Generalized Quadrangles and Associated Structures

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Thesis submitted for the degree of
Doctor of Philosophy
in
Pure Mathematics
at
The University of Adelaide
(Faculty of Mathematical and Computer Sciences)

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August 21, 1997

Contents

Abstract

Our aim in this thesis has been to consider questions concerning the relationship between a Generalized Quadrangle (GQ) and various substructures, with a view to proving characterisation and classification results. We also lay the groundwork for new GQ construction methods, although no new GQs are constructed here.

In Chapter 1 we introduce preliminary concepts and results required for the rest of the thesis, involving graphs, quadrics, geometries, GQs, algebraic topology on a simplicial complex and covers of geometries.

Chapter 2 contains a detailed investigation of the ovoid $K1(\sigma)$ of $Q(4,q)$ constructed by Kantor in [30], including construction of non-elation rosettes of $Q(4,q)$ containing only $K1(\sigma)$ ovoids and rosettes containing both $K1(\sigma)$ ovoids and elliptic quadric ovoids.

In Chapter 3 we show that if $\mathcal{S}$ is a GQ of order (s,s^2) and $\mathcal{S}'$ is a subquadrangle of order s doubly subtended in $\mathcal{S}$, then the subtended ovoid/rosette structure is a Semi-Partial Geometry (SPG). A new SPG is constructed from a GQ of Kantor ([31]) and a $Q(4,q)$ subquadrangle. For a q-clan GQ $\mathcal{S}$, q even, Payne constructed a family of subquadrangles $\mathcal{S}_\alpha$ of order q ([45]). We derive the algebraic conditions under which $\mathcal{S}_\alpha$ is doubly subtended in $\mathcal{S}$, and hence gives an SPG.

In Chapter 4 it is shown that if q is even a non-classical GQ of order (q,q^2) containing a subquadrangle isomorphic to $Q(4,q)$ implies the existence of a new ovoid of $\mathrm{PG}(3,q)$. Also, by a homology calculation, it is shown that if $\mathcal{S}$ is a GQ of order (q,q^2), q odd, such that $\mathcal{S}$ contains a $Q(4,q)$ subquadrangle, with each ovoid of $Q(4,q)$ subtended by $\mathcal{S}$ an elliptic quadric ovoid, then $\mathcal{S}$ is isomorphic to $Q(5,q)$.

In Chapter 5 we show a GQ $\mathcal{S}$ of order s with a regular point (∞) gives rise to a cover of the affine plane constructed from $\mathcal{S}$ and (∞), as in [49, 1.3.1]. Given an affine plane π of order s and an s-fold cover of π satisfying special conditions we construct a GQ of order s with a regular point. If the cover of π is algebraic the condition on the cover is interpreted in cohomological terms; we investigate these for the remainder of the Chapter 5.

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Preface

Generalized Quadrangles (GQs) were introduced by J. Tits in 1959 ([66]) in his work on triality. Tits was also responsible for the construction of the first non-classical GQ of order (s, s^2), the GQ $T_3(\theta)$ of order (q, q^2) where $q = 2^e$, e odd, and θ is the Suzuki-Tits ovoid of $\mathrm{PG}(3, q)$, which first appeared in [19]. After the classical GQ $Q(5, q)$ of order (q, q^2) and $T_3(\theta)$, the next GQ of order (s, s^2) was constructed by Kantor via the group coset geometry method ([29]). A number of new GQs of order (s, s^2) followed using the group coset geometry method of Kantor, all of which, except the Roman GQ of Thas and Payne [65], arise from q-clans (see [43, 45] and [31]). In all cases the parameter s is a prime power.

Relatively little is known about subquadrangles of the known GQs of order (s, s^2) although there are a number of examples of subquadrangles of order s. The GQ $Q(5, q)$ has subquadrangles isomorphic to $Q(4, q)$; the GQ of Kantor in [31] and the Roman GQ both have subquadrangles isomorphic to $Q(4, q)$, while the GQs derived from q-clans for q even have subquadrangles isomorphic to the GQ $T_2(\mathcal{O})$, for $\mathcal{O}$ some oval of $\mathrm{PG}(2, q)$. This thesis is largely devoted to investigating the combinatorics of a subquadrangle of order s of a GQ of order (s, s^2).

A result of Payne and Thas ([49]) shows that if $\mathcal{S}$ is a GQ of order (s, s^2) and $\mathcal{S}'$ is a subquadrangle of order s, then each point of $\mathcal{S} \setminus \mathcal{S}'$ is collinear to the $s^2 + 1$ points of an ovoid of $\mathcal{S}'$. A line of $\mathcal{S} \setminus \mathcal{S}'$ contains s such points of $\mathcal{S} \setminus \mathcal{S}'$ and the corresponding set of q ovoids of $\mathcal{S}'$ has the property that any two elements intersect in a fixed point (the point where the line meets $\mathcal{S}'$). A suggestion of Christine O'Keefe and Tim Penttila was to study these "rosettes" of ovoids of a GQ $\mathcal{S}'$ of order s (defined combinatorially as a set of s ovoids meeting pairwise in a fixed point) to see what they revealed about GQs of order (s, s^2) containing $\mathcal{S}'$ as a subquadrangle. A result in this direction had already been proved by Payne and Thas ([65, VII.1.]) showing that the GQ $Q(5, q)$ is characterised by the existence of a $Q(4, q)$ subquadrangle with each point of $Q(5, q) \setminus Q(4, q)$ being collinear to an elliptic quadric ovoid of $Q(4, q)$.

An obvious first step in considering ovoids and rosettes of GQs of order s is to consider ovoids and rosettes of $Q(4, q)$. For q even the only known non-classical ovoid of $Q(4, q)$ is for the case $q = 2^e$, e odd, and is isomorphic to the Tits ovoid of $W(q)$. In Chapter 4 rosettes of these "Tits" ovoids of $Q(4, q)$ are used to show that the existence of a non-classical GQ of order (q, q^2), q even, containing a subquadrangle isomorphic to $Q(4, q)$ implies the existence of a new ovoid of $\mathrm{PG}(3, q)$. (Note that the latter is fairly unlikely!).

For q odd there are a number of non-classical ovoids of $Q(4, q)$ known, the most "structured" of these being the $K1(\sigma)$ ovoid constructed by Kantor in [30]. Chapter 2 is devoted to investigating the properties of this ovoid and the rosettes of $Q(4, q)$ containing it.

The other obvious problem for q odd it to show that $Q(5, q)$ is characterised by the existence of a $Q(4, q)$ subquadrangle with each point of $Q(5, q) \setminus Q(4, q)$ being collinear to an elliptic quadric ovoid of $Q(4, q)$. Consideration of this problem led to considering the general combinatorial setting: that is, a GQ $\mathcal{S}$ of order (s, s^2) with a subquadrangle $\mathcal{S}'$ of order s such that if the points of an ovoid of $\mathcal{S}'$ are collinear with a point of $\mathcal{S} \setminus \mathcal{S}'$, then they are collinear with exactly two points of $\mathcal{S} \setminus \mathcal{S}'$. A consequence of a result of Bose and Shrikhande ([8], see [49, 1.2.4]) is that any set of points of an ovoid of $\mathcal{S}'$ may be collinear with at most two points of $\mathcal{S} \setminus \mathcal{S}'$ and so $\mathcal{S}'$ is a combinatorially "extreme" subquadrangle. It was observed by Metz (see [18]) and Hirschfeld and Thas (see [26]) that in the case where $\mathcal{S} = Q(5, q)$ and $\mathcal{S}' = Q(4, q)$ that the ovoids and rosettes of $Q(4, q)$ associated with the points and lines of $\mathcal{S} \setminus \mathcal{S}'$ form a geometry which is a Semi-Partial Geometry (SPG). In Chapter 3 it is shown that this is also true in general for any such $\mathcal{S}$ and $\mathcal{S}'$ described above. A new SPG is constructed from a GQ of Kantor in [31] and a $Q(4, q)$ subquadrangle.

In this "extreme" case the relationship between the SPG and the geometry of $\mathcal{S} \setminus \mathcal{S}'$ is a cover, and more specifically an algebraic cover. The question of how many "different" GQs of order (s, s^2) give rise to a given SPG in this manner becomes a problem in cohomology. In Chapter 4 this approach allows us to prove [65, VII.1] in the case where q is odd.

This idea of the geometry of a "substructure" of a GQ being related to the geometry of the

GQ "outside" the substructure is explored again in Chapter 5. In this case the GQ is of order s with a regular point (∞), and the substructure is an affine plane of order s, as defined in [49, 1.3.1.]. It is shown that such a GQ always gives rise to a cover of the associated affine plane (and also a distance-regular cover of the complete graph on s^2 vertices). The rest of Chapter 5 is spent exploring the idea of algebraic covers of an affine plane and their relation to GQs of order s with a regular point. This seems a promising approach to constructing new GQs of order s, although none, as yet, have been found.

Acknowledgements

I would like to thank my supervisor Christine O'Keefe for her steadfast and unwavering support, on both academic and personal grounds, during the course of my PhD. Without her patience, dedication to the ideals of research and standing in the academic community my PhD. would have been much the poorer.

Thanks also to the geometry seminar at the University of Adelaide, especially including Wen-Ai Jackson, Catherine Quinn and Sue Barwick.

Particular thanks also to the now scattered Combinatorial Computing Research Group of the University of Western Australia: Tim Penttila, Gordon Royle, Ivano Pinneri and Nicholas Hamilton (in no particular order). All extremely gifted mathematicians and more importantly all extremely generous, friendly and cultured people.

During the course of my PhD. many geometers made their way to Adelaide and I was grateful for the opportunity to meet and interact with them. In particular I would like to thank Frank De Clerck, Jef Thas, Stan Payne and Bill Cherowitzo for their visits to Adelaide.

I am greatly in debt to my family, parents Barbara and Graham and siblings Anna and David, for their love and unqualified support at all times.

For the (almost) countless, privileged hours spent in his company during the four years of my PhD. I would like to thank Paul McCann. I am inestimably grateful to Shane Richards who despite putting up with my domestic habits for nearly three years still remains a true friend. Special thanks to Catherine Quinn, Deb Brown (my big sister), Dave Standingford, David Beard, Keith Martin, Steve Cox, Wen-Ai Jackson and Sue Barwick for their friendship during my PhD.

Finally, I would like to acknowledge the financial support of an Australian Postgraduate Research Award.

Chapter 1

Preliminaries

In this chapter we give definitions and results that will be used throughout the thesis. We will assume that reader is familiar with the theories of finite fields and projective spaces over finite fields; an excellent and thorough introduction to projective spaces over finite fields is given in [24]. For notational convenience we will associate a point of the projective space $\mathrm{PG}(n, q)$ with the homogeneous coordinate vector that represents it.

1.1 Graphs

A **graph** G is an ordered pair (V, E) where V is a non-empty set of elements called **vertices** and E is a set of unordered pairs of distinct vertices, called **edges**. Two vertices of G, u and v, that are contained in E are said to be **adjacent** and we write $u \sim v$.

A **path** of G is a finite, ordered set of vertices $v_0, v_1, \ldots, v_n$ such that $v_i \sim v_{i+1}$ for $0 \leq i \leq n-1$. The path is said to have **length** n. Two vertices u and v are said to be **connected** if there exists a path $v_0, v_1, \ldots, v_n$ such that $v_0 = u$ and $v_n = v$. A graph G is **connected** if each pair of vertices of G is connected.

A path $v_0, v_1, \ldots, v_n$ is said to be **closed** if $v_0 = v_n$. Alternatively, we may think of a closed path as a set of edges of G such that each vertex of G appears an even number of times in edges in the path. This set of edges is called the **edge set** of the path. A **circuit** of G is a closed path such that no edge is repeated. We will represent the circuit $v_0, v_1, \ldots, v_n$ with $v_0 = v_n$ by $(v_0, v_1, \ldots, v_{n-1})$. For convenience we will often consider two circuits that differ only by a cyclic permutation of their vertices to be the same circuit. An **elementary** circuit is a circuit $v_0, v_1, \ldots, v_{n-1}, v_0$ such that $v_i \neq v_j$ for $0 \leq i, j \leq n-1$, $i \neq j$. The elementary circuit C is said to be **induced** if each vertex of C is adjacent to exactly two others in C.

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. The graph G' is a **subgraph** of G if $V' \subseteq V$ and $E' = \{(u, v) : u, v \in V' \text{ and } (u, v) \in E\}$. If v is a vertex of G, then G_v is the subgraph of G with vertex set $\{u : u \sim v\}$. In general, if $A \subseteq V$ with $A = \{v_1, v_2, \ldots, v_n\}$, then G_A is the subgraph of G with vertex set $\{u : u \sim v_i \text{ for } i = 1, \ldots, n\}$. A **clique** of G is defined to be a complete subgraph of G. A **connected component** of G is a maximal connected subgraph.

A connected graph $G = (V, E)$ is **regular** if for every vertex $u \in V$, $|\{v \in V : u \sim v\}| = k$, for some fixed k. The number k is called the **valence** of the graph G.

Let u and v be two vertices of the connected graph $G = (V, E)$ and let $u = u_0, u_1, \ldots, u_{n-1}, u_n = v$ be a path of shortest length from u to v. The **distance** between u and v is defined to be n and we denote this by $d(u, v) = n$ or $d(v, u) = n$. The **diameter** of G is defined to be the number $d = max\{d(u, v) : u, v \in V\}$. If u is a vertex of G, then let $G_i(u)$ be the set $\{v : d(u, v) = i\}$. Thus $G_0(u) = \{u\}$ and $G_1(u) = G_u$.

A connected graph $G = (V, E)$ is called **distance-regular** if there exists integers b_i, c_i $(i \geq 0)$ such that for any two vertices $u, v \in V$, with $i = d(u, v)$, there are precisely c_i vertices contained in $G_{i-1}(u)$ that are adjacent to v and b_i vertices contained in $G_{i+1}(u)$ that are adjacent to v. The sequence $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, where d is the diameter of G, is called the **intersection array** of G. Note that it follows that G is regular with valence $k = b_0$ and that $b_d = c_0 = 0$ and $c_1 = 1$. For more information on distance-regular graphs see [9] or [5]. A distance-regular graph of diameter 2 is called a **strongly regular** graph. If $G = (V, E)$ is a strongly regular graph, then it has parameters v, k, μ, λ, where $v = |V|$, k is the valence of G, $\mu = c_2$ and $\lambda = k - b_1 - 1$.

For any connected graph $G = (V, E)$ of diameter $d > 1$ we denote by G_i (for $i = 1, 2, \ldots, d$) the graph with vertex set V and with vertices u and v adjacent if $d(u, v) = i$ in G. The graph G is **antipodal** if G_d is a disjoint union of cliques.

1.2 Quadrics

In this section we introduce quadrics of $\mathrm{PG}(3, q)$; for a more detailed treatment see [24, Chapters 5, 6 and 7], [25, Chapters 15, 16, 17 and 20] and [27, Chapter 22].

A **quadric** $\mathcal{Q}_n$ of $\mathrm{PG}(n, q)$ is the set of points of $\mathrm{PG}(n, q)$ whose homogeneous coordinates satisfy a quadratic form $F(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j$. If F cannot be reduced to a form in fewer than $n + 1$ variables by a linear transformation, then F is **non-degenerate** and $\mathcal{Q}_n$ is **non-singular**. If n is even, then every non-singular quadric in $\mathrm{PG}(n, q)$ is projectively equivalent to the **parabolic** quadric, denoted by $\mathcal{P}_n$. If n is odd, then $\mathrm{PG}(n, q)$ has exactly two projectively distinct non-singular quadrics, the **elliptic** quadric, denoted by $\mathcal{E}_n$ and the **hyperbolic** quadric, denoted by $\mathcal{H}_n$. Canonical forms for the non-singular quadrics are as follows:

$$
\begin{aligned}
\mathcal{P}_n &= V(x_0^2 + x_1 x_2 + \ldots + x_{n-1} x_n), \\
\mathcal{E}_n &= V(f(x_0, x_1) + x_2 x_3 + \ldots + x_{n-1} x_n), \\
\mathcal{H}_n &= V(x_0 x_1 + x_2 x_3 + \ldots + x_{n-1} x_n),
\end{aligned}
$$

where f is an irreducible binary quadratic form.

If we allow F to be **degenerate**, then the quadric defined by F may be singular, that is, a cone $\Pi_k \mathcal{Q}_s$. This cone is the join of the vertex Π_k, a subspace of $\mathrm{PG}(n, q)$ of dimension k and a non-singular quadric $\mathcal{Q}_s$ in the subspace Π_s of dimension s, with Π_k and Π_s skew and $k + s = n - 1$.

Suppose that $\mathcal{Q}_n$ is a non-singular quadric of $\mathrm{PG}(n, q)$ defined by the quadratic form $F(x_0, x_1, \ldots, x_n)$. Then the **bilinear form** β associated with F is given by $\beta(X, Y) = F(X+Y) - F(X) - F(Y)$. The bilinear form gives the geometry of the quadric, as the following lemma shows.

Lemma 1.2.1 *[27, Lemma 22.3.1] Let $Y \in \mathcal{Q}_n$. Then*

(i) *If $X \notin \mathcal{Q}_n$, then $\beta(X, Y) = 0 \Longleftrightarrow \langle X, Y \rangle$ is tangent to $\mathcal{Q}_n$.*

(ii) *If $X \in \mathcal{Q}_n$, then $\beta(X, Y) = 0 \Longleftrightarrow \langle X, Y \rangle \subset \mathcal{Q}_n$.*

(iii) $\beta(X, Y) \neq 0 \Longleftrightarrow |\langle X, Y \rangle \cap \mathcal{Q}_n| = 2$.

Lemma 1.2.2 *[27, Lemma 22.3.3] If q and n are not both even, the correspondence $Y \longleftrightarrow V(\beta(Y, X))$ is a polarity. For q odd, the set of self-polar points is $\mathcal{Q}_n$. For q even, the polarity is null and every point in $\mathrm{PG}(n, q)$ is self-polar.*

We will denote the action of the polarity by $\perp$. If Y is a point of $\mathrm{PG}(3, q)$, then the polar hyperplane $V(\beta(Y, X))$ is called the **tangent space** or **tangent hyperplane** of Y. For details of the action of the polarity on subspaces of $\mathrm{PG}(n, q)$, see [27, Theorem 22.7.2].

Now we consider the symmetries of $\mathcal{Q}_n$, where q and n are not both even. Let Q be a point of $\mathrm{PG}(n, q) \setminus \mathcal{Q}_n$ and let $\mu_Q : \mathcal{Q}_n \to \mathcal{Q}_n$ be defined as follows. For P in $\mathcal{Q}_n$,

$$
\mu_Q(P) = \begin{cases} P & \text{if } \langle P, Q \rangle \text{ is a tangent to } \mathcal{Q}_n \\ P' & \text{if } \langle P, Q \rangle \text{ meets } \mathcal{Q}_n \text{ at } P' \neq P. \end{cases}
$$

The map μ_Q may be extended to an automorphism of $\mathrm{PG}(n,q)$ (fixing $\mathcal{Q}_n$) given by

$$X \mapsto X - \frac{\beta(X,Q)}{F(Q)}Q$$

([27, Lemma 22.6.3]).

1.3 An introduction to geometries

An **(incidence) geometry** is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called **points** and **lines**, respectively, and for which $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation. In addition, we require that two distinct points of $\mathcal{S}$ are incident with at most one line and two distinct lines of $\mathcal{S}$ are incident with at most one point. Two points of $\mathcal{S}$ that are incident with a common line are said to be **collinear** and two lines of $\mathcal{S}$ that are incident with a common point are **concurrent**.

The **dual** of a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ is the geometry $\mathcal{S}^\wedge = (\mathcal{B}, \mathcal{P}, I)$. So, the points of $\mathcal{S}^\wedge$ are the lines of $\mathcal{S}$, the lines of $\mathcal{S}^\wedge$ are the points of $\mathcal{S}$ and the incidence relation between $\mathcal{P}$ and $\mathcal{B}$ is the same in both $\mathcal{S}$ and $\mathcal{S}^\wedge$.

An **automorphism** of a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ is a bijection $\phi : \mathcal{P} \cup \mathcal{B} \to \mathcal{P} \cup \mathcal{B}$, such that ϕ maps points to points, lines to lines and preserves incidence. By preserving incidence we mean that if $P \in \mathcal{P}$ and $\ell \in \mathcal{B}$, then $P \; I \; \ell$ if and only if $\phi(P) \; I \; \phi(\ell)$. The **point graph** of $\mathcal{S}$ is the graph with vertex set $\mathcal{P}$ and edge-set the set of collinear pairs of points of $\mathcal{S}$.

The geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ is a **subgeometry** of $\mathcal{S}$ if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. If $\mathcal{S}'$ is a subgeometry of $\mathcal{S}$, then we write $\mathcal{S}' \subseteq \mathcal{S}$. If $\mathcal{S} \neq \mathcal{S}'$, then we say that $\mathcal{S}'$ is a **proper subgeometry** of $\mathcal{S}$ and write $\mathcal{S}' \subset \mathcal{S}$. If $\mathcal{S}' \subset \mathcal{S}$, then it follows that $\mathcal{P}' \neq \mathcal{P}$ or $\mathcal{B}' \neq \mathcal{B}$.

1.3.1 Examples of geometries

A (finite) **partial geometry** (PG) is an incidence structure $\mathcal{W} = (\mathcal{P}, \mathcal{B}, I)$, satisfying the following axioms:

(i) Each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If X is a point and ℓ is a line not incident with X, then the number of pairs $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \; I \; m \; I \; Y \; I \; \ell$ is a constant α ($\alpha \geq 0$).

The integers s, t and α are called the **parameters** of $\mathcal{W}$. It follows that $|\mathcal{P}| = (s+1)(st+\alpha)/\alpha$, $|\mathcal{B}| = (t+1)(st+\alpha)/\alpha$ and that the point graph of $\mathcal{W}$ is strongly regular. Partial geometries were introduced by Bose in [7] to investigate strongly regular graphs.

A (finite) **semi-partial geometry** (SPG) is an incidence structure $\mathcal{T} = (\mathcal{P}, \mathcal{B}, I)$, satisfying the following axioms:

(i) Each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If X is a point and ℓ is a line not incident with X, then the number of pairs $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \; I \; m \; I \; Y \; I \; \ell$ is either 0 or a constant α ($\alpha \geq 0$);

(iv) For any pair of non-collinear points (X, Y) there are μ $(\mu \geq 0)$ points Z such that Z is collinear with both X and Y.

The integers s, t, α, μ are the **parameters** of $\mathcal{T}$. It follows that $|\mathcal{P}| = 1 + (t+1)s(\mu + t(s-\alpha+1))/\mu$ and $|\mathcal{B}| = \mu(t+1)/(s+1)$. Note that axiom (iii) of a semi-partial geometry is a generalisation of axiom (iii) for a partial geometry, while axiom (iv) forces the point graph of a semi-partial geometry to be strongly regular. Thus, partial geometries are a subclass of semi-partial geometries. The point graph of T has parameters

$$v = 1 + \frac{(t+s)s(\mu + t(s-\alpha+1))}{\mu}, \ k = (t+1)s, \ , \mu, \ \lambda = s - 1 + t(\alpha - 1).$$

Semi-partial geometries were first introduced by Debroey and Thas in [18]. For more information on partial geometries and semi-partial geometries see [17].

An important class of partial geometries (and hence semi-partial geometries) is the case $\alpha = 1$, the generalized quadrangles. A (finite) **generalized quadrangle** (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$, satisfying the following axioms:

(i) Each point is incident with $1+t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line;

(ii) Each line is incident with $1+s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point;

(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \ I \ m \ I \ Y \ I \ \ell$.

It follows that $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$. We say that $\mathcal{S}$ has order (s,t) while if $s = t$, then we say that $\mathcal{S}$ has order s. A subgeometry of a GQ $\mathcal{S}$ that is also a GQ is called a **subquadrangle** or **subGQ**. Generalized quadrangles were introduced as part of a larger class of geometries called generalized n-gons, by Tits in [66]. For a complete list of the known GQs see [64].

1.4 Generalized Quadrangles (GQs)

1.4.1 Combinatorics of GQs

In this section we give some combinatorial definitions and properties of GQs. For more details see [49].

Let $\mathcal{S}$ be a GQ of order (s,t). Given two points X, Y of $\mathcal{S}$, we write $X \sim Y$ if X and Y are collinear, or if $X = Y$, otherwise we write $X \not\sim Y$. Dually, for two lines ℓ, m of $\mathcal{S}$, we write $\ell \sim m$ if ℓ and m are concurrent, or if $\ell = m$, otherwise we write $\ell \not\sim m$.

For $X \in \mathcal{P}$, define $X^\perp = \{Y \in \mathcal{P} : Y \sim X\}$. The set $X^\perp$ is called the **trace** of X. The trace of a pair of distinct points X and Y, is the set $X^\perp \cap Y^\perp$ and is denoted $\{X,Y\}^\perp$. The size of $\{X,Y\}^\perp$ is $s+1$ or $t+1$, according as $X \sim Y$ or $X \not\sim Y$. For $X \neq Y$, the **span** of the pair (X,Y) is $\{X,Y\}^{\perp\perp} = \{U \in \mathcal{P} : U \in Z^\perp \text{ for all } Z \in \{X,Y\}^\perp\}$, that is, $\{X,Y\}^{\perp\perp} = (\{X,Y\}^\perp)^\perp$. In general, if $\mathcal{P}' \subset \mathcal{P}$, then $\mathcal{P}'^\perp = \cap_{X \in \mathcal{P}'} X^\perp$ and $\mathcal{P}'^{\perp\perp} = (\mathcal{P}'^\perp)^\perp$.

If $X \sim Y$, $X \neq Y$, or if $X \not\sim Y$ and $|\{X,Y\}^{\perp\perp}| = t+1$, we say that the pair (X,Y) is **regular**. The point X is **regular** if (X,Y) is regular for each $Y \in \mathcal{P}$ such that $Y \neq X$.

A **triad** (of points) of $\mathcal{S}$ is a triple of distinct, pairwise non-collinear points.

An **ovoid** of a GQ $\mathcal{S}$ of order (s,t) is a set θ of points such that each line of $\mathcal{S}$ is incident with precisely one point of θ. It follows that θ has $st+1$ points. Dually, a **spread** of $\mathcal{S}$ is a set Ω of $st+1$ lines of $\mathcal{S}$, such that each point of $\mathcal{S}$ is incident with exactly one line of Ω.

A **rosette** based at a point X of $\mathcal{S}$ is a set $\mathcal{R}$ of ovoids with pairwise intersection $\{X\}$ and such that $\{\theta \setminus \{X\} : \theta \in \mathcal{R}\}$ is a partition of the points of $\mathcal{S}$ not collinear with X. The point X is called the **base point** of $\mathcal{R}$. It follows that a rosette $\mathcal{R}$ has s ovoids. A rosette is **homogeneous** if all of its ovoids are of the same isomorphism class in $\mathcal{S}$. A rosette that isn't homogeneous is **inhomogeneous**.

Let the GQ $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ of order (s', t') be a subGQ of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$. If $\mathcal{S}'$ is a proper subGQ of $\mathcal{S}$, that is $\mathcal{S}' \neq \mathcal{S}$, and $\ell \in \mathcal{B}$ then by [49, 2.1], exactly one of the following occurs: (i) $\ell \in \mathcal{B}'$; (ii) $\ell \notin \mathcal{B}'$ and ℓ is incident with a unique point $X \in \mathcal{P}'$, in which case we say ℓ is **tangent** to $\mathcal{S}'$ at X; or (iii) $\ell \notin \mathcal{B}'$ and ℓ is incident with no point of $\mathcal{P}'$, in which case we say ℓ is **external** to $\mathcal{S}'$. Dually, we may define **external** points and **tangent** points.

Lemma 1.4.1 ([61], [41], see [49, 2.2.1]) Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (s,t) with a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ of order (s, t'). Each point of $\mathcal{S}$, external to $\mathcal{S}'$, is collinear with the $1 + st'$ points of an ovoid of $\mathcal{S}'$.

An ovoid arising as in Lemma 1.4.1 is said to be **subtended** by P, or just **subtended**. The ovoids of $\mathcal{S}'$ subtended by the points in $\mathcal{P} \setminus \mathcal{P}'$ are said to be the ovoids **subtended by** $\mathcal{S}$ or just the **subtended** ovoids.

If ℓ is a line of $\mathcal{S}$ that is tangent to $\mathcal{S}'$ then each of the s points incident with ℓ and not contained in $\mathcal{P}'$, is external to $\mathcal{S}'$. Furthermore, the set of s ovoids subtended by these points is a rosette of $\mathcal{S}'$. To see this, first observe that if $X, Y \; I \; \ell$ and $X, Y \in \mathcal{P} \setminus \mathcal{P}'$, then the ovoids θ_X and θ_Y, subtended by X and Y respectively, are both incident with the point $\ell \cap \mathcal{S}'$. Also, if θ_X and θ_Y are both incident with a further point Z of $\mathcal{S}'$, then X, Y, Z form a triangle. We say that the rosette constructed as above is **subtended** by the line ℓ, or just **subtended**.

We define **subtended spreads** and **subtended rosettes** of $\mathcal{S}'$ dually.

1.4.2 The classical GQs

The classical GQs are those GQs corresponding to the classical groups and are as follows (see [49, Section 3.1]).

1(a) The GQ $Q(3,q)$ of order $(q,1)$ arises as the points and lines of the non-singular hyperbolic quadric in $\mathrm{PG}(3,q)$, given by the equation $x_0x_1 + x_2x_3 = 0$. Since $t = 1$, the structure of $Q(3,q)$ is trivial.

1(b) The GQ $Q(4,q)$ of order q arises as the points and lines of the non-singular (parabolic) quadric in $\mathrm{PG}(4,q)$, given by the equation $x_0^2 + x_1x_2 + x_3x_4 = 0$.

1(c) The GQ $Q(5,q)$ of order (q,q^2) arises as the points and lines of the non-singular elliptic quadric in $\mathrm{PG}(5,q)$ given by the equation $x_0^2 + ax_0x_5 + bx_5^2 + x_1x_2 + x_3x_4 = 0$, where $x_0^2 + ax_0x_5 + bx_5^2$ is an irreducible binary quadratic form.

Note that $Q(5,q)$ contains subquadrangles isomorphic to $Q(4,q)$ (see [49, 3.5(a)]).

2(a) The GQ $H(3,q^2)$ of order (q^2,q) arises as the points and lines of the non-singular hermitian variety of $\mathrm{PG}(3,q^2)$, given by the equation $x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0$.

2(b) The GQ $H(4,q^2)$ of order (q^2,q^3) arises as the points and lines of the non-singular hermitian variety of $\mathrm{PG}(4,q^2)$, given by the equation $x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1} = 0$.

3(a) The GQ $W(q)$ of order q arises as the absolute points and absolute lines of a symplectic polarity in $\mathrm{PG}(3,q)$, with canonical bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$.

It has been shown, independently by Buekenhout and Lefèvre in [11] and Olanda in [39, 40], that the classical GQs are precisely the GQs $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ such that $\mathcal{P}$ is a subset of the pointset of some projective space $\mathrm{PG}(n,q)$ and $\mathcal{B}$ is a set of lines of $\mathrm{PG}(n,q)$.

1.4.3 The non-classical GQ of Tits

The following constructions of the non-classical GQs $T_2(\mathcal{O})$ and $T_3(\theta)$ are due to Tits and first appeared in [19].

Let $\Sigma = \mathrm{PG}(3,q)$ and $H = \mathrm{PG}(2,q)$ be a hyperplane of Σ. Further, let $\mathcal{O}$ be an oval of H. Define $T_2(\mathcal{O})$ to be the following incidence structure:

Points:	(i)	The points of $\Sigma \setminus H$
	(ii)	The planes of Σ, not H, meeting H in a tangent to $\mathcal{O}$
	(iii)	The symbol (∞)
Lines:	(a)	Lines of Σ, not contained in H, that meet $\mathcal{O}$
	(b)	Points of $\mathcal{O}$
Incidence:	(i)(a)	Incidence in Σ
	(ii)(a)	Containment in Σ
	(iii)(a)	None
	(i)(b)	None
	(ii)(b)	Incidence in Σ
	(iii)(b)	All

Let $\Sigma' = \mathrm{PG}(4,q)$ and $H' = \mathrm{PG}(3,q)$ be a hyperplane of Σ'. Further, let θ be an ovoid of H'. Define $T_3(\theta)$ to be the following incidence structure:

Points:	(i)	The points of $\Sigma' \setminus H'$
	(ii)	The hyperplanes of Σ', not H', meeting θ in a unique point
	(iii)	The symbol (∞)
Lines:	(a)	Lines of Σ', not contained in H', that meet θ
	(b)	Points of θ
Incidence:	(i)(a)	Incidence in Σ'
	(ii)(a)	Containment in Σ'
	(iii)(a)	None
	(i)(b)	None
	(ii)(b)	Incidence in Σ'
	(iii)(b)	All

1.4.4 Isomorphisms between GQ

In this section we give some isomorphisms between the GQs defined in Section 1.4.2 and Section 1.4.3. For proofs of the isomorphisms, see the cited result in [49].

Theorem 1.4.2 *[49, 3.2.1] The GQ $Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) is self-dual if and only if q is even.*

Theorem 1.4.3 *[49, 3.2.2] The GQ $T_2(\mathcal{O})$ is isomorphic to $Q(4,q)$ if and only if $\mathcal{O}$ is an irreducible conic; it is isomorphic to $W(q)$ if and only if q is even and $\mathcal{O}$ is a conic.*

Theorem 1.4.4 *[49, 3.2.3] The GQ $Q(5,q)$ is isomorphic to the dual of $H(3,q^2)$.*

Theorem 1.4.5 *[49, 3.2.4] The GQ $T_3(\theta)$ is isomorphic to $Q(5,q)$ if and only if θ is an elliptic quadric of $\mathrm{PG}(3,q)$.*

1.4.5 The q-clan GQs

In this section we give a summary of results on the q-clan construction of GQs of order (q,q^2). First consider the group coset construction of a GQ as introduced by Kantor [29]. Let G be a finite group of order s^2t, $s \geq 2$, $t \geq 2$ and suppose that G admits two families of subgroups

$\mathcal{F} = \{S_0, \ldots, S_t\}$, $\mathcal{F}^\star = \{S_0^\star, \ldots, S_t^\star\}$ such that $|S_i| = s$, $|S_i^\star| = st$ and $S_i \subseteq S_i^\star$. If $\mathcal{F}$ and $\mathcal{F}^\star$ satisfy:

$$\begin{array}{lll} K1 & S_iS_j \cap S_k = 1 & \text{for } k \neq i,j \text{ and } i \neq j \text{ and} \\ K2 & S_i^\star \cap S_j = 1 & \text{for } i \neq j, \end{array}$$

then $\mathcal{F}$ is a 4-*gonal family* for G. The following point-line geometry $\mathcal{S}(G, \mathcal{F})$ is a GQ of order (s, t).

$$\begin{array}{rll} \text{Points}: & \text{(i)} & \text{Elements of } G \\ & \text{(ii)} & \text{Right cosets, } S_i^\star g,\ i = 0, \ldots, t,\ g \in G \\ & \text{(iii)} & (\infty) \\ \text{Lines}: & \text{(a)} & \text{Right cosets } S_i g,\ i = 0, \ldots, t,\ g \in G \\ & \text{(b)} & \text{Symbols } [S_i],\ i = 0, \ldots, t. \end{array}$$

A point g of type (*i*) is incident with each line S_ig. A point $S_i^\star g$ of type (*ii*) is incident with $[S_i]$ and with each line $S_ih \subset S_i^\star g$. The point (∞) is incident with each line $[S_i]$ of type (*b*). These are all of the incidence relations.

We now consider q-clans and give the construction of a 4-gonal family from a q-clan, the development of which is due to Payne [43, 45] and Kantor [31]. For q a prime power a q**-clan** is a set $\mathcal{C} = \{A_t : t \in \mathrm{GF}(q)\}$ of 2×2 matrices over $\mathrm{GF}(q)$ such that for distinct $s, t \in \mathrm{GF}(q)$, $(a, b)(A_s - A_t)(a, b)^T = 0$ has only the trivial solution $a = b = 0$. We can normalise a q-clan such that for q odd $A_t = \begin{pmatrix} x_t & y_t \\ y_t & z_t \end{pmatrix}$ and for q even $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$.

Now consider the particular group $G = \{(\alpha, c, \beta) : \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$ and define a binary operation on G by

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta').$$

This binary operation makes G into a group where $(\alpha, c, \beta)^{-1} = (-\alpha, \alpha\beta^T - c, -\beta)$. Let $\mathcal{C}$ be a normalised q-clan and let $K_t = A_t + A_t^T$ for $t \in \mathrm{GF}(q)$. Define the following subgroups of G:

$$\begin{array}{rcll} A(\infty) & = & \{(\overline{0}, 0, \beta) : \beta \in \mathrm{GF}(q)^2\} & \\ A^\star(\infty) & = & \{(\overline{0}, c, \beta) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} & \\ A(t) & = & \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) : \alpha \in \mathrm{GF}(q)^2\} & \text{for } t \in \mathrm{GF}(q) \\ A^\star(t) & = & \{(\alpha, c, \alpha K_t) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} & \text{for } t \in \mathrm{GF}(q), \end{array}$$

where $\overline{0}$ is the identity element of $\mathrm{GF}(q)^2$. Then $\mathcal{F}(\mathcal{C}) = \{A(t) : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ is a 4-gonal family for G which gives rise to a GQ $\mathcal{S}(G, \mathcal{F}(\mathcal{C}))$ of order (q^2, q). We will denote $\mathcal{S}(G, \mathcal{F}(\mathcal{C}))$ by $\mathcal{S}(\mathcal{C})$.

The following three theorems are results on the automorphism group of a q-clan GQ.

Theorem 1.4.6 [48, III.(1)] Suppose G_1, G_2 are groups and $\mathcal{F}$ is a 4-gonal family for G_1. If $\theta : G_1 \to G_2$ is a group isomorphism or a group anti-isomorphism, then $\mathcal{S}(G_1, \mathcal{F})$ and $\mathcal{S}(G_2, \theta(\mathcal{F}))$ are isomorphic GQs.

(A group anti-isomorphism is a one-to-one and onto map $\theta : G_1 \to G_2$ such that $\theta(gg') = \theta(g')\theta(g)$ for all $g, g' \in G_1$.)

In particular, if we have a group automorphism of G then it induces an automorphism of $\mathcal{S}(G, \mathcal{F})$.

Theorem 1.4.7 [48, IV.1.] Let $\mathcal{C}$ be a normalised q-clan, q odd, with $A_0 = K_0 = 0$. Let $\mathcal{S}(\mathcal{C}) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be the associated GQ of order (q^2, q). Let Θ be a collineation of $\mathcal{S}(\mathcal{C})$ which fixes (∞), $[A(\infty)]$, and $(0,0,0)$. Then there is

a permutation $t \mapsto \bar{t}$ of the elements of $\mathrm{GF}(q)$,
a nonzero element λ of $\mathrm{GF}(q)$,
an automorphism σ of $\mathrm{GF}(q)$ and ,
a 2×2 matrix $D \in GL(2,q)$

for which the following holds

$$(D^{-1})^T \left(A_{\bar{t}} - A_{\bar{0}}\right) D^{-1} - \lambda A_t^{\sigma} \text{ is a skew symmetric matrix (with 0 diagonal)}$$
$$\text{for all } t \in \mathrm{GF}(q). \tag{1.4.1}$$

Moreover, Θ is induced by an automorphism of G, and is given by

$$\Theta : (\alpha, c, \beta) \mapsto (\alpha^{\sigma}\lambda^{-1}D^{-T}, \lambda^{-1}c^{\sigma} + \lambda^{-2}\alpha^{\sigma}D^{-T}A_{\bar{0}}D^{-1}(\alpha^{\sigma})^T, \beta^{\sigma}D + \alpha^{\sigma}\lambda^{-1}D^{-T}K_{\bar{0}}).$$

Conversely, given D, σ, λ and $t \mapsto \bar{t}$ as just described, the Θ as above is a collineation of $\mathcal{S}(\mathcal{C})$. When q is odd, and if each $A_t \in \mathcal{C}$ is symmetric, 1.4.1 may be replaced by

$$A_{\bar{t}} = \lambda D^T A_t^{\sigma} D + A_{\bar{0}} \quad \text{for all} \quad t \in \mathrm{GF}(q).$$

The automorphism Θ is denoted by $\Theta(\pi, \lambda, \sigma, D)$.

The automorphisms in Theorem 1.4.7 all fix the line $[A(\infty)]$, in the following theorem we consider automorphisms not fixing $[A(\infty)]$.

Theorem 1.4.8 [48, III.5.] If $\mathcal{C} = \{A_t : t \in \mathrm{GF}(q)\}$ is a q-clan, q odd, with $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $\mathcal{C}' = \{A_t^{-1} : 0 \neq t \in \mathrm{GF}(q)\} \cup \{A_0\}$ is a q-clan with $\mathcal{S}(\mathcal{C}) \cong \mathcal{S}(\mathcal{C}')$. In fact, the switch from $\mathcal{C}$ to $\mathcal{C}'$ just amounts to interchanging the roles of $A(\infty)$ and $A(0)$ in the coordinatisation of $\mathcal{S}$.

Note that from the proof of Theorem 1.4.8 in [48] the isomorphism in Theorem 1.4.8 is given by the group anti-automorphism $\theta : G \to G : (\alpha, c, \beta) \mapsto (\beta, 2c, 2\alpha)$. Thus the subgroup $\mathcal{Q}_1 = (\mathrm{GF}(q) \times 0) \times \mathrm{GF}(q) \times (\mathrm{GF}(q) \times 0)$ is fixed under the isomorphism.

1.4.6 Ovoids and rosettes of GQs

Ovoids of $Q(4,q)$

Let $Q(4,q) \subset \mathrm{PG}(4,q)$, for q odd or even, and let Σ be a hyperplane of $\mathrm{PG}(4,q)$ such that $\Sigma \cap Q(4,q) = \theta$ is a non-singular elliptic quadric of Σ. Then θ is an ovoid of $Q(4,q)$ and an ovoid that arises in such a way is called **classical**.

If q is even, then $Q(4,q)$ is isomorphic to $W(q)$ and so an ovoid of $Q(4,q)$ is equivalent to an ovoid of $W(q)$. If $W(q) \subset \mathrm{PG}(3,q)$, then an ovoid of $W(q)$ is also an **ovoid** of $\mathrm{PG}(3,q)$, that is a set of q^2+1 points of $\mathrm{PG}(3,q)$, no three collinear. Further, any ovoid of $\mathrm{PG}(3,q)$ is an ovoid of a GQ $W(q)'$, where $W(q)' \subset \mathrm{PG}(3,q)$ and $W(q)'$ is isomorphic to $W(q)$ ([55]). Two ovoids of $W(q)$ are isomorphic under an automorphism of $W(q)$ if and only if they are isomorphic under an automorphism of $\mathrm{PG}(3,q)$ ([2, Prop. 1]). Thus, the distinct classes of ovoids of $W(q)$ are exactly the projectively distinct ovoids of $\mathrm{PG}(3,q)$.

If $q = 2^e$, e even, then every known ovoid of $\mathrm{PG}(3,q)$ is classical. For $q = 2$, it is straightforward to show that the only ovoids of $\mathrm{PG}(3,q)$ (and hence of $W(q)$) are the classical ovoids. Further, $\mathrm{PG}(3,q)$ has only the classical ovoid when $q = 4$, [4] (see also [25, 16.1.7]) and $q = 16$, [36]. If $q = 2^e$, e odd, $e = 2h+1$ say, and $e \geq 3$, then in addition to the classical ovoid $\mathrm{PG}(3,q)$

also contains the Tits ovoid. The canonical form of the Tits ovoid is the following set of points of PG$(3,q)$.

$$\left\{(1, s^\sigma + st + t^{\sigma+2}, s, t) : s, t \in \mathrm{GF}(q)\right\} \cup \{(0,1,0,0)\}$$

where σ is the automorphism of GF(q) such that $\sigma : x \mapsto x^{2^{h+1}}$. For $q = 8$, PG$(3,q)$ contains only the classical ovoid and the Tits ovoid, which was shown with the assistance of a computer by Fellegara in [20] and without the use of a computer by Penttila and Praeger in [52]. For $q = 32$, PG$(3,q)$ contains only the classical ovoid and the Tits ovoid [37]. For more information on ovoids of PG$(3,q)$ see [35, 25] and for a discussion of the connection between ovoids of $Q(4,q)$ and ovoids (and spreads) of PG$(3,q)$ see [49, 3.2.1, 3.4.1].

We now consider ovoids of $Q(4,q)$ when q is odd. As we have seen, $Q(4,q)$ contains the classical ovoids and further there are three non-classical ovoids of $Q(4,q)$. The first two were constructed by Kantor in [30]. The ovoid $K1$ exists for the case $q = p^h$, with p an odd prime and $h \geq 2$. In [30] $K1$ is defined to be the set of points

$$\left\{(1, y, z, -my^\sigma, -z + my^{\sigma+1}) : y, z \in \mathrm{GF}(q)\right\} \cup \{(0,0,0,0,1)\}$$

of $Q(4,q)$ with the form $x_0x_5 + x_1x_4 + x_2x_3$, where m is a non-square element of GF(q) and σ is a non-trivial automorphism of GF(q). In this thesis we shall indicate that a $K1$ ovoid has associated field automorphism σ by calling it a $K1(\sigma)$ ovoid. The ovoid $K2$ of $Q(4,q)$ exists for the case $q = 3^{2r+1}$ with $r \geq 1$. The third type of non-classical ovoid of $Q(4,q)$ was constructed by Thas and Payne ([65]) for $q = 3^h$ with $h \geq 3$. The ovoid of Thas and Payne arises as a subtended ovoid of the embedding of $Q(4,q)$ as a subquadrangle in the dual of the Roman GQ ([65]). For $q = 3, 5$ or 7, every ovoid of $Q(4,q)$ is classical [10, 34, 16, 14, 15, 33]. For more information on ovoids of $Q(4,q)$ and ovoids $Q(2n,q)$ in general, see [38].

Elation rosettes of $Q(4,q)$

We saw in Section 1.4.4 that the GQ $Q(4,q)$ is dual to $W(q)$ and that for q even, $Q(4,q)$ is also isomorphic to $W(q)$. We shall use this and elations of the ambient PG$(3,q)$ of $W(q)$ to construct rosettes of $Q(4,q)$.

An **elation** of $PG(n,q)$ is an automorphism of $PG(n,q)$ that fixes a given hyperplane pointwise and a given point on the hyperplane linewise. The hyperplane is the **axis** of the elation and the point is the **centre** of the elation. For more details on elations see [1].

Lemma 1.4.9 *[1, VIII.5 lemma 2] Let P be a point and π a plane of $PG(3,q)$ with $P \in \pi$ and let ℓ be a line of $PG(3,q)$ such that $P \in \ell$ and $\ell \not\subseteq \pi$. The group consisting of the elations with centre P and axis π, acts regularly on the set of points $l \setminus P$.*

Let ψ be an ovoid of $Q(4,q)$ and suppose that q is even. Since $Q(4,q)$ and $W(q)$ are isomorphic GQs for q even, there is an ovoid θ of $W(q)$ that is the image of ψ under the isomorphism from $Q(4,q)$ to $W(q)$. We will use specific elations of PG$(3,q)$ to construct rosettes containing θ.

Lemma 1.4.10 *[59, p. 71] Let σ be an elation of the ambient $PG(3,q)$ of $W(q)$ with centre P and axis $P^\perp$, then σ can be naturally restricted to an automorphism of $W(q)$.*

Given an ovoid θ of $W(q)$ containing a point P we will now consider a description of θ that will be useful in the work that follows. The points of $W(q)$ collinear with P are exactly the points of the plane $P^\perp$ and so the the q^2 points of $\theta \setminus \{P\}$ are in the set $PG(3,q) \setminus P^\perp$. Also, no two of these points span a line of PG$(3,q)$ that is incident with P, since in that case θ contains three collinear points, contradicting the definition of an ovoid in $PG(3,q)$. Thus each of the q^2 lines of $PG(3,q) \setminus P^\perp$ incident with P is incident with exactly one further point of θ.

Lemma 1.4.11 *Let P be a point of $W(q)$ and θ an ovoid of $W(q)$ such that $P \in \theta$. An elation σ which has centre P and axis $P^{\perp}$ maps θ to an ovoid of $W(q)$ containing P. Furthermore the group consisting of the set of such elations, acting on the set of ovoids of $W(q)$, has a rosette based at P and containing θ as the orbit containing θ.*

Proof: By Lemma 1.4.10, the elation σ induces an automorphism of $W(q)$ and fixes P, hence θ^{σ} is an ovoid of $W(q)$ containing P. Recall that θ has one point per line through P not in $P^{\perp}$ and the elation group acts regularly on the points of such lines, not P. So the q ovoids which are images of θ under the action of the group of such elations have P as their pairwise intersection and partition the points of $PG(3,q) \setminus P^{\perp}$. Thus the orbit of θ under this group is a rosette of $W(q)$ based at P.

□

We will call the rosette based at P, containing θ and generated as in Lemma 1.4.11 an **elation rosette** with respect to P and θ.

For q even the only known ovoids of $W(q)$ are the elliptic quadric ovoids and the Tits ovoids. In Chapter 4 we shall see that if θ is either an elliptic quadric ovoid or a Tits ovoid of $W(q)$ and P is a point of θ, then the elation rosette $\mathcal{R}$ constructed in Lemma 1.4.11 is the unique homogeneous rosette containing θ and with base point P.

Now, let θ be an ovoid of $Q(4,q)$ and let q be odd or even. Since $Q(4,q)$ and $W(q)$ are dual GQs, there exists a spread S_{θ} of $W(q)$ such that S_{θ} is the image of θ under the duality from $Q(4,q)$ to $W(q)$. Let ℓ be a fixed line of S_{θ} and P a point incident with ℓ. The plane $P^{\perp}$ contains ℓ and consequently, every other line of S_{θ} meets $P^{\perp}$ in a point.

We introduce the term **rosette** of spreads of a GQ to mean the dual of a rosette of ovoids.

Lemma 1.4.12 *Let ℓ be a line of $W(q)$ and S a spread of $W(q)$ such that $\ell \in S$. Let P be a point of $W(q)$ incident with ℓ. An elation σ which has centre P and axis $P^{\perp}$ maps S to a spread of $W(q)$ containing ℓ. Furthermore, the group consisting of the set of such elations, acting on the set of spreads of $W(q)$, has a rosette of spreads based at ℓ and containing S as the orbit containing S.*

Proof: We consider the action of the elation σ on the lines of S. Since ℓ is contained in the axis of σ it follows that ℓ is fixed by σ. If $m \in S$ with $m \neq \ell$, then $m \cap P^{\perp}$ is a point $\{Q\}$, say and $< m, P >$ is a plane π_m, say. Since π_m contains P, it is fixed by σ and so $\sigma(m)$ is a line contained in π_m, containing Q, but not P. By Lemma 1.4.9 the group of q elations with centre P and axis $P^{\perp}$ map m onto the q distinct lines of π_m which contain Q, but not P. Thus if σ and σ' are two elations with centre P and axis $P^{\perp}$, then $\sigma(m) = \sigma'(m)$ if and only if $\sigma = \sigma'$. If m' is a line of S such that $m' \neq \ell, m$, meeting $P^{\perp}$ in the point Q', then $Q \neq Q'$ and so no elation with centre P and axis $P^{\perp}$ maps m onto m'. Thus, the orbit of S under the group of elations with centre P and axis $P^{\perp}$ has size q and has the property that any two spreads in the orbit intersect in exactly ℓ.

□

If θ is an ovoid of $Q(4,q)$ and P is a fixed point of θ, then by dualising Lemma 1.4.12 we construct $q+1$ homogeneous rosettes containing θ and based at P. We will call these rosettes, elation rosettes. Thus, if q is even, there are $q+1$ elation rosettes containing θ and based at P, from Lemma 1.4.12, and 1 elation rosette containing θ and based at P from Lemma 1.4.11. However, for both known examples of ovoids of $Q(4,q)$, where q is even, the elliptic quadric ovoid and the ovoid isomorphic to the Tits ovoid of $W(q)$, there is a unique homogeneous rosette containing a given ovoid of that type and based at a given point. (Suppose that θ is an elliptic quadric ovoid of $Q(4,q)$ with ambient hyperplane Σ and that θ' is a second elliptic quadric ovoid with ambient hyperplane Σ'. If $\theta \cap \theta' = \{P\}$, then $\pi = \Sigma \cap \Sigma'$ is the unique tangent plane to θ at P in Σ. Thus the only elliptic quadric ovoids of $Q(4,q)$ that intersect θ in exactly P are the q elliptic quadric ovoids whose ambient hyperplane contains π. This is the unique rosette containing θ with base point P. In the case where θ is isomorphic to a Tits ovoid of $W(q)$ see Lemma 4.1.7.) Thus the $q+2$ elation rosettes are identical.

In the case where q is odd, there are $q+1$ elation rosettes containing θ and based at P. For θ an elliptic quadric ovoid, the $q+1$ elation rosettes are identical (by the same argument presented for the q even case in the preceding paragraph). In Chapter 2 we shall show that if θ is a $K1$ ovoid, then there exist a point P such that the $q+1$ elation rosettes based at P are distinct. We shall also construct homogeneous rosettes of $K1$ ovoids, that are not elation rosettes.

1.5 Algebraic topology on a simplicial complex

In this section we introduce the concept of a simplicial complex and the homology and cohomology groups of the simplicial complex over an abelian group. Useful introductory texts to algebraic topology are [23], [58], [57] and [32]. We will not be concerned with the more general topological aspects of simplicial complexes only the calculation of their homology and cohomology, for which the results that follow will suffice.

1.5.1 Modules

Let R be a ring with identity $1 \neq 0$. A **left R-module** A is an additive abelian group together with a function $p: R \times A \to A$, written $p(r,a) = ra$, such that

$$\begin{aligned} (r+r')a &= ra + r'a, \\ (rr')a &= r(r'a), \\ r(a+a') &= ra + ra', \\ 1a &= a \end{aligned}$$

where $r, r' \in R$ and $a, a' \in A$. It follows that $0a = 0$ and $(-1)a = -a$.

We can define right R-modules in the obvious manner. Since we will not be using right R-modules in what follows, we will refer to a left R-module as an R-module, or simply a module if the ring R is understood from context. If $\mathbb{F}$ is a field, then an $\mathbb{F}$-module is a vector space over $\mathbb{F}$.

Note that for any abelian group A we may find a ring R such that A is an R-module. As an example, if A is any abelian group, then we may turn it into a $\mathbb{Z}$-module. If m is any positive integer, and a any element of A, then define $ma = a + a + \ldots + a$ (m times). If we also define $(-m)a = -(ma)$ then this makes A a $\mathbb{Z}$-module.

A subset S of an R-module A is a **submodule**, if S is closed under addition and if $r \in R$, $s \in S$ imply $rs \in S$; then S itself is an R-module. As S is a subgroup of A, we may form the factor group A/S, which is also an R-module, since $r(b+S) = rb + S$. The R-module A/S is called a **quotient module**.

If A and B are both R-modules, then $\alpha: A \to B$ is an **R-module homomorphism** if

$$\alpha(a+a') = \alpha a + \alpha a', \quad \alpha(ra) = r(\alpha(a))$$

where $r \in R$ and $a \in A$.

We define $Hom_R(A,B)$ to be the set of all R-module homomorphisms from A into B. If $f \in Hom_R(A,B)$ and $t \in R$, then we define $tf: A \to B$ by $(tf)(a) = t(f(a))$ for all $a \in A$. If the ring R is commutative, then we have that

$$(tf)(ra) = t(f(ra)) = tr(f(a)) = rt(f(a)) = r[(tf)(a)],$$

which shows that $tf \in Hom_R(A,B)$ and so $Hom_R(A,B)$ is an R-module. In particular, if $A = B$ and R is a field, then $Hom_R(A,A)$ is the space of linear functionals of the vector space A, that is the dual vector space.

1.5.2 Simplicial complexes, homology and cohomology

A **simplicial complex** $\Gamma = (V, S)$ consists of a set V of **vertices** and a set S of finite non-empty subsets of V called **simplexes** such that

(a) Any set consisting of exactly one vertex is a simplex.

(b) Any non-empty subset of a simplex is a simplex.

A simplicial complex $\Gamma' = (V', S')$ is a subcomplex of $\Gamma = (V, S)$ if $V' \subseteq V$ and $S' \subseteq S$ and a **proper** subcomplex if $\Gamma' \neq \Gamma$.

A simplex containing exactly $q+1$ vertices is called a **q-simplex**. The **dimension** of a q-simplex is defined to be q. The **dimension** of the simplicial complex Γ, denoted by $dim\Gamma$, is defined to be equal to -1 if Γ is empty, equal to n if Γ contains an n-simplex but no $(n+1)$-simplex and equal to ∞ if Γ contains n-simplexes for all $n \geq 0$. The simplicial complex Γ is said to be **finite** if it contains only a finite number of simplexes.

If s and s' are simplexes of Γ and $s' \subseteq s$, then s' is called a **face** of s (a **proper face** if $s' \neq s$) and if s' is a p-simplex, it is called a **p-face** of s.

A simplex becomes **oriented**, or is given an **orientation**, when we specify a particular ordering of its vertices. Two orderings of the vertices which agree up to an even permutation of the vertices determine the same orientation. If σ is the simplex s with a particular orientation, then we represent s with the opposite orientation by $-\sigma$.

Let R be a finite ring and Γ a simplicial complex. Let $S_q(\Gamma, R)$ be the R-module whose elements are the formal linear combinations $\sum_\sigma r_\sigma \sigma$, where σ runs through the oriented q-simplexes of Γ and r_σ is an element of R. The operations of the R-module are defined as follows:

$$\begin{aligned} \sum_\sigma r_\sigma \sigma + \sum_\sigma r'_\sigma \sigma &= \sum_\sigma (r_\sigma + r'_\sigma)\sigma \\ r\left(\sum_\sigma r_\sigma \sigma\right) &= \sum_\sigma (r r_\sigma)\sigma \qquad \text{for } r \in R. \end{aligned}$$

Now let B be a list of the q-simplexes of Γ with each simplex given an orientation. Then since each oriented q-simplex is either σ or $-\sigma$ for a unique element σ of B, it follows that each element of $S_q(\Gamma, R)$ can be written uniquely in the form

$$\sum_{\sigma \in B} r_\sigma \sigma + \sum_{\sigma \in B} r'_\sigma(-\sigma)$$

where r_σ and r'_σ are both elements of R.

Let $S'_q(\Gamma, R)$ be the submodule of $S_q(\Gamma, R)$ generated by elements of the form $r\sigma + r(-\sigma)$, where $r \in R$ and σ is an oriented q-simplex. We define $C_q(\Gamma, R)$ to be the quotient module $S_q(\Gamma, R)/S'_q(\Gamma, R)$. An element of the R-module $C_q(\Gamma, R)$ is called a **q-chain** of Γ over R. Essentially, in $C_q(\Gamma, R)$ we are allowed to replace $r\sigma$ by $-r(-\sigma)$ wherever we please. If Γ is non-empty, then we define $C_{-1}(\Gamma, R) = 0$ and if Γ has finite dimension n we define $C_q(\Gamma, R) = 0$ for $q > n$.

The **q-th boundary operator** is an R-homomorphism $\partial_q : C_q(\Gamma, R) \to C_{q-1}(\Gamma, R)$. Let $s = (P_0 P_1 \ldots P_q)$ be an oriented q-simplex of Γ (and so also a q-chain), then the action of ∂_q on s is

$$\partial_q(s) = \sum_{i=0}^{q} (-1)^i (P_0 P_1 \ldots P_{i-1} P_{i+1} \ldots P_q).$$

The $(q-1)$-simplex $\{P_0 P_1 \ldots P_{i-1} P_{i+1} \ldots P_q\}$ is the face of s **opposite** P_i. Note that $\partial_q(s)$ is independent of the ordering of the vertices we use to represent the orientation of s. If we take an even permutation, T, on the ordering of the vertices of s, we may consider the permutation as the product of a permutation, T_1, fixing the vertex P_i in the i-th position, and a permutation,

T_2, shifting P_i r positions. In terms of the formula for ∂_q, T_1 induces a permutation on the vertices of the face of s opposite P_i, while T_2 multiplies the oriented face opposite P_i by a factor of $(-1)^r$. Since T is even, T_1 and T_2 are either both odd or both even. In either case, the orientation of the face of s opposite to P_i, as it appears in $\partial_q(s)$ is preserved.

We extend ∂_q to act on q-chains of Γ over R by letting it act linearly, that is,

$$\partial_q(r_\sigma \sigma) = r_\sigma \partial_q(\sigma) \quad \text{and} \quad \partial_q(\sigma + \sigma') = \partial_q(\sigma) + \partial_q(\sigma').$$

where r_σ is an element of R and σ and σ' are oriented q-simplexes of Γ. The $(q-1)$-chain $\partial_q(s)$ is called the **boundary** of s. Note that since $\partial_q(\sigma) + \partial_q(-\sigma) = 0$ for any q-simplex σ. The operator ∂_q does not depend on which representation of a q-chain we choose.

