Preface

The subject of this thesis is situated in the field of finite geometry and the theory of finite groups. The work can be seen as consisting of two main parts; the first part treats the construction of new examples of (α, β)-geometries and the second part focuses on the characterisation of a particular class of (α, β)-geometries, namely (0,2)-geometries, which satisfy Moufang-like conditions.

In order to make the reader familiar with the objects that appear in this thesis, the first chapter gives an overview of the definitions and fundamental results concerning incidence structures in a finite projective space. In the second chapter, we study the relation between distance-regular graphs and (α, β)-geometries. First we give necessary and sufficient conditions for the neighbourhood geometry of a distance-regular graph to be an (α, β)-geometry and we discuss some examples. Next we observe that properties of certain regular two-graphs allow us to construct $(0, \alpha)$-geometries on the corresponding Taylor graph.

In the third chapter, we introduce the concept of a $(0, \alpha)$-regulus and we prove that it yields $(0, \alpha)$-geometries with a distance-regular point graph. We also give two infinite classes of examples.

Some partial geometries, a special class of (α, β)-geometries which have a strongly regular point graph, can be constructed from perp-systems in a projective plane. The fourth chapter is devoted to characterise some partial geometries arising from perp-systems. In chapter five, we change perspective and concentrate on $(0, 2)$-geometries. Here we use the special properties of a $(0, 2)$-geometry to define Moufang-like conditions, in a similar way as is done for general polygons. These definitions are explored and some partial classification results are obtained.

The work that is presented in this thesis could not have been possible without the help of many people. First of all I would like to express my deep gratitude to my supervisor, Prof. Dr. F. De Clerck for his excellent guidance. He encouraged and motivated me when it seemed to me I had tried all possibilities, believed in me and gave me the opportunity to do this research. It was a pleasure to work with him. In that sense I would also like to thank my co-supervisor, Prof. Dr. H. Van Maldeghem for his patience and good ideas for solving problems. I would like to thank Prof. Dr. Matthew Brown for helping me carry out my research. I also want to thank him for reading my results and for providing very helpful comments. I am grateful to the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT) for its financial support. Next, I want to say thank to Dr. E. Kuijken who helped me a great deal in my research. It was a privilege to work in the Department of Pure Mathematics and Computer Algebra at Ghent University, which supported me during this period. I would like to thank my colleagues with whom I shared pleasant moments and who supported me in moments of despondency, especially Deirdre Luyckx who read my reports in Dutch. Last but not least, I would like to thank my parents for encouraging me and always believing in me, and Philippe, my husband and source of my enthusiasm, for his patience and understanding.

Cristina Tonesi
March 2005

Contents

Chapter 1

Introduction

The aim of this chapter is to introduce the basic notions that are needed to understand the problems that will be handled in the further chapters.

1.1 Incidence structures

An *incidence structure* is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ consisting of a set $\mathcal{P}$ of *points*, a set $\mathcal{B}$ of *blocks* and a symmetric *incidence relation* $\mathrm{I} \subseteq (\mathcal{P}\times\mathcal{B})\cup(\mathcal{B}\times\mathcal{P})$. The sets $\mathcal{P}$ and $\mathcal{B}$ will always be finite here. The *dual* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the incidence structure $\mathcal{S}^D = (\mathcal{B}, \mathcal{P}, \mathrm{I})$. We will sometimes identify a block of an incidence structure with the set of points incident with it. We remark that we will use expressions like "a point *on* a block L" or "a block *through* the point p".

An incidence structure is called a *partial linear space* if any two blocks of the incidence structure are incident with at most one common point; in this case blocks are usually called *lines*. Points of a partial linear space are called *collinear* if they are incident with a common line. Dually, lines of a partial linear space are called *concurrent* if they are incident with a common point. We note that a point is collinear with itself and dually that a line is concurrent with itself.

An incident point-line pair in a partial linear space is a *flag*, while a non-incident point-line pair is an *antiflag*. The *incidence number* of an antiflag $\{p, L\}$ is the number of points incident with the line L and collinear with the point p. If each line is incident with a constant number $s + 1$ of points and each point is incident with a constant number $t + 1$ of lines, the partial linear space is said to have *order* (s, t), or order s if $s = t$.

An incidence structure can be represented by its *incidence matrix*. This is a matrix N of which the rows (columns) are indexed by the points (blocks); the entry n_{pL} is 1 if the point p is incident with the line L and 0 otherwise. Clearly dualising corresponds to transposing the incidence matrix. An *isomorphism* between incidence structures is a bijection between the point sets together with a bijection between the block sets such that incidence is preserved by the bijections and their inverses. An *automorphism* of an incidence structure is an isomorphism from the incidence structure to itself. An incidence structure is called *self-dual* if it is isomorphic to its dual; the isomorphism is a *duality* of the incidence structure.

1.2 Collineations and polarities

Consider the n-dimensional projective space $\mathrm{PG}(n,q)$ over the finite field $\mathrm{GF}(q)$. A *collineation* of $\mathrm{PG}(n,q)$, $n \geq 2$, is a permutation of the set of all subspaces of $\mathrm{PG}(n,q)$, which preserves inclusion between subspaces of $\mathrm{PG}(n,q)$. Every collineation is induced by a semilinear transformation of the underlying vector space V. For $n = 1$, a collineation of $\mathrm{PG}(1,q)$ is defined as being induced by a semilinear transformation of the underlying vector space V. Hence a collineation, with respect to a given coordinate system in $\mathrm{PG}(n,q)$, is described by a non-singular $(n+1)\times(n+1)$ matrix A over the finite field $\mathrm{GF}(q)$ and an automorphism θ of $\mathrm{GF}(q)$. The group of all collineations of $\mathrm{PG}(n,q)$ is denoted by $\mathrm{P\Gamma L}(n+1,q)$. The collineations of $\mathrm{PG}(n,q)$ for which θ is the identity, are called *projectivities* and they also form a group, denoted by $\mathrm{PGL}(n+1,q)$.

A *correlation* of $\mathrm{PG}(n,q)$ is a permutation of the set of all subspaces of $\mathrm{PG}(n,q)$ which reverses inclusion between spaces of $\mathrm{PG}(n,q)$. We remark that every semilinear mapping of the underlying vector space V onto its dual V^D induces a correlation. We also note that a correlation sends a subspace of dimension k onto a subspace of dimension $n-k-1$. If $n = 1$, a correlation is defined as being induced by a semilinear mapping of the underlying 2-dimensional vector space V onto its dual. If α is a correlation of $\mathrm{PG}(n,q)$ such that α^2 is the identity, then α is called a *polarity*.

A correlation α of $\mathrm{PG}(n,q)$, with respect to a given coordinate system in $\mathrm{PG}(n,q)$, can also be described by a non-singular $(n+1)\times(n+1)$ matrix A over the finite field $\mathrm{GF}(q)$ and an automorphism θ of $\mathrm{GF}(q)$. One can show that if α is a polarity then θ^2 is the identity. According to the conditions on θ and A, we obtain different types of polarities. For more information on collineations and general properties about $\mathrm{PG}(n,q)$ we refer to [41, 46].

1.3 Finite polar spaces

1.3.1 Definition

Polar spaces were introduced by Veldkamp [77] and have been thoroughly studied, see for instance [14], [12], [16] or [41].

A *finite polar space of rank n*, $n \geq 3$ consists of a finite set $\mathcal{P}$ of elements which are called *points*, and a set of *subspaces* which are subsets of $\mathcal{P}$, such that the following axioms are satisfied.

PS1 A subspace together with the subspaces contained in it is isomorphic to a projective space $\mathrm{PG}(d, q)$, $-1 \leq d \leq n-1$, q a prime power. Such a subspace is said to have *dimension d*.

PS2 The intersection of two subspaces is a subspace.

PS3 If π is a subspace of dimension $n-1$ and $p \in \mathcal{P} \setminus \pi$, then there is a unique subspace π' of dimension $n-1$ which contains p and intersects π in an $(n-2)$-dimensional subspace. The points of $\pi \cap \pi'$ are exactly the points r of π such that there exists a one-dimensional subspace containing p and r.

PS4 There exist disjoint $(n-1)$-dimensional subspaces.

Distinct points x and y are said to be *collinear* if there exists a one-dimensional subspace containing x and y. The subspaces of maximal dimension $n-1$ are also called *generators* and the integer $n-1$ is also referred to as the *projective index* of the polar spaces. The subspaces of a finite polar space will be called *totally singular* or *totally isotropic* subspaces.

A *singular polar space* $\mathcal{P}$ in $\mathrm{PG}(n, q)$ is a cone $\pi\mathcal{P}'$, with π a d-dimensional subspace of $\mathrm{PG}(n, q)$, and with $\mathcal{P}'$ a non-singular classical polar space in some $\mathrm{PG}(n-d-1, q) \subset \mathrm{PG}(n, q)$ disjoint from π.

1.3.2 Examples of finite polar spaces

1. A *quadric* in $\mathrm{PG}(n, q)$, $n \geq 1$, is a set of points the coordinates of which satisfy a quadratic equation of the form

$$\sum_{\substack{i,j=1 \\ i \leq j}}^{n} a_{ij} X_i X_j = 0,$$

with not all a_{ij} equal to 0.

A quadric is *singular* if there exists a change of coordinate system which

reduces its equation to one with less than $n+1$ variables; otherwise the quadric is said to be *non-singular*. We note that the definition of singular quadric is coherent with the definition of singular polar space. We mention some results on the classification of non-singular quadrics. In $\mathrm{PG}(2n,q)$ there is, up to collineations, only one non-singular quadric, called the *parabolic quadric*, denoted by $Q(2n,q)$. In $\mathrm{PG}(2n+1,q)$ there are, up to collineations, exactly two non-singular quadrics, the *elliptic quadric*, denoted by $Q^-(2n+1,q)$, and the *hyperbolic quadric*, denoted by $Q^+(2n+1,q)$. For each of the three cases, the quadratic equation can be reduced to *canonical form.*
The parabolic quadric $Q(2n,q)$ is determined by the canonical equation of the form

$$x_0^2 + x_1x_2 + \cdots + x_{2n-1}x_{2n} = 0.$$

The hyperbolic quadric $Q^+(2n+1,q)$ is determined by the canonical equation of the form

$$x_0x_1 + x_2x_3 + \cdots + x_{2n}x_{2n+1} = 0.$$

The elliptic quadric $Q^-(2n+1,q)$ is determined by the canonical equation of the form

$$f(x_0,x_1) + x_2x_3 + \cdots + x_{2n}x_{2n+1} = 0,$$

where $f(x_0,x_1)$ is an irreducibile quadratic polynomial over $\mathrm{GF}(q)$.
For $n \geq 3$ and q a prime power, consider one of the non-singular quadrics $Q^+(2n-1,q)$, $Q(2n,q)$ or $Q^-(2n+1,q)$. The points and projective subspaces on the quadric form a polar space of rank n.

2. A *hermitian variety* in $\mathrm{PG}(n,q^2)$, $n \geq 1$, is a set of points the coordinates of which satisfy an equation of the form

$$\sum_{i,j=0}^{n} a_{ij}X_iX_j^q = 0,$$

with not all a_{ij} equal to 0 and $a_{ji} = a_{ij}^q$ for all $i,j = 0,1,\cdots,n$.
A hermitian variety is called *singular* if there exists a change of the coordinate system which reduces its equation to one with less than $n+1$ variables; otherwise a hermitian variety is said to be *non-singular*. We note that also the definition of a singular hermitian variety is coherent with the definition of singular polar space.
In $\mathrm{PG}(n,q^2)$, there is, up to collineations, only one non-singular hermitian variety, denoted by $H(n,q^2)$.

For $n \geq 3$ and q a prime power, consider a non-singular hermitian variety $H(2n-1, q^2)$ or $H(2n, q^2)$. The points and projective subspaces on the hermitian variety form a polar space of rank n.

3. Let π be a symplectic polarity of $\mathrm{PG}(2n-1, q)$, $n \geq 3$, q a prime power. Then the set of points of $\mathrm{PG}(2n-1, q)$ together with the set of totally isotropic subspaces of $\mathrm{PG}(2n-1, q)$ with respect to π form a polar space of rank n, usually written as $W(2n-1, q)$.

In [70] Tits proved that the above examples are (up to isomorphism) the only polar spaces of rank at least 3. If q is even and $n \geq 3$, the polar spaces $Q(2n, q)$ and $W(2n-1, q)$ are isomorphic. The number of points on each of the polar spaces is

$$\begin{aligned}
|Q^+(2n-1, q)| &= (q^{n-1}+1)(q^n-1)/(q-1), \\
|Q(2n, q)| &= (q^{2n}-1)/(q-1), \\
|Q^-(2n+1, q)| &= (q^n-1)(q^{n+1}+1)/(q-1), \\
|H(2n-1, q^2)| &= (q^{2n}-1)(q^{2n-1}+1)/(q^2-1), \\
|H(2n, q^2)| &= (q^{2n+1}+1)(q^{2n}-1)/(q^2-1), \\
|W(2n-1, q)| &= (q^{2n}-1)/(q-1).
\end{aligned}$$

1.4 Graphs

1.4.1 Terminology

A *graph* is a pair (V, E) consisting of a *vertex* set V and a set E of *edges* which are unordered pairs of vertices. Only graphs on a finite vertex set will be considered here. Two distinct vertices x and y are said to be *adjacent* if the pair $\{x, y\}$ is an edge. Using the standard notation, we will denote adjacent vertices x and y by $x \sim y$ and non-adjacent vertices x and y by $x \not\sim y$. We note that $x \not\sim x$. A graph with an empty edge set is *void*, and a graph in which all pairs of vertices are edges is *complete*, it has n vertices and $n(n-1)/2$ edges, and is denoted by K_n. The void graphs and the complete graphs are called *trivial* graphs. A vertex of a graph is *isolated* if it is adjacent to no other vertex. The *complement* $\overline{\Gamma}$ of a graph $\Gamma = (V, E)$ is the graph with vertex set V in which the edges are the unordered pairs of vertices that are not in E. The *subgraph* of a graph (V, E) *induced on* a subset W of V is the graph (W, F), where F consists of all edges of (V, E) which are contained in W. A set of mutually adjacent vertices is called a *clique*, while a set of mutually non-adjacent vertices is a *coclique*. If the vertex set of a graph can be partitioned in cocliques, it is *multipartite*; in particular the vertex set of a *bipartite* graph is the disjoint union of two cocliques. The *line graph* $L(\Gamma)$ of a graph $\Gamma = (V, E)$ has E as a vertex set; two edges of Γ are adjacent in $L(\Gamma)$ if and only if they have a vertex of Γ in common. The *incidence graph* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ has $\mathcal{P} \cup \mathcal{L}$ as a vertex set, adjacency being incidence; it is obviously bipartite. An *isomorphism* between two graphs (V, E) and (W, F) is a pair of bijections θ_1 between the vertex sets and θ_2 between the edge sets such that a vertex $x \in V$ is incident with an edge $e \in E$ if and only if $\theta_1(x) \in W$ is a vertex incident with the egde $\theta_2(e) \in F$. An *automorphism* of a graph is an isomorphism from the graph to itself.

Let x and y be (not necessarily distinct) vertices of a graph. A *path* of *length* $n \geq 1$ from x to y is an $(n+1)$-tuple $(x = x_0, x_1, \cdots, x_{n-1}, x_n = y)$ of vertices such that $x_i \sim x_{i+1}$ for all $i \in \{0, \cdots, n-1\}$ and $x_i \neq x_{i+2}$ for all $i \in \{0, \cdots, n-2\}$ (if $n \geq 2$). A *circuit* is a path of length at least one (hence at least three) from a vertex to itself. For completeness, a path of length zero from a vertex to itself is defined as just that vertex. The *girth* of a graph is the length of the shortest circuit. A graph is *connected* if there exists a path between any two distinct vertices, otherwise it is called *disconnected*. The *distance* between two distinct vertices is the length of the shortest path between them. The *diameter* of a connected graph is the maximal distance between two of its vertices. In a graph Γ with diameter d, the set of vertices at distance i from a vertex x is written as $\Gamma_i(x)$, for $i \in \{1, \cdots, d\}$. Clearly $\Gamma_1(x)$ is the set of vertices adjacent to x. We remark that $\Gamma_1(x)$ is often denoted by $\Gamma(x)$. A graph is *regular* of *degree* or *valency* k $(k > 0)$, or *k-regular*, if each vertex is adjacent to k vertices.

A convenient way to represent graphs is by their $(0, 1)$ *adjacency matrix*. This is the symmetric matrix A of which the rows and corresponding columns are indexed by the vertices of the graph; the entry a_{ij} is 1 if $i \sim j$ and 0 otherwise.

1.4.2 Moore graphs

A *Moore graph* (see for instance [7, Section 6.7]) is a regular graph of valency k and diameter d such that

$$v = 1 + k + k(k-1) + \cdots + k(k-1)^{d-1}.$$

It can be shown that girth g of a Moore graph is odd and satisfies $g = 2d+1$. The reason why the Moore graphs are interesting is that the number of vertices of a Moore graph is the maximal possible value for a regular graph with valency k and diameter d. In [60] Singleton proved that a connected graph with diameter d and girth $2d+1$ is necessarily regular and moreover is a Moore graph. In [24] it is shown that a Moore graph with valency $k=2$ is a polygon and a Moore graph with valency $k \geq 3$ has diameter 2 and $k \in \{3, 7, 57\}$. In the last case these graphs are the graphs with girth 5 (which means that they have no 3-cycles, no 4-cycles, but they do have 5-cycles) and with k^2+1 vertices. A Moore graph with $k=57$ is not known to exist, while there is a unique Moore graph for $k=3$ and $k=7$, see Section 1.7.7.

1.4.3 Distance-regular graphs

A connected graph Γ with diameter $d \geq 2$ is *distance-regular* if there exist integers b_i, $i \in \{0, \cdots, d-1\}$, and c_i, $i \in \{1, \cdots, d\}$, such that the following conditions are satisfied.

drg1 Γ is regular with valency b_0.

drg2 For any two vertices x and y at distance $i \in \{1, \cdots, d-1\}$ in Γ the vertex y is adjacent to precisely c_i vertices in $\Gamma_{i-1}(x)$ and to precisely b_i vertices in $\Gamma_{i+1}(x)$.

drg3 For any two vertices x and y at distance d the vertex y is adjacent to precisely c_d vertices in $\Gamma_{d-1}(x)$.

The sequence $\{b_0, \cdots, b_{d-1}; c_1, \cdots, c_d\}$ is known as the *intersection array* of Γ, and its elements are called the *intersection numbers*. Clearly $c_1 = 1$ holds. We note that the numbers $a_i = b_0 - b_i - c_i$ $(1 \leq i \leq d-1)$ are the numbers of vertices in $\Gamma_i(x)$ adjacent to a vertex $y \in \Gamma_i(x)$. In the same way $a_d = b_0 - c_d$ is the number of vertices in $\Gamma_d(x)$ adjacent to a vertex $y \in \Gamma_d(x)$.

For later purposes, we mention the definition of an antipodal r-cover graph. First of all let Γ be a distance-regular graph and let Γ_i be the graph, also called the *distance-i graph*, with vertex set the vertex set of Γ and with edges the pairs of vertices which are at distance i in Γ for $i = 1, \cdots, d$. A graph Γ is *antipodal* if the distance-d graph Γ_d is a disjoint union of cliques. In this case, we define a new graph $\overline{\Gamma}$ whose vertices are the maximal cliques of Γ_d, and two vertices are adjacent if their union contains an edge of Γ. If each vertex $\gamma \in \Gamma$ has the same valency as the vertex of $\overline{\Gamma}$ which is the maximal clique containing γ, then Γ is called an *antipodal covering graph* of $\overline{\Gamma}$. If all maximal cliques of Γ_d have the same size r then Γ is an *antipodal r-cover* of $\overline{\Gamma}$.

The standard reference on distance-regular graphs is [7].

Some examples of distance-regular graphs

1. Let n and e be integers with $n \geq 4$ and $2 \leq e \leq n-2$, and let X be a set with n elements. The vertices of the *Johnson graph* $J(n,e)$ (see [7, Section 9.1]) are the e-subsets of X, and two vertices are adjacent if and only if they intersect in an $(e-1)$-set. The graph $J(n,e)$ has diameter $d := \min\{e, n-e\}$, is distance-regular with intersection numbers $b_i = (e-i)(n-e-i)$ (for $0 \leq i \leq d-1$) and $c_i = i^2$ for $(1 \leq i \leq d)$.

2. Let q be any prime power, and let $V(n,q)$ denote the n-dimensional vector space over $\mathrm{GF}(q)$, with $n \geq 4$. Choose an integer e such that $2 \leq e \leq n-2$. The vertices of the *Grassmann graph* $G(n,e,q)$ (see [7, Section 9.3]) are the e-dimensional subspaces of $V(n,q)$, and two vertices are adjacent if and only if they intersect in an $(e-1)$-dimensional subspace. The diameter of $G(n,e,q)$ is $d := \min\{e, n-e\}$. The graph $G(n,e,q)$ is a distance-regular graph with intersection numbers

$$b_i = q^{2i+1} \begin{bmatrix} e-i \\ 1 \end{bmatrix}_q \begin{bmatrix} n-e-i \\ 1 \end{bmatrix}_q \quad (0 \leq i \leq d-1),$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q^2 \quad (1 \leq i \leq d),$$

where the number

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{\prod_{i=0}^{m-1}(q^{n-i}-1)}{\prod_{i=1}^{m}(q^i-1)}$$

is called the *q-ary Gaussian binomial coefficient*. We note that the q-ary Gaussian binomial coefficient is precisely the number of m-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$.

1.4.4 Strongly regular graphs

A distance-regular graph of diameter two is called a *strongly regular graph* and it is denoted by $\mathrm{srg}(v,k,\lambda,\mu)$. Here v is the number of vertices, $k = b_0$, $\lambda = b_0 - b_1 - 1 = a_1$ and $\mu = c_2$. The complement of an $\mathrm{srg}(v,k,\lambda,\mu)$ is an $\mathrm{srg}(v, v-k-1, v-2k+\mu-2, v-2k-\lambda)$. As the strongly regular graph and its complement should be connected, we have to assume $0 < \mu < k < v-1$. The *Bose-Mesner algebra* of a strongly regular graph Γ is the 3-dimensional algebra generated by I, J (J being the $v \times v$-matrix with all of its entries equal to 1) and its adjacency matrix A.

In the following theorem, the most important necessary conditions for the existence of a strongly regular graph are summarized. For the proofs and for more information on strongly regular graphs we refer to [7, 45].

Theorem 1.1 *If* Γ *is a* $\mathrm{srg}(v, k, \lambda, \mu)$*, then the following holds.*

1. $k(k - \lambda - 1) = \mu(v - k - 1)$.
2. *If* A *is the adjacency matrix of* Γ*, then* $AJ = kJ$*,*

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

and A *has three eigenvalues* k*,* r *and* l *such that*

$$r, l = \frac{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} \qquad r > l,$$

with multiplicities respectively

$$1, \quad f = \frac{-k(l+1)(k-l)}{(k+rl)(r-l)}, \quad g = \frac{k(r+1)(k-r)}{(k+rl)(r-l)}.$$

Clearly, f *and* g *must be integers, yielding an extra condition on the parameters. The eigenvalues* r *and* l *are sometimes called, the* restricted eigenvalues*.*

3. *The eigenvalues* $r > 0$ *and* $l < 0$ *are both integers, except for one family of graphs, the so-called conference graphs. A conference graph is an* $\mathrm{srg}(2k+1, k, \frac{k}{2} - 1, \frac{k}{2})$*. Hence the eigenvalues are* $r = \frac{-1+\sqrt{v}}{2}$ *and* $l = \frac{-1-\sqrt{v}}{2}$*. By a theorem of Seidel and van Lint [73] the number* $2k+1$ *of vertices can be written as a sum of two squares.*
4. *The two Krein conditions.*

$$\begin{aligned}(r+1)(k+r+2rl) &\leq (k+r)(l+1)^2;\\ (l+1)(k+l+2rl) &\leq (k+l)(r+1)^2.\end{aligned}$$

5. *The two absolute bounds.*
   - $v \leq \frac{1}{2}f(f+3)$*; if there is no equality in the first Krein condition then* $v \leq \frac{1}{2}f(f+1)$*;*
   - $v \leq \frac{1}{2}g(g+3)$*; if there is no equality in the second Krein condition then* $v \leq \frac{1}{2}g(g+1)$*.*

6. *The claw bound. If* $\mu \neq l^2$ *and* $\mu \neq l(l+1)$*, then the following inequality holds:* $2(r+1) \leq l(l+1)(\mu+1)$.

7. *The Hoffman bound.*

   - *If* C *is a clique of* Γ*, then* $|C| \leq 1 - \frac{k}{l}$*, with equality if and only if every vertex* $x \notin C$ *has the same number of neighbours in* C *(this number is then* $-\frac{\mu}{l}$*).*
   - *If* C *is a coclique of* Γ*, then* $|C| \leq v(1-\frac{k}{l})^{-1}$*, with equality if and only if every vertex* $x \notin C$ *has the same number of neighbours in* C *(this number is then* $-l$*).*

Some examples of strongly regular graphs

1. The Johnson graph of diameter 2, also called the *triangular graph* $T(n)$ $(n \geq 4)$, is an $\mathrm{srg}\big(\frac{1}{2}n(n-1), 2(n-2), n-2, 4\big)$. It is also the line graph of the complete graph K_n. The complement $\overline{T(5)}$ of the triangular graph $T(5)$ which will be denoted by Pe(10), is better known as the *Petersen graph*. This graph is the unique srg(10, 3, 0, 1). It can also be constructed by taking the 10 points of a Desargues configuration as vertex set, and two vertices being adjacent if they are not on a line of the Desargues configuration. The Petersen graph is the Moore graph with $k = 3$.

2. Consider the 3-dimensional projective space PG(3, 2). It can be shown that there exists a bijection between the 35 lines of PG(3, 2) and the 35 unordered triples in a 7-set such that lines intersect if and only if the corresponding triples have exactly one element in common. By relying on this property, the following strongly regular graph can be constructed. The 15 points of PG(3, 2), together with the 35 lines of PG(3, 2) form the vertex set of the *Hoffman-Singleton graph*, which is denoted by HoS(50). Points are never adjacent, two lines are adjacent whenever the corresponding triples are disjoint, and a point is adjacent with a line whenever they are incident. It is possible to prove that HoS(50) is a srg(50, 7, 0, 1) (see [43]). This graph is uniquely defined by its parameters and it does not belong to a known infinite family. We remark that the Hoffman-Singleton graph is the Moore graph with $k = 7$. A lot of other constructions of this graph are known.

1.4.5 Latin square graphs $L_m(n)$, $(m \geq 3)$

It is the aim of this subsection to construct the lattice graph which will appear further in this thesis. In order to illustrate it, we first define an orthogonal scheme.

Orthogonal schemes

A *latin square* L *of order* m is a $m \times m$ square of the set $X = \{1, \cdots, m\}$ such that on each row and each column, each element of X occurs exactly once. Two latin squares L_1 and L_2 on the same set X are called *orthogonal* if and only if $(L_1 \times L_2)_{ij}$ are all different.

Let $\mathcal{M}$ be a set of $m-2$ mutually orthogonal latin squares of order n, with elements $1, 2, \cdots, n$. Let $\mathcal{O}_m(n)$ be the scheme with m rows and n^2 columns, constructed as follows. On the first row, we write one after each other, n times 1, n times 2, $\cdots$, n times n. On the second row we write one after each other, $1, 2, \cdots, n$, and we repeat this n times. If a^k_{ij} is the element on the i-th row and j-th column of the latin square L_k $(1 \leq k \leq m-2)$, then a^k_{ij} is written on the $(k+2)$-nd row and on the column with first element i and second element j. In other words, the elements of L_k are written on the $(k+2)$nd-row, in the given order. This scheme $\mathcal{O}_m(n)$ has the property that for each two rows a and b, any ordered pair (i, j) occurs exactly once (i on the row a and j on the row b). This is a direct consequence of the fact that the latin squares are mutually orthogonal. A scheme with this property is called an *othogonal scheme with* n^2 *columns and* m *rows.* Conversely, it is easy to check that the existence of an orthogonal scheme with n^2 elements and m rows implies the existence of at least $m-2$ mutually latin squares of order n.

Description of the Latin square graph $L_m(n)$

The graph $L_m(n)$ is defined as follows. The vertices of the graph are the n^2 columns of $\mathcal{O}_m(n)$. Two vertices j and l are adjacent if and only if there exists a row i such that $a_{il} = a_{ij}$. As $\mathcal{O}_m(n)$ is an orthogonal scheme, there exists at most one row i such that $a_{il} = a_{ij}$.

Theorem 1.2 *The graph $L_m(n)$, $m \geq 3$, is a strongly regular graph with parameters $v = n^2$, $k = m(n-1)$, $\lambda = n-2+(m-1)(m-2)$, $\mu = m(m-1)$.*

We remark that a complete set of mutually orthogonal squares of order n has $n-1$ elements.

Theorem 1.3 ([9]) *A graph with the parameters of $L_m(n)$ (i.e. a pseudo-latin square graph) is a latin square graph if $n > \frac{1}{2}(m-1)(m^3-m^2+m+2)$.*

We note that this theorem holds also for $m = 2$.

1.4.6 Graphs associated with partial linear spaces

Let $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. The *point graph* (respectively *block graph*) of $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ has $\mathcal{P}$ (respectively $\mathcal{L}$) as a vertex set, and adjacency is collinearity (respectively concurrency). We use "block graph" instead of "line graph" to avoid confusion with the notion of the line graph of a graph (see Subsection 1.4.1). The adjacency matrix of a partial linear space is by definition the adjacency matrix of the point graph of this geometry.

For later purposes, we mention the concepts of *point diameter*, *line diameter* and *point-line diameter* of the incidence graph of a partial linear space (see also [11]). They indicate the maximal possible distance between two vertices corresponding to two points, two lines and a point and a line, respectively. Clearly point and line diameters are even, while point-line diameters are odd. The difference between the point or line diameter and the point-line diameter of the incidence graph of a partial linear space is 1; the point and line diameter differ by two or are equal.

Let N be the incidence matrix of a partial linear space having order (s, t). Then the adjacency matrix of its point graph is $A = NN^T - (t+1)I$ and the adjacency matrix of the block graph is $C = N^TN - (s+1)I$. As NN^T and N^TN have the same non-zero eigenvalues, it is possible to determine the eigenvalues of A from those of C or conversely.

1.5 Classical generalized hexagons

In this section, generalized hexagons are introduced. The reason why we focus on generalized hexagons and their ovoids and spreads, is that they will appear in Section 2.1.4.

1.5.1 Definitions

An *ordinary k-gon*, $k \geq 3$, is an incidence structure of vertices and edges of a graph consisting of one circuit of length k. An *ordinary 2-gon* is an incidence structure with two points and two blocks in which each point is incident with each block.

A *generalized hexagon* is an incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines such that the following axioms are satisfied:

GH1 Γ contains no ordinary k-gon for $2 \leq k \leq 5$.

GH2 Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary hexagon in Γ.

GH3 There exits an ordinary heptagon in Γ.

If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ only satisfies the conditions **GH1** and **GH2**, then it is called a *weak generalized hexagon*. A *thick line* of a weak generalized hexagon is a line of Γ which contains at least three points of Γ.

In [74, Corollary 1.5.3] it is shown that in a generalized hexagon, both the number of points incident with a line and the number of lines incident with a point are independent of the choice of a line, respectively point. For a finite generalized hexagon Γ, these numbers are constants, which are denoted by $s+1$, respectively $t+1$, and we say that Γ has *order* (s, t). The *dual* of a generalized hexagon of order (s, t) is again a generalized hexagon of order (t, s).

Up to duality, only two classes of examples of generalized hexagons of order (s, t), $s \neq 1 \neq t$, are known: the *split Cayley hexagon* $H(q)$, which has order (q, q), and the *twisted triality hexagon* $H(q^3, q)$, which has order (q^3, q). In these examples, q is a prime power. Both of them are called *classical* because they live on quadrics (classical objects) in projective spaces. They were introduced by Tits in [69].

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized hexagon and let $x, y \in \mathcal{P} \cup \mathcal{L}$. We remark that the maximal distance between two elements is 6 and two elements at distance 6 are said to be *opposite*.

Tits' description of the split Cayley hexagon $H(q)$

The following description of the split Cayley hexagon $H(q)$ was obtained for an arbitrary field, but here we consider the finite field $\mathrm{GF}(q)$. Let $Q(6, q)$ be the non-singular parabolic quadric in $\mathrm{PG}(6, q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The points of $H(q)$ are the points of $Q(6, q)$ and the lines of $H(q)$ are exactly the lines of $H(q)$ whose Grassmann coordinates (see [41, Chapter 24]) satisfy the following six linear equations:

$$p_{12} = p_{34},\ p_{54} = p_{32},\ p_{20} = p_{35},\ p_{65} = p_{30},\ p_{01} = p_{36},\ p_{46} = p_{31}.$$

This gives a complete description of $H(q)$, embedded in the quadric $Q(6, q)$. This embedding is also called the *standard embedding of* $H(q)$. In its standard embedding, $H(q)$ has the property that for a given point p, the points which are collinear with p in $H(q)$ are exactly the points of a fixed plane through

p lying on the quadric. We note that the split Cayley hexagon $H(q)$ is self-dual if and only if $q = 3^h$, $h \in \mathbb{N} \setminus \{0\}$, and admits a polarity if and only if $q = 3^{2h+1}$, $h \in \mathbb{N}$.

1.5.2 Ovoids

An *ovoid* of a generalized hexagon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a set $\mathcal{O}$ of mutually opposite points such that each point of Γ not in $\mathcal{O}$ is collinear with a unique point of $\mathcal{O}$. A *spread* of a generalized hexagon is the dual of an ovoid. In [52] it is proved that a polarity of a generalized polygon yields an ovoid and a spread.

Theorem 1.4 *The set of absolute points (lines) with respect to a polarity of a generalized n-gon, n even and $n \geq 4$, is an ovoid (a spread).*

In particular an ovoid of the split Cayley hexagon $H(q)$ is a set of $q^3 + 1$ points of $H(q)$ which are pairwise opposite in $H(q)$. Dually a spread of $H(q)$ is a set of $q^3 + 1$ pairwise opposite lines of $H(q)$.

For any prime power q the generalized hexagon $H(q)$ contains a particular spread which is constructed as follows in [66]. Let Π be a hyperplane of $\mathrm{PG}(6, q)$ which intersects the quadric $Q(6, q)$ underlying $H(q)$ in a $Q^-(5, q)$, and let $\mathcal{S}_H$ be the set of lines of $H(q)$ which are contained in Π. One can show that $\mathcal{S}_H$ is a spread, which is called a *hermitian spread* of $H(q)$. Another geometric construction of $\mathcal{S}_H$ can be found in [74]; in [17] it is described group-theoretically. The *hermitian ovoid* arises by dualising the hermitian spread $\mathcal{S}_H$ of $H(q)$, q a power of 3.

We note that the split Cayley hexagon $H(q)$ admits a polarity if and only if $q = 3^{2h+1}$, $h \in \mathbb{N}$; this polarity is unique up to conjugacy with respect to the automorphism group of $H(q)$. By Theorem 1.4 it yields an ovoid $\mathcal{O}_R$ and a spread $\mathcal{S}_R$ of $H(q)$ which are called the *Ree–Tits ovoid* and the *Ree–Tits spread*, respectively. The points of the Ree–Tits ovoid have in $\mathrm{PG}(6, q)$ coordinates $(1, 0, 0, 0, 0, 0, 0)$ and:

$$(-a^{4+2s} - aa'^s - a^{s+1}a''^s + a'^2 + a''^{s+1} - a^{3+s}a'' + aa'a'', -a'', -a, -a' + aa'', 1,$$

$$a''^s - a^{3+s} + 2aa' - a^2a'', -a^{3+2s} - a'^s - a^s a''^s + a'a''),$$

where $s = 3^{h+1}$, and $a, a', a'' \in \mathrm{GF}(q)$.

1.6 Linear representation of a set $\mathcal{K}$ of PG(n,q)

Let $\mathcal{K}$ be a set of points in PG(n,q), and embed PG(n,q) as a hyperplane Π_∞ in a projective space PG$(n+1,q)$. Define an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ of points and lines in the following way. The point set $\mathcal{P}$ is the set of all points of PG$(n+1,q)$ not in Π_∞. The line set $\mathcal{L}$ consists of all lines of PG$(n+1,q)$ which intersect $\mathcal{K}$ in a point, but are not contained in Π_∞. Incidence is the natural incidence of PG$(n+1,q)$. This incidence structure is called a *linear representation* of $\mathcal{K}$ and is denoted by $T_n^*(\mathcal{K})$. We will give below some examples of special sets $\mathcal{K}$ which will yield geometries $T_n^*(\mathcal{K})$ with special properties, for instance geometries $T_n^*(\mathcal{K})$ with a strongly regular point graph.

1.7 (α,β)-geometries

1.7.1 Definitions

An (α,β)-*geometry of order* (s,t) is a connected incidence structure $\mathcal{S}$ satisfying the following conditions.

1. $\mathcal{S}$ is a partial linear space of *order* (s,t).

2. Each antiflag of $\mathcal{S}$ has an incidence number either α or β.

Although the concept of an (α,β)-geometry was commonly known for special values of α and β, the general definition appeared for the first time in [29].

If $\alpha\neq\beta$, then we will always assume that $\alpha<\beta$, unless in the case $\alpha=0$, where we will often speak of $(0,\alpha)$-geometries instead of $(0,\beta)$-geometries. An (α,β)-geometry is *strongly regular* if its point graph is a strongly regular graph (see [37]). We will give more information on strongly regular (α,β)-geometries in Section 1.10

Although the terminology of (α,β)-geometries has not been used until recently, several particular (α,β)-geometries have been the subject of a great deal of research in the past.

1.7.2 Buekenhout-Shult geometries

A *Buekenhout-Shult geometry* is defined as an $(1,s+1)$-geometry. In [14] Buekenhout and Shult proved that an $(1,s+1)$-geometry is essentially a polar space.

1.7.3 Copolar spaces

J. I. Hall investigated the $(0, s)$-geometries, which are called *copolar spaces* [36]. He was able to classify the so called *reduced $(0, s)$-geometries*; these are $(0, s)$-geometries $\mathcal{S}$ such that for every two distinct points x and y of $\mathcal{S}$, $\Gamma(x) \neq \Gamma(y)$, where $\Gamma(\mathcal{S})$ is the point graph of $\mathcal{S}$. Copolar spaces are related to another special class of (α, β)-geometries, namely the semipartial geometries. In Section 1.7.7, we will define semipartial geometries and will say more about the relation between them and copolar spaces.

1.7.4 Generalized quadrangles

Another special class of (α, β)-geometries that was previously studied is the class of generalized quadrangles introduced by Tits [69]. A *generalized quadrangle* is an (α, β)-geometry such that $\alpha = \beta = 1$. On generalized quadrangles there is an extensive literature. In particular we refer here to "*Finite generalized quadrangles*" by Payne and Thas [53], which is a standard work about this topic and in which other useful references can be found.

Some examples of generalized quadrangles

The classical generalized quadrangles are the classical polar spaces of rank 2. In particular, they arise from quadrics, hermitian varieties and symplectic polarities. The points and lines on a quadric $Q^+(3, q)$, $Q(4, q)$ or $Q^-(5, q)$, q a prime power, form a generalized quadrangle of order $(q, 1)$, q and (q, q^2), respectively. In the theory of the generalized quadrangles one usually omits the upper index in $Q^+(3, q)$ and $Q^-(5, q)$ and hence uses the notation $Q(3, q)$ and $Q(5, q)$. The points and the lines on a hermitian variety $H(3, q^2)$ or $H(4, q^2)$, q a prime power, form a generalized quadrangle of order (q^2, q), respectively (q^2, q^3). Finally the points of $\mathrm{PG}(3, q)$, q a prime power, and the lines which are totally isotropic with respect to a symplectic polarity of $\mathrm{PG}(3, q)$ form a generalized quadrangle of order q which is denoted by $W(3, q)$ or $W(q)$. There exist isomorphisms between certain (dual) classical generalized quadrangles: $Q(4, q)$ is isomorphic to the dual of $W(q)$, $Q(5, q)$ is isomorphic to the dual of $H(3, q^2)$, $W(q)$ is self dual if and only if q is even. For the proofs of these well-known facts we refer to [53].
We proceed with another example of generalized quadrangles. A *hyperoval* of a projective plane $\mathrm{PG}(2, q)$, q even, is a set of $q + 2$ points no three collinear. Consider now a hyperoval $\mathcal{O}$ in $\mathrm{PG}(2, q)$, q even, embed $\mathrm{PG}(2, q)$ as a hyperplane π_∞ in a projective space $\mathrm{PG}(3, q)$. Then the linear representation $T_2^*(\mathcal{O})$ of $\mathcal{O}$ is a generalized quadrangle of order $(q-1, q+1)$. We observe that

$T_2^*(\mathcal{O})$ is an example of a non-classical generalized quadrangle, for $q > 3$. For more information, we refer to [53]. A generalized quadrangle of order $(2, t)$, has $t = 1, 2$ or 4. Generalized quadrangles with $t = 1$ are necessarily grids, hence we do not mention them anymore. There is a unique generalized quadrangle of order 2, namely $W(2)$ and there is a unique generalized quadrangle of order $(2, 4)$, namely $Q(5, 2)$. A generalized quadrangle of order $(3, t)$ has $t \in \{1, 3, 5, 9\}$. A generalized quadrangle of order 3 is isomorphic to either $W(3)$ or its dual $Q(4, 3)$. A generalized quadrangle of order $(3, 5)$ is isomorphic to $T_2^*(\mathcal{O})$, $q = 4$. A generalized quadrangle of order (3,9) is isomorphic to $Q(5, 3)$. A generalized quadrangle of order $(4, t)$ has necessarily $t \in \{1, 2, 4, 6, 8, 11, 12, 16\}$. A generalized quadrangle of order 4 is isomorphic to $W(4)$.

1.7.5 Partial quadrangles

A generalisation of the class of the generalized quadrangles is the class of the partial quadrangles. A *partial quadrangle* is a strongly regular $(0, 1)$-geometry (see [15]).
An *ovoid* of the 3-dimensional space $\mathrm{PG}(3, q)$, $q > 2$, is a set of $q^2 + 1$ points no three collinear. If $q = 2$, an ovoid of $\mathrm{PG}(3, 2)$ is a set of 5 points no 4 of which are coplanar. Consider now an ovoid $\mathcal{O}$ in $\mathrm{PG}(3, q)$ and embed $\mathrm{PG}(3, q)$ as a hyperplane Π_∞ in a projective space $\mathrm{PG}(4, q)$. Then the linear representation $T_3^*(\mathcal{O})$ of $\mathcal{O}$ is a partial quadrangle of order $(q-1, q^2)$ (see [15]).

1.7.6 Partial geometries

Partial geometries were introduced by Bose in the paper [6] as a generalisation of generalized quadrangles. A *partial geometry* with parameters s, t, α, which we denote by $\mathrm{pg}(s, t, \alpha)$, is an (α, β)-geometry $\mathcal{S}$ of order (s, t) such that $\alpha = \beta$ (> 0). Here $s, t \geq 1$ and $1 \leq \alpha \leq \min\{s + 1, t + 1\}$. The dual of a partial geometry $\mathrm{pg}(s, t, \alpha)$ is a partial geometry $\mathrm{pg}(t, s, \alpha)$. Partial geometries with $\alpha = s+1$ are $2-(v, s+1, 1)$ *designs*. (Dually, a $\mathrm{pg}(s, t, t + 1)$ is a dual design). If a partial geometry has $\alpha = t$, it is called a (*Bruck*) *net* of *order* $s + 1$ and *degree* $t + 1$ (see [9]). Its dual is called a *dual* (*Bruck*) *net*. For later purposes, we mention that one can show that the existence of a net of order $s + 1$ and degree $t + 1$ is equivalent to the existence of a set $\mathcal{M}$ of $t - 1$ mutually orthogonal latin squares of order $s + 1$, hence of an orthogonal scheme $\mathcal{O}_{t+1}(s + 1)$ (for the description see 1.4.5).

The point graph of a partial geometry $\mathrm{pg}(s, t, \alpha)$ with $\alpha \leq s$ is a strongly regular graph

$$\text{srg}((s+1)(st/\alpha+1),(t+1)s,s-1+t(\alpha-1),(t+1)\alpha)$$

with restricted eigenvalues $r = s-\alpha$ and $l = -t-1$. The requirement $\alpha \leq s$ is needed to exclude partial geometries in which any two points are collinear and which therefore have a complete point graph. We note that a line of a partial geometry, seen as the set of points incident with it, is a (maximal) clique of size $s+1$ in the point graph. A strongly regular graph which has the above parameters, for some $s,t,\alpha \in \mathbb{N}\setminus\{0\}$, $s,t \geq 1$, $1 \leq \alpha \leq \min\{s,t+1\}$, is called *pseudo-geometric*. If it is really the point graph of a partial geometry, then the graph is called *geometric*. One can show that the parameters of the (pseudo-) geometric graph uniquely determine the parameters of the (putative) partial geometry it supports.

Some examples of partial geometries

1. An important class of partial geometries is the class of generalized quadrangles (see Section 1.7.4).

2. A *maximal arc* $\mathcal{K}$ of a projective plane π is a non-empty set of k points in the plane such that any line of π intersects $\mathcal{K}$ in 0 or d points. The number d is called the *degree* of the maximal arc. It easily follows that $k = (q+1)(d-1)+1$ and either $d = q+1$ or $d \mid q$ (we refer to [12, Chapter 7] for more information on maximal arcs). A maximal arc $\mathcal{K}$ of degree d is called *non-trivial* if $1 < d < q$. In [1] Ball, Blokhuis and Mazzocca proved that there exists no non-trivial maximal arc in a desarguesian projective plane of order q if q is odd. For known examples and constructions of maximal arcs, we refer to Denniston [34], Thas [65], Mathon [51], and Hamilton and Mathon [38].
   Let $\mathcal{K}$ be a maximal arc of degree d in a projective plane π of order q. Define $\mathcal{P}$ to be the set of points of π which are not contained in $\mathcal{K}$, let $\mathcal{L}$ be the set of lines of π which are incident with d points of $\mathcal{K}$, and let I be the incidence of π. Then the incidence structure $\mathcal{S}(\mathcal{K}) := (\mathcal{P},\mathcal{L},\mathrm{I})$ is a partial geometry with parameters $t = q - q/d$, $s = q-d$, $\alpha = q - q/d - d + 1$. This infinite family was constructed by Thas [64, 65] and independently by Wallis [78].
   As there exist maximal arcs of degree $d = 2^m$, whenever $0 < m < h$, in (not necessarily desarguesian) projective planes of order 2^h, there exist partial geometries $\mathcal{S}(\mathcal{K})$ with parameters $s = 2^h - 2^m$, $t = 2^h - 2^{h-m}$, $\alpha = (2^m-1)(2^{h-m}-1)$.

3. Let $\mathcal{K}$ be a maximal arc of degree d in $\mathrm{PG}(2,q)$, q a prime power. Embed $\mathrm{PG}(2,q)$ as hyperplane π_∞ in $\mathrm{PG}(3,q)$. Then the linear representation $T_2^*(\mathcal{K})$ of $\mathcal{K}$ is a partial geometry with parameters $t=(q+1)(d-1)$, $s=q-1$, $\alpha=d-1$. This infinite family was constructed for the first time by Thas [64, 65].

4. Consider the n-dimensional projective space $\mathrm{PG}(n,q) := \Sigma$ and let H be a fixed $(n-2)$-dimensional projective subspace of Σ. Define $\mathcal{P}$ to be the set of all points of Σ not in H, and let $\mathcal{L}$ be the set of all lines of Σ, skew to H. Then $H_q^n := (\mathcal{P},\mathcal{L},\mathrm{I})$, where I is the incidence of $\mathrm{PG}(n,q)$, is a partial geometry with parameters $s=q$, $t=q^{n-1}-1$, $\alpha=q$. We note that H_q^n is a dual net.

5. Van Lint and Schrijver [72] constructed the following sporadic partial geometry. Let β be a primitive element of $\mathrm{GF}(3^4)$. Then $\gamma=\beta^{16}$ is a primitive 5-th root of the unity. Let $\mathcal{P}=\mathrm{GF}(3^4)$, let $\mathcal{L}$ be the set

$$\{b, 1+b, \gamma+b, \gamma^2+b, \gamma^3+b, \gamma^4+b | b \in \mathrm{GF}(3^4)\},$$

and let the incidence I be the natural one, namely inclusion. Then $\mathcal{S} := (\mathcal{P},\mathcal{L},\mathrm{I})$ is a partial geometry with parameters $s=t=5$ and $\alpha=2$.
Another construction of this geometry is given in [18]. Let C be the *ternary repetition code* of length 6, i.e.

$$C=\{(0,0,0,0,0,0),(1,1,1,1,1,1),(2,2,2,2,2,2)\}.$$

Any coset of C in $\mathrm{GF}(3)^6$ has a well-defined *type* i, that is the sum i modulo 3 of the coordinates of any vector in the coset. Let $\mathcal{A}_i$ be the set of cosets of type i. Define a tripartite graph Γ joining the coset $C+v$ to the coset $C+v+w$ for each vector w of weight 1. Any element in $\mathcal{A}_i$ has 6 neighbours in $\mathcal{A}_{i+1}$ and 6 in $\mathcal{A}_{i+2}$ (indices taken mod 3). Define $\mathcal{P}$ to be $\mathcal{A}_i$, the set of cosets of type i, and let $\mathcal{L}$ be the set $\mathcal{A}_{i+1}$ of cosets of type $i+1$. Then $\mathcal{S} := (\mathcal{P},\mathcal{L},\mathrm{I})$, where I is the incidence defined by the adjacency in Γ, is the partial geometry of van Lint-Schrijver.

6. Haemers [35] constructed another sporadic proper partial geometry. It has parameters $s=4$, $t=17$ and $\alpha=2$. The point graph Γ however was known before (see for instance [45]). This graph Γ is constructed as follows. The vertices of Γ are the 175 edges of the Hoffman-Singleton graph HoS(50). Two vertices of Γ are adjacent whenever the corresponding edges of HoS(50) have distance two in the line graph

(that is the two edges are disjoint and there exists an edge connecting both). One can prove that this graph is a srg$(175, 72, 20, 36)$; moreover it is a pseudo-geometric graph with parameters $s = 4$, $t = 17$, and $\alpha = 2$. Haemers proved that Γ is indeed geometric. First of all we remark that a line of the partial geometry will be a set of five disjoint edges pairwise at distance two in the Hoffman-Singleton graph HoS(50). It is easy to see that in a Petersen graph there are six such sets, that are 1-factors. If we can find 105 Petersen graphs in the Hoffman-Singleton graph HoS(50), then we have the right number of lines. However there are more than 105 Petersen graphs in HoS(50). Haemers was able to find a good subset of 105 special Petersen graphs in the Hoffman-Singleton graph HoS(50), such that every pentagon of HoS(50) is contained in exactly one such special Petersen graph. We note that any two edges at distance two in HoS(50) are in a unique pentagon, so in a unique special Petersen graph, and they define a unique 1-factor in this graph. As in every Petersen graph of HoS(50) there are six 1-factors, every special Petersen graph yields six lines of the geometry. In other words, the incidence structure of 175 vertices of Γ and the 630 1-factors of the special Petersen graphs of HoS(50) has the property that any two adjacent vertices define a unique line. This is enough to conclude that the pseudo-geometric graph Γ is indeed geometric.

For more on partial geometries, including a complete list of known examples, we refer to [25] and [29].

1.7.7 Semipartial geometries

Semipartial geometries were introduced by Debroey and Thas [33]. A *semipartial geometry* spg(s, t, α, μ) is a $(0, \alpha)$-geometry $\mathcal{S}$ of order (s, t) $(\alpha > 0)$ such that for any two non-collinear points of $\mathcal{S}$ there are μ points collinear with both. Here $s, t \geq 1$, $1 \leq \alpha \leq \min\{s + 1, t + 1\}$ and $1 \leq \mu \leq (t + 1)\alpha$. We note that $\mu = (t + 1)\alpha$ implies that $\mathcal{S}$ is a partial geometry. We see that the semipartial geometries generalise both the partial geometries and the partial quadrangles. A *proper* semipartial geometry is a semipartial geometry that is not a partial geometry. The dual of a proper semipartial geometry is again a semipartial geometry if and only if $s = t$, while the dual of a partial geometry is trivially always a partial geometry [33]. The point graph of a semipartial geometry spg(s, t, α, μ) with $\alpha \leq s$ is a strongly regular graph

$$\mathrm{srg}(1 + (t + 1)s(\mu + t(s + 1 - \alpha))/\mu, (t + 1)s, s - 1 + t(\alpha - 1), \mu)\,.$$

A strongly regular graph having such parameters, for some $s, t, \alpha, \mu \in \mathbb{N} \setminus \{0\}$ with $s, t \geq 1$, $1 \leq \alpha \leq \min\{s, t+1\}$, $1 \leq \mu < (t+1)\alpha$, is called *pseudo-semigeometric*; if it is the point graph of a semipartial geometry it is *semigeometric*. A line of a semipartial geometry, seen as the set of points incident with it, is a clique of size $s+1$ which is not necessarily of maximal size. Contrary to the case of partial geometries, the parameters of a (pseudo-) semigeometric graph do not determine the parameters of the (putative) semipartial geometry in general.
For more on semipartial geometries, we refer to [25] and [29].

Some examples of proper semipartial geometries

1. Let U be a set of cardinality n. Let $\mathcal{P}$ be the set of pairs, let $\mathcal{L}$ be the set of unordered triples of U, and let I be the inclusion relation. Then $U_{2,3}(n) := (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = n-3$, $\mu = 4$ [33]. The point graph of this geometry is the triangular graph $T(n)$.

2. Let $\mathcal{Q}$ be a (non-singular) quadric in $\mathrm{PG}(2n-1, 2)$. Let $\mathcal{P}$ be the set of points not on the quadric $\mathcal{Q}$, let $\mathcal{L}$ be the set of non-intersecting lines of $\mathcal{Q}$, and let I be the incidence of $\mathrm{PG}(2n-1, 2)$. Then $\mathcal{S} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = 2^{2n-3} - \varepsilon 2^{n-2} - 1$, $\mu = 2^{2n-3} - \varepsilon 2^{n-1}$, where $\varepsilon = +1$ for the hyperbolic quadric and $\varepsilon = -1$ for the elliptic quadric (we will denote this geometry by $NQ^+(2n-1, 2)$ and $NQ^-(2n-1, 2)$, respectively). This was remarked by H. Wilbrink [unpublished].

3. Consider the n-dimensional projective space $\mathrm{PG}(n, q) := \Sigma$ and H to be a fixed $(n-2)$-dimensional projective subspace of Σ. Define $\mathcal{P}$ to be the set of all lines of Σ, skew to H, and let $\mathcal{L}$ be the set of planes of Σ, intersecting H in exactly one point. Then $H_q^{n^*} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$, where I is the incidence of $\mathrm{PG}(n, q)$, is a semipartial geometry with parameters $s = q^2 - 1$, $t = (q^{n-1} - 1)/(q-1) - 1$, $\alpha = q$ and $\mu = q(q+1)$.

4. Let $\mathcal{S}$ be a generalized quadrangle embedded in a projective space $\mathrm{PG}(n, q)$, hence $\mathcal{S}$ is classical and $n = 3, 4$ or 5 [13]. Let p be a point of $\mathrm{PG}(n, q)$ and Π be a hyperplane of $\mathrm{PG}(n, q)$, not containing p. Let $\mathcal{P}_1$ be the projection of the point set of $\mathcal{S}$ from p onto Π and $\mathcal{P}_2$ be the set of points of Π on a tangent through p at $\mathcal{S}$. Let $\mathcal{P}_p = \mathcal{P}_1 \setminus \mathcal{P}_2$, let $\mathcal{L}_p$ consist of the lines of Π with q points in $\mathcal{P}_p$, and let the incidence I_p be the incidence inherited from the projective space. For the generalized quadrangles $\mathcal{S} = Q(5, q)$, embedded in $\mathrm{PG}(5, q)$ and $\mathcal{S} = H(4, q^2)$,

embedded in $\mathrm{PG}(4,q^2)$, the incidence structure $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ is a semipartial geometry.
If $\mathcal{S} = H(4,q^2)$ and p is a point on $H(4,q^2)$, then the semipartial geometry $\mathcal{S}_p$ is an $\mathrm{spg}(q^2-1, q^3, q, q^2(q^2-1))$; it is a $T_2^*(\mathcal{U})$ where $\mathcal{U}$ is a hermitian curve in $\mathrm{PG}(2,q^2)$. On the other hand if p is not on $H(4,q^2)$, then $\mathcal{S}_p$ is an $\mathrm{spg}(q^2-1, q^3, q+1, q(q+1)(q^2-1))$; this example is due to Thas [29].
If $\mathcal{S} = Q(5,q)$ and p is a point on the quadric, then $\mathcal{S}_p$ is a partial quadrangle $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an elliptic quadric in $\mathrm{PG}(3,q)$. However if p is not on the quadric $Q(5,q)$, then $\mathcal{S}_p$ is an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$; this construction is due to Hirschfeld and Thas [40]. Following De Clerck [25] this semipartial geometry is denoted by $\mathrm{TQ}(4,q)$.

5. Let Γ be a Moore graph. Define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines as follows. The point set $\mathcal{P}$ is the set of vertices of Γ, the line set $\mathcal{L}$ is the set $\{\Gamma(x) | x \in \mathcal{P}\}$, with $\Gamma(x)$ the set of vertices adjacent to x, and incidence I is the natural incidence relation. Then it is shown in [33] that $\mathcal{S}$ is a semipartial geometry with parameters $s = t = \alpha = k-1$, $\mu = (k-1)^2$, with $k \in \{3, 7, 57\}$. This geometry will be denoted by $\overline{M(k)}$.

6. A *Baer subspace* of an n-dimensional projective space of order q^2 is an n-dimensional subgeometry of order q. Consider the n-dimensional space $\Sigma_\infty = \mathrm{PG}(n,q^2)$ and let $\mathcal{B}$ be a Baer subspace of Σ_∞. Then $T_n^*(\mathcal{B})$ is an $\mathrm{spg}(q^2-1, (q^{n+1}-1)/(q-1)-1, q, q(q+1))$. We note that this geometry is isomorphic to $H_q^{(n+2)^*}$.

7. Consider the n-dimensional projective space $\mathrm{PG}(n,q)$, $n \geq 4$. Define $\mathcal{P}$ to be the set of all lines of $\mathrm{PG}(n,q)$, and let $\mathcal{L}$ be the set of all planes of $\mathrm{PG}(n,q)$. Then $\mathrm{LP}(n,q) := (\mathcal{P}, \mathcal{L}, \mathrm{I})$, where I is the incidence of $\mathrm{PG}(n,q)$, is an $\mathrm{spg}(q(q+1), (q^{n-1}-1)/(q-1)-1, q+1, (q+1)^2)$.

8. Let σ be a symplectic polarity of $\mathrm{PG}(2n+1,q)$, $n \geq 1$. Define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines as follows. The point set $\mathcal{P}$ is the set of points of $\mathrm{PG}(2n+1,q)$, the line set $\mathcal{L}$ is the set of lines which are not totally isotropic (that is *hyperbolic*) with respect to σ, and incidence I is the incidence relation of $\mathrm{PG}(2n+1,q)$. Then $\overline{W(2n+1,q)} := (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is an $\mathrm{spg}(q, q^{2n}-1, q, q^{2n}(q-1))$.

For more examples of semipartial geometries, we also refer to [25] and [29].

As has been mentioned in Section 1.7.3, semipartial geometries are related to copolar spaces. In the following theorem, the connection between these two geometries is given.

Theorem 1.5 ([36]) *If $\mathcal{S}$ is a finite reduced $(0, s)$-geometry, $s \geq 2$, then $\mathcal{S}$ is a dual net or it is isomorphic to one of the following semipartial geometries:*

1. $\overline{M(k)}$, $k \in \{2, 3, 7, 57\}$,
2. $U_{2,3}(n)$,
3. $\overline{W(2n+1, q)}$,
4. $NQ^{\pm}(2n-1, 2)$.

A characterisation of (semi)partial geometries

For later purposes, we mention that it is possible to give characterisation theorems for (semi)partial geometries (and also some other geometries) which satisfy the diagonal axiom. We start with the definition of the axiom of Pasch (also known as axiom of Veblen-Young). Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a (semi)partial geometry. We say that $\mathcal{S}$ satisfies the *axiom of Pasch* if for each two lines $L_1, L_2 \in \mathcal{L}$ which intersect in a point x and for any two lines $M_1, M_2 \in \mathcal{L}$ such that x is not incident with M_1, M_2, whenever M_1 and M_2 intersect both L_1 and L_2 they mutually intersect. The dual of the axiom of Pasch is called the *diagonal axiom*.

For a generalized quadrangle both the Pasch axiom and the diagonal axiom are satisfied in a trivial way. The only known partial geometry with $\alpha \notin \{1, s+1, t+1\}$ and satisfying the axiom of Pasch is the dual net H_q^n. In [68] it was proved that this dual net is the only dual net that satisfies the axiom of Pasch.

In [32] the semipartial geometries satisfying the axiom of Pasch or the diagonal axiom are studied. In this paper, Debroey proved the following characterisation theorems of the semipartial geometry $H_q^{n^*}$.

Theorem 1.6 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a (proper) semipartial geometry with parameters s, t, α $(t > \alpha > 1)$ and $\mu = \alpha(\alpha+1)$. If $\mathcal{S}$ satisfies the diagonal axiom, then $\mathcal{S}$ is isomorphic to $H_q^{n^*}$.*

Theorem 1.7 *Let $\mathcal{S}$ be a proper semipartial geometry with $\mu = \alpha^2$.*

1. *If $\alpha = t+1$, then $\mathcal{S} \cong U_{2,3}(n)$.*
2. *If $2 < \alpha = s$, then $\alpha = t \in \{1, 2, 6, 56\}$ and $\mathcal{S} \cong \overline{M(t+1)}$.*

3. If $2 < \alpha < s$, then $\mathcal{S} \cong \mathrm{LP}(n, q)$.

Remark

In [79] Wilbrink and Brouwer have shown that all proper semipartial geometries with $\mu = \alpha^2$ and $2 \leq \alpha < s$ satisfy the diagonal axiom. Moreover, they proved that up to possibly a finite number of exceptions, all (proper) semipartial geometries with $\mu = \alpha(\alpha + 1)$ satisfy the diagonal axiom.

1.8 Distance-regular geometries

As mentioned before, the point graph of a (semi)partial geometry is a strongly regular graph.

Let $\mathcal{S}$ be an incidence structure, let Φ be its incidence graph with point diameter $2d \geq 4$. $\mathcal{S}$ is called a *distance-regular geometry* with diameter d if the following axioms hold.

DRG1 $\mathcal{S}$ is a partial linear space of order (s, t).

DRG2 There exist integers α_{2i-1}, $1 \leq i \leq d$, such that for any point p and line L of $\mathcal{S}$ which are at distance $2i - 1$ in Φ there are precisely α_{2i-1} points incident with L and at distance $2i - 2$ in Φ from p.

DRG3 There exist integers t_{2i}, $1 \leq i \leq d$, such that for any two points p and q of $\mathcal{S}$ which are at distance $2i$ in Φ there are precisely $t_{2i} + 1$ lines incident with q and at distance $2i - 1$ in Φ from p.

Lemma 1.8 *The point graph Γ of a distance-regular geometry $\mathcal{S}$ is distance-regular with diameter d and intersection numbers*

$$\begin{aligned} b_0 &= (t+1)s, \\ b_i &= (t - t_{2i})(s + 1 - \alpha_{2i+1}), \ 1 \leq i \leq d-1, \\ c_i &= (t_{2i} + 1)\alpha_{2i-1}, \ 1 \leq i \leq d. \end{aligned} \tag{1.1}$$

Proof. We note that $\alpha_1 = 1$ and $t_2 = 0$. It is clear that the point graph Γ is regular with valency $(t + 1)s$ and has diameter d. Vertices x and y of Γ at distance i, $1 \leq i \leq d$, correspond to points of $\mathcal{S}$ which are at distance $2i$ in the incidence graph Φ. The axioms tell us that there are $(t_{2i} + 1)\alpha_{2i-1}$ vertices of Γ which are adjacent to y and at distance $i - 1$ in Γ from x. If $i \leq d - 1$ there are $(t - t_{2i})(s + 1 - \alpha_{2i+1})$ vertices of Γ adjacent to y and at distance $i + 1$ in Γ from x. We conclude that Γ is distance-regular with diameter d and intersection numbers given by the above formulae. □

An (α, β)-geometry $\mathcal{S}$ which is distance-regular with diameter $d \geq 3$ is necessarily a $(0, \alpha)$-geometry. In fact in this case it is always possible to find a point and a line which are at distance 5 in the incidence graph Γ. If a point p and a line L of $\mathcal{S}$ are at distance 5, then there is no line incident with p which is concurrent with L. This means that α or β has to be equal to 0. A distance-regular geometry with $d = 2$ and $t_4 = t$ is a partial geometry $\mathrm{pg}(s, t, \alpha_3)$; if $d = 2$ and $t_4 < t$, we have a semipartial geometry $\mathrm{spg}(s, t, \alpha_3, (t_4 + 1)\alpha_3)$. Distance-regular geometries with $\alpha_{2i-1} = 1$ for all $i \in \{1, \cdots, d\}$ are *regular near polygons*. Near polygons were introduced by Shult and Yanushka in [59]. We will call a distance-regular geometry *proper* if it is neither a (semi)partial geometry nor a regular near polygon.

In the most general case, a distance-regular geometry is described by $2d$ relevant parameters: s, t, α_{2i-1} for $i \in \{2, \cdots, d\}$, and $t_{2i} + 1 \in \{2, \cdots, d\}$. In (1.1), the equation for c_1 is always satisfied (being $\alpha_1 = 1$ and $t_2 = 0$); leaving us with only $2d - 1$ conditions. Hence the intersection numbers of a distance-regular graph Γ do not necessarily determine the parameters of a putative distance-regular geometry with Γ as a point graph.

In Chapter 3, we give some examples of distance-regular geometries.

1.9 SPG-reguli

SPG-reguli were introduced by Thas [67]. An *SPG-regulus* of $\mathrm{PG}(n, q)$ is a set $\mathcal{R}$ of m-dimensional subspaces $\pi_1, \pi_2 \cdots, \pi_r$, $r > 1$, of $\mathrm{PG}(n, q)$, satisfying:

SPG1 $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

SPG2 If $\mathrm{PG}(m + 1, q)$ contains $\pi_i \in \mathcal{R}$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $\mathcal{R} \setminus \{\pi_i\}$. If $\mathrm{PG}(m + 1, q)$ has no point in common with $\pi_i \in \mathcal{R}$ for all $i \neq j$, then it is called a *tangent* $(m + 1)$-space of $\mathcal{R}$ at π_i.

SPG3 If the point x of $\mathrm{PG}(n, q)$ is not contained in an element of $\mathcal{R}$, then it is contained in a constant number θ ($\theta \geq 0$) of tangent $(m + 1)$-spaces of $\mathcal{R}$.

The reason why SPG-reguli are interesting, is the fact that a semipartial geometry can be constructed from every SPG-regulus. The following construction is described in [67].

Let $\mathcal{R}$ be an SPG-regulus of $\mathrm{PG}(n, q)$, consisting of m-dimensional subspaces, and embed $\mathrm{PG}(n, q)$ as a hyperplane Π_∞ in a projective space $\mathrm{PG}(n + 1, q)$. Define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines in the following way. The point set $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n+1,q)$, not in Π_∞. The line set $\mathcal{L}$ consists of all $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ which contain an element of $\mathcal{R}$, but are not contained in Π_∞. Incidence is the natural incidence of $\mathrm{PG}(n+1,q)$. Then it is shown in [67] that $\mathcal{S}$ is a semipartial geometry with parameters $s = q^{m+1}-1$, $t = r-1$, $\alpha = \alpha$ and $\mu = (r-\theta)\alpha$.

The geometry $\mathcal{S}$ is also called the *generalized linear representation* of the SPG-regulus $\mathcal{R}$; we will sometimes denote this geometry by $T^*_{n,m}(\mathcal{K})$.

1.10 Strongly regular (α,β)-geometries

Strongly regular (α,β)-geometries were introduced by N. Hamilton and R. Mathon in [37]. An (α,β)-geometry is called *strongly regular* if its point graph is a strongly regular graph. Hence from the conditions of existence of a strongly regular graph, one can deduce similar conditions for existence of strongly regular (α,β)-geometries (see Theorem 1.1). Some of these are summarized in the next theorem.

Theorem 1.9 ([37]) *Let $\mathcal{S}$ be a strongly regular (α,β)-geometry with parameters (s,t). Then the following conditions hold.*

1. *The point graph has 3 eigenvalues, one of which is k, the other two $u_1 < u_2$ and their multiplicities f_1, f_2, satisfy*

$$u_i^2 + (\mu-\lambda)u_i + (\mu-k) = 0, f_1+f_2 = v-1, u_1f_1+u_2f_2 = -k$$

2. *For $v > 5$, one of the following occurs:*
   - *(a) $v = 2k+1$, $\lambda = k/2-1$ and $\mu = k/2$, where v is the sum of two squares and the eigenvalues u_1 and u_2 are $(1\pm\sqrt{v})/2$.*
   - *(b) u_1 and u_2 are integers with $u_1 < 0 < u_2$. Hence $(\mu-\lambda)^2+4(k-\mu)$ is a square and $(u_1-u_2) \mid (u_2(v-1)+k)$.*

3. *$\beta-\alpha \mid \beta(v-s-1)-st(s+1)$.*

4. *$v \le b$ and $s \le t$.*

5. *If $\beta = s+1$, then $(s+1-\alpha) \mid (t+1)(s-\alpha)(t-c)$, where c is a constant equal to $(st+s-\lambda-1)/(s+1-\alpha)$.*

1.10.1 Strongly regular (α, β)-reguli

Similarly as Thas defined SPG-reguli in order to construct semipartial geometries, Hamilton and Mathon [37] introduced the concept of strongly regular (α, β)-reguli which yield strongly regular (α, β)-geometries. A *strongly regular* (α, β)*-regulus* of $\mathrm{PG}(n, q)$ is a set $\mathcal{R}$ of m-dimensional subspaces $\pi_1, \pi_2 \cdots, \pi_r$, $r > 1$, of $\mathrm{PG}(n, q)$, satisfying:

1. $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

2. If $\mathrm{PG}(m+1, q)$ contains $\pi_i \in \mathcal{R}$, then it has a point in common with either α or β spaces in $\mathcal{R} \setminus \{\pi_i\}$. Such an $(m+1)$-dimensional subspace that meets α (respectively β) elements of $\mathcal{R} \setminus \{\pi_i\}$ is said to be an α-secant (respectively β-secant) to $\mathcal{R}$ at π_i.

3. If a point x of $\mathrm{PG}(n, q)$ is contained in an element π_i of $\mathcal{R}$, then it is contained in a constant number p of α-secant $(m+1)$-dimensional spaces on elements of $\mathcal{R} \setminus \{\pi_i\}$.

4. If a point x of $\mathrm{PG}(n, q)$ is contained in no element of $\mathcal{R}$ then it is contained in a constant number $\bar{p}$ of α-secant $(m+1)$-dimensional spaces of $\mathcal{R}$.

We note that if $\alpha = 0$, then a strongly regular (α, β)-regulus is an SPG-regulus. A strongly regular (α, β)-geometry can be constructed from a strongly regular (α, β)-regulus in the same way as the construction of a semipartial geometry from an SPG-regulus. For the sake of completeness we recall this construction.

Let $\mathcal{R}$ be a strongly regular (α, β)-regulus of $\mathrm{PG}(n, q)$, consisting of m-dimensional subspaces, and embed $\mathrm{PG}(n, q)$ as a hyperplane Π_∞ in a projective space $\mathrm{PG}(n+1, q)$. Define an incidence structure $T^*_{n,m}(\mathcal{R}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines in the following way. The point set $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n+1, q)$, not in Π_∞. The line set $\mathcal{L}$ consists of all $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1, q)$ which contain an element of $\mathcal{R}$, but are not contained in Π_∞. Incidence is the natural incidence of $\mathrm{PG}(n+1, q)$. Then it is shown in [67] that $T^*_{n,m}(\mathcal{R})$ is a strongly regular (α, β)-geometry with parameters $s = q^{m+1} - 1$, $t = r - 1$.

1.11 Seidel switching and two-graphs

1.11.1 Seidel switching

Let $\Gamma = (V, E)$ be a graph. Let $\{V_1, V_2\}$ be a partition of the vertex set V of a graph Γ, and construct a graph Γ' from Γ by interchanging edges and non-edges between a vertex in V_1 and a vertex in V_2, while leaving (non-) edges inside V_1 or inside V_2 unchanged. This process, which is called *Seidel switching* with respect to the partition $\{V_1, V_2\}$ of V, also *switching*, was introduced by Seidel in [73]. Graphs which can be constructed from each other by switching are called *switching equivalent*; this is indeed an equivalence relation, and the equivalence classes are known as *switching classes*.

1.11.2 Two-graphs

A *two-graph* is a pair (V, Δ) consisting of a finite *vertex set* V and a set Δ of *coherent* (unordered) *triples* of vertices such that each set of four vertices contains an even number of coherent triples. Two-graphs were introduced by Higman; [57] and [62] are excellent surveys. A two-graph (V, Δ) without coherent triples is called *void*, and a two-graph in which every triple of vertices is coherent is *complete*; both are *trivial* two-graphs. The *complement* of a two-graph (V, Δ) is the two-graph on V in which the coherent triples are the *incoherent* triples of (V, Δ). The *sub-two-graph* of a two-graph (V, Δ) *induced on* a subset W of V is the two-graph (W, Θ), where Θ consists of all coherent triples of (V, Δ) which are contained in W. A set C of vertices in which each triple is (in)coherent is called a(n) *(in)coherent set* of the two-graph. An *isomorphism* between two-graphs (V, Δ) and (W, Θ) is a pair of bijections θ_1 between the vertex sets and θ_2 between the sets of coherent triples such that if $x \in \delta$, with $x \in V$ and $\delta \in \Delta$, then $\theta_1(x) \in \theta_2(\delta)$, with $\theta_1(x) \in W$ and $\theta_2(\delta) \in \Theta$ and also for inverse. An *automorphism* of a two-graph is an isomorphism from the two-graph to itself.

The following theorem explains the connection between two-graphs and switching classes of graphs.

Theorem 1.10 ([61]) *A switching class of graphs uniquely determines a two-graph on the same vertex set, and conversely a two-graph uniquely leads to a unique switching class of graphs on the same vertex set.*

A two-graph is said to be *regular* with parameters n and a if it has n vertices and each pair of vertices is contained in a coherent triples. Let (V, Δ) be a regular two-graph with parameters n and a, and choose a vertex ω. Define a graph on $V \setminus \{\omega\}$ as follows: two vertices x and y are adjacent if and only if $\{\omega, x, y\}$ is a coherent triple in (V, Δ). This graph is called the *descendant* of (V, Δ) with respect to ω, and it is an $\mathrm{srg}(n-1, a, (3a-n)/2, a/2)$. On the other hand, let Γ be a strongly regular graph $\mathrm{srg}(v, k, \lambda, k/2)$, add an isolated vertex, and define a triple of vertices to be coherent if and only if it has an odd number of edges. Then a regular two-graph with parameters $n = v + 1$ and $a = k$ is obtained. More information on two-graphs can be found in [57] and [62].

The reason why the regular two-graphs are introduced, is that distance-regular geometries can be constructed from certain regular two-graphs. The construction relies on properties concerning the relation between regular two-graphs and Taylor graphs. This will be investigated in Chapter 2.

Chapter 2

Distance-regular graphs and (α, β)-geometries

In this chapter, we will study the relation between distance-regular graphs and (α, β)-geometries in two different ways. We construct (α, β)-geometries from distance-regular graphs satisfying certain conditions on the intersection numbers and describe some (classes of) examples. On the other hand, properties of certain regular two-graphs allow us to construct $(0, \alpha)$-geometries on the corresponding Taylor graphs (which form a special class of distance-regular graphs). This will be the main result of this chapter; it appeared in [49].

2.1 Neighbourhood geometries of distance-regular graphs

2.1.1 Definitions

Let Γ be a graph, and define an incidence structure $\mathcal{S}(\Gamma)$ as follows. The points are the vertices of Γ, the lines are also the vertices of Γ (for convenience we put them between square brackets), and incidence is adjacency in Γ. In particular, if x is any vertex of Γ, the point x of $\mathcal{S}(\Gamma)$ is not incident with the line $[x]$. Following [50], we call $\mathcal{S}(\Gamma)$ the *neighbourhood geometry* of Γ. If Γ is not bipartite, then $\mathcal{S}(\Gamma)$ is uniquely defined, and is self-polar by the mapping $x \mapsto [x]$. If Γ is bipartite, then $\mathcal{S}(\Gamma)$ is disconnected with two components and up to a duality uniquely defined by Γ (see also Theorem 2.3).

In [33], it is shown that the neighbourhood geometry of a Moore graph , which is a $\mathrm{srg}(k^2+1, k, 0, 1)$, is an $\mathrm{spg}(k-1, k-1, k-1, (k-1)^2)$.

Actually, if the neighbourhood geometry of a strongly regular graph $\mathrm{srg}(v, k, \lambda, \mu)$ is a partial linear space, then necessarily $\lambda = 1$ and $\mu = 0$.

In this section we want to investigate under which conditions the neighbourhood geometry of a distance-regular graph with diameter $d \geq 3$ is an (α, β)-geometry.

In Section 1.7, we define (α, β)-geometries as connected geometries. It is of course possible to assume that (α, β)-geometries are not connected. In the next theorems the (α, β)-geometries mentioned, need not to be connected, see also Theorem 2.3.

Theorem 2.1 *Let Γ be a distance-regular graph with diameter 3, and let $\mathcal{S}(\Gamma)$ be its neighbourhood geometry. Then $\mathcal{S}(\Gamma)$ is a partial linear space if and only if $c_2 = 1$ and $b_1 \in \{k-1, k-2\}$. Moreover, it is an (α, β)-geometry if and only if the intersection array of Γ is of one of the following types.*

- $\{k, k-1, k-1; 1, 1, c_3\}$*; then* $\alpha = 0$, $\beta = c_3$.
- $\{k, k-1, k-1-c_3; 1, 1, c_3\}$*; then* $\alpha = 0$, $\beta = c_3$.
- $\{k, k-2, k-1-a_2; 1, 1, k\}$*; then* $\alpha = a_2 + 1$, $\beta = k$.
- $\{k, k-2, k-c_3; 1, 1, c_3\}$*; then* $\alpha = c_3$, $\beta = k$.

Proof. First we want to deduce conditions on the geometry $\mathcal{S}(\Gamma)$ such that it is a partial linear space. Let p and q be two distinct points on a line $[r]$. If p is at distance 2 from q in Γ, then r has to be the unique vertex which is adjacent to p and q; hence $c_2 = 1$. If p is at distance 1 from q in Γ, then the edge $\{p, q\}$ is in a unique triangle; hence $a_1 = 1$. If all collinear points correspond to vertices at distance 2 in the graph, then $a_1 = 0$. Conversely, the neighbourhood geometry of a distance-regular graph with $c_2 = 1$ and $a_1 \in \{0, 1\}$ (or equivalently $b_1 \in \{k-1, k-2\}$) is a partial linear space. Clearly $\mathcal{S}(\Gamma)$ has order $(k-1, k-1)$, where k is the valency of the graph Γ.

Now we want to determine the possible incidence numbers of antiflags of $\mathcal{S}(\Gamma)$. Consider an antiflag $\{p, [q]\}$. If $p = q$, then obviously there are no vertices that are adjacent to q and at distance 2 from p. If $a_1 = 0$, no vertex adjacent to p is in a triangle containing p; if $a_1 = 1$, all k vertices adjacent to p are in a unique triangle containing p. Hence the number of points incident with $[p]$ and collinear with p is 0 if $a_1 = 0$, and k if $a_1 = 1$. If p is at distance 2 from q, then there are a_2 vertices which are adjacent to q and at distance 2 from p. The point corresponding to the unique vertex r adjacent to p and q is collinear with p if and only if $a_1 = 1$; it is also incident with $[q]$. Consequently the incidence number of $\{p, [q]\}$ is a_2 if $a_1 = 0$, and

a_2+1 if $a_1 = 1$. Finally, if p is at distance 3 from q, then there are c_3 vertices adjacent to q and at distance 2 from p, and no vertices adjacent to p and q. It follows that the possible incidence numbers of antiflags of $\mathcal{S}(\Gamma)$ are 0, a_2 and c_3 if $a_1 = 0$, and k, a_2+1 and c_3 if $a_1 = 1$. We conclude that $\mathcal{S}(\Gamma)$ is an (α, β)-geometry if and only if two of these incidence numbers are equal. □

Theorem 2.2 *Let Γ be a distance-regular graph with diameter $d > 3$, and let $\mathcal{S}(\Gamma)$ be its neighbourhood geometry. Then $\mathcal{S}(\Gamma)$ is a (possibly disconnected) (α, β)-geometry if and only if the intersection array of Γ is of one of the following types.*

- $\{k, k-1, k-1, b_3, \cdots, b_{d-1}; 1, 1, c_3, \cdots, c_d\}$
- $\{k, k-1, k-1-c_3, b_3, \cdots, b_{d-1}; 1, 1, c_3, \cdots, c_d\}$

In both cases $\alpha = 0$ and $\beta = c_3$.

Proof. In the same way as in Theorem 2.1 we can prove that $\mathcal{S}(\Gamma)$ is a partial linear space of order $(k-1, k-1)$ if and only if $c_2 = 1$ and $a_1 \in \{0, 1\}$. In order to determine the possible incidence numbers of antiflags of $\mathcal{S}(\Gamma)$, we consider an antiflag $\{p, [q]\}$. Since $d > 3$ we may suppose that p is at distance at least 4 from q. Then there are no points adjacent with q and at distance at most 2 from p; hence $\{p, [q]\}$ has incidence number 0. Using the results obtained in Theorem 2.1 for the cases where p is at distance at most 3 from q, we conclude that the incidence numbers of antiflags of $\mathcal{S}(\Gamma)$ are 0, a_2 and c_3 if $a_1 = 0$, and k, a_2+1, c_3 and 0 if $a_1 = 1$. It follows that $\mathcal{S}(\Gamma)$ is an (α, β)-geometry if and only if two of these incidence numbers are equal. As $0 < c_3 < k$, the case $a_1 = 1$ cannot occur. □

Theorem 2.3 *Let Γ be a distance-regular graph having diameter at least three, and let $\mathcal{S}(\Gamma)$ be its neighbourhood geometry. Then $\mathcal{S}(\Gamma)$ is the disjoint union of two connected components if and only if Γ is bipartite.*

Proof. Suppose that the graph Γ is bipartite. Then the vertex set of Γ can be partitioned into two cocliques denoted by A and B. This means that for any flag $\{p, [q]\}$ of $\mathcal{S}(\Gamma)$, $p \in A$ and $q \in B$, or $p \in B$ and $q \in A$. As a consequence, $\mathcal{S}(\Gamma)$ is the disjoint union of two components which are mutually dual. Conversely suppose that the geometry $\mathcal{S}(\Gamma)$ is the disjoint union of two components $\mathcal{A}$ and $\mathcal{B}$. Let A_p, A_l, B_p and B_l denote the sets of vertices of Γ corresponding to the points of $\mathcal{A}$, the lines of $\mathcal{A}$, the points of $\mathcal{B}$ and the lines of $\mathcal{B}$, respectively. We note that there are no flags of $\mathcal{S}(\Gamma)$

containing an element of A_p and an element of B_l, or an element of B_p and an element of A_l. Suppose that $A_p \cap A_l \neq \emptyset$. Then there exists at least one vertex x which corresponds to a point x of $\mathcal{A}$ and to a line $[x]$ of $\mathcal{A}$. Each vertex y adjacent to x corresponds to a point y incident with $[x]$ and to a line $[y]$ incident with x, and hence is also contained in $A_p \cap A_l$. Since the graph Γ is connected, the set $A_p \cap A_l$ is the whole vertex set of Γ, a contradiction. This implies $A_p \cap A_l = \emptyset$. Using the same arguments we obtain $B_p \cap B_l = \emptyset$. Obviously $A_p \cap B_p = \emptyset = A_l \cap B_l$. Since $A_p \cup B_p = A_l \cup B_l$ is the whole vertex set of Γ, we obtain $A_p = B_l$ and $A_l = B_p$. As there are no edges inside $A_p = B_l$ and inside $A_l = B_p$, Γ is bipartite. □

Although the conditions in Theorems 2.1 and 2.2 may seem quite restrictive, there are actually some distance-regular graphs satisfying them, thus yielding (α, β)-geometries.

2.1.2 Odd graphs

Let X be a set having $2m + 1$ elements, with $m \geq 3$. The vertices of the *Odd graph* (see [7, Section 9.1]) are the m-subsets of X; two vertices are adjacent if and only if they are disjoint. The neighbourhood geometry of this graph can be described as follows: the points are the $(m + 1)$-subsets of X (or equivalently, the complements of the m-subsets), the lines are the m-subsets, and incidence is (reverse) containment. The point graph of this $(0, 2)$-geometry is the distance-regular *Johnson graph* $J(2m + 1, m + 1)$ (explained on page 8) which has diameter $m + 1$. The vertices are indeed m-subsets of X, and two vertices are adjacent if and only if they intersect in an $(m - 1)$-set.

In fact, every Johnson graph $J(n, e)$ supports a $(0, 2)$-geometry consisting of the e-subsets and the $(e-1)$-subsets of a set with n elements, see [48]. The *Grassmann graph* $G(n, e, q)$ (see Section 1.4.3 of Chapter 1) can be considered as a “thicker” versions of the Johnson graph $J(n, e)$; in [48] it is shown that it supports a distance-regular $(0, q + 1)$-geometry.

2.1.3 Graphs from an alternating form in $V(2, q)$

Let $V(2, q)$ denote the two-dimensional vector space over $\mathrm{GF}(q)$, q even, and suppose that f is a non-degenerate alternating form in $V(2, q)$. Choose an element $b \in \mathrm{GF}(q) \setminus \{0\}$, and define a graph as follows. The vertices are the non-zero vectors in $V(2, q)$, and two vertices u and v are adjacent if and only if $f(u, v) = b$. This graph is distance-regular with diameter 3 and intersection array $\{q, q - 2, 1; 1, 1, q\}$; the construction comes from [7, Section 12.5]. The vector space $V(2,q)$ can also be seen as an affine plane $\mathrm{AG}(2,q)$; suppose that the point $\infty \in \mathrm{AG}(2,q)$ corresponds to the zero vector. For any non-zero vector u, the set $\{v \in V(2,q) \mid f(u,v) = b\}$ corresponds to a line of $\mathrm{AG}(2,q)$ not containing ∞. One easily checks that distinct non-zero vectors give rise to distinct lines. This means that we can describe the neighbourhood geometry of this graph as follows: the points are the points of $\mathrm{AG}(2,q)$ different from ∞, the lines are the lines of $\mathrm{AG}(2,q)$ not through ∞, and incidence is natural. This is a $(q-1,q)$-geometry with $s = t = q-1$. We note that the construction in $\mathrm{AG}(2,q)$ also works for odd q. The point graph of the geometry is the complete multipartite graph on $q+1$ sets of $q-1$ vertices.

2.1.4 Some sporadic examples

- The *dodecahedron* is a distance-regular graph with diameter 5 and intersection array $\{3,2,1,1,1;1,1,1,2,3\}$. It satisfies the conditions in Theorem 2.2 and yields a $(0,1)$-geometry with $s = t = 2$.

- The generalized hexagon of order $(2,1)$ (see [74]) can be described as the dual of the double of the projective plane $\mathrm{PG}(2,2)$: its points are the flags of $\mathrm{PG}(2,2)$, its lines are the points and lines of $\mathrm{PG}(2,2)$, and incidence is (reverse) containment. The point graph of this structure is distance-regular with diameter 3 and intersection array $\{4,2,2;1,1,2\}$; it satisfies the conditions of Theorem 2.1 and yields a $(2,4)$-geometry with $s = t = 3$. To obtain a nice description, we note that the points and lines of $\mathrm{PG}(2,2)$ not incident with any element of a fixed flag form a quadrangle, which we call the complement of the flag. If $\{p,L\}$ and $\{p,M\}$ are flags which share the point p, then L is a diagonal in the complement of $\{p,M\}$. If $\{p,L\}$ and $\{q,L\}$ are flags which have the line L in common, then p is the intersection of two opposite edges of the complement of $\{q,L\}$. This yields the following description for the neighbourhood geometry: the points are the flags of $\mathrm{PG}(2,2)$, and the lines are the quadrangles of $\mathrm{PG}(2,2)$. A flag and a quadrangle are incident if they have no elements in common, and the point of the flag is on two lines of the quadrangle or the line of the flag contains two points of the quadrangle.

- The vertices of the *Coxeter graph* (see [21], [22] and [7, Section 12.3]) are the antiflags of $\mathrm{PG}(2,2)$; two antiflags are adjacent if together they cover all points of $\mathrm{PG}(2,2)$. This graph is distance-regular with diameter 4 and intersection array $\{3,2,2,1;1,1,1,2\}$. It satisfies the conditions of Theorem 2.2; we obtain a $(0,1)$-geometry with $s = t = 2$.

To find a nice description for this geometry, we define the complement of an antiflag as the set of points and lines which do not belong to the antiflag and are not incident with any of its elements. These points and lines form a triangle. If two antiflags $\{p, L\}$ and $\{q, M\}$ together cover all points of PG$(2, 2)$, then p (respectively L) is a point (respectively a line) of the triangle which is the complement of $\{q, M\}$. Hence we can describe the neighbourhood geometry as follows: the points are the antiflags of PG$(2, 2)$, the lines are the triangles of PG$(2, 2)$, and incidence is (reverse) containment.

- The *Biggs–Smith graph* is described in [4], [76] and also in [7, Section 13.4]. It is a distance-regular graph with diameter 7 and intersection array $\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$. As it satisfies the conditions of Theorem 2.2, it yields a $(0, 1)$-geometry with $s = t = 2$.

- The *Perkel graph* (see [54], [55] and [7, Section 13.3]) is a distance-regular graph with diameter 3 which is uniquely determined by its intersection array $\{6, 5, 2; 1, 1, 3\}$ (see [19]). It satisfies the conditions of Theorem 2.2, yielding a $(0, 3)$-geometry with $s = t = 5$.

The neighbourhood geometries of the dodecahedron, the Coxeter graph and the Biggs–Smith graph are also discussed in [75].

2.2 Distance-regular geometries on Taylor graphs

A *Taylor graph* is a distance-regular graph with diameter 3 and an intersection array of the form $\{k, \mu, 1; 1, \mu, k\}$. It is an antipodal double cover of the complete graph on $k + 1$ vertices (see [7]). In [63] it is proved that there is a bijective correspondence between two-graphs and antipodal double covers of complete graphs; regular two-graphs correspond to Taylor graphs. We will use our knowledge of certain regular two-graphs to construct distance-regular geometries on the corresponding Taylor graphs.

First we describe how to obtain a Taylor graph from a regular two-graph and conversely. Let (Ω, Δ) be a regular two-graph with parameters n and a, and choose a vertex ∞. Define a graph as follows: its vertices are the elements x^+ and x^-, for $x \in \Omega$. For all $x \in \Omega \setminus \{\infty\}$, the vertex x^+ (respectively x^-) is adjacent to the vertex ∞^+ (respectively ∞^-). For all $x, y \in \Omega \setminus \{\infty\}$, $\{x^+, y^+\}$ and $\{x^-, y^-\}$ are edges if and only if $\{\infty, x, y\}$ is coherent in (Ω, Δ), while $\{x^+, y^-\}$ and $\{x^-, y^+\}$ are edges if and only if $\{\infty, x, y\}$ is not coherent in

(Ω, Δ). One can show that this is a Taylor graph with $k = n-1$ and $\mu = n-2-a$. The construction does not depend on the choice of the vertex $\infty \in \Omega$.

Conversely, let Γ be a Taylor graph with intersection array $\{k, \mu, 1; 1, \mu, k\}$. One verifies that for each vertex of Γ there is a unique vertex at distance three from it. Let Ω denote the set of pairs of vertices of Γ which are at mutual distance 3. Consider a 6-set X which is the union of three elements of Ω. Then the subgraph of Γ induced on X is either a hexagon or the disjoint union of two triangles. Call a triple of elements of Ω coherent if and only if the subgraph induced on the corresponding set of 6 vertices of Γ is the disjoint union of two triangles. One verifies that Ω together with the set of coherent triples so defined is a two-graph. The distance-regularity of Γ implies that this two-graph is regular with parameters $n = k+1$ and $a = k-1-\mu$.

A coherent set in a two-graph yields two disjoint cliques of the same size in the corresponding Taylor graph. In some cases, we can find an appropriate set of maximal coherent sets in a regular two-graph which yields a partial linear space on the corresponding Taylor graph.

2.2.1 Hermitian two-graphs

Let H be a non-degenerate hermitian form of $\mathrm{PG}(2, q^2)$, q an odd prime power, and let $\mathcal{U}$ be the corresponding hermitian curve. The vertices of the *hermitian two-graph* $\mathcal{H}(q)$ are the points of $\mathcal{U}$; a triple $\{x, y, z\}$ of vertices is coherent if and only if the expression $H(x,y)H(y,z)H(z,x)$ is a square or a non-square in $\mathrm{GF}(q^2)$ according as $q \equiv 3 \pmod 4$ or $q \equiv 1 \pmod 4$. One shows that $\mathcal{H}(q)$ is regular with parameters $n = q^3+1$ and $a = (q-1)(q^2+1)/2$ (see [61]). For any line L of $\mathrm{PG}(2, q^2)$ containing $q+1$ points of $\mathcal{U}$ the set $L \cap \mathcal{U}$ is a coherent set in $\mathcal{H}(q)$. Choose a point x in $L \cap \mathcal{U}$, and let $\mathcal{H}'(q)$ denote the descendant of $\mathcal{H}(q)$ with respect to x. We note that $\mathcal{H}'(q)$ is an $\mathrm{srg}(q^3, (q-1)(q^2+1)/2, (q-1)^3/4-1, (q-1)(q^2+1)/4)$. One easily checks that the restricted eigenvalue l equals to $-(q^2+1)/2$ (see Theorem 1.1). The set $(L \cap \mathcal{U}) \setminus \{x\}$ is a clique in $\mathcal{H}'(q)$ of order $q = 1 - k/l$, hence the order of this clique meets the Hoffman bound (see Theorem 1.1), implying that any vertex of $\mathcal{H}'(q)$ not in this clique is adjacent to precisely $(q-1)/2$ vertices in it (see [42] for more details).

The Taylor graph Γ corresponding to $\mathcal{H}(q)$ has intersection array

$$\{q^3, \frac{(q+1)(q^2-1)}{2}, 1; 1, \frac{(q+1)(q^2-1)}{2}, q^3\}.$$

We will show that the $(q+1)$-secants of $\mathcal{U}$ give rise to a $(0, (q+1)/2)$-geometry with Γ as a point graph. Each $(q+1)$-secant of $\mathcal{U}$ yields two disjoint $(q+1)$-

cliques in Γ; it is obvious that two cliques constructed in this way cannot have more than one vertex in common. As any point of $\mathcal{U}$ is contained in q^2 lines of $\mathrm{PG}(2, q^2)$ which contain $q+1$ points of $\mathcal{U}$, we find a partial linear space of order (q, q^2-1) on the vertex set of Γ.

Let x be a point of $\mathcal{U}$ and let L be a line through x containing $q+1$ points of $\mathcal{U}$. Then one of the cliques of Γ corresponding to L contains the vertex x^+, and the other one contains x^-. As x^+ and x^- are at distance 3, x^+ is not adjacent to any of the vertices of the clique containing x^- and conversely.

Now let L be a $(q+1)$-secant of $\mathcal{U}$ and let x be a point on $\mathcal{U}$ but not on L. Without loss of generality, we may choose the vertex ∞ appearing in the construction of a Taylor graph from a regular two-graph to be a point of $L \cap \mathcal{U}$. Considering the descendant of $\mathcal{H}(q)$ with respect to ∞, we see that $\{\infty, x, y\}$ is coherent for $(q-1)/2$ points $y \in (L \cap \mathcal{U}) \setminus \{\infty\}$, and non-coherent for the remaining $(q+1)/2$ points $y \in (L \cap \mathcal{U}) \setminus \{\infty\}$. Let L^+ denote the clique in Γ corresponding to L which contains ∞^+. Then the vertex x^+ is adjacent to $(q+1)/2$ vertices of L^+, namely ∞^+ and the $(q-1)/2$ vertices y^+ such that $\{\infty, x, y\}$ is coherent in $\mathcal{H}(q)$. The vertex x^- is adjacent to the $(q+1)/2$ vertices y^+ of L^+ such that $\{\infty, x, y\}$ is not coherent. This implies that the partial linear space is a $(0, (q+1)/2)$-geometry. As any two adjacent vertices in Γ are contained in a unique clique arising from a $(q+1)$-secant of $\mathcal{U}$, Γ is the point graph of this geometry. The remaining parameters are easily calculated from s, t, α_3, b_i ($i \in \{0,1,2\}$) and c_i ($i \in \{1,2,3\}$); we find $t_4 + 1 = q^2 - 1$, $\alpha_5 = q$, $t_6 + 1 = q^2$.

2.2.2 Ree two-graphs

Assume $q = 3^{2h+1}$, $h \in \mathbb{N}$, and let $\mathcal{O}_R$ be the Ree–Tits ovoid of the generalized hexagon $H(q)$ (see Section 1.5), considered as an ovoid of $Q(6,q)$. Let π denote the orthogonal polarity associated with $Q(6,q)$. Define a triple $\{x, y, z\}$ of points of $\mathcal{O}_R$ to be coherent if and only if the three-dimensional subspace $\langle x, y, z\rangle^\pi$ intersects $Q(6,q)$ in a hyperbolic quadric $Q^+(3,q)$. The two-graph thus obtained is called the *Ree two-graph* $\mathcal{R}(q)$; it is regular with the same parameters as the hermitian two-graph $\mathcal{H}(q)$. Like the hermitian curve, which is a $2-(q^3+1, q+1, 1)$ design, also called a *unital*, the Ree–Tits ovoid can serve as the point set of a unital, and the blocks of the unital are coherent sets in the Ree two-graph. Consequently, a distance-regular $(0, (q+1)/2)$-geometry on the corresponding Taylor graph can be constructed in a similar way as above.

2.2.3 Symplectic two-graphs

Let $V(2m, 2)$, with $m \geq 2$, denote the $2m$-dimensional vector space over GF(2), equipped with a non-degenerate alternating form f. The vertex set of the *symplectic two-graph* $\Sigma(2m, 2)$ is $V(2m, 2)$, and a triple $\{x, y, z\}$ of vertices is coherent if and only if $f(x, y) + f(y, z) + f(z, x) \equiv 0 \pmod 2$. This two-graph is regular with parameters $n = 2^{2m}$ and $a = 2^{2m-1} - 2$. If the zero vector is deleted from $V(2m, 2)$, the projective space $\mathrm{PG}(2m-1, 2)$ is obtained, and the alternating form f becomes a symplectic polarity. The symplectic polar space $W(2m-1, 2)$ always contains a spread $\mathcal{S}$, which consists of $2^m + 1$ subspaces of (projective) dimension $m-1$. In $V(2m, 2)$, $\mathcal{S}$ becomes a set $\mathcal{S}'$ of $2^m + 1$ subspaces of dimension m. The set $\mathcal{T}$ consisting of all elements of $\mathcal{S}'$ and all their translates has $2^m(2^m + 1)$ elements. Each element of $\mathcal{T}$ is a coherent set in $\Sigma(2m, 2)$, and no two of them have more than one point in common. Consider an element $X \in \mathcal{T}$, and let $x \in X$. Then the clique $X \setminus \{x\}$ in the descendant of $\Sigma(2m, 2)$ with respect to x meets the Hoffman bound, implying that a vertex outside $X \setminus \{x\}$ is adjacent to $2^{m-1} - 1$ vertices in it. Reasoning similar to the one in the hermitian case yields a $(0, 2^{m-1})$-geometry on the Taylor graph corresponding to $\Sigma(2m, 2)$. Its parameters are $s = 2^m - 1$, $t = 2^m$, $\alpha_3 = 2^{m-1}$, $t_4 + 1 = 2^m - 1$, $\alpha_5 = 2^m - 1$, $t_6 = 2^m$.

Chapter 3

Distance-regular $(0, \alpha)$-reguli

Distance-regular $(0, \alpha)$-reguli, the objects of study in this chapter, will be introduced as a means to give rise to $(0, \alpha)$-geometries with a distance-regular point graph.

In [67] Thas defined *SPG-reguli*, see also Section 1.9, as certain sets of subspaces of a projective space which yield semipartial geometries. This was generalised in [37] to *strongly regular* (α, β)*-reguli* (see also 1.10) giving rise to (α, β)-geometries with a strongly regular point graph. Here we will go one step further: we will introduce distance-regular $(0, \alpha)$-reguli and obtain distance-regular geometries with $\alpha_3 = \alpha$.

In this chapter, distance-regular $(0, \alpha)$-reguli will be introduced, and we will describe two infinite classes of examples of related distance-regular geometries, one of which is a generalisation of the well-known semipartial geometry $T_n^*(\mathcal{B})$ arising from a Baer subspace $\mathrm{PG}(n, q)$ in $\mathrm{PG}(n, q^2)$.

Most of this chapter is contained in [30].

3.1 Definitions and a general result

Let $\mathcal{R}$ be a set of m-dimensional subspaces of $\mathrm{PG}(n, q)$, $|\mathcal{R}| > 1$. We call a point $z \in \mathrm{PG}(n, q)$ of *type* i if i is the smallest integer such that z lies in an i-dimensional subspace spanned by $i + 1$ points in distinct elements of $\mathcal{R}$. Let d be the greatest number such that there exists a point of type $d - 1$ in $\mathrm{PG}(n, q)$. Suppose that $\mathcal{R}$ satisfies the following conditions.

1. Any two distinct elements of $\mathcal{R}$ have an empty intersection.
2. If an $(m+1)$-dimensional subspace through some $\Sigma \in \mathcal{R}$ contains points of type $i-1$ and points of type $i-2$ (not in Σ if $i = 2$), then it contains exactly α_{2i-1} points of type $i - 2$ (not in Σ if $i = 2$), $2 \leq i \leq d$.

3. If z is a point of type $i-1$, then there are exactly $t_{2i}+1$ elements $\Sigma \in \mathcal{R}$ such that the $(m+1)$-dimensional subspace $\langle z, \Sigma \rangle$ contains points of type $i-1$ and points of type $i-2$ (not in Σ if $i=2$), $2 \leq i \leq d$.

4. Each point of $\mathrm{PG}(n,q)$ is of type i for an $i \in \{0, \ldots, d-1\}$.

Then $\mathcal{R}$ is called a *distance-regular* $(0,\alpha)$*-regulus*, with $\alpha := \alpha_3$. The numbers α_{2i-1} and $t_{2i}+1$ $(2 \leq i \leq d)$ are called the *parameters* of $\mathcal{R}$. We will always assume that $\alpha_3 \leq q^{m+1}-1$, which implies that there exists at least one point of type 1.

In the following theorem, we will show that a distance-regular $(0,\alpha)$-regulus gives rise, via the generalised linear representation of $\mathcal{R}$, to a distance-regular $(0,\alpha)$-geometry $T^*_{n,m}(\mathcal{R})$.

Theorem 3.1 *Let $\mathcal{R}$ be a distance-regular $(0,\alpha)$-regulus in $\mathrm{PG}(n,q)$, the elements of $\mathcal{R}$ being m-dimensional subspaces. Let α_{2i-1} and $t_{2i}+1$ $(2 \leq i \leq d)$ denote the parameters of $\mathcal{R}$ (with $\alpha_3 = \alpha$). Embed $\mathrm{PG}(n,q)$ as a hyperplane Π_∞ in $\mathrm{PG}(n+1,q)$, and define an incidence structure $T^*_{n,m}(\mathcal{R}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows.*

- *The point set $\mathcal{P}$ is the set of points of $\mathrm{PG}(n+1,q)$ which do not lie in Π_∞.*

- *The line set $\mathcal{L}$ is the set of $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ that meet Π_∞ in an element of $\mathcal{R}$.*

- *Incidence I is the one induced by $\mathrm{PG}(n+1,q)$.*

Then $T^*_{n,m}(\mathcal{R})$ is a distance-regular $(0,\alpha)$-geometry with parameters $s = q^{m+1}-1$, $t = |\mathcal{R}|-1$, α_{2i-1} and $t_{2i}+1$ $(2 \leq i \leq d)$.

Proof. Let $L, M \in \mathcal{L}$ such that L and M contain the same element $\Sigma \in \mathcal{R}$. Then L and M have no point of $\mathrm{PG}(n+1,q) \setminus \Pi_\infty$ in common. Now let $L, M \in \mathcal{L}$ such that L and M contain distinct elements Σ and Σ', respectively, of $\mathcal{R}$. If $L \cap M$ contains a line, then this line meets Π_∞ in a point contained in Σ and in Σ', a contradiction. It follows that $T^*_{n,m}(\mathcal{R})$ is a partial linear space. Clearly the number of points on a line of $T^*_{n,m}(\mathcal{R})$ is q^{m+1}, and the number of lines on a point is $|\mathcal{R}|$, so the order of the partial linear space is $(q^{m+1}-1, |\mathcal{R}|-1)$.

We will prove by induction that two points x and y of $T^*_{n,m}(\mathcal{R})$ are at distance $2i$ in the incidence graph of $T^*_{n,m}(\mathcal{R})$ if and only if the point $\langle x, y \rangle \cap \Pi_\infty$ is of type $i-1$ $(1 \leq i \leq d)$. This obviously holds for $i=1$. Now take $i \geq 2$, and suppose that for all $j < i$ two points x and y are at distance $2j$

if and only if $\langle x, y\rangle$ meets Π_∞ in a point of type $j-1$. Let $x, y \in \mathcal{P}$ such that x is at distance $2i$ from y. Then there exists a point z such that z is at distance $2i-2$ from x and at distance 2 from y. Since z is collinear with y, $\langle z, y\rangle$ meets Π_∞ in a point w of type 0. By the induction hypothesis, $\langle x, z\rangle$ intersects Π_∞ in a point v of type $i-1$. In $\mathrm{PG}(n+1, q)$, consider the plane π spanned by x, y and z. Then π meets Π_∞ in a projective line ℓ that contains w, v, and a point u on $\langle x, y\rangle$ which is of type at least $i-1$. Since v is a point of type $i-2$, it lies in an $(i-2)$-dimensional subspace Π_{i-2} spanned by $i-1$ points in distinct elements of $\mathcal{R}$. Let Σ denote the element of $\mathcal{R}$ containing w. If Σ intersects Π_{i-2}, then it is possible to find a point a at distance at most $2i-4$ from x and at distance 2 from z such that $\langle z, a\rangle$ intersects Σ. As y is also a point of $\langle z, \Sigma\rangle$, y is at distance 2 from a and hence at distance at most $2i-2$ from x, a contradiction to the assumption. Consequently Σ is disjoint from Π_{i-2}, and the $(i-1)$-dimensional subspace $\Pi_{i-1} := \langle w, \Pi_{i-2}\rangle$ is spanned by i points in distinct elements of $\mathcal{R}$. This implies that u is of type $i-1$.

Conversely let $x, y \in \mathcal{P}$ such that the line $\langle x, y\rangle$ meets $\mathrm{PG}(n, q)$ in a point u of type $i-1$. Then x is at distance at least $2i$ from y, and u is contained in an $(i-1)$-dimensional subspace Π_{i-1} spanned by i points in distinct elements of $\mathcal{R}$. Let Π_{i-2} be the $(i-2)$-dimensional subspace of Π_{i-1} spanned by $i-1$ of these points in elements of $\mathcal{R}$, and let w be a point in $\Pi_{i-1} \setminus \Pi_{i-2}$ belonging to an element of $\mathcal{R}$. The point u cannot be contained in Π_{i-2} as it is of type $i-1$; hence the line $\langle u, w\rangle$ meets Π_{i-2} in a point v of type at most $i-2$. Consider the plane π given by the span of the lines $\langle x, y\rangle$ and $\langle u, w\rangle$. The line joining x and v meets the line joining y and w in a point z. By the induction hypothesis, z is at distance 2 from y and at distance at most $2i-2$ from x. We conclude that x is at distance $2i$ from y.

Let $p \in \mathcal{P}$ and let $L \in \mathcal{L}$ such that p is at distance $2i-1$ from L in the incidence graph of $T^*_{n,m}(\mathcal{R})$, with $2 \leq i \leq d$. This means that some points on L are at distance $2i$ from x and some are at distance $2i-2$ from x. Hence the $(m+2)$-dimensional subspace Π_{m+2} spanned by p and L meets Π_∞ in an $(m+1)$-dimensional subspace Π_{m+1} containing an element of $\mathcal{R}$ and points of types $i-1$ and $i-2$. The number of points of type $i-2$ in Π_{m+1} (and not in $L \cap \mathrm{PG}(n, q)$ if $i = 2$) is α_{2i-1}. If u is such a point, the line $\langle u, p\rangle$ intersects L in a unique point which is at distance $2i-2$ from p, so there are α_{2i-1} points incident with L and at distance $2i-2$ from p. Similarly, let p and q be two points at distance $2i$ ($2 \leq i \leq d$) in the incidence graph, or equivalently, suppose that $\langle p, q\rangle$ meets $\mathrm{PG}(n, q)$ in a point u of type $i-1$. The point u is contained in $t_{2i}+1$ $(m+1)$-dimensional subspaces $\langle u, \Sigma\rangle$, $\Sigma \in \mathcal{R}$, which contain points of type $i-1$ and points of type $i-2$ (outside Σ if $i = 2$). If $\langle u, \Sigma\rangle$ is such a subspace, then $\langle q, \Sigma\rangle$ is a line of $T^*_{n,m}(\mathcal{R})$ which is at distance

$2i-1$ from p. Hence there are $t_{2i}+1$ lines incident with q and at distance $2i-1$ from p. This proves the theorem. □

Remark

In [31] S. De Winter, introduces elation semipartial geometries and translation semipartial geometries (extending the concept of elation and translation generalized quadrangles, see [53]). He proves among other things that the semipartial geometry from an SPG-regulus is a translation semipartial geometry. It is possible to extend this theory for the distance-regular geometries arising from distance-regular $(0,\alpha)$-reguli, but we will omit this.

3.2 Spreads of parabolic quasi-quadrics

In this subsection, we will describe the first infinite class of examples. In order to obtain the desired examples, we need to overview the definition of a quasi-quadric and some results deriving from it which are sufficient for our construction of a distance-regular $(0,\alpha)$-regulus.

3.2.1 Quasi-quadrics

Definition

Quasi-quadrics were introduced by De Clerck, Hamilton, O'Keefe and Pentilla [28]. A *quasi-quadric* in a projective space $\mathrm{PG}(n,q)$ is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric in that space. Of course, non-degenerate quadrics themselves are examples of quasi-quadrics, but many other examples exist. In particular a *parabolic quasi-quadric* with *nucleus* n in $\mathrm{PG}(2m,q)$, $m \geq 2$ and q even, is a set $\mathcal{Q}$ of $(q^{2m}-1)/(q-1)$ points such that each line through the point n intersects $\mathcal{Q}$ in a unique point and each hyperplane not on n contains either $(q^m+1)(q^{m-1}-1)/(q-1)$ or $(q^m-1)(q^{m-1}+1)/(q-1)$ points of $\mathcal{Q}$. This means that $\mathcal{Q}$ has the same intersection numbers with hyperplanes as $Q(2m,q)$ if q is even. An example of a parabolic quasi-quadric is a non-singular parabolic quadric $\mathcal{Q}$ with nucleus n, since each hyperplane not on n meets $\mathcal{Q}$ in a non-singular elliptic or hyperbolic quadric. In [28] also elliptic and hyperbolic quasi-quadrics are defined, but we will not use them in this thesis.

Constructing parabolic quasi-quadrics

For the sake of completeness we give a construction of a parabolic quasi-quadric, as it appeared in [28].

Let $Q(2m,q)$ be a non-degenerate parabolic quadric in $\mathrm{PG}(2m,q)$, q even, $m>1$. Let Σ_k be a subspace of dimension k contained in $Q(2m,q)$, $k<m-1$. The tangent space $\Sigma_k^\perp$ at Σ_k to the quadric $Q(2m,q)$ is then of dimension $2m-k-1$. The factor space $\Sigma_k^\perp/\Sigma_k$ has (even) dimension $2m-2k-2$. It follows that $\Sigma_k^\perp \cap Q(2m,q)$ is the cone $\Sigma_k\mathcal{Q}$ with vertex Σ_k and base $\mathcal{Q}$, being a non-degenerate parabolic quadric $Q(2m-2k-2,q)$ in some subspace $\Sigma_{2m-2k-2}$ of $\Sigma_k^\perp$ of dimension $2m-2k-2$ disjoint from Σ_k.

Suppose that $\mathcal{Q}$ has nucleus n'. Let $\mathcal{Q}'$ be a parabolic quasi-quadric in $\Sigma_{2m-2k-2}$ with the same parameters as $\mathcal{Q}$ and with the same nucleus n'. We then call the set $Q(2m,q)-\Sigma_k\mathcal{Q}\cup\Sigma_k\mathcal{Q}'$ a *pivotted set* of $Q(2m,q)$ with respect to Σ_k. We note that the size of a pivotted set is the same as the size of $Q(2m,q)$.

Theorem 3.2 *Every pivotted set of $Q(2m,q)$, q even, is a parabolic quasi-quadric with the same intersection numbers as those of $Q(2m,q)$.*

Proof. We will show that every hyperplane of $\mathrm{PG}(2m,q)$ not on the nucleus n of $Q(2m,q)$ meets the pivotted set in either $|Q^-(2m-1,q)|$ or $|Q^+(2m-1,q)|$ points. We note that every line through n will contain a unique point of the pivotted set.

Let Σ_{2m-1} be a hyperplane of $\mathrm{PG}(2m,q)$ that does not contain n. Then Σ_{2m-1} meets $\Sigma_k^\perp$ in a hyperplane Σ_{2m-k-2} of $\Sigma_k^\perp$. There are two cases to consider.

(i) Assume $\Sigma_k \not< \Sigma_{2m-k-2}$. Then $\Sigma_k\cap\Sigma_{2m-k-2}$ is a hyperplane Σ_{k-1} of Σ_k. Dimensional arguments then show that $\Sigma_{2m-k-2}\cap\Sigma_k\mathcal{Q}$ is a cone with vertex Σ_{k-1} and base a parabolic quadric $Q(2m-2k-2,q)$. Similarly, it follows that $\Sigma_{2m-k-2}\cap\Sigma_k\mathcal{Q}'$ is a cone with vertex Σ_{k-1} projecting a set projectively equivalent to $\mathcal{Q}'$. It follows that Σ_{2m-1} meets $Q(2m,q)$ and the pivotted set in the same number of points.

(ii) Assume $\Sigma_k<\Sigma_{2m-k-2}$. We note that the nucleus n of $Q(2m,q)$ is contained in the subspace $\langle n',\Sigma_k\rangle$. Hence Σ_{2m-k-2} meets $\Sigma_{2m-2k-2}$ in a hyperplane of $\Sigma_{2m-2k-2}$ not on n'. It follows that $\Sigma_{2m-k-2}\cap\mathcal{Q}$ is either a $Q^-(2m-2k-3,q)$ or a $Q^+(2m-2k-3,q)$. Hence we have two cases to consider.

(a) Assume $\Sigma_{2m-k-2}\cap\Sigma_k^\perp=\Sigma_kQ^-(2m-2k-3,q)$. Now Σ_{2m-1} meets $Q(2m,q)$ in either a $Q^-(2m-1,q)$ or a $Q^+(2m-1,q)$. But $Q^+(2m-1,q)$ does not contain a surface isomorphic to $\Sigma_kQ^-(2m-2k-3)$ [41, Corollary

2 to Theorem 22.8.3]. So in this case it must be that $\Sigma_{2m-1} \cap Q(2m,q)$ is isomorphic to a $Q^-(2m-1,q)$.

Since the intersection sizes with respect to hyperplanes not on the nucleus are the same for $\mathcal{Q}$ and $\mathcal{Q}'$, it follows that $|\Sigma_{2m-1} \cap \mathcal{Q}'|$ is equal to either

$$|Q^-(2m-1,q)| - |\Sigma_k Q^-(2m-2k-3,q)| + |\Sigma_k Q^-(2m-2k-3,q)|$$

or

$$|Q^-(2m-1,q)| - |\Sigma_k Q^-(2m-2k-3,q)| + |\Sigma_k Q^+(2m-2k-3,q)|$$

which, with some algebra, is easily shown to be either $|Q^-(2m-1,q)|$ or $|Q^+(2m-1,q)|$.
(b) Assume $\Sigma_{2m-k-2} \cap \Sigma_k^\perp = \Sigma_k Q^+(2m-2k-3,q)$. As before Σ_{2m-1} meets $Q(2m,q)$ in either a $Q^-(2m-1,q)$ or a $Q^+(2m-1,q)$. But $Q^-(2m-1,q)$ does not contain a surface isomorphic to $\Sigma_k Q^+(2m-2k-3)$ [41, Corollary 2 to Theorem 22.8.3]. So in this case it must be that $\Sigma_{2m-1} \cap Q(2m,q)$ is isomorphic to a $Q^+(2m-1,q)$.

Similarly to case (a), $|\Sigma_{2m-1} \cap \mathcal{Q}'|$ is given by

$$|Q^+(2m-1,q)| - |\Sigma_k Q^+(2m-2k-3,q)| + |\Sigma_k Q^-(2m-2k-3,q)|$$

or

$$|Q^+(2m-1,q)| - |\Sigma_k Q^+(2m-2k-3,q)| + |\Sigma_k Q^+(2m-2k-3,q)|$$

which can be shown to be either $|Q^-(2m-1,q)|$ or $|Q^+(2m-1,q)|$. □

3.2.2 An example of a distance-regular $(0,\alpha)$-regulus

In this subsection, we will illustrate the first method to construct distance-regular $(0,\alpha)$-reguli. For this purpose, we introduce the definition of a spread of a quasi-quadric.

A *spread* $\mathcal{R}$ of a parabolic quasi-quadric in $\mathrm{PG}(2m,q)$ with nucleus n is a set of q^m+1 subspaces of dimension $m-1$ forming a partition of $\mathcal{Q}$. Assume that the number of points of $\mathcal{Q}$ in an m-dimensional subspace which contains an element Σ of $\mathcal{R}$ and does not contain n is a constant. Then we will show that $\mathcal{R}$ is a distance-regular $(0,q^{m-1})$-regulus. Obviously the elements of $\mathcal{R}$ are two by two disjoint. The points of type 0 are the points of $\mathcal{Q}$. Any point of $\mathrm{PG}(2m,q) \setminus (\mathcal{Q} \cup \{n\})$ is contained in an m-dimensional subspace containing an element of $\mathcal{R}$ and not containing n, and hence is of type 1. The unique point of type 2 is n. In order to find α_3, we count in two ways the pairs consisting of a point z of type 1 and a subspace $\langle z, \Sigma\rangle$, $\Sigma \in \mathcal{R}$, which does not contain n. This yields

$$(q^{2m}-1)q^m = (q^m+1)\left(\frac{q^{m+1}-1}{q-1}-1\right)(q^m-\alpha_3),$$

and we find $\alpha_3 = q^{m-1}$. If z is a point of type 1, then there is a unique $\Sigma \in \mathcal{R}$ such that $\langle z, \Sigma\rangle$ contains n, implying $t_4+1 = q^m$. The parameter α_5 is the number of points of type 1 in an m-dimensional subspace $\langle n, \Sigma\rangle$ with $\Sigma \in \mathcal{R}$, so $\alpha_5 = q^m - 1$. Finally, n is contained in $t_6 + 1 = q^m + 1$ subspaces $\langle n, \Sigma\rangle$ with $\Sigma \in \mathcal{R}$. We summarise the above in the following theorem.

Theorem 3.3 *Let $\mathcal{Q}$ be a parabolic quasi-quadric with nucleus n in $\mathrm{PG}(2m, q)$, $m \geq 2$, which admits a spread $\mathcal{R}$. If the number of points of $\mathcal{Q}$ in an m-dimensional subspace containing an element Σ of $\mathcal{R}$ and not containing n is a constant, then $\mathcal{R}$ is a distance-regular $(0, q^{m-1})$-regulus with parameters $\alpha_3 = q^{m-1}$, $t_4+1 = q^m$, $\alpha_5 = q^m-1$ and $t_6+1 = q^m+1$.*

The most straightforward example of a parabolic quasi-quadric admitting a spread is the parabolic quadric $Q(2m, q)$ for q even. The extra condition involving m-dimensional subspaces is automatically satisfied here. For q even, there exist other parabolic quasi-quadrics [28], but it is not known to us whether they admit spreads. For q odd, we do not know any examples.

By Lemma 1.8, the point graph of the distance-regular geometry arising from a spread of a parabolic quasi-quadric has diameter 3 and intersection array $\{q^{2m}-1, q^{2m-1}(q-1), 1; 1, q^{2m-1}, q^{2m}-1\}$. Using a characterisation result (see [7, Section 4.2]) on antipodal graphs (see 1.4.3), one sees that it is antipodal if and only if $q = 2$; in this case the graph is a Taylor graph (see Section 2.2). If the parabolic quasi-quadric is a parabolic quadric $Q(2m, 2)$ in $\mathrm{PG}(2m, 2)$, $m \geq 2$, then the geometry can also be described using the symplectic two-graph, as is done in Section 2.2.3. We note that we rely on the result concerning the isomorphism between the polar spaces $Q(2n, q)$ and $W(2n-1, q)$, with q even, which were briefly discussed in Section 1.3.

3.2.3 Characterisation of $(0, \alpha)$-reguli with a nucleus

In this subsection, we will show that a parabolic quasi-quadric with nucleus n of $\mathrm{PG}(2m, q)$, $q \geq 2$, can be constructed from a particular class of distance-regular $(0, \alpha)$-reguli.

Define a distance-regular $(0, \alpha)$-regulus *with nucleus* n to be a distance-regular $(0, \alpha)$-regulus consisting of $(m-1)$-dimensional subspaces in $\mathrm{PG}(2m, q)$, $m \geq 2$, such that n is the only point of type 2 and there are no points of higher type. The structures satisfying the hypotheses of Theorem 3.3 are easily seen to be distance-regular $(0, \alpha)$-reguli with a nucleus. However, the converse is also true.

Theorem 3.4 *Suppose that $\mathcal{R}$ is a distance-regular $(0, \alpha)$-regulus with nucleus n in $\mathrm{PG}(2m, q)$, $m \geq 2$. Then the set $\tilde{\mathcal{R}}$ consisting of all points contained in the elements of $\mathcal{R}$ is a parabolic quasi-quadric with nucleus n.*

Proof. First we determine the number of elements of $\mathcal{R}$ by counting the points of each type in $\mathrm{PG}(2m, q)$. Obviously there are $|\mathcal{R}|(q^m - 1)/(q - 1)$ points of type 0 and there is one point of type 2. For each $\Sigma \in \mathcal{R}$, the $q^m - 1$ points in $\langle n, \Sigma\rangle \setminus (\{n\} \cup \Sigma)$ are of type 1. If two subspaces $\langle n, \Sigma\rangle$ and $\langle n, \Sigma'\rangle$, with $\Sigma, \Sigma' \in \mathcal{R}$ and $\Sigma \neq \Sigma'$, had more than one point in common, either there would exist a line through n intersecting two distinct elements of $\mathcal{R}$ or Σ and Σ' would intersect. Hence $\langle n, \Sigma\rangle \cap \langle n, \Sigma'\rangle = \{n\}$, and there are $|\mathcal{R}|(q^m - 1)$ points of type 1. As each point of $\mathrm{PG}(2m, q)$ is of type 0, 1 or 2, we have $|\mathcal{R}|(q^m - 1)/(q - 1) + |\mathcal{R}|(q^m - 1) + 1 = (q^{2m+1} - 1)/(q - 1)$, which yields $|\mathcal{R}| = q^m + 1$.

Now we count in two ways the pairs consisting of a point z of type 1 and a subspace $\langle z, \Sigma\rangle$, $\Sigma \in \mathcal{R}$, which does not contain n. This yields

$$(q^{2m} - 1)q^m = (q^m + 1)\left(\frac{q^{m+1} - 1}{q - 1} - 1\right)(q^m - \alpha),$$

and we find $\alpha = q^{m-1}$.

As $\tilde{\mathcal{R}}$ is the set of points contained in the elements of $\mathcal{R}$, $|\tilde{\mathcal{R}}| = |\mathcal{R}|(q^m - 1)/(q - 1) = (q^{2m} - 1)(q - 1)$, and so any line through n intersects $\tilde{\mathcal{R}}$ in a unique point. If H is a hyperplane containing an element Σ of $\mathcal{R}$ and not containing n, then any m-dimensional subspace in H through Σ contains q^{m-1} points of type 0 outside Σ. This gives us $|H \cap \tilde{\mathcal{R}}| = (q^m - 1)/(q - 1) + ((q^m - 1)/(q - 1))q^{m-1} = (q^m - 1)(q^{m-1} + 1)/(q - 1)$. A hyperplane H which does not contain any element of $\mathcal{R}$ intersects every element of $\mathcal{R}$ in an $(m-2)$-dimensional subspace, implying $|H \cap \tilde{\mathcal{R}}| = (q^m + 1)(q^{m-1} - 1)/(q - 1)$. Consequently $\tilde{\mathcal{R}}$ is a parabolic quasi-quadric with nucleus n. □

3.3 Subspaces over subfields

In this subsection,we will describe a second method to construct examples of distance-regular $(0, \alpha)$-reguli.

3.3.1 Description

Let q' be any prime power, let $q = {q'}^h$ with $h \in \mathbb{N} \setminus \{0, 1\}$, and consider a subgeometry $\mathrm{PG}(n, q')$ in $\mathrm{PG}(n, q)$, $n \geq 1$. We will show that the point set $\mathcal{R}$ of $\mathrm{PG}(n, q')$ is a distance-regular $(0, q')$-regulus.

The points of type 0 in $\mathrm{PG}(n, q)$ are the points of $\mathrm{PG}(n, q')$ (i. e. the elements of $\mathcal{R}$). The points of type $i > 0$ are the points z of $\mathrm{PG}(n, q) \setminus \mathrm{PG}(n, q')$ for which i is the smallest integer such that z is contained in a $\mathrm{PG}(i, q)$ intersecting $\mathrm{PG}(n, q')$ in a $\mathrm{PG}(i, q')$. Suppose that a point z of type $i > 0$ is contained in at least two i-dimensional subspaces Π_i^1 and Π_i^2 of $\mathrm{PG}(n, q)$ intersecting $\mathrm{PG}(n, q')$ in an i-dimensional subspace ${\Pi'_i}^1$, respectively ${\Pi'_i}^2$, over $\mathrm{GF}(q')$. Let H' be a hyperplane of $\mathrm{PG}(n, q')$ which contains ${\Pi'_i}^1$, but intersects ${\Pi'_i}^2$ in an $(i-1)$-dimensional subspace Π'_{i-1}. Then the extension H of H' to $\mathrm{GF}(q)$ is a hyperplane of $\mathrm{PG}(n, q)$ which contains Π_i^1 (and hence z) and the extension Π_{i-1} of Π'_{i-1} over $\mathrm{GF}(q)$. As z is of type i, it is not contained in Π_{i-1}, so $\langle z, \Pi_{i-1}\rangle = \Pi_i^2$. Since $\langle z, \Pi_{i-1}\rangle \subset H$, this implies that ${\Pi'_i}^2 = \Pi_i^2 \cap \mathrm{PG}(n, q')$ is contained in $H' = H \cap \mathrm{PG}(n, q')$, a contradiction. We conclude that a point of type $i > 0$ is contained in a unique $\mathrm{PG}(i, q)$ intersecting $\mathrm{PG}(n, q')$ in a $\mathrm{PG}(i, q')$.

Suppose that a 1-secant L of $\mathrm{PG}(n, q')$ contains a point x of type $i > 0$ and a point y of type $j > i$. The i-dimensional subspace Π_i over $\mathrm{GF}(q)$ through x which intersects $\mathrm{PG}(n, q')$ in an i-dimensional subspace Π'_i over $\mathrm{GF}(q')$ cannot contain y and hence intersects L only in x. The $(i+1)$-dimensional subspace $\Pi_{i+1} := \langle y, \Pi_i\rangle$ over $\mathrm{GF}(q)$ intersects $\mathrm{PG}(n, q')$ in the $(i+1)$-dimensional subspace $\Pi'_{i+1} := \langle L \cap \mathrm{PG}(n, q'), \Pi'_i\rangle$ over $\mathrm{GF}(q')$. This means that the point y is of type $i+1$. It follows that all 1-secants of $\mathrm{PG}(n, q')$ containing y and containing points of type i lie in Π_{i+1}. Moreover, if x is a point of type i on such a 1-secant, then the unique $\mathrm{PG}(i, q)$ through x which intersects $\mathrm{PG}(n, q')$ in a $\mathrm{PG}(i, q')$ is contained in Π_{i+1}. These facts assure that we can use induction on the dimension: if we restrict to a subspace $\mathrm{PG}(i, q)$ intersecting $\mathrm{PG}(n, q')$ in a $\mathrm{PG}(i, q')$, we obtain precisely the model in i dimensions.

Now we can prove that $\mathcal{R}$ is indeed a distance-regular $(0, q')$-regulus, and we will calculate the parameters. As any line of $\mathrm{PG}(n, q)$ intersecting $\mathrm{PG}(n, q')$ in at least two points is a $(q'+1)$-secant of $\mathrm{PG}(n, q')$, we find $\alpha_3 = q'$. To calculate α_{2i-1}, $i \geq 3$, we consider a 1-secant L of $\mathrm{PG}(n, q')$ containing points of type $i-1$ and points of type $i-2$. Let y be one of the points of type $i-1$ on L, and let Π_{i-1} denote the unique $(i-1)$-dimensional subspace over $\mathrm{GF}(q)$ through y which intersects $\mathrm{PG}(n, q')$ in an $(i-1)$-dimensional subspace Π'_{i-1} over $\mathrm{GF}(q')$. If Π'_{i-2} is any $(i-2)$-dimensional subspace of Π'_{i-1} not containing the point $L \cap \mathrm{PG}(n, q')$, then the extension Π_{i-2} of Π'_{i-2}

over GF(q) is contained in Π_{i-1} and hence intersects L in a point of type $i-2$. As all points of type $i-2$ on L lie in an $(i-2)$-dimensional subspace of Π_{i-1} intersecting PG(n,q') in a PG$(i-2,q')$, we find $\alpha_{2i-1} = {q'}^{i-1}$.

A point of type 1 lies on precisely one $(q'+1)$-secant of PG(n,q'), implying $t_4+1 = q'+1$. Let y be a point of type $i-1 \geq 2$. All 1-secants through y containing points of type $i-1$ and points of type $i-2$ are contained in the unique $(i-1)$-dimensional subspace Π_{i-1} of PG(n,q) through y which intersects PG(n,q') in a PG$(i-1,q')$. On the other hand, any 1-secant through y in Π_{i-1} contains points of type $i-2$, so $t_{2i}+1 = ({q'}^i-1)/(q'-1)$.

Let $T_n^*(\mathcal{R})$ denote the distance-regular geometry arising from $\mathcal{R}$. Then the order of $T_n^*(\mathcal{R})$ is $(s,t) = (q-1, ({q'}^{n+1}-q')/(q'-1))$. If ${q'}^n < q$, then $t_{2i}+1 > t+1$ for $i > n+1$, implying that points of $T_n^*(\mathcal{R})$ cannot be at distance greater than $2(n+1)$ in the incidence graph. On the other hand, $\alpha_{2n+1} = {q'}^n < q = s+1$, so points at distance $2(n+1)$ do occur. Therefore the point graph has diameter $n+1$. If ${q'}^h = q$ for some $h \in \{2,\ldots,n\}$, then $\alpha_{2i+1} > s+1$ for $i > h$, so the maximal distance between a point and a line is $2h+1$; this distance does occur because $t_{2h}+1 = ({q'}^h-1)/(q'-1) < t+1$. As $\alpha_{2h+1} = {q'}^h = q = s+1$, the point diameter of the incidence graph is $2h$, and the diameter of the point graph is h. One can also obtain the diameter d of the point graph by calculating the number of points of all types; then d equals the largest i for which the number of points of type $i-1$ is positive.

The arguments above lead to the following theorem.

Theorem 3.5 *Let q' be any prime power, let $q = {q'}^h$ with $h \in \mathbb{N} \setminus \{0,1\}$, and consider a* PG$(n,q')$ *in* PG(n,q)*, $n \geq 1$. The points of* PG(n,q') *form a distance-regular $(0,q')$-regulus with parameters $\alpha_{2i-1} = {q'}^{i-1}$ and $t_{2i}+1 = ({q'}^i-1)/(q'-1)$, $2 \leq i \leq d$, $d := \min\{n+1,h\}$.*

Remarks

1. Using Lemma 1.8, one verifies that the intersection numbers of the point graph of $T_n^*(\mathcal{R})$ are

$$b_i = \frac{({q'}^{n+1}-{q'}^i)(q-{q'}^i)}{q'-1}, \quad 0 \leq i \leq d-1,$$
$$c_i = \frac{({q'}^i-1){q'}^{i-1}}{q'-1}, \quad 1 \leq i \leq d.$$

2. If $n=1$, then $T_n^*(\mathcal{R})$ is a net of order q and degree $q'+1$. The points and lines are all points and $q(q'+1)$ lines, respectively, of a desarguesian affine plane AG$(2,q)$. The parallel classes are defined by the points of a subline PG$(1,q')$ of the line at infinity of the affine plane.

3. If $q' = \sqrt{q}$, then $T_n^*(\mathcal{R})$ is the semipartial geometry $T_n^*(\mathcal{B})$ described in [29].

3.3.2 A characterisation

Let $\mathcal{R}$ be a distance-regular $(0,\alpha)$-regulus consisting of m-dimensional subspaces in $\mathrm{PG}(n,q)$. $\mathcal{R}$ is called *α-geometric* [3] (see also [27]) if for any two elements Σ and Σ' of $\mathcal{R}$ the $(2m+1)$-dimensional subspace $\langle\Sigma,\Sigma'\rangle$ contains precisely $\alpha+1$ elements of $\mathcal{R}$. If $m=0$, then this property always holds since $\mathcal{R}$ intersects every line in 0, 1 or $\alpha+1$ points. Now let $m>0$ and suppose that an element Σ'' of $\mathcal{R}$ intersects the $(2m+1)$-dimensional subspace $\langle\Sigma,\Sigma'\rangle$, with Σ, $\Sigma'\in\mathcal{R}$. Choose a point $z\in\Sigma''\cap\langle\Sigma,\Sigma'\rangle$, and consider the $(m+1)$-dimensional subspace $\langle z,\Sigma\rangle$. As each element of $\mathcal{R}$ different from Σ and contained in $\langle\Sigma,\Sigma'\rangle$ intersects $\langle z,\Sigma'\rangle$, and $\langle z,\Sigma\rangle$ intersects precisely α elements of $\mathcal{R}$ different from Σ, we find that Σ'' must be contained in $\langle\Sigma,\Sigma'\rangle$.

Let $\mathcal{R}$ be a distance-regular $(0,\alpha)$-regulus consisting of m-dimensional subspaces in $\mathrm{PG}(n,q)$, and suppose that there are no points of type at least two. In this case $\mathcal{R}$ is an SPG-regulus and the corresponding geometry $T_{n,m}^*(\mathcal{R})$ is a semipartial geometry. In [31], De Winter proves that if $t_4=\alpha\notin\{1,q^{m+1}-1,q^{m+1}\}$, then $\mathcal{R}$ is α-geometric, $\alpha=q^{(m+1)/2}$, and $T_{n,m}^*(\mathcal{R})$ is isomorphic to the semipartial geometry $T_N^*(\mathcal{B})$ arising from a Baer subspace in $\mathrm{PG}(N,q^{m+1})$, with $N=(n+1)/(m+1)-1$. This theorem can be generalised as follows.

Theorem 3.6 *Let $\mathcal{R}$ be a distance-regular $(0,\alpha)$-regulus which consists of m-dimensional subspaces in $\mathrm{PG}(n,q)$, $n>2m+1$. Suppose that $t_4=\alpha\notin\{1,q^{m+1}-1,q^{m+1}\}$. Then α is a power of q, $\mathrm{GF}(\alpha)$ is a subfield of $\mathrm{GF}(q^{m+1})$, and the geometry $T_{n,m}^*(\mathcal{R})$ arising from $\mathcal{R}$ is isomorphic to the one arising from $\mathrm{PG}(\frac{n+1}{m+1}-1,\alpha)$ in $\mathrm{PG}(\frac{n+1}{m+1}-1,q^{m+1})$.*

Proof.

STEP 1. Inspired by the proof in [31], we will show that $\mathcal{R}$ is α-geometric.

First of all we observe the following. Let p be a point of type 1 and let Σ_0 be an element of $\mathcal{R}$ not containing p. If the $(m+1)$-dimensional subspace $\langle p,\Sigma_0\rangle$ which does not contain points of type 2 intersects α elements, $\Sigma_1,\cdots,\Sigma_\alpha$ of $\mathcal{R}$ distinct from Σ_0, then for every point p' of type 1 in $\langle p,\Sigma_0\rangle\setminus(\Sigma_0\cup p)$ the $(m+1)$-dimensional subspaces $\langle p',\Sigma_0\rangle,\langle p',\Sigma_1\rangle,\cdots,\langle p',\Sigma_\alpha\rangle$ do not contain points of type higher than type 1.

First we assume that $\alpha<q^{m+1}-q^m$. Let p be a point of type 1 and suppose that $\Sigma_1,\cdots,\Sigma_\alpha$ are α elements of $\mathcal{R}$ intersecting $\langle p,\Sigma_0\rangle$. We prove

that $\langle p, \Sigma_0\rangle$ is completely contained in $\langle \Sigma_i, \Sigma_j\rangle$, with Σ_i, Σ_j, $i, j \in \{1, \ldots, d\}$, two distinct elements of $\mathcal{R}$. Since $\alpha < q^{m+1} - q^m$, there are more than q^m points of type 1 in $\langle p, \Sigma_0\rangle$. Hence the points of type 1 generate the space $\langle p, \Sigma_0\rangle$ implying that $\langle p, \Sigma_0\rangle$ is completely contained in $\langle \Sigma_i, \Sigma_j\rangle$. Using the same argument for any element Σ of $\mathcal{R}$, we can conclude that $\mathcal{R}$ is α-geometric.

Now we assume that $\alpha \geq q^{m+1} - q^m$ and $q \geq 3$. Let p be a point of type 1 and let $\Sigma_0, \Sigma_1, \cdots, \Sigma_\alpha$ elements of $\mathcal{R}$ such that $\langle p, \Sigma_i\rangle$ for $i \in \{0, 1, \cdots, \alpha\}$ contain only points of type 0 or 1. Choose p_1 a point of type 1 in $\langle p, \Sigma_0\rangle$, with $p_1 \neq p$. We may assume that $\Sigma_1 \cap \langle p, p_1\rangle = \emptyset$. It follows that $|\{\Sigma_1, \cdots, \Sigma_\alpha\}| = \alpha \geq q^{m+1} - q^m = q^m(q-1) > q-1$. If $m = 1$, then $\langle \Sigma_1, p, p_1\rangle$ is a 3-dimensional subspace and every line in $\{\Sigma_0, \Sigma_1, \cdots, \Sigma_\alpha\}$ has a point in common with $\langle p, \Sigma_1\rangle$ as well as with $\langle p_1, \Sigma_1\rangle$. This implies that all these lines lie in $\langle \Sigma_1, p, p_1\rangle$. Hence $\mathcal{R}$ is α-geometric. If $m > 1$, choose a point p_2 of type 1 in $\langle p, \Sigma_1\rangle$. As $\alpha \geq q^{m+1} - q^m$, then we may assume that $\Sigma_2 \cap \langle p, p_1, p_2\rangle = \emptyset$. If $m = 2$, then $\langle \Sigma_2, p, p_1, p_2\rangle$ is a 5-dimensional subspace and every plane in $\{\Sigma_0, \Sigma_1, \cdots, \Sigma_\alpha\}$ intersects each of the spaces $\langle p, \Sigma_2\rangle$, $\langle p_1, \Sigma_2\rangle$ and $\langle p_2, \Sigma_2\rangle$. This implies that all these planes lie in $\langle \Sigma_2, p, p_1, p_2\rangle$. Hence $\mathcal{R}$ is α-geometric. Now suppose that $m > 2$. Since $\alpha \geq q^{m+1} - q^m > q^m + q^{m-1} + \cdots + q + 1 - m - 1$, we can continue this process till we have chosen a point p_m of type 1 in $\langle p, \Sigma_m\rangle$ and an m-dimensional space Σ_m disjoint from $\langle p_1, \cdots, p_m\rangle$. Consider the space $\langle \Sigma_m, p_1, \cdots, p_m\rangle$, then every element of $\{\Sigma_0, \cdots, \Sigma_\alpha\} \setminus \{\Sigma_m\}$ intersects the spaces $\langle p, \Sigma_m\rangle$, $\langle p_1, \Sigma_m\rangle$, $\cdots$, $\langle p_m, \Sigma_m\rangle$ which implies that Σ_i is contained in $\langle \Sigma_m, p, p_1, \cdots, p_m\rangle$ for every $i \in \{0, \cdots, \alpha\}$ and that $\mathcal{R}$ is α-geometric.

Now we assume that $q = 2$ and that $\alpha > 2^{m+1} - 2^m$. Define x to be such that $\alpha = 2^{m+1} - x$ (implying $x < 2^m$). We note that we assumed that $\alpha \neq 2^{m+1}$, hence $x > 0$. Define z to be such that $2^z \geq x > 2^{z-1}$ (implying $z \leq m$, $z \geq 1$ and $\alpha < 2^{m+1} - 1$). Finally define k to be such that $kz \geq m > (k-1)z$ (implying $k \geq 1$). Let $p, \Sigma_0, \cdots, \Sigma_\alpha$ as before. We observe that the spaces $\langle p, \Sigma_0\rangle, \cdots, \langle p, \Sigma_\alpha\rangle$ do not contain points of type $i \geq 2$. The number of points of type 1 in $\langle p, \Sigma_0\rangle$ equals $2^{m+1} - \alpha = x > 2^{z-1} > 2^{z-1} - 1$. This means that the points of type 1 are not all in a $(z-1)$-dimensional subspaces and hence they generate at least a z-dimensional subspace. Let π_1 be a z-dimensional subspace of $\langle p, \Sigma_0\rangle$ containing p generated by points of type 1. Suppose there exists an element of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ disjoint from π_1. Without loss of generality let such an element be Σ_1. Then we choose an z-dimensional subspace β of $\langle p, \Sigma_1\rangle$ containing p generated by points of type 1. In this way we can define a $2z$-dimensional subspace $\pi_2 := \langle \pi_1, \beta\rangle$. If there exists an element of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ disjoint from π_2, then, using the same argument as before, we are able to construct a $3z$-dimensional subspace π_3. Now we assume that π_a is an az-dimensional subspace intersecting every element of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ and that π_{a-1} is an $(a-1)z$-dimensional subspace not intersecting every element of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$. We remark that $a \geq 2$ since $z < m+1$ (which implies that $\mid \pi_1 \cap \Sigma_0 \mid = 2^z < \alpha$), a elements of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ intersect π_a in a $(z-1)$-dimensional subspace and the remaining $\alpha + 1 - a$ elements of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ have at least one point in common with π_a. As a consequence, we have

$$(2^{az+1} - 1) - a(2^z - 1) - (1 + az) \geq \alpha + 1 - a.$$

If we replace x by 2^z (which implies that α will be replaced by $2^{m+1} - 2^z$), this yields

$$2^{az+1} - (a-1)2^z - 2 - (z-2)a \geq 2^{m+1} + 1 > 2^{m+1}.$$

It follows that $az + 1 > m + 1$ which implies that $a \geq k$.

First we assume that $a > k$. Then $(a-1)z \geq m$. Since $(a-1)z$ is the dimension of π_{a-1}, we are able to find an m-dimensional subspace in π_{a-1} generated by $m+1$ points $p, p_1, \cdots, p_m$ of type 1, such that there exists an element Σ_i of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ which is disjoint from $\langle p, p_1, \cdots, p_m \rangle$ and such that every $\Sigma_j \in \{\Sigma_0, \cdots, \Sigma_\alpha\} \setminus \{\Sigma_i\}$ intersects each of the spaces $\langle p, \Sigma_i \rangle, \langle p_1, \Sigma_i \rangle, \cdots, \langle p_m, \Sigma_i \rangle$, for every $j \in \{0, \cdots, \alpha\}$. It follows that $\mathcal{R}$ is α-geometric.

Now we assume that $a = k$. Then $(k-1)z < m \leq kz$. Since $(k-1)z$ is the dimension of π_{a-1} and kz is the dimension of π_a, we are able to find an m-dimensional subspace π generated by points of type 1 such that $\pi_{a-1} \subset \pi \subseteq \pi_a$. Define $u := m - (a-1)z > 0$. Suppose that every element of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ intersects π. Since $a - 1$ elements of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ intersect π in an $(z-1)$-dimensional subspace and one intersects π in an (at least) $(u-1)$-dimensional subspace, it follows

$$(2^{m+1} - 1) - (a-1)(2^z - 1) - (2^u - 1) - (m+1) \geq \alpha + 1 - a.$$

If we replace $\alpha + 1$ by $2^{m+1} - 2^z$, then we obtain

$$2^z(2-a) + 2a - 2^u - m - 2 > 0.$$

Since $a \geq 2$, we write a as $2 + \delta$ with $\delta \geq 0$. This yields

$$\delta(2 - 2^z) + 2 - 2^u - m > 0,$$

a contradiction because $z \geq 1$ and $u \geq 1$. This means that there exists an element of $\{\Sigma_0, \cdots, \Sigma_\alpha\}$ not intersecting π. As before we can conclude that $\mathcal{R}$ is α-geometric.

Finally we assume that $\alpha = 2^{m+1} - 2^m$. Let p be a point of type 1 and let X be the set of points of type 1 in $\langle p, \Sigma_0 \rangle$. Furthermore let $l := \Sigma_1 \cap \langle p, \Sigma_0 \rangle$ and $l_1 := \Sigma_2 \cap \langle p, \Sigma_0 \rangle$. Choose $x_1 \in X \setminus \langle l, l_1 \rangle$, with x_1 different from p. It follows that $\langle x_1, \Sigma_1 \rangle \cap \Sigma_2 \neq \emptyset$. Let $l_2 := \langle x_1, \Sigma_1 \rangle \cap \Sigma_2$. If $m = 1$, then l_1, l_2 lie in $\langle \Sigma_0, \Sigma_1 \rangle$ which implies that $\Sigma_2 \subset \langle \Sigma_0, \Sigma_1 \rangle$. If $m > 1$, then we pick a point x_2 in $X \setminus \langle l, l_1, x_1 \rangle$. It follows that $\langle x_2, \Sigma_1 \rangle \cap \langle x_1, l_1, \Sigma_1 \rangle = \Sigma_1$. Also here we have $\Sigma_2 \cap \langle x_2, \Sigma_1 \rangle \neq \emptyset$, and so we define $l_3 := \langle x_1, \Sigma_1 \rangle \cap \Sigma_2$. Since $\alpha = 2^{m+1} - 2^m$, we are able to continue this process till we have chosen a point $x_m \in X \setminus \langle l, l_1, x_1, \cdots, x_{m-1} \rangle$ and defined $l_{m+1} := \Sigma_2 \cap \langle x_m, \Sigma_1 \rangle$, for $i = 3, \cdots, m$. In this way we find that $\Sigma_2 = \langle l_1, \cdots, l_{m+1} \rangle$ which implies that $\Sigma_2 \subset \langle \Sigma_0, \Sigma_1 \rangle$. Hence $\mathcal{R}$ is α-geometric.

STEP 2. Now consider the incidence structure $\mathcal{S}$ with point set $\mathcal{R}$ which has the $(2m+1)$-dimensional subspaces $\langle \Sigma, \Sigma' \rangle$, for Σ, $\Sigma' \in \mathcal{R}$, as lines. The fact that $\mathcal{R}$ is α-geometric implies that the natural incidence is well-defined. We will show that this incidence structure is a projective space.

First we assume that $n > 3m+2$, and we verify the Veblen–Young axioms for projective spaces of dimension at least three. It is clear that two dinstict points of $\mathcal{S}$ lie in exactly one common line of $\mathcal{S}$. Since $\alpha \geq 2$, each line of $\mathcal{S}$ contains at least three points of $\mathcal{S}$. The elements of $\mathcal{R}$ span $\mathrm{PG}(n, q)$ and $n > 3m + 2$, so there exist three non-collinear points in $\mathcal{S}$. Let L and M be two dinstict lines of $\mathcal{S}$ intersecting in the point p, and let S and T be lines of $\mathcal{S}$ which meet L and M in distinct points different from p. Then L, M, S and T are $(2m + 1)$-dimensional subspaces of $\mathrm{PG}(n, q)$, and S and T are contained in $\langle L, M \rangle$. It follows that the dimension of $S \cap T$ is at least m. Suppose that $S \cap T$ contains a point z of type 1. Then there are at least $2\alpha + 1$ subspaces $\langle z, \Sigma \rangle$ containing points of type 0 and points of type 1, namely the ones for which $\Sigma \subseteq S$ or $\Sigma \subseteq T$. This contradicts $t_4 = \alpha$, so $S \cap T$ contains only points of type 0. The points necessarily belong to the same element of $\mathcal{R}$, otherwise $S = T$. Hence S and T intersect in a point of $\mathcal{S}$. We conclude that $\mathcal{S}$ is isomorphic to the structure of points and lines of a projective space $\mathrm{PG}(n', \alpha)$, where $n' \geq 3$ since $n > 3m + 2$.

Now assume that $n \leq 3m+2$. As $n > 2m+1$, and hence $\mathrm{PG}(n, q)$ contains as least 3 disjoint m-dimensional subspaces, we only have to consider the case $n = 3m + 2$. One sees that any two points of $\mathcal{S}$ are incident as subspaces of $\mathrm{PG}(n, q)$ with exactly one common point. Because $\alpha \geq 2$, we can always find four points no three of which are collinear. As a consequence, $\mathcal{S}$ is a projective plane of order α.

In order to prove that $\mathcal{S}$ is desarguesian, we choose a point p of $\mathcal{S}$, and consider three lines L_1, L_2 and L_3 incident with it. Let a_i and b_i be distinct points on L_i different from p, for $i \in \{1, 2, 3\}$. Let $u := \langle a_1, a_2 \rangle \cap \langle b_1, b_2 \rangle$, $v := \langle a_1, a_3 \rangle \cap \langle b_1, b_3 \rangle$ and $w := \langle a_2, a_3 \rangle \cap \langle b_2, b_3 \rangle$. Then we have to show that u, v and w are collinear in $\mathcal{S}$. Recall that the points of $\mathcal{S}$ are m-dimensional subspaces in $\mathrm{PG}(3m+2,q)$. Choose a point p' of $\mathrm{PG}(3m+2,q)$ in p. Then there exists a line L'_i of $\mathrm{PG}(3m+2,q)$ which contains p' and intersects a_i and b_i in points a'_i and b'_i, respectively, for $i \in \{1,2,3\}$. Define $u' := \langle a'_1, a'_2\rangle \cap \langle b'_1, b'_2\rangle$, $v' := \langle a'_1, a'_3\rangle \cap \langle b'_1, b'_3\rangle$ and $w' := \langle a'_2, a'_3\rangle \cap \langle b'_2, b'_3\rangle$. As $u' \in \langle a_1, a_2\rangle \cap \langle b_1, b_2\rangle$, we have $u' \in u$; similary $v' \in v$ and $w' \in w$. If L'_1, L'_2 and L'_3 lie together in a plane of $\mathrm{PG}(3m+2,q)$, then u', v' and u' are collinear because all planes in $\mathrm{PG}(3m+2,q)$ are desarguesian. If L'_1, L'_2 and L'_3 span a 3-dimensional subspace of $\mathrm{PG}(3m+2,q)$, then u', v' and w' are collinear because they lie on the intersection line of the planes $\langle a_1,a_2,a_3\rangle$ and $\langle b_1,b_2,b_3\rangle$. Since the line $\langle u',v'\rangle$ is contained in $\langle u,v\rangle$ and $w' \in \langle u',v'\rangle \cap w$, we find $w \cap \langle u,v\rangle \neq \emptyset$, and consequently $w \subseteq \langle u,v\rangle$. We conclude that α is a prime power and that $\mathcal{S}$ is isomorphic to $\mathrm{PG}(2,\alpha)$.

Actually, from now on we may assume that q is a prime. Indeed, assume that $q = p^h$, $h > 1$; an $(n+1)$-dimensional vector space $V(n+1,q)$ over $\mathrm{GF}(q)$, can be seen as a vector space $V(h(n+1),p)$. The first vector space defines a $\mathrm{PG}(n,q)$, while the latter one defines a $\mathrm{PG}(h(n+1)-1,p)$. Moreover every subspace $\mathrm{PG}(m,q)$ of $\mathrm{PG}(n,q)$, yields a subspace $\mathrm{PG}(h(m+1)-1,p)$ of $\mathrm{PG}(h(n+1)-1,p)$. This mapping is also called *field reduction*. Notice that the image of a distance-regular $(0,\alpha)$-regulus under field reduction is again a distance-regular $(0,\alpha)$-regulus with the property that the geometries arising from both are isomorphic.

