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On Moufang Sets and Applications

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Preface

The main subject of this thesis is the study of Moufang sets. They were introduced some years ago by J. Tits as tools to classify twin buildings—this work has been successfully completed by B. Mühlherr (and in Mühlherr's approach, Moufang sets play indeed a central role).

In the first chapter some necessary notions will be defined; we start with generalized polygons. The specific geometries in which we work can always be seen (to some extent) as (arising from) a generalized polygon. The aim of this thesis is to get some knowledge about the automorphism groups of the geometries we work with. For this, one notion is particularly necessary: the Moufang condition of a generalized polygon, this obliges the generalized polygon in which we work to be very symmetric. J. Tits introduced the Moufang condition in the appendix of [37]. The last notion we get acquainted with in this chapter is the Moufang set: Moufang sets are the Moufang buildings of rank 1. They are the axiomatization of the permutation groups generated by two opposite root groups (belonging to opposite roots $\mathcal{R}_0$ and $\mathcal{R}_\infty$) in a Moufang building of rank at least 2, acting on the set of roots $\mathcal{R}$ such that $\mathcal{R} \cup \mathcal{R}_0$ or $\mathcal{R} \cup \mathcal{R}_\infty$ form an apartment.

The second chapter is dedicated to some characterization of generalized quadrangles with the Moufang property. Generalized quadrangles were introduced by J. Tits in the appendix of [34]. The half Moufang condition was introduced by J. A. Thas, S. E. Payne and H. Van Maldeghem in [29], where the equivalence with the Moufang condition in the finite case was shown. Later on, R. Weiss and H. Van Maldeghem defined the k-Moufang condition for generalized polygons [43] and J. A. Thas, S. E. Payne and H. Van Maldeghem proved in [30] that the 3-Moufang property is equivalent to the Moufang condition for finite generalized quadrangles.

Not so long ago, K. Tent [23] proved in general that the half Moufang condition is equivalent to the Moufang condition. Next, we found that, again in general, the 3-Moufang condition is equivalent to the Moufang condition (for generalized quadrangles). In chapter 2 we will further weaken the Moufang condition. We will introduce a condition that is weaker than both the half Moufang condition and the 3-Moufang condition, and therefore we will call it the *half* 3-*Moufang condition*. Also this condition will be equivalent to the Moufang condition. This result is proved using Moufang sets.

The third chapter shows that if a spherical BN-pair is split, the splitting has to be unique. The notion of a BN-pair is introduced by J. Tits in [37]. If G is a group with a BN-pair, then there is a unique associated spherical building *Omega* (having G as an automorphism group) where the group B is associated with a unique maximal flag, or *chamber*, C of Ω, and the group N is associated with a unique appartment of Ω. The split conditon now immediately translates into the property that there exist a nilpotent group U which is normal in B such that U acts transitively on the set of chambers of Ω *opposite* C. Hence it makes sense to call U a *transitive normal nilpotent subgroup* of B.

Recently it was shown that a *split* BN-pair of spherical and irreducible rank 2 is essentially equivalent to the so-called *Moufang condition* for the associated generalized polygon (see [25], [27], [28]; the finite case was already treated back in the 1970s in [10], [11]). The uniqueness of such a splitting says that if (G, B, N) defines a generalized n-gon Ω for $n > 2$, and if there is a normal nilpotent subgroup U of B such that $B = U(B \cap N)$, then Ω is a Moufang polygon and G contains all the appropriate root groups, i.e., U necessarily coincides with the standard unipotent subgroup U^+ of B, which is a product of root groups. We want to show this uniqueness result for all Moufang sets arising from higher rank Moufang buildings as permutation groups generated by opposite root groups, and use this result to prove that the splittings of all split BN-pairs of higher rank are unique. In addition we show the uniqueness result also for some well known Moufang sets arising from diagram automorphisms of some rank two buildings, more specifically for the Suzuki groups and the Ree groups in characteristic 3.

In the last chapter we investigate the Ree Geometry and its automorphism group. Every algebraic group of relative rank one gives rise to a Moufang set, but those with root groups of nilpotency class two also give rise to an additional geometric structure on that Moufang line, according to J. Tits [38], and we will call the resulting geometry a *Moufang geometry of rank* 1. J. Tits then asked whether this additional structure is rich enough to recover the algebraic group. More precisely, is the automorphism group of this geometric structure contained in the automorphism group of the corresponding algebraic group?

In chapter 4 we consider the Moufang lines defined by the Ree groups. Here, the root groups have nilpotency class 3, and this situation does not occur with algebraic groups. In fact, until recently these groups were the only known split BN-pairs of rank 1 whose root groups exceeded nilpotency class 2; in [17], a second class of such examples was discovered (with nilpotency class 3 again). We will show that the Moufang geometry of rank one constructed this way is rich enough to recover the group and the rank 2 geometry from which it was created.

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Contents

Chapter 1

Preliminaries

1.1 Geometries

A *geometry* (of rank 2) is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ where $\mathcal{P}, \mathcal{L}$ are disjoint non-empty sets, $\mathtt{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a relation, called the *incidence* relation such that every element of $\mathcal{P} \cup \mathcal{L}$ is incident with at least one element of $\mathcal{P} \cup \mathcal{L}$. The elements of $\mathcal{P}$ are called *points*, the elements of $\mathcal{L}$ are *lines* and the elements of $\mathtt{I}$ are (maximal) *flags* in Γ. A geometry Γ is called *thick* if all point rows and all line pencils have cardinalities at least 3. An element which is incident with at least three elements is sometimes called thick itself.

An ordinary polygon is a geometry consisting of n different points x_{2i} and n different lines x_{2i+1}, $i \in \{0..n-1\}$, such that $x_{2i-1}\mathtt{I}x_{2i}\mathtt{I}x_{2i+1}$ for $i \in \{1..n-1\}$ and $x_{2n-1}\mathtt{I}x_0\mathtt{I}x_1$.

1.2 Generalized polygons

Let $n \geq 2$ be a natural number. A *generalized n-gon* is a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ such that the following axioms are satisfied

GP_1 Γ contains no ordinary k-gon (as a subgeometry) for $2 \leq k < n$.

GP_2 Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary n-gon in Γ, which we call an apartment.

GP_3 There exists an ordinary $(n+1)$-gon (as a subgeometry) in Γ.

If only the first two conditions are satisfied, we call this geometry a *weak* generalized polygon. It turns out that the third axiom is equivalent with the generalized polygon being thick. A finite generalized n-gon is said to have order (s,t) if every line is incident with $s+1$ points and every point is incident with $t+1$ lines.

All weak polygons can be described in terms of (thick) generalized polygons. For this, we first need to define the multiple $m\Gamma$ of a geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ for any natural number $m \geq 1$, so we define a graph on the set

$$W = \mathcal{P} \cup \mathcal{L} \cup (\mathtt{I} \times \{1, 2, \ldots, m-1\})$$

by specifying adjacency $*$ as follows:

$$p * (p, L, 1) * (p, L, 2) * \ldots * (p, L, m-1) * L$$

for every incident pair $(p, L) \in \mathtt{I} \subseteq \mathcal{P} \times \mathcal{L}$. The graph $(W, *)$ is bipartite and hence determines a geometry $m\Gamma$ up to duality; by insisting that the points of Γ should also be points of $m\Gamma$, the geometry $m\Gamma$ is determined uniquely.

Theorem 1.2.1 (Tits [36]). *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a weak generalized n-gon with $n \geq 3$. Then we have precisely one of the following cases:*

(i) *Γ is an ordinary n-gon.*

(ii) *Γ is obtained from an ordinary n-gon $x_0 \mathtt{I} x_1 \mathtt{I} \ldots \mathtt{I} x_{2n} = x_0$ by inserting a non-zero number of (mutually disjoint) paths of length n from x_0 to x_n. Then x_0 and x_n are the only thick elements of Γ, and Γ does not have an order (see [41]).*

(iii) *There exists a divisor d of n, with $d < n$ and a (thick) generalized $\frac{n}{d}$-gon Γ' (with $\frac{d}{n}$ possibly equal to 2, but not to 1) such that $\Gamma \equiv d\Gamma'$ or $\Gamma \equiv d\Gamma'^D$, the dual of $d\Gamma'$. There is also a bijection from the set of points and lines of Γ' onto the set of thick elements of Γ which maps elements at distance i onto elements at distance id.*

In a generalized polygon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ a *flag* is an incident point-line pair. A *path* of length m from x to y, with x and y in $\mathcal{P} \cup \mathcal{L}$, is a sequence $(x = x_0, x_1, \ldots, x_m = y)$ with $x_{i-1} \mathtt{I} x_i$ for $i \in \{1, \ldots, m\}$. The *distance* between two elements x and y is the length of the shortest path between those two elements. We denote this by $\mathrm{d}(x, y)$. If the distance between two elements is maximal (i.e. n in a generalized n-gon), these elements are called *opposite*. If the distance between two elements is 2, we are dealing with *collinear points* resp. *concurrent lines*. We denote collinearity with the symbol $\sim$.

If two elements x_i and x_j are not opposite, there exists a unique element x incident with x_j such that $\mathrm{d}(x_i, x) < \mathrm{d}(x_i, x_j)$. This element x is called the *projection* of x_i onto x_j. We denote this by $\mathrm{proj}_{x_j}(x_i)$.

We will now briefly explain the geometric structure of some generalized n-gons.

Generalized digons A generalized 2-gon is a geometry in which every point is incident with every line.

If, for instance, we fix one line L in a projective 3-space V, we can define $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ where $\mathcal{P}$ consists of the points on L, $\mathcal{L}$ equals the set of planes through L and incidence is natural, $\mathcal{S}$ is then a generalized digon. This example shows that generalized digons appear in abundance in real—mathematical—life.

Generalized triangles Generalized triangles are exactly projective planes: every two points are incident with exactly one line, and every two lines are incident with exactly one point. There also exist four points, of which no three points are incident with the same line. To find out more about projective planes, we refer to [15].

The best known projective planes are those defined over a (skew) field $\mathbb{K}$; the *Desarguesian planes.* This is one way of seeing this geometry: The points are of three types: the pairs $(x, y) \in \mathbb{K} \times \mathbb{K}$, the elements (m), $m \in \mathbb{K}$ and a symbol (∞). The lines are dually also of three types: the pairs $[m, k] \in \mathbb{K} \times \mathbb{K}$, the elements $[x]$, $x \in \mathbb{K}$ and the symbol $[\infty]$. Incidence is defined as follows; the point (∞) is incident with $[\infty]$ and $[x]$, for all $x \in \mathbb{K}$; the point (m), $m \in \mathbb{K}$, is incident with $[\infty]$ and $[m, k]$ for all $k \in \mathbb{K}$; the point (x, y) $x, y \in \mathbb{K}$ is incident with $[x]$ and with $[m, k]$ if and only if $mx + y = k$.

Generalized quadrangles One of the most complete references on *finite* generalized quadrangles is [18]. In that book a generalized quadrangle is defined as a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfying the following axioms:

GQ_1 Each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line.

GQ_2 Each line is incident with $1 + s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point.

GQ_3 For every non-incident point-line pair (p, L), there is a unique point q and a unique line M such that $p \mathtt{I} M \mathtt{I} q \mathtt{I} L$.

The last axiom will sometimes be referred to as the *Main Axiom* for generalized quadrangles. Note that these axioms are equivalent with the previous definition,; see, for example, [41].

The classical generalized quadrangles arise from pseudo-quadratic forms, we distinguish quadrics, Hermitian varieties and symplectic quadrangles: the points and lines of the quadrics with Witt-index 2 form generalized quadrangles. When defined over a finite field $GF(q)$, the hyperbolic quadric

$Q^+(3,q)$ in $PG(3,q)$, the parabolic quadric $Q(4,q)$ in $PG(4,q)$ and the elliptic quadric $Q^-(5,q)$ in $PG(5,q)$ define finite generalized quadrangles with order $(q,1)$,(q,q) and (q,q^2), respectively.

The points and lines of the Hermitian varieties with Witt-index 2 form generalized quadrangles. When defined over a finite field $GF(q^2)$, $H(3,q^2)$ and $H(4,q^2)$ form generalized quadrangles with order (q^2,q) and (q^2,q^3), resp.

A symplectic polarity τ over $PG(3,\mathbb{K})$ gives rise to a generalized quadrangle $W(\mathbb{K})$ (of order (q,q) in the finite case) by considering the points of the projective space along with the totally isotropic lines of $PG(3,\mathbb{K})$.

Generalized hexagons There are two well-known classes of classical generalized hexagons, they are not classical from the point of view of group theory (no classical group is naturally associated with a generalized hexagon) but they live naturally on classical objects like quadrics (in particular $Q^+(7,\mathbb{K})$) and there is a great similarity with symplectic quadrangles.

We start with describing the D_4-geometry arising from $Q^+(7,\mathbb{K})$, the quadric in $PG(7,\mathbb{K})$ with Witt-index 4. This quadric has as characteristic property that every plane contained in it is itself contained in exactly 2 *generators*, i.e. 3-spaces lying completely in the quadric. Those 3-spaces can be subdivided in two subsets in the following way: they belong to the same subset, if and only if they have an odd-dimensional intersection. We can denote these subsets $\mathcal{G}_1$ and $\mathcal{G}_2$. The D_4-geometry $\Omega(\mathbb{K})$ attached to $Q^+(7,\mathbb{K})$ is defined as follows; there are four different types of elements. The set $\mathcal{P}^{(0)}$ of 0-points is the set of points of $Q^+(7,\mathbb{K})$, the lines are the lines of $Q^+(7,\mathbb{K})$ and we denote this set by $\mathcal{L}$. In the set $\mathcal{P}^{(1)}$ lie the 1-points which are the elements of $\mathcal{G}_1$ and $\mathcal{P}^{(2)}$ consists of the elements of $\mathcal{G}_2$ called 2-points. Incidence is symmetrized containment for $\mathcal{P}^{(i)}$ and $\mathcal{L}$, also for 0-points and j-points, j being 1 or 2. A 1-point is incident with a 2-point if the corresponding 3-spaces meet in a plane of $Q^+(7,\mathbb{K})$. The key property of this geometry is that every permutation of the set $\{\mathcal{P}^{(0)},\mathcal{P}^{(1)},\mathcal{P}^{(2)}\}$ defines a geometry which is isomorphic to $\Omega(\mathbb{K})$.

A *triality* of $\Omega(\mathbb{K})$ is a map

$$\theta:\mathcal{L}\rightarrow\mathcal{L},\, \mathcal{P}^{(0)}\rightarrow\mathcal{P}^{(1)},\, \mathcal{P}^{(1)}\rightarrow\mathcal{P}^{(2)},\, \mathcal{P}^{(2)}\rightarrow\mathcal{P}^{(0)}$$

preserving incidence in $\Omega(\mathbb{K})$ and such that θ^3 is the identity. *Absolute* i-points are i-points which are incident with their image under θ, absolute lines are lines which are fixed under θ. The geometry $\Gamma^{(i)}$ with point set $\mathcal{P}^{(i)}_{abs}$ the set of absolute i-points, with line set $\mathcal{L}_{abs}$ the set of absolute lines and with natural incidence gives rise to a generalized hexagon provided that there is at least one absolute i-point, for some $i \in \{0, 1, 2\}$, that every absolute i-point is incident with at least 2 absolute lines and that there exists a cycle $(L_0, L_1, \ldots, L_d = L_0)$, $d > 2$, of absolute lines with L_i concurrent with L_{i+1}.

For every field $\mathbb{K}$ there exists a triality that produces a *split Cayley* generalized hexagon $\mathsf{H}(\mathbb{K})$. the reason for that name is that this hexagon can also be constructed using a split Cayley algebra over $\mathbb{K}$, see for instance [20],[21]. If the triality uses a field automorphism σ of order 3, this triality produces a *twisted triality* generalized hexagon. The twisted triality hexagon has an *ideal* subhexagon isomorphic to $\mathsf{H}(\mathbb{K})$. This means that all the lines of H($\mathbb{K}$) through a point of the subhexagon are also lines of the subhexagon. For the explicit coordinates of these classical hexagons embedded in a projective space, we refer to the appendix.

Generalized octagons There is one known class of generalized octagons, namely the *Ree-Tits* octagons. Normally the construction of this geometry is given using the (B, N)-pair in the Chevalley Groups of type ${}^2F_4$ (the Ree groups of characteristic 2). It is however possible to construct such an octagon using a polarity over the building F_4 (see [35] and [41]). The field $\mathbb{K}$ over which F_4 is defined has characteristic 2. If $\mathbb{K}$ is a perfect field, an explicit embedding in $PG(24, \mathbb{K})$ now exists due to Coolsaet [3].

Restriction on the parameters for finite n-gons The theorem of Feit and Higman [9] implies that finite generalized n-gons exist only for a few values of n. The idea of the proof is to calculate the multiplicity of a certain eigenvalue of the matrix $M = A^t A$, where A is an incidence matrix of the finite n-gon, and to require that this must be a positive integer.

Theorem 1.2.2 (Feit and Higman [9]). *If $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ is a finite n-gon of order (s, t), s and t not both 1, then only the following n-gons are possible:*

- $n = 2$
- $n = 3$ *and* $s = t$
- $n = 4$
- $n = 6$ *and* st *is a square*
- $n = 8$ *and* $2st$ *is a square*
- $n = 12$ *and either* $s = 1$ *or* $t = 1$

1.3 Moufang generalized polygons

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I})$ be two geometries. An isomorphism or a collineation of Γ onto Γ' is a pair of bijections $\alpha : \mathcal{P} \to \mathcal{P}'$, $\beta : \mathcal{L} \to \mathcal{L}'$ preserving incidence i.e. $p^\alpha \mathtt{I} L^\beta$ if and only if $p \mathtt{I} L$ for $p \in \mathcal{P}$ and $L \in \mathcal{L}$.

A group element g is said to act *freely* on a set S, if the group generated by g does not contain any nontrivial element that fixes some $x \in S$. A group acts *freely* on S if only the identity fixes some $x \in S$.

Let $\gamma = (x_1, \ldots, x_{n-1})$ be a panel, i.e. a path of length $n-2$ of the generalized n-gon Γ. The group G of collineations of Γ fixing every element incident with at least one element of γ — we call such collineations γ-*elations* — acts freely on the set of apartments (i.e. ordinary n-gons) through γ. If this action is transitive, we say that γ is a *Moufang path*. If all paths of length $n-2$ are Moufang, we speak of a Moufang generalized polygon. The group of γ-elations is called a *root group* of the Moufang generalized polygon

Restrictions on existence of Moufang n-gons The Moufang generalized polygons have been classified in [40]. This paragraph is very much based on chapter 16 of that book. Every Moufang generalized polygon Γ determines an $(n+1)$-tuple $(U_{[1,n]}, U_1, U_2, \ldots, U_n)$, U_i are root groups acting on Γ while $U_{[i,j]}$ is the group generated by the root groups $U_i, U_{i+1}, \ldots, U_j$. They all satisfy certain axioms.

$\mathcal{M}_1$ $[U_i, U_j] \leq U_{[i+1,j-1]}$ for $1 \leq i < j \leq n$.

$\mathcal{M}_2$ The product map from $U_1 \times U_2 \times \cdots \times U_n$ to $U_{[1,n]}$ is bijective.

$\mathcal{M}_3$ There exists a subgroup $\tilde{U}_0$ of $\mathrm{Aut}(U_{[1,n-1]})$ such that for each $a_n \in U_n^\times$ there exists $\mu(a_n) \in \tilde{U}_0^\times \tilde{a}_n \tilde{U}_0^\times$ such that $U_j^{\mu(a_n)} = U_{n-j}$ for $1 \leq j \leq n-1$ and, for some $e_n \in U_n^\times$, $\tilde{U}_j^{\mu(e_n)} = \tilde{U}_{n-j}$ for $j = 0$ and $j = n$. For $a_n \in U_n$, $\tilde{a}_n$ denotes the image of a_n in $\tilde{U}_n$, the subgroup of $\mathrm{Aut}(U_{[1,n-1]})$ induced by U_n by conjugation.

$\mathcal{M}_4$ There exists a subgroup $\tilde{U}_{n+1}$ of $\mathrm{Aut}(U_{[2,n]})$ such that for each $a_1 \in U_1^\times$ there exists $\mu(a_1) \in \tilde{U}_{n+1}^\times \tilde{a}_1 \tilde{U}_{n+1}^\times$ such that $U_j^{\mu(a_1)} = U_{n+2-j}$ for $2 \leq j \leq n$ and, for some $e_1 \in U_1^\times$, $\tilde{U}_j^{\mu(e_1)} = \tilde{U}_{n+2-j}$ for $j = 1$ and $j = n+1$. For $a_1 \in U_1$, $\tilde{a}_1$ denotes the image of a_1 in $\tilde{U}_1$, the subgroup of $\mathrm{Aut}(U_{[2,n]})$ induced by U_1 by conjugation.

To classify the Moufang generalized polygons, all possible structures for $(U_{[1,n]}, U_1, U_2, \ldots, U_n)$ satisfying these axioms are examined for every possible

n. We will just give the different root groups and their mutual commutation relations since they determine the structure of $U_{[1,n]}$ completely. The groups U_i will be parametrized by some underlying algebraic structure S. We denote the isomorphism from S to U_i with x_i.

Generalized triangles Let A be an alternative division ring. Let U_1, U_2, U_3 be three groups all parametrized by the additive group of A. Let $\mathcal{T}(A)$ denote the graph defined by the relations

$$[x_1(t), x_3(u)] = x_2(tu)$$

for all $t, u \in A$. These are the Moufang generalized triangles.

Generalized quadrangles There are 6 different classes of generalized quadrangles; the quadrangles of involutory type, those of quadratic form type, the quadrangles of indifferent type, those of pseudoquadratic form type, the exceptional quadrangles of type E_6, E_7, E_8 and the exceptional quadrangles of type F_4. We will describe them one by one, the algebraic structures over which these quadrangles are defined will be described in Chapter 3.

quadrangles of involutory type Let (K, K_0, σ) be an involutory set. Let U_1 and U_3 be groups parametrized by the group K_0 and let U_2 and U_4 be groups parametrized by the additive group of K. Let $\mathcal{Q}_{\mathcal{I}}(K, K_0, \sigma)$ denote the graph defined by the relations

$$[x_2(a), x_4(b)^{-1}] = x_3(a^\sigma b + b^\sigma a) \text{ and}$$
$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(a^\sigma ta)$$

for all $t \in K_0$ and $a, b \in K$.

quadrangles of quadratic form type Let (K, L_0, q) be an anisotropic quadratic space with $L_0 = 0$ and let f denote the bilinear form associated with q. Let U_1 and U_3 be groups parametrized by the additive group of K and let U_2 and U_4 be groups parametrized by L_0. Let $\mathcal{Q}_{\mathcal{Q}}(K, L_0, q)$ denote the graph defined by the relations

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a, b)) \text{ and}$$
$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tq(a))$$

for all $t \in K$ and $a, b \in L_0$.

quadrangles of indifferent type Let (K, K_0, L_0) be an indifferent set. Let U_1 and U_3 be groups parametrized by K_0. Let U_2 and

U_4 be groups parametrized by L_0. Let $\mathcal{Q}_{\mathcal{D}}(K, K_0, L_0)$ denote the graph defined by the relations

$$[x_1(t), x_4(a)] = x_2(t^2a)x_3(ta)$$

for all $t \in K_0$ and $a \in L_0$.

quadrangles of pseudoquadratic form type Let (K, K_0, σ, L_0, q) be an anisotropic pseudo-quadratic space, let f denote the skew-hermitian form associated with q and let T be the group

$$T = \{(a, t) \in L_0 \times K \mid q(a) - t \in K_0\}$$

with

$$(a, t) \cdot (b, u) = (a + b, t + u + f(b, a)).$$

Let U_1 and U_3 be groups parametrized by T and let U_2 and U_4 be groups parametrized by the additive group of K. Let $\mathcal{Q}_{\mathcal{P}}(K, K_0, \sigma, L_0, q)$ denote the graph defined by the relations

$$[x_1(a, t), x_3(b, u)^{-1}] = x_2(f(a, b)),$$
$$[x_2(v), x_4(w)^{-1}] = x_3(0, v^\sigma w + w^\sigma v) \text{ and}$$
$$[x_1(a, t), x_4(v)^{-1}] = x_2(tv)x_3(av, v^\sigma tv)$$

for all $(a, t), (b, u) \in T$ and all $v, w \in K$.

quadrangles of type E_6, E_7 and E_8. Let (K, L_0, q) be a quadratic space of type E_6, E_7 or E_8, let f denote the bilinear form associated with q. Choose an element ϵ in L_0, replace q by $q/q(\epsilon)$ and choose a norm splitting map T of q. Let X_0 and the map $(a, v) \mapsto av$ from $X_0 \times L_0$ to X_0 be such that for some $b \in X_0^*$

$$b \cdot T(v) = (b \cdot T(\epsilon)) \cdot v$$

for all $v \in L_0$. Let h, θ, g and ϕ be the maps given in [**?**, Tit-Wei:02], h from $X_0 \times X_0$ to L_0, θ from $X_0 \times L_0$ to L_0, g from $X_0 \times X_0$ to K and ϕ from $X_0 \times L_0$ to K. Let S be the group with underlying set $X_0 \times K$ and multiplication given by

$$(a, s) \cdot (b, t) = (a + b, s + t + g(a, b))$$

for all $a, b \in x_0$ and all $s, t \in K$. Let U_1 and U_3 be groups parametrized by S and let U_2 an U_4 be groups parametrized by L_0. Let $\mathcal{Q}_\epsilon(K, L_0, q)$ denote the graph defined by the relations

$$[x_1(a, t), x_3(b, s)^{-1}] = x_2(h(a, b))$$
$$[x_2(u), x_4(v)^{-1}] = x_3(0, f(u, v)) \text{ and}$$
$$[x_1(a, t), x_4(v)^{-1}] = x_2(\theta(a, v) + tv)x_3(av, tq(v) + \phi(a, v))$$

for all $(a,t),(b,s) \in S$ and all $u,v \in L_0$. These quadrangles are independent of the choice of the element ϵ and the norm splitting map T.

quadrangles of type F_4 Let (K, L_0, q) be a quadratic space of type F_4 and let $R_0 = Rad(q)$. Choose an element $\rho \in R_0^*$, replace q by $q/q(\rho)$ and let $F = q(R_0)$. Choose a complement S_0 of R_0 in L_0 and a norm splitting $(E, \{v_1, v_2\})$ of the restriction of q to S_0 with constants s_1, s_2 such that $s_1 s_2 \in F$. Let $W_0 = E \oplus E$ and let q_1 denote the restriction of q to S_0. We coordinatize W_0 with respect to the ordered basis (v_1, v_2) so that

$$q_1(u,v) = s_1 N(u) + s_2 N(v)$$

for all $(u,v) \in X_0$ and then identify L_0 with $W_0 \oplus F$ as in [**?**, Tit-Wei:02] so that $q(b,s) = q_1(b) + s$ for all $(b,s) \in w_0 \oplus F$. Let $D = E^2 F$, let $X_0 = D \oplus D$, let q_2 be the quadratic form on X_0 as a vector space over F given by

$$q_2(x,y) = s_1^{-1} s_2 N(x) + s_1^{-3} s_2 N(y)$$

for all $(x.y) \in X_0$, let $\hat{q}$ be the quadratic form on $X_0 \oplus K$ given by

$$\hat{q}(a,t) = q_2(a) + t^2$$

for all $(a,t) \in X_0 \oplus K$. Let f_1 and f_2 denote the bilinear forms on W_0 and X_0 associated with q_1 and q_2. Let Θ and Υ be the maps from $X_0 \times W_0$ to K and to F given by

$$\Theta((x,y),(u,v)) = (\alpha(\bar{x}v + \beta y\bar{(v)}), xu + \beta y\bar{u})$$
$$\Upsilon((x,y),(u,v)) = (y\bar{u}^2 + \alpha\bar{y}v^2, \beta^{-2}(xu^2 + \alpha\bar{x}v^2))$$

for all $(x,y) \in X_0$ and all $(u,v) \in W_0$, and let ν and ψ be the maps from $X_0 \times W_0$ to K and to F given by

$$\nu((x,y),(u,v)) = \alpha(\beta^{-1}(xu\bar{v} + \bar{x}\bar{u}v) + y\bar{u}\bar{v} + \bar{y}uv)$$
$$\psi((x,y),(u,v)) = \alpha(x\bar{y}u^2 + \bar{x}y\bar{u}^2 + \alpha(xy\bar{v}^2 + \bar{x}\bar{y}v^2))$$

for all $(x,y) \in X_0$ and $(u,v) \in W_0$. Now let U_1 and U_3 be groups parametrized by $X_0 \oplus K$ and let U_2 and U_4 be groups parametrized by $W_0 \oplus F$. Let $\mathcal{Q}_{\mathcal{F}}(K, L_0, q)$ be the graph defined by the relations

$$\begin{aligned}
[x_1(a,t), x_4(b,s)] =& x_2(\Theta(a,b) + tb, \hat{q}(a,t)s + \psi(a,b)) \\
& \cdot x_3(\Upsilon(a,b) + sa, q(b,s)t + \nu(a,b)), \\
[x_1(a,t), x_3(a',t')] =& x_2(0, f_2(a,a')) \\
[x_2(b,s), x_4(b',s')] =& x_3(0, f_1(b,b'))
\end{aligned}$$

for all $(a,t),(a',t') \in X_0 \oplus K$ and $(b,s),(b',s') \in W_0 \oplus F$. These quadrangles are independent of the choice of the element ρ, the complement S_0 and the norm splitting $(E,\{v_1,v_2\})$.

Generalized hexagons Let $(J,F,N,\sharp,T,\times,1)$ be an hexagonal system. The functions $T,N,\times$ and the element 1 are all uniquely determined by J,F and the adjoint map $\sharp$. Let U_1,U_3 and U_5 be groups parametrized by J and let U_2,U_4 and U_6 be groups parametrized by the additive group of F. Let $\mathcal{H}(J,F,\sharp)$ denote the graph defined by the relations

$$\begin{aligned}
&[x_1(a),x_3(b)] = x_2(T(a,b)),\\
&[x_3(a),x_5(b)] = x_4(T(a,b)),\\
&[x_1(a),x_5(b)] = x_2(-T(a^\sharp,b))x_3(a\times b)x_4(T(a,b^\sharp)),\\
&[x_2(t),x_6(u)] = x_4(tu) \text{ and}\\
&[x_1(a),x_6(t)] = x_2(-tN(a))x_3(ta^\sharp)x_4(t^2N(a))x_5(-ta)
\end{aligned}$$

for all $a,b \in J$ and $t,u \in F$.

Generalized octagons Let (K,σ) be an octagonal set and let $K_\sigma^{(2)}$ be the group with underlying set $K \times K$ and the multiplication given by

$$(t,u)\cdot(s,v) = (t+s+u^\sigma v, u+v)$$

for all $t,u,s,v \in K$. Let U_1,U_3,U_5 and U_7 be groups parametrized by the additive group of K and let U_2,U_4,U_6 and U_8 be groups parametrized by $K_\sigma^{(2)}$. Let $x_i(t) = x_i(t,0)$ and $y_i(u) = x_i(0,u)$ for all $t,u \in K$ and all even i. Let $V_i = \{x_i(t) \mid t \in K\}$ for all even i. Let $\mathcal{S}$ denote the set consisting of the following relations:

$$
\begin{aligned}
&[U_1, U_2] = [U_1, U_3] = [U_1, V_4] = [U_1, U_5] = [V_2, U_4] = [u_2, U_6] = 1\\
&[x_1(t), y_4(u)] = x_2(tu),\\
&[x_1(t), x_6(u)] = x_4(tu),\\
&[x_1(t), y_6(u)^{-1}] = x_2(t^\sigma u)x_3(tu^\sigma)x_4(tu^{\sigma+1})\\
&[x_1(t), x_7(u)] = x_3(t^\sigma u)x_5(tu^\sigma),\\
&[x_1(t), x_8(u)] = x_2(t^{\sigma+1}u)x_3(t^{\sigma+1}u^\sigma)y_4(t^\sigma u)x_5(t^{\sigma+1}u^2)y_6(tu)^{-1}x_7(tu^\sigma),\\
&[x_1(t), y_8(u)^{-1}] =\\
&y_2(tu)x_3(t^{\sigma+1}u^{\sigma+2})y_4(t^\sigma u^{\sigma+1})^{-1}x_5(t^{\sigma+1}u^{2+2\sigma})x_6(t^{\sigma+1}u^{2\sigma+3})x_7(tu^{\sigma+2}),\\
&[y_2(t), y_4(u)] = x_3(tu),\\
&[x_2(t), x_8(u)] = x_4(t^\sigma u)x_5(tu)x_6(tu^\sigma),\\
&[x_2(t), y_8(u)^{-1}] = x_3(tu)x_4(t^\sigma u^{\sigma+1})x_6(tu^{\sigma+2}) \text{ and}\\
&[y_2(t)^{-1}, y_8(u)^{-1}] = x_3(t^{\sigma+1}u)y_4(t^\sigma u)^{-1}y_6(tu^\sigma)x_7(tu^{\sigma+1}).
\end{aligned}
$$

for all $t, u \in K$. For each i, let τ_i denote the permutation of $\mathbb{Z}$, which sends each $x \in \mathbb{Z}$ to $2i - x$ and let $N = \langle \tau_0, \tau_1 \rangle$. For each relation $r \in \mathcal{S}$, let $I(r)$ denote the set of indices which appear in r. For each $r \in \mathcal{S}$ and each $\rho \in N$ mapping $I(r)$ onto a subset of $\{1, 2, \ldots, 8\}$, we let r^ρ denote the relation obtained from r by replacing each index by its image under ρ. Let $\mathcal{S}_0$ be the set of all such relations r^ρ for $r \in \mathcal{S}$. The set $\mathcal{S}_0$ has a unique extension to a set of relations involving $[a_i, a_j]$ for all $(i, j) \in I$ and all $(a_i, a_j) \in u_i \times u_j$ such that the resulting graph is a Moufang polygon.

Let $\mathcal{O}(K, \sigma)$ denote the graph defined by these relations

Theorem 1.3.1 (Tits and Weiss [40]). *A Moufang n-gon Γ exists only if*

- $n = 3$ *and* $\Gamma \cong \mathcal{T}(A)$ *is a projective plane defined over some alternative division ring* A.
- $n = 4$ *and* Γ *is of involutory, quadratic form, indifferent or pseudo-quadratic form type or it is of* E_6, E_7, E_8 *or* F_4*-type.*
- $n = 6$ *and* $\Gamma \cong \mathcal{H}(J, F, \#)$ *for some hexagonal system* $(J, F, \#)$.
- $n = 8$ *and* $\Gamma \cong O(K, \sigma)$ *for some octogonal set* (K, σ).

1.4 Moufang sets

A *Moufang set* is a system $\mathcal{M} = (X, (U_x)_{x\in X})$ consisting of a set X and a family of groups of permutations of X indexed by X itself (for which we write the action of a permutation on a point on the right, using exponential notation) satisfying the following conditions.

MS_1 U_x fixes $x \in X$ and acts sharply transitively on $X\backslash\{x\}$.

MS_2 In the full permutation group of X, each U_x normalizes the set of subgroups $\{U_y \mid y \in X\}$.

The groups U_x will be called *root groups*. The elements of U_x are often called *root elations*. If U_x is abelian for some $x \in X$, then it is abelian for all $x \in X$ and we call the Moufang set a *translation Moufang set*.

The first article in which a theory concerning Moufang sets was built in a systematic way is [5]. In this article the Moufang set is constructed out of a group U together with some permutation τ. They also found a connection between Jordan division algebras and the now known Moufang sets with abelian root groups.

Moufang sets can be found inside Moufang generalized polygons: consider 2 opposite points p and q inside a Moufang n-gon, then the panels P_i, for which $\{p\} \cup P_i \cup \{q\}$ represents a *root* i.e. a path of length n, along with their corresponding root groups form a Moufang set. Also polarities give rise to Moufang sets. In one of the following chapters we shall work with the Ree-Tits Moufang set which is such a Moufang set.

If $(X, (U_x)_{x\in X})$ is a Moufang set, and $Y \subseteq X$, then Y, $|Y| > 2$, induces a *sub Moufang set* if, for each $y \in Y$, the stabilizer $(U_y)_Y$ acts sharply transitively on $Y \setminus \{y\}$. In this case $(Y, ((U_y)_Y)_{y\in Y})$ is a Moufang set.

The group S generated by the U_x, for all $x \in X$, is called the *little projective group* of the Moufang set. A permutation of X that normalizes the set of subgroups $\{U_y \mid y \in X\}$, is called an *automorphism* of the Moufang set. The set of all automorphisms of the Moufang set is a group G, called the *full projective group* of the Moufang set. Any group H, with $S \leq H \leq G$, is called a *projective group* of the Moufang set. Projective groups can be recognized as follows:

Lemma 1.4.1. (i) *The little projective group S of a Moufang set $(X, (U_x)_{x\in X})$ acts doubly transitively on the set X.*

(ii) *A permutation group H (acting on X) is a projective group if and only if $U_x \trianglelefteq H_x$, for every $x \in X$.*

Since Moufang sets deal with sharply transitive actions, we quickly review the basics of those actions. These statements and their proofs may be found in every text book on basic group theory.

Lemma 1.4.2. *Let (G, X) be a sharply transitive group.*

(i) *We can identify X with G and the action of $g \in G$ on the element $x \in X = G$ is given by right multiplication xg.*

(ii) *Suppose some permutation u commutes with every element of $G = X$. Then u acts freely on X and the action of u is given by left multiplication with some element $h \in G$ (so u maps every $x \in X$ to hx).*

(iii) *If a transitive permutation group H centralizes G, then H acts sharply transitively on X and there is an isomorphism $\varphi : H \to G$ such that the action of $h \in H$ on $G = X$ is given by $h : x \mapsto h^{-\varphi}x$. If in particular G is abelian, then $H = G$.*

(iv) *Suppose the sharply transitive permutation groups H and G normalize each other. Then either G and H have a nontrivial intersection, or G and H centralize each other.*

Chapter 2

Moufang Conditions for Generalized Quadrangles

The Moufang condition is a well-known and useful characterization of generalized polygons. In 1991 S. E. Payne, J. A. Thas and H. Van Maldeghem wondered if there existed weaker versions of this condition. They came up with the half Moufang condition and proved that this condition is equivalent with the Moufang notion for finite generalized quadrangles. In this chapter we will define a bunch of Moufang-like conditions. We prove that the half 3-Moufang property is equivalent with the Moufang condition, as a consequence the half Moufang property and the 3-Moufang property are equivalent with the Moufang condition for generalized quadrangles. At the end of the chapter we give a schematic overview of Moufang-like conditions which are equivalent with the Moufang property and the Moufang-like conditions for which finiteness is required to obtain equivalence with the Moufang condition.

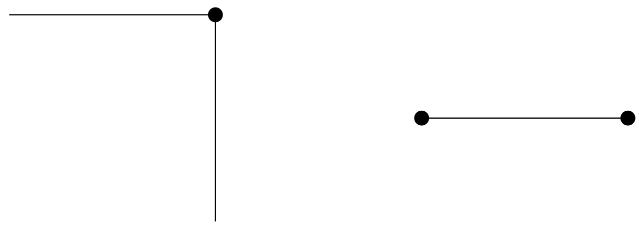
2.1 Subquadrangles and Moufang-like conditions

A subquadrangle is a subgeometry of a generalized quadrangle which itself is a generalized quadrangle. We say that a subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ is *full* if for every line L of $\mathcal{S}'$ all points on L in $\mathcal{S}$ also lie in $\mathcal{S}'$. Dually a subquadrangle is *ideal* if a point is incident with the same lines in $\mathcal{S}'$ as in $\mathcal{S}$.

For a set S of points (resp lines), we denote the set of points collinear (resp lines concurrent) with every element of S with $S^{\perp}$. The set of points collinear (lines concurrent) with every element of $S^{\perp}$ is denoted by $S^{\perp\perp}$, this is the *span* of S.

Since we have to work a lot with permutation groups here, the following notation will come in handy: The group acting on the generalized quadrangle and fixing the elements of a set X will be denoted by G_X, the group fixing all elements at distance i from at least one element of X, will be denoted by $G_X^{[i]}$. Sometimes we will write $G_z^{[x,y]}$ when we mean $G_{x,y}^{[1]} \cap G_z$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ be a generalized quadrangle. An *apartment* is an ordinary quadrangle in $\mathcal{S}$. It consists of four points and four lines. Recall that a *root* is a set of five "consecutive" elements of an apartment. Hence a root contains either two points x_1, x_2 and three lines L_1, L_2, L_3, with $L_1 \mathtt{I} x_1 \mathtt{I} L_2 \mathtt{I} x_2 \mathtt{I} L_3$, or dually, it contains three points and two lines. It will be useful to distinguish between these dual notions. Therefore, we will call the former a *root*, and the latter (dually) a *dual root*. A root without its extremal lines will be called a *panel*, more precisely, the *interior (panel)* of the root. Similar but dual definitions exist for *dual panels*. So a panel is a set of two distinct collinear points, together with the joining line.

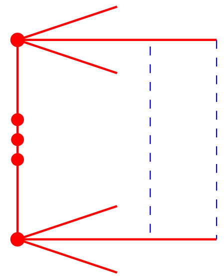


dual panel panel

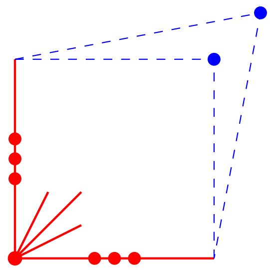
Note that the Main axiom of GQs implies that every pair of flags of $\mathcal{S}$ is contained in at least one apartment of $\mathcal{S}$.

We repeat the definition of the Moufang property but now for generalized quadrangles.

Let $\mathcal{S}$ be a thick generalized quadrangle with full collineation group $G = \mathrm{Aut}(\mathcal{S})$. Let π be a panel. Then we say that π *has the Moufang property* (or, equivalently, π is a *Moufang panel*) if for some root α with interior π, the group $G_\pi^{[1]}$ of collineations (called *root elations* and, dually, *dual root elations*) fixing every element incident with any element of π acts transitively on the set of apartments containing α.



root−elation

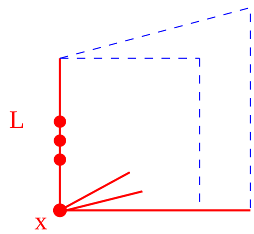


dual root−elation

Let $\pi = \{x, L, y\}$, with $x, y \in \mathcal{P}$ and $L \in \mathcal{B}$. Let M be any line through x distinct from L. Suppose π is a Moufang panel, and let α be a root with interior π such that $G_\pi^{[1]}$ acts transitively on the set of apartments containing α. Let u, u' be two points on M distinct from x. Suppose the unique line of α incident with y and unequal L is N. Then N is opposite M and we may consider the distinct points $u_1 = \mathrm{proj}_N u$ and $u_1' = \mathrm{proj}_N u'$. Using the Main axiom of GQs, it is quite easy to see that u_1 and u_1' determine unique apartments Σ and Σ', respectively, containing α. Hence there is a collineation $\theta \in G_\pi^{[1]}$ mapping Σ to Σ', and hence mapping u_1 to u_1'. But since M is fixed under θ by assumption, θ maps u to u', and hence, for every line M' through y distinct from L, θ maps the (unique) apartment containing π, u and M' to the (unique) apartment containing π, u' and M'. We have shown that the definition of Moufang panel is independent of the root involved in that definition.

If every panel and every dual panel of the GQ $\mathcal{S}$ has the Moufang property, then we say that $\mathcal{S}$ *has the Moufang property*, or that $\mathcal{S}$ is a *Moufang* GQ. If every panel is a Moufang panel, or if every dual panel is a Moufang dual panel, then we say that $\mathcal{S}$ is a *half Moufang* GQ. If $\mathcal{S}$ is a Moufang GQ, then the group generated by all root elations and dual root elations is called the *little projective group of* $\mathcal{S}$.

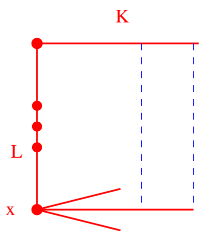
Let $\{x, L\}$ be a flag of the GQ $\mathcal{S}$. Let $M \mathtt{I} x$ and $y \mathtt{I} L$ such that y is not incident with M. As we will see later, the group $G_{x,L}^{[1]}$ of collineations fixing all lines through x and fixing all points on L acts freely on the set of apartments containing $\{x, y, L, M\}$. We call the flag $\{x, L\}$ a *Moufang flag* if the group $G_{x,L}^{[1]}$ acts transitively (and hence sharply transitively) on the set of apartments containing $\{x, y, L, M\}$. As before, one shows easily that this definition is independent of the chosen line M through x, $M \neq L$, and of the chosen point y on L, $y \neq x$.



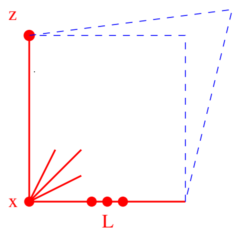
Moufang flag

We see that the definition of a Moufang flag is a self-dual one, hence there is no need to introduce something like a "dual Moufang flag". If every flag of the GQ $\mathcal{S}$ is a Moufang flag, then we call $\mathcal{S}$ *3-Moufang*, where the number 3 refers to the length of the sequence (y, L, x, M) as a path in the incidence graph of $\mathcal{S}$ (which is the graph $(\mathcal{P} \cup \mathcal{B}, \mathtt{I})$). The 3-Moufang condition is a completely self-dual one, and so one might think that a notion of "half 3-Moufang" cannot be defined. But actually, one can take a kind of greatest common divisor of the 3-Moufang condition and the half Moufang condition to obtain the following definition of a half 3-Moufang generalized quadrangle.

Let $\{x, L\}$ be a flag of the GQ $\mathcal{S}$. Let $z\mathtt{I}M\mathtt{I}x$ and $K\mathtt{I}y\mathtt{I}L$ such that $z \neq x$, $K \neq L$ and y is not incident with M. As before, the group $G^{[1]}_{x,L} \cap G_z$ acts freely on the set of apartments containing $\{y, L, x, M, z\}$, and the group $G^{[1]}_{[x,L]} \cap G_K$ acts freely on the set of apartments containing $\{K, y, L, x, M\}$. When, for every choice of $z \sim x$, $G^{[1]}_{x,L} \cap G_z$ acts transitively on the set of apartments containing $\{y, L, x, M, z\}$, we say that the flag $\{x, L\}$ is *half* 3-*Moufang at* x, while a transitive action of $G^{[1]}_{[x,L]} \cap G_K$ on the set of apartments containing $\{K, y, L, x, M\}$, for any choice for K concurrent with L, defines $\mathcal{S}$ to be *half* 3-*Moufang at* L. The GQ $\mathcal{S}$ is called *half* 3-*Moufang* if either every flag $\{x, L\}$ of $\mathcal{S}$ is half 3-Moufang at x, or if every flag $\{x, L\}$ of $\mathcal{S}$ is half 3-Moufang at L.



Half 3−Moufang at L



Half 3−Moufang at x

Now let x be any point of the GQ $\mathcal{S}$. If the group $G^{[x]}$ of collineations fixing all lines through x acts transitively on the set of points of $\mathcal{S}$ opposite

x (here, this is not necessarily a regular action!), then we say that x is a *center of transitivity*. Dually, we define an *axis of transitivity*. If all points are centers of transitivity, and all lines are axes of transitivity, then we say that $\mathcal{S}$ is *a* 2-*Moufang* GQ. If either all points are centers of transitivity, or all lines are axes of transitivity, then we call $\mathcal{S}$ a *half* 2-*Moufang* GQ.

Let $\{x, L\}$ again be a flag of the GQ $\mathcal{S}$, with x a point and L a line. Another flag $\{y, M\}$ is called *opposite* $\{x, L\}$ if the point y is opposite x and the line M is opposite L. If the group $G_{x,L}$ of all collineations of $\mathcal{S}$ fixing the flag $\{x, L\}$ acts transitively on the set of flags opposite $\{x, L\}$, then we say that $\{x, L\}$ is a *transitive flag*. If all flags are transitive, then we say that $\mathcal{S}$ is a 1-*Moufang* GQ, or, equivalently, that $\mathcal{S}$ satisfies the *Tits condition*, or that $\mathcal{S}$ is a *Tits* GQ. The last two names are motivated by their group theoretic counterparts, the *Tits systems*. If $\mathcal{S}$ satisfies the Tits condition, then the corresponding collineation group G acts transitively on the set of ordered pairs of opposite flags of $\mathcal{S}$.

For convenience, we will sometimes also refer to the (half) 4-Moufang condition when we mean a (half) Moufang GQ.

2.2 Some straightforward implications

2.2.1 Root groups are semi-regular

In a generalized n-on, the group $G_{x_0,\ldots,x_{n+1}} \cap G^{[1]}_{x_i,x_{i+1}}$ contains only the identity. This is proved in [**?**, Tit-Wei:02]3.7), but it is also an immediate corollary of [**?**, Mal:98]:

Theorem 2.2.1. *Let g be a collineation of a generalized n-gon Γ with n even. Then g is the identity if and only if g fixes an apartment Σ, all points on a certain line L of Σ and all lines through a certain point p of Γ with $\mathrm{d}(p, L)$ relatively prime to n (this happens in particular when p is incident with L).*

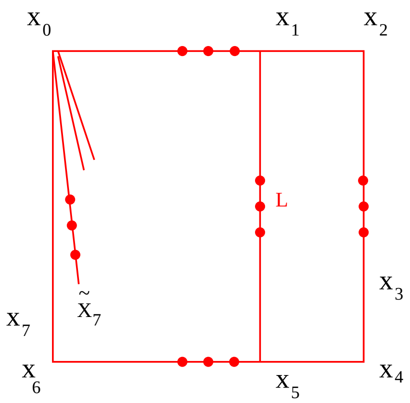
We will provide a proof for one particular case in order to get acquainted with the necessary reasonings. From here on we will denote $\Sigma = (x_0, \ldots, x_8 = x_0)$ where points are labeled with x_i for i even.

Theorem 2.2.2. *The group $G_{x_0,\ldots,x_5} \cap G^{[1]}_{x_0,x_1}$ acting on the generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ contains only the identity.*

Proof. Any collineation g in this group has to fix the apartment Σ through $(x_0, x_1, x_2, x_3, x_4, x_5)$ since $\mathrm{d}(x_0, x_5) < 4$.

So if we look at the fixed structure $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathtt{I})$ the first 2 axioms of a generalized quadrangle are immediately satisfied. If we now look a point

p and a line L not incident with p, we know that there is a unique point y collinear with p on L in $\mathcal{S}$, the collineation g can map this point only to a point collinear with p on L, hence y is fixed by g and therefore belongs to $\mathcal{S}'$. In the same way the unique line M in $\mathcal{S}$ concurrent with L through p has to belong to $\mathcal{S}'$. In $\mathcal{S}$, every point on a fixed line opposite x_1 is fixed by g, since it is the unique projection of some point of $x-1$ on this particular line. The points on fixed lines concurrent with x_1 are also fixed by g, say we consider such a line L: either this line is opposite to x_5— whose points are all fixed by g— or L is concurrent with both x_1 and x_5, in that case let's observe any line through x_0 distinct from x_1 or x_7. This line is opposite to both x_5 and L, so g fixes every point on L. This proves that g fixes a full generalized quadrangle, the dual reasoning leads to the conclusion that g must be the identity. □



g fixes $\Gamma_1(x_1)$, $\Gamma_1(x_5)$, $\Gamma_1(\tilde{x}_7)$ and thus $\Gamma_1(L)$

2.2.2 i-Moufang quadrangles and half i-Moufang quadrangles

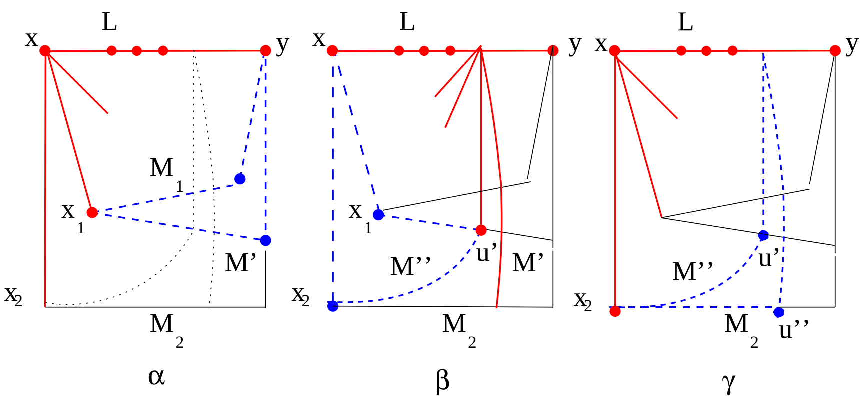
For $2 \leq i \leq 4$ every i-Moufang quadrangle is half i-Moufang: for i even, we only demand that one half of the paths are i-Moufang while for $i = 3$ the transitivity of $G_{x_7,x_0,x_1,x_2} \cap G^{[1]}_{x_0,x_1}$ on the set of apartments through (x_7, x_0, x_1, x_2) a fortiori implies transitivity on the apartments through $(x_6, x_7, x_0, x_1, x_2)$ or on the apartments through $(x_7, x_0, x_1, x_2, x_3)$.

2.2.3 Half i-Moufang quadrangles and half $(i-1)$-Moufang quadrangles

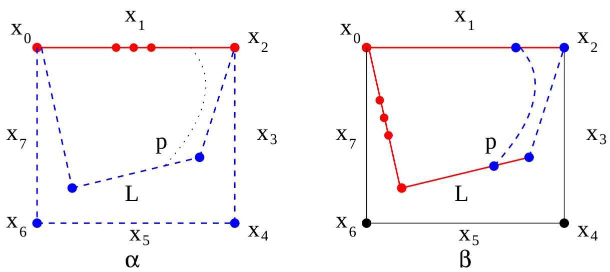
The half Moufang condition implies the half 3-Moufang condition which implies the half 2-Moufang property and thus the 1-Moufang condition.

From half Moufang to half 3-Moufang If all dual panels are Moufang, then every flag is Moufang at its point, in the same way Moufang panels give rise to flags which are Moufang at their line.

From half 3-Moufang to half 2-Moufang If, for a fixed line L, all flags $\{p, L\}$ are half 3-Moufang at p, the line L is an axis of transitivity: Indeed, consider 2 lines M_1 and M_2 both opposite L, let x and y be two arbitrary points on L, then we can denote $x_i := \text{proj}_{M_i} x$ and $y_i := \text{proj}_{M_i} y$. Since x_1 is not incident with the line yy_2 there is a unique line M' through x_1 concurrent with the line yy_2. We can define α to be the unique collineation in $G^{[1]}_{x,L} \cap G_{x_1}$ mapping M_1 onto M'. Now we look for a collineation fixing all the points of L and mapping x_1 to x_2; so we may assume that $x_1 \neq x_2$. If $xx_1 \neq xx_2$ there is a unique line M'' through x_2 concurrent with M', their intersection point u' is collinear with a unique point u on L, the collineation $\beta \in G^{[1]}_{u,L} \cap G_{u'}$ mapping M' onto M'' indeed maps x_1 onto x_2. If $xx_1 = xx_2$ we can first map x_1 onto x_1^g with $g \in G^{[L,y]}_{y_1{}^\alpha}$ and then apply β, again x_1 is mapped onto x_2. If we finally consider the projection of u onto the line x_2y_2 and call it u'' then it is clear that there is a unique collineation $\gamma \in G^{[1]}_{x,L} \cap G_{x_2}$ mapping u' onto u''. Now the composition $\alpha\beta\gamma$ fixes all points on L and maps M_1 onto M_2. Since M_i , $i = 1, 2$ were arbitrarily chosen, it is clear that L is an axis of transitivity.

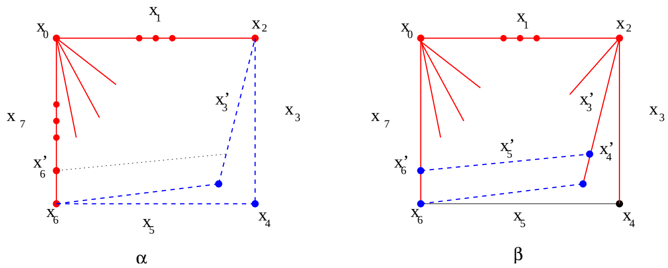


The half 2-Moufang condition implies the 1-Moufang property We consider the apartment Σ and a flag $\{p, L\}$ opposite to $\{x_0, x_1\}$, we claim that there is a collineation of the generalized quadrangle fixing $\{x_0, x_1\}$ and mapping $\{x_4, x_5\}$ onto $\{p, L\}$ if every line is an axis of transitivity. Of course, there is a collineation α fixing every point on x_1 and mapping the line x_5 onto L. If we compose this collineation with a collineation β fixing all points on x_7^α and mapping x_3^α onto the unique line through p concurrent with x_2, we have what we wanted.



2.2.4 i-Moufang quadrangles and $(i-1)$-Moufang quadrangles

From Moufang to 3-Moufang Consider 2 apartments Σ and Σ' both going through the path (x_7, x_0, x_1, x_2), the apartment Σ' being the cycle $(x_0, x_1, x_2, x_3', x_4', x_5', x_6', x_7, x_8 = x_0)$. The collineation $\alpha \in G^{[1]}_{x_7,x_0,x_1}$ mapping x_3 to x_3' followed by $\beta \in G^{[1]}_{x_0,x_1,x_2}$ mapping x_6 onto x_6' is a collineation fixing all lines through x_0, all points on x_1 and it maps Σ onto Σ'.



From 3-Moufang to 2-Moufang Since every flag $\{x, L\}$ is Moufang at its point, every line L is an axis of transitivity. Dually, every flag $\{x, L\}$ is Moufang at its line, so every point is a center of transitivity.

From 2-Moufang to 1-Moufang If half 2-Moufang quadrangles are 1-Moufang, then surely 2-Moufang quadrangles are 1-Moufang.

2.3 Some properties of Moufang sets

Recall that a Moufang set is a set X, together with a group U_x for every $x \in X$ such that the groups U_x permute $X \setminus \{x\}$, and such that all the root groups U_x are conjugate. A first question that can be asked is: do we really need all the root groups in the definition? This is one answer.

Theorem 2.3.1. *Let X ($|X| > 2$) be a set and let $a, b \in X$ be distinct. Let U_a and U_b be two permutation groups acting on X such that U_a fixes a and acts sharply transitively on $X \setminus \{a\}$, while U_b fixes b and acts sharply transitively on $X \setminus \{b\}$. Then U_a and U_b are root groups of at most one common Moufang set. Also, they are root groups of a Moufang set if and only if for each $u \in U_a^\times$, there exists $v \in U_b$ such that $U_a^v = U_b^u$, if and only if U_a is normal in G_a, U_b is normal in G_b, and U_a is conjugate to U_b in $G := \langle U_a, U_b \rangle$.*

Proof. Suppose first that $(X, (U_x)_{x\in X})$ is a Moufang set. For given $u \in U_a \setminus \{\mathrm{id}\}$, define $v \in U_b$ by $a^v = b^u$. Then (MS2) implies that $U_a^v = U_{a^v} = U_{b^u} = U_b^u$. By the double transitivity of G, U_a can be mapped to U_b using some conjugation of G. Also, since G normalizes the U_x, the stabilizer G_a of a must normalize U_a, hence $U_a \trianglelefteq G_a$ and likewise $U_b \trianglelefteq G_b$.

Now suppose that for each $u \in U_a^\times$ there exists $v \in U_b$ such that $U_a^v = U_b^u$. Clearly $a^v = b^u$. This motivates the following notation. For each $x \in X \setminus \{a, b\}$, denote by u_x the unique element of U_a mapping b to x and let v_x be the unique element of U_b mapping a to x. Define $U_x := U_a^{v_x} = U_b^{u_x}$. Then we claim that $(X, (U_x)_{x\in X})$ is a Moufang set. Indeed, (MS1) is obvious, so consider (MS2). We have to show that, for all $x \in X$, the group U_x permutes by conjugation the groups U_y, $y \in X$. For $x \in \{a, b\}$, this is trivial. If $x \notin \{a, b\}$, then we can write an arbitrary element of U_x as $v_x^{-1} u v_x$, with $u \in U_a$, and since each of v_x^{-1}, u and v_x preserves by conjugation $\{U_y \parallel y \in X\}$, also $v_x^{-1} u v_x$ does. It is now also clear that U_a and U_b are root groups of at most one common Moufang set.

Now suppose that U_a is normal in G_a, U_b is normal in G_b, and U_a is conjugate to U_b in $G = \langle U_a, U_b \rangle$. Let $u \in U_a^\times$ be arbitrary, and let $v \in U_b$ be such that $b^u = a^v$. Then $a^{vu^{-1}} = b$. Since U_a and U_b are conjugate in G, there exists $g \in G$ such that $U_a^g = U_b$. Clearly $a^g = b$, as a and b are unambiguously determined by U_a and U_b, respectively. This implies that $vu^{-1}g^{-1} \in G_a$, and so, by assumption, $U_a^{vu^{-1}g^{-1}} = U_a$, hence $U_a^v = U_a^{gu} = U_b^u$. The previous paragraph completes the proof of the proposition. □

The condition that for each $u \in U_a^\times$ there exists $v \in U_b$ such that $U_a^v = U_b^u$ is used by Timmesfeld [32] to define rank one groups. But note that a rank one group has the additional condition that all root groups are nilpotent.

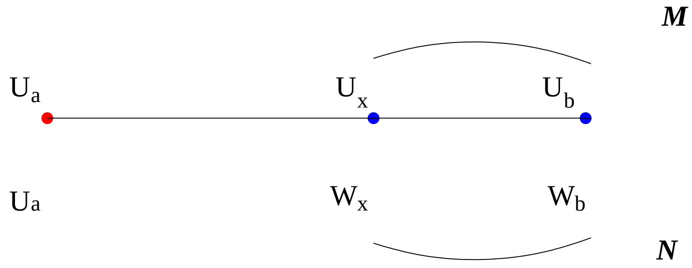
For a set X, elements $a, b \in X$ (with $|X| > 2$), and permutation groups U_a, U_b of X such that U_a (respectively, U_b) fixes a (respectively, b) and acts sharply transitively on $X \setminus \{a\}$ (respectively, $X \setminus \{b\}$), we call (X, U_a, U_b) a *Moufang triple* if, for $G = \langle U_a, U_b \rangle$, U_a is normal in G_a, U_b is normal in G_b, and U_a is conjugate to U_b in G.

Proposition 2.3.1 tells us something about the possibility for two root groups to be contained in the same Moufang set. Now we want to look at the situation where we have two Moufang sets acting on the same set X and sharing at least one root group, and we want to find conditions under which these Moufang sets are the same. More precisely, we have the following result.

Theorem 2.3.2. *Let $(X, (U_x)_{x \in X})$ be a Moufang set, and let $a, b \in X$ be distinct. Moreover, let W_b be a permutation group acting on X, fixing b and acting sharply transitively on $X \setminus \{b\}$. Suppose that (X, U_a, W_b) is a Moufang triple. If $U_b \cap W_b$ is nontrivial, or if U_b and W_b normalize each other, then $W_b = U_b$. If U_b and W_b centralize each other, then $U_b = W_b$ and the Moufang set is a translation Moufang set.*

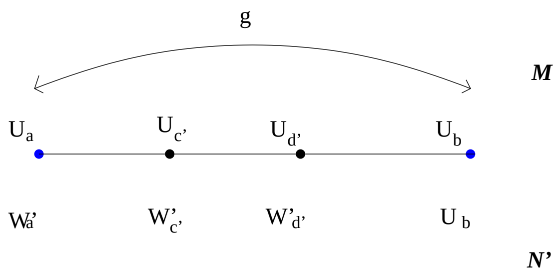
Proof. Let $\mathcal{M}$ be the Moufang set determined by U_a and U_b, and let $\mathcal{N} = (X, (W_x)_{x \in X})$ be the one determined by U_a and W_b, with $U_a = W_a$. First suppose that $v \in U_b \cap W_b$ is nontrivial. Then $W_{a^v} = W_a^v = U_a^v = U_{a^v}$ is a root group in both $\mathcal{M}$ and $\mathcal{N}$, and since these Moufang sets are both determined by U_a and U_{a^v}, they must coincide. Hence $U_b = W_b$.

Now suppose that U_b centralizes W_b. Then by lemma 1.4.2, the action of W_b is opposite the action of U_b. Take arbitrary elements $u \in U_a$, $v \in U_b$ and $w \in W_b$. Then $[v, w] = \mathrm{id}$, hence $[v^u, w^u] = \mathrm{id}$ and we see that the action of W_{b^u} is opposite the action of U_{b^u}. If $|X| = 3$, then clearly $U_b = W_b$ is commutative and the result follows. Hence we may assume $|X| \geq 4$, so that we can take two distinct elements c, d in $X \setminus \{a, b\}$. We then see that $\mathcal{N}$ is determined by the permutation groups W_c and W_d the actions of which are opposite those of U_c and U_d, respectively.



Since U_a acts sharply transitively on $X \setminus \{a\}$ and it fixes both $\mathcal{M}$ and $\mathcal{N}$, all W_x are opposite U_x for $x \neq a$

By the double transitivity of the little projective group $G(\mathcal{M})$ of $\mathcal{M}$, there is a permutation $g \in G(\mathcal{M})$ interchanging a with b. Since $[U_b, W_b] = \{\mathrm{id}\}$, we also have $[U_b^g, W_b^g] = \{\mathrm{id}\}$. Hence, if we denote U_a^{opp} by V_a, then, since $U_b^g = U_{b^g} = U_a$, and since W_b^g acts sharply transitively on $X \setminus \{a\}$, we obtain $W_b^g = V_a$. If $c^{g^{-1}} = c'$ and $d^{g^{-1}} = d'$, then similarly, one easily shows that $W_{c'}^g = W_c$ and $W_{d'}^g = W_d$.



g fixes $\mathcal{M}$ and maps $\mathcal{N}$ onto another Moufang Set $\mathcal{N}' = (X, (W_x')_{x \in X})$, where all W_x' are opposite U_x for $x \neq b$.

Hence the conjugate $\mathcal{N}^g$ of $\mathcal{N}$ contains the root groups W_c and W_d and thus coincides with $\mathcal{N}$. Comparing the root groups fixing b, we see that $U_b = W_b$, and $[U_b, W_b] = \{\mathrm{id}\}$ implies that U_b is abelian. Hence $\mathcal{M} = \mathcal{N}$ is a translation Moufang set.

Finally suppose that U_b and W_b normalize each other. If they share a nontrivial permutation of X, then by the first part of the proof $U_b = W_b$. If they intersect trivially, then by Lemma 1.4.2(iv) they centralize each other, and the second part of our proof implies again $U_b = W_b$. □

2.4 Half 3-Moufang quadrangles are Moufang

2.4.1 Main idea

If all flags in $\mathcal{S}$ are Moufang at their point, it is fairly easy to construct a collineation g fixing a root $(x_7, x_0, x_1, x_2, x_3)$, all points on x_1 and moving Σ to an arbitrary apartment through that root. Let x be any point on x_1, $x \neq x_0$ and choose $y \sim x$ not lying on x_1, we state that the action of $G^{[1]}_{x_1,x} \cap G_y$ on the lines through x_0 is independent of our choice for x and y. This allows a collineation g' fixing all elements incident with the panel x_0, x_1, x_2. In other words, every panel is Moufang and we have a half Moufang quadrangle. Since a half Moufang quadrangle gives rise to flags which are Moufang at their lines, we can apply a dual reasoning and conclude that half 3-Moufang quadrangles are Moufang. The proof is also written down [13].

2.4.2 Proof

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ be a thick generalized quadrangle with automorphism group G, satisfying the half 3-Moufang Condition. More precisely, we assume that for all dual roots $\{y_0, y_1, y_2, y_3, y_4\}$, with $y_0 \mathtt{I} y_1 \mathtt{I} \dots \mathtt{I} y_4$, and with $y_0, y_2, y_4 \in \mathcal{P}$, the group $G^{[y_2,y_3]}_{y_0}$ acts transitively on the apartments containing $y_0, \dots, y_4$.

Our first aim is to show that $\mathcal{S}$ is half Moufang — more precisely, that all panels are Moufang.

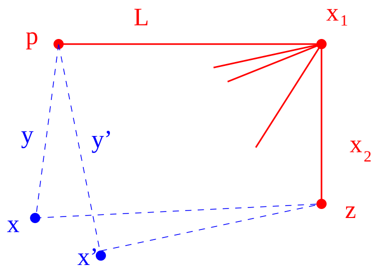
As in the previous section, we fix some apartment

$$\Sigma := \{x_0, x_1, \dots, x_7\}, \quad x_0 \mathtt{I} x_1 \mathtt{I} \cdots \mathtt{I} x_7 \mathtt{I} x_0,$$

where $x_0 \in \mathcal{P}$, and where we read the subscripts modulo 8. We prove some lemmas, under the assumptions just stated.

Lemma 2.4.1. *All sequences (x, y, y', z), with $x \mathtt{I} y \mathtt{I} y' \mathtt{I} z$, $x \in \mathcal{B}$, $x \neq y'$, and $y \neq z$, form a single orbit under G. In particular, all groups $G^{[x,y]}_z$ are conjugate.*

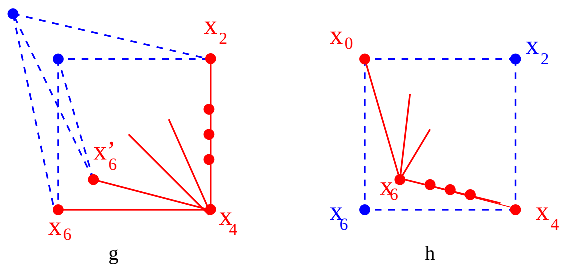
Proof. We already know that G acts transitively on the set of flags of $\mathcal{S}$ so for any given $(p, L) \in \mathcal{P} \times \mathcal{B}$, with $p \mathtt{I} L$, it suffices to prove that the group $G_{p,L}$ acts transitively on the set of flags $\{x, y\}$, with $x \mathtt{I} y \mathtt{I} p$, $x \neq p$ and $y \neq L$. So let $\{x, y\}$ and $\{x', y'\}$ be two such flags. First suppose that $y \neq y'$. Then we choose arbitrarily some element z in $\{x, x'\}^\perp \setminus \{p\}$. Set $x_1 := \mathrm{proj}_L z$ and $x_2 := \mathrm{proj}_z L$. There is a collineation $u \in G^{[x_1,x_2]}_p$ mapping x to x', and hence y to y'.



If $y = y'$, then we consider any flag $\{y'', x''\}$, with $x''\mathtt{I}y''\mathtt{I}p''$, $y \neq y'' \neq L$, and $z'' \neq p$, apply twice the previous paragraph and get the result. □

Lemma 2.4.2. *If the span $\{x, y\}^{\perp\perp}$ of some opposite points x, y contains at least 3 elements, then all dual panels of $\mathcal{S}$ are Moufang.*

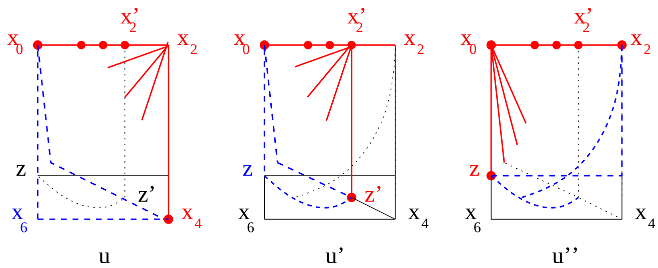
Proof. We may assume without loss of generality that $\{x, y\} = \{x_2, x_6\}$. Let $x'_6 \in \{x_2, x_6\}^{\perp\perp}$, with $x_2 \neq x'_6 \neq x_6$. Let x'_5 denote the line incident with x_4 and x'_6. As $G^{[x_3,x_4]}_{x_6}$ fixes $\{x_2, x_6\}$, the span $\{x_2, x_6\}^{\perp\perp}$ has to be stabilized as a set, but as the lines through x_4 are fixed as well, this implies that the span is fixed pointwise, and hence in particular x'_6 is fixed. Consider an arbitrary element $g \in G^{[x_3,x_4]}_{x_6}$ and choose an element $h \in G^{[x'_5,x'_6]}_{x_0}$ mapping x_2 to x_6 (h exists by the half 3-Moufang assumption on the dual root $\{x_0, x_0x'_6, x'_6, x'_5, x_4\}$). The commutator $[g, h]$ clearly belongs to $G^{[x_4,x'_5,x'_6]}_{x_6}$ and hence it is trivial. Consequently $g = g^h \in G^{[x_3,x_4,x_5]}$. □



Let Ω denote the set of lines incident with x_0.

Lemma 2.4.3. *Let x range over the set of points incident with x_1, $x \neq x_0$, and let y range over the set of points not on x_1 but collinear with x. If the action of $G^{[x_1,x]}_y$ on Ω is independent of x and y, then all panels of $\mathcal{S}$ are Moufang.*

Proof. It suffices to show that there is an element $g \in G^{[x_0,x_1,x_2]}$ mapping x_6 to an arbitrary point $z \neq x_0$ on x_7. Let us start with an arbitrary nontrivial collineation $u \in G_{x_4}^{[x_1,x_2]}$. Then there is a unique point z' on x_5^u collinear with z. Hence, if we denote by x_2' the unique point on x_1 collinear with z', then the collineation $u' \in G_{z'}^{[x_1,x_2']}$ mapping x_7^u to x_7 maps x_6^u to z. The composition uu' fixes all points on x_1 and — by assumption — it also fixes all lines incident with x_0, since the action of u on Ω must be the inverse of the action of u' on Ω. Moreover, uu' maps x_6 to z. Also, the action of uu' on the set of lines through x_2 is the same as the action of u' on that set (since u fixes every line through x_2). Interchanging now the roles of x_0 and x_2, we see that the collineation $u'' \in G_z^{[x_0,x_1]}$ mapping $x_3^{u'}$ back to x_3 has an action on the lines through x_2 that is inverse to the action of uu' on that set. This implies that $uu'u'' \in G^{[x_0,x_1,x_2]}$. Since $uu'u''$ maps x_6 to z, the assertion follows. □

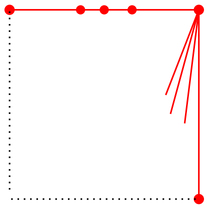


So, with the notation of the previous lemma, we will first fix x, vary y, and prove independence of the appropriate action; afterwards, we vary x and make particular choices for the points y.

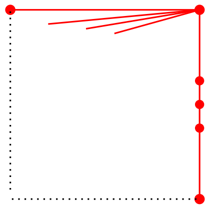
Lemma 2.4.4. *Let y be any point not on x_1 collinear with x_2. Then the action of $G_y^{[x_1,x_2]}$ on Ω is independent of y.*

Proof. First we note that we may assume y to be incident with x_3. Indeed, this follows immediately from the fact that the group $G_{x_6}^{[x_0,x_1]}$ acts transitively on the lines through x_2 distinct from x_1, and so any group $G_z^{[x_1,x_2]}$, with z any point not on x_1 but collinear with x_2, can thus be seen as a conjugate of some $G_y^{[x_1,x_2]}$, with $y \mathtt{I} x_3$, under a collineation which fixes all lines through x_0.

Now, if the action of $G_y^{[x_1,x_2]}$ on Ω were not independent of the choice of y, with y incident with x_3, then we may assume that for some $y \mathtt{I} x_3$, $y \neq x_2$, the action of the group $G_1 := G_y^{[x_1,x_2]}$ on Ω differs from the action of the group $G_2 := G_{x_0}^{[x_2,x_3]}$ on Ω. Since this statement does not involve x_4, x_5, x_6 and x_7, we may rename x_4 as y and hence assume $G_1 = G_{x_4}^{[x_1,x_2]}$. Note that both G_1 and G_2 act faithfully on Ω.



G 1



G 2

Suppose first that there is an element $u \in G_1 \cup G_2$ such that u commutes with every element of $G_1 \cup G_2$. We claim that G_1 and G_2 must have the same action on Ω. Indeed, if not, then there is a collineation $g_1 \in G_1$ such that its action on Ω is not induced by any element of G_2. Let $g_2 \in G_2$ be such that g_2 maps $x_7^{g_1}$ back to x_7. Noting that $(x_6^u)^{g_1 g_2} = (x_6^{g_1 g_2})^u = x_6^u$, we see that $g_1 g_2 \in G_{x_6, x_6^u}^{[x_2]}$. If x_6^u were not contained in $\{x_2, x_6\}^{\perp\perp}$, then $g_1 g_2$ would fix at least three points on some line through x_2, implying that $g_1 g_2$ would fix an ideal subGQ. This contradicts the fact that $g_1 g_2$ does not fix all lines through x_0. Hence we have a point span of at least three elements. Lemma 2.4.2 now implies that $\mathcal{S}$ is half Moufang with respect to dual roots, i.e., $G_1 = G_2$.

Hence we may assume that the centralizer of $G_1 \cup G_2$ in $G_1 \cup G_2$ is trivial. Note that G_1 and G_2 normalize each other. We claim that G_1 cannot have a commutative action on Ω. Indeed, if G_1 were commutative, then also G_2 would be commutative (as by Lemma 2.4.1 the groups G_1 and G_2 are conjugate, and they both act faithfully on Ω). If only the identity in G_1 has the same action on Ω as some element of G_2, then G_1 and G_2 centralize each other, contradicting the assumption that the centralizer of $G_1 \cup G_2$ in $G_1 \cup G_2$ is trivial (alternatively, by Proposition 1.4.2(iii) two abelian groups acting regularly on a set Ω and centralizing each other must have the same action on Ω). Hence there is some nontrivial element c_1 in G_1 having the same action on Ω as an element c_2 in G_2. Now c_1 centralizes G_1 since G_1 is commutative. Since c_1 and c_2 have the same action on Ω, this implies that $[c_2, G_1]$ acts trivially on Ω. Since $[c_2, G_1] \leq G_1 \cap G_2$, this implies $[c_2, G_1] = \{\mathrm{id}\}$. Consequently both c_1, c_2 centralize $G_1 \cup G_2$, again a contradiction with our assumptions. The claim is proved.

Next we claim that only the identity in G_1 has the same action on Ω as some element of G_2. Indeed, suppose by way of contradiction that there is a $u_1 \in G_1^\times$ inducing the same action on Ω as some $u_2 \in G_2$. Since u_1 cannot lie in the center of $G_1 \cup G_2$, we may suppose there is a $g \in G_1 \cup G_2$ such that the commutator $[u_1, g] \neq \mathrm{id}$ (and this is equivalent to the assumption that the action on Ω of that commutator is nontrivial). Suppose $g \in G_2$ — the case $g \in G_1$ is similar, if one interchanges the roles of x_0 and x_4, noting that the action of G_1 and G_2 on Ω is permutation equivalent with their action on the set of lines through x_4. Consider an arbitrary $h \in G_{x_6}^{[x_0,x_1]}$; then g^h induces the same action on Ω as g. It is clear that all the commutators $[u_1, g]$, $[u_2, g]$ and $[u_2, g^h]$ induce the same action on Ω, and each of them fixes all points of x_3. This easily implies $[u_1, g] = [u_2, g] = [u_2, g^h] =: u$ (use Theorem 2.2.1). Since $[u_2, g^h]$ fixes the line x_3^h pointwise and since h is arbitrary, we see that u, which is not trivial, fixes all points collinear with x_2. So, the image of x_6 under u must lie in the span of x_2 and x_6 which forces the generalized quadrangle to be half Moufang with respect to dual roots by Lemma 2.4.2. But then again $G_1 = G_2$, a contradiction.

Hence the regular actions of G_1 and G_2 on Ω normalize each other and share only the identity. This implies that they centralize each other, and the actions on Ω are opposite, see Proposition 1.4.2(iii) and (iv).

We conclude that, for arbitrary $y \mathtt{I} x_3$, $y \neq x_2$, the action of $G_y^{[x_1,x_2]}$ on Ω either is the same as the action of G_2 on Ω, or it is opposite.

Suppose both situations occur, so for some $y \mathtt{I} x_3$, $y \neq x_2$, the action of $G_1 = G_y^{[x_1,x_2]}$ on Ω is opposite the action of G_2 on Ω, and for some $z \mathtt{I} x_3$, $z \neq x_2$, the action of $G_3 := G_z^{[x_1,x_2]}$ on Ω is the same as the action of G_2 on Ω. Since $G_1 \cap G_2$ is trivial, no nontrivial element of G_2 can fix all points on x_1. This implies that $G_2 \cap G_3$ is trivial. But G_2 and G_3 normalize each other, hence they centralize each other. This means that the action of G_3 on Ω — which is the same as the action of G_2 on Ω — centralizes the action of G_2 on Ω, hence this action is commutative! This contradicts a previous claim.

We conclude that all actions of $G_y^{[x_1,x_2]}$ on Ω, $y \mathtt{I} x_3$, $y \neq x_2$, either are the same as the action of G_2 on Ω, or are opposite. In particular, the action is independent of y.

The lemma is proved. □

Lemma 2.4.5. *If x_2' is an arbitrary point on x_1, $x_2' \neq x_0$, and x_4' is the unique point on x_5 collinear with x_2', then the action of $G_{x_4}^{[x_1,x_2]}$ on Ω coincides with the action of $G_{x_4'}^{[x_1,x_2']}$ on Ω.*

Proof. Let U_2 be the permutation group acting on Ω given by the action of $G_{x_4}^{[x_1,x_2]}$. Define U_2' as the permutation group on Ω given by the action of $G_{x_4'}^{[x_1,x_2']}$. We assume that $U_2 \neq U_2'$ and seek a contradiction.

Let U_6 be the permutation group acting on Ω defined by $G_{x_4}^{[x_6,x_7]}$, and note that by Lemma 2.4.4, this action is the same as the one induced by $G_{x_4'}^{[x_6,x_7]}$ on Ω. Now notice that $G_{x_4}^{[x_1,x_2]}$ is normal in G_{x_0,x_2,x_4}, that $G_{x_4}^{[x_6,x_7]}$ is normal in G_{x_0,x_6,x_4}, and that $G_{x_4}^{[x_1,x_2]}$ is conjugate to $G_{x_4}^{[x_6,x_7]}$ in the group H generated by both (indeed, if g is nontrivial in $G_{x_4}^{[x_6,x_7]}$, then there is a unique element $h \in (G_{x_4}^{[x_1,x_2]})^g$ mapping x_3 to x_5 and hence x_2 to x_6 and x_1 to x_7; then $(G_{x_4}^{[x_1,x_2]})^h = G_{x_4}^{[x_6,x_7]}$). Consequently, U_2 and U_6 are conjugate in $K := \langle U_2, U_6 \rangle$, $U_2 \trianglelefteq K_{x_1}$ and $U_6 \trianglelefteq K_{x_7}$ (although K is defined as a permutation group acting (only) on Ω, it is clear what we mean with K_{x_1} and K_{x_7}). By Lemma 2.3.1, (K, Ω) defines a Moufang set with root groups U_2 and U_6. Likewise, (K', Ω), with $K' = \langle U_2', U_6 \rangle$, defines a Moufang set with root groups U_2' and U_6. We now want to apply Theorem 2.3.2 to obtain a contradiction. So we show that U_2 and U_2' normalize each other.

Choose arbitrary nontrivial $u_2 \in U_2$ and $u_2' \in U_2'$, these permutations being induced by the collineations $g \in G_{x_4}^{[x_1,x_2]}$ and $g' \in G_{x_4'}^{[x_1,x_2']}$, respectively. Then $g^{g'}$ belongs to $G_{x_4^{g'}}^{[x_1,x_2]}$, which has the same action on Ω as $G_{x_4}^{[x_1,x_2]}$ by Lemma 2.4.4. Hence $u_2^{u_2'} \in U_2$ and U_2' normalizes U_2. Similarly, U_2 normalizes U_2'.

Now Theorem 2.3.2 leads to a contradiction. □

Lemmas 2.4.3, 2.4.4 and 2.4.5 complete the proof of the following theorem.

Theorem 2.4.6. *If every flag of a* GQ is half 3-Moufang at its point, then all panels are Moufang.

2.4.3 Corollaries

We have the following easy corollary, see also Tent [25] and Haot and Van Maldeghem [12].

Corollary 2.4.7. *Every half Moufang* GQ is a Moufang GQ. Hence every half 3-Moufang GQ is a Moufang GQ.

Proof. Assume all dual panels are Moufang. Then obviously all flags are half 3-Moufang at their points. According to Theorem 2.4.6, every panel is Moufang. □

Corollary 2.4.8. *Every 3-Moufang* GQ is Moufang.

Proof. By the previous theorem, this follows immediately from the fact that every 3-Moufang quadrangle is half 3-Moufang. □

2.5 Other characterizations

2.5.1 Generalized quadrangles

For completeness' sake, we state some other results concerning Moufang generalized quadrangles without proof: the first one was observed by H. Van Maldeghem

[42], the second one is due to K. Thas and H. Van Maldeghem [31], the third one we owe to K. Tent and H. Van Maldeghem [28].

Theorem 2.5.1. *Every 2-Moufang generalized quadrangle* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ *is a Moufang generalized quadrangle.*

Theorem 2.5.2. *The half 2-Moufang condition is equivalent to the Moufang condition in the finite case.*

Theorem 2.5.3. *Fong-Seitz generalized quadrangles are Moufang.*

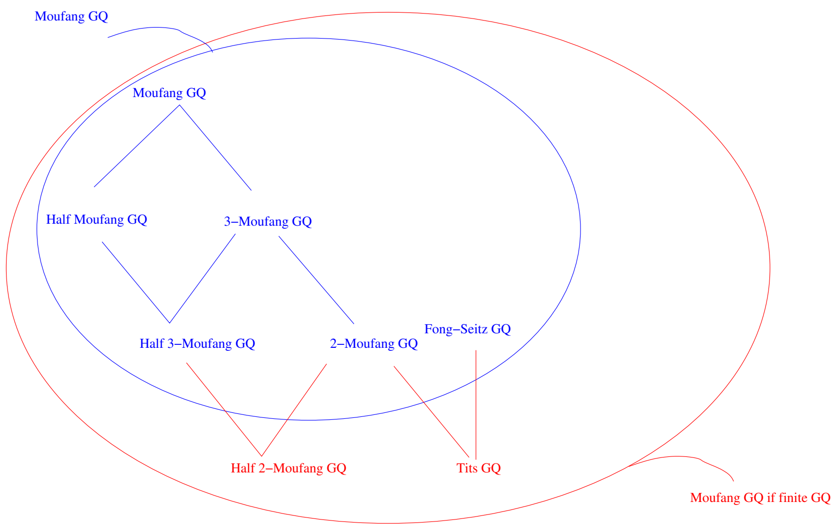
A Fong-Seitz generalized quadrangle is a 1-Moufang generalized quadrangle for which its BN-pair has a splitting. What this exactly means will become clear in the next chapter.

2.5.2 Generalized n-gons

The following observation was made by K. Tent [26], it is especially interesting for generalized hexagons: it implies that the Moufang property is equivalent with the k-Moufang property ($1 < k \leq n$).

Theorem 2.5.4. *If* Γ *is a* 2*-Moufang generalized* n*-gon for* $n \leq 6$*, then* Γ *is Moufang.*

2.6 Schematic Overview of the Moufang Conditions in a Generalized Quadrangle



Chapter 3

BN-pairs

In this chapter we consider all Moufang sets arising from a pair of opposite root groups in a Moufang building of rank 2, and the Moufang sets corresponding to the Suzuki groups and the Ree groups. In all these cases (except for one well understood exception), the (natural) root groups are the only transitive nilpotent normal subgroups U of the point stabilizers G_x.

Since this chapter tells us something about the behaviour of root groups in Moufang buildings, we first define buildings, and we continue with the closely related notion of a BN-pair. The split BN-pairs of rank 1 are the already defined Moufang sets. We have a classification of the Moufang sets coming from Moufang buildings of higher rank and we will use it in order to prove that the splitting of a BN-pair of rank 1 is unique. We also conclude with the statement that every split BN-pair of higher rank has a unique splitting.

We note that the fact that the uniqueness of U of rank 1 implies the classification of split BN-pairs of higher rank was also noticed independently by Timmesfeld [33], who proved a similar result for a rather restricted class of BN-pairs of rank 2, namely only those appearing as proper residues in an irreducible spherical BN-pair of higher rank. The same result has been shown very recently for all special Moufang sets, i.e. Moufang sets with abelian root groups. This was done by by De Medts, Tent and Segev [6].

Finally, we note that all finite Moufang sets are classified. In the case where the little projective group (see below for a definition) is not sharply 2-transitive, one has either $\mathsf{PSL}_2(q)$, $q \geq 4$, $\mathsf{PSU}_3(q)$, $q \geq 3$, $\mathsf{Sz}(q) \cong {}^2\mathsf{B}_2(q)$, $q \geq 8$, and $\mathsf{Ree}(q) \cong {}^2\mathsf{G}_2(q)$, for appropriate prime powers q. This has been shown by Hering, Kantor and Seitz [14] (odd characteristic) and Shult [22] (even characteristic).

3.1 Some definitions concerning BN-pairs

In this section we encounter the definition of a building and we define (split) BN-pairs, both notions were first described in [37]. We also describe the μ-actions of a Moufang set, actions preserving the Moufang set and fixing two elements (0 and ∞) of the Moufang set.

3.1.1 Buildings

A *simplicial complex* (S, X) is a set S together with a set X of subsets of S, such that the union of X is S and such that every subset of every element of X itself belongs to X. The elements of X are called the *simplices* or *faces* ; the maximal elements of X are called the *chambers* of (S, X). If we remove one element from a chamber, then we talk about a *panel*. *Adjacent* chambers are chambers which meet in a panel. A *chamber complex* is a simplicial complex (S, X) such that all chambers are finite and have the same cardinality, and such that every two chambers can be joined by a sequence of consecutively adjacent chambers, a so-called (non-stammering) *gallery*. A chamber complex is called *thick* if every panel is in at least three chambers; it is called thin if every panel is in exactly two chambers. A *chamber subcomplex* (S', X') of a chamber complex is a chamber complex with $S' \subseteq S$ and $X' = \{A \in X : A \subseteq S'\}$.

Let Δ be a (chamber) complex and let $\mathcal{A}$ be a set of subcomplexes of Δ. The pair $(\Delta, \mathcal{A})$ is called a *building*, of which the elements of $\mathcal{A}$ are called *apartments*, if the following conditions hold:

B_1 Δ is thick.

B_2 The elements of $\mathcal{A}$ are thin chamber complexes.

B_3 Any two elements of Δ belong to an apartment.

B_4 If two apartments Σ and Σ' contain two elements $A, A' \in \Delta$, there exists an isomorphism from Σ onto Σ' which leaves invariant A, A' and all their faces.

We already met buildings of rank 2, these are the generalized polygons. the chambers are flags and apartments are ordinary polygons. We also know some buildings of rank one. The Moufang sets are the rank one buildings satisfying the Moufang condition, and every set is an abstract building of rank 1. To read more about buildings, the main reference is of course [37], other books introducing buildings are [2][44].

3.1.2 BN-pairs

A *BN-pair* in a group G is a system (B, N) consisting of two subgroups of G such that

BN_0 B and N generate G.

BN_1 $B \cap N = H \trianglelefteq N$.

BN_2 The group $W = N/H$ is a Coxeter group with standard generating set S (of involutions) such that the following two relations hold for any $s \in S$ and any $w \in W$:

BN_2' $sBwB \subseteq BwB \cup BswB$.

BN_2'' $sBs \nsubseteq B$.

The group W is called the *Weyl group* of the BN-pair. A BN-pair is called *split* whenever there exists a subgroup U of B which is nilpotent and such that $U \cdot H = B$, with $H = B \cap N$.

Starting from a building, we can always construct a BN-pair as follows: B is the stabilizer of a chamber C in a building, N stands for the stabilizer of one (arbitrary) fixed apartment through C, so H must be the pointwise stabilizer of this apartment. The Weyl group is then the full automorphism group of this apartment. The group U is a group fixing C and acting transitively on the chambers opposite C, we say U is a transitive normal nilpotent subgroup of B.

The analogue with generalized polygons is quite direct: B fixes a flag, N an arbitrary n-gon through that flag. W is isomorphic to the dihedral group D_{2n}. The transitive nilpotent normal subgroup can for example be a Sylow p-subgroup of the little projective group in a finite Moufang n-gon, with p a divisor of its order. In general $U^+ := U_{[1,n]}$ (see 1.3) is such a transitive nilpotent normal subgroup of B.

3.1.3 Split BN-pairs and Moufang sets

Moufang sets always have a BN-pair: for any two elements x and y in X, we can define B as the stabilizer of x, and N as the stabilizer of the set $\{x, y\}$. Conversely, we can build a Moufang generalized polygon out of every BN-pair of rank 2 as follows. Suppose there is no non-trivial normal subgroup of G contained in B and replace N by NH_1, where

$$H_1 = \bigcap_{w \in N} wBw^{-1}$$

Let $G_i = B \cup Bm_iB$ for $i = 1$and 2 where m_i are distinct elements of S which generate S. G_1 and G_2 are subgroups of G. Let V_1 and V_2 denote the set of their right cosets and let Ξ denote the bipartite graph with vertex set $V_1 \cup V_2$ where $\{G_1a, G_2b\}$ is an edge if and only if $G_1a \cap G_2b \neq \emptyset$. By [37], Ξ is a generalized polygon, G acts faithfully on Ξ by right multiplication, B acts transitively on the set of apartments containing $\{G_1, G_2\}$ and the orbit of this edge under the group N forms an apartment.

For buildings of rank one, the split condition translates beautifully into Moufang sets. For a given projective group G we say a subgroup V_x of G_x is a *unipotent subgroup of* G if

US_1 V_x acts transitively on $X \setminus \{x\}$.

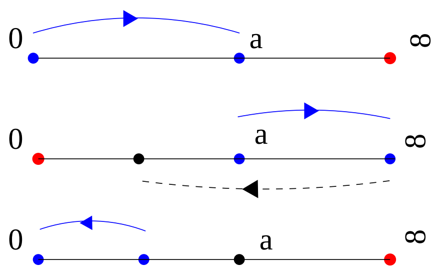
US_2 $V_x \trianglelefteq G_x$.

US_3 V_x is nilpotent.

The group V_x is a transitive normal nilpotent subgroup of the Moufang set. In all known examples, the root groups U_x are nilpotent, hence they are also transitive nilpotent subgroups of the G_x. The main goal of this chapter is to show that they are the unique transitive nilpotent normal subgroups. From now on, we will denote the standard unipotent subgroups of a Moufang set (i.e. the root groups) with U_x^+.

3.1.4 An important automorphism of a Moufang Set

Later on, we will encounter so-called μ-actions, these are actions generated by the root groups U_0 and U_∞, a simple μ-action switches (0) and (∞), while a double μ-action fixes them. If we define t_a as the element of U_∞ mapping 0 to a while $t'_a \in U_0$ maps a onto ∞, the simple μ-action μ_a is the composition $t_at'_at_b{}^{-1}$ where b is the image of (∞) under t'_a. The double μ-action is then $\mu_1^{-1}\mu_a$, we will denote this double μ-action shortly with h_a.



3.2 Classification of Moufang sets arising from Moufang polygons

3.2.1 Projective lines over skew fields

Let $\mathbb{K}$ be any skew field. Put X equal to the set of vector lines of the 2-dimensional (left) vector space $V(2,\mathbb{K})$ over $\mathbb{K}$. After a suitable coordinatization, let 0 denote the vector line spanned by $(1,0)$ and ∞ the vector line spanned by $(0,1)$. Then, with regard to the usual (right) action of matrices on vectors (and hence on vector lines), we define U_0^+ as the group of matrices $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, for all $k \in \mathbb{K}$. The group U_∞^+ consists of the matrices $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \in \mathbb{K}$.

The little projective group is here $PSL_2(\mathbb{K})$ in its natural action. We denote this Moufang set as $\mathcal{M}\mathsf{PL}(\mathbb{K})$ and call it *the projective line over* $\mathbb{K}$. The corresponding set is sometimes denoted by $\mathsf{PG}(1,\mathbb{K})$.

Let us compute the μ-actions for the pair (U_∞^+, U_0^+). One obtains

$$\mu_{(1,a)} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}.$$

So $\mu_{(1,a)} : \mathbb{K}(x,y) \mapsto \mathbb{K}(ya^{-1}, -xa)$. The double μ-action is now easy: $h_{(1,a)} := \mu_{((1,1),(1,a))} : \mathbb{K}(x,y) \mapsto \mathbb{K}(xa^{-1}, ya)$.

Identifying $\mathbb{K}(1,k)$ with $k \in \mathbb{K}$ and $\mathbb{K}(0,1)$ with the element ∞, we may also write the above actions as follows:

$$\begin{aligned}
&U_\infty^+ = \{u : x \mapsto x + a, \infty \mapsto \infty \mid a \in \mathbb{K}\},\\
&U_0^+ = \{u : x \mapsto (x^{-1} + a^{-1})^{-1}, -a \mapsto \infty, \infty \mapsto a, 0 \mapsto 0 \mid a \in \mathbb{K}\},\\
&\mu_a : x \mapsto -ax^{-1}a, 0 \leftrightarrow \infty,\\
&h_a : x \mapsto axa.
\end{aligned}$$

We will refer to this as the *non-homogeneous representation.*

3.2.2 Projective lines over alternative division rings

An *alternative division ring* is a ring $\mathbb{A}$ with identity 1 in which the following laws hold:

(ADR_1) For each non-zero element a, there exists an element b such that $b \cdot ac = c$ and $ca \cdot b = c$ for all $c \in \mathbb{A}$.

(ADR_2) $(ab \cdot a)c = a(b \cdot ac)$ and $b(a \cdot ca) = (ba \cdot c)a$ for all $a, b, c \in \mathbb{A}$.

(ADR_3) $ab \cdot ca = a(bc \cdot a) = (a \cdot bc)a$ for all $a, b, c \in \mathbb{A}$.

An alternative division ring is associative if and only if it is a skew field. The only non-associative alternative division rings are the Cayley-Dickson division rings, so every two elements of an alternative division ring are contained in a sub skew field.

As, by (ADR_1), each element a of $\mathbb{A}$ has a unique inverse a^{-1}, we may define the root groups U_0^+ and U_∞^+ in the same way as we did for the skew fields in the non-homogeneous representation. We then also obtain the same results for the simple and double μ-actions (and note that expressions like axa are unambiguous by (ADR_2)). The corresponding Moufang set is denoted by $\mathcal{M}\mathsf{PL}(\mathbb{A})$ and called *the projective line over* $\mathbb{A}$.

3.2.3 Polar lines

Let $\mathbb{K}$ be a skew field and let σ be an involution of $\mathbb{K}$ (so $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in \mathbb{K}$). Define $\mathbb{K}_\sigma = \{a + a^\sigma \mid a \in \mathbb{K}\}$ and $\mathrm{Fix}_{\mathbb{K}}(\sigma) = \{a \in \mathbb{K} \mid a^\sigma = a\}$. Let K_0 be an additive subgroup of $\mathbb{K}$ such that

(IS_1) $\mathbb{K}_\sigma \subseteq K_0 \subseteq \mathrm{Fix}_{\mathbb{K}}(\sigma)$,

(IS_2) $a^\sigma K_0 a \subset K_0$ for all $a \in \mathbb{K}$,

(IS_3) $1 \in K_0$.

Then $(\mathbb{K}, K_0, \sigma)$ is called an *involutory set*. The restriction of $\mathcal{M}\mathrm{PL}(\mathbb{K})$ to $K_0 \cup \{\infty\}$ in the non-homogeneous representation is well defined and is a Moufang set, called a *polar line*, denoted by $\mathcal{M}\mathrm{PL}(\mathbb{K}, K_0, \sigma)$. Hence, again, the root group actions and the μ-actions can be copied from the non-homogeneous representation of projective lines over a skew field given above.

3.2.4 Hexagonal Moufang sets

The notion of a hexagonal system is essentially equivalent to the notion of a quadratic Jordan division algebra of degree three.

A *hexagonal system* is a tuple $(\mathbb{J}, \mathbb{F}, \mathsf{N}, \#, \mathsf{T}, \times, 1)$, where $\mathbb{F}$ is a commutative field, $\mathbb{J}$ is a vector space over $\mathbb{F}$, N is a function from $\mathbb{J}$ to $\mathbb{F}$ called the *norm*, $\#$ is a function from $\mathbb{J}$ to itself called the *adjoint*, T is a symmetric bilinear form on $\mathbb{J}$ called the *trace*, $\times$ is a symmetric bilinear map from $\mathbb{J} \times \mathbb{J}$ to $\mathbb{J}$ and 1 is a distinguished element of $\mathbb{J} \setminus \{0\}$ called the *identity* such that for all $t \in \mathbb{F}$ and all $a, b, c \in \mathbb{J}$, the following identities hold.

- $(ta)^\# = t^2 a^\#$,

- $\mathsf{N}(ta) = t^3\mathsf{N}(a)$,
- $\mathsf{T}(a \times b, c) = \mathsf{T}(a, b \times c)$,
- $(a + b)^\# = a^\# + a \times b + b^\#$,
- $\mathsf{N}(a + b) = \mathsf{N}(a) + \mathsf{T}(a^\#, b) + \mathsf{T}(a, b^\#) + \mathsf{N}(b)$,
- $\mathsf{T}(a, a^\#) = 3\mathsf{N}(a)$,
- $a^{\#\#} = \mathsf{N}(a)a$,
- $a^\# \times (a \times b) = \mathsf{N}(a)b + \mathsf{T}(a^\#, b)a$,
- $a^\# \times b^\# + (a \times b)^\# = \mathsf{T}(a^\#, b)b + \mathsf{T}(a, b^\#)a$,
- $1^\# = 1$,
- $b = \mathsf{T}(b, 1) \cdot 1 - 1 \times b$,
- $\mathsf{N}(a) = 0$ if and only if $a = 0$.

If we define the *inverse* a^{-1} of an arbitrary nonzero $a \in \mathbb{J}$ as $a^{-1} = \mathsf{N}(a)^{-1}a^\#$, then we can define the Moufang set $\mathcal{M}\mathsf{H}(\mathbb{J})$ related to $\mathbb{J}$ in exactly the same way as before for the projective line over a field $\mathbb{K}$, in its non-homogeneous representation. The double μ-actions are given by $\mu(1, a) : x \mapsto T(a, x)a - a^\# \times x,\ \infty \mapsto \infty$. These Moufang sets are called *hexagonal Moufang sets*.

Hexagonal systems are classified by the work of various people. We refer to [40] for more details.

3.2.5 Orthogonal Moufang sets

Let $\mathbb{K}$ be a commutative field and let L_0 be vector space over $\mathbb{K}$. An *anisotropic quadratic form* q on L_0 is a function from L_0 to $\mathbb{K}$ such that

(AQF_1) $q(ta) = t^2q(a)$ for all $t \in \mathbb{K}$ and all $a \in L_0$,

(AQF_2) the function $f : L_0 \times L_0 \to \mathbb{K}$ given by $f(a, b) = q(a+b) - q(a) - q(b)$, for all $a, b \in L_0$, is bilinear,

(AQF_3) $q^{-1}(0) = \{0\}$.

The map f is called the *bilinear form associated with* q. Now embed L_0 in a vector space L over $\mathbb{K}$ as a codimension 2 subspace; hence we may put $L = \mathbb{K} \times L_0 \times \mathbb{K}$, and we define X as the set of all vector lines $\mathbb{K}(x_-, v_0, x_+)$ in the vector space $\mathbb{K} \times L_0 \times \mathbb{K}$) such that $x_-x_+ = q(v_0)$. Then $U^+_{(0,0,1)}$ consists of the maps u_w, $w \in L_0$, fixing $\mathbb{K}(0,0,1)$ and mapping $\mathbb{K}(1, v, q(v))$ onto $\mathbb{K}(1, v+w, q(v+w))$. Likewise, $U^+_{(1,0,0)}$ consists of the maps u'_w, $w \in L_0$, fixing $\mathbb{K}(1,0,0)$ and mapping $\mathbb{K}(q(v), v, 1)$ onto $\mathbb{K}(q(v+w), v+w, 1)$. This defines a Moufang set, called an *orthogonal Moufang set over* K, and denoted by $\mathcal{M}\mathsf{O}(\mathbb{K}, q)$.

One calculates that u_w maps $\mathbb{K}(q(v), v, 1)$ onto the vector line $\mathbb{K}(q(z), z, 1)$, with

$$z = q(v)q(q(v)w + v)^{-1}(q(v)w + v).$$

If L_0 has dimension 1, then we may put $q(x) = x^2$ and $\mathcal{M}\mathsf{O}(\mathbb{K}, q)$ is isomorphic with the projective line $\mathcal{M}\mathsf{PL}(\mathbb{K})$. If L_0 has dimension 2, then q defines a field extension $\mathbb{F}$ of $\mathbb{K}$ and $\mathcal{M}\mathsf{O}(\mathbb{K}, q)$ is isomorphic with $\mathcal{M}\mathsf{PL}(\mathbb{F})$.

This class of Moufang sets also comprises the ones related to indifferent sets (with the terminology of [40], see [39]).

3.2.6 Hermitian Moufang sets

Let $(\mathbb{K}, K_0, \sigma)$ be an involutory set, let L_0 be a right vector space over $\mathbb{K}$ and let q be a function from L_0 to $\mathbb{K}$. Then q is an *anisotropic pseudo-quadratic form* on L_0 with respect to K_0 and σ if there is a skew-hermitian form (with respect to σ) f on L_0 such that

$(APQF_1)$ $q(a+b) \equiv q(a) + q(b) + f(a,b) \pmod{K_0}$,

$(APQF_2)$ $q(at) \equiv t^\sigma q(a) t \pmod{K_0}$
for all $a, b \in L_0$ and all $t \in \mathbb{K}$,

$(APQF_3)$ $q(a) \equiv 0 \pmod{K_0}$ only for $a = 0$.

An *anisotropic pseudo-quadratic space* is a quintuple $(\mathbb{K}, K_0, \sigma, L_0, q)$ such that $(\mathbb{K}, K_0, \sigma)$ is an involutory set, L_0 is a right vector space over $\mathbb{K}$ and q is an anisotropic pseudo-quadratic form on L_0 with respect to K_0 and σ.

Let $(\mathbb{K}, K_0, \sigma, L_0, q)$ be some anisotropic pseudo-quadratic space and let f denote the corresponding skew-hermitian form. Following (11.24) of [40], let $(T, \cdot)$ denote the group $\{(a,t) \in L_0 \times K \mid q(a) - t \in K_0\}$ with $(a,t)\cdot(b,u) = (a+b, t+u+f(b,a))$ and choose $(a,t) \in T \setminus \{(0,0)\}$ and $s \in \mathbb{K} \setminus \{0\}$. Then we may put $X = T \cup \{\infty\}$, and the group U^+_∞ is given by the right action of T on itself. The double μ-action is given by $\mu((0,1),(a,t)) =: h_{(a,t)} : (b,v) \mapsto ((b - at^{-1}f(a,b))t^\sigma, tvt^\sigma)$.

These Moufang sets are called *Hermitian Moufang sets*.

3.2.7 An exceptional Moufang set of type E_7

There is a Moufang set corresponding with an algebraic group of absolute type E_7 and which also arises from an exceptional Moufang quadrangle of type E_8.

Let $\mathbb{K}$ be a commutative field, let L_0 be a vector space over $\mathbb{K}$ and let q be a quadratic form on L_0 with associated bilinear form f. We say that $(\mathbb{K}, L_0, q)$ is a *quadratic space.*

A *norm splitting* of $(\mathbb{K}, L_0, q)$ is a triple $(E, \cdot, \{v_1, \dots, v_d\})$ such that;

- $E/\mathbb{K}$ is a separable quadratic extension,
- $\cdot$ is a scalar multiplication from $E \times L_0$ to L_0 extending the scalar multiplication from $\mathbb{K} \times L_0$ and
- $\{v_1, \dots, v_d\}$ is a basis of L_0 over E (with respect to $\cdot$) and

$$q(t_1 \cdot v_1 + \cdots + t_d \cdot v_d) = s_1 N(t_1) + \cdots + s_d N(t_d)$$

  for all $t_1, \dots, t_d \in E$, where $s_i = q(v_i)$ for all $i \in \{1, \dots, d\}$ and N is the norm of the extension $E/\mathbb{K}$.

A *norm splitting map* of q is an automorphism T of L_0 such that for some $\alpha \in \mathbb{K}$ ($\alpha = 0$ if and only if $\mathrm{char}(\mathbb{K}) \neq 2$) and for some $\beta \in \mathbb{K} \setminus \{0\}$)

(i) $q(T(v)) = \beta q(v)$

(ii) $f(v, T(v)) = \alpha q(v)$

(iii) $(T^2 + \alpha T + \beta)(v) = 0$, $\forall v \in L_0$

Let $(\mathbb{K}, L_0, q)$ be a quadratic space, then

- $(\mathbb{K}, L_0, q)$ is of *type* E_6 if q is anisotropic, $dim_{\mathbb{K}} L_0 = 6$ and q has a norm splitting.
- $(\mathbb{K}, L_0, q)$ is of *type* E_7 if q is anisotropic, $dim_{\mathbb{K}} L_0 = 8$ and q has a norm splitting $(E, \{v_1, \dots, v_4\})$ with constants $s_1, \dots, s_4$ such that $s_1 \dots s_4 \notin N(E)$.
- $(\mathbb{K}, L_0, q)$ is of *type* E_8 if q is anisotropic, $dim_{\mathbb{K}} L_0 = 12$ and q has a norm splitting $(E, \{v_1, \dots, v_6\})$ with constants $s_1, \dots, s_6$ such that $-s_1 \dots s_6 \in N(E)$.

Starting from this quadratic space of type E_k ($k = 6, 7$ or 8), we take a base point ϵ such that $q(\epsilon) = 1$ (if necessary, we replace q by $q/q(\epsilon)$) and let X_0 be a vector space of dimension 2^{k-3}. We then define some map h from $X_0 \times X_0$ to L_0 which is bilinear over $\mathbb{K}$, its explicit description is stated in [40],(13.18) but we will not need it. We also define a bilinear form g on X_0 as a vector space over $\mathbb{K}$:

$$g(a, b) = f(h(b, a), \delta)$$

for all $a, b \in X_0$, where $\delta = \epsilon/2$ if $char(\mathbb{K}) \neq 2$ and $\delta = T(\epsilon)/f(\epsilon, T(\epsilon))$ if $char(\mathbb{K}) = 2$.

Following (16.6) of [40], let $(S, \cdot)$ be the group with underlying set $X_0 \times \mathbb{K}$ and with group operation

$$(a, t) \cdot (b, u) = (a + b, t + u + g(a, b))$$

for all $(a, t), (b, u) \in S$. Then the group $U^+ := U^+_\infty$ is isomorphic to S, and acts in a natural way on S itself by right multiplication.

At last we define a function θ from $X_0 \times L_0$ to L_0. (For an explicit description, see [**?**, Tit-Wei:02]. Now we can set $\pi(a) = \theta(a, \epsilon)$ and $\pi(a, t) = \pi(a) + t\epsilon$.

3.2.8 Suzuki-Tits Moufang sets

Let $\mathbb{K}$ be a field of characteristic 2, and denote by $\mathbb{K}^2$ its subfield of all squares. Suppose that $\mathbb{K}$ admits some Tits endomorphism θ, i.e., an endomorphism θ is such that $(x^\theta)^\theta = x^2$, for all $x \in \mathbb{K}$. Let $\mathbb{K}^\theta$ denote the image of $\mathbb{K}$ under θ. Let L be a vector space over $\mathbb{K}^\theta$ contained in $\mathbb{K}$, such that $\mathbb{K}^\theta \subseteq L$ and such that $L \setminus \{0\}$ is closed under taking multiplicative inverse. For a unique standard notation, we also assume that L generates $\mathbb{K}$ as a ring. The *Suzuki-Tits Moufang set* $\mathcal{M}\mathsf{Sz}(\mathbb{K}, L, \theta)$ can be defined as the action of a certain subgroup of the centralizer of a polarity of a mixed quadrangle $\mathsf{Q}(\mathbb{K}, \mathbb{K}^\theta; L, L^\theta)$ on the corresponding set of absolute points. A more precise and explicit description goes as follows.

Let X be the following set of points of $\mathsf{PG}(3, \mathbb{K})$, given with coordinates with respect to some given basis:

$$X = \{(1, 0, 0, 0)\} \cup \{(a^{2+\theta} + aa' + a'^\theta, 1, a', a) \mid a, a' \in L\}.$$

Let $(x, x')_\infty$ be the collineation of $\mathsf{PG}(3, \mathbb{K})$ determined by

$$(x_0\ x_1\ x_2\ x_3) \mapsto (x_0\ x_1\ x_2\ x_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^{2+\theta} + xx' + x'^\theta & 1 & x' & x \\ x & 0 & 1 & 0 \\ x^{1+\theta} + x' & 0 & x^\theta & 1 \end{pmatrix},$$

and let $(x,x')_0$ be the collineation of $\mathsf{PG}(3,\mathbb{K})$ determined by

$$(x_0\ x_1\ x_2\ x_3) \mapsto (x_0\ x_1\ x_2\ x_3)\begin{pmatrix} 1 & x^{2+\theta}+xx'+x'^{\theta} & x & x' \\ 0 & 1 & 0 & 0 \\ 0 & x^{1+\theta}+x' & 1 & x^{\theta} \\ 0 & x & 0 & 1 \end{pmatrix}.$$

The group $\mathsf{Sz}(\mathbb{K},L,\theta)$ is generated by the subgroups

$$U_\infty^+ = \{(x,x')_\infty \mid x,x' \in L\} \text{ and } U_0^+ = \{(x,x')_0 \mid x,x' \in L\}.$$

Both subgroups U_∞^+ and U_0^+ indeed act on X, as an easy computation shows, and they act sharply transitively on $X \setminus \{(1,0,0,0)\}$ and $X \setminus \{(0,1,0,0)\}$, respectively. Moreover, it can be checked easily that $(U_0^+)^{(x,x')_\infty} = (U_\infty^+)^{(y,y')_0}$, with

$$y = \frac{x'}{x^{2+\theta}+xx'+x'^{\theta}} \text{ and } y' = \frac{x}{x^{2+\theta}+xx'+x'^{\theta}}.$$

It now follows from Theorem 2.3.1 that we indeed obtain a Moufang set. When emphasizing one particular point, namely $(\infty) := (1,0,0,0)$, we can write $(a,a') := (a^{2+\theta}+aa'+a'^{\theta},1,a',a)$, and the unique element of U_∞^+ that maps $(0,0)$ to (b,b') is given by $(b,b')_\infty : (a,a') \mapsto (a+b, a'+b'+ab^{\theta})$. The root group U_∞^+ is given by the set $\{(a,a')_\infty \mid a,a' \in L\}$ with operation $(a,a')_\infty \oplus (b,b')_\infty = (a+b,a'+b'+ab^{\theta})_\infty$.

We remark that, if $L = \mathbb{K}$, then the Moufang set can also be obtained from a Moufang octagon, unlike the case $L \neq \mathbb{K}$.

3.2.9 Ree-Tits Moufang sets

Let $\mathbb{K}$ be a field of characteristic 3, and denote by $\mathbb{K}^3$ its subfield of all third powers. Suppose that $\mathbb{K}$ admits some Tits endomorphism θ, i.e., an endomorphism θ is such that $(x^\theta)^\theta = x^3$, for all $x \in \mathbb{K}$. Let $\mathbb{K}^\theta$ denote the image of $\mathbb{K}$ under θ. The *Ree-Tits Moufang set* $\mathcal{M}\mathsf{Ree}(\mathbb{K},\theta)$ can be defined as the action of a certain subgroup of the centralizer of a polarity of a mixed Moufang hexagon $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$ on the corresponding set of absolute points. A more precise and explicit description based on Section 7.7 of [41] goes as follows.

For $a,a',a'' \in \mathbb{K}$, we put

$$\begin{aligned} f_1(a,a',a'') &= -a^{4+2\theta} - aa''^{\theta} + a^{1+\theta}a'^{\theta} + a''^2 + a'^{1+\theta} - a'a^{3+\theta} - a^2a'^2, \\ f_2(a,a',a'') &= -a^{3+\theta} + a'^{\theta} - aa'' + a^2a', \\ f_3(a,a',a'') &= -a^{3+2\theta} - a''^{\theta} + a^{\theta}a'^{\theta} + a'a'' + aa'^2. \end{aligned}$$

Let X be the following set of points of $\mathsf{PG}(6,\mathbb{K})$, given with coordinates with respect to some given basis:

$$\begin{aligned}X =&\{(1,0,0,0,0,0,0)\}\cup\\ &\{(f_1(a,a',a''),-a',-a,-a'',1,f_2(a,a',a''),f_3(a,a',a''))\mid a,a',a''\in\mathbb{K}\}.\end{aligned}$$

Let $(x,x',x'')_\infty$ be the collineation of $\mathsf{PG}(6,\mathbb{K})$ determined by $\bar{x}\mapsto$

$$\bar{x}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ p & 1 & 0 & -x & 0 & x^2 & -x''-xx'\\ q & x^\theta & 1 & x'-x^{1+\theta} & & r & s\\ x'' & 0 & 0 & 1 & 0 & x & -x'\\ f_1(x,x',x'') & -x' & -x & -x'' & 1 & f_2(x,x',x'') & f_3(x,x',x'')\\ x'-x^{1+\theta} & 0 & 0 & 0 & 0 & 1 & -x^\theta\\ x & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

with $\bar{x}=(x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)$ and

$$\begin{aligned} p &= x^{1+\theta}-x'^\theta-xx''-x^2x',\\ q &= x''^\theta+x^\theta x'^\theta+x'x''-xx'^2-x^{2+\theta}x'-x^{1+\theta}x''-x^{3+2\theta},\\ r &= x''-xx'+x^{2+\theta},\\ s &= x'^2-x^{1+\theta}x'-x^\theta x'',\end{aligned}$$

and put $(x,x',x'')_0 := (x,x',x'')_\infty^g$, with g the collineation of $\mathsf{PG}(6,\mathbb{K})$ determined by

$$\bar{x}\mapsto\bar{x}\begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&-1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\end{pmatrix}.$$

The group $\mathsf{Ree}(\mathbb{K},\theta)$ is generated by the subgroups

$$U_\infty^+=\{(x,x',x'')_\infty\mid x,x',x''\in\mathbb{K}\}\text{ and }U_0^+=\{(x,x',x'')_0\mid x,x',x''\in\mathbb{K}\}.$$

Both subgroups U_∞^+ and U_0^+ indeed act on X, as the reader can verify with a straightforward, but tedious computation, and they act regularly on

$X \setminus \{(1,0,0,0,0,0,0)\}$ and $X \setminus \{(0,0,0,0,1,0,0)\}$, respectively. Moreover, it can be checked that $(U_0^+)^{(x,x',x'')_\infty} = (U_\infty^+)^{(y,y',y'')_0}$, with

$$\begin{aligned} y &= -\frac{f_3(x,x',x'')}{f_1(x,x',x'')}, \\ y' &= -\frac{f_2(x,x',x'')}{f_1(x,x',x'')}, \\ y'' &= -\frac{x''}{f_1(x,x',x'')}. \end{aligned}$$

It now follows that we indeed obtain a Moufang set. When emphasizing one particular point, namely $(\infty) := (1,0,0,0,0,0,0)$, we can write, following 7.7.7 of [41],

$$(a,a',a'') := (f_1(a,a',a''), -a', -a, -a'', 1, f_2(a,a',a''), f_3(a,a',a'')),$$

and the unique element of U_∞^+ that maps $(0,0,0)$ to (b,b',b'') is given by

$$(b,b',b'')_\infty : (a,a',a'') \mapsto (a+b, a'+b'+ab^\theta, a''+b''+ab'-a'b-ab^{1+\theta}).$$

The root group U_∞^+ is now the set $\{(a,a',a'')_\infty \mid a,a',a'' \in \mathbb{K}\}$ with operation

$$(a,a',a'')_\infty \oplus (b,b',b'')_\infty = (a+b, a'+b'+ab^\theta, a''+b''+ab'-a'b-ab^{1+\theta})_\infty.$$

3.3 The uniqueness of splittings of some split BN-pairs of rank 1

3.3.1 General idea

We investigate the action of an automorphism φ lying in a transitive nilpotent subgroup U_∞ of the stabilizer of (∞) in a projective group of every described Moufang set.

Lemma 3.3.1. *If U_x and V_x are two transitive normal nilpotent subgroups of G_x, then so is U_xV_x.*

Proof. The transitivity and the fact that U_xV_x is normal, is obvious. The nilpotency of U_xV_x follows from the normality of U_x and V_x in G_x and Fitting's Theorem [1]:

$$[uv, u'v'] = [u,v']^v[v,v'][u,u']^{vv'}[v,u']^{v'}$$

□

So to prove the uniqueness of the splitting, we may assume that U_∞ contains U_∞^+ and has a non-trivial stabilizer $(U_\infty)_0$.

Since U_∞^+ acts regularly on $X \setminus \{\infty\}$, we may assume the automorphism ϕ fixes both 0 and ∞. The nilpotency of U_∞ ensures that its center is not trivial, but since φ fixes 0 it fixes also its orbit under the center of U_∞, our main goal now is to show that $Z(U_\infty)$ is big enough, so that φ has to be the identity.

A distinction is made between translation Moufang sets (for which there is always a sub-Moufang set isomorphic to a projective line or a polar line) and the other Moufang sets.

For one well understood case, the group U_∞^+ is not the unique transitive nilpotent group. All the Moufang sets arising from residues of Moufang buildings and from polarities in Moufang buildings are discussed here (except for one Moufang set arising from a polarity in the F_4-quadrangle). This result is proven [7].

3.3.2 Proof

Moufang sets with commutative root groups The Moufang sets introduced in Section 3.2 that have commutative root groups are those isomorphic to a sub Moufang set of a projective line over a (skew) field (which we shall refer to as *semi projective lines (over a (skew) field)*, the hexagonal and orthogonal Moufang sets, and the projective lines over proper alternative division rings. We treat all these cases simultaneously. Note that we also include some other type of Moufang sets with commutative root groups, and contained in a projective line over a field in characteristic 2, see [17], not explicitly mentioned here.

So let $\mathcal{M} = (X, (U_x^+)_{x\in X})$ be a Moufang set with commutative root groups as in the previous paragraph. Each of these Moufang sets is defined using (or "over") a field $\mathbb{K}$. We choose arbitrarily two elements of X and call them 0 and ∞. Then U_∞^+ is an abelian group and we denote the composition law in this group by $+$. Let G be any projective group of $\mathcal{M}$.

Suppose U_∞ is a second unipotent subgroup of G, contained in G_∞. Since the product of two normal nilpotent subgroups is nilpotent (see Lemma 3.3.1), we may assume without loss of generality that $U_\infty^+ \lneqq U_\infty$. Since U_∞^+ acts sharply transitively on the set $X \setminus \{\infty\}$, there exists $\varphi \in U_\infty$ fixing some element of X, and we may assume without loss of generality that φ fixes 0.

Let z be a nontrivial element of the center of U_∞, and let $u \in U_\infty^+$ be such that zu fixes 0. Since zu centralizes U_∞^+, it must fix X pointwise, hence $z = u^{-1}$. In all cases, the three elements $0, 0^z$ and ∞ are contained in a semi projective line over a field $\mathbb{F}$. Without loss of generality, we may put

0^z equal to the multiplicative identity element 1 of $\mathbb{F}$. Indeed, we pass to the new multiplication $a \cdot b = a(0^z)^{-1}b$ if necessary. Moreover, in the case of hexagonal Moufang sets and projective lines over alternative division rings, we may assume that 1 is the identity element of the division algebra (this amounts to passing to an isotopic algebra; the Moufang sets do not change). Hence z maps 0 to 1, i.e., $u : x \mapsto x + 1$. Now let $a \in X \setminus \{0, 1, \infty\}$ be arbitrary. Then $0, 1, \infty$ and a are contained in a semi projective line over some skew field $\mathbb{F}'$. The restriction to the points of X of the addition with respect to ∞ in $\mathbb{F}'$ coincides with the $+$ of our root group. Since we are in a skew field now, the double μ-actions are well defined; in particular a^2 is well defined (and one can check that it is independent of the chosen sub Moufang set by considering the intersection of all of them). Since the center is a characteristic subgroup of U_∞, it is normal in G_∞, and since G contains the little projective group, it contains all double μ-actions with respect to $(U_\infty^+, U_0^+, 1, a)$, for every $a \in X \setminus \{0, \infty\}$. It follows that $x \mapsto x + a^2$ also belongs to the center of U_∞, and hence φ fixes all squares in X.

First suppose that the characteristic of $\mathbb{K}$ is not equal to 2. Then every $a \in X$ can be written as $a = \frac{1}{4}((a+1)^2 - (a-1)^2)$. Note that $a+1, a-1 \in X$, and so $x \mapsto x + (a+1)^2 - (a-1)^2$ belongs to the center of U_∞. Applying now the double μ-action with respect to $(U_\infty^+, U_0^+, 1, 1/2)$, we see that $x \mapsto x + a$ belongs to the center of U_∞, and hence φ must fix all $a \in X$, a contradiction.

Next suppose that $\mathbb{K}$ has characteristic 2. Define $U_\infty^{[0]} := U_\infty$, $U_\infty^{[j]} := [U_\infty, U_\infty^{[j-1]}]$ for $j \geq 1$ and take i such that $U_\infty^{[i]}$ does not act freely on $X \setminus \{\infty\}$, but $U_\infty^{[i+1]}$ does (i exists by nilpotency of U_∞). We may assume that $\varphi \in U_\infty^{[i]}$. We prove some properties of φ.

Observation 3.3.2. *The map φ is additive, i.e., for all $a, b \in X$, we have $(a + b)^\varphi = a^\varphi + b^\varphi$.*

Proof. Denote $U_\infty^+ \ni t_a : x \mapsto x + a$, $a \in X \setminus \{\infty\}$. We have $(t_a t_b)^\varphi = t_a^\varphi t_b^\varphi$ and $0^{t_a^\varphi} = a^\varphi$, so $t_a^\varphi = t_{a^\varphi}$. We get $(t_{a+b})^\varphi = (t_a t_b)^\varphi = t_a^\varphi t_b^\varphi = t_{a^\varphi} t_{b^\varphi}$. Taking the image of 0 , we obtain the result. □

Observation 3.3.3. *For all $a, b \in X$ such that $\{0, 1, a, b, \infty\}$ is contained in a semi projective line over some skew field $\mathbb{L}$ with the property that φ fixes the multiplicative identity 1, we have $(aba)^\varphi = a^\varphi b^\varphi a^\varphi$ (where juxtaposition is multiplication in the skew field $\mathbb{L}$). Consequently, $(a^{-1})^\varphi = (a^\varphi)^{-1} =: a^{-\varphi}$.*

Proof. Denote the element of U_0^+ mapping a to ∞ by t'_a, and use the notation t_a of the previous proof, too. By definition the double μ-action $\mu_a := x \mapsto axa$ is equal to the product $t_1 t'_1 t_1 t_a t'_a t_a$. As before, $t_a^\varphi = t_{a^\varphi}$ and $t'^\varphi_a = t'_{a^\varphi}$. We now have, remembering that φ fixes 1:

$$\begin{aligned}(aba)^\varphi &= b^{\mu_a\varphi} = b^{t_1t_1't_1t_at_a't_a\varphi} = (b^\varphi)^{(t_1t_1't_1t_at_a't_a)^\varphi}\\ &= (b^\varphi)^{t_1t_1't_1t_{a^\varphi}t'_{a^\varphi}t_{a^\varphi}} = (b^\varphi)^{\mu_{a^\varphi}} = a^\varphi b^\varphi a^\varphi.\end{aligned}$$

So the first assertion is proved. Now put $b = a^{-1}$ and the second assertion follows. □

For every $b \in X$, we have $[\varphi, t_b] = t_{b+b^\varphi}$ and by nilpotency of U_∞ and the fact that φ cannot centralize U_∞^+, there exists $b \in X$ with $b \neq b^\varphi$ such that $[\varphi, t_{b+b^\varphi}] = 1$ and $t_{b+b^\varphi} \neq 1$. So we have $(b+b^\varphi)^\varphi = b + b^\varphi$, implying $b^{\varphi^2} = b$.

Now $[\varphi, U^{[i]}]$ acts freely on $X \setminus \{\infty\}$. Denote as above, for $a \in X$, the double μ-action $x \mapsto axa$ by μ_a. Then $[\mu_a\varphi^{-1}\mu_a^{-1}, \varphi^{-1}]$ acts freely on $X\setminus\{\infty\}$, and since both φ and μ_a fix 0, we get $[\mu_a\varphi^{-1}\mu_a^{-1}, \varphi^{-1}] = \mathrm{id}$. Now we claim that in all cases except for $\mathcal{M}$ orthogonal, the set $\{0, 1, b, b^\varphi, \infty\}$ is contained in a semi projective line over some skew field $\mathbb{F}$. This is trivial if $\mathcal{M}$ is itself a semi projective line. If it is a projective line over a proper alternative division ring, then this follows from the fact that every two elements in such a division ring generate an associative division ring. If $\mathcal{M}$ is a hexagonal Moufang set, then use [(30.6) and (30.17)] of [40]. The claim follows. If $\mathcal{M}$ is an orthogonal Moufang set, then, as is noted in [4], $\{0, b, b^\varphi, \infty\}$ is contained in a sub Moufang set isomorphic to a projective line over a field, which we can also denote by $\mathbb{F}$ (and which is isomorphic to a quadratic extension of $\mathbb{K}$). If this sub Moufang set does not contain the element 1 chosen before, then we can re-choose it as $b + b^\varphi$. It is fixed under φ.

We now calculate, using the multiplication in $\mathbb{F}$, and taking into account $b^{\varphi^2} = b$, and Observation 3.3.3,

$$\begin{aligned}b^{-1} &= (b^{-1})^{[\mu_b\varphi^{-1}\mu_b^{-1},\varphi^{-1}]} = b^{\varphi\mu_b^{-1}\varphi\mu_b\varphi^{-1}\mu_b^{-1}\varphi}\\ &= (b^{-1})^{\varphi^{-1}}b^{\varphi^{-2}}(b^{-1})^{\varphi^{-1}}bb^{-1}b(b^{-1})^{\varphi^{-1}}b^{\varphi^{-2}}(b^{-1})^{\varphi^{-1}}\\ &= cbc,\end{aligned}$$

where $c = b^{-\varphi}bb^{-\varphi}$. So we have $cbc = b^{-1}$, which implies $(cb)^2 = 1$. Since $\mathrm{char}\mathbb{K} = 2$, we obtain $cb = 1$. Hence $1 = b^{-1}cb^2$. But $b^2 = (b^2)^\varphi = (b^\varphi)^2$ (since φ fixes all squares and then use the first assertion of Observation 3.3.3), and we obtain $1 = b^{-1}b^{-\varphi}bb^\varphi$, resulting in $bb^\varphi = b^\varphi b$.

But now $(b+b^\varphi)^2 = b^2 + (b^\varphi)^2 + bb^\varphi + b^\varphi b = b^2 + b^2 = 0$, hence $b = b^\varphi$, a contradiction. Hence φ is already the identity and $U_\infty = U_\infty^+$.

Hermitian Moufang sets Let $\Xi = (\mathbb{K}, K_0, \sigma, L_0, q)$ be a proper anisotropic pseudo-quadratic space as defined before (see also (11.17) of [40]), with corresponding skew-hermitian form $f : L_0 \times L_0 \to \mathbb{K}$. By (21.16) of [40], we may assume that q is non-degenerate, i.e. $\{a \in L_0 \mid f(a, L_0) = 0\} = 0$. Let $(T, \cdot)$ be as in Subsection 3.2.6. Then the group U_∞^+ is isomorphic to T, and acts in a natural way on T itself by right multiplication; we will write $\tau_{(a,t)}$ for the element of U_∞^+ mapping $(b, v) \in T$ to $(b, v) \cdot (a, t)$. Then $Z(U_\infty^+) = \{\tau_{(0,t)} \mid t \in K_0\}$. We will also write T^* for $T \setminus \{(0, 0)\}$. In general, we write a superscript $*$ when we delete the 0-element of a set (0-vector, additive identity,...).

As before, let $U_\infty^+ \lneqq U_\infty$. For convenience we shall write $U = U_\infty$ and $U^+ = U_\infty^+$. Also, put $B := G_\infty$. Since $U^+ \trianglelefteq B$ and $U \leq B$, we have that $Z(U^+) \trianglelefteq B$ and $Z(U^+) \trianglelefteq U$. Let $\tilde{U} := U/Z(U^+)$, $\tilde{U}^+ := U^+/Z(U^+)$, and $\tilde{B} := B/Z(U^+)$. Then $\tilde{U}^+ \lneqq \tilde{U}$, and $\tilde{U}$ is a non-trivial nilpotent group; in particular, $\tilde{Z} := Z(\tilde{U}) \neq 1$. For every $a \in L_0$, we let $\tau_a := \tau_{(a,q(a))} Z(U^+) \in \tilde{U}^+$; then the map $a \mapsto \tau_a$ is an isomorphism from $(L_0, +)$ to $\tilde{U}^+$. Note that $\tilde{U}^+ \neq 1$ by the properness of Ξ. The natural action of B on T induces an action of $\tilde{B}$ on L_0. Since $\tilde{U}^+$ acts regularly on L_0, there exists an element φ in $\tilde{U} \setminus \tilde{U}^+$ fixing $0 \in L_0$. Then φ fixes the orbit $0^{\tilde{Z}}$ elementwise.

Since $[\tilde{U}^+, \tilde{U}] \leq \tilde{U}^+$, it follows from the nilpotency of $\tilde{U}$ that there exists a non-trivial element $\tau \in \tilde{U}^+ \cap \tilde{Z}$. Moreover, $\tilde{Z} \trianglelefteq \tilde{B}$; for every $(a, t) \in T^*$, the mapping

$$\mu_{a,t} : b \mapsto \big(b - at^{-1} f(a, b)\big) t^\sigma ,$$

for all $b \in L_0$, belongs to $\tilde{B}$. (See (33.13) of [40].) Let $F := \{c \in L_0 \mid \tau_c \in \tilde{Z}\}$. Then F is a non-trivial additive subgroup of L_0 such that $\mu_{(a,t)}(F) \subseteq F$ for all $(a, t) \in T^*$. If we can now show that $F = L_0$, then it would follow that $\varphi = 1$, which is a contradiction; hence it would follow that $U = U^+$, which is what we want to obtain. We will see that there is one exception for which there really exists $U \neq U^+$.

We start by making some observations about the maps $\mu_{(a,t)}$. Let $b \in L_0^*$ be fixed. If $(a, t) \in T^*$ is such that $f(a, b) = 0$, then we have

$$\mu_{(a,t)}(b) = bt^\sigma ; \tag{3.1}$$

in particular, if $t \in K_0$, then $\mu_{(0,t)}(b) = bt^\sigma = bt$, since $K_0 \leq \mathrm{Fix}_K(\sigma)$, and hence F is closed under right multiplication by K_0.

Lemma 3.3.4. *If F is a non-trivial K-subspace of L_0, then $F = L_0$.*

Proof. Suppose that F is a non-trivial K-subspace of L_0.

Let $b \in F^*$ be fixed, let $a \in L_0^*$ be arbitrary, and let $t = q(a)$; then $(a, t) \in T^*$. If $f(a, b) \neq 0$, then $b - \mu_{(a,t)}(b)t^{-\sigma} = at^{-1} f(a, b) \in F$, and hence $a \in F$. So assume that $f(a, b) = 0$. Since q is non-degenerate, there exists a $c \in L_0$ such that $f(c, b) \neq 0$, and hence also $f(a + c, b) = f(c, b) \neq 0$. Hence $c \in F$ and $a + c \in F$, so also in this case we have that $a = (a + c) - c \in F$. □

If K_0 generates $\mathbb{K}$ (as a ring), then it follows from the fact that F is closed under right multiplication by K_0, that F is a $\mathbb{K}$-subspace of L_0. So we may assume that K_0 does not generate $\mathbb{K}$ as a ring. By (23.23) in [40], this implies that K_0 is a commutative field, and either $\mathbb{K}/K_0$ is a separable quadratic extension and σ is the non-trivial element of $\mathrm{Gal}(\mathbb{K}/K_0)$, or $\mathbb{K}$ is a quaternion division algebra over K_0 and σ is the standard involution of $\mathbb{K}$. Let N and T denote the (reduced) norm and trace of $\mathbb{K}/K_0$, respectively.

Assume first that $\dim_{\mathbb{K}} L_0 = 1$; we will, in fact, identify L_0 and $\mathbb{K}$ in this case. Let $\rho := q(1) \in \mathbb{K} \setminus K_0$; then $q(t) + K_0 = t^\sigma \rho t + K_0 = t^\sigma(\rho + K_0)t$ for all $t \in \mathbb{K}$. Also, $f(1,1) = \gamma := \rho - \rho^\sigma$, and hence $f(s,t) = s^\sigma \gamma t$ for all $s, t \in \mathbb{K}$. One can now compute that

$$\begin{aligned}\mu_{(t,t^\sigma(\rho+c)t)}(s) &= (s - t(t^\sigma(\rho+c)t)^{-1} f(t,s))(t^\sigma(\rho+c)t)^\sigma \\ &= (1 - (tt^{-1})(\rho+c)^{-1}((t^\sigma)^{-1}t^\sigma)\gamma) s t^\sigma (\rho+c)^\sigma t^\sigma \\ &= (\rho+c)^{-1}(\rho + c + \rho - \rho^\sigma) s t^\sigma (\rho+c)^\sigma t \\ &= (\rho+c)^{-1}(\rho+c)^\sigma s t^\sigma (\rho+c)^\sigma t\end{aligned}$$

for all $s, t \in \mathbb{K}^*$ and all $c \in K_0$. Since $\mathsf{N}(\rho + c) = (\rho+c)(\rho+c)^\sigma \in K_0$, it follows that, for all $s \in F^*$, $(\rho^\sigma + c)^2 s t^\sigma (\rho^\sigma + c)t \in F$ as well, and hence

$$r^2 s t^\sigma r t \in F\,, \quad \text{for all } r \in \langle 1, \rho\rangle_{K_0} \text{ and all } t \in \mathbb{K}\,. \tag{3.2}$$

Suppose first that $\mathbb{K}/K_0$ is a separable quadratic extension; then $\mathbb{K}$ is commutative, and $\mathbb{K} = \langle 1, \rho\rangle_{K_0}$. Hence, by (3.2), $r^3 s \in F$ for all $r \in \mathbb{K}$. If $K_0 = \mathrm{GF}(2)$, then $\mathbb{K} = \mathrm{GF}(4)$, and then $r^3 \in K_0$ for all $r \in \mathbb{K}$ (this is the case which will lead to the exception). So assume that $|K_0| \geq 3$, and suppose that $\mathbb{K}^3 \subseteq K_0$. Since $K = K_0(\rho)$ is a quadratic extension field of K_0, we have $\rho^2 = a\rho + b$ for some $a, b \in K_0$. Then $\rho^3 = (a^2 + b)\rho + ab$, hence $a^2 + b = 0$, and therefore $\rho^2 - a\rho + a^2 = 0$. If $\mathrm{char}(\mathbb{K}) = 3$, then this would imply $(\rho + a)^2 = 0$ and thus $\rho = -a \in K_0$, a contradiction. If $\mathrm{char}(\mathbb{K}) \neq 3$, then $(\rho + t)^3 - \rho^3 - 1 = 3\rho t(\rho + t) \in K_0$, and therefore $\rho(\rho + t) \in K_0$ for all $t \in K_0^*$. Choose a $t \in K_0 \setminus \{0, -1\}$; then $\rho = \rho(\rho + (t+1)) - \rho(\rho + t) \in K_0$, again a contradiction. We conclude that $\mathbb{K}^3 \not\subseteq K_0$, and hence $F = \mathbb{K}$.

Suppose now that $\mathbb{K}$ is a quaternion division algebra over K_0; in particular, K_0 is an infinite commutative field. If we consider (3.2) with $r = \rho + c$ for some $c \in K_0 \setminus \{0\} = Z(\mathbb{K})^*$, subtract the same expression with $r = \rho$ and $r = c$, and divide by c, then we get that

$$\rho(\rho + 2c)\mathsf{N}(t)s + (c + 2\rho)st^\sigma \rho t \in F\,,$$

for all $c \in K_0^*$. If $\mathrm{char}(\mathbb{K}) = 2$, then it follows that $\rho^2 \mathsf{N}(t)s + cst^\sigma \rho t \in F$, for all $c \in K_0^*$, and hence $st^\sigma \rho t \in F$ for all $t \in \mathbb{K}$. If $\mathrm{char}(\mathbb{K}) \neq 2$, then we write

$\rho^2 = a\rho + b$ with $a, b \in K_0$; if we take $r = \rho - a/2$ in (3.2), then we obtain that $st^\sigma \rho t \in F$ for all $t \in \mathbb{K}$ since $r^2 \in K_0^*$ and $st^\sigma(a/2)t \in sK_0 \subseteq F$. So we have shown that, in all characteristics, F is invariant under right multiplication by elements of the set $K_0 \cup \{t^\sigma \rho t \mid t \in \mathbb{K}\}$. It remains to show that the subring generated by $K_0 \cup \{t^\sigma \rho t \mid t \in \mathbb{K}\}$ is $\mathbb{K}$. Suppose that

$$K_1 := \langle K_0 \cup \{t^\sigma \rho t \mid t \in \mathbb{K}\}\rangle_{\text{ring}} \neq \mathbb{K}\,.$$

Since every subring of $\mathbb{K}$ containing K_0 is a K_0-vector space of dimension 1, 2 or 4, and since $\rho \notin K_0$, we must have $\dim_{K_0} K_1 = 2$; hence we can find a $t \in \mathbb{K} \setminus K_1$ for which $\mathsf{T}(t) = 0$ and $\mathsf{T}(\rho t) \neq 0$. Then $t^\sigma = -t$ and $\rho t = -t^\sigma \rho^\sigma + r$ for some $r \in K^*$; hence

$$t^\sigma \rho t = -t(-t^\sigma \rho^\sigma + r) = t^\sigma t \cdot \rho^\sigma - t \cdot r \notin K_1\,,$$

a contradiction. So $K_1 = \mathbb{K}$, and hence $F = \mathbb{K}$ in this case as well.

Now suppose that $\dim_K L_0 \geq 2$. If $\mathbb{K}$ is a quaternion division algebra over K_0 or if $\mathbb{K}$ is a quadratic extension field over K_0 with $K_0 \neq \mathrm{GF}(2)$, then it follows from the result in dimension 1 that F is a $\mathbb{K}$-subspace of L_0, and hence $F = L_0$ by Lemma 3.3.4. It only remains to consider the case where $K_0 = \mathrm{GF}(2)$ and $\mathbb{K} = \mathrm{GF}(4)$.

Let $b \in F \setminus \{0\}$ be arbitrary. Since $\dim_K L_0 \geq 2$, there exists an $a \in L_0^*$ such that $f(a, b) = 0$; by (3.1), $bq(a)^\sigma \in F$. Since q is anisotropic, $q(a)^\sigma \notin K_0$, and it thus follows that $bK \in F$. This shows that F is a $\mathbb{K}$-subspace of L_0, and we can again conclude by Lemma 3.3.4 that $F = L_0$.

We will now describe the exception. So let $K_0 = \mathrm{GF}(2)$, let $\mathbb{K} = \mathrm{GF}(4)$, and let $\dim_{\mathbb{K}} L_0 = 1$; we will again identify L_0 and $\mathbb{K} = \mathrm{GF}(4)$. Then $\rho := q(1)$ is one of the two elements in $\mathbb{K} \setminus K_0$, and $f(1, 1) = \gamma := \rho - \rho^\sigma = 1$; hence $f(s, t) = s^\sigma t$ for all $s, t \in \mathbb{K}$. Then $U^+ \cong T$ is a group of order 8. In the case that the projective group is $\mathsf{P\Sigma U}(3, 2)$, we have $B_+ = T \cdot \mathrm{Gal}(\mathbb{K}/K_0)$, which is a group of order 16. If we take $U = B_+$, then U is of course a normal subgroup of B_+, but U is also nilpotent (since it is a 2-group) and transitive (since U^+ is already transitive), giving us the desired exception to the Main Theorem.

Exceptional Moufang sets of type E_7 We now consider the case of the Moufang sets arising from a Moufang quadrangle of type E_6, E_7 or E_8. In fact, we have already handled E_6 and E_7, since these correspond to Hermitian Moufang sets, but our approach does not make any distinction between these three cases.

Let $(\mathbb{K}, L_0, q)$ be a quadratic space of type E_6, E_7 or E_8 as defined in Section 3.2.7, with corresponding bilinear form $f : L_0 \times L_0 \to \mathbb{K}$ and with base point $\epsilon \in L_0^*$. Let X_0 be the vector space over $\mathbb{K}$ and $(a, v) \mapsto av$ be the map from $X_0 \times L_0 \to X_0$ as defined in [40](13.9). Let h be the bilinear map from $X_0 \times X_0$ to L_0 defined in [40](13.18) and (13.19), let g be the bilinear map from $X_0 \times X_0$ to $\mathbb{K}$ defined in (13.26) of [40], and let π be the map from X_0 to L_0 as defined in [40](13.28). Moreover, let $\pi(a, t) := \pi(a) + t\epsilon$ for all $(a, t) \in X_0$. Following (16.6) of [40], let $(S, \cdot)$ be the group with underlying set $X_0 \times \mathbb{K}$ and with group operation

$$(a, t) \cdot (b, u) = (a + b, t + u + g(a, b))$$

for all $(a, t), (b, u) \in S$. Then the group $U^+ := U_\infty^+$ is isomorphic to S, and acts in a natural way on S itself by right multiplication; we will write $\tau_{(a,t)}$ for the element of U^+ mapping $(b, v) \in S$ to $(b, v) \cdot (a, t)$. Then $Z(U^+) = \{\tau_{(0,t)} \mid t \in \mathbb{K}\}$.

Let U be a second unipotent subgroup in G_∞, and assume, as before, $U^+ \leq U$.

Exactly as in section 3.3.2, we let $\tilde{U} := U/Z(U^+)$, $\tilde{U}^+ := U^+/Z(U^+)$, $\tilde{B} := B/Z(U^+)$, and $\tilde{Z} := Z(\tilde{U}) \neq 1$. For every $a \in X_0$, we let $\tau_a := \tau_{(a,0)}Z(U^+) \in \tilde{U}^+$; then the map $a \mapsto \tau_a$ is an isomorphism from $(X_0, +)$ to $\tilde{U}^+$. The natural action of U on S induces an action of $\tilde{U}$ on X_0. Since $\tilde{U}^+$ acts regularly on X_0, there exists an element φ in $\tilde{U} \setminus \tilde{U}^+$ fixing $0 \in X_0$, and hence fixing the orbit $0^{\tilde{Z}}$ elementwise. Again, there exists a non-trivial element $\tau \in \tilde{U}^+ \cap \tilde{Z}$. For every $(a, t) \in S^*$, the mapping

$$\mu_{a,t} : b \mapsto b\pi(a, t) + ah(b, a) - \frac{f(h(b, a), \pi(a, t))}{q(\pi(a, t))} a\pi(a, t)\,,$$

for all $b \in X_0$, belongs to $\tilde{B}$. (The computation of this expression requires some calculation, similar to the other cases in [40](Chapter 33). Observe also that $q(\pi(a, t)) \neq 0$ by (13.49)[40].) Let $F := \{c \in X_0 \mid \tau_c \in \tilde{Z}\}$. Then F is a non-trivial additive subgroup of X_0 such that $\mu_{(a,t)}(F) \subseteq F$ for all $(a, t) \in S^*$. We will again show that $F = X_0$ to obtain the required contradiction.

First of all, observe that it follows from the fact that $\mu_{(0,t)}(b) = tb$ for all $t \in \mathbb{K}$ and all $b \in X_0$ that F is a $\mathbb{K}$-subspace of X_0.

Lemma 3.3.5. *Let $b \in X_0^*$. If $b \in F$, then $b\pi(b) \in F$.*

Proof. Let $b \in F$. Then, for all $t \in \mathbb{K}$, also $\mu_{b,t}(b) \in F$, that is,

$$\mu_{b,t}(b) = \left(1 - \frac{f(h(b, b), \pi(b, t))}{q(\pi(b, t))}\right) b\pi(b, t) + bh(b, b) \in F\,. \qquad (3.3)$$

Note that $h(b, b) = 2\pi(b)$ if $\mathrm{char}(\mathbb{K}) \neq 2$ and that $h(b, b) = f(\pi(b), \epsilon)\epsilon$ if $\mathrm{char}(\mathbb{K}) = 2$, by [40]((13.28) and (13.45)). Also observe that we have already shown that $b \cdot s\epsilon \in F$ for all $s \in \mathbb{K}$.

Assume first that $\mathrm{char}(\mathbb{K}) \neq 2$. Then it follows from (3.3) that

$$\left(3 - \frac{f(2\pi(b), \pi(b,t))}{q(\pi(b,t))}\right) b\pi(b) \in F\,,$$

for all $t \in \mathbb{K}$, and it is easily checked that this expression is zero if and only if $q(\pi(b)) = 3t^2$. Choose any t for which $q(\pi(b)) \neq 3t^2$; then it follows that $b\pi(b) \in F$ since F is a $\mathbb{K}$-subspace of X_0.

Now assume that $\mathrm{char}(\mathbb{K}) = 2$. It now follows from (3.3) that

$$\left(1 + \frac{f(f(\pi(b), \epsilon)\epsilon, \pi(b,t))}{q(\pi(b,t))}\right) b\pi(b) \in F\,,$$

for all $t \in \mathbb{K}$, and this expression is zero if and only if

$$t^2 + f(\pi(b), \epsilon)t + q(\pi(b)) + f(\pi(b), \epsilon)^2 = 0\,.$$

This quadratic equation has at most 2 solutions; let t be any element of K which is not a solution of this equation. Then it follows that $b\pi(b) \in F$ in this case as well. □

Lemma 3.3.6. *Let $b \in X_0^*$. If there exist elements $s, t \in \mathbb{K}$, not both zero, such that $b(s\pi(b) + t\epsilon) \in F$, then $b \in F$.*

Proof. Let $b \in X_0^*$ and $s, t \in \mathbb{K}$ (not both zero) be such that $b(s\pi(b) + t\epsilon) \in F$. If $s = 0$, then $t \neq 0$, and then $tb \in F$, hence $b \in F$. So assume that $s \neq 0$; then $b\pi(b, s^{-1}t) \in F$. Assume without loss of generality that $s = 1$. It is shown in the proof of (13.67) in [40] that $\pi(b\pi(b,t)) = q(\pi(b,t))\pi(b)$. By (13.49) [40], $q(\pi(b,t)) \neq 0$. If we now apply Lemma 3.3.5 on the element $b\pi(b,t) \in F$, then we get that $b\pi(b,t)\pi(b) \in F$, and since $\overline{\pi(b,t)} = f(\epsilon, \pi(b,t))\epsilon - \pi(b,t)$, it also follows that $b\pi(b,t)\overline{\pi(b,t)} \in F$. But $b\pi(b,t)\overline{\pi(b,t)} = q(\pi(b,t))b$ by [40, (13.7)], so $b \in F$, and we are done. □

As in [40, (13.42)], we define $P(a,b) := f(h(a,b), \epsilon)$ for all $a, b \in X_0$; then P is an alternating bilinear form, which is non-degenerate. (This form is called F in [40], but we choose P to avoid confusion with our set F.)

Lemma 3.3.7. *Let $a, b \in X_0^*$. If $b \in F$ and $P(b,a) \neq 0$, then $a \in F$.*

Proof. Let $a \in X_0^*$ and let $b \in F$ such that $P(b,a) \neq 0$. Then for all $s, t \in \mathbb{K}$, we have that $\mu_{a,t}(b) - \mu_{a,s}(b) \in F$. It follows that

$$\frac{f(h(b,a), \pi(a,s))}{q(\pi(a,s))} a\pi(a,s) - \frac{f(h(b,a), \pi(a,t))}{q(\pi(a,t))} a\pi(a,t) \in F\,,$$

for all $s, t \in \mathbb{K}$. Let $x := f(h(b,a), \pi(a)) \in \mathbb{K}$ and let $y := P(b,a) \in \mathbb{K}^*$; then this can be rewritten as

$$\left(\frac{x+sy}{q(\pi(a,s))} - \frac{x+ty}{q(\pi(a,t))}\right) a\pi(a) + \left(s\frac{x+sy}{q(\pi(a,s))} - t\frac{x+ty}{q(\pi(a,t))}\right) a \in F\,.$$

By [40, (13.41)], a and $a\pi(a)$ are linearly independent. On the other hand, since $y \neq 0$, there exists only one element $s \in \mathbb{K}$ for which $x + sy = 0$. If we now choose $s \neq t$ such that $x + sy \neq 0$ and $x + ty \neq 0$, then the expression above cannot be zero, and hence we have found constants $c, d \in \mathbb{K}$, not both zero, such that $a(c\pi(a) + d\epsilon) \in F$. It follows from Lemma 3.3.6 that $a \in F$, which is what we had to show. □

We are now in a position to show that $X_0 = F$. We already know that F is non-trivial, so choose some fixed element $b \in F^*$. Now let $c \in X_0^*$ be arbitrary. If $P(b,c) \neq 0$, then $c \in F$ by Lemma 3.3.7. If $P(b,c) = 0$, then choose an element $a \in X_0$ such that $P(b,a) \neq 0$ (such an element exists since P is non-degenerate). But now the elements a and $a + c$ both satisfy the hypotheses of Lemma 3.3.7, and hence they both belong to F. It follows that also $c = (a + c) - a$ belongs to F, and hence we have shown that $X_0 = F$.

Suzuki-Tits Moufang sets We start with some observations. We use the notation of Subsection 3.2.8.

Observation 3.3.8. *The mapping $x \mapsto x^{1+\theta}$ induces a permutation of L. Also, the Tits endomorphism $x \mapsto x^\theta$ is a bijection from L onto L^θ.*

Proof. Indeed, if $x \in L$, then $x^\theta \in L\theta \subseteq \mathbb{K}^\theta$, so $x^{1+\theta} = x^\theta x \in \mathbb{K}^\theta L = L$. Moreover, for given nonzero $u \in L$, the element $u^{\theta-1}$ is mapped onto $\frac{u^\theta}{u}.\frac{u^2}{u^\theta} = u$. Since $u^{-1} \in L$, also $u^{\theta-1} = u^\theta u^{-1} \in L$. The mapping $x \mapsto x^{1+\theta}$ is injective since $x \mapsto x^{\theta-1}$ is its inverse.

If $x^\theta = y^\theta$, then applying θ, we get $x^2 = y^2$, so $x = y$. □

Observation 3.3.9. *For each nonzero $t \in L^\theta$, the mapping h_t fixing (∞) and mapping (a, a') onto $(ta, t^{1+\theta}a')$ belongs to $\mathsf{Sz}(\mathbb{K}, L, \theta)$.*

Proof. This follows from a calculation similar to one culminating in the formulae of (33.17) of [40], using the matrices in Subsection 3.2.8. □

Observation 3.3.10. *For $|\mathbb{K}| = 2$, every projective group of $\mathcal{M}\mathsf{Sz}(\mathbb{K}, \mathbb{K}, \mathrm{id})$ is isomorphic to the little projective group G. Also, in this Moufang set the stabilizer G_∞ related to (∞) is isomorphic to U_∞^+ and hence this Moufang set has unique transitive nilpotent normal subgroups.*

Proof. This readily follows from the well known fact that, in this case, the Moufang set is a Frobenius group of order 20 acting on 5 elements, and that this group is a maximal subgroup of the full symmetric group on five letters. ☐

From now on, we may assume that $|\mathbb{K}| \geq 8$. The following observation is well known for the classical case $L = \mathbb{K}$.

Observation 3.3.11. *The center Z_∞^+ of U_∞^+ coincides precisely with the set of elements of U_∞^+ of order less than or equal to* 2*. The orbit of* $(0,0)$ *under Z_∞^+ is equal to* $\{(0,a') \mid a' \in L\}$*, while the orbit of* (∞) *under the center Z_0^+ of U_0^+ is equal to* $\{(a,0) \mid a \in L^*\} \cup \{(\infty)\}$*.*

Proof. An easy and straightforward computation shows that $Z_\infty^+ = \{(0,a')_\infty \mid a' \in L\}$, and also that $(a,a')_\infty$ has order two if and only if $a = 0$ and $a' \neq 0$. Using the matrices of Subsection 3.2.8, one now sees that $Z_0^+ = \{(0,x')_0 \mid x \in L\}$, but the element $(0,a^{-1-\theta})_0$ maps $(1,0,0,0)$ to $(1,(a^{-1-\theta})^\theta,0,a^{-1-\theta})$, which coincides with $(a^{2+\theta},1,0,a) = (a,0)$. ☐

The fact that Suzuki-Tits Moufang sets have a unique splitting will strongly depend on the following lemma.

Lemma 3.3.12. *Let φ be an automorphism of the Moufang set $\mathcal{M}\mathsf{Sz}(\mathbb{K},L,\theta)$ fixing* (∞) *and all elements* $(0,a')$ *with $a' \in L^\theta$. Then φ is necessarily the identity.*

Proof. By the definition of automorphism, the permutation φ normalizes U_∞^+, and hence also Z_∞^+. Likewise, it normalizes Z_0^+. Using Observation 3.3.11, this immediately implies that φ stabilizes the sets $\{(0,a') \mid a \in L\}$ and $(a,0) \mid a \in L$. Hence we may write $(a,0)^\varphi = (a^{\varphi_1},0)$, with φ_1 a permutation of L fixing 0, and $(0,a')^\varphi = (0,a'^{\varphi_2})$, with φ_2 a permutation of L fixing L^θ pointwise. Since φ fixes $(0,0)$, we may interpret the foregoing formulae as conjugation of elements of U_∞^+ with φ. Hence, we obtain

$$(a,a')_\infty^\varphi = (a,0)_\infty^\varphi \oplus (0,a')_\infty^\varphi = (a^{\varphi_1},a'^{\varphi_2})_\infty.$$

We now use the fact that φ induces an automorphism of $U^+\infty$ by conjugation. The equality $(a,0)_\infty^\varphi \oplus (b,0)_\infty^\varphi = (a+b,ab^\theta)_\infty^\varphi$ translates implies

$$a^{\varphi_1}(b^{\varphi_1})^\theta \quad = \quad (ab^\theta)^{\varphi_2} \tag{3.4}$$

Putting $a = 1$, and taking into account that $b^\theta \in L^\theta$ is fixed by φ_2, we see that $1^{\varphi_1}(b^{\varphi_1})^\theta = b^\theta$. Putting $b = 1$, this implies $1^{\varphi_1}(1^{\varphi_1})^\theta = 1$, hence $1^{\varphi_1} = 1$

by Observation 3.3.8. The previous equality now gives us $(b^{\varphi_1})^\theta = b^\theta$. Again using Observation 3.3.8 we conclude $\varphi_1 = \mathrm{id}$.

Now putting $b = 1$ in Equation (3.4), we deduce $a^{\varphi_1} = a^{\varphi_2}$. The assertion now follows. □

Theorem 3.3.13. *Let G be an arbitrary projective group of $\mathcal{M}\mathsf{Sz}(\mathbb{K}, L, \theta)$, and let U_∞ be a unipotent subgroup of G. Then $U_\infty \equiv U_\infty^+$.*

Proof. We may assume $U_\infty^+ \leq U_\infty$. Let $u \in Z(U_\infty)$. Then u acts fixed point freely on $X \setminus \{(\infty)\}$, and it commutes with every element of U_∞^+. Identifying the element (a, a') with the group element $(a, a')_\infty$, and noting that the action of U_∞^+ can hence be identified with the right action on itself, the action of u can be described as left action on U_∞^+. So, if u maps $(0, 0)$ onto (c, c'), then we may write $u : (a, a')_\infty \mapsto (c, c')_\infty \oplus (a, a')_\infty$. Hence, if $c \neq 0$, then the map $\varphi : (a, a')_\infty \mapsto (c, c')_\infty \oplus (a, a')_\infty \oplus (c, c' + c^{1+\theta})_\infty$ is nontrivial, belongs to U_∞ and fixes all elements of the form $(0, a')$, with $a' \in L$. This contradicts Lemma 3.3.12.

So $c = 0$. Considering the isomorphic Moufang set $\mathcal{M}\mathsf{Sz}(\mathbb{K}, Lc'^{-1}, \theta)$, we may assume that $c' = 1$. Since the center of U_∞ is invariant under each mapping h_t, $t \in L^\theta$. Observation 3.3.9 implies that $(0, t^\theta)_\infty \in Z(U_\infty)$. If $U_\infty \neq U_\infty^+$, then there exists a nontrivial element $\varphi \in U_\infty$ fixing $(0, 0)$. Since φ commutes with $(0, t^\theta)$, $t \in L$, it fixes all elements $(0, t^\theta)$, with $t \in L$. Lemma 3.3.12 shows that φ is the identity, a contradiction. Hence U_∞ must coincide with U_∞^+.

The theorem is proved. □

Ree-Tits Moufang sets We start again with some observations, using the notation of Subsection 3.2.9.

Observation 3.3.14. *The mapping $x \mapsto x^{2+\theta}$ is a permutation of $\mathbb{K}$, inducing a permutation of $\mathbb{K}^2$. Also, the Tits endomorphism $x \mapsto x^\theta$ is a bijection from $\mathbb{K}$ onto $\mathbb{K}^\theta$. Finally, the set $\{t^{1+\theta} \mid t \in \mathbb{K}\}$ contains $\mathbb{K}^2$.*

Proof. The inverse of $x \mapsto x^{2+\theta}$ is given by $x \mapsto x^{2-\theta}$, for $x \neq 0$, and $0 \mapsto 0$.

Also, if $x^\theta = y^\theta$, then applying θ, we get $x^3 = y^3$, so $x = y$.

Finally, for any $x \in \mathbb{K}$, the element $(x^{-1+\theta})^{1+\theta}$ is the arbitrary but prescribed square $x^2 \in \mathbb{K}^2$, which proves the last assertion. □

Observation 3.3.15. *For each nonzero $t \in \mathbb{K}$, the mapping h_t fixing (∞) and mapping (a, a', a'') onto $(t^{\theta-1}a, t^2a', t^{1+\theta}a'')$ belongs to $\mathsf{Ree}(\mathbb{K}, \theta)$.*

Proof. The subgroups $\{(0, x', 0)_\infty \mid x' \in \mathbb{K}\} \leq U_\infty^+$ and $\{(0, x', 0)_0 \mid x' \in \mathbb{K}\} \leq U_0^+$ preserve the set $\{(0, a', 0) \mid a' \in \mathbb{K}\} \cup \{(\infty)\}$, inducing a Moufang set $\mathcal{M}'$ isomorphic to a projective line over $\mathbb{K}$. Using the matrices above related to the mapping $(0,x',0)_\infty$ and $(0,x',0)_0$, one now calculates that the mapping $(0,a',0)\mapsto(0,t^2a',0)$, for any $t\in\mathbb{K}^*$, which belongs to $\mathcal{M}'$, acts on X as h_t. □

Observation 3.3.16. *The center Z_∞^+ of U_∞^+ consists precisely of the elements $(0,0,a'')_\infty$, with $a''\in\mathbb{K}$. Also, the elements of U_∞^+ of order less than or equal to 3 form a subgroup $V_\infty^+=\{(0,a',a'')\mid a',a''\in\mathbb{K}\}$ which coincides precisely with the commutator subgroup $[U_\infty^+,U_\infty^+]$, and also with the set of elements $u\in U_\infty^+$ satisfying $[u,U_\infty^+]\le Z_\infty^+$. The orbit of $(0,0,0)$ under Z_∞^+ is equal to $\{(0,0,a'')\mid a''\in\mathbb{K}\}$, while the orbit of (∞) under the center Z_0^+ of U_0^+ is equal to $\{(a,0,-a^{2+\theta})\mid a\in\mathbb{K}^*\}\cup\{(\infty)\}$.*

Proof. The first assertion follows from an easy and straightforward computation using the operation $\oplus$ introduced above.

The second assertion follows from the identities

$$(a,a',a'')_\infty\oplus(a,a',a'')_\infty\oplus(a,a',a'')_\infty=(0,0,-a^{2+\theta})_\infty$$

and

$$[(a,a',a'')_\infty,(b,b',b'')_\infty]=(0,ab^\theta-a^\theta b,ab^{1+\theta}-a^{1+\theta}b+a^\theta b^2-a^2b^\theta+a'b-ab')_\infty,$$

and from the following two claims: (1) for arbitrary $a\in\mathbb{K}$, the identity $ab^\theta-a^\theta b=0$, for all $b\in\mathbb{K}$, implies $a=0$, and (2) the additive subgroup A of $\mathbb{K}$ generated by the elements $ab^\theta-a^\theta b$, for $a,b\in\mathbb{K}$, coincides with $\mathbb{K}$ itself.

We prove Claim (1). Putting $b=1$, Observation 3.3.14 implies $a=1$, a contradiction since $b^\theta-b=0$ is not an identity in $\mathbb{K}$. We now prove Claim (2). Putting $a\ne b$, we see that A is nontrivial. Let $x\in A$, $x\ne0$, with $x=ab^\theta-a^\theta b$, for some $a,b\in\mathbb{K}$. Substituting ta and tb for a and b, respectively, with $t\in\mathbb{K}^*$ arbitrary, we see that $t^{1+\theta}x\in A$. Observation 3.3.14 implies that, for all $k\in\mathbb{K}$, the element xk^2 belongs to A. For arbitrary $y\in\mathbb{K}$, we now have

$$y=x(x^{-1}-y)^2-x(x^{-1})^2-xy^2\in A.$$

The claim is proved.

The explicit form (using matrices as in Subsection 3.2.9) of $(0,0,a'')_0=(0,0,a'')_\infty^g$ shows that

$$\begin{aligned}(\infty)^{(0,0,a'')_0}&=(-f_3(0,0,a'')f_1(0,0,a'')^{-1},-f_2(0,0,a'')f_1(0,0,a'')^{-1},-a''f_1(0,0,a'')^{-1}),\\&=(a''^{\theta-2},0,-a''^{-1}),\end{aligned}$$

and the last assertion follows by putting $a''=a^{-2-\theta}$. □

We need one more observation before we can prove the analogue of Lemma 3.3.12 for Ree-Tits Moufang sets.

Observation 3.3.17. *Let φ be an automorphism of the Moufang set $\mathcal{M}\mathsf{Ree}(\mathbb{K},\theta)$ fixing (∞) and $(0,0,0)$. Then φ stabilizes the set $\{(0,a',0) \mid a' \in \mathbb{K}\}$.*

Proof. Let $a' \in \mathbb{K}^*$ be arbitrary and let (b,b',b'') be the image of $(0,a',0)$ under φ. Then $(0,a',0)_\infty^\varphi = (b,b',b'')_\infty$. Since $(0,a',0)_\infty \in [U_\infty^+, U_\infty^+]$, also $(b,b',b'')_\infty$ belongs to the commutator subgroup. It follows that $b=0$. This argument means in fact that (b,b',b'') must belong to the orbit of $(0,0,0)$ under $[U_\infty^+, U_\infty^+]$. Now we remark that $(0,-b'^{-1},0)_0$ maps (∞) onto $(0,b',0)$. Hence, similarly as above, $(0,b',b'')$ must belong to the orbit of (∞) under $[U_0^+, U_0^+]$. Using the same technique as in the proof of the previous observation, one shows that this orbit consists of, besides (∞), the elements $(-f_3(0,x',x'')f_1(0,x',x'')^{-1}, -f_2(0,x',x'')f_1(0,x',x'')^{-1}, -x''f_1(0,x',x'')^{-1})$, for $x',x'' \in \mathbb{K}$. Such an element also belongs to the orbit of $(0,0,0)$ under $[U_\infty^+, U_\infty^+]$ if and only if $f_3(0,x',x'')=0$, hence if and only if $x''^\theta = x'x''$. If $x''=0$, then the assertion follows. If $x'' \neq 0$, then $x' = x''^{\theta-1}$ and we have $f_1(0,x',x'') = x''^2 + x''^{(\theta-1)(\theta+1)} = -x''^2$, hence $(0,b',b'') = (0,x''^{1-\theta},x''^{-1})$, for some $x'' \in \mathbb{K}^*$. In this case, the image of $(0,-a',0)$ must be equal to, in view of $(0,-a',0)_\infty = (0,a',0)_\infty^{-1}$, the element $(0,-x''^{1-\theta},-x''^{-1})$. But then

$$-x''^{1-\theta} = (-x'')^{1-\theta},$$

a contradiction. □

The fact that Ree-Tits Moufang sets have a unique splitting strongly depends on the following lemma.

Lemma 3.3.18. *Let φ be an automorphism of the Moufang set $\mathcal{M}\mathsf{Ree}(\mathbb{K},\theta)$ fixing (∞) and all elements $(0,0,a'')$ with $a'' \in \mathbb{K}$. Then φ is necessarily the identity.*

Proof. By assumption, we have $(0,0,a'')^\varphi = (0,0,a'')$, for all $a'' \in \mathbb{K}$. By Observation 3.3.17, there is a permutation φ_1 of $\mathbb{K}$ such that $(0,a',0)^\varphi = (0,a'^{\varphi_1},0)$, for all $a' \in \mathbb{K}$. Now, by definition of automorphism of a Moufang set, φ normalizes U_0^+, and hence also its center Z_0^+. Using Observation 3.3.16, this implies that there is a permutation φ_2 of $\mathbb{K}$ such that $(a,0,-a^{2+\theta})^\varphi = (a^{\varphi_2},0,-(a^{\varphi_2})^{2+\theta})$.

This implies

$$\begin{aligned}(a,a',a'')_\infty^\varphi &= (a,0,-a^{2+\theta})_\infty^\varphi \oplus (0,a',0)_\infty^\varphi \oplus (0,0,a''+a^{2+\theta}-aa')_\infty^\varphi \quad (3.5)\\ &= (a^{\varphi_2},a'^{\varphi_1},a''-(a^{\varphi_2})^{2+\theta}+a^{\varphi_2}a'^{\varphi_1}+a^{2+\theta}-aa')_\infty. \quad (3.6)\end{aligned}$$

Let $a, b \in \mathbb{K}$ be arbitrary. Equating the second positions of $(a,0,0)^\varphi_\infty \oplus (b,0,0)^\varphi_\infty$ and $(a+b, ab^\theta, -ab^{1+\theta})^\varphi_\infty$, we obtain, using the general formulae (3.6),

$$a^{\varphi_2}(b^{\varphi_2})^\theta \quad = \quad (ab^\theta)^{\varphi_1}, \tag{3.7}$$

for all $a, b \in \mathbb{K}$.

Similarly, equating the third positions of $(0,c,0)^\varphi_\infty \oplus (d,0,0)^\varphi_\infty$ and $(d,c,-cd)^\varphi_\infty$, we obtain, again using the general formulae (3.6),

$$-(d^{\varphi_2})^{2+\theta} + d^{2+\theta} - d^{\varphi_2}c^{\varphi_1} \quad = \quad cd - (d^{\varphi_2})^{2+\theta} + d^{\varphi_2}c^{\varphi_1} + d^{2+\theta},$$

for all $c, d \in \mathbb{K}$, which implies

$$cd \quad = \quad c^{\varphi_1}d^{\varphi_2}, \tag{3.8}$$

for all $c, d \in \mathbb{K}$. Putting $a = b = 1$ in Equation (3.7), we see that $1^{\varphi_2}(1^{\varphi_2})^\theta = 1^{\varphi_1}$, which implies, in view of Equation (3.8) with $c = d = 1$, that $(1^{\varphi_2})^{2+\theta} = 1$. Consequently, Observation 3.3.14 shows $1^{\varphi_2} = 1$. Putting $d = 1$ in Equation (3.8), we now see $c = c^{\varphi_1}$, for all $c \in \mathbb{K}$, so φ_1 is the identity. The same Equation (3.8), now again with general $d \in \mathbb{K}$, now also shows that φ_2 is the identity. Formula (3.6) now implies that φ is trivial. □

Theorem 3.3.19. *Let G be an arbitrary projective group of $\mathcal{M}\mathsf{Ree}(\mathbb{K}, \theta)$, and let $U_\infty \leq G_\infty$ be a unipotent subgroup of G. Then $U_\infty \equiv U^+_\infty$.*

Proof. We may assume $U^+_\infty \leq U_\infty$. Let $u \in Z(U_\infty)$. If u maps $(0,0,0)$ onto (c, c', c''), then, similarly as in the beginning of the proof of Theorem 3.3.13, u can be presented as $u : (a, a', a'')_\infty \mapsto (c, c', c'')_\infty \oplus (a, a', a'')_\infty$. Hence, if $(c, c') \neq (0,0)$, then the map $\varphi : (a, a', a'')_\infty \mapsto (c, c', c'')_\infty \oplus (a, a', a'')_\infty \oplus (-c, -c' + c^{1+\theta}, -c'' + cc' - c^{2+\theta})_\infty$ belongs to U_∞ and fixes all elements of the form $(0,0,a'')$, with $a' \in L$. This contradicts Lemma 3.3.12.

So we may assume that $(c, c') = (0,0)$. Then $u = (0,0,c'')_\infty$, for some $c'' \in \mathbb{K}$. Since the center of U_∞ is invariant under each mapping h_t, $t \in \mathbb{K}$, Observation 3.3.15 implies that $(0,0,t^{1+\theta}c'')_\infty \in Z(U_\infty)$. Hence by Observation 3.3.14 $(0,0,k^2c'')_\infty \in Z(U_\infty)$, for all $k \in \mathbb{K}$. For arbitrary $x \in \mathbb{K}$, we see that

$$(0,0,x)_\infty = (0,0,(x - c''^{-1})c'')_\infty \oplus (0,0,x^2c'')^{-1}_\infty \oplus (0,0,(c''^{-1})^2c'')^{-1}_\infty,$$

which implies $Z(U_\infty) = Z^+_\infty$. Standard group theory now implies that φ fixes all elements $(0,0,x)$, with $x \in \mathbb{K}$. Lemma 3.3.18 shows that φ is the identity, a contradiction. Hence U_∞ must coincide with U^+_∞.

The theorem is proved. □

3.4 The uniqueness of splittings for BN-pairs of rank 2

In this section, we prove that generalized n-gons for which the BN-pairs are split, have a unique splitting. The strategy of our proof is use the fact that $U^{+*} = U^*$ for the rank 1 Moufang sets. For $n = 3, 6$, we will study all possibilities; for $n = 4$, we will only need "half" of them (see below for more details). The upshot of our investigations will in any case be that φ fixes every vertex adjacent to any vertex v^* of Σ (for $n \neq 4$), or to any vertex v^* of fixed type (for $n = 4$).

3.4.1 The generalized triangles and hexagons

Suppose that Ω is a generalized n-gon with $n \in \{3, 6\}$ corresponding to a split Tits system (G, B, N) of rank 2, with some transitive normal nilpotent subgroup U of B. Remember that we may assume that $U^+ \leq U$. If $U \neq U^+$, then there is some nontrivial collineation $\varphi \in U$ fixing the standard apartment Σ pointwise. Let v^* be as before. If the corresponding (split) Tits system of rank 1 is defined by one of the structures dealt with in the previous three subsections, then φ automatically fixes all elements of Ω adjacent to v^*. If $n = 3, 6$, then clearly this is true for all vertices v^* in Σ (see [40](17.2),(17.5)); hence φ is the identity. Consequently $U = U^+$.

3.4.2 The generalized quadrangles

In this case, we put the set of vertices of Σ equal to $\{x_0, x_1, \ldots, x_7\}$, with subscripts modulo 8, and such that x_i and x_{i+1} are adjacent for all i. To fix the ideas, we may think of x_0 as a point, and then x_1 is a line of Ω. We put $C = \{x_0, x_1\}$.

By the classification of Moufang quadrangles (see [40]) we may suppose that the rank 1 Tits systems related to x_{2i+1} (for any integer i) are commutative. Also by that same classification result, we may assume that this rank 1 group corresponds either to a skew field (Moufang quadrangles of *involution type*, of *quadratic form type*, and of *pseudo-quadratic form type*), or to a quadratic form of Witt index 1 (Moufang quadrangles of *exceptional types* E_6, E_7, E_8, F_4), or to an indifferent set in characteristic 2 (*indifferent* or *mixed* Moufang quadrangles). Hence, if $U \neq U^+$, and if φ is a nontrivial element of U fixing Σ pointwise, then φ fixes all points incident with one of x_{2i+1}. We now show that φ is necessarily the identity, showing that $U = U^+$. Henceforth we assume $\varphi \neq 1$ and we seek a contradiction.

The conjugate of U under a collineation that maps the chamber C to another chamber C' is denoted by $U[C']$ (and we have obviously $U = U[C]$).

We denote by $\{1\} = U^{[\ell]} \trianglelefteq U^{[\ell-1]} \trianglelefteq U^{[\ell-2]} \trianglelefteq \cdots \trianglelefteq U^{[0]} = U$ the ascending central series of U (and U is nilpotent of class ℓ).

Step 1 *For every chamber $C' = \{x_i, x_{i+1}\}$, we have $\varphi \in U[C']$.*

Indeed, let g be a nontrivial elation in $G^{[1]}_{x_1,x_2,x_3}$ which commutes with φ (g exists as $[\varphi, G^{[1]}_{x_1,x_2,x_3}] \leq G^{[1]}_{x_1,x_2,x_3} \leq U$ and as U is nilpotent). Hence φ fixes x_7^g. Now let $h \in G^{[1]}_{x_5,x_6,x_7}$ be such that $x_1^h = x_7^g$. Since φ fixes x_1^h, we have $[\varphi, h] = 1$, and hence $[\varphi, hg^{-1}] = 1$. Consequently $\varphi = \varphi^{hg^{-1}} \in U^{hg^{-1}} = U[\{x_0, x_7\}]$. A similar argument now shows that $\varphi \in U[\{x_6, x_7\}]$. Continuing like this, the assertion follows. This implies that φ has to fix a thick full subquadrangle.

Step 2 *Suppose $y \sim x_2$ is not fixed by φ, and let $u \in G^{[1]}_{x_3,x_4,x_5}$ be such that $y^u = y^\varphi$. Then $u \notin G^{[2]}_{x_4}$.*

Under the stated assumptions, choose $g \in G^{[1]}_{x_4,x_5,x_6} \setminus \{1\}$ arbitrarily. Then let $v \in G^{[1]}_{x_1,x_2,x_3}$ be such that $y^{\varphi g^{-1} v}$ is fixed under φ (this can be accomplished by putting $y^{\varphi g^{-1} v}$ equal to the unique vertex adjacent to $x_2^{\varphi g^{-1} v}$ at distance 2 from x_7). Then $\alpha := [\varphi^{-1}, g^v] \in G^{[1]}_{x_3,x_4,x_5}$. But evaluating y^α, we see that $y^\alpha = y^u$, hence $\alpha = u$. So α is not the identity and hence φ cannot fix ${x_5}^v$ (if it did, then φ would fix the line x_5^v pointwise and α would fix all elements incident with x_6^v, a contradiction). This now implies that $\alpha = u$ cannot fix all elements incident with x_5^v, and so u is not a central elation.

Step 3 *Let y be an arbitrary vertex adjacent to x_0 but different from x_1. Let $u \in U^{[i]}$ be a non-central elation in $G^{[1]}_{y,x_0,x_1}$ with i maximal. Then $[u, \varphi] = 1$.*

Indeed, it is clear that $[\varphi, u]$ is a central elation by minimality of i, and that $(x_3^u)^{[u,\varphi]} = (x_3^u)^\varphi$. The previous step implies that a central elation in $G^{[1]}_{x_0}$ mapping some line through x_2 onto its image under φ has to be the identity, hence $[u, \varphi] = 1$.

We can now finish the proof.

Let $u' \in G^{[1]}_{y',x_0,x_1}$ be noncentral and contained in $U^{[i]}$, with i maximal, and with y' some line through x_0 different from x_1. We may assume that u' does not fix all points on x_7. Then $[u', \varphi] = 1$ by Step 3.

We claim that we can re-choose y' in such a way that it is not fixed under φ, i.e., we claim that there exists $y \sim x_0$ and a noncentral elation

$u \in G^{[1]}_{y',x_0,x_1} \cap U^{[i]}$ such that $y^\varphi \neq y$. Indeed, let $y \sim x_0$ with $y^\varphi \neq y$ and let $v \in G^{[1]}_{x_1,x_2,x_3}$ with $(y')^v = y$. Note that $u' \notin G^{[1]}_y$, hence $[u', v^{-1}] \notin G^{[1]}_{y'}$ and in particular $[u', v^{-1}] \neq 1$. Now $u = u'^v \in G^{[1]}_{y,x_0,x_1}$ is an elation belonging to $U^{[i]}$. It remains to show that u is noncentral. Since $u' \notin G^{[1]}_y$, we have that $u'^v \notin G^{[1]}_{y^v}$. The claim is proved.

So we assume that $u \in G^{[1]}_{y,x_0,x_1}$ and $y^\varphi \neq y$. Let $y \sim y_2 \sim y_3 \sim x_4$, and let $w \in G^{[1]}_{y,y_2,y_3}$ be such that $x_1^w = x_7$. Then $u^w \in Z_i(U[x_7, x_0])$, and hence $[u^w, \varphi] = 1$ (using Steps 1 and 3). But then also $[[u,w],\varphi] = 1$. Notice that $[u,w] \in G^{[1]}_{y_2,y,x_0} \setminus \{1\}$, because the action on the points incident with x_1 is nontrivial. Since $G^{[1]}_{y_2,y,x_0} \cap (G^{[1]}_{y_2,y,x_0})^\varphi = \{1\}$ $(y \neq y^\varphi)$, it is impossible that $[[u,w],\varphi] = 1$. This contradiction proves that φ has to fix every $y \sim x$ and thus has to be the identity.

3.4.3 The generalized octagons

Although this fact follows directly from [25], we could also give an alternative proof here. Indeed, if $n = 8$, then for one bipartition class of the vertices v^*, the corresponding rank 1 Tits system is defined by a field (see [40](17.7)), and hence the fixed elements of Ω under φ form, up to duality, a thick ideal suboctagon (for terminology, see [41]). By Proposition 5.9.13 of [41] (originally due to M. Joswig and H. Van Maldeghem [16]), φ is the identity.

3.5 The uniqueness of splittings for BN-pairs of rank $n > 2$

Since every Moufang set arising from root groups of buildings of higher rank has a unique splitting, the buildings themselves also have a unique splitting: Let us consider a building Ω where B fixes a unique chamber C and N stabilizes some apartment Σ containing C. Let U^+ be the group generated by all elations related to roots in Σ which contain the vertices of C. Then it is well known that U^+ is a transitive normal nilpotent subgroup of B. We now have the following result.

Theorem 3.5.1. *Let G be a group having a BN-pair of rank ≥ 2 acting faithfully on the corresponding building, and supppose that U is a normal nilpotent subgroup of B such that $UH = B$, where $H = B \cup N$. Then $U = U^+$.*

Proof. We assume that (G, B, N) is an irreducible spherical BN-pair of rank at least 3. Let Ω be the corresponding building, let Σ be the apartment fixed by N and let C be the chamber fixed by B. Further, let Σ^+ be a half apartment in the apartment Σ containing C. Let φ be an arbitrary element of U fixing all chambers contained in Σ^+. Let P be a panel in the interior of Σ^+ and let R be a flag of corank 2 contained in P. Consider the stabilizer B_R of R in B, and the stabilizer U_R of R in U. Clearly $U_R \trianglelefteq B_R$ and U_R is nilpotent. Let C_R be the unique chamber containing R nearest to C (the *projection* of C onto R in building language), and let C_R^* be the chamber in Σ containing R opposite C_R in the residue of R. Let C_R' be any chamber containing R opposite C_R in the residue of R. There is an apartment Σ' containing C_R' and C (by the very definition of a building), and hence there exists $u \in U$ mapping Σ to Σ'. Clearly u fixes R and maps C_R^* onto C_R'. Hence U_R is transitive. So, if we denote by K the kernel of the action of G_R on the residue of R, then U_RK/K is a splitting of the rank 2 BN-pair $(G_R/K, B_RK/K, N_RK/K)$ (with obvious notation). It follows from the results in this section that φ fixes all chambers containing P, i.e., φ is a root elation by definition. Hence U coincides with the standard unipotent subgroup U_+ and the theorem is proved. □

Chapter 4

Moufang Lines defined by the Ree Group

In order to investigate the Moufang set associated with the Ree group, we turn to the mixed hexagons. These hexagons are defined over a field of characteristic 3 admitting a Tits endomorphism θ, and they allow a polarity ρ. The absolute points under this polarity, together with the automorphisms of the mixed hexagon commuting with ρ, form the *Ree-Tits Moufang set.* We call these mixed hexagons Ree hexagons, because of their close relation with the Ree groups in characteristic 3.

These Ree groups have root groups of nilpotency class 3. As a consequence, the Moufang geometries of rank one that we will define corresponding to the Ree groups will have *dimension* 2. This means that we will have two types of *blocks* in our geometry. In this chapter we prove that every automorphism of such a geometry is an automorphism of the corresponding Ree group, by writing down explicitly the automorphisms of this geometry. Therefore we start with the coordinatization of the mixed hexagon and the coordinates of this mixed hexagon in its natural embedding in $PG(6, \mathbb{K})$. We proceed with the definition and the coordinates of the Ree-Tits ovoid, which allows us then to construct the Ree geometry. When all this is defined, we turn to the automorphisms of the Ree geometry. As an application, we conclude with the proof that an automorphism of the little projective group of the generalized hexagon fixing the absolute points under ρ has to fix the absolute lines as well.

4.1 The coordinatization of the Ree hexagon

In this section, we give two coordinatizations of the Ree hexagon. We start with the coordinatization with respect to one flag $((\infty),[\infty])$. This coordinatization was first carried out by De Smet and Van Maldeghem for (finite) generalized hexagons in [8]. For a detailed description of the coordinatization theory for other generalized polygons we refer to [41]. The second coordinatization is in fact the natural embedding of the mixed hexagon in $PG(6,\mathbb{K})$.

4.1.1 Hexagonal sexternary rings corresponding to the mixed hexagon

In [41] a coordinatization theory with respect to a flag $((\infty),[\infty])$ is described, it is a generalization of Hall's coordinatization for generalized triangles. Here we describe explicitly the coordinatization of the Ree hexagon. Let $\mathbb{K}$ be a field of characteristic 3 and let $\mathbb{K}'$ be a subfield such that $\mathbb{K}^3 \leq \mathbb{K}' \leq \mathbb{K}$. The hexagonal sexternary ring $\mathcal{R} = (\mathbb{K},\mathbb{K}',\Psi_1,\Psi_2,\Psi_3,\Psi_4)$ with

$$\begin{cases} \Psi_1(k,a,l,a',l',a'') = a^3k + l, \\ \Psi_2(k,a,l,a',l',a'') = a^2k + a' + aa'', \\ \Psi_3(k,a,l,a',l',a'') = a^3k^2 + l' + kl, \\ \Psi_4(k,a,l,a',l',a'') = -ak + a'', \end{cases}$$

defines the Ree hexagon $\mathsf{H}(\mathbb{K},\mathbb{K}')$ in which points and lines are certain i-tuples of elements of $\mathbb{K}\cup\mathbb{K}'$ ($i \leq 5$) and where incidence is defined as follows:

- If the number of coordinates of a point p differs by at least 2 from the number of coordinates of a line L, then p and L are not incident.
- If the number i_p of coordinates of a point p differs by exactly 1 from the number i_L of coordinates of a line L, then p is incident with L if and only if p and L share the first i coordinates, where i is the smallest among i_p and i_L.
- If $i_p = i_L \neq 5$, then p is incident with L if and only if $p = (\infty)$ and $L = [\infty]$.
- A point p with coordinates (a,l,a',l',a'') is incident with a line $[k,b,k',b',k'']$

if and only if

$$\begin{cases} \Psi_1(k,a,l,a',l',a'') = k'' = a^3k + l, \\ \Psi_2(k,a,l,a',l',a'') = b' = a^2k + a' + aa'', \\ \Psi_3(k,a,l,a',l',a'') = k' = a^3k^2 + l' + kl, \\ \Psi_4(k,a,l,a',l',a'') = b = -ak + a'', \end{cases}$$

if and only if

$$\begin{cases} \Phi_1(a,k,b,k',b',k'') = a'' = ak + b, \\ \Phi_2(a,k,b,k',b',k'') = l' = a^3k^2 + k' + kk'', \\ \Phi_3(a,k,b,k',b',k'') = a' = a^2k + b' - ab, \\ \Phi_4(a,k,b,k',b',k'') = l = -a^3k + k''. \end{cases}$$

4.1.2 The embedding of the Ree hexagon in $PG(6,\mathbb{K})$

The Ree hexagon has a natural embedding in $PG(6,\mathbb{K})$. Root-elations on this hexagon are elements of $PSL_7(\mathbb{K})$.
We write α for $-al' + a'^2 + a''l + aa'a''$ and β for $l - aa' - a^2a''$.

Coordinates in $\mathsf{H}(\mathbb{K},\mathbb{K}')$	Coordinates in $PG(6,\mathbb{K})$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(\alpha,-a'',-a,-a'+aa'',1,\beta,-l'+a'a'')$

Coordinates in $\mathsf{H}(\mathbb{K},\mathbb{K}')$	Points generating this line
$[\infty]$	(∞) and (0)
$[k]$	(∞) and $(k,0)$
$[a,l]$	(a) and $(a,l,0)$
$[k,b,k']$	(k,b) and $(k,b,0)$
$[a,l,a',l']$	(a,l,a') and $(a,l,a',l',0)$
$[k,b,k',b',k'']$	(k,b,k',b') and $(0,k'',b',k'+kk'',b)$

The $([\infty],(\infty),[0],(0,0),[0,0,0])$-elation mapping $[0,0]$ onto $[0,L]$ has the following corresponding matrix in $PG(6,\mathbb{K})$:

$$M_L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -L & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $((\infty),[\infty],(0),[0,0],(0,0,0))$-elation mapping $(0,0)$ onto $[0,B]$ has the following corresponding matrix in $PG(6,\mathbb{K})$:

$$M_L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & B & 0 & 0 & B^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -B \\ 0 & -B & 0 & 0 & 1 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.2 The Ree-Tits Ovoid

We start from the Ree hexagon $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$ where θ is a Tits-endomorphism of $\mathbb{K}$. This hexagon allows a polarity, and the absolute points under this polarity will form an ovoid of the Ree hexagon: the Ree-Tits ovoid. We fix the polarity ρ which maps the hat-rack (i.e. the apartment through the flags $((\infty),[\infty])$ and $((0,0,0,0,0),[0,0,0,0,0]))$ onto its dual and the point (1) onto the line $[1]$. The polarity takes the form:

$$(a,l,a',l',a'')^\rho = [a^\theta, l^{\theta^{-1}}, a'^\theta, l'^{\theta^{-1}}, a''^\theta];$$
$$[k,b,k',b',k'']^\rho = (k^{\theta^{-1}}, b^\theta, k'^{\theta^{-1}}, b'^\theta, k''^{\theta^{-1}}).$$

for all $a,a',a'',b,b' \in \mathbb{K}$ and $k,k',k'',l,l' \in \mathbb{K}'$.

Now the point (a,l,a',l',a'') is absolute for ρ if and only if it is incident with its image, this leads to the following conditions:

$$\begin{cases} l = a''^\theta - a^{\theta+3}, \\ l' = a^{2\theta+3} + a'^\theta + a^\theta a''^\theta. \end{cases}$$

4.2.1 Coordinates of the Ree-Tits ovoid in the projective space embedding

We associate the triple $(a, a'', a' - aa'')$ with the point $(a, a''^{\theta} - a^{3+\theta}, a', a^{3+2\theta} + a'^{\theta} + a^{\theta}a''^{\theta}, a'')$ Now, for $a, a', a'' \in \mathbb{K}$, we put

$$\begin{aligned} f_1(a, a', a'') &= -a^{4+2\theta} - aa''^{\theta} + a^{1+\theta}a'^{\theta} + a''^2 + a'^{1+\theta} - a'a^{3+\theta} - a^2a'^2, \\ f_2(a, a', a'') &= -a^{3+\theta} + a'^{\theta} - aa'' + a^2a', \\ f_3(a, a', a'') &= -a^{3+2\theta} - a''^{\theta} + a^{\theta}a'^{\theta} + a'a'' + aa'^2. \end{aligned}$$

So the set of absolute points can be described in $PG(6, \mathbb{K})$ by

$$\begin{aligned} X =& \{(1, 0, 0, 0, 0, 0, 0)\} \cup \\ & \{f_1(a, a', a''), -a', -a, -a'', 1, f_2(a, a', a''), f_3(a, a', a''))|a, a', a'' \in \mathbb{K}\}. \end{aligned}$$

The action of an automorphism of this ovoid fixing one point (∞) as an element of $PSL_7(\mathbb{K})$ has already been given in paragraph 3.2.3 of the previous chapter.

4.2.2 Compact notation

As before, we associate the triple $(a, a'', a' - aa'')$ with the point $(a, a''^{\theta} - a^{3+\theta}, a', a^{3+2\theta} + a'^{\theta} + a^{\theta}a''^{\theta})$. The set of absolute points under the polarity is now

$$X = \{\infty\} \cup \{(a, a', a'')\}\}.$$

The automorphisms of the ovoid fixing the point (∞) act as follows on the remaining points (x, x', x''):

$$(x, x', x'') \cdot (y, y', y'') = (x + y, x' + y' + xy^{\theta}, x'' + y'' + xy' - x'y - xy^{\theta+1})$$

We obtain the Ree-Tits Moufang line. The Ree groups arise as (simple subgroups of the) centralizers of polarities in these hexagons. We can see the Ree-Tits ovoid and its automorphism group embedded in the Ree hexagon as a representation of the Ree-Tits Moufang set, this Moufang set has nilpotency class 3, so it can define a geometry of dimension 2. The upcoming results are valid for Ree groups over not necessarily perfect fields.

4.3 The Ree Geometry

4.3.1 Construction

As already mentioned, the Ree groups have root groups of nilpotency class 3. As a consequence, the Moufang geometries of rank one that we will define corresponding to the Ree groups will have *dimension* 2. this means that we will have two types of *blocks* in our geometry and one type of blocks is a refinement of the other type.

Starting from the Moufang set $(X, (U_x)_{x\in X})$ it is clear that X defines the set of points $\mathcal{P}$ of our Ree geometry $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$. The circles arise as orbits of a point y under the center $Z(U_x)$ for some point $x \in \mathcal{P}$ together with that point x, this particular point x is then called the *gnarl* of this circle. So every point and gnarl defines a circle in a unique way. The spheres are again a point x together with the orbit of some point y, but the orbit is now defined under the group $[U_x, U_x]$. The point x is the gnarl of the sphere. The circles and spheres together form the block set $\mathcal{B}$ of $\mathcal{G}$.

Let us be more concrete now and look for the coordinates of the circles and spheres which have (∞) for gnarl: The group U_∞ acts as follows on the points (x, x', x''):

$$(x, x', x'').(u, u', u'') = (x + u, x' + u' + xu^\theta, x'' + u'' + xu' - x'u - xu^{\theta+1})$$

The group U'_∞ consisting of the commutators of U_∞ is precisely the set $\{(0, u', u'')|u', u'' \in \mathbb{K}\}$. Indeed, computing an arbitrary commutator, we get

$$[(u_1, u'_1, u''_1), (u_2, u'_2, u''_2)] = (0, u_1u_2^\theta - u_2u_1^\theta, u'_1u_2 - u_1u'_2 - u_1u_2^{1+\theta} + u_2u_1^{1+\theta})$$

These elements generate the set $\{(0, u', u'') \mid u', u'' \in \mathbb{K}\}$.

The center of U_∞ is the set $\{(0, 0, u'')|u'' \in \mathbb{K}\}$, we can see immediately that such an element commutes with every element of U_∞. The commutator of an element $(0, u'_1, u''_1) \in U'_\infty$ and $(u_2, u'_2, u''_2) \in U_\infty$ is

$$\begin{aligned}[(0, u'_1, u''_1), (u_2, u'_2, u''_2)] &= (0, 0, u'_1u_2)\\ &= (0, 0, u'')\end{aligned}$$

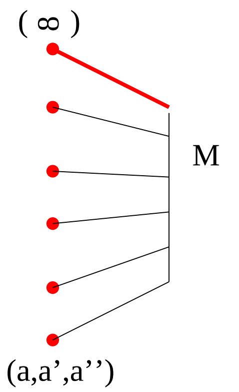
Now, since the circles having (∞) as gnarl are the orbits of a point (a, a', a'') under the group $\{(0, 0, x)|x \in \mathbb{K}\}$, all circles with (∞) as gnarl look like this:

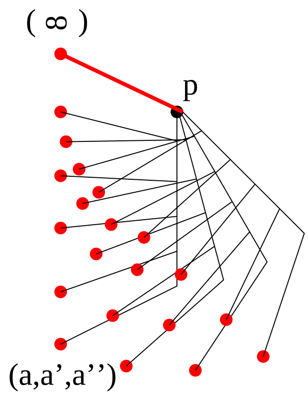
$$\{(a, a', a'' + x)|x \in \mathbb{K}\} \cup \{(\infty)\} = \{(a, a', t)|t \in \mathbb{K}\} \cup (\infty).$$

The spheres with gnarl (∞) have the following coordinates:

$$\{a, a' + x', a'' + x'' + ax'|x', x'' \in \mathbb{K}\} \cup \{\infty\} = \{(a, t', t'')|t', t'' \in \mathbb{K}\} \cup \{\infty\}$$

We defined the Ree-Tits unital in the mixed hexagon $\mathsf{H}(\mathbb{K}, \mathbb{K}')$. In this hexagon, we can "see" the circles and spheres as follows: a circle is the set of absolute points at distance 3 from a non-absolute line M, not going through an absolute point, the unique absolute point for which its corresponding absolute line intersects M is the gnarl of the circle. A sphere is the set of absolute points not opposite some point p, with p lying on an absolute line. The unique absolute point at distance 2 from p is the gnarl of the sphere.





4.3.2 Derived structure at (∞)

We define the structure $\mathcal{G}' = (\mathcal{P}', \mathcal{B}', \mathtt{I})$ where $\mathcal{P}' = \mathcal{P} \setminus \{(\infty)\}$, and $\mathcal{B}'$ is the set of blocks of $\mathcal{G}$ going through (∞) minus (∞). In order to know the coordinates of the circles through (∞) we first write down the coordinates of the circles with gnarl (∞). As we saw earlier they look like this:

$$\{(a, a', t) | t \in \mathbb{K}\} \cup (\infty).$$

Removing the point infinity gives us the *vertical line* $L_{a,a'}$. We now compute the coordinates of the circle with gnarl $(0, 0, 0)$ through (∞). When this is done, its image under $(a, a', a'') \in U_\infty$ is the circle through (∞) with gnarl (a, a', a''). Let $(x, x', x'') \in U_\infty$, g and $(x, x', x'')_0 := (x, x', x'')^g$ be collineations of $\mathsf{PG}(6, \mathbb{K})$ as in 3.2.9, then we can write $(x, x', x'')_0 \in U_0 = U_{(0,0,0)}$ as follows :
$\bar{x} = (x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6) \mapsto$

$$\bar{x} \cdot \begin{pmatrix} 1 & f_2(x,x',x'') & f_3(x,x',x'') & x'' & f_1(x,x',x'') & -x' & -x \\ 0 & 1 & -x^\theta & 0 & x' - x^{1+\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 & 0 \\ 0 & -x & x' & 1 & -x'' & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x^2 & -x'' - xx' & x & p & 1 & 0 \\ 0 & r & s & -x' + x^{1+\theta} & q & x^\theta & 1 \end{pmatrix},$$

The point (∞) can be identified with $(1,0,0,0,0,0,0)$, so its orbit under $Z(U_0)$ is the set:

$$\begin{aligned}\{(1, f_2(0,0,x''), f_3(0,0,x''), x'', f_1(0,0,x''), 0,0)\} &= \{(1,0,-x''^{\theta}, x'', x''^2, 0, 0)\}\\ &= \{(x,0,-x^{2+\theta})\} \cup \{(\infty)\}\end{aligned}$$

with $x'' \in \mathbb{K}$ and x being $x''^{-2-\theta}$. The image of this circle under $(a,a',a'') \in U_\infty$ is the set

$$\{(a+x, a'+a^{\theta}x, a''+(a'-a^{1+\theta})x - x^{2+\theta}) | x \in \mathbb{K}\} \cup \{(\infty)\}.$$

Removing the point (∞) gives us the *ordinary line* $C_{(a,a',a'')}$

As for circles, we consider the spheres with gnarl (∞) and the other spheres through (∞) separately: The spheres with gnarl (∞) have coordinates:

$$\{(a,t',t'')|t',t'' \in \mathbb{K}\} \cup \{(\infty)\}.$$

Removing the point (∞) gives us the *vertical plane* P_a.

The orbit of (∞) under U_0' is the set:

$$\begin{aligned}&\{(1, f_2(0,x',x''), f_3(0,x',x''), x'', f_1(0,x',x''), -x', 0)\}\\ &\qquad = \{(1, x'^{\theta}, -x''^{\theta} + x'x'', x'', x''^2 + x'^{1+\theta}, -x', 0)\}\\ &\qquad = \{(\frac{x''^{\theta} - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^{\theta}}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}})\} \cup \{(\infty)\}\end{aligned}$$

with $x', x'' \in \mathbb{K}$ The image of this sphere under $(a,a',a'') \in U_\infty$ is the set

$$\{(\frac{x''^{\theta} - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^{\theta}}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}}) \cdot (a,a',a'')\} \cup \{(\infty)\}\}$$

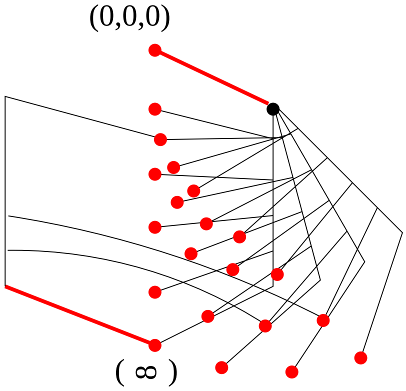
Removing the point infinity gives us the ordinary plane $S_{(a,a',a'')}$

Lemma 4.3.1. *A sphere contains only its obvious circles*

Proof. A sphere cannot contain a circle completely if the gnarl of the circle is not the gnarl of the sphere: Let us consider such a circle, because of the 2-transitivity of the Ree-Tits Moufang set we may assume without loss of generality that this circle's gnarl is (∞) while the gnarl of the sphere is $(0,0,0)$. Translating this in coordinates, there should exist $a, a' \in \mathbb{K}$ such that for every x, (a,a',x) can be written as

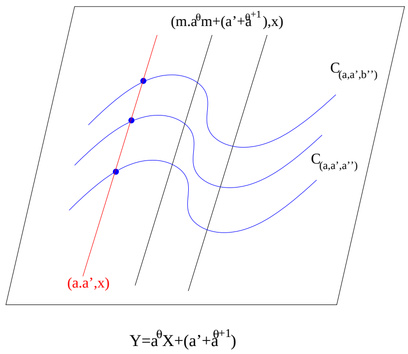
$$\left(\frac{x''^{\theta} - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^{\theta}}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}}\right)$$

with $x', x'' \in \mathbb{K}$. As an easy computation will reveal, this is not possible. □



4.3.3 Parallellism in this derived structure

First we remark that every ordinary line $C_{(a,a',a'')}$ lies completely in the affine plane with equation $Y = a^\theta X + (a' - a^{1+\theta})$. We say that two ordinary lines C_1 and C_2 are parallel if all vertical lines intersecting C_1 intersect C_2 — in that case the two ordinary lines lie in the same affine plane — or if there is no vertical line intersecting both ordinary lines — which implies that the ordinary lines lie in parallel, but disjoint, affine planes —.



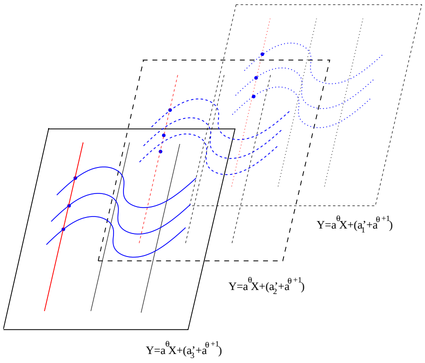
Two circles lying in the same projective plane

We can see that two ordinary lines $C_{(a,a',a'')}$ (coming from the circle through (∞) with gnarl (a, a', a'')) and $C_{(b,b',b'')}$ are parallel if and only if $a = b$. Indeed, a vertical line meeting the ordinary line $C_{(a,a',a'')}$ must lie in the affine plane $Y = a^\theta X + (a' - a^{1+\theta})$, so any vertical line meeting both

$C_{(a,a',a'')}$ and $C_{(b,b',b'')}$ must lie in the intersection of

$$\begin{cases} Y = a^\theta X + (a' - a^{1+\theta}) \\ Y = b^\theta X + (b' - b^{1+\theta}). \end{cases}$$

This has a unique solution if and only if $a \neq b$.



One class of parallel circles

4.4 Automorphism group of the Ree geometry

4.4.1 General idea

We consider an automorphism φ of the Ree geometry. Without loss of generality we may assume that φ fixes both (∞) and $(0, 0, 0)$. Our automorphism has to preserve the parallellism relation we just defined since a vertical plane cannot be mapped onto an ordinary plane. This condition translates algebraically into φ having the following action: $(x, y, z)^\varphi = (\ell x^\sigma, \ell^{1+\theta} y^\sigma, \ell^{2+\theta} z^\sigma)$. This action preserves the structure of the Ree-Tits Moufang set.

4.4.2 Proof

Lemma 4.4.1. *There is no automorphism of the Ree geometry Δ fixing (∞) and mapping some vertical line onto an ordinary line.*

Any automorphism of Δ maps spheres onto spheres and circles onto circles. If one sphere S_1 is mapped onto another sphere S_2, the gnarl of the first sphere has to be mapped onto the gnarl of the second. Indeed, the gnarl of

S_1 is exactly the intersection of any two circles lying completely in S_1. Those two circles are mapped onto two circles lying completely in S_2, hence their intersection is the gnarl of S_2. Now, any circle C of Δ lies completely in exactly one sphere. The gnarl of C is the gnarl of this corresponding sphere, so we obtain that gnarls of circles have to be mapped onto gnarls of circles. The lemma is proved.

The previous lemma now easily implies that φ must preserve the gnarls of the blocks of Δ. Since the Ree group acts doubly transitively on the points of Δ, we may also assume that φ fixes the points (∞) and $(0,0,0)$. Consequently, φ fixes the set of vertical lines. Therefore the points (a,a',z_1) and (a,a',z_2) are mapped on the same vertical line. If we represent φ as follows:

$$\varphi : (x,y,z) \mapsto (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$$

then both g_1 and g_2 have to be independent of z.

The mapping φ preserves the parallellism relation between ordinary lines, since the number of vertical lines meeting two circles (i.e. none, one or all) is preserved under φ. This translates to g_1 being independent of y. Indeed, two points (a,y_1,z_1) and (a,y_2,z_2) being the gnarls of two parallel ordinary lines are mapped onto two gnarls of parallel ordinary lines, which implies that $g_1(a,y_1) = g_1(a,y_2)$ for every choice for y_1 and y_2.

The point $(0,0,0)$ is fixed by φ, so the affine plane $Y = 0$ — which is the unique affine plane containing both $C_{(0,0,0)}$ and $L_{(0,0)}$— is fixed by φ. The plane $Y = c_1$ gets mapped onto the plane $Y = c_2$, so $g_2(x,c_1) = g_2(0,c_1)$ for every choice for x.

It follows that there are two permutations α and β of $\mathbb{K}$ such that $(x,y,z)^\varphi = (x^\alpha, y^\beta, g_3(x,y,z))$. Since φ preserves gnarls, it maps the ordinary line $C_{(a,b,c)}$ onto the ordinary line $C_{(a^\alpha,b^\beta,g_3(a,b,c))}$. Now notice that the point (x,y,z) can only be contained in the ordinary line $C_{(a,b,c)}$ if $y = b + a^\theta(x-a)$. Expressing that the point $(a+x,y,z)$ lies on the circle $C_{(a,b,c)}$ if and only if its image under β lies in $C_{(a,b,c)\beta}$ now shows that, for all $a,b,x \in \mathbb{K}$,

$$(b + a^\theta x)^\beta = b^\beta + (a^\alpha)^\theta((x+a)^\alpha - a^\alpha). \tag{4.1}$$

The permutation β is clearly additive:$(b + a^\theta x)^\beta = b^\beta + (a^\theta x)^\beta$. Put $\ell = 1^\alpha$. Then we see, by setting $a = 1$ and $b = 0$ in the equation (4.1) above, that

$$x^\beta = \ell^\theta((x+1)^\alpha - 1^\alpha), \tag{4.2}$$

so α is additive if and only if $(x+1)^\alpha = x^\alpha + 1^\alpha$. Plugging in $x = m-1$ in (4.2) we have that $(m-1)^\beta = \ell^\theta(m^\alpha - 1^\alpha)$. Because of the additivity of β we have on the other side that $(m-1)^\beta = m^\beta + (-1)^\beta = \ell^\theta((1+m)^\alpha - 2\cdot 1^\alpha)$, so α is additive as well.

We now have that $x^\beta = \ell^\theta x^\alpha$, $\forall x \in \mathbb{K}$. We can define the bijection $\sigma : \mathbb{K} \to \ell^{-1}\mathbb{K} : y \mapsto y^\sigma = \ell^{-1}y^\alpha$ (note that $1^\sigma = 1$). Plugging in these identities in equation (4.1) yields

$$(b + a^\theta x)^\sigma = b^\sigma + (a^\sigma)^\theta x^\sigma,$$

for all $a, b, x \in \mathbb{K}$. Putting $a = 1$, we see that σ is additive; putting $b = 0$ and $x = 1$, we see that σ commutes with θ. Putting $b = 0$, we see that $(xy)^\sigma = x^\sigma y^\sigma$ for $x \in \mathbb{K}^\theta$ and $y \in \mathbb{K}$. We may view σ as an automorphism of $\mathbb{K}$ by defining $x^\sigma = t$ if and only if $(x^\theta)^\sigma = t^\theta$ (and this is well defined and agrees on $\mathbb{K}$ since θ is injective on $\mathbb{K}^\theta$). Now the action of φ on a point $(x, y, \cdot)$ is given by $(x, y, \cdot)^\varphi = (\ell x^\sigma, \ell^{1+\theta} y^\sigma, \cdot)$, for all $x, y \in \mathbb{K}$.

Let us now investigate what $g_3(x, y, z)$ looks like.

The point p with coordinates $(a - \frac{a'}{a^\theta}, 0, a'' + (a' - a^{1+\theta})(\frac{-a'}{a^\theta}) - (\frac{-a'}{a^\theta})^{2+\theta})$ lies on both $C_{(a,a',a'')}$ and on the circle with gnarl $(0, 0, a'' + \frac{(a^{1+\theta}-a')^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}})$, so its image under φ lies on the circle with gnarl $(\ell a^\sigma, \ell^{1+\theta}a'^\sigma, g_3(a, a', a''))$ and on the circle with gnarl $(0, 0, g_3(0, 0, a'' + \frac{(a^{1+\theta}-a')^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}}))$ which leads to

$$\begin{cases} g_3(a,a',a'') = g_3(a - \frac{a'}{a^\theta}, 0, a'' - \frac{(a'-a^{1+\theta})a'}{a^\theta} + (\frac{a'}{a^\theta})^{2+\theta}) + \ell^{2+\theta}(\frac{a'^2}{a^\theta} - aa' - \frac{a'^{2+\theta}}{a^{3+2\theta}})^\sigma \\ g_3(a - \frac{a'}{a^\theta}, 0, a'' - \frac{(a'-a^{1+\theta})a'}{a^\theta} + (\frac{a'}{a^\theta})^{2+\theta}) = g_3(0,0,a'' + \frac{(a^{1+\theta}-a')^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}}) - (\ell(a - \frac{a'}{a^\theta})^\sigma)^{2+\theta} \end{cases}$$

Putting these two equations together we get:

$$g_3(a,a',a'') = g_3(0,0,a''+\frac{(a'-a^{1+\theta})^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}}) - \ell^{2+\theta}\left(\frac{(a'-a^{1+\theta})^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}}\right)^\sigma.$$

for every $a \in \mathbb{K}\backslash\{0\}$ and $a' \in \mathbb{K}$. The point $(0, a', a'')$ lies on every circle with gnarl $(A, a', a'' + a'A - A^{2+\theta})$, this implies for every choice of A:

$$\begin{cases} g_3(0,a',a'') = g_3(0,0,a'' - a'^\theta \frac{a'+A^{1+\theta}}{A^{2+\theta}}) - \ell^{2+\theta}(-a'^\theta \frac{a'+A^{1+\theta}}{A^{2+\theta}})^\sigma, \\ g_3(0,a',a'') = g_3(0,0,a'' + a'^\theta \frac{a'+A^{1+\theta}}{A^{2+\theta}}) - \ell^{2+\theta}(a'^\theta \frac{a'+A^{1+\theta}}{A^{2+\theta}})^\sigma. \end{cases}$$

The second equation arises if we use $-A$ instead of A.

Subtracting both previous equations, we get:

$$g_3(0,0,a'' - a'^\theta\frac{a'+A^{1+\theta}}{A^{2+\theta}}) - g_3(0,0,a'' + a'^\theta\frac{a'+A^{1+\theta}}{A^{2+\theta}}) = \ell^{2+\theta}(a'^\theta\frac{a'+A^{1+\theta}}{A^{2+\theta}})^\sigma$$

Setting $a' = (B^{2-\theta})^{1+\theta}$ and $A = B^{2-\theta}$ we have

$$g_3(0,0,a''+B) - g_3(0,0,a''-B) = \ell^{2+\theta}(-B)^{\sigma}.$$

Finally, we can set $B = a''$, for a general a'', and this implies

$$\begin{cases} g_3(0,0,-a'') = \ell^{2+\theta}(-a'')^{\sigma}, & \text{and} \\ g_3(a,a',a'') = \ell^{2+\theta}a''^{\sigma}. \end{cases}$$

Now the action of φ on a point (x,y,z) is given by $(x,y,z)^{\varphi} = (\ell x^{\sigma}, \ell^{1+\theta}y^{\sigma}, \ell^{2+\theta}z^{\sigma})$.

This action preserves the original Ree-Tits Moufang set The proof is complete. □

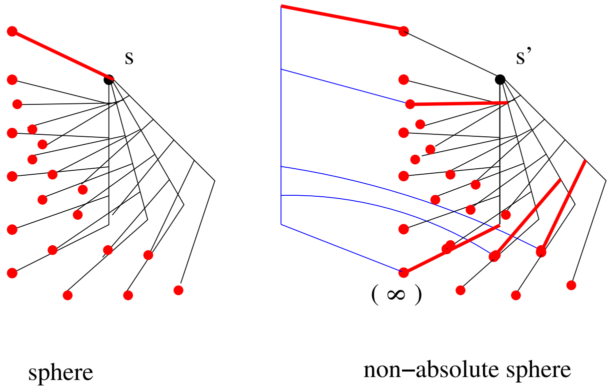
Remark Recently, K. Struyve proved that the geometry defined Moufang set arising from the polarity of the F_4-quadrangle — the other known Moufang set whose root groups have nilpotency class 3— has an automorphism group lying completely in the automorphism group of the corresponding Moufang set.

4.5 Absolute points and lines of some polarity in the Ree hexagon

Corollary 4.5.1. *If an collineation σ of a Moufang hexagon fixes all absolute points of some polarity, it fixes all absolute lines as well.*

Proof. By Theorem 7.3.4 and Theorem 7.7.2 of [41], any polarity ρ of a Moufang hexagon is associated to a Ree group, so it is an automorphism of the Ree hexagon. Now, the explicit form of the set of absolute points of a polarity on page 343 of [41] implies very easily that the circle with gnarl (∞) containing $(0,0,0)$ is precisely the set of absolute points at distance 3 (in the incidence graph) from the (unique) line L of the hexagon at distance 2 from $(\infty)^{\rho}$ and 3 from $(0,0,0)$. Hence, it is now easy to see that the set of circles coincides with the sets of absolute points at distance 3 from a given line of the hexagon. We say that this given line corresponds in the hexagon to our circle. Likewise, the set of spheres coincides with the sets of absolute points at distance 4 or less from a given point, contained in some absolute line. This given point is referred to as the *special point*.

We want to show now that every automorphism σ of the little projective group of Γ stabilizing the Ree-Tits ovoid also stabilises its image under the polarity ρ. If this were not the case, we would have an automorphism σ of the little projective group of Γ mapping spheres onto *non-absolute spheres*: sets of absolute points at distance 4 or less from a given (special) point p, but now with p not contained in some absolute line. The (absolute) spheres and the non-absolute spheres however do not have the same geometric properties:

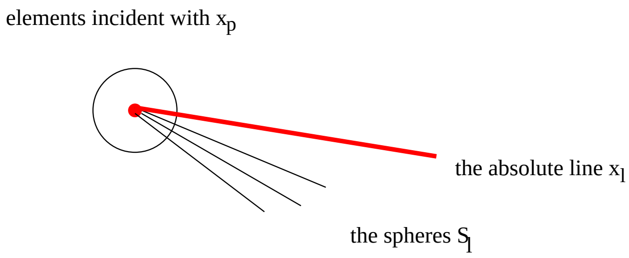


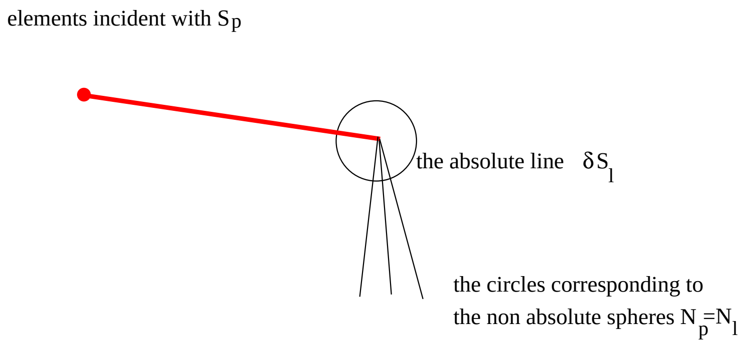
A sphere, on the one hand, contains only its obvious circles: the circles corresponding to lines in the hexagon going through the special point. A non-absolute sphere, on the other hand, contains not only circles corresponding to lines through the special point; at least one other circle also lies completely in this set. The circles corresponding to lines through the special point all have different gnarls, the absolute lines corresponding to these gnarls lie at distance 3 from the special point. The image of our special point is now a non-absolute line at distance 3 from all these gnarls and at distance 3 from the special point, hence it corresponds to a circle lying completely in our non-absolute sphere without going through the special point.

□

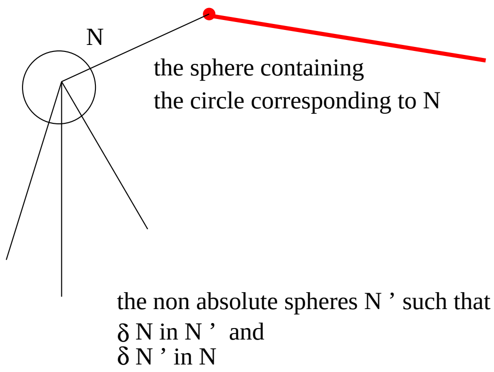
4.6 Construction of the generalized hexagon out of the Ree geometry

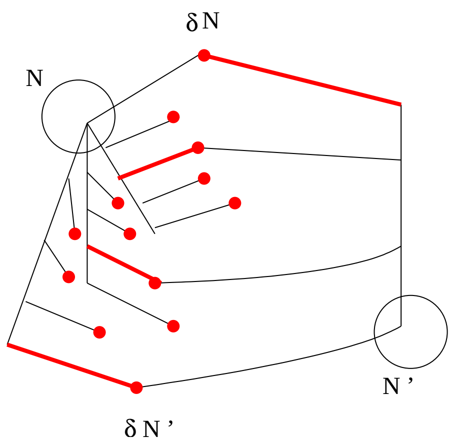
We already defined points, circles and spheres in the Ree geometry, the *non-absolute spheres* can be defined as follows: for the unique (absolute) sphere with special point x going through y, we consider the unique circle C with gnarl x going through y. For every point $p \neq x$ on this circle, the circle with gnarl p going through x is completely contained in the new sphere, and these are all the points of the new sphere. Note that these circles form a set of parallel ordinary lines in the derived structure at x, for which there are vertical lines intersecting all these circles. The circle C is the circle *corresponding* to the non-absolute sphere. We will denote the gnarl of a sphere S with ∂S, the special point of a non-absolute sphere gets the same notation: ∂N. We can now construct the Ree hexagon. We define the following geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. Both the point set $\mathcal{P}$ and the line set $\mathcal{L}$ of Γ are the union of the points, the spheres and the new spheres of the Ree Geometry. Hence if x is a point of the Ree-Tits ovoid, then x can be viewed as a point or as a line of Γ. To distinguish them we write x_p or x_l if we view x as an element of $\mathcal{P}$ or $\mathcal{L}$, respectively. The same holds for spheres and non-absolute spheres. We now define incidence in Γ. A point x_p, with x a point of the Ree geometry is incident with a line y_l, y also being a point of the Ree geometry, if and only if $x = y$. A point x_p (line x_l), x a point of the Ree geometry, is incident with a line S_l (point S_p), S being a sphere, if and only if $\partial S = x$. A point x_p (line x_l) is never incident with a line N_l (point N_p), if x is a point and N is a non-absolute sphere of the Ree geometry. A point S_p is never incident with a line S'_l for two spheres S, S'. A point S_p (line S_l) is incident with a line N_l (point N_p), for a sphere S and a non-absolute sphere N if and only if the circle corresponding to N lies completely in S. A point N_p is incident with a line N'_l, for N, N' new spheres of the Ree geometry if and only if the circle corresponding to N' lies completely in N such that $\partial N \in N'$, $\partial N' \in N$ and $\partial N \neq \partial N'$.


elements incident with x_p
the absolute line x_l
the spheres S_l


elements incident with S_p
the absolute line δS_l
the circles corresponding to
the non absolute spheres $N_p=N_l$

elements incident with N_p





Appendix A

Appendix

A.1 Embedding of the classical hexagons

The points of $Q(7,\mathbb{K})$ can be viewed as 8-tuples $(x_0, x_1, \ldots, x_7)$, up to a scalar multiple, with elements in $\mathbb{K}$ and satisfying the relation

$$x_0x_4 + x_1x_5 + x_2x_6 + x_3x_7 = 0$$

Let V denote the projective space $PG(7,\mathbb{K})$. The trilinear form $\mathcal{T} : V \times V \times V \to \mathbb{K}$ with the following explicit description:

$$\begin{aligned}\mathcal{T}(x,y,z) = &\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} + \begin{vmatrix} x_4 & x_5 & x_6 \\ y_4 & y_5 & y_6 \\ z_4 & z_5 & z_6 \end{vmatrix} \\ &+ x_3(z_0y_4 + z_1y_5 + z_2y_6) + x_7(y_0z_4 + y_1z_5 + y_2z_6) \\ &+ y_3(x_0z_4 + x_1z_5 + x_2z_6) + y_7(z_0x_4 + z_1x_5 + z_2x_6) \\ &+ z_3(y_0x_4 + y_1x_5 + y_2x_6) + z_7(x_0y_4 + x_1y_5 + x_2y_6) \\ &- x_3y_3z_3 - x_7y_7z_7\end{aligned}$$

gives us an expression to decide whether a pair of points of $Q(7,\mathbb{K})$ represents an incident (0-point,1-point)-pair in $\Omega(\mathbb{K})$. This is the case if and only if the linear form $T(x, y, z')$ is identically zero in z', and similarly for any cyclic permutation of the letters x, y, z. So if we want to know which 1-point is associated with $y = (y_0, y_1, \ldots, y_7)$ we just have to look for all the 0-points incident with y (this will be a 3-space).

Now let σ be an automorphism of $\mathbb{K}$ of order 1 or 3, then the map

$$\tau_\sigma : \mathcal{P}^{(i)} \to \mathcal{P}^{(i+1)} : (x_j)_{j\in J} \mapsto (x_j^\sigma)_{j\in J}$$

$i = 0, 1, 2 \bmod 3$, $J = \{0, 1, \ldots, 7\}$ preserves incidence in $\Omega(\mathbb{K})$. It gives rise to a twisted triality hexagon if the order of σ is 3 and to a split-Cayley hexagon if σ represents the identity.

The points of the twisted triality hexagon are:

Coordinates in $\mathsf{T}(\mathbb{K}', \sigma)$	Coordinates in $PG(7,\mathbb{K}')$
(∞)	$(1, 0, 0, 0, 0, 0, 0, 0)$
(a)	$(a, 0, 0, 0, 0, 0, 1, 0)$
(k, b)	$(b, 0, 0, 0, 0, 1, -k, 0)$
(a, l, a')	$(-l - aa', 1, 0, a^\sigma, 0, a^{\sigma+\sigma^2}, -a', -a^{\sigma^2})$
(k, b, k', b')	$(k' + bb', k, 1, -b^\sigma, 0, b', b^{\sigma+\sigma^2} - b'k, b^{\sigma^2})$
(a, l, a', l', a'')	$(-al' + a'^{\sigma+\sigma^2} + a''l + aa'a'', -a'', -a, a'^{\sigma^2} - a^\sigma a'',$ $1, l + (aa')^\sigma + (aa')^{\sigma^2} - a^{\sigma+\sigma^2}a'', -l' + a'a'', a^{\sigma^2}a'' - a'^\sigma)$

The points of the split Cayley hexagon hexagon all lie in the hyperplane of $PG(7,\mathbb{K})$ with equation $x_3+x_7=0$. That is why we can embed this hexagon in $PG(6,\mathbb{K})$, Setting $\alpha = -al' + a'^2 + a''l + aa'a''$ and $\beta = l + 2aa' - a^2a''$, we obtain:

Coordinates in $\mathsf{H}(\mathbb{K})$	Coordinates in $PG(6,\mathbb{K})$
(∞)	$(1,0,0,0,0,0,0)$
(a)	$(a,0,0,0,0,0,1)$
(k,b)	$(b,0,0,0,0,1,-k)$
(a,l,a')	$(-l-aa',1,0,-a,0,a^2,-a')$
(k,b,k',b')	$(k'+bb',k,1,b,0,b',b^2-b'k)$
(a,l,a',l',a'')	$(\alpha,-a'',-a,-a'+aa'',1,\beta,-l'+a'a'')$

The Ree hexagon can be described as follows: Let $\mathbb{K}$ be a field of characteristic 3 and let $\mathbb{K}'$ be a subfield such that $\mathbb{K}^3 \leq \mathbb{K}' \leq \mathbb{K}$. If we restrict in the coordinatization of $H(\mathbb{K})$ the set R_2 to $\mathbb{K}'$, then all operations are still well defined, and we obtain a subhexagon $\mathsf{H}(\mathbb{K},\mathbb{K}')$, which we call a Ree hexagon. The coordinates are the same as for the split Cayley hexagon, but now k, k', l and l' must lie in the subfield $\mathbb{K}'$.

Appendix B

Nederlandse Samenvatting

B.1 Nederlandse samenvatting

B.1.1 Definities

Veralgemeende Veelhoeken

Een *meetkunde* van rang 2 is een drietal $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ met $\mathcal{P}$ en $\mathcal{L}$ niet lege, disjuncte verzamelingen, en waarbij de incidentierelatie $\mathtt{I} \subseteq \mathcal{P} \times \mathcal{L}$ een relatie voorstelt waarbij elk element van $\mathcal{P} \cup \mathcal{L}$ incident is met minstens 1 element van $\mathcal{P} \cup \mathcal{L}$. De elementen van $\mathcal{P}$ noemen we *punten* terwijl $\mathcal{L}$ rechten bevat. Een *gewone veelhoek*, bijvoorbeeld, is een meetkunde bestaande uit n verschillende punten x_{2i} en n verschillende rechten x_{2i+1}, $i \in \{0, \ldots, n-1\}$ zodat $x_{2i-1}\mathtt{I}x_{2i}\mathtt{I}x_{2i+1}$ voor $i \in \{1, \ldots, n-1\}$ en $x_{2n-1}\mathtt{I}x_0\mathtt{I}x_1$.

Een *veralgemeende n-hoek* (voor n een natuurlijk getal groter dan 1) is een meetkunde $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ die voldoet aan de volgende drie axioma's:

VV_1 Γ bevat geen gewone k-hoeken (als deelmeetkunde) voor $2 \leq k < n$,

VV_2 Elke 2 elementen x en y in $\mathcal{P} \cup \mathcal{L}$ zijn bevat in een gewone n-hoek in Γ. Zo'n gewone n-hoek noemen we een appartement van Γ.

VV_3 Er bestaat een gewone $(n+1)$-hoek, als deelmeetkunde van Γ

We zeggen dat een eindige veralgemeende veelhoek de orde (s, t) heeft als er op elke rechte $s + 1$ punten liggen en er door elk punt $t + 1$ rechten gaan. In deze thesis worden veralgemeende vierhoeken en veralgemeende zeshoeken onderzocht: alles wat er te weten valt over *eindige* veralgemeende vierhoeken staat in [18], hoewel daar een licht andere definitie gegeven wordt voor een veralgemeende vierhoek, komt die overeen met onze gehanteerde definitie in het geval dat er meer dan 2 rechten door een punt gaan en er meer dan 2 punten op een rechte liggen. Voor veralgemeende zeshoeken beperken we ons tot een klassiek geval, we hanteren hier ook de coördinatizatie van de Ree-zeshoek; de zelfduale deelzeshoek van de Split-Cayley zeshoek.

Niet elke $n \in \setminus\{0, 1\}$ laat een eindige veralgemeende n-hoek toe; Feit en Higman bewezen in [9] dat een eindige n-hoek van orde (s, t) (s en t verschillend van 1) enkel bestaan als

- $n = 2$,
- $n = 3$ en $s = t$,
- $n = 4$,
- $n = 6$ met st een kwadraat,
- $n = 8$ met $2st$ een kwadraat.

Moufang veralgemeende veelhoeken

Voor elke natuurlijke $n > 1$ zijn er veralgemeende n-hoeken als we niet eisen dat de orde eindig is. We kunnen echter wel de oneindige veralgemeende veelhoeken classificeren als we een transitiviteitsvoorwaarde opleggen, dat is dan de *Moufang voorwaarde*. We beschouwen daarvoor een pad $\gamma = (x_1, \ldots, x_{n-1})$ van $n-2$ elementen van $\mathcal{P} \cup \mathcal{L}$ waarvoor $x_i \mathtt{I} x_{i+1}$ voor $i \in \{1, \ldots, n-2\}$ samen met de verzameling van collineaties van Γ die elk element incident met minstens 1 element van γ fixeren. Als diens actie op de appartementen door γ transitief is noemen we γ een *Moufang pad*. Als alle paden van lengte $n-2$ Moufang zijn, dan voldoet de veralgemeende veelhoek aan de Moufang voorwaarde. Tits en Weiss schreven in 2002 de classificatie van de Moufang veelhoeken neer in [40]. Daarin staat dat een Moufang veralgemeende n-hoek Γ slechts bestaat als

- $n = 3$ en $\Gamma \cong \mathcal{T}(A)$ is projectief vlak gedefinieerd over een alternatieve delingsring A.

- $n = 4$ en Γ is van involutorisch, kwadratisch , indifferentieel of pseudokwadratisch type ofwel is het van type E_6, E_7, E_8 of F_4.
- $n = 6$ en $\Gamma \cong \mathcal{H}(J, F, \#)$ voor een hexagonaal systeem $(J, F, \#)$.
- $n = 8$ en $\Gamma \cong O(\mathbb{K}, \sigma)$ voor een octagonale verzameling $(\mathbb{K}, \sigma)$.

De Moufang verzamelingen waarover deze veralgemeende veelhoeken gedefinieerd zijn worden beschreven in hoofdstuk 4 van deze thesis, in hoofdstuk 1 staat heel kort beschreven wat we bedoelen met die soorten veelhoeken. Voor meer informatie over die classificatie verwijzen we —uiteraard— naar [40] zelf.

Moufang verzamelingen

Moufang verzamelingen zijn gegroeid uit de gebouwentheorie; de residu's afkomstig van Moufang gebouwen van hogere rang vormen een verzameling waarop een familie van unipotente groepen werkt. Die groepen voldoen aan welbepaalde transitiviteitseigenschappen waardoor de Moufang set een zekere structuur in zich heeft.

Een *Moufang verzameling* is een systeem $\mathcal{M} = (X, (U_x)_{x \in X})$ bestaande uit een verzameling X en een familie van permutatiegroepen (we noteren de actie van een permutatie op een punt exponentieel, zodat de samenstelling van links naar rechts wordt toegepast) van X geïndexeerd door X zelf die voldoet aan de volgende voorwaarden:

MS_1 U_x fixeert $x \in X$ and werkt scherp transitief op $X \backslash \{x\}$.

MS_2 In de volledige permutatiegroup over X normalizeert elke U_x de verzameling van deelgroepen $\{U_y | y \in X\}$.

De groepen U_x noemen we *wortelgroepen*. The elementen van U_x noemen we dan *wortel elaties*. Als U_x een commutatieve groep is voor een $x \in X$, dan is die commutatief voor alle $x \in X$ en spreken we van een *translatie Moufang verzameling*.

De groep S voortgebracht door de groepen U_x, voor alle $x \in X$, noemen we de *kleine projectieve groep* van de Moufang verzameling. Een permutatie van X dat de verzameling van deelgroepen $\{U_y \parallel y \in X\}$ normalizeert, is een *automorfisme* van de Moufang verzameling. De verzameling van alle automorfismes van de Moufang verzameling is een groep G, die we de *volledige projectieve groep* van de Moufang verzameling noemen. Elke groep H, met $S \leq H \leq G$, noemen we een *projectieve group* van de Moufang verzameling.

De Moufang verzameling worden in het tweede hoofdstuk beschouwd als residu's van veralgemeende vierhoeken. Doordat er maar 2 reguliere acties op een verzameling bestaan, kunnen we afleiden dat de veralgemeende vierhoek Moufang moet zijn als hij aan een (ogenschijnlijk) minder strenge transitiviteitsvoorwaarde voldoet. Moufang verzameling worden soms ook gespleten BN-paren van rang 1 genoemd, dit houdt in dat er een group U bestaat die nilpotent is en waarvoor $U \cdot H = B$, met $H = B \cap N$, B is hier de stabilizator van een punt van X (een vlag van het gebouw van rang 1) en N fixeert twee punten (een appartement van het gebouw). In het derde hoofdstuk tonen we aan dat die splijting uniek is voor de Moufang verzameling die we reeds kennen: zij die afkomstig zijn van wortelgroepen uit gebouwen van hogere rang en de Suzuki-Tits en Ree-Tits Moufang verzameling. Uit Moufang sets vallen er ook nieuwe meetkundes te construeren (als ze tenminste niet abels zijn); ze werden geconstrueerd door Tits in 1996 [38]. Tits vroeg zich dan af of de automorfismegroep van deze meetkundige structuur bevat was in de automorfismegroep van de overeenkomstige algebraïsche groep, m.a.w. bevat deze nieuwe meetkunde genoeg informatie om de oorspronkelijke algebraïsche groep terug te reconstrueren? Dit kon hijzelf aantonen voor bepaalde klassen van algebraïsche groepen. In hoofdstuk 4 wordt aangetoond dat de Ree-Tits meetkunde een automorfismegroep heeft die volledig bevat is in de automorfismegroep van de veralgemeende zeshoek waarop hij leeft.

B.1.2 Moufang voorwaarden voor veralgemeende vierhoeken

We weten reeds dat als een veralgemeende veelhoek aan de Moufang voorwaarde voldoet, de verzameling van γ-elaties, voor ieder pad γ van lengte $n-2$, transitief werkt op de appartementen door γ, Deze voorwaarde kan op verschillende manieren afgezwakt worden, zo kunnen we de verzameling van γ-elaties beschouwen voor ieder pad γ van lengte $i-2$, als die transitief blijkt te werken op de appartementen door γ spreken we van een een veralgemeende vierhoek die *i-Moufang* is. Voor paden van even lengte kunnen we eigenlijk 2 soorten beschouwen: zij die als uiterste elementen punten hebben en zij die als uiterste elementen rechten hebben. Als de γ-elaties transitief werken voor slechts 1 soort pad met lengte $i-2$, spreken we van een *half i-Moufang vierhoek*

Ook bestaan er *half* 3-*Moufang* veralgemeende vierhoeken, ze vertegenwoordigen de grootste gemene deler van half (4-)Moufang vierhoeken en 3-Moufang veralgemeende vierhoeken. Beschouw een vlag $\{x, L\}$ van de veralgemeende vierhoek $\mathcal{S}$. Beschouw een willekeurig punt z collineair met x (dat niet op L ligt) en noem de rechte die x met z verbindt M. Beschouw analoog een rechte K concurrent met L die niet door x gaat en noem het snijpunt $y = K \cap L$. De groep $G^{[1]}_{x,L} \cap G_z$ werkt semi-regulier op de verzameling van appartementen die $\{x, y, z, L, M\}$ bevatten, net zoals de groep $G^{[1]}_{[x,L]} \cap G_K$ semi-regulier werkt op de appartementen door $\{x, y, L, M, K\}$. Wanneer voor elke keuze voor $z \sim x$, de groep $G^{[1]}_{x,L} \cap G_z$ transitief blijkt te werken op de verzameling appartementen door $\{x, y, z, L, M\}$, zeggen we dat de vlag $\{x, L\}$ *half* 3-*Moufang ten opzichte van* x is, terwijl een transitieve actie of $G^{[1]}_{[x,L]} \cap G_K$ op de verzameling appartementen door $\{x, y, K, L, M\}$ voor elke keuze voor K concurrent met L, van $\mathcal{S}$ een veralgemeende vierhoek maakt die *half* 3-*Moufang ten opzichte van* L is. De veralgemeende vierhoek $\mathcal{S}$ noemen we *half* 3-*Moufang* als ofwel elke vlag $\{x, L\}$ van $\mathcal{S}$ half 3-Moufang is ten opzichte van x, ofwel als elke vlag $\{x, L\}$ van $\mathcal{S}$ half 3-Moufang is ten opzichte van L.

De meest natuurlijke vraag luidt nu: Zijn deze afzwakkingen van de Moufang voorwaarde wel echte afzwakkingen, of zijn ze er eigenlijk mee equivalent? Uiteindelijk konden we hetvolgende aantonen.

Stelling B.1.1. *Alle half 3-Moufang vierhoeken voldoen aan de Moufang voorwaarde*

In grote lijnen ziet het bewijs er als volgt uit. Als alle vlaggen in $\mathcal{S}$ Moufang zijn ten opzichte van hun punt, kunnen we een collineatie g construeren die de wortel $(x_7, x_0, x_1, x_2, x_3)$ fixeert, alle punten op x_1 vasthoudt en het standaardappartement $\Sigma = (x_0, x_1, \ldots, x_7, x_0)$ afbeeldt op een willekeurig appartement door deze wortel. Beschouw een willekeurig punt x op x_1, $x \neq x_0$ en kies een punt y collineair met x dat niet op x_1 ligt. We stellen dat de actie van $G^{[1]}_{x_1,x} \cap G_y$ op de rechten door x_0 onafhankelijk is van onze keuzes voor x and y. Dit geeft ruimte om een collineatie g' te construeren die alle elementen incident met x_0, x_1, x_2 fixeert. Met andere woorden, de helft van de paden van lengte 2 is Moufang en we hebben een half Moufang vierhoek. Aangezien zo'n half Moufang vierhoek vlaggen bezit die Moufang zijn ten opzichte van hun rechten, kunnen we een duale redenering toepassen om te besluiten dat een half 3 Moufang vierhoek aan de Moufang voorwaarde moeten voldoen.

Opmerking Ondertussen heeft K. Tent in [26] aangetoond dat veralgemeende n-hoeken die 2-Moufang zijn moeten voldoen aan de Moufang voorwaarde voor $n \leq 6$.

Ook hebben K. Thas en H. Van Maldeghem in [31] de equivalentie van de half-2 Moufang voorwaarde met de Moufang voorwaarde aangetoond voor *eindige* veralgemeende vierhoeken.

B.2 De uniciteit van de splijting van eindige BN-paren

B.2.1 BN-paren

Een *BN-paar* in een groep G is een systeem (B, N) bestaande uit twee deelgroepen van G zodat

BN_0 B and N de volledige groep G voortbrengen

BN_1 $B \cap N = H \trianglelefteq N$

BN_2 de groep $W = N/H$ een verzameling S van generatoren bevat zodat aan de volgende relaties voldaan is voor elke $s \in S$ en voor elke $w \in W$

BN_2' $sBwB \subseteq BwB \cup BswB$

BN_2'' $sBs \nsubseteq B$

The groep W heet de *Weyl groep* van het BN-paar. Een BN-pair is *gespleten* wanneer er een deelgroep U van B bestaat die nilpotent is en zodat $U \cdot H = B$.

In een veralgemeende veelhoek is B de stabilizator van een vlag, N staat dan voor de stabilisator van een (willekeurige) appartement door die vlag, dus moet H de puntsgewijze stabilisator van dit appartement zijn. De Weyl groep is dan de volledige automorfisme groep van dit appartement gezien als een gewone veelhoek .

We weten reeds (dank zij K.Tent [24]) dat de wortelgroepen van een Moufang set nilpotent zijn, dus kennen we al zeker 1 splijting van onze Moufang set. Nu is het de vraag of er andere zijn.

B.2.2 Splijtingen van BN-paren

Stelling B.2.1. *Voor alle Moufang sets waar we weet van hebben zijn de mogelijke splijtingen gekend, in slechts 1 geval is de splijting niet uniek.*

Een schets van ons bewijs ziet er als volgt uit: We onderzoeken de actie van een automorfisme φ in een transitieve nilpotente deelgroep U_∞ van de stabilizator van (∞) in een projectieve groep van elke beschreven Moufang set. Aangezien U_∞^+ regulier werkt op $X \setminus \{\infty\}$, mogen we aannemen dat dit automorfisme zowel 0 als ∞ fixeert. Het feit dat U_∞ nilpotent is verzekert ons van het feit dat het centrum van U_∞ niet triviaal is, maar aangezien φ het punt 0 fixeert, moet het ook diens baan onder het centrum van U_∞ fixeren, en ons doel is nu om aan te tonen dat $Z(U_\infty)$ groot genoeg is, zodat φ de identiteit moet zijn.

We maken een onderscheid tussen translatie Moufang sets (waarvoor er altijd een deel-Moufang set bestaat die isomorf is met een projectieve rechte of een polaire rechte) en de andere Moufang sets.

Voor 1 welbepaald geval is de groep U_∞^+ niet de unieke transitieve nilpotente groep, namelijk voor de kleinst mogelijke hermitische Moufang set. De Moufang set is dan de hermitische kromme in $PG(2,4)$, bestaande uit 9 punten, samen met zijn automorfismegroep, de groep U_∞^+ werkt regulier op de punten van deze kromme en heeft dus orde acht, het veldautomorfisme σ van $GF(4)$ induceert een automorfisme van de hermitische kromme, de groep U voortgebracht door U_∞^+ en σ is dan nilpotent en $U \cdot H = B$.

Voor Moufang veelhoeken en Moufang gebouwen met rang groter dan 2 kunnen we dankzij de uniciteit van de splijtingen van Moufang verzameling aantonen dat ook zij een unieke splijting bevatten.

B.3 Ree-Tits Moufang Verzamelingen

De *Ree groepen* werden ontdekt door Ree [19]; hij definieerde die echter over eindige velden die perfect moesten zijn. In [35], gaf Tits een constructie die ook werkte voor niet-perfecte eindige velden zolang die maar een *Tits endomorfisme* toelieten. Een andere constructie van Tits (die niet publiceerd werd, maar die wel beschreven staat in Sectie 7.7 van [41]) gebruikt bepaalde Moufang veralgemeende zeshoeken van gemixte type (we noemen die zeshoeken de *Ree zeshoeken* die gedefinieerd worden over velden met karakteristiek 3. De Ree groepen komen dan tevoorschijn als (simpele deelgroepen van de) centralizatoren van polariteiten in deze zeshoeken. We zien de Ree-Tits ovoïde en zijn automorfismengroep die ingebed is in de Ree zeshoek dan als een permutatievoorstelling van de Ree-Tits Moufang set; deze Moufang set is nilpotent van lengte 3, dus definieert het een meetkunde met dimensie 2. Dit betekent dat we twee soorten blokken in onze meetkunde hebben, de ene soort zal een verfijning zijn van de blokken van het andere type.

Vertrekkend van de Moufang verzameling $(X, (U_x)_{x\in X})$ is het duidelijk dat X puntenverzameling $\mathcal{P}$ van de Ree meetkunde $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ definieert, de cirkels komen dan tevoorschijn als banen van een punt y onder het centrum $Z(U_x)$ voor een punt $x \in \mathcal{P}$ samen met dat punt x; dit punt x noemen we dan de *gnarl* van die cirkel. Op die manier definieert elk punt-gnarl koppel op unieke wijze een cirkel. De sferen zijn opnieuw banen van y maar nu onder de groep $[U_x, U_x]$. Het punt x is dan de gnarl van de sfeer. De cirkels en de sferen vormen samen de blokkenverzameling $\mathcal{B}$ van $\mathcal{G}$.

Stelling B.3.1. *De automorfismegroep van de Ree meetkunde* $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$ *is volledig bevat in de automorfismegroep van de veralgemeende Ree zeshoek.*

B.3.1 Idee achter het bewijs

We beschouwen een willekeurige afbeelding φ die de structuur van de Ree-Tits Moufang verzameling vasthoudt. Zonder verlies van algemeenheid mogen we aannemen dat φ zowel (∞) als $(0, 0, 0)$ vasthoudt. We kunnen nu de afgeleide structuur $\mathcal{G}_\infty$ beschouwen: onze nieuwe puntenverzameling is $\mathcal{P}$ maar dan zonder het punt (∞), en enkel de blokken door (∞) worden nog beschouwd, al wordt hier het punt (∞) van afgehaald. De cirkels die oorspronkelijk gnarl (∞) hadden worden onze *vertikale rechten*, de andere cirkels noemen we *gewone rechten*. Analoog spreken we van een *vertikaal vlak* als die in de Ree meetkunde een cirkel met gnarl (∞) voorstelde en we definiëren de andere sferen door (∞) als *gewone vlakken*. We kunnen een parallellisme relatie definiëren op de gewone rechten: als voor twee gewone rechten er precies 1 vertikale rechte bestaat die beide gewone rechten snijdt, dan zijn de gewone rechten niet parallel, in de andere gevallen zijn die wel parallel. Het voordeel van de afgeleide struktuur is dat hierin al die objecten en de voorwaarde voor parallellisme algebraïsch vertaald kunnen worden. Ons automorfisme φ moet nu de parallellisme relatie bewaren, en omdat φ de gnarl van een cirkel C op de gnarl van diens beeld moet afbeelden moet φ de volgende vorm hebben: $(x, y, z)^\varphi = (\ell x^\sigma, \ell^{1+\theta} y^\sigma, \ell^{2+\theta} z^\sigma)$. Die actie bewaart de structuur van de veralgemeende zeshoek.

Opmerking Met behulp van deze cirkels en sferen kunnen we ook aantonen dat elke automorfisme van de Ree zeshoek die de absolute punten bewaart noodzakelijkerwijs de absolute rechten moet bewaren.

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