The following lemma is an exercise in algebra:

Lemma 1.5.1 $\partial_q \circ \partial_{q+1} = 0.$

Proof: Let $s = (P_0, P_1, \dots, P_{q+1})$ be a oriented $(q+1)$-simplex of Γ. Now

$$\partial_q \cdots \partial_{q+1}(s) = \sum_{0 \le i < j \le q+1} k_{ij}(P_0 P_1 \dots P_{i-1} P_{i+1} \dots P_{j-1} P_{j+1} \dots P_{q+1}),$$

where k_{ij} is some element of R for each i, j, $0 \le i < j \le q+1$. We now show that every $k_{ij} = 0$. The oriented $(q-1)$-simplex $(P_0 P_1 \dots P_{i-1} P_{i+1} \dots P_{j-1} P_{j+1} \dots P_{q+1})$ appears in $\partial_q \cdots \partial_{q+1}(s)$ either by ∂_{q+1} "removing" the vertex P_i and ∂_q removing the vertex P_j, or vice versa. Hence

$$k_{ij} = (-1)^i(-1)^{j-1} + (-1)^j(-1)^i = 0.$$

Thus $\partial_q \circ \partial_{q+1}(s) = 0$ for any oriented $(q+1)$-simplex s and it follows that $\partial_q \circ \partial_{q+1} = 0$.

□

A q-chain s such that $\partial_q(s) = 0$ is called a **q-cycle** and if $s = \partial_{q+1}(s')$ for some $(q+1)$-chain s', then s is called a **q-boundary**. Two q-chains that differ by a boundary are called **homologous** and a q-cycle that is homologous to the zero q-chain is called **null homologous**. The set of q-cycles form an R-module $(ker(\partial_q))$ denoted by $Z_q(\Gamma, R)$ and the set of q-boundaries also from an R-module $(im(\partial_{q+1}))$ which is denoted by $B_q(\Gamma, R)$. If s is a q-boundary with $s = \partial_{q+1}(s')$, then $\partial_q(s) = \partial_q \circ \partial_{q+1}(s') = 0$ by Lemma 1.5.1 and so $B_q(\Gamma, R)$ is a subgroup of $Z_q(\Gamma, R)$. The quotient group $Z_q(\Gamma, R)/B_q(\Gamma, R)$ is called the **q-th homology group** of Γ over R, and is denoted $H_q(\Gamma, R)$. Note that $H_{-1}(\Gamma, R) = 0$ and if Γ has finite dimension n, then $H_q(\Gamma, R) = 0$ for $q > n$.

Let A be an R-module, then $Hom_R(C_q(\Gamma, R), A)$ is the abelian group of R-homomorphisms from $C_q(\Gamma, R)$ to A. We will denote $Hom_R(C_q(\Gamma, R), A)$ by $C^q(\Gamma, A)$, where A is understood to be an R-module. Any element of $C^q(\Gamma, A)$ is called a **q-cochain** of Γ into A.

The q-th coboundary operator is a group homomorphism $\delta^q : C^q(\Gamma, A) \to C^{q+1}(\Gamma, A)$. Let c be a q-cochain and $s = (P_0 P_1 \dots P_{q+1})$ an oriented $q+1$-simplex of Γ. Then the action of $\delta^q c$ on s is defined as

$$\delta^q c(s) = \sum_{i=0}^{q+1} (-1)^i c(P_0 P_1 \dots P_{i-1} P_{i+1} \dots P_{q+1}).$$

By linearity this determines the action of $\delta^q c$ on all $q+1$-chains. The $q+1$-cochain $\delta^q c$ is called the **coboundary** of c.

We have an analogous result to Lemma 1.5.1 for δ^q:

Lemma 1.5.2 $\delta^{q+1} \circ \delta^q = 0.$

The following lemma links the boundary and coboundary operators:

Lemma 1.5.3 *Let s be a $(q+1)$-chain and c a q-cochain. Then*

$$\delta^q c(s) = c(\partial_{q+1} s).$$

Proof: First let $s = (P_0P_1 \dots P_{q+1})$ be an oriented $(q+1)$-simplex of Γ. Then

$$\partial_{q+1}(s) = \sum_{1=0}^{q+1} (-1)^i (P_0P_1 \dots P_{i-1}P_{i+1} \dots P_{q+1}),$$

and so

$$c(\partial_{q+1}(s)) = \sum_{1=0}^{q+1} (-1)^i c(P_0P_1 \dots P_{i-1}P_{i+1} \dots P_{q+1})$$

since c is an R-homomorphism. Now by definition

$$\delta^q c(s) = \sum_{1=0}^{q+1} (-1)^i c(P_0P_1 \dots P_{i-1}P_{i+1} \dots P_{q+1})$$

and so $\delta^q c(s) = c(\partial_{q+1}(s))$ for any oriented $(q+1)$-simplex s. From this it follows that $\delta^q c(s) = c(\partial_{q+1}(s))$ for any $(q+1)$-chain s.

$\square$

A q-cochain c such that $\delta^q c = 0$ is called a **q-cocycle** and if $c = \delta^{q-1}c'$ for some $(q-1)$-cochain c', then c is called a **coboundary**. Two q-cochains that differ by a boundary are called **cohomologous**. $Z^q(\Gamma, A)$ is the group of q-cocycles $(ker(\delta^q))$ and $B^q(\Gamma, A)$ is the group of q-coboundaries $(im(\delta^{q-1}))$. Lemma 1.5.2 tells us that $B^q(\Gamma, A)$ is a subgroup of $Z^q(\Gamma, A)$. The quotient group $Z^q(\Gamma, A)/B^q(\Gamma, A)$ is the **q-th cohomology group** of Γ over A, and is denoted $H^q(\Gamma, A)$.

In the special case where Γ is finite and $R = A = \mathbb{F}$ for some field $\mathbb{F}$ we have the following theorem:

Theorem 1.5.4 *Let Γ be a non-empty, finite simplicial complex and $\mathbb{F}$ a field. Then*

$$H_q(\Gamma, \mathbb{F}) \cong H^q(\Gamma, \mathbb{F}) \quad \text{for all } q \geq -1.$$

Proof: Let the dimension of Γ be n, which is necessarily finite. If $q = -1$ or $q > n$ then $H_q(\Gamma, \mathbb{F}) = H^q(\Gamma, \mathbb{F}) = 0$.

Now suppose that $0 \leq q \leq n$. Since Γ is finite and $\mathbb{F}$ is a field, $C_q(\Gamma, \mathbb{F})$ is a vector space over $\mathbb{F}$, of dimension k, say, and $C^q(\Gamma, \mathbb{F})$ is the dual space of $C_q(\Gamma, \mathbb{F})$. $Z_q(\Gamma, \mathbb{F})$ and $B_q(\Gamma, \mathbb{F})$ are subspaces of $C_q(\Gamma, \mathbb{F})$. We will denote the respective dimensions of $Z_q(\Gamma, \mathbb{F})$ and $B_q(\Gamma, \mathbb{F})$ by k_1 and k_2.

Now

$$\begin{aligned}
\text{annihilator}(B_q(\Gamma, \mathbb{F})) &= \{c \in C^q(\Gamma, \mathbb{F}) : c(s) = 0 \text{ for all } s \in B_q(\Gamma, \mathbb{F})\} \\
&= \{c \in C^q(\Gamma, \mathbb{F}) : c(\partial^{q+1}s') = 0 \text{ for all } s' \in C_{q+1}(\Gamma, \mathbb{F})\} \\
&= \{c \in C^q(\Gamma, \mathbb{F}) : \delta^q c(s') = 0 \text{ for all } s' \in C_{q+1}(\Gamma, \mathbb{F})\} \\
&= Z^q(\Gamma, \mathbb{F})
\end{aligned}$$

Thus $Z^q(\Gamma, \mathbb{F})$ is a subspace of $C^q(\Gamma, \mathbb{F})$ with dimension $k - k_2$. By a similar argument we obtain that annihilator$(B^q(\Gamma, \mathbb{F})) = Z_q(\Gamma, \mathbb{F})$ which implies that the dimension of $B^q(\Gamma, \mathbb{F})$ is $k - k_1$. Now, the dimension of $H_q(\Gamma, \mathbb{F}) = k_1 - k_2$ and the dimension of $H^q(\Gamma, \mathbb{F}) = (k - k_2) - (k - k_1) = k_1 - k_2$. Since $H_q(\Gamma, \mathbb{F})$ and $H^q(\Gamma, \mathbb{F})$ are vector spaces over $\mathbb{F}$ with the same dimension, we have that $H_q(\Gamma, \mathbb{F}) \cong H^q(\Gamma, \mathbb{F})$.

$\square$

1.5.3 Cohomology groups of a simplicial complex

Recall, that in the previous section we defined $C^q(\Gamma, A)$ to be the group $Hom_R(C_q(\Gamma, A))$ where A was understood to be an R-module. By linearity, if c is any element of $Hom_R(C_q(\Gamma, A))$, then c is determined by how it maps the oriented q-simplexes. If σ is an oriented q-simplex of Γ, then the only restriction on $c(\sigma)$ is that $c(\sigma) = -c(-\sigma)$. At this point we observe that the abelian group $C^q(\Gamma, A)$ is independent of the ring R. So for any simplicial complex Γ and any abelian group A we will define the q-th cohomology group of Γ into A, without reference to any ring, or to A as an R-module. The group $C^q(\Gamma, A)$ is defined to be the group of alternating functions from the oriented q-simplexes into A (where a function c is alternating if $c(\sigma) = -c(-\sigma)$ for all oriented q-simplexes σ). We then define the coboundary operator and cohomology group as in Section 1.5.2.

1.5.4 The simplicial complex of a graph

Let G be a graph with vertex set V and edge set E. We define Γ_G to be the simplicial complex which has as 0-simplexes the vertices of G, 1-simplexes the edges of G, 2-simplexes the complete subgraphs on 3 vertices and in general, has as q-simplexes the complete subgraphs on $q+1$ vertices.

If Γ is a simplicial complex, then the **q-skeleton** Γ^q of Γ is the simplicial complex consisting of all p-simplexes of Γ for $p \leq q$. The 1-skeleton of Γ is the simplicial complex of 0-simplexes and 1-simplexes of Γ, which is a graph. The 1-skeleton of Γ_G is G.

1.5.5 Homology and cohomology over $\mathbb{Z}_2$

Suppose that Γ is a simplicial complex and that R is a ring with even characteristic, that is $-r = r$ for any $r \in R$ and hence $(-r)(-\sigma) = r\sigma$ for any $r \in R$ and oriented q-simplex σ. Recall, from the construction of $C_q(\Gamma, R)$, that if σ is an oriented q-simplex, then the q-chain represented by σ is the same as that represented by $\sigma + (\sigma + (-\sigma)) = (1+1)(\sigma) + (-\sigma) = -\sigma$. In other words, the orientation of the simplex is unimportant and $C_q(\Gamma, R)$ may be considered as the R-module of formal linear sums of q-simplexes (unoriented) of Γ, over R.

In the case where $R = \mathbb{Z}_2$, we may represent $C_q(\Gamma, R)$ as follows. The elements of $C_q(\Gamma, R)$ are the subsets of the set of q-simplexes and the abelian group operation is symmetric difference on the subsets. Given an element s of $C_q(\Gamma, R)$, as a sum of q-simplexes, the subset of q-simplexes corresponding to s contains those q-simplexes whose coefficient in s is 1. The boundary of a q-simplex s is the set of $(q-1)$-faces of s and the boundary of a q-chain σ is the symmetric difference of the sets of $(q-1)$-faces of the q-simplexes of σ. Note, that from the definition of an R-module, we know that $1.\sigma = \sigma$ and $0.\sigma = 0$ for any $\sigma \in C_q(\Gamma, R)$.

If we consider the cohomology group $C^q(\Gamma, A)$ where A is a $\mathbb{Z}_2$-module, it should be noted that A has characteristic 2, since for $a \in A$ we have that $a + a = 1 \cdot a + 1 \cdot a = (1+1) \cdot a = 0 \cdot a = 0$.

In particular, we now consider the vector space $H_1(\Gamma, \mathbb{Z}_2)$, for some finite simplicial complex Γ. Let σ be a 1-cycle of Γ, then $\partial_1(\sigma) = 0$ and so each 0-simplex of Γ is contained in an even number of 1-simplexes of σ (the number may be 0). Let σ' be a 1-boundary of Γ with $\sigma' = \partial_2(\sigma'')$, then σ' is the symmetric difference of the sets of 1-faces of the 2-simplexes of σ''.

Let $\sigma = \{s_1, s_2, \ldots, s_n\}$ be a 1-cycle of Γ. We say that σ is an **elementary** 1-cycle of Γ, if each vertex of Γ appears in exactly none or two 1-simplexes of σ. We say that σ is **induced** if for any 1-simplex s such that $\partial_1(s) = \{P, Q\}$ with $P \in \partial_1(s_i)$, $Q \in \partial_1(s_j)$, then $s \in \sigma$. If σ is an induced 1-cycle that is not the boundary of a 2-simplex, we say that σ is a **proper** induced 1-cycle.

Lemma 1.5.5 *A 1-cycle of Γ may be written as the sum of elementary 1-cycles, with no common 1-simplexes.*

Proof: Let σ be a 1-cycle of Γ, which we represent as the set $\{s_1, s_2, \ldots, s_n\}$, where the s_i are distinct 1-simplexes of Γ. Let $i \in \{1, 2, \ldots, n\}$ and let $\partial_1(s_i) = \{P_1, P_2\}$, where P_1 and P_2 are 0-simplexes of Γ. Since σ is a 1-cycle, there is an i_1 such that $1 \leq i_1 \leq n$ and $\partial_1(s_{i_1}) = \{P_2, P_3\}$, for some 0-simplex P_3 of Γ. Similarly, there is an i_2 such that $1 \leq i_2 \leq n$ and $\partial_1(s_{i_2}) = \{P_3, P_4\}$, for some 0-simplex P_4 of Γ. Continuing this process will eventually yield an i_r such that $1 \leq i_r \leq n$ and $\partial_1(s_{i_r}) = \{P_{r+1}, P_1\}$. The 1-cycle $\sigma_i = \{s_i, s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$

is an elementary 1-cycle. If we now choose $s_j \in \sigma \setminus \sigma_i$, we can generate a elementary 1-cycle contained in σ and disjoint to σ_i. Continuing this process gives σ as the sum of elementary 1-cycles.

□

Now we prove a lemma showing that we may break down an elementary 1-cycle into induced 1-cycles.

Lemma 1.5.6 *An elementary* 1*-cycle of* Γ *can be written as the sum of induced* 1*-cycles of* Γ.

Proof: Let $\sigma = \{s_1, s_2, \dots, s_n\}$ be an elementary 1-cycle of Γ such that $\partial_1(s_i) = \{P_i, P_{i+1}\}$ for $1 \leq i \leq n-1$ and $\partial_1(s_n) = \{P_n, P_1\}$. If σ is an induced 1-circuit, then there is no more to show, so suppose that σ is not induced. Since σ is not induced we may assume, without loss of generality, that for some i, $i \neq 1, 2, n$ there exists a 1-simplex s with $\partial_1(s) = \{P_1, P_i\}$. Then σ may be written as the sum of the elementary 1-cycles $\{s_1, s_2, \dots, s_{i-1}, s\}$ and $\{s_i, s_{i+1}, \dots, s_n, s\}$, both of which have fewer elements than σ. We continue this process until we have written σ as sum of induced 1-cycles.

□

Combining Lemma 1.5.5 and Lemma 1.5.6 gives us the following result.

Corollary 1.5.7 *A* 1*-cycle of* Γ *can be expressed as the sum of induced* 1*-cycles of* Γ.

Thus we have the following theorem.

Theorem 1.5.8 $H^1(\Gamma, \mathbb{Z}_2)$ *is trivial if and only if each induced* 1*-cycle of* Γ *is a* 1*-boundary.*

The above definitions and results make more intuitive sense in the graph Γ^1, the 1-skeleton of Γ. Recall that a 2-simplex of Γ is a triangle in Γ^1, a 1-simplex an edge and a 0-simplex a vertex of Γ^1. In Γ^1 a 1-cycle of Γ is a set of edges such that each vertex appears in an even number of edges, that is, the set of edges of a circuit of Γ^1. An elementary 1-cycle is the set of edges of a elementary circuit of Γ^1 and an induced 1-cycle is the set of edges of an induced circuit of Γ^1. Lemma 1.5.5 says that every circuit may be decomposed into elementary circuits, while Lemma 1.5.6 says that any elementary circuit may be decomposed into induced circuits. Finally, Theorem 1.5.8 says that $H^1(\Gamma, \mathbb{Z}_2)$ is trivial if and only if each induced circuit of the graph Γ^1 can be decomposed into triangles.

1.6 Covers of a graph and covers of a geometry

In this section we introduce the cover of a graph and the cover of a geometry, as in [12] (see also [56]).

Let G be a graph, then an m**-fold cover of** G is a pair $(\overline{G}, p)$ where $\overline{G}$ is a graph and p is a map from $\overline{G}$ to G satisfying:

(i) For any vertex $P \in G$, $p^{-1}(P)$ consists of m pairwise non-adjacent vertices

(ii) For any edge e of G, $p^{-1}(e)$ consists of m disjoint edges

(iii) For any non-edge $\{P, Q\}$ of G, $p^{-1}(\{P, Q\})$ is a graph with no edges.

If G is the point graph of a geometry $\mathcal{S}$ and $(\overline{G}, p)$ satisfies

(iv) For any line ℓ of $\mathcal{S}$, if $\mathcal{P}_\ell = \{P \in \mathcal{P} : P \ I \ \ell\}$, then $p^{-1}(\mathcal{P}_\ell)$ consists of m disjoint complete graphs

then we can form a geometry $\overline{\mathcal{S}}$ with pointset the vertices of $\overline{G}$ and lines (as sets of points) defined to be the complete graphs from (iv). The map p naturally induces a map from $\overline{\mathcal{S}}$ to $\mathcal{S}$, which, introducing an abuse of notation, we also call p. The pair $(\overline{\mathcal{S}}, p)$ is called an m**-fold cover of** $\mathcal{S}$. The geometry $\overline{\mathcal{S}}$ will be called the **covering geometry** and the map p the

covering map. We will often take the existence of the map p to be understood and call $\overline{\mathcal{S}}$ an m-fold cover of $\mathcal{S}$. Any element of $p^{-1}(P)$ will be called a **cover** of P and similarly, any element of $p^{-1}(\ell)$ a cover of ℓ,

Note that if (iv) is satisfied, and ℓ is a line of $\mathcal{S}$, then (i) and (ii) imply that each line in the set $p^{-1}(\ell)$ has the same size as ℓ (as a set of points). Also, if $P\ I\ \ell$, then each point in the set $p^{-1}(P)$ is incident with exactly one element of $p^{-1}(\ell)$. This means that if ℓ (as a set of points) is $\{P_1, P_2, \ldots, P_{s+1}\}$, then each line of the set $p^{-1}(\ell)$ has the form $\{P'_1, P'_2, \ldots, P'_{s+1}\}$, where $P'_i \in p^{-1}(P_i)$, for $i = 1, 2, \ldots, s+1$. That is, P' incident with ℓ' in $\overline{\mathcal{S}}$ implies that $p(P')$ is incident with $p(\ell')$ in $\mathcal{S}$.

If $(\overline{\mathcal{S}}, p)$ is an m-fold cover of the geometry $\mathcal{S}$, $\mathcal{S}^\wedge$ is the dual geometry of $\mathcal{S}$ and $\overline{\mathcal{S}}^\wedge$ is the dual geometry of $\overline{\mathcal{S}}$, then it is relatively straight-forward to show that $(\overline{\mathcal{S}}^\wedge, p)$ is an m-fold cover of $\overline{\mathcal{S}}^\wedge$.

Let G be a graph, Γ_G the simplicial complex of G and A an additive abelian group of order m. To simplify matters, we will identify a vertex of Γ_G with the corresponding vertex of G, and a 1-simplex of Γ_G with the corresponding ordered pair of adjacent vertices of G. Recall that a 1-cochain c of Γ_G into A is a map from the set of ordered pairs of adjacent vertices of G to A, such that $c(P, Q) = -c(Q, P)$ for $\{P, Q\}$ any pair of adjacent vertices of G. An **algebraic** m-fold cover of G over A is an m-fold cover of G, $(\overline{G}, p)$ where $\overline{G}$ is the graph with

$$\begin{array}{ll} \text{Vertex Set:} & \{(P, \alpha) :\ P \in G,\ \alpha \in A\} \\ \text{Adjacency:} & (P, \alpha) \sim (Q, \beta) \text{ if } P \sim Q \text{ and } c(P, Q) = \alpha - \beta \end{array}$$

and p is the map $p((P, \alpha)) = P$. Any 1-cochain c defines an m-fold cover of G in the above way. If G is the point graph of a geometry $\mathcal{S}$ and $(\overline{\mathcal{S}}, p)$ is an m-fold cover of $\mathcal{S}$, then we say that $(\overline{\mathcal{S}}, p)$ is an **algebraic** m-fold cover of $\mathcal{S}$.

Let G be the point graph of a geometry $\mathcal{S}$ and let $(\overline{G}, p)$ be an algebraic m-fold cover of G. We investigate conditions under which condition (iv) above is satisfied.

Let ℓ be a line of $\mathcal{S}$ such that $\mathcal{P}_\ell = \{P_1, P_2, \ldots, P_{s+1}\}$, then (P_1, α) is collinear to the set of points $\{(P_2, \alpha - c(P_1, P_2)), (P_3, \alpha - c(P_1, P_3)), \ldots, (P_{s+1}, \alpha - c(P_1, P_{s+1})\}$. Thus, $p^{-1}(\mathcal{P}_\ell)$ consists of m disjoint complete graphs if and only if the m complete graphs have vertex sets

$$\{(P_1, \alpha), (P_2, \alpha - c(P_1, P_2)), \ldots, (P_{s+1}, \alpha - c(P_1, P_{s+1}))\} \quad \text{for } \alpha \in A.$$

This is true if and only if

$$(P_i, \alpha - c(P_1, P_i)) \sim (P_j, \alpha - c(P_1, P_j)) \text{ for all } P_i, P_j \text{ where } i \neq j,\ i, j \neq 1 \text{ and } \alpha \in A.$$

Writing δ for the first coboundary operator δ^1, we see that this is true if and only if

$$\delta c(P_1, P_i, P_j) = 0 \quad \text{for all } P_i, P_j \text{ where } i \neq j \text{ and } i, j \neq 1.$$

Equivalently,

$$\begin{aligned} \delta c(P_i, P_j, P_k) &= c(P_i, P_j) - c(P_i, P_k) + c(P_j, P_k) \\ &= (c(P_1, P_j) - c(P_1, P_i)) - (c(P_1, P_k) - c(P_1, P_i)) + (c(P_1, P_k) - c(P_1, P_j)) \\ &= 0, \end{aligned}$$

for all P_i, P_j where i, j, k are distinct and $i, j, k \neq 1$. Thus $(\overline{G}, p)$ gives rise to an algebraic m-fold cover of $\mathcal{S}$ if and only if

$$\delta c(P, Q, R) = 0 \text{ for all distinct collinear points } P, Q, R.$$

We will call $(\overline{G}, p)$ and $(\overline{\mathcal{S}}, p)$ the algebraic m-fold covers of G and $\mathcal{S}$ respectively, **defined** by c, or say that c **defines** $(\overline{G}, p)$ and $(\overline{\mathcal{S}}, p)$ respectively.

If $(\overline{\mathcal{S}}, p)$ and $(\overline{\mathcal{S}}', p')$ are two algebraic m-fold covers of the geometry $\mathcal{S}$, defined by c and c' respectively, such that $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}}'$ are isomorphic geometries, then we say that $(\overline{\mathcal{S}}, p)$ and $(\overline{\mathcal{S}}', p')$ are **equivalent**. Where $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}}'$ are understood to be covers of $\mathcal{S}$, we say that c and c' are **equivalent**. Note that if c and c' are cohomologous, with $c' = c + \delta b$, where δ is the first coboundary operator δ^1, then c and c' are equivalent, and the map $i :\ (P, \alpha) \mapsto (P, \alpha + b(P))$ is an isomorphism from $\overline{\mathcal{S}}$ to $\overline{\mathcal{S}}'$.

Chapter 2

The Kantor ovoid $K1(\sigma)$

In this chapter we investigate the $K1(\sigma)$ ovoid of Kantor constructed in [30]. The $K1(\sigma)$ ovoid exists for $q = p^h$, where p is an odd prime, $h \geq 2$ and σ is a non-trivial automorphism of $\mathrm{GF}(q)$. For this chapter we will assume that q is of the above form. In Section 2.1 we investigate the connection between the $K1(\sigma)$ ovoid and the q-clan (see Section 1.4.5) of Kantor ([31, (4.1) Theorem]), via the work of Thas on flocks ([63]). In Section 2.2 and Section 2.3 we consider the symmetry properties of the $K1(\sigma)$ ovoid and the intersections of the $K1(\sigma)$ ovoid with the elliptic quadric ovoids of $Q(4,q)$. Section 2.5 contains some characterisations of the $K1(\sigma)$ ovoid, while Section 2.6 is dedicated to constructing rosettes of $Q(4,q)$ containing a $K1(\sigma)$ ovoid.

Let the non-singular quadric $Q^+(5,q)$ be given by the equation $x_0x_5 + x_1x_4 + x_2x_3 = 0$ in $\mathrm{PG}(5,q)$. In section 5 of [30] Kantor constructs a family of ovoids of $Q^+(5,q)$, given by the set of points

$$\{(0,0,0,0,0,1)\} \cup \{(1, y, z, z^\tau, -my^\sigma, -z^{\tau+1} + my^{\sigma+1}) : y, z \in \mathrm{GF}(q)\}$$

where σ and τ are automorphisms of $\mathrm{GF}(q)$, not both trivial, and m is a fixed non-square element of $\mathrm{GF}(q)$.

In [30], Kantor makes the comment that if we let $\tau = 1$, then the above ovoid of $Q^+(5,q)$ is contained in a hyperplane section of $Q^+(5,q)$ (with equation $x_2 = x_3$) and hence is an ovoid of the GQ $Q(4,q)$. In this case, $Q(4,q)$ is given by the equation $x_0x_4 + x_1x_3 + x_2^2 = 0$ and the ovoid comprises the following set of points

$$\{(0,0,0,0,1)\} \cup \{(1, y, z, -my^\sigma, -z^2 + my^{\sigma+1}) : y, z \in \mathrm{GF}(q)\}$$

Any ovoid of $Q(4,q)$ that is isomorphic to this set of points is a $K1(\sigma)$ ovoid of $Q(4,q)$.

2.1 Flocks and Ovoids

If we represent the above $K1(\sigma)$ ovoid as the intersection of $Q(4,q)$ with the variety defined by the equation $mx_1^\sigma + x_0^{\sigma-1}x_3 = 0$, then it may be viewed as a perturbation of the classical elliptic quadric ovoid that is the intersection of $Q(4,q)$ with the hyperplane with equation $mx_1 + x_3 = 0$. To put the $K1(\sigma)$ ovoid into a broader geometrical context, we will consider the connection between ovoids of $Q(4,q)$ and flocks of a quadratic cone of $\mathrm{PG}(3,q)$.

Let $\mathcal{C}$ be a conic of $\mathrm{PG}(2,q)$ and let $\mathcal{K}$ be the quadratic cone with vertex a point V of $\mathrm{PG}(3,q) \setminus \mathrm{PG}(2,q)$ and base $\mathcal{C}$. A **flock** of $\mathcal{K}$ is a set of q disjoint conics that partition the points of $\mathcal{K} \setminus \{V\}$ (see [21] and [19, p.254]). Note that we can equivalently define a flock to be the set of planes containing these q conics. We will swap between the two definitions according to which is more convenient. The following result, proved independently by Thas and Walker ([63] and [67], respectively), shows how we may construct an ovoid of $Q^+(5,q)$ from a flock of $\mathrm{PG}(3,q)$.

Theorem 2.1.1 *Let $\mathcal{K}$ be a quadratic cone of* $\mathrm{PG}(3,q)$ *and* $\mathcal{F}$ *a flock of* K*. Then* $\mathcal{F}$ *gives rise to an ovoid of the Klein quadric* $Q^+(5,q)$*.*

Proof: ([63]) Consider the flock $\mathcal{F} = \{\mathcal{C}_1, \ldots, \mathcal{C}_q\}$ (of conics) of the quadratic cone $\mathcal{K}$ with vertex V. Embed $\mathcal{K}$ into $\mathcal{Q} = Q^+(5,q)$ and let $\perp$ denote the polarity of $\mathcal{Q}$. Note $\mathcal{C}_i^\perp$ is a conic on $\mathcal{Q}$ ([27, Theorem 22.7.2]).

If π_i is the plane containing $\mathcal{C}_i$ then $\pi_i \cap \pi_j$ is an exterior line to $\mathcal{Q}$, for $i \neq j$, and so $\langle \pi_i^\perp, \pi_j^\perp \rangle$ is a non-singular three dimensional elliptic section of $\mathcal{Q}$. Hence no two points of $\mathcal{C}_i^\perp \cup \mathcal{C}_j^\perp$ are on a common line of $\mathcal{Q}$. Now $\langle \pi_i, \pi_j \rangle$ is the section containing $\mathcal{K}$ and so $\pi_i^\perp \cap \pi_j^\perp$ is a tangent to $\mathcal{Q}$. Thus $\mathcal{C}_i^\perp \cap \mathcal{C}_j^\perp = \{V\}$. So now the set $\mathcal{C}_1^\perp \cup \ldots \cup \mathcal{C}_q^\perp$ is a set of q^2+1 points, no two on a common line of $\mathcal{Q}$, which is an ovoid of $\mathcal{Q}$.

□

Corollary 2.1.2 *Let $\mathcal{K}$ be a quadratic cone in* $\mathrm{PG}(3,q)$ *and let* $\mathcal{F} = \{\pi_1, \ldots, \pi_q\}$ *a flock (of planes) of* $\mathcal{K}$*. If there is a point* $P \in \mathrm{PG}(3,q)$ *such that* P *is an exterior point to* $\mathcal{K}$ *and* $P \in \pi_i$ *for* $i = 1, \ldots, q$*, then* $\mathcal{F}$ *gives rise to an ovoid of* $Q(4,q)$*.*

Proof: By the construction outlined in Theorem 2.1.1, the flock $\mathcal{F}$ gives an ovoid $\mathcal{O}$ of $Q^+(5,q)$. If the planes of $\mathcal{F}$ have a common exterior point $\mathcal{P}$, then the conic planes that form $\mathcal{O}$ are contained in $P^\perp$; a common parabolic section of $Q^+(5,q)$. Such a section is isomorphic to $Q(4,q)$ and since $\mathcal{O}$ is a set of q^2+1 points of $Q(4,q)$, no 3 collinear, $\mathcal{O}$ is an ovoid of $Q(4,q)$.

□

In [63, Section 2.5.3] Thas constructs a flock of a quadratic cone of $\mathrm{PG}(3,q)$ from a q-clan. We now consider the flock $\mathcal{F}_\sigma$ constructed in this manner from the q-clan of Kantor given in [31] and derive the corresponding ovoid of $Q^+(5,q)$. We will show that the ovoid of $Q^+(5,q)$ corresponding to the flock $\mathcal{F}_\sigma$ is an ovoid of $Q(4,q)$, for $Q(4,q) \subset Q^+(5,q)$ and, in fact, is a $K1(\sigma)$ ovoid of $Q(4,q)$. Let $\mathcal{K}$ be a quadratic cone of $\mathrm{PG}(3,q)$ given by the equation $x_0x_1 = x_2^2$ (and so with vertex $V = (0,0,0,1)$), then the planes of the Kantor flock $\mathcal{F}_\sigma$ are

$$\pi_t: \quad tx_0 - mt^\sigma x_1 + x_3 = 0,$$

where $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$, $\sigma \neq$ identity, m is a fixed non-square of $\mathrm{GF}(q)$ and $t \in \mathrm{GF}(q)$.

Now embed the $\mathrm{PG}(3,q)$ containing $\mathcal{K}$ into the $\mathrm{PG}(5,q)$ containing $Q^+(5,q)$, given by the equation $x_0x_1 + x_2x_3 + x_4x_5 = 0$, so that $\mathcal{K}$ occurs as the intersection of $Q^+(5,q)$ and

$$\Sigma = \{(x_0, x_1, x_2, x_3, x_4, x_5) : \ x_2 + x_3 = 0 \text{ and } x_5 = 0\}.$$

Thus, in this representation, $\mathrm{PG}(3,q)$ is in coordinates (x_0, x_1, x_2, x_4), the cone $\mathcal{K}$ has equation $x_0x_1 - x_2^2 = 0$ and π_t is the intersection of Σ with the hyperplane of $\mathrm{PG}(5,q)$ with equation $tx_0 - mt^\sigma x_1 + x_4 = 0$.

If we let $\perp$ represent the polarity of the $Q^+(5,q)$, then

$$\pi_t^\perp = \langle (-mt^\sigma, t, 0, 0, 0, 1), (0,0,1,1,0,0), (0,0,0,0,1,0) \rangle.$$

The point $(0,0,1,-1,0,0)$ is on the plane π_t for all $t \in \mathrm{GF}(q)$ and so the ovoid constructed from $\mathcal{F}_\sigma$ is contained in the hyperplane $(0,0,1,-1,0,0)^\perp$, which has equation $x_2 = x_3$. The corresponding section of $Q^+(5,q)$ is the $Q(4,q)$ with equation $x_0x_1 + x_2^2 + x_4x_5 = 0$.

Since $x_2 = x_3$, we suppress the x_3 coordinate. In $\mathrm{PG}(4,q)$ with homogeneous coordinates $(x_0, x_1, x_2, x_4, x_5)$; $Q(4,q)$ has equation $x_0x_1 + x_2^2 + x_4x_5 = 0$ and

$$\begin{aligned} \pi_t^\perp &= \langle (0,0,0,1,0), (0,0,1,0,0), (-mt^\sigma, t, 0, 0, 1) \rangle \\ &= \{(x_0, x_1, x_2, x_4, x_5) : \ x_0 + mt^{\sigma-1}x_1 = 0 \text{ and } x_1 - tx_5 = 0\}. \end{aligned}$$

Now $\pi_t^\perp \cap \mathcal{Q} = \{(-mt^\sigma, t, z, -z^2 + mt^{\sigma+1}, 1) : \ z \in \mathrm{GF}(q)\} \cup \{(0,0,0,1,0)\}$ is a conic, $\mathcal{C}_t$, say, and so the ovoid is

$$\theta_\sigma = \{(-mt^\sigma, t, z, -z^2 + mt^{\sigma+1}, 1) : \ t, z \in \mathrm{GF}(q)\} \cup \{(0,0,0,1,0)\}.$$

This is exactly the form of the (canonical) $K1(\sigma)$ ovoid of Kantor as given in Kantor [30].

The line $\ell_\sigma = \langle(0,0,1,0,0),(0,0,0,1,0)\rangle$ is a tangent to the $Q(4,q)$ and has the property that $\ell \subset \pi_t^\perp$ for $i = 1,\ldots,q$. The line ℓ_σ is known as the **special tangent line** to θ_σ. The point $X_\sigma = (0,0,0,1,0)$ has the property that $X_\sigma \in \pi_t^\perp \cap Q(4,q)$ for $i = 1,\ldots,q$. In fact, $X_\sigma = \mathcal{C}_t \cap \mathcal{C}_{t'}$, for $t,t' \in \mathrm{GF}(q)$ and $t \neq t'$. The point X_σ is known as the **special point** of the ovoid θ_σ. Any element of the set $\{\pi_t^\perp : t \in \mathrm{GF}(q)\}$ will be called a **plane** of θ_σ.

Now if $\mathcal{F}_\sigma$ is the Kantor flock defined by fixed non-square m and field automorphism σ and $\mathcal{F}'_\sigma$ is the Kantor flock defined by fixed non-square m' and field automorphism σ, then by the proof of [31, Theorem (4.1)(i)] and the fundamental theorem of q-clan geometry ([47]), it follows that $\mathcal{F}_\sigma$ and $\mathcal{F}'_\sigma$ are projectively equivalent. That is, there is an automorphism of Σ that fixes $\mathcal{K}$ and maps $\mathcal{F}_\sigma$ onto $\mathcal{F}'_\sigma$. This is the case if and only if there is an automorphism of $\mathrm{PG}(5,q)$ that fixes $Q^+(5,q)$, $Q(4,q)$ and maps the ovoid corresponding to $\mathcal{F}_\sigma$ onto the ovoid corresponding to $\mathcal{F}'_\sigma$. Hence the ovoids are isomorphic in $Q(4,q)$. This means that the isomorphism class of $K1(\sigma)$ ovoids does not depend on the non-square m.

Similarly, if $\mathcal{F}_\sigma$ and $\mathcal{F}_\tau$ are two Kantor flocks, defined by the same fixed non-square m and field automorphisms σ and τ, respectively, then by the proof of [31, Theorem (4.1)(i)] and the fundamental theorem of q-clan geometry ([47]), it follows that $\mathcal{F}_\sigma$ and $\mathcal{F}'_\sigma$ are projectively equivalent if and only if $\tau = \sigma$ or σ^{-1}. By a similar argument to above, the ovoids corresponding to $\mathcal{F}_\sigma$ and $\mathcal{F}_\tau$, respectively, are isomorphic in $Q(4,q)$ if and only if $\tau = \sigma$ or σ^{-1}.

As we shall see in Chapter 3, the Kantor flock $\mathcal{F}_\sigma$ is interesting not only because it gives the $K1(\sigma)$ ovoid but also because the q-clan GQ associated with $\mathcal{F}_\sigma$ contains a $Q(4,q)$ subGQ, in which the subtended ovoids are $K1(\sigma)$ ovoids.

2.2 Symmetry properties of the $K1(\sigma)$ ovoid

Recall in Section 2.1 that θ_σ was an ovoid of $Q(4,q)$ In this case $Q(4,q)$ was the non-singular quadric in $\mathrm{PG}(4,q)$, in coordinates (x_0,x_1,x_2,x_4,x_5), defined by the equation $x_0x_1+x_2^2+x_4x_5 = 0$, lying on the Klein quadric $Q^+(5,q)$ in $\mathrm{PG}(5,q)$. Since from this section onwards we will not be using the embedding of $Q(4,q)$ in $Q^+(5,q)$, we define $Q(4,q)$ to be the non-singular quadric in $\mathrm{PG}(4,q)$, in coordinates (x_0,x_1,x_2,x_3,x_4), defined by the equation $x_0x_1+x_2^2+x_3x_4 = 0$. In this representation we let

$$\begin{array}{rcl}
\theta_\sigma & = & \{(-mt^\sigma, t, z, -z^2+mt^{\sigma+1}, 1) : t,z \in \mathrm{GF}(q)\} \cup \{(0,0,0,1,0)\} \\
X_\sigma & = & (0,0,0,1,0) \\
\ell_\sigma & = & \langle(0,0,0,1,0),(0,0,1,0,0)\rangle \\
\pi_t & : & x_0 + mt^{\sigma-1}x_1 = 0 \\
 & : & x_1 - tx_4 = 0, \text{ for } t \in \mathrm{GF}(q).
\end{array}$$

Note that we are now using π_t for the plane $\pi_t^\perp$ in Section 2.1, and so we will let $\mathcal{C}_t$ denote the conic that is the intersection of π_t and $Q(4,q)$. As in Section 2.1 any plane π_t for $t \in \mathrm{GF}(q)$ (that contains the special line ℓ_σ and meets θ_σ in a conic) is called a **plane** of θ_σ.

In this section we will investigate particular elements of the stabiliser of θ_σ, in the group of $Q(4,q)$ and what they tell us about the symmetry properties of θ_σ. By the construction of the $K1(\sigma)$ ovoid from the flock $\mathcal{F}_\sigma$, we can see that any element of the stabiliser of θ_σ, in the group of $Q(4,q)$, must necessarily fix the point X_σ and the line ℓ_σ.

For the calculations that follow it will be useful to consider the point $P_t \in \pi_t$ with the coordinates $(-mt^\sigma, t, 0, 0, 1)$, so $\pi_t = \langle \ell_\sigma, P_t\rangle$. For $t \in \mathrm{GF}(q) \setminus \{0\}$, let S_t be the automorphism of $\mathrm{PG}(4,q)$ defined by

$$S_t(x_0,x_1,x_2,x_3,x_4) = (t^\sigma x_0, tx_1, \sqrt{t^{\sigma+1}}x_2, t^{\sigma+1}x_3, x_4).$$

Note that S_t fixes $Q(4,q)$, X_σ and ℓ_σ. Also

$$\begin{aligned} S_t(P_{t'}) &= (-mt^\sigma t'^\sigma, t', 0, 0, 1) \\ &= (-m(tt')^\sigma, tt', 0, 0, 1) \\ &= P_{tt'}, \end{aligned}$$

which means that $S_t : \pi_{t'} \longmapsto \pi_{tt'}$ and thus S_t fixes θ_σ.

For $t \in \mathrm{GF}(q)$ let T_t be the automorphism of $\mathrm{PG}(4,q)$ defined by

$$T_t(x_0, x_1, x_2, x_3, x_4) = (-mt^\sigma x_4 + x_0, tx_4 + x_1, x_2, -tx_0 + mt^\sigma x_1 + x_3 + mt^{\sigma+1}x_4, x_4).$$

The automorphism T_t fixes $Q(4,q)$, X_σ and ℓ_σ. Also

$$\begin{aligned} T_t(P_{t'}) &= (-mt^\sigma - mt'^\sigma, t + t', 0, -, 1) \\ &= (-m(t+t')^\sigma, t+t', 0, -, 1) \\ &\in \pi_{t+t'}, \end{aligned}$$

which means that $T_t : \pi_{t'} \longmapsto \pi_{t+t'}$ and thus T_t fixes θ_σ.

From the automorphisms T_t it is immediately obvious that the group of θ_σ is transitive on the planes $\pi_{t'}$. However, by using both the S and T automorphisms we can say more than this, since

$$\begin{aligned} T_t S_{t'-t}(\pi_0, \pi_1) &= T_t(\pi_0, \pi_{t'-t}) \\ &= (\pi_t, \pi_{t'}) \end{aligned}$$

Thus the group of θ_σ is transitive on the ordered pairs of planes $(\pi_t, \pi_{t'})$, $t, t' \in \mathrm{GF}(q)$ and $t \neq t'$. This is equivalent to saying that the Kantor flock $\mathcal{F}_\sigma$ is doubly transitive.

Now we consider the orbits of the group of the canonical $K1(\sigma)$ ovoid θ_σ on the points of θ_σ. One orbit consists of the special point X_σ. Suppose that P and P' are two points of the set $\mathcal{C}_t \setminus \{X_\sigma\}$. Let the point Q be the intersection of the line ℓ_σ with the line $\langle P, P' \rangle$. Recall from Section 1.2 the symmetry μ_Q, where for $R \in Q(4,q)$ we have

$$\mu_Q(R) = \begin{cases} R & \text{if } \langle R, Q \rangle \text{ is tangent to } Q(4,q) \\ R' & \text{if } \langle R, Q \rangle \text{ meets } Q(4,q) \text{ at } R' \neq R, \end{cases}$$

is the automorphism of $\mathrm{PG}(4,q)$ that fixes $Q(4,q)$. In fact fixes each subspace of $Q(4,q)$ containing Q. Thus μ_Q fixes ℓ_σ, X_σ, $\pi_{t'}$ for $t' \in \mathrm{GF}(q)$ and hence the ovoid θ_σ. We also have that $\mu_Q(P) = (P')$. Now suppose that P and P' are two distinct points of θ_σ, where $\langle P, \ell_\sigma \rangle = \pi_t$ and $\langle P', \ell_\sigma \rangle = \pi_{t'}$ for some $t, t' \in \mathrm{GF}(q)$ with $t \neq t'$. Since we have transitivity on the planes of θ_σ there exists U, an element of the stabiliser of θ_σ, that maps π_t to $\pi_{t'}$. Hence, $U(P)$ is an element of $\mathcal{C}_{t'} \setminus \{X_\sigma\}$ and by the above may be mapped onto P' by an element of the stabiliser of θ_σ (possibly 0). We may summarise the above in the following theorem.

Theorem 2.2.1 *Let θ_σ be the canonical $K1(\sigma)$ ovoid of $Q(4,q)$. Let X_σ be the special point of θ_σ and let $\{\pi_t : t \in \mathrm{GF}(q)\}$ be the set of planes of θ_σ through the special line ℓ_σ meeting θ_σ in a conic. The group of θ_σ is 2-transitive on the planes of θ_σ and has two orbits on the points of θ_σ. The two orbits on the points of θ_σ are $\{X_\sigma\}$ and $\theta_\sigma \setminus \{X_\sigma\}$.*

2.3 Intersections of θ_σ and elliptic quadrics on $Q(4,q)$, containing X_σ

In this section we investigate the intersection of the canonical $K1(\sigma)$ ovoid θ_σ and the elliptic quadric ovoids of $Q(4,q)$, given that the special point of θ_σ, X_σ, is contained in the intersection. Let Σ be a three dimensional projective subspace of $\mathrm{PG}(4,q)$ that intersects the $Q(4,q)$ in a non-singular elliptic quadric $\mathcal{O}$. Suppose that $X_\sigma \in \mathcal{O}$, then we calculate the intersection of θ_σ and $\mathcal{O}$. We separate this intersection problem into the case where $\ell_\sigma \subset \Sigma$ and the case where $\ell_\sigma \not\subset \Sigma$.

2.3.1 Case 1: $\ell_\sigma \subset \Sigma$

Now since $\ell_\sigma \subset \Sigma$ it follows that if Σ contains a point P of θ_σ, then it contains the whole plane of θ_σ containing P. Thus the intersection of θ_σ and the elliptic quadric $\Sigma \cap Q(4,q)$ is the union of a number of the conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ (possibly 0).

Now let $\Sigma_{t,t'} = \langle \pi_t, \pi_{t'} \rangle$, then $\Sigma_{t,t'}$ is a three dimensional projective subspace that intersects $Q(4,q)$ in an elliptic quadric. The equation of $\Sigma_{t,t'}$ is

$$(t-t')x_0 + m(t^\sigma - t'^\sigma)x_1 + (mt'^\sigma t - mt^\sigma t')x_4 = 0.$$

Using the 2-transitivity on the planes of θ_σ, we need only consider the elliptic quadric $\mathcal{O}_{0,1}$ on $Q(4,q)$ that is the intersection of $Q(4,q)$ and the three-dimensional subspace $\Sigma_{0,1}$, which has equation $x_0 + mx_1 = 0$. We now ask how many of the planes π_t are contained in $\Sigma_{0,1}$, which will in turn tell us the intersection size of the θ_σ and $\mathcal{O}_{0,1}$. Now

$$\begin{aligned} \pi_t \subset \Sigma_{0,1} &\iff P_t \in \Sigma_{0,1} \\ &\iff -mt^\sigma + mt = 0 \\ &\iff t^\sigma = t, \end{aligned}$$

and so $\pi_t \subset \Sigma_{0,1}$ if and only if $t \in Fix(\sigma)$, where $Fix(\sigma)$ is the fixed field of the automorphism σ. Let $|Fix(\sigma)| = k$, then $|\Sigma_{0,1} \cap K1(\sigma)| = kq+1$ and consequently

$$|\Sigma_{t,t'} \cap \theta_\sigma| = kq+1,$$

where $t, t' \in \mathrm{GF}(q)$ with $t \neq t'$. That is, if a three-dimensional subspace of $\mathrm{PG}(4,q)$ contains two elements of the set $\{\pi_t : t \in \mathrm{GF}(q)\}$ (and so necessarily meets $Q(4,q)$ in an elliptic quadric), then it contains exactly k. Thus there are $\binom{q}{2}/\binom{k}{2} = q(q-1)/k(k-1)$ elliptic quadric ovoids of $Q(4,q)$ intersecting the θ_σ in k conic planes.

For a given π_t there are $(q-1)/(k-1)$ three-dimensional subspaces of $\mathrm{PG}(4,q)$ containing π_t and containing k elements of the set $\{\pi_t : t \in \mathrm{GF}(q)\}$. In total there are $(q-1)/2$ three-dimensional subspaces of $\mathrm{PG}(4,q)$ containing π_t and meeting $Q(4,q)$ in an elliptic quadric and so there are $(q-1)(1/2 - 1/(k-1))$ elliptic quadrics ovoids of $Q(4,q)$ meeting θ_σ in exactly the conic $\mathcal{C}_t = \pi_t \cap Q(4,q)$, that is, with an intersection size of $q+1$.

This leaves

$$\begin{aligned} & \frac{q(q-1)}{2} - \frac{q(q-1)}{k(k-1)} - q(q-1)(\frac{1}{2} - \frac{1}{k-1}) \\ = \quad & q(q-1)\left[\frac{1}{2} - \frac{1}{k(k-1)} - \frac{1}{2} + \frac{1}{k-1}\right] \\ = \quad & \frac{q(q-1)}{k} \end{aligned}$$

elliptic quadrics ovoids of $Q(4,q)$ which intersect θ_σ in exactly X_σ.

2.3.2 Case 2: $\ell_\sigma \not\subset \Sigma$ but $X_\sigma \in \Sigma$

Now suppose that $X_\sigma \in \mathcal{O}$ but that $\ell_\sigma \not\subset \Sigma$. For $t \in \mathrm{GF}(q)$, let $\ell_t = \pi_t \cap \Sigma$. Now $X_\sigma \in \ell_t$ and $\ell_t \neq \ell_\sigma$ so it follows that ℓ_t is a secant line to the conic $\mathcal{C}_t = \pi_t \cap Q(4,q)$. Thus we have that $|\theta_\sigma \cap \mathcal{O}| = q+1$. There are $q^2(q-1)/2$ elliptic quadric ovoids of $Q(4,q)$ containing X_σ and $q(q-1)/2$ of these sections have an ambient three space containing ℓ_σ. Thus there are

$$\frac{q^2(q-1)}{2} - \frac{q(q-1)}{2} = \frac{q(q-1)^2}{2}$$

elliptic quadric ovoids of $Q(4,q)$ intersecting θ_σ in the above way.

We summarise the intersection sizes of the two cases above in the following theorem

Theorem 2.3.1 *Let θ_σ be the canonical $K1(\sigma)$ ovoid of $Q(4,q)$ and let X_σ be the special point of θ_σ. Suppose that Σ is a hyperplane of $\mathrm{PG}(4,q)$ that intersects $Q(4,q)$ in a non-singular elliptic quadric $\mathcal{O}$ and that $X_\sigma \in \mathcal{O}$. If $\ell_\sigma \subset \Sigma$, then $|\theta_\sigma \cap \mathcal{O}| = q+1$ or $k(q+1)$, where k is the size of the fixed field of σ. If $\ell_\sigma \not\subset \Sigma$, then $|\theta_\sigma \cap \mathcal{O}| = q+1$.*

2.4 More properties of the $\Sigma_{t,t'}$

In this section we investigate some more properties of the set of three dimensional subspaces $\{\Sigma_{t,t'} : t,t' \in \mathrm{GF}(q) \text{ with } t \neq t'\}$, where $\Sigma_{t,t'}$ has equation $(t-t')x_0 + m(t^\sigma - t'^\sigma)x_1 + (mt'^\sigma t - mt^\sigma t')x_4 = 0$.

Now in Section 2.3.1 we saw that

$$\pi_{t''} \in \Sigma_{0,1} \Longleftrightarrow t'' \in Fix(\sigma),$$

and since $S_{1/(t'-t)}T_{-t}(\Sigma_{t,t'}) = \Sigma_{0,1}$, it follows that

$$\pi_{t''} \in \Sigma_{t,t'} \Longleftrightarrow S_{1/(t'-t)}T_{-t}(\pi_{t''}) \in \Sigma_{0,1}.$$

Thus we have

$$\pi_{t''} \in \Sigma_{t,t'} \Longleftrightarrow \frac{t''-t}{t'-t} \in Fix(\sigma).$$

First we determine when the intersection of $\Sigma_{t,t'}$ and $\Sigma_{t'',t'''}$ is a plane of θ_σ (recall π is a plane of θ_σ if it is an element of the set $\{\pi_t : t \in \mathrm{GF}(q)\}$). To do this we first consider the conditions under which $\Sigma_{0,1} \cap \Sigma_{t,t'}$ is a plane of θ_σ. Now by the above

$$\Sigma_{0,1} \cap \Sigma_{t,t'} = \pi_{t''} \Longleftrightarrow t'' \in Fix(\sigma) \text{ and } \frac{t''-t}{t'-t} \in Fix(\sigma).$$

Now

$$\begin{aligned} t'', \frac{t''-t}{t'-t} \in Fix(\sigma) &\Longleftrightarrow \left(\frac{t''-t}{t'-t}\right)^\sigma = \frac{t''-t}{t'-t} \text{ and } t'' \in Fix(\sigma), \\ &\Longleftrightarrow (t''-t^\sigma)(t'-t) = (t'^\sigma - t^\sigma)(t''-t) \text{ and } t'' \in Fix(\sigma), \\ &\Longleftrightarrow t'' = \frac{tt'^\sigma - t't^\sigma}{t'^\sigma - t^\sigma - t' + t} \in Fix(\sigma). \end{aligned}$$

Thus it follows that $\Sigma_{0,1} \cap \Sigma_{t,t'}$ is a plane of θ_σ if and only if

$$\frac{tt'^\sigma - t't^\sigma}{t'^\sigma - t^\sigma - t' + t} \in Fix(\sigma).$$

The conditions under which the intersection $\Sigma_{t,t'}$ and $\Sigma_{t'',t'''}$ is a plane of θ_σ can be determined by using the transitivity of the group of θ_σ on the set $\{\Sigma_{s,s'} : s,s' \in \mathrm{GF}(q) \text{ and } s \neq s'\}$.

Note that for $|\sigma| = 2$, $(tt'^\sigma - t't^\sigma)/(t'^\sigma - t^\sigma - t' + t)$ is fixed by σ and so $\Sigma_{t,t'} \cap \Sigma_{t',t''}$ is always a plane of θ_σ.

Second we investigate the elliptic quadric ovoids of $Q(4,q)$ whose ambient three space contains k planes of θ_σ. Since $\pi_t \subset \Sigma_{0,1}$ if and only if $t \in Fix(\sigma)$ it follows that $\pi_t \subset T_{t'}(\Sigma_{0,1})$ if and only if $t \in Fix(\sigma) + t'$. Thus the additive cosets of $Fix(\sigma)$ partition the planes of θ_σ into q/k sets of size k, each set contained in a distinct three-dimensional subspace of $\mathrm{PG}(4,q)$.

Now $\Sigma_{0,1} : x_0 + mx_1 = 0$ meets $Q(4,q)$ in the elliptic quadric with the equation $-mx_1^2 + x_2^2 + x_3x_4 = 0$. The tangent plane at X_σ to this elliptic quadric is the plane $\pi' : x_4 = 0$. Thus π' has equations $x_0 + mx_1 = 0$ and $x_4 = 0$, and $\pi' = \langle(-m,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0)\rangle$. Since $T_{t'}(-m,1,0,0,0) = (-m,1,0,mt^\sigma + mt, 0)$, which is an element of π', it follows that $T_{t'}$ fixes π'. Thus for $t,t' \in \mathrm{GF}(q)$, with $t \neq t'$, the hyperplanes $T_t(\Sigma_{0,1})$ and $T_{t'}(\Sigma_{0,1})$ meet in the plane π'. We generalise this set of hyperplanes to give the following definition.

Definition 2.4.1 *Let θ be a $K1(\sigma)$ ovoid of $Q(4,q)$. Let τ be a set of q/k distinct hyperplanes of* $\mathrm{PG}(4,q)$*, each meeting $Q(4,q)$ in a non-singular elliptic quadric and each containing the special tangent line of θ. Suppose that any two elements of τ intersect in a fixed plane π', that is tangent to $Q(4,q)$. If each plane of θ is a subspace of an element of τ, then we say that the set τ is a* **trade** *of θ. The plane π' is called the* **base plane** *of the trade τ.*

Note that it follows from Theorem 2.3.1 that if τ is a trade of the $K1(\sigma)$ ovoid θ, then each element of τ contains exactly k planes of θ.

Note also that since any two elements of a trade intersect in a fixed plane tangent to $Q(4,q)$, it follows that the set of q/k elliptic quadric ovoids of $Q(4,q)$ formed by the intersection of the trade with $Q(4,q)$ meet pairwise in a fixed point. Suppose that ρ is a set of q/k $K1(\sigma)$ ovoids meeting pairwise in a fixed point and with a common trade τ. In Section2.6 we shall see that if $\mathcal{R}$ is a rosette containing ρ, then we may take $\mathcal{R}$, "trade" ρ for the set of elliptic quadric ovoids corresponding to τ, and form a new rosette.

Our original model for the definition of a trade of a $K1(\sigma)$ ovoid of $Q(4,q)$ was to take the canonical ovoid θ_σ and consider the set $\tau = \{T_t(\Sigma_{0,1}) : t \in \mathrm{GF}(q)\}$ and the plane π' with equations $x_4 = 0$ and $x_0 + mx_1 = 0$. By transitivity on the planes of θ_σ it follows that every hyperplane of the form $\Sigma_{t,t'}$ is contained in a trade of θ_σ. Now consider the point $(-m,1,0,0,0) \in \pi' \setminus \ell_\sigma$. We have that $(-m,1,0,0,0) \in \Sigma_{tt'}$ if and only if $t-t' \in Fix(\sigma)$ and so $(-m,1,0,0,0)$ is contained in exactly q/k of the $\Sigma_{t,t'}$. Consequently, the plane π' is contained in exactly q/k of the $\Sigma_{t,t'}$, and so is the base plane for exactly one trade of θ_σ, namely the trade $\tau = \{T_t(\Sigma_{0,1}) : t \in \mathrm{GF}(q)\}$. Now since for any element $T_t(\Sigma_{0,1})$ of τ, π' is the unique plane of $T_t(\Sigma_{0,1})$ that is tangent to $Q(4,q)$ at X_σ, it follows that $T_t(\Sigma_{0,1})$ is contained in exactly one trade of θ_σ. Hence it follows that every hyperplane of the form $\Sigma_{t,t'}$ is contained in exactly one trade of θ_σ (the unique trade with base plane the tangent plane to $\Sigma_{t,t'} \cap Q(4,q)$ at X_σ).

We conclude this section with a summary theorem.

Theorem 2.4.2 *Let $\theta_\sigma = \{(-mt^\sigma, t, z, -z^2 + mt^{\sigma+1}, 1) : t, z \in \mathrm{GF}(q)\} \cup \{(0,0,0,1,0)\}$ be the canonical $K1(\sigma)$ ovoid of $Q(4,q)$, where $Q(4,q)$ is defined by the equation $x_0x_1 + x_2^2 + x_3x_4 = 0$, with special point $X_\sigma = (0,0,0,1,0)$. Let $\{\Sigma_{t,t'} : t,t' \in \mathrm{GF}(q)\}$ be the set of hyperplanes of* $\mathrm{PG}(4,q)$ *where $\Sigma_{t,t'} : (t-t')x_0 + m(t^\sigma - t'^\sigma)x_1 + (mt'^\sigma t - mt^\sigma t')x_4 = 0$. Each hyperplane $\Sigma_{t,t'}$ meets $Q(4,q)$ in a non-singular elliptic quadric and contains $k = |Fix(\sigma)|$ planes of θ_σ. Further*

(i) If $t - t' \notin Fix(\sigma)$ then the plane $\Sigma_{0,1} \cap \Sigma_{t,t'}$ is a plane of θ_σ if and only if

$$\frac{tt'^\sigma - t't^\sigma}{t'^\sigma - t^\sigma - t' + t} \in Fix(\sigma).$$

(ii) Each $\Sigma_{t,t'}$ is contained in a unique trade of θ_σ, with base plane $X_\sigma^\perp \cap \Sigma_{t,t'}$. The plane $X_\sigma^\perp \cap \Sigma_{t,t'}$ is the base plane for exactly one trade of θ_σ.

2.5 Characterisations of the $K1(\sigma)$ ovoid

In this section we give three characterisations of the $K1(\sigma)$ ovoid. The first two are in terms of the conic structure of the ovoid. The third is a characterisation in terms of the trade structure of the $K1(\sigma)$ ovoid.

Theorem 2.5.1 *([63]) Let θ be a set of points of $Q(4,q)$ consisting of q conics, $\mathcal{C}_1, \ldots, \mathcal{C}_q$ intersecting pairwise in a point X, that is, $\mathcal{C}_i \cap \mathcal{C}_j = \{X\}$ for $i,j = 1,2,\ldots,q$, $i \neq j$. Let $\pi_1, \ldots, \pi_q$ be the planes of* $\mathrm{PG}(4,q)$ *such that $\mathcal{C}_i \subset \pi_i$, for $i = 1,\ldots,q$. Suppose that ℓ is a fixed line of* $\mathrm{PG}(4,q)$ *such that $\pi_i \cap \pi_j = \ell$ for $i,j \in \mathrm{GF}(q)$, $i \neq j$ and that no hyperplane of* $\mathrm{PG}(4,q)$ *contains all the π_i. Suppose also that for each pair $i,j \in \mathrm{GF}(q)$, $i \neq j$ the hyperplane $\langle \pi_i, \pi_j \rangle$ intersects $Q(4,q)$ in a non-singular elliptic quadric. Then the set θ is isomorphic to the $K1(\sigma)$ ovoid.*

Proof: Suppose that we have such a set of points θ. Embed $Q(4,q)$ in $Q^+(5,q)$, which has polarity denoted by $\perp$. Then $\pi_1^\perp, \dots, \pi_q^\perp$ forms a flock $\mathcal{F}$ (of planes) of the cone $\ell^\perp \cap Q^+(5,q)$. Since θ is not contained in a hyperplane of $\mathrm{PG}(4,q)$, and so it follows that $\mathcal{F}$ is non-linear (that is, the planes of the $\mathcal{F}$ do not intersect pairwise in a fixed line). Each plane $\pi_i^\perp$ contains the exterior point $\mathrm{PG}(4,q)^\perp$ and so by [63, Section 1.5.6] $\mathcal{F}$ is a Kantor flock $\mathcal{F}_\sigma$ for some field automorphism σ and so θ is a $K1(\sigma)$ ovoid.

□

Note that the automorphism is only determined up to the size of its fixed field, which is equal to the number of elements of the set $\{\pi_1, \pi_2 \dots, \pi_q\}$ contained in the space $\langle \pi_i, \pi_j \rangle$.

Theorem 2.5.2 *([63] and [22], see [6, Lemma 15]) Let θ be a non-classical ovoid of $Q(4,q)$ consisting of the q conics $\mathcal{C}_1, \dots, \mathcal{C}_q$, that meet pairwise in a fixed point X. Then θ is isomorphic to the $K1(\sigma)$ ovoid, for some $\sigma \in aut(\mathrm{GF}(q))$.*

Proof: In general let π_i be the plane containing $\mathcal{C}_i$. Then for any $i, j \in \mathrm{GF}(q)$ with $i \neq j$, $\mathcal{C}_i \cup \mathcal{C}_j$ is a set of $2q+1$ points of $Q(4,q)$, no two collinear. Suppose that $\pi_i \cap \pi_j = \{X\}$ for some $i, j \in \mathrm{GF}(q)$ with $i \neq j$.

If $\perp$ represents the polarity of $Q(4,q)$, then $\pi_j^\perp$ is a line of $\mathrm{PG}(4,q)$, which is either a secant to $Q(4,q)$ or external to $Q(4,q)$. Let H_j be the set of three-dimensional subspaces of $\mathrm{PG}(4,q)$ containing π_j and intersecting $Q(4,q)$ in a non-singular hyperbolic quadric. If $\pi_j^\perp$ is a secant line to $Q(4,q)$, then $|H_j| = (q-1)/2$ and if $\pi_j^\perp$ is an external to $Q(4,q)$, then $|H_j| = (q+1)/2$. Now let Σ and Σ' be two distinct elements of H_j. Since the intersection of π_i and π_j is a single point, it follows that $\langle \pi_i, \pi_j \rangle$ is $\mathrm{PG}(4,q)$ and that neither Σ nor Σ' contains π_i. Thus $\ell = \Sigma \cap \pi_i$ and $\ell' = \Sigma \cap \pi_j$ are both lines of π_i, and in fact distinct lines of π_i since if $\ell = \ell'$, then $\Sigma = \Sigma'$. Now since X is incident with both ℓ and ℓ' and π_i contains a unique line that is tangent to $\mathcal{C}_i$ at X, it follows that at least one of ℓ and ℓ' is a secant to $\mathcal{C}_i$. Thus, without loss of generality we may suppose that Σ contains a point of $\mathcal{C}_i \setminus \{X\}$, say Y. Since $Y \notin \mathcal{C}_j$ and $\langle \pi_j, Y \rangle$ meets $Q(4,q)$ in a non-singular hyperbolic quadric, it follows that Y is collinear with two points of $\mathcal{C}_j$, which gives a contradiction. Thus $\pi_i \cap \pi_j$ is a line tangent to $Q(4,q)$ at X for all $i, j \in \mathrm{GF}(q)$, with $i \neq j$. Further if we let ℓ_1 be the (unique) tangent to $\mathcal{C}_1$ at X, it follows by the above that $\pi_1 \cap \pi_j = \ell_1$ for $i = 2, \dots, q$. Thus ℓ is contained in all of the planes π_i and so is the pairwise intersection of any two of them. Since $\mathcal{C}_i \cup \mathcal{C}_j$ is a set of points with no two collinear it follows that $\langle \pi_i, \pi_j \rangle$ intersects $Q(4,q)$ in a non-singular elliptic quadric. Thus by Theorem 2.5.1, the ovoid θ is isomorphic to the $K1(\sigma)$ ovoid for some $\sigma \in aut(\mathrm{GF}(q))$.

□

Theorem 2.5.3 *Let $\mathcal{Q} = Q(4,q)$, let ℓ be a line tangent to $\mathcal{Q}$ and let $X = \ell \cap \mathcal{Q}$. Let $\sigma \in aut(\mathrm{GF}(q))$ and $k = |Fix(\sigma)|$. Let $\mathcal{S} = \left\{\tau_{1,1}, \tau_{1,2}, \dots, \tau_{1,\frac{q}{k}}, \tau_{2,1}, \dots, \tau_{2,\frac{q}{k}}, \dots, \tau_{\frac{q-1}{k-1},\frac{q}{k}}\right\}$ be a set of $q(q-1)/k(k-1)$ hyperplanes of $\mathrm{PG}(4,q)$, each element of which contains ℓ and meets $\mathcal{Q}$ in a non-singular elliptic quadric. Suppose that the following two conditions hold:*

(i) $\tau_{i,j} \cap \tau_{i,j'} = \alpha_i$ is a tangent plane to $\mathcal{Q}$ at X and $\alpha_i \neq \alpha_k$ if $i \neq k$.

(ii) For each $\tau_{i,j}$ there are k sets of the following form:

$$\left\{\tau_{i,j}, \tau_{i_2,j_2}, \dots, \tau_{i_{\frac{q-1}{k-1}}, j_{\frac{q-1}{k-1}}}\right\}$$

such that elements of each set intersect pairwise in the same plane of $\mathrm{PG}(4,q)$, which intersects $Q(4,q)$ in a conic and is contained in no other $\tau_{i',j'}$.

Then the points of the conics in (ii) form an ovoid of $\mathcal{Q}$ isomorphic to the $K1(\sigma)$ ovoid.

Proof: Since each pair of sets of hyperplanes in (ii) intersect in exactly one element, it follows that the span of any two planes in (ii) is a hyperplane intersecting $\mathcal{Q}$ in an elliptic quadric. Hence if we can show that there are q conics in (ii), then by Theorem 2.5.2 we have a $K1(\sigma)$ ovoid. If we let the number of planes be a then counting the incident (plane, $\tau_{i,j}$) pairs for the planes of the conics in (ii) yields

$$a \cdot \frac{q-1}{k-1} = \frac{q}{k} \cdot \frac{q-1}{k-1} \cdot k$$

and so $a = q$.

□

Note that in Theorem 2.5.3 the trades of the $K1(\sigma)$ ovoid are the sets $\{\tau_{i,j} : j = 1, 2, \ldots, q/k\}$ for $i = 1, 2, \ldots, q/k$.

2.6 Rosettes containing $K1(\sigma)$ ovoids

In this section we will construct a number of types of rosettes of $Q(4,q)$ that contain $K1(\sigma)$ ovoids. We will consider the two types of elation rosettes of $K1(\sigma)$ ovoids (where the basepoint is a non-special point and a special point, respectively), non-elation rosettes consisting entirely of $K1(\sigma)$ ovoids and rosettes containing both $K1(\sigma)$ ovoid and elliptic quadric ovoids.

We will conclude the section by constructing all of such rosettes of the GQ $Q(4,9)$, with the aid of a computer search conducted by Gordon Royle ([54]).

2.6.1 Elation rosettes of $K1(\sigma)$ ovoids

Let θ be an ovoid of $Q(4,q)$, q odd, and P a fixed point of θ. We recall the construction of the $q+1$ elation rosettes containing θ and with base point P, from Section 1.4.6. In the dual of $Q(4,q)$, $W(q)$, θ is a spread $\mathtt{S}$ of $W(q)$ and P a line ℓ of $\mathtt{S}$. If Q is any point of $W(q)$ (and so of PG$(3,q)$) incident with ℓ, then $W(q)$ admits the elations of PG$(3,q)$ with axis $Q^\perp$ and centre Q (where $\perp$ is the polarity of $W(q)$). The image of $\mathtt{S}$ under this group of elations is a rosette of spreads of $W(q)$. Since there are $q+1$ points of $W(q)$ incident with ℓ, it follows that there are $q+1$ such rosettes. Dualising, these are the $q+1$ elation rosettes containing θ and with base point P. Now for the case where θ is an elliptic quadric ovoid there is a unique rosette $\mathcal{R}$ of elliptic quadric ovoids containing θ and with basepoint P. If π is the tangent plane to θ at P, then $\mathcal{R}$ is the set of elliptic quadric ovoids for which π is also a tangent plane. Thus the $q+1$ elation rosettes containing θ and based at P coincide. In Chapter 4 (Lemma 4.1.7) we will see that a similar result holds if θ is a Tits-type ovoid (that is, if the image of θ under an isomorphism from $Q(4,q)$ to $W(q)$ is a Tits ovoid). This, however, is not true in general for elation rosettes of $Q(4,q)$, since if θ is a fixed $K1(\sigma)$ ovoid and P a non-special point of θ, then elation rosettes containing θ and with base point P are not all identical:

Theorem 2.6.1 *Suppose that θ is a $K1(\sigma)$ ovoid of $Q(4,q)$ and that P is a point of $Q(4,q)$ such that $P \in \theta$, but that P is not the special point of θ. Then the $q+1$ elation rosettes containing θ and with base point P are distinct. Further, any two such rosettes intersect in exactly θ.*

Proof: We will work in $W(q)$ the dual of $Q(4,q)$. Let $\mathtt{S}$ be the dual of θ, ℓ be the dual of P, ℓ_∞ the dual of the special point of θ and let $\perp$ represent the polarity (symplectic) of $W(q)$. If Q is a point of ℓ, then let E_Q be the group of elations of PG$(3,q)$ with axis $Q^\perp$ and centre Q. Let $\mathcal{R}_Q$ be the elation rosette of spreads $\{T(\mathtt{S}) : T \in E_Q\}$, containing $\mathtt{S}$ and with base line ℓ.

Now $Q^\perp$ is a plane of PG$(3,q)$ not containing ℓ_∞ and so $Q^\perp \cap \ell_\infty$ is a point of PG$(3,q)$, Q_∞ say. Since Q_∞ is contained in the axis of the elation group E_Q, it is fixed by the elements of E_Q and so the special line of each element of $\mathcal{R}_Q$ contains Q_∞. Further, since ℓ_∞, the special line of $\mathtt{S}$, is not contained in the plane $Q^\perp$, the axis of the elations in E_Q, it follows that each element of $\mathcal{R}_Q$ has a distinct special line. Thus the set of special lines of spreads of $\mathcal{R}_Q$ is the set of q lines of $W(q)$, not ℓ, incident with Q_∞ (note that the lines in this set are all contained in the plane $Q_\infty^\perp$). If Q and Q' are two distinct point of ℓ, then clearly the sets of special lines of the corresponding elation rosettes $\mathcal{R}_Q$ and $\mathcal{R}_{Q'}$ are concurrent at different points of $W(q)$, and so $\mathcal{R}_Q$ and $\mathcal{R}_{Q'}$ are distinct.

Now suppose that Q and Q' are two distinct points of ℓ such that $\mathcal{R}_Q$ and $\mathcal{R}_{Q'}$ contain a common spread $\mathtt{S}'$. Let the intersection of ${Q'}^\perp$ and ℓ_∞ be the point Q'_∞. Now if $Q_\infty = Q'_\infty$, then $\{Q_\infty, Q, Q'\}$ is a triangle in $W(q)$, and so $Q_\infty \neq Q'_\infty$. By the above, since $\mathtt{S}'$ is an element of both $\mathcal{R}_Q$ and $\mathcal{R}_{Q'}$, it follows that the special line of $\mathtt{S}'$ is incident with both Q_∞ and Q'_∞. This means that the special line of $\mathtt{S}'$ is $\langle Q_\infty, Q'_\infty\rangle$, but this line is ℓ_∞, the special line of $\mathtt{S}$. Thus $\mathtt{S} \cong \mathtt{S}'$.

□

Now we consider the elation rosettes containing a fixed $K1(\sigma)$ ovoid θ and with base point P, where P is the special point of θ. We will start with elations in $W(q)$ the dual of $Q(4,q)$ and dualise (via the Klein quadric) to get the elation rosettes of the $K1(\sigma)$ on its special point. The first result we prove is that all of the elation rosettes coincide.

Theorem 2.6.2 *Let θ be a $K1(\sigma)$ ovoid of $Q(4,q)$ and P the special point of θ. Then the $q+1$ elation rosettes of $Q(4,q)$ containing θ and with base point P, coincide.*

Proof: Let $\mathcal{Q}' = Q^+(5,q)$ be the Klein quadric with equation $x_0x_1 + x_2x_3 + x_4x_5 = 0$. We will consider the case where $Q(4,q)$ is the intersection of $\mathcal{Q}'$ and the hyperplane of $\mathrm{PG}(5,q)$ with equation $x_2 = x_3$. In this setting we abuse notation by saying the canonical $K1(\sigma)$ ovoid θ_σ has the form

$$\theta_\sigma = \{(-mt^\sigma, t, z, z, -z^2 + mt^{\sigma+1}, 1) : t, z \in \mathrm{GF}(q)\} \cup \{(0,0,0,0,1,0)\}$$

with special point $X_\sigma = (0,0,0,0,1,0)$ and special tangent line $\ell_\sigma = \langle (0,0,0,0,1,0), (0,0,0,1,0,0) \rangle$. Let $\mathtt{S}_{\theta_\sigma}$ be the dual spread of θ_σ in $W(q)$ and ℓ_{X_σ} the dual line of X_σ in $W(q)$. We will first establish the form of the elations of $\mathrm{PG}(3,q)$, with centre $Q \in \ell_{X_\sigma}$ and axis $Q^\perp$ (where $\perp$ denotes the polarity of $W(q)$), as automorphisms of $Q(4,q)$.

If $\ell = \langle (x_0,x_1,x_2,x_3), (y_0,y_1,y_2,y_3) \rangle$ is a line of $\mathrm{PG}(3,q)$, then the Klein correspondence λ, mapping lines of $\mathrm{PG}(3,q)$ to points of $\mathcal{Q}'$, sends ℓ to the point

$$(\ell_{01}, \ell_{23}, \ell_{02}, \ell_{31}, \ell_{03}, \ell_{12}) \text{ where } \ell_{ij} = x_iy_j - x_jy_i$$

(see [25, Chapter 15]). The Klein correspondence defines a duality from $W(q)$, defined by the equation $x_0y_2 - x_2y_0 - x_3y_1 + x_1y_3 = 0$, to $Q(4,q) \subset \mathcal{Q}'$. We shall denote the bilinear form associated with $W(q)$ by $\beta((x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)) = x_0y_2 - x_2y_0 - x_3y_1 + x_1y_3$.

Now the special line of the spread $\mathtt{S}_{\theta_\sigma}$ is

$$\begin{aligned} \ell_{X_\sigma} &= \lambda^{-1}(X_\sigma) \\ &= \lambda^{-1}(0,0,0,0,1,0) \\ &= \langle (1,0,0,0), (0,0,0,1) \rangle. \end{aligned}$$

If U is a point of $W(q)$, then the group of elations with centre U and axis $U^\perp$ is $\{t_{a,U} : V \mapsto V + a\beta(U,V)U \mid a \in \mathrm{GF}(q)\}$. The automorphism of $Q(4,q)$ equivalent to $t_{a,U}$ is the restriction of the automorphism $\lambda t_{a,U}\lambda^{-1}$ of $\mathcal{Q}$ to $Q(4,q)$. If we let $U_s = (s,0,0,1)$ and $U_\infty = (1,0,0,0)$, then $\ell_{X_\sigma} = \{U_s : s \in \mathrm{GF}(q)\} \cup \{U_\infty\}$.

Now $t_{a,U_s}(V) = V + a(-sv_2 + v_1)U_s = (v_0 - as^2v_2 + asv_1, v_1, v_2, v_3 - asv_2 + av_1)$ which acts on the Plucker coordinates by

$$\begin{aligned}
\ell_{01} &\mapsto (x_0 - as^2x_2 + asx_1)y_1 - x_1(y_0 - as^2y_2 + asy_1) \\
&\quad = \ell_{01} + as^2\ell_{12}, \\
\ell_{02} &\mapsto (x_0 - as^2x_2 + asx_1)y_2 - x_2(y_0 - as^2y_2 + asy_1) \\
&\quad = \ell_{02} + as\ell_{12}, \\
\ell_{03} &\mapsto (x_0 - as^2x_2 + asx_1)(y_3 - asy_2 + ay_1) - (x_3 - asx_2 + ax_1)(y_0 - as^2y_2 + asy_1), \\
&\quad = \ell_{03} - as\ell_{02} + a\ell_{01} - as^2\ell_{23} - as\ell_{31} \\
\ell_{12} &\mapsto x_1y_2 - x_2y_1 \\
&\quad = \ell_{12}, \\
\ell_{31} &\mapsto (x_3 - asx_2 + ax_1)y_1 - x_1(y_3 - asy_2 + ay_1) \\
&\quad = \ell_{31} + as\ell_{12}, \\
\ell_{23} &\mapsto x_2(y_3 - asy_2 + ay_1) - (x_3 - asx_2 + ax_1)y_2 \\
&\quad = \ell_{23} - a\ell_{12}.
\end{aligned}$$

Thus

$$\begin{aligned} &\lambda t_{a,U_s}\lambda^{-1}(x_0,x_1,x_2,x_3,x_4,x_5) \\ &= (x_0 + as^2x_5, x_1 - ax_5, x_2 + asx_5, x_3 + asx_5, x_4 + ax_0 - as^2x_1 - asx_2 - asx_3, x_5), \end{aligned}$$

When we restrict this to to $Q(4,q)$ we have the automorphism

$$\begin{aligned} &E_{a,s}(x_0,x_1,x_2,x_2,x_4,x_5) \\ &= (x_0 + as^2x_5, x_1 - ax_5, x_2 + asx_5, x_2 + asx_5, x_4 + ax_0 - as^2x_1 - 2asx_2, x_5). \end{aligned}$$

We now repeat the above for the elations t_{a,U_∞}.

Now $t_{a,U_\infty}(V) = V + a(-v_2)U_\infty = (v_0 - av_2, v_1, v_2, v_3)$ and so acts on the Plucker coordinates by

$$(\ell_{01}, \ell_{02}, \ell_{03}, \ell_{12}, \ell_{31}, \ell_{23}) \mapsto (\ell_{01} + a\ell_{12}, \ell_{02}, \ell_{03} - a\ell_{23}, \ell_{12}, \ell_{31}, \ell_{23}),$$

that is

$$\lambda t_{a,U_\infty} \lambda^{-1}(x_0, x_1, x_2, x_3, x_4, x_5) = (x_0 + ax_5, x_1, x_2, x_3, x_4 - ax_1, x_5).$$

When restricted to to $Q(4,q)$ we have the automorphism

$$E_{a,\infty}(x_0, x_1, x_2, x_2, x_4, x_5) = (x_0 + ax_5, x_1, x_2, x_2, x_4 - ax_1, x_5).$$

The $q+1$ elation rosettes of $Q(4,q)$ containing θ_σ and with special point $X_\sigma = (0,0,0,0,1,0)$ are

$$\begin{aligned} \mathcal{R}_s &= \{E_{a,s}(\theta_\sigma) : a \in \mathrm{GF}(q)\} \quad \text{for } s \in \mathrm{GF}(q) \\ \mathcal{R}_\infty &= \{E_{a,\infty}(\theta_\sigma) : a \in \mathrm{GF}(q)\}. \end{aligned}$$

We now show that $\mathcal{R}_\infty = \mathcal{R}_s$ for any $s \in \mathrm{GF}(q)$. Consider the ovoid $E_{a,s}(\theta_\sigma) \in \mathcal{R}_s$, then

$$\begin{aligned} E_{a,s}(\theta_\sigma) = \; & \{(-mt^\sigma + as^2, t - a, z + as, z + as, -z^2 + mt^{\sigma+1} - mat^\sigma - as^2t - 2asz, 1) : \\ & t, z \in \mathrm{GF}(q)\} \cup \{(0,0,0,0,1,0)\} \end{aligned}$$

Now let $a' = as^2 - ma^\sigma$, then consider

$$E_{a',\infty}(\theta_\sigma) = \{(-mt^\sigma + a', t, z, z, -z^2 + mt^{\sigma+1} - a't, 1) : z, t \in \mathrm{GF}(q)\} \cup \{(0,0,0,0,1,0)\}$$

$= \{(-mt^\sigma + ma^\sigma + a', t - a, z, z, -z^2 + ma^{\sigma+1} - mat^\sigma - ma^\sigma t + mt^{\sigma+1} + aa' - a't,$
$1) : z, t \in \mathrm{GF}(q)\} \cup \{(0,0,0,0,1,0)\}$ by replacing the parameter t by $t - a$,

$= \{(-mt^\sigma + as^2, t - a, z + as, z + as, -z^2 - 2asz - a^2s^2 + ma^{\sigma+1} - mat^\sigma - ma^\sigma t +$
$mt^{\sigma+1} - aa' - a't, 1) : z, t \in \mathrm{GF}(q)\} \cup \{(0,0,0,0,1,0)\}$ by replacing z by $z + as$,

$= \{(-mt^\sigma + as^2, t - a, z + as, z + as, -z^2 - 2asz - mat^\sigma + mt^{\sigma+1} - as^2t, 1)$
$: z, t \in \mathrm{GF}(q)\} \cup \{(0,0,0,0,1,0)\}$ by substituting $a' = as^2 - ma^\sigma$,

$= E_{a,s}(\theta_\sigma)$.

Thus each ovoid of $\mathcal{R}_s$ is contained in $\mathcal{R}_\infty$, and so $\mathcal{R}_s = \mathcal{R}_\infty$ for all $s \in \mathrm{GF}(q)$. From this it follows that the $q+1$ elation rosettes coincide.

□

We now consider $Q(4,q)$ to be the quadric in $\mathrm{PG}(4,q)$ defined by the equation $x_0x_1 + x_2^2 + x_3x_4 = 0$ and the canonical $K1(\sigma)$ ovoid θ_σ to be

$$\theta_\sigma = \{(-mt^\sigma, t, z, -z^2 + mt^{\sigma+1}, 1) : t, z \in \mathrm{GF}(q)\} \cup \{(0,0,0,1,0)\}$$

with special point $X_\sigma = (0,0,0,1,0)$ and special tangent line $\ell_\sigma = \langle (0,0,0,1,0), (0,0,1,0,0) \rangle$. In this setting the elations derived in the proof of Theorem 2.6.2 have the form

$$\begin{aligned} E_{a,s}(x_0, x_1, x_2, x_3, x_4) &= (x_0 + as^2x_4, x_1 - ax_4, x_2 + asx_4, x_3 + ax_0 - as^2x_1 - 2asx_2, x_4), \\ E_{a,\infty}(x_0, x_1, x_2, x_3, x_4) &= (x_0 + ax_4, x_1, x_2, x_3 - ax_1, x_4) \end{aligned}$$

for $a, s \in \mathrm{GF}(q)$.

Let $\mathcal{R}_\sigma$ be the elation rosette of $Q(4,q)$ containing θ_σ and with base point X_σ, the special point of θ_σ. In the following we investigate the structure of $\mathcal{R}_\sigma$ by calculating the action of the elation group $\{E_{a,0} : a \in \mathrm{GF}(q)\}$ on $\mathcal{R}_\sigma$.

Theorem 2.6.3 *Let θ_1 be a $K1(\sigma)$ ovoid of $Q(4,q)$ with special point P. Let $\mathcal{R} = \{\theta_1, \theta_2, \dots, \theta_q\}$ be the (unique) elation rosette of $Q(4,q)$ containing θ_1 and based at P. If Ω_i is the set $\{\pi' : \pi'$ is a base plane of a trade of $\theta_i\}$, then $\Omega_i = \Omega_j$, for all $1 \leq i, j \leq q$ with $i \neq j$. Further, if τ_i is any trade of θ_i and τ_j is any trade of θ_j, then τ_i and τ_j are either equal or disjoint.*

Proof: We prove the result for the case where $Q(4,q)$ is defined by the equation $x_0x_1 + x_2^2 + x_3x_4 = 0$ and $\theta_1 = \theta_\sigma$, the canonical $K1(\sigma)$ ovoid.

The rosette $\mathcal{R}$ is generated as the set of images of θ_σ under the group of elations $\{E_{a,0} : a \in \mathrm{GF}(q)\}$, that is $\mathcal{R} = \{E_{a,0}(\theta_\sigma) : a \in \mathrm{GF}(q)\}$. Let $\mathcal{B}_a$ be the set of base planes of trades of the ovoid $E_{a,0}(\theta_\sigma)$. We first show that $\mathcal{B}_a = \mathcal{B}_{a'}$ for all $a, a' \in \mathrm{GF}(q)$ with $a \neq a'$.

Recall from Theorem 2.4.2(ii) that the three-dimensional subspace of $\mathrm{PG}(4,q)$ $\Sigma_{t,t'}$ (which contains $k = |Fix(\sigma)|$ planes of θ_σ) is contained in a unique trade of θ_σ and the base plane of this trade is $X_\sigma^\perp \cap \Sigma_{t,t'}$. Now since the group $\{S_\eta : \eta \in \mathrm{GF}(q)^*\}$ (where S_η is the automorphism of $Q(4,q)$ fixing θ_σ, introduced in Section 2.2), also fixes $X_\sigma^\perp$ and is transitive on the set of $\Sigma_{t,t'}$, it follows that the set of base planes of trades of θ_σ is

$$\mathcal{B}_0 = \{\overline{\pi}_\eta = S_\eta(\Sigma_{0,1} \cap X_\sigma^\perp) = \Sigma_{0,\eta} \cap X_\sigma^\perp : \eta \in \mathrm{GF}(q)^*\}.$$

Note that the size of this set is $(q-1)/(k-1)$ and that the plane $\overline{\pi}_\eta$ has equations

$$\begin{aligned} \overline{\pi}_\eta \quad : \quad & \eta x_0 + m\eta^\sigma x_1 = 0 \\ & x_4 = 0. \end{aligned}$$

Now let $A_\eta \in \overline{\pi}_\eta \backslash \ell_\sigma$ be the point with coordinates $(-m\eta^{\sigma-1}, 1, 0, 0, 0)$, and so $\overline{\pi}_\eta = \langle \ell_\sigma, A_\eta \rangle$. The point $E_{a,0}(A_\eta) = (-m\eta^{\sigma-1}, 1, 0, -ma\eta^{\sigma-1}, 0) \in \overline{\pi}_\eta$ and $E_{a,0}(\ell_\sigma) = \ell_\sigma$ from which it follows that $E_{a,0}(\overline{\pi}_\eta) = \overline{\pi}_\eta$. Since $\mathcal{B}_a$ is the set of base planes of the ovoid $E_{a,0}(\theta_\sigma) \in \mathcal{R}$, it follows that

$$\begin{aligned} \mathcal{B}_a &= E_{a,0}(\mathcal{B}_0) \\ &= \{E_{a,0}(\overline{\pi}_\eta) : \eta \in \mathrm{GF}(q)^*\} \\ &= \{\overline{\pi}_\eta : \eta \in \mathrm{GF}(q)^*\} \\ &= \mathcal{B}_0, \end{aligned}$$

and so $\mathcal{B}_a = \mathcal{B}_{a'}$ for all $a, a' \in \mathrm{GF}(q)$ with $a \neq a'$.

Now let τ be a trade of $E_{a,0}(\theta_\sigma)$, with base plane $\overline{\pi}$ and τ' a trade of $E_{a',0}(\theta_\sigma)$, with base plane $\overline{\pi}'$. We show that τ and τ' are either equal or disjoint. If Σ is an element of τ and τ', then it contains both $\overline{\pi}$ and $\overline{\pi}'$. Now $\overline{\pi}$ and $\overline{\pi}'$ are both planes tangent to $Q(4,q)$ at the point X_σ, but Σ contains a unique plane tangent to $\Sigma \cap Q(4,q)$ at X_σ, thus $\overline{\pi} = \overline{\pi}'$. So if τ and τ' have different base planes, then they are disjoint. We now consider the case where τ and τ' share the same base plane, that is $\overline{\pi} = \overline{\pi}'$. Since $\mathcal{B}_a = \mathcal{B}_0$ for all $a \in \mathrm{GF}(q)$, it follows that $\overline{\pi} = \overline{\pi}' \in \mathcal{B}_0$, that is $\overline{\pi} = \overline{\pi}_\eta \in \mathcal{B}_0$ for some $\eta \in \mathrm{GF}(q)^*$. Now by the above $E_{a,0}(\overline{\pi}_\eta) = E_{a',0}(\overline{\pi}_\eta) = \overline{\pi}_\eta$, and so if τ_η is the unique trade of θ_σ with base plane $\overline{\pi}_\eta$, then $\tau = E_{a,0}(\tau_\eta)$ and $\tau' = E_{a',0}(\tau_\eta)$. Recall from Section 2.2 that the trade of θ_σ with base plane $\overline{\pi}_1$ (that is, the trade containing $\Sigma_{0,1}$) is $\tau_1 = \{T_\alpha(\Sigma_{0,1}) : \alpha \in \mathrm{GF}(q)\}$ where T_α is the automorphism of $Q(4,q)$ fixing θ_σ introduced in Section 2.2. Now since $\overline{\pi}_\eta = S_\eta(\overline{\pi}_1)$ it follows that

$$\begin{aligned} \tau_\eta &= S_\eta(\tau_1) \\ &= \{S_\eta T_\alpha(\Sigma_{0,1}) : \alpha \in \mathrm{GF}(q)\} \\ &= \{\Sigma_{\eta\alpha, \eta(\alpha+1)} : \alpha \in \mathrm{GF}(q)\} \text{ (see Section 2.2)} \\ &= \{[\eta, m\eta^\sigma, 0, 0, m\eta^{\sigma+1}(\alpha - \alpha^\sigma)] : \alpha \in \mathrm{GF}(q)\} \end{aligned}$$

Now $E_{a,0}([\eta, m\eta^\sigma, 0, 0, m\eta^{\sigma+1}(\alpha - \alpha^\sigma)]) = [\eta, m\eta^\sigma, 0, 0, m\eta^\sigma(\eta\alpha - \eta\alpha^\sigma + a)]$, and so

$$\tau = E_{a,0}(\tau_\eta) = \{[\eta, m\eta^\sigma, 0, 0, m\eta^\sigma(\eta\alpha - \eta\alpha^\sigma + a)] : \alpha \in \mathrm{GF}(q)\}.$$

Similarly

$$\tau' = E_{a,0}(\tau_\eta) = \{[\eta, m\eta^\sigma, 0, 0, m\eta^\sigma(\eta\alpha - \eta\alpha^\sigma + a')] : \alpha \in \mathrm{GF}(q)\}.$$

Consequently $\tau = \tau'$ if $a - a' = \eta(\beta - \beta^\sigma)$ for some $\beta \in \mathrm{GF}(q)$ and τ and τ' are disjoint if $a - a' \neq \eta(\beta - \beta^\sigma)$ for all $\beta \in \mathrm{GF}(q)$.

□

If π' is a base plane of a trade of an ovoid of $\mathcal{R}$, then π' is called a **base plane** of $\mathcal{R}$ and any trade of an ovoid of $\mathcal{R}$ is called a **trade** of $\mathcal{R}$.

Corollary 2.6.4 *Let θ_1 be a $K1(\sigma)$ ovoid of $Q(4,q)$ with special point P. Let $\mathcal{R} = \{\theta_1, \theta_2, \ldots, \theta_q\}$ be the (unique) elation rosette of $Q(4,q)$ containing θ_1 and based at P. If π' is a base plane of $\mathcal{R}$, then the set of trades of $\mathcal{R}$, with base plane π' partitions the set of hyperplanes that contain π' and intersect $Q(4,q)$ in a non-singular elliptic quadric.*

If π' is a base plane of $\mathcal{R}$, then the set $\mathcal{T}$ of trades of $\mathcal{R}$ with base plane π' as base plane is called a **tower** of $\mathcal{R}$ and π' is called the **base** of the tower $\mathcal{T}$.

2.6.2 Rosettes, trades and towers

In this section we will use the ideas of a trade of a $K1(\sigma)$ ovoid and a tower of an elation rosette of $K1(\sigma)$ ovoids to construct a variety of rosettes of $Q(4,q)$, each of which contains $K1(\sigma)$ ovoids.

The general idea of the constructions is as follows. Let $\mathcal{S}$ be a GQ of order s. Let $\mathcal{R} = \{\theta_1, \theta_2, \ldots, \theta_s\}$ and $\mathcal{R}' = \{\theta'_1, \theta'_2, \ldots, \theta'_s\}$ be two rosettes of ovoids of $\mathcal{S}$, that have the same base point P. Suppose that as sets of points

$$\theta'_1 \cup \theta'_2 \ldots \cup \theta'_r = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_r,$$

then both $\{\theta'_1, \theta'_2, \ldots, \theta'_r, \theta_{r+1}, \theta_{r+2}, \ldots, \theta_s\}$ and $\{\theta_1, \theta_2, \ldots, \theta_r, \theta'_{r+1}, \theta'_{r+2}, \ldots, \theta'_s\}$ are rosettes of $\mathcal{S}$ with base point P.

Now we consider the particular case where $\mathcal{S} = Q(4,q)$ and $\mathcal{R}$ is an elation rosette of $K1(\sigma)$ ovoids of $Q(4,q)$ with base point P, where P is the special point of the rosettes of $\mathcal{R}$. Let $\mathcal{T}$ be a tower of $\mathcal{R}$ with base π and let $\mathcal{R}' = \{\theta'_1, \theta'_2, \ldots, \theta'_q\}$ be the rosette of elliptic quadric ovoids with tangent plane π. Let $\tau = \{\theta_1, \theta_2, \ldots, \theta_{q/k}\} \in \mathcal{T}$ be a trade of $\mathcal{R}$ such that, without loss of generality, the elements of τ intersect the $Q(4,q)$ in the elliptic quadric ovoids $\theta'_1, \theta'_2, \ldots, \theta'_k$ (where $k = |Fix(\sigma)|$) and the k ovoids of $\mathcal{R}$ that have τ as a trade are $\theta_1, \theta_2, \ldots, \theta_k$. Then the sets of ovoids $\{\theta'_1, \theta'_2, \ldots, \theta'_k, \theta_{k+1}, \theta_{k+2}, \ldots, \theta_q\}$ and $\{\theta_1, \theta_2, \ldots, \theta_k, \theta'_{k+1}, \ldots, \theta'_q\}$, are both a rosette of ovoids of $Q(4,q)$ with base point P, containing both $K1(\sigma)$ ovoids and elliptic quadric ovoids.

We can generalise these construction in the following way. Let the elements of the tower $\mathcal{T}$ of $\mathcal{R}$ be $\tau_1, \tau_2, \ldots, \tau_{q/k}$. Let the set of k ovoids of $\mathcal{R}$ that have τ_i as a trade be denoted $\overline{\tau_i}$. Then the set of ovoids $\rho_1 \cup \rho_2 \cup \ldots \cup \rho_{q/k}$, where $\rho_i = \tau_i$ or $\overline{\tau_i}$ for each i, is a rosette of ovoids of $Q(4,q)$ with base point P. Note that this construction includes both $\mathcal{R}$ and $\mathcal{R}'$. If π and π' are two distinct tangent planes to the $Q(4,q)$ at P and θ and θ' are two elliptic quadric ovoids with π and π' as tangent planes, respectively, then θ and θ' intersect in a conic. Thus, in the above construction all the trades τ_i must come from the same tower of $\mathcal{R}$.

A natural extension of the above construction provides a construction of rosettes of $Q(4,q)$ containing all $K1(\sigma)$ ovoids, but that are not elation rosettes.

Theorem 2.6.5 *Let $\mathcal{R}$ be an elation rosette of $K1(\sigma)$ ovoids of $Q(4,q)$ and let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_{q/k}\}$ be a tower of $\mathcal{R}$. Let ρ_i be a set of q/k $K1(\sigma)$ ovoids contained in a common elation rosette (not necessarily $\mathcal{R}$), and sharing the trade τ_i. Then the set of ovoids $\rho_1 \cup \rho_2 \cup \ldots \cup \rho_{q/k}$ is a rosette of $Q(4,q)$.*

We are interested in determining whether rosettes constructed as in Theorem 2.6.5 exist. We do this by looking for $K1(\sigma)$ ovoids, not contained in $\mathcal{R}$ that share a trade with $\mathcal{R}$. This is the direction we pursue in the next two sections. To complete this section we summarise all of the above constructions of rosettes.

Theorem 2.6.6 *Let $\mathcal{R}$ be an elation rosette of $K1(\sigma)$ ovoids of $Q(4,q)$ and let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_{q/k}\}$ be a tower of $\mathcal{R}$. Let ρ_i be either*

1. *a set of q/k $K1(\sigma)$ ovoids contained in a common elation rosette (not necessarily $\mathcal{R}$), and sharing the trade τ_i; or*
2. *the set of q/k elliptic quadric ovoids of $Q(4,q)$ whose elements are the intersection of the elements of τ_i with $Q(4,q)$.*

Then the set of ovoids $\rho_1 \cup \rho_2 \cup \ldots \cup \rho_{q/k}$ is a rosette of $Q(4,q)$.

$K1(\sigma)$ ovoids sharing a trade and with the same special tangent line

Our aim in this section is to find examples of $K1(\sigma)$ ovoids that have the same special tangent line as the canonical $K1(\sigma)$ ovoid θ_σ and share a trade with θ_σ. From Section 2.6.1 we have the example of the ovoids of the elation rosette $\mathcal{R}_\sigma$ containing θ_σ and with base point X_σ. Here we calculate whether there are any examples of $K1(\sigma)$ ovoids sharing a trade and the special tangent line with θ_σ, that are not in $\mathcal{R}_\sigma$.

We first consider collineations of $\mathrm{PG}(4,q)$ that fix $Q(4,q)$ (with equation $x_0x_1+x_2^2+x_3x_4 = 0$) and ℓ_σ, the special tangent line of θ_σ, which map θ_σ onto $K1(\sigma)$ ovoids with special tangent line ℓ_σ. Now suppose that θ is a $K1(\sigma)$ ovoid with special tangent line ℓ_σ and such that θ and θ_σ share the trade τ. Since by Theorem 2.2.1 the group of θ_σ is 2-transitive on the planes of θ_σ, we may assume that τ is the trade of θ_σ containing $\Sigma_{0,1}$. Also, the group of θ is 2-transitive on the planes of θ, so we consider collineations of $\mathrm{PG}(4,q)$ that fix $Q(4,q)$, $\Sigma_{0,1}$ and ℓ_σ.

Let $\Upsilon = T\alpha$ be the collineation of $\mathrm{PG}(4,q)$ consisting of the automorphic collineation α (we will use α to represent both the automorphic collineation of $\mathrm{PG}(4,q)$ and the automorphism of $\mathrm{GF}(q)$) and the homography T, which has matrix $[t_{ij}]$ where $0 \le i,j \le 4$. From this point we will associate the homography T with the matrix $[t_{ij}]$. Now we require that Υ fixes $\Sigma_{0,1}$, ℓ_σ and X_σ, and hence fixes $\Sigma_{0,1}^\perp = (m,1,0,0,0)$, $X_\sigma = (0,0,0,1,0)$ and ℓ_σ; so

$$\begin{aligned}\Upsilon(0,0,0,1,0) &= (0,0,0,1,0),\\ \Upsilon(m,1,0,0,0) &= (m,1,0,0,0) \text{ and}\\ \Upsilon(0,0,1,0,0) &= (0,0,1,a,0)\end{aligned}$$

for some $a \in \mathrm{GF}(q)$. Now we also require that Υ fixes $\ell_\sigma^\perp = [0,0,1,0,0] \cap [0,0,0,0,1]$. The intersection of ℓ_σ with $Q(4,q)$ is the line pair $\ell_1 = \langle(0,0,0,1,0),(1,0,0,0,0)\rangle$ and $\ell_2 = \langle(0,0,0,1,0),(0,1,0,0,0)\rangle$. We will first consider the case where $\Upsilon(\ell_1) = \ell_1$ and $\Upsilon(\ell_2) = \ell_2$. Thus

$$\begin{aligned}\Upsilon(1,0,0,0,0) &= (1,0,0,b,0) \text{ and}\\ \Upsilon(0,1,0,0,0) &= (0,1,0,c,0)\end{aligned}$$

for some $b, c \in \mathrm{GF}(q)$. Thus T has the form:

$$T = \begin{pmatrix} t_{00} & 0 & 0 & 0 & t_{04}\\ 0 & t_{00}m^{\alpha-1} & 0 & 0 & t_{14}\\ 0 & 0 & t_{22} & 0 & t_{24}\\ bt_{00} & ct_{00}m^{\alpha-1} & at_{22} & t_{33} & t_{34}\\ 0 & 0 & 0 & 0 & t_{44}\end{pmatrix}, \text{ such that } mb+c=0.$$

We also require that Υ fixes $Q(4,q)$. Since $(x_0x_1+x_2^2+x_3x_4)^\alpha = x_0^\alpha x_1^\alpha + (x_2^\alpha)^2 + x_3^\alpha x_4^\alpha$, it follows that the automorphic collineation α fixes $Q(4,q)$. Thus to find the conditions under which Υ

fixes $Q(4,q)$ we find the conditions under which T fixes $Q(4,q)$. These conditions are:

$$\begin{aligned} m^{\alpha-1}t_{00}^2 = t_{22}^2 &= t_{33}t_{44}, \\ t_{00}t_{14} + bt_{00}t_{44} &= 0, \\ t_{00}t_{04}m^{\alpha-1} + t_{00}t_{44}m^{\alpha-1}c &= 0, \\ 2t_{22}t_{24} + at_{22}t_{44} &= 0, \\ t_{04}t_{14} + t_{24}^2 + t_{34}t_{44} &= 0. \end{aligned}$$

We normalise by letting $t_{44} = 1$ and let $t_{00} = s \in \mathrm{GF}(q) \setminus \{0\}$. Thus, T has the form

$$T = \begin{pmatrix} s & 0 & 0 & 0 & -c \\ 0 & sm^{\alpha-1} & 0 & 0 & -b \\ 0 & 0 & \pm\gamma s & 0 & -a/2 \\ bs & csm^{\alpha-1} & \pm a\gamma s & m^{\alpha-1}s^2 & -bc - a^2/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.6.1}$$

for $b, c \in \mathrm{GF}(q)$ such that $mb + c = 0$, $s \in \mathrm{GF}(q) \setminus \{0\}$ and $\gamma = m^{(\alpha-1)/2}$.

If G_σ is the group fixing $Q(4,q)$, $\Sigma_{0,1}$ and ℓ_σ, then the subgroup of G_σ consisting of the elements of G_σ that fix ℓ_1 (and hence also fix ℓ_2), is a subgroup of index 2 in G_σ. We will denote this subgroup of G_σ by G'_σ. The group G'_σ consists of collineations of the form $T\alpha$ where T is of the form in 2.6.1. If we define

$$W(x_0, x_1, x_2, x_3, x_4) = (mx_1, x_0/m, x_2, x_3, x_4),$$

then W is a collineation of $\mathrm{PG}(4,q)$ fixing $Q(4,q)$, $\Sigma_{0,1}$ and ℓ_σ and swapping ℓ_1 and ℓ_2. Thus we have that

$$G_\sigma = G'_\sigma \cup WG'_\sigma.$$

We are now interested in elements of G_σ that fix the trade τ of θ_σ that contains $\Sigma_{0,1}$, since the image of θ_σ under such a collineation is a $K1(\sigma)$ ovoid for which τ is also a trade. The group of such elements is the stabiliser of τ in G_σ which we will denote by $(G_\sigma)_\tau$. Using the description of τ in the proof of Theorem 2.6.3 we may easily show that τ consists of the hyperplanes of $\mathrm{PG}(4,q)$ with coordinates $[1, m, 0, 0, u]$ where $u = m(t^\sigma - t)$ for some $t \in \mathrm{GF}(q)$. First we consider the action of $\Upsilon \in G'_\sigma$ on the polar point of the hyperplane $[1, m, 0, 0, u]$ which is given by

$$\begin{aligned} \Upsilon(m, 1, 0, u, 0) &= T(m^\alpha, 1, 0, u^\alpha, 0) \\ &= (sm^\alpha, sm^{\alpha-1}, 0, m^{\alpha-1}s^2u^\alpha, 0) \\ &\equiv (m, 1, 0, su^\alpha, 0) \end{aligned}$$

Thus the collineation $\Upsilon = T\alpha \in G'_\sigma$ (where T has the form of 2.6.1) fixes the trade τ if and only if for each $t \in \mathrm{GF}(q)$ there is a $t' \in \mathrm{GF}(q)$ such that $s(m(t^\sigma - t))^\alpha = m(t'^\sigma - t')$. That is, for each $t \in \mathrm{GF}(q)$ there is a $t' \in \mathrm{GF}(q)$ such that

$$sm^\alpha(t^{\alpha\sigma} - t^\alpha) = m(t'^\sigma - t') \tag{2.6.2}$$

This is equivalent to the single condition

$$sm^{\alpha-1}\{t^\sigma - t : \ t \in \mathrm{GF}(q)\} = \{t^\sigma - t : \ t \in \mathrm{GF}(q)\}. \tag{2.6.3}$$

Solving this problem in general is messy so we will instead consider a subset of the solutions, which will correspond to a subgroup of the stabiliser of τ in G'_σ.

We consider the elements $\Upsilon = T\alpha$ of G'_σ (so T is of the form of 2.6.1) such that $sm^{\alpha-1} \in Fix(\sigma)$ from which it follows that Υ satisfies 2.6.3. Let two such collineations be $\Upsilon = T\alpha$ and $\Upsilon' = T'\alpha'$. Now $(T\alpha)(T'\alpha') = (TT'^\alpha)(\alpha\alpha')$ (where T'^α is the matrix obtained by operating on each element of T' with α) and

$$(ss'^\alpha)m^{\alpha\alpha'-1} = (sm^{\alpha-1})(s'm^{\alpha'-1})^\alpha.$$

Now $Fix(\sigma)$ is the unique subfield of GF(q) of order k, so $Fix(\sigma)^\alpha = Fix(\sigma)$ which implies that $(s'm^{\alpha'-1})^\alpha$ is an element of $Fix(\sigma)$. Thus $(sm^{\alpha-1})(s'm^{\alpha'-1})^\alpha \in Fix(\sigma)$ and

$$H'_\sigma = \{\Upsilon = T\alpha : \Upsilon \in G'_\sigma \text{ and } sm^{\alpha-1} \in Fix(\sigma)\}$$

is a subgroup of G'_σ. Now for any automorphism α of GF(q) there are $k-1$ possibilities for s such that $sm^{\alpha-1} \in \text{GF}(q)$, which gives a total of $h(k-1)$ pairs (s,α). Further in the matrix T b is any element of GF(q), c is dependent on b and a may be any element of GF(q). Hence the order of G'_σ is $2(k-1)hq^2$.

Note that the elements of the set $\{t^\sigma - t : t \in \text{GF}(q)\}$ are in a one-to-one correspondence to the additive cosets of $Fix(\sigma)$. Thus the set has size q/k and the maximum number of solutions, in x, to $x\{t^\sigma - t : t \in \text{GF}(q)\} = \{t^\sigma - t : t \in \text{GF}(q)\}$ is q/k. But if $x \in Fix(\sigma)$, then x is clearly a solution so there are at least k solutions. In the case where $k = q/k$ the maximum number of solutions in x is equal to the number of solutions where $x \in Fix(\sigma)$. Thus every $\Upsilon \in G'_\sigma$ that satisfies 2.6.3 is an element of H'_σ, that is, H'_σ is the full stabiliser of τ in G'_σ. Note that $k = q/k$ if and only if σ^2 is the identity automorphism of GF(q).

Now $W(m,1,0,0,0) = (m,1,0,0,0)$ and so the trade $\tau = \{[1,m,0,0,u] : u = m(t^\sigma - t) \text{ for some } t \in \text{GF}(q)\}$ is fixed by W. Since H'_σ is a subgroup of G'_σ each element of H'_σ fixes ℓ_1 and it follows that the set WH'_σ is a set of elements of G_σ that swap ℓ_1 and ℓ_2. We now show that $H_\sigma = H'_\sigma \cup WH'_\sigma$ is a subgroup of G_σ by displaying that multiplication amongst elements of H_σ is closed.

Let h and h' be two elements of H_σ. If $h, h' \in H'_\sigma$, then $hh' \in H'_\sigma \subset H_\sigma$. If $h \in WH'_\sigma$, say $h = WT\alpha$ and $h \in H'_\sigma$, then $hh' = W(T\alpha h') \in WH'_\sigma \subset H_\sigma$. If $h, h' \in WH'_\sigma$, say $h = WT\alpha$ and $h' = WT'\alpha'$ where

$$T = \begin{pmatrix} s & 0 & 0 & 0 & -c \\ 0 & sm^{\alpha-1} & 0 & 0 & -b \\ 0 & 0 & \pm\gamma s & 0 & -a/2 \\ bs & csm^{\alpha-1} & \pm a\gamma s & m^{\alpha-1}s^2 & -bc - a^2/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } mb + c = 0 \text{ and } \gamma = m^{(\alpha-1)/2},$$

then

$$WTW^\alpha = \begin{pmatrix} s & 0 & 0 & 0 & -mb \\ 0 & sm^{\alpha-1} & 0 & 0 & -c/m \\ 0 & 0 & \pm\gamma s & 0 & -a/2 \\ cs/m & bsm^\alpha & \pm a\gamma s & m^{\alpha-1}s^2 & -bc - a^2/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and so $(WTW^\alpha)\alpha$ is an element of H'_σ. Thus $hh' = (WT\alpha)(WT'\alpha') = (WTW^\alpha)(\alpha T'\alpha') = (WTW^\alpha)\alpha(T'\alpha') \in H'_\sigma \subset H_\sigma$. Finally, if $h \in H'_\sigma$, $h = T\alpha$ say and $h' \in WH'_\sigma$, $h' = WT'\alpha'$ say, then

$$\begin{aligned} hh' &= (T\alpha)(WT'\alpha') \\ &= (W^2T\alpha)(WT'\alpha') \quad \text{since } W^2 \text{ is the identity} \\ &= W(WT\alpha)(WT'\alpha') \\ &\in WH'_\sigma \subset H_\sigma. \end{aligned}$$

We have shown that multiplication of elements of H_σ is closed and so H_σ is a subgroup of $(G_\sigma)_\tau$. The order of H_σ is $4h(k-1)q^2$ where $q = p^h$.

Note that in the case where σ^2 is the identity H_σ is the full stabiliser of τ in G_σ, that is, $H_\sigma = (G_\sigma)\tau$.

Now H_σ is a group of collineations of PG$(4,q)$ fixing $Q(4,q)$, $\Sigma_{0,1}$, ℓ_σ and the trade τ of θ_σ that contains $\Sigma_{0,1}$. Thus the image of θ_σ under any element of H_σ is a $K1(\sigma)$ ovoid of $Q(4,q)$

that share the trade τ with θ_σ. Now if θ_1 and θ_2 are $K1(\sigma)$ ovoids with special tangent line ℓ_σ and with the common trade τ, then there exists an element of $(G_\sigma)_\tau$ mapping θ_1 to θ_2 (this follows from the 2-transitivity of the group of a $K1(\sigma)$ ovoid on its planes). Thus we may use the orbit-stabiliser theorem on $(G_\sigma)_\tau$ to determine the number of distinct $K1(\sigma)$ ovoids with special tangent line ℓ_σ and with τ as a trade.

In the case where σ^2 is the identity, since $H_\sigma = (G_\sigma)_\tau$ it follows that

$$\begin{aligned}&\#\text{ distinct ovoids with special tangent line } \ell_\sigma \text{ and trade } \tau\\ &= \frac{|H_\sigma|}{|\text{Stabiliser of } \theta_\sigma \text{ in } H_\sigma|}\end{aligned} \tag{2.6.4}$$

In the case where σ^2 is not the identity, since H_σ is a subgroup of $(G_\sigma)_\tau$, it follows that

$$\begin{aligned}&\#\text{ distinct ovoids with special tangent line } \ell_\sigma \text{ and trade } \tau\\ &= \frac{|(G_\sigma)_\tau|}{|\text{Stabiliser of } \theta_\sigma \text{ in } (G_\sigma)_\tau|}\\ &\geq \frac{|H_\sigma|}{|\text{Stabiliser of } \theta_\sigma \text{ in } (G_\sigma)_\tau|}\end{aligned} \tag{2.6.5}$$

Now if $\Upsilon \in G_\sigma$ fixes θ_σ, then since Υ fixes $\Sigma_{0,1}$ it follows that Υ also fixes τ, that is, $\Upsilon \in (G_\sigma)_\tau$. Thus the stabiliser of θ_σ in G_σ is equal to the stabiliser of θ_σ in $(G_\sigma)_\tau$. Consequently we calculate the stabiliser of θ_σ in G_σ.

First we consider $\Upsilon = T\alpha \in G'_\sigma$. Now Υ fixes θ_σ if and only if it fixes the set of planes of θ_σ. Recall from Section 2.2 that the plane π_t of θ_σ may be written as $\langle \ell_\sigma, (-mt^\sigma, t, 0, 0, 1)\rangle$, and so since Υ fixes ℓ_σ, it follows that

$$\begin{aligned}\Upsilon(\pi_t) &= \Upsilon(\langle \ell_\sigma, (-mt^\sigma, t, 0, 0, 1)\rangle)\\ &= \langle \ell_\sigma, (-m^\alpha t^{\sigma\alpha} s - c, sm^{\alpha-1}t^\alpha - b, -, -, 1)\rangle\end{aligned}$$

Now recall from Section 2.2 that the plane π_η of θ_σ is defined by the equations $x_0 + m\eta^{\sigma^{-1}}x_1 = 0$ and $x_1 - \eta x_4 = 0$, and so $T\alpha(-mt^\sigma, t, 0, 0, 1) \in \pi_t$ if and only if

$$\eta = sm^{\alpha-1}t^\alpha - b \text{ and } -m^\alpha t^{\sigma\alpha}s - c + m(sm^{\alpha-1}t^\alpha - b)^\sigma = 0.$$

Combined, these equations imply that the $\Upsilon(\pi_t)$ is a plane of θ_σ if and only if

$$t^{\sigma\alpha}(-m^\alpha s + s^\sigma m^{(\alpha-1)\sigma+1}) - c - mb^\sigma = 0 \tag{2.6.6}$$

Now, Υ fixes the set of planes of θ_σ if and only if 2.6.6 is satisfied for all $t \in \mathrm{GF}(q)$. This is equivalent to the polynomial $x^{\sigma\alpha}(-m^\alpha + s^\sigma m^{(\alpha-1)\sigma+1}) - c - mb^\sigma$ being identically zero. This in turn holds, if and only if the following two equations are satisfied

$$-m^\alpha s + s^\sigma m^{(\alpha-1)\sigma+1} = 0 \tag{2.6.7}$$

$$-c - mb^\sigma = 0 \tag{2.6.8}$$

Since b and c are related by $mb + c = 0$, 2.6.8 is equivalent to $b^\sigma = -b$. If $b^\sigma = -b$, then $b^{2\sigma} = b^2$ and so $b^2 \in Fix(\sigma)$. Now suppose that x is a non-zero element of $\mathrm{GF}(q)$, such that $x^\sigma = x$. If $y^2 = x$, then $y^{2\sigma} = y^2$ and so either $y^\sigma = y$ or $y^\sigma = -y$. Since the other square root of x is $-y$, it follows that if $y^\sigma = y$, then $(-y)^\sigma = (-y)$ and if $y^\sigma = -y$, then $(-y)^\sigma = -(-y)$. Thus, for x a square such that $x^\sigma = x$, either both square roots of x are in $Fix(\sigma)$, or both are in the set $\{b : b^\sigma = -b\} \subset \mathrm{GF}(q) \setminus Fix(\sigma)$. Since $Fix(\sigma)$ is a field of order k, there are $(k-1)/2$ elements of $Fix(\sigma)$ that are square in $\mathrm{GF}(q)$ and have square roots in $Fix(\sigma)$. If $q = p^h$ and $k = p^r$, then the other $(k-1)/2$ non-zero elements of $Fix(\sigma)$ are square in $\mathrm{GF}(q)$ if and only if 2 divides (h/r). When 2 divides (h/r) there are $(k-1)$ non-zero solutions to $b^\sigma = -b$ and including the solution $b = 0$ gives a total of k. If 2 does not divide (h/r), then the only solution to the equation $b^\sigma = -b$ is $b = 0$.

Now solving equation 2.6.7,

$$\begin{array}{rrcl} & -m^\alpha s + s^\sigma m^{(\alpha-1)\sigma+1} & = & 0 \\ \iff & -m^\alpha s + (sm^{\alpha-1})^\sigma m & = & 0 \\ \iff & (sm^{\alpha-1})^\sigma = sm^{\alpha-1}. & & \end{array}$$

Thus, 2.6.7 is satisfied if and only if $sm^{\alpha-1} \in \mathrm{GF}(q)$. Hence if 2 divides h/r then there are $2qk(h(k-1))$ elements of G'_σ that fix θ_σ and if 2 does not divide h/r, then there are $2qh(k-1)$ elements of G'_σ that fix θ_σ.

Having calculated which elements of G'_σ fix θ_σ, we now calculate which elements of WG'_σ fix θ_σ.

Now

$$\begin{array}{rcl} W(\pi_t) & = & W(\langle \ell_\sigma, (-mt^\sigma, t, 0, 0, 1)\rangle) \\ & = & \langle \ell_\sigma, (mt, -t^\sigma, 0, 0, 1)\rangle \end{array}$$

and so W fixes θ_σ if and only if

$$t - t^{\sigma^2} = 0 \text{ for all } t \in \mathrm{GF}(q).$$

Thus, in the case where σ^2 is the identity automorphism of $\mathrm{GF}(q)$, W fixes θ_σ and so the set of elements of WG'_σ that fix θ_σ is $\{W\Upsilon : \Upsilon \in G'_\sigma \text{ and } \Upsilon(\theta_\sigma) = \theta_\sigma\}$. Hence if σ^2 is the identity (and so 2 divides h/r) there are $4qk(h(k-1))$ elements of G_σ that fix θ_σ.

In the case where σ^2 is not the identity, if $\Upsilon = T\alpha \in G'_\sigma$, then

$$\begin{array}{rcl} W\Upsilon(\pi_t) & = & WT\alpha(\langle \ell_\sigma, (-mt^\sigma, t, 0, 0, 1)\rangle) \\ & = & \langle \ell_\sigma, (sm^\alpha t^\alpha - mb, -m^{\alpha-1}t^{\sigma\alpha}s - c/m, -, -, 1)\rangle \end{array}$$

and so $W\Upsilon$ fixes θ_σ if and only if

$$\begin{array}{rrcll} & sm^\alpha t^\alpha - mb + m(-m^{\alpha-1}t^{\sigma\alpha}s - \frac{c}{m})^\sigma & = & 0 & \text{for all } t \in \mathrm{GF}(q) \\ \iff & sm^\alpha t^\alpha - mb + m\left(-m^{\alpha-1}st^{\sigma^2\alpha} - \frac{c^\sigma}{m^\sigma}\right) & = & 0 & \text{for all } t \in \mathrm{GF}(q) \\ \iff & -m^\alpha st^{\sigma\alpha} + m^\alpha st^\alpha - mb - \frac{c^\sigma}{m^{\sigma-1}} & = & 0 & \text{for all } t \in \mathrm{GF}(q). \end{array}$$

Thus, $W\Upsilon$ fixes θ_σ if and only if the polynomial

$$-m^\alpha sx^{\sigma\alpha} + m^\alpha sx^\alpha - mb - \frac{c^\sigma}{m^{\sigma-1}}$$

is identically zero. This, in turn, yields the equations

$$-m^\alpha s = 0 \text{ and } -mb - \frac{c^\sigma}{m^{\sigma-1}}.$$

Since $s \neq 0$ these equations are never satisfied.

Hence if σ^2 is not the identity and 2 divides h/r, then there are $2qkh(k-1)$ elements of G_σ that fix θ_σ and if 2 does not divide h/r, then there are $2qh(k-1)$ elements of G_σ that fix θ_σ

Theorem 2.6.7 *Let θ be a $K1(\sigma)$ ovoid of $Q(4,q)$ with special tangent line ℓ and let τ be a trade of θ. The number of $K1(\sigma)$ ovoids, with special tangent line ℓ, that have τ as a trade is:*

at least	$2q$	*if*	*2 does not divide h/r,*
at least	$2(q/k)$	*if*	*$2 \mid (q/k)$ and $\sigma^2 \neq$ identity,*
exactly	q/k	*if*	*$\sigma^2 =$ identity.*

Proof: Recall that the stabiliser of θ_σ in $(G_\sigma)_\tau$ is equal to the stabiliser of θ_σ in G_σ. Thus using 2.6.4 and 2.6.5, $|H_\sigma| = 4h(k-1)q^2$ and the size of the stabiliser of θ_σ in G_σ calculated above we have the result.

□

We are now able to determine the conditions under which there exists two $K1(\sigma)$ ovoids with the same special tangent line and a common trade but not in a common elation rosette.

Corollary 2.6.8 *Let σ be a non-identity automorphism of* GF(q)*, q odd. If σ^2 is the identity, then any two $K1(\sigma)$ ovoids of $Q(4,q)$ with the same special tangent line and sharing a trade are contained in a common elation rosette. If σ^2 is not the identity, then for any $K1(\sigma)$ ovoid of $Q(4,q)$ θ and trade τ of θ, there exists at least one $K1(\sigma)$ ovoid θ', not equal to θ, such that θ' has the same special tangent line as θ, has τ as a trade but is not contained in a common elation rosette with θ.*

Proof: Let $\mathcal{R}$ be the rosette of $Q(4,q)$ containing θ and with base point the special point of θ. Then there are q/k ovoids of $\mathcal{R}$ (including θ) with τ as a trade (and necessarily with the same special tangent line as θ). Applying Theorem 2.6.7 gives the result.

□

$K1(\sigma)$ ovoids sharing a trade but with different special tangent lines

In this section we find $K1(\sigma)$ ovoids that share the trade $\tau = \{[1,m,0,0,u] : u = m(t^\sigma - t) \text{ for some } t \in \mathrm{GF}(q)\}$ (the trade of θ_σ containing $\Sigma_{0,1}$, as in Section 2.6.2) with θ_σ but not the special tangent line with θ_σ. We do this by finding automorphisms of $Q(4,q)$ that fix τ, but do not fix ℓ_σ.

If A is a point of $\pi \setminus \{X_\sigma\}$ then recall the symmetry μ_A of $Q(4,q)$ from Section 1.2 and Section 2.2. The symmetry μ_A fixes X_σ and every subspace of PG$(4,q)$ that contains A. Thus it follows that $\mu_A(\theta_\sigma)$ is a $K1(\sigma)$ ovoid with special point X_σ with τ as a trade and with special tangent line $\mu_A(\ell_\sigma)$. Now ℓ_σ is such that $\ell_\sigma^\perp$ is a plane intersecting $Q(4,q)$ in a pair of lines (as observed in Section 2.6.2). There are $(q+1)/2$ such tangents, containing X_σ, in π and we may easily show that the set of symmetries $\{\mu_A : A \in \pi\setminus\{X_\sigma\}\}$ is transitive on these tangents. Thus there are (at least) $(q-1)/2$ $K1(\sigma)$ ovoids (not including θ_σ) of the form $\mu_A(\theta_\sigma)$, $A \in \pi \setminus \{X_\sigma\}$, that share the trade τ with θ_σ. If $\mathcal{R}_A$ is the elation rosette containing $\mu_A(\theta_\sigma)$ and with base point X_σ, then there are q/k ovoids in $\mathcal{R}_A$ with τ as a trade, each with special tangent line $\mu_A(\ell_\sigma)$. Thus there are (at least) $(q-1)q/(2k)$ $K1(\sigma)$ ovoids that share the trade τ with θ_σ, but not the special tangent line ℓ_σ. Thus we have the following theorem.

Theorem 2.6.9 *Let θ be a $K1(\sigma)$ ovoid of $\mathcal{Q}(4,q)$ with special tangent line ℓ. Let τ be a trade of θ. Then there are (at least) $(q-1)q/(2k)$ $K1(\sigma)$ ovoids of $Q(4,q)$ that share the trade τ with θ, but not the special tangent line ℓ.*

2.6.3 Rosettes of $Q(4,9)$

In this section we consider rosettes containing $K1(\sigma)$ ovoids of $Q(4,9)$, where σ is the non-identity automorphism of GF(9) (that is, $x^\sigma = x^3$ and σ^2 is the identity). Penttila and Royle have shown ([53]) that there are only two isomorphism classes of ovoids of $Q(4,9)$, the elliptic quadric ovoids and the $K1(\sigma)$ ovoids. Hence if $\{\theta_1, \theta_2, \ldots, \theta_9\}$ is a rosette of $Q(4,9)$, then for each $i = 1,2,\ldots,9$ the ovoid θ_i is either and elliptic quadric ovoid or a $K1(\sigma)$ ovoid. We know that if θ_i is an elliptic quadric ovoid for $i = 1,2,\ldots,9$, then the rosette is an elation rosette. Otherwise the rosette must contain a $K1(\sigma)$ ovoid, without loss of generality the ovoid θ_1. A computer search by Gordon Royle ([54]) gives us the following result on such rosettes.

Theorem 2.6.10 *[54] Let θ_1 be a fixed $K1(\sigma)$ ovoid of $Q(4,9)$ with special point X. Then every rosette $\{\theta_1, \theta_2, \ldots, \theta_9\}$ is included in the following table.*

Base point	Isomorphism Type	# rosettes	$(K1(\sigma), EQ)$	Size of orbits on ovoids
Fixed non-special point	I	2	(9,0)	9
	II	8	(9,0)	9
Special point	A	32	(6,3)	3,6
	B	48	(9,0)	3,6
	C	48	(9,0)	3,6
	D	4	(3,6)	3,6
	E	8	(6,3)	3,6
	F	1	(9,0)	9

In the above table the first column indicates whether the base point of the rosette is the special point X of θ_1, or not. The second column indicates the different isomorphism classes of rosettes containing θ_1, while the third gives the number of rosettes in each isomorphism class. The fourth column gives the number of $K1(\sigma)$ and elliptic quadric ovoids in a rosette of a particular isomorphism type. The final column gives the orbit sizes for the action of the group of a rosette (in a particular isomorphism class) on the ovoids of the rosette.

We will see that all of these rosettes may be constructed either as in Theorem 2.6.1 or by the method shown in Theorem 2.6.6 and the work in Section 2.6.2.

By Theorem 2.6.1, if P is a fixed non-special point of θ_1, then there are $q+1=10$ elation rosettes containing θ_1 with base point P. These are precisely the rosettes of type I and II.

In the current context Theorem 2.6.6 says the following: let X be the special point of the $K1(\sigma)$ ovoid θ_1 and $\mathcal{R}$ the elation rosette containing θ_1 and with base point X. Let $\mathcal{T}=\{\tau_1,\tau_2,\tau_3\}$ be a tower of $\mathcal{R}$ (see Section 2.6.1) such that τ_1 is a trade of θ_1. Let ρ_1 be the set of ovoids of $\mathcal{R}$ that have τ_1 as a trade and ρ_i, for $i=2,3$ be either

1. a set of 3 $K1(\sigma)$ ovoids contained in a common elation rosette (not necessarily $\mathcal{R}$), and sharing the trade τ_i; or
2. the set of 3 elliptic quadric ovoids of $Q(4,9)$ whose elements are the intersection of the elements of τ_i with $Q(4,9)$.

Then the set of ovoids $\rho_1\cup\rho_2\cup\rho_3$ is a rosette of $Q(4,9)$. We will call these different types of sets of ovoids sets of type 1 or 2, as enumerated above.

Note that $\mathcal{R}$ has $(q-1)/(k-1)=4$ towers and that the group of θ_1 is transitive on these towers.

The rosette of type F is the (unique) elation rosette $\mathcal{R}$, containing θ_1 and with base point X, the special point of θ_1.

Recall in Section 2.6.2 that we found $(q+1)/2=5$ possible sets of type 1 for ρ_2, including the set of 3 ovoids of $\mathcal{R}$ for which τ_2 is a trade. Each of these sets has the property that the ovoids it contains have a common special tangent line. Also the special tangent line corresponding to each of these sets is distinct (the special tangent lines are those tangent lines in the base plane of $\mathcal{T}$ that have a polar with hyperbolic character). A similar result holds for sets of type 1 for ρ_3.

Now the rosette $\rho_1\cup\tau_2\cup\tau_3$ is of type D and the 4 towers give the 4 rosettes of type D.

If ρ_2 is the set of type 1 contained in $\mathcal{R}$, then $\rho_1\cup\rho_2\cup\tau_3$ is a rosette of type E. Similarly, the rosette $\rho_1\cup\tau_2\rho_3$ with $\rho_2\subset\mathcal{R}$. The 4 towers give us the 8 rosettes of type E.

If ρ_2 is a set of type 1 not contained in $\mathcal{R}$, then $\rho_1\cup\rho_2\cup\tau_3$ is a rosette of type A and there are 4 such rosettes. Similarly, there are 4 rosettes of the form $\rho_1\cup\tau_2\cup\rho_3$ of type A and the 4 towers give us the 32 rosettes of type A.

Now suppose that ρ_2 and ρ_3 are sets of type 1. Let ℓ_i be the special tangent line of the ovoids in ρ_i. There are two possibilities: either two elements of $\{\ell_1,\ell_2,\ell_3\}$ coincide, or they are all distinct (if all three coincide then we have the elation rosette of type F). These 2 possibilities correspond to the rosettes of type B and C.

2.7 Remarks

With a little more computational effort it may be possible to classify the rosettes of $Q(4,q)$ such that each ovoid of the rosette is either an elliptic quadric ovoid of a $K1(\sigma)$ ovoid (for fixed σ). Also of interest is considering rosettes of $Q(4,q)$ containing other ovoids of $Q(4,q)$, to try and determine whether there is a GQ of order (q,q^2) containing $Q(4,q)$ as a subquadrangle and subtending rosettes of these ovoids. Similarly for q even since the q-clan GQs all possess subquadrangles of order q isomorphic to $T_2(\mathcal{O})$ for some oval $\mathcal{O}$ of $\mathrm{PG}(2,q)$, it would be interesting to investigate rosettes of the GQ $T_2(\mathcal{O})$.

Chapter 3

SPGs and GQs of order (r, r^2)

3.1 SPGs from GQs of order (r, r^2)

Consider the SPG (constructed by Metz see [18] and by Hirschfeld and Thas [26]) with parameters $s = q-1$, $t = q^2$, $\alpha = 2$, $\mu = 2q(q-1)$, where q is a prime power. The construction due to Metz as follows: let $\mathcal{Q} = Q(4,q)$ and let $\mathcal{P}$ be the set of three dimensional, non-singular elliptic quadrics contained in $\mathcal{Q}$. Let a **bundle** of $\mathcal{Q}$ be a set of q elements of $\mathcal{P}$ that meet pairwise in a common point. Let $\mathcal{B}$ be the set of bundles of $\mathcal{Q}$. Define incidence $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ to be containment. Since each bundle is a set of q elliptic quadric ovoids of $\mathcal{Q}$ sharing a common tangent plane at the point where the elliptic quadrics intersect, and two elements of $\mathcal{P}$ that are tangent are incident with exactly one common bundle, it follows that the structure $\mathcal{T} = (\mathcal{P}, \mathcal{B}, I)$ is a SPG with the above parameters.

Now consider the GQ $\mathcal{S} = Q(5,q)$. $\mathcal{S}$ contains $\mathcal{Q}$ as a subGQ and the subtended ovoids are exactly the elements of $\mathcal{P}$. Moreover, the subtended rosettes are the bundles and $\mathcal{T}$ is the incidence structure obtained by taking subtended ovoids as points and subtended rosettes as lines. This relation between $Q(5,q)$, $Q(4,q)$ and the SPG $\mathcal{T}$ depends on the combinatorics of the situation, rather than the specific geometry. In this section we generalise the Metz/ Hirschfeld and Thas construction of an SPG by using a GQ of order (r, r^2) and a subquadrangle of order r (with particular combinatorial properties) in the place of $Q(5,q)$ and $Q(4,q)$, resulting in Theorem 3.1.7. In Section 3.1.1 we construct an algebraic 2-fold cover of the SPG constructed in Theorem 3.1.7 and use this in Section 3.2 to solve the isomorphism problem for such SPGs. In Section 3.3 we construct a new SPG from a GQ constructed by Kantor in [31]. In Section 3.4 we investigate the q-clan GQs, q even and the subquadrangles of order q constructed by Payne in [45]. We determine the conditions under which a q-clan GQ, with such a subquadrangle, gives rise to an SPG by the method of Theorem 3.1.7.

To begin with we state and prove a lemma which is implicit in [49, Chapter 2]. The lemma proves important for the work in this chapter and in Chapter 4. Recall from Section 1.4.1 that if $\mathcal{S}$ is a GQ with a subGQ $\mathcal{S}'$, then each line of $\mathcal{S}$ is a line of $\mathcal{S}'$, is incident with exactly one point of $\mathcal{S}'$ or is incident with no point of $\mathcal{S}'$. In the last two cases the line is called tangent and external (to $\mathcal{S}'$), respectively. Dually we define tangent points and external points.

Lemma 3.1.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (r, r^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ a subquadrangle of $\mathcal{S}$ of order r. Then each point of $\mathcal{P} \setminus \mathcal{P}'$ is an external point to $\mathcal{S}'$ and each line of $\mathcal{B} \setminus \mathcal{B}'$ is a tangent line to $\mathcal{S}'$.*

Proof: If $\ell \in \mathcal{B}'$, then it follows that each point of $\mathcal{S}$ incident with ℓ is a point of $\mathcal{S}'$. Thus a point of $\mathcal{P} \setminus \mathcal{P}'$ is incident with no line of $\mathcal{S}'$ and hence is external to $\mathcal{S}'$. By [49, 2.2.1] each external point P is collinear with exactly $1 + r^2$ points of $\mathcal{S}'$, that is, each line incident with P is incident with exactly one point of $\mathcal{S}'$. Thus each line of $\mathcal{B} \setminus \mathcal{B}'$ is a tangent line to $\mathcal{S}'$.

□

Corollary 3.1.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (r, r^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ a subquadrangle of $\mathcal{S}$ of order r. Then each point of $\mathcal{P}\backslash\mathcal{P}'$ subtends an ovoid of $\mathcal{S}'$ and each line of $\mathcal{B}\backslash\mathcal{B}'$ subtends a rosette of $\mathcal{S}'$ (as in Section 1.4.1).*

Now we are interested in determining the intersection of subtended ovoids. To this end we recall a result of Bose and Shrikhande, interpreted in the GQ context.

Lemma 3.1.3 *([8], see [49, 1.2.4]) If $\mathcal{S}$ is a GQ of order (r, r^2) and $\{X, Y, Z\}$ is a triad of $\mathcal{S}$, then $|\{X, Y, Z\}^\perp| = r + 1$.*

Corollary 3.1.4 *Let $\mathcal{S}$ be a GQ of order (r, r^2) and $\mathcal{S}'$ a subGQ of order r. A subtended ovoid of $\mathcal{S}'$ is subtended by at most two points of $\mathcal{S}$. Further, if an ovoid θ is subtended by two points X, X', then the size of the intersection of θ with any other subtended ovoid θ_Y, $Y \neq X, X'$, is determined: if $Y \sim X$ or $Y \sim X'$, then $|\theta \cap \theta_Y| = 1$ and if $Y \not\sim X, X'$, then $|\theta \cap \theta_Y| = r + 1$.*

Proof: Suppose that an ovoid θ is subtended by three points X, Y, Z. These three points are necessarily pairwise non-collinear and so form a triad of $\mathcal{S}$. Since $|\{X, Y, Z\}^\perp| \geq r^2 + 1$ we have a contradiction of Lemma 3.1.3. Thus any ovoid may be subtended by at most two points.

Now suppose that θ is subtended by exactly two points X and X'. Let θ_Y be the ovoid subtended by the point Y, $Y \neq X, X'$. Suppose that $Y \sim X$ or $Y \sim X'$; without loss of generality we may suppose that $Y \sim X$. Thus θ_X and θ_Y are contained in the rosette subtended by the line $\langle X, Y\rangle$ and so $\theta_X \cap \theta_Y = \{P\}$ for some point P and $|\theta \cap \theta_Y| = |\{X, X', Y\}^\perp| = 1$. Suppose now that $Y \not\sim X, X'$, then $\{X, X', Y\}$ is a triad of $\mathcal{S}$ and so $|\theta \cap \theta_Y| = |\{X, X', Y\}^\perp| = r + 1$.

□

If a GQ $\mathcal{S}$ of order (r, r^2) has a subGQ $\mathcal{S}' \subset \mathcal{S}$ of order r such that each subtended ovoid of $\mathcal{S}'$ is subtended by exactly two points of $\mathcal{S}$, then we say that $\mathcal{S}'$ is **doubly subtended** in $\mathcal{S}$. In some sense a doubly subtended subquadrangle is an 'extreme' subquadrangle, so it is not surprising that we get some nice geometry from it. At this stage we introduce a slight abuse of notation. If X is a point of $\mathcal{S}$, then we denote this by $X \in \mathcal{S}$. If X is a point of $\mathcal{S}$ but not a point of $\mathcal{S}'$, then we denote this $X \in \mathcal{S} \setminus \mathcal{S}'$.

We might now ask how many subtended rosettes can two subtended ovoids have in common. Let θ_X and θ_Y be two subtended ovoids; subtended by X and Y, respectively, where $\theta_X \neq \theta_Y$.

There are three cases to consider: (i) neither θ_X nor θ_Y is subtended by a second point, (ii) θ_X is subtended by another point X' and θ_Y is not subtended by a second point, (iii) θ_X and θ_Y are each subtended by a second point, say X' and Y' respectively. For case (i) θ_X and θ_Y are contained in exactly one common subtended rosette if $X \sim Y$ and in none otherwise. In case (ii) Y may be collinear with at most one of X, X' since otherwise $Y \in \{X, X'\}^\perp \subset S'$ which contradicts $Y \in \mathcal{S} \setminus \mathcal{S}'$. Thus for case (ii) θ_X and θ_Y have one common rosette if $Y \sim X$ or $Y \sim X'$ (not both) and none otherwise. For case (iii), to each unordered, incident pair taken from $\{X, X', Y, Y'\}$, there corresponds a subtended rosette containing θ_X and θ_Y. So there may be none, one or two subtended rosettes containing θ_X and θ_Y. Note that it is possible that two distinct lines subtend the same rosette.

We have already seen that a GQ $\mathcal{S}$ of order (r, r^2) with a subGQ $\mathcal{S}'$ of order r that has all subtended ovoids of $\mathcal{S}$ being subtended twice is a special and also extremal case of a subGQ of $\mathcal{S}$. We now give a result which shows the relationship between double subtending and the existence of a particular type of involution of the GQ $\mathcal{S}$. The idea for the construction of the involution comes from Thas [62]. For the following we will denote the ovoid subtended by a point X by θ_X.

Lemma 3.1.5 *Let $\mathcal{S}$ be a GQ of order (r, r^2) and $\mathcal{S}'$ a subGQ of order r. Then $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$ if and only if there exists a non-identity involution of $\mathcal{S}$ that fixes $\mathcal{S}'$ pointwise.*

Proof: First, let τ be an involution of $\mathcal{S}$ that fixes $\mathcal{S}'$ pointwise. We first show that τ fixes no point of $\mathcal{S} \setminus \mathcal{S}'$. Suppose that $X \in \mathcal{S} \setminus \mathcal{S}'$ and $\tau(X) = X$. Consider a line ℓ such that $X \in \ell$ and let $\ell \cap \mathcal{S}' = P$, say. Now $\tau(P) = P$ and so $\tau(\ell) = \ell$. Now let $R \in \ell$, $R \neq P$. Then $\theta_R = \theta_{\tau(R)}$ but $\tau(R) \in \ell$ and so $R = \tau(R)$. Thus ℓ is fixed pointwise by τ. Now consider a point $Y \in \mathcal{S} \setminus \mathcal{S}'$, $Y \not\sim X$. If $\theta_Y = \theta_X$ then since an ovoid may only be subtended twice and $\tau(Y)$ subtends θ_Y we must have that $\tau(Y) = Y$. If $\theta_Y \neq \theta_X$, then there exists a point R' such that $R' \in \ell$, $R' \in \mathcal{S} \setminus \mathcal{S}'$ and $Y \sim R'$. However, R' is fixed by τ as it is collinear with X and so every point collinear with R' is fixed, so τ also fixes Y. Thus τ fixes every point of $\mathcal{S}$ and so is the identity, which is a contradiction. Hence τ fixes no point in $\mathcal{S} \setminus \mathcal{S}'$.

Thus we can now say that for any point $X \in \mathcal{S} \setminus \mathcal{S}'$ we have $\tau(X) = X' \neq X$ and so θ_X is subtended by the distinct points X, X'.

Now, suppose that $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$. If $X \in \mathcal{S} \setminus \mathcal{S}'$ then let X' be the second point subtending θ_X. Now define τ to be the following map:

$$\begin{aligned} \tau: \quad & X \mapsto X' \quad X \in S \setminus \mathcal{S}' \\ & X \mapsto X \quad X \in \mathcal{S}'. \end{aligned}$$

Consider points $P, Q \in \mathcal{S}$, with $P \sim Q$. If $P, Q \in \mathcal{S}'$ then $P = \tau(P) \sim \tau(Q) = Q$. If $P \in \mathcal{S}'$, $Q \in \mathcal{S} \setminus \mathcal{S}'$ then $P \in \theta_Q = \theta_{Q'} = \theta_{\tau(Q)}$ and so $P \sim \tau(Q)$, that is $\tau(P) \sim \tau(Q)$.

Now if $P, Q \in \mathcal{S} \setminus \mathcal{S}'$ then $|\theta_P \cap \theta_Q| = |\theta_{\tau(P)} \cap \theta_{\tau(Q)}| = 1$; so by Corollary 3.1.4 we have that $\tau(Q) \sim P$ or $\tau(Q) \sim \tau(P)$. Since $Q \sim P$ it must be that $\tau(Q) \not\sim P$ and so $\tau(P) \sim \tau(Q)$. Thus τ is an automorphism of $\mathcal{S}$ and clearly an involution.

□

Corollary 3.1.6 *If $\mathcal{S}$ is a GQ of order (r, r^2) that has a doubly subtended subGQ $\mathcal{S}'$ of order r, then for each incident point, subtended ovoid pair (X, θ) there exists a unique subtended rosette $\mathcal{R}$, containing θ and with basepoint X.*

Proof: If θ is subtended by the points Y and Y', then the only subtended rosettes containing θ and with base point X, are those subtended by the lines $\langle X, Y \rangle$ and $\langle X, Y' \rangle$. However $\langle X, Y \rangle$ is the image of $\langle X, Y' \rangle$ under the involution constructed in Lemma 3.1.5, and vice-versa. Since the involution in Lemma 3.1.5 fixes the subquadrangle $\mathcal{S}'$ pointwise (and linewise), the rosette subtended by $\langle X, Y \rangle$ and the rosette subtended by $\langle X, Y' \rangle$ are the same.

□

Now we show that if a subGQ is doubly subtended, then we get an SPG from its subtended ovoid/rosette structure.

Theorem 3.1.7 *Let $\mathcal{S}$ be a GQ of order (r, r^2) containing a subGQ $\mathcal{S}'$ of order r, such that $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$. Consider the following incidence structure $\mathcal{T}$:*

Points : Subtended ovoids of $\mathcal{S}'$.
Lines : Subtended rosettes of $\mathcal{S}'$.
Incidence : Containment.

Then $\mathcal{T}$ is a SPG with parameters $s = r - 1$, $t = r^2$, $\alpha = 2$ and $\mu = 2r(r-1)$.

Proof: A rosette contains r ovoids, thus $s = r - 1$. By Corollary 3.1.6 there are $r^2 + 1$ subtended rosettes containing a subtended ovoid θ_X, that is, $t = r^2$.

Now consider a subtended rosette $\mathcal{R}$ with basepoint P and not containing the ovoid θ_X. Recall that ovoids of $\mathcal{R}$ partition the points of $\mathcal{S}'$ that are not collinear with P. Suppose that $P \in \theta_X$. Then $\theta_X \subset \mathcal{S}' \setminus P^\perp$. Let n_1 and n_{r+1} be the number of ovoids of $\mathcal{R}$ that meet θ_X in 1 and $r+1$ points, respectively. Then we have the following equations:

$$\begin{aligned} n_{r+1} \cdot r + n_1 \cdot 0 &= r^2 \\ n_{r+1} + n_1 &= r. \end{aligned}$$

Solving simultaneously gives $n_{r+1} = r$ and $n_1 = 0$, that is, θ_X meets each ovoid in $\mathcal{R}$ in $r+1$ points.

Suppose now that $P \notin \theta_X$. Then θ_X has $r^2 - r$ points non-collinear with P and so we have the following equations:

$$\begin{aligned} n_{r+1} \cdot (r+1) + n_1 &= r^2 - r \\ n_{r+1} + n_1 &= r \end{aligned}$$

Solving simultaneously we have $n_{r+1} = r - 2$ and $n_1 = 2$.

In terms of $\mathcal{T}$ the above means that if we have a non-incident point/line pair (A, ℓ) in $\mathcal{T}$ there are 0 or 2 point/line pairs (B, m) such that $A\ I\ m\ I\ B\ I\ l$.

Now consider two subtended ovoids of $\mathcal{S}'$, say θ_X and θ_Y, such that $|\theta_X \cap \theta_Y| = r+1$. Let $\Omega = \theta_X \setminus \theta_Y$. By Corollary 3.1.6, for each $Q \in \Omega$ there exists exactly one subtended rosette $\mathcal{R}$, with base point Q and containing θ_X. By the above, we see that there are two subtended rosettes containing θ_Y and an ovoid in $\mathcal{R}$, and so two subtended ovoids that are contained in a subtended rosette with θ_X and contained in a distinct subtended rosette with θ_Y. This is true for all points in Ω and so there are $2(r^2 - r)$ subtended ovoids that are contained in a rosette with both θ_X and θ_Y. In $\mathcal{T}$, this means that given two non-collinear points A and B there are $2r(r-1)$ points collinear to both A and B.

Thus $\mathcal{T}$ is an SPG with the parameters as required.

□

Corollary 3.1.8 *Let $\mathcal{S}$ be a GQ of order (r, r^2) containing a subGQ $\mathcal{S}'$ of order r such that there exists a non-identity involution of $\mathcal{S}$ that fixes $\mathcal{S}'$ pointwise. Then there is an associated SPG with parameters $s = r-1$, $t = r^2$, $\alpha = 2$ and $\mu = 2r(r-1)$.*

3.1.1 Algebraic 2-fold covers of SPGs and the GQ condition

Suppose that $\mathcal{S}$ is a GQ of order (r, r^2) and $\mathcal{S}'$ is a subquadrangle of order r that is doubly subtended. Let $\mathcal{T}$ be the SPG constructed from $\mathcal{S}$ and $\mathcal{S}'$ as in Theorem 3.1.7. In this section we show that if $\overline{\mathcal{T}}$ is the geometry of $\mathcal{S} \setminus \mathcal{S}'$ (that is, the subgeometry of $\mathcal{S}$ consisting of the points and lines not in $\mathcal{S}'$), then there exists a map p from $\overline{\mathcal{T}}$ to $\mathcal{T}$ such that $(\overline{\mathcal{T}}, p)$ is a 2-fold cover of $\mathcal{T}$. In fact the 2-fold cover is an algebraic 2-fold cover. We then show that we may reconstruct $\mathcal{S}$ from $\mathcal{S}'$ and $\overline{\mathcal{T}}$. Generalising this construction we construct a GQ of order (r, r^2) from $\mathcal{S}'$, $\overline{\mathcal{T}}$ and an algebraic 2-fold cover of $\overline{\mathcal{T}}$, provided the cover has a property called the GQ condition.

Let $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{T}$ be as above and let $(\overline{\mathcal{T}}, p)$ be an algebraic 2-fold cover of $\mathcal{T}$ over $\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is the unique abelian group of order 2), defined by the 1-cochain c. Then $(\overline{\mathcal{T}}, p)$ is said to satisfy the **GQ condition** if for each set $\{P, Q, R\}$ of pairwise collinear points of $\mathcal{T}$

$$\delta c(P,Q,R) = c(P,q) - c(P,R) + c(Q,R) = 0 \Longleftrightarrow P, Q, R \text{ are collinear points of } \mathcal{T}. \quad (3.1.1)$$

Theorem 3.1.9 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (r, r^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I)$ a doubly subtended subGQ of order r. Let $\mathcal{T}$ be the SPG constructed from $\mathcal{S}$ and $\mathcal{S}'$, as in Theorem 3.1.7 and let $\overline{\mathcal{T}}$ be the geometry $(\mathcal{P} \setminus \mathcal{P}', \mathcal{B} \setminus \mathcal{B}', I)$. Let Θ be the set of subtended ovoids of $\mathcal{S}'$ and represent $\mathcal{P} \setminus \mathcal{P}'$ as the set $\{(\theta, 0), (\theta, 1) : \theta \in \Theta\}$. Let c be the 1-cochain (of the simplicial complex of the point graph of $\mathcal{T}$) defined by*

$$c(\theta_i, \theta_j) = \begin{cases} 0 & \textit{if } |\theta_i \cap \theta_j| = 1 \textit{ and } (\theta_i, 0) \textit{ and } (\theta_j, 0) \textit{ are collinear.} \\ 1 & \textit{if } |\theta_i \cap \theta_j| = 1 \textit{ and } (\theta_i, 0) \textit{ and } (\theta_j, 0) \textit{ are not collinear.} \end{cases}$$

Then c defines an algebraic 2-fold cover of $\mathcal{T}$ with covering geometry $\overline{\mathcal{T}}$. Furthermore, c satisfies the GQ condition.

Proof: Let θ_1 and θ_2 be two collinear points of $\mathcal{T}$ and so $|\theta_1 \cap \theta_2| = 1$. Clearly, $c(\theta_1, \theta_2) = c(\theta_2, \theta_1)$ and so c is a 1-cochain. Since they are collinear points of $\mathcal{T}$, θ_1 and θ_2 are two subtended ovoids of $\mathcal{S}'$ contained in a common subtended rosette, that is subtended by two lines of $\mathcal{S}$. The point labelled $(\theta_1, 0)$ is incident with one of these lines and $(\theta_1, 1)$ is incident with the other, and similarly for $(\theta_2, 0)$ and $(\theta_2, 1)$. Thus (θ_1, α) is collinear with (θ_2, β) if and only if $c(\theta_1, \theta_2) = \alpha + \beta$. Thus c defines an algebraic 2-fold cover of the point graph of $\mathcal{T}$.

To show that c defines an algebraic 2-fold cover of the geometry $\mathcal{T}$ we need to show that $\delta c(\theta_1, \theta_2, \theta_3) = 0$ whenever θ_1, θ_2 and θ_3 are collinear points of $\mathcal{T}$. Let θ_1, θ_2 and θ_3 be three collinear points of $\mathcal{T}$, then they are contained in a common subtended rosette $\mathcal{R}$ of $\mathcal{S}'$. Now $(\theta_1, 0)$ is collinear with $(\theta_2, c(\theta_1, \theta_2))$ and $(\theta_1, 0)$ with $(\theta_3, c(\theta_1, \theta_3))$. Since $\langle(\theta_1, 0), (\theta_2, c(\theta_1, \theta_2))\rangle$ and $\langle(\theta_1, 0), (\theta_3, c(\theta_1, \theta_3))\rangle$ both subtend the rosette $\mathcal{R}$, it follows that $(\theta_2, c(\theta_1, \theta_2))$ and $(\theta_3, c(\theta_1, \theta_3))$ are collinear and so $\delta c(\theta_1, \theta_2, \theta_3) = 0$. Thus c defines a cover of the geometry $\mathcal{T}$.

If θ_1, θ_2 and θ_3 are pairwise collinear but not incident with a common line of $\mathcal{T}$, then it follows that they are not contained in a common subtended rosette of $\mathcal{S}'$. Thus $(\theta_1, 0)$, $(\theta_2, c(\theta_1, \theta_2))$ and $(\theta_3, c(\theta_1, \theta_3))$ are not incident with a common line of $\overline{\mathcal{T}}$ and so $(\theta_2, c(\theta_1, \theta_2))$ and $(\theta_3, c(\theta_1, \theta_3))$ are not collinear since this would be a triangle in $\mathcal{S}$. Hence, $\delta c(\theta_1, \theta_2, \theta_3) = 1$ and c satisfies the GQ condition.

□

Given the notation of Theorem 3.1.9, consider the following description of $\mathcal{S}$.

Points	(i)	Points of $\mathcal{S}'$.
	(ii)	Points of $\overline{\mathcal{T}}$.
Lines	(a)	Lines of $\mathcal{S}'$.
	(b)	$\ell \cup P$ where ℓ is a line of $\overline{\mathcal{T}}$ and P the basepoint of the subtended rosette covered by ℓ.
Incidence	(i),(a)	as in $\mathcal{S}'$.
	(i),(b)	A point P of type (i) is incident with a line $\ell \cup Q$ of type (b) if and only if $P = Q$.
	(ii),(a)	None.
	(ii),(b)	A point P of type (ii) is incident with a line $\ell \cup Q$ of type (b) if and only if P is incident with ℓ in $\overline{\mathcal{T}}$

Now suppose that in the above incidence structure instead of using the algebraic 2-fold cover of $\mathcal{T}$ from Theorem 3.1.9 we use an arbitrary algebraic 2-fold cover of $\mathcal{T}$. The following theorem specifies the conditions under which this new incidence structure is a GQ.

Theorem 3.1.10 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (r, r^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I)$ a doubly subtended subGQ of order r. Let $\mathcal{T}$ be the SPG constructed from $\mathcal{S}$ and $\mathcal{S}'$, as in Theorem 3.1.7 and let $(\overline{\mathcal{T}}, p)$ be an algebraic 2-fold cover of $\mathcal{T}$ defined by the 1-cochain c. Define $\mathcal{W}$ to be the following incidence structure:*

Points	(i)	Points of $\mathcal{S}'$.
	(ii)	Points of $\overline{\mathcal{T}}$.
Lines	(a)	Lines of $\mathcal{S}'$.
	(b)	$\ell \cup P$ where ℓ is a line of $\overline{\mathcal{T}}$ and P the basepoint of the subtended rosette covered by ℓ.
Incidence	(i),(a)	as in $\mathcal{S}'$.
	(i),(b)	A point P of type (i) is incident with a line $\ell \cup Q$ of type (b) if and only if $P = Q$.
	(ii),(a)	None.
	(ii),(b)	A point P of type (ii) is incident with a line $\ell \cup Q$ of type (b) if and only if P is incident with ℓ in $\overline{\mathcal{T}}$

Then $\mathcal{S}$ is a GQ of order (r, r^2) if and only if c satisfies the GQ condition. In this case $\mathcal{S}$ contains $\mathcal{S}'$ as a subquadrangle and $\mathcal{S}'$ is doubly subtended by $\mathcal{S}$. The SPG constructed from $\mathcal{S}$ and $\mathcal{S}'$ as in Theorem 3.1.7 is $\mathcal{T}$.

Proof: Any line of $\mathcal{S}'$ is incident with $r+1$ points of $\mathcal{S}'$ and so $r+1$ points of $\mathcal{W}$. A line $\ell \cup P$ of type (b) is incident with P and with the r points of $\overline{\mathcal{T}}$ incident with ℓ. Thus each line of $\mathcal{S}$ is incident with $r+1$ points.

Let Q be a point of type (i), then Q is incident with $r+1$ lines of $\mathcal{S}'$. There are $(r^2 - r)/2$ subtended rosettes that have Q as a basepoint and so there are $r^2 - r$ lines of $\mathcal{S}$ of type (b) that are incident with Q. Thus Q is incident with $r^2 + 1$ lines of $\mathcal{S}$. By Corollary 3.1.6 each subtended ovoid of $\mathcal{S}'$ is contained in $r^2 + 1$ subtended rosettes and so each type (ii) point of $\mathcal{S}$ is incident with $r^2 + 1$ lines of $\mathcal{W}$.

We check the third GQ axiom for each non-incident point/line pair, (P, ℓ) of $\mathcal{S}$. If P is of type (i) and ℓ is of type (a), then since $\mathcal{S}'$ is a GQ the property holds. Let P be of type (i) and $\ell \cup Q$ of type (b). If P and Q are collinear then there is no ovoid of $\mathcal{S}'$ containing both

P and Q. Thus, Q is the unique point of $\ell \cup Q$ that is collinear with P. If P is not collinear with Q then P is contained in a unique ovoid in the rosette subtended by ℓ. There is a unique subtended rosette containing this ovoid and with basepoint P.

Let P be of type (ii) and ℓ of type (a). The ovoid θ, corresponding to P meets ℓ in exactly one point X. There is a unique subtended rosette containing θ and with basepoint X and thus a unique line of type (b) containing P and X.

Let P be of type (ii) and let $\ell \cup Q$ of type (b). Let θ be the ovoid of $\mathcal{S}'$ corresponding to P and $R = \{\theta_1, \ldots, \theta_r\}$ the subtended rosette of $\mathcal{S}'$ corresponding to ℓ. Without loss of generality suppose that $P = (\theta, 0)$. There are two possibilities for ℓ, either

$\ell = \{(\theta_1, 0), (\theta_2, c(\theta_1, \theta_2)), \ldots, (\theta_r, c(\theta_1, \theta_r))\}$ or

$\ell = \{(\theta_1, 1), (\theta_2, c(\theta_1, \theta_2) + 1), \ldots, (\theta_r, c(\theta_r, \theta_1) + 1)\}$. Suppose that $\theta \in R$ and that without loss of generality $\theta = \theta_1$. Then since $(\theta, 0)$ is not incident with ℓ we have that $\ell = \{(\theta_1, 1), (\theta_2, c(\theta_1, \theta_2) + 1), \ldots, (\theta_r, c(\theta_1, \theta_r) + 1)\}$ and $(\theta, 0)$ is collinear with none of the points on ℓ. Thus Q is the unique point on $\ell \cup Q$ that is collinear with P. Now suppose that $\theta \notin R$ and that without loss of generality $\ell = \{(\theta_1, 0), (\theta_2, c(\theta_1, \theta_2)), \ldots, (\theta_r, c(\theta_r, \theta_1))\}$. If $Q \in \theta$ then θ meets each of the θ_i in $r + 1$ points and is contained in a unique subtended rosette with Q as the basepoint, which gives a unique line incident with P and a point of $\ell \cup Q$. If $Q \notin \theta$, then there are two ovoids of R that meet θ in precisely one point. Without loss of generality let these ovoids be θ_1 and θ_2. Now $(\theta_1, 0)$ is collinear to $(\theta_2, c(\theta_1, \theta_2))$ (on ℓ) and $(\theta_1, 1)$ is collinear to $(\theta_2, c(\theta_1, \theta_2) + 1)$, while $(\theta, 0)$ is collinear to exactly one point of the form $(\theta_1, -)$ and one of the form $(\theta_2, -)$. So $(\theta, 0)$ is collinear to exactly to one point on $\ell \cup Q$ if and only if either $(\theta, 0)$ is collinear to $(\theta_1, 0)$ and $(\theta_2, c(\theta_1, \theta_2) + 1)$ or $(\theta, 0)$ is collinear to $(\theta_1, 1)$ and $(\theta_2, c(\theta_1, \theta_2))$. This occurs if and only if $c(\theta, \theta_2) = c(\theta, \theta_1) + c(\theta_1, \theta_2) + 1$. That is, if and only if

$$\begin{aligned} \delta c(\theta, \theta_1, \theta_2) &= c(\theta, \theta_1) + c(\theta, \theta_2) + c(\theta_1, \theta_2) \\ &= c(\theta, \theta_1) + (c(\theta, \theta_1) + c(\theta_1, \theta_2) + 1) + c(\theta_1, \theta_2) \\ &= 1. \end{aligned}$$

This is precisely the GQ condition. Thus $\mathcal{W}$ is a GQ of order (r, r^2) if and only if c satisfies the GQ condition.

Now suppose that $\mathcal{W}$ is a GQ of order (r, r^2). If X is a point of type (ii) of $\mathcal{S}$, then let θ_X be the ovoid of $\mathcal{S}'$ that is covered by X. The set of lines of $\mathcal{W}$ incident with X meets $\mathcal{S}'$ in the set of basepoints of subtended rosettes containing θ_X. So, in $\mathcal{W}$, X subtends the ovoid θ_X in $\mathcal{S}'$. It then follows that a line $\ell \cup P$ of $\mathcal{W}$ subtends the rosette that is covered by the line ℓ of $\overline{\mathcal{T}}$. Thus $\mathcal{S}'$ is doubly subtended in $\mathcal{W}$ and subtended ovoid/rosette structure is $\mathcal{T}$.

□

We will study this construction in greater depth in Chapter 4.

Note that in Theorem 3.1.10 since c defines an algebraic 2-fold cover of $\mathcal{T}$, it follows from Section 1.6 that c satisfies one half of the GQ condition

$$\delta c(\theta_1, \theta_2, \theta_3) = c(\theta_1, \theta_2) - c(\theta_1, \theta_3) + c(\theta_2, \theta_3) = 0 \text{ if } \theta_1, \theta_2, \theta_3 \text{ are distinct collinear points of } \mathcal{T}.$$

In fact by Theorem 3.1.10 any 1-cochain satisfying the GQ condition, defines an algebraic 2-fold cover of $\mathcal{T}$ and the construction in Theorem 3.1.10 is a GQ.

3.2 Isomorphisms of SPGs

In this section we determine when two SPGs, constructed from the double subtending process, are isomorphic. We also calculate the group of such an SPG.

Theorem 3.2.1 *Let $\mathcal{W}$ and $\mathcal{S}$ be two GQs of order (r, r^2) and let $\mathcal{W}'$ and $\mathcal{S}'$ be subGQs of $\mathcal{W}$ and $\mathcal{S}$ respectively, of order r. Let $\mathcal{W}'$ and $\mathcal{S}'$ be doubly subtended in $\mathcal{W}$ and $\mathcal{S}$ and let the SPGs constructed as in Theorem 3.1.7 be $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$. The SPGs $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$ are isomorphic if and only if there exists an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$ that induces an isomorphism from $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$.*

Proof: First, let $c_{\mathcal{S}}$ define an algebraic 2-fold cover of $\mathcal{T}_{\mathcal{S}}$, as in Theorem 3.1.9, and let $i : \mathcal{T}_{\mathcal{W}} \to \mathcal{T}_{\mathcal{S}}$ be an isomorphism. If θ and θ' are two points of $\mathcal{T}_{\mathcal{W}}$, then we may easily show that the function $c_{\mathcal{W}} = c_{\mathcal{S}}(i(\theta), i(\theta'))$ defines an algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}}$, that satisfies the GQ condition. Let $(\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}, p_{\mathcal{W}})$ be the algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}}$ defined by $c_{\mathcal{W}}$, $(\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{S}}}, p_{\mathcal{S}})$ the algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}}$ defined by $c_{\mathcal{S}}$ and $\overline{\mathcal{S}}$ the GQ of order (r, r^2) constructed from $(\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}, p_{\mathcal{W}})$ and $\mathcal{W}'$ as in Theorem 3.1.10. Now, let $\bar{i}$ be the map from the pointset of $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ to the pointset of $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{S}}}$, which acts by $(\theta, \alpha) \mapsto (i(\theta), \alpha))$, for θ a point of $\mathcal{T}_{\mathcal{W}}$ and $\alpha \in \mathbb{Z}_2$. If the lines of $\mathcal{T}_{\mathcal{W}}$ are considered as sets of points of $\mathcal{T}_{\mathcal{W}}$, then $\bar{i}$ induces an isomorphism from $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ to $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{S}}}$, which we also denote by $\bar{i}$. We show that $\bar{i}$ may be extended to an isomorphism from $\overline{\mathcal{S}}$ to $\mathcal{S}$.

Let ℓ and m be two skew (that is, non-concurrent) lines of $\overline{\mathcal{S}}$ that are tangent to $\mathcal{W}'$. A line of $\overline{\mathcal{S}}$ that is tangent to $\mathcal{W}'$ is said to be a **transversal** to ℓ and m if it is concurrent to both ℓ and m.

Let P be a point of $\mathcal{W}'$ incident with ℓ, then P is the unique point of $\mathcal{W}'$ that is incident with ℓ. Let Q be the unique point of $\mathcal{W}'$ that is incident with m. Now if P and Q are not collinear, then by the third GQ axiom one point of the set $\ell \setminus \{P\}$ is collinear with Q and each of the remaining $r-1$ points of $\ell \setminus \{P\}$ is collinear with a unique point of $m \setminus \{Q\}$. Thus there are $r-1$ transversals to ℓ and m in this case. If P is collinear to Q (with the line incident with P and Q necessarily a line of $\mathcal{W}'$), then each point of $\ell \setminus \{P\}$ is collinear with a unique point of $m \setminus \{Q\}$. Thus there are r transversals to ℓ and m.

Now if ℓ and m are not skew, but meet in a point of $\mathcal{W}'$, that is $P = Q$, then the third GQ axiom decrees that there are no transversals to ℓ and m.

Let ℓ' and m' be the lines of $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ such that $\ell = \ell' \cup \{P\}$ and $m = m' \cup \{Q\}$. Let a line of $\mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}$ be a transversal to ℓ' and m' if it is concurrent to both ℓ' and m'. Consider the incidence structure which has pointset $\{\mathcal{L} : \mathcal{L} \text{ is a set of } q^2 - q \text{ pairwise skew lines of } \mathcal{T}_{\mathcal{W}}^{c_{\mathcal{W}}}\}$ and points $\mathcal{L}_1$ and $\mathcal{L}_2$ collinear if ℓ_1 and ℓ_2 have exactly r transversals, for all $\ell_1 \in \mathcal{L}_1$ and $\ell_2 \in \mathcal{L}_2$ (lines are the maximal sets of pairwise collinear points). The above calculations show that this incidence structure is $\mathcal{W}'$.

Thus $\bar{i}$ may be extended to an isomorphism from $\overline{\mathcal{S}}$ to $\mathcal{S}$. The restriction of $\bar{i}$ to $\mathcal{W}'$ is an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$ that induces i from $\mathcal{T}_{\mathcal{W}}$ to $\mathcal{T}_{\mathcal{S}}$.

Now suppose that there exists an isomorphism from $\mathcal{W}'$ to $\mathcal{S}'$ that takes $T_{\mathcal{W}}$ to $T_{\mathcal{S}}$. Since such an isomorphism maps ovoids to ovoids, rosettes to rosettes and preserves inclusion of an ovoid in a rosette, it induces an isomorphism from $T_{\mathcal{W}}$ to $T_{\mathcal{S}}$.

□

As a corollary of Theorem 3.2.1 we state the automorphism group of an SPG arising from the double subtending process.

Corollary 3.2.2 *Let $\mathcal{S}$ be a GQ of order (r, r^2) and $\mathcal{S}'$ a subGQ of order r. Let $\mathcal{S}'$ be doubly subtended in $\mathcal{S}$, with SPG $\mathcal{T}$ constructed as in Theorem 3.1.7. The automorphism group of $\mathcal{T}$ is the stabiliser of $\mathcal{T}$ in the automorphism group of $\mathcal{S}'$.*

Proof: From the proof of Theorem 3.2.1, if $\mathcal{T} = \mathcal{T}_{\mathcal{W}} = \mathcal{T}_{\mathcal{S}}$ and i is an automorphism of $\mathcal{T}$, then there is an automorphism of $\mathcal{S}'$ that induces i. Also, any automorphism of $\mathcal{S}'$ that fixes $\mathcal{T}$ induces an automorphism of $\mathcal{T}$. Since any point of $\mathcal{S}'$ may be expressed as the intersection of two ovoids that are points of $\mathcal{T}$, any automorphism of $\mathcal{S}'$ that induces the identity on $\mathcal{T}$ must be the identity. So, if we consider the group of $\mathcal{S}'$ that fixes $\mathcal{T}$ acting on $\mathcal{T}$ by the automorphism it induces, the action is faithful and so it is the group of $\mathcal{T}$.

□

3.3 A GQ of Kantor, an ovoid of Kantor and a new SPG

In [31] Kantor constructed the q-clan $\mathcal{C}_\sigma$, which has associated GQ $\mathcal{S}(\mathcal{C}_\sigma)$ of order (q^2, q) (see Section 1.4.5 for details and references). In this section we investigate the connection between the GQ $\mathcal{S}(\mathcal{C}_\sigma)$ and the ovoid θ_σ constructed by Kantor in [30] and studied in Chapter 2 of this document. The dual GQ $\mathcal{S}(\mathcal{C}_\sigma)$ possesses a subquadrangle isomorphic to $Q(4,q)$ and we show that each subtended ovoid of the $Q(4,q)$ subquadrangle is isomorphic to θ_σ. We also show that the $Q(4,q)$ subquadrangle is doubly subtended in $\mathcal{S}(\mathcal{C}_\sigma)$. Since the subtended ovoids are isomorphic to θ_σ, Theorem 3.2.1 shows that the SPG constructed as in Theorem 3.1.7 is distinct from the Metz/Hirschfeld and Thas SPG constructed in the classical case. Hence we have a new SPG. Note that the work in this section relies heavily on the work of Payne and Rogers in [48].

Consider the q-clan

$$\mathcal{C}_\sigma = \left\{ A_t = \begin{pmatrix} t & 0 \\ 0 & -mt^\sigma \end{pmatrix} \; : \; t \in \mathrm{GF}(q) \right\}$$

where q is an odd prime power, m a fixed non-square of $\mathrm{GF}(q)$ and $\sigma \in Aut(GF(q))$, as constructed in [31]. Recall from Section 1.4.5 that if G is the group with elements $\{(\alpha, c, \beta) : \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$ and operation

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta'),$$

then the family of subgroups of G (of order q^2)

$$\begin{aligned} A(\infty) &= \{(\overline{0}, 0, \beta) : \beta \in \mathrm{GF}(q)^2\} \\ A(t) &= \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) : \alpha \in \mathrm{GF}(q)^2\} \quad \text{for } t \in \mathrm{GF}(q), \text{ where } K_t = A_t + A_t^T \\ &= \{(\alpha, \alpha A_t \alpha^T, 2\alpha A_t) : \alpha \in \mathrm{GF}(q)^2\} \quad \text{for } t \in \mathrm{GF}(q) \end{aligned}$$

is a 4-gonal family for G (see Section 1.4.5) which we denote by $\mathcal{F}(\mathcal{C}_\sigma)$. The family of subgroups

$$\begin{aligned} A^\star(\infty) &= \{(\overline{0}, c, \beta) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} \\ A^\star(t) &= \{(\alpha, c, 2\alpha A_t) : c \in \mathrm{GF}(q), \alpha \in \mathrm{GF}(q)^2\} \quad \text{for } t \in \mathrm{GF}(q) \end{aligned}$$

is denoted by $\mathcal{F}^\star(\mathcal{C}_\sigma)$. The GQ of order (q^2, q) constructed from the 4-gonal family $\mathcal{F}(\mathcal{C}_\sigma)$ is denoted by $\mathcal{S}(\mathcal{C}_\sigma)$. Note that Kantor observes that $\mathcal{S}(\mathcal{C}_\sigma) \cong \mathcal{S}(\mathcal{C}_{\sigma^{-1}})$.

In [31] Kantor observes that if $\mathcal{Q}_1 = (\mathrm{GF}(q) \times 0) \times \mathrm{GF}(q) \times (\mathrm{GF}(q) \times 0)$ and $\mathcal{Q}_2 = (0 \times \mathrm{GF}(q)) \times \mathrm{GF}(q) \times (0 \times \mathrm{GF}(q))$ are subgroups of G, then for $i = 1$ or 2, $\mathcal{F}_i = \{A_i(t) = A(t) \cap \mathcal{Q}_i : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ is a 4-gonal family for $\mathcal{Q}_i$ giving rise to an $Sp(4,q)$ subquadrangle, that is, a subquadrangle of $\mathcal{S}(\mathcal{C}_\sigma)$ isomorphic to $W(q)$. We saw in Lemma 1.4.1 that if $\mathcal{W}$ is a GQ of order (r, r^2) with $\mathcal{W}'$ a subquadrangle of order r, then each point of $\mathcal{W}$, external to $\mathcal{W}'$, subtends an ovoid of $\mathcal{W}'$. Dually, $\mathcal{W}^\wedge$ is a GQ of order (r^2, r) with $(\mathcal{W}')^\wedge$ a subquadrangle of order r and each line external to $(\mathcal{W}')^\wedge$ subtends a spread of $(\mathcal{W}')^\wedge$. If $\mathcal{W}'$ is doubly subtended in $\mathcal{W}$, then we say that $(\mathcal{W}')^\wedge$ is doubly subtended in $\mathcal{W}^\wedge$. So, we are interested in the subtended spreads of the GQ determined by $\mathcal{F}_1$ and $\mathcal{F}_2$. It can be shown using Theorem 1.4.7 ([48, IV.1.]) that there exists an isomorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ that maps the subquadrangle determined by $\mathcal{F}_1$ to the subquadrangle determined by $\mathcal{F}_2$. Since this is the case, we will consider only the subquadrangle determined by $\mathcal{F}_1$, which we will referred to as $W(q)$ from here on.

Firstly, we show that the subtended spreads of $W(q)$ are pairwise isomorphic. We do this by showing that the group of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$ is transitive on the lines of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$. The lines of $\mathcal{S}(\mathcal{C}_\sigma)$ that are external to $W(q)$ are $A(\infty)g$, for $g \in G$, such that $A(\infty)g \cap \mathcal{Q}_1$ is empty, and $A(t)g$, for $t \in \mathrm{GF}(q)$ and $g \in G$, such that $A(t)g \cap \mathcal{Q}_1$ is empty. The following lemma deals with the external lines of the form $A(\infty)g$.

Lemma 3.3.1 *The stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on lines external to $W(q)$ that have the form $A(\infty)g$.*

Proof: Let $A(\infty)g$ and $A(\infty)g'$ be two external lines of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$. So $g = ((g_1, g_2), g_3, g_4, (g_4, g_5))$ with $g_2 \neq 0$ and $g = ((g_1', g_2'), g_3', g_4, (g_4', g_5'))$ with $g_2' \neq 0$. Our aim is to find an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$ and maps $A(\infty)g$ to $A(\infty)g'$.

Recall from Theorem 1.4.6 that an automorphism of the group G naturally induces an automorphism of the GQ $\mathcal{S}(\mathcal{C}_\sigma)$. We first find an automorphism of G that maps g to an element of G of the form $((g_1', g_2'), -, (-, -))$, such that the corresponding induced automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ fixes $W(q)$. For $z, k \in \mathrm{GF}(q)$ let

$$\begin{array}{lcl} T_z & : & (\alpha, c, \beta) \mapsto ((\alpha_1 + z, \alpha_2), c, (\beta_1, \beta_2)) \text{ where } \alpha = (\alpha_1, \alpha_2) \text{ and } \beta = (\beta_1, \beta_2) \\ S_k & : & (\alpha, c, \beta) \mapsto (k\alpha, k^2 c, k\beta) \end{array}$$

Now T_z and S_k are automorphisms of G; hence induce automorphisms of $\mathcal{S}(\mathcal{C}_\sigma)$ which we also denote by T_z and S_k. Clearly, T_z and S_k both fix $\mathcal{Q}_1$ and hence fix $W(q)$. Choosing $z = g_1' g_2 / g_2'$ and $k = g_2'/g_2$, we have that

$$S_k T_z : \; g \mapsto ((g_1', g_2'), -, (-, -)),$$

as required. So, without loss of generality we may assume that $g = ((g_1', g_2'), g_3, (g_4, g_5))$.

Let $x = g_3' + (g_4 - g_4')g_1' + (g_5 - g_5')g_2'$, then the automorphism Θ_x of G defined by

$$\Theta_x : \; (\alpha, c, \beta) \mapsto (\alpha, c + x - g_3, \beta)$$

maps g to $((g_1', g_2'), x, (g_4, g_5))$. A straight-forward calculation shows that $\Theta_x g$ and g' are in the same coset of $A(\infty)$, and so $A(\infty)\Theta x = A(\infty)g'$. Thus, if Θ_x induces the automorphism Θ_x of $\mathcal{S}(\mathcal{C}_\sigma)$, then Θ_x maps $A(\infty)g$ to $A(\infty)g'$.

□

Lemma 3.3.2 *The stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on lines external to $W(q)$.*

Proof: First we show that the stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on the set $\{[A(t)] : t \in \mathrm{GF}(q)\}$. Let $s \in \mathrm{GF}(q)$ and let π be a permutation of $\mathrm{GF}(q)$, $\pi : t \mapsto \overline{t} = t + s$. Then

$$\begin{aligned} A_{\overline{t}} = A_{t+s} &= \begin{pmatrix} t+s & 0 \\ 0 & -m(t+s)^\sigma \end{pmatrix} \\ &= \begin{pmatrix} t & 0 \\ 0 & -mt^\sigma \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & -ms^\sigma \end{pmatrix} \\ &= A_t + A_s = A_t + A_{\overline{0}} \end{aligned}$$

So by Theorem 1.4.7 there is an automorphism Θ of $\mathcal{S}(\mathcal{C}_\sigma)$ with $\lambda = 1$, σ the identity, $D = I$ (see Theorem 1.4.7 for notation) and associated permutation π. By Theorem 1.4.7 we have that

$$\begin{array}{lcl} \Theta & : & T_s([A(t)]) \mapsto [A(t+s)] \\ & : & T_s(\alpha, c, \beta) \mapsto (\alpha, c + \alpha A_s \alpha^T, \beta + 2\alpha A_s) \end{array}$$

Now Θ fixes $W(q)$ and if we let s vary over $\mathrm{GF}(q)$, then we have the desired transitivity on $\{[A(t)] : t \in \mathrm{GF}(q)\}$.

From this it can be shown that any line external to $W(q)$, of the form $A(t)g$ may be mapped to an external line of the form $A(0)g'$, for some g'. We now wish to show to construct an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$ and swaps $[A(\infty)]$ and $[A(0)]$ (and hence maps a line of the form $A(0)g$ to one of the form $A(\infty)g'$ and vice versa). Consider the q-clan $\mathcal{C}_\sigma' = \{A_t^{-1} : t \in \mathrm{GF}(q) \setminus \{0\}\} \cup \{A_0\}$. Now if $S = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ then the automorphism of G given by $(\alpha, c, \beta) \mapsto (\alpha S^{-1}, c, \beta S)$ induces an isomorphism from $\mathcal{S}(\mathcal{C}_\sigma')$ to $\mathcal{S}(\mathcal{C}_\sigma)$ which maps $([A(t)], [A(0)], [A(\infty)])$ to $([A(t^{-1})], [A(0)], [A(\infty)])$ for $t \in \mathrm{GF}(q) \setminus \{0\}$. Composing this isomorphism with that in Theorem 1.4.8 yields an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ mapping $[A(\infty)] \leftrightarrow [A(0)]$ and fixing $W(q)$.

Thus any line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$, that has the form $A(t)g$ may be mapped to a line of the form $A(\infty)g'$ by an automorphism of $\mathcal{S}(\mathcal{C}_\sigma)$ fixing $W(q)$. Lemma 3.3.1 then implies that the stabiliser of $W(q)$ in $\mathcal{S}(\mathcal{C}_\sigma)$ is transitive on the lines of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$.

$\square$

We now show that each spread of $W(q)$ subtended by a line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$, is dual to a $K1(\sigma)$ ovoid of $Q(4,q)$. That is, under the duality from $W(q)$ to $Q(4,q)$ any subtended spread of $W(q)$ is mapped to a $K1(\sigma)$ ovoid of $Q(4,q)$. Given Lemma 3.3.2, if one subtended spread of $W(q)$ is dual to a $K1(\sigma)$ ovoid, then all subtended spreads must be.

The $K1(\sigma)$ ovoid of Kantor ([30]) was studied in detail in Chapter 2, but for our present purposes we require only the canonical form of the $K1(\sigma)$ ovoid. We will let $Q(4,q)$ be defined by the equation $x_0x_4 + x_1x_3 + x_2^2 = 0$ and let ϕ_σ be the canonical $K1(\sigma)$ ovoid, where

$$\phi_\sigma = \{(0,0,0,0,1)\} \cup \left\{(1, y, z, -my^\sigma, -z^2 + my^{\sigma+1}) \text{ for } y, z \in \mathrm{GF}(q)\right\}$$

Here m is the same fixed non-square and σ the same automorphism of $\mathrm{GF}(q)$ used in the definition of $\mathcal{C}_\sigma$. The ovoid ϕ_σ may also be written as the intersection of $Q(4,q)$ with the variety defined by the equation $mx_1^\sigma + x_0^{\sigma-1}x_3 = 0$.

Note that this form is slightly different to the canonical form for $Q(4,q)$ and the $K1(\sigma)$ ovoid presented in Chapter 2.

The GQ $W(q)$ may be represented as the set of absolute points and lines of a symplectic polarity in $\mathrm{PG}(3,q)$. The canonical form of $W(q)$ in $\mathrm{PG}(3,q)$ is given by the polarity which has associated bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$. In the following lemma we give an explicit isomorphism between the $W(q)$ as a subquadrangle of $\mathcal{S}(\mathcal{C}_\sigma)$ and the canonical representation in $\mathrm{PG}(3,q)$.

Lemma 3.3.3 Let $\mathcal{F}_1 = \{A_1(t) = A(t) \cap \mathcal{Q}_1 : t \in \mathrm{GF}(q) \cup \{\infty\}\}$ be the 4-gonal family for $W(q)$ as above. Let $W(q)'$ be the GQ arising in $\mathrm{PG}(3,q)$ as the set of absolute points and absolute lines of the symplectic polarity with associated bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$. Then the map ρ is an isomorphism from $W(q)$ to $W(q)'$ where ρ acts as follows:

$$\begin{array}{rcl}
(\infty) & \mapsto & (0,1,0,0) \\
[A_1(t)] & \mapsto & \langle (0,0,2t,1), (0,1,0,0) \rangle \\
[A_1(\infty)] & \mapsto & \langle (0,1,0,0), (0,0,1,0) \rangle \\
((g_1,0), g_2, (g_3,0)) & \mapsto & (1, 2g_2 - g_1g_3, g_3, g_1) \\
A_1(t)((g_1,0), g_2, (g_3,0)) & \mapsto & \langle (1, 2g_2 - g_1g_3, g_3, g_1), (0, 2g_1t - g_3, 2t, 1) \rangle \\
A_1(\infty)((g_1,0), g_2, (g_3,0)) & \mapsto & \langle (1, 2g_2 - g_1g_3, g_3, g_1), (0, g_1, 1, 0) \rangle \\
A_1^\star(t)((g_1,0), g_2, (g_3,0)) & \mapsto & (0, 2tg_1 - g_3, 2t, 1) \\
A_1^\star(\infty)((g_1,0), g_2, (g_3,0)) & \mapsto & (0, 2g_1, 1, 0)
\end{array}$$

where $t, g_1, g_2, g_3 \in \mathrm{GF}(q)$.

Theorem 3.3.4 *Let $W(q)$ be the subquadrangle of $\mathcal{S}(\mathcal{C}_\sigma)$ given by $\mathcal{F}_1$. Then, each spread of $W(q)$ subtended by a line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$ is dual to a $K1(\sigma)$ ovoid of $Q(4,q)$.*

Proof: Given Lemma 3.3.2, we may take our favourite fixed line of $\mathcal{S}(\mathcal{C}_\sigma)$ external to $W(q)$ and find its subtended spread $\mathtt{S}$. Let $\ell = A(\infty)g$ be the external line where $g = ((0,1), 0, (0,0))$. Then the points on $A(\infty)g$ are

$$\{((0,1), u_2, (u_1, u_2)) : u_1, u_2 \in \mathrm{GF}(q)\} \cup \{A^\star(\infty)g\}$$

where $A^\star(\infty)g = \{((0,1), u_3, (u_4, u_5)) : u_3, u_4, u_5 \in \mathrm{GF}(q)\}$.

For each point on $A(\infty)g$ there is a unique line of $W(q)$ incident with it, which is a line of the subtended spread. For the point $A^\star(\infty)$ the corresponding line is $[A(\infty)]$. Now let

$h(u_1,u_2) = ((0,1),u_2,(u_1,u_2))$ for $u_1,u_2 \in \mathrm{GF}(q)$, then for each pair (u_1,u_2) we need to find $t \in \mathrm{GF}(q)$ such that $A(t)h(u_1,u_2)$ is a line of $W(q)$. Now

$$\begin{aligned} A(t)h(u_1,u_2) &= \Big\{\big((v_1,v_2+1),v_1^2t - mv_2^2t^\sigma + u_2 - 2mv_2t^\sigma,(2v_1t+u_1,-2mv_2t^\sigma+u_2)\big) \\ &\quad : v_1,v_2 \in \mathrm{GF}(q)\Big\} \end{aligned}$$

and is a line of $W(q)$ if it contains any point of $W(q)$ (in which case it contains $q+1$ points of $W(q)$). This occurs if, as a coset of $A(t)$ in G, $A(t)h(u_1,u_2)$ contains an elements of $\mathcal{Q}_1$ (in which case it contains q elements of $\mathcal{Q}_1$). So $v_2 = -1$ and $-2mv_2t^\sigma = -u_2$ which implies $t = -\left(\frac{u_2}{2m}\right)^{\sigma^{-1}}$.

Thus the spread of $W(q)$ subtended by $\mathcal{S}(\mathcal{C}_\sigma)$ has the form:

$$\mathtt{S} = \begin{cases} [A(\infty)] \\ A\left(-\left(\frac{u_2}{2m}\right)^{\sigma^{-1}}\right) h(u_1,u_2) \quad u_1,u_2 \in \mathrm{GF}(q) \end{cases}$$

To express the spread in the group coset representation of $W(q)$ without reference to $\mathcal{S}(\mathcal{C}_\sigma)$, we need a representative of the coset $A\left(-\left(\frac{u_2}{2m}\right)^{\sigma^{-1}}\right) h(u_1,u_2)$ that is in $\mathcal{Q}_1$, say $((0,0),u_2 + 2mt^\sigma,(u_1,0)) = ((0,0),u_2/2,(u_1,0))$. Thus the spread is

$$\mathtt{S} = \begin{cases} [A(\infty)] \\ A\left(-\left(\frac{u_2}{2m}\right)^{\sigma^{-1}}\right) ((0,0),\frac{u_2}{2},(u_1,0)) \quad u_1,u_2 \in \mathrm{GF}(q). \end{cases}$$

By using the isomorphism in Lemma 3.3.3, $\mathtt{S}$ has the following form in $W(q)'$:

$$\mathtt{S} = \begin{cases} \{x_0 = 0; x_3 = 0\} = \langle (0,1,0,0),(0,0,1,0)\rangle \\ \langle (1,u_2,u_1,0),(0,-u_1,2t,1)\rangle, \qquad t = -\left(\frac{u_2}{2m}\right)^{\sigma^{-1}} \quad \text{for } u_1,u_2 \in \mathrm{GF}(q). \end{cases}$$

We use Plucker coordinates and the Klein correspondence (see [25, Chapter 15]) to give a duality from $W(q)'$ to $Q(4,q)$. Thus, in $Q(4,q)$ the spread $\mathcal{S}$ becomes an ovoid θ, say, which has the form:

$$\theta = \begin{cases} (0,0,0,0,1) \\ (1,2t,-u_1,u_2,-2u_2t-u_1^2), \quad t = -\left(\frac{u_2}{2m}\right)^{\sigma^{-1}} \quad \text{for } u_1,u_2 \in \mathrm{GF}(q). \end{cases}$$

Now since

$$\begin{aligned} m(2t)^\sigma + 1^{\sigma-1}u_2 &= m\left(2\left(\frac{-u_2}{2m}\right)^{\sigma^{-1}}\right)^\sigma + u_2 \\ &= m\left(\frac{-u_2}{m}\right) + u_2 \\ &= 0, \end{aligned}$$

it follows that every point of θ satisfies the equation $mx_1^\sigma + x_0^{\sigma-1}x_3 = 0$. Thus θ is the canonical $K1(\sigma)$ ovoid ϕ_σ.

□

Theorem 3.3.5 *There is an involution of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$.*

Proof: Consider the automorphism $\Theta_D = \Theta(\pi, \lambda, \rho, D)$ of $\mathcal{S}(\mathcal{C}_\sigma)$, using the notation of Theorem 1.4.7, where π is the identity permutation on GF(q), $\lambda = 1$, ρ the identity automorphism of GF(q) and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2,q)$. Then

$$\Theta_D : (\alpha, c, \beta) \mapsto ((\alpha_1, -\alpha_2), c, (\beta_1, -\beta_2)) \text{ where } \alpha = (\alpha_1, \alpha_2) \text{ and } \beta = (\beta_1, \beta_2).$$

Thus Θ_D is an involution of $\mathcal{S}(\mathcal{C}_\sigma)$ that fixes $W(q)$.

□

Corollary 3.3.6 *The SPG constructed by the GQ $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ and the subGQ isomorphic to $Q(4,q)$, as in Theorem 3.1.7, is not isomorphic to the known SPG with parameters $s = q-1$, $t = q^2$, $\alpha = 2$ and $\mu = 2q(q-1)$.*

Proof: The GQ $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ of order (q, q^2) has a subquadrangle of order q isomorphic to $Q(4,q)$ such that there is a collineation of $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ that fixes this subquadrangle pointwise. Thus, by Corollary 3.1.8 we have an SPG and since each subtended ovoid of the subquadrangle is non-classical, by Theorem 3.2.1 it is not isomorphic to the known SPG of that order.

□

It would be interesting to determine the configuration in $Q(4,q)$ of the special points of the $K1(\sigma)$ ovoids that are points of the SPG (that is, subtended in $Q(4,q)$ by $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$). One possible conjecture is that the special points form a line ℓ of $Q(4,q)$ and that the set of rosettes with base point $P \in \ell$ are the elation rosettes with special tangent line in the polar plane of ℓ (with respect to $Q(4,q)$). This sort of information would be important in using the techniques of Section 4.2 to try to characterise the GQ $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ by the $Q(4,q)$ subquadrangle.

3.4 SPG from q-clan GQs, q even

Let $\mathcal{S}$ be a GQ of order (q^2, q), q even, constructed from a q-clan, as in Section 1.4.5 and reprised in Section 3.3. For any such GQ $\mathcal{S}$ Payne ([45]) has constructed a family of subquadrangles, $\{\mathcal{S}_\alpha : \alpha \in \mathrm{GF}(q)\}$, of order q. In this section we investigate whether any of the subquadrangles $\mathcal{S}_\alpha$ is doubly subtended in $\mathcal{S}$ and so leads to an SPG via the construction in Theorem 3.1.7 (note that we are considering the dual of the situation in Theorem 3.1.7). In Section 3.4.1 we determine the algebraic conditions on the q-clan under which $\mathcal{S}_\alpha$ is doubly subtended in $\mathcal{S}$. In Section 3.4.2 we use the algebraic conditions generated in Section 3.4.1 to show that of the known q-clan GQs, q even, only in the classical case $\mathcal{S} \cong Q(5,q)$, is any $\mathcal{S}_\alpha$ doubly subtended in $\mathcal{S}$.

3.4.1 Algebraic conditions

In this section we outline the construction of the subquadrangles $\mathcal{S}_\alpha$ and then look for an automorphism of $\mathcal{S}$ that fixes $\mathcal{S}_\alpha$. By Lemma 3.1.5 such an automorphism exists if and only if $\mathcal{S}_\alpha$ is doubly subtended in $\mathcal{S}$.

The following derivation of the subquadrangles $\mathcal{S}_\alpha$ is taken from [46].

For $q = 2^e$ a q-clan $\mathcal{C}$ may be normalised to a q-clan $\mathcal{C}'$ giving a GQ isomorphic to that given by $\mathcal{C}$, such that $\mathcal{C}'$ has the form $\{A_t : t \in \mathrm{GF}(q)\}$, where

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix} \quad \text{(with trace}(\delta) = 1\text{); and } A_t = \begin{pmatrix} x_t & t \\ 0 & y_t \end{pmatrix}, t \in \mathrm{GF}(q).$$

From this point we will assume that a q-clan $\mathcal{C}$ has the above normalised form.

We now introduce a group isomorphic to G where the subgroups generating the subGQs will be easier to recognise. In Section 1.4.5 4-gonal families are constructed for the group

$$G = \{(\alpha, c, \beta) : \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$$

with the group operation

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta').$$

The subgroups of the 4-gonal family are then:

$$\begin{aligned}
A(\infty) &= \{(0, 0, \beta) : \beta \in \mathrm{GF}(q)^2\} \\
A(t) &= (\alpha, \alpha A_t \alpha^T, \alpha K_t) : \alpha \in \mathrm{GF}(q)^2\} \\
A^\star(\infty) &= \{(0, c, \beta) \in \overline{G} : c \in \mathrm{GF}(q), \beta \in \mathrm{GF}(q)^2\} \\
A^\star(t) &= \{\alpha, c, \alpha K_t) : \alpha \in \mathrm{GF}(q)^2\} \\
&\ \text{where } K_t = A_t + A_t^T.
\end{aligned}$$

Now, let $E = \mathrm{GF}(q^2)$, such that $E = \mathrm{GF}(q)(\mu)$ where $\mu^2 + \mu + \delta$ (where δ is an element of $\mathrm{GF}(q)$ with $trace(\delta) = 1$). We will associated the element $a + b\mu$ of E with the element (a, b) of $\mathrm{GF}(q^2)$. Let $x \mapsto \overline{x} = x^q$ and define $\alpha \circ \beta = \alpha\overline{\beta} + \overline{\alpha}\beta$. We now define a new group

$$\overline{G} = \{(\alpha, c, \beta) : \alpha, \beta \in E, c \in \mathrm{GF}(q)\}$$

with operation

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \sqrt{\beta \circ \alpha'}, \beta + \beta').$$

If $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\zeta : G \to \overline{G} : (\alpha, c, \beta) \mapsto (\alpha, \sqrt{c}, \beta P)$ is an isomorphism with $\zeta^{-1} : \overline{G} \to G : (\alpha, c, \beta) \mapsto (\alpha, c^2, \beta P)$.

Thus the q-clan $\mathcal{C}$ gives the following 4-gonal family $\overline{\mathcal{F}}$.

$$\begin{aligned}
\overline{A}(\infty) &= \{(0, 0, \beta) : \beta \in E\} \\
\overline{A}(t) &= (\alpha, \sqrt{\alpha A_t \alpha^T}, y_t \alpha) : \alpha \in E\} \\
\overline{A}^\star(\infty) &= \{(0, c, \beta) : c \in \mathrm{GF}(q), \beta \in E\} \\
\overline{A}^\star(t) &= \{\alpha, c, y_t \alpha) : \alpha \in E\}
\end{aligned}$$

Consider the following subgroup of $\overline{G}$:

$$\overline{G}_\alpha = \{(a\alpha, c, b\alpha) \in G : a, b, c \in \mathrm{GF}(q)\} \text{ where } \alpha \in E$$

The group operation of $\overline{G}$ restricts to $\overline{G}_\alpha$ in the following way:

$$\begin{aligned}
(a\alpha, c, b\alpha) \cdot (a'\alpha, c', b'\alpha) &= (a\alpha + a'\alpha, c + c' + \sqrt{b\alpha \circ a'\alpha}, b\alpha + b'\alpha) \\
&= ((a + a')\alpha, c + c', (b + b')\alpha)
\end{aligned}$$

So $\overline{G}_\alpha$ is an elementary abelian group of order q^3, and in fact $\overline{G}_\alpha \cong \mathrm{GF}(q)^3$ under component-wise addition.

Now consider the following subgroups of $\overline{G}_\alpha$:

$$\begin{aligned}
\overline{A}_\alpha(\infty) &= A(\infty) \cap G_\alpha &&= \{(0, 0, b\alpha) : b \in \mathrm{GF}(q)\} \\
\overline{A}_\alpha(t) &= A(t) \cap G_\alpha &&= \{(a\alpha, a\sqrt{\alpha A_t \alpha^T}, at\alpha) : a \in \mathrm{GF}(q)\} \\
\overline{A}^\star_\alpha(\infty) &= A^\star(\infty) \cap G_\alpha &&= \{(0, c, b\alpha) : b, c \in \mathrm{GF}(q)\} \\
\overline{A}^\star_\alpha(t) &= A^\star(t) \cap G_\alpha &&= \{(a\alpha, c, at\alpha) : a, c \in \mathrm{GF}(q)\}.
\end{aligned}$$

The family $\overline{\mathcal{F}}_\alpha = \left\{\overline{A}_\alpha(t) : t \in \mathrm{GF}(q) \cup \{(\infty)\}\right\}$ is a 4-gonal family for a subGQ, which we will denote by $\mathcal{S}_\alpha$.

At this point we make the following definition from [48]

Definition 3.4.1 *Let $A, B \in GL(2,q)$ such that $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then*

$$\mathbf{A} \equiv \mathbf{B} \text{ if } x = a,\ y = d,\ z + y = b + c.$$

Given this definition it can be shown that, for q even, [48, IV.1] is equivalent to the following theorem.

Theorem 3.4.2 Let $\mathcal{C} = \{A_t : t \in \mathrm{GF}(q)\}$ be a q-clan with $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let Θ be a collineation of the GQ $\mathcal{S} = S(G, \mathcal{F}(\mathcal{C}))$ derived from $\mathcal{C}$ which fixes the point (∞), the line $[A(\infty)]$ and the point $(0,0,0)$. Then the following must exist:

(i) A permutation $\pi : t \mapsto t'$ of $\mathcal{F}$.

(ii) $\lambda \in \mathrm{GF}(q)$, $\lambda \neq 0$.

(iii) $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$.

(iv) $D \in GL(2,q)$ for which $A_{t'} \equiv \lambda D^T A_t^\sigma D - A_{0'}$, for all $t \in \mathrm{GF}(q)$.

Conversely, given σ, D, λ and a permutation $\pi : t \mapsto t'$, satisfying the above conditions, the following automorphism Θ of G induces a collineation of $\mathcal{S}$, fixing (∞),$[A(\infty)]$ and $(0,0,0)$.

$$\begin{aligned}
&\Theta = \Theta(\pi, \lambda, \sigma, D) : G \to G : \\
&(\alpha, c, \beta) \mapsto (\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-\frac{1}{2}}c^\sigma + \lambda^{-1}\sqrt{\alpha^\sigma D^{-T} A_{0'} D^{-1} (\alpha^\sigma)^T}, \beta^\sigma PDP + \lambda^{-1} y_{0'} \alpha^\sigma D^{-T}).
\end{aligned}$$

Theorem 3.4.3 *Let $C = \{A_t : t \in \mathrm{GF}(q)\}$, q even, be a q-clan normalised as follows:*

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix} \quad (\textit{with } \mathrm{trace}(\delta) = 1); \textit{ and } A_t = \begin{pmatrix} x_t & t \\ 0 & y_t \end{pmatrix}, t \in \mathrm{GF}(q).$$

Let $\mathcal{S}$ be the GQ of order (q^2, q) defined by $\mathcal{C}$ and let $\mathcal{S}_\alpha$ be the subquadrangle of $\mathcal{S}$ defined by the subgroup $\overline{G}_\alpha$ of $\overline{G}$, where $\alpha = (\alpha_1, \alpha_2)$. Then $\mathcal{S}_\alpha$ defines an SPG, as in Theorem 3.1.7, if and only if

$$\alpha_1^2 x_t + \alpha_2^2 y_t + (\alpha_1^2 + \delta\alpha_2^2)t = 0 \textit{ for all } t \in \mathrm{GF}(q) \setminus \{0, 1\}.$$

Proof: By Lemma 3.1.5, it suffices to find conditions under which $\mathcal{S}$ admits a non-identity involution Θ that fixes the subGQ $\mathcal{S}_\alpha$ pointwise.

Now (∞), $[A(\infty)]$ and $(0,0,0)$ are all contained in the subGQ S_α and so Θ must fix these. Thus Θ has the form of the automorphisms of $\mathcal{S}$ given by Theorem 3.4.2. Using the notation of Theorem 3.4.2 we have that, $A_{0'} = A_0$ (since $[A(0)] \in \mathcal{S}_\alpha$ and so if fixed) and $A_{t'} \equiv \lambda D^T A_t^\sigma D$, for some $\lambda \in \mathrm{GF}(q)$, $\lambda \neq 0$, $\sigma \in Aut(\mathrm{GF}(q))$ and $D \in GL(2,q)$. Since $\mathcal{S}_\alpha$ is fixed by θ it follows that $A_{t'} \equiv A_t$ and so

$$A_t \equiv \lambda D^T A_t^\sigma D \tag{3.4.2}$$

The subgroup $G_\alpha = \{(a\alpha, c, b\alpha) : a, b, c \in \mathrm{GF}(q)\}$ is fixed element-wise by Θ. But

$$\Theta : (0, c, 0) \mapsto (0, \lambda^{-\frac{1}{2}}c^\sigma, 0),$$

thus $\lambda^{-\frac{1}{2}}c^\sigma = c$ for all $c \in \mathrm{GF}(q)$. Letting $c = 1$ implies that $\lambda = 1$ and so $c^\sigma = c$, that is σ fixes $\mathrm{GF}(q)$ and hence σ is the identity automorphism on $\mathrm{GF}(q)$.

Now we know that $\Theta:\ (\alpha, c, \beta) \mapsto (\alpha D^{-T}, c, \beta PDP)$ so D^2 is the identity matrix, I_2 (since Θ is an involution). We also know that $A_t \equiv D^T A_t D$. If $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $D^2 = I_2$ gives us the following equations

$$\begin{aligned} a^2 + bc &= 1 && (3.4.3)\\ ab + bd &= 0 && (3.4.4)\\ ac + cd &= 0 && (3.4.5)\\ bc + d^2 &= 1 && (3.4.6) \end{aligned}$$

As D is non-singular, equations 3.4.3 and 3.4.4 imply that $a = d$, in which case equations 3.4.4 and 3.4.5 are satisfied.

Now

$$\begin{aligned} D^T A_1 D &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a & a + c\delta \\ b & b + d\delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 + ac + c^2\delta & ab + ad + cd\delta \\ ab + bc + cd\delta & b^2 + bd + d^2\delta \end{pmatrix} \end{aligned}$$

and so by 3.4.2 we have

$$\begin{aligned} a^2 + ac + c^2\delta &= 1 && (3.4.7)\\ b^2 + bd + d^2\delta &= \delta && (3.4.8)\\ (ab + ad + cd\delta) + (ab + bc + cd\delta) &= 1 \\ \Rightarrow ad + bc = |D| &= 1 && (3.4.9) \end{aligned}$$

Note that since $a = d$ 3.4.3, 3.4.6 and 3.4.9 are equivalent.

Since Θ fixes $\mathcal{S}_\alpha$ we have that

$$\Theta:\ (a\alpha, c, b\alpha) \mapsto (a\alpha D^{-T}, c, b\alpha PDP) = (a\alpha, c, b\alpha),$$

which gives us the following equations:

$$\begin{aligned} \alpha D^{-T} &= \alpha \\ \alpha PDP &= \alpha. \end{aligned}$$

Now

$$D^{-1} = \frac{1}{ad + bc} \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \text{ by 3.4.9}$$

and so $D^{-T} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$.

Now $PDP = D^{-T}$, and so the two equations in D and α are identical, giving us the equations

$$\begin{aligned} d\alpha_1 + b\alpha_2 &= \alpha_1 && (3.4.10)\\ c\alpha_1 + a\alpha_2 &= \alpha_2, && (3.4.11) \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$. Now if $\alpha_1 = 0$ then $\alpha_2 \neq 0$ and it follows that $b = 0$, $a = d = 1$ and $c = 0$ or $1/\delta$. If $c = 0$, then $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and Θ is the identity automorphism of $\mathcal{S}$.

If $\alpha_1 \neq 0$ and $\alpha_2 = 0$, then it follows that $c = 0$, $a = d = 1$ and $b = 0$ or 1. In the case where $b = 0$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and Θ is the identity automorphism of $\mathcal{S}$.

Now if $\alpha_1, \alpha_2 \neq 0$, then we may rewrite 3.4.10 and 3.4.11 as follows

$$a + b\frac{\alpha_2}{\alpha_1} \quad = \quad 1 \tag{3.4.12}$$

$$c\frac{\alpha_1}{\alpha_2} + a \quad = \quad 1, \tag{3.4.13}$$

and so $b = c\alpha_1^2/\alpha_2^2$. Substituting this value for b into equation 3.4.3 yields $a = d = 1 + c\alpha_1/\alpha_2$. Given these formulae for a, b, d in terms of c, we may rewrite 3.4.7 as

$$\begin{aligned} 1 + c^2\frac{\alpha_1^2}{\alpha_2^2} + c + c^2\frac{\alpha_1}{\alpha_2} + c^2\delta \quad &= \quad 1 \\ \Longleftrightarrow c\left(1 + c\left(\frac{\alpha_1^2}{\alpha_2^2} + \frac{\alpha_1}{\alpha_2} + \delta\right)\right) \quad &= \quad 0. \end{aligned} \tag{3.4.14}$$

Note that since $trace(\delta) = 1$, it follows that $x^2+x+\delta$ is irreducible, and so $\alpha_1^2/\alpha_2^2+\alpha_1/\alpha_2+\delta \neq 0$.

Equation 3.4.14 is satisfied if and only if $c = 0$ or

$$c \;=\; \frac{1}{\dfrac{\alpha_1^2}{\alpha_2^2} + \dfrac{\alpha_1}{\alpha_2} + \delta} \;=\; \frac{\alpha_2^2}{\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2}.$$

Rewriting 3.4.8 yields

$$\begin{aligned} c^2\frac{\alpha_1^4}{\alpha_2^4} + c\frac{\alpha_1^2}{\alpha_2}\left(1 + c\frac{\alpha_1}{\alpha_2}\right) + \left(1 + c^2\frac{\alpha_1^2}{\alpha_2^2}\right)\delta \quad &= \quad \delta \\ \Longleftrightarrow \qquad c^2\frac{\alpha_1^4}{\alpha_2^4} + c\frac{\alpha_1^2}{\alpha_2^2} + c^2\frac{\alpha_1^3}{\alpha_2^3} + c^2\delta\frac{\alpha_1^2}{\alpha_1^2} \quad &= \quad 0 \\ \Longleftrightarrow \qquad c\left(1 + c\left(\frac{\alpha_1^2}{\alpha_2^2} + \frac{\alpha_1}{\alpha_2} + \delta\right)\right) \quad &= \quad 0, \end{aligned}$$

which is precisely 3.4.14.

If $c = 0$, then $b = 0$ and $a = d = a$, that is D is the identity matrix and θ the identity automorphism of $\mathcal{S}$. If $c = \alpha_2^2/(\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2)$, then

$$b = \frac{\alpha_1^2}{\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2}, \qquad a = d = 1 + \frac{\alpha_1\alpha_2}{\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2} = \frac{\alpha_1^2 + \delta\alpha_2^2}{\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2}.$$

Now notice that these formulae also give the previously determined values for a, b, c and d in the case where $\alpha_1 = 0$, $\alpha_2 \neq 0$ and D is not the identity matrix and the case where $\alpha_1 \neq 0$, $\alpha_2 = 0$ and D is not the identity matrix.

We now need to check whether when the possibilities for D that we have calculated satisfy the property 3.4.2.

$$\begin{aligned} & D^T A_t D \\ = \quad & \frac{1}{(\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2)^2}\begin{pmatrix} \alpha_1 + \delta\alpha_2^2 & \alpha_2^2 \\ \alpha_1^2 & \alpha_1^2 + \delta\alpha_2^2 \end{pmatrix}\begin{pmatrix} x_t & t \\ 0 & y_t \end{pmatrix}\begin{pmatrix} \alpha_1 + \delta\alpha_2^2 & \alpha_1^2 \\ \alpha_2^2 & \alpha_1^2 + \delta\alpha_2^2 \end{pmatrix} \\ = \quad & \frac{1}{(\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2)^2}\begin{pmatrix} (\alpha_1^2 + \delta\alpha_2^2)x_t & (\alpha_1^2 + \delta\alpha_2^2)t + \alpha_2^2 y_t \\ \alpha_1^2 x_t & \alpha_1^2 t + (\alpha_1^2 + \delta\alpha_2^2)y_t \end{pmatrix}\begin{pmatrix} \alpha_1^2 + \delta\alpha_2^2 & \alpha_1^2 \\ \alpha_2^2 & \alpha_1^2 + \delta\alpha_2^2 \end{pmatrix} \\ = \quad & \frac{1}{(\alpha_1^2 + \alpha_1\alpha_2 + \delta\alpha_2^2)^2}\times \end{aligned}$$

$$\begin{pmatrix} (\alpha_1^4+\delta^2\alpha_2^4)x_t+\alpha_2^4y_t+(\alpha_1^2+\delta\alpha_2^2)\alpha_2^2t & (\alpha_1^2+\delta\alpha_2^2)\alpha_1^2x_t+(\alpha_1^4+\delta^2\alpha_2^4)t+(\alpha_1^2+\delta\alpha_2^2)\alpha_2^2y_t \\ (\alpha_1^2+\delta\alpha_2^2)\alpha_1^2x_t+\alpha_1^2\alpha_2^2t+(\alpha_1^2+\delta\alpha_2^2)\alpha_2^2y_t & \alpha_1^4x_t+(\alpha_1^4+\delta^2\alpha_2^4)y_t+(\alpha_1^2+\delta\alpha_2^2)\alpha_1^2t \end{pmatrix}$$

Now $D^TA_tD \equiv A_T$ if and only if the following three equations are satisfied

$$\begin{aligned} x_t &= \frac{(\alpha_1^4+\delta^2\alpha_2^4)x_t+\alpha_2^4y_t+(\alpha_1^2+\delta\alpha_2^2)t\alpha_2^2}{(\alpha_1^2+\alpha_1\alpha_2+\delta\alpha_2^2)^2} \\ y_t &= \frac{\alpha_1^4x_t+(\alpha_1^4+\delta^2\alpha_2^4)y_t+(\alpha_1^2+\delta\alpha_2^2)\alpha_1^2t}{(\alpha_1^2+\alpha_1\alpha_2+\delta\alpha_2^2)^2} \\ t &= \frac{(\alpha_1^4+\alpha_1^2\alpha_2^4+\delta^2\alpha_2^4)t}{(\alpha_1^2+\alpha_1\alpha_2+\delta\alpha_2^2)^2} = t \end{aligned}$$

Note that the third of these equations always holds. Rewriting the first of the three equations yields

$$\begin{aligned} x_t(\alpha_1^4+\alpha_1^2\alpha_2^2+\delta^2\alpha_2^4)+(\alpha_1^4+\delta^2\alpha_2^4)x_t+\alpha_2^4y_t+\alpha_2^2(\alpha_1^2+\delta\alpha_2^2)t &= 0 \\ \Longleftrightarrow \qquad \alpha_1^2\alpha_2^2x_t+\alpha_2^4y_t+\alpha_2^2(\alpha_1^2+\delta\alpha_2^2)t &= 0 \\ \Longleftrightarrow \qquad \alpha_2^2(\alpha_1^2x_t+\alpha_2^2y_t+(\alpha_1^2+\delta\alpha_2^2)t) &= 0 \end{aligned}$$

and similarly the second of the three equations may be rewritten as

$$\alpha_1^2(\alpha_1^2x_t+\alpha_2^2y_t+(\alpha_1^2+\delta\alpha_2^2)t)=0.$$

Now if $\alpha_1 \neq 0$, then the second equation yields

$$\alpha_1^2x_t+\alpha_2^2y_t+(\alpha_1^2+\delta\alpha_2^2)t=0$$

and if $\alpha_2 \neq 0$, then the first equation also gives this. Thus the conditions that we require are

$$\alpha_1^2x_t+\alpha_2^2y_t+(\alpha_1^2+\delta\alpha_2^2)t=0 \quad \text{for all } t \in \mathrm{GF}(q)\setminus\{0,1\}. \tag{3.4.15}$$

Note that the normalised form of the q-clan ensures that the equation in 3.4.15 is always satisfied for $t=0$ and $t=1$.

□

3.4.2 Examples

In this section we consider the examples of q-clan GQs, q even. We show that if $\mathcal{S}$ is a known example that has a doubly subtended subquadrangle $\mathcal{S}_\alpha$, for some $\alpha \in \mathrm{GF}(q)^2 \setminus \{(0,0)\}$, then $\mathcal{S}$ is the classical GQ, that is $\mathcal{S} \cong H(3,q^2) \cong Q(5,q)^\wedge$. In this case, $\mathcal{S}_\alpha$ is doubly subtended for all $\alpha \in \mathrm{GF}(q)^2 \setminus \{(0,0)\}$.

Classical (see [44] for example): The GQs exist for all $q=2^e$. The q-clan is

$$\left\{\begin{pmatrix} t & t \\ 0 & at \end{pmatrix} : t \in \mathrm{GF}(q)\right\} \text{ where } trace(a)=1,$$

and so the condition on α in Theorem 3.4.3 is equivalent to the polynomial in t given by

$$\alpha_1^2t+\alpha_2^2at+(\alpha_1^2+a\alpha_2^2)t$$

begin identically zero for $(\alpha_1,\alpha_2) \neq (0,0)$. This is clearly always the case.

Fisher-Thas-Walker (see [63]): The GQs exist for $q \equiv -1 \pmod 3$, so for the q even case this is $q = 2^e$, e odd and $e > 1$. The q-clan is

$$\left\{ \begin{pmatrix} \sqrt{t} & t \\ 0 & \sqrt{t^3} \end{pmatrix} : t \in \mathrm{GF}(q) \right\},$$

that is $x_t = \sqrt{t}$, $y_t = \sqrt{t^3}$ and $\delta = 1$, using the notation of Theorem 3.4.3. The condition for the existence of an SPG is that the polynomial in t given by

$$\alpha_1^2 \sqrt{t} + \alpha_2^2 \sqrt{t^3} + (\alpha_1^2 + \alpha_2^2) t$$

is identically zero for fixed α_1, α_2, $(\alpha_1, \alpha_2) \neq (0,0)$. Clearly, this is never the case.

Payne [45]: The GQs exist for $q = 2^e$, e odd and $e > 1$. The q-clan is

$$\left\{ \begin{pmatrix} t^{1/3} & t \\ 0 & t^{1/5} \end{pmatrix} : t \in \mathrm{GF}(q) \right\},$$

and so the condition from Theorem 3.4.3 is equivalent to the polynomial in t given by

$$\alpha_1^2 t + \alpha_2^2 t^5 + (\alpha_1^2 + \alpha_2^2) t^3$$

being identically zero for $(\alpha_1, \alpha_2) \neq 0$, which is clearly never the case.

Subiaco [13]: The GQs exist for $q = 2^e$ and $q \geq 8$. The q-clan is given by

$$\left\{ \begin{pmatrix} f(t)^2 & t \\ 0 & a^2 g(t)^2 \end{pmatrix} : t \in \mathrm{GF}(q) \right\},$$

where

$$\begin{aligned} f(t) &= \frac{d^2(t^4 + t) + d^2(1 + d + d^2)(t^3 + t^2)}{(t^2 + dt + 1)^2} + t^{1/2}, \\ g(t) &= \frac{d^4 t^4 + d^3(1 + d^2 + d^4)t^3 + d^3(1 + d^2)t}{(d^2 + d^5 + d^{1/2})(t^2 + dt + 1)^2} + \frac{d^{1/2}}{d^2 + d^5 + d^{1/2}} t^{1/2}, \text{ and} \\ a &= \frac{d^2 + d^5 + d^{1/2}}{d(1 + d + d^2)}, \end{aligned}$$

with $d \in \mathrm{GF}(q)$ such that $d^2 + d + 1 \neq 0$ and $trace(1/d) = 1$.

The condition from Theorem 3.4.3 becomes

$$\alpha_1^2 f(t)^2 + \alpha_2^2 a^2 g(t)^2 + (\alpha_1^2 + \delta \alpha_2^2) t = 0 \quad \text{for all } t \in \mathrm{GF}(q),$$

which is equivalent to the polynomial in t given by

$$\left[\left(\frac{dt^4 + d^3(1 + d^2 + d^4)t^3 + d^3(1 + d^2)t}{d(1 + d + d^2)} \right)^2 + \frac{td(t^2 + dt + 1)^4}{d^2(1 + d + d^2)^2} \right] \alpha_2^2$$

$$+ \left[(d^2(t^4 + t) + d^2(1 + d + d^2)(t^3 + t^2))^2 + t(t^2 + dt + 1)^4 \right] \alpha_1^2 + (\alpha_1^2 + \delta \alpha_2^2) t (t^2 + dt + 1)^4$$

being identically zero. If we equate the coefficient of t^2 to zero we obtain

$$\begin{aligned} \alpha_1^2 (d^4) + \alpha_2^2 \frac{d^6(1 + d^4)}{d^2(1 + d^2 + d^4)} &= 0 \\ \iff \quad \alpha_1^2 (1 + d^2 + d^4) + \alpha_2^2 (1 + d^4) &= 0 \end{aligned}$$

and by equating the coefficient of t^6 to zero we obtain

$$\begin{array}{rrcl} & \alpha_1^2 d^4(1+d^2+d^4)+\alpha_2^2\dfrac{d^6(1+d^4+d^8)}{d^2(1+d+d^2)} & = & 0 \\ \Longleftrightarrow & (\alpha_1^2+\alpha_2^2)(1+d^2+d^4) & = & 0 \\ \Longleftrightarrow & \alpha_1 & = & \alpha_2. \end{array}$$

Substituting $\alpha_1 = \alpha_2$ into the t^2 coefficient equation yields $\alpha_1^2 d^2 = 0$, which means that $\alpha_1 = \alpha_2 = 0$.

Chapter 4

Characterisations of GQs of order (r, r^2)

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (r, r^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I)$ a subquadrangle of $\mathcal{S}$ of order r. In Chapter 3 (Lemma 3.1.1) we observed that every point of $\mathcal{P} \setminus \mathcal{P}'$ is external to $\mathcal{S}'$ and so subtends an ovoid of $\mathcal{S}'$, while every line of $\mathcal{B} \setminus \mathcal{B}'$ is tangent to $\mathcal{S}'$ and so subtends a rosette of ovoids of $\mathcal{S}'$ (see Section 1.4.1 for definitions). Two questions naturally arise at this point. The first is: given $\mathcal{S}'$, a GQ of order r what ovoid/rosette structures of $\mathcal{S}'$ may be subtended by a GQ $\mathcal{S}$ of order (r, r^2) that contains σ' as a subquadrangle? This is particularly interesting in the light of Theorem 3.1.10 which says that in the case where $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$ (that is, each ovoid of $\mathcal{S}'$ that is subtended by $\mathcal{S}$ is subtended by exactly two points of $\mathcal{S}$), that $\mathcal{S}$ can be reconstructed from $\mathcal{S}'$, the subtended ovoid/rosette structure (an SPG in this case) and an algebraic 2-fold cover of the subtended ovoid/rosette structure. If we choose the 'right' ovoid/rosette structure of $\mathcal{S}'$ can we 'naturally' construct a GQ $\mathcal{S}$ of order (r, r^2) from $\mathcal{S}'$ and this ovoid/rosette structure?

The second question is: given $\mathcal{S}$, a GQ of order (r, r^2) containing a subquadrangle $\mathcal{S}'$ of order r, does the subtended ovoid/rosette structure of $\mathcal{S}'$ characterise the GQ $\mathcal{S}$, or is it possible to find another GQ $\overline{\mathcal{S}}$ of order (r, r^2), not isomorphic to $\mathcal{S}$, but with $\mathcal{S}'$ as a subquadrangle and subtending the same ovoid/rosette structure in $\mathcal{S}'$ as $\mathcal{S}$? A result in this direction, which served as the inspiration for the work in this chapter, is the following theorem due to Thas and Payne [65, VII.1].

Theorem 4.0.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (q, q^2), q even, having a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, q)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_x$ consisting of all of the points collinear with a given point x of $\mathcal{S} \setminus \mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

In the first section of this chapter we consider the question: which ovoid/rosette structures of $W(q)$, q even, (recall $W(q)$ is isomorphic to $Q(4, q)$ for q even) may be subtended by a GQ of order (q, q^2)? As a result we are able to strengthen the characterisation of Theorem 4.0.4.

In the second section of this chapter we consider a GQ $\mathcal{S}$ of order (r, r^2) containing a doubly subtended subquadrangle $\mathcal{S}'$ of order r. We formulate the question of whether $\mathcal{S}$ can be characterised by $\mathcal{S}'$ and the associated SPG (a generalisation of the case $Q(4, q) \subset Q(5, q)$ in Theorem 4.0.4) as a cohomological problem, the solution of which allows us to prove the equivalent result to Theorem 4.0.4 for q odd.

4.1 Characterisations of $Q(5, q)$, q even

Recall that for q even the GQ $W(q)$ is isomorphic to $Q(4, q)$ ([49, 3.2.1]). If θ is an ovoid of $W(q)$, then θ is also an ovoid of the ambient $\mathrm{PG}(3, q)$ of $W(q)$, and if θ' is an ovoid of $\mathrm{PG}(3, q)$, then there is an automorphism T of $\mathrm{PG}(3, q)$ such that $T(\theta')$ is an ovoid of $W(q)$ ([55]). Note that it is not true in general that any ovoid of $\mathrm{PG}(3, q)$ is an ovoid of $W(q)$.

For $q = 2^e$, e even, the only known ovoid of $W(q)$ is the elliptic quadric ovoid of $\mathrm{PG}(3,q)$ while for $q = 2^e$, e odd, there is also the Tits ovoid (see Section 1.4.6 for the explicit form of the Tits ovoid).

Now if $Q(4,q)$ is the non-singular quadric of $\mathrm{PG}(4,q)$, P a point of $Q(4,q)$ and Σ a hyperplane of $\mathrm{PG}(4,q)$ not containing P, then the projection of points and lines of $Q(4,q)$ from P onto Σ is the geometry $W(q)$ and Σ is the ambient $\mathrm{PG}(3,q)$ of this $W(q)$ (see the proof of [49, 3.2.1]). From this isomorphism it follows that an elliptic quadric ovoid of $Q(4,q)$ is isomorphic to an elliptic quadric ovoid of $W(q)$. Thus Theorem 4.0.4 is equivalent to the following.

Theorem 4.1.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (q, q^2), q even, having a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I)$ isomorphic to $W(q)$. If in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ subtended by a point $X \in \mathcal{P} \setminus \mathcal{P}'$ is an elliptic quadric ovoid, then $\mathcal{S}$ is isomorphic to $Q(5,q)$.*

4.1.1 The intersection and subtending of ovoids of $W(q)$, $q = 2^e$, e odd

In this section we look at the size of the intersection of a given Tits ovoid of $W(q)$ with elliptic quadrics of $W(q)$. We show that if $W(q)$ is a subquadrangle of a GQ $\mathcal{S}$ of order (q, q^2) then the set of ovoids of $W(q)$ subtended by $\mathcal{S}$ cannot contain both a Tits ovoid and an elliptic quadric and cannot be all Tits ovoids.

Theorem 4.1.2 *[3, Theorem 1(a)] The intersection of an elliptic quadric of $W(q)$ with a Tits ovoid of $W(q)$ consists of either $q + \sqrt{2q} + 1$ points or $q - \sqrt{2q} + 1$ points.*

For the proof of the following theorem recall the preliminary section on graphs, Section 1.1.

Theorem 4.1.3 *[50] Let $W(q)$ be a subquadrangle of a GQ $\mathcal{S}$ of order (q, q^2). Let N be the graph with vertices the set of ovoids of $W(q)$ subtended by $\mathcal{S}$ and with two vertices adjacent if their corresponding ovoids meet in exactly one point. Then N is connected.*

Proof: Consider the graph M with vertices the points of $\mathcal{S}$ not in $W(q)$ and two vertices adjacent if their corresponding points are collinear in $\mathcal{S}$. Let P and Q be two vertices of M. If P and Q are collinear in $\mathcal{S}$, then they are adjacent and therefore connected. Otherwise there exist two lines of S, l and m, on P and Q respectively, such that l and m meet $W(q)$ in P' and Q' respectively. We can assume that $P' \neq Q'$. By the third GQ axiom we may choose $R \in m$ such that R is not collinear with P' in $\mathcal{S}$. Also by the third GQ axiom there exists a point $T \in l$, $T \neq P'$ such that T is collinear with R in $\mathcal{S}$. Thus P, T, R, Q is a path of M and hence P and Q are connected. We have shown, therefore, that the graph M is connected.

Now we consider the graph N. Let θ and ϕ be two ovoids of $W(q)$ subtended by points P and Q, respectively. If P and Q in $\mathcal{S}$ are collinear, then by Lemma 3.1.1 the line z joining them is a tangent to $W(q)$ at some point U. The line z subtends a rosette based at U containing θ and ϕ. Thus θ and ϕ intersect in exactly one point and are adjacent as vertices of N. If P and Q are not collinear in $\mathcal{S}$, then they are connected as vertices of M, since M is connected. Let $P, C_1, ..., C_n, Q$ be the path of M connecting P and Q and $\theta, \psi_1, ..., \psi_n, \phi$ the ovoids subtended by the elements of the path. Since P and C_1 are collinear in $\mathcal{S}$, it follows that θ and ψ_1 intersect in exactly one point and are hence connected in N. Similarly, C_i is connected in N to C_{i+1} for $1 \leq i \leq n-1$ and also C_n is connected to ϕ in N. Thus $\theta, \psi_1, ..., \psi_n, \phi$ is a path connecting θ and ϕ. As these were arbitrary points of the graph N, it follows that N is connected.

□

Recall from Section 1.4.1 that a rosette $\mathcal{R}$ of a GQ of order r is **homogeneous** if any two ovoids of $\mathcal{R}$ are isomorphic, and **inhomogeneous** otherwise. We extend this definition to the following.

Definition 4.1.4 *Let $\mathcal{S}$ be a GQ of order (r, r^2) and $\mathcal{S}'$ a subquadrangle of $\mathcal{S}$ of order r. $\mathcal{S}'$ is* **homogeneous** *in $\mathcal{S}$ if the ovoids of $\mathcal{S}'$ subtended by $\mathcal{S}$ are all isomorphic. If this is not the case $\mathcal{S}'$ is said to be* **inhomogeneous** *in $\mathcal{S}$.*

Theorem 4.1.5 *[50] Let $W(q)$ be a subquadrangle of a GQ $\mathcal{S}$ of order (q, q^2) and suppose that $W(q)$ is inhomogeneous in $\mathcal{S}$. Then the set of ovoids of $W(q)$ subtended by $\mathcal{S}$ cannot be contained in the union of the set of Tits ovoids of $W(q)$ and the set of elliptic quadrics of $W(q)$.*

Proof: Suppose that the set of subtended ovoids is contained in union of the set of Tits ovoids of $W(q)$ and the set of elliptic quadrics of $W(q)$. In other words, that each subtended ovoid is either a Tits ovoid or an elliptic quadric. Then since $\mathcal{S}'$ is inhomogeneous in $\mathcal{S}$, it follows that there occurs at least one ovoid of each type. Now from Theorem 4.1.3 the graph N is connected, which implies that there exists a Tits ovoid θ and an elliptic quadric ϕ of $W(q)$ that are connected vertices of N. Thus there exists a path of N of the form $\theta = C_0, C_1, C_2, ..., C_{n-1}, C_n = \phi$, where C_i is either a Tits ovoid or an elliptic quadric. If j is the smallest number such that C_j is an elliptic quadric then C_{j-1} is a Tits ovoid, and so by Theorem 4.1.2 they intersect in either exactly $q + \sqrt{2q} + 1$ or $q - \sqrt{2q} + 1$ points. But C_j and C_{j-1} are adjacent in N and so the corresponding ovoids of $W(q)$ intersect in exactly one point; a contradiction.

□

We now consider the case where $\mathcal{S}$ is a GQ of order (q, q^2) containing $W(q)$ as a subquadrangle, such that each subtended ovoid of $W(q)$ is a Tits ovoid.

Let $\mathcal{G}$ be the graph with vertices the Tits ovoids of $W(q)$ and with two vertices adjacent if they intersect in exactly one point of $W(q)$. We denote the automorphism group of $\mathcal{G}$ by $aut(\mathcal{G})$.

Note that an automorphism of $W(q)$ maps a Tits ovoid to a Tits ovoid and preserves collinearity, thus inducing an automorphism of $\mathcal{G}$. Suppose that T is an arbitrary, non-trivial, automorphism of $W(q)$ and let P be a point such that $Q = T(P) \neq P$. Now consider a Tits ovoid θ of $W(q)$ such that $P \in \theta$ and $Q \notin \theta$. Since $Q \in T(\theta)$ but $Q \notin \theta$, it follows that $\theta \neq T(\theta)$. Thus the induced action of any group fixing $W(q)$ is faithful (that is, the only element of the group that induces the identity automorphism on $\mathcal{G}$ is the identity element of the group). In particular, the induced action of the group $Sp(4, q)$ on $\mathcal{G}$ is faithful and so $Sp(4, q)$ is isomorphic to a group of automorphisms of $\mathcal{G}$. Now the Suzuki group $Sz(q)$ is the stabiliser of the Tits ovoid in $Sp(4, q)$ ([25, Corollary 16.4.7]) and an element of the conjugacy class of $Sz(q)$ in $Sp(4, q)$ induces a group of automorphisms on $\mathcal{G}$ which is isomorphic to $Sz(q)$. This induced group fixes a vertex in $\mathcal{G}$ corresponding to the ovoid fixed by the particular conjugate of $Sz(q)$ in $Sp(4, q)$. For simplicity we will use $Sp(4, q)$ to denote the group of automorphisms of $\mathcal{G}$ induced by $Sp(4, q)$ and $Sz(q)$ to denote the group of automorphisms of $\mathcal{G}$ induced by a conjugate of $Sz(q)$ fixing a particular Tits ovoid θ. We introduce a slight abuse of notation by using θ to refer to a particular ovoid $PG(3, q)$ and also the corresponding vertex of $\mathcal{G}$.

Lemma 4.1.6 *Let $\mathcal{G}$ be the graph with vertices the Tits ovoids of $W(q)$ and with two vertices adjacent if they intersect in exactly one point of $W(q)$. Then $\mathcal{G}$ is a connected graph.*

Proof: Let $aut(\mathcal{G})_\theta$ denote the stabiliser of θ in $aut(\mathcal{G})$; then we have $Sz(q) \subseteq aut(\mathcal{G})_\theta$ and $Sp(4, q) \subseteq aut(\mathcal{G})$. By Lemma 1.4.11 we know that θ lies in an elation rosette of $W(q)$, and so is in a non-trivial connected component $\mathcal{C}$ of $\mathcal{G}$, with stabiliser denoted by $aut(\mathcal{G})_C$.

Thus $Sz(q) \subseteq aut(\mathcal{G})_\mathcal{C} \cap Sp(4, q) \subseteq Sp(4, q)$. Now since $\mathcal{C}$ is non-trivial it contains a vertex ϕ such that $\phi \neq \theta$. The subgroup $aut(\mathcal{G})_\phi$ of $aut(\mathcal{G})$ that fixes ϕ also preserves adjacency in $\mathcal{G}$ and so fixes $\mathcal{C}$. Furthermore, it contains a subgroup isomorphic to $Sz(q)$, induced by a conjugate of $Sz(q)$ in $Sp(4, q)$. Thus since $Sz(q)$ and its conjugate that fixes the ovoid ϕ are both in $aut(\mathcal{G})_\mathcal{C}$ and they are not identical in $Sp(4, q)$ we have that $aut(\mathcal{G})_\mathcal{C} \cap Sp(4, q) \neq Sz(q)$. Finally $Sz(q)$ is maximal in $Sp(4, q)$, so $aut(\mathcal{G})_\mathcal{C} \cap Sp(4, q) = Sp(4, q)$, and as $Sp(4, q)$ is transitive on $\mathcal{G}$ we have $\mathcal{C} = \mathcal{G}$. Thus $\mathcal{G}$ is connected.

□

Lemma 4.1.7 *Let θ and ϕ be two Tits ovoids of $W(q)$ such that $\theta \cap \phi = \{P\}$. Then θ and ϕ lie in exactly one common rosette of Tits ovoids based at P, namely the elation rosette with respect to P and θ (or equivalently P and ϕ). Furthermore, for every point $Q \in \theta$ there is exactly one* *rosette of Tits ovoids based at Q which contains θ, namely the elation rosette with respect to Q and θ.*

Proof: The number of Tits ovoids that intersect θ in exactly one point is $(q-1)(q^2+1)$ ([3] Theorem 5(b)). By considering the elation rosette with respect to θ and P for all points $P \in \theta$, we obtain $(q-1)(q^2+1)$ Tits ovoids intersecting θ in exactly one point. Thus the ovoids intersecting θ in exactly one point are precisely those in elation rosettes containing θ, and there are only $q-1$ Tits ovoids intersecting θ in a given point P. Thus since $|\theta \cap \phi| = 1$, θ and ϕ lie in a common elation rosette: the unique rosette of Tits ovoids containing θ and ϕ. It also follows that the elation rosette with respect to θ and Q is the only rosette of Tits ovoids containing θ and based at Q.

□

Theorem 4.1.8 *There does not exist a GQ of order (q, q^2) such that $W(q)$ is homogeneous in the GQ and all subtended ovoids are Tits ovoids.*

Proof: Suppose such a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ exists. Let θ be a vertex of the graph $\mathcal{G}$ subtended by $X \in \mathcal{P}$. Let ϕ be a vertex of $\mathcal{G}$ adjacent to θ, so that $|\phi \cap \theta| = 1$, and let $\phi \cap \theta = \{P\}$. Now by Lemma 4.1.7 there is a unique elation rosette R of Tits ovoids based at P and containing the ovoids θ and ϕ. The tangent joining P and X in S subtends a rosette R' in $W(q)$. By assumption, all subtended ovoids are Tits ovoids, so R' is a rosette of Tits ovoids based at P and containing θ. Thus by Lemma 4.1.7 $R = R'$ and ϕ is a subtended ovoid of $W(q)$.

We have shown that if one vertex $\mathcal{G}$ corresponds to a subtended ovoid of $W(q)$ then so does each adjacent vertex. Since $\mathcal{G}$ is connected by Lemma 4.1.6 it follows that if one vertex of $\mathcal{G}$ corresponds to a subtended ovoid then all of the vertices do. Thus all Tits ovoids of $W(q)$ are subtended. $W(q)$ contains $q^2(q+1)(q^2-1)$ Tits ovoids ($= |[Sp(4,q) : Sz(q)]|$), but there are only $q^4 - q^2$ points of S subtending ovoids, so we have a contradiction and the theorem is proved.

□

4.1.2 Applications to $Q(5, q)$, q even

We now apply the work of Section 4.1.1 to extend the characterisation of $Q(5, q)$ in Theorem 4.1.1 to include Tits ovoids of $W(q)$.

Theorem 4.1.9 *[65, VII.1] If a GQ $\mathcal{S}$ of order (q, q^2), q even, has a subquadrangle isomorphic to $W(q)$ and subtends only ovoids isomorphic to the elliptic quadric, then $\mathcal{S}$ is isomorphic to the classical GQ $Q(5, q)$.*

Now using the work in this chapter we can extend this characterisation to include Tits ovoids.

Theorem 4.1.10 *If a GQ $\mathcal{S}$ of order (q, q^2), q even, has a subquadrangle isomorphic to $W(q)$ and each subtended ovoid is either an elliptic quadric or a Tits ovoid then $\mathcal{S}$ is isomorphic to the classical GQ $Q(5, q)$.*

Proof: By Theorem 4.1.5 if $W(q)$ is a subquadrangle of a GQ $\mathcal{S}$ of order (q, q^2) there cannot be a mixture of Tits ovoids and elliptic quadric ovoids subtended. By Theorem 4.1.8 the subtended ovoids are not all Tits ovoids and thus all the subtended ovoids are elliptic quadric ovoids. Hence by Theorem 4.1.9 $\mathcal{S}$ is isomorphic to the classical GQ $Q(5, q)$.

□

Corollary 4.1.11 *Suppose q is even and that the only ovoids of $W(q)$ are Tits ovoids and the elliptic quadric ovoids. If $\mathcal{S}$ is a GQ of order (q, q^2) containing a subquadrangle isomorphic to $W(q)$, then $\mathcal{S}$ is isomorphic to the classical GQ $Q(5, q)$.*

Corollary 4.1.12 *If there exists a GQ $\mathcal{S}$ of order (q, q^2) such that $\mathcal{S}$ is not isomorphic to $Q(5, q)$ and $\mathcal{S}$ contains a subquadrangle isomorphic to $W(q)$, then there is a new ovoid of $PG(3, q)$.*

Corollary 4.1.13 *For $q \leq 32$, q even, if $\mathcal{S}$ is a GQ of order (q, q^2) containing a subquadrangle isomorphic to $W(q)$, then $\mathcal{S}$ is isomorphic to the classical GQ $Q(5, q)$.*

Proof: For $q = 2$ the GQs of order 2 and $(2, 4)$ are unique ([49], 5.2.3 and 5.3.2 (ii)). By [4] ($q = 4$), [20] ($q = 8$), [36] ($q = 16$) and [37] ($q = 32$), the only ovoids in $W(q)$ for $4 \leq q \leq 32$ and q even are Tits ovoids and elliptic quadric ovoids.

□

4.2 GQs of order (r, r^2) with a doubly subtended subquadrangle of order r

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order (r, r^2) and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I)$ a subquadrangle of order r. Recall from Chapter 3 that $\mathcal{S}'$ is said to be *doubly subtended* in $\mathcal{S}$ if each ovoid of $\mathcal{S}'$ subtended by a point of $\mathcal{P} \setminus \mathcal{P}'$ is subtended by exactly two points of $\mathcal{P} \setminus \mathcal{P}'$. Recall from Theorem 3.1.7 that there is an associated SPG $\mathcal{T}$. In this section we will investigate the number of embeddings of $\mathcal{S}'$ in a GQ of order (r, r^2) such that $\mathcal{S}'$ is doubly subtended in the GQ, with associated SPG $\mathcal{T}$.

Suppose that $\mathcal{T}_{\mathcal{W}'}$ and $\mathcal{T}_{\mathcal{S}'}$ are SPGs constructed by the subtending of the GQs $\mathcal{W}'$ and $\mathcal{S}'$, of order r, in the GQs $\mathcal{W}$ and $\mathcal{S}$, of order (r, r^2) (note that in Chapter 3 we denoted these SPGs by $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{S}}$ respectively) . Let $(\mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}}, p_{\mathcal{W}})$ and $(\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}, p_{\mathcal{S}})$ be the algebraic 2-fold covers of $\mathcal{T}_{\mathcal{W}'}$ and $\mathcal{T}_{\mathcal{S}'}$, given by $\mathcal{W}$ and $\mathcal{S}$ respectively, as in Theorem 3.1.9. Recall from Section 1.1 that if G is a graph of diameter d and G_d is the graph with the same vertex set as G and two vertices adjacent if they are at distance d in G, then G is *antipodal* if G_d is the disjoint union of cliques. Now if $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I)$, then $\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}} = (\mathcal{P} \setminus \mathcal{P}', \mathcal{B} \setminus \mathcal{B}', I)$ (this is the definition in Theorem 3.1.9), from which we may easily show that the point graph of $\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$ has diameter 3. We can also show that the point graph of $\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$ is antipodal, with two points antipodal if they subtend the same ovoid of $\mathcal{S}'$, or (equivalently) if they cover the same point of $\mathcal{T}_{\mathcal{S}'}$.

If there exists an isomorphism from $\mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}}$ to $\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$, then the isomorphism must induce a graph isomorphism on the corresponding point graphs. Since the antipodal nature of the graphs must be preserved under an isomorphism, it follows that a set of antipodal vertices of the point graph of $\mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}}$ maps onto a set of antipodal vertices of the point graph of $\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$. Thus we have an induced bijection from the pointset of $\mathcal{T}_{\mathcal{W}'}$ onto the pointset of $\mathcal{T}_{\mathcal{S}'}$. Further, we can easily show that this bijection induces an isomorphism from the geometry $\mathcal{T}_{\mathcal{W}'}$ to the geometry $\mathcal{T}_{\mathcal{S}'}$. We say that this isomorphism is **induced** by the isomorphism from $\mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}}$ to $\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$. Note that Section 5.3.2 contains a more detailed exposition of the fact that an isomorphism of covering geometries induces an isomorphism of the geometries being covered. Although Section 5.3.2 deals with different geometries to those considered here, the general arguments are still applicable.

The following result is a consequence of the proof of Theorem 3.2.1.

Theorem 4.2.1 *Let $\mathcal{W}$ and $\mathcal{S}$ be two GQs of order (r, r^2) and let $\mathcal{W}'$ and $\mathcal{S}'$ be subGQs of $\mathcal{W}$ and $\mathcal{S}$ respectively, of order r. Let $\mathcal{W}'$ and $\mathcal{S}'$ be doubly subtended in $\mathcal{W}$ and $\mathcal{S}$ and let the SPGs constructed as in Theorem 3.1.7 be $\mathcal{T}_{\mathcal{W}'}$ and $\mathcal{T}_{\mathcal{S}'}$. Let the algebraic 2-fold covers of $\mathcal{T}_{\mathcal{W}'}$ and $\mathcal{T}_{\mathcal{S}'}$ determined by $\mathcal{W}$ and $\mathcal{S}$ as in Theorem 3.1.9 be $(\mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}}, p_{\mathcal{W}})$ and $(\mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}, p_{\mathcal{S}})$, defined by the 1-cochains $c_{\mathcal{W}}$ and $c_{\mathcal{S}}$ respectively. If $i : \mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}} \to \mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$ is an isomorphism, then i extends uniquely to an an isomorphism $\bar{i} : \mathcal{W} \to \mathcal{S}$. Further, $\bar{i}(\mathcal{W}') = \mathcal{S}'$ and i and $\bar{i}$ induce the same isomorphism from $\mathcal{T}_{\mathcal{W}'}$ to $\mathcal{T}_{\mathcal{S}'}$.*

Proof: From the discussion above it follows that the isomorphism $i : \mathcal{T}_{\mathcal{W}'}^{c_{\mathcal{W}}} \to \mathcal{T}_{\mathcal{S}'}^{c_{\mathcal{S}}}$ induces a unique isomorphism from $\mathcal{T}_{\mathcal{W}'}$ to $\mathcal{T}_{\mathcal{S}'}$. In the proof of Theorem 3.2.1 it was shown that given i and the induced isomorphism from $\mathcal{T}_{\mathcal{W}'}$ to $\mathcal{T}_{\mathcal{S}'}$ there is a unique isomorphism $\bar{i}$ from $\mathcal{W}$ to $\mathcal{S}$, taking $\mathcal{W}'$ to $\mathcal{S}'$ and inducing the same isomorphism from $\mathcal{T}_{\mathcal{W}'}$ to $\mathcal{T}_{\mathcal{S}'}$ as i.

$\square$

Theorem 4.2.1 means that, in order to characterise a GQ of order (r, r^2) in terms of a doubly subtended subGQ of order r, we need to consider the isomorphism classes of covering geometries of algebraic 2-fold covers of the associated SPG. Note that Theorem 4.2.1 does not preclude the possibility that given an SPG $\mathcal{T}_{\mathcal{W}'}$, two covers $(\mathcal{T}^c_{\mathcal{W}'}, p)$ and $(\mathcal{T}^{c'}_{\mathcal{W}'}, p')$ which give rise to the GQs $\mathcal{S}$ and $\mathcal{W}$ of order (r, r^2), as in Theorem 3.1.10, and $\mathcal{T}^c_{\mathcal{W}'}$ not isomorphic to $\mathcal{T}^{c'}_{\mathcal{W}'}$ that $\mathcal{S} \cong \mathcal{W}$. In this case $\mathcal{S}$ and $\mathcal{W}$ are isomorphic but there is no isomorphism from $\mathcal{S}$ to $\mathcal{W}$ that maps $\mathcal{W}'$ to $\mathcal{W}'$ and $\mathcal{T}_{\mathcal{W}'}$ and $\mathcal{T}_{\mathcal{W}'}$.

Suppose that $\mathcal{T}_{\mathcal{W}'}$ is an SPG constructed by the GQ $\mathcal{W}'$ of order r being doubly subtended in a GQ $\mathcal{W}$ of order (r, r^2). We now consider isomorphisms between algebraic 2-fold covers of $\mathcal{T}_{\mathcal{W}'}$ such that the defining 1-cochain satisfies the GQ condition (3.1.1); that is, covers that give rise to a GQ of order (r, r^2), as in Theorem 3.1.10. Let $(\mathcal{T}^c_{\mathcal{W}'}, p)$ and $(\mathcal{T}^{c'}_{\mathcal{W}'}, p')$ be two algebraic 2-fold covers of $\mathcal{T}_{\mathcal{W}'}$, defined by c and c' respectively. Let $\overline{U} : \mathcal{T}^{c'}_{\mathcal{W}'} \to \mathcal{T}^c_{\mathcal{W}'}$ be an isomorphism. If (P, α) is a point of $\mathcal{T}^{c'}_{\mathcal{W}'}$, then $\overline{U}$ acts by $(P, \alpha) \mapsto (U(P), \mathrm{u}_P(\alpha))$, where U is the automorphism of $\mathcal{T}_{\mathcal{W}'}$ induced by $\overline{U}$ and u_P is a permutation of $\mathbb{Z}_2$, such that

$$c'(P, Q) = \alpha + \beta \Longleftrightarrow c(U(P), U(Q)) = \mathrm{u}_P(\alpha) + \mathrm{u}_Q(\beta). \tag{4.2.1}$$

Now there are only two permutations on $\mathbb{Z}_2$, the identity permutation and the permutation that swaps 0 and 1. Let b be the 0-cochain defined by $b(P) = 0$ if u_P is the identity and $b(P) = 1$ if u_P swaps 0 and 1. Thus $\mathrm{u}_P(\alpha) = \alpha + b(P)$. So, rewriting the condition 4.2.1 U and the permutations u_P, we have that

$$c'(P, Q) = \alpha + \beta \Longleftrightarrow c(U(P), U(Q)) = \alpha + b(P) + \beta + b(Q).$$

That is, $c'(P, Q) = c(U(P), U(Q)) + \delta^0 b(P, Q)$. Conversely, if b is a 0-cochain and U an automorphism of $\mathcal{T}_{\mathcal{W}}$, we define $c'(P, Q) = c(U(P), U(Q)) + \delta^0 b(P, Q)$. Then the map $(P, \alpha) \mapsto (U(P), b(P) + \alpha)$ is an isomorphism from $\mathcal{T}^{c'}_{\mathcal{W}'}$ to $\mathcal{T}^c_{\mathcal{W}'}$.

The above work gives us the following theorem.

Theorem 4.2.2 *Let $\mathcal{W}$ be a GQ of order (r, r^2) and let $\mathcal{W}'$ be a subGQ of $\mathcal{W}$ of order r. Let $\mathcal{W}'$ be doubly subtended in $\mathcal{W}$ and the SPG constructed as in Theorem 3.1.7 be $\mathcal{T}_{\mathcal{W}'}$. Let $(\mathcal{T}^c_{\mathcal{W}'}, p)$ and $(\mathcal{T}^{c'}_{\mathcal{W}'}, p')$ be two algebraic 2-fold covers of $\mathcal{T}_{\mathcal{W}'}$, defined by c and c' respectively, such that c and c' satisfy the GQ condition 3.1.1. Then $\mathcal{T}^c_{\mathcal{W}'}$ and $\mathcal{T}^{c'}_{\mathcal{W}'}$ are isomorphic if and only if $c'(P, Q) = c(U(P), U(Q)) + \delta^0 b(P, Q)$, where U is an automorphism of $\mathcal{T}_{\mathcal{W}'}$ and b a 0-cochain. In this case the map $i : \mathcal{T}^{c'}_{\mathcal{W}'} \to \mathcal{T}^c_{\mathcal{W}'}$ that acts by $(P, \alpha) \mapsto (U(P), \alpha + b(P))$ is an isomorphism.*

Proof: Let (P, α) and (Q, β) be points of $\mathcal{T}_{\mathcal{W}'}$. Then $i(P, \alpha) = (U(P), b(P) + \alpha)$, and $i(Q, \beta) = (U(Q), b(Q) + \beta)$ and $i(P, \alpha)$ and $i(Q, \beta)$ are collinear if and only if

$$c(U(P), U(Q)) = \alpha + \beta + b(P) + b(Q).$$

This is the case if and only if $c(U(P), U(Q)) + \delta^0(P, Q) = c'(P, Q) = \alpha + \beta$, which is true if and only if (P, α) and (Q, β) are collinear.

$\square$

Recall from Section 1.6 that two 1-cochains c and c' are said to be **equivalent** if $\mathcal{T}^c_{\mathcal{W}'}$ and $\mathcal{T}^{c'}_{\mathcal{W}'}$ are isomorphic, that is, if and only if $c'(P, Q) = c(U(P), U(Q)) + \delta^0(P, Q)$, for some automorphism U of $\mathcal{T}_{\mathcal{W}'}$ and 1-coboundary $\delta^0 b$. Note that if b is a 0-cochain, then the 1-cochains c and $c + \delta^0 b$ are equivalent.

Having established the form of the 1-cochains which give isomorphic covers of a given SPG, we now consider automorphisms of a cover. From the above, we know that an automorphism of $\mathcal{T}^c_{\mathcal{W}'}$ has the form $(P, \alpha) \mapsto (U(P), \alpha + b(P))$ where $c(U(P), U(Q)) = c(P, Q) + \delta^0 b(P, Q)$, for U an automorphism of $\mathcal{T}_{\mathcal{W}'}$ and b a 0-cochain. Such an automorphism U is said to be **admitted** by $\mathcal{T}^c_{\mathcal{W}'}$. The automorphisms of $\mathcal{T}_{\mathcal{W}'}$ admitted by $\mathcal{T}^c_{\mathcal{W}'}$ form a subgroup of the automorphism group of $\mathcal{T}_{\mathcal{W}'}$, which we denote by $aut(\mathcal{T}_{\mathcal{W}'})_c$.

4.2.1 Introducing the cohomology

Suppose that $\mathcal{T}_{\mathcal{W}'}$ is an SPG constructed by the GQ $\mathcal{W}'$ of order r being doubly subtended in a GQ $\mathcal{W}$ of order (r, r^2). Recall that if G is the point graph of $\mathcal{T}_{\mathcal{W}'}$ and Γ_G the simplicial complex of G, then $c \in C^1(\Gamma_G, \mathbb{Z}_2)$ (that is, c is a 1-cochain) satisfies the GQ condition 3.1.1 if and only if the following condition holds

$$\delta^1 c(P, Q, R) = 0 \Longleftrightarrow P, Q, R \text{ are distinct collinear points of } \mathcal{T}_{\mathcal{W}'}.$$

Since we are working over $\mathbb{Z}_2$, if $\delta^1 c(P, Q, R)$ is not 0 then it must be 1, and hence every 1-cochain that satisfies the GQ condition has the same coboundary. Thus if c is a particular 1-cochain satisfying the GQ condition, then the set of all 1-cochains satisfying the GQ condition is

$$c + Z^1(\Gamma_G, \mathbb{Z}_2),$$

where $Z^1(\Gamma_G, \mathbb{Z}_2)$ is the set of 1-cocycles (that is, 1-cochains with a zero coboundary). So, given one algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}'}$ satisfying the GQ condition, we can find all others by adding on 1-cocycles. The question we now answer is: what is the relationship between the set of covers equivalent to c and the first cohomology group $H^1(\Gamma_G, \mathbb{Z}_2)$?

Recall from Section 1.5.2 that the first cohomology group is given by $H^1(\Gamma_G, \mathbb{Z}_2) = Z^1(\Gamma_G, \mathbb{Z}_2)/B^1(\Gamma_G, \mathbb{Z}_2)$; that is it's the group that results from factoring the 1-coboundaries out of the 1-cocycles. Thus an element of $H^1(\Gamma_G, \mathbb{Z}_2)$ is a set of 1-cocycles of the form $z + B^1(\Gamma_G, \mathbb{Z}_2) = \{z + \delta^0 b : \delta^0 b \in B^1(\Gamma_G, \mathbb{Z}_2)\}$, where z is a fixed 1-cocycle. Since two 1-cochains whose difference is a 1-coboundary are equivalent, it follows that any two elements of $c + (z + B^1(\Gamma_G, \mathbb{Z}_2))$ are equivalent.

We now consider the conditions under which two 1-cochains equivalent to c are in a common element of $c + H^1(\Gamma_G, \mathbb{Z}_2)$. Let c' and c'' be two 1-cochains equivalent to c, such that

$$\begin{aligned} c'(P, Q) &= c(S(P), S(Q)) + \delta^0 b'(P, Q) \text{ and} \\ c''(P, Q) &= c(T(P), T(Q)) + \delta^0 b''(P, Q), \end{aligned}$$

where S and T are automorphisms of $\mathcal{T}_{\mathcal{W}'}$ and $\delta^0 b'$ and $\delta^0 b''$ are 1-coboundaries. The 1-cochains c' and c'' are in the same element of $c + H^1(\Gamma_G, \mathbb{Z}_2)$ if and only if their sum is a 1-coboundary. Now

$$c'(P, Q) + c''(P, Q) = c(S(P), S(Q)) + c(T(P), T(Q)) + \delta^0 b'(P, Q) + \delta^0 b''(P, Q), \tag{4.2.2}$$

but $\delta^0 b'(P, Q) + \delta^0 b''(P, Q)$ is a 1-coboundary. So 4.2.2 holds if and only if

$$c(S(P), S(Q)) + c(T(P), T(Q)) = \delta^0 b(P, Q)$$

for some 1-coboundary $\delta^0 b$. This is the case if and only if

$$c(P, Q) + c(S^{-1} \circ T(P), S^{-1} \circ T(Q)) = \delta^0 b(S^{-1} \circ T(P), S^{-1} \circ T(Q)), \tag{4.2.3}$$

but since $\delta^0 b(S^{-1} \circ T(P), S^{-1} \circ T(Q))$ is a 1-coboundary, it follows that 4.2.3 holds if and only if $S^{-1} \circ T \in aut(\mathcal{T}_{\mathcal{W}'})_c$. This is the case if and only if S and T are in the same left coset of $aut(\mathcal{T}_{\mathcal{W}'})_c$ in $aut(\mathcal{T}_{\mathcal{W}'})$. Thus, we have a one-to-one correspondence from the set of left cosets of $aut(\mathcal{T}_{\mathcal{W}'})_c$ in $aut(\mathcal{T}_{\mathcal{W}'})$ to the set of elements of $H^1(\Gamma_G, \mathbb{Z}_2)$ (note that the correspondence need not be onto). Note that if $h \in H^1(\Gamma_G, \mathbb{Z}_2)$ corresponds to the left coset H of $aut(\mathcal{T}_{\mathcal{W}'})_c$ in $aut(\mathcal{T}_{\mathcal{W}'})$, then every element of h acts by

$$c(S(P), S(Q)) + \delta b(P, Q),$$

for some $S \in H$ and 1-coboundary δb.

From the above we have the following theorem.

Theorem 4.2.3 *Let $\mathcal{W}$ and $\mathcal{S}$ be two GQs of order (r, r^2) and let $\mathcal{W}'$ be a subGQ of order r of both $\mathcal{W}$ and $\mathcal{S}$. Suppose that $\mathcal{W}'$ is doubly subtended in both $\mathcal{W}$ and $\mathcal{S}$ and that $\mathcal{T}_{\mathcal{W}'}$ is the SPG constructed from both $\mathcal{W}$ and $\mathcal{T}$, as in Theorem 3.1.7. Let the 2-fold algebraic cover of $\mathcal{T}_{\mathcal{W}'}$ constructed from $\mathcal{W}$ as in Theorem 3.1.9 be $(\mathcal{T}^c_{\mathcal{W}'}, p)$ and let the 1-cochain c define $(\mathcal{T}^c_{\mathcal{W}'}, p)$. Let Γ_G be the simplicial complex of the point graph of $\mathcal{T}^c_{\mathcal{W}'}$. If $|aut(\mathcal{T}_{\mathcal{W}'}) : aut(\mathcal{T}_{\mathcal{W}'})_c| = |H^1(\Gamma_G, \mathbb{Z}_2)|$, then there exists an isomorphism $i : \mathcal{W} \to \mathcal{S}$, such that $i(\mathcal{W}') = \mathcal{W}'$.*

Proof: Let h be an element of $H^1(\Gamma_G, \mathbb{Z}_2)$. Since $|aut(\mathcal{T}_{\mathcal{W}'}) : aut(\mathcal{T}_{\mathcal{W}'})_c| = |H^1(\Gamma_G, \mathbb{Z}_2)|$, by the correspondence above it follows that if c' is any 1-cochain in the set $c + h$, then c' acts by $c'(P, Q) = c(S(P), S(Q)) + \delta^0 b(P, Q)$ for some 1-coboundary $\delta^0 b$ and some automorphism S of $\mathcal{T}_{\mathcal{W}'}$ contained in the coset of $aut(\mathcal{T}_{\mathcal{W}'})_c$ corresponding to h. Thus c is equivalent to c'. Since this holds for any $h \in H^1(\Gamma_G, \mathbb{Z}_2)$ it follows that any 1-cochain of the form $c+$ 1-cocycle is equivalent to c. These are precisely the 1-cochains of Γ_G over $\mathbb{Z}_2$ that satisfy the GQ condition. Thus if $(\mathcal{T}^{c'}_{W'}, p_{\mathcal{S}})$ is the algebraic 2-fold cover of $\mathcal{T}_{\mathcal{W}'}$ constructed from $\mathcal{S}$, then it follows that c and c' are equivalent and by Theorem 4.2.1 we have the existence of the isomorphism i.

□

If we let $aut(\mathcal{W}')_{\mathcal{W}}$ denote the subgroup of $aut(\mathcal{W}')$ that is induced by $aut(\mathcal{W})$, then Theorem 4.2.3 has the following corollary.

Corollary 4.2.4 *Let $\mathcal{W}$ and $\mathcal{S}$ be two GQs of order (r, r^2) and let $\mathcal{W}'$ be a subGQ of order r of both $\mathcal{W}$ and $\mathcal{S}$. Suppose that $\mathcal{W}'$ is doubly subtended in both $\mathcal{W}$ and $\mathcal{S}$ and that $\mathcal{T}_{\mathcal{W}'}$ is the SPG constructed from both $\mathcal{W}$ and $\mathcal{T}$, as in Theorem 3.1.7. Let the 2-fold algebraic cover of $\mathcal{T}_{\mathcal{W}'}$, constructed from $\mathcal{W}$ as in Theorem 3.1.9, be $(\mathcal{T}^c_{\mathcal{W}'}, p)$ and let the 1-cochain c define $(\mathcal{T}^c_{\mathcal{W}'}, p)$. Let Γ_G be the simplicial complex of the point graph of $(\mathcal{T}^c_{\mathcal{W}'}, c)$. If $|aut(\mathcal{T}_{\mathcal{W}'}) : aut(\mathcal{W}')_{\mathcal{W}}| = |H^1(\Gamma_G, \mathbb{Z}_2)|$, then there exists an isomorphism $i : \mathcal{W} \to \mathcal{S}$, such that $i(\mathcal{W}') = \mathcal{W}'$.*

Proof: We show that $aut(\mathcal{T}_{\mathcal{W}'}) \cong aut(\mathcal{W}')_{\mathcal{W}}$, and so the result follows from Theorem 4.2.3. Let T be an element of $aut(\mathcal{W}')_{\mathcal{W}}$ and let $\overline{T}$ be an automorphism of $\mathcal{W}$ that induces T on $\mathcal{W}'$. Thus $\overline{T}$ induces an element of $aut(T_{\mathcal{W}'})_c$. Let $i : aut(\mathcal{W}')_{\mathcal{W}} \to aut(T_{\mathcal{W}'})_c$ be the map taking an element of $aut(\mathcal{W}')_{\mathcal{W}}$ to the (unique) element of $aut(\mathcal{T}_{\mathcal{W}'})$ it induces. We can easily show that i is a group homomorphism, so now we show that it is also both one-to-one and onto and hence an isomorphism. By Corollary 3.2.2 any element S of $aut(\mathcal{T}_{\mathcal{W}'})_c$ is induced by a unique element of $aut(\mathcal{W}')$, and so i is one-to-one. Now any element S of $aut(\mathcal{T}_{\mathcal{W}'})_c$ is induced by an automorphism of $\mathcal{T}^c_{\mathcal{W}'}$ (by definition), and so by Theorem 4.2.1 there is an element of $aut(\mathcal{W}')_{\mathcal{W}}$ inducing S. Hence i is onto and so is an isomorphism, that is, $aut(\mathcal{T}_{\mathcal{W}'}) \cong aut(\mathcal{W}')_{\mathcal{W}}$.

□

4.2.2 Calculating the homology

This section depends heavily on the definitions and results of Section 1.5 and Section 1.1. In Section 4.2.1 we established that characterising a GQ of order (r, r^2) by a doubly subtended subGQ of order r depends on the order of the first cohomology group of the simplicial complex of the point graph of the subtended SPG. By Theorem 1.5.4, we know that the first cohomology group, over a field, is isomorphic to the first homology group. In this section we will work towards calculating the first homology group of the simplicial complex of the point graph of the subtended SPG.

From Section 1.5.5, the fact that we are calculating homology over $\mathbb{Z}_2$ means that we may represent a 1-chain as the set of 1-simplexes with a coefficient of 1 in the 1-chain. The sum of two 1-chains is the symmetric difference of the corresponding sets of 1-simplexes.

Let $\mathcal{W}$ be a GQ of order (r, r^2) with a subGQ $\mathcal{W}'$ of order r, and suppose that $\mathcal{W}'$ is doubly subtended in $\mathcal{W}$, we will denote the associated SPG by $\mathcal{T}$. Let G be the point graph of $\mathcal{T}$ and Γ_G the simplicial complex of G.

Recall from the discussion after Theorem 1.5.8 that the 1-cycles of Γ_G are the edge sets of circuits of G. The elementary 1-cycles of Γ_G are the edge sets of elementary circuits of G

and the induced 1-cycles of Γ_G are the edge sets of the induced circuits of G. We use this representation of 1-cycles of Γ_G as it is more intuitive, and leads us to results that will allow us to calculate the homology directly from G.

A 1-boundary of Γ_G is a triangle in the graph G. In much of the work that follows in this section we will be selectively replacing an edge e_1 of the edge set of a circuit of G with two edges e_2 and e_3 of G, where $\{e_1, e_2, e_3\}$ is a triangle of G. This is equivalent to adding the boundary of a 2-simplex to a 1-cycle of Γ_G, which produces a homologous 1-cycle. We use this technique to manipulate 1-cycles into more convenient forms.

In the following theorem we show that any induced 1-cycle of Γ_G is homologous to a sum of induced 1-cycles, each of which consists of four 1-simplexes. A circuit of G with edge set that is a 1-cycle of Γ_G consisting of four 1-simplexes will be called a **four-circuit** of G. If in addition the 1-cycle is induced, then we call the corresponding circuit and **induced four-circuit**. In other words, an induced four-circuit is a circuit of length four such that "opposite" vertices of the circuit are not adjacent. Recall from Section 1.1 that if $X_1, X_2, \dots, X_n, X_{n+1}$ is a circuit of G of length n (and so $X_{n+1} = X_1$) then we represent it by $(X_1, X_2, \dots, X_n)$.

Recall from Section 1.5 that two 1-cycles of Γ_G are **homologous** if their difference is a 1-boundary and that any 1-cycle that is homologous to the zero 1-cycle (that is, any 1-boundary) is **null homologous**.

Theorem 4.2.5 *Let σ be an induced 1-cycle of Γ_G consisting of at least four 1-simplexes. Then there exist induced 1-cycles $\sigma_1, \sigma_2, \dots, \sigma_r$ such that each σ_i consists of four 1-simplexes and σ is homologous to the sum of the σ_i.*

Proof: Let σ consist of n 1-simplexes; we proceed by induction on n. If $n = 4$, then the result is immediate. If $n \geq 5$, then let σ be the edge set of the circuit $(X_1, X_2, \dots, X_n)$. Recall from the construction of $\mathcal{T}$ (Theorem 3.1.7) that the X_i are ovoids of $\mathcal{W}'$, and that X_i and X_j are adjacent (in G) when the ovoids they represent intersect in exactly one point. Now suppose that $X_1 \cap X_2 = \{P\}$ and that $\mathcal{R}$ is the rosette, with basepoint P, containing both X_1 and X_2. By the proof of Theorem 3.1.7, if X_4 does not contain P then X_4 intersects exactly two ovoids of $\mathcal{R}$ in one point, and if X_4 does contain P then X_4 does not intersect any ovoid of $\mathcal{R}$ in exactly one point. Suppose first that P is not a point of X_4. Since σ is an induced circuit and we have assumed that $n \geq 5$, it follows that X_4 is adjacent to neither X_1 nor X_2. Thus there exists an ovoid X, $X \neq X_1, X_2$, such that X is in $\mathcal{R}$ and X intersects X_4 in exactly one point. Hence the induced 1-cycle σ may be expressed as the sum of the 1-cycles that are the edge sets of the circuits $(X_1, X, X_4, X_5, \dots, X_n)$, (X_1, X_2, X) and (X_2, X_3, X_4, X). If X_3 is adjacent to X, then (X_2, X_3, X) and (X_4, X_3, X) are triangles in G and so the 1-cycle corresponding to (X_2, X_3, X_4, X) is null homologous. If X_3 is not adjacent to X, then (X_2, X_3, X_4, X) is an induced four-circuit. Now the induced 1-cycle σ contains n 1-simplexes, and the 1-cycles corresponding to the circuits $(X_1, X, X_4, X_5, \dots, X_n)$ and (X_2, X_3, X_4, X) contain $n-1$ and four 1-simplexes, respectively. Thus the induced 1-cycle σ is homologous to the sum of an elementary 1-cycle containing $n-1$ 1-simplexes and a 1-cycle containing four 1-simplexes (both of which may or may not be induced). By the proof of Lemma 1.5.6 we can write both of these 1-cycles as the sum of induced 1-cycles, each consisting of fewer than n 1-simplexes.

Now suppose that P is a point of the ovoid X_4. Let Q be a point contained in the ovoid X_1 but in neither X_3 nor X_4. By Corollary 3.1.6, there is a (unique) subtended rosette $\mathcal{R}'$, containing X_1 and having basepoint Q. Since X_3 does not contain the basepoint of $\mathcal{R}'$, it follows, from the proof of Theorem 3.1.7, that X_3 is adjacent to two ovoids in $\mathcal{R}'$. If we let one such ovoid be X', then σ can be expressed as the sum of the 1-cycles corresponding to the circuits (X_1, X_2, X_3, X') and $(X_1, X', X_3, X_4, \dots, X_n)$. Now, since X_4 does not contain the basepoint of the rosette $\mathcal{R}'$, and X_4 is not adjacent to X_1, there is a second ovoid X'', contained in $\mathcal{R}'$, such that X_4 is adjacent to X''. The 1-cycle corresponding to the circuit $(X_1, X', X_3, X_4, \dots, X_n)$ can be expressed as the sum of the 1-cycles corresponding to the circuits (X', X_3, X_4, X''), $(X_1, X', X'^{\prime})$ and $(X_1, X'', X_4, \dots, X_n)$. The circuit $c = (X_1, X'', X_4, \dots, X_n)$ has length $n-1$, but is not necessarily induced. If σ_c is the 1-cycle corresponding to the circuit c, then σ_c contains $n-1$ 1-simplexes and is not necessarily induced. If σ_c is not induced, then by the proof of Lemma 1.5.6, we may write σ_c as the sum of induced 1-cycles, each consisting of fewer than n 1-simplexes. Thus the 1-cycle σ may be written as the sum of induced circuits each of which contains fewer than n 1-simplexes. Thus, σ is homologous to the sum of induced circuits each of which has at least four and at most $n-1$ 1-simplexes. The result follows by induction.

□

By using Theorem 1.5.8 and Theorem 4.2.5 we have the following.

Theorem 4.2.6 *Let $\mathcal{W}$ be a GQ of order (r, r^2) with a subGQ $\mathcal{W}'$ of order r. Suppose that $\mathcal{W}'$ is doubly subtended in $\mathcal{W}$, with associated SPG $\mathcal{T}$. Let G be the point graph of $\mathcal{T}$ and Γ_G the simplicial complex of G. Then $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial if and only if each induced 1-cycle of Γ_G, consisting of four 1-simplexes, is a 1-boundary (that is, null homologous).*

Proof: By Theorem 1.5.8 $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial if and only if every induced 1-cycle is null homologous and by Theorem 4.2.5 every induced 1-cycle of Γ_G is null homologous if and only if every induced 1-cycle consisting of four 1-simplexes is null homologous.

□

Given Theorem 4.2.6 we now need a method for showing that a given induced four-circuit is null homologous. The following lemma provides this.

Lemma 4.2.7 *Let σ be an induced 1-cycle of Γ_G and let $c = (X, Y, Z, W)$ be the corresponding induced four-circuit of G. If $G_{\{X,Z\}}$ is connected, then σ is null homologous.*

Proof: Suppose that $G_{\{X,Z\}}$ is connected and let $Y, v_0, v_1, \dots, v_n, W$ be a path connecting Y and W in $G_{\{X,Z\}}$. Then σ is equal to the sum of the 1-cycles corresponding to the circuits (X, Y, v_0), (Z, Y, v_0), (X, v_0, v_1), (Z, v_0, v_1), $\dots$ (X, v_{n-1}, v_n), (Z, v_{n-1}, v_n), (X, v_n, W) and (Z, v_n, W). Since each of these 1-cycles is a 1-boundary, it follows that c is null homologous.

□

4.3 Application to $Q(5, q)$

In this section we apply the work of the previous section to the classical GQ $Q(5, q)$ where q is odd and show that $Q(5, q)$ is characterised by a $Q(4, q)$ subquadrangle and the subtended ovoid/rosette structure. Recall from Section 3.1 that $Q(5, q)$ contains subGQs, of order q, isomorphic to $Q(4, q)$. Each such subGQ is doubly subtended in $Q(5, q)$ and the set of such subGQs form a single orbit under the action of the group of $Q(5, q)$ [27, Theorem 22.6.6]. The set of subtended ovoids of $Q(4, q)$ is exactly the set of elliptic quadric ovoids of $Q(4, q)$. If we use the notation of the previous section, that is, $\mathcal{W} = Q(5, q)$, $\mathcal{W}' = Q(4, q)$ with SPG $\mathcal{T}$, then every automorphism of $Q(4, q)$ fixes $\mathcal{T}$ and so by Corollary 3.2.2 $aut(\mathcal{T}_{\mathcal{W}'}) \cong aut(\mathcal{W}') \cong aut(\mathcal{T}_{\mathcal{W}'})_{\mathcal{W}'}$. Since every automorphism of $Q(4, q)$ is induced by an automorphism of $Q(5, q)$ we have $aut(\mathcal{W}')_{\mathcal{W}} = aut(\mathcal{W}')$. Thus by Theorem 4.2.3 $Q(5, q)$ is characterised by having $Q(4, q)$ as a doubly subtended subGQ with associated SPG $\mathcal{T}$, if and only if $H^1(\Gamma_G, \mathbb{Z}_2)$ is trivial, which is the case if and only if $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial, by Theorem 1.5.4.

As in Section 4.2.2 let G be the point graph of the SPG $\mathcal{T}$ and Γ_G the simplicial complex of G. To show $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial we will employ Lemma 4.2.7 and so need a "user-friendly" representation of the graph G. For q odd under the polarity of $Q(4, q)$ the ambient three-dimensional subspace of an elliptic quadric ovoid maps to a non-singular point of $\mathrm{PG}(4, q)$ and two elliptic quadric ovoids are adjacent in G if and only if the corresponding polar points span a tangent to $Q(4, q)$. In this polar representation, a rosette is the set of non-singular points on a tangent to $Q(4, q)$ (note that if one non-singular point on a line tangent to $Q(4, q)$ has a polar hyperplane that intersects $Q(4, q)$ in an elliptic quadric ovoid, then so do all the other non-singular points on the line). In the following section we show that $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial for q odd.

For q even the theory of Section 4.2.2 also applies and the most obvious representation to use seems to be that given in [26]. The calculation, however, would seem to be more involved than that for the q odd case, and certainly longer than the proof of the same result by Thas and Payne in [65].

4.3.1 Explicit homology calculation for $Q(4,q)$, q odd

Let $\mathcal{T}$ be the SPG with pointset the elliptic quadric ovoids of $Q(4,q)$ and lineset the rosettes of elliptic quadric ovoids. Let G be the point graph of $\mathcal{T}$ and Γ_G the simplicial complex of G. In this section we show that $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial. We show that for each induced four-circuit (X_1, X_2, X_3, X_4) of G either the subgraph $G_{\{X_1,X_3\}}$ or the subgraph $G_{\{X_2,X_4\}}$ is connected. By Lemma 4.2.7 this implies that each induced 1-cycle of Γ_G over $\mathbb{Z}_2$ that consists of four 1-simplexes is null homologous. By Theorem 4.2.6 this implies that $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial.

Let $\mathcal{Q} = Q(4,q)$ be the non-singular (parabolic) quadric of $\mathrm{PG}(4,q)$ defined by $Q(X) = x_0^2 + x_1x_2 + x_3x_4 = 0$, where X has coordinates $(x_0, x_1, x_2, x_3, x_4)$. The bilinear form associated with $\mathcal{Q}$ is given by $\beta(x,y) = 2x_0y_0 + x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3$ and if $\perp$ denotes the polarity associated with $\mathcal{Q}$, then $(x_0, x_1, x_2, x_3, x_4)^\perp = [2x_0, x_2, x_1, x_4, x_3]$ and $[a_0, a_1, a_2, a_3, a_4]^\perp = (a_0/2, a_2, a_1, a_4, a_3)$. If X and Y are two points of $\mathcal{Q}$ such that the line $\langle X, Y\rangle$ is singular (that is, $\beta(X,Y) = 0$), then we write $X \sim Y$. If X is a point of $\mathrm{PG}(4,q)$ not on the quadric $\mathcal{Q}$, and Y is a second point of $\mathrm{PG}(4,q)$ (which may be either singular or non-singular) such that the line $\langle X, Y\rangle$ is tangent to $\mathcal{Q}$, then we say that X and Y are **cotangent** and write $X \frown Y$.

It will be useful in what follows to have an algebraic condition that determines when two points of $\mathrm{PG}(3,q)$ are cotangent with respect to $\mathcal{Q}$. Let X and Y be two points of $\mathrm{PG}(4,q)$ such that X is not contained in $\mathcal{Q}$. Then the set of points incident with the line $\langle X, Y\rangle$ but not equal to X is $\{\lambda X + Y : \lambda \in \mathrm{GF}(q)\}$. The points X and Y are cotangent if and only if the line $\langle X, Y\rangle$ intersects $\mathcal{Q}$ in precisely one point; that is, the equation $Q(\lambda X + Y) = 0$ has exactly one solution in λ. Now

$$\begin{aligned} Q(\lambda X + Y) &= \beta(\lambda X, Y) + Q(\lambda X) + Q(Y) \\ &= Q(X)\lambda^2 + \beta(X,Y)\lambda + Q(Y). \end{aligned}$$

This quadratic in λ has a unique solution, $\lambda = -\beta(X,Y)/(2Q(X))$, if and only if the discriminant of the quadratic is zero. That is,

$$\beta(X,Y)^2 - 4Q(X)Q(Y) = 0.$$

We define $\Delta(X,Y) = \beta(X,Y)^2 - 4Q(X)Q(Y)$.

If X is a non-singular point of $\mathrm{PG}(4,q)$, then the lines tangent to $\mathcal{Q}$ containing X form a quadric T_X. The quadric T_X is a cone with vertex X and base $X^\perp \cap \mathcal{Q}$. The cone T_X is called the **tangency cone** of X.

If X is a non-singular point of $\mathrm{PG}(3,q)$, then the hyperplane $X^\perp$ intersects $\mathcal{Q}$ in a non-singular hyperbolic quadric if and only if $Q(X)$ is a non-zero square of $\mathrm{GF}(q)$, and in a non-singular elliptic quadric if and only if $Q(X)$ is a non-square of $\mathrm{GF}(q)$. If Σ_1 and Σ_2 are two hyperplanes of $\mathrm{PG}(4,q)$, such that $\Sigma_1 \cap \mathcal{Q}$ and $\Sigma_2 \cap \mathcal{Q}$ are both non-singular elliptic quadrics, then the two elliptic quadrics intersect in exactly one point if and only if the line $\langle \Sigma_1^\perp, \Sigma_2^\perp\rangle$ is tangent to $\mathcal{Q}$. That is, $\Sigma_1^\perp$ and $\Sigma_2^\perp$ are cotangent. Given this, we have the following representation of G:

Vertex set: $\{X \in \mathrm{PG}(4,q) : Q(X) \text{ is a non-square of } \mathrm{GF}(q)\}$.

Adjacency: Cotangency.

In this representation of the points of $\mathcal{T}$ the lines of $\mathcal{T}$ are the lines of $\mathrm{PG}(4,q)$ that are tangent to $\mathcal{Q}$ and have a polar plane intersecting $\mathcal{Q}$ in exactly one point (the base point of the rosette corresponding to the line of $\mathcal{T}$). Recall that $\mathcal{T}$ is an SPG with parameter $\alpha = 2$ and so if (P,ℓ) is a non-incident point/line pair of $\mathcal{T}$, then P is collinear to exactly 0 or 2 points of ℓ. Recall from the proof of Theorem 3.1.7 that P is collinear to 0 points of ℓ when it is an ovoid containing the base point of the rosette corresponding to ℓ, and collinear to two points of ℓ when it does not. We state the equivalent result in the current setting, which will prove useful for the calculations that follow.

Lemma 4.3.1 *Let ℓ be a line of* $\mathrm{PG}(4,q)$ *tangent to $\mathcal{Q}$ at the point Q, such that the polar space of each non-singular point of ℓ intersects $\mathcal{Q}$ in a non-singular elliptic quadric. Let P be a point of* $\mathrm{PG}(4,q)$ *such that $P^\perp$ intersects $\mathcal{Q}$ in a non-singular elliptic quadric and $P \notin \ell$. Then one of the following must be the case*

(i) *There are exactly two points incident with ℓ and cotangent to P, both of which are not on $\mathcal{Q}$; or*

(ii) *The point $Q \in \mathcal{Q}$ is the only point incident with ℓ that is cotangent to P.*

Proof: Consider the plane $\pi = \langle \ell, P\rangle$. If π intersects $\mathcal{Q}$ in a conic, then P is an exterior point to the conic and there are exactly two points of ℓ cotangent to P, both of which are non-singular with respect to $\mathcal{Q}$. The other possibilities for π are that it intersects $\mathcal{Q}$ in a line pair (where the lines intersect in Q) or that it intersects $\mathcal{Q}$ in the unique point Q. In either case, π contains a unique tangent on P and Q is the unique point of ℓ cotangent to P.

□

We now consider four-circuits of G. If $c = (X_1, X_2, X_3, X_4)$ and $c' = (X_1', X_2', X_3', X_4')$ are two four-circuits of G and S is a collineation of $\mathrm{PG}(4,q)$ that fixes $\mathcal{Q}$, such that $S : (X_1, X_2, X_3, X_4) \mapsto (X_1', X_2', X_3', X_4')$, then we say that S maps c to c' and write $S(c) = c'$. Now, any collineation of $\mathrm{PG}(4,q)$ that fixes $\mathcal{Q}$ also fixes both the vertex set of G and the set of tangents to $\mathcal{Q}$ and so preserves cotangency amongst the vertices of G. Thus S induces an automorphism of the graph G. From this we have the following result.

Lemma 4.3.2 *Let c and c' be two four-circuits of G and let S be a collineation of* $\mathrm{PG}(4,q)$ *such that S fixes $\mathcal{Q}$ and $S(c) = c'$. Let the edge sets of c and c' be the 1-cycles σ and σ' of Γ_G, respectively. Then σ is null homologous if and only if σ' is null homologous.*

Proof: Suppose that σ is null homologous, so that $\mathcal{S}$ is the boundary of some 2-chain $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$, where each Δ_i is a 2-simplex of Γ_G. Each Δ_i is a triangle in G and since $\sigma = \partial(\{\Delta_1, \Delta_2, \ldots, \Delta_n\})$ it follows that the edge set of c is the symmetric difference of the edge sets of the triangles $\Delta_1, \Delta_2, \ldots, \Delta_n$. This property is preserved by any automorphism of G and, since the above discussion shows that S induces an automorphism of G, it follows that σ' is null homologous.

□

By Theorem 4.2.6, to prove that $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial we need to show that the edge set of every induced four-circuit is a null homologous 1-cycle. By Lemma 4.3.2, we can do this by directly showing that the edge set of a given induced four-circuit of G is null homologous or by showing it is isomorphic in $\mathrm{PG}(4,q)$ to an induced four-circuit with a null homologous edge set. We proceed by using the elements of the group of $\mathcal{Q}$ in $\mathrm{PG}(4,q)$ to choose canonical forms for the induced four-circuits of G, and then apply Lemma 4.2.7 to the canonical induced four-circuits.

The following result gives a canonical form for the induced four-circuits of G.

Lemma 4.3.3 *Let c be an induced four-circuit of G. Then c is isomorphic to an induced four-circuit of the form $(O, A, B, C(\lambda))$ where $O = (0,1,\eta,0,0)$ for η a fixed non-square of* $\mathrm{GF}(q)$, *$A = (0,1,\eta,0,1)$, $C(\lambda) = (0,1,\eta,\lambda,0)$, for some $\lambda \in \mathrm{GF}(q) \setminus \{0, -4\eta\}$ and B is a non-singular point of* $\mathrm{PG}(4,q)$ *such that $\Delta(O,B) \neq 0$.*

Proof: Let $c = (X_1, X_2, X_3, X_4)$ and let the plane $\langle X_1, X_2, X_4\rangle = \pi$. Since c is an induced four-circuit X_2 and X_4 are not adjacent in G. Thus the lines $\langle X_1, X_2\rangle$ and $\langle X_1, X_4\rangle$ are tangents at distinct points of $\mathcal{Q}$ and the line $\langle X_2, X_4\rangle$ is not a tangent to $\mathcal{Q}$. From this it follows that π intersects $\mathcal{Q}$ in a conic and that $\pi^\perp$ is a line exterior to $\mathcal{Q}$. By [27, Theorem 22.6.6] we know that the group of $\mathcal{Q}$ is transitive on such planes, so we may assume that $\pi = [0,\eta,-1,0,0] \cap [1,0,0,0,0]$, where η is a fixed non-square of $\mathrm{GF}(q)$. Now let $\langle X_1, X_2\rangle \cap \mathcal{Q} = P_1$, $\langle X_1, X_4\rangle \cap \mathcal{Q} = P_2$ and $X_2^\perp \cap \pi \cap \mathcal{Q} = \{P_1, P_3\}$. By [24, Lemma 7.2.3 Corollary 8] we know that the group of $\mathcal{Q}$ is three transitive on the singular points incident with π. So we may assume that $P_1 = (0,0,0,0,1)$, $P_2 = (0,0,0,1,0)$ and $P_3 = (0,-1/(2\eta),-1/2,1,-1/(4\eta))$. This means that X_1 is the intersection of the tangent on $(0,0,0,0,1)$ incident with π and the tangent on $(0,0,0,1,0)$ incident with π. That is, $X_1 = O = (0,1,\eta,0,0)$. Similarly $X_2 = A$ and $X_4 = C(\lambda)$ for some $\lambda \in \mathrm{GF}(q)\setminus\{0\}$. Now the circuit c is induced if and only if X_3 is a non-singular point of $\mathrm{PG}(4,q)$ such that $\Delta(O,X_3) \neq 0$ and $\Delta(A, C(\lambda)) \neq 0$. Since

$$\begin{aligned}\Delta(A,C(\lambda)) &= \Delta((0,1,\eta,0,1),(0,1,\eta,\lambda,0))\\ &= (2\eta+\lambda)^2 - 4(\eta)(\eta)\\ &= \lambda(4\eta+\lambda),\end{aligned}$$

we have the required result.

□

We now consider induced four-circuits of the form $(O,A,B,C(\lambda))$ (using the notation of Lemma 4.3.3) where $\lambda \neq 0,-4\eta$. First we need to establish for a given λ, $\lambda \neq 0, -4\eta$, the possible points B. Since $(O\ A\ B\ C(\lambda))$ is a four-circuit if and only if $B \frown A$, $B \frown C(\lambda)$ and B is non-singular, then, for a fixed λ, $\lambda \neq 0, -4\eta$, the set of possible points B is $\left(T_A \cap T_{C(\lambda)}\right) \setminus \mathcal{Q}$.

It is straight-forward to show that the quadric T_A has equation

$$x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{x_3^2}{4\eta} + \frac{x_1x_2}{2} - \frac{x_1x_3}{2} - \frac{x_2x_3}{2\eta} + x_3x_4 = 0,$$

and the quadric $T_{C(\lambda)}$ has equation

$$x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{\lambda^2 x_4^2}{4\eta} + \frac{x_1x_2}{2} - \frac{\lambda x_1x_4}{2} - \frac{\lambda x_2x_4}{2\eta} + x_3x_4 = 0.$$

We will now show that for $\lambda \neq 0, -4\eta$, there exist two hyperplanes of $\mathrm{PG}(4,q)$, $\Sigma_1(\lambda)$ and $\Sigma_2(\lambda)$, such that $\Sigma_1(\lambda) \cap \mathcal{Q} = \mathcal{E}_1(\lambda)$ and $\Sigma_2(\lambda) \cap \mathcal{Q} = \mathcal{E}_2(\lambda)$ are both non-singular elliptic quadrics and $T_A \cap T_{C(\lambda)} = \mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda)$.

Let $\Sigma_1(\lambda)$ be the hyperplane of $\mathrm{PG}(4,q)$ with coordinates $[0,0,0,1,-\lambda]$. Since $\Sigma_1(\lambda)$ contains neither A nor $C(\lambda)$ it meets T_A in a non-singular elliptic quadric, and similarly for $T_{C(\lambda)}$. The equation for the section of T_A in $\Sigma_1(\lambda)$ is that of T_A with the substitution $x_3 = \lambda x_4$:

$$x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{\lambda^2 x_4^2}{4\eta} + \frac{x_1x_2}{2} - \frac{\lambda x_1x_4}{2} - \frac{\lambda x_2x_4}{2\eta} + \lambda x_4^2 = 0,$$

which is the same as the equation for the section of $T_{C(\lambda)}$ in $\Sigma_1(\lambda)$, that is $\Sigma_1(\lambda) \cap T_A = \Sigma_1(\lambda) \cap T_{C(\lambda)}$. Let $\mathcal{E}_1(\lambda)$ be the non-singular elliptic quadric $\Sigma_1(\lambda) \cap T_A = \Sigma_1(\lambda) \cap T_{C(\lambda)}$.

Similarly, let $\Sigma_2(\lambda)$ be the hyperplane of $\mathrm{PG}(4,q)$ with coordinates $[0,2\eta,2,1,\lambda]$. The hyperplane $\Sigma_2(\lambda)$ contains neither A nor $C(\lambda)$ and the intersection of T_A with $\Sigma_2(\lambda)$ has equation:

$$\begin{aligned}& x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{(-2\eta x_1 - 2x_2 - \lambda x_4)^2}{4\eta} + \frac{x_1x_2}{2}\\ & -\frac{x_1(-2\eta x_1 - 2x_2 - \lambda x_4)}{2} - \frac{x_2(-2\eta x_1 - 2x_2 - \lambda x_4)}{2\eta} + x_3x_4 = 0\\ \Longleftrightarrow\quad & x_0^2 + (-\frac{\eta}{4} - \eta + \eta)x_1^2 + (-\frac{1}{4\eta} - \frac{1}{\eta} + \frac{1}{\eta})x_2^2 - \frac{\lambda^2}{4\eta}x_4^2 + (\frac{1}{2} - 2 + 1 + 1)x_1x_2\\ & +(-\lambda + \frac{\lambda}{2})x_1x_4 + (-\frac{\lambda}{\eta} + \frac{\lambda}{2\eta})x_2x_4 + x_3x_4 = 0\\ \Longleftrightarrow\quad & x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{\lambda^2 x_4^2}{4\eta} + \frac{x_1x_2}{2} - \frac{\lambda x_1x_4}{2} - \frac{\lambda x_2x_4}{2\eta} + x_3x_4 = 0.\end{aligned}$$

Thus the intersection of T_A with $\Sigma_2(\lambda)$ is the same as the intersection of $T_{C(\lambda)}$ with $\Sigma_2(\lambda)$. Let this non-singular elliptic quadric section be $\mathcal{E}_2(\lambda)$.

The intersection of the quadrics $\mathcal{E}_1(\lambda)$ and $\mathcal{E}_2(\lambda)$ is contained in the plane $\pi_\lambda = \Sigma_1(\lambda)\cap\Sigma_2(\lambda)$, which has equations $x_3 = \lambda x_4$ and $2\eta x_1 + 2x_2 + x_3 + \lambda x_4 = 0$. Substituting the first equation of π_λ into the second yields $\eta x_1 + x_2 + \lambda x_4 = 0$, or $\eta x_1 + x_2 + x_3 = 0$, which are the equations of $C(\lambda)^\perp$ and $A^\perp$ respectively. So $\pi_\lambda \subset C(\lambda)^\perp, A^\perp$ and in fact $\pi_\lambda = C(\lambda)^\perp \cap A^\perp$. Thus $\left(T_A \cap T_{C(\lambda)}\right) \cap \mathcal{Q} = \pi_\lambda \cap \mathcal{Q}$, and so $\mathcal{E}_1(\lambda) \cap \mathcal{E}_2(\lambda) = \mathcal{E}_1(\lambda) \cap \mathcal{Q} = \mathcal{E}_2(\lambda) \cap \mathcal{Q}$.

Using the above we now prove the following result.

Lemma 4.3.4 *Let $\lambda \in \mathrm{GF}(q)\setminus\{0, -4\eta\}$ and let $\Sigma_1(\lambda)$ and $\Sigma_2(\lambda)$ be the hyperplanes of $\mathrm{PG}(4,q)$ with the equations $x_3 - \lambda x_4 = 0$ and $2\eta x_1 + 2x_2 + x_3 + \lambda x_4 = 0$, respectively. Let $\mathcal{E}_1(\lambda)$ and $\mathcal{E}_2(\lambda)$ be the non-singular elliptic quadrics, $\Sigma_1(\lambda)\cap\mathcal{Q}$ and $\Sigma_2(\lambda)\cap\mathcal{Q}$, respectively. If $A = (0,1,\eta,0,1)$ and $C(\lambda) = (0,1,\eta,\lambda,0)$, then $T_A \cap T_{C(\lambda)} = \mathcal{E}_1(\lambda)\cup\mathcal{E}_2(\lambda)$ and $\mathcal{E}_1(\lambda)\cap\mathcal{Q} = \mathcal{E}_2(\lambda)\cap\mathcal{Q} = \mathcal{E}_1(\lambda)\cap\mathcal{E}_2(\lambda)$.*

Proof: We have already established that $\mathcal{E}_1(\lambda) \cap \mathcal{Q} = \mathcal{E}_2(\lambda) \cap \mathcal{Q} = \mathcal{E}_1(\lambda) \cap \mathcal{E}_2(\lambda)$ and that $\mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda) \subseteq T_A \cap T_{C(\lambda)}$, so we must now show that $T_A \cap T_{C(\lambda)} \subseteq \mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda)$.

Let P be a point of $T_A \cap T_{C(\lambda)}$. If $P \in \mathcal{Q}$, then by the above $T_A \cap T_{C(\lambda)} \cap \mathcal{Q} = \pi_\lambda \cap \mathcal{Q} = \mathcal{E}_1(\lambda) \cap \mathcal{E}_2(\lambda)$ and so $P \in \mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda)$. If $P \notin \mathcal{Q}$, then consider the line $\ell = \langle P, A\rangle$ tangent to $\mathcal{Q}$. By Lemma 4.3.1, since $C(\lambda)$ is cotangent to P and $P \notin \mathcal{Q}$, it follows that $C(\lambda)$ is cotangent to exactly two points of ℓ, both of which are non-singular with respect to $\mathcal{Q}$ and one of which is P. We also know that $C(\lambda)$ cotangent to the points $P_1 = \ell \cap \mathcal{E}_1(\lambda)$ and $P_2 = \ell \cap \mathcal{E}_1(\lambda)$. Since $\mathcal{E}_1(\lambda) \cap \mathcal{E}_2(\lambda) \subset \mathcal{Q}$, it follows that P_1 and P_2 are distinct and so either $P = P_1$ or P_2. Thus $P \in \mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda)$.

□

From Lemma 4.2.7, if c is the induced four-circuit $(O, A, B, C(\lambda))$ of G, for $B \in (\mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda)) \setminus \mathcal{Q}$, and the subgraph $G_{\{A,C(\lambda)\}}$ of G is connected, then the edge set of c is null homologous. Thus if $G_{\{A,C(\lambda)\}}$ is connected, then all the canonical induced four-circuits of G have null homologous edge sets. In the next section we show that $G_{\{A,C(\lambda)\}}$ is connected.

4.3.2 The graph $G_{\{A,C(\lambda)\}}$, $\lambda \neq 0, -4\eta$

The subgraph $G_{\{A,C(\lambda)\}}$ of G has vertex set the set of points $(\mathcal{E}_1(\lambda) \cup \mathcal{E}_2(\lambda)) \setminus \mathcal{Q} = (\mathcal{E}_1(\lambda) \cup \mathcal{E}_1(\lambda)) \setminus (\mathcal{E}_1(\lambda) \cap \mathcal{E}_2(\lambda))$ of $\mathrm{PG}(4,q)$. Two vertices of this set are adjacent if they are cotangent. We now prove a result which simplifies the task of showing that $G_{\{A,C(\lambda)\}}$ is connected.

Lemma 4.3.5 *Let the graph $\Gamma_1(\lambda)$ be the subgraph of G with vertex set $\mathcal{E}_1(\lambda) \setminus \mathcal{Q}$. If $\Gamma_1(\lambda)$ is connected then $G_{\{A,C(\lambda)\}}$ is connected.*

Proof: Suppose that $\Gamma_1(\lambda)$ is connected and that X and Y are two vertices of $G_{\{A,C(\lambda)\}}$. If X and Y are both vertices of $\Gamma_1(\lambda)$, then there is a path connecting them in $\Gamma_1(\lambda)$ and hence in $G_{\{A,C(\lambda)\}}$. Now suppose that X is a vertex of $\Gamma_1(\lambda)$ and that Y is not. That is, X is a point of $\mathcal{E}_1(\lambda) \setminus \mathcal{Q}$ and Y is a point of $\mathcal{E}_2(\lambda) \setminus \mathcal{Q}$. Consider the line $\langle Y, A\rangle$ that is tangent to $\mathcal{Q}$. Since $\langle Y, A\rangle$ contains a point of $\mathcal{E}_2 \setminus \mathcal{Q}$ it also contains a (distinct) point of $\mathcal{E}_1(\lambda)$, Y' say. As X and Y' are connected in $\Gamma_1(\lambda)$, it follows that X and Y are connected in $G_{\{A,C(\lambda)\}}$. A similar argument shows that if neither X nor Y are vertices of $\Gamma_1(\lambda)$, then they are connected in $G_{\{A,C(\lambda)\}}$.

□

We now prove that $\Gamma_1(\lambda)$ is connected. The first step is to prove some results on the group of $\Gamma_1(\lambda)$.

Lemma 4.3.6 *The stabiliser of $\mathcal{E}_1(\lambda)$ in the group of $\mathcal{Q}$ is transitive on the points of $\mathcal{E}_1(\lambda) \setminus \mathcal{Q}$.*

Proof: Let X and Y be two points of $\mathcal{E}_1(\lambda) \setminus \mathcal{Q}$. Since $\mathcal{E}_1(\lambda)$, $\langle X, Y\rangle$ and the plane π_λ are contained in the hyperplane $\Sigma_1(\lambda)$ of $\mathrm{PG}(4,q)$, it follows that the tangent $\langle X, Y\rangle$ meets the plane π_λ in a point P. Recall from Section 4.3.1 that π_λ was the plane such that $\mathcal{E}_1(\lambda)\cap\mathcal{Q} = \mathcal{Q}\cap\pi_\lambda$ and so $X, Y \notin \pi_\lambda$ and by Lemma 4.3.1 $P \notin \mathcal{Q}$. Since P is not on $\mathcal{Q}$, we consider the automorphism μ_P of $\mathcal{Q}$ (see Section 1.2). Now from Section 4.3.1 we have that $\pi_\lambda \subset A^\perp, C(\lambda)^\perp$ and so since $P \in \pi_\lambda$, it follows that μ_P fixes both $A^\perp$ and $C(\lambda)^\perp$. Thus μ_P fixes $T_A \cap T_{C(\lambda)}$. Also since $P \in \Sigma_1(\lambda)$ it follows that μ_P fixes $\Sigma_1(\lambda)$ and so μ_P fixes $\mathcal{E}_1(\lambda)$. Since μ_P fixes the lien $\langle X, Y\rangle$ and $\mathcal{E}_1(\lambda)$ it also fixes $\langle X, Y\rangle \cap \mathcal{E}_1(\lambda) = \{X, Y\}$. If μ_P fixes X, then it must also fix Y. Now μ_P fixes X if and only if

$$\mu_P(X) = X - \frac{\beta(X,P)}{Q(P)}P = X,$$

which is the case if and only if $\beta(X, P) = 0$. That is, $X \in P^\perp$. So if μ_P fixes both X and Y, then the line $\langle X, Y\rangle$ is contained in $P^\perp$, which implies that $P \in P^\perp$. But this is a contradiction as $P \notin \mathcal{Q}$. Thus $\mu_P(X) = Y$.

□

Corollary 4.3.7 *The automorphism group of the graph $\Gamma_1(\lambda)$ is transitive on the vertices of $\Gamma_1(\lambda)$.*

Proof: Any automorphism of $\mathcal{Q}$ fixing $\mathcal{E}_1(\lambda)$ induces an automorphism of $\Gamma_1(\lambda)$.

□

Corollary 4.3.8 *The automorphism group of $\Gamma_1(\lambda)$ is transitive on the connected components of $\Gamma_1(\lambda)$.*

We now proceed to show that $\Gamma_1(\lambda)$ has a connected component containing more than half the vertices of $\Gamma_1(\lambda)$, since by Corollary 4.3.8 this means that $\Gamma_1(\lambda)$ is connected.

Now if X and Y are two points of $\mathcal{E}_1(\lambda) \setminus \mathcal{Q}$, then they are cotangent if and only if the line $\langle X, Y\rangle$ is tangent to the quadric $\Sigma_1(\lambda) \cap \mathcal{Q}$. That is, the graph $\Gamma_1(\lambda)$ can be defined from $\mathcal{E}_1(\lambda) \setminus \mathcal{Q}$ and $\Sigma_1(\lambda) \cap \mathcal{Q}$. We first define an isomorphism from $\Sigma_1(\lambda)$ to $\mathrm{PG}(3, q)$ which will make calculating the connected components of $\Gamma_1(\lambda)$ a little easier.

Let $U : \Sigma_1(\lambda) \to \mathrm{PG}(3, q)$ (where $\mathrm{PG}(3, q)$ has coordinates x_0, x_1, x_2, x_3) be such that $U(x_0, x_1, x_2, \lambda x_4, x_4) = (x_0, x_1, x_2, x_4)$. If $\mathcal{E}_1(\lambda)' = U(\mathcal{E}_1(\lambda))$, $\mathcal{Q}' = U(\Sigma_1(\lambda) \cap \mathcal{Q})$ and $\pi_\lambda = U(\pi_\lambda)$, then $\mathcal{E}_1(\lambda)'$, $\mathcal{Q}'$ and π'_λ have equations

$$\begin{aligned}
\mathcal{E}_1(\lambda)' \quad &: \quad x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{\lambda^2 x_3^2}{4\eta} + \frac{x_1x_2}{2} - \frac{\lambda x_1 x_3}{2} - \frac{\lambda x_2 x_3}{2\eta} + \lambda x_3^2 &= 0 \\
\mathcal{Q}' \quad &: \quad x_0^2 + x_1x_2 + \lambda x_3^2 &= 0 \\
\pi'_\lambda \quad &: \quad \eta x_1 + x_2 + \lambda x_3 &= 0
\end{aligned}$$

If X is the point of $\mathrm{PG}(3, q)$ with coordinates (x_0, x_1, x_2, x_3), then let $Q'(X) = x_0^2 + x_1x_2 + \lambda x_3^2$. The bilinear form of $\mathrm{PG}(3, q)$ associated with $\mathcal{Q}'$ is $\beta'(X, Y) = 2x_0y_0 + x_1y_2 + x_2y_1 + 2\lambda x_3y_3$, and we will denote the polarity associated with $\mathcal{Q}'$ by $\perp'$. If X and Y are two distinct, non-singular points of $\mathrm{PG}(3, q)$, then X and Y are cotangent with respect to $\mathcal{Q}'$ if and only if $\Delta'(X, Y) = 0$ where $\Delta'(X, Y) = (\beta'(X, Y))^2 - 4Q'(X)Q'(Y)$. Using the above, an alternative representation of $\Gamma_1(\lambda)$ is

Vertex set: $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$.

Adjacency: Cotangency with respect to $\mathcal{Q}'$.

At this point we note that, as in Theorem 4.3.6, if X and Y are two points of $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ and $P = \langle X, Y\rangle \cap \pi'_\lambda$, then μ_P fixes both $\mathcal{E}_1(\lambda)'$ and $\mathcal{Q}'$, as well as interchanging X and Y.

Now $O \in \mathcal{E}_1(\lambda) \setminus \mathcal{Q}$ and so $O' = (0, 1, \eta, 0)$ is the image of O under U, $O' = (0, 1, \eta, 0)$ and $O' \in \mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$. We now determine the points of $\mathcal{E}_1(\lambda)'$ that are cotangent to O'. Let $\pi_{O'}$ be the plane of $\mathrm{PG}(3,q)$ with equation $2\eta x_1 + 2x_2 + \lambda x_3 = 0$ and note that $O' \notin \pi_{O'}$. The points of $\pi_{O'}$ that lie in $\mathcal{E}_1(\lambda)'$ have equation

$$\begin{aligned}
x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \frac{(2\eta x_1 + 2x_2)^2}{4\eta} + \frac{x_1x_2}{2} - \frac{(-2\eta x_1 - 2x_2)x_1}{2} & \\
-\frac{(-2\eta x_1 - 2x_2)x_2}{2\eta} + \frac{(2\eta x_1 + 2x_2)^2}{\lambda} &= 0 \\
\iff \quad x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} - \eta x_1^2 - \frac{x_2^2}{\eta} - 2x_1x_2 + \frac{x_1x_2}{2} + \eta x_1^2 + x_1x_2 + x_1x_2 & \\
+\frac{x_2^2}{\eta} + \frac{(2\eta x_1 + 2x_2)^2}{\lambda} &= 0 \\
\iff \quad x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} + \frac{x_1x_2}{2} + \frac{(2\eta x_1 + 2x_2)^2}{\lambda} &= 0.
\end{aligned}$$

The points of $\pi_{O'}$ that are cotangent to O' are those that satisfy the equation

$$\begin{aligned}
\Delta'\left(O', \left(x_0, x_1, x_2, -\frac{2\eta x_1}{\lambda} - \frac{2x_2}{\lambda}\right)\right) &= 0 \\
\iff \quad (\eta x_1 + x_2)^2 - 4\eta\left(x_0^2 + x_1x_2 + \frac{(2\eta x_1 + 2x_2)^2}{\lambda}\right) &= 0 \\
\iff \quad x_0^2 - \frac{\eta x_1^2}{4} - \frac{x_2^2}{4\eta} + \frac{x_1x_2}{2} + \frac{(2\eta x_1 + 2x_2)^2}{\lambda} &= 0.
\end{aligned} \tag{4.3.4}$$

Thus the set of points of $\pi_{O'}$ that are cotangent to O' is exactly the intersection of $\mathcal{E}_1(\lambda)'$ with $\pi_{O'}$. Now the set of lines incident with O' that are tangent to $\mathcal{Q}'$ form a quadratic cone $T'_{O'}$ (regardless of whether $\mathcal{Q}'$ is a hyperbolic or elliptic quadric), and so it follows that the set of points of $\pi_{O'}$ cotangent to O' is the conic $T'_{O'} \cap \pi_{O'} = \mathcal{C}_{O'}$. Also since the point O' is on exactly $q+1$ tangents to $\mathcal{Q}'$, it follows that the set of points of $\mathcal{E}_1(\lambda)$, that are cotangent to O', is exactly $\mathcal{C}_{O'}$.

Note that the discussion preceding Lemma 4.3.4 shows that $\mathcal{E}_1(\lambda)' \cap \mathcal{Q}' = \pi'_\lambda \cap \mathcal{Q}'$. Thus the points of $\mathcal{C}_{O'}$ that are also on $\mathcal{Q}'$ are precisely the points of $\mathcal{E}_1(\lambda)'$ on the line $\pi_{O'} \cap \pi'_\lambda$. Now any point of $\mathcal{E}_1(\lambda)'$ on the line $\pi_{O'} \cap \pi'_\lambda$ must satisfy the equations, $x_2 = -\eta x_1$ and $x_3 = 0$. Since η is a non-square of $\mathrm{GF}(q)$ there is no point that satisfies these equations. Hence $\mathcal{C}_{O'}$ contains no points of $\mathcal{Q}'$.

By the proof of Lemma 4.3.6 the stabiliser of $\mathcal{E}_1(\lambda)'$ in the group of $\mathcal{Q}'$ is transitive on the points of $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ and so the above results for O' apply for all $X \in \mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$.

Lemma 4.3.9 *Let X be a point of $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ (for $\lambda \neq 0, -4\eta$). Then the set of points of $\mathcal{E}_1(\lambda)'$ that are cotangent to X form a conic, $\mathcal{C}_X$ in the plane π_X. Furthermore, $\mathcal{C}_X$ contains no points of $\mathcal{Q}'$.*

We now calculate the conditions under which $\pi_X = \pi_Y$ for $X, Y \in \mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$.

Lemma 4.3.10 *Let $X \in \mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$. If $\lambda \neq -8\eta$, then for each $Y \in \mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ and $X \neq Y$ it follows that $\pi_X \neq \pi_Y$. If $\lambda = -8\eta$, then $Y = (\pi_X)^{\perp'}$ is the unique point of $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ such that $\pi_X = \pi_Y$ and $X \neq Y$.*

Proof: We show that the result holds for the case $X = O'$ and so the general result holds by the transitivity on the points of $\mathcal{E}_1(\lambda)'$ (Lemma 4.3.6).

Now it follows from the proof of Lemma 4.3.6 that if $Y \in \mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ and $\pi'_\lambda \cap \langle Y, O' \rangle = P$, then μ_P fixes $\mathcal{Q}'$ and $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$, and interchanges O' and Y. Hence $\pi_{O'} = \pi_Y$ if and only if $\mu_P(\pi_{O'}) = \pi_{O'}$. We also know from the proof of Lemma 4.3.6 that any point $P \in \pi'_\lambda \setminus \mathcal{Q}'$ has the property that μ_P fixes $\mathcal{Q}'$ and $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$ and either fixes O' of maps it to a different point of $\mathcal{E}_1(\lambda)' \setminus \mathcal{Q}'$. Thus we look for points P of $\pi'_\lambda \setminus \mathcal{Q}'$ such that μ_P fixes $\pi_{O'}$ but not O'.

Now μ_P fixes $\pi_{O'}$ if and only if μ_P fixes the point $(\pi_{O'})^{\perp'}$ and this is the case if and only if $P \in \pi_{O'}$ or $P = (\pi_{O'})^{\perp'}$. In the first case P is any point of $\pi_{O'} \cap \pi'_\lambda$ which is a line with equations $2\eta x_1 + 2x_2 + \lambda x_3 = 0$ and $\eta x_1 + x_2 + \lambda x_3 = 0$, that is $\eta x_1 + x_2 = x_3 = 0$. However, the polar plane of O' with respect to $\mathcal{Q}'$ has equation $\eta x_1 + x_2 = 0$ and so contains the line $\pi_{O'} \cap \pi'_\lambda$. Thus if $P \in \pi_{O'} \cap \pi'_\lambda$, then it is also contained in the polar plane of O', and so μ_P fixes O'.

For the second case we note that $P = \pi_{O'}^{\perp'} = (0, 4, 4\eta, 1)$, which lies on π'_λ if and only if $\lambda = -8\eta$. In this case, $\mu_P(\pi_{O'}) = \pi_{O'}$ and $\mu_P(O') \neq O'$.

□

Given Lemma 4.3.9 and Lemma 4.3.10, we are now equipped to show that $\Gamma_1(\lambda)$ is connected.

Theorem 4.3.11 *The graph $\Gamma_1(\lambda)$ is connected.*

Proof: We show that the connected component of $\Gamma_1(\lambda)$ containing O' has size greater than $|\Gamma_1(\lambda)|/2 = (q^2 - q)/2$. Recall from Lemma 4.3.9 that the vertices at distance 1 to X form the conic $\mathcal{C}_X$ in the plane π_X of $\mathrm{PG}(3, q)$. We now find a lower bound for the number of vertices at distance 0, 1 or 2 to O'.

We separate the proof into two cases, $\lambda = -8\eta$ and $\lambda \neq -8\eta$ (note that we still have the restriction that $\lambda \neq 0, -4\eta$).

Suppose that $\lambda = -8\eta$. To obtain a lower bound for the number of points at distance 2 to O' we consider the points at distance 1 to a point $X \in \mathcal{C}_{O'}$, that is, the points of the conic $\mathcal{C}_X$. Since $\lambda = -8\eta$ we have that $O', (\pi_{O'})^{\perp'} \in \mathcal{C}_X$. Since there are at most two lines contained in $\pi_{O'}$ that are incident with X and tangent to $\mathcal{Q}'$, it follows that $|\mathcal{C}_{O'} \cap \mathcal{C}_X| \leq 2$. Thus there at least $q-3$ of the points in the set $\mathcal{C}_X \setminus \{(\pi_{O'})^{\perp'}\}$ that are at distance 2 to O'. Now suppose that $Y \in \mathcal{C}_{O'} \setminus \{X\}$, then either $\pi_Y \neq \pi_X$ and $\mathcal{C}_X \cap \mathcal{C}_Y = \{O', (\pi_{O'})^{\perp'}\}$, or $\pi_X = \pi_Y$ and $\mathcal{C}_X = \mathcal{C}_Y$. Thus if $Z \in \mathcal{C}_X$ and is at distance 2 to O', and not equal to $(\pi_{O'})^{\perp'}$, then Z is at distance 1 to at most one other point of $\mathcal{C}_{O'}$. Hence a lower bound on the number of points at distance 2 to O' is $(q+1)(q-3)/2 + 1$ and so a lower bound for the number of points at distance 0, 1 or 2 to O' is:

$$\frac{(q+1)(q-3)}{2} + 1 + (q+1) + 1 \;=\; \frac{(q+1)(q-1)}{2} + 2 > \frac{q^2-q}{2} \;=\; \frac{|\Gamma_1(\lambda)|}{2}.$$

This shows that $\Gamma_1(\lambda)$ is connected for $\lambda = -8\eta$.

Now suppose that $\lambda \neq -8\eta$. As above we consider a point $X \in \mathcal{C}_{O'}$. By an argument above there are at most two points at distance 1 to X in $\mathcal{C}_{O'}$. Hence at least $q-2$ of the points of $\mathcal{C}_X$ are at distance 2 to O'. Now let Z be a point at distance 2 to O'. Since $\pi_Z \neq \pi_{O'}$, it follows that $\pi_Z \cap \pi_{O'}$ is a line and $|\mathcal{C}_Z \cap \mathcal{C}_{O'}| \leq 2$, that is, any point at distance 2 to O' is at distance 1 to at most two points of $\mathcal{C}_{O'}$. Hence a lower bound for the number of points at distance 2 to O' is $(q+1)(q-2)/2$. Thus a lower bound for the number of points at distance 0, 1 or 2 to O' is

$$\frac{(q+1)(q-2)}{2} + (q+1) + 1 \;=\; \frac{(q+1)q}{2} + 1 > \frac{q^2-q}{2} \;=\; \frac{|\Gamma_1(\lambda)|}{2}.$$

This shows that $\Gamma_1(\lambda)$ is connected for $\lambda \neq -8\eta$.

□

We now have the following string of consequences of Theorem 4.3.11.

Corollary 4.3.12 *The graph $G_{\{A, C(\lambda)\}}$ is connected.*

Proof: By Lemma 4.3.5.

□

Theorem 4.3.13 *$H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial.*

Proof: By Theorem 4.2.6 $H_1(\Gamma_G, \mathbb{Z}_2)$ is trivial if and only if the edge set of each induced four-circuit of G is null homologous. By Lemma 4.3.2 and Lemma 4.3.3 the edge set of every induced four-circuit of G is null homologous if all the canonical induced four-circuits have null homologous edge sets (where the canonical induced four-circuit are the induced four-circuits $(O, A, B, C(\lambda))$ for some $\lambda \in \mathrm{GF}(q) \setminus \{0, -4\eta\}$). By Lemma 4.2.7 if $G_{\{A,C(\lambda)\}}$ is connected, then the edge set of $(O, A, B, C(\lambda))$ is null homologous. Thus Corollary 4.3.12 proves the result.

□

Theorem 4.3.13 has the following corollary:

Theorem 4.3.14 *Let $\mathcal{S}$ be a GQ of order (q, q^2), q odd, and $\mathcal{S}'$ a subquadrangle of $\mathcal{S}$ of order q isomorphic to $Q(4, q)$. If each ovoid of $\mathcal{S}'$ subtended by $\mathcal{S}$ is isomorphic to an elliptic quadric ovoid, then $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

Proof: Since $H_1(\Gamma_G, \mathbb{Z}_2) = H^1(\Gamma_G, \mathbb{Z}_2)$ is trivial, the result follows from Theorem 4.2.3.

□

Combining Theorem 4.3.14 with the corresponding result for q even given in [65] we have the following result for general q.

Theorem 4.3.15 *Let $\mathcal{S}$ be a GQ of order (q, q^2) and $\mathcal{S}'$ a subquadrangle of $\mathcal{S}$ of order q isomorphic to $Q(4, q)$. If each ovoid of $\mathcal{S}'$ subtended by $\mathcal{S}$ is isomorphic to an elliptic quadric ovoid, then $\mathcal{S}$ is isomorphic to $Q(5, q)$.*

4.4 Remarks

Theorem 4.3.14 now makes it possible to prove results similar to those in Section 4.1 for the case where q is odd. In the case where $q = 9$ Penttila and Royle have shown that the only ovoids of $Q(4, q)$ are $K1(\sigma)$ ovoids, where $\sigma : x \mapsto x^3$, and the elliptic quadric ovoid ([53]). Further (still for $q = 9$) Penttila has shown that there is no GQ of order (q, q^2) with a subquadrangle isomorphic to $Q(4, q)$ that subtends both an elliptic quadric ovoid and a $K1(\sigma)$ ovoid ([51]). Thus, by Theorem 4.3.14, a new GQ $\mathcal{S}$ of order $(9, 81)$ with a subquadrangle isomorphic to $Q(4, 9)$ must subtend an ovoid-rosette configuration in $Q(4, 9)$ where every subtended ovoid is a $K1(\sigma)$ ovoid. Using the work in Section 4.2 it may be possible to rule out the existence of a new GQ subtending the SPG found in Chapter 3 by characterising $\mathcal{S}(\mathcal{C}_\sigma)^\wedge$ (as in Chapter 3) by its doubly subtended $Q(4, 9)$ subquadrangle. Of course it may also be possible to prove this characterisation for $q = p^h$, where p is an odd prime and $h > 1$, using the work of Section 4.2, although it is likely to be a more difficult problem than the classical case solved in Section 4.3.

Chapter 5

Affine planes and GQs of order s

In this section we study the connections between affine planes and GQs of order s containing a regular point. In particular, it is known that if $\mathcal{S}$ is a GQ of order s, with a regular point, then there is an associated affine plane of order s ([49, 1.3.1]). The GQ defines an s-fold cover of the affine plane and we investigate the relationships between the GQ, the affine plane and the associated cover of the affine plane.

To begin we consider the classical case, for q even. Let $\mathcal{Q} = Q(4,q)$, the non-singular (parabolic) quadric in $\mathrm{PG}(4,q)$ and let I denote incidence between points and lines of $\mathcal{Q}$. For such a quadric there is a unique point $N \notin \mathcal{Q}$, called the nucleus of $\mathcal{Q}$. The point N has the property that every tangent space to $\mathcal{Q}$ contains N. In addition, every line of $\mathrm{PG}(4,q)$ incident with N is either tangent to $\mathcal{Q}$ or is external to $\mathcal{Q}$ [27, Lemma 22.3.3(ii) and Lemma 22.3.1(i)].

Let V be a fixed point of $\mathcal{Q}$. By [49, 3.3.1.(i)] every point of $Q(4,q)$, q even, is regular, and so it follows that V is regular. Let $V^\perp$ denote the tangent space to $\mathcal{Q}$ at V, then $V^\perp \cap \mathcal{Q}$ is a quadratic cone $\mathcal{K}$ with vertex V [27, Lemma 22.7.3]. If X is any point of $\mathcal{Q} \setminus V^\perp$, that is, a point of $\mathcal{Q}$ not collinear to V in $\mathcal{Q}$, then $X^\perp \cap V^\perp$ is a plane of $\mathrm{PG}(4,q)$ that intersects $\mathcal{K}$ in a conic, $\mathcal{C}_X$. We say that X **subtends** the conic $\mathcal{C}_X$ in $\mathcal{K}$.

Let ℓ be a line of $\mathcal{Q}$ such that V is not incident with ℓ. Then there exists a unique point/line pair (Y,m) such that $V\ I\ m\ I\ Y\ I\ \ell$. If Z is a point of the set $\ell \setminus \{Y\}$ (where ℓ is considered as the set of points incident with it), then by the above, Z subtends a conic $\mathcal{C}_Z$ in $\mathcal{K}$. If Z_1 and Z_2 are two distinct point of $\ell \setminus \{Y\}$, then the tangent spaces $Z_1^\perp$ and $Z_2^\perp$ intersect in the plane $\langle \ell, N\rangle$ of $\mathrm{PG}(4,q)$. Thus, $\mathcal{C}_{Z_1} \neq \mathcal{C}_{Z_2}$, and in fact $\mathcal{C}_{Z_1} \cap \mathcal{C}_{Z_2} = \{Y\}$. Hence each point of $\ell \setminus \{Y\}$ subtends a distinct conic and the line $\langle N, Y\rangle$ is a tangent to each of these conics. We say that the tangent $\langle N, Y\rangle$ is **subtended** by ℓ.

Let π be the incidence structure with pointset the set of conics of $\mathcal{K}$ whose plane contains N, lineset the tangents to $\mathcal{Q}$ contained in $V^\perp$, except the tangent $\langle V, N\rangle$, and incidence inherited from $\mathrm{PG}(4,q)$. Then π is an affine plane, and in fact π is the affine plane obtained by removing the line $\left(\dfrac{\langle N,V\rangle}{N}\right)^D$ from the projective plane $\left(\dfrac{V^\perp}{N}\right)^D$, and so $\pi \cong \mathrm{AG}(2,q)$.

If the line m of $V^\perp$ is a line of π, then it is determined by the unique point of $\mathcal{Q}$ incident with it, since all such lines are incident with N. Thus an equivalent description of π is to redefine the lineset as the set $\mathcal{K} \setminus \{V\}$, keep the pointset as the set of conics of $V^\perp$ whose plane contains N, and keep incidence as incidence in $\mathrm{PG}(4,q)$. This is a specific case of the following theorem.

Theorem 5.0.1 *[49, 1.3.1] Let X be a regular point of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ of order (s,t). Then the incidence structure with pointset $X^\perp \setminus \{X\}$, lineset the set of spans $\{Y,Z\}^{\perp\perp}$, where $Y, Z \in X^\perp \setminus \{X\}$, $Y \not\sim Z$ and with natural incidence, is the dual of a net of order s and degree $t+1$. If $s = t \geq 2$ it is a dual affine plane of order s.*

5.1 Covering affine planes

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order s and (∞) a point of $\mathcal{S}$ that is regular. Let $\pi = \pi_{(\infty)}$ be the affine plane constructed from $\mathcal{S}$ and (∞) as in Theorem 5.0.1. We will define a map p from $\mathcal{S} \setminus (\infty)^\perp$ to π that defines an s-fold cover of π.

Let X be a point of $\mathcal{P} \setminus (\infty)^\perp$. Then $\{X, (\infty)\}^\perp$ is a point of π. Since (∞) is regular it follows that $|\{X, (\infty)\}^{\perp\perp}| = s+1$, and if $Y \in \mathcal{P} \setminus (\infty)^\perp$ then $\{Y, (\infty)\}^\perp = \{X, (\infty)\}^\perp$ if and only if $Y \in \{X, (\infty)\}^{\perp\perp}$. So if we define the map p from $\mathcal{P} \setminus (\infty)^\perp$ into the pointset of π by $p(X) = \{X, (\infty)\}^\perp$, then p has the property that for any point $\{Y, (\infty)\}^\perp$ of π, the set $p^{-1}(\{Y, (\infty)\}^\perp) = \{Y, (\infty)\}^{\perp\perp}$ consists of s pairwise non-adjacent points of $\mathcal{P} \setminus (\infty)^\perp$.

Now suppose that $\{X, (\infty)\}^\perp$ and $\{Y, (\infty)\}^\perp$ are two distinct points of π. Then $\{X, (\infty)\}^\perp \cap \{Y, (\infty)\}^\perp = Z$ for some point $Z \in (\infty)^\perp$. Now each of the s lines of $\mathcal{S}$ incident with Z, but not incident with (∞), is incident with exactly one point of $\{X, (\infty)\}^{\perp\perp} \setminus \{(\infty)\}$ and exactly one point of $\{Y, (\infty)\}^{\perp\perp} \setminus \{(\infty)\}$. Also, any line of $\mathcal{S}$ that contains one point of $\{X, (\infty)\}^{\perp\perp} \setminus \{(\infty)\}$ and one point of $\{Y, (\infty)\}^{\perp\perp} \setminus \{(\infty)\}$ must contain Z. Thus the set $p^{-1}(\{\{X, (\infty)\}^\perp, \{Y, (\infty)\}^\perp\})$ consists of s disjoint, incident pairs of points of $\mathcal{P} \setminus (\infty)^\perp$.

If Γ is the subgraph of the point graph of $\mathcal{S}$ defined on the vertex set $\mathcal{P} \setminus (\infty)^\perp$, then thus far we have shown that (Γ, p) satisfies requirements (i) and (ii) of an s-fold cover of the point graph of π (see Section 1.6). Since any two points of π are collinear, requirement (iii) is satisfied vacuously; so (Γ, p) is an s-fold cover of the point graph of π. We now show that this gives rise to an s-fold cover of π.

Let P be a fixed line of π: that is, a point of $(\infty) \setminus (\infty)^\perp$. Then the set $\{p^{-1}(\{X, (\infty)\})^\perp : \ P \text{ is incident with } \{X, (\infty)\}^\perp \text{ in } \pi\} = \{X \in \mathcal{P} \setminus (\infty)^\perp : X \ I \ P\}$. Now if Γ_p is the subgraph of Γ defined on the vertex set $\{X \in \mathcal{P} \setminus (\infty)^\perp : X \ I \ P\}$, then the s lines of $\mathcal{S}$ incident with P, but not (∞), divide Γ_p into s disjoint complete graphs, each of size s. Thus the map p satisfies condition (iv) in Section 1.6, and so we have the following theorem.

Theorem 5.1.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order s with a regular point (∞). Let π be the affine plane with lineset $(\infty)^\perp \setminus (\infty)$, pointset the set of spans $\{X, Y\}^{\perp\perp}$, where $X, Y \in (\infty)^\perp \setminus (\infty)$, $X \not\sim Y$ and natural incidence. Let $\overline{\pi}$ be the geometry with pointset $\mathcal{P} \setminus (\infty)^\perp$, lineset the set of lines of $\mathcal{S}$ not incident with (∞), and incidence as in $\mathcal{S}$. Let $p : \overline{\pi} \to \pi$ be such that: $p(X) = \{X, (\infty)\}^\perp$ for $X \in \mathcal{P} \setminus (\infty)^\perp$, and for ℓ a line of $\mathcal{S}$ not incident with (∞), $p(\ell) = Y$, where Y is the unique point such that $Y \ I \ \ell$ and $Y \sim (\infty)$. Then $(\overline{\pi}, p)$ is an s-fold cover of π.*

Note that in Theorem 5.1.1 p denotes the map induced on $\overline{\pi}$ by the map p in the discussion preceding Theorem 5.1.1.

At this point it is natural to consider under what conditions we can "reverse" the covering process in Theorem 5.1.1 and construct a GQ of order s, with a regular point, from an s-fold cover of an affine plane of order s.

Theorem 5.1.2 *Let π be an affine plane of order s and $(\overline{\pi}, p)$ an s-fold cover of π. Let $\mathcal{S}$ be the following incidence structure*

Points:	(i)	Lines of π
	(ii)	Points of $\overline{\pi}$
	(iii)	(∞)
Lines:	(a)	Parallel classes of π
	(b)	Lines of $\overline{\pi}$
Incidence I:	(i)	(a) Containment
		(b) The s covers of the point of type (i)
	(ii)	(a) None
		(b) Incidence in $\overline{\pi}$
	(iii)	(a) All
		(b) None

The geometry $\mathcal{S}$ is a GQ of order s if and only if for each non-incident point-line pair (P, ℓ), where P is of type (ii) and ℓ is of type (b) and $p(P)$ is not incident with $p(\ell)$ in π, there is a unique incident point-line pair (Q, m) such that $P\ I\ m\ I\ Q\ I\ \ell$. Further, if $\mathcal{S}$ is a GQ, then the point (∞) is regular.

Proof: In Section 1.6 it was noted that if ℓ is a line of π and is incident with the set of points $\{P_1, P_2, \ldots, P_{s+1}\}$, then each element of $p^{-1}(\ell)$, as a set of points, has the form $\{P_1', P_2', \ldots, P_{s+1}'\}$, where $p(P_i') = P_i$ for $i = 1, 2, \ldots, s+1$. We will be using this fact, implicitly, throughout this proof.

Let ℓ be a line of type (a) (that is, a parallel class of π). Then, in $\mathcal{S}$, ℓ is incident with the s lines of π it contains as a parallel class of π and the point (∞), $s+1$ in total. Let ℓ be a line of type (b) (that is, a line of $\overline{\pi}$). Then, in $\mathcal{S}$, ℓ is incident with the s points incident with it in $\overline{\pi}$ and the point of type (i) $p(\ell)$, $s+1$ in total. We now show that two distinct lines of $\mathcal{S}$ intersect in at most one point. Let ℓ and m be two lines of $\mathcal{S}$, with $\ell \neq m$. If ℓ and m are both of type (a), then they are concurrent at the unique point (∞). If ℓ is of type (a) and m is of type (b), then they are concurrent if and only if the line m (as a line of $\overline{\pi}$) is a cover of the line n of π, which is contained in the parallel class m. In this case n is the unique point incident with both ℓ and m. If ℓ and m are both of type (b) and cover the same line n of π, then they are concurrent at the point n and no other point of $\mathcal{S}$. If ℓ and m cover different lines of π and X is a point of $\mathcal{S}$ such that $X\ I\ \ell, m$, then X is a point of $\overline{\pi}$. Since $\overline{\pi}$ is a geometry there is at most one such point.

Let m be a point of type (i) (that is, a line of π). Then, in $\mathcal{S}$, m is incident with the s lines of type (b), $p^{-1}(m)$, and the parallel class of π containing it, $s+1$ in total. Let P be a point of type (ii) (that is, a point of $\overline{\pi}$). Then, in $\mathcal{S}$, P is incident with the $s+1$ lines of $\overline{\pi}$ incident with it in $\overline{\pi}$. The point (∞) is incident with the $s+1$ lines of type (a). Now let X be a point of type (i) and Y a point of type (ii). Any line of $\mathcal{S}$ incident with both X and Y must be a line of $\overline{\pi}$, covering X (as a line of π) and incident with Y in $\overline{\pi}$. Since it is a property of the cover $(\overline{\pi}, p)$ that the s covers of m are pairwise non-concurrent, it follows that m is incident with at most one such line. It is straight-forward to show for the other cases that two points of $\mathcal{S}$ are incident with at most one common line.

It now remains to check the GQ incidence axiom. Let (P, ℓ) be a non-incident point-line pair of $\mathcal{S}$. Suppose P is of type (i) and ℓ is of type (a). Let $\mathcal{L}$ be the parallel class of π containing P. Then $((\infty), \mathcal{L})$ is the unique point-line pair such that $P\ I\ \mathcal{L}\ I\ (\infty)\ I\ \ell$. Now suppose that P is of type (i) and ℓ is of type (b). Then $p(\ell)$ is a line of π such that $P \neq p(\ell)$. If P and $p(\ell)$ belong to the same parallel class of π, $\mathcal{L}$ say, then $(p(\ell), \mathcal{L})$ is the unique point-line pair such that $P\ I\ \mathcal{L}\ I\ p(\ell)\ I\ \ell$. If P and $p(\ell)$ intersect in the point Q, say, then there is a unique point Q' of π such that $Q'\ I\ \ell$ and $p(Q') = Q$. There is also a unique line m of $\overline{\pi}$ such that $p(m) = P$ and $Q'\ I\ m$. Now suppose that P is of type (ii) and ℓ is of type (a). Let m be the unique line of π such that $p(P) \in m$ and $m \in \ell$ (as a parallel class). Then there is a unique element m' of $p^{-1}(m)$, such that P is incident with m in $\overline{\pi}$ and (m, m') is the unique point-line pair such that $P\ I\ m'\ I\ m\ I\ \ell$. Now suppose that P is (∞) and ℓ is of type (b). Let $\mathcal{L}$ be the parallel class of π containing $p(\ell)$, then $(\mathcal{L}, p(\ell))$ is the unique point-line pair such that $P\ I\ \mathcal{L}\ I\ p(\ell)\ I\ \ell$.

The only cases that we haven't yet considered are where P is (∞) and ℓ is of type (a), in which case P and ℓ are always incident, and where P is of type (ii) and ℓ is of type (b). In this second case, if P and ℓ are non-incident in $\mathcal{S}$ and $p(P)$ is incident with $p(\ell)$ in π, then there exists a unique element of $p^{-1}(p(\ell))$, m say, such that $P\ I\ m$. Then $(p(\ell), m)$ is the unique point-line pair such that $P\ I\ m\ I\ p(\ell) = p(m)\ I\ \ell$.

This leaves the case where P is of type (ii), ℓ is of type (b) and $p(P)$ is not incident with $p(\ell)$ in π. Hence $\mathcal{S}$ is a GQ if and only if pairs of this type satisfy the GQ incidence axiom.

If $\mathcal{S}$ is a GQ, then consider the non-collinear pair of points of $\mathcal{S}$, $\{(\infty), X\}$. It follows that X must be a point of type (ii). Now $\{X, (\infty)\}^{\perp\perp} = p^{-1}(p(X)) \cup (\infty)$ and so $|\{X, (\infty)\}^{\perp\perp}| = s+1$. Thus the pair $\{(\infty), X\}$ is regular and therefore (∞) is also regular.

□

Note that if π is an affine plane of order s and $(\overline{\pi}, p)$ is an s-fold cover of π such that the incidence structure $\mathcal{S}$ defined in Theorem 5.1.2 is a GQ with regular point (∞), then the s-fold cover of π constructed as in Theorem 5.1.1 is $(\overline{\pi}, p)$.

Let π be an affine plane of order s and $(\overline{\pi}, p)$ an s-fold cover of π giving rise to a GQ of order s as in Theorem 5.1.2. If Γ is the point graph of π (a complete graph on s^2 vertices) and $\overline{\Gamma}$ the point graph of $\overline{\pi}$, then $(\overline{\Gamma}, p)$ is an s-fold cover of Γ. We now prove that $\overline{\Gamma}$ is antipodal and distance-regular. (See Section 1.1 for the definition of an antipodal graph and a distance regular graph.)

Theorem 5.1.3 *Let $\mathcal{S}$ be a GQ of order s, with a regular point (∞), and let π be the affine plane constructed from $\mathcal{S}$ and (∞) as in Theorem 5.0.1. Let $(\overline{\pi}, p)$ be the s-fold cover of π constructed in Theorem 5.1.1. If Γ is the point graph of $\overline{\pi}$, then Γ is an antipodal distance-regular graph, with intersection set $\{s^2-1, s(s-1), 1; 1, s^2, s^2-1\}$.*

Proof: For the purposes of this proof we will use $\sim$ to denote adjacency in the graph $\overline{\Gamma}$ and $d(X,Y)$ to denote the distance between two vertices X and Y of $\overline{\Gamma}$.

First observe that Γ is a regular graph with valence s^2-1. Now let X and Y be two non-collinear points of $\overline{\pi}$. If $p(X) \neq p(Y)$ then, in $\mathcal{S}$, $\{(\infty), X\}^\perp \neq \{(\infty), Y\}^\perp$. Let Z be a point of the set $\{(\infty), X\}^\perp \setminus \{(\infty), Y\}^\perp$ and ℓ the line $\langle Z, X\rangle$. Now $Y \notin \ell$ and Y is not collinear to Z since $Z \notin \{(\infty), Y\}^\perp$. So it follows from the GQ incidence axiom that there exists a point $Z' \in \ell \setminus \{Z\}$ collinear to both Y and X. Hence $d(X,Y) = 2$. If $p(X) = p(Y)$, then $\{(\infty), X\}^\perp = \{(\infty), Y\}^\perp$ in $\mathcal{S}$ and so in $\mathcal{S}$ we have that $\{X,Y\}^\perp = \{(\infty), X\}^\perp \subset (\infty)^\perp$. Thus there are no points of $\overline{\pi}$ collinear to both X and Y and so $d(X,Y) > 2$. If Z is a point of $\overline{\pi}$ collinear to X, then Z is not collinear to Y and $p(Z) \neq p(Y)$. Hence $d(Y,Z) = 2$ and therefore $d(X,Y) = 3$. Thus, the diameter of $\overline{\Gamma}$ is 3. We now consider the graph $\overline{\Gamma}_3$, which is defined on the same vertex set as $\overline{\Gamma}$ and has two vertices adjacent if they are at distance 3 in $\overline{\Gamma}$. If P is a point of π, then the set $p^{-1}(P)$, as vertices of Γ_3, is a clique and so $\overline{\Gamma}$ is an antipodal graph.

We now show that $\overline{\Gamma}$ is distance-regular by calculating its intersection array. Let X and Y be two distinct vertices of $\overline{\Gamma}$. The parameter b_0 is the valence of $\overline{\Gamma}$, s^2-1, and $c_0 = 0$. Now suppose that $d(X,Y) = 1$ (so $X \sim Y$), Z is a vertex of $\overline{\Gamma}$ and $X \sim Z$. If, as a point of $\mathcal{S}$, Z is not incident with the line spanned by X and Y (considered as points of $\mathcal{S}$), then $Y \not\sim Z$, since otherwise $\{X, Y, Z\}$ is a triangle in $\mathcal{S}$; and so $d(Y,Z) = 2$. Hence, the vertices of $\overline{\Gamma}$ at distance 1 to X and distance 2 to Y are precisely the points of $\mathcal{S} \setminus (\infty)^\perp$ that are collinear to X but not incident with the line $\langle X, Y\rangle$. Thus $b_1 = s(s-1)$. The parameter $c_1 = 1$. Now suppose that $d(X,Y) = 2$. Since $p(X)$ and $p(Y)$ are adjacent in the point graph of π, then it follows that Y is adjacent to a unique element of the set $p^{-1}(p(X))$, that is $b_2 = 1$. The set of vertices at distance 1 to both X and Y is the set of points $\{X,Y\}^\perp \setminus (\infty)^\perp$ of $\mathcal{S}$. Since $|\{X,Y\}^\perp \cap (\infty)^\perp| = 1$, it follows that $c_2 = s$. Finally, suppose that $d(X,Y) = 3$. If Z is a vertex adjacent to X, then $d(Y,Z) \neq 1$ and $d(Y,Z) \neq 3$, since in the case $d(Y,Z) = 3$ the point Z is antipodal to Y and so must also be antipodal to X. Thus every point adjacent to X in $\overline{\Gamma}$ is at distance 2 to Y, and so $c_3 = s^2 - 1$.

□

5.2 Algebraic covers and GQs

In Theorem 5.1.2 we constructed a geometry $\mathcal{S}$ from an affine plane π of order s and $(\overline{\pi}, p)$, an s-fold cover of π. The geometry $\mathcal{S}$ was a GQ of order s if and only if $\overline{\pi}$ satisfied certain conditions. In this section we consider the case where $(\overline{\pi}, p)$ is an algebraic s-fold cover of π over the abelian group A, defined by the 1-cochain c. We establish the conditions on the 1-cochain c under which $\mathcal{S}$ is a GQ of order s and rewrite Theorem 5.1.2 for the case where $(\overline{\pi}, p)$ is an algebraic s fold cover of π over A. We will then show that every known GQ of order s, with a regular point, can be constructed via an algebraic cover of $\mathrm{AG}(2,q)$.

Let π be an affine plane of order s and $(\overline{\pi}, p)$ and algebraic s-fold cover of π over the abelian group A, defined by the 1-cochain c. The 1-cochain c is said to satisfy the **GQ condition** if

$$\delta c(P,Q,R) = 0 \Longleftrightarrow P,Q,R \text{ are distinct, collinear points of } \pi. \tag{5.2.1}$$

In the following theorem we shall prove that this condition on c is equivalent to the conditions on $(\overline{\pi}, p)$ under which the geometry $\mathcal{S}$ in Theorem 5.1.2 is a GQ of order s.

Note that in 5.2.1 and for the rest of the chapter we have adopted the convention of representing any coboundary operator by δ and allowing the context to determine which coboundary operator it is. For example, in 5.2.1 δ is the coboundary operator δ^1.

Theorem 5.2.1 *Let π be an affine plane of order s and let $(\overline{\pi}, p)$ be an algebraic s-fold cover of π over the abelian group A, defined by the 1-cochain c. Let $\mathcal{S}$ be the incidence structure defined from π and $(\overline{\pi}, p)$ as in Theorem 5.1.2. Then, $\mathcal{S}$ is a GQ if and only if the 1-cochain c satisfies the GQ condition 5.2.1. Further, if $\mathcal{S}$ is a GQ, then the point (∞) is regular.*

Proof: First we observe that since $(\overline{\pi}, p)$ is an algebraic s-fold cover of π and is defined by c, it follows from Section 1.6 that if P,Q,R are distinct collinear points of π, then $\delta c(P,Q,R) = 0$. Thus we are required to show that $\mathcal{S}$ is a GQ if and only if $\delta c(P,Q,R) = 0$ implies that P,Q,R are distinct collinear points of π: equivalently, if P,Q,R is a triple of distinct non-collinear points of π, then $\delta c(P,Q,R) \neq 0$.

Now recall from Theorem 5.1.2 that $\mathcal{S}$ is a GQ if and only if each non-incident point-line pair $(\overline{P}, \ell)$, where $\overline{P}$ is a point of type (ii) and ℓ is a line of type (b) and $p(\overline{P})$ is not incident with $p(\ell)$ in π, satisfies the GQ incidence axiom (axiom (iii)).

Let $(\overline{P}, \ell)$ be such a point-line pair, then $\overline{P} = (P, \alpha)$ for some $P \in \pi$ and $\alpha \in A$ (that is, $P = p(\overline{P})$) such that $P \notin p(\ell)$. If $p^{-1}(p(\ell)) = \{\ell_1, \ell_2, \ldots, \ell_s\}$ and (without loss of generality) $\ell = \ell_1$, then for each $i = 1, 2, \ldots, s$ the point-line pair $((P,\alpha), \ell_i)$ is a non-incident point-line pair, where $\overline{P} = (P, \alpha)$ is a point of type (ii), ℓ_i a line of type (b) and $P \notin p(\ell_i)$. If $p(\ell_1) = \{Q_1, Q_2, \ldots, Q_s\}$, then (P,α) is collinear to exactly one cover of each of the Q_i. In particular, (P, α) is collinear to $(Q_i, \alpha - c(P,Q_i))$ for each $i = 1, 2, \ldots, s$. Thus the s non-incident point-line pairs $((P,\alpha),\ell_1)$, $((P,\alpha),\ell_2)$, $\ldots$, $((P,\alpha),\ell_s)$ each satisfy the GQ incidence axiom if and only if each line ℓ_i is incident with a unique point of the form $(Q_j, \alpha - c(P,Q_j))$. Now each point $(Q_j, \alpha - c(P,Q_j))$ is incident with exactly one line of $p^{-1}(p(\ell_1))$ and so each line ℓ_i is incident with a unique point of the form $(Q_j, \alpha - c(P,Q_j))$ if and only if no line of $p^{-1}(p(\ell_1))$ is incident with two points of $(Q_1, \alpha - c(P,Q_1)), \ldots, (Q_s, \alpha - c(P,Q_s))$. Now if the points $(Q_i, \alpha - c(P,Q_i))$ and $(Q_j, \alpha - c(P,Q_j))$ are collinear, then they span a line of $p^{-1}(p(\ell))$, and so no line of $p^{-1}(p(\ell_1))$ is incident with two points of $(Q_1, \alpha - c(P,Q_1)), \ldots, (Q_s, \alpha - c(P,Q_s))$ if and only if no two points of $(Q_1, \alpha - c(P,Q_1)), \ldots, (Q_s, \alpha - c(P,Q_s))$ are collinear. That is

$$(Q_i, \alpha - c(P,Q_i)) \not\sim (Q_j, \alpha - c(P,Q_j)) \text{ for all } Q_i, Q_j \text{ such that } i,j = 1, \ldots, s \text{ with } i \neq j.$$

Thus the point-line pairs $((P,\alpha),\ell_1), \ldots, ((P,\alpha),\ell_s)$ all satisfy the GQ incidence axiom if and only if

$$\delta c(P,Q_i,Q_j) \neq 0 \text{ for all } Q_i, Q_j \text{ such that } i,j = 1, \ldots, s \text{ with } i \neq j.$$

Hence every non-incident point-line pair $(\overline{P}, \ell)$, where $\overline{P}$ is a point of type (ii) and ℓ is a line of type (b) and $p(\overline{P})$ is not incident with $p(\ell)$ in π, satisfies the GQ incidence axiom if and only if

$$\delta c(P,Q,R) \neq 0 \text{ for all distinct non-collinear triples } P,Q,R \text{ of points of } \pi\ .$$

From this and our observation at the start of the proof, it follows $\mathcal{S}$ is a GQ if and only if the 1-cochain c satisfies the GQ condition 5.2.1.

By Theorem 5.1.2 if $\mathcal{S}$ is a GQ then the point (∞) is regular.

□

Theorem 5.2.1 demonstrates that if c is a 1-cochain on the simplicial complex of the point graph of an affine plane π of order s and c satisfies the GQ condition, then c defines an algebraic

s-fold cover of π. Further, this algebraic s-fold cover gives rise to a GQ $\mathcal{S}$ of order s by the construction in Theorem 5.1.2. In this case the GQ $\mathcal{S}$ is said to be **defined** by c and we say that c **defines** $\mathcal{S}$.

The only known examples of GQs of order s containing a regular point are: $W(q)$ which has all points regular by [49, 3.2.1 and 3.3.1], $T_2(\mathcal{O})$ where $\mathcal{O}$ is an oval of $\mathrm{PG}(2,q)$, q even, which has regular point (∞) by [49, 3.3.2 (i)] (for the construction of $T_2(\mathcal{O})$ see Section 1.4.3) and the dual $T_2(\mathcal{O})^\wedge$ of $T_2(\mathcal{O})$, where $\mathcal{O}$ is not a translation oval of $\mathrm{PG}(2,q)$, which has regular points corresponding to the lines of type (b) of $T_2(\mathcal{O})$ by [49, 3.2.2]. It should be noted that for q even $W(q) \cong Q(4,q)$ ([49, 3.2.2]) and that $Q(4,q) \cong T_2(\mathcal{O})$ where $\mathcal{O}$ is a non-degenerate conic ([49, 3.2.2]. Also, by [49, 12.5.2], the GQ $T_2(\mathcal{O})^\wedge$ is isomorphic to the GQ $T_2(\mathcal{O}')$, for some $\mathcal{O}'$ if and only if $\mathcal{O}$ is a translation oval. In this case, $T_2(\mathcal{O})^\wedge \cong T_2(\mathcal{O})$. If $\mathcal{O}$ is a translation oval, then $T_2(\mathcal{O})$ has $q+2$ regular points, while if $\mathcal{O}$ is not a translation oval, then $T_2(\mathcal{O})$ has a unique regular point (∞), by [49, 12.4.5 (ii)].

In the following two sections we will derive algebraic q-fold covers of $\mathrm{AG}(2,q)$ which give rise to $W(q)$ and $T_2(\mathcal{O})$, respectively, by the construction in Theorem 5.2.1. In the second case the regular point is (∞). In Section 5.6 we will consider the regular points of $T_2(\mathcal{O})^\wedge$ dual to the lines of type (b) of $T_2(\mathcal{O})$ and derive corresponding algebraic covers.

5.2.1 A cover associated with the GQ $W(q)$

Consider $W(q)$ defined as the set of absolute points and absolute lines of a symplectic polarity of $\mathrm{PG}(3,q)$. If P is a point of $W(q)$, then P is a regular point of $W(q)$. The $q+1$ lines of $W(q)$ that contain P form a pencil of lines, through P, on the polar plane π_P of P ([49, 3.1.1(iii)]).

Now let P be a fixed point of $W(q)$. We will determine the affine plane π constructed from $W(q)$ and the regular point P via Theorem 5.0.1, and also calculate the cover of π constructed from $W(q)$, as in Theorem 5.1.1. The lines of π are the points of $P^\perp \setminus \{P\}$, which is the set of points $\pi_P \setminus \{P\}$ of $\mathrm{PG}(3,q)$. The points of π are the sets of points $\{P,X\}^\perp$, where $X \in W(q) \setminus P^\perp$. If π_X is the polar plane of X, then $\{P,X\}^\perp$ is equal to the set of points on the line $\pi_P \cap \pi_X$ (note that $P \notin \pi_X$). Hence, $\pi = (\pi_P \setminus \{P\})^D \cong \mathrm{AG}(2,q)$.

If ℓ is a point of π (and so ℓ is a line of π_P), then the point X of $W(q) \setminus P^\perp$ is a cover of ℓ if and only if $\ell \subset \pi_X$. This is the case if and only if X is incident with the image of ℓ under the symplectic polarity. Thus if ℓ' is the image of ℓ under the symplectic polarity then $P \in \ell'$ and the q covers of ℓ are the points of $\ell' \setminus \{P\}$. If Q is a line of π, and so a point of π_P, then it is covered by the set of lines of $W(q)$, not $\langle P,Q\rangle$, that are contained in the plane π_Q.

Now we introduce coordinates. Let $W(q)$ have the canonical bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0$. If $X = (x_0,x_1,x_2,x_3)$, then π_X has plane coordinates $[-x_1,x_0,-x_3,x_2]$. If $P = (0,1,0,0)$, then $\pi_P = [1,0,0,0]$, π has pointset $\{[0,1,x_1,x_2] \cap \pi_P : x_1,x_2 \in \mathrm{GF}(q)\}$ and the covers of the point $[0,1,x_1,x_2] \cap \pi_P$ of π are the elements of the set $\{(1,\lambda,x_2,-x_1) : \lambda \in \mathrm{GF}(q)\}$. If we denote $[0,1,x_1,x_2] \cap \pi_P$ by (x_1,x_2), then π assumes the canonical form of $\mathrm{AG}(2,q)$. In addition, we denote $(1,\lambda,x_2,-x_1)$ by the pair $((x_1,x_2),\lambda)$, where $(x_1,x_2) \in \mathrm{AG}(2,q)$ and $\lambda \in \mathrm{GF}(q)$. With this notation the covering map (defined as in Theorem 5.1.1) $(1,\lambda,x_2,-x_1) \mapsto [0,1,x_1,x_2] \cap \pi_P$ becomes $((x_1,x_2),\lambda) \mapsto (x_1,x_2)$. If we can find a 1-cochain c (on the simplicial complex of the point graph of $\mathrm{AG}(2,q)$ and over the abelian group $(\mathrm{GF}(q),+)$) such that $((x_1,x_2),\alpha) \sim ((y_1,y_2),\beta)$ if and only if $(x_1,x_2) \sim (y_1,y_2)$ and $c((x_1,x_2),(y_1,y_2)) = \alpha - \beta$, then the q-fold cover of π, defined by $W(q)$ as in Theorem 5.1.1, is an **algebraic** q-fold cover.

Now $(1,x_1,x_2,x_3)$ and $(1,y_1,y_2,y_3)$ are collinear points of $W(q)$ if and only if $y_1 - x_1 + x_2y_3 - x_3y_2 = 0$. So $(1,x_1,x_2,x_3) \sim (1,y_1,y_2,y_3)$ if and only if $\beta - \alpha - x_2y_1 + x_1y_2 = 0$ and hence $((x_1,x_2),\alpha) \sim ((y_1,y_2),\beta)$ if and only if $\alpha - \beta = x_1y_2 - x_2y_1$. Now the map $c((x_1,x_2),(y_1,y_2)) = x_1y_2 - x_2y_1$ is symmetric in (x_1,x_2) and (y_1,y_2), so it follows that c is a 1-cochain and is such that the q-fold cover of π constructed from $W(q)$ in Theorem 5.1.1 is an algebraic cover defined by c. Note that from Theorem 5.2.1 c satisfies the GQ condition 5.2.1. The GQ constructed from $\mathrm{AG}(2,q)$ and c in Theorem 5.2.1 is $W(q)$.

5.2.2 A cover associated with the GQ $T_2(\mathcal{O})$, q even

Let $\Sigma = \mathrm{PG}(3,q)$, q even, and let $H = \mathrm{PG}(2,q)$ be a plane of Σ. Let $\mathcal{O}$ be an oval of H with nucleus N. Let $T_2(\mathcal{O})$ be the GQ, of order q, constructed from Σ, H and $\mathcal{O}$, as in Section 1.4.3.

First we calculate the affine plane π constructed from $T_2(\mathcal{O})$ and its regular point (∞), and the q-fold cover of π defined by $T_2(\mathcal{O})$, as in Theorem 5.1.1. Consider a point X of $T_2(\mathcal{O})$ that is not an element of $(\infty)^\perp$. Then X is a point of type (ii) of $T_2(\mathcal{O})$, that is, a point of $\Sigma \setminus H$. The set $\{(\infty), X\}^\perp$ consists of the points of type (ii) of $T_2(\mathcal{O})$ that are collinear with X. That is, planes of Σ containing X and meeting H in a tangent to $\mathcal{O}$. Thus $\{(\infty), X\}^\perp$ is the pencil of planes of Σ on the line $\langle N, X\rangle$, and this is the point of π that is covered by X. Let ℓ be a line of $T_2(\mathcal{O})$ not incident with (∞). Then ℓ is a line of type (a) of $T_2(\mathcal{O})$: that is, a line of Σ not contained in H and incident with a point of $\mathcal{O}$. The line ℓ is incident with a unique point of $(\infty)^\perp$ and this point is the point of type (ii) $\langle \ell, N\rangle$. Hence $\langle \ell, N\rangle$ is the line of π covered by ℓ.

Thus π is the incidence structure with pointset the pencils of planes of Σ about lines of the form $\langle N, X\rangle$, where $X \in \Sigma \setminus H$; lineset the set of planes of Σ, not including H, that contain N; and incidence inherited from Σ. If we let the point of π that is the pencil of planes about the line $\langle N, X\rangle$, where $X \in \Sigma \setminus H$, be represented by the line $\langle N, X\rangle$, then π is isomorphic to the incidence structure

$$\begin{aligned} \text{Points} &: \text{ Lines } \langle N, X\rangle \text{ with } X \in \Sigma \setminus H \\ \text{Lines} &: \text{ Planes of } \Sigma, \text{ not } H, \text{ containing } N \\ \text{Incidence} &: \text{ Inherited from } \Sigma. \end{aligned}$$

For this description of π, the point $X \in \Sigma \setminus H$ of $T_2(\mathcal{O})$ covers the point $\langle N, X\rangle$ of π, and the line ℓ of type (a) of $T_2(\mathcal{O})$) covers the line $\langle \ell, N\rangle$ of π. Given this form of π we have

$$\pi \cong \frac{\Sigma}{N} \setminus \frac{H}{N} \cong \mathrm{AG}(2,q).$$

We now introduce coordinates. Let H be the plane of $\Sigma = \mathrm{PG}(3,q)$ with equation $x_0 = 0$. We may assume that

$$\mathcal{O} = \{(0,1,t,F(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,0,1,0)\},$$

where F is a polynomial satisfying the conditions of [24, 8.4.2 Theorem]. In this case the nucleus N is the point $(0,0,0,1)$. The affine plane π has pointset $\{\langle N, (1,x_1,x_2,x_3)\rangle : (1,x_1,x_2,x_3) \in \Sigma \setminus H\}$ and the covers of $\langle N, (1,x_1,x_2,x_3)\rangle$ are the points $\langle N, (1,x_1,x_2,x_3)\rangle \setminus \{N\}$. That is, the elements of the set $\{(1,x_1,x_2,\lambda) : \lambda \in \mathrm{GF}(q)\}$. If we denote the point $\langle N, (1,x_1,x_2,x_3)\rangle$ of π by (x_1,x_2), then π assumes the canonical form of $\mathrm{AG}(2,q)$. In addition, we denote the point $(1,x_1,x_2,\lambda)$ of $\Sigma \setminus H$ by the pair $((x_1,x_2),\lambda)$, where (x_1,x_2) is a point of $\mathrm{AG}(2,q)$ and $\lambda \in \mathrm{GF}(q)$. With this notation the covering map onto π given by $\langle N, (1,x_1,x_2,x_3)\rangle \mapsto (1,x_2,x_2,\lambda)$ becomes $((x_1,x_2),\lambda) \mapsto (x_1,x_2)$. If we can find a 1-cochain c (on the simplicial complex of the point graph of $\mathrm{AG}(2,q)$ over the abelian group $(\mathrm{GF}(q),+)$) such that $((x_1,x_2),\alpha) \sim ((y_1,y_2),\beta)$ if and only if $c((x_1,x_2),(y_1,y_2)) = \alpha + \beta$, then the q-fold cover of π defined by $T_2(\mathcal{O})$, as in Theorem 5.1.1, is an **algebraic** q-fold cover.

Now $(1,x_1,x_2,x_3)$ and $(1,y_1,y_2,y_3)$ are collinear points of $T_2(\mathcal{O})$ if and only if $\langle (1,x_1,x_2,x_3),(1,y_1,y_2,y_3)\rangle$ intersects H in a point of $\mathcal{O}$. The point of intersection of the line $\langle (1,x_1,x_2,x_3),(1,y_1,y_2,y_3)\rangle$ with H is the point $(0,x_1+y_1,x_2+y_2,x_3+y_3)$, and this is a point of $\mathcal{O}$ if and only if

$$x_3 + y_3 = \begin{cases} 0 & \text{if } x_1 = y_1 \\ (x_1+y_1)F\left(\frac{x_2+y_2}{x_1+y_1}\right) & \text{if } x_1 \neq y_1 \end{cases}.$$

Now $((x_1,x_2),\alpha) \sim ((y_1,y_2),\beta)$ if and only if $(1,x_1,x_2,x_3) \sim (1,y_1,y_2,y_3)$, which is the case if and only if

$$\alpha + \beta = \begin{cases} 0 & \text{if } x_1 = y_1 \\ (x_1+y_1)F\left(\frac{x_2+y_2}{x_1+y_1}\right) & \text{if } x_1 \neq y_1 \end{cases}.$$

If we define the map

$$c((x_1,x_2),(y_1,y_2)) = \begin{cases} 0 & \text{if } x_1 = y_1 \\ (x_1+y_1)F\left(\frac{x_2+y_2}{x_1+y_1}\right) & \text{if } x_1 \neq y_1 \end{cases},$$

then since c is symmetric in (x_1,x_2) and (y_1,y_2) it is a 1-cochain. Further, $((x_1,x_2),\alpha) \sim ((y_1,y_2),\beta)$ if and only if $c((x_1,x_2),(y_1,y_2)) = \alpha+\beta$. Thus the q-fold cover of π constructed from $T_2(\mathcal{O})$ in Theorem 5.1.1 is an algebraic cover defined by c. Note that Theorem 5.2.1 implies that c satisfies the GQ condition 5.2.1. The GQ constructed from $\mathrm{AG}(2,q)$ and c in Theorem 5.2.1 is $T_2(\mathcal{O})$.

5.2.3 A "geometric" construction of a cover of $T_2(\mathcal{O})$, with regular point (∞)

Let $\mathcal{O}$ be an oval of $\mathrm{PG}(2,q)$, q even, with canonical form

$$\mathcal{O} = \{(1,t,F(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\},$$

where F is a polynomial satisfying the conditions of [24, 8.4.2 Theorem]. In this case, the nucleus of $\mathcal{O}$ is the point $(0,0,1)$.

Let $\mathrm{AG}(2,q)$ be the subgeometry of $\mathrm{PG}(2,q)$ with pointset $\{(x_1,x_2,1) : x_1,x_2 \in \mathrm{GF}(q)\}$, and denote the point $(0,0,1)$ by O. Note that the line at infinity of $\mathrm{AG}(2,q)$ contains the points $(1,0,0)$ and $(0,1,0)$ of $\mathcal{O}$, which correspond to the parallel classes $\{x_2 = d : d \in \mathrm{GF}(q)\}$ and $\{x_1 = d : d \in \mathrm{GF}(q)\}$ of $\mathrm{AG}(2,q)$ respectively. If X and Y are two distinct points of $\mathrm{AG}(2,q)$, then consider the line $\ell_{XY} = \langle O, X-Y\rangle$ of $\mathrm{AG}(2,q)$. If ℓ_{XY} is neither the line given by the equation $x_1 = 0$, nor that given by $x_2 = 0$, then ℓ_{XY} contains a point of $\mathcal{O}$, say Φ_{XY}. Let $X = (x_1,x_2,1)$, $Y = (y_1,y_2,1)$ and define the 1-cochain c as follows:

$$c(X,Y) = \begin{cases} 0 & \text{if } x_1 = y_1 \text{ or } x_2 = y_2 \\ r & \text{where } x_1 \neq y_1,\ x_2 \neq y_2 \text{ and } X - Y = r.\Phi_{XY} \end{cases}.$$

Now, let X, Y and $Z = (z_1,z_2,1)$ be three distinct points of $\mathrm{AG}(2,q)$ and suppose that the sets $\{x_1,y_1,z_1\}$ and $\{x_2,y_2,z_2\}$ both consist of three distinct elements. In this case X, Y and Z are collinear if and only if $\Phi_{XY} = \Phi_{XZ} = \Phi_{YZ}$. Now $\Phi_{XY} = \Phi_{XZ} = \Phi_{YZ}$ if and only if

$$\begin{aligned} O &= (X-Y) - (X-Z) + (Y-Z) \\ &= c(X,Y)\Phi_{XY} - c(X,Z)\Phi_{XZ} + c(Y,Z)\Phi_{YZ} \\ &= \delta c(X,Y,Z)\Phi_{XY}, \end{aligned}$$

which holds if and only if $\delta c(X,Y,Z) = 0$.

Now suppose that $x_1 = y_1$. Then X, Y, Z are collinear if and only if $x_1 = y_1 = z_1$, which holds if and only if $\delta c(X,Y,Z) = 0$. Similarly, in the case where $x_2 = y_2$, X, Y, Z are collinear if and only if $\delta c(X,Y,Z) = 0$.

Thus for X,Y,Z any three distinct points of $\mathrm{AG}(2,q)$, X,Y,Z are collinear if and only if $\delta c(X,Y,Z) = 0$.

Now suppose that $x_1 \neq y_1$, $x_2 \neq y_2$ and that $\Phi_{XY} = (1/F(t), t/F(t), 1)$, for some $t \in \mathrm{GF}(q)$. Then it follows that $X - Y = r.\Phi_{XY}$, where r and t satisfy the equations

$$\begin{aligned} x_1 + y_1 &= \frac{r}{F(t)} \quad \text{and} \\ x_2 + y_2 &= \frac{rt}{F(t)}. \end{aligned}$$

Then it follows that $t = \dfrac{x_2+y_2}{x_1+y_1}$, $r = (x_1+y_1)F\left(\dfrac{x_2+y_2}{x_1+y_1}\right)$ and so

$$c(X,Y) = (x_1+y_1)F\left(\frac{x_2+y_2}{x_1+y_1}\right).$$

If $x_1 = y_1$, or $x_1 = y_2$, then $c(X,Y) = 0$. Thus, c is precisely the 1-cochain that was derived in Section 5.2.2.

5.3 Equivalence of covers of π

5.3.1 Normalised algebraic covers

Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover of π, defined by the 1-cochain c. It was noted in Section 1.6 that if $c' = c + \delta b$ for any 1-coboundary δb and if $(\overline{\pi}', p')$ is the algebraic s-fold cover of π defined by c', then the geometries $\overline{\pi}$ and $\overline{\pi}'$ are isomorphic. Thus, by judiciously choosing the 1-coboundary δb, we may assume a normalised form for the 1-cochain c.

Let O be a fixed point of π. The 1-cochain c is **normalised** with respect to O if $c(O, P) = 0$ for all $P \in \pi \setminus \{0\}$. If $(\overline{\pi}, c)$ is any algebraic s-fold cover of $\overline{\pi}$, then define $c' = c + \delta b$, where $b(P) = c(O, P)$ if $P \neq O$ and $b(O) = 0$. The 1-cochain c' is normalised with respect to O and is called the form of c normalised with respect to O. Suppose that $c + \delta b$ and $c + \delta b'$ are both normalised with respect to the point O. Then $\delta b(O, P) = \delta b'(O, P) = 0$ for all points P of π such that $P \neq O$. Let $b(O) - b'(O) = k$, then it follows that $b(P) - b'(P) = k$ for all points P of π such that $P \neq O$. Thus $\delta b = \delta b'$ and so the normalised form of a 1-cochain with respect to a fixed point of π is unique.

Now, let $\mathcal{L}$ be a fixed parallel class of π and let ℓ be an element of $\mathcal{L}$. Then the simplicial complex Γ_ℓ of the point graph of ℓ is a subcomplex of the simplicial complex Γ of the point graph of π. Thus the 1-cochain c of Γ over the abelian group A induces a 1-cochain on Γ_ℓ over A, c_ℓ say. Since c defines a cover of the geometry π it follows that $\delta c(P, Q, R) = 0$ for P, Q, R any three distinct points of ℓ. Thus c_ℓ is a 1-cocycle and since the point graph of ℓ is complete, it follows from [12, Proposition 2.2] that $H^1(\Gamma_\ell, A)$ is trivial and so c_ℓ is a 1-coboundary, δb_ℓ say. Now let $c' = c + \delta b$ be a 1-cochain on Γ over A such that if $P \in \ell$, where $\ell \in \mathcal{L}$, then $b(P) = b_\ell(P)$. Thus, c' has the property that $c'(P, Q) = 0$ if $\langle P, Q \rangle \in \mathcal{L}$. In this case we say that c' is **normalised** with respect to the parallel class $\mathcal{L}$.

5.3.2 Equivalent algebraic covers of π

Let π be an affine plane of order s and let A be an abelian group of order s. We will represent A as an additive group with identity element 0. Let $(\overline{\pi}, p)$ and $(\overline{\pi}', p')$ be two algebraic s-fold covers of π defined by c and c' respectively, such that c and c' both satisfy the GQ condition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', I')$ be the GQs of order s constructed from $(\overline{\pi}, p)$ and $(\overline{\pi}', p')$, respectively, as in Theorem 5.1.2. Let the point of type (iii) of $\mathcal{S}$ be denoted by (∞) and the point of type (iii) of $\mathcal{S}'$ be denoted by $(\infty)'$. We are naturally interested in the conditions under which $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic. A first step is to determine the conditions under which there is an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$ that maps (∞) to $(\infty)'$. This is the objective of the current section.

Theorem 5.3.1 *Let $i : \mathcal{S} \to \mathcal{S}'$ be an isomorphism such that $i : (\infty) \mapsto (\infty)'$. If $\overline{i}$ is the map obtained by restricting i to the subgeometry $\overline{\pi}$, then $\overline{i} : \overline{\pi} \to \overline{\pi}'$ is an isomorphism.*

Proof: The pointset of $\overline{\pi}$ is $\mathcal{P} \setminus (\infty)^\perp$ and the lineset is $\mathcal{B} \setminus \{\ell : (\infty)\ I\ \ell\}$. Since i maps (∞) to $(\infty)'$, it also maps $\overline{\pi}$ to $\overline{\pi}'$. As $\overline{\pi}$ and $\overline{\pi}'$ are subgeometries of $\mathcal{S}$ and $\mathcal{S}'$ respectively, it follows that i induces an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$.

□

Recall from Section 1.6 that if there exists an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$, then we say c and c' are **equivalent**. If there exists an isomorphism $i : \mathcal{S} \to \mathcal{S}'$, such that $i : (\infty) \mapsto (\infty)'$, we say that the 1-cochains c and c' are **GQ equivalent**.

Our eventual aim is to determine the conditions under which two 1-cochains that satisfy the GQ condition are GQ equivalent. First we show that any isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ may be "completed" to an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$.

Lemma 5.3.2 *Let $\overline{i} : \overline{\pi} \to \overline{\pi}'$ be an isomorphism. Then for each point P of π, $\overline{i} : \{(P, \alpha) : \alpha \in A\} \mapsto \{(P', \alpha) : \alpha \in A\}$, for some point P' of π. Furthermore, the map $T : P \mapsto P'$ defines an automorphism of π.*

Proof: Since $\bar{i}$ is an isomorphism, it induces an isomorphism from the point graph of $\overline{\pi}$ to the point graph of $\overline{\pi}'$. By Theorem 5.1.3 the point graph of $\overline{\pi}$ and the point graph of $\overline{\pi}'$ are both antipodal graphs, with the sets of antipodal points being $\{(P,\alpha) : \alpha \in A\}$ for both graphs. Thus, for P a point of π, $\bar{i} : \{(P,\alpha) : \alpha \in A\} \mapsto \{(P',\alpha) : \alpha \in A\}$, for some point P' of π. Now let ℓ be a line of $\overline{\pi}$ represented by the set $\{(P_1,\alpha_1),(P_2,\alpha_2),\dots,(P_s,\alpha_s)\}$ of points of $\overline{\pi}$ incident with ℓ. Then the image of ℓ under $\bar{i}$ is the set of points $\{(T(P_1),\alpha_1'),(T(P_2),\alpha_2'),\dots,(T(P_s),\alpha_s')\}$ for some $\alpha_i' \in A$, where $i = 1,2,\dots,s$. Since $\{(T(P_1),\alpha_1'),(T(P_2),\alpha_2'),\dots,(T(P_s),\alpha_s')\}$ is a line of $\overline{\pi}$ it follows that $\{P_1',P_2',\dots,P_s'\}$ is a line of π, ℓ' say. Thus, $\bar{i}$ extends to a well defined map on the lines of π, mapping ℓ to ℓ'.

Now from Section 1.6 we know that if the point (P,α) of $\overline{\pi}$ (or $\overline{\pi}'$ respectively) is incident with the line ℓ of $\overline{\pi}$ (or $\overline{\pi}'$ respectively), then the point P of π is incident with the line $p(\ell)$. Further if (P,m) is an incident point-line pair of π, then there exists an incident point-line pair $((P,\alpha),\ell)$ of $\overline{\pi}$ (or $\overline{\pi}'$ respectively) such that $m = p(\ell)$ (or $m = p'(\ell)$ respectively). In fact for each $\alpha \in A$ there is a unique such element l of $p^{-1}(m)$. So suppose that (P,m) is a incident point-line pair of π and $((P,\alpha),\ell)$ is an incident point-line pair of $\overline{\pi}$ such that $m = p(\ell)$. Now $\bar{i}((P,\alpha),\ell) = ((P',\alpha'),\ell')$ for some $\alpha' \in A$ and $((P',\alpha'),\ell')$ is an incident point-line pair of $\overline{\pi}'$, from which it follows that $(P',p'(\ell'))$ is an incident point-pair of π. Now $p(\bar{i}(\ell)) = m'$ (where m' is the image of m under the map on π induced by $\bar{i}$) and so the map on π induced by $\bar{i}$ preserves incidence and is therefore an automorphism of π.

□

Theorem 5.3.3 *Let $\bar{i} : \overline{\pi} \to \overline{\pi}'$ be an isomorphism. Then there exists a unique isomorphism $i : \mathcal{S} \to \mathcal{S}'$, such that $i : (\infty) \mapsto (\infty)'$ and i induces $\bar{i}$, as in Theorem 5.3.1.*

Proof: Let $\bar{i}$ induce the collineation T on π, as in Lemma 5.3.2. Consider the map $i : \mathcal{S} \to \mathcal{S}'$ defined as follows. If ℓ is a point of $\mathcal{S}$ of type (i) (that is, a line of π), then $i(\ell) = T(\ell)$. If (P,α) is a point of $\mathcal{S}$ of type (ii) (that is, a point of $\overline{\pi}$), then $i((P,\alpha)) = \bar{i}((P,\alpha))$. Let $i : (\infty) \mapsto (\infty)'$.

If $\mathcal{L}$ is a line of $\mathcal{S}$ of type (a) (that is, a parallel class of π), then $i(\mathcal{L}) = T(\mathcal{L})$. Finally, if $\bar{\ell}$ is a line of $\mathcal{S}$ of type (b) (that is, a line of $\overline{\pi}$), then $i(\bar{\ell}) = \bar{i}(\bar{\ell})$.

The map i is an isomorphism of the form required. Since T determines the action of i on $(\infty)^\perp$, $\bar{i}$ determines the action of i on $\mathcal{S} \setminus (\infty)^\perp$ and $i : (\infty) \mapsto (\infty)'$, it follows that i is the unique isomorphism from $\mathcal{S}$ to $\mathcal{S}'$, mapping (∞) to $(\infty)'$ that induces $\bar{i}$ for $\overline{\pi}$ to $\overline{\pi}'$. □

By Theorem 5.3.1 and Theorem 5.3.3, we have the following corollary.

Corollary 5.3.4 *Let $(\overline{\pi},p)$ and $(\overline{\pi}',p')$ be two algebraic s-fold covers of π, defined by c and c', respectively, such that c and c' both satisfy the GQ condition. Then c and c' are GQ equivalent if and only if they are equivalent.*

We want to determine when $\mathcal{S} \cong \mathcal{S}'$ by an isomorphism that maps (∞) to $(\infty)'$ and by Corollary 5.3.4 this occurs if and only if c and c' are equivalent. Our aim now is to determine the conditions under which c and c' are equivalent. The first step is to determine the conditions under which there exists an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces a given collineation T of π.

If $\bar{i}$ is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces the collineation T of π, then $\bar{i}$ must act on the points of $\overline{\pi}$ by

$$\bar{i} : (P,\alpha) \mapsto (T(P), t_P(\alpha)) \text{ for } P \in \pi \text{ and } \alpha \in A,$$

where $\{t_P : P \in \pi\}$ is some set of permutations on A.

Now suppose that $\bar{i}$ is a map from the pointset of $\overline{\pi}$ to the pointset of $\overline{\pi}'$, such that $\bar{i} : (P,\alpha) \mapsto (T(P), t_P(\alpha))$, where t_P is an automorphism on A. If we determine the conditions under which this map is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$, then since, by the above, all isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ have this form, we will have determined all isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$. The map $\bar{i}$ is an isomorphism if and only if

$$(P,\alpha) \sim (Q,\beta) \text{ in } \overline{\pi} \iff (T(P), t_P(\alpha)) \sim (T(Q), t_Q(\beta)) \text{ in } \overline{\pi}',$$

for all $P, Q \in \pi$, $P \neq Q$ and $\alpha, \beta \in A$. Now $(P, \alpha) \sim (Q, \beta)$ in $\overline{\pi}$ if and only if $\beta = \alpha - c(P, Q)$, and so $\overline{i}$ is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ if and only if

$$(T(P), t_P(\alpha)) \sim (T(Q), t_Q(\alpha - c(P, Q)) \text{ for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A, \tag{5.3.2}$$

This is the case if and only if

$$c'(T(P), T(Q)) = t_P(\alpha) - t_Q(\alpha - c(P, Q)) \text{ for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A. \tag{5.3.3}$$

We now consider the case where c is normalised with respect to the point O of π and c' is normalised with respect to the point $T(O)$ of π.

For k a constant element of A, the map $(P, \alpha) \mapsto (P, \alpha + k)$ is an automorphism of $\overline{\pi}'$. Thus it follows that there is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces T if and only if there is an isomorphism $\overline{i} : (P, \alpha) \mapsto (T(P), t_P(\alpha))$, such that $t_O(0) = 0$. By 5.3.3 such an isomorphism exists if and only if

$$c'(T(P), T(Q)) = t_P(\alpha) - t_Q(\alpha - c(P, Q)) \text{ for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A. \tag{5.3.4}$$

Substituting $Q = O$ we have that $t_P(\alpha) = t_O(\alpha)$ for all $P \in \pi$ and $\alpha \in A$, and so 5.3.4 is equivalent to

$$c'(T(P), T(Q)) = t_O(\alpha) - t_O(\alpha - c(P, Q)) \text{ for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A. \tag{5.3.5}$$

By substituting $\alpha = 0$ into this expression it follows that

$$\begin{array}{rrcll} & c'(T(P), T(Q)) & = & -t_O(-c(P, Q)) & \text{for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A \\ \iff & -c'(T(P), T(Q)) & = & t_O(c(Q, P)) & \text{for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A \\ \iff & c'(T(Q), T(P)) & = & t_O(c(Q, P)) & \text{for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A \\ \iff & c'(T(P), T(Q)) & = & t_O(c(P, Q)) & \text{for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A. \end{array}$$

Thus with $c'(T(P), T(Q)) = t_O(c(P, Q))$ 5.3.5 is equivalent to

$$t_O(\alpha) = t_O(\alpha - c(P, Q)) + t_O(c(P, Q)) \text{ for all } P, Q \in \pi,\ P \neq Q \text{ and } \alpha \in A. \tag{5.3.6}$$

If we can show that t_O maps onto A, then t_O must be an automorphism of A. Let ℓ be a line of π such that $O \in \ell$ and X a point of π not incident with ℓ. Now if $c(X, Y) = c(X, Y')$, for Y, Y' distinct points of ℓ, then

$$\delta c(X, Y, Y') = c(X, Y) - c(X, Y') + c(Y, Y^p) = c(X, Y) - c(X, Y') = 0. \tag{5.3.7}$$

However, X, Y, Z are not collinear in π and so 5.3.7 contradicts the fact that c satisfies the GQ condition 5.2.1. Thus $\{c(X, Y) : Y \in \ell\} = A$. Hence t_O maps onto A and is an automorphism of A.

Thus we have established that there exists an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces the collineation T on π if and only if

$$c'(T(P), T(Q)) = \sigma(c(P, Q)), \text{ where } \sigma \text{ is an automorphism of } A. \tag{5.3.8}$$

In this case $\overline{i} : (P, \alpha) \mapsto (T(P), \sigma(\alpha))$ is such an isomorphism.

In the following lemma we restate this result and calculate all isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ that induce the collineation T on π.

Lemma 5.3.5 *Let π be an affine plane of order s. Let $(\overline{\pi}, p)$ and $(\overline{\pi}', p')$ be two algebraic s-fold covers of π over the abelian group A, defined by the 1-cochains c and c' respectively, such that c and c' satisfy the GQ condition. Let T be a collineation of π and suppose that c and c' are* *normalised with respect to the points O and $T(O)$ respectively. Then there exists an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces T if and only if there exists a unique automorphism σ of A such that*

$$c'(T(P),T(Q)) = \sigma(c(P,Q)).$$

In this case, the full set of isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ that induce T on π is $\{\bar{i}_k : k \in A\}$, where $\bar{i}_k(P,\alpha) = (T(P),\sigma(\alpha)+k)$.

Proof: The condition for the existence of an isomorphism that induces T comes from 5.3.8, and we are also furnished with an example of such an isomorphism, namely $\bar{i}$ where $\bar{i} : (P,\alpha) \mapsto (T(P),\sigma(\alpha))$. The isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ that induce T on π are exactly those of the form $\tau\bar{i}$, where τ is an automorphism of $\overline{\pi}'$ inducing the identity on π. Thus we now calculate the set of such τ. Since τ induces the identity on π, it is of the form $\tau : (P,\alpha) \mapsto (P,\tau_P(\alpha))$, where τ_P is a permutation on A. Now a map of this form is an automorphism of π if and only if

$$(P,\alpha) \sim (Q,\beta) \Longleftrightarrow (P,t_P(\alpha)) \sim (Q,t_Q(\beta)) \text{ for } P,Q \in \pi,\ P \neq Q \text{ and } \alpha,\beta \in A.$$

This is the case if and only if $\alpha - \beta = \tau_P(\alpha) - \tau_Q(\beta)$ for all $P,Q \in \pi$, $P \neq Q$ and $\alpha,\beta \in A$. Let O be a fixed point of π and let $\tau_O(0) = k$. Then $\tau_P(\alpha) = \alpha + k$, for all points P, not O and for all $\alpha \in A$. To determine the permutation τ_O note that $\tau_O(\alpha) = \alpha - \beta + \tau_P(\beta) = \alpha + k$. Thus we have the full set of isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ that induce T on π.

□

We now generalise to the case where c and c' are any pair of 1-cochains satisfying the GQ condition. Let T be a collineation of π, $c + \delta b$ the form of c normalised with respect to O and $c' + \delta b'$ the form of c' normalised with respect to $T(O)$. Let $\overline{\pi}_b$ be the covering geometry of the algebraic s-fold cover of π defined by $c + \delta b$. The map from $\overline{\pi}$ to $\overline{\pi}_b$ acting by $(P,\alpha) \mapsto (P,\alpha + b(P))$ is an isomorphism. Similarly, let $\overline{\pi}'_{b'}$ be the covering geometry of the algebraic s-fold cover of π defined by $c' + \delta b'$. Then the map from $\overline{\pi}'$ to $\overline{\pi}'_{b'}$ acting by $(P,\alpha) \mapsto (P,\alpha + b'(P))$ is an isomorphism.

Now there is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces T on π if and only if there is an isomorphism from $\overline{\pi}_b$ to $\overline{\pi}'_{b'}$, and by Lemma 5.3.5 this is the case if and only if

$$\begin{aligned}&c'(T(P),T(Q)) = \sigma(c(P,Q)) + \sigma(\delta b(P,Q)) - \delta b'(T(P),T(Q)) \text{ for all } P,Q \in \pi,\ P \neq Q,\\ &\text{for some } \sigma \in aut(A).\end{aligned} \tag{5.3.9}$$

Note that by Lemma 5.3.5 the automorphism σ of A is unique.

We now show that 5.3.9 holds if and only if

$$\begin{aligned}&c'(T(P),T(Q)) = \sigma(c(P,Q)) + \delta d(P,Q) \text{ for all } P,Q \in \pi,\ P \neq Q,\\ &\text{for some } \sigma \in aut(A) \text{ and 1-coboundary } \delta d.\end{aligned} \tag{5.3.10}$$

Note that the map $\sigma(\delta b(P,Q)) - \delta b'(T(P),T(Q))$ is a 1-coboundary. It follows that 5.3.9 satisfies 5.3.10.

Now suppose that δd is a 1-coboundary that acts by $\delta d(P,Q) = c'(T(P),T(Q)) - \sigma(c(P,Q))$, for some $\sigma \in aut(A)$. That is, the 1-cochains c and c' satisfy 5.3.10. Then define $\delta d'$ to be the 1-coboundary that acts by

$$\delta d'(P,Q) = \sigma^{-1}(\delta d(P,Q) + \delta b'(T(P),T(Q))).$$

Thus $\delta d(P,Q) = \sigma(\delta d'(P,Q)) - \delta b'(T(P),T(Q))$ and so

$$\sigma(\delta d'(P,Q)) - \delta b'(T(P),T(Q)) = c'(T(P),T(Q)) - \sigma(c(P,Q)).$$

From this it follows that

$$c'(T(P),T(Q)) + \delta b'(T(P),T(Q)) = \sigma(c(P,Q) + \delta d'(P,Q)).$$

Now $c' + \delta b'$ is normalised with respect to $T(O)$ and so $c + \delta d'$ is normalised with respect to O. Hence $\delta d' = \delta b$. Substituting $\delta d' = \delta b$ into the definition of δd yields

$$\delta d(P,Q) = \sigma(\delta b(P,Q)) - \delta b'(T(P),T(Q)),$$

which when substituted into 5.3.10 yields 5.3.9. Hence we have that 5.3.10 is equivalent 5.3.9 and thus holds if and only if there is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces T on π. Note that we observed that σ is the unique automorphism of A such that 5.3.9 holds, and so since δd is determined by c, c' and σ, it follows that $(\sigma, \delta d)$ is the unique pair such that 5.3.10 holds.

Lemma 5.3.5 and the above discussion give the following result.

Lemma 5.3.6 *Let π be an affine plane of order s. Let $(\overline{\pi}, p)$ and $(\overline{\pi}', p')$ be two algebraic s-fold covers of π over the abelian group A, defined by the 1-cochains c and c' respectively, such that c and c' satisfy the GQ condition. Let T be a collineation of π. Then there exists an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces T if and only if there exists a unique automorphism σ of A and unique 1-coboundary δb, such that*

$$c'(T(P),T(Q)) = \sigma(c(P,Q)) + \delta b(P,Q).$$

In this case, the full set of isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ that induce T on π is $\{\bar{i}_k : k \in A\}$, where $\bar{i}_k(P,\alpha) = (T(P), \sigma(\alpha) + k + d(P))$.

Proof: From the above discussion there is an isomorphism from $\overline{\pi}$ to $\overline{\pi}'$ that induces T on π if and only if there is a unique pair $(\sigma, \delta d)$ with $\sigma \in aut(A)$ and δd a 1-coboundary such that

$$c'(T(P),T(Q)) = \sigma(c(P,Q)) + \delta d(P,Q) \text{ for all } P, Q \in \pi,\ P \neq Q.$$

Now δb normalises c with respect to the point O and $\delta b'$ normalises c' with respect to the point $T(O)$. If $\overline{\pi}_b$ and $\overline{\pi}'_{b'}$ are the covering geometries defined by $c + \delta b$ and $c' + \delta b'$, then by Lemma 5.3.5 the full group of isomorphisms from $\overline{\pi}_b$ to $\overline{\pi}'_{b'}$ that induce T on π is $\{(P,\alpha) \mapsto (T(P), \sigma(\alpha) + k) : k \in A\}$. Since $(P,\alpha) \mapsto (P, \alpha + b(P))$ is an isomorphism from $\overline{\pi}$ to $\overline{\pi}_b$ and $(P,\alpha) \mapsto (P, \alpha - b'(P))$ is an isomorphism from $\overline{\pi}'_{b'}$ to $\overline{\pi}'$, it follows that the full group of isomorphisms from $\overline{\pi}$ to $\overline{\pi}'$ that induce T on π is

$$\{(P,\alpha) \mapsto (T(P), \sigma(\alpha + b(P)) - b'(P) + k) : k \in A\} = \{(P,\alpha) \mapsto (T(P), \sigma(\alpha) + d(P) + k) : k \in A\}.$$

□

Lemma 5.3.6 allows to calculate all of the 1-cochains equivalent to a fixed 1-cochain c (such that c satisfies the GQ condition). In the next section we will use this to calculate the stabiliser of the regular point (∞) in the group of the GQ $\mathcal{S}$ constructed from c (as in Theorem 5.2.1).

Theorem 5.3.7 *Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover of π, over the abelian group A, defined by the 1-cochain c, such that c satisfies the GQ condition. The set of 1-cochains equivalent to c is*

$$\{c' : c'(P,Q) = \sigma(c(T(P),T(Q)) + \delta b(T(P),T(Q)),\ \textit{for all } P,Q \in \pi, P \neq Q$$
$$\textit{where } \sigma \in aut(A) \textit{ and } T \in aut(\pi)\}$$

Proof: This result follows directly from Lemma 5.3.6.

□

If σ is an automorphism of A, then let c_σ be the 1-cochain that acts by $c_\sigma(P,Q) = \sigma(c(P,Q))$, and if T is a collineation of π, then let c_T be the 1-cochain that acts by $c_T(P,Q) = c(T(P),T(Q))$. Thus if c is a 1-cochain and c' is the 1-cochain such that $c'(P,Q) = \sigma(c(T(P),T(Q))) + \delta b(P,Q)$, then $c' = (c_T)_\sigma + \delta b$.

Now by [12, Proposition 2.2], since the point graph of π is complete, it follows that two 1-cochains are cohomologous if and only if their difference is a 1-coboundary. Thus $c' = (c_T)_\sigma + \delta b$ if and only if $\delta c' = \delta(c_T)_\sigma$. Note that in this case $\delta c'(P,Q,R) = \sigma(\delta c_T(P,Q,R))$. Given this we have the following corollary to Theorem 5.3.7.

Corollary 5.3.8 *Let π be an affine plane of order s. Let $(\overline{\pi}, p)$ and $(\overline{\pi}', p')$ be algebraic s-fold covers of π, over the abelian group A, defined by the 1-cochains c and c' respectively, such that c and c' satisfy the GQ condition. Then c and c' are equivalent if and only if there exists an automorphism σ of A, and T collineation of π, such that $\delta c' = \delta(c_T)_\sigma$.*

5.4 The automorphism group of $\overline{\pi}$ and the group of $\mathcal{S}$ fixing (∞)

Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover of π over the abelian group A, defined by the 1-cochain c, such that c satisfies the GQ condition. Let $\mathcal{S}$ be the GQ constructed from $(\overline{\pi}, p)$ as in Theorem 5.1.2. In this section we show that the automorphism group of $\overline{\pi}$ is isomorphic to the subgroup of the automorphism group of $\mathcal{S}$ that fixes (∞). Then we calculate the automorphism group of the geometry $\overline{\pi}$.

If we consider Theorem 5.3.1 and Theorem 5.3.3 for the case where $c = c'$ (so $\overline{\pi} = \overline{\pi}'$ and $\mathcal{S} = \mathcal{S}'$) and $(\infty) = (\infty)'$, then the automorphisms of $\mathcal{S}$ fixing (∞) are "equivalent" to those fixing $\overline{\pi}$. More precisely:

Theorem 5.4.1 *Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover over the abelian group A, defined by the 1-cochain c, such that c satisfies the GQ condition. Let $\mathcal{S}$ be the GQ of order s constructed from $(\overline{\pi}, p)$, as in Theorem 5.1.2. If $aut(\mathcal{S})_{(\infty)}$ denotes the subgroup of the group of $\mathcal{S}$ fixing (∞), and $aut(\overline{\pi})$ denotes the group of $\overline{\pi}$, then*

$$aut(\mathcal{S})_{(\infty)} \cong aut(\overline{\pi}).$$

Proof: Let $\phi : aut(\mathcal{S})_{(\infty)} \to aut(\overline{\pi})$ be the map that takes an element of $aut(\mathcal{S})_{(\infty)}$ to the element of $aut(\overline{\pi})$ it induces as in Theorem 5.3.1 (with $c = c'$). Then ϕ is a homomorphism. By Theorem 5.3.3 ϕ is one-to-one and onto and so is an isomorphism.

□

In the previous section it was shown that if $\overline{i}$ is an automorphism from $\overline{\pi}$ to $\overline{\pi}'$, then $\overline{i}$ induces a collineation of π (Lemma 5.3.2). In the case where $c = c'$, $\overline{i}$ is an automorphism of $\overline{\pi}$ and the collineation of π that is induced by $\overline{i}$ is said to be **admitted** by c. In the following theorem we determine the conditions under which a collineation of π is admitted by c.

Theorem 5.4.2 *Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover of π over the abelian group A, defined by the 1-cochain c, such that c satisfies the GQ condition. Let T be a collineation of π. Then the following statements are equivalent*

1. *The collineation T is admitted by c.*
2. *There exists a unique automorphism σ of A such that*
$$\delta c_T = \delta c_\sigma.$$
3. *There is a unique 1-coboundary, δb and a unique automorphism σ_T of A such that*
$$c_T = c_\sigma + \delta b.$$

Proof: The collineation T is admitted by c if and only if there is an automorphism of $\overline{\pi}$ that induces T. By setting $c = c'$ in Lemma 5.3.6, it follows that 1 is equivalent to 3 By [12, Proposition 2.2], it follows that 2 is equivalent to 3.

□

Note that the set of collineations of π that are admitted by c forms a group, which will be denoted by $aut(\pi)_c$ and is called the subgroup of $aut(\pi)$ that is **admitted** by c.

We now calculate the automorphism group of the geometry $\overline{\pi}$.

Theorem 5.4.3 *Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover of π, over the abelian group A, defined by the 1-cochain c, such that c satisfies the GQ condition. Then the full automorphism group $aut(\overline{\pi})$ of the geometry $\overline{\pi}$ comprises the elements*

$$(P, \alpha) \mapsto (T(P), \sigma_T(\alpha) + k + b_T(P)),$$

where $k \in A$, $T \in aut(\pi)_c$ and $(\sigma_T, \delta b_T)$ is the unique pair, such that $c_T = c_{(\sigma_T)} + \delta b_T$.

Proof: Use $c = c'$ in Lemma 5.3.6.

□

In the proof of Theorem 5.3.3 the unique element of $aut(\mathcal{S})_{(\infty)}$ that induces a given element of $aut(\overline{\pi})$ was explicitly constructed from the element of $aut(\overline{\pi})$. Combining this construction and Theorem 5.4.3 enables the explicit construction of the subgroup of the automorphism group of $\mathcal{S}$ that fixes (∞).

5.4.1 Elations and symmetries about (∞)

Let π be an affine plane of order s and $(\overline{\pi}, p)$ an algebraic s-fold cover over the abelian group A, defined by c, such that c satisfies the GQ condition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be the GQ of order s constructed from π, $\overline{\pi}$ and p as in Theorem 5.1.2. In this section we consider particular types of elements of the group $aut(\mathcal{S})_{(\infty)}$, called elations and symmetries. The terminology and definitions in this section are taken from [49, Chapter 8].

A **whorl** of $\mathcal{S}$ about a point P is an automorphism of $\mathcal{S}$ that fixes each line incident with P. An **elation** of $\mathcal{S}$ about P is a whorl of $\mathcal{S}$ about P, that fixes no point of $\mathcal{P} \setminus P^{\perp}$. If θ is a whorl of $\mathcal{S}$ about (∞), then θ induces an automorphism θ' of π that fixes each parallel class of π. (We can show this by using Theorem 5.3.1 and Lemma 5.3.2, or it is relatively straight-forward to verify directly). If Π is the projective completion of π with line at infinity ℓ_∞, then by [28, Theorem 4.9], the extension of θ' to Π is either an elation of a homology of Π and hence either a translation or a dilatation of π.

A **symmetry** of $\mathcal{S}$ about P is an automorphism of $\mathcal{S}$ that fixes each point of $P^{\perp}$. Any symmetry about P is an elation about P. Now by [49, Chapter 8] the set of symmetries about P forms a group, the order of which must divide s. In the case where the order of the symmetry group is s, the point P is called a **centre of symmetry** (and must be regular). By Theorem 5.4.3, the group of symmetries about (∞) is $\{(P, \alpha) \mapsto (P, \alpha + k) : k \in A\}$ and so (∞) is a centre of symmetry of $\mathcal{S}$. Note that any symmetry about (∞) induces the identity automorphism on π. A **symmetry** about a line ℓ of $\mathcal{S}$ is defined dually (that is, as an automorphism of σ that fixes each line of $\ell^{\perp}$). A line whose symmetry group has maximal order s is an **axis of symmetry**. Suppose that ℓ is a line of $\mathcal{S}$ incident with (∞) and S is a symmetry about ℓ. Now S fixes (∞) linewise and ℓ pointwise, and so S induces a collineation of π that fixes each parallel class of π and fixes the parallel class of π corresponding to ℓ linewise. Thus by [28, Theorem 4.9] S induces a translation on π. Note that this implies that each symmetry about a line incident with (∞) is an elation about (∞), although the converse of this does not necessarily hold. The line ℓ is an axis of symmetry if and only if π has a full translation group, of order s, fixing the parallel class of π corresponding to ℓ linewise, and c admits this group.

The GQ $\mathcal{S}$ is an **Elation Generalized Quadrangle** (EGQ), with basepoint P, if G is a group of elations about P acting regularly on the points of $\mathcal{P} \setminus P^{\perp}$. Since G acts regularly on the points of $\mathcal{P} \setminus P^{\perp}$ it is a maximal group of elations about P.

The GQ $\mathcal{S}$ is a **Translation Generalized Quadrangle** (TGQ), with basepoint P, if $\mathcal{S}$ is an EGQ with basepoint P, such that the elation group G contains a full group of s symmetries about each line through P.

Theorem 5.4.4 *The GQ $\mathcal{S}$ is a TGQ with base point (∞) if and only if π is a translation plane and c admits the translation group of π.*

Proof: Suppose that $\mathcal{S}$ is a TGQ with base point (∞). Recall from the above discussion that if ℓ is a line of $\mathcal{S}$ incident with (∞), then ℓ is an axis of symmetry if and only if π has a full translation group, of order s, fixing the parallel class of π corresponding to ℓ linewise, and c admits this group. Thus if each line incident with (∞) is an axis of symmetry, then π is a translation plane and c admits the full translation group of π.

Now if π is a translation plane and c admits the full translation group G, then by Theorem 5.4.3 the group

$$\{(P,\alpha)\mapsto(T(P),\sigma_T(\alpha)+b_T(P)+k)\,:\,T\in G,\ k\in A \text{ and } c_T=c_{(\sigma_T)}+\delta b_T\}$$

is a group fixing $\overline{\pi}$. By Theorem 5.3.3 this group is isomorphic to a group of $\mathcal{S}$ that is an elation group about (∞) acting regularly on the points of $\mathcal{P}\setminus(\infty)^\perp$. Thus $\mathcal{S}$ is an EGQ with base point (∞).

Let ℓ be a line of $\mathcal{S}$ incident to (∞) and G_ℓ the full group of translations of π fixing the parallel class of π corresponding to ℓ. By Theorem 5.4.3 the group

$$\{(P,\alpha)\mapsto(T(P),\sigma_T(\alpha)+b_T(P)+k)\,:\,T\in G_\ell,\ k\in A \text{ and } c_T=c_{(\sigma_T)}+\delta b_T\}$$

is a group fixing $\overline{\pi}$, which by Theorem 5.3.3 extends to a group of symmetries of $\mathcal{S}$ about ℓ. Thus ℓ is an axis of symmetry. Since this is true for each line incident with (∞), it follows that $\mathcal{S}$ is a TGQ with base point (∞).

□

The discussion of TGQ was introduced for GQ of order s by Thas in [60], and technical aspects were developed by Payne in [42], before it assumed the form given in [49, Chapter 8].

5.4.2 Example: The subgroup of $A\Gamma L(3,q)$ admitted by $x_1y_2-x_2y_1$

Consider the classical affine plane $\mathrm{AG}(2,q)$ with pointset $\{(x_1,x_2)\,:\,x_1,x_2\in\mathrm{GF}(q)\}$ and lines given by the equations $x_2=mx_1+d$ and $x_1=d$, where $m,d\in\mathrm{GF}(q)$. Then

$$\begin{aligned}
aut(\mathrm{AG}(2,q)) &= A\Gamma L(2,q)\\
&= \{T\,:\,T(x_1,x_2)=(a+bx_1^\rho+cx_2^\rho,d+ex_1^\rho+fx_2^\rho),\\
&\qquad \text{such that } a,b,c,d,e,f\in\mathrm{GF}(q) \text{ and } bf-ec\neq 0\},
\end{aligned}$$

which may be derived from the fundamental theorem of projective geometry (see [24, 2.1(ii)]).

From Section 5.2.1 the algebraic q-fold cover of $\mathrm{AG}(2,q)$ giving rise to $W(q)$ is defined by the 1-cochain c, where $c((x_1,x_2),(y_1,y_2))=x_1y_2-x_2y_1$. If $T\in aut(\mathrm{AG}(2,q))$, then

$$\begin{aligned}
c(T(x_1,x_2),T(y_1,y_2)) &= (a+bx_1^\rho+cx_2^\rho)(d+ey_1^\rho+fy_2^\rho)-(d+ex_1^\rho+fx_2^\rho)(a+by_1^\rho+cy_2^\rho)\\
&= (bf-ce)(x_1y_2-x_2y_1)^\rho+(bd-ae)(x_1-y_1)^\rho\\
&\quad +(cd-af)(x_2-y_2)^\rho\\
&= (bf-ce)c(x_1,x_2),(y_1,y_2))^\rho+((bd-ae)x_1^\rho+(cd-af)x_2^\rho)\\
&\quad -((bd-ae)y_1^\rho+(cd-af)y_2^\rho).
\end{aligned}$$

That is, $c_T=c_{(\sigma_T)}+\delta b_T$, where $\sigma_T(x)=(bf-ec)x^\rho$ and $\delta b_T(x_1,x_2)=(bd-ae)x_1^\rho+(cd-af)x_2^\rho$. Thus, by Theorem 5.4.2 we have that

$$aut(\mathrm{AG}(2,q))_c=A\Gamma L(3,q).$$

5.4.3 Example: Covers of $\mathrm{AG}(2,q)$, q even, of the form $(x_1+y_1)F\left(\dfrac{x_2+y_2}{x_1+y_1}\right)$

Let π be an affine plane of order s and $(\overline{\pi},p)$ an algebraic s-fold cover of π over the abelian group A, defined by the general 1-cochain c, (that is, c need not satisfy the GQ condition). Let

T be a collineation of π and σ an automorphism of A. Recall from Section 5.4 that c_T denotes the 1-cochain which acts by $c_T(X,Y) = c(T(X),T(Y))$ and that c_σ denotes the 1-cochain which acts by $c_\sigma(X,Y) = \sigma(c(X,Y))$.

The collineation T is said to be **admitted** by c if there exists an automorphism σ of A and a 1-coboundary δb such that $c_T = c_\sigma + \delta b$. If T is admitted by c, then it follows that $\delta c_T = \delta c_\sigma$ and so $\delta c_T = 0$ if and only if $\delta c = 0$. Note that if c satisfies the GQ condition, then by Theorem 5.4.2 this definition of admitted agrees with that given in Section 5.4.

We will now give an example showing how the definition of T being admitted by a general 1-cochain c may be used to help determine if c satisfies the GQ condition. Let $q = 2^e$, $\pi = \mathrm{AG}(2,q)$, $A = \mathrm{GF}(q)$ and for $X = (x_1, x_2)$, $Y = (y_1, y_2)$ define

$$c(X,Y) = \begin{cases} (x_1+y_1)F\left(\dfrac{x_2+y_2}{x_1+y_1}\right) & \text{if } x_1 \neq y_1 \\ 0 & \text{if } x_1 = y_1, \end{cases}$$

for some function F on $\mathrm{GF}(q)$. We know from Section 5.2.2 and Section 5.2.3 that if F has the property that $\mathcal{O} = \{(1,t,F(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\}$ is an oval of $\mathrm{PG}(2,q)$ with nucleus $(0,0,1)$, then c satisfies the GQ condition. We wish to determine all functions F such that c satisfies the GQ condition.

Let $T_{(a,b)}$ be the collineation of $\mathrm{AG}(2,q)$ that acts by translating the points of $\mathrm{AG}(2,q)$ by (a,b). That is $T_{(a,b)}(x_1,x_2) = (x_1+a, x_2+b)$.

Now given the points $X = (x_1,x_2)$ and $Y = (y_1,y_2)$ of $\mathrm{AG}(2,q)$ such that $x_1 \neq y_1$, we have

$$\begin{aligned} c(T_{(a,b)}(X), T_{(a,b)}(Y)) &= c((x_1+a, x_2+b),(y_1+a,y_2+b)) \\ &= (x_1+y_1+a+a)F\left(\frac{x_2+y_2+b+b}{x_1+y_1+a+a}\right) \\ &= c(X,Y). \end{aligned}$$

If $x_1 = y_1$, then $x_1 + a = y_1 + a$ and so

$$c(T_{(a,b)}(X), T_{(a,b)}(Y)) = 0 = c(X,Y).$$

Thus $T_{(a,b)}$ is admitted by c for all $a, b \in \mathrm{GF}(q)$. Now c satisfies the GQ condition if and only if for X, Y, Z distinct points of $\mathrm{AG}(2,q)$

$$\delta c(X,Y,Z) = 0 \Longleftrightarrow X, Y, Z \text{ are collinear.}$$

Now

$$\begin{aligned} \delta c(X,Y,Z) &= \delta c(T_X(X), T_X(Y), T_X(Z)) \\ &= \delta c(O, T_X(Y), T_X(Z)), \text{ where } O = (0,0), \end{aligned}$$

since T_X is admitted by c. So in this case, the GQ condition on c is equivalent to

$$\delta c(O,X,Y) = 0 \Longleftrightarrow O, X, Y \text{ are collinear.}$$

If O, X, Y are collinear, either $x_1 = y_1 = 0$, in which case $\delta c(O,X,Y) = 0$, or $\dfrac{x_2}{x_1} = \dfrac{y_2}{y_1} = \dfrac{x_2+y_2}{x_1+y_1} = m$, where the line collinear with O, X, Y has slope m. In this case

$$\delta c(O,X,Y) = x_1 F\left(\frac{x_2}{x_1}\right) + y_1 F\left(\frac{y_2}{y_1}\right) + (x_1+y_1)F\left(\frac{x_2+y_2}{x_1+y_1}\right) = 0.$$

Now suppose that O, X, Y are not collinear and that $x_1 = 0$ (so $y_1 \neq 0$). Then

$$\begin{aligned} \delta c(O,X,Y) &= y_1 F\left(\frac{y_2}{y_1}\right) + y_1 F\left(\frac{x_2+y_2}{y_1}\right) \\ &= y_1\left(F\left(\frac{y_2}{y_1}\right) + F\left(\frac{x_2+y_2}{y_1}\right)\right). \end{aligned}$$

Thus, $\delta c(O,X,Y) \neq 0$ if and only if F is a permutation of $\mathrm{GF}(q)$. A similar result holds for $y_1 = 0$ and the case where $x_1 = y_1 \neq 0$.

Now suppose that $x_1 \neq y_1$ and $x_1, y_1 \neq 0$. Then

$$\delta c(O,X,Y) = x_1 F\left(\frac{x_2}{x_1}\right) + y_1 F\left(\frac{y_2}{y_1}\right) + (x_1+y_1) F\left(\frac{x_2+y_2}{x_1+y_1}\right),$$

so $\delta c(O,X,Y) \neq 0$ if and only if

$$\frac{F\left(\frac{x_2}{x_1}\right) + F\left(\frac{x_2+y_2}{x_1+y_1}\right)}{\frac{x_1 y_2 + y_1 x_2}{x_1(x_1+y_1)}} \neq \frac{F\left(\frac{y_2}{y_1}\right) + F\left(\frac{x_2+y_2}{x_1+y_1}\right)}{\frac{x_1 y_2 + y_1 x_2}{y_1(x_1+y_1)}},$$

where $x_1 y_2 + x_2 y_1 \neq 0$, since O, X, Y are non-collinear (this is the 1-cochain derived from $W(q)$ in Section 5.2.1). Now setting

$$u = \frac{x_2}{x_1},\ t = \frac{x_2+y_2}{x_1+y_1},\ s = \frac{y_2}{y_1},$$

we have that

$$\delta c(O,X,Y) \neq 0 \Longleftrightarrow \frac{F(u)+F(t)}{u+t} \neq \frac{F(u)+F(s)}{u+s}.$$

These conditions that we have determined for the function F are exactly those under which $\mathcal{O} = \{(1,t,F(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\}$ is an oval of $\mathrm{PG}(2,q)$ with nucleus $(0,0,1)$. Thus the only 1-cochains of the form $(x_1+y_1)F\left(\frac{x_2+y_2}{x_1+y_1}\right)$ satisfying the GQ condition are precisely those that define $T_2(\mathcal{O})$, with regular point (∞) as derived in Section 5.2.2. (Note that in Section 5.2.2 we supposed that F had the form as in [24, 8.4.2 Theorem]. All that the construction required, however, was that F was such that the set of points $\mathcal{O}$ formed an oval).

5.5 Automorphisms of A and coboundaries associated with collineations of π

In this section we consider 1-cochains that define an s-fold cover of an affine plane π of order s over an abelian group A, and admit a particular subgroup G of $aut(\pi)$. We investigate conditions under which such a 1-cochain satisfies the GQ condition.

Now suppose that π is an affine plane of order s, $(\overline{\pi}, p)$ an algebraic s-fold cover of π, over the abelian group A, defined by the 1-coboundary c such that c satisfies the GQ condition. Recall that if T is a collineation of π, then T is admitted by c if there is a (unique) pair $(\sigma_T, \delta b_T)$, such that

$$c(T(P), T(Q)) = \sigma_T(c(P,Q)) + \delta b_T(P,Q).$$

Let c be normalised with respect to a fixed point O of π. If P and Q are any two points of π such that $O \in \langle P, Q\rangle$, then $c(P,Q) = 0$. Further, if T is a collineation of π admitted by c, then $c(T(P),T(Q)) = \delta b_T(P,Q)$. So c is at least partially determined by the admitted collineations and their associated 1-coboundaries. Now we establish conditions under which we may construct a 1-cochain satisfying the GQ condition from collineations of π and associated 1-coboundaries.

If S and T are collineations of π admitted by c then

$$\begin{aligned}
c(S\circ T(P), S\circ T(Q)) &= \sigma_S(c(T(P),T(Q)) + \delta b_S(T(P),T(Q)) \\
&= \sigma_S(\sigma_T(c(P,Q)) + \delta b_T(P,Q)) + \delta b_S(T(P),T(Q)) \\
&= \sigma_S \circ \sigma_T(c(P,Q)) + \delta b_T(P,Q) + \delta b_S(T(P),T(Q)),
\end{aligned}$$

and hence $\sigma_{S\circ T} = \sigma_S \circ \sigma_T$ and $\delta b_{S\circ T} = (\delta b_T)_\sigma + (\delta b_S)_T$. Note also that the 1-coboundary associated with the identity collineation of π is the zero 1-cochain, and that if $T(O) \in \langle P, Q\rangle$, then $c(P,Q) = \delta_T(T^{-1}(P), T^{-1}(Q))$. We incorporate these ideas into the following theorem.

Theorem 5.5.1 *Let π be an affine plane of order s, O a fixed point of π and A an additive abelian group of order s. Let G be a group of automorphisms of π that acts transitively on the points of π. For each $T \in G$ let δb_T be a 1-coboundary on the simplicial complex of the point graph of π and σ_T an automorphism of A. Suppose that δb_T and σ_T satisfy the following conditions*

1. *If I denotes the identity collineation of π, then $\delta b_I = 0$*
2. *$\sigma_{S\circ T} = \sigma_S \circ \sigma_T$ and $\delta b_{S\circ T} = (\sigma_S)\delta b_T + (\delta b_S)T$, for all $S, T \in G$*
3. *If $S(O), T(O) \in \langle P, Q\rangle$ then $\delta b_T(T^{-1}(P), T^{-1}(Q)) = \delta b_S(S^{-1}(P), S^{-1}(Q))$.*

If we define $c(P,Q) = \delta b_T(T^{-1}(P), T^{-1}(Q))$ for $T \in G$ such that $T(O) \in \langle P, Q\rangle$, then c defines an algebraic s-fold cover of π over A. Further, c is normalised with respect to the point O and admits all elements of the group G with $c_T = c_{(\sigma_T)} + \delta b_T$. Finally, if δb_T satisfies the following condition

4. *If $T(O) \in \langle P, Q\rangle$, then*
$$\delta b_T(T^{-1}(P), T^{-1}(Q)) = 0 \text{ if and only if } O \in \langle P, Q\rangle,$$

then c satisfies the GQ condition.

Proof: Let P and Q be any pair of distinct points of π. Since there is an element T of G such that $T(O) = P$, it follows that $c(P,Q)$ is defined. By condition 3 c is well defined.

Now since $O \in \langle O, P\rangle$, it follows that $c(O,P) = \delta b_I(O,P) = 0$. Hence c is normalised with respect to the point O.

We now wish to show that c admits each element of G, so we consider the 1-cochain c_T for $T \in G$. Let P and Q be two distinct points of π and suppose that $S(O) \in \langle P, Q\rangle$ for some $S \in G$. Then by repeatedly using condition 2 we have the following:

$$\begin{aligned}
c(T(P), T(Q)) &= \delta b_S(S^{-1}\circ T(P), S^{-1}\circ T(Q)) \\
&= \delta b_{(S\circ S^{-1}\circ T)}(P,Q) - \sigma_S\left(\delta b_{(S^{-1}\circ T)}(P,Q)\right) \\
&= \delta b_T(P,Q) - \sigma_S\left(\sigma^{-1}_{(T^{-1}\circ S)}\left(\delta b_{(T^{-1}\circ S)\circ(S^{-1}\circ T)}(P,Q)\right.\right. \\
&\quad \left.\left. -\delta b_{(T^{-1}\circ S)}(S^{-1}\circ T(P), S^{-1}\circ T(Q))\right)\right) \\
&= \delta b_T(P,Q) - \sigma_T\left(-\delta b_{(T^{-1}\circ S)}(S^{-1}\circ T(P), S^{-1}\circ T(Q))\right) \\
&= \delta b_T(P,Q) + \sigma_T(c(P,Q)).
\end{aligned}$$

Thus, $c_T = c_{(\sigma_T)} + \delta b_T$, for all $T \in G$ and so c admits each element of G.

We now show that c defines an algebraic s-fold cover of π over A. Let P, Q, R be three distinct collinear points of π such that P, Q, R are incident with the line ℓ of π. Since G is transitive on the points of π there is an element T of G such that $T(O) \in \ell$. In this case let $T \in G$ be such that $T(O) = P$. Then we have

$$\begin{aligned}
\delta c(P,Q,R) &= c(P,Q) - c(P,R) + c(Q,R) \\
&= \delta b_T(T^{-1}(P), T^{-1}(Q)) - \delta b_T(T^{-1}(P), T^{-1}(R)) + \delta b_T(T^{-1}(Q), T^{-1}(R)) \\
&= 0,
\end{aligned}$$

and so c defines an algebraic s-fold cover of π over A.

Now suppose that the δb_T satisfy condition 4, which is equivalent to $c(P,Q) \neq 0$ if $O \notin \langle P,Q\rangle$. We show that c satisfies the GQ condition. Let P,Q,R be three distinct non-collinear points of π. Let $\ell = \langle P,Q\rangle$ and T be an element of G such that $T(O) = P$. Then

$$\begin{aligned}\delta c(P,Q,R) &= \sigma_T(\delta c(T^{-1}(P),T^{-1}(Q),T^{-1}(R))) \text{ since } T \text{ is admitted by } c\\ &= \sigma_T(\delta c(O,T^{-1}(Q),T^{-1}(R)))\\ &= \sigma_T(c(T^{-1}(Q),T^{-1}(R))),\end{aligned}$$

and so by 4 we have $c(T^{-1}(P),T^{-1}(Q)) \neq 0$. Thus $\delta c(P,Q,R) \neq 0$ and c satisfies the GQ condition.

□

Note that in Theorem 5.5.1 we do not actually need G to be transitive on the pointset of π for c to be defined for all pairs of points of π. In fact we could relax the condition to require that the orbit of G on the pointset of π containing O have a point incident with each line of π and still have c well defined and defining an algebraic s-fold cover of π. However, this would entail strengthening condition 4 to force c to satisfy the GQ condition.

5.6 Covers associated with $T_2(\mathcal{O})^\wedge$

Let $\mathcal{O}$ be an oval of PG$(2,q)$, q even, with nucleus N. In Section 5.2.2 we calculated the algebraic q-fold cover of AG$(2,q)$ arising from $T_2(\mathcal{O})$ and its regular point (∞). From [49, 3.3.2(i)], all lines of type (b) of $T_2(\mathcal{O})$ are regular (lines of type (b) are the points of $\mathcal{O}$), thus in the dual GQ $T_2(\mathcal{O})^\wedge$, the points of type (b) of $T_2(\mathcal{O})$ are regular points. In this section we show that for each of these regular points of $T_2(\mathcal{O})^\wedge$ there is an associated affine plane isomorphic to AG$(2,q)$ and an algebraic q-fold cover of AG$(2,q)$. For each of these $q+1$ algebraic q-fold covers of AG$(2,q)$ we calculate an explicit 1-cochain which defines the cover.

We represent $T_2(\mathcal{O})^\wedge$ by swapping the labels point and line in the construction of $T_2(\mathcal{O})$ (see Section 1.4.3). Thus we will have points of type (a) and (b) and lines of type (i), (ii) and (iii). The regular points we are considering are $q+1$ points of type (b).

Let $T_2(\mathcal{O})$ be constructed from $\Sigma = $ PG$(3,q)$, $H = $ PG$(2,q)$ contained in Σ and the oval $\mathcal{O}$ of H, which has nucleus N. Let X be a point of type (b) in $T_2(\mathcal{O})^\wedge = (\mathcal{P},\mathcal{B},I)$ and π_X the affine plane of order q constructed from $T_2(\mathcal{O})^\wedge$ and X as in Theorem 5.0.1. The lines of $T_2(\mathcal{O})^\wedge$ incident with X are the planes of Σ meeting H in the line $\langle N,X\rangle$ and the line of type (i), which we will denote by $[\infty]$. The points of $X^\perp$ are X, the lines of Σ incident with X but not contained in H, and the points of $\mathcal{O}\setminus\{X\}$.

Let $\ell \in \mathcal{P}\setminus X^\perp$. Then ℓ is a point of type (a) of $T_2(\mathcal{O})^\wedge$ not incident with X. That is, a line of Σ such that $\ell \not\subset H$ and ℓ is incident with a point of $\mathcal{O}\setminus\{X\}^\perp$, Y say. In $T_2(\mathcal{O})^\wedge$, $\{X,\ell\}$ consists of the point Y of type (b) and the q points of type (a) that are the q lines of Σ contained in the plane $\langle \ell, X\rangle$, incident with X but not contained in H. This set, $\{X,\ell\}^\perp$, is a point of π and the q points of $T_2(\mathcal{O})^\wedge$ that cover $\{X,\ell\}$ are the q points of type (a) that are lines of the plane $\langle X,\ell\rangle$, incident with Y but not X.

Let H' be a line of type (ii) of $T_2(\mathcal{O})^\wedge$ such that X is not incident with H'. Then H' is a plane of Σ containing a unique point of $\mathcal{O}\setminus\{X\}$, Z say. Now $Z \in X^\perp$ (since $Z\ I\ [\infty]\ I\ X$) and so Z is the line of π_X covered by H'. The q lines of $T_2(\mathcal{O})^\wedge$ that cover Z (as a point of π_X) are the q lines of type (ii) that are planes of Σ meeting H in the line $\langle N,Z\rangle$.

Let P be a line of type (i) of $T_2(\mathcal{O})^\wedge$ such that $P \notin X^\perp$. That is, P is a point of $\Sigma\setminus H$. The line of π_X that is covered by P is the line $\langle P,X\rangle$, and the q lines of $T_2(\mathcal{O})^\wedge$ covering the line $\langle P,X\rangle$ are the q points $\langle P,X\rangle\setminus\{X\}$ of type (i).

Now a point of π_X is a set of q coplanar lines, so we will identify the point of π_X with this plane. We will identify a line of π_X which is a point of $Y \in \mathcal{O}\setminus\{X\}$ with the line $\langle Y,X\rangle$, and a line of π_X that is a line of Σ, not contained in H and incident with X, will be identified with itself. With these identifications π_X is the incidence structure

Points: Planes of Σ, not H, containing X, but not N.

Lines: Lines of Σ containing X, but not N.

Incidence: Inherited from Σ.

Hence,

$$\pi_X \cong \left(\frac{\Sigma}{X} \setminus \frac{\langle X, N\rangle}{X}\right)^D \cong \mathrm{AG}(2,q).$$

Now we introduce coordinates by letting $\Sigma = \mathrm{PG}(3,q)$, H the plane with equation $x_0 = 0$ and letting $\mathcal{O}$ have the canonical form

$$\mathcal{O} = \{(1, t, F(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\},$$

where F is a polynomial, as in [24, 8.4.2 Theorem]. In this case, the nucleus of $\mathcal{O}$ is the point $(0,0,1)$.

We consider the various possibilities for the point X and construct 1-cochains corresponding to the q-fold covers of $\mathrm{AG}(2,q)$.

5.6.1 Case (a) $X = (0,0,1,0)$

The pointset of π_X is the set of planes of Σ containing $(0,0,1,0)$, not including H. That is, the planes with coordinates $[x_1, x_2, 0, 1]$ with $x_1, x_2 \in \mathrm{GF}(q)$. Note that if we let the point $[x_1, x_2, 0, 1]$ of π_X be represented by the pair (x_1, x_2), then π_X assumes the canonical form of $\mathrm{AG}(2,q)$.

The plane $[x_1, x_2, 0, 1]$ contains X and a second point of Y of $\mathcal{O}$. Now the point $(0,1,t,F(t))$ of $\mathcal{O} \setminus \{X\}$ is contained on the plane $[x_1, x_2, 0, 1]$ of Σ if and only if $t = F^{-1}(x_2)$ (F^{-1} is defined since F is a permutation of $\mathrm{GF}(q)$), and so $Y = (0, 1, F^{-1}(x_2), x_2)$. The covers of the point (x_1, x_2) of $\mathrm{AG}(2,q)$ are the lines of Σ contained in the plane $[x_1, x_2, 0, 1]$ that are incident with Y, but not X. In order to show that the s-fold cover of π_X defined by $T_2(\mathcal{O})^\wedge$ is algebraic we must first represent the covers of $[x_1, x_2, 0, 1]$ as the set $\{((x_1, x_2), \alpha) : \alpha \in \mathrm{GF}(q)\}$ and then define an appropriate 1-cochain. Now the set of covers of (x_1, x_2) may be described as the intersection of the plane $[x_1, x_2, 0, 1]$ with the set of planes $\{[\alpha, F^{-1}(x_2), 1, 0] : \alpha \in \mathrm{GF}(q)\}$. So let $((x_1, x_2), \alpha)$ represent the cover of (x_1, x_2), $[x_1, x_2, 0, 1] \cap [\alpha, F^{-1}(x_2), 1, 0]$.

Consider the points of $T_2(\mathcal{O})^\wedge$ represented by $((x_1, x_2), \alpha)$ and $((y_1, y_2), \beta)$. If $x_2 \neq y_2$, then $((x_1, x_2), \alpha)$ and $((y_1, y_2), \beta)$ are collinear in $T_2(\mathcal{O})^\wedge$ if and only if $[x_1, x_2, 0, 1] \cap [\alpha, F^{-1}(x_2), 1, 0]$ and $[y_2, y_1, 0, 1] \cap [\beta, F^{-1}(y_2), 1, 0]$ are concurrent lines of Σ. This is the case if and only if

$$\alpha + \beta = \frac{x_1 + y_1}{x_2 + y_2}\left(F^{-1}(x_2) + F^{-1}(y_2)\right).$$

If $x_1 = y_1$, then $((x_1, x_2), \alpha)$ and $((y_1, y_2), \beta)$ are incident with the same point of type (b) of $T_2(\mathcal{O})^\wedge$, and are collinear as points of $T_2(\mathcal{O})^\wedge$ if and only if the plane they span contains the point N. This is the case if and only if $\alpha + \beta = 0$.

Thus the 1-cochain

$$c_\infty((x_1, x_2), (y_1, y_2)) = \begin{cases} \dfrac{x_1 + y_1}{x_2 + y_2}\left(F^{-1}(x_2) + F^{-1}(y_2)\right) & \text{for } x_2 \neq -y_2 \\ 0 & \text{for } x_2 = y_2 \end{cases}$$

defines the q-fold cover of $\mathrm{AG}(2,q)$ constructed from $T_2(\mathcal{O})^\wedge$ and the regular point $(0,0,1,0)$.

5.6.2 Case (b) $X = (0,1,t,F(t))$, $t \in \mathrm{GF}(q)$

At this point we note that the only property of F that we used in deriving c_∞ was that $\mathcal{O} = \{(0,1,s,F(s)) : s \in \mathrm{GF}(q)\} \cup \{(0,0,1,0)\}$ is an oval with nucleus $(0,0,0,1)$. Suppose that we can find a collineation T of $H = \mathrm{PG}(2,q)$ that maps $\mathcal{O}$ onto an oval $\mathcal{O}'$, such that $(0,1,t,F(t))$ is mapped to $(0,0,1,0)$ and $(0,0,0,1)$ is fixed. Then $T(\mathcal{O}) = \mathcal{O}' = \{(0,1,u,G(u)) : u \in \mathrm{GF}(q)\} \cup \{(0,0,1,0)\}$, for some polynomial G. Since T induces an isomorphism from $T_2(\mathcal{O})^\wedge$ to $T_2(\mathcal{O}')^\wedge$, it follows from Corollary 5.3.4 that the derivation in Section 5.6.1 gives a 1-cochain equivalent to that defining the q-fold cover given by $T_2(\mathcal{O})^\wedge$ and the regular point $(0,1,t,F(t))$. In what follows we define such collineations and calculate the corresponding 1-cochains. Note that we still assume that F satisfies the conditions of [24, 8.4.2 Theorem].

Now let T be the collineation of H that acts by $T(x_0,x_1,x_2,x_3) = (x_0, x_2 + tx_1, x_1, x_3 + F(t)x_1)$. Then $T(0,0,0,1) = (0,0,0,1)$, $T(0,1,t,F(t)) = (0,1,t,F(t))$ and

$$\begin{aligned}
T(\mathcal{O}) &= \{(0,s+t,1,F(s)+F(t)) : s \in \mathrm{GF}(q)\} \cup \{(0,1,0,0)\} \\
&= \left\{\left(0,1,\frac{1}{s+t},\frac{F(s)+F(t)}{s+t}\right) : s \in \mathrm{GF}(q) \setminus \{t\}\right\} \\
&= \left\{\left(0,1,u,\frac{\frac{1}{u}+F(t)}{\frac{1}{u}}\right) : u \in \mathrm{GF}(q) \setminus \{0\}\right\} \cup \{(0,1,0,0),(0,0,1,0)\} \\
&= \{(0,1,u,f_t(u)) : u \in \mathrm{GF}(q)\} \cup \{(0,0,1,0)\},
\end{aligned}$$

where

$$f_t(u) = \begin{cases} (F\left(\frac{1}{u}+t\right) + F(t))/\frac{1}{u} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}.$$

Now if we define the permutation f on $\mathrm{GF}(q)$ by

$$f(u) = \begin{cases} \frac{1}{u} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases},$$

which may be written as the polynomial $f(x) = x^{q-2}$, then $f_t = F_t \circ f$ where F_t is the permutation $s \mapsto (F(t+s)+F(t))/s$ defined in [24, 8.4.2 Theorem].

Given the above, we have that

$$c_t((x_1,x_2),(y_1,y_2)) = \begin{cases} \dfrac{x_1+y_1}{x_2+y_2}\left(f_t^{-1}(x_2) + f_t^{-1}(y_2)\right) & \text{if } x_2 \neq -y_2 \\ 0 & \text{if } x_2 = y_2 \end{cases},$$

is a 1-cochain that defines the q-fold cover of $\mathrm{AG}(2,q)$ given by $T_2(\mathcal{O})^\wedge$ and the regular point $(0,1,t,F(t))$.

By a similar approach to that taken in Section 5.4.3, it is possible to show that if

$$c((x_1,x_2),(y_1,y_2)) = \begin{cases} \dfrac{x_1+y_1}{x_2+y_2}(G(x_2)+G(y_2)) & \text{if } x_2 \neq y_2 \\ 0 & \text{if } x_2 = y_2 \end{cases}$$

is a 1-cochain of the simplicial complex of the point graph of $\mathrm{AG}(2,q)$, then c satisfies the GQ condition if and only if the set $\{(1,t,G(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\}$ is an oval of $\mathrm{PG}(2,q)$.

5.7 Remarks

The work in this chapter raises many questions and possibilities for future research. An obvious line of enquiry is to investigate 1-cochains of the simplicial complex of the point graph of a non-desarguesian affine plane π of order s. Any such 1-cochain that satisfies the GQ condition gives rise to a GQ of order s that is new, since all of the known GQs of order s with a regular point have associated affine plane $\mathrm{AG}(2,s)$ (as in Theorem 5.0.1) for s a prime power. (This was shown for $W(q)$ in Section 5.2.1. For $T_2(\mathcal{O})$ it was shown in Section 5.2.2 that the affine plane associated with the regular point (∞) is $\mathrm{AG}(2,q)$. For $T_2(\mathcal{O})^\wedge$ the affine plane associated with each of the regular points of type (b) was shown to be $\mathrm{AG}(2,q)$ in Section 5.6. Note that these results are already well known! (See [49, Chapter 8], for example). By [49, 3.3.3 (i)] the only regular points of $T_2(\mathcal{O})^\wedge$ are the points of type (b) and by [49, 12.4.6] if $T_2(\mathcal{O})$ has a regular point not (∞), then $\mathcal{O}$ is a translation oval and so by [49, 12.5.3 (i)] is isomorphic to $T_2(\mathcal{O}')^\wedge$ for some oval $\mathcal{O}'$. Thus the work in Sections 5.2.1, 5.2.2 and 5.6 covers all regular points of known GQ of order s).

By Sections 5.2.1, 5.2.2 and 5.6 and by Theorem 5.3.7 we know all 1-cochains of the simplicial complex of the point graph of $\mathrm{AG}(2,q)$ that give rise to the known GQ of order q. It would be interesting to investigate the 1-cochains to see if others satisfying the GQ condition could be found. A possible approach is to consider a particular group G of $A\Gamma L(3,q)$ and see it it is possible to find a 1-cochain admitting G and satisfying the GQ condition by the construction in Theorem 5.5.1.

It was observed in Section 1.6 that an s-fold cover of a geometry is equivalent to an s-fold cover of the dual geometry. Thus the construction in Theorem 5.1.2 can be equivalently rewritten in terms of an s-fold cover of the dual affine plane. Presumably the algebraic s-fold covers of an affine plane π are equivalent to the algebraic s-fold covers of the dual geometry of π, and so the corresponding theories regarding GQs of order s with a regular point are presumably also equivalent. However, it may be that covering the dual affine plane offers some computational advantages. Note that even though the point graph of a dual affine plane is not complete it is relatively straight-forward to show that the first cohomology group of the corresponding simplicial complex is trivial, and so the cohomological aspects of the theory remain simple.

In Section 5.1 the theory of GQs with a regular point and covers of the associated affine plane was introduced for a general cover rather than an algebraic cover. It would be interesting to try and find other "natural" constructions of covers, apart from the algebraic covers, and investigate those with respect to the construction in Theorem 5.1.2.

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