Consider two distinct lines L and M in a plane $\Pi := \mathrm{PG}(2,\alpha)$ of $\mathcal{S}$. Without loss of generality suppose that L and M intersect in the element Σ_0 of $\mathcal{R}$. Further choose two points of $\mathcal{S}$, say $\Sigma_1 \neq \Sigma_2$, respectively $\Sigma_3 \neq \Sigma_4$, on L, respectively on M, distinct from Σ_0. Finally let a be a point of $\mathrm{PG}(n,q)$ contained in Σ_0. Then there is a unique line L', respectively M', of $\mathrm{PG}(n,q)$ through a intersecting Σ_1 and Σ_2, respectively Σ_3 and Σ_4. We consider the plane $\pi := \langle L', M'\rangle$ of $\mathrm{PG}(n,q)$. Denote by $\mathcal{P}'$ the set of points b of π for which there exists some $\Sigma \in \mathcal{R}$ such that $b = \Sigma \cap \pi$ (notice that by the fact that $\mathcal{R}$ is α-geometric an element of $\mathcal{R}$ can intersect π in at most one point). Clearly such Σ must be a point of the plane Π. It easily follows (using an analogous argument to that used to show that $u' \in u$ in the proof that $\mathcal{S}$ was desarguesian) that $\mathcal{P}'$ is the point set of a subplane of π, which, by the assumption that q is prime, must coincide with π. Using the fact that $\mathcal{R}$ is α-geometric it now follows that $\mathcal{P}'' = \{\Sigma \in \mathcal{R} \mid \Sigma \cap \pi \neq \emptyset\}$ is the point set of a subplane of order $q = p$ of Π. Consequently $\mathrm{GF}(p)$ is a subfield of $\mathrm{GF}(\alpha)$. Next choose a point $a' \neq a$ in Σ_0. Let L'', respectively M'', be the unique line of $\mathrm{PG}(n,q)$ through a' intersecting Σ_1 and Σ_2, respectively Σ_3 and Σ_4. In an analogue way as above we find a subplane of order q of Π, whose point set must clearly coincide with $\mathcal{P}''$. Hence the line L'' intersects exactly the same elements of $\mathcal{R}$ as L'. From this we can conclude that the m-regulus, determined by any three elements of $\mathcal{R}$, which are contained in a $(2m+1)$-dimensional subspace of $\mathrm{PG}(n,q)$, is completely contained in $\mathcal{R}$. The elements of $\mathcal{R}$ span a subspace of dimension $(n'+1)(m+1)-1$. We also know that the elements of $\mathcal{R}$ span $\mathrm{PG}(n,q)$ (otherwise there are points which have no type), so we obtain $n' = (n+1)/(m+1) - 1$.

From the above we can now conclude that the geometry $T^*_{n,m}(\mathcal{R})$ is a $(0,\alpha)$-geometry satisfying the diagonal axiom. Cuypers [23] has characterised these geometries (generalising a theorem of Wilbrink and Brouwer [79], see also [5]). He proves (Corollary 2.5 of [23]) that the $s+1$ points on a line of the $(0,\alpha)$-geometry can be regarded as points of an affine space (being of dimension $m+1$ in our notation) over $\mathrm{GF}(q)$ and that $\mathrm{GF}(\alpha)$ is a subfield of $\mathrm{GF}(q^{m+1})$. We may conclude that $T^*_{n,m}(\mathcal{R})$ arising from $\mathcal{R}$ is isomorphic to the one arising from $\mathrm{PG}(\frac{n+1}{m+1}+1, \alpha)$ in $\mathrm{PG}(\frac{n+1}{m+1}+1, q^{m+1})$. $\square$

We note that the restriction $\alpha \notin \{1, q^{m+1}-1, q^{m+1}\}$ is not too strong. These cases correspond to the (dual) designs, the reduced copolar spaces (see 1.7.3) and the distance-regular $(0,1)$-geometries.

3.3.3 Graphs of bilinear forms

Let q be any prime power, and let m and d be integers greater than 1. Consider a projective space $\mathrm{PG}(m+d-1,q)$ containing a fixed $(m-1)$-dimensional subspace H. The vertices of the *bilinear forms graph* $H_q(m,d)$ are the $(d-1)$-dimensional subspaces of $\mathrm{PG}(m+d-1,q)$ which are disjoint from H; two vertices are adjacent if and only if they intersect in a $(d-2)$-dimensional subspace. It can be shown that $H_q(m,d)$ is distance-regular with diameter $D := \min\{m,d\}$ and intersection numbers

$$\begin{aligned} b_i &= \frac{(q^d-q^i)(q^m-q^i)}{q-1},\ 0 \le i \le D-1, \\ c_i &= \frac{(q^i-1)q^{i-1}}{q-1},\ 1 \le i \le D. \end{aligned}$$

If we consider $H_q(m,d)$ and dualise in $\mathrm{PG}(m+d-1,q)$, we obtain $H_q(d,m)$, so interchanging m and d yields an isomorphic graph.

This graph can also be defined by taking the set $M_{d\times m}(q)$ of all $d\times m$ matrices over $\mathrm{GF}(q)$ as vertex set, where $A, B \in M_{d\times m}(q)$ are adjacent if and only if the rank of $A-B$ is 1. For more details on the bilinear forms graphs we refer to [7].

The point graph of the distance-regular $(0,q')$-geometry constructed in Subsection 3.3.1 has the same parameters as the bilinear forms graph $H_{q'}(n+$

$1, h)$. We will prove that it is indeed $H_{q'}(n+1, h)$. First we need a rather technical lemma.

Lemma 3.7 *Let $n \geq 1$, let q' be any prime power, and let $q = q'^h$ with $h \in \mathbb{N}\backslash\{0, 1\}$. By field reduction, the projective space* $\mathrm{PG}(n, q)$ *becomes a* $\mathrm{PG}((n+1)h-1, q')$. *There exists an n-dimensional subspace H of* $\mathrm{PG}((n+1)h-1, q')$ *which intersects every $(h-1)$-dimensional subspace of* $\mathrm{PG}((n+1)h-1, q')$ *corresponding to a point of* $\mathrm{PG}(n, q)$ *in at most one point. Moreover, there exists an $((n+1)(h-1)-1)$-dimensional subspace K in* $\mathrm{PG}((n+1)h-1, q')$ *which satisfies the following conditions.*

1. *$H \cap K = \emptyset$.*
2. *K contains no $(h-1)$-dimensional subspace corresponding to a point of* $\mathrm{PG}(n, q)$.
3. *If an $(h-1)$-dimensional subspace corresponding to a point of* $\mathrm{PG}(n, q)$ *intersects H in a point, then it intersects K in an $(h-2)$-dimensional subspace.*

Proof. We introduce coordinates for $\mathrm{PG}((n+1)h-1, q')$. Let γ be a primitive element of $\mathrm{GF}(q)$; then $\{\gamma^{q'^i} \mid i = 0, \ldots, h-1\}$ is a basis for $\mathrm{GF}(q)$ over $\mathrm{GF}(q')$. Also define

$$\gamma^{q'^i}\gamma^{q'^j} =: \sum_{k=0}^{h-1} n_k^{ij}\gamma^{q'^k},$$

where $n_k^{ij} \in \mathrm{GF}(q')$ for all $i, j, k \in \{0, \ldots, h-1\}$. Let $x = (x_0, \ldots, x_n)$ denote a point of $\mathrm{PG}(n, q)$; to fix the ideas, assume $x_0 = 1$. Define $x_i =: \sum_{j=0}^{h-1} x_{ij}\gamma^{q'^j}$ with $x_{ij} \in \mathrm{GF}(q')$ for all $i, j \in \{0, \ldots, h-1\}$. The $(h-1)$-dimensional subspace π_x of $\mathrm{PG}((n+1)h-1, q')$ corresponding to x is spanned by the points having as coordinates the $(n+1)h$-tuples over $\mathrm{GF}(q')$ corresponding to $\gamma^{q'^j}(1, x_1, \ldots, x_n)$, $j \in \{0, \ldots, h-1\}$. As

$$\begin{aligned}
\gamma^{q'^j} x_i &= \sum_{k=0}^{h-1} x_{ik}\gamma^{q'^k}\gamma^{q'^j} \\
&= \sum_{k=0}^{h-1} x_{ik} \sum_{l=0}^{h-1} n_l^{jk}\gamma^{q'^l} \\
&= \sum_{l=0}^{h-1} \left(\sum_{k=0}^{h-1} x_{ik} n_l^{jk}\right)\gamma^{q'^l},
\end{aligned}$$

the subspace π_x is spanned by the points

$$(y^{(j)}_{00}, \ldots, y^{(j)}_{0,h-1}; \ldots; y^{(j)}_{n0}, \ldots, y^{(j)}_{n,h-1}),$$

$j \in \{0, \ldots, h-1\}$, where $y^{(j)}_{0j} = 1$, $y^{(j)}_{0l} = 0$ if $l \neq j$, and $y^{(j)}_{il} = \sum_{k=0}^{h-1} x_{ik} n_l^{jk}$ for all $i \in \{1, \ldots, n\}$ and $l \in \{0, \ldots, h-1\}$. Without loss of generality, we may choose H to be the subspace spanned by the points $(0, \ldots; 1, 0, \ldots, 0; \ldots, 0)$ with a one on the 0-th position of the i-th block, $i \in \{0, \ldots, n\}$, and zeroes elsewhere. Let K be the subspace spanned by the points $(0, \ldots; 0, \ldots, 0, 1, 0, \ldots, 0; \ldots, 0)$ which have a one on the j-th position in the i-th block, $i \in \{0, \ldots, n\}$, $j \in \{1, \ldots, h-1\}$, and zeroes elsewhere. It is obvious that K has dimension $((n+1)(h-1)-1)$ and is disjoint from H, and that it cannot contain an $(h-1)$-dimensional subspace corresponding to a point of $\mathrm{PG}(n,q)$. Now let $(x_{00}, 0, \ldots, 0; \ldots; x_{n0}, 0, \ldots, 0)$ be a point of H. This point is contained in the $(h-1)$-dimensional subspace π_x of $\mathrm{PG}((n+1)h-1, q')$ corresponding to the point $x = (x_{00}, \ldots, x_{n0})$ of $\mathrm{PG}(n,q)$. The intersection of K with the subspace π_x of $\mathrm{PG}((n+1)h-1, q')$ corresponding to x is the $(h-2)$-dimensional subspace spanned by the points $(0, \ldots, 0, x_{00}, 0, \ldots, 0; \ldots; 0, \ldots, 0, x_{n0}, 0, \ldots, 0)$ where x_{i0} is on the j-th position in the i-th block for all $i \in \{0, \ldots, n\}$, $j \in \{1, \ldots, h-1\}$. We conclude that K satisfies the conditions. □

Theorem 3.8 *The point graph of the distance-regular* $(0,q')$*-geometry arising from a* $\mathrm{PG}(n,q')$ *in* $\mathrm{PG}(n,q)$*, with* $n \geq 1$ *and* ${q'}^h = q$*,* $h \in \mathbb{N} \setminus \{0,1\}$*, is isomorphic to the bilinear forms graph* $H_{q'}(n+1,h)$*.*

Proof. Consider all objects in the distance-regular $(0,q')$-geometry arising from a $\mathrm{PG}(n,q')$ in $\mathrm{PG}(n,q)$ over the field $\mathrm{GF}(q')$. The space $\mathrm{PG}(n+1,q)$ in which the geometry lives, becomes a $\mathrm{PG}((n+2)h-1,q')$, and the n-dimensional subspace at infinity becomes a $\mathrm{PG}((n+1)h-1,q')$ which we call Π_∞. The space $\mathrm{PG}(n,q')$ corresponds to an n-dimensional subspace H of Π_∞ which intersects every $(h-1)$-dimensional subspace arising from a point of $\mathrm{PG}(n+1,q)$ in at most one point. Let K be an $((n+1)(h-1)-1)$-dimensional subspace of Π_∞ which satisfies the conditions in Lemma 3.7 with respect to H. Choose an $(n+h)$-dimensional subspace V of $\mathrm{PG}((n+2)h-1,q')$ which intersects Π_∞ in H. Let X denote an $(h-1)$-dimensional subspace of $\mathrm{PG}((n+2)h-1,q')$ which corresponds to a point of $\mathrm{PG}(n+1,q)$ not in the n-dimensional subspace at infinity. Then X is disjoint from Π_∞. Projection of X onto V from K yields an $(h-1)$-dimensional subspace of V disjoint from H. Suppose that two $(h-1)$-dimensional subspaces X and Y corresponding to points of $\mathrm{PG}(n+1,q)$ are projected onto the same $(h-1)$-dimensional subspace of V disjoint from H. Then the $(2h-1)$-dimensional subspace $\langle X, Y\rangle$, which corresponds to a line in $\mathrm{PG}(n+1, q)$, would intersect K in an $(h-1)$-dimensional subspace corresponding to a point of $\mathrm{PG}(n+1, q)$, a contradiction. This means that the projection is a bijection between the point set of the distance-regular $(0, q')$-geometry and the set of $(h-1)$-dimensional subspaces of V disjoint from H. The latter is precisely the vertex set of the bilinear forms graph $H_{q'}(n+1, h)$.

Let X and Y denote $(h-1)$-dimensional subspaces of $\mathrm{PG}((n+2)h-1, q')$ corresponding to collinear points of the geometry. Then $Z := \langle X, Y\rangle \cap \Pi_\infty$ contains a point z of H, and hence intersects K in an $(h-2)$-dimensional subspace. Consequently $\langle K, Z\rangle$ has dimension $(n+1)(h-1)$, and cannot intersect H in more than a point. Let X' and Y' denote the projections of X and Y from K onto V. As $X' \neq Y'$ and

$$\begin{aligned} \langle X', Y'\rangle \cap H &\subseteq \langle K, X, Y\rangle \cap H \\ &= \langle K, Z, X\rangle \cap H \\ &= \langle K, Z\rangle \cap H \\ &= \langle Z \cap H\rangle = \{z\}, \end{aligned}$$

the dimension of $\langle X', Y'\rangle$ must be h, or equivalently, $X' \cap Y'$ is $(h-2)$-dimensional. Thus we see that distinct collinear points of the geometry are adjacent vertices in the bilinear forms graph $H_{q'}(n+1, h)$. As $H_{q'}(n+1, h)$ and the point graph of the geometry have the same valency, the theorem is proved. □

Remarks

1. Theorem 3.8 generalises the well-known fact that the semipartial geometry $T_n^*(\mathcal{B})$ is isomorphic to $H_q^{(n+2)*}$, (see 1.7.7).

2. In the proof of Theorem 3.6 we have used the result of Cuypers [23], characterising $(0, \alpha)$-geometries satisfying the diagonal axiom. Actually, in his paper, he proves, using results proved by Huang in [44], that a distance-regular graph Γ with the same intersection array as $H_q(m, d)$, with $m \geq 2d \geq 6$ and $q \geq 4$, is the point graph of a $(0, \alpha)$-geometry satisfying the diagonal axiom. He is then using arguments similar as in [79] to prove that the geometry is uniquely defined and hence that Γ is isomorphic to $H_q(m, d)$.

3. The geometry constructed in Subsection 3.3.1, seen as a geometry on the bilinear forms graph, is also known as an *attenuated space* [56].

Chapter 4

Characterisations of partial geometries arising from perp-systems

As was mentioned in the first chapter (see Section 1.9), Thas [67] proved that a semipartial geometry can be constructed from every SPG regulus. Inspired by this work, De Clerck, Delanote, Hamilton and Mathon introduced in [26] the concept of a perp-system, which is a special type of SPG-regulus consisting of m-dimensional spaces in $\mathrm{PG}(n, q)$, with no tangent spaces. Hence it yields a partial geometry. From the conditions on the perp-system it follows that $2m + 1 \leq n \leq 3m + 2$ (see further). If $n = 2m + 1$, then the partial geometry is a net. R. Mathon found by computer search a perp-system in $\mathrm{PG}(5, 3)$ yielding a new $\mathrm{pg}(8, 20, 2)$, details can be found in [26]. No example is known for n not equal to one of the bounds. In this chapter we investigate the existence of perp-systems yielding partial geometries with $\alpha = 2, 3$ or 4.

The next session is based on the paper by De Clerck, Delanote, Hamilton and Mathon [26]. We have collected only those main results on which we will rely to prove our small results.

4.1 Overview of known results

4.1.1 Definition and bounds

This section is taken from [26].

Definition

Let ρ be a polarity of $\mathrm{PG}(n, q)$. Define a *partial perp-system* $\mathcal{R}(m)$ to be any set $\{\pi_1, \ldots, \pi_k\}$ of k ($k > 1$) mutual disjoint m-dimensional subspaces of $\mathrm{PG}(n,q)$ such that no π_i^ρ meets an element of $\mathcal{R}(m)$. Hence each π_i is non-singular with respect to ρ. Note that $n \geq 2m+1$.

De Clerck, Delanote, Hamilton and Mathon show that there is an upper bound on the size of a partial perp-system.

Theorem 4.1 *Let $\mathcal{R}(m)$ be a partial perp-system of* $\mathrm{PG}(n,q)$ *equipped with a polarity ρ. Then*

$$|\mathcal{R}(m)| \leq \frac{q^{\frac{n-2m-1}{2}}(q^{\frac{n+1}{2}}+1)}{q^{\frac{n-2m-1}{2}}+1}. \tag{4.1}$$

This upper bound inspires the following definition.

Definition
If a partial perp-system $\mathcal{R}(m)$ of $\mathrm{PG}(n,q)$ meets the upper bound of (4.1), then it is called a *perp-system*.

Results
In [26] the following results are obtained.

1. Every point p of $\mathrm{PG}(n,q)$ not contained in an element of $\mathcal{R}(m)$ is incident with $q^{\frac{n-2m-1}{2}}$ $(n-m-1)$-dimensional spaces π^ρ.

2. If n is even then equality in (4.1) implies that a perp-system can only exist if q is a square.

3. If $\mathcal{R}(m)$ is a perp-system of $\mathrm{PG}(n,q)$, then

$$2m+1 \leq n \leq 3m+2. \tag{4.2}$$

4. If $n = 2m+1$, then a perp-system of $\mathrm{PG}(n,q)$ contains $\frac{q^{m+1}+1}{2}$ elements. In this case q has to be odd and every point not contained in an element of the perp-system is incident with exactly one space π^ρ ($\pi \in \mathcal{R}(m)$), which is an m-dimensional space.

5. If $n = 3m+2$, then a perp-system contains

$$q^{\frac{m+1}{2}}(q^{m+1} - q^{\frac{m+1}{2}} + 1)$$

elements. Hence if m is even then q has to be a square.

4.1.2 Perp-systems and partial geometries

In [26] the following theorem is proved.

Theorem 4.2 *Let $\mathcal{R}(m)$ be a perp-system of* $\mathrm{PG}(n,q)$ *equipped with a polarity ρ, then $\mathcal{R}(m)$ is an SPG-regulus with no tangent spaces.*

As explained in Section 1.9, the generalized linear representation of an SPG-regulus with no tangent spaces is a partial geometry, yielding in this case a partial geometry

$$\mathrm{pg}\left(q^{m+1}-1, \frac{q^{\frac{n-2m-1}{2}}(q^{\frac{n+1}{2}}+1)}{q^{\frac{n-2m-1}{2}}+1}-1, \frac{q^{m+1}-1}{q^{\frac{n-2m-1}{2}}+1}\right).$$

4.1.3 Examples

In this subsection, we recall some examples of perp-systems as they appeared in [26]. First we note that examples are only known for $n = 2m+1$ and $n = 3m+2$.

Perp-systems in $\mathrm{PG}(2m+1,q)$

Assume that $n = 2m+1$, then the general linear representation of a perp-system $\mathcal{R}(m)$ in $\mathrm{PG}(2m+1,q)$ yields a

$$\mathrm{pg}\left(q^{m+1}-1, \frac{q^{m+1}-1}{2}, \frac{q^{m+1}-1}{2}\right),$$

which is a Bruck net of order q^{m+1} and degree $(q^{m+1}+1)/2$.

Note that q is odd and that a net of order q^{m+1} and degree $(q^{m+1}+1)/2$ coming from a perp system $\mathcal{R}(m)$ in $\mathrm{PG}(2m+1,q)$ is in fact a net that is embeddable in an affine plane of order q^{m+1}. Assume that Φ is an m-spread of $\mathrm{PG}(2m+1,q)$, which is a set of m-dimensional subspaces of $\mathrm{PG}(2m+1,q)$ partitioning the point set $\mathrm{PG}(2m+1,q)$, hence $|\Phi| = q^{m+1}+1$. Taking half of the elements of Φ, the general linear representation of them yields a net with the requested parameters. However this does not immediately imply that there exist a polarity ρ such that these $(q^{m+1}+1)/2$ elements form a perp-system with respect to ρ. However examples do exist. Take in $\mathrm{AG}(2,q^{m+1})$ only those lines with slope a square. Let ν be any non-square in $\mathrm{GF}(q^{m+1})$, then the mapping $x \mapsto \nu x$ ($x \neq 0$) extended with $0 \mapsto \infty \mapsto 0$ is an involution and hence a polarity on the line at infinity $\mathrm{PG}(1,q^{m+1})$ of $\mathrm{AG}(2,q^{m+1})$. It can be shown that this yields a perp-system $\mathcal{R}'(m)$ in $\mathrm{PG}(2m+1,q)$, see [26].

Perp-systems in $\mathrm{PG}(3m+2,q)$

We consider now the case $n = 3m+2$. The partial geometry corresponding to a perp-system $\mathcal{R}(m)$ in $\mathrm{PG}(3m+2,q)$ is a

$$\mathrm{pg}\left(q^{m+1}-1,(q^{m+1}+1)(q^{\frac{m+1}{2}}-1),q^{\frac{m+1}{2}}-1\right).$$

Such a partial geometry has the parameters of a partial geometry which is the linear representation $T_2^*(\mathcal{K})$ of a maximal arc $\mathcal{K}$ of degree $q^{\frac{m+1}{2}}$ in a projective plane $\mathrm{PG}(2,q^{m+1})$. As we will see in what follows, there do exist partial geometries related to perp-systems and isomorphic to a $T_2^*(\mathcal{K})$ while there exist partial geometries coming from perp-systems $\mathcal{R}(m)$ in $\mathrm{PG}(3m+2,q)$ that are not isomorphic to a $T_2^*(\mathcal{K})$.

We proceed with the known examples of perp-systems.

1. A perp-system $\mathcal{R}(0)$ in $\mathrm{PG}(2,q^2)$ equipped with a polarity ρ is equivalent to a self-polar maximal arc $\mathcal{K}$ of degree q in the projective plane $\mathrm{PG}(2,q^2)$; i.e. for every point $p \in \mathcal{K}$, the line p^ρ is an exterior line w.r.t. $\mathcal{K}$. The partial geometry $T_2^*(\mathcal{K})$ is a $\mathrm{pg}(q^2-1,(q^2+1)(q-1),q-1)$. In [26] it is proved that in $\mathrm{PG}(2,q^2)$ there exists a self-polar maximal arc of degree q for all even q.
   Note that, by expanding over a subfield, it is possible to obtain an SPG-regulus (with no tangent spaces) from a maximal arc $\mathcal{K}$, but the corresponding partial geometry is isomorphic to $T_2^*(\mathcal{K})$. Hence a self-polar maximal arc of degree q^n in $\mathrm{PG}(2,q^{2n})$ is a perp-system $\mathcal{R}(0)$ in $\mathrm{PG}(2,q^{2n})$, but also yields a perp-system $\mathcal{R}(n-1)$ in $\mathrm{PG}(3n-1,q^2)$ and a perp-system $\mathcal{R}(2n-1)$ in $\mathrm{PG}(6n-1,q)$.

2. A perp-system $\mathcal{R}(1)$ in $\mathrm{PG}(5,q)$ equipped with a polarity ρ will yield a $\mathrm{pg}(q^2-1,(q^2+1)(q-1),q-1)$. Mathon found by computer search such a system M in $\mathrm{PG}(5,3)$ yielding a $\mathrm{pg}(8,20,2)$.
   The set M can be represented as follows. A point of $\mathrm{PG}(5,3)$ is given as a triple abc where a, b and c are in the range 0 to 8. Taking the base 3 representation of each digit then gives a vector of length 6 over $\mathrm{GF}(3)$. Each of the following columns of 4 points corresponds to a line of the set in $\mathrm{PG}(5,3)$.

300	330	630	310	610	440	540	470	570	713	813	343	843
100	103	203	201	101	707	137	134	404	831	531	741	351
700	763	563	821	421	387	827	684	254	157	657	407	717
400	433	833	511	711	247	377	514	674	344	144	184	264

373	773	723	823	353	453	383	583
451	641	381	671	881	761	571	461
177	267	867	187	537	347	227	217
704	424	174	564	214	224	834	654

This set M is the unique perp-system with respect to a symplectic polarity in $\mathrm{PG}(5,3)$ but also with respect to an elliptic orthogonal polarity. The set has many interesting properties.

(i) The stabilizer of the set $\tilde{M}$, the point set covered by elements of M in $\mathrm{PG}(5,3)$ has order 120 and has two orbits on $\tilde{M}$ of size 24 and 60, being the points of 6 and 15 lines, respectively. The group of the $\mathrm{pg}(8,20,2)$ has order 120*729, acts transitively on the points and has two orbits on the lines. Since each line of M generates a spread of lines in $\mathrm{pg}(8,20,2)$ it contains a parallelism. The subgroup of M fixing this parallelism is isomorphic to S_5.

(ii) There are 7 solids S_i in $\mathrm{PG}(5,3)$ which contain 3 lines of M each. The S_i meet in a common line L (disjoint from the lines of M).

(iii) Every point of $\mathrm{PG}(5,3)\setminus M$ is incident with a unique line with 3 points in M. These 280 lines meet each of the 21 lines of M 40 times and each pair of lines 4 times, hence forming a 2-(21,3,4) design.

(iv) The set M contains exactly 21 lines of $\mathrm{PG}(5,3)$, these lines form a partial spread. $\mathrm{PG}(5,3)\setminus M$ contains exactly 21 solids of $\mathrm{PG}(5,3)$, these solids intersect mutually in a line, and there are exactly 3 solids through any point of $\mathrm{PG}(5,3)\setminus M$. An exhaustive computer search by R. Mathon established that any set of 21 solids in $\mathrm{PG}(5,3)$ satisfying the above properties is isomorphic to the complement of the set M.

Note that the related partial geometry $\mathrm{pg}(8,20,2)$ has the same parameters as the linear representation $T_2^*(\mathcal{K})$ of a maximal arc $\mathcal{K}$ of degree 3 in $\mathrm{PG}(2,9)$ which can not exist by [1], but that was already proved for this small case by Cossu [20].

4.2 A characterisation of the partial geometry of Mathon

Theorem 4.3 *Suppose that* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ *is a partial geometry* $\mathrm{pg}(s,t,2)$, *with* $t>2$, *which is a generalized linear representation of a perp-system* $\mathcal{R}(m)$. *Then* $\mathcal{S}$ *is isomorphic to the partial geometry* $\mathrm{pg}(8,20,2)$ *of Mathon.*

Proof. Assume that $\mathcal{R}(m)$ consists of m-dimensional subspaces of $\mathrm{PG}(n,q)$, $n > 2m+1$. Since

$$2 = \alpha = \frac{q^{m+1}-1}{q^{\frac{n-2m-1}{2}}+1},$$

$$q^{m+1}-1 = 2q^{\frac{n-2m-1}{2}}+2,$$

which implies that

$$q^{m+1} - 2q^{\frac{n-2m-1}{2}} = 3. \tag{4.3}$$

Assume that $q = p^h$, p prime. From (4.3) it follows that $p = 3$. Substituting $q = 3^h$ in (4.3), we obtain

$$3^{h(m+1)} - 2 \cdot 3^{h(\frac{n-2m-1}{2})} = 3. \tag{4.4}$$

Note that $n \leq 3m+2$, and hence $n-2m-1 \leq m+1$, so we can write (4.3) as

$$3^{\frac{h(n-2m-1)}{2}-1}(3^{\frac{h(4m+3-n)}{2}}-2) = 1$$

Both factors at the left side are positive integers with product 1 hence

$$\frac{h(n-2m-1)}{2} = \frac{h(4m+3-n)}{2} = 1,$$

which implies that $h = 1$, $m = 1$ and $n = 5$ and we can conclude that the perp-system $\mathcal{R}(m)$ consists of 21 lines of $\mathrm{PG}(5,3)$.

Now assume that $\mathcal{R}(1) = \{L_0, \cdots, L_{20}\}$ and denote by $L_i^{\perp}$ the image of L_i under the action of the polarity $\perp$, $i = 0, 1, \cdots, 20$. Let $\widetilde{\mathcal{R}(1)}$ be the set consisting of all points contained in the elements of $\mathcal{R}(1)$. The average number of 3-dimensional subspaces $L_i^{\perp}$ through a point of $\mathrm{PG}(5,3) \setminus \widetilde{\mathcal{R}(1)}$ equals

$$\frac{|\ \mathcal{R}(1)\ | \cdot |\ \mathrm{PG}(3,3)\ |}{|\ \mathrm{PG}(5,3) \setminus \widetilde{\mathcal{R}(1)}\ |}.$$

After dualising, this means that the average number of elements of $\mathcal{R}(1)$ in a hyperplane not containing any $L_i^{\perp}$, $i = 0, 1, \cdots, 20$, is also 3. Now we assume that there exists a hyperplane Π which contains at least an arbitrary element L_0 of $\mathcal{R}(1)$. Then Π does not contain any $L_i^{\perp}$, because the intersection in $\mathrm{PG}(5,3)$ of any two 3-dimensional subspaces is at least a line. If Π contains also another k elements of $\mathcal{R}(1)$ different from L_0, then the number of points of $\widetilde{\mathcal{R}(1)}$ in $\Pi \setminus L_0$ is $4k + (20-k) = 3k + 20$. On the other hand each of the thirteen planes of Π through L_0 contains exactly two points of $\widetilde{\mathcal{R}(1)}$ not on L_0. Hence $3k + 20 = 26$, which implies that $k = 2$. This means that each hyperplane not containing any $L_i^\perp$, $i = 0, 1, \cdots, 20$, contains 0 or 3 elements of $\mathcal{R}(1)$. As the average number of these hyperplanes is also 3, it is clear that the number of elements of $\mathcal{R}(1)$ contained in a hyperplane which does not contain any $L_i^\perp$, $i = 0, 1, \cdots, 20$, does not equal 0. According to the computer result of Mathon (see (iv) on page 65), we conclude that $\widetilde{\mathcal{R}(1)}$ is isomorphic to the set consisting of all points contained in the elements of the perp-system constructed by Mathon.

We still have to show that $\mathcal{R}(1)$ determines the unique partition of $\widetilde{\mathcal{R}(1)}$ in disjoint lines. To that end, we assume that there exists a line M of $\mathrm{PG}(5,3)$ which intersects four elements L_0, L_1, L_2, L_3 of $\mathcal{R}(1)$. Then the plane $\langle L_0, M\rangle$ contains at least three other points of $\widetilde{\mathcal{R}(1)}$ not on L_0. This is a contradiction and we conclude that $\mathcal{R}(1)$ is isomorphic to the perp-system defined by Mathon. □

4.3 On partial geometries with $\alpha = 3$ arising from perp-systems

In this subsection, we investigate the partial geometries with $\alpha = 3$ arising from a perp-system and which are not nets.

Let $\mathcal{R}(m)$ be a perp-system of $\mathrm{PG}(n,q)$, $n > 2m+1$, consisting of m-dimensional subspaces of $\mathrm{PG}(n,q)$. Since

$$3 = \frac{q^{m+1} - 1}{q^{\frac{n-2m-1}{2}} + 1},$$

we obtain

$$4 = q^{m+1} - 3q^{\frac{n-2m-1}{2}}. \tag{4.5}$$

This can be written as

$$q^{\frac{n-2m-1}{2}}(q^{\frac{4m+3-n}{2}} - 3) = 4 \tag{4.6}$$

Both factors at the left side are positive integers, as the second factor cannot be 2 the only possibility is

$$q^{\frac{n-2m-1}{2}} = 4 \quad \text{and} \quad q^{\frac{4m+3-n}{2}} = 4$$

This yields the following possible solutions for the triple (q, m, n): $(16, 0, 2)$, $(4, 1, 5)$, $(2, 3, 11)$. In the three cases the partial geometries is a $\mathrm{pg}(15, 51, 3)$. An example is given by the linear representation of a self-polar maximal arc of degree 4 in PG$(2,16)$, which yields by expanding over the subfield GF(4), a perp-system $\mathcal{R}(1)$ in PG$(5,4)$, and by expanding over the subfield GF(2), a perp-system $\mathcal{R}(3)$ in PG$(11,2)$.

We do not know whether there exist other perp-systems $\mathcal{R}(1)$ in PG$(5,4)$ and $\mathcal{R}(3)$ in PG$(11,2)$.

4.4 On partial geometries with $\alpha = 4$ arising from perp-systems

As in the previous subsection, here we study the partial geometries with $\alpha = 4$ arising from a perp-system and which are not nets.

Let $\mathcal{R}(m)$ be a perp-system of PG(n,q), $n > 2m+1$, consisting of m-dimensional subspaces of PG(n,q), yielding a pg$(s,t,4)$ then

$$5 = q^{m+1} - 4q^{\frac{n-2m-1}{2}}. \tag{4.7}$$

Similarly to the former case, we can rewrite this as

$$q^{\frac{n-2m-1}{2}}(q^{\frac{4m+3-n}{2}} - 4) = 5 \tag{4.8}$$

The only solution being

$$q^{\frac{n-2m-1}{2}} = 5 \quad \text{and} \quad q^{\frac{4m+3-n}{2}} = 5,$$

yielding two possible solutions $(q,m,n) = (5,1,5)$ or $(25,0,2)$. As a maximal arc of degree 5 does not exist in PG$(2,25)$, we can conclude that a perp-system yielding a partial geometry with $\alpha = 4$, which is not a net, consists of a set of 105 lines of PG$(5,5)$ and the corresponding geometry, if it exists, is a pg$(24,104,4)$. We have no idea whether such a perp-system exists.

Chapter 5

A group theoretical approach to $(0, 2)$-geometries

The ideas for the following chapter were inspired by the extensive literature on the groups of the projective planes and generalized polygons (see [46] and [74] as general references). Much work has been done on the study of the groups of geometries by considering particular, very symmetrical, automorphisms of the geometry. For example, there are two ways in which one can define a group related to a generalized polygon. Either one looks at the *automorphism group* (sometimes called the *collineation group*), or one considers the group of *projectivities*. The latter is the group of permutations of all points on a line arising from the bijections between the point sets of two opposite lines given by a pair of points being not opposite. The existence and special properties of the relation "being opposite" is essential in this context. Consequently, the notion of "group of projectivities" has only been considered for generalized polygons (as a generalization of this notion for projective planes; more generally, one can consider spherical or twin buildings, but, to our knowledge, there has been very little, done in this direction in the literature).

In this chapter, we will investigate the automorphism group of $(0, 2)$-geometries, and we will observe that $(0, 2)$-geometries also have a special geometric property that enables one to define a group of projectivities in a very natural way. It also allows us to characterise some classical nets by means of that group.

Moreover, the geometry of $(0, 2)$-geometries permits us to define Moufang-like conditions. We introduce these conditions, develop some theory, and prove some characterisation theorems.

It is a valuable exercise to compare the notions introduced in the present chapter with the existing notions for generalized polygons; we therefore refer to [71, 74]. We note that generalized polygons were introduced by Jacques Tits [69] and are arguably the most important rank 2 incidence geometries, see [29]. Almost all other rank 2 geometries are modelled after the generalized polygons by weakening some axioms. The generalized polygons earn their status partly because of the properties of their automorphism groups and groups of projectivities. In this respect, the present chapter shows that $(0,2)$-geometries are also fundamental. Unfortunately, no complete classification theorem has yet proved; however many partial results are available.

The results in this chapter will appear in a joint paper with M. R. Brown and H. Van Maldeghem, see [8].

5.1 Perspectivities, projectivities and their duals

Much can be proved about the groups of axial collineations and groups of central collineations of $(0,2)$-geometries by defining and studying projectivities of such geometries. In this section, we use the special properties of a $(0,2)$-geometry to define groups of projectivities, in a similar way as is done for generalized polygons ([47]).

We start with the following definitions.

Definitions

Let $\mathcal{S}$ be a $(0,2)$-geometry. In this chapter, Γ will be the incidence graph of $\mathcal{S}$. Recall, that for a point or line x and a natural number i, we denote by $\Gamma_i(x)$ the set of vertices of the graph Γ at distance i from x. Let L and M be concurrent lines of $\mathcal{S}$ intersecting in the point x. Recall that by definition, for every line L and every point x of $\mathcal{S}$ not incident with L, i.e. (x,L) is an antiflag, there are either exactly two lines incident with x and concurrent with L, or no line is incident with x and concurrent with L. This property implies that for every point $y \neq x$ on L there is a unique point $y^{\pi_{L,M}} \neq x$ on M collinear with y. If we thus define the mapping $\pi_{L,M} : \Gamma_1(L) \rightarrow \Gamma_1(M)$, with $x^{\pi_{L,M}} = x$, then $\pi_{L,M}$ is a bijection with inverse $\pi_{M,L}$. Such a bijection is called a *perspectivity*, and the composition of two or more perspectivities $\pi_{L_1,L_2}\pi_{L_2,L_3}\ldots\pi_{L_{n-1},L_n}$, for lines $L_1, L_2, \ldots, L_n$, with L_i concurrent with L_{i+1}, for $i = 1, \cdots, n-1$, is called a *projectivity*. If $L_1 = L_n$, the projectivity is called a *self-projectivity of* L_1 and the set of all self-projectivities of a line L forms a permutation group under the usual composition, called the *group of projectivities of* L and denoted $\Pi(L)$.

Proposition 5.1 *The group* $\Pi(L)$ *of projectivities of* L *is independent of the line* L.

Proof. Let L and M be two lines of $\mathcal{S}$. First we suppose that L and M intersect in the point p. Then clearly $\Pi(L)^{\pi_{L,M}} = \Pi(M)$. Then, considering a path of consecutive concurrent lines joining L and M, the result follows by connectivity of the block graph of $\mathcal{S}$. □

Since we have proved that the group $\Pi(L)$ is independent of the choice of L, we denote this group by $\Pi(\mathcal{S})$, which is called the *group of projectivities of* $\mathcal{S}$.

As usual, one can also restrict to the self-projectivities that are composed of an even number of perspectivities. We thus obtain the *special group of projectivities* of the line L and denote this group by $\Pi^+(\mathcal{S})$. It is a subgroup of index 1 or 2 of $\Pi(\mathcal{S})$.

Theorem 5.2 $\Pi^+(\mathcal{S})$ *is a normal subgroup of* $\Pi(\mathcal{S})$.

Proof. It suffices to prove that the group $\Pi^+(\mathcal{S})$ is normalized by $\Pi(\mathcal{S})$. Let σ be an element of $\Pi(\mathcal{S})$, not in $\Pi^+(\mathcal{S})$; hence σ and its inverse σ^{-1} are the product of an odd number of perspectivities. Let τ be an element of $\Pi^+(\mathcal{S})$; thus τ is the product of an even number of perspectivities. Hence the mapping $\sigma^{-1}\tau\sigma$ is the product of an even number of perspectivities. □

If we consider perspectivities between lines of $\mathcal{S}$ containing a fixed point x, then, for $L \mathrm{I} x$, we obtain the *restricted group of projectivities of* L *relative to* x, and denote this by $\Pi_x(\mathcal{S})$. As the notation suggests, it is independent of the chosen line L. This can be proved similarly to Proposition 5.1. We can also consider the *special restricted* group $\Pi_x^+(\mathcal{S})$ of all elements of $\Pi_x(\mathcal{S})$ that are the composition of an even number of perspectivities between lines containing x (and so not necessarily equal to $\Pi_x(\mathcal{S}) \cap \Pi^+(\mathcal{S})$). Notice that both $\Pi_x(\mathcal{S})$ and $\Pi_x^+(\mathcal{S})$ in general depend on the choice of x.

Remark
All previous notions may be dualised; then we speak of *dual perspectivities* and of the *(special) (restricted) group of dual projectivities.*

5.2 Collineations

It is the aim of this section to give some definitions that are needed to define Moufang conditions for $(0, 2)$-geometries.

Definitions

We recall here some well-known definitions. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. A permutation $\theta : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ that induces a graph automorphism in the incidence graph $\Gamma = (\mathcal{P} \cup \mathcal{L}, \mathrm{I})$ will be called a *correlation of* $\mathcal{S}$. If the correlation θ maps at least one point to a point, then it is a *collineation*. The group of all collineations of $\mathcal{S}$ is denoted by $\mathrm{Aut}\mathcal{S}$; this group may be viewed as a permutation group, either on $\mathcal{P}$, or on $\mathcal{L}$, or on $\mathcal{P} \cup \mathcal{L}$.

If θ is a collineation of $\mathcal{S}$ fixing all points on some line L, then we call L *an axis* of θ. Dually, one defines a *center* of θ.

The following lemma will be responsible for the existence of a rather natural notion of "elation" in $(0,2)$-geometries.

Lemma 5.3 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a* $(0,2)$*-geometry, and let* θ *be a collineation with some axis* L *and some center* x*. If* $x\mathrm{I}L$ *or* $x \in \Gamma_3(L)$*, then* θ *is the identity. If* $x \in \Gamma_5(L)$*, then* θ *has order at most* 2.

Proof. First suppose that $x\mathrm{I}L$. Consider an arbitrary point y collinear with x. We claim that y is fixed under θ. Indeed, this is trivial if $y\mathrm{I}L$. Otherwise, there is a unique point $z\mathrm{I}L$, $z \neq x$, collinear with y. Since y is the unique point on xy collinear with z and different from x, the claim follows. Now, every line M through y meets a unique fixed line through x different from xy; hence y is a center for θ. By connectivity, every point of $\mathcal{S}$ is a center, and so θ is trivial.

Now suppose $x \in \Gamma_3(L)$ and let M be a line through x meeting L in, say, the point y. The line M is fixed and since every point on M is collinear with a unique point of L different from y, we deduce that M is an axis. The assertion now follows from the first paragraph.

Finally, suppose $x \in \Gamma_5(L)$, let $y \in \Gamma_2(x) \cap \Gamma_3(L)$, and let $z \in \Gamma_2(y) \cap \Gamma_1(L)$. There are exactly two points y, y' on xy collinear with z. Hence θ^2 fixes both y and y'. As above, this implies that y is a center of θ^2. The second paragraph of our proof now shows that θ^2 is the identity. □

Remark

In general, using similar arguments as above, one can prove that, if $x \in \Gamma_{2n+1}(L)$, and if x is a center of θ and L an axis, then the order of θ is a divisor of 2^{n-1} $(n \geq 1)$.

Definitions

We call a line L of the $(0,2)$-geometry $\mathcal{S}$ an *axis of transitivity* if, for some point $x\mathrm{I}L$, the group of collineations $G_{[L]}$ with axis L acts transitively on

$\Gamma_1(x) \setminus \{L\}$. Dually, one defines a *center of transitivity*. A $(0,2)$-geometry in which all points are centers of transitivity will be called a $(0,2)$-*geometry with central transitivity*. Dually, one has the notion of $(0,2)$-*geometries with axial transitivity*. An axis of transitivity is called an *elation line* (dually, *elation point*) if, for some point $x\mathrm{I}L$, there is a group $E_{[L]}$ of collineations with axis L acting regularly on $\Gamma_1(x) \setminus \{L\}$. Finally, an elation line is a *Moufang line* (dually, a *Moufang point*) if, for some point $x\mathrm{I}L$, there is a group U_L of collineations with axis L acting regularly on $\Gamma_1(x) \setminus \{L\}$ and such that U_L is normal in $(\mathrm{Aut}\mathcal{S})_L$. If every point and line are Moufang, then $\mathcal{S}$ is called a *Moufang* $(0,2)$-geometry.

A *Moufang set* $\mathcal{M} = (X, G; U_x : x \in X)$ consists of a set X, a permutation group G acting faithfully on X, and for each $x \in X$ a subgroup U_x of the stabilizer G_x of x in G such that

- each U_x is a normal subgroup of G_x and acts regularly on $X \setminus \{x\}$;
- the family $\mathcal{U} := \{U_x : x \in X\}$ is a conjugacy class of subgroups in G;
- the group G is generated by $\mathcal{U}$.

The following lemma is straightforward.

Lemma 5.4 *Let x and y be two collinear Moufang points of the $(0,2)$-geometry $\mathcal{S}$, with corresponding groups U_x and U_y. If $G := \langle U_x, U_y \rangle$, then $\mathcal{M}_{xy} := (\Gamma_1(xy), G; U_x^G)$ is a Moufang set.*

Lemma 5.5 *If a $(0,2)$-geometry $\mathcal{S}$ has two centers of transitivity, x and y and two axes of transitivity, L and M, such that $x\mathrm{I}L\mathrm{I}y\mathrm{I}M$, then $\mathcal{S}$ is a $(0,2)$-geometry with both axial and central transitivity.*

Proof. If s or t equals to 1, the result is trivial; so we suppose that each point and each line is incident with at least three elements.

We first claim that we can map y to any point collinear with y, and M to any line through y. Using $G_{[x]}$ and $G_{[y]}$ we see that all points of L are in the orbit of y. Using $G_{[L]}$ and $G_{[M]}$, the claim follows.

Now let z be an arbitrary point, and let i be such that $z \in \Gamma_i(y)$ (i exists by connectivity). Let $y' \in \Gamma_{i-2}(z) \cap \Gamma_2(y)$. By the previous claim, there is a collineation θ mapping y' to y; hence $z^\theta \in \Gamma_{i-2}(y)$. An inductive argument on i now shows that z is a center of transitivity.

The dual argument completes the proof of the lemma. □

5.3 Some algebraic notions

In this section, we introduce some important algebraic notions which will be useful tools in the sequel.

Some words on loops

A set Q with a binary operation $(x, y) \mapsto x \circ y$ is called a *quasigroup* if for any given $a, b \in Q$ the equations $a \circ y = b$ and $x \circ a = b$ have precisely one solution. If a quasigroup Q has an element 1 with $1 \circ x = x \circ 1 = x$, then it is called a *loop* and 1 is the unit element of Q. For more on loops, we refer to [2] and [10].

Burnside's Theorem

For later purposes, we mention the following result obtained by Burnside.

Theorem 5.6 *Let (X, G) be a permutation group. Suppose that G has k orbits on the set X. Then*

$$k \mid G \mid = \sum_{g \in G} \mid Fix(g) \mid$$

Some words on the Frobenius group

Suppose that (X, G) is a permutation group satisfying the following properties:

- G acts transitively but not regularly on X;
- every non-trivial element of G has at most one fixed point in X.

Then (X, G) is called a *Frobenius group.* If $N \subset G$ is the subset of G consisting of the identity and all those elements of G which act fixed point free on X, then N is called the *Frobenius kernel* of G. To conclude this subsection, we give an important theorem in the field of group theory.

Theorem 5.7 *The Frobenius kernel N of a Frobenius group (X, G) is a normal subgroup of G acting regularly on X.*

5.4 Examples

In this section, we gather some examples of $(0,2)$-geometries with emphasis on the cases with centers and/or axes of transitivity, elation points and/or elation lines, and Moufang points and/or Moufang lines.

5.4.1 Some linear representations

Let $\mathcal{K}$ be a set of points of the projective space $\mathrm{PG}(d,q)$, $d > 0$ and q any prime power, with the property that any line of $\mathrm{PG}(d,q)$ meets $\mathcal{K}$ in either $0, 1$ or 3 points. We also assume that $\mathcal{K}$ spans $\mathrm{PG}(d,q)$ linearly. Consider the linear representation $T_d^*(\mathcal{K})$ of $\mathcal{K}$ (see Section 1.6). Assume that $T_d^*(\mathcal{K})$ is connected. If $\{p, L\}$ is an antiflag of $T_d^*(\mathcal{K})$, then, since any line of $\mathrm{PG}(d,q)$ meets $\mathcal{K}$ in either $0, 1$ or 3 points, the number of points incident with the line L and collinear with the point p is either 0 or 2. Hence $T_d^*(\mathcal{K})$ is a $(0,2)$-geometry (actually, with diameter at least $2d+2$). All points are Moufang points, as is easily verified. Each line of $T_d^*(\mathcal{K})$ is an axis of transitivity if the collineations of $\mathrm{PG}(d,q)$ leaving $\mathcal{K}$ invariant act 2-transitively on $\mathcal{S}$. In the special case where $d = 1$ and hence $|\mathcal{K}| = 3$ for the geometry to be connected, the geometry $T_1^*(\mathcal{K})$ is a net of order q and degree 3. This net has the property that every line is a Moufang line. In this case we will also denote $T_1^*(\mathcal{K})$ by $\Gamma_{1,q}$.

As has been seen in Section 1.6, there are a lot of sets $\mathcal{K}$ known, but we mention here two special cases. One case is when $\mathcal{K}$ is the point set of a projective subspace $\mathrm{PG}(d,2)$ arising from $\mathrm{PG}(d,q)$, q even, by restricting coordinates from $\mathrm{GF}(q)$ down to $\mathrm{GF}(2)$. Another special case arises for $q = 3^e$, $e > 1$, with $\mathcal{K}$ the point set of an affine subspace of $\mathrm{PG}(d,q)$ isomorphic to $\mathrm{AG}(d,3)$ by first deleting a hyperplane of $\mathrm{PG}(d,q)$ and then restricting the coordinates from $\mathrm{GF}(q)$ down to $\mathrm{GF}(3)$.

5.4.2 Some generalizations of linear representations with symmetry

The case $d = 1$

The net $\Gamma_{1,q}$ generalises to any net of degree 3 (which is equivalent to a Latin square (see Section 1.7.6)). Here we give a particular generalisation that has a large collineation group.

Let G be an arbitrary group containing at least three elements. Then the points of the geometry $\mathcal{S}_G$ are all pairs of elements in G. The lines consist of the sets $H_a := \{(g,a) : g \in G\}$, $V_a := \{(a,g) : g \in G\}$ and $D_a := \{(g,ga) : g \in G\}$, for all $a \in G$ (where H, V, D stand for horizontal, vertical and diagonal, respectively). The direct product $G \times G$ acts (on the right) in a natural way on $\mathcal{S}_G$ as a regular permutation group on the point set. Moreover, for each $a \in G$, the involutory mapping $(x,y) \mapsto (ay^{-1}x, ay^{-1}a)$ maps H_b to $H_{ab^{-1}a}$, maps V_b onto $D_{ab^{-1}}$ and maps D_b onto $V_{ab^{-1}}$. Hence it fixes the line H_a pointwise and interchanges V_e, where e is the identity element of G, with D_a. So the line H_a is a Moufang line, and by symmetry, every line V_a is also a Moufang line. It follows that all lines are Moufang lines.

Suppose now that $\mathcal{S}_G$ admits a nontrivial collineation θ fixing a point. By the sharply transitive action of $G \times G$ on the point set of $\mathcal{S}_G$ we may assume that θ fixes the point (e,e). By the above observation that each line is a Moufang line, we may assume that θ fixes all lines through (e,e). Hence θ induces a permutation σ on G via the action $(x,e)^\theta = (x^\sigma, e)$ of θ on the points of the line H_e. We note that $e^\sigma = e$.

Now the vertical line V_a, $a \in G \setminus \{e\}$, is mapped under θ to a vertical line (as every other line meets V_e, which is fixed by θ); hence V_a is mapped onto V_{a^σ}. Similarly the diagonal line $D_{a^{-1}b}$ is mapped onto another diagonal line which must then be $D_{((b^{-1}a)^\sigma)^{-1}}$. Hence the point (a,b), which is the intersection of V_a with $D_{a^{-1}b}$, is mapped onto the point $(a^\sigma, a^\sigma((b^{-1}a)^\sigma)^{-1})$. In view of the fact that horizontal lines must be mapped onto (horizontal) lines, the second coordinate, namely $a^\sigma((b^{-1}a)^\sigma)^{-1}$, is independent of a. Putting $a = b$, we see that it must be equal to b^σ, and we obtain the identity $a^\sigma = b^\sigma(b^{-1}a)^\sigma$, from which it follows that θ is an automorphism of G.

Hence $\mathcal{S}_G$ has centers of transitivity if and only if G admits a transitive automorphism group. In the finite case this is equivalent to G being elementary abelian.

More generally, if in the above construction we allow G to be a quasigroup, then $\mathcal{S}_G$ is a net of degree 3. Further, any net of degree 3 may be constructed in such a manner and in particular with G a loop (see [2] or [10], for instance). For any such net $\mathcal{S}$ a *translation* is a collineation of $\mathcal{S}$ which fixes each of the three parallel classes of $\mathcal{S}$ and each line of one of the parallel classes. The parallel class fixed elementwise is called the *axis* of the translation. If the group of translations with a fixed axis acts transitively on the set of points incident with one of the lines of the axis, then the axis is called *transitive* (not to be confused with an axis of transitivity defined earlier). A collineation β of $\mathcal{S}$ which fixes each of the parallel classes is called a *homology* if all elements of $\langle\beta\rangle$, different from the identity, have exactly one fixed point x which is called the *centre* of β. If the group of homologies with centre x acts transitively on the points, different from x, on a line incident with x, then x is called a *transitive centre* (not to be confused with an centre of transitivity defined earlier). Any elation point or Moufang point of $\mathcal{S}$ can be shown to be a transitive centre of $\mathcal{S}$. In [2] a Lenz classification for loops and nets is given in terms of transitive axes and transitive centres. The paper [2] also contains many other results connecting the collineation group of a net to the algebraic structure of a loop giving rise to the net.

The affine case

Let G be a group of exponent 3, that is, a group in which every non-identity element has order 3. Let n be a positive integer. We define a geometry $\mathcal{S}_{n,G}$ as follows. The point set of $\mathcal{S}_{n,G}$ is the Cartesian product $G \times G \times \cdots \times G$ ($n+1$ factors). For each pair of nonnegative integers (k,ℓ), with $k+\ell \leq n$, and each n-tuple $(a_1, a_2, \ldots, a_n)$ of elements of G, the set $\{(g, ga_1, ga_2, \ldots, ga_k, g^{-1}a_{k+1}, \ldots, g^{-1}a_{k+\ell}, a_{k+\ell+1}, \ldots, a_n) : g \in G\}$ and every set obtained from this one by permuting the coordinates, but leaving the first coordinate fixed, is a line of $\mathcal{S}_{n,G}$.

If G is elementary abelian, then we obtain exactly the linear representation related to the affine space $\mathrm{AG}(n,3)$ inside $\mathrm{PG}(n,|G|)$. However there exist nonabelian groups of exponent 3, and they give rise to new $(0,2)$-geometries with a transitive group of collineations, and with distance regular point graph. The smallest example arises as the multiplicative group of upper diagonal 3×3 matrices over $\mathrm{GF}(3)$ with 1 on each diagonal entry.

5.4.3 Other representations

Consider the projective space $\mathrm{PG}(d,q)$ embedded as a hyperplane Π_∞ in $\mathrm{PG}(d+1,q)$. Let $\mathcal{K}$ be a set of disjoint n-dimensional subspaces of Π_∞, $0 \leq n < d$. Let $T^*_{d,n}(\mathcal{K})$ be the generalized linear representation of $\mathcal{K}$, see page 26 for the definition. Then $T^*_{d,n}(\mathcal{K})$ is a $(0,2)$-geometry if and only if $\mathcal{K}$ satisfies the following condition.

(C02) For every pair of members $S,T \in \mathcal{K}$, and for every point $x \in T$, there exists a unique element $U \in \mathcal{K} \setminus \{S,T\}$ that meets $\langle S,x\rangle$ nontrivially.

We will assume that the elements of $\mathcal{K}$ generate $\mathrm{PG}(d,q)$, and we will call $\mathcal{K}$ a $(0,2)$-*representation set* of $\mathrm{PG}(d,q)$.

Before we proceed to examples of this structure, we briefly show that $T^*_{d,n}(\mathcal{K})$ is a $(0,2)$-geometry if and only if the condition (C02) is satisfied.

It is trivial to see that $T^*_{d,n}(\mathcal{K})$ is a partial linear space. Let $p \in \mathcal{P}$ and let $L \in \mathcal{L}$ such that p is not incident with L. The $(n+2)$-dimensional subspace Σ spanned by p and L intersects Π_∞ in an $(n+1)$-dimensional subspace H_∞ containing an element S of $\mathcal{K}$ that defines the line L. Let $T \neq S$ be an element of $\mathcal{K}$. Suppose that T intersects H_∞ in a point x. Then in Σ the projective line $\langle p,x\rangle$ spanned by p and x intersects L in a point y. Hence T defines a line M of $T^*_{d,n}(\mathcal{K})$ which contains the projective line $\langle p,x\rangle$ and intersects L in the point y. This means that p and y are collinear in $T^*_{d,n}(\mathcal{K})$. According to the condition (C02), there exists a unique element $U \in \mathcal{K} \setminus \{S,T\}$ that meets $\langle S, x\rangle$ in a point z, not x. If we repeat the above argument for the point z, then it follows that p is collinear with exactly two points on L. Now suppose that T and H_∞ are disjoint, and let M be a line of $T^*_{d,n}(\mathcal{K})$, defined by T and p. If M intersects L in a point y, then the projective line $\langle p, y\rangle$ lies in Σ and hence $\langle p, y\rangle$ intersects H_∞ in a point x. But we also have obtained that the projective line $\langle p, y\rangle$ intersects T, a contradiction. We conclude that $T^*_{d,n}(\mathcal{K})$ is a $(0,2)$-geometry. Conversely, one sees by similar arguments that any $(0,2)$-geometry satisfies (C02).

As an example, consider a spread $\mathcal{K}$ of the generalized quadrangle $Q(4,2)$ naturally embedded in $\mathrm{PG}(4,2)$. Then $\mathcal{K}$ satisfies condition (C02). We will denote the corresponding $(0,2)$-geometry by $T^*_{4,1}(Q(4,2))$.

We note that $T^*_{d,n}(\mathcal{K})$ is a semipartial geometry if and only if for each point x of $\mathrm{PG}(d,q)$ not in any member of $\mathcal{K}$, there are a constant number of members $S \in \mathcal{K}$ such that $\langle x, S\rangle$ intersects two other members of $\mathcal{K}$ nontrivially. In this case the $(0,2)$-representation set is indeed an SPG-regulus (see Section 1.9).

5.5 Some classification results

Let $\mathcal{S}$ be a semipartial geometry $\mathrm{spg}(s,t,2,6)$, hence every two noncollinear points are collinear with exactly $\alpha(\alpha+1)$ points ($\alpha = 2, \mu = 6$). By Theorem 1.6, any proper semipartial geometry $\mathrm{spg}(s,t,2,6)$, satisfying the diagonal axiom is isomorphic to the geometry H_2^{n*}, with point set the set of lines of $\mathrm{PG}(n,2)$ which do not intersect a $\mathrm{PG}(n-2,2)$ H, and as line set the set of planes of $\mathrm{PG}(n,2)$ which intersect H in exactly one point (and with natural incidence). This geometry has both central and axial transitivity.

We note that, as briefly discussed in Section 1.7.7 (see page 20), Wilbrink and Brouwer ([79]) proved that, up to possibly a finite number of exceptions, a proper semipartial geometry with $\mu = \alpha(\alpha+1)$ satisfies the diagonal axiom. In the case $\alpha = 2$, the only exception is $s = t = 28$.

In the rest of this section we are interested in characterisation theorems from group theoretical point of view, for partial geometries $\mathrm{pg}(s,2,2)$, hence for nets of degree 3.

Theorem 5.8 *If $\mathcal{S}$ is a net of degree 3 and of order $s+1$, s odd, with central and axial transitivity, then $\mathcal{S}$ satisfies the diagonal axiom.*

Proof. Let p be any point of $\mathcal{S}$, and let A, B, C be the three lines incident with p. Define a graph G with vertices the set $\Gamma_2(p)$ as follows. Two elements of $\Gamma_2(p)$ are adjacent if they are collinear, but not lying together on one of the lines A, B, C. Let x be any point on A, different from p. Then x is adjacent with exactly two elements of $\Gamma_2(p)$, by the fact that $\alpha = 2$. Hence G consists of disjoint polygons, in particular, $3n$-gons, for fixed natural n (for the definition see Section 1.5). Indeed, by the fact that p is a center of transitivity, all these polygons can be mapped onto each other. Now, since A is an axis of transitivity, we can fix exactly n points of such a polygon, preserving it globally. This is impossible if $n > 2$, since an element of a finite dihedral group that is the group of automorphisms of an n-polygon, can have at most two fixed points on the corresponding polygon. Hence $n \in \{1, 2\}$. If $n = 2$, then s is even, contradicting the hypothesis. Hence $n = 1$, which clearly implies that $\mathcal{S}$ satisfies the diagonal axiom. □

Corollaries

Thas and De Clerck [68] proved that if a net satisfies the diagonal axiom it is isomorphic to the dual of H_q^n (see Section 1.7.7 for the construction). Hence we can formulate the following corollary.

Corollary 5.9 *If $\mathcal{S}$ is a net of degree 3 and of order $s+1$, s odd, with central and axial transitivity, then $\mathcal{S}$ is embeddable in a desarguesian affine plane of order $q = s + 1 = 2^h$. Hence $\mathcal{S}$ arises from $\mathrm{AG}(2, s+1)$ by deleting $s - 1$ parallel classes of lines.*

Proof. By the theorem of Thas and De Clerck, there exists a positive integer $n = h + 1$, such that $\mathcal{S}$ is isomorphic to the dual of H_2^n and hence $s + 1 = 2^{n-1} = 2^h$. The rest of the claim easily follows from the construction of H_2^n. □

For completeness we translate the above result to the equivalent results on loops.

Corollary 5.10 *Let G be a loop of even order and let $\mathcal{S}_G$ be the net of degree 3 constructed from G. If $\mathcal{S}_G$ has central and axial transitivity, then G is an elementary abelian group of order 2^h for some $h \geq 1$.*

Proof. Apply Corollary 17.3 of [2] to Theorem 5.8. □

Remark

Although the results in the Corollaries 5.9 and 5.10 are probably also true for nets of degree 3 and odd order $s+1$, we are not able to prove this. However we can prove the following characterisation theorem under the more restrictive assumption that the net is a Moufang net. We can of course restrict ourselves to the case s even.

Theorem 5.11 *Let $\mathcal{S}$ be a Moufang net of degree 3 and order $s+1$, s even, then $\mathcal{S}$ is isomorphic to $\mathcal{S}_H$, for an abelian group H of odd order.*

Proof. We start by applying Lemma 5.4. So let x and y be two collinear Moufang points of $\mathcal{S}$, and denote the corresponding groups U_x and U_y. If $G := \langle U_x, U_y \rangle$, then $\mathcal{M}_{xy} := (\Gamma_1(xy), G; U_x^G)$ is a Moufang set. Since $\mathcal{S}$ is a net, there are three parallel classes of lines, and we may call them *horizontal, vertical* and *diagonal*, respectively. We also may assume that the line xy is horizontal. Since the groups U_x and U_y fix both the vertical and horizontal class of lines, the group G also fixes each type of parallel class of lines.

Suppose that $G_{x,y}$, the stabilizer of both x and y in G, is nontrivial. We claim that $G_{x,y}$ fixes some point of xy different from x and y. Assume, by way of contradiction, that $G_{x,y}$ does not fix any point on xy except for x and y. There are exactly two points of $\mathcal{S}$, say z_1 and z_2, collinear with both x and y, and not incident with xy (since $\mathcal{S}$ is a net of degree 3). If z_1 were not collinear with z_2, then there would be a point u on xz_2 different from both x and z_2, collinear with z_1, and fixed under $G_{x,y}$. But u would be collinear with a point u' on xy different from both x and y (indeed, $u' \neq y$ since otherwise y is collinear with three points on xz_2), and u' would be fixed under $G_{x,y}$, a contradiction to our assumption. Hence z_1 and z_2 are collinear. It now follows easily that $\mathcal{S}$ satisfies the diagonal axiom. Hence, by [68], $\mathcal{S}$ is isomorphic to the dual of H_2^n and hence $s+1 = 2^{n-1}$, which is even, a contradiction.

Hence, we may suppose that $G_{x,y}$ is trivial. The classification of finite Moufang sets (see [58, 39]) now implies easily that the group G is a sharply 2-transitive group. Let H be the Frobenius kernel of G. Then H preserves both the vertical and the diagonal class of lines. From Burnside's Theorem it easily follows that H fixes every horizontal line.

Indeed, suppose H has k orbits on the set of s horizontal lines distinct from xy. Remember that H has order $s+1$ and all nontrivial elements of H are conjugate. Hence all nontrivial elements of H fix equally many horizontal lines, say m. By Burnside's result, the average number of horizontal lines distinct from xy fixed by a nontrivial element of H is equal to $\frac{k}{s}(s+1)-1$, and since this has to be equal to the integer m, we conclude that $k \geq s$, implying $k = s$ and the claim follows.

Let θ be the nontrivial collineation of $\mathcal{S}$ with axis the unique diagonal line L through x (and swapping the horizontal and vertical lines). Then H^θ fixes all vertical lines and preserves the other two classes of lines. Hence $[H, H^\theta]$ is trivial (because each element of that commutator fixes all vertical and all horizontal lines). Now, for $h \in H$, it is easy to see that hh^θ stabilizes the line xy; hence if we write the group $\langle H, H^\theta \rangle$ as $H \times H^\theta$, and if we identify a point z of $\mathcal{S}$ with the group element of $H \times H^\theta$ taking x to z, then the set $\{(h,h) : h \in H\}$ is a diagonal line.

Clearly, the point $(x,y) \in H \times H^\theta$ is mapped onto the point (xa, yb) by the collineation $(a,b) \in H \times H^\theta$. It follows that, for all $a \in H$, the sets $\{(h, a^\theta) : h \in H\}$, $\{(a, h^\theta) : h \in H\}$ and $\{(h, h^\theta a^\theta) : h \in H\}$ represent all lines of $\mathcal{S}$. We conclude that $\mathcal{S}$ is isomorphic to $\mathcal{S}_H$. □

Remarks

1. From the proof of the above theorem it also easily follows that, if G is a group acting sharply 2-transitively on a set Ω, and if H is the Frobenius kernel, then $\mathcal{S}_H$ is a Moufang net.

2. Let $\mathcal{S}$ be a Moufang net of degree 3 and order $s+1$, s odd, then as all sharply 2-transitive groups are in this case isomorphic to $\mathrm{AGL}_1(s+1)$, the group G in the proof of the above theorem is isomorphic to $\mathrm{AGL}_1(s+1)$ and hence $\mathcal{S}$ is isomorphic to $\mathcal{S}_H$, with H the elementary abelian group of order $s+1$. Hence if s is odd, then the Moufang net admits a unique "Moufang structure". For s is even, as there might be different sharply 2-transitive groups of a given order, some Moufang nets admit different "Moufang structures".

5.6 Perspectivities

In the previous section we have proved that a Moufang net of degree 3 has even order and is embeddable in a desarguesian affine plane of that order. It is easy to see that for such a net, the group of projectivities of a line is a Frobenius group. Indeed, the group is transitive because the projectivity $A \to B \to C \to A$ for a triangle A, B, C interchanges the two intersection points $A \cap B$ and $A \cap C$.

Conversely, suppose a net of degree 3 and even order has a group of projectivities which is a Frobenius group. Let A, B, C be as above, and let D, E be such that A, D, E form a triangle, with $A \cap B = A \cap D$ and $A \cap C = A \cap E$, with $D \neq B$ and $E \neq C$. Let X be the unique line through $B \cap C$ distinct from both B, C. If we assume that $\mathcal{S}$ does not satisfy the diagonal axiom, then X, D, E form a triangle. The projectivity $X \to D \to E \to X$ has an involutory pair and a fixed point (namely, $B \cap C$), which is impossible for a Frobenius group acting on an even number of points.

Hence we have proved the following theorem.

Theorem 5.12 *If $\mathcal{S}$ is a net of degree 3 and even order $s+1$ and if the group of projectivities of $\mathcal{S}$ is a Frobenius group, then $\mathcal{S}$ is embeddable in a desarguesian affine plane of order $s+1=2^h$.*

We note that our definition of projectivity group for a $(0,2)$-geometry is in the case of a net of degree 3, equivalent to that of Barlotti and Strambach in [2]. In [2] there are many more interesting results on the groups of projectivities of nets.

5.7 Moufang $(0,2)$-geometries arising from $(0,2)$-representation sets

For the moment it is not feasible to classify all Moufang $(0,2)$-geometries arising from a linear representation, or arising from a $(0,2)$-representation set. However, there is one subclass that we can handle. We begin with a lemma.

Lemma 5.13 *Let $\mathcal{K}$ be a $(0,2)$-representation set of $\mathrm{PG}(5n-1,q)$, $n \geq 1$, consisting of $(2n-1)$-dimensional subspaces. Then q is even.*

Proof. Consider two distinct elements S,T of $\mathcal{K}$. For x a point of T there exists by definition a unique $U \in \mathcal{K} \setminus \{S,T\}$ meeting $\langle S,x\rangle$ non-trivially and hence a unique element $U \in \mathcal{K} \setminus \{S,T\}$ such that $x \in \langle S,U\rangle$. It follows that the sets $\langle S,U\rangle \cap T$, $U \in \mathcal{K} \setminus \{S,T\}$, form a partition of T. Since the dimension of $\langle S,U\rangle \cap T$ is at least $n-1$ for $U \in \mathcal{K} \setminus \{S,T\}$, it follows that either the dimension of $\langle S,U\rangle \cap T$ is exactly $n-1$ for all $U \in \mathcal{K} \setminus \{S,T\}$ and $|\mathcal{K}| = (q^n+1)+2 = q^n+3$; or that $|\mathcal{K}| = 3$ and the elements of $\mathcal{K}$ are contained in $\langle S,T\rangle$. In the latter case the elements of $\mathcal{K}$ do not generate $\mathrm{PG}(5n-1,q)$ and so do not form a $(0,2)$-representation set of $\mathrm{PG}(5n-1,q)$.

Now, if we project $\mathcal{K}$ from S onto a $(3n-1)$-dimensional subspace $\mathrm{PG}(3n-1,q)$ skew to S, then we obtain a set $\mathcal{K}'$ of q^n+2 subspaces of $\mathrm{PG}(3n-1,q)$, each of dimension $2n-1$ with the properties

(DA1) two distinct elements of $\mathcal{K}'$ intersect in an $(n-1)$-dimensional subspace, and

(DA2) three distinct elements of $\mathcal{K}'$ meet in the empty set.

Consider distinct $S',T' \in \mathcal{K}'$ and put $R = S' \cap T'$. Let x be any point of $\mathrm{PG}(3n-1,q)$ not contained in $S' \cup T'$, and put $R^* = \langle x,R\rangle$. From (DA1) and (DA2) above it follows that each point of R^* is contained in either 0 or

2 elements of $\mathcal{K}'$. Also, every member of $\mathcal{K}'$ distinct from both S' and T' intersects the n-dimensional space R^* in a point (since, if the intersection contained a line, this line would meet R nontrivially, contradicting (DA2)). We now see that the number of elements of $\mathcal{K}'$ distinct from S' and T' is even, hence the lemma. □

A representation set as in the previous lemma will be called *tight*.

We now have the following classification.

Theorem 5.14 *Let $\mathcal{K}$ be a tight $(0,2)$-representation set of lines in $\mathrm{PG}(4,q)$. Suppose that all lines of the corresponding $(0,2)$-geometry $\mathcal{S}$ are Moufang and that the corresponding groups are induced by linear collineations of $\mathrm{PG}(4,q)$. Then $q=2$ and $\mathcal{S}$ is isomorphic to $T^*_{4,1}(Q(4,2))$.*

Proof. By Lemma 5.4, the action of the groups related to two intersecting Moufang lines induced on $\mathcal{K}$ defines a Moufang set on $\mathcal{K}$, with $|\mathcal{K}| = q+3$ odd by Lemma 5.13. By the classification of finite Moufang sets in [58, 39], either $q+2$ is a prime power, implying $q=2$, or $q+3$ is a prime power (and there is a sharply 2-transitive action on $\mathcal{K}$). In any case, the number $(q+3)(q+2)$ must divide the order of the linear collineation group of $\mathrm{PG}(4,q)$, which is $q^{10}(q^4+q^3+q^2+q+1)(q^3+q^2+q+1)(q^2+q+1)(q+1)(q-1)^4$. Since q is even, $q+3$ does not have any nontrivial divisor in common with q, $q-1$ or $q+1$. Furthermore, the only possible nontrivial common divisors of $q+3$ with q^2+q+1, q^3+q^2+q+1 and $q^4+q^3+q^2+q+1$ are 7, 5 and 61, respectively. Hence either $q=2$ or $q=4$. If $q=2$, then the result follows readily from the fact that $\mathrm{PGL}(5,2)$ admits only one conjugacy class of elements of order 5.

Now let $q=4$. By the transitive action on $\mathcal{K}$, there is an element θ of order 5 cyclically permuting the elements of $\mathcal{K}$. Since the number of points, 341, is equal to 5 modulo 7, there are at least 5 fixed points; dually, there are at least 5 fixed hyperplanes of $\mathrm{PG}(4,4)$. It is easy to see that there are exactly five fixed points, and that they are incident with a common line L (otherwise θ is the identity). Dually, θ fixes a plane π and all five hyperplanes through it. The plane π and the line L are skew. Evidently, no member of $\mathcal{K}$ meets π or L. Since $\langle\theta\rangle$ is the Frobenius kernel of a sharply 2-transitive group G acting on $\mathcal{K}$, the stabilizer in G of an element of $\mathcal{K}$ in G fixes L and π; hence it stabilizes the projection of $\mathcal{K}$ from L onto π. But in π, there is no group of order 7 cyclically permuting transitively six lines and fixing one. □

Theorem 5.15 *Let $\mathcal{K}$ be a tight $(0,2)$-representation set of $(2n-1)$-dimensional subspaces in $\mathrm{PG}(5n-1,q)$, $n \geq 2$. Suppose that all lines of*

the corresponding $(0, 2)$*-geometry are Moufang and that all corresponding groups are induced by linear collineations of* $\mathrm{PG}(5n-1, q)$*. Then we have the following cases:*
1. $n = 2$, $q = 2$ *(in* $\mathrm{PG}(9, 2)$*);*
2. $n = 2$, $q = 4$ *(in* $\mathrm{PG}(9, 4)$*);*
3. $n = 3$, $q = 2$ *(in* $\mathrm{PG}(14, 2)$*);*
4. $n = 4$, $q = 2$ *(in* $\mathrm{PG}(19, 2)$*).*

Proof. We have already proved that q is even. As in the previous proof, there is an induced Moufang set on $\mathcal{K}$, and it must arise from a sharply 2-transitive group. Let F be the Frobenius kernel of that group. Then all nontrivial elements of F are mutually conjugate.

Suppose that F is not of prime order. We claim that F fixes a point x. If not, then every element of F acts freely on the point set of $\mathrm{PG}(5n-1, q)$, and hence F is contained in a Singer cycle. But then F is cyclic, a contradiction. The claim follows. Similarly, F fixes a line through x. We can continue this argument until we obtain that F fixes a maximal flag. But then F is contained in a Borel subgroup, which is the normalizer of a Sylow 2-group, and hence the unique prime p that divides $|F|$ also divides $q(q-1)$. Since p is odd, p divides $q-1$, and this contradicts the fact that p also divides q^n+3.

So we have shown that $|F| = q^n + 3 = p$ is a prime. Consequently $q^n + 3$ divides some number $q^i - 1$, for some i, with $n+1 \leq i \leq 5n$. We have now to distinguish between $4n < i \leq 5n$, $3n < i \leq 4n$, $2n < i \leq 3n$ and $n < i \leq 2n$.

We give the details of the case $4n < i \leq 5n$, which is the most involved one. The other cases are similar.

Put $i = 5n - k$. Then, modulo $q^n + 3$, the number $q^i - 1$ is equal to $81q^{n-k} - 1$, and this must be $0 \pmod{q^n + 3}$. Clearly, this first implies

$$81q^{n-k} - 1 \geq q^n + 3,$$

hence $k \leq 6$ if $q = 2$, or $k \leq 3$, if $q = 4$, or $k \leq 2$ if $q = 8$, or $k = 1$ if $q \geq 16$, and $k = 0$ if $q \geq 128$.

In any case, the number q^n+3 divides $81q^n - q^k$, hence it divides $243+q^k$. Since q^k is always a power of 2 and is at most 2^6, we have that q^n+3 divides 244, 245, 247, 251, 259, 275, or 307. Consequently q^n+3 is smaller than 260. The primes of the form $2^j + 3$ not exceeding 259 are $5, 7, 11, 19, 67$ and 131. Since $k < n$, the only possibilities are $(q, n, k) = (2, 2, 1)$, $(q, n, k) = (4, 2, 1)$ and $(q, n, k) = (2, 4, 2)$. These give rise to Cases 1,2 and 4, respectively.

Similarly, the Case $3n < i \leq 4n$ gives rise to $(q, n, k) = (2, 3, 2)$, which is Case 3, and $2n < i \leq 3n$ implies $(q, n, k) = (2, 2, 0)$, which is Case 1 again. Finally, $n < i \leq 2n$ yields $(q, n, k) = (2, 2, 1)$, which is again Case 1. □

Bijlage A

Nederlandse samenvatting

In deze bijlage wordt een overzicht in vogelvlucht gegeven van de bijdrage die door dit proefschrift wordt geleverd tot de theorie van afstandsreguliere meetkunden en een groeptheoretische karakterisering van $(0, 2)$-meetkunden. We behandelen hier enkel de structuren die centraal staan in ons proefschrift.

A.1 Inleiding

Eerst vermelden we een aantal meetkundige structuren die meermaals aan bod komen in dit werk. De meeste van deze meetkunden zijn *eindige incidentiestructuren* die formeel gedefinieerd worden als drietallen $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ (de puntenverzameling) en $\mathcal{B}$ (de blokkenverzameling) eindige verzamelingen zijn en waarbij $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ een (symmetrische) incidentierelatie is. Een incidentiestructuur heet een *partieel lineaire ruimte* als twee verschillende blokken incident zijn met hoogstens één gemeenschappelijk punt.

A.1.1 Afstandsreguliere grafen

Een samenhangend graaf Γ met diameter $d \geq 2$ heet *afstandsregulier* als er getallen b_i, $i \in \{0, \cdots, d-1\}$ en c_i, $i \in \{1, \cdots, d\}$ bestaan zodanig dat aan de volgende voorwaarden voldaan is.

drg1 Γ is regulier met valentie b_0;

drg2 voor elke twee toppen x en y van Γ op afstand $i \in \{1, \cdots, d-1\}$ liggen c_i (respectievelijk b_i) van de toppen adjacent met y op afstand $i-1$ (respectievelijk $i+1$) van x;

drg3 voor elke twee toppen x en y van Γ op afstand d liggen c_d van de toppen adjacent met y op afstand $d-1$ van x.

Het is duidelijk dat $c_1 = 1$. De rij $\{b_0 \cdots, b_{d-1}; c_1, \cdots, c_d\}$ heet *intersectierij*, en de getallen b_i en c_i noemen we de *intersectiegetallen*. Een sterk regulier graaf $\mathrm{srg}(v, k, \lambda, \mu)$ is afstandsregulier met intersectierij $\{k, k-1-\lambda; 1, \mu\}$ en vice versa. Afstandsreguliere grafen worden uitgebreid behandeld in [7].

A.1.2 (α, β)-meetkunden

Een *(α, β)-meetkunde van orde* (s, t) is een eindige samenhangende incidentiestructuur $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ van punten en rechten, die voldoet aan de volgende axioma's.

1. $\mathcal{S}$ is een partieel lineaire ruimte.

2. Elke anti-vlag van $\mathcal{S}$ heeft incidentiegetallen ofwel α ofwel β.

Indien geëist wordt dat er voor elk niet-incident punt-rechte paar (x, L) juist $\alpha = \beta$ (> 0) punten van L bestaan die collineair zijn met x, dan bekomt men de definitie van een *partiële meetkunde*. Partiële meetkunden werden ingevoerd door R.C. Bose [6] en worden in het vervolg aangeduid met de notatie $\mathrm{pg}(s, t, \alpha)$. Partiële meetkunden met $\alpha = 1$ zijn beter bekend als *veralgemeende vierhoeken* en werden gedefinieerd door J. Tits [69]. Als $\alpha = s + 1$, vinden we een *design*. Een partiële meetkunde waarvoor $\alpha = t$, wordt ook wel een *(Bruck)-net van de orde* $s+1$ *en de graad* $t+1$ genoemd. Het puntgraaf van een $\mathrm{pg}(s, t, \alpha)$ met $\alpha \leq s$ is een sterk regulier graaf

$$\mathrm{srg}((s+1)(st/\alpha+1), (t+1)s, s-1+t(\alpha-1), (t+1)\alpha)\,.$$

De voorwaarde $\alpha \leq s$ is nodig om partiële meetkunden uit te sluiten waarin elke twee punten collineair zijn en waarvan het puntgraaf bijgevolg is. Een sterk regulier graaf dat dergelijke parameters heeft, voor zekere positieve natuurlijke getallen s, t en α met $\alpha \leq \min\{s, t+1\}$, heet *pseudo-meetkundig*. Als het werkelijk het puntgraaf van een partiële meetkunde is, wordt het *meetkundig* genoemd. In Paragraaf 1.7.6 pagina 17 worden voorbeelden van partiële meetkunden besproken.

Een *semi-partiële meetkunde* met parameters s, t, α, μ is een $(0, \alpha)$-meetkunde van orde (s, t) $(\alpha > 0)$ zodanig dat er voor elke twee niet-collineaire punten μ $(\mu > 0)$ punten bestaan die collineair zijn met beide punten. Hierbij geldt $s, t \geq 1$, $1 \leq \alpha \leq \min\{s+1, t+1\}$ en $1 \leq \mu \leq (t+1)\alpha$.

Merk op dat als $\mu = (t+1)\alpha$, we een partiële meetkunde hebben. Een *eigenlijke* semi-partiële meetkunde is een semi-partiële meetkunde die geen partiële meetkunde is. Semi-partiële meetkunden werden ingevoerd door Debroey en Thas [33] en worden in het vervolg aangeduid met de notatie $\mathrm{spg}(s,t,\alpha,\mu)$.
Het puntgraaf van een semi-partiële meetkunde $\mathrm{spg}(s,t,\alpha,\mu)$ met $\alpha \leq s$ is een

$$\mathrm{srg}(1+(t+1)s(\mu+t(s+1-\alpha))/\mu,(t+1)s,s-1+t(\alpha-1),\mu)\,.$$

Een sterk regulier graaf dat dergelijke parameters heeft, voor zekere positieve natuurlijke getallen $s,t,\alpha,\mu \in \mathbb{N}\setminus\{0\}$ with $s,t \geq 1$, $1 \leq \alpha \leq \min(s,t+1)$, $1 \leq \mu < (t+1)\alpha$, heet *pseudo-semi-meetkundig*. Als het werkelijk het puntgraaf van een semi-partiële meetkunde is, wordt het *semi-meetkundig* genoemd. In Paragraaf 1.7.7 pagina 20 worden voorbeelden van semi-partiële meetkunden beschreven.

Een karakterisering van (semi-)partiële meetkunden

Als eerste stap definiëren we het *diagonaalaxioma* voor een meetkunde $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$.

Diagonaalaxioma Als $x_1 \,\mathrm{I}\, L \,\mathrm{I}\, x_2$, $x_1 \neq x_2$, $y_1 \not\mathrm{I}\, L \not\mathrm{I}\, y_2$, en als x_i collineair is met y_j voor elke $i,j \in \{1,2\}$, dan is y_1 collineair met y_2.

We beperken ons in de hier gegeven opsomming tot de meetkunden die in het kader van deze thesis van belang zijn.

Zij H een $(n-2)$-dimensionale deelruimte van $\mathrm{PG}(n+1,q)$ en beschouw de volgende incidentiestructuur $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$. De puntenverzameling bestaat uit de punten van $\mathrm{PG}(n,q)$ niet in H, de rechtenverzameling bestaat uit alle rechten van $\mathrm{PG}(n,q)$ die geen enkel punt gemeen hebben met H; incidentie is gewoon de incidentie van $\mathrm{PG}(n,q)$. De aldus geconstrueerde partiële meetkunde heeft parameters $s = \alpha = q$, $t = q^{n-1}-1$ (en is dus het duale van een Bruck net) en wordt in het vervolg aangeduid met de notatie H_q^n. Beschouw nu de volgende incidentiestructuur $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$. De puntenverzameling bestaat uit de rechten van $\mathrm{PG}(n,q)$ disjunct van H; de rechtenverzameling bestaat uit alle vlakken van $\mathrm{PG}(n,q)$ die H in juist één punt snijden; incidentie is gewoon de incidentie van $\mathrm{PG}(n,q)$. De aldus geconstrueerde semi-partiële meetkunde heeft parameters $s = q^2-1$, $t = (q^{n-1}-1)/(q-1)-1$, $\alpha = q$ en $\mu = q(q-1)$ en wordt in het vervolg aangeduid met de notatie $H_q^{n^*}$. Merk op dat de dualen van H_q^n enerzijds en

$H_q^{n^*}$ anderzijds, beiden voldoen aan het diagonaalaxioma. In [32] onderzocht Debroey de semi-partiële meetkunden die aan het diagonaalaxioma voldoen. De resultaten voor dergelijke semi-partiële meetkunden worden door de volgende stellingen gegeven.

Stelling A.1 (Theorem 1.6) *Zij* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *een (eigenlijke) semi-partiële meetkunde met parameters* $s, t, \alpha\,(t > \alpha > 1)$ *en* $\mu = \alpha(\alpha+1)$*. Als* $\mathcal{S}$ *aan het diagonaalaxioma voldoet, dan is* $\mathcal{S}$ *isomorf met* $H_q^{n^*}$*.*

Stelling A.2 (Theorem 1.7) *Zij* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *een eigenlijke semi-partiële meetkunde met* $\mu = \alpha^2$*.*

1. *Als* $\alpha = t + 1$*, dan* $\mathcal{S} \cong U_{2,3}(n)$*.*

2. *Als* $2 < \alpha = s$*, dan* $\alpha = t \in \{1, 2, 6, 56\}$ *en* $\mathcal{S} \cong \overline{M(t+1)}$*.*

3. *Als* $2 < \alpha < s$*, dan* $\mathcal{S} \cong \mathrm{LP}(n, q)$*.*

A.1.3 Afstandsreguliere meetkunden

Men kan zich afvragen of er partieel lineaire ruimten bestaan die een afstandsregulier puntgraaf hebben, net zoals sommige sterk reguliere grafen het puntgraaf van een (semi-)partiële meetkunde zijn. Om de axioma's voor een dergelijke incidentiestructuur op te stellen, hebben we de begrippen *punt-diameter, rechte-diameter* en *punt-rechte-diameter* nodig. Dit zijn de maximale afstanden in het incidentiegraaf van een partieel lineaire ruimten tussen twee toppen die respectievelijk twee punten, twee rechten, en een punt en een rechte voorstellen. Punt- en rechte-diameter zijn even, terwijl de punt-rechte-diameter altijd oneven is. Het verschil tussen de punt-rechte-diameter en de punt- of rechte-diameter is 1, terwijl punt- en rechte-diameter ofwel gelijk zijn, ofwel 2 verschillen.

Zij $\mathcal{S}$ een incidentiestructuur met incidentiegraaf Φ die aan de volgende voorwaarden voldoet.

DRG1 $\mathcal{S}$ is een partieel lineaire ruimte van orde (s, t).

DRG2 De punt-diameter van Φ is $2d \geq 4$.

DRG3 Er bestaan getallen α_{2i-1}, $1 \leq i \leq d$, zodanig dat voor elke punt p en voor elke recht L op afstand $2i - 1$ van p in Φ precies α_{2i-1} van de punten incident met L op afstand $2i - 2$ van p liggen.

DRG4 Er bestaan getallen t_{2i}, $1 \leq i \leq d$, zodanig dat voor elke twee punten p en q op afstand $2i$ in Φ precies $t_{2i} + 1$ van de rechten incident met q op afstand $2i - 1$ van p liggen.

Het is niet moeilijk aan te tonen dat het puntgraaf van $\mathcal{S}$ afstandsregulier is met diameter d en intersectiegetallen

$$\begin{aligned} b_0 &= (t+1)s, \\ b_i &= (t - t_{2i})(s + 1 - \alpha_{2i+1}), 1 \leq i \leq d-1, \\ c_i &= (t_{2i} + 1)\alpha_{2i-1}, 1 \leq i \leq d. \end{aligned}$$

Een afstandsreguliere meetkunde met $d = 2$ en $t_4 = t$ is een partiële meetkunde $\mathrm{pg}(s, t, \alpha_3)$; als $d = 2$ en $t_4 < t$, dan hebben we een semi-partiële meetkunde $\mathrm{spg}(s, t, \alpha_3, (t_4 + 1)\alpha_3)$. Afstandsreguliere meetkunden waarin $\alpha_{2i-1} = 1$ voor elke $i \in \{1, \cdots, d\}$ zijn *reguliere schierveelhoeken* (zie [59]). Een *eigenlijke* afstandsreguliere meetkunde is noch een (semi-) partiële meetkunde, noch een reguliere schierveelhoek.

In het algemeen volgen de parameters van een mogelijke afstandsreguliere meetkunde niet eenduidig uit de parameters van het afstandsreguliere graaf. Men kan wel beperkingen op de parameters afleiden. Een telargument leert dat de parameters van afstandsreguliere meetkunden voldoen aan $\alpha_{2i-3} \leq \alpha_{2i-1} \leq s$ en $t_{2i-2} \leq t_{2i} \leq t$ voor elke $i \in \{2, \cdots, d\}$. Als de punt-rechte-diameter van het incidentiegraaf $2d - 1$ is, dan is $t_{2i} = t$, zo niet is $t_{2i} < t$. Andere parameterbeperkingen volgen uit de beperkingen op de intersectiegetallen van afstandsreguliere grafen, zie [7].

A.2 Afstandsreguliere grafen en (α, β)-meetkunden

We onderzoeken het verband tussen afstandsreguliere grafen en (α, β)-meetkunden. Hierbij wordt in twee richtingen gewerkt. Enerzijds zoeken we naar voorwaarden waaraan de meetkunde van een afstandsregulier graaf moet voldoen opdat het een (α, β)-meetkunde zou zijn. Daarnaast beschrijven we een aantal voorbeelden. Anderzijds concentreren we ons op de studie van eigenschappen van bepaalde reguliere grafen die karakteriserend zijn om $(0, \alpha)$-meetkunden uit Taylor grafen te kunnen construeren.

A.2.1 Omgevingsmeetkunden van afstandsreguliere grafen

Zij Γ een graaf, en definieer een incidentiestructuur $\mathcal{S}(\Gamma)$ als volgt. De punten zijn de toppen van Γ, de rechten zijn ook de toppen van Γ (die tussen rechte haakjes worden gezet), en de incidentie is de adjacentie in Γ. In het bijzonder, als x een top van Γ is, dan is het punt x van $\mathcal{S}(\Gamma)$ niet incident met de rechte $[x]$. We noemen $\mathcal{S}(\Gamma)$ de *omgevingsmeetkunde* van Γ. Als Γ niet tweedelig is, dan is $\mathcal{S}(\Gamma)$ op unieke wijze gedefinieerd, en ze is zelf-polair door de afbeelding $x \mapsto [x]$. Als Γ tweedelig is, dan is $\mathcal{S}(\Gamma)$ niet-samenhangend en op dualiteit na uniek gedefinieerd.

Als logisch vervolg op deze constructiemethode zoeken we naar voorwaarden die voldaan moeten zijn opdat een omgevingsmeetkunde van een afstandsregulier graaf met diameter $d \geq 3$ een (α, β)-meetkunde zou zijn.

In Paragraaf A.1.3 definiëren we (α, β)-meetkunden als samenhangende meetkunden. Het is ook mogelijk te veronderstellen dat (α, β)-meetkunden niet samenhangend zijn. De (α, β)-meetkunden uit de volgende stellingen mogen ook niet-samenhangend zijn (zie ook Stelling A.5).

Stelling A.3 (Theorem 2.1) *Zij Γ een afstandsregulier graaf met diameter 3, en zij $\mathcal{S}(\Gamma)$ een omgevingsmeetkunde. Dan is $\mathcal{S}(\Gamma)$ een partieel lineaire ruimte als en slechts als $c_2 = 1$ en $b_1 \in \{k-1, k-2\}$. Bovendien is $\mathcal{S}(\Gamma)$ een (α, β)-meetkunde als en slechts als de intersectierij van Γ van één van de volgende types is.*

- $\{k, k-1, k-1; 1, 1, c_3\}$*; dan* $\alpha = 0$, $\beta = c_3$.
- $\{k, k-1, k-1-c_3; 1, 1, c_3\}$*; dan* $\alpha = 0$, $\beta = c_3$.
- $\{k, k-2, k-1-a_2; 1, 1, k\}$*; dan* $\alpha = a_2 + 1$, $\beta = k$.
- $\{k, k-2, k-c_3; 1, 1, c_3\}$*; dan* $\alpha = c_3$, $\beta = k$.

Stelling A.4 (Theorem 2.2) *Zij Γ een afstandsregulier graaf met diameter $d > 3$, en zij $\mathcal{S}(\Gamma)$ een omgevingsmeetkunde. Dan is $\mathcal{S}(\Gamma)$ een (α, β)-meetkunde (het kan zijn dat ze niet-samenhagend is)als en slechts als de intersectierij van Γ van één van de volgende types is.*

- $\{k, k-1, k-1, b_3, \cdots, b_{d-1}; 1, 1, c_3, \cdots, c_d\}$.
- $\{k, k-1, k-1-c_3, b_3, \cdots, b_{d-1}; 1, 1, c_3, \cdots, c_d\}$.

In beide gevallen geldt $\alpha = 0$ *en* $\beta = c_3$.

Stelling A.5 (Theorem 2.3) *Zij Γ een afstandsregulier graaf met diameter $d > 3$, en zij $\mathcal{S}(\Gamma)$ een omgevingsmeetkunde. Dan bestaat $\mathcal{S}(\Gamma)$ uit twee onderling niet verbonden stukken als en slechts als Γ tweedelig is.*

Hoewel de voorwaarden van Stellingen A.3 en A.4 waaraan een graaf Γ moet voldoen opdat de corresponderende omgevingsmeetkunde $\mathcal{S}(\Gamma)$ een (α, β)-meetkunde zou zijn, tamelijk sterk zijn, vinden we voorbeelden van dergelijke grafen.

Oneven grafen

Zij X een verzameling met $2m + 1$ elementen. De toppen van het *Oneven graaf* (zie [7, Section 9.1]) zijn de deelverzamelingen met m elementen; twee toppen zijn adjacent als en slechts als ze disjunct zijn. De omgevingsmeetkunde van dit graaf kan beschreven worden als volgt. De punten zijn de deelverzamelingen met $m + 1$ elementen, de rechten zijn de deelverzamelingen met m elementen, en de incidentie is de inclusie. We merken op dat het puntgraaf van deze $(0, 2)$-meetkunde het afstandsreguliere *Johnson graaf* $J(2m + 1, m + 1)$ is (zie [7, Section 9.1]).

Grafen uit een symplectische vorm

Zij $V(2, q)$ een twee-dimensionale vectorruimte over $\mathrm{GF}(q)$, q even, en veronderstel dat f een niet-ontaarde symplectische vorm van $V(2, q)$ is. Kies een element $b \in \mathrm{GF}(q) \setminus \{0\}$, en definieer een graaf als volgt. De toppen zijn de elementen van $V(2, q) \setminus \{0\}$; twee toppen u en v zijn adjacent als en slechts als $f(u, v) = b$. Dit graaf is afstandsregulier met diameter 3 en intersectierij $\{q, q - 2, 1; 1, 1, q\}$ (voor de constructie zie [7, Section 12.5]). De vectoren van de vectorruimte $V(2, q)$ kunnen worden beschouwd als de punten van een affien vlak $\mathrm{AG}(2, q)$; veronderstel dat het punt $\infty \in \mathrm{AG}(2, q)$ correspondeert met de nulvector. Voor elke niet-nulvector v correspondeert de verzameling $\{v \in V(2, q) \mid f(u, v) = b\}$ met een rechte van $\mathrm{AG}(2, q)$ niet door ∞. Men kan aantonen dat elke rechte van $\mathrm{AG}(2, q)$ met precies één niet-nulvector overeenkomt. Op deze manier kan de omgevingsmeetkunde uit dit graaf als volgt worden beschreven. De punten zijn de punten van $\mathrm{AG}(2, q)$ verschillend van ∞, de rechten zijn de rechten van $\mathrm{AG}(2, q)$ niet door ∞, en de incidentie is de natuurlijke incidentie. Zo bekomt men een $(q - 1, q)$-meetkunde met $s = t = q - 1$. We merken op dat deze constructie ook voor q oneven werkt. Het puntgraaf van deze meetkunde is het complement van $q + 1$ onderling niet verbonden kopieën van K_{q-1}.

Sporadische voorbeelden

- De *dodecaëder* is een afstandsregulier graaf met diameter 5 en intersectierij $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$. Het voldoet aan de voorwaarden van Stelling A.4 en de meetkunde die ermee overeenkomt, is een $(0, 1)$-meetkunde met $s = t = 2$.

- De veralgemeende zeshoek van orde $(1, 2)$ kan worden beschreven als het duaal van het dubbel van het projectieve vlak $\mathrm{PG}(2, 2)$: de punten zijn de vlaggen van $\mathrm{PG}(2, 2)$; de rechten zijn de punten en de rechten van $\mathrm{PG}(2, 2)$, en de incidentie is de inclusie. Het puntgraaf van deze structuur is afstandsregulier met diameter 3 en intersectierij $\{4, 2, 2; 1, 1, 2\}$; het voldoet aan de voorwaarden van Stelling A.3 en de geassocieerde meetkunde is een $(2, 4)$-meetkunde met $s = t = 3$. Om een mooie beschrijving van deze meetkunde te bekomen, merken we op dat de punten en de rechten van $\mathrm{PG}(2, q)$ niet incident met een element van een bepaalde vlag een unieke vierhoek vormen, die we het complement van de vlag noemen. Als de vlaggen $\{p, L\}$ en $\{p, M\}$ een punt p gemeen hebben, dan is L de diagonaal van het complement van $\{p, M\}$. Als de vlaggen $\{p, L\}$ en $\{q, L\}$ de rechte L gemeen hebben, dan is het punt p het snijpunt van twee overstaande zijden van het complement van $\{q, L\}$. Nu kunnen we een mooier model voor de omgevingsmeetkunde geven: de punten zijn de vlaggen van $\mathrm{PG}(2, 2)$ en de rechten zijn de vierhoeken van $\mathrm{PG}(2, 2)$. Een vlag en een vierhoek zijn incident als zij geen element gemeen hebben, en het punt van de vlag ligt op twee rechten van de vierhoek of de rechte van de vlag bevat twee punten van de vierhoek.

- De toppen van de *Coxeter-graaf* zijn de antivlaggen van $\mathrm{PG}(2, 2)$ (zie [21], [22] en [7, Section 12.3]); twee antivlaggen zijn adjacent als hun unie de puntenverzameling van $\mathrm{PG}(2, 2)$ is. Dit graaf is afstandsregulier met diameter 4 en intersectierij $\{3, 2, 2, 1; 1, 1, 1, 2\}$. Het voldoet aan de voorwaarden van stelling 2.2; we krijgen een $(0, 1)$-meetkunde met $s = t = 2$. Om tot een mooie beschrijving van deze meetkunde te komen, definiëren we het complement van een antivlag als de verzameling van punten en rechten die niet tot de antivlag behoren en niet met elkaar incident zijn. Deze rechten en punten bepalen een unieke driehoek. Als de unie van twee antivlaggen $\{p, L\}$ en $\{q, M\}$ de puntenverzameling van $\mathrm{PG}(2, 2)$ is, dan is p (respectievelijk L) een punt (respectievelijk een rechte) van de driehoek die het complement van $\{q, M\}$ is. Dus kan de omgevingsmeetkunde als volgt worden beschreven: de punten zijn de antivlaggen van $\mathrm{PG}(2, 2)$, de rechten zijn de driehoeken van PG$(2, 2)$, en incidentie is (wederzijdse) inclusie.

- Het *Biggs–Smith-graaf* is in [4], [76] en ook in [7, Section 13.4] beschreven. Het is een afstandsregulier graaf met diameter 7 en intersectierij $\{3, 2, 2, 2, 1, 1, \ 1; 1, 1, 1, 1, 1, 1, 3\}$. Het voldoet aan de voorwaarden van stelling 2.2 en de geassocieerde meetkunde is een $(0, 1)$-meetkunde met $s = t = 2$.

- Het *Perkel-graaf* is (zie [54], [55] en [7, Section 13.3]) is een afstandsregulier graaf met diameter 3 dat door zijn intersectierij $\{6, 5, 2; 1, 1, 3\}$ uniek bepaald is (zie [19]). Dit graaf voldoet aan de voorwaarden van Stelling A.4 en de geassocieerde meetkunde is een $(0, 3)$-meetkunde $s = t = 5$.

A.2.2 Afstandsreguliere meetkunden uit Taylor-grafen

Een *tweegraaf* is een paar (Ω, Δ) dat bestaat uit een eindige *toppenverzameling* Ω en een verzameling Δ van (ongeordende) *coherente drietallen*, zodanig dat elke verzameling van vier toppen een even aantal coherente drietallen bevat. Een verzameling $X \subset \Omega$ heet *coherent* als alle drietallen die in X bevat zijn, tot Δ behoren. Een tweegraaf is *regulier* als elk paar toppen bevat is in hetzelfde aantal coherente drietallen. Dit aantal a en het aantal toppen n noemen we de *parameters* van het tweegraaf. Zij (Ω, Δ) een regulier tweegraaf met parameters n en a, en kies een top ω. Definieer een graaf op $\Omega \setminus \{\omega\}$ als volgt: twee toppen x en y zijn adjacent als en slecht als $\{\omega, x, y\}$ coherent is in (Ω, Δ). Het graaf wordt het *afgeleide graaf* van (Ω, Δ) ten opzichte van ω genoemd; het is een sterk regulier graaf srg$(n - 1, a, (3a - n)/2, a/2)$. Men kan deze constructie ook omkeren: voeg een geïsoleerde top toe aan een sterk regulier graaf srg$(v, k, \lambda, k/2)$, en definieer een drietal toppen coherent als en slecht als het een oneven aantal bogen heeft. Dan is het tweegraaf dat overeenkomt met dit nieuw graaf regulier met parameters $n = v + 1$ and $a = k$.

Een *Taylor-graaf* is een afstandsregulier graaf met diameter 3 en intersectierij $\{k, \mu, 1; 1, \mu, k\}$. Het is een antipodale dubbele bedekking van het complete graaf op $k + 1$ toppen (zie [7]).

Uit een regulier tweegraaf kan een Taylor-graaf geconstrueerd worden en omgekeerd. Deze constructie gaat als volgt. Zij (Ω, Δ) een regulier tweegraaf met parameters n en a, en kies een top ∞. Definieer nu een graaf als volgt: de toppen zijn de elementen x^+ en x^-, voor $x \in \Omega$. Voor elke $x \in \Omega \setminus \{\infty\}$, is de top x^+ (respectievelijk x^-) adjacent met de top ∞^+ (respectievelijk ∞^-).

Voor elke $x, y \in \Omega \setminus \{\infty\}$, zijn $\{x^+, y^+\}$ en $\{x^-, y^-\}$ bogen als en slecht als $\{\infty, x, y\}$ coherent is in (Ω, Δ), terwijl $\{x^+, y^-\}$ en $\{x^-, y^+\}$ bogen zijn als en slecht als $\{\infty, x, y\}$ niet coherent is in (Ω, Δ). Men kan aantonen dat dit een Taylor-graaf is met $k = n - 1$ en $\mu = n - 2 - a$. We merken op dat de constructie onafhankelijk is van de keuze van $\infty \in \Omega$.

Omgekeerd gaan we uit van de veronderstelling dat Γ een Taylor-graaf is met intersectierij $\{k, \mu, 1; 1, \mu, k\}$. Men kan bewijzen dat voor elke top van Γ er een unieke top bestaat die op afstand drie ervan ligt. Noteer met Ω de verzameling van paren van toppen van Γ die op afstand drie van elkaar liggen. Beschouw een 6-verzameling X die uit de unie van drie elementen van Ω bestaat. Dan is het deelgraaf van Γ geïnduceerd op X ofwel een zeshoek ofwel de disjuncte unie van twee driehoeken. Noem een drietal van elementen van Ω coherent als en slechts als het deelgraaf geïnduceerd op de corresponderende verzameling van 6 toppen van Γ de disjuncte unie is van twee driehoeken. Men kan aantonen dat het graaf dat bestaat uit Ω als toppenverzameling en de verzameling van coherente drietallen als bogenverzameling, een tweegraaf is. Door gebruik te maken van het feit dat Γ afstandsregulier is, bewijzen we dat dit tweegraaf regulier is met parameters $n = k + 1$ and $a = k - 1 - \mu$.

Een coherente verzameling in een regulier tweegraaf geeft twee disjuncte cliques in het overeenkomstige Taylor-graaf. Om een meetkunde met dit Taylor-graaf als puntgraaf te kunnen construeren, zoeken we coherente verzamelingen in een tweegraaf die elkaar in hoogstens één punt snijden.

We toetsen de zopas beschreven constructie aan gekende voorbeelden van reguliere tweegrafen die over een geschikte coherente verzameling beschikken.

Hermitische twee-grafen

Zij H een niet-ontaarde hermitische vorm van $\mathrm{PG}(2, q^2)$, q een oneven priemmacht, en noem de overeenkomstige hermitische kromme $\mathcal{U}$. De toppenverzameling van het *hermitische tweegraaf* $\mathcal{H}(q)$ is $\mathcal{U}$; een drietal $\{x, y, z\}$ is coherent als en slechts als $H(x, y)H(y, z)H(z, x)$ een kwadraat (respectievelijk niet-kwadraat) is in $\mathrm{GF}(q^2)$, $q \equiv 3 \pmod 4$ (respectievelijk $q \equiv 1 \pmod 4$). Men kan aantonen dat $\mathcal{H}(q)$ regulier is met parameters $n = q^3 + 1$ en $a = (q - 1)(q^2 + 1)/2$ (zie [61]). Voor elke rechte L van $\mathrm{PG}(2, q^2)$ die $q + 1$ punten van $\mathcal{U}$ bevat, is de verzameling $L \cap \mathcal{U}$ coherent in $\mathcal{H}(q)$. Kies een punt x in $L \cap \mathcal{U}$ en duid met $\mathcal{H}'(q)$ het afgeleide graaf van $\mathcal{H}(q)$ ten opzichte van x aan. Dan is $(L \cap \mathcal{U}) \setminus \{x\}$ een clique in $\mathcal{H}'(q)$ die aan de Hoffman-grens voldoet; dat betekent dat elke top van $\mathcal{H}'(q)$ die niet in deze clique ligt, adjacent is met precies $2^{m-1} - 1$ toppen ervan.

Het Taylor-graaf Γ dat met $\mathcal{H}(q)$ overeenkomt, heeft intersectierij

$$\{q^3, \frac{(q+1)(q^2-1)}{2}, 1; 1, \frac{(q+1)(q^2-1)}{2}, q^3\}.$$

We bewijzen dat de $(q+1)$-secanten van $\mathcal{U}$ kunnen dienen als rechten in een $(0,(q+1)/2)$-meetkunde met Γ als puntgraaf. Deze observatie ligt aan de basis van het bewijs dat de meetkunde met als toppen de toppen van het Taylor-graaf en als rechten de $(q+1)$-secanten van $\mathcal{U}$ een afstandsreguliere $(0,(q+1)/2)$-meetkunde is met parameters $s = q$, $t = q^2 - 1$, $\alpha_3 = (q+1/2)$, $t_4 + 1 = q^2 - 1$, $\alpha_5 = q$, $t_6 + 1 = q^2$.

Ree-tweegrafen

Veronderstel dat $q = 3^{2h+1}$, $h \in \mathbb{N}$, en zij $\mathcal{O}_R$ de Ree-Tits-ovoïde van de veralgemeende zeshoek $H(q)$ (zie [74]), beschouwd als een ovoïde van $Q(6,q)$. Zij π de orthogonale polariteit die geassocieerd is met $Q(6,q)$. Definieer een drietal $\{x,y,z\}$ van punten van $\mathcal{O}_R$ coherent als en slechts als de driedimensionale ruimte $\langle x,y,z\rangle^\pi$ de kwadriek $Q(6,q)$ snijdt in een hyperbolische kwadriek $Q^+(3,q)$. Het tweegraaf dat zo geconstrueerd wordt, heet het *Ree-tweegraaf* $\mathcal{R}(q)$; het is regulier met dezelfde parameters als het hermitische tweegraaf $\mathcal{H}(q)$. Net als de hermitische kromme, kan de Ree-Tits-ovoïde dienen als puntverzameling van een unitaal, en de blokken van de unitaal zijn coherente verzamelingen in het Ree-tweegraaf. Aan de hand van de bovenstaande methode kan een afstandsreguliere $(0,(q+1)/2)$-meetkunde uit het corresponderende Taylor-graaf worden geconstrueerd.

Symplectische twee-grafen

Noteer met $V(2m,2)$, met $m \geq 2$, de $2m$-dimensionale vectorruimte over GF(2), en duid met f een niet-ontaarde alternerende vorm aan. De toppenverzameling van het *symplectische tweegraaf* $\Sigma(2m,2)$ is $V(2m,2)$, en een drietal $\{x,y,z\}$ van toppen is coherent als en slechts als $f(x,y) + f(y,z) + f(z,x) \equiv 0 \pmod 2$. Dit tweegraaf is regulier met parameters $n = 2^{2m}$ en $a = 2^{2m-1} - 2$. Als de nulvector uit $V(2m,2)$ verwijderd wordt, dan levert dit de projectieve ruimte $\mathrm{PG}(2m-1,2)$ op, en de niet-ontaarde symplectische vorm f wordt een symplectische polariteit. De corresponderende symplectische polaire ruimte $W(2m-1,2)$ bevat steeds een spread $\mathcal{S}$ die uit $2^m + 1$ deelruimten van (projectieve) dimensie $m-1$ bestaat. In $V(2m,2)$, wordt $\mathcal{S}$ een verzameling $\mathcal{S}'$ van 2^m+1 m-dimensionale deelruimten. De verzameling $\mathcal{T}$ die uit alle elementen van $\mathcal{S}'$ en alle mogelijke verschuivingen van de elementen van $\mathcal{S}'$ bestaat, heeft $2^m(2^m+1)$ elementen.

De elementen van $\mathcal{T}$ zijn coherente verzamelingen in $V(2m, 2)$ die elkaar in hoogstens één punt snijden. Beschouw een element $X \in \mathcal{T}$, en zij $x \in X$. Dan voldoet de clique $X \setminus \{x\}$ in het afgeleide graaf van $\Sigma(2m, 2)$ ten opzichte van x aan de Hoffman grens; dat betekent dat een top niet in $X \setminus \{x\}$ adjacent is met $2^{m-1} - 1$ toppen in $X \setminus \{x\}$. Door een gelijkaardige redenering als in het hermitische geval, vinden we een $(0, 2^{m-1})$-meetkunde uit het Taylor-graaf dat met $\Sigma(2m, 2)$ overeenkomt, met parameters $s = 2^m - 1$, $t = 2^m$, $\alpha_3 = 2^{m-1}$, $t_4 + 1 = 2^m - 1$, $\alpha_5 = 2^m - 1$, $t_6 = 2^m$.

A.3 Afstandsreguliere $(0, \alpha)$-meetkunden

Gezien het succes van de methode om semi-partiële meetkunden te construeren aan de hand van een SPG-regulus [67] die ermee geassocieerd wordt, veralgemeenden Hamilton en Mathon SPG-reguli tot sterk reguliere (α, β)-reguli [37] van waaruit sterk reguliere (α, β)-meetkunden opgebouwd kunnen worden. Hier pogen we deze techniek te veralgemenen tot de constructie van $(0, \alpha)$-meetkunden. Daarom ontwikkelen we een gelijkaardige methode.

Zij $\mathcal{R}$ een verzameling van m-dimensionale deelruimten van $\mathrm{PG}(n, q)$, $|\mathcal{R}| > 1$. We noemen een punt $z \in \mathrm{PG}(n, q)$ van *type* i als i het kleinste reële getal is zodanig dat z in een i-dimensionale ruimte ligt die door $i + 1$ punten van $i + 1$ verschillende elementen van $\mathcal{R}$ opgespannen is. Stel dat d het maximale getal is zodanig dat er een punt van type $d - 1$ in $\mathrm{PG}(n, q)$ bestaat. Veronderstel dat $\mathcal{R}$ aan de volgende voorwaarden voldoet.

1. De elementen van $\mathcal{R}$ zijn twee aan twee disjunct.

2. Indien een $(m + 1)$-dimensionale ruimte door een element Σ van $\mathcal{R}$, punten van type $i - 1$ en punten van type $i - 2$ (buiten Σ als $i = 2$) bevat, dan bevat het precies α_{2i-1} punten van type $i - 2$ (buiten Σ als $i = 2$), $2 \leq i \leq d$.

3. Indien z een punt van type $i - 1$ is, dan bestaan er precies $t_{2i} + 1$ elementen $\Sigma \in \mathcal{R}$ zodanig dat de $(m + 1)$-dimensionale ruimte $\langle z, \Sigma \rangle$ punten van type $i - 1$ en punten van type $i - 2$ (buiten Σ als $i = 2$), $2 \leq i \leq d$ bevat.

Dan noemt men $\mathcal{R}$ een *afstandsreguliere* $(0, \alpha)$*-regulus* met $\alpha := \alpha_3$. De getallen α_{2i-1} en $t_{2i} + 1$ $(2 \leq i \leq d)$ heten *parameters* van $\mathcal{R}$. We veronderstellen steeds dat $\alpha_3 \leq q^{m+1} - 1$ zodat er minstens punten van type 1 bestaan.

Stelling A.6 (Theorem 3.1) *Zij $\mathcal{R}$ een afstandsreguliere $(0, \alpha)$-regulus in $\mathrm{PG}(n, q)$, waarvan de elementen m-dimensionaal zijn. Bed $\mathrm{PG}(n, q)$ als een hypervlak Π_∞ in een $\mathrm{PG}(n+1, q)$ in en definieer een incidentiestructuur $T^*_{n,m}(\mathcal{R}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ van punten en rechten als volgt.*

- *De puntenverzameling $\mathcal{P}$ bestaat uit alle punten van $\mathrm{PG}(n+1, q)$ die niet in Π_∞ liggen.*
- *De rechtenverzameling $\mathcal{L}$ bestaat uit alle $(m+1)$-dimensionale deelruimten van $\mathrm{PG}(n+1, q)$ die Π_∞ snijden in een element van $\mathcal{R}$.*
- *De incidentie I is gewoon de incidentie van $\mathrm{PG}(n+1, q)$.*

Dan is $T^*_{n,m}(\mathcal{R})$ een afstandsreguliere $(0, \alpha)$-meetkunde met parameters $s = q^{m+1} - 1$, $t = |\mathcal{R}| - 1$, α_{2i-1} en $t_{2i} + 1$ $(2 \leq i \leq d)$.

De zopas beschreven meetkunde $T^*_{n,m}(\mathcal{R})$ wordt een *veralgemeende lineaire representatie van $\mathcal{R}$* genoemd.

De vraag naar het bestaan van afstandsreguliere $(0, \alpha)$-meetkunden die geconstrueerd kunnen worden uit een afstandsreguliere $(0, \alpha)$-regulus kan voorlopig alleen beantwoord worden in twee gevallen die hieronder worden behandeld.

A.3.1 Afstandsreguliere meetkunden uit spreads van parabolische quasi-kwadrieken

Een *parabolische quasi-kwadriek* met *kern n* in $\mathrm{PG}(2m, q)$, q even, is een verzameling $\mathcal{Q}$ van $(q^{2m} - 1)/(q - 1)$ punten zodanig dat elke rechte door het punt n, $\mathcal{Q}$ in een uniek punt snijdt en elk hypervlak niet door n ofwel $(q^m + 1)(q^{m-1} - 1)/(q - 1)$ ofwel $(q^m - 1)(q^{m-1} + 1)/(q - 1)$ punten van $\mathcal{Q}$ bevat. Een *spread* van $\mathcal{Q}$ is een verzameling van q^m+1 $(m-1)$-dimensionale deelruimten die een partitie van $\mathcal{Q}$ definiëren. Dan kunnen we de volgende stelling aantonen.

Stelling A.7 (Theorem 3.3) *Zij $\mathcal{Q}$ een parabolische quasi-kwadriek met kern n in $\mathrm{PG}(2m, q)$, $m \geq 2$ die een spread $\mathcal{R}$ bevat. Als het aantal punten van $\mathcal{Q}$ in een m-dimensionale ruimte die een element Σ van $\mathcal{R}$ bevat en die het punt n niet bevat, constant is, dan is $\mathcal{R}$ een afstandsreguliere $(0, q^{m-1})$-regulus met parameters $\alpha_3 = q^{m-1}$, $t_4 + 1 = q^m$, $\alpha_5 = q^m - 1$ en $t_6 + 1 = q^m + 1$.*

Tenslotte beogen we een karakterisering van deze klasse van afstandsreguliere $(0, \alpha)$-reguli, maar om dit te kunnen bewijzen, moet eerst een bijkomende definitie ingevoerd worden.

Een afstandsreguliere $(0, \alpha)$-regulus *met kern* n in $\mathrm{PG}(2m, q)$ $m \geq 2$, is een afstandsreguliere $(0, \alpha)$-regulus $\mathcal{R}$ waarvan de elementen $(m-1)$-dimensionale deelruimten zijn zodanig dat n het unieke punt van type 2 is en $\mathcal{R}$ geen punt van hoger type bevat. Men kan dan aantonen dat structuren die aan veronderstellingen van de stelling A.7 voldoen, afstandsreguliere $(0, \alpha)$-reguli met kern n zijn.

Aan de hand van de bovenstaande definitie, kunnen we de volgende karakteriserende stelling formuleren.

Stelling A.8 (Theorem 3.4) *Zij* $\mathcal{R}$ *een afstandsreguliere* $(0, \alpha)$*-regulus met kern* n *in* $\mathrm{PG}(2m, q)$, $m \geq 2$. *Dan is verzameling* $\tilde{\mathcal{R}}$ *die uit de unie van alle punten van de elementen van* $\mathcal{R}$ *bestaat, een parabolische quasi-kwadriek met kern* n.

A.3.2 Afstandsreguliere meetkunden uit deelruimten over een deelveld

De constructie van de semi-partiële meetkunde $T_n^*(\mathcal{B})$ met $\mathcal{B}$ een Baer deelruimte $\mathrm{PG}(n, q)$ van $\mathrm{PG}(n, q^2)$ kan worden veralgemeend tot afstandsreguliere meetkunden.

Stelling A.9 (Theorem 3.5) *Zij* q' *een priemmacht, zij* $q = {q'}^h$ *met* $h \in \mathbb{N} \setminus \{0, 1\}$, *en beschouw een* $\mathrm{PG}(n, q')$ *in* $\mathrm{PG}(n, q)$, $n \geq 1$. *Dan is de verzameling die uit alle punten van* $\mathrm{PG}(n, q')$ *bestaat, een afstandsreguliere* $(0, q')$*-regulus met parameters* $\alpha_{2i-1} = {q'}^{i-1}$ *en* $t_{2i} + 1 = ({q'}^i - 1)/(q' - 1)$, $2 \leq i \leq d$, $d := \min\{n + 1, h\}$.

Als logisch vervolg kan men zich afvragen of deze klasse van afstandsreguliere $(0, \alpha)$-reguli ook gekarakteriseerd kan worden. Om een karakteriserende stelling te kunnen formuleren, moet eerst een definitie ingevoerd worden.

Een afstandsreguliere $(0, \alpha)$-regulus waarvan de elementen m-dimensionale deelruimten van $\mathrm{PG}(n, q)$ zijn, wordt α*-meetkundig* [3] (zie ook [27]) genoemd als en slechts als voor elke twee elementen Σ en Σ' van $\mathcal{R}$, de $(2m + 1)$-dimensionale deelruimte $\langle \Sigma, \Sigma' \rangle$ juist $\alpha + 1$ elementen van $\mathcal{R}$ bevat.

Aan de hand van deze definitie bekomen we de volgende karakterisering.

Stelling A.10 (Theorem 3.6) *Zij* $\mathcal{R}$ *een afstandsreguliere* $(0, \alpha)$*-regulus waarvan de elementen* m*-dimensionale deelruimten van* $\mathrm{PG}(n, q)$ *zijn,* $n > 2m + 1$. *Veronderstel dat* $t_4 = \alpha \notin \{1, q^{m+1} - 1, q^{m+1}\}$. *Dan is* α

een macht van q, $\mathrm{GF}(\alpha)$ is een deelveld van $\mathrm{GF}(q^{m+1})$, en de meetkunde $T^*_{n,m}(\mathcal{R})$ opgebouwd vanuit $\mathcal{R}$ is isomorf met de meetkunde geconstrueerd uit $\mathrm{PG}(\frac{n+1}{m+1} - 1, \alpha)$ in $\mathrm{PG}(\frac{n+1}{m+1} - 1, q^{m+1})$.

Als afsluiting van dit onderwerp vermelden we nog een aantal stellingen waaruit volgt dat het puntgraaf van de afstandsreguliere $(0, q')$-meetkunde geconstrueerd uit de bovenvermelde afstandsreguliere $(0, q')$-regulus isomorf is met het graaf van bilineaire vormen $H_{q'}(n + 1, h)$.
Zij q een priemmacht, en zij m en d twee gehele getallen groter dan 1. Beschouw een projectieve ruimte $\mathrm{PG}(m + 1, q)$ die een vaste $(m - 1)$-dimensionale deelruimte H bevat. De toppen van het *graaf van bilineaire vormen* $H_q(m, d)$ zijn de $(d-1)$-dimensionale deelruimten van $\mathrm{PG}(m+d-1, q)$ die disjunct zijn van H; twee toppen zijn adjacent als en slechts als ze snijden in een $(d - 2)$-dimensionale deelruimte. Men kan aantonen dat $H_q(m, d)$ afstandsregulier is met diameter $D := \min\{m, d\}$ en intersectierij

$$\begin{aligned} b_i &= \frac{(q^d - q^i)(q^m - q^i)}{q-1}, \ 0 \le i \le D - 1, \\ c_i &= \frac{(q^i - 1)q^{i-1}}{q-1}, \ 1 \le i \le D. \end{aligned}$$

Lemma A.1 (Lemma 3.7) *Zij $n \ge 1$, zij q' een priemmacht, en zij $q = {q'}^h$ met $h \in \mathbb{N}\backslash\{0, 1\}$. Beschouw de projectieve ruimte $\mathrm{PG}(n, q)$ over het deelveld $\mathrm{GF}(q')$ zodat $\mathrm{PG}(n, q)$ beschouwd kan worden als een $\mathrm{PG}((n + 1)h - 1, q')$. Veronderstel dat H een n-dimensionale deelruimte van $\mathrm{PG}((n+1)h-1, q')$ is die elke $(h-1)$-dimensionale deelruimte van $\mathrm{PG}((n+1)h-1, q')$, die met een punt van $\mathrm{PG}(n, q)$ overeenkomt, in hoogstens één punt snijdt. Dan bestaat er een $((n+1)(h-1)-1)$-dimensionale deelruimte K van $\mathrm{PG}((n+1)h-1, q')$ die aan de volgende voorwaarden voldoet.*

1. *$H \cap K = \emptyset$.*
2. *K bevat geen $(h - 1)$-dimensionale deelruimte die met een punt van $\mathrm{PG}(n, q)$ overeenkomt.*
3. *Als een $(h-1)$-dimensionale deelruimte die met een punt van $\mathrm{PG}(n, q)$ overeenkomt, H in een punt snijdt, dan snijdt ze K in een $(h - 2)$-dimensionale deelruimte.*

Dit lemma ligt aan de basis van de volgende stelling.

Stelling A.11 (Theorem 3.8) *Het puntgraaf van de afstandsreguliere $(0, q')$-meetkunde geconstrueerd vanuit $\mathrm{PG}(n, q')$ in $\mathrm{PG}(n, q)$, met $n \ge 1$ en ${q'}^h = q$, $h \in \mathbb{N}\backslash\{0, 1\}$, is isomorf met het graaf van de bilineaire vormen $H_{q'}(n+1, h)$.*

A.4 Karakterisering van partiële meetkunden opgebouwd vanuit een perp-systeem

Zij ρ een polariteit van $\mathrm{PG}(n,q)$. Definieer een *partieel perp-systeem* [26] $\mathcal{R}(m)$ als een verzameling $\{\pi_1,\ldots,\pi_k\}$ van k ($k>1$) twee aan twee disjuncte m-dimensionale deelruimten van $\mathrm{PG}(n,q)$, zodanig dat geen π_i^ρ een punt gemeen heeft met een element van $\mathcal{R}(m)$. Als de kardinaliteit van $\mathcal{R}(m)$ maximaal is, dan wordt $\mathcal{R}(m)$ een *perp-systeem* genoemd.

Zoals uitgelegd in Paragraaf 1.9 pagina 25 is de veralgemeende lineaire representatie van een SPG-regulus die geen raakruimtes heeft, een semi-partiële meetkunde die in dit geval een partiële meetkunde is:

$$\mathrm{pg}\left(q^{r+1}-1,\frac{q^{\frac{n-2r-1}{2}}(q^{\frac{n+1}{2}}+1)}{q^{\frac{n-2r-1}{2}}+1}-1,\frac{q^{r+1}-1}{q^{\frac{n-2r-1}{2}}+1}\right).$$

In dit hoofdstuk wordt aangetoond dat een aantal van de partiële meetkunden, opgebouwd vanuit een perp-systeem, gekarakteriseerd kunnen worden, namelijk de partiële meetkunde $\mathrm{pg}(8,20,2)$ van Mathon [26] en de partiële meetkunden $\mathrm{pg}(s,t,3)$ en $\mathrm{pg}(s,t,4)$, die geen netten zijn. We bekomen de volgende resultaten.

Stelling A.12 (Theorem 4.3) *Veronderstel dat $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ een partiële meetkunde $\mathrm{pg}(s,t,\alpha)$ is, $t>\alpha$, die een veralgemeende lineaire representatie van een perp-systeem $\mathcal{R}(m)$ is.*

1. *Als $\alpha=2$, dan is $\mathcal{S}$ isomorf met de partiële meetkunde $\mathrm{pg}(8,20,2)$ van Mathon.*
2. *Als $\alpha=3$, dan is $\mathcal{S}$ een $\mathrm{pg}(15,51,3)$.*
3. *Als $\alpha=4$, dan is $\mathcal{S}$ een $\mathrm{pg}(24,104,4)$.*

Merk op dat het niet bekend is of er een $\mathrm{pg}(24,104,4)$ bestaat.

A.5 Groeptheoretische karakterisering van $(0,2)$-meetkunden

In de uitgebreide literatuur over groepen van projectieve vlakken en veralgemeende veelhoeken (zie [46] en [74]), werden veel resultaten bekomen door bijzondere automorfismen van de meetkunde te beschouwen. In dit hoofdstuk hebben wij een analoge studie doorgevoerd voor meetkunden met $\alpha=2$.

A.5.1 Perspectiviteiten, projectiviteiten en hun dualen

We kunnen veel te weten komen over groepen van axiale collineaties en over groepen van centrale collineaties van (semi-)partiële meetkunden met $\alpha = 2$ door de projectiviteiten van deze meetkunden te bestuderen. De motivatie voor dit deel van het onderzoek is de veelheid aan resultaten die op het terrein van veralgemeende veelhoeken met de Moufang-eigenschap gekend zijn ([74] en [47]).

Veronderstel dat $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ een $(0,2)$-meetkunde is en dat L en M twee rechten van $\mathcal{S}$ zijn, incident met hetzelfde punt p. Dankzij de definitie van $(0,2)$-meetkunde, bestaan er voor elk punt x en voor elke rechte L van $\mathcal{S}$ die niet incident zijn, nul of twee rechten door x die L snijden. Dat betekent dat er voor elk punt $y \neq x$ op L een uniek punt $y^{\pi_{L,M}} \neq x$ op M bestaat, dat collineair is met y. Op deze manier kunnen we een bijectie $\pi_{L,M} : \Gamma_1(L) \rightarrow \Gamma_1(M)$, met $x^{\pi_{L,M}} = x$, definiëren zodanig dat $\pi_{M,L}$ de inverse van $\pi_{L,M}$ is. Een dergelijke afbeelding wordt een *perspectiviteit* genoemd. Beschouwen wij een rij van rechten $L_1, \ldots, L_n$, zodanig dat de rechte L_i de rechte L_{i+1} snijdt voor $i = 1, \ldots, n-1$, dan kunnen we door perspectiviteiten te combineren een bijectie verkrijgen van de puntenverzameling van L_1 naar de puntenverzameling van L_n. Een dergelijke bijectie wordt een *projectiviteit* genoemd. Veronderstellen we dat $L_1 = L_n$, dan noemen we deze projectiviteit een *zelf-projectiviteit van* L_1; de volledige groep van zelf-projectiviteiten van L_1 wordt de *projectiviteitengroep van* L_1 genoemd en wordt genoteerd als $\Pi(L_1)$. Eerst hebben we de volgende algemene resultaten bewezen.

Stelling A.13 (Proposition 5.1) *Zij* L *een rechte van* $\mathcal{S}$. *De projectiviteitengroep van* L *is onafhankelijk van de rechte* L.

Dankzij dit resultaat wordt deze groep in het vervolg aangeduid met de notatie $\Pi(\mathcal{S})$, en de *projectiviteitengroep van* $\mathcal{S}$ genoemd wordt. Aangezien deze groep een permutatiegroep is, kunnen we de *even* deelgroep van deze groep beschouwen, die met de notatie $\Pi_+(\mathcal{S})$ aangeduid wordt. $\Pi_+(\mathcal{S})$ is een deelgroep van $\Pi(\mathcal{S})$ van index ten hoogste 2. In het bijzonder levert dat het volgende resultaat op.

Stelling A.14 (Theorem 5.2) $\Pi_+(\mathcal{S})$ *is een normaaldeler van* $\Pi(\mathcal{S})$.

A.5.2 Collineaties

In de uitgebreide literatuur over groepen van projectieve vlakken en veralgemeende veelhoeken (zie [46] en [74]), werden veel resultaten bekomen door bijzondere automorfismen van de meetkunde te beschouwen. Wij hebben een analoge studie doorgevoerd voor $(0,2)$-meetkunden.

Definities
Zij $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ een $(0,2)$-meetkunde. Als een permutatie $\theta : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ een graafautomorfisme op het incidentiegraaf $(\mathcal{P} \cup \mathcal{L}, \mathrm{I})$ definieert, dan wordt θ een *correlatie* van $\mathcal{S}$ genoemd. Als de correlatie θ minstens één punt op een ander punt afbeeldt, dan noemen we θ een *collineatie*. De collineatiegroep van $\mathcal{S}$ wordt aangeduid met de notatie $\mathrm{Aut}\mathcal{S}$. Als een collineatie θ van $\mathcal{S}$ een rechte L puntsgewijs fixeert, dan wordt de rechte L de *as* van θ genoemd. Duaal wordt een punt dat rechte-gewijs gefixeerd wordt, een *centrum* van θ genoemd.

Aan de hand van de bovenstaande definities, kunnen we het volgende lemma bewijzen.

Lemma A.2 (Lemma 5.3) *Zij θ een collineatie met as L en centrum x van een $(0,2)$-meetkunde $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. Als $x\mathrm{I}L$ of $x \in \Gamma_3(L)$, dan is θ triviaal. Als $x \in \Gamma_5(L)$, dan heeft θ orde ten hoogste 2.*

Definities
Een rechte L van een $(0,2)$-meetkunde $\mathcal{S}$ wordt een *as van transitiviteit* genoemd als de collineatiegroep $G_{[L]}$ met as L transitief werkt op $\Gamma_1(x) \setminus \{L\}$, voor een punt $x\mathrm{I}L$. Duaal kunnen we het *centrum van transitiviteit* definiëren. Een as van transitiviteit wordt een *elatie-rechte* genoemd (in het duale geval, *elatie-punt*) als er een collineatiegroep met as L $E_{[L]}$ bestaat die regulier werkt op $\Gamma_1(x) \setminus \{L\}$, voor een punt $x\mathrm{I}L$. Als er voor een punt $x\mathrm{I}L$ een collineatiegroep $U_{[L]}$ met as L bestaat, die regulier werkt op $\Gamma_1(x) \setminus \{L\}$ en zodanig dat $U_{[L]}$ een normaaldeler van $(\mathrm{Aut}\mathcal{S})_L$ is, dan wordt L een *Moufang-rechte* genoemd. Op een duale manier kunnen we een *Moufang-punt* definiëren. Als elk punt en elke rechte van $\mathcal{S}$ Moufang zijn, dan zeggen we dat $\mathcal{S}$ een *Moufang-$(0,2)$-meetkunde*.

Een *Moufang-verzameling* $\mathcal{M} = (X, G; U_x : x \in X)$ bestaat uit een verzameling X, een permutatiegroep G die getrouw werkt op X, en voor elk element $x \in X$ een deelgroep U_x van de stabilisator G_x van x in G zodanig dat de volgende eigenschappen gelden:

- elke U_x is een normaaldeler van G_x en werkt regulier op $X \setminus \{x\}$;
- de familie $\mathcal{U} := \{U_x : x \in X\}$ is een conjugatieklasse van deelgroepen van G;

- de groep G wordt voortgebracht door $\mathcal{U}$.

We bekwamen het volgende lemma.

Lemma A.3 (Lemma 5.4) *Veronderstel dat x en y twee collineaire Moufang-punten van de $(0,2)$-meetkunde $\mathcal{S}$ zijn en dat U_x en U_y twee groepen zijn die respectievelijk met x en y overeenkomen. Als $G := \langle U_x, U_y \rangle$, dan is $\mathcal{M}_{xy} := (\Gamma_1(xy), G; U_x^G)$ een Moufang-verzameling.*

Als elk punt van een $(0,2)$-meetkunde $\mathcal{S}$ een centrum van transitiviteit is, dan zeggen we dat $\mathcal{S}$ een $(0,2)$-*meetkunde met centrale transitiviteit* is. Als het duale geval gelt, dan zeggen we dat $\mathcal{S}$ een $(0,2)$-*meetkunde met axiale transitiviteit* is.

Lemma A.4 (Lemma 5.5) *Veronderstel dat een $(0,2)$-meetkunde $\mathcal{S}$ twee centra van transitiviteit x en y en twee assen van transitiviteit L en M heeft zodanig dat $x\mathrm{I}L\mathrm{I}y\mathrm{I}M$. Dan is $\mathcal{S}$ is een $(0,2)$-meetkunde met axiale en centrale transitiviteit.*

A.5.3 Voorbeelden

In het licht van de bovenstaande definities beschouwen we voorbeelden van $(0,2)$-meetkunden en we onderzoeken of ze aan één van deze definities voldoen.

1. Zij $\mathcal{K}$ een niet-triviale puntenverzameling van $\mathrm{PG}(d,q)$, $d > 0$ en q een priemmacht, zodanig dat elke rechte van $\mathrm{PG}(d,q)$ $\mathcal{K}$ in $0, 1$ of 3 punten snijdt. De lineaire representatie $T_d^*(\mathcal{K})$ van de verzameling $\mathcal{K}$ is een $(0,2)$-meetkunde waarvan alle punten en alle rechten Moufang zijn.

2. Zij G een willekeurige groep die uit minstens drie elementen bestaat. Beschouw een incidentiestructuur $\mathcal{S}_G$ van punten en rechten als volgt. De punten van $\mathcal{S}_G$ zijn alle paren van elementen van G. De rechtenverzameling bestaat uit alle verzamelingen $H_a := \{(g,a) : g \in G\}$, $V_a := \{(a,g) : g \in G\}$ en $D_a := \{(g, ga) : g \in G\}$, voor elk $a \in G$. Het kan bewezen worden dat alle rechten van deze meetkunde Moufang zijn en dat $\mathcal{S}_G$ centra van transitiviteit heeft als en slechts als de automorfismengroep die ermee overeenkomt, transitief is. We merken op dat $\mathcal{S}_G$ een net van graad 3 is, als G een quasi-groep is.

3. Zij G een groep van macht 3 en zij n een positief geheel getal. Beschouw de meetkunde $\mathcal{S}_G$ van punten en rechten die geconstrueerd kan worden als volgt. De puntenverzameling bestaat uit het Cartesiaans product

$G \times G \times \cdots \times G$ ($n+1$ factoren), de rechten van $\mathcal{S}_G$ zijn de verzamelingen $\{(g, ga_1, ga_2, \ldots, ga_k, g^{-1}a_{k+1}, \ldots, g^{-1}a_{k+\ell}, a_{k+\ell+1}, \ldots, a_n) : g \in G\}$ waarbij (k, ℓ) met $k + \ell \leq n$ niet-negatieve gehele getallen zijn en de n-tuples $(a_1, a_2, \ldots, a_n)$ uit elementen van G bestaan. Als G niet abels is, dan is $\mathcal{S}_G$ een $(0, 2)$-meetkunde die een transitieve collineatiegroep en een afstandsregulier graaf heeft.

4. Veronderstel dat $\mathcal{K}$ een verzameling is van disjuncte n-dimensionale deelruimten van $\mathrm{PG}(d, q)$, $0 \leq n < d$. De veralgemeende lineaire representatie $T^*_{d,n}(\mathcal{K})$ is een $(0, 2)$-meetkunde als en slechts als $\mathcal{K}$ aan de volgende voorwaarde voldoet.

   (C02) Voor elke twee elementen $S, T \in \mathcal{K}$ en voor elk punt $x \in T$ bestaat er een uniek element $U \in \mathcal{K} \setminus \{S, T\}$ dat $\langle S, x \rangle$ snijdt.

   Elke verzameling $\mathcal{K}$ die aan (C02) voldoet, noemen we $(0, 2)$-*regulus*. Een voorbeeld hiervan is een spread $\mathcal{K}$ van de veralgemeende vierhoek $Q(4, 2)$ ingebed in $\mathrm{PG}(4, 2)$ waarvoor aan de voorwaarde (C02) voldaan is.
   Merk op dat $T^*_{d,n}(\mathcal{K})$ een semi-partiële meetkunde is als en slechts als voor elk punt x van $\mathrm{PG}(d, q)$, niet in een element van $\mathcal{K}$ liggend, er een constant aantal elementen $S \in \mathcal{K}$ bestaat zodanig dat $\langle S, x \rangle$ twee andere elementen van $\mathcal{K}$ snijdt.

A.5.4 Enkele classificatieresultaten

In deze paragraaf worden enkele classificatieresultaten bewezen vanuit groeptheoretisch perspectief voor netten van de graad 3 en orde $s + 1$.

Stelling A.15 (Theorem 5.8) *Als $\mathcal{S}$ een net van de graad 3 en de orde $s + 1$, s oneven, met centrale en axiale transitiviteit is, dan voldoet $\mathcal{S}$ aan het diagonaalaxioma.*

Een opmerkelijke eigenschap van netten die door Thas en De Clerck [68] bewezen werd, is dat een net dat aan het diagonaalaxioma voldoet, isomorf is met het duale van H^n_q (zie A.1.2 voor de constructie van H^n_q). Aan de hand van deze observatie, kunnen we het volgende resultaat formuleren.

Gevolg A.16 (Corollary 5.9) *Als $\mathcal{S}$ een net is van de graad 3 en de orde $s + 1$, s oneven, dan kan $\mathcal{S}$ ingebed worden in een desarguesiaans affien vlak van de orde $q = s + 1 = 2^h$. Bijgevolg ontstaat $\mathcal{S}$ uit $\mathrm{AG}(2, s + 1)$ door $s - 1$ parallelklassen van rechten te laten vallen.*

Voor de volledigheid vertalen we Stelling A.15 naar de equivalente resultaten betreffende loops.

Gevolg A.17 (Corollary 5.10) *Zij* G *een loop van even orde en* $\mathcal{S}_G$ *het net van de graad 3 dat uit* G *geconstrueerd kan worden. Als* $\mathcal{S}_G$ *centrale en axiale transitiviteit heeft, dan is* G *een elementair abselse groep van de orde* 2^h *voor een zekere* $h \leq 1$*.*

Hoewel de resultaten van de gevolgen A.16 en A.17 waarschijnlijk ook gelden voor netten van de graad 3 en de orde $s+1$, s even, kunnen we dit niet bewijzen. Niettemin kunnen we de volgende karakteriserende stelling formuleren aan de hand van de strengere voorwaarde dat het net een Moufang-net is. Uiteraard kunnen we ons beperken tot het geval s even.

Stelling A.18 (Theorem 5.11) *Als* $\mathcal{S}$ *een Moufang-net is van de graad 3 en de orde* $s+1$*,* s *even, dan is* $\mathcal{S}$ *isomorf met* $\mathcal{S}_H$*, waarbij* H *een abelse groep is van oneven orde.*

A.5.5 Moufang-$(0,2)$-meetkunden uit $(0,2)$-reguli

We beogen een karakterisering van een klasse van Moufang-$(0,2)$-meetkunden, namelijk de Moufang-$(0,2)$-meetkunden ontstaan uit een $(0,2)$-regulus. Om dit te kunnen doen, moet eerst het volgende resultaat bewezen worden.

Lemma A.5 (Lemma 5.13) *Zij* $\mathcal{K}$ *een* $(0,2)$*-regulus waarvan de elementen* $(2n-1)$*-dimensionale deelruimten van* $\mathrm{PG}(5n-1,q)$*,* $n \geq 1$*, zijn. Dan is* q *even.*

Een $(0,2)$-regulus zoals in het bovenstaande lemma heet *mager*.

Aan de hand van het bovenstaande lemma, kunnen we de volgende karakteriserende stellingen formuleren.

Stelling A.19 (Theorem 5.14) *Zij* $\mathcal{K}$ *een magere* $(0,2)$*-regulus van rechten in* $\mathrm{PG}(4,q)$*. Veronderstel dat de* $(0,2)$*-meetkunde* Γ *die ermee overeenkomt, Moufang is en dat de corresponderende groep door de collineatiegroep van* $\mathrm{PG}(4,q)$ *geïnduceerd wordt. Dan is* $q=2$ *en is* Γ *isomorf met de veralgemeende lineaire representatie van een spread van* $Q(4,2)$*.*

Stelling A.20 (Theorem 5.15) *Zij* $\mathcal{K}$ *een magere* $(0,2)$*-regulus van* $(2n-1)$*-dimensionale deelruimten van* $\mathrm{PG}(5n-1,q)$, $n \geq 2$. *Veronderstel dat de* $(0,2)$*-meetkunde die ermee overeenkomt, Moufang is en dat de corresponderende groep door de collineatiegroep van* $\mathrm{PG}(5n-1,q)$ *geïnduceerd wordt. Dan hebben we één van de volgende gevallen.*

- $n = 2$, $q = 2$.
- $n = 2$, $q = 4$.
- $n = 3$, $q = 2$.
- $n = 4$, $q = 2$.

Bibliography

[1] A. Ball, S. Blokhuis and F. Mazzocca. Maximal arcs in Desarguesian planes of odd order do not exist. *Combinatorica*, 17(1):31–41, 1997.

[2] A. Barlotti and K. Strambach. The geometry of binary systems. *Adv. in Math.*, 49(1):1–105, 1983.

[3] A. Beutelspacher. Partial spreads in finite projective spaces and partial designs. *Math. Z.*, 145(3):211–229, 1975.

[4] N. L. Biggs and D. H. Smith. On trivalent graphs. *Bull. London Math. Soc.*, 3:155–158, 1971.

[5] G. Bonoli and N. Melone. A characterization of Grassmann and attenuated spaces as $(0, \alpha)$-geometries. *European J. Combin.*, 24(5):489–498, 2003.

[6] R. C. Bose. Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.*, 13:389–419, 1963.

[7] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989.

[8] M. R. Brown, C. Tonesi, and H. Van Maldeghem. A group theoretic approach to $(0, 2)$-geometries. Submitted for publication in *Finite Geometries, Groups and Computation*, 2004.

[9] R. H. Bruck. Finite nets. II. Uniqueness and imbedding. *Pacific J. Math.*, 13:421–457, 1963.

[10] R. H. Bruck. What is a loop? In *Studies in modern algebra. Studies in Mathematics*, volume 2, pages 59–99. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

[11] F. Buekenhout. (g, d^*, d)-gons. In *Finite Geometries (Pullman, Wash., 1981)*, volume 82 of *Lecture Notes in Pure and Appl. Math.*, pages 93–111. Dekker, New York, 1983.

[12] F. Buekenhout, editor. *Handbook of Incidence Geometry*. North-Holland, Amsterdam, 1995. Buildings and foundations.

[13] F. Buekenhout and C. Lefèvre. Generalized quadrangles in projective spaces. *Arch. Math. (Basel)*, 25:540–552, 1974.

[14] F. Buekenhout and E. Shult. On the foundations of polar geometry. *Geometriae Dedicata*, 3:155–170, 1974.

[15] P. J. Cameron. Partial quadrangles. *Quart. J. Math. Oxford Ser. (2)*, 26:61–73, 1975.

[16] P. J. Cameron. *Projective and Polar Spaces*, volume 13 of *QMW Maths Notes*. Queen Mary and Westfield College School of Mathematical Sciences, London, 1991.

[17] P. J. Cameron, J. A. Thas, and S. E. Payne. Polarities of generalized hexagons and perfect codes. *Geometriae Dedicata*, 5(4):525–528, 1976.

[18] P. J. Cameron and J. H. van Lint. On the partial geometry pg(6, 6, 2). *J. Combin. Theory Ser. A*, 32(2):252–255, 1982.

[19] K. Coolsaet and J. Degraer. A computer assisted proof of the uniqueness of the Perkel graph. *Designs, Codes and Cryptography*, 34 (2-3):155–171, 2005.

[20] A. Cossu. Su alcune proprietà dei $\{k, n\}$-archi di un piano proiettivo sopra un corpo finito. *Rend. Mat. e Appl. (5)*, 20:271–277, 1961.

[21] H. S. M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc.*, 56:413–455, 1950.

[22] H. S. M. Coxeter. My graph. *Proc. London Math. Soc. (3)*, 46(1):117–136, 1983.

[23] H. Cuypers. Two remarks on Huang's characterization of the bilinear forms graphs. *European J. Combin.*, 13(1):33–37, 1992.

[24] R. M. Damerell. On Moore graphs. *Proc. Cambridge Philos. Soc.*, 74:227–236, 1973.

[25] F. De Clerck. Partial and semipartial geometries: an update. *Discrete Math.*, 267(1-3):75–86, 2003. Combinatorics 2000 (Gaeta).

[26] F. De Clerck, M. Delanote, N. Hamilton, and R. Mathon. Perp-systems and partial geometries. *Adv. Geom.*, 2(1):1–12, 2002.

[27] F. De Clerck, H. Gevaert, and J. A. Thas. Translation partial geometries. In *Combinatorics '86 (Trento, 1986)*, volume 37 of *Ann. Discrete Math.*, pages 117–135. North-Holland, Amsterdam, 1988.

[28] F. De Clerck, N. Hamilton, C. M. O'Keefe, and T. Penttila. Quasi-quadrics and related structures. *Australas. J. Combin.*, 22:151–166, 2000.

[29] F. De Clerck and H. Van Maldeghem. Some classes of rank 2 geometries. In *Handbook of incidence geometry*, pages 433–475. North-Holland, Amsterdam, 1995.

[30] F. De Clerck, S. De Winter, E. Kuijken, and C. Tonesi. Distance-regular $(0, \alpha)$-reguli. To appear in *Designs, Codes and Cryptography.*

[31] S. De Winter. Elation and translation semipartial geometries. *J. Combin. Theory Ser. A*, 108(2):313–330, 2004.

[32] I. Debroey. Semipartial geometries satisfying the diagonalaxiom. *J. Geom.*, 13(2):171–190, 1979.

[33] I. Debroey and J. A. Thas. On semipartial geometries. *J. Combin. Theory Ser. A*, 25(3):242–250, 1978.

[34] R. H. F. Denniston. Some maximal arcs in finite projective planes. *J. Combinatorial Theory*, 6:317–319, 1969.

[35] W. Haemers. A new partial geometry constructed from the Hoffman-Singleton graph. In *Finite geometries and designs (Proc. Conf., Chelwood Gate, 1980)*, volume 49 of *London Math. Soc. Lecture Note Ser.*, pages 119–127. Cambridge Univ. Press, Cambridge, 1981.

[36] J. I. Hall. Classifying copolar spaces and graphs. *Quart. J. Math. Oxford Ser. (2)*, 33(132):421–449, 1982.

[37] N. Hamilton and R. Mathon. Strongly regular (α, β)-geometries. *J. Combin. Theory Ser. A*, 95(2):234–250, 2001.

[38] N. Hamilton and R. Mathon. More maximal arcs in Desarguesian projective planes and their geometric structure. *Adv. Geom.*, 3(3):251–261, 2003.

[39] C. Hering, W. M. Kantor, and G. M. Seitz. Finite groups with a split BN-pair of rank 1. I. *J. Algebra*, 20:435–475, 1972.

[40] J. W. P. Hirschfeld and J. A. Thas. Sets of type $(1, n, q+1)$ in PG(d, q). *Proc. London Math. Soc. (3)*, 41(2):254–278, 1980.

[41] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[42] A. J. Hoffman. On eigenvalues and colorings of graphs. In *Graph Theory and its Applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969)*, pages 79–91. Academic Press, New York, 1970.

[43] A. J. Hoffman and R. R. Singleton. On Moore graphs with diameters 2 and 3. *IBM J. Res. Develop.*, 4:497–504, 1960.

[44] T. Huang. A characterization of the association schemes of bilinear forms. *European J. Combin.*, 8(2):159–173, 1987.

[45] X. L. Hubaut. Strongly regular graphs. *Discrete Math.*, 13(4):357–381, 1975.

[46] D. R. Hughes and F. C. Piper. *Projective planes.* Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 6.

[47] N. Knarr. Projectivities of generalized polygons. *Ars Combin.*, 25(B):265–275, 1988. Eleventh British Combinatorial Conference (London, 1987).

[48] E. Kuijken. *A study of incidence structures and codes related to regular two-graphs.* PhD thesis, Ghent University, 2003.

[49] E. Kuijken and C. Tonesi. Distance-regular graphs and (α, β)-geometries. To appear in *Journal of Geometry.*

[50] C. Lefèvre-Percsy, N. Percsy, and D. Leemans. New geometries for finite groups and polytopes. *Bull. Belg. Math. Soc. Simon Stevin*, 7(4):583–610, 2000.

[51] R. Mathon. New maximal arcs in Desarguesian planes. *J. Combin. Theory Ser. A*, 97(2):353–368, 2002.

[52] S. E. Payne. Symmetric representations of nondegenerate generalized n-gons. *Proc. Amer. Math. Soc.*, 19:1321–1326, 1968.

[53] S. E. Payne and J. A. Thas. *Finite generalized quadrangles*. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[54] M. Perkel. *On finite groups acting on polygonal graphs*. PhD thesis, University of Michigan, 1977.

[55] M. Perkel. Bounding the valency of polygonal graphs with odd girth. *Canad. J. Math.*, 31(6):1307–1321, 1979.

[56] D. K. Ray-Chaudhuri and A. P. Sprague. A combinatorial characterization of attenuated spaces. *Utilitas Math.*, 15:3–29, 1979.

[57] J. J. Seidel. A survey of two-graphs. In *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo I*, pages 481–511. Atti dei Convegni Lincei, No. 17. Accad. Naz. Lincei, Rome, 1976.

[58] E. Shult. On a class of doubly transitive groups. *Illinois J. Math.*, 16:434–445, 1972.

[59] E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata*, 9(1):1–72, 1980.

[60] R. Singleton. There is no irregular Moore graph. *Amer. Math. Monthly*, 75:42–43, 1968.

[61] D. E. Taylor. *Some topics in the theory of finite groups*. PhD thesis, Univ. Oxford, 1971.

[62] D. E. Taylor. Regular 2-graphs. *Proc. London Math. Soc. (3)*, 35(2):257–274, 1977.

[63] D. E. Taylor and R. Levingston. Distance-regular graphs. In *Combinatorial mathematics (Proc. Internat. Conf. Combinatorial Theory, Australian Nat. Univ., Canberra, 1977)*, volume 686 of *Lecture Notes in Math.*, pages 313–323. Springer, Berlin, 1978.

[64] J. A. Thas. Construction of partial geometries. *Simon Stevin*, 46:95–98, 1972/73.

[65] J. A. Thas. Construction of maximal arcs and partial geometries. *Geometriae Dedicata*, 3:61–64, 1974.

[66] J. A. Thas. Polar spaces, generalized hexagons and perfect codes. *J. Combin. Theory Ser. A*, 29(1):87–93, 1980.

[67] J. A. Thas. Semipartial geometries and spreads of classical polar spaces. *J. Combin. Theory Ser. A*, 35(1):58–66, 1983.

[68] J. A. Thas and F. De Clerck. Partial geometries satisfying the axiom of Pasch. *Simon Stevin*, 51(3):123–137, 1977/78.

[69] J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Etudes Sci. Publ. Math.*, 2:13–60, 1959.

[70] J. Tits. *Buildings of spherical type and finite BN-pairs*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 386.

[71] J. Tits and R. M. Weiss. *Moufang polygons*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[72] J. H. van Lint and A. Schrijver. Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields. *Combinatorica*, 1(1):63–73, 1981.

[73] J. H. van Lint and J. J. Seidel. Equilateral point sets in elliptic geometry. *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, 28:335–348, 1966.

[74] H. Van Maldeghem. *Generalized polygons*, volume 93 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1998.

[75] H. Van Maldeghem. Slim and bislim geometries. *Topics in Diagram Geometry*, 12:227–254, 2003.

[76] H. Van Maldeghem and V. Ver Gucht. Some properties of the Biggs-Smith geometry. Preprint.

[77] F. D. Veldkamp. Polar geometry. I, II, III, IV, V. *Nederl. Akad. Wetensch. Proc. Ser. A 62; 63 = Indag. Math. 21 (1959), 512-551*, 22:207–212, 1959.

[78] W. D. Wallis. Configurations arising from maximal arcs. *J. Combinatorial Theory Ser. A*, 15:115–119, 1973.

[79] H. A. Wilbrink and A. E. Brouwer. A characterization of two classes of semipartial geometries by their parameters. *Simon Stevin*, 58(4):273–288, 1984.