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A study of buildings of low rank

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Preface

As one can see, the title of this thesis is: 'A study of buildings of low rank'. The theory of buildings is developed in the early '60s by Jacques Tits. The aim was to study various important classes of simple groups, such as the simple algebraic groups, classical groups, groups of mixed type and Frobenius-twisted Chevalley groups in a geometric way.

Why of low rank? Jacques Tits proved two important classifications for certain classes of buildings. One for spherical buildings of rank at least 3 in 1974 ([44]), and one for affine buildings of rank at least 4 in 1986 ([47]). The spherical buildings of rank 2 and affine buildings of rank 3 cannot be classified. However these cases do not lose their importance because of this, because they still have strong geometric properties and have a much richer behaviour.

We have obtained various characterizations and constructions of such buildings of low rank. One can find the results explained in more detail at the beginning of Chapters 2, 3 and 4.

I want to end this preface with some words of thanks. First of all, I would like to thank my supervisor Hendrik Van Maldeghem. He suggested me lots of interesting mathematical problems, while at the same time he gave me the freedom to work on problems I liked. Another person who deserves special thanks is my co-supervisor Koen Thas for his interest in my activities, the many mathematical discussions, reading my manuscripts, and improving my mathematical writing skills. For other non-mathematical things he was a great help and friend too the past years.

I also thank my family, my friends and my colleagues for pleasant times on many occasions. I especially give thanks to my parents for supporting me in what I am doing, and Jeroen Schillewaert, whom I shared an office with during the last three years, for being a great friend and providing a healthy competition in many areas.

Finally, I acknowledge the Fund for Scientific Research - Flanders (FWO - Vlaanderen) for financial support and making this Ph.D. possible.

Contents

Chapter 1

Preliminaries

In this first chapter, we define buildings and additional concepts needed in the later chapters.

1.1 Simplicial complexes

The first thing we will define are simplicial complexes, which is the kind of object buildings are.

1.1.1 Definitions

A *simplicial complex* S on a set X is a set of finite subsets of X such that for each subset $x \in S$ and $y \subset x$, we also have that $y \in S$. We also ask that each singleton of X is in S. The elements of X are called the *vertices*, the elements of S are called simplices. We will always assume that the order of simplices is bounded.

A *maximal simplex* of a simplicial complex S on X, is a simplex of S not contained in a larger simplex. Two maximal simplices of the same order are called *adjacent* if they share a simplex of order one less.

A *type function* of a symplicial complex S on X, is a function t from X to some set I, such that no two different elements which have the same image under t can be in the same simplex. The image under t of an element (set) is called the *type* of that element (set).

A *morphism* from a simplicial complex S on X to a simplicial complex S' on X' is a map ϕ from X to X' which maps simplices in S to simplices in S'. A morphism ϕ is an *isomorphism* if there exists a morphism ϕ' from the simplicial complex S' on X' to the simplicial complex S on X, such that $\phi' \circ \phi$ is the identity on X. An *automorphism* is an isomorphism from a simplicial complex to itself. The automorphisms of a simplicial complex form a group: the *automorphism group* of the simplicial complex.

1.1.2 Chamber complexes

A simplicial complex is a *chamber complex* if for each two maximal simplices C and D there is a sequence $(C_0 = C, C_1, \ldots, C_i = D)$ of maximal simplices, such that each two subsequent maximal simplices are adjacent. In this case the maximal simplices are called *chambers*. Note that this implies that all the chambers have the same order. The simplices of order one less than the chambers are called *panels*.

A chamber complex is *thin* if each panel is in exactly two chambers. It is *thick* if each panel lies in at least 3 chambers.

1.1.3 Convexity

A *gallery* in a chamber complex is a sequence of chambers $(C_0, C_1, \ldots, C_i)$, such that each two subsequent chambers share at least a panel. The *length* of a gallery is the number of chambers in the sequence minus one. The *distance* between two chambers is the minimal length of a gallery between the two chambers.

The *product* $\mathrm{proj}_B A$ of a simplex A with a simplex B (the order of the simplices matter, so $\mathrm{proj}_B A$ is not equal $\mathrm{proj}_A B$), is the intersection of all the last chambers in galleries of minimal length, starting with a chamber containing A, and ending with a chamber containing B. (The minimal length considered here is the minimal length over all such possible chambers.)

A sub simplicial complex S' of a simplicial complex S is *convex*, if for every two simplices A and B in S', the product of A with B is again in S'.

Remark 1.1.1 The notion of product (which can be found in [1]) is also known as the 'projection' of the simplex A on B. However, we will not use this since it can lead to confusion with the notion of projection for generalized polygons (see Section 1.6.2).

1.2 Geometries

A *pre incidence geometry* is a tuple $(X, \Delta, \mathrm{tp}, \mathtt{I})$, where X is called the set of *elements*, Δ the set of *types*, tp is a surjective map from X to Δ, and $\mathtt{I}$ the *incidence relation*, consisting of (unordered) pairs of elements in X such that no such pair has the same image under the type function.

The function tp is called the *type function*. The *type* of an element is its image under the type function. Two elements are called *incident* if the pair they define is an element of the incidence relation (instead of $\{x, y\} \in \mathtt{I}$, we will use the notation $x\mathtt{I}y$). A *flag* is a set of elements such that each two (different) elements in the set are incident. It is easily seen that the set of all flags forms a simplicial complex (called the *flag complex*) with a type function on the set of elements. The *rank* of a pre incidence geometry is the order of the set of types.

The *type* $\mathrm{tp}(F)$ of a flag F is the set of types of its elements. A pre incidence geometry is an *incidence geometry* if each maximal flag has type Δ. A *residue* of a flag F is the geometry obtained by restricting the elements to those distinct of F and incident with all elements of F.

A morphism (ϕ, ψ) of one incidence geometry $(X, \Delta, \mathrm{tp}, , \mathtt{I})$ to another $(X', \Delta', \mathrm{tp}', \mathtt{I}')$ consists of two maps $\phi : X \to X'$ and $\psi : \Delta \to \Delta'$ such that for all $x, y \in X$ it holds that $\mathrm{tp}'(\phi(x)) = \psi(\mathrm{tp}(x))$ and $x\mathtt{I}y \Rightarrow \phi(x)\mathtt{I}'\phi(y)$. Isomorphisms and automorphisms are then defined in the usual way.

In most cases we will give the different types specific names - such as: points, blocks, lines, planes, circles, spheres ... In addition we will adopt common linguistic expressions such as *points lie on a line, lines go through points* to describe incidence. Points on a block (or line) will be called *collinear*, blocks (or lines) through a point *concurrent*. If two elements are collinear or concurrent, then we say they are *adjacent*. If for two adjacent elements x and y there exists a unique z such that $x\mathtt{I}z\mathtt{I}y$, then we will denote z by xy.

Further elaborating this point of view, one often denotes a rank 2 incidence geometry as $(\mathcal{P}, \mathcal{L}, \mathtt{I})$, where $\mathcal{P}$ (called the points) together with $\mathcal{L}$ (called lines, blocks ...) form the elements, subdivided by type.

1.3 Coxeter complexes

1.3.1 Coxeter matrices, groups, systems and diagrams

A *Coxeter matrix* is an $n \times n$-matrix M such that $m_{ii} = 1$ for $i \in \{1, \dots, n\}$ and $m_{ij} = m_{ji} \in \{2, 3, \dots, +\infty\}$ for $i, j \in \{1, \dots, n\}$ and $i \neq j$.

The *Coxeter group* arising from this matrix M is the group W with generators $S = \{s_1, \dots, s_n\}$ and relations $(s_i s_j)^{m_{ij}} = e$, with e the identity element of W. The *Coxeter system* is the group together with the set of generators: (W, S). Note that the elements in S are involutions.

Remark 1.3.1 It is possible that Coxeter systems with a different number of generators still give rise to isomorphic Coxeter groups. The following two Coxeter matrices are examples of this:

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}. \tag{1.1}$$

Most often, instead of using a Coxeter matrix to define things, one uses a *Coxeter diagram*. This diagram consists of n vertices, one for each generator in S. If for two different generators s_i and s_j it holds that $m_{ij} = 2$, then there is nothing drawn between the associated vertices; if $m_{ij} = 3$, then one draws a single edge, if $m_{ij} = 4$, a double edge. If $m_{ij} > 4$ one draws an edge and labels it with m_{ij}.

The Coxeter system is *irreducible* if this diagram is connected, and *reducible* if it is not. We will always assume that a Coxeter system is irreducible. The reducible cases can be viewed as direct products of irreducible cases.

Remark 1.3.2 In the literature triple edges are sometimes used for $m_{ij} = 5$, but also sometimes for $m_{ij} = 6$ (in the context of Lie algebras). In order to avoid confusion, we will not use triple edges.

1.3.2 Coxeter complexes

Let J be a subset of $\{1, \dots, n\}$; the generators s_j with $j \in J$ generate a sub Coxeter group W_J. One now obtains a simplicial complex (called the *Coxeter complex* modeled on (W, S)) in the following way.

- The set of vertices consists of all left cosets of sub Coxeter groups W_J with $|J| = |S| - 1$.
- The set of simplices consists of all left cosets of sub Coxeter groups W_J with $J \subseteq \{1, \ldots, n\}$. A vertex lies in a simplex if the coset associated with the simplex is a subset of the coset associated with the vertex.

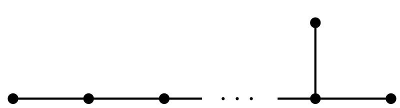
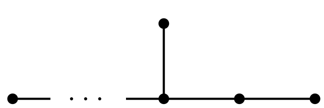
The Coxeter complex forms a thin chamber complex with as chambers the left cosets of the trivial subgroup $\{e\}$. So the chambers correspond to the elements of W. The group W (with left action) forms an automorphism group of this Coxeter complex.

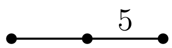
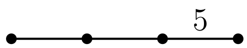
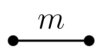
Spherical Coxeter complexes

A *spherical Coxeter complex* is a Coxeter complex which is finite. If this is the case, the associated Coxeter group W can be realized as a *finite reflection group* of a real vector space V, which is a finite group generated by reflections defined by hyperplanes of the vector space of dimension $|S|$ (a hyperplane of a vector space contains the zero vector by definition). In addition, the generators S of the Coxeter group W will correspond to the generating reflections of the finite reflection group.

The hyperplanes corresponding to the generators in S and their conjugates in W, will subdivide V in cones corresponding to the chambers of the Coxeter complex (see Section 1.8.2 for more details). If we consider the intersection of these cones with the unit sphere in V, one gets a tesselation of the sphere, whence the name 'spherical Coxeter complex'.

The (irreducible) spherical Coxeter diagrams corresponding to spherical Coxeter complexes have been classified:

- A_n: $(n \geq 1)$
- C_n: $(n \geq 2)$
- D_n:  $(n \geq 4)$
- E_n:  $(n = 6, 7, 8)$

- F_4:
- H_3: 
- H_4: 
- $\mathsf{I}_2(m)$:  $(m \geq 5)$

The subscript n denotes the number of nodes in the diagram. The case C_n is sometimes denoted as B_n, the case $\mathsf{I}_2(6)$ often as G_2. This difference in notation stems from the theory of (crystallographic) root systems, where these different notations correspond to essentially different (crystallographic) root systems. However, the Coxeter systems defined by the root systems do not exhibit this difference.

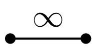
An important notion for spherical Coxeter complexes is opposition. Let (W, S) be a spherical Coxeter system. The finite group W has a unique 'longest' group element w_0 (longest in terms of shortest representation as word with letters the generators S). This element is an involution and is called the *opposition involution*. The induced action as an automorphism of the corresponding spherical Coxeter simplex can be interpreted as the point reflection across the centre of the sphere formed by the complex. Two simplices of a spherical Coxeter are said to be *opposite* if they are interchanged by the opposition involution.

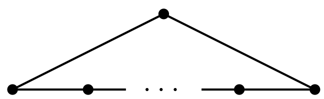
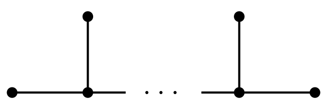
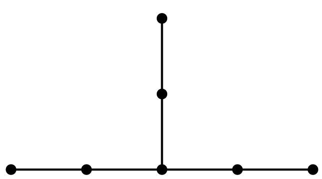
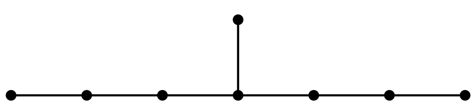
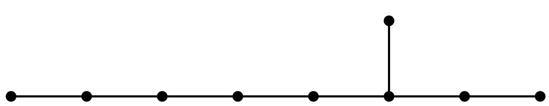
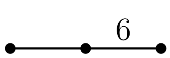
Affine Coxeter complexes

A second interesting class of Coxeter complexes are the *affine Coxeter complexes*. These are not finite, but the associated Coxeter group contains a normal abelian subgroup such that the corresponding quotient group is finite.

The Coxeter group W associated to the affine Coxeter complex can again be realized as a group acting on a real affine space of dimension $|S| - 1$ generated by reflections, but this time not all the associated hyperplanes share the same point. Because of this we now get a tesselation of the affine space instead. The normal abelian subgroup of which we spoke in the previous paragraph is formed by the elements of W corresponding to translations of the affine space.

The (affine) Coxeter diagrams corresponding to affine Coxeter complexes also have been classified:

- $\widetilde{\mathsf{A}}_1$: 

- $\widetilde{\mathsf{A}}_n$:  $(n \geq 2)$
- $\widetilde{\mathsf{B}}_n$:  $(n \geq 3)$
- $\widetilde{\mathsf{C}}_n$: $(n \geq 2)$
- $\widetilde{\mathsf{D}}_n$:  $(n \geq 4)$
- $\widetilde{\mathsf{E}}_6$: 
- $\widetilde{\mathsf{E}}_7$: 
- $\widetilde{\mathsf{E}}_8$: 
- $\widetilde{\mathsf{F}}_4$:
- $\widetilde{\mathsf{G}}_2$: 

The subscript n denotes the number of nodes minus one.

