_g|_p \geq q^5$ which contradicts Lemma 2.11. In the latter case, we find that $|SL(3,q) : M_1|$ is even and this case can be excluded.

We have covered all possible odd-index maximal subgroups in $G_2(q)$.

2.10.6 Case: $L = F_4(q)$

Referring to [GLS94, Table 4.5.1], we see that $F_4(q)$ contains an involution g such that $C_L(g)$ contains $Spin(9,q)$. There is one such conjugacy class of involutions in L and so $n_g = q^8(q^8 + q^4 + 1)$.

This implies that $|v|_p \leq q^8$ and hence that $|L_\alpha|_p \geq q^{16}$. Then Lemma 2.38 implies that $|N_L(F_4(q_0))|_p \leq |F_4(q_0)|_p = q_0^{24}$. Since $q = q_0^a$ where a is an odd prime, $q_0^{24} \leq q^8$ and so L_α does not lie in $|N_L(F_4(q_0))$.

The list in [LS85, Table 1] contains one maximal subgroup M such that $|M|_p \geq q^{16}$. Then $M \cong 2.\Omega(9,q)$, L_α must contain a Sylow p-subgroup of M since $|L : M|_p = q^{16}$. Furthermore, $p \equiv 1(3)$. Now the parabolic subgroups of $\Omega(9,q)$ have even index, hence we may conclude that $L_\alpha = M$ and $v = q^8(q^8 + q^4 + 1)a$ for some integer a. Lemma 2.10 implies that $a \neq 1$ and hence $a \geq 7$.

Now suppose $r_g \geq \frac{1}{2}q^4(q^4 - 1)$. Then $\frac{n_g}{r_g} \leq 2q^4(q^4 + 3) < \frac{7}{3}q^8$. Then $d_g < \frac{14}{3}q^8$ and $v < 7q^{16}$ which is a contradiction. Also r_g is clearly greater than 1. Thus there is an involution $g \in 2.\Omega(9,q)$ such that

$$1 < |2.\Omega(9,q) : C_{2.\Omega(9,q)}(g)| < \frac{1}{2}q^4(q^4 - 1).$$

Now let B be the central subgroup of L_α of order 2, so that $L_\alpha/B \cong P\Omega(9,q)$. Let $h = Bg$ an involution in $P\Omega(9,q)$. Then we must have

$$|\Omega(9,q) : C_{\Omega(9,q)}(h)| < \frac{1}{2}q^4(q^4 - 1).$$

Examining [GLS94, Table 4.5.1] we see that all involution centralizers in $\Omega(9,q)$ have index at least $\frac{1}{2}q^4(q^4 - 1)$. Hence we have a contradiction.

Proposition 2.37 is now proven.

2.11 L is an exceptional group of Lie type in characteristic 2

In this section we prove the following proposition:

Proposition 2.39. *Suppose G has a minimal normal subgroup L where L is an exceptional group of Lie type in characteristic 2 or that G has a unique component L such that $L^\dagger$ is isomorphic to $E_6(q)$ or ${}^2E_6(q)$ where $q = 2^a$. Then G does not act transitively on a projective plane.*

We have nine possibilities for L and, by Tits' Lemma [Sei73, 1.6], we know that L_α must lie in a parabolic subgroup M of L. We demonstrate that this is impossible, generally by showing a contradiction with Lemma 2.7.

2.11.1 Case: $L = {}^3D_4(q)$; $G_2(q), q > 2$

In each case, for any parabolic subgroup M, $|L : M|$ is divisible by $(q^4+q^2+1)(q+1)$. If $q \equiv 1(3)$ then $|L : M|$ is divisible by $q+1 \equiv 2(3)$, while if $q \equiv 2(3)$ then 9 divides $|L : M|$. Thus M cannot contain L_α (Lemma 2.7) and we are done.

2.11.2 Case: $L = {}^2B_2(q), q > 2; {}^2F_4(q)', F_4(q), E_7(q), E_8(q)$

Examining the indices of the parabolic subgroups M in L in these cases, we find that they are nearly always divisible by $q^m + 1$ for some even integer m. Since $q^m + 1 \equiv 2(3)$ these cases are excluded. We deal with the exceptions which are as follows:

1. $L = E_7(q)$ and M is of type E_6. Then $|L : M|$ is divisible by $(q^5 + 1)(q^9 + 1)$. If $q \equiv 1(3)$ then $q^5 + 1 \equiv 2(3)$ and if $q \equiv 2(3)$ then 9 divides $|L : M|$. Both of these are impossible hence M cannot contain L_α.

2. $L = E_7(q)$ and M is of type D_6. Then $|L : M|$ is divisible by $(q^8+q^4+1)(q^{12}+q^6+1)$ which is in turn divisible by 9. Hence M cannot contain L_α.

3. $L = E_7(q)$ and M is of type $D_5 \times A_1$. Then $|L : M|$ is divisible by $(q^5+1)(q^8+q^4+1)$. If $q \equiv 1(3)$ then $q^5 + 1 \equiv 2(3)$ and if $q \equiv 2(3)$ then 9 divides $|L : M|$. Both of these are impossible hence M cannot contain L_α.

Note that Kantor's argument to exclude the last two cases ($L = E_7(q)$ and M of type D_6 or $D_5 \times A_1$) when the action is primitive is incorrect[Kan87].

2.11.3 Case: $L^\dagger = E_6^\epsilon(q)$

As in Subsection 2.11.2 we need only examine the parabolic subgroups M in L which are not divisible by q^m+1 for some even integer m. There are two possibilities:

1. $L^\dagger = E_6^+(q)$ and M is of type D_5. Then $|L:M| = (q^6+q^3+1)(q^8+q^4+1)(q^2+q+1)$. For $q \equiv 1(3)$, $|L:M|$ is divisible by 9 hence M cannot contain L_α. Thus we assume that $q \equiv 2(3)$ and so L is simple.

   Now we know that $M' := [q^{16}].\Omega_{10}^+(q) \le L_\alpha \le M \cong [q^{16}] : (\Omega_{10}^+(q)H)$ where H is the Cartan subgroup of L. This is because all parabolic subgroups of $\Omega_{10}^+(q)$ have index divisible by $q^4+1 \equiv 2(3)$.

   By [AS76, (15.1),(15.5)], L contains an involution g such that $C_L(g) = [q^{21}] : SL(6,q)$ and so $n_g = (q^6+q^3+1)(q^8+q^4+1)(q^8-1)$. Now if $r_g \ge (q^6+q^3+1)(q^8-1)$ then $\frac{n_g}{r_g} \le (q^4+1)^2-(q^4+1)+1$ and so $d_g \le (q^4+1)^2+(q^4+1)+1$. But then $\frac{n_g}{r_g}d_g < |L:M|$ which is a contradiction. Thus, for all $h \in L_\alpha$ conjugate in G to g, $|K : C_K(h)| < (q^6+q^3+1)(q^8-1)$.

   Now $\Omega_{10}^+(q) \not\le C_L(g)$. Furthermore the only maximal subgroups of $\Omega_{10}^+(q)$ with index less than $(q^6+q^3+1)(q^8-1)$ are the parabolic subgroups and $Sp_8(q)$. All but one type of parabolic subgroups have index divisible by q^3+1. Since q^3+1 does not divide into n_g, there must be $h \in L_\alpha$ conjugate in G to g such that $C_K(h)$ lies in either $[q^{16}].([q^8] : \frac{1}{2}((q-1)\times SO_8^+(q)))$ or $[q^{16}].Sp_8(q)$.

   Consider the first possibility. Now $SO_8^+(q) \not\le C_L(g)$ and so
   $$r_g \ge P(SO_8^+(q))\frac{|\Omega_{10}^+(q)|}{|[q^8] : \frac{1}{2}((q-1)\times SO_8^+(q))|}.$$
   Using the value for $P(SO_8^+(q))$ given in [KL90, Table 5.2.A] we conclude that $r_g > (q^6+q^3+1)(q^8-1)$ which is impossible.

   Similarly $Sp_8^+(q) \not\le C_L(g)$ and so
   $$r_g \ge P(Sp_8^+(q))\frac{|\Omega_{10}^+(q)|}{|Sp_8^+(q))|}.$$
   Once again we find that $r_g > (q^6+q^3+1)(q^8-1)$ which is impossible.

2. $L^\dagger = E_6^-(q)$ and M is of type ${}^2D_4(q)$. Then $|L:M|$ is divisible by $(q^5+1)(q^9+1)$; we exclude this possibility in the same way as in Subsection 2.11.2, when $L = E_7(q)$ and M is of type E_6.

This concludes the proof of Proposition 2.39. Theorem A is now also proven.

Chapter 3

$PSL(3,q)$ acting line-transitively on linear spaces

"What is line? It is life."
Jean Cocteau, *"The Difficulty of Being"*

We present a partial classification of those spaces $\mathcal{S}$ on which an almost simple group G with socle $PSL(3,q)$ acts line-transitively. The statement of our theorem is as follows:

Theorem B. *Suppose that $PSL(3,q) \trianglelefteq G \leq AutPSL(3,q)$ and that G acts line-transitively on a finite linear space $\mathcal{S}$. Then one of the following holds:*

- *$S = PG(2,q)$, the Desarguesian projective plane, and G acts 2-transitively on points;*
- *$PSL(3,q)$ is point-transitive but not line-transitive on $\mathcal{S}$. Furthermore, if G_α is a point-stabilizer in G then $G_\alpha \cap PSL(3,q) \cong PSL(3,q_0)$ where $q = q_0^a$ for some integer a.*

Note that, in the case where $\mathcal{S}$ is a projective plane, Theorem A implies Theorem B. Note too that line-transitive linear spaces with $k=3$ or $k=4$ have been completely classified in [Cla76, KS84, BDD$^+$90, CS89, Li95]. Hence we need to consider the situation when $\mathcal{S}$ is not a projective plane and $k \geq 5$.

One result should be mentioned which has an important bearing on our work here: Camina, Neumann and Praeger have classified the line-transitive actions of $PSL(2,q)$ although this result has not been published. The result is as follows:

Theorem 3.1. *Let $G = PSL(2, q), q \geq 4$ and suppose that G acts line-transitively on a linear space $\mathcal{S}$. Then one of the following holds:*

- *$G = PSL(2, 2^a), a \geq 3$ acting transitively on $\mathcal{S}$, a Witt-Bose-Shrikhande space. Here Π is the set of dihedral subgroups of G of order $2(q+1)$ and Λ is the set of involutions $t \in G$ with the incidence relation being inclusion.*
- *$\mathcal{S} = PG(2, 2)$, $G = PSL(2, 7)$ and the action is 2-transitive.*

This chapter is devoted to proving Theorem B and the structure of the chapter is as follows: The first two sections outline some background lemmas concerning linear spaces. Section 3.3 gives background information about $PSL(3, q)$. In Section 3.4 we reduce the proof to the situation when $PSL(3, q)$ is transitive upon the lines of the space $\mathcal{S}$. This reduction makes use of the notion of **exceptionality** of permutation representations, the relevance of which was pointed out by Dr Peter Neumann. The remaining sections are devoted to the situation when $PSL(3, q)$ is line-transitive, under different assumptions about significant primes.

The following notation will hold, unless stated otherwise, throughout the chapter. We will take G to be a group acting on a regular linear space $\mathcal{S}$ with parameters b, v, k, r. We will write α to be a point of $\mathcal{S}$ with G_α to be the stabilizer of α in the action of G. Similarly $\mathfrak{L}$ is a line of $\mathcal{S}$ and $G_{\mathfrak{L}}$ is the corresponding line-stabilizer.

3.1 Known Lemmas

We list here some well-known lemmas which we will use later. The first lemma is proved easily by counting.

Lemma 3.2. *1. $b = \frac{v(v-1)}{k(k-1)} \geq v$ (Fisher's inequality);*

2. $r = \frac{v-1}{k-1} \geq k$;

Lemma 3.3. *[CNP03, Lemma 6.5] Let p be an odd prime divisor of v.*

1. If $b = \frac{3}{2}v$ then $p = 5$ and $25 \nmid v$, or $p \equiv 1, 2, 4$ or $8(15)$;

2. If $b = 2v$ then $p \equiv 1(4)$.

For the remainder of this section assume that G acts line-transitively on the linear space $\mathcal{S}$.

Theorem 3.4. *[CG84, Theorem1] If $k \big| v$ then G is flag-transitive.*

Lemma 3.5. *[CS89, Lemma 4] If g is an involution of G and g fixes no points, then $k \big| v$. In particular, G is flag-transitive.*

Lemma 3.6. *[CS89, Lemma 2] Let $\mathfrak{L}$ be a line in $\mathcal{S}$ and let $T \leq G_{\mathfrak{L}}$. Assume that T satisfies the following two conditions:*

1. *$|Fix_{\Pi}(T) \cap \mathfrak{L}| > 1$;*
2. *if $U \leq G_{\mathfrak{L}}$ and $|Fix_{\Pi}(U) \cap L| > 1$ and U is conjugate to T in G, then U is conjugate to T in $G_{\mathfrak{L}}$.*

Then either $Fix_{\Pi}(T) \subseteq \mathfrak{L}$ or the induced linear space on $Fix_{\Pi}(T)$ is regular and $N_G(T)$ acts line-transitively on the space.

Lemma 3.7. *[CS00, Lemma 2.2] Let g be an involution in G and assume that there exists N, $N \lhd G$ such that $|G : N| = 2$ with $g \notin N$. Then N acts line-transitively also.*

Note that Lemma 3.7 allows us to conclude that if $PGL(2, q)$ acts transitively on the lines of a linear space $\mathcal{S}$ then $PSL(2, q)$ also acts transitively on the lines of $\mathcal{S}$ and so that space is known.

Our next result provides the framework for our analysis of the line-transitive actions of $PSL(3, q)$. Since $\mathcal{S}$ is not a projective plane then, by Fisher's inequality $b > v$ and since $b = v(v-1)/(k(k-1))$, there must be some prime p that divides both $v - 1$ and b. We shall refer to such a prime as a *significant* prime.

Lemma 3.8. *[CNP03, Lemma 6.1] Suppose that $\mathcal{S}$ is not a projective plane and let p be a significant prime. Let P be a Sylow p-subgroup of G_α. Then P is a Sylow p-subgroup of G and G_α contains the normalizer $N_G(P)$.*

Lemma 3.9. *[CNP03, Lemma 6.3] Let H, K be subgroups such that*

$$G_\alpha \leq H < K \leq G$$

and let $c = |K : H|$. Then r divides $\frac{1}{2}(c-1)k$ and b divides $\frac{1}{2}(c-1)v$.

Corollary 3.10. *[CNP03, Corollary 6.4] Let H, K be as in Lemma 3.9.*

1. *Let*
$$c_0 = \gcd\{(c-1) \mid c = |K:H|, \text{ where } G_\alpha \leq H < K \leq G\}.$$
Then r divides $\frac{1}{2}c_0k$ and b divides $\frac{1}{2}c_0v$.

2. *There cannot be groups H, K such that $G_\alpha \leq H < K \leq G$ and $|K:H| = 2$.*

3. *If there are groups H, K such that $G_\alpha \leq H < K \leq G$ and $|K:H| = 3$ then $\mathcal{S}$ is a projective plane.*

3.2 New Lemmas

We state a series of lemmas which will be used in our analysis of the actions of $PSL(3,q)$. The first is a generalization of the Fisher inequality to non-regular linear spaces.

3.2.1 General linear spaces

Lemma 3.11. *In any linear space $\mathcal{S}$, not necessarily regular, Fisher's inequality holds: $b \geq v$.*

Proof. We need to prove the statement under the assumption that the number of points in a line is not a constant. Let c be the maximum number of points on a line of $\mathcal{S}$. Since any two points lie on a unique line we know that

$$b \geq \frac{\binom{v}{2}}{\binom{c}{2}} = \frac{v(v-1)}{c(c-1)}.$$

Thus if $c(c-1) \leq v-1$ then we are finished. Assume to the contrary from this point on. We split into two cases:

1. **Suppose that $(c-1)^2 \geq v$.** If there are two lines of size c then there are at least $(c-1)^2$ lines between them and we are done. Now consider the two largest lines of size c and $c-a, 1 \leq a < c$.

   Then

   - $b > (c-1)(c-a-1)+2$ since there are at least this many lines between the points of the two largest lines;
   - $b > \frac{(v-c)c}{c-a-1}$ since there are at least this many lines joined to the line with c points.

Now we may assume that $\frac{(v-c)c}{c-a-1} < v$ since otherwise we are done. This inequality implies that $v < \frac{c^2}{a+1}$. Similarly we can assume that $(c-1)(c-a-1)+2 < v$ since otherwise the result holds. This implies that $(c-1)(c-a-1)+2 < \frac{c^2}{a+1}$ and so $c < a+2+\frac{3}{a}$.

For $a \geq 3$ this implies that $c \leq a+2$. But then all but one line must contain precisely 2 points. Clearly $b \geq v$ in this case.

If $a \leq 2$ then $c \leq 5$. If $a = 2$ and $c = 5$ then $v < \frac{c^2}{3} \leq 8$. But the number of lines connecting the points of the two largest lines is at least 8 and the result holds. If $a = 2$ and $c = 4$ then all but one line must contain precisely 2 points and once again the result holds. If $a = 1$ then $(c-1)(c-2)+2 < \frac{c^2}{2}$ and so $c = 3$. But, again, this means that all but one line must contain precisely 2 points and the result holds.

2. **Suppose that** $(c-1)^2 < v \leq c(c-1)$**.** Note that $v > 2$ implies that $c > 2$. Let r_α be the number of lines incident with a point α. If $r_\alpha \geq c$ for all α then, let f be the number of flags:

$$vc \leq f \leq bc.$$

Thus $v \leq b$ as required. Assume then that there exists a point α such that $r_\alpha \leq c-1$. Observe that every line not passing through α must have be incident with at most r_α points. Remove α and any lines which are incident with only α and one other point. Then $v > c$ and we still have a linear space, S^*. S^* has $v-1$ points, at most b lines, and the maximum number of points on a line is $c-1$. This implies that,

$$b_S \geq b_{S^*} \geq \frac{\binom{v-1}{2}}{\binom{c-1}{2}} = \frac{(v-1)(v-2)}{(c-1)(c-2)}.$$

Thus we are finished so long as $(c-1)(c-2) < v-2$. But $(c-1)^2 < v$ gives us this inequality since $c \geq 3$.

All cases are proved and the result stands. □

3.2.2 Regular linear spaces

We return to our assumption that $\mathcal{S}$ is a regular linear space.

Lemma 3.12. *Let $g \in G$ be an involution. Then g fixes at least $(v-1)/k$ lines.*

Proof. If g has no fixed point then g fixes $v/k \geq (v-1)/k$ lines. If g has a fixed point, α, then let m be the number of fixed lines through α. By definition, g moves the rest of the lines through α. Apart from α these lines contain $v - m(k-1) - 1$ points. None of these points is fixed hence every one of these points lies on a fixed line. Thus the number of lines fixed by g is at least

$$m + \frac{v - m(k-1) - 1}{k} = \frac{v + m - 1}{k} \geq \frac{v-1}{k}$$

lines as required. □

Lemma 3.13. *Let g be an involution which is an automorphism of a linear space $\mathcal{S}$. Suppose that $\mathcal{S}$ has a constant number of points on a line, k, and that g fixes d_l lines and d_p points. Then, either*

- $d_l \geq d_p$; *or*
- $v = k^2$.

Proof. We know that if $\mathcal{S}$ is a projective plane then the result holds since the permutation character on points and lines is the same [Dem97, 4.1.2]. Now suppose that $\mathcal{S}$ is not a projective plane and split into two cases:

1. **Suppose that $d_p \leq k$.** Assume that $d_l < d_p$. We know, by Lemma 3.12, that g fixes at least $\frac{v-1}{k}$ lines. Then

$$\begin{aligned} d_l < d_p \quad &\Longrightarrow \quad \frac{v-1}{k} < k \\ &\Longrightarrow \quad v - 1 < k^2. \end{aligned}$$

   Then, since $(k-1)\big|(v-1)$, we must have $\frac{v-1}{k-1} \leq k+1$. If $\frac{v-1}{k-1} \leq k$ then $b \leq v$ and so $b = v$ and $\mathcal{S}$ is a projective plane. If $\frac{v-1}{k-1} = k+1$ then $v = k^2$ as given.

2. **Suppose that $d_p > k$.** Then the fixed points and lines of g form a linear space. We may appeal to Lemma 3.11.

□

Lemma 3.14. *Suppose that $b = \frac{c}{d}v$ where $(c,d) = 1$. Then the significant primes are exactly those which divide c.*

Proof. By definition a prime is significant if it divides b and $v-1$. Then we just use the fact that

$$\frac{c}{d}v = b = \frac{v(v-1)}{k(k-1)} = \frac{(v-1)/(k-1)}{k}v.$$

□

Lemma 3.15. *Let $H < G_\alpha$. If $N_G(H) \not\leq G_\alpha$ then H is in $G_{\mathfrak{L}}$ for some line $\mathfrak{L}$.*

Proof. Simply take $g \in N_G(H)\backslash G_\alpha$. Then $H^g = H$ is contained in G_α and $G_{\alpha g}$. Hence H fixes the line joining α and αg. □

3.2.3 Line-transitive linear spaces

Throughout this section we assume that G acts line-transitively on $\mathcal{S}$.

Lemma 3.16. *Let g be an involution of G and write $n_g = |g^G|$ for the size of a conjugacy class of involutions in G. Let $r_g = |g^G \cap G_{\mathfrak{L}}|$ be the number of such involutions in a line-stabilizer $G_{\mathfrak{L}}$. Then the following inequality holds:*

$$\frac{n_g(v-1)}{br_g} \leq k \leq \frac{r_g v}{n_g} + 1.$$

Proof. Count pairs of the form $(\mathfrak{L}, g)$ where $\mathfrak{L}$ is a line and g is an involution fixing $\mathfrak{L}$, in two different ways. Then

$$|\{(\mathfrak{L}, g)\}| = br_g \geq n_g c$$

where c is the minimum number of lines fixed by an involution. Now, by the previous lemma, $c \geq \frac{v-1}{k}$ thus we have

$$r_g \geq \frac{n_g c}{b} \geq \frac{n_g(v-1)}{bk} = \frac{n_g(k-1)}{v}.$$

This implies two inequalities:

$$k - 1 \leq \frac{r_g v}{n_g} \quad , \quad k \geq \frac{n_g(v-1)}{br_g}$$

and the result follows. □

Lemma 3.17. *Suppose that $|G_\alpha| = \frac{c}{d}|G_{\mathfrak{L}}|$ where $(c, d) = 1$. Then the significant primes are exactly those which divide c.*

Proof. Simply use the fact that $v = |G|/|G_\alpha|$, $b = |G|/|G_{\mathfrak{L}}|$ and refer to Lemma 3.14. □

Lemma 3.18. *Suppose that p^a is a prime power dividing $v - 1$ and that p does not divide into $|G|$. Then p^a divides $k(k - 1)$.*

Proof. Since p does not divide $|G|$, p cannot divide into b. Since $b = \frac{v(v-1)}{k(k-1)}$ and p^a divides into $v - 1$ we must have p^a dividing into $k(k - 1)$. □

We will often repeatedly use Lemma 3.18, with different primes, to exclude the possibility of a particular group, G, acting line-transitively on a space with a particular number of points, v. Our method for doing this usually involves showing that any line size k must be too large to satisfy Fisher's inequality (Lemma 3.2).

3.3 Background Information on $PSL(3, q)$

First a word about notation: As in the previous chapter, we will sometimes precede the structure of a subgroup of a projective group with ˆ which means that we are giving the structure of the pre-image in the corresponding linear group. We will also refer to elements of this linear group in terms of matrices under the standard modular representation.

3.3.1 Subgroup information

We need information about the subgroups of $PSL(3, q)$, $PSL(2, q)$ and $GL(2, q)$.

Theorem 3.19. *[Kle87, Mit11, Blo67b, Har25] The maximal subgroups of $PSL(3, q)$ are among the following list. Conditions given are necessary for existence and maximality but not sufficient. The first three types are all maximal for $q \geq 5$.*

	Description	**Notes**
1	$\hat{}[q^2] : GL(2, q)$	*two $PSL(3, q)$-conjugacy classes*
2	$\hat{}(q - 1)^2 : S_3$	*one $PSL(3, q)$-conjugacy class*
3	$\hat{}(q^2 + q + 1).3$	*one $PSL(3, q)$-conjugacy class*
4	$PSL(3, q_0).(q - 1, 3, b)$	$q = q_0^b$ *where b is prime*
5	$PSU(3, q_0)$	$q = q_0^2$
6	A_6	*q odd*
7	$3^2.SL(2, 3)$	*q odd*
8	$3^2.Q_8$	*q odd*
9	$SO(3, q)$	*q odd*
10	$PSL(2, 7)$	*q odd*

In general we will refer to maximal subgroups of $PSL(3,q)$ as being of type x, where x is a number between 1 and 10 corresponding to the list above.

Referring to [Blo67b, Kle87] we state the following lemma:

Lemma 3.20. *Suppose that H is a subgroup of $PSL(3,q)$ lying in a maximal subgroup of type 4 or 5 and H does not lie in any other maximal subgroup of $PSL(3,q)$. Then one of the following holds:*

- *H has a cyclic normal subgroup of index less than or equal to 3.*
- *H contains $PSL(3,q_1)$ with index less than or equal to 3. Here $q = q_1^c$, c an integer.*
- *H contains $PSU(3,q_1)$ with index less than or equal to 3. Here $q = q_1^c$, c an integer.*
- *H is isomorphic to $A_6.2$ or A_7 and $q = 5^a, a$ even.*

We will need information about the subgroups of $PSL(2,q)$. These have already been listed in Theorem 2.24. Note also the comments following that theorem regarding the conjugacy classes of subgroups in $PSL(2,q)$.

Information about the subgroups of $GL(2,q)$ has also already been listed in Theorem 2.25. This will be of use in analysing the subgroups of the parabolic subgroups of $PSL(3,q)$. Note that when we refer to subgroups of type 5 in $GL(2,q)$ we will use q_0 for the size of our subfield, rather than r as stated in Theorem 2.25 (throughout this chapter the letter r is reserved for the number of lines through a point of a linear space.)

Note finally that we will write μ for $(q-1,3)$.

3.3.2 Involutions and 2-transitive actions

We will use the fact that $PSL(3,q)$ contains a single conjugacy class of involutions. This class is of size $q^2(q^2+q+1)$ for q odd and of size $(q^2-1)(q^2+q+1)$ for q even.

We need a list of the 2-transitive actions of $PSL(2,q)$. The list is classical, see for instance [Ban71]:

Lemma 3.21. *Let G have socle $PSL(2,q), q \geq 4$ and suppose it acts 2-transitively on a set Ω then one of the following holds:*

1. $PSL(2,q) \trianglelefteq G \leq P\Gamma L(2,q)$, $|\Omega| = q+1$;

2. $G = PSL(2,11)$, $|\Omega| = 11$;

3. $G = P\Gamma L(2,8)$, $|\Omega| = 28$;

4. $PSL(2,7) \trianglelefteq G \leq PGL(2,7)$, $|\Omega| = 7$;

5. *G has socle* $PSL(2,5) \simeq A_5$, $|\Omega| = 5$;

6. *G has socle* $PSL(2,9) \simeq A_6$, $|\Omega| = 6$.

Note that this list covers all of the 2-*homogeneous* actions of groups with socle $PSL(2,q)$; here a 2-*homogeneous action* of a group G on a set Ω is one which is transitive on unordered pairs of elements of Ω.

Note too that $PSL(2,2) \simeq S_3$ and $PSL(2,3) \simeq A_4$ also have 2-homogeneous actions of degree 3 and, for $PSL(2,3)$, of degree 4.

3.3.3 The subgroup D

We define D to be the centre of a Levi complement of a particular parabolic subgroup. Typically D is the projective image of

$$\left\{ \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a \in F_q \right\}.$$

Suppose that $G = PSL(3,q)$ acts line-transitively on a linear space. Since D normalizes a Sylow t-subgroup of $PSL(3,q)$ for many different t, D often lies inside a point-stabilizer G_α. Furthermore, since D has a large normalizer, $\hat{}GL(2,q)$, by Lemma 3.15, D often lies inside a line-stabilizer, $G_{\mathfrak{L}}$.

We exploit this fact using Lemma 3.6 since if D satisfies the conditions given in the lemma and the fixed points of D are not collinear then we induce a line-transitive action of $PGL(2,q)$ on a linear space. All such actions on a non-trivial linear space are known. In the event that the fixed set is a trivial linear space (that is, $k=2$) line-transitivity is equivalent to 2-homogeneity on points and these actions are also all well-known.

We need information about the occurrence of D in various subgroups and about how G-conjugates of D intersect. We state the relevant facts in a series of lemmas in which we refer to the following matrix equation. DET is the value required to give the matrix determinant 1 and $a \in GF(q)$.

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} \begin{pmatrix} \frac{1}{DET} & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{pmatrix}$$
$$\implies \begin{pmatrix} \frac{r}{a^2} & \frac{s}{a^2} & \frac{t}{a^2} \\ ua & va & wa \\ xa & ya & za \end{pmatrix} = \begin{pmatrix} \frac{r}{DET} & se+tg & sf+th \\ \frac{u}{DET} & ve+wg & vf+wh \\ \frac{x}{DET} & ye+zg & yf+zh \end{pmatrix}$$

Lemma 3.22. *The $PSL(3,q)$ conjugates of D intersect trivially.*

Proof. In the equation above, we can take $f = g = 0$ and suppose that $e = h$ and they are both equal to a or a^{-1}. Then $DET = e^2$. In order for the left columns to be equal in the above, we must have either $a^3 = 1$ or $u = x = 0$. For the top rows to be equal in the above, we must have $a^3 = 1$ or $s = t = 0$. The first option in each case corresponds to the trivial intersection in $PSL(3,q)$. The second possibility means that we are conjugating by the normalizer of our diagonal element, hence our conjugate of D is D itself. The result is proved. $\square$

Next we wish to analyse which $PSL(3,q)$-conjugates of D lie in the Levi complement $\hat{\ }GL(2,q)$ of a particular parabolic subgroup. We will use Theorem 2.25 which lists the subgroups of $GL(2,q)$. Our notation will be consistent with that theorem, in particular referring to a subgroup H of $\hat{\ }GL(2,q)$ as being *of type y* if the pre-image of H in $SL(3,q)$ is of type y in $GL(2,q)$. In the statement of the following lemma, a will be a primitive element of $GF(q)$.

Lemma 3.23. *Let $U : \hat{\ }GL(2,q)$ be a parabolic subgroup of $PSL(3,q)$, $q > 7$, U an elementary abelian p-group. We can choose $\hat{\ }GL(2,q)$ conjugate to*

$$C_G(D) = \hat{\ }\left\{ \begin{pmatrix} \frac{1}{DET} & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{pmatrix} : \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL(2,q), DET = eh - fg \right\}.$$

Let H be a maximal subgroup of $\hat{\ }GL(2,q)$ in $PSL(3,q)$.

1. *If H is of type 2 in $\hat{\ }GL(2,q)$ then some $\hat{\ }GL(2,q)$ conjugate of H contains one individual conjugate, and two families of conjugates, of D, generated by*

the projective images of the following matrices, for $f \in GF(q)$:

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & f & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & f & \frac{1}{a^2} \end{pmatrix}.$$

2. *If H is of type 3 in $\hat{}GL(2,q)$ then H contains only D.*

3. *If H is of type 4 in $\hat{}GL(2,q)$ then some $\hat{}GL(2,q)$-conjugate of H contains three conjugates of D, generated by the projective images of the following matrices:*

$$\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}.$$

4. *If H is of type 5 in $\hat{}GL(2,q)$ then one of the following holds:*
   - *H contains only D;*
   - *$H \geq SL(2,q)$;*
   - *$H \geq SL(2,q_0)$ where $q = (q_0)^2$ and $q_0 = 3, 4$ or 7.*

5. *If H is of type 6 or 7 in $\hat{}GL(2,q)$ then one of the following holds:*
   - *H contains only the central copy of D;*
   - *$q = 13, 16$ or 19.*

Proof. 1. Suppose H is maximal of type 2 in $\hat{}GL(2,q)$. Assume $g = 0$ in the equations above and consider the middle column. We must have one of the following:

- $e = a, s = 0$;
- $e = \frac{1}{a^2}, v = y = 0$

Consider the left column. We must have one of the following:

- $\frac{1}{DET} = \frac{1}{a^2}, u = x = 0$;
- $\frac{1}{DET} = a, r = 0$

Combining these, we must have one of the following:

- $\frac{1}{DET} = a = h, e = \frac{1}{a^2}, v = y = r = t = 0$;
- $\frac{1}{DET} = \frac{1}{a^2}, e = a = h, u = x = s = f = 0$;
- $\frac{1}{DET} = a = e, h = \frac{1}{a^2}, r = s = 0$.

These correspond to the options as listed above.

2. Suppose H is maximal of type 3 in $\hat{\ }GL(2, q)$. Then H has a single normal subgroup of size $\frac{q-1}{\mu}$. This is D.

3. Suppose H is maximal of type 4 in $\hat{\ }GL(2, q)$. If $e = h = 0$ in the equations above then, by examining the right and middle column, we can conclude that $f = \frac{a^2}{g}$. Then we must have $DET = -a^2$. But we know, by considering the left columns, that $DET = a^2$ or $\frac{1}{a}$. For $q > 7$ this is impossible. Thus D has only diagonal conjugates in this case.

   So take $f = g = 0$. Consider the middle columns: Either $e = a$ or $v = y = 0$. Assume the latter and, considering the top row, we must have $s \neq 0$ and $e = \frac{1}{a^2}$. Furthermore, looking at the left columns, we must have $x \neq 0 \neq u$ and so $DET = \frac{1}{a}$. This forces h to be a and we have one of the subgroups described.

   Now suppose that $e = a$. Then, looking at the right column, we can have $h = a$ and we get D back again or $w = z = 0$. So assume the latter. The top right corner gives us that $h = \frac{1}{a^2}$. Then $u = x \neq 0$ and we have $DET = \frac{1}{a}$. Again we have one of the subgroups described.

4. Suppose H is maximal of type 5 in $\hat{\ }GL(2, q)$. Take $q_0 < q$ and suppose that $H =< \hat{\ }SL(2, q_0), D >$ or $H =< \hat{\ }SL(2, q_0), D > .2$. Clearly $D \trianglelefteq H$ and, if H contains any other conjugate of D, then H/D must contain an element of order $\frac{q-1}{\mu}$. But an element of $\hat{\ }SL(2, q_0)$ has order at most $q_0 + 1$. Thus $\frac{q-1}{2\mu} \leq q_0 + 1$. If $q_0 < q$ then we must have $q = (q_0)^2$ and $q_0 = 3, 4$ or 7.

5. Suppose H is maximal of type 6 or 7 in $\hat{\ }GL(2, q)$. Clearly $D \trianglelefteq H$ then, if H contains any other conjugate of D, E we must have $DE \leq H$. Then $E \cong DE/D \leq H/D \leq PGL(2, q)$. Then H/D is isomorphic to A_4, S_4, A_5 or $A_4.2$. The largest order of any element these groups is 6. Thus $|E| = \frac{q-1}{\mu} \leq 6$ as required.

□

Corollary 3.24. *A subgroup of $PSL(3,q)$ of type 3 contains only the 3 diagonal conjugates of D as listed above for H of type 4 in $\hat{}GL(2,q)$.*

Proof. We know, by the above proof for H of type 4 in $\hat{}GL(2,q)$, that the generator of D is not conjugate to any element $X.Y$ where X is diagonal, Y an involutory permutation matrix. The only other non-diagonal possibility is if Y is a permutation matrix of order 3. But then $X.Y$ has order 3 and $\frac{q-1}{\mu} \neq 3$ for all q. □

3.4 Reducing to the Simple Case

Suppose a group G acts line-transitively on a linear space $\mathcal{S}$; suppose furthermore that B is a normal subgroup in G which is not line-transitive on $\mathcal{S}$; finally suppose that $|G:B| = t$, a prime. This means that, for a line $\mathfrak{L}$ of $\mathcal{S}$, $G_{\mathfrak{L}} = B_{\mathfrak{L}}$. We have two possibilities:

- **Suppose that B is point-transitive on $\mathcal{S}$.** Then let α and β be members of Π, the set of points of $\mathcal{S}$. Let $\mathfrak{L}$ be the line connecting them. Then, since $G_{\alpha,\beta} \leq G_{\mathfrak{L}}$ and $B_{\alpha,\beta} \leq B_{\mathfrak{L}}$, we know that $G_{\alpha,\beta} = B_{\alpha,\beta}$.

  We know furthermore that $|G_\alpha : B_\alpha| = t$, hence we may conclude that, for all pairs of points α and β, $|B_\alpha : B_{\alpha,\beta}| < |G_\alpha : G_{\alpha,\beta}|$. In other words, considering B and G as permutation groups on Π, the only common orbital of B and G is the diagonal. In this situation we say that the triple (G, B, Π) is **exceptional** (after [GMS03].)

- **Suppose that B is not point-transitive on $\mathcal{S}$.** Then, by the Frattini argument, $G = N_G(P)B$ for all $P \in Syl_pB$ where p is any prime dividing into $|B|$. If $G_\alpha \geq N_G(P)$ then B is point-transitive which is a contradiction. Thus, by Lemma 3.15, if a Sylow p-subgroup of B stabilizes a point then it also stabilizes a line.

  Now let $b_B = |B : B_{\mathfrak{L}}|$, $v_B = |B : B_\alpha|$. Then primes dividing into b_B are a subset of the primes dividing into v_B. Furthermore $b = tb_B$ and $v = tv_B$. Thus primes dividing into b are a subset of the primes dividing into v. Thus there are no significant primes and $\mathcal{S}$ is a projective plane.

Now suppose that $PSL(3,q) \trianglelefteq G \leq AutPSL(3,q)$ and G acts line-transitively on a space $\mathcal{S}$ which is not a projective plane. Suppose furthermore that $PSL(3,q)$ is not line-transitive on $\mathcal{S}$. Then there exist groups G_1, G_2 such that $PSL(3,q) \trianglelefteq G_1 \trianglelefteq G_2 \leq G \leq AutPSL(3,q)$ where $|G_2 : G_1|$ is a prime, G_1 is not line-transitive on $\mathcal{S}$ while G_2 is. By the above argument, (G_2, G_1, Π) is an exceptional triple and [GMS03, Theorem 1.5] implies that G_α only lies inside maximal subgroups of type $PSL(3, q_0)$ where $q = q_0^a, a > 3$.

Appealing to Lemma 3.20 we have only four possibilities for G_α. The last two on the list lie inside $PSU(3, q_0)$ with $q = q_0^2$ and so can be excluded. The first can be excluded similarly. Hence Theorem B holds in this situation for $q > 2$.

When $q = 2$, $PSL(3,2) \cong PSL(2,7)$. Since $AutPSL(2,7) = PGL(2,7)$, Lemma 3.7 implies, under the given suppositions, that $PSL(2,7)$ is line-transitive on $\mathcal{S}$. Then Theorem 3.1 implies that Theorem B holds in this situation.

In order to prove Theorem B it is now sufficient to proceed under the assumptions of the following hypothesis. Our aim is to show that this hypothesis leads to a contradiction. We will need to consider different possibilities for a linear space $\mathcal{S}$ having a significant prime dividing $|PSL(3,q)| = q^3(q-1)^2(q+1)(q^2+q+1)/\mu$ where $\mu = (q-1, 3)$.

Hypothesis. *Suppose that $G = PSL(3,q)$ acts line-transitively but does not act flag-transitively on a linear space $\mathcal{S}$ which is not a projective plane. Let b, v, k, r be the parameters of the space. Let D be the subgroup of $PSL(3,q)$ as defined in the previous section. We suppose, by Lemma 3.5, that every involution of $PSL(3,q)$ fixes a point. Finally we assume that $q > 2$.*

3.5 Preliminary Cases

3.5.1 Significant prime: $t|q^2+q+1, t \neq 3$

Suppose first that some $t|q^2+q+1, t \neq 3$ a significant prime. By Lemma 3.8 $G_\alpha \geq \hat{}(q^2+q+1).3$ which is the normalizer of a Sylow t-subgroup of $PSL(3,q)$. Now $\hat{}(q^2+q+1).3$ is maximal in $PSL(3,q)$ for $q \neq 4$ and so, in this case, $G_\alpha = \hat{}(q^2+q+1).3$ This is a contradiction since then G_α doesn't contain any involution, contradicting our Hypothesis.

When $q = 4$ the only other possibility is that $G_\alpha = PSL(2,7)$ and $v = 120$. Then $17|v-1$ and by Lemma 3.18, $k \geq 17$ which contradicts Fisher's inequality (Lemma 3.2).

3.5.2 Significant prime: $t = p$

Suppose now that p is significant. By Lemma 3.8 $G_\alpha \geq \hat{}[q^3] : (q-1)^2$, a Borel subgroup, which is the normalizer of a Sylow p-subgroup of $PSL(3,q)$. Then G_α is either a Borel subgroup or a parabolic subgroup of $PSL(3,q)$.

In the latter case the action of G is 2-transitive on points and hence flag-transitive. Thus this case is already covered.

When G_α is a Borel subgroup $v = (q^2+q+1)(q+1)$ and, by Corollary 3.10, b divides into $\frac{1}{2}q(q+1)(q^2+q+1)$. This implies that $r > k > q+1$. Then $r = \frac{v-1}{k-1} < q^2+q+1$.

Consider the set of lines through the point α. These lines contain all points of $\mathcal{S}$ and so the points of $\mathcal{S}\backslash\{\alpha\}$ can be thought of as making up a rectangle with dimensions r by $k-1$. The area of this rectangle (that is, the number of points in the rectangle) is $v - 1 = r(k-1) = q^3 + 2q^2 + 2q$.

Now G_α has six orbits of size $1, q, q, q^2, q^2$ and q^3. Each of these orbits forms a rectangle of points in $\mathcal{S}\backslash\{\alpha\}$. Thus we have a rectangle of area q^3+2q^2+2q made out of rectangles of area q, q, q^2, q^2 and q^3 with integer dimensions. We investigate this situation.

Suppose the rectangle of area q has dimensions $p^a \times p^b$ where $p^a \leq \sqrt{q} \leq p^b$. Then either the width of the large rectangle is p^b (which is impossible since the rectangle has dimensions r by $k-1$ and $r > k-1 > q$) or there are other rectangles with side length p^a or, possibly $2p^a$ which make up the total width. The possibilities are as follows:

Other rectangles	Total width
q	$2p^b$
q^2	$\frac{q^2}{p^a} + p^b = (q+1)p^b$
$q^2, p=2$	$\frac{q^2}{2p^a} + p^b = (q+2)\frac{p^b}{2}$
q^2, q	$\frac{q^2}{p^a} + 2p^b = (q+2)p^b$
q^2, q^2	$2\frac{q^2}{p^a} + p^b = (2q+1)p^b$
$q^2, q^2, p=2$	$\frac{2q^2}{2p^a} + p^b = (q+1)p^b$
q^2, q^2, q	$2\frac{q^2}{p^a} + 2p^b = (2q+2)p^b$
q^3	$\frac{q^3}{p^a} + p^b = (q^2+1)p^b$
$q^3, p=2$	$\frac{q^3}{2p^a} + p^b = (q^2+2)\frac{p^b}{2}$

All possibilities involving another rectangle of area q^3 result in a dimension at least as big as $(q^2+2)\frac{p^b}{2}$. This must be less than q^2+q+1 and so $q \leq 4$. These possibilities can be ruled out using Lemma 3.18.

Now the dimensions of the rectangles both divide into $v-1=q(q^2+2q+2)$. The only possibility is that the width is $2p^b$. In fact the width is at least $q+1$ and so equals $2q$ and $p=2$. If $k-1=2q$ then $k=2q+1$ divides into $v(v-1)$ and so divides into $(q^2+q+1)(q^2+2q+2)$. This is impossible. If $r=2q$ then, in order for our action to be intransitive on flags we must have the rectangle of area q^3 having width at most q. Then $k-1 \geq q^2+1$. But $k=\frac{1}{2}(q^2+2q+4)$ which is a contradiction. Thus this possibility is excluded.

Remark. *Note that we have excluded the possibility that G_α is a parabolic or a Borel subgroup, no matter what prime is significant.*

3.5.3 Small Cases

We seek to rule out the cases where $q<8$, thus $q=3,4,5,7$.

Suppose first that $q=3$ and $G=PSL(3,3)$ acts on a linear space S. Now $|PSL(3,3)|=3^3.2^4.13$. We need only consider the possibility that 2 is uniquely significant.

Since G_α contains a Sylow 2-subgroup of $PSL(3,3)$, G_α lies in a parabolic subgroup. G_α must be a proper subgroup of the parabolic subgroup and so $v=13.3^a$ where $a=1,2,3$. If $a=3$ then 25 divides into $v-1$, if $a=2$ then 29 divides into $v-1$, if $a=1$ then 19 divides into $v-1$. Appealing to Lemma 3.18 we have lower bounds for $k-1$ of, in each case, 25, 29 and 19. But then $k(k-1)>v$ which contradicts Fisher's inequality (Lemma 3.2).

Thus we can conclude that any line-transitive action of $PSL(3,3)$ on a linear space is flag-transitive as required.

Now consider the case where $q=4$. Then $PSL(3,4)$ has order $2^6.3^2.5.7$ and there are several cases to consider:

1. If 5 is significant then G_α contains the normalizer of a Sylow 5-subgroup, which is a maximal subgroup of $\hat{}GL(2,q)$ of order $\frac{2(q^2-1)}{3}$. Then the possible candidates for G_α containing this subgroup are:
   - maximal subgroup, H, of $\hat{}GL(2,q)$ of order $\frac{2(q^2-1)}{3}$. Then $v=2016$ and $v-1$ is divisible by 13 and 31 which must divide into $k(k-1)$. By Lemma 3.18, we must have $k\geq 156$ which is a contradiction.

- $\hat{}GL(2,q)$. Then $v = 336$. Then $v-1$ is divisible by 67 hence $k \geq 67$ which is a contradiction.
- $A.H$ where A is elementary abelian of order q^2. Then $v = 126$. Now $k-1$ must divide into $v-1$, hence $k-1 = 1, 5, 25$ or 125. The only valid possibility is that $k-1 = 5$. But then $k|v$ and hence any line-transitive action is flag-transitive which is a contradiction.

2. Suppose that 3 is uniquely significant. Then G_α is divisible by 18. This implies that G_α lies in a subgroup of $PSL(3,4)$ of type 2,5 or 6. Thus G_α must have order divisible by 18 and dividing into 72. So $G_\alpha = 18, 36$ or 72. Then $v = 1120, 560$ or 280. All of these cases can be excluded by examining primes dividing $v-1$ and appealing to Lemma 3.18.

Now suppose that $q = 5$. Then $|PSL(3,5)| = 5^3.2^5.3.31$ and there are several cases to consider:

1. Suppose that 2 is significant. Then 32 divides $|G_\alpha|$. If we go through the possibilities for $v > 31$ we find that, in all cases, there exists a prime p dividing $v-1$ and not $|PSL(3,5)|$ which is such that if $k \geq p$ then $k(k-1) > v$ which is a contradiction of Fisher's inequality. For $v \leq 31$ the only possible stabilizers are parabolic subgroups which are already excluded.
2. Thus suppose that 3 is uniquely significant. Then $b = 3v$ or $b = \frac{3}{2}v$. Now $3|(q+1)$ implies that $H.2 \leq G_\alpha < q^2 : GL(2,q)$, where H is cyclic of order $q^2 - 1 = 24$. Going through these seven possibilities $(31\big|v\big|7750, v > 31)$ we find that, in all cases but one, a large prime dividing $v-1$ exists which rules out any valid action. The remaining case is $|G_\alpha| = 48$. In this case $v = 7750$ and our two possible values for b are 23250 and 11625. In both cases, no integer k exists for which $b = \frac{v(v-1)}{k(k-1)}$.

Thus we can conclude that any line-transitive action of $PSL(3,5)$ on a linear space is flag-transitive as required. Now suppose that $q = 7$ and $G = PSL(3,7)$ acts on a linear space S.Then $|PSL(3,7)| = 2^5.3^2.7^3.19$ and there are several cases to consider:

1. Suppose that 2 is significant. Then 32 divides $|G_\alpha|$ and hence G_α lies in a parabolic subgroup. Once again, we conclude that $57\big|v\big|57.3.7^3, v > 57$. Going through these seven possibilities we find that, in all cases, there are large primes

dividing $v-1$ and not dividing $|G|$. We can use Lemma 3.18 to exclude all cases.

2. Suppose that 3 is significant. Then the Sylow 3-subgroup is of order 9 and hence is normal in $PSU(3,q) = 3^2 : Q_8$. Since H is not strictly contained in any other group, $G_\alpha = H$ and $v = 26068$. But then the prime 8689 divides $v-1$ and $k \geq 8689$ which contradicts Fishers' inequality.

Thus we can conclude that any line-transitive action of $PSL(3,7)$ on a linear space is flag-transitive as required.

3.5.4 Remaining Cases

We wish to enumerate the remaining cases that we need to examine. One case in particular is worth mentioning now: When q is odd and when both 2 and $3|(q-1)$ are significant primes.

The only maximal subgroups which have index not divisible by 2 and 3 in this case are those of type 2 and 4. Suppose that G_α lies in a subgroup M of type 2, Without loss of generality the diagonal subgroup normalized by the group of permutation matrices isomorphic to S_3. Now D normalizes a Sylow 2-subgroup of M. In addition $Q \in Syl_3 G$ is conjugate to $H : C_3$ where H is a diagonal subgroup, C_3 a group of permutation matrices. Q does not normalize D hence G_α contains at least two conjugates of D. Since these intersect trivially, by Lemma 3.22, these generate a subgroup of index dividing μ in the diagonal subgroup. Our group G_α must therefore be the full subgroup of type 2.

If G_α is contained in a subgroup, M, of type 4 then in order to contain an element of order $\frac{q-1}{\mu}$, $M = PSL(3, q_0), q = (q_0)^2$. But then the index of M in G is even which is a contradiction.

Thus the cases which we need to examine are:

	Significant primes t	**Possible stabilizers**
I	$\exists t\big\|(q+1), t \neq 2$	$\hat{}(q^2-1).2 \leq G_\alpha < q^2 : \hat{}GL(2,q)$
II	$\exists t\big\|(q-1), t \neq 2, 3$ OR $2, 3\big\|(q-1)$ both significant	$G_\alpha = \hat{}(q-1)^2 : S_3$
III	$3\big\|(q-1)$ is uniquely significant	G_α is a subgroup of a maximal subgroup of type 2, 4, 5 or 8
IV	$2\big\|(q-1)$ is uniquely significant	G_α is a subgroup of a maximal subgroup of types 1, 2 or 4

3.6 Case I: $\exists t\big|(q+1), t \neq 2$ significant

In this case G_α contains a subgroup H of order $2(q^2-1)/\mu$ which itself has a cyclic subgroup of size $(q^2-1)/\mu$ and G_α lies inside a parabolic subgroup of G.

Now observe that H lies inside a copy of $\hat{}GL(2,q)$ and that $\hat{}GL(2,q)$ normalizes an elementary abelian subgroup, U, of $PSL(3,q)$, of order q^2. In its conjugation action on the non-identity elements of U, $\hat{}GL(2,q)$ has stabilizers of order $q(q-1)$. Thus our group H must, if it normalizes any subgroup of U, normalize a subgroup of order $1+x(q+1)$ for some integer x. Now for such a value to divide q^2, as required, x must be 0 or $q-1$.

Thus $G_\alpha = \hat{}A.B$ where A is trivial or of size q^2 and $H \leq B \leq GL(2,q)$. Now, in the characteristic 2 case, $GL(2,q) = PSL(2,q) \times (q-1)$ and $H = D_{2(q+1)} \times (q-1)$. Since $D_{2(q+1)}$ is maximal in $PSL(2,q)$ for all even $q \geq 8$, we know that $B = H$ or $B = GL(2,q)$. In the odd characteristic case, $GL(2,q) =< -I > .(PSL(2,q) \times (\frac{q-1}{2})).2$ and $H =< -I > .(H \times (\frac{q-1}{2})).2$. Now, for all odd $q > 9$, $D_{2(q+1)}$ is maximal in $GL(2,q)$ and, once again we conclude that $B = H$ or $B = GL(2,q)$.

We need to consider the case where $q = 9$ and $H < B < GL(2,q)$. In fact this case cannot occur since the only proper subgroup of $PSL(2,9)$ containing D_{10} is A_5, but $< -I > .(A_5 \times (\frac{q-1}{2}))$ is not normalized by any element of $GL(2,q)$ of non-square determinant.

Thus we can summarize the cases that we need to examine:

1. $G_\alpha = U.\hat{}(q^2-1).2$ where $U = [q^2]$;

2. $G_\alpha = \hat{}GL(2,q)$;

3. $G_\alpha = \hat{}((q^2-1).2)$.

Note that we exclude the case where $G_\alpha = \hat{}U : GL(2,q)$, as then G_α is maximal parabolic and this case is already excluded. We will consider the remaining cases in turn.

Remark. *These cases also arise when $2\big|(q+1)$ is the only significant prime (see Section 3.9). The arguments given below are general and apply in that situation as well.*

3.6.1 Case 1: $G_\alpha = U.(\hat{}(q^2-1).2)$.

Now we know that $v = \frac{1}{2}(q^2+q+1)q(q-1)$ and, since G_α lies inside a parabolic subgroup, we can appeal to Corollary 3.10 to observe that

$$b\Big|\frac{1}{8}(q^2+q+1)q(q-1)(q+1)(q-2) \text{ and } b\Big|\frac{1}{4}(q^2+q+1)q(q-1)(q+1)q.$$

Thus $b\big|\frac{1}{4(2,q-1)}(q^2+q+1)q(q-1)(q+1)$ and so $4(2,q-1)q^2(q-1)/\mu$ divides $|G_{\mathfrak{L}}|$. For $q > 7$ this means that $G_{\mathfrak{L}}$ lies in a parabolic subgroup. Observe that we can presume that $U : D$ lies in $G_{\mathfrak{L}}$ for some $\mathfrak{L}$ since $U : D$ lies in G_α and is normalized by the full parabolic subgroup (Lemma 3.15).

Suppose that U is non-normal in $G_{\mathfrak{L}} = \hat{}A.B$ where A is an elementary abelian p-group and $B \leq GL(2,q)$. Then $G_{\mathfrak{L}}$ must lie in a parabolic subgroup which is not conjugate to $N_G(U)$ and $|U \cap A| = q$. If $A\backslash U$ is non-empty then U acts by conjugation on these elements with an orbit, Ω, of size q. Then $U \cap A$ and Ω lie inside A and generate q^2 elements. Hence we must have A of size q or q^2. The latter would make $|G_{\mathfrak{L}}| \geq 4q^3(q-1)/\mu$ which is larger than $|G_\alpha|$ which is a contradiction. Hence we conclude that $|A| = q$.

Since A is normal in $G_{\mathfrak{L}}$ we must have $G_{\mathfrak{L}}$ a subgroup of a Borel subgroup. However in this case U is normal in $G_{\mathfrak{L}}$. This is a contradiction.

Hence we have U normal in $G_{\mathfrak{L}}$. Furthermore there are no other G-conjugates of U in $G_{\mathfrak{L}}$, since $U \cap U^g$ is trivial for all g in $G\backslash N_G(U)$. Hence we may appeal to Lemma 3.6. Then either $U : \hat{}GL(2,q)$ acts line-transitively on the fixed set of U, which is itself a linear space, or this fixed set lies completely in one line. In the first case, such an action of $U : \hat{}GL(2,q)$ has a kernel $U : \hat{}D$ and corresponds to a line-transitive action of $PGL(2,q)$ with stabilizer a dihedral group $D_{2(q+1)}$.

Examining the results of line-transitive and 2-transitive actions of $PGL(2,q)$ we find that there is one such action to consider. We have q even and $PGL(2,q)$ acts line-transitively upon a Witt-Bose-Shrikhande space with line-stabilizer an elementary abelian group of order q. In $PSL(3,q)$ this corresponds to $G_{\mathfrak{L}}$ having order $\frac{q^3(q-1)}{\mu}$ and $b=(q-1)(q+1)(q^2+q+1)$. Then we must have,

$$\begin{aligned} & k(k-1) = \frac{v(v-1)}{b} = \frac{1}{4}q(q^3-q^2+q-2) \\ \implies \quad & 2k(2k-2) = q^4-q^3+q^2-2q. \end{aligned}$$

Now observe that,

$$(q^2-\frac{1}{2}q+1)(q^2-\frac{1}{2}q-1) < q^4-q^3+q^2-2q < (q^2-\frac{1}{2}q+2)(q^2-\frac{1}{2}q).$$

Thus this case is excluded.

We can assume therefore that the set of fixed points of U lies completely in one line. This fixed set has size $\frac{1}{2}q(q-1)$ and thus k is at least this large. Now the subgroups conjugate to U intersect trivially. Thus U lying in $G_{\mathfrak{L}}$ has orbits on the points of $\mathfrak{L}$ of size 1 ($\frac{1}{2}q(q-1)$ such) or q^2 (for q odd) or $\frac{q^2}{2}$ (for q even.)

If $k \geq q^2+\frac{1}{2}q(q-1)$ then $k(k-1)>v$ which is a contradiction. If $k=\frac{1}{2}q(q-1)$ then $k-1=\frac{1}{2}(q+1)(q-2)$ divides into $v-1=\frac{1}{2}(q+1)(q^3-q^2+q-2)$. This is possible only for $q \leq 4$ which is a contradiction. Thus we are left with the possibility that q is even and $k=\frac{1}{2}q(q-2)$. Once again $k-1$ dividing into $v-1$ implies that $q \leq 4$.

3.6.2 Case 2: $G_\alpha = \hat{\ }GL(2,q)$

Since $v=q^2(q^2+q+1)$ and G_α lies inside a parabolic subgroup, we can appeal to Corollary 3.10 to observe that

$$b\Big|\frac{1}{2}q^2(q^2+q+1)(q-1)(q+1) \ \text{ and } \ b\Big|\frac{1}{2}q^2(q^2+q+1)(q+1)q.$$

Thus $b\big|\frac{1}{2}q^2(q^2+q+1)(q+1)$ and so $2q(q-1)^2/\mu$ divides $|G_{\mathfrak{L}}|$.

This implies that, for $q>7$, $G_{\mathfrak{L}}$ lies in a parabolic subgroup or $q=16$. When $q=16$ we find that the prime 4111 divides into v-1=69888 which, using Lemma 3.18, contradicts Lemma 3.2.

Thus $G_{\mathfrak{L}}$ lies in a parabolic subgroup and we write $G_{\mathfrak{L}} = \hat{\ }A.B$ as usual. If $A=\{1\}$ then we must have $G_{\mathfrak{L}} = \hat{\ }B \leq \hat{\ }GL(2,q)$. Examining the subgroups of $GL(2,q)$ given in Theorem 2.25 we find that $|G_{\mathfrak{L}}|$ is divisible by $\frac{|GL(2,q)|}{2\mu}$. Now if $\mu=3$ and 3 is significant then G_α does not lie in a parabolic subgroup. Hence we must have $|G_{\mathfrak{L}}| = \frac{1}{2}|G_\alpha|$ with 2 uniquely significant. But then Lemma 3.3 implies that any prime dividing into v must be equivalent to $1(4)$. Now in our current situation any significant prime divides into $\frac{q+1}{2}$ thus 2 is not a significant prime; this is a contradiction.

Now if $1 \neq g \in A$ then $|C_{PSL(3,q)}(g)| = q^3(q-1)/\mu$. Thus B must act on the non-trivial elements of A with orbits of size divisible by $q-1$. Thus $|A| = q$ or q^2.

If $|A| = q^2$ then $|G_{\mathfrak{L}}| \geq 2q^2(q-1)^2/\mu > |G_\alpha|$ which cannot happen. If $|A| = q$ then $p = 2$ (since, if p is odd, B must act on the non-trivial elements of A with orbits of size divisible by $2(q-1)$.) For $q > 4$ we must have B either maximal in $GL(2,q)$ of type 4 or a subgroup of the Borel subgroup of $GL(2,q)$. In the first case $\hat{}B$ has orbits of size at least $2(q-1)$ on the non-identity elements of A, thus this case can be excluded.

If B lies inside a Borel subgroup of $GL(2,q)$ then $B = B_1.B_2$ where $2 < B_1$ and $B_2 = (q-1)^2$. In fact we must have $|B| = \frac{q(q-1)^2}{\mu}$ since B_2 acts by conjugation on the non-identity elements of B_1 with orbits of size $q-1$. Hence $|G_{\mathfrak{L}}| = \frac{q^2(q-1)^2}{\mu}$ and $b = q(q+1)(q^2+q+1)$. Hence we must have

$$k(k-1) = q^4 + q^2 - q.$$

Now observe that,

$$q^2(q^2-1) < q^4 + q^2 - q < (q^2+1)q^2.$$

Thus this case is excluded.

3.6.3 Case 3: $G_\alpha = \hat{}(q^2-1).2$

Since $v = \frac{1}{2}q^3(q^2+q+1)(q-1)$ and G_α lies inside a parabolic subgroup, we can appeal to Corollary 3.10 to observe that b divides into both

$$\frac{1}{4}q^3(q^2+q+1)(q-1)(q+1)q \text{ and } \frac{1}{8}q^3(q^2+q+1)(q-1)(q+1)(q^3-2q^2+2q-2).$$

Thus $b\big|\frac{1}{4(2,q-1)}q^3(q^2+q+1)(q-1)(q+1)$ and so $4(2,q-1)(q-1)/\mu$ divides $|G_{\mathfrak{L}}|$.

To begin with note that all cases where $11 < q \leq 16$ and $q = 9, 19, 25, 31, 37, 64$ can be ruled out using Lemma 3.18. When $q = 11$, Lemma 3.18 leaves one possibility, namely that $k = 444$. But then b is not an integer and so this situation can be excluded. When $q = 8$, Lemma 3.18 leaves one possibility, namely that $k = 171$. But then $k-1$ does not divide into $v-1$ and so this situation too can be excluded.

Using these facts, and recalling that $4(2,q-1)(q-1)/\mu$ divides $|G_{\mathfrak{L}}| < |G_\alpha|$, we can exclude the possibility that $G_{\mathfrak{L}}$ lies in a subgroup of $PSL(3,q)$ of type 3-10. Hence we assume that $q \geq 17$ and $G_{\mathfrak{L}}$ lies inside a subgroup of type 1 or 2 for the rest of this section.

Now $D < G_\alpha$ and, by Lemma 3.15, D lies in $G_{\mathfrak{L}}$ for some line $\mathfrak{L}$. We refer to Lemma 3.6 to split our investigation into three cases:

- **Case 3.A:** All G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and the fixed set of D is a linear-space acted on line-transitively by $\hat{}GL(2,q)$, the normalizer of D.

- **Case 3.B:** All G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and the fixed points of D, of which there are $\frac{1}{2}q(q-1)$, lie on one line;

- **Case 3.C:** $G_{\mathfrak{L}}$ contains at least two $G_{\mathfrak{L}}$-conjugacy classes of G-conjugates of D.

Case 3.A

This situation corresponds to a line-transitive action of $PGL(2,q)$ with stabilizer $D_{2(q+1)}$. Then Theorem 3.1 implies that $p = 2$ and the fixed set of D is a Witt-Bose-Shrikhande space. The corresponding line-stabilizer in $PGL(2,q)$ has size q and so $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)}{\mu}$ in $PSL(3,q)$. Suppose that $|G_{\mathfrak{L}}| = \frac{q(q-1)}{\mu}$ and so

$$\begin{aligned} & k(k-1) = \frac{v(v-1)}{b} = \frac{1}{4}(q^6 - q^5 + q^4 - 2q^3 + 2q^2 - 2q) \\ \implies \quad & (2k)(2k-2) = q^6 - q^5 + q^4 - 2q^3 + 2q^2 - 2q. \end{aligned}$$

But now observe that

$$(q^3 - \frac{1}{2}q^2 + \frac{3}{8}q + 2)(q^3 - \frac{1}{2}q^2 + \frac{3}{8}q) < 2k(2k-2) < (q^3 - \frac{1}{2}q^2 + \frac{3}{8}q)(q^3 - \frac{1}{2}q^2 + \frac{3}{8}q - 2).$$

For $q > 16$ this gives a contradiction.

The only other possibility is that $|G_{\mathfrak{L}}| = \frac{2q(q-1)}{\mu}$ and $[q] \times \frac{q-1}{\mu} = G_{\mathfrak{L}} \cap C_G(D)$. This implies that $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $PSL(3,q)$.

Now $[q] \times \frac{q-1}{\mu}$ is normal in $G_{\mathfrak{L}}$ and so $[q]$ is normal in $G_{\mathfrak{L}}$ and $G_{\mathfrak{L}}$ lies inside a Borel subgroup of $PSL(3,q)$. Then D acts on the normal subgroup of $G_{\mathfrak{L}}$ of order $2q$. Furthermore D centralizes at most q of these elements and has orbits on the rest of size at least $\frac{q-1}{\mu}$. These orbits intersect cosets of $[q] \trianglelefteq C_G(D) \cap G_{\mathfrak{L}}$ with a size of at most 1. This gives a contradiction.

Case 3.B

Observe that all $PSL(3,q)$-conjugates of D intersect trivially. Observe too that all elements of G_α are of form TS where $T \in \hat{}(q^2-1)$ and S^2 lies in D. Then $(TS)^2$ lies in D and hence if E is some other conjugate of D then $E \cap G_\alpha$ is of size at most $(2, q-1)$. Thus the orbits of D on $\mathfrak{L}$, a line which it fixes, are either of size $\frac{q-1}{(2,q-1)\mu}$ or of size 1 and there are $\frac{1}{2}q(q-1)$ of these. We conclude that k is a multiple of $\frac{q-1}{(2,q-1)\mu}$.

Now we find that $(v-1, |G|) = \frac{q+1}{(2,q-1)}$. Since $\frac{q-1}{(2,q-1)\mu}\Big|k$ and $b = \frac{v(v-1)}{k(k-1)}$ divides into $|G|$ then $b\Big|\frac{\mu}{2}(q^2+q+1)q^3(q+1)$.

Thus, for $q \not\equiv 1(3)$, $|G_\mathfrak{L}| = 2(q-1)^2 \geq 512$. If $q \equiv 1(3)$ then $|G_\mathfrak{L}| = \frac{2}{9}(q-1)^2.a$ where $a = 1, 2$ or 3.

Suppose first that p is odd. Consider the possibility that $G_\mathfrak{L}$ lies inside a subgroup of type 2 and not in a parabolic subgroup. So $G_\mathfrak{L}$ is a subgroup of $\hat{}(q-1)^2 : S_3$ and must have either 3 or S_3 on top. The former case is impossible as then b does not divide into $\frac{\mu}{2}(q^2+q+1)q^3(q+1)$. Now $G_\mathfrak{L} = \hat{}(A \times A) : S_3$ or $(\frac{A}{\mu} \times \frac{A}{\mu}) : S_3$. Then, since $G_\mathfrak{L}$ must contain a subgroup conjugate to D, we find that $G_\mathfrak{L} = (\frac{q-1}{\mu} \times \frac{q-1}{\mu}) : S_3$, $\mu = 3$ or $G_\mathfrak{L} = \hat{}(q-1)^2 : S_3$. The latter case violates Fisher's inequality and can be excluded. In the former case $G_\mathfrak{L}$ contains at most $q+2$ involutions. Appealing to Lemma 3.16, we observe that

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{1}{2}q(q+2)(q-1) + 1.$$

This means that $b = \frac{v(v-1)}{k(k-1)} > q^5(q-3)$ which is a contradiction.

Thus $G_\mathfrak{L}$ lies inside a parabolic subgroup; in fact $G_\mathfrak{L}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$. In order for Fisher's inequality to hold, we must have one of the following cases:

- $b = \frac{1}{2}q^3(q^2+q+1)(q+1)$ and $|G_\mathfrak{L}| = \frac{2(q-1)^2}{\mu}$. Thus $G_\mathfrak{L}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 4 (in which case $G_\mathfrak{L}$ contains more than one $G_\mathfrak{L}$-conjugacy class of G-conjugates of D which is a contradiction) or $G_\mathfrak{L}$ is isomorphic to a subgroup of type 6 or 7. This latter case requires that $2(q-1)$ divides into 24 or 60. These possibilities have already been excluded.

- $b = \frac{3}{4}q^3(q^2+q+1)(q+1)$. Hence $|G_\mathfrak{L}| = \frac{4}{9}(q-1)^2$ and $q \equiv 7(12)$. Thus $G_\mathfrak{L}$ is isomorphic to a subgroup of type 6 or 7 in $\hat{}GL(2,q)$ and $\frac{4(q-1)}{3}$ must divide 24 or 60. This is impossible.

- $b = \frac{3}{2}q^3(q^2+q+1)(q+1)$. Then $|G_{\mathfrak{L}}| = \frac{2}{9}(q-1)^2$ and $q \equiv 1(3)$. Thus $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 4, 6 or 7.

  If $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 4 then $r_g \leq \frac{q+8}{3}$. Using Lemma 3.16 we see that

  $$k \geq \frac{n_g(v-1)}{br_g} > q^2(q-9).$$

  Since $(k-1)^2 < v$ this implies that

  $$q^4(q-9)^2 < \frac{1}{2}q^3(q^2+q+1)(q-1)$$

  which means that $q < 31$. Then $q = 25$, but this possibility has already been excluded using Lemma 3.18.

  If $G_{\mathfrak{L}}$ is isomorphic to a subgroup of $\hat{}GL(2,q)$ of type 6 or 7 then we require that $\frac{2(q-1)}{3}$ divides into 24 or 60. Hence $q = 31$ or 37. These possibilities have already been excluded.

If $p = 2$ then, in order for Fisher's inequality to hold and so that $4(q-1)/\mu$ divides into $|G_{\mathfrak{L}}|$, we have $|G_{\mathfrak{L}}| = \frac{4}{9}(q-1)^2$ and $q \equiv 1(3)$. Thus $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $PSL(3,q)$ and $G_{\mathfrak{L}} = \hat{}A.B$ as usual.

If A is trivial then $G_{\mathfrak{L}}$ is a subgroup of type 2 in $\hat{}GL(2,q)$. Then $G_{\mathfrak{L}}$ has a normal 2-group and, by Schur-Zassenhaus, $G_{\mathfrak{L}}$ also contains a subgroup of size $\frac{(q-1)^2}{9}$. This subgroup has orbits in its conjugation action on 2-elements of $G_{\mathfrak{L}}$ of size at least $\frac{q-1}{3}$. This implies that $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)^2}{9}$ which is a contradiction.

If A is non-trivial then $G_{\mathfrak{L}}$ must have orbits in its conjugation action on non-identity elements of A of size at least $\frac{q-1}{3}$. Once again this implies that $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)^2}{9}$ which is a contradiction.

Case 3.C

Now consider the possibility that $G_{\mathfrak{L}}$ contains at least two $G_{\mathfrak{L}}$-conjugacy classes of G-conjugates of D.

Suppose first that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}(q-1)^2 : S_3$ and does not lie in a parabolic subgroup. We know that q is odd since $4(2,q-1)(q-1)/\mu$ divides into $|G_{\mathfrak{L}}|$. Since $G_{\mathfrak{L}}$ is not in a parabolic subgroup we must have a non-trivial part of S_3 on top, of order 3 or 6. Thus all G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate which is a contradiction.

Thus we may conclude that $G_{\mathfrak{L}}$ is in a parabolic subgroup. Write $G_{\mathfrak{L}} = \hat{}A.B$ as usual. If A is trivial then, referring to Lemma 3.23, we conclude that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2,q)$ of type 2,4 or 5. If $G_{\mathfrak{L}}$ is of type 5 then $q = 49$ and this can be ruled out using Lemma 3.18.

If $G_{\mathfrak{L}}$ is of type 2 and not of type 4 then it must contain non-trivial p-elements. Some conjugate of D in $G_{\mathfrak{L}}$ must have orbits in its conjugation action on these elements of size $\frac{q-1}{\mu}$. Thus $A_1 : \frac{q-1}{\mu} \leq |G_{\mathfrak{L}}|$ where A_1 is a p-group of size divisible by q. We will consider this possibility together with the case when A is non-trivial.

So suppose that A is non-trivial. Now either all G-conjugates of D in $G_{\mathfrak{L}}$ lie in $C_G(A)$ or else $|A| \geq q$. Consider the first possibility. In this case $A : D$ and $A : E$ lie inside $C_G(A)$ where E is a G-conjugate of D. Now $C_G(A) \leq C_G(g)$ for g an element or order p. Since $C_G(g) \cong [q^3] : \frac{q-1}{\mu}$, we know that D and E are conjugate in $C_G(A) \cap G_{\mathfrak{L}}$ by Schur-Zassenhaus. This is a contradiction and so we assume that $|A| \geq q$; thus, in both cases that we have considered so far, $Q : D \leq G_{\mathfrak{L}}$ where Q is a p-group of order divisible by q.

Now let E be a G-conjugate of D in $G_{\mathfrak{L}}$ which is not $G_{\mathfrak{L}}$-conjugate to D. Suppose $E \cap (Q : D)$ is non-trivial and $1 \neq h \in E \cap (Q : D)$. Then h lies inside a $Q : D$-conjugate of D by applying Sylow theorems to $Q : D$. But this is impossible since Lemma 3.22 implies that either $E = D$ or $E \cap D$ is trivial. Hence $|G_{\mathfrak{L}}| \geq \frac{q(q-1)^2}{\mu^2} > |G_\alpha|$ which is also impossible.

Finally we must consider the possibility that $G_{\mathfrak{L}}$ is of type 2 in $\hat{}GL(2,q)$; that is, $G_{\mathfrak{L}}$ is a subgroup of $\hat{}(q-1)^2 : 2$. We must have q odd since $4(2, q-1)(q-1)/\mu$ divides into $|G_{\mathfrak{L}}|$. Furthermore the G-conjugates of D in $\hat{}(q-1)^2 : 2$ normalize each other and so $\frac{(q-1)^2}{\mu^2}$ divides into $|G_{\mathfrak{L}}|$. There are three possibilities to consider:

- $G_{\mathfrak{L}} \leq \hat{}(q-1)^2$. In this case $G_{\mathfrak{L}}$ contains at most 3 involutions. Appealing to Lemma 3.16, we observe that
$$k \leq \frac{r_g v}{n_g} + 1 = \frac{3}{2}q(q-1) + 1.$$
This is too small to satisfy $b = \frac{v(v-1)}{k(k-1)}$ hence we have a contradiction.

- $G_{\mathfrak{L}} = (\frac{q-1}{\mu} \times \frac{q-1}{\mu}) : 2$. Then $G_{\mathfrak{L}}$ contains $\frac{q+8}{3}$ involutions. Once again using Lemma 3.16, we observe that
$$k \leq \frac{r_g v}{n_g} + 1 = \frac{1}{6}q(q+8)(q-1) + 1.$$
But this is too small to satisfy $b = \frac{v(v-1)}{k(k-1)}$ hence we have a contradiction.

- $G_{\mathfrak{L}} = \hat{}((q-1)\times(q-1)) : 2$. Then $G_{\mathfrak{L}}$ contains $q+2$ involutions and we have that,
$$k \leq \frac{r_g v}{n_g} + 1 = \frac{1}{2}q(q+2)(q-1)+1.$$
Once again this is too small to satisfy $b = \frac{v(v-1)}{k(k-1)}$.

Hence we may conclude that no line-transitive actions exist with primes dividing $q+1$ significant.

3.7 $G_\alpha = \hat{}(q-1)^2 : S_3$

In this case $v = \frac{1}{6}q^3(q+1)(q^2+q+1)$ and any significant prime t must divide into $q-1$.

Note first that, by using Lemma 3.18, we can assume that $q > 25$ and that $q \neq 31, 37, 43, 49, 64, 109$ or 271. Furthermore a conjugate of D lies in G_α and D is normalized by $\hat{}GL(2,q)$. Thus, by Lemma 3.15, a conjugate of D lies inside $G_{\mathfrak{L}}$. We split into three cases:

- **Case A**: A G-conjugate of D is normal in $G_{\mathfrak{L}}$ and $G_{\mathfrak{L}}$ contains no other G-conjugates of D;

- **Case B**: A G-conjugate of D is normal in $G_{\mathfrak{L}}$ and $G_{\mathfrak{L}}$ contains other G-conjugates of D. Thus $|G_{\mathfrak{L}}|$ is divisible by $(\frac{q-1}{\mu})^2$ and so b divides into $6\mu v$;

- **Case C**: All G-conjugates of D in $G_{\mathfrak{L}}$ are non-normal in $G_{\mathfrak{L}}$.

We examine these possibilities in turn.

3.7.1 Case A

In this case we know, by Lemma 3.6, that either $\hat{}GL(2,q)$ acts line-transitively on the linear-space which is the fixed set of D or all fixed points of D lie on a single line. The first possibility cannot occur however as this would correspond to $PGL(2,q)$ acting line-transitively on a linear-space (possibly having $k=2$ and so being a 2-homogeneous action) with line-stabilizer a dihedral group of size $2(q-1)$ which is impossible. Hence we may assume that all fixed points of D lie on a single line. There are $\frac{1}{2}q(q+1)$ of these.

If E is some other conjugate of D then $E \cap G_\alpha$ is of size at most 2. We conclude that $k = \frac{1}{2}q(q+1)+n\frac{q-1}{2\mu}$ for some integer n. This implies that $k-1$ is divisible by $\frac{q-1}{2\mu}$. Now, since $v-1 = \frac{q-1}{2}\frac{q^5+3q^4+5q^3+6q^2+6q+6}{3}$, we observe that $b\big|(q^5+3q^4+5q^3+6q^2+6q+6)v$. Now, for p odd, $(|G|, q^5+3q^4+5q^3+6q^2+6q+6)$ is a power of 3, hence 3 is the only significant prime and $3|q-1$. For $p=2$, $(|G|, q^5+3q^4+5q^3+6q^2+6q+6)$ is divisible, at most, by the primes 2 and 3. However we know that 2 is not a significant prime here thus, again, 3 is the only significant prime. Note that $q^5+3q^4+5q^3+6q^2+6q+6$ is divisible by 27 if and only if $q \equiv 28(81)$. Thus, if 3^a is the highest power of 3 in $q-1$ then $a \neq 3$ implies that $b|27v$. If $a=3$ then we know already that $b|81v$.

This case will be completed below.

3.7.2 Case A and B

Now we examine the remaining possibilities of Case A along with Case B. Thus $G_{\mathfrak{L}} < \hat{}GL(2,q)$ and one of the following holds:

- $q \equiv 28(81)$, $\frac{2(q-1)^2}{81}$ divides into $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ contains precisely one G-conjugate of D;
- $q \equiv 1(3)$, $\frac{2(q-1)^2}{27}$ divides into $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ contains precisely one G-conjugate of D;
- $\frac{(q-1)^2}{\mu^2}$ divides into $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ contains more than one G-conjugate of D.

Observe also that $k(k-1) = \frac{v(v-1)}{b}$ is even and that

$$|v(v-1)|_2 = \frac{(q,2)}{4}|q^3(q+1)(q-1)|_2.$$

Thus if p is odd then we need $|G_{\mathfrak{L}}|$ divisible by $8(q-1)/\mu$.

Suppose that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2,q)$ of type 6 or 7. Since $q > 25$, Lemma 3.23 implies that $G_{\mathfrak{L}}$ contains at most one conjugate of D. Thus $\frac{2(q-1)}{9}$ must divide 24 or 60 or $\frac{2(q-1)}{27}$ divides 24 or 60 and $q \equiv 28(81)$. The prime powers we need to check are, therefore, 13, 19, 31, 37, 109 and 271. These cases are already all excluded.

If $G_{\mathfrak{L}}$ lies inside a group of type 3 then $G_{\mathfrak{L}}$ contains at most one conjugate of D and either $q \cong 28(81)$ and $\frac{2(q-1)}{27}$ divides into 4 or $\frac{2(q-1)}{9}$ divides into 4. Both yield values for q which are less than 25 and so can be excluded.

Suppose that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2,q)$ of type 5, $G_{\mathfrak{L}} \cong \hat{} < SL(2,q_0), V >$. Then $\frac{(q-1)^2}{81}$ divides into $2q_0(q_0^2-1)\frac{q_0-1}{3}$ and so $q-1$ divides into $54(q_0^2-1)$. For $q \geq q_0^3$ we find that this is impossible for $q_0 > 2$. If $q_0 = 2$ then $q < 32$ and so all cases have been excluded. For $q = q_0$, $|G_{\mathfrak{L}}| < |G_\alpha|$ implies a contradiction. For $q = q_0^2$, $|G_{\mathfrak{L}}| < |G_\alpha|$ implies that $\sqrt{q} \leq 5$ and all possibilities have been excluded.

Suppose that $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $\hat{}GL(2,q)$ and not of type 4. Then $|G_{\mathfrak{L}}|$ is divisible by p for $q = p^a$, integer a. If $|G_{\mathfrak{L}}|$ is divisible by $\frac{(q-1)^2}{\mu^2}$ then $G_{\mathfrak{L}}$ has orbits on the non-identity elements of its normal p-Sylow subgroup divisible by $\frac{q-1}{\mu}$. Thus $G_{\mathfrak{L}}$ contains the entire Sylow p-subgroup of $\hat{}GL(2,q)$ and $|G_{\mathfrak{L}}| \geq q\frac{(q-1)^2}{\mu^2}$; this implies that $q < 6\mu$ which is impossible. So assume that $3\big|(q-1)$ is the only significant prime. If $\frac{2(q-1)^2}{81}$ divides into $|G_{\mathfrak{L}}|$ we must have $p = 2$ and $G_{\mathfrak{L}} = \hat{}A : B$ where A is a non-trivial 2-group. Then $q \geq 2^a$ and $q-1$ has a primitive prime divisor s greater than 3 and $\frac{s(q-1)}{3}$ divides into $|B|$. Then B acts on the non-identity elements of A by conjugation with orbits of size divisible by s and so $|A| = q$. Thus $|G_{\mathfrak{L}}|$ is divisible by $\frac{q(q-1)s}{3}$ which means s must be 5 and so $q = 16$. This is already excluded.

We are left with the possibility that $G_{\mathfrak{L}}$ is a subgroup of $\hat{}GL(2,q)$ of type 4. If 2 is significant then p is odd and $G_{\mathfrak{L}}$ contains at most 3 involutions since $G_{\mathfrak{L}} \leq \hat{}(q-1)^2$. By Lemma 3.16 we know that $k \leq \frac{3v}{n} + 1 = \frac{1}{2}q(q+1)+1$. This is inconsistent with our value for b. If 2 is not significant then $|G_{\mathfrak{L}}| = 2|D|e$ where e is a constant dividing $q-1$. Then the number of involutions in $G_{\mathfrak{L}}$ is at most $e+3$. We appeal to Lemma 3.16 to conclude that,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{(e+3)(q+1)q}{6} + 1.$$

Thus,

$$\frac{3(q-1)}{e} = \frac{b}{v} = \frac{v-1}{k(k-1)} \geq \frac{6(q^6+2q^5+2q^4+q^3-6)}{(e+3)^2q^2(q^2+3q+2)} > \frac{6q^2}{(e+3)^2}.$$

This implies that $\frac{(e+3)^2}{e} > 2q$ and so $e + 15 > 2q$. Since $e < q$ this must mean that $q < 15$ which is a contradiction.

3.7.3 Case C

Finally we consider the possibility that no conjugate of D is normal in $G_{\mathfrak{L}}$. We must have at least two conjugates of D in $G_{\mathfrak{L}}$ and so $|G_{\mathfrak{L}}| > \frac{(q-1)^2}{\mu^2}$.

Suppose first that $G_{\mathfrak{L}}$ lies in a parabolic subgroup. Then $G_{\mathfrak{L}} = \hat{}A.B$ where A is an elementary abelian p-group, $B \leq GL(2,q)$.

Suppose that A is trivial and refer to Lemma 3.23. Then $G_{\mathfrak{L}}$ lies in a subgroup of $\hat{}GL(2,q)$ of types 2, 4 or 5. If $G_{\mathfrak{L}}$ lies in a subgroup of type 5 then $G_{\mathfrak{L}} \geq SL(2,q)$ in which case $|G_{\mathfrak{L}}| > |G_\alpha|$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies in a subgroup of $\hat{}GL(2,q)$ of type 4 then conjugates of D in $G_{\mathfrak{L}}$ normalize each other and so $\frac{(q-1)^2}{\mu^2}$ divides into $|G_{\mathfrak{L}}|$. In this case some conjugate of D must be normal in $G_{\mathfrak{L}}$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies in a subgroup of $\hat{}GL(2,q)$ of type 2 then we must have p dividing $|G_{\mathfrak{L}}|$ otherwise all conjugates of D are normal in $G_{\mathfrak{L}}$. But then some conjugate of D acts by conjugation on the non-trivial elements of the normal p-subgroup with orbits of size $\frac{q-1}{\mu}$. Thus q divides $|G_{\mathfrak{L}}|$ and $G_{\mathfrak{L}}$ has a normal subgroup Q of size q. We will deal with this situation at the end of the section.

Thus A is non-trivial. Suppose that all conjugates of D in $G_{\mathfrak{L}}$ centralize all elements of A. Then these conjugates lie in a subgroup of order $q^3(q-1)/\mu$. Now if $G_{\mathfrak{L}} \cap C_G(A)$ only contains p-elements centralized by D then $G_{\mathfrak{L}} \cap C_G(A)$ contains only one conjugate of D. By our supposition this means that $G_{\mathfrak{L}}$ contains only one conjugate of D which is a contradiction. Thus $G_{\mathfrak{L}} \cap C_G(A)$ contains p-elements not centralized by D. Then the normal p-subgroup of $G_{\mathfrak{L}} \cap C_G(A)$ has size $|A| + n\frac{q-1}{\mu}$ for some n. Thus $G_{\mathfrak{L}} \geq Q : D$ for a p-group Q of size at least q.

If a conjugate of D in $G_{\mathfrak{L}}$ does not act trivially in its action on elements of A then A must be of order divisible by q. Once again $G_{\mathfrak{L}} \geq Q : D$ where $|Q| \geq q$. We deal with this situation at the end of the section.

Now suppose that $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 2. In order for there to be two conjugates, D and E, of D in $G_{\mathfrak{L}}$ we must have D, E in $\hat{}(q-1)^2$. Hence $\frac{(q-1)^2}{\mu^2} \big| |G_{\mathfrak{L}}|$. For D, E to be non-normal, we must have $G_{\mathfrak{L}} \geq (\frac{q-1}{\mu} \times \frac{q-1}{\mu}) : 3$. If 2 is significant then p is odd and $G_{\mathfrak{L}} \leq \hat{}(q-1)^2 : 3$ and $G_{\mathfrak{L}}$ contains at most 3 involutions. By Lemma 3.16, we know that $k \leq \frac{3v}{n} + 1 = \frac{1}{2}q(q+1) + 1$. This is inconsistent with our value for b. If 2 is not significant then $G_{\mathfrak{L}} = (\frac{q-1}{3} \times \frac{q-1}{3}) : S_3$ and $b = 3v$.

When p is odd, $G_{\mathfrak{L}}$ contains at most $q+2$ involutions and, by Lemma 3.16, this implies that $k \leq \frac{(q+2)v}{q^2(q^2+q+1)} + 1$. We therefore conclude that

$$k(k-1) \leq \frac{q(q+1)(q+2)(q+3)(q^2+2)}{36}.$$

However this implies that $\frac{b}{v} = \frac{v-1}{k(k-1)} > 4$ which is a contradiction.

When $p = 2$, $G_{\mathfrak{L}}$ contains at most $q-1$ involutions and we find that $k(k-1) \leq \frac{1}{36}q^3(q^3+6)$. Once again $\frac{b}{v} = \frac{v-1}{k(k-1)} > 4$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 4 or 5 then we have two possibilities. If $G_{\mathfrak{L}} = A_6.2$ or A_7 then, in order to satisfy $|G_{\mathfrak{L}}| > \frac{(q-1)^2}{\mu^2}$, we must have $q = 25$. This has already been excluded. If $G_{\mathfrak{L}}$ contains a subgroup of index less than or equal to 3 isomorphic to $PSU(3,q_0)$ or $PSL(3,q_0)$ where $q = q_0^a$ then we require that $q_0^3(q_0^2-1)(q_0^3-1) < 6(q-1)^2$. Thus we need $q \geq q_0^4$. This implies that either $\frac{q-1}{\mu}$ does not divide into $|G_{\mathfrak{L}}|$ or that $q = 64$. Both cases give contradictions.

If $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 6,7,8 or 10 then $\frac{(q-1)^2}{\mu^2} < 360$. This implies that $q \leq 19$ or $q \equiv 1(3)$ and $q \leq 49$. All of these cases have been excluded already.

If $G_{\mathfrak{L}}$ is in a group of type 9 then $|G_{\mathfrak{L}}| < |G_\alpha|$ implies that $G_{\mathfrak{L}}$ is a proper subgroup. Since $|G_{\mathfrak{L}}| > \frac{(q-1)^2}{\mu^2}$ we must have $G_{\mathfrak{L}} \leq [q] : (q-1)$. Thus $G_{\mathfrak{L}} = A : B$ where $A \leq [q]$, $B \leq (q-1)$. All conjugates of B in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and B contains a conjugate of D. Thus $\frac{q-1}{\mu}$ divides into $|B|$. Since B acts semi-regularly on the non-trivial elements of A this means that $|A| = q$. Once more we conclude that $G_{\mathfrak{L}}$ has a normal subgroup of order q.

We have reduced all cases to the situation where $G_{\mathfrak{L}} \geq Q : D$ where Q is a p-group of order divisible by q. Observe that all conjugates of D in $Q : D$ are $G_{\mathfrak{L}}$ conjugate. If $G_{\mathfrak{L}}$ contains E, another G-conjugate of D which is not $G_{\mathfrak{L}}$-conjugate, then $E \cap (Q : D)$ is trivial; hence $|G_{\mathfrak{L}}| \geq \frac{q(q-1)^2}{\mu^2}$ which is too large. Thus all G-conjugates of D in $G_{\mathfrak{L}}$ are $G_{\mathfrak{L}}$-conjugate and we can apply Lemma 3.6 as in Case A. As in Case A this implies that 3 is uniquely significant and either $2\frac{(q-1)^2}{81}\big||G_{\mathfrak{L}}|, q \equiv 28(81)$ or $2\frac{(q-1)^2}{27}\big||G_{\mathfrak{L}}|, q \equiv 1(3)$. If p is odd then this means that either $q < 81$ and $q \equiv 28(81)$ or $q < 27$ and $q \equiv 1(3)$. If $p = 2$ then this means that either $q < 162$ and $q \equiv 28(81)$ or $q < 54$ and $q \equiv 1(3)$. All such possibilities have already been excluded.

Hence we may conclude that no new line-transitive action of $PSL(3,q)$ exists where $G_\alpha = \hat{}(q-1)^2 : S_3$.

Remark. *The argument in this section deals with Case II in our analysis of significant primes.*

3.8 Case III: $3|q-1$ is uniquely significant

In this case G_α lies inside a subgroup of $PSL(3,q)$ of type 2, 4, 5 or 8.

3.8.1 Case 1: G_α is a proper subgroup of a group of type 2

Then $G_\alpha = A.B$ where $B = C_3$ or S_3 and $A = \hat{}(u \times u)$ (this structure for A follows since it is normalized by C_3.) We can conclude, using Corollary 3.10, that $B = S_3$. Now observe that $A.2$ lies inside a copy of $\hat{}GL(2,q)$, hence is centralized by $Z(\hat{}GL(2,q))$. Thus, by Lemma 3.15, $A.2$ lies in $G_{\mathfrak{L}}$. Thus $|G_{\mathfrak{L}}| = 2|A|$ or $|G_{\mathfrak{L}}| = 4|A|$ while $b\big|3v$. When $p = 2$ we know that $v-1$ is odd. Since $k(k-1)$ is even and $\frac{b}{v} = \frac{v-1}{k(k-1)}$, this means that $|G_{\mathfrak{L}}| = 4|A|$ and $b = \frac{3}{2}v$.

Consider first the case where $b = \frac{3}{2}v$. Then $\frac{b}{v} = \frac{3}{2} = \frac{v-1}{k(k-1)}$ and so

$$k(k-1) = \frac{2}{3}(v-1) = \frac{1}{9u^2}[q^8 - q^6 - q^5 + q^3 - 6u^2].$$

Now observe that, for $q > 8$,

$$[\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)+\frac{1}{2}][\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)-\frac{1}{2}] > \frac{2}{3}(v-1);$$

$$[\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)\frac{1}{3}][\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q)-\frac{2}{3}] < \frac{2}{3}(v-1).$$

Since $\frac{1}{3u}(q-1)(q^3+q^2+\frac{1}{2}q) = \frac{1}{6}a$ for some integer a, this is a contradiction. Thus p is odd and $b = 3v$.

Now suppose that 4 does not divide into u. Then $|G_\alpha|_2 \leq 8$ while $|G|_2 \geq 16$, hence $v-1$ is odd. This implies that $|b|_2 < |v|_2$ which is a contradiction. Hence $12|u$.

Now $G_{\mathfrak{L}} = \hat{}(u \times u).2 < \hat{}(q-1)^2 : 2 < \hat{}GL(2,q)$ and so contains at most $u+3$ involutions. We appeal to Lemma 3.16 to observe that,

$$k \leq \frac{(u+3)q(q+1)(q-1)^2}{6u^2} + 1.$$

We can conclude therefore that, for $u \geq 12$,

$$k(k-1) \leq \frac{q^2(q+1)^2(q-1)^4(u+3)(u+4)}{36u^4}.$$

This is strictly smaller than $\frac{v-1}{3}$ which is a contradiction.

3.8.2 Case 2: G_α is a subgroup of type 4 or 5

We refer to Lemma 3.20. Consider first the possibility that G_α is isomorphic to $A_6.2$ or A_7 and $p = 5$. We exclude $q = 25$ using Lemma 3.18.

Observe that, since 3 divides $q-1$, there is a group of order 3 normal in a group isomorphic to $\hat{}(q-1)^2$. Hence a line-stabilizer contains a subgroup of order 3 or else contains the group $\hat{}(q-1)^2$ (by Lemma 3.15). The latter possibility is not possible, hence we may assume that $3\big||G_\mathfrak{L}|$. We may therefore conclude that $b = 3v$ or $b = \frac{3}{2}v$.

Now suppose that m is an integer dividing v and $b = \frac{3}{x}v$ where x is 1 or 2. We have that

$$\begin{aligned} \frac{v-1}{k(k-1)} &= \frac{3}{x} \\ \implies \quad & 3k(k-1) + x \equiv 0 \ (mod\ m) \\ \implies \quad & 36k^2 - 36k + 12x \equiv 0 \ (mod\ m) \\ \implies \quad & 9(2k-1)^2 \equiv 9 - 12x \ (mod\ m) \end{aligned}$$

Thus $9-12x$ is a square modulo m and m is not divisible by 3. If $G_\alpha = A_6.2$ then we know that 25 divides v. For both values of x we find that $9-12x$ is not a square modulo 25.

Thus we assume that either $G_\alpha = PSL(3, q_0), q = q_0^a, 3\big|q_0 - 1, a \not\equiv 0(mod\ 3)$; or $G_\alpha = PSU(3, q_0), q = q_0^a, 3\big|q_0 + 1, a \not\equiv 0(mod\ 6)$.

Then in the first instance we have a subgroup of G_α, $\hat{}(q_0-1)^2$; in the second instance we have a subgroup of G_α, $\hat{}(q_0+1)^2$. Such subgroups are normal in the subgroup of $PSL(3,q)$, $\hat{}(q-1)^2$. Thus these subgroups of G_α lie in $G_\mathfrak{L}$ and we may conclude that $b\big|3v$. Once again when $p = 2$ we know that $v-1$ is odd and so $b = \frac{3}{2}v$.

We know that $q_0^3\big||G_\mathfrak{L}|$, hence $G_\mathfrak{L}$ is not a subgroup of a group of type 2,3,6,7,8 or 10. If $G_\mathfrak{L}$ is a subgroup of a group of type 9 then $\frac{(q_0^3\pm1)}{3}\big|(q^2-1)$. Since $q = q_0^a, a \not\equiv 0(3)$ we must have $q_0 = 2$ and $G_\alpha = PSU(3,2)$. But then $|G_\alpha| = 72$ which is the same size as in Case 1 with $u = 6$. The arguments given there exclude both $b = 3v$ and $b = \frac{3}{2}v$.

If $G_\mathfrak{L}$ is only a subgroup of a group of type 4 or 5 then either $G_\mathfrak{L} = A_6.2$ or A_7 (and 25 divides into v which is a contradiction), or $G_\mathfrak{L}$ is one of $PSL(3,q_1)$ or $PSU(3,q_1)$. Since $b|3v$ we must have $q_0 = q_1$ and $\frac{q_0^3+1}{q_0^3-1}$ equal to 3 or $\frac{3}{2}$. This is impossible.

Thus $G_{\mathfrak{L}}$ is a subgroup of a parabolic subgroup. Then we require that $(q_0^3 \pm 1)\big|(q^2-1)(q-1)$. This implies that $q_0 = 2$ which can be excluded as in Case 1 setting u to be 6.

3.8.3 Case 3: G_α is a maximal subgroup of type 8

Note that p is odd here and, using Lemma 3.18, $q \geq 43$. Here $G_\alpha \cong 3^2.Q_8$ and $|q-1|_3 = 3$. Observe that, since 3 divides $q-1$, there is a group of order 3 normal in a group isomorphic to $\hat{}(q-1)^2$ and so, by Lemma 3.15, $3 \leq G_{\mathfrak{L}}$. Thus $b\big|3v$. Now G_α has the same size as G_α in Case 1 with $u = 6$. The arguments given there exclude both $b = 3v$ and $b = \frac{3}{2}v$ and we are done.

Thus we have ruled out all possible actions of line-transitive actions of $PSL(3,q)$ where 3 is the unique significant prime.

3.9 Case IV: $2|q-1$ is uniquely significant

In this case G_α either lies in a parabolic subgroup or in a subgroup of $PSL(3,q)$ of type 2 or 4. Since D normalizes a Sylow 2-subgroup of $PSL(3,q)$, we know that G_α contains D for some α. Furthermore, by Lemma 3.15, either $G_\alpha \geq \hat{}GL(2,q)$ or $D < G_{\mathfrak{L}}$.

3.9.1 Case 1: G_α is a subgroup of a group of type 4 only

In this case $G_\alpha = PSL(3,q_0)$ or $PSL(3,q_0).3$ for some q_0 where $q = q_0^a$, a odd. Then $D < G_{\mathfrak{L}}$ and so $\frac{q-1}{\mu}$ divides into $3|PSL(3,q_0)|$. We must have $q = q_0^3$. But then $PSL(3,q_0)$ does not contain an element of order $\frac{q_0^3-1}{\mu}$ and so $D \not< PSL(3,q_0)$ and this case is also excluded.

3.9.2 Case 2: G_α lies inside a group of type 2

Here G_α is non-maximal, $q \equiv 1(4)$ and G_α contains a cyclic subgroup of order $q-1/\mu$. We have two possibilities:

1. $G_\alpha = A:2$ where $A \leq \hat{}(q-1)^2$ and $|A| = a\frac{q-1}{\mu}$. Then A is proper normal in $\hat{}(q-1)^2$ for $a < q-1$ and proper normal in $\hat{}(q-1)^2 : S_3$ for $a = q-1$. Thus we may conclude, by Lemma 3.15, that $G_{\mathfrak{L}} = A$. We can conclude that $G_{\mathfrak{L}}$ contains at most 3 involutions.

2. We suppose that $3|(q-1)$ and $G_\alpha = (\frac{q-1}{3} \times \frac{q-1}{3}) : S_3$. In this case, $(\frac{q-1}{3} \times \frac{q-1}{3})$ is normal in $\hat{}(q-1)^2$ and hence lies in $G_{\mathfrak{L}}$. We can conclude that $|G_{\mathfrak{L}}| = 3(\frac{q-1}{3})^2$ and $G_{\mathfrak{L}}$ contains at most 9 involutions.

Consider the first case. Since $G_{\mathfrak{L}}$ contains at most 3 involutions, we may appeal to Lemma 3.16 to give,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{3q(q+1)(q-1)}{2a} + 1.$$

This implies that,

$$k(k-1) < \frac{9}{4a^2} q^3 (q+1)^2 (q-1).$$

Now we know that $k(k-1) = \frac{v-1}{2}$. Thus

$$\frac{v-1}{2} = \frac{q^3(q^2+q+1)(q+1)(q-1) - 2a}{4a} < \frac{9}{4a^2} q^3 (q+1)^2 (q-1).$$

Hence $q < \frac{9}{a}$ which is impossible.

We move on to the next possibility: $H = (\frac{q-1}{3} \times \frac{q-1}{3})$ lies inside $G_{\mathfrak{L}}$ with index 3. Now H contains 3 involutions, hence $G_{\mathfrak{L}}$ must contain at most 9 involutions. Once again we appeal to Lemma 3.16 to give,

$$k \leq \frac{r_g v}{n_g} + 1 = \frac{9q(q+1)}{2} + 1.$$

This gives,

$$\frac{v-1}{2} = k(k-1) < \frac{41q^2(q+1)^2}{2}.$$

Given our value for v we may conclude that,

$$q^3(q^2+q+1)(q+1) - 2 < 41q^2(q+1)^2.$$

This is only true for $q \leq 7$ which is impossible.

3.9.3 Case 3: G_α lies in a parabolic subgroup

Now, for P a parabolic subgroup, $|G : P| = q^2 + q + 1$. By Lemma 3.9 this means that any significant prime must divide $\frac{1}{2}q(q+1)$. Since 2 is uniquely significant, we may conclude that $q \equiv 3(4)$ and $b \big| \frac{1}{2}(q+1)v$. We write $G_\alpha = A.B$ where A is an elementary abelian p-group and $B \leq \hat{}GL(2,q)$.

Suppose $q \equiv 3(8)$. Then, by Lemma 3.9, $b = 2v$. Then, by Lemma 3.3, any prime m dividing into v must be equivalent to $1(4)$. Since $p \equiv 3(4)$ we have q^3

dividing into $|G_\alpha|$. Thus $A = [q^2]$ and $B \geq \hat{}SL(2,q)$. However this means that $A.B$ is normal in the full parabolic subgroup. Hence, by Lemma 3.15, either $G_{\mathfrak{L}} \geq G_\alpha$ (which is impossible) or G_α is the full parabolic subgroup. This case has already been excluded.

Thus $qequiv7(8)$ and B is a subgroup of $\hat{}GL(2,q)$ of type 3 or 5. Consider the case where B is a subgroup of $\hat{}GL(2,q)$ of type 3. We examine the possible situations here:

1. Suppose that B is maximal in $\hat{}GL(2,q)$, i.e. $|B| = 2(q^2-1)/\mu$. Then B acts by conjugation on the non-trivial elements of A with orbits divisible by $q+1$. Thus $|A| = q^2$ or 1. Since 2 is uniquely significant, $A < G_{\mathfrak{L}}$. This is the same situation as in Subsections 3.6.1 and 3.6.3; precisely the same arguments as in those sections allow us to exclude the situation here.

2. Suppose that B is non-maximal in $\hat{}GL(2,q)$. Then B contains a cyclic group C which is normal in $\hat{}(q^2-1)$, hence lies in $G_{\mathfrak{L}}$. Furthermore $|A|\,||G_{\mathfrak{L}}|$ since 2 is uniquely significant. Thus $|G_{\mathfrak{L}}| = |A|.|C|$ and $G_\alpha = 2|A|.|C|$ and so $b = 2v$. However in this case, by Lemma 3.3, any prime m dividing into v must be equivalent to $1(4)$. Here though $p \equiv 3(4)$ and p divides into v. This is a contradiction.

Now consider the possibility that B is of type 5. Since $q \equiv 7(8)$, we must have $q = p^a$ where a is odd and so $B = \hat{} < SL(2,q_0), V >$.

Suppose first that $q = q_0$ and so $B \geq \hat{}SL(2,q)$ and either A is trivial or $A = [q^2]$.

If A is trivial then either $B \lhd \hat{}GL(2,q)$ or $B = \hat{}GL(2,q)$. The first option implies that $G_{\mathfrak{L}} \geq G_\alpha$ (which is impossible). The latter option is the same as in Subsection 3.6.2; precisely the same arguments as in that section allow us to exclude the situation here.

If on the other hand A is non-trivial then $A = [q^2]$ and so G_α is either the full parabolic subgroup (this possibility is already excluded) or G_α is normal in the full parabolic subgroup and $G_{\mathfrak{L}} \geq G_\alpha$ (which is impossible). Thus both possibilities are excluded when $q = q_0$. We assume that $q = q_0^a$, a is odd, $a \geq 3, p \equiv 7(8)$ and $D < G_{\mathfrak{L}}$.

Now observe that $A. < V >$ is a split extension by Schur-Zassenhaus. So we can take V to be in G_α. Furthermore G_α must contain a conjugate of D. Then, since $q \geq q_0^3$, $< V > \cong \frac{q-1}{\mu}$ is G-conjugate to D. The G-conjugates of D split into two conjugacy classes inside the parabolic subgroup with centralizers isomorphic to $\hat{}[q]:(q-1)^2$ and $\hat{}GL(2,q)$. If we factor out the unipotent subgroup of the maximal parabolic then we see that, in G_α/A, $<V>A$ is centralized by $SL(2,q_0)$ and so $<V>$ must be centralized in the maximal parabolic by $\hat{}GL(2,q)$. This means that $<V>$ acts by conjugation on the non-identity elements of A with orbits of size $\frac{q-1}{\mu}$. In fact B has orbits of length a multiple of $\frac{(q_0+1)(q-1)}{\mu}$ on the non-trivial elements of A. Thus $|A|=q^2$ or $|A|=1$.

Now note that, since $b|\frac{1}{2}v(q+1)q$, we know that $\frac{2q_0(q_0-1)(q-1)}{\mu}||G_{\mathfrak{L}}|$. Thus $G_{\mathfrak{L}}$ lies inside a subgroup of $PSL(3,q)$ of type 1 or 4.

If $G_{\mathfrak{L}}$ lies in a subgroup of $PSL(3,q)$ of type 9 then $G_{\mathfrak{L}}=SO(3,q)$. If A is trivial then $|G_{\mathfrak{L}}|>|G_\alpha|$ which is a contradiction. If A is non-trivial then q^2 divides into $|G_{\mathfrak{L}}|$ which is a contradiction.

If $G_{\mathfrak{L}}$ lies in a subgroup of $PSL(3,q)$ of type 4 then $G_{\mathfrak{L}}=PSL(3,q_1)$ or $PSL(3,q_1).3$. Since $D<G_{\mathfrak{L}}$ we must have $q\leq q_1^2$. But $q=p^a$ where a is odd which is a contradiction.

Thus $G_{\mathfrak{L}}$ lies inside a parabolic subgroup of $PSL(3,q)$. So $G_{\mathfrak{L}}=A_1.B_1$ where A_1 is elementary abelian and $B_1\leq \hat{}GL(2,q)$. Then $\frac{2(q_0-1)(q-1)}{\mu}$ divides into $|B_1|$ and B_1 is of type 4, 5, 6 or 7.

If B_1 is of type 5 then we must have $B_1\geq SL(2,q_0)$. Since $D<A_1.B_1$ we require that B_1 contains a cycle of length $\frac{q-1}{2\mu}$ and so $B_1\geq <SL(2,q_0),\frac{q-1}{2\mu}>$. If A is trivial then $|B_1|\geq\frac{1}{2}|G_\alpha|$ which is a contradiction. If $A=[q^2]$ then A_1 must be non-trivial and B_1 has orbits on the non-trivial elements of A_1 of size a multiple of $\frac{(q_0+1)(q-1)}{\mu}$. Thus $|A_1|=q^2$ and $|G_{\mathfrak{L}}|\geq\frac{1}{2}|G_\alpha|$. By Lemma 3.3, $p\equiv 1(4)$ which is a contradiction.

If B_1 is of type 4, 6 or 7 then q_0 divides into $|A_1|$ and $G_{\mathfrak{L}}=A_1.B_1$ is a split extension. Furthermore A is trivial since q^2q_0 cannot divide into $|G_{\mathfrak{L}}|$.

In the case of types 6 and 7, B_1 must centralize EA_1 in $G_{\mathfrak{L}}/A_1$ where E is a conjugate of D. Thus E has an orbit on the non-trivial elements of A_1 of size a multiple of $\frac{(q-1)}{\mu}$. Thus $|A_1|\geq q$. But $|G_\alpha|<q_0^3\frac{q-1}{\mu}$ and $|G_{\mathfrak{L}}|>q\frac{q-1}{\mu}$ which is impossible.

We are left with the possibility that B_1 is of type 4 and take D to be in $G_{\mathfrak{L}}$. Suppose first that DA_1 is central in $B_1=G_{\mathfrak{L}}/A_1$. Since $q+1$ does not divide into b, $|B_1|_2\geq 2|(q-1)^2|_2$. This implies that D is centralized in the full parabolic by $\hat{}GL(2,q)$ and D has orbits on A_1 of size a multiple of $\frac{(q-1)}{\mu}$. If, on the other hand, DA_1 is not central in $B_1=G_{\mathfrak{L}}/A_1$ then it is not normal either and $|B_1|$ is divisible by $2(\frac{q-1}{\mu})^2$ Then $G_{\mathfrak{L}}$ has orbits on the non-trivial elements of A_1 of size a multiple of $\frac{(q-1)}{\mu}$. Thus in either case $|A_1| \geq q$. But $|G_\alpha| < q_0^3 \frac{q-1}{\mu}$ and $|G_{\mathfrak{L}}| > q\frac{q-1}{\mu}$ which is impossible.

This deals with all the cases where 2 is a uniquely significant prime. We conclude that $PSL(3,q)$ has no line-transitive actions in this case.

We have now dealt with all possibilities for line-transitive actions of $PSL(3,q)$ on finite linear spaces. Our proof of Theorem B is complete.

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