1.3.3 Adjacency and roots

Suppose we have again a Coxeter complex modeled on (W, S), and that we have two chambers C and D sharing a panel. These two chambers correspond to two elements g_C and g_D in W. As they share a panel they are in the left coset of a subgroup $\{e, s_i\}$

for some i. So $g_D = g_C s_i$. We then say that these two chambers are *i-adjacent*. The involutory automorphism $g_C s_i g_C{}^{-1}$ maps the chambers C and D to each other.

Now consider the set R of all chambers for which the distance to the chamber C is strictly less than the distance to D. Analogously define R' as the set of chambers closer to D than to C. These two sets partition the set of chambers in the Coxeter complex. The union of all the chambers in such a set forms a convex subcomplex of the Coxeter complex, which we shall call a *root*. Note that $g_C s_i g_C{}^{-1}$ maps the roots to each other. The simplicial subcomplex fixed by this mapping is called the *wall* of the root.

1.4 Buildings

A *weak building* is a simplicial complex Λ, with a set A of subcomplexes called *apartments*, such that:

(B0) Each apartment is a Coxeter complex.

(B1) Each two simplices of Λ are contained in an apartment.

(B2) If two apartments Σ and Σ' share two simplices A and B, then there exists an isomorphism from Σ to Σ' fixing the vertices in A and B.

A weak building is a chamber complex; if it is thick, we call it a *building*. We will always assume that the Coxeter complex is irreducible - the reducible cases can be thought of as direct products of irreducible buildings. The *roots* of the building are the roots of its apartments. The type of the Coxeter complexes formed by the apartments, will be called the *type* of the building.

One can prove that (weak) buildings are flag complexes of (unique) geometries of rank $|S|$. The types of the elements of this geometry (or equivalently the vertices of the simplicial complex) correspond to the nodes of the diagram.

Remark 1.4.1 If we only would want to define buildings, then due to the thickness condition one can significantly weaken condition (B0) and only ask that the apartments are thin chamber complexes.

The notions *morphism*, *isomorphism* and *automorphism* for buildings are the same for the associated simplicial complex, but with the added condition that apartments are mapped to apartments.

1.5 Galleries in buildings

In Section 1.1.3 we defined galleries in chamber complexes. Since buildings are chamber complexes, galleries of chambers in buildings are also defined. Note that because of Axioms [B1] and [B2] of buildings, the notion of *i-adjacency* can be extended to chambers in the building, and sharing a panel.

Combining this, one can associate a word with letters the generators S of the Coxeter system (W, S), by concatenating for each two subsequent chambers in the galleries the generator s_i, if those two chambers are i-adjacent. This word can also be interpreted as a group element of W.

The following lemma is well-known in the theory of buildings (see for example [28, p. 28]):

Lemma 1.5.1 *A gallery between two chambers has the shortest length possible between those two chambers, if and only if the associated word has no shorter representation in the Coxeter group W.*

Also one can prove that this word viewed as group element of W does not depend on which gallery between the two chambers is considered. This provides some sort of distance function between chambers, the *Weyl distance*.

Remark 1.5.2 There is another way to define buildings, where one of the axioms is exactly the above lemma. In fact, [28] uses this approach, and then shows equivalence with the definition we used.

1.6 Some interesting cases

There are many types of buildings, in this section we look at some interesting cases.

1.6.1 Rank one

Here the building is just a set, the chambers are the elements, and the apartments all the pairs of elements. On its own this is not an interesting case, but it becomes interesting and useful if we add some Moufang-like condition, see Section 1.8.1.

1.6.2 Rank two

Here the Coxeter group is a (finite or infinite) dihedral group of order $2m_{12}$.

First suppose we are in the finite case, and that we have a dihedral group of order $2n$. Then the Coxeter complexes are flag complexes of ordinary n-gons. The buildings are the flag complexes of geometries called 'generalized polygons' (we will often omit 'generalized' if the context is clear).

A *generalized n-gon* ($n \in \mathbb{N}$, $n \geq 2$) $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a rank 2 geometry consisting of a *point set* $\mathcal{P}$, a *line set* $\mathcal{L}$ (with $\mathcal{P} \cap \mathcal{L} = \emptyset$), and *incidence relation* $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$ satisfying the following axioms.

(GP1) Every element is incident with at least three other elements.

(GP2) For every pair of elements $x, y \in \mathcal{P} \cup \mathcal{L}$, there exists a sequence $x_0 = x, x_1, \ldots, x_{k-1}$, $x_k = y$, with $x_{i-1} \mathtt{I} x_i$ for $1 \leq i \leq k$ and with $k \leq n$.

(GP3) The sequence in (GP2) is unique whenever $k < n$.

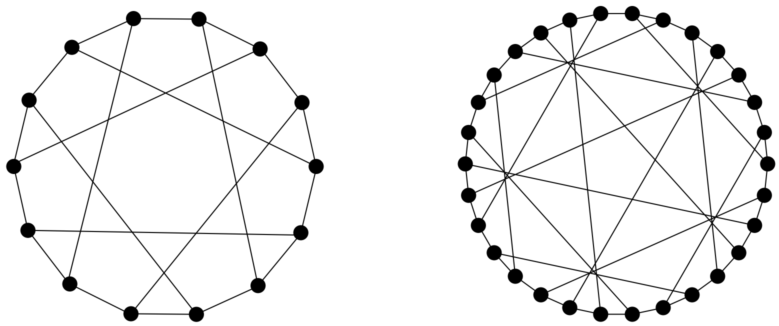
Note that this definition is self-dual; it is invariant under interchanging the notions point and line. If we weaken Axiom (GP1) to 'at least two other elements', then we call the geometry a *weak generalized n-gon*.

A *path* of a generalized polygon is a sequence of elements,such that each two subsequent elements are incident. The *length* of such a path and the *distance* d between two elements (not chambers) are now defined in a similar fashion as for galleries in Section 1.1.3. A path is *closed* if the last element of the sequence equals the first, and is *nonstammering* if each for each element of the sequence, the two neighbours are different.

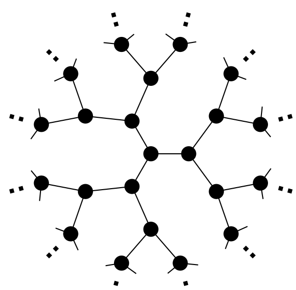
Two elements at maximal distance n are said to be *opposite*. If two elements are not opposite, then the unique element incident with y closest to x is the *projection of x on y*.

The apartments correspond to the nonstammering closed paths of length $2n$, i.e. the ordinary n-gons in the geometry. The stammering closed paths of length $2n$ will be called *degenerate apartments*.

A generalized 3-gon is the same as a projective plane. Below are the smallest building of type A_2 (the flag complex of the projective plane $\mathsf{PG}(2,2)$), and the smallest building of type B_2 (the flag complex of the symplectic quadrangle 2).



Now suppose we are in the infinite case. In this case the Coxeter diagram of the building is of type $\widetilde{\mathsf{A}}_1$, and the buildings are the trees without endpoints, and such that each vertex has at least three neighbours. The smallest such building is shown below.



Collineations and dualities

We now take a closer look at the automorphisms of the spherical rank 2 buildings and the corresponding generalized polygons. These break down in two classes, *collineations*, which map points to points and lines to lines, and *dualities*, which map points to lines and lines to points, both preserving incidence.

A duality from a polygon to itself of order 2 is called a *polarity*. An *absolute* element of a polarity of a generalized polygon is an element incident with the image of that element under the polarity.

The set of absolute points of a polarity of a $2n$-gon forms an *ovoid* of the $2n$-gon, which is a set $\mathcal{O}$ of mutually opposite points, such that every element of the $2n$-gon lies at distance at most n from a certain element of $\mathcal{O}$.

1.6.3 Spherical buildings

Spherical buildings are buildings with spherical (and so finite) Coxeter complexes. Let Λ be a such a building and α a root of it. The *root group* U_α is the set of all automorphisms of the building which fixes α and all chambers sharing a panel with two different chambers of α. One says that the spherical building is *Moufang* if for each root α, the group U_α acts transitively on the set of apartments containing α. Furthermore it can be shown that, if this is the case, then the group U_α acts sharply transitively on the set of these apartments.

Now suppose the rank of the building is 3 or greater; then J. Tits proved in [44] that it (which are pure geometric objects) satisfies the Moufang condition, and that it can be classified. Roughly speaking such buildings correspond to three types of groups - classical, algebraic and mixed groups. This is perhaps the most important result in the theory of buildings.

Spherical Moufang buildings of rank 2, i.e. generalized n-gons, only occur for $n = 3, 4, 6$ and 8 (see [45], [46] and [64]). A consequence of this is that no buildings of type H_3 or H_4 exist, as they would lead to the existence of Moufang generalized 5-gons.

Remark 1.6.1 The Moufang property can be defined for all types of buildings, but it is omitted here as we will only need it in the spherical case (where the definition is less elaborate).

Opposition and subapartments

We define 2 simplices of a spherical building to be *opposite* if they are opposite in an apartment (a spherical Coxeter complex) which contains them both. Existence of such an apartment is implied by (B1), the independence of which apartment is chosen by (B2).

1.6.4 Affine buildings

Affine buildings are the buildings with affine Coxeter complexes. The more general concept of affine apartment system will be discussed later on.

1.7 Residues of buildings

In Section 1.2 we already discussed residues of flags of geometries. As buildings are flag complexes of geometries, residues of simplices of buildings also make sense. These residues are again buildings, where the corresponding Coxeter diagram will be the diagram of the original building with the nodes corresponding to the elements of the flag (equivalently the vertices of the simplex) erased.

1.8 Related objects

1.8.1 Moufang sets

As we have seen above, buildings of rank one are trivial structures. But by adding a Moufang-like condition these become very interesting. Many examples arise from higher rank buildings.

Let X be a set (with $|X| \geq 3$), with for each $x \in X$ a group U_x (we call the *root groups*) acting on X while fixing x. Then $(X, (U_x)_{x \in X})$ is a *Moufang set* if the following two conditions are met:

- For every $x \in X$, U_x acts regularly on $X \backslash \{x\}$.
- The set of all root groups is normalized by the group $G^\dagger$ generated by all the root groups.

The group $G^\dagger$ is called the *little projective group*, and is obviously 2-transitive. If it is sharply 2-transitive, we say the Moufang set is *improper*, otherwise we call it *proper*. The *full projective group* is the group of all elements of $\mathrm{Sym}(X)$ that leave the set of root groups invariant.

Geometries defined by Moufang sets

Let $(X, (U_x)_{x \in X})$ be a Moufang set as above. For a certain $x \in X$, let V_x be a nontrivial subgroup of U_x such that V_x is a normal subgroup of the stabilizer $G^\dagger_x$. For any $y \in X$, we can now define a similar subgroup $V_y = V_x^g \trianglelefteq G^\dagger_y$, with $g \in G^\dagger$ such that $x^g = y$ (this is possible by 2-transitivity). The condition on V_x makes it so that V_y is independent of the choice of g.

The *Moufang building of rank one defined on* X *by* $(U_x)_{x \in X}$ *relative to* $(V_x)_{x \in X}$ is the rank 2 geometry $(X, \Lambda, \in \text{ or } \ni)$ with as points the elements of X, and blocks Λ the subsets of X of the form $\{x\} \cup \{y^v | v \in V_x\}$. The element x of such a block is called the *gnarl* of the block (in the cases we will consider, the gnarl of a block will turn out to be unique).

It is clear that the little projective group will induce automorphisms of this geometry $(X, \Lambda, \in \text{ or } \ni)$. If one shows that all the automorphisms of (X, Λ) arise in this way, then the study of the Moufang set will be equivalent with the study of the geometry (this idea has been proposed by Tits in [49] and [50], see also [61]). Such results are obtained in Chapter 2 for the Ree-Tits Moufang set.

Good candidates for the choice of V_x are the centers and derived groups of the root groups.

1.8.2 $\mathbb{R}$-Buildings

Definitions

Let $(\overline{W}, S)$ be a spherical irreducible Coxeter system. So $\overline{W}$ is presented by the set S of involutions subject to the relations which specify the order of the products of every pair of involutions (see Section 1.3.2). This group has a natural action on a real vector space V of dimension $|S|$. Let $\mathbf{A}$ be the affine space associated to V, which we call the *model space*. We define W to be the group generated by $\overline{W}$ and the translations of the model space.

Let $\mathcal{H}_0$ be the set of hyperplanes of V corresponding to the axes of the reflections in S and all their conjugates. Let $\mathcal{H}$ be the set of all translates of all elements of $\mathcal{H}_0$. The elements of $\mathcal{H}$ are called *walls* and the (closed) half spaces they bound are called *half-apartments* or *roots*. A *vector sector* is the intersection of all roots that (1) are bounded by elements of $\mathcal{H}_0$, and (2) contain a given point x that does not belong to any element of $\mathcal{H}_0$. The bounding walls of these roots will be referred to as the *side-walls* of the vector sector. A vector sector can also be defined as the topological closure of a connected component of $V \setminus (\cup \mathcal{H}_0)$. Any translate of a vector sector is a *sector*, with corresponding translated *side-walls*. A *sector-facet* is an intersection of a given sector with a finite number of its side-walls. The latter number can be zero, in which case the sector-facet is the sector itself; if this number is one, then we call the sector-facet a *sector-panel*. The intersection of a sector with all its side-walls is a point which is called the *source* of the sector, and of every sector-facet defined from it. This source is unique due to the irreducibility of the Coxeter system.

An $\mathbb{R}$-*building* of type (W, S) (also called an *affine apartment system*) (definition by Jacques Tits as can be found in [28] by Mark Ronan, along with some historic background) is an object $(\Lambda, \mathcal{F})$ consisting of a set Λ together with a collection $\mathcal{F}$ of injections of $\mathbf{A}$ into Λ called *charts* obeying the five conditions below. The image of $\mathbf{A}$ under a chart $f \in \mathcal{F}$ will be called an *apartment*, and the image of a sector, half-apartment, ... of $\mathbf{A}$ under a certain $f \in \mathcal{F}$ will be called a *sector*, *half-apartment*, ... of Λ.

(A1) If $w \in W$ and $f \in \mathcal{F}$, then $f \circ w \in \mathcal{F}$.

(A2) If $f, f' \in \mathcal{F}$, then $X = f^{-1}(f'(\mathbf{A}))$ is closed and convex in $\mathbf{A}$, and $f|_X = f' \circ w|_X$ for some $w \in W$.

(A3) Any two points of Λ lie in a common apartment.

The last two axioms allow us to define a function $\mathsf{d} : \Lambda \times \Lambda \to \mathbb{R}^+$ such that for any $a, b \in \mathbf{A}$ and $f \in \mathcal{F}$, $\mathsf{d}(f(a), f(b))$ is equal to the Euclidean distance between a and b in $\mathbf{A}$.

(A4) Any two sectors contain subsectors lying in a common apartment.

(A5$'$) Given $f \in \mathcal{F}$ and a point $\alpha \in \Lambda$, there is a retraction $\rho : \Lambda \to f(\mathbf{A})$ such that the preimage of α is $\{\alpha\}$ and such that for each $\beta, \gamma \in \Lambda : \mathsf{d}(\rho(\beta), \rho(\gamma)) \leq \mathsf{d}(\beta, \gamma)$.

Besides the original paper [47] of J. Tits, an important article is the one of Anne Parreau ([24]). In the latter she describes many structural properties of $\mathbb{R}$-buildings. Also she introduces some alternative definitions, including the following one: we again ask (A1), (A2), (A3) and (A4) to be satisfied, but replace (A5$'$) by d being a distance function, and

(A5) If we have three apartements, each two apartments of which share a half-apartment, then the intersection of all three is nonempty.

We call $|S|$, which is also equal to $\dim \mathbf{A}$, the *dimension* of $(\Lambda, \mathcal{F})$. We will usually denote $(\Lambda, \mathcal{F})$ briefly by Λ, by slight abuse of notation.

Spherical buildings from $\mathbb{R}$-buildings

One can associate spherical buildings of type $(\overline{W}, S)$ to $\mathbb{R}$-buildings in two ways. The first way to do so is to construct the building at infinity. Two sector-facets of Λ will be called *parallel* if the distance between them is bounded. Due to the triangle inequality this is an equivalence relation. The equivalence classes (named *facets at infinity*) form a spherical building Λ_∞ of type $(\overline{W}, S)$ called the *building at infinity* of $(\Lambda, \mathcal{F})$. The chambers of Λ_∞ are the equivalence classes of parallel sectors. An apartment Σ of Λ corresponds to an apartment Σ_∞ of Λ_∞ in a bijective way. The *direction* of a facet is the parallel class it belongs to. Another way to define equivalence classes is the following: two sector-facets are *asymptotic* if they have a sub sector-facet in common of the same dimension as the original two. Two asymptotic sector-facets are necessarily parallel, for sectors these two notions are identical.

A second way to construct a spherical building is to look at the 'local' structure instead of the one at infinity. Let α be a point of Λ, and F, F' two sector-facets with source α. Then these two facets will *locally coincide* if their intersection is a neighbourhood of α in both F and F'. This relation forms an equivalence relation defining *germs of facets* as equivalence classes (notation $[F]_\alpha$). These germs form a (possibly weak) building $[\Lambda]_\alpha$ of type $(\overline{W}, S)$, called the *residue* at α (this notion is different, but slightly related to the previously defined residues). If Σ is an apartment containing α, then $[\Sigma]_\alpha$ will be used to denote the corresponding apartment in $[\Lambda]_\alpha$. If we speak about a *germ* in $[\Lambda]_\alpha$ without further specifying which kind of facet it is derived from, we mean a germ of a sector.

The following lemma by Anne Parreau will prove to be an important tool in our proofs.

Lemma 1.8.1 (Parreau [24], Proposition 1.8) *Let x be a chamber of the building at infinity Λ_∞ and C a sector with source $\alpha \in \Lambda$. Then there exists an apartment Σ containing an element of the germ $[C]_\alpha$ and such that Σ_∞ contains x.*

This has also an interesting corollary.

Corollary 1.8.2 (Parreau [24], Corollary 1.9) *Let α be a point of Λ and F_∞ a facet of the building at infinity. Then there is a unique facet $F' \in F_\infty$ with source α.*

The unique facet of the previous corollary will be denoted by $(F_\infty)_\alpha$ or F_α.

This introduction of germs allows us to state an additional alternative definition from [24], which replaces (A3) and (A5') by the following stronger version of (A3).

(A3') Any two germs lie in a common apartment.

Affine buildings form a special case of $\mathbb{R}$-buildings; they will be referred to as the 'discrete case' of $\mathbb{R}$-buildings. The type of the spherical building at infinity of an affine building is the ‘type of affine Coxeter system without the tilde’, keeping in mind that $\mathsf{I}_2(6)$ equals G_2 and B_n equals C_n (see Section 1.3.2). This implies that the types of the possible spherical buildings at infinity of an affine building are restricted (in particular only generalized n-gons with $n = 3$, 4 or 6 are possible as building at infinity of an affine building). For $\mathbb{R}$-buildings, every type of spherical building at infinity is possible (except H_3 or H_4, as there do not exist such spherical buildings), by the classical examples and free constructions discussed in the next paragraph.

If the dimension of Λ is at least 3, then Λ_∞ is a spherical Moufang building and, in principle, Λ is known, see [47]. For the dimension 2 case, so with a generalized n-gon at infinity, there exist free constructions for the discrete case by M. Ronan in [27] (with $n = 3$, 4 or 6), and nondiscrete constructions for all n by A. Berenstein and M. Kapovich ([6]). These constructions imply that a classification for the dimension 2 case is impossible.

Also $\mathbb{R}$-buildings can be generalized. They form a special case of Λ-buildings, where Λ is an ordered abelian group. For more information see [4].

Trees associated to walls and panels at infinity

With a wall M of an $\mathbb{R}$-building one can associate a direction at infinity (by taking the direction of all sector-facets it contains). This direction M_∞ at infinity will be a wall of the spherical building at infinity.

Let m (respectively π) be a wall (resp. a sector-panel contained in the wall m) of the building at infinity. Let $T(m)$ be the set of all walls M of the $\mathbb{R}$-building with $M_\infty = m$, and $T(\pi)$ the set of all asymptotic classes of sector-panels in the parallel class π.

One can define charts (and so also apartments) from $\mathbb{R}$ to $T(m)$ (resp. $T(\pi)$). First choose M (resp. D) a wall (resp. a sector-panel contained in M) of the model space, such that there exists some chart f such that $f(M)_\infty = m$ and $f(D) \in \pi$. One can identify the model space $\mathbf{A}$ with the product $\mathbb{R} \times M$. For every chart $g \in \mathcal{F}$ of the $\mathbb{R}$-building $(\Lambda, \mathcal{F})$ such that $g(M)_\infty = m$ (resp. $f(D) \in \pi$), one defines a chart g' as follows: if $x \in \mathbb{R}$, then $g'(r)$ is the wall $g(\{r\} \times M)$ (resp. the asymptotic class containing $g(\{r\} \times D)$).

These two constructions yield $\mathbb{R}$-buildings with a rank one building at infinity, such buildings are better know as $\mathbb{R}$-*trees* (or shortly trees when no confusion can arise). The following theorem shows the connection between the above two constructions.

Theorem 1.8.3 *If π is a panel in some wall m, then for each asymptotic class D of sector-panels with direction π, there is a unique wall M in the direction m containing a* *representative of D. The map $D \mapsto M$ is an isomorphism from the $\mathbb{R}$-tree $T(\pi)$ to the $\mathbb{R}$-tree $T(m)$.*

These constructions will be generalized in Section 4.12.

CAT(0)-spaces

For now suppose that (X, d) is some metric space, not necessarily an $\mathbb{R}$-building. A *geodesic* is a subset of the metric space X isometric to a closed interval of real numbers. The metric space (X, d) is a *geodesic metric space* if each two points of X can be connected by a geodesic. From (A3) it follows that $\mathbb{R}$-buildings are geodesic metric spaces.

Let x, y and $z \in X$ be three points in a geodesic metric space (X, d). Because of the triangle inequality we can find three points $\bar{x}, \bar{y}$ and $\bar{z}$ in the Euclidean plane $\mathbb{R}^2$ such that each pair of points have the same distance as the corresponding pair in x, y, z. The triangle formed by the three points is called a *comparison triangle* of x, y and z. Consider a point a on a geodesic between x and y, so we have that $\mathsf{d}(x, y) = \mathsf{d}(x, a) + \mathsf{d}(a, y)$ (note that the geodesic, and so also the point a, is not necessarily unique). We now can find a point $\bar{a}$ on the line through $\bar{x}$ and $\bar{y}$ such that the pairwise distances in $\bar{x}, \bar{y}, \bar{a}$ are the same as in x, y, a. If the distance between z and a is smaller than the distance between $\bar{z}$ and $\bar{a}$, we say that the geodesic metric space (X, d) is a *CAT(0)-space*. Roughly this should be thought of as the space having nonpositive curvature.

The metric spaces formed by $\mathbb{R}$-buildings are examples of CAT(0)-spaces. Complete CAT(0)-spaces (*complete* meaning that all Cauchy sequences converge) have several nice properties, such as:

Theorem 1.8.4 *A nonempty bounded subset of a complete CAT(0)-space X has an unique 'center'.*

The following direct corollary of the above lemma is known as the Bruhat-Tits theorem.

Corollary 1.8.5 *Let G be a group of isometries of a complete CAT(0)-space (X, d). If G stabilizes a nonempty bounded subset of X, then G fixes some point in X.*

Although all discrete $\mathbb{R}$-buildings form complete metric spaces, this is not true in general. We will take a closer look at this problem in Section 4.10.

Remark 1.8.6 The notion of completeness has also another meaning when used for $\mathbb{R}$-buildings, in the sense of 'the complete system of apartments'. However, there will be no confusion possible as we will not use this other notion in this thesis.

1.9 Some additional concepts

We end the introduction by defining two minor concepts which appear at various chapters of this thesis.

1.9.1 Tits endomorphisms

Let $\mathbb{K}$ be a field with finite characteristic p. The *Frobenius endomorphism* is the map $x \mapsto x^p$. A *Tits endomorphism* is then an endomorphism, such that applying it twice gives the Frobenius endomorphism. When the field $\mathbb{K}$ is a finite field of characteristic p, then every field element is a p-th power, so $\mathbb{K}^p = \mathbb{K}$ (one says that the field is *perfect*). A finite field of characteristic p admits a Tits endomorphism if and only if the order of $\mathbb{K}$ is an odd power of p.

1.9.2 Nets

A *net* is a rank 2 geometry $(\mathcal{P}, \mathcal{B}, \mathtt{I})$, consisting of points $\mathcal{P}$, blocks $\mathcal{B}$ and incidence relation $\mathtt{I}$, such that for each point $p \in \mathcal{P}$ and block $B \in \mathcal{B}$, there exists exactly one block B' incident with p, parallel with B (where 2 blocks are *parallel* if the points incident with the two blocks are either completely the same or disjoint).

It can be shown that the parallelism of blocks in a net forms an equivalence relation, defining *parallel classes*.

Chapter 2

'Rank one' case, or Moufang sets

The Ree groups in characteristic 3 (defined by Ree in [26]) and their generalizations over nonperfect fields (by Tits [42]) provide examples of Moufang sets. The root groups of these Moufang sets have nilpotency class 3. This is a rather rare phenomenon; indeed, until recently, these were the only known Moufang sets with this property (a second class was discovered and constructed in [23]). Associated with each Ree group is a geometry (called a *unital* in the finite case), where each pair of points lies on exactly one line (in the finite case a $2-(q^3+1, q+1, 1)$-design), see [19]. This geometry can be viewed as the geometry of involutions in a Ree group, since the blocks are in one-to-one correspondence with a conjugacy class of involutions (in the finite case there is only one conjugacy class). In this way, Ree groups can be better understood in that several properties become more geometric and intuitive through this geometry.

In this chapter we introduce another geometry for each Ree group, inspired by the general construction of geometries associated to 'wide' Moufang sets (for this construction see Section 1.8.1) as proposed by Tits in one of his lectures: 'wide' here means that the unipotent subgroups are not abelian. In fact, this construction is the counterpart for Ree groups of the inversive planes for Suzuki groups (see also the next chapter and [61]). The structure of the geometries that we will introduce is probably slightly more involved than that of the 'unitals', but they have the major advantage that the automorphism groups of the corresponding Ree groups are their full automorphism groups (and this is our Main Result below), a result that is not yet proved for the unitals. This result contributes to Tits' programme of characterizing all 'wide' Moufang sets in this way. As an application, we can show that every collineation of a Moufang hexagon of mixed type permuting the absolute points of a polarity, centralizes that polarity (or, equivalently, also permutes the absolute lines). This, in turn, means that the set of absolute points of any polarity of any Moufang hexagon (necessarily of mixed type) determines the polarity completely and unambiguously. Combined with other results of the author ([34]) and H. Van Maldeghem ([59]), this provides the answer to the aformentioned question for all Moufang polygons.

The 'new' geometries also have a number of interesting combinatorial properties, but we will not concentrate on these, though it would be worthwhile to perform an investigation in that direction.

Every Ree group is the centralizer of a certain outer involution of a Dickson group of type G_2 over a field of characteristic 3 admitting a Tits endomorphism. A geometric way to see this is to consider the associated Moufang generalized hexagon, which is of mixed type. Then the outer involution is a polarity, and the associated Ree group acts doubly transitively on the absolute points of that polarity. That is essentially the way we are going to define and use the Ree groups. These Moufang hexagons are called *Ree hexagons* in [59] precisely for that reason.

Hence, in order to investigate the Moufang sets associated with the Ree groups, we turn to the Ree hexagons, which, as follows from our remarks above, are defined over a field of characteristic 3 admitting a Tits endomorphism θ, and they allow a polarity ρ. The absolute points under this polarity, together with the automorphisms of the mixed hexagon commuting with ρ, form the *Ree-Tits Moufang set*. Since we will need an explicit description of the absolute points of ρ, we will use coordinates. These will be introduced in Section 2.1. We define the Ree geometries in Section 2.3 and state our main results and main application in Section 2.4 (but we formulate our main results also below in rough terms). The rest of the chapter is then devoted to the proofs.

Since the Ree groups have root groups of nilpotency class 3 (at least, if the base field is large enough), the Ree geometries that we will define have rank 3. This means that we will have two types of blocks in our geometry. In this chapter we prove that every automorphism of such a geometry is an automorphism of the corresponding Ree group, by writing down explicitly the automorphisms of this geometry. But we also do slightly better and prove that the same conclusion holds when restricting to one type of blocks. We call these geometries *truncated Ree geometries*. Hence, loosely speaking, we may write our main result as follows:

> *The full automorphism group of a (truncated) Ree geometry is induced by the full collineation group of the corresponding Ree hexagon.*

The results in this chapter are joint work together with Fabienne Haot and Hendrik Van Maldeghem, and are accepted for publication in *Forum Math.*

2.1 Coordinatization of the Ree hexagon

In this section, we present two coordinatizations of the mixed hexagons, of which the Ree hexagons are a special case. These coordinatizations can at the same time serve as a definition of these structures. We start with the coordinatization with respect to one flag $\{(\infty), [\infty]\}$ (which was first carried out by De Smet and Van Maldeghem for (finite) generalized hexagons in [12]). For a detailed description of the coordinatization theory for other generalized polygons we refer to [59]. The second coordinatization follows in fact from the natural embedding of the mixed hexagons in $\mathsf{PG}(6,\mathbb{K})$.

2.1.1 Hexagonal sexternary rings for mixed hexagons

In [59] a coordinatization theory with respect to a flag $\{(\infty), [\infty]\}$ is described. It is a generalization of the coordinatization of Hall for generalized triangles. Here we explicitly describe the coordinatization of the mixed hexagon. Let $\mathbb{K}$ be a field of characteristic 3. Let $\mathbb{K}'$ be a subfield of $\mathbb{K}$ containing the subfield $\mathbb{K}^3$ (so $\mathbb{K}^3 \leq \mathbb{K}' \leq \mathbb{K}$). We consider a hexagonal sexternary ring $\mathcal{R} = (\mathbb{K}, \mathbb{K}', \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ with

$$\begin{cases} \Psi_1(k,a,l,a',l',a'') = a^3k + l, \\ \Psi_2(k,a,l,a',l',a'') = a^2k + a' + aa'', \\ \Psi_3(k,a,l,a',l',a'') = a^3k^2 + l' + kl, \\ \Psi_4(k,a,l,a',l',a'') = -ak + a'', \end{cases}$$

where $a, a', a'' \in \mathbb{K}$ and $k, l, l' \in \mathbb{K}'$. This defines the *mixed hexagon* $\mathsf{H}(\mathbb{K},\mathbb{K}')$ as follows. The points and lines are the i-tuples of elements of $\mathbb{K} \cup \mathbb{K}'$ ($i \leq 5$) with alternately an entry in $\mathbb{K}$ and one in $\mathbb{K}'$, and for points (lines) the last entry is supposed to be in $\mathbb{K}$ ($\mathbb{K}'$), except when $i = 0$, in which case we denote the point by (∞) and the line by $[\infty]$ (we generally use round brackets for points and square brackets for lines). Incidence is defined as follows:

- If the number of coordinates of a point p differs by at least 2 from the number of coordinates of a line L, then p and L are not incident.
- If the number i_p of coordinates of a point p differs by exactly 1 from the number i_L of coordinates of a line L, then p is incident with L if and only if p and L share the first i coordinates, where i is the smallest among i_p and i_L.

- If $i_p = i_L \neq 5$, then p is incident with L if and only if $p = (\infty)$ and $L = [\infty]$.

- A point p with coordinates (a, l, a', l', a'') is incident with a line $[k, b, k', b', k'']$ (with $b, b' \in \mathbb{K}$ and $k', k'' \in \mathbb{K}'$) if and only if

$$\begin{cases} \Psi_1(k, a, l, a', l', a'') = k'', \\ \Psi_2(k, a, l, a', l', a'') = b'', \\ \Psi_3(k, a, l, a', l', a'') = k', \\ \Psi_4(k, a, l, a', l', a'') = b'. \end{cases}$$

Suppose now that our field $\mathbb{K}$ (which has characteristic 3) has a Tits endomorphism θ; then the specific choice $\mathbb{K}' = \mathbb{K}^\theta$ gives a *Ree hexagon*.

2.1.2 The embedding of mixed hexagons in $\mathsf{PG}(6, \mathbb{K})$

The mixed hexagons (and then also the Ree hexagons) have natural embeddings in $\mathsf{PG}(6, \mathbb{K})$. Indeed, $\mathsf{H}(\mathbb{K}, \mathbb{K}')$ is a substructure of the split Cayley hexagon $\mathsf{H}(\mathbb{K})$, which has itself a natural embedding in $\mathsf{PG}(6, \mathbb{K})$ as discovered and described by Tits in [41], see also Chapter 2 of [59]. All these embeddings are *full*, meaning that all points of $\mathsf{PG}(6, \mathbb{K})$ incident with a line of the mixed hexagon are points of the mixed hexagon). Here, we content ourselves with the table below translating the above coordinates to the projective coordinates. We refer to Chapter 3 of [59] for details and proofs.

We write α for $-al' + a'^2 + a''l + aa'a''$ and β for $l - aa' - a^2a''$.

Coordinates in $\mathsf{H}(\mathbb{K}, \mathbb{K}')$	Coordinates in $\mathsf{PG}(6, \mathbb{K})$
(∞)	$(1, 0, 0, 0, 0, 0, 0)$
(a)	$(a, 0, 0, 0, 0, 0, 1)$
(k, b)	$(b, 0, 0, 0, 0, 1, -k)$
(a, l, a')	$(-l - aa', 1, 0, -a, 0, a^2, -a')$
(k, b, k', b')	$(k' + bb', k, 1, b, 0, b', b^2 - b'k)$
(a, l, a', l', a'')	$(\alpha, -a'', -a, -a' + aa'', 1, \beta, -l' + a'a'')$

Coordinates in $\mathsf{H}(\mathbb{K},\mathbb{K}')$	Points generating this line
$[\infty]$	(∞) and (0)
$[k]$	(∞) and $(k,0)$
$[a,l]$	(a) and $(a,l,0)$
$[k,b,k']$	(k,b) and $(k,b,0)$
$[a,l,a',l']$	(a,l,a') and $(a,l,a',l',0)$
$[k,b,k',b',k'']$	(k,b,k',b') and $(0,k'',b',k'+kk'',b)$

The subgroup of $\mathsf{PSL}_7(\mathbb{K})$ stabilizing $\mathsf{H}(\mathbb{K},\mathbb{K}')$ is denoted by $\mathsf{G}_2(\mathbb{K},\mathbb{K}')$ and is simple (a mixed group of type G_2, see [44]).

2.2 The Ree-Tits ovoid

We start from the Ree hexagon $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$, with θ as above a Tits-endomorphism of $\mathbb{K}$. This hexagon allows a polarity. The absolute points under this polarity form an ovoid of the Ree hexagon - the *Ree-Tits ovoid*, see Chapter 7 of [59]. We denote the polarity, which we can choose in such a way that it fixes the flags $\{(\infty),[\infty]\}$ and $\{(0,0,0,0,0),[0,0,0,0,0]\}$ and maps the point (1) onto the line $[1]$, by ρ. It has the following actions:

$$
\begin{aligned}
(a,l,a',l',a'')^\rho &= [a^\theta, l^{\theta^{-1}}, a'^\theta, l'^{\theta^{-1}}, a''^\theta];\\
[k,b,k',b',k'']^\rho &= (k^{\theta^{-1}}, b^\theta, k'^{\theta^{-1}}, b'^\theta, k''^{\theta^{-1}}),
\end{aligned}
$$

for all $a,a',a'',b,b'\in\mathbb{K}$ and $k,k',k'',l,l'\in\mathbb{K}^\theta$.

Now the point (a,l,a',l',a'') is absolute for ρ if and only if it is incident with its image. This leads to the following conditions:

$$
\begin{cases}
l = a''^\theta - a^{\theta+3},\\
l' = a^{2\theta+3} + a'^\theta + a^\theta a''^\theta.
\end{cases}
$$

Coordinates of the Ree-Tits ovoid in $\mathsf{PG}(6,\mathbb{K})$. — Instead of using the 5-tuple $(a, a''^\theta - a^{3+\theta}, a', a^{3+2\theta} + a'^\theta + a^\theta a''^\theta, a'')$, we now will use the shorter notation $(a, a'', a' - aa'')$. Note that every triple in $\mathbb{K}^3$ now corresponds to a point of the ovoid. Now, for $a,a',a''\in\mathbb{K}$, we put

$$
\begin{aligned}
f_1(a,a',a'') &= -a^{4+2\theta} - aa''^\theta + a^{1+\theta}a'^\theta + a''^2 + a'^{1+\theta} - a'a^{3+\theta} - a^2a'^2,\\
f_2(a,a',a'') &= -a^{3+\theta} + a'^\theta - aa'' + a^2a',\\
f_3(a,a',a'') &= -a^{3+2\theta} - a''^\theta + a^\theta a'^\theta + a'a'' + aa'^2.
\end{aligned}
$$

So the set of absolute points in $\mathsf{PG}(6, \mathbb{K})$ can be described by

$$\begin{aligned}\mathcal{P} = &\{(1,0,0,0,0,0,0)\} \cup \\ &\{(f_1(a,a',a''), -a', -a, -a'', 1, f_2(a,a',a''), f_3(a,a',a'')) \mid a, a', a'' \in \mathbb{K}\}.\end{aligned}$$

Compact notation. — As before, we associate the triple $(a, a'', a' - aa'')$ with the point $(a, a''^{\theta} - a^{3+\theta}, a', a^{3+2\theta} + a'^{\theta} + a^{\theta}a''^{\theta})$. The set of absolute points under the polarity is now

$$\mathcal{P} = \{(\infty)\} \cup \{(a, a', a'') \mid a, a', a'' \in \mathbb{K}\}.$$

On this ovoid there acts a Moufang set. The elements of the root group U_∞ of this Moufang set (fixing the point (∞)), act as follows on the remaining points (x, x', x''): the unipotent element that fixes (∞) and maps $(0,0,0)$ to (y, y', y'') maps (x, x', x'') to

$$(x, x', x'') \cdot (y, y', y'') = (x + y, x' + y' + xy^{\theta}, x'' + y'' + xy' - x'y - xy^{\theta+1}),$$

and this action can also be seen as the multiplication inside U_∞, see Chapter 7 of [59].

In this way we obtain the *Ree-Tits Moufang set*. The (simple) Ree groups arise as (simple subgroups of the) centralizers of polarities in these hexagons. More exactly, the Ree group $\mathsf{R}(\mathbb{K}, \theta)$ is defined as the centralizer in $\mathsf{G}_2(\mathbb{K}, \mathbb{K}^{\theta})$ of the outer automorphism ρ. This group is simple if $|\mathbb{K}| > 3$ and the multiplicative group of $\mathbb{K}$ is generated by all squares together with -1, see [26]. In any case, the group generated by the root groups is simple, provided $|\mathbb{K}| > 3$, and it coincides with the derived group $\mathsf{R}'(\mathbb{K}, \theta)$. For $|\mathbb{K}| = 3$, $\mathsf{R}(\mathbb{K}, \theta) = \mathsf{R}(3)$ is isomorphic to $\mathsf{P\Gamma L}_2(8)$ and contains $\mathsf{PSL}_2(8)$ as a simple subgroup of index 3.

We can see the Ree-Tits ovoid and its automorphism group embedded in the Ree hexagon as a representation of the Ree-Tits Moufang set. Henceforth, we will denote by $\mathcal{P}$ the Ree-Tits ovoid, and by U_x, $x \in \mathcal{P}$, the root group fixing x in the Ree-Tits Moufang set over the field $\mathbb{K}$ with associated Tits endomorphism θ.

We will also need the explicit form of a generic element of the root group $U_{(0,0,0)}$, which we shall briefly denote by U_0. This is best given by the action on coordinates in the projective space. Such a generic element $u^{(0,0,0)}_{(x,x',x'')}$ then looks like (x, x', x'' are arbitrary in $\mathbb{K}$):

$\bar{x} = (x_0\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6) \mapsto$

$$\bar{x} \cdot \begin{pmatrix} 1 & f_2(x,x',x'') & f_3(x,x',x'') & x'' & f_1(x,x',x'') & -x' & -x \\ 0 & 1 & -x^\theta & 0 & x'-x^{1+\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 & 0 \\ 0 & -x & x' & 1 & -x'' & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x^2 & -x''-xx' & x & p & 1 & 0 \\ 0 & r & s & -x'+x^{1+\theta} & q & x^\theta & 1 \end{pmatrix},$$

where

$$\begin{cases} p = x^{3+\theta} - x'^\theta - xx'' - x^2x', \\ q = x''^\theta + x^\theta x'^\theta - xx'^2 - x^{2+\theta}x' - x^{1+\theta}x'' - x^{3+2\theta}, \\ r = x'' - xx' + x^{2+\theta}, \\ s = x'^2 - x^{1+\theta}x' - x^\theta x'', \end{cases}$$

see Section 9.2.4 of [38].

Remark 2.2.1 An explicit construction (with detailed proofs) of the Ree group acting on the Ree hexagon can be found in [11].

We are now ready to define the Ree geometries.

2.3 The Ree geometry

As already mentioned, the Ree groups have root groups of nilpotency class 3 (if $|\mathbb{K}| > 3$, see below for a calculation). So applying the construction in Section 1.8.1 using the subgroups $[U_x, U_x]$ and $[[U_x, U_x], U_x]$, gives us two types of blocks in our geometry, and blocks of one type are subsets of the others (the last group is the same as the center $Z(U_x)$ when $|\mathbb{K}| > 3$, see further in this section). When $|\mathbb{K}| = 3$ the Ree group has nilpotency class 2, but one can consider similar subgroups as above (see below). In order to distinguish the two types of blocks, we will call the 'smallest' ones *circles* (notation $\mathcal{C}$), and the others *spheres* (notation $\mathcal{S}$). All the blocks, regardless of the type, will be denoted by $\mathcal{B}$, and the points by $\mathcal{P}$. In this way we have constructed the *Ree geometry* $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \in \text{ or } \ni)$. We can define two further geometries by restricting the set of blocks.

We call the geometries $\mathcal{G}_\mathcal{C} = (\mathcal{P}, \mathcal{C}, \in \text{ or } \ni)$ and $\mathcal{G}_\mathcal{S} = (\mathcal{P}, \mathcal{S}, \in \text{ or } \ni)$ the *truncated Ree geometries.*

Let us be more concrete now and look for the coordinates of the circles and spheres which have (∞) for gnarl.

We first claim that, if $|\mathbb{K}| > 3$, then the group $U'_\infty = [U_\infty, U_\infty]$ is precisely $\{(0, u', u'') \mid u', u'' \in \mathbb{K}\}$. Indeed, computing an arbitrary commutator, we get

$$[(u_1, u'_1, u''_1), (u_2, u'_2, u''_2)] = (0, u_1 u_2^\theta - u_2 u_1^\theta, u'_1 u_2 - u_1 u'_2 - u_1 u_2^{1+\theta} + u_2 u_1^{1+\theta}).$$

Noting that $(0, x', 0) \cdot (0, 0, x'') = (0, x', x'')$, we only have to show that $(0, x', 0) \in U'_\infty$, for all $x' \in \mathbb{K}$, and that $(0, 0, x'') \in U'_\infty$, for all $x'' \in \mathbb{K}$. Putting $u_1 = u''_1 = u'_2 = u''_2 = 0$, $u'_1 = 1$ and $u_2 = x''$ in the above commutator, we see that $(0, 0, x'') \in U_\infty$. Now let $x' \in \mathbb{K}$ be arbitrary. Since $|\mathbb{K}| > 3$, there exists an element $t \in \mathbb{K}$ with $t^3 - t \neq 0$. Put $k = t^3 - t$ and let $y = x' k^{-\theta}$. Putting $u'_1 = u'_2 = u''_1 = u''_2 = 0$ and $(u_1, u_2) = (y, t^3)$, respectively $(u_1, u_2) = (t^\theta y, 1)$, we obtain $(0, t^{3\theta} y - t^3 y^\theta, 0) \in U'_\infty$ and $(0, t^\theta y - t^3 y^\theta, 0) \in U'_\infty$. Multiplying the former with the inverse of the latter, we see that $(0, x', 0) \in U'_\infty$, proving our claim.

If $|\mathbb{K}| = 3$, then U'_∞ has order 3 and coincides with the center (see below). In this case, for the construction of the Ree geometry, we will substitute U'_x by the subgroup of U_x generated by the elements of order 3, and we will denote it, with abuse of notation, by U'_x (but there will be no confusion possible), since for $|\mathbb{K}| > 3$, the derived group coincides with the group generated by elements of order 3 (as one can check easily).

The center of U_∞ is the subgroup $\{(0, 0, u'') \mid u'' \in \mathbb{K}\}$. Indeed, this follows from the explicit form of the multiplication in U_∞ by standard arguments. Since the commutator of an element $(0, u'_1, u''_1) \in U'_\infty$ and $(u_2, u'_2, u''_2) \in U_\infty$ is

$$\begin{aligned}[(0, u'_1, u''_1), (u_2, u'_2, u''_2)] &= (0, 0, u'_1 u_2) \\ &= (0, 0, u''),\end{aligned}$$

with u'' essentially arbitrary, we see that the second group in the normal series $U''_\infty = [U_\infty, [U_\infty, U_\infty]]$ coincides with the center $Z(U_\infty)$ when $|\mathbb{K}| > 3$.

When $|\mathbb{K}| = 3$, the group U''_∞ will be the subgroup consisting only of the identity. Again, for the construction, we will substitute U''_x by the subgroup $\{(0, 0, u'') \mid u'' \in \mathbb{K}\}$ of U_x in this case.

Now, since the circles having (∞) as gnarl are the orbits of a point (a, a', a'') under the group $\{(0, 0, x) \mid x \in \mathbb{K}\}$, union with $\{(\infty)\}$, these circles are given by

$$\{(a, a', a'' + x) \mid x \in \mathbb{K}\} \cup \{(\infty)\} = \{(a, a', t) \mid t \in \mathbb{K}\} \cup \{(\infty)\}.$$

The spheres with gnarl (∞) have the following description:

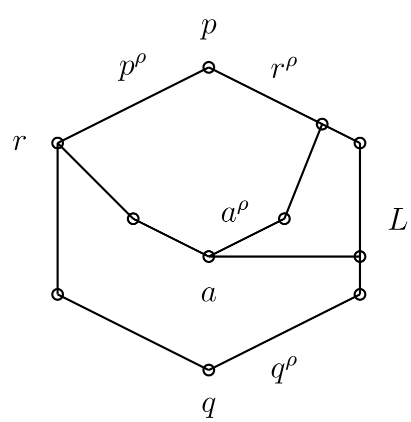
$$\{a, a' + x', a'' + x'' + ax' \mid x', x'' \in \mathbb{K}\} \cup \{(\infty)\} = \{(a, t', t'') \mid t', t'' \in \mathbb{K}\} \cup \{(\infty)\}.$$

We can now interpret the algebraic description of a circle and a sphere with gnarl (∞) in the corresponding Ree hexagon $\mathsf{H}(\mathbb{K}, \mathbb{K}^\theta)$. The points at distance 3 from the nonabsolute line $[0, 0]$ are (∞) and all the points of the form $(0, 0, a', l', a'')$ with $a', l', a'' \in \mathbb{K}$. The absolute points in this set are exactly the points in the circle with gnarl (∞) and containing $(0, 0, 0)$. From this it follows that each circle is the set of absolute points at distance 3 from a nonabsolute line M, not going through an absolute point. The unique absolute point for which its corresponding absolute line intersects M is the gnarl of the circle. With similar reasoning, one sees that each sphere is the set of absolute points not opposite some nonabsolute point p, with p lying on an absolute line. The unique absolute point at distance 2 from p is the gnarl of the sphere. Conversely, every such set is a circle or sphere, respectively. It follows now easily that the gnarl of a circle and of a sphere is unique. These gnarls will play a prominent role in our proofs.

As an application we make the following important observation.

Lemma 2.3.1 *A sphere contains only circles with the same gnarl. Also, the point set of a sphere, except for its gnarl, is partitioned by the circles contained in the sphere.*

Proof. Let us consider a sphere and circle, and assume that this sphere's gnarl is the absolute point p while the gnarl of the circle is a different absolute point q. The flags $\{p, p^\rho\}$ and $\{q, q^\rho\}$ determine a unique apartment Σ containing both flags, and because both flags are absolute, ρ will stabilize Σ. Denote the unique line in Σ at distance 2 from q^ρ and at distance 3 from p with L, and the projection of q on p^ρ with r. Let a be a third absolute point on the circle different from both p and q.



Because a lies on the circle with gnarl q through p, a lies at distance 3 from L. Similarly a also lies at distance 4 from r because of the definition of a sphere. The last statement implies that a^ρ lies at distance 4 from the line r^ρ. This line r^ρ intersects the line L, so the point a and the lines L, r^ρ, a^ρ are contained in an ordinary 5-gon, which contradicts the definition of a generalized hexagon. This proves the first assertion.

For the second assertion, we just consider the circles defined by the nonabsolute lines of $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$ through the point defining the sphere in question. $\square$

2.4 Results on Ree geometries

Given the construction of the circles and spheres in the corresponding Ree hexagon $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$, it is clear that every collineation of $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$ that commutes with the polarity ρ induces a collineation of the Ree geometry and its truncations. Our main results now say that also the converse holds. More precisely:

Main Result 2.4.1 *The full automorphism group of the Ree geometry $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \in$ or $\ni)$ is the centralizer of ρ in the full collineation group of $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$.*

Likewise, we will show:

Main Result 2.4.2 *The full automorphism groups of the truncated Ree geometries $\mathcal{G}_\mathcal{C} = (\mathcal{P}, \mathcal{C}, \in$ or $\ni)$ and $\mathcal{G}_\mathcal{S} = (\mathcal{P}, \mathcal{S}, \in$ or $\ni)$ coincide with the centralizer of ρ in the full collineation group of $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$.*

As a main consequence we will be able to show:

Main Corollary 2.4.3 *The stabilizer of a Ree-Tits ovoid in the full collineation group of $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$ coincides with the centralizer of the corresponding polarity in the full collineation group of $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$. Consequently, any polarity is determined by its set of absolute points.*

The latter was already announced in [59] as Theorem 7.7.9, but not proved there. Combined with results in [59] and [34], one directly obtains:

Corollary 2.4.4 *Each automorphism of a Moufang n-gon with a polarity stabilizing the set of absolute points of that polarity, also stabilizes the set of absolute lines and centralizes that polarity, except if either $n = 3$, the projective plane is Pappian, the characteristic of the underlying field is 2, and the polarity is not Hermitian (i.e., there is no twisting field automorphism), or if $n = 4$ and the generalized quadrangle is the smallest symplectic quadrangle* $\mathsf{W}(2)$.

We will now prove these results.

2.5 Auxiliary tools

Before we can begin with the actual proof, we need to introduce some additional terminology and tools.

2.5.1 The derived geometry at (∞)

We define the structure $\mathcal{G}' = (\mathcal{P}', \mathcal{B}', \in \text{ or } \ni)$, where $\mathcal{P}' = \mathcal{P} \setminus \{(\infty)\}$, and $\mathcal{B}'$ is the set of blocks of $\mathcal{G}$ going through (∞), with (∞) removed. We call this the *derived geometry at* (∞), inspired by a similar concept in the theory of designs. In order to know the coordinates of the circles through (∞), we first write down the coordinates of the circles with gnarl (∞). As we saw earlier, these are the sets

$$\{(a, a', t) \mid t \in \mathbb{K}\} \cup \{(\infty)\}, \text{ with } a, a' \in \mathbb{K}.$$

Removing the point (∞) gives us the *vertical line* $L_{a,a'}$. We now compute the coordinates of the circle with gnarl $(0,0,0)$ through (∞). The point (∞) is identified with $(1,0,0,0,0,0,0)$, so its orbit under $Z(U_0)$ (using the elements $u^{(0,0,0)}_{(0,0,x'')}$ defined above) is the set

$$\begin{aligned}\{(1, f_2(0,0,x''), f_3(0,0,x''), x'', f_1(0,0,x''), 0, 0) \mid x'' \in \mathbb{K}\}\\ = \{(1, 0, -x''^{\theta}, x'', x''^2, 0, 0) \mid x'' \in \mathbb{K}\}.\end{aligned}$$

Putting $x = x''^{-2-\theta}$ (and hence $x'' = x^{-2+\theta}$), adding the gnarl and deleting the point (∞), we obtain the set $\{(x, 0, -x^{2+\theta}) \mid x \in \mathbb{K}\}$. The image of this set under $(a, a', a'') \in U_\infty$ is the set

$$\{(a + x, a' + a^{\theta}x, a'' + (a' - a^{1+\theta})x - x^{2+\theta}) \mid x \in \mathbb{K}\},$$

which we call the *ordinary line* $C_{(a,a',a'')}$ (*with gnarl* (a,a',a'')). Note that unlike the vertical lines, these are not affine lines.

Just as we did for circles, we consider the spheres with gnarl (∞) and the other spheres through (∞) separately.

The spheres with gnarl (∞) are the sets $\{(a,t',t'') \mid t',t'' \in \mathbb{K}\} \cup \{(\infty)\}$, with $a \in \mathbb{K}$. Removing the point (∞) gives us the *vertical plane* P_a.

The orbit of (∞) under U_0', using the elements $u^{(0,0,0)}_{(0,x',x'')}$, is the set

$$\begin{aligned}\{(1, f_2(0,x',x''), f_3(0,x',x''), x'', f_1(0,x',x''), -x', 0) \mid x', x'' \in \mathbb{K}\}\\ = \{(1, x'^{\theta}, -x''^{\theta} + x'x'', x'', x''^2 + x'^{1+\theta}, -x', 0) \mid x', x'' \in \mathbb{K}\}\\ = \left\{\left(\frac{x''^{\theta} - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^{\theta}}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}}\right) \mid \mathbb{K}\times\mathbb{K} \ni (x',x'') \neq (0,0)\right\} \cup \{(\infty)\}.\end{aligned}$$

Note that $x''^2 \neq -x'^{1+\theta}$ is equivalent with $(x',x'') \neq (0,0)$. Adding $(0,0,0)$, the image of this sphere under $(a,a',a'') \in U_\infty$ is the set

$$\left\{\left(\frac{x''^{\theta} - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^{\theta}}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}}\right) \cdot (a,a',a'') \mid \mathbb{K}\times\mathbb{K} \ni (x',x'') \neq (0,0)\right\} \cup \{(a,a',a''), (\infty)\}.$$

Removing the point (∞) gives us the *ordinary plane* $S_{(a,a',a'')}$ (*with gnarl* (a,a',a'')). Again note that these are not affine planes, unlike vertical planes.

Notice that points of vertical planes have constant first coordinate, while the points of an ordinary line never have constant first coordinate. This provides an algebraic proof of Lemma 2.3.1.

2.5.2 Parallelism in the derived structure

We consider the set of points (x,x',x'') as an affine space in the standard way, and call the planes *affine planes*. We assume that the coordinates are given with respect to a basis with axes X, Y, Z.

First we remark that every ordinary line $C_{(a,a',a'')}$ completely lies in the affine plane with equation $Y = a^{\theta}X + (a' - a^{1+\theta})$. We say that two ordinary lines C_1 and C_2 are *parallel* if all vertical lines intersecting C_1 intersect C_2 — in that case the two ordinary lines lie in the same affine plane of the aforementioned form — or if there is no vertical line intersecting both ordinary lines — which implies that the ordinary lines lie in parallel, but disjoint, affine planes of the above form.

We claim that two ordinary lines $C_{(a,a',a'')}$ and $C_{(b,b',b'')}$ are parallel if and only if $a = b$. Indeed, a vertical line meeting the ordinary line $C_{(a,a',a'')}$ must lie in the affine plane $Y = a^\theta X + (a' - a^{1+\theta})$, so any vertical line meeting both $C_{(a,a',a'')}$ and $C_{(b,b',b'')}$ must lie in the intersection of

$$\begin{cases} Y = a^\theta X + (a' - a^{1+\theta}), \\ Y = b^\theta X + (b' - b^{1+\theta}). \end{cases}$$

This has a unique solution if and only if $a \neq b$, proving our claim.

We have the following lemma.

Lemma 2.5.1 *The gnarls of the ordinary lines of the parallel class of $C_{(a,a',a'')}$ are exactly the points of the vertical plane P_a.*

Proof. The above says that the set of gnarls of the lines of the parallel class of $C_{(a,a',a'')}$ is given by $\{(a, t', t'') \mid t', t'' \in \mathbb{K}\}$, which is exactly P_a. $\square$

2.5.3 Ree unitals

In Section 2.9, we will use the Ree unitals mentioned in the introduction. We do not need a formal definition, nor a complete description of them, but only the following facts about these geometries (for a proof of these facts or a more detailed description, see Chapter 7 of [59]):

- the set of points is the same as of the Ree geometries,
- two different points are joined by exactly one block of the Ree unital,
- the block through (∞) and $(a, 0, a'')$, with a and $a'' \in \mathbb{K}$, is given by $\{(\infty)\} \cup \{(a, t, a'' - at) | t \in \mathbb{K}\}$,
- the Ree group $\mathsf{R}(\mathbb{K}, \theta)$ acting on the Ree geometries stabilizes the Ree unital (together with the previous fact, this can be used to define the Ree unital).

If B is a unital block containing (∞), then we will call the set $B \setminus \{(\infty)\}$ an *affine unital block*.

2.6 Automorphism group of the Ree geometry

General idea. — We consider an automorphism φ of the Ree geometry. Without loss of generality we may assume that φ fixes both (∞) and $(0,0,0)$ (because of the 2-transitivity induced by the Moufang set). We will prove that φ must preserve gnarls, and this will imply that it has to preserve the parallelism we just defined. We then compute the algebraic form of φ and conclude that it can be extended to $\mathsf{H}(\mathbb{K},\mathbb{K}^\theta)$.

Lemma 2.6.1 *The automorphism φ maps the gnarl of any sphere onto the gnarl of the image of the sphere, and it maps the gnarl of any circle onto the gnarl of the image of the circle under φ.*

Proof. Any automorphism of Δ maps spheres onto spheres and circles onto circles, since every circle is properly contained in a sphere, but no sphere is properly contained in any circle or sphere. Since the gnarl of a sphere is exactly the intersection of all circles contained in it (by Lemma 2.3.1), and there are at least two such circles, φ preserves gnarls of spheres. But then φ must also preserve the gnarls of these circles. $\square$

Since φ fixes the points (∞) and $(0,0,0)$, it acts on the derived structure $\mathcal{G}'$, and the previous lemma implies that φ fixes the set of vertical lines. Therefore the points (a,a',z_1) and (a,a',z_2) are mapped on the same vertical line. If we represent φ as follows:

$$\varphi:(x,y,z)\mapsto(g_1(x,y,z),g_2(x,y,z),g_3(x,y,z)),$$

then both g_1 and g_2 have to be independent of z, and we write $g_i(x,y,z)=g_i(x,y)$, $i=1,2$.

The mapping φ preserves the parallel relation between ordinary lines, since the number of vertical lines meeting two circles (i.e. none, one or all) is preserved under φ. This translates to g_1 being independent of y. Indeed, two points (a,y_1,z_1) and (a,y_2,z_2) being the gnarls of two parallel ordinary lines are mapped onto two gnarls of parallel ordinary lines, which implies that $g_1(a,y_1)=g_1(a,y_2)$ for every choice for y_1 and y_2.

The point $(0,0,0)$ is fixed by φ, so the affine plane $Y=0$ — which is the unique affine plane containing both $C_{(0,0,0)}$ and $L_{0,0}$, and which consists of the union of vertical lines all meeting $C_{(0,0,0)}$ — is fixed by φ. The plane $Y=c_1$ — which is also a union of vertical lines — must necessarily be mapped onto a plane $Y=c_2$. So $g_2(x,c_1)=g_2(0,c_1)$ for every choice of $x\in\mathbb{K}$.

It follows that there are two permutations α and β of $\mathbb{K}$ such that $(x,y,z)^\varphi$ is equal to $(x^\alpha,y^\beta,g_3(x,y,z))$. Since φ preserves gnarls, it maps the ordinary line $C_{(a,b,c)}$ onto the ordinary line $C_{(a^\alpha,b^\beta,g_3(a,b,c))}$. Now notice that the point (x,y,z) can only be contained in the ordinary line $C_{(a,b,c)}$ if $y=b+a^\theta(x-a)$. Expressing that the point $(a+x,y,z)$ lies on the circle $C_{(a,b,c)}$ if and only if its image under φ lies in $C^\varphi_{(a,b,c)}$ shows that, for all $a,b,x\in\mathbb{K}$,

$$(b+a^\theta x)^\beta = b^\beta + (a^\alpha)^\theta((x+a)^\alpha - a^\alpha). \tag{2.1}$$

Putting $b=0$, and noting that $0^\alpha=0^\beta=0$, we see that $(a^\alpha)^\theta((x+a)^\alpha-a^\alpha)=(a^\theta x)^\beta$, which implies, by substituting this back in Equation (2.1), that $(b+a^\theta x)^\beta=b^\beta+(a^\theta x)^\beta$. So β is additive. Put $\ell=1^\alpha$. Then we see, by setting $a=1$ and $b=0$ in Equation (2.1) above, that

$$x^\beta = \ell^\theta((x+1)^\alpha - 1^\alpha), \tag{2.2}$$

so α is additive if and only if $(x+1)^\alpha=x^\alpha+1^\alpha$. Plugging in $x=m-1$ in Equation (2.2) we have that $(m-1)^\beta=\ell^\theta(m^\alpha-1^\alpha)$. Because of the additivity of β we have on the other hand that $(m-1)^\beta=m^\beta+(-1)^\beta=\ell^\theta((1+m)^\alpha-2\cdot 1^\alpha)$. So α is additive as well.

We now have that $x^\beta=\ell^\theta x^\alpha$. Define the bijection $\sigma:\mathbb{K}\to\ell^{-1}\mathbb{K}:y\mapsto y^\sigma=\ell^{-1}y^\alpha$ (note that $1^\sigma=1$). Plugging in these identities in Equation (2.1) yields

$$(b+a^\theta x)^\sigma = b^\sigma + (a^\sigma)^\theta x^\sigma,$$

for all $a,b,x\in\mathbb{K}$. Putting $a=1$, we see that σ is additive; putting $b=0$ and $x=1$, we see that σ commutes with θ. Putting $b=0$, we see that $(xy)^\sigma=x^\sigma y^\sigma$ for $x\in\mathbb{K}^\theta$ and $y\in\mathbb{K}$. If $x,y\in\mathbb{K}$, then

$$((xy)^\sigma)^\theta = ((xy)^\theta)^\sigma = (x^\theta y^\theta)^\sigma = (x^\theta)^\sigma(y^\theta)^\sigma = (x^\sigma)^\theta(y^\sigma)^\theta = (x^\sigma y^\sigma)^\theta,$$

and the injectivity of θ implies that σ is an automorphism of $\mathbb{K}$. Now the action of φ on a point (x,y,z) is given by $(x,y,z)^\varphi=(\ell x^\sigma,\ell^{1+\theta}y^\sigma,g_3(x,y,z))$, for all $x,y,z\in\mathbb{K}$.

Let us now investigate what $g_3(x,y,z)$ looks like.

The point p with coordinates $(a-\frac{a'}{a^\theta},0,a''+(a'-a^{1+\theta})(\frac{-a'}{a^\theta})-(\frac{-a'}{a^\theta})^{2+\theta})$ lies on both $C_{(a,a',a'')}$ and the ordinary line with gnarl $(0,0,a''+\frac{(a^{1+\theta}-a')^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}})$. So its image under φ lies on the ordinary line with gnarl $(\ell a^\sigma,\ell^{1+\theta}a'^\sigma,g_3(a,a',a''))$ and on the ordinary line with gnarl $(0,0,g_3(0,0,a''+\frac{(a^{1+\theta}-a')^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}}))$. This leads to

$$\begin{cases} g_3(a-\frac{a'}{a^\theta},0,a''-\frac{(a'-a^{1+\theta})a'}{a^\theta}+(\frac{a'}{a^\theta})^{2+\theta}) = g_3(a,a',a'') - \ell^{2+\theta}(\frac{a'^2}{a^\theta}-aa'-\frac{a'^{2+\theta}}{a^{3+2\theta}})^\sigma, \\ \\ g_3(a-\frac{a'}{a^\theta},0,a''-\frac{(a'-a^{1+\theta})a'}{a^\theta}+(\frac{a'}{a^\theta})^{2+\theta}) = g_3(0,0,a''+\frac{(a^{1+\theta}-a')^{1+\theta}+a'^{1+\theta}}{a^{2+\theta}}) - (\ell(a-\frac{a'}{a^\theta})^\sigma)^{2+\theta}. \end{cases}$$

Putting these two equations together we get :

$$g_3(a,a',a'') = g_3\left(0,0,a'' + \frac{(a'-a^{1+\theta})^{1+\theta} + a'^{1+\theta}}{a^{2+\theta}}\right) - \ell^{2+\theta}\left(\frac{(a'-a^{1+\theta})^{1+\theta} + a'^{1+\theta}}{a^{2+\theta}}\right)^{\sigma},$$

for every $a \in \mathbb{K}\backslash\{0\}$ and $a', a'' \in \mathbb{K}$. We want to extend this equation to one with $a = 0$. To this end, we note that the point $(0, a', a'')$ lies on every circle with gnarl $(A, a' + A^{1+\theta}, a'' + a'A - A^{2+\theta})$, with $A \in \mathbb{K}$. We now only consider $A \neq 0$. Then we take the image under φ and obtain that

$$g_3(0,a',a'') = g_3(A, a' + A^{1+\theta}, a'' + a'A - A^{2+\theta}) - \ell^{2+\theta}(Aa' - A^{2+\theta})^{\sigma}.$$

We can now use the above expression for $g_3(a,a',a'')$ for $a \neq 0$ to express $g_3(0,a',a'')$ in terms of $g_3(0,0,z)$, for some $z \in \mathbb{K}$. We rewrite $g_3(0,a',a'')$ in this form, substitute $a' = B^{\theta-1}$ and $A = B^{2-\theta}$, and obtain after a tedious calculation

$$g_3(0, B^{\theta-1}, a'') = g_3(0,0,a''-B) + \ell^{2+\theta}B^{\sigma},$$

for all $B \in \mathbb{K} \setminus \{0\}$, and all $a'' \in \mathbb{K}$. Substituting $-B$ for B, we see that $g_3(0,0,a''-B) = g_3(0,0,a''+B) + \ell^{2+\theta}B^{\sigma}$. We may now put $a'' = -B$ and obtain finally that $g_3(0,0,B) = \ell^{2+\theta}B^{\sigma}$. Plugging this into the formulae above for $g_3(a,a',a'')$, $a \neq 0$, and $g_2(0,a',a'')$, we see that $g_3(a,a',a'') = \ell^{2+\theta}a''^{\sigma}$, for all $a, a', a'' \in \mathbb{K}$.

So the action of φ on a point (x,y,z) is given by $(x,y,z)^{\varphi} = (\ell x^{\sigma}, \ell^{1+\theta}y^{\sigma}, \ell^{2+\theta}z^{\sigma})$, with σ and θ commuting automorphisms of $\mathbb{K}$. This action is the restriction to Ω of the collineation of $\mathsf{H}(\mathbb{K},\mathbb{K}^{\theta})$ defined by the following mapping on the points and lines with five coordinates:

$$\begin{cases} (a,l,a',l',a'') \mapsto (\ell a^{\sigma}, \ell^{\theta+3}l^{\sigma}, \ell^{\theta+2}a'^{\sigma}, \ell^{2\theta+3}l'^{\sigma}, \ell^{\theta+1}a''^{\sigma}), \\ [k,b,k',b',k''] \mapsto [\ell^{\theta}k^{\sigma}, \ell^{\theta+1}b^{\sigma}, \ell^{2\theta+3}k'^{\sigma}, \ell^{\theta+2}b'^{\sigma}, \ell^{\theta+3}k''^{\sigma}]. \end{cases}$$

The proof of Main Result 2.4.1 is complete. □

2.7 Automorphism group of the truncated Ree geometry $\mathcal{G}_{\mathcal{C}}$

General idea. — Let $\mathcal{G}_{\mathcal{C}} = (\mathcal{P}, \mathcal{C}, \in \text{ or } \ni)$ be the truncated Ree geometry, with $\mathcal{C}$ the set of circles. We first prove that gnarls of circles have to be mapped onto gnarls of circles.

Then we use the result from the previous section to prove that the automorphism group of $\mathcal{G}_\mathcal{C}$ is equal to the automorphism group of the Ree geometry $\mathcal{G}$.

We denote by ${\mathcal{G}_\mathcal{C}}'$ the derived geometry in (∞) (so the point set is $\mathcal{P} \setminus \{(\infty)\}$ and the blocks are the vertical and ordinary lines, as defined in Section 2.5.1).

Lemma 2.7.1 *The full group G of automorphisms of ${\mathcal{G}_\mathcal{C}}'$ has two orbits on the lines, which are the vertical and the ordinary lines.*

Proof. It is clear that G acts transitively on both the set of vertical lines and the set of ordinary lines (as G contains the corresponding Ree group), so we only have to exclude the possibility of one orbit. We suppose this is the case and derive a contradiction.

Consider, as before, the point set $\mathcal{P} \setminus \{(\infty)\}$ as a 3-dimensional affine space with point set $\{(a, a', a'')|a, a', a'' \in \mathbb{K}\}$. We project it on the 2-dimensional space $\{(a, a', 0)|a, a' \in \mathbb{K}\}$ by the standard projection map $(a, a', a'') \mapsto (a, a', 0)$. The projection of a vertical line $L_{a,a'}$ is the point $(a, a', 0)$, and the projection of an ordinary line $C_{(a,a',a'')}$ is the affine line $Y = a^\theta X + (a' - a^{1+\theta})$. All these affine lines coming from the projections of ordinary lines form the line set of a net $\mathcal{N}$, and a parallel class of ordinary lines is projected to a parallel class in this net.

Let L be a vertical line and M a vertical or ordinary line disjoint from L. If M is a vertical line, then the projection of L and M are two points. If there exists an ordinary line such that the projection contains both points, then translating this back to the lines means that through each point of L there is an (ordinary) line intersecting M (by varying the third coordinate a''). If, on the other hand, there is no projection of an ordinary line containing both points, then there is no (ordinary) line intersecting both L and M.

If M is an ordinary line, then the projection of M is a certain affine line with equation $Y = a^\theta X + (a' - a^{1+\theta})$. As no projection of an ordinary line is of the form $X = c$ with $c \in \mathbb{K}$ a constant, there are points of M through which no (ordinary) line passes which also intersects L (because we would have projections of the form $X = c$). Also, there obviously are ordinary lines whose projection contains the projection of L and intersect the projection of M. The set of ordinary lines projected to this projection forms a subset of a parallel class exactly one member of which intersects both L and M. We conclude that there exist lines intersecting both L and M, but not through each point of M.

In the above two paragraphs we proved that we can tell a vertical line from an ordinary line if one vertical line is given. Using the hypothesis that there is only one orbit on the lines, this implies that there is an equivalence relation on the lines which is preserved by

G. One of the equivalence classes is obviously the set of vertical lines. By transitivity it follows that through each point of $\mathcal{G}_{\mathcal{C}}'$ there is exactly one line of a given equivalence class. We now claim that the other classes are the parallel classes of ordinary lines. Indeed, if an ordinary line $C_{(a,a',a'')}$ lies in a certain equivalence class, then all lines $C_{(a,a',k)}$ with $k \in \mathbb{K}$ lie in this class, because there is a vertical line through each point of $C_{(a,a',a'')}$ intersecting $C_{(a,a',k)}$. It is implied that two lines are in the same equivalence class if they are projected to the same affine line. Since two intersecting affine lines can be viewed as the projection of two intersecting ordinary lines, two of these subsets are parallel if and only if the corresponding affine lines are parallel. This implies that the equivalence classes are subpartitions of the parallel classes. But since through each point there has to be a line of each equivalence class, the latter must coincide with a parallel class.

Now consider the ordinary line $C_{(0,0,0)}$ and its parallel class π. We can conjugate the center of $U_{(\infty)}$ to obtain an automorphism $\phi \in G$ that fixes the ordinary lines in π, acts freely on the points of such a line, fixes the equivalency classes, and maps $(0,0,0)$ to $(1,0,-1)$.

Let (x,x',x'') be an arbitrary point of $\mathcal{G}_{\mathcal{C}}'$. This point lies on the ordinary line $C_{(0,x',b)} = \{(t,x',b+x't-t^{2+\theta}) \mid t \in \mathbb{K}\}$ for $t = x$ with $b := x'' - x'x + x^{2+\theta}$. As this ordinary line is an element of π, the point $(x,x',x'')^\phi$ also lies on this line. Hence there exists an $f_{x',b}(x) \in \mathbb{K}$ such that $(x,x',x'')^\phi = (f_{x',b}(x), x', b + x' f_{x',b}(x) - f_{x',b}(x)^{2+\theta})$. Notice that the middle coordinate is always fixed.

The vertical line $L_{x,x'} = \{(x,x',t) \mid t \in \mathbb{K}\}$ must be mapped to another vertical line $L_{f_{x',b}(x),x'} = \{(f_{x',b}(x),x',t) \mid t \in \mathbb{K}\}$. From this it follows that the function f is independent of the last coordinate. As both the first and second coordinate are independent of the last, it follows that ϕ induces an automorphism ϕ' on the net $\mathcal{N}$, mapping $(x,x',0)$ to $(f_{x',b}(x),x',0)$. Now ϕ' also fixes every parallel class of $\mathcal{N}$ (the parallel class coming from π is even fixed linewise), and maps $(0,0,0)$ to $(1,0,0)$ (because $(0,0,0)^\phi = (1,0,-1)$). It is now easy to see that this implies $f_{x',b}(x) = x+1$. This gives us the following explicit formula for ϕ:

$$\begin{aligned}\phi : (x,x',x'') &\mapsto (x+1, x', x'' - x'x + x^{2+\theta} + x'(x+1) - (x+1)^{2+\theta}) \\ &\mapsto (x+1, x', x'' + x' + x^{2+\theta} - (x+1)^{2+\theta}).\end{aligned}$$

The image of the ordinary line $C_{(1,1,0)} = \{(1+t, 1+t, -t^{2+\theta}) \mid t \in \mathbb{K}\}$, using the formula for ϕ, is:

$$C^{\phi}_{(1,1,0)} = \{(t-1, t+1, -t^{2+\theta} - t^2 + t^{1+\theta} + t) \mid t \in \mathbb{K}\}.$$

The latter has to coincide with a certain ordinary line $C_{(1,a',a'')} = \{(1+s, a'+s, a''+(a'-1)s - s^{2+\theta}) \mid s \in \mathbb{K}\}$ (because the parallel class is preserved), with $a', a'' \in \mathbb{K}$. This yields the following system of equalities:

$$\begin{cases} t-1 = 1+s, \\ t+1 = a'+s, \\ -t^{2+\theta} - t^2 + t^{1+\theta} + t = a'' + (a'-1)s - s^{2+\theta}, \end{cases}$$

which simplifies to:

$$\begin{cases} s = t+1, \\ a' = 0, \\ t = a'' + 1 - t^{\theta}. \end{cases}$$

If $t = 0$ the last equation gives us $a'' = -1$, but if we use $t = 1$, we obtain $a'' = 1$, which is a contradiction since a'' is a constant. It follows that the hypothesis of one orbit is false. □

The following corollary follows directly:

Corollary 2.7.2 *Gnarls of circles are mapped onto gnarls of circles.*

Using the above and Lemma 2.5.1, one can reconstruct the spheres, giving the following result (which is part of Main Result 2.4.2):

Corollary 2.7.3 *The automorphism group of $\mathcal{G}_C$ is equal to that of $\mathcal{G}$.*

2.8 Absolute points and lines of polarities in the Ree hexagon

We now show our Main Corollary in the formulation below. We note that our proof will not use the full strength of our results proved so far. Indeed, we will only use Corollary 2.7.2. The last few lines of the proof can be deleted if we use Main Result 2.4.1.

Corollary 2.8.1 *If a collineation σ of a Moufang hexagon stabilizes the set of all absolute points of some polarity, then it stabilizes the set of all absolute lines as well.*

Proof. By Theorem 7.3.4 and Theorem 7.7.2 of [59], any polarity ρ of a Moufang hexagon is associated to a Ree group, so it is a polarity of the associated Ree hexagon.

As mentioned before, a circle C of the Ree geometry is the set of absolute points at distance 3 from a line M, not going through an absolute point. The collineation σ maps this set to the set of absolute points at distance 3 from M^σ, which is again a circle since M^σ clearly is not incident with any absolute point (as σ stabilizes the set of absolute points). It follows that σ induces an automorphism of $\mathcal{G}_\mathcal{C}$. The gnarl of C is the absolute point x such that the corresponding absolute line x^ρ intersects M. Corollary 2.7.2 now implies that the absolute line $(x^\sigma)^\rho$ intersects M^σ. As $(x^\rho)^\sigma$ also contains x^σ and intersects M^σ, it follows that $(x^\sigma)^\rho = (x^\rho)^\sigma$. This means that the absolute line x^ρ is mapped to another absolute line. Varying C we now see that the set of all absolute lines is stabilized by σ. $\square$

2.9 Automorphism group of the truncated Ree geometry $\mathcal{G}_\mathcal{S}$

General idea. — Let $\mathcal{G}_\mathcal{S} = (\mathcal{P}, \mathcal{S}, \in \text{ or } \ni)$ be the truncated Ree geometry with $\mathcal{S}$ the set of spheres. We again prove that gnarls of spheres have to be mapped onto gnarls of spheres. As a consequence one can recognize certain automorphisms of the Ree geometry generating the Ree group. Using this the circles can be reconstructed giving us the full Ree geometry $\mathcal{G}$ and its automorphism group.

We denote by $\mathcal{G}_\mathcal{S}{}'$ the derived geometry in (∞) (so the point set is $\mathcal{P} \setminus \{(\infty)\}$ and the blocks are the vertical and ordinary planes, as defined in Section 2.5.1).

We start with some small observations:

Lemma 2.9.1 *A vertical plane and an ordinary plane always intersect.*

Proof. By transitivity we can suppose that the vertical plane is given by

$$P_a = \{(a, t', t'') \mid t', t'' \in \mathbb{K}\}, \text{ with } a \in \mathbb{K}$$

while the ordinary plane can be represented by $S_{(0,0,0)}$, which is the set

$$\left\{\left(\frac{x''^\theta - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^\theta}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}}\right) \mid \mathbb{K} \times \mathbb{K} \ni (x', x'') \neq (0,0)\right\} \cup \{(0,0,0)\}.$$

If $a = 0$, then $(0,0,0) \in P_a \cap S_{(0,0,0)}$. If $a \neq 0$, then putting $x' = 0$ and $x'' = a^{-2-\theta}$ in the formula of $S_{(0,0,0)}$ gives the point $(a, 0, -a^{2+\theta})$, which is also a point of P_a. $\square$

Lemma 2.9.2 *The intersection of* P_0 *and* $S_{(0,0,0)}$ *is given by the set* $\{(0,t,0) \mid t \in K\} \cup \{(0,t^{\theta-1},t) \mid t \in \mathbb{K}\backslash\{0\}\}$.

Proof. Using the representations of $P_0 = \{(0,t,t') \mid t,t' \in \mathbb{K}\}$ and $S_{(0,0,0)} =$

$$\left\{\left(\frac{x''^\theta - x'x''}{x''^2 + x'^{1+\theta}}, \frac{-x'^\theta}{x''^2 + x'^{1+\theta}}, \frac{-x''}{x''^2 + x'^{1+\theta}}\right) \mid \mathbb{K} \times \mathbb{K} \ni (x',x'') \neq (0,0)\right\} \cup \{(0,0,0)\},$$

we see that the points of the intersection are determined by the equation $x''^\theta - x'x'' = 0$. The solutions of this equation are given by $x'' = 0$ or $x' = x''^{\theta-1}$. The first set of solutions gives us $\{(0,t,0) \mid t \in K\}$, the second $\{(0,t^{\theta-1},t) \mid t \in \mathbb{K}\backslash\{0\}\}$. □

Note that P_0 is the disjoint union of affine unital blocks. Indeed, the affine blocks $\{(0,t,b) \mid t \in \mathbb{K}\}$, with $b \in \mathbb{K}$, partition P_0. It is now clear that the intersection of $S_{(0,0,0)}$ and P_0 contains exactly one affine unital block, and all other affine unital blocks in P_0 share exactly one point with that intersection.

Lemma 2.9.3 *The ordinary planes* $S_{(0,0,0)}$ *and* $S_{(0,a',a'')}$, *with* $a', a'' \in \mathbb{K}$, *intersect.*

Proof. Since $(0,a',a'') \in U_\infty$ maps P_0 to itself and $S_{(0,0,0)}$ to $S_{(0,a',a'')}$, it follows from the paragraph preceding this lemma that $P_0 \cap S_{(0,a',a'')}$ contains an affine unital block B. But from that same paragraph also follows that B shares a point with $P_0 \cap S_{(0,0,0)}$. That point is hence contained in $S_{(0,0,0)} \cap S_{(0,a',a'')}$. □

The above lemmas now allow us to prove the following analogue to Lemma 2.7.1.

Lemma 2.9.4 *The full group* G *of automorphisms of* ${\mathcal{G}_\mathcal{S}}'$ *has two orbits on the planes, which are the vertical and the ordinary planes.*

Proof. As with the case of points and circles, it suffices to prove that the planes can not be all in one orbit. So suppose this is the case.

We call two vertical or ordinary planes *parallel* if they are disjoint or equal. By the transitivity assumption on the planes and Lemma 2.9.1, for each point p (different from (∞)) and plane P, there is exactly one plane Q parallel to P and containing p. Let ϖ be the parallel class containing $S_{(0,0,0)}$. Because U_∞ preserves parallelism and acts regularly on the ordinary planes, the stabilizer V of ϖ in U_∞ acts regularly on the planes in ϖ and $S_{(a,a',a'')} \in \varpi$ if and only if $(a,a',a'') \in V$.

Let $g = (a,a',a'') \in U_\infty$ be a nontrivial element of V. Then, in view of Lemma 2.9.3, a has to be different from 0. But as V is a group, $g^3 = (0,0,-a^{2+\theta})$ is also a nontrivial element of V, which does have as first coordinate 0, so the hypothesis is false. □

Lemma 2.9.5 *In $\mathcal{G}_{\mathcal{S}}{}'$ the affine unital blocks are (geometric) invariants.*

Proof. We will denote the intersection of a vertical plane through the point p with the ordinary plane with gnarl p by W_p. The sets W_p are invariants of the geometry by virtue of Lemma 2.9.4. Lemma 2.9.2 implies that the affine unital block through p is contained in W_p.

By transitivity, it suffices to construct the affine unital block B through $(0,0,0)$. Let $p \in W_{(0,0,0)}$ be a point different from $(0,0,0)$. If p lies on B, then $W_{(0,0,0)} \cap W_p$ contains B itself and so at least 4 points (as $|\mathbb{K}| > 3$). Now suppose $p \notin B$, so $p = (0, k^{\theta-1}, k)$ for a certain $k \in \mathbb{K}$ different from 0. Using $(0, k^{\theta-1}, k)$ as an element of U_∞ and Lemma 2.9.2, we calculate that $W_p = \{(0, t + k^{\theta-1}, k) \mid t \in \mathbb{K}\} \cup \{(0, t^{\theta-1} + k^{\theta-1}, t + k) \mid t \in \mathbb{K}\backslash\{0\}\}$. The intersection $W_{(0,0,0)} \cap W_p$ contains two obvious intersection points on the affine unital blocks contained in either $W_{(0,0,0)}$ and W_p. To look for more intersection points we need to investigate whether or not it is possible to have $(0, t^{\theta-1} + k^{\theta-1}, t + k) = (0, s^{\theta-1}, s)$ for certain $s, t \in \mathbb{K}\backslash\{0\}$. Equality on the third coordinate gives us $t + k = s$, the second gives us

$$\begin{aligned} s^{\theta-1} = t^{\theta-1} + k^{\theta-1} &\Leftrightarrow (t+k)^{\theta-1} = t^{\theta-1} + k^{\theta-1} \\ &\Leftrightarrow t^{2-\theta} = -k^{2-\theta}. \end{aligned}$$

If we raise both hand sides of the last equation to the power $2+\theta$, then we obtain $t = -k$, implying $s = 0$, a contradiction.

Hence in this case we have that $|W_{(0,0,0)} \cap W_p| = 2$. This allows us to recognize the points of the affine unital block through $(0,0,0)$ as those for which $|W_{(0,0,0)} \cap W_p| > 2$. $\square$

Lemma 2.9.6 *In $\mathcal{G}_{\mathcal{S}}$, the circles of $\mathcal{G}$ are invariants.*

Proof. Let p and q be two different points of $\mathcal{G}_{\mathcal{S}}$, and let G be the full automorphism group of $\mathcal{G}_{\mathcal{S}}$. Then we first want to determine the elements of G which fix p and all the blocks of the unital through p, within the sphere with gnarl p through q. We will denote this group by $G_{[p,q[}$.

By 2-transitivity we can suppose that $p = (\infty)$ and $q = (0,0,0)$. The aim is to prove that $G_{[(\infty),(0,0,0)[} = \{(0,t,0) \mid t \in \mathbb{K}\} =: H$. It is easy to see that these automorphisms satisfy the needed properties and act transitively (even regularly) on the points of the affine unital block B through $(0,0,0)$. Suppose there is another automorphism g which satisfies these properties. Then, possibly by composing with a suitable element of H, we may assume that g fixes $(0,0,0)$. This implies that the sphere with gnarl $(0,0,0)$ through (∞) is also fixed. By Lemma 2.9.2 the points $(0,k^{\theta-1},k)$ with $k \in \mathbb{K}\backslash\{0\}$ are also fixed, so also the blocks through $(0,0,0)$ in the sphere with gnarl $(0,0,0)$ through (∞), which makes the situation symmetric in both points. We can also let the fixed points of the form $(0,k^{\theta-1},k)$ play the role of $(0,0,0)$, which yields the fixed points $(0,k_1^{\theta-1}+k_2^{\theta-1}+\cdots+k_n^{\theta-1},k_1+k_2+\cdots+k_n)$ with $k_i \in \mathbb{K}\backslash\{0\}$, by repeating the argument. Choosing $n=3$ and $k_1=-k_2=k_3=k$ with $k \in \mathbb{K}\backslash\{0\}$ gives us the fixed points $(0,0,k)$ for all $k \in \mathbb{K}$.

Interchanging the roles of (∞) and $(0,0,0)$, we get the fixed points $(k,0,-k^{2+\theta})$ (to calculate these observe that $(0,0,k)$ are the points different from (∞) on the circle with gnarl (∞) through $(0,0,0)$, interchanging gives us the points different from $(0,0,0)$ on the circle with gnarl $(0,0,0)$ through (∞)). If we let a fixed point $(0,0,l)$ with $l \in \mathbb{K}$ play the role of $(0,0,0)$, we obtain that all the points of the form $(k,0,l)$ with $k,l \in K$ are fixed points. On each affine unital block lies a point of this form, so all affine unital blocks are fixed, and by symmetry also the blocks of the Ree unital through $(0,0,0)$. It follows that all points are fixed points, and that g is the identity.

The above proves that $G_{[p,q[}$ is a subgroup of the root group U_p and hence, if $|\mathbb{K}|>3$, also a subgroup of the simple Ree group $\mathsf{R}'(\mathbb{K},\theta)$. The group K generated by all groups of the form $G_{[p,q[}$ is a normal subgroup of this Ree group (indeed, if g is an automorphism of $\mathcal{G}_{\mathcal{S}}$, then $G_{[p,q[}^g = G_{[p^g,q^g[}$). So by simplicity, K coincides with $\mathsf{R}'(\mathbb{K},\theta)$. Now, by [10], the root groups of K are the unique unipotent subgroups of K. Hence we can recover these root groups and consequently also the circles constructed from these root groups.

If $|\mathbb{K}|=3$, then K is a normal subgroup of the Ree group $\mathsf{R}(3)$ over the field with 3 elements. But the groups $G_{[p,q[}$ do not belong to the simple Ree group. Hence it is easy to see that K coincides with the Ree group $\mathsf{R}(3)$ and, as above, we can again recover the circles. $\square$

We have proved :

Corollary 2.9.7 *The automorphism group of $\mathcal{G}_{\mathcal{S}}$ coincides with that of $\mathcal{G}$.*

This completes the proof of Main Result 2.4.2.

Chapter 3

'Rank two' case, or generalized polygons

During my Ph.D. studies, I obtained various results about generalized polygons, which can roughly be put in two categories: mixed quadrangles and generalized inversive planes, and embeddings of quadrangles in buildings of type F_4.

Mixed quadrangles. — In 1974, Jacques Tits [44] introduced what he called *groups of mixed type*, as a certain generalization of algebraic groups. This was motivated by the fact that certain spherical buildings arise from such groups, and Tits classified all spherical buildings of rank at least three in [44].

Roughly, the groups of mixed type of rank 2 arise when the weight of the edge of the rank 2 Coxeter diagram is equal to the characteristic of the underlying field. Indeed, in the commutation relation of the root groups, the weight w of the edge turns up as a coefficient, and as a power (if the diagram is included in a rank 3 diagram, then only the cases $w \in \{1, 2, 3\}$ occur). If the corresponding term does not vanish (i.e., if in the underlying field w is not equal to 0), then we are in the generic case where we are able to distinguish long and short roots (by the commutation relations, but also by the geometry of the corresponding building). However, if $w = 0$, i.e., if the characteristic of the underlying field is equal to w, then the commutation relations become much more symmetric, allowing for diagram automorphisms. If the field is perfect, not much extra happens since the symmetry is then *up to the field Frobenius automorphism* $x \mapsto x^w$, and we only obtain an extra group automorphism (diagram automorphism). However, if the field is not perfect, then this 'duality' is not surjective anymore, and we obtain the peculiar situation in which the rank 2 geometry 'looks' symmetric, but isn't. Technically, the duality maps the geometry *into* itself, but not *onto*. In other words, the geometry (building) is isomorphic to the dual of a subgeometry. On the algebraic level, we obtain an infinite descending chain of algebraic structures, each one containing the next one, and the first one parameterizing the chambers in a certain panel. Since we have two different types of panels, we have two such chains (which are mapped onto each other by the duality). The strange thing is now that 'interlacing' subchains define subgeometries and the corresponding automorphism groups are the groups of mixed type. If the original chains consist of fields, then the interlacing chains may consist of fields, too, but also of vector spaces. The latter only happens for $w = 2$.

In this chapter, we study the case $w = 2$ in a geometric way. This is the case where the Coxeter diagram has a weight 2 edge, hence a double bond. Geometrically, this is the case of the (Moufang) generalized quadrangles. In the (algebraically) split case, we have a symplectic quadrangle over some field $\mathbb{K}$. If $\mathbb{K}$ has characteristic 2 and is perfect, then this generalized quadrangle, denoted by $\mathsf{W}(\mathbb{K})$, is self-dual. If $\mathbb{K}$ has characteristic 2 and is not perfect, then we are in the mixed case. There are two types of panels here, and hence two different parametrizations. Any point row is parametrized by $\mathbb{K} \cup \{\infty\}$, while any line pencil is parametrized by $\mathbb{K}^2 \cup \{\infty\}$ (here, $\mathbb{K}^2$ is the field of squares of $\mathbb{K}$). We obtain two chains $\mathbb{K} \supseteq \mathbb{K}^2 \supseteq \mathbb{K}^4 \supseteq \cdots$ and $\mathbb{K}^2 \supseteq \mathbb{K}^4 \supseteq \mathbb{K}^8 \supseteq \cdots$. An interlacing chain may look like $\mathbb{K}' \supseteq \mathbb{K}'^2 \supseteq \mathbb{K}'^4 \supseteq \cdots$, with $\mathbb{K}'$ a field satisfying $\mathbb{K}^2 \subseteq \mathbb{K}' \subseteq \mathbb{K}$. But we may also substitute $\mathbb{K}$ in the first chain by a vector space L over $\mathbb{K}'$ contained in $\mathbb{K}$, and $\mathbb{K}'$ in the second chain by a vector space L' over $\mathbb{K}^2$ contained in $\mathbb{K}'$. This is the most general case that can occur. We denote the corresponding (Moufang) quadrangle by $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$.

The quadrangle $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ has an interesting geometric property. Indeed, all its points and lines are *regular* (for precise definitions, see below). Moreover, the dual nets associated with the regular elements also satisfy some regularity properties. In a very weak form one can say that these dual nets satisfy a certain Little Desargues Axiom. We will show that this axiom, together with the regularity of points and lines, characterizes all quadrangles of mixed type. In order to answer the question of the geometric difference between the cases where both / exactly one / none of L and L' are fields, we consider the Veblen & Young Axiom in these dual nets. We will show that if a generalized quadrangle has enough regular points and lines, and if the dual nets related to the regular points satisfy the Axiom of Veblen & Young, then the quadrangle is of mixed type and L' is a field.

These results hold in both the infinite and finite case. But in the finite case there are no proper mixed quadrangles since a finite field is always perfect. All the results of the present chapter that are also valid in this improper mixed case are actually well known for finite quadrangles, but some of our proofs give rise to alternative arguments. As an example we mention that Theorem 3.5.6 immediately implies that, if a finite generalized quadrangle of order q has an ovoid of regular points, then all corresponding projective planes are classical.

These results, and the ones in the next subsection, were obtained in a joint work with Hendrik Van Maldeghem, see [35].

Generalized Suzuki-Tits inversive planes. — Another feature of the mixed quadrangles is that certain of them admit *polarities*, i.e., dualities of order 2. In this case, the centralizer of that polarity in the little projective group of the quadrangle is a (generalized) Suzuki group. The set of elements fixed under a polarity can be structured to a geometry which is called a *generalized inversive plane* in [61]. The main result of [61] says that the automorphism groups of these generalized inversive planes are essentially the (generalized) Suzuki groups. In the present chapter, we use the above characterizations of the mixed quadrangles to axiomatize the generalized inversive planes corresponding to the generalized Suzuki groups. In the perfect case, this has already been done by Hendrik Van Maldeghem in [58]. So we relax the axioms of [58] to deal with the more general case of imperfect fields (using the Veblen & Young Axiom) and vector spaces (using the Little Desargues Axiom). As a corollary, these new results let us simplify the characterization for the perfect case in [58] by removing one axiom.

Embeddings of quadrangles in buildings of type F_4. — The first examples of generalized polygons mainly arose as *embeddings* in projective spaces, i.e., the points of the polygon are some points of a projective space, while the lines of the polygon can be identified with some lines of the projective space, and the incidence relation is the natural one. The mixed quadrangles and the hexagons mentioned in the above subsection and the previous chapter are examples of such embeddings. If the embedding is 'nice', then it automatically inherits beautiful symmetry properties from the projective space, see [13, 18, 32, 33, 40]. 'Nice' could mean that the lines of the polygon through any point are contained in a certain subspace of the projective space (plane, hyperplane), or that the points not opposite a given point in the polygon do not span the entire projective space, or just a bound on the dimension of the projective space together with the fact that all points of the projective space on any line of the polygon belong to the polygon. In particular, the previous references contain characterizations and classifications of the 'nice' embeddings of the Moufang generalized quadrangles and hexagons.

However, not all Moufang polygons admit an embedding as considered above. The notable examples are the exceptional Moufang quadrangles and their duals, the duals of some embeddable classical Moufang quadrangles, and the duals of the exceptional Moufang hexagons and of the Ree-Tits octagons. These exceptional polygons geometrically come forward in a different way: they do not arise from 'forms' of a projective space, but from 'forms' of buildings of exceptional type and rank at least 4. All types arise: $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4$. In this chapter, we take a closer look at the situation of F_4 (called metasymplectic spaces from a geometric point of view). This case is the least 'algebraic' of the lot. Similar as explained above, characteristic 2 is a special case for buildings of type F_4 (which contain an edge of weight 2). This leads to the existence of groups and buildings of mixed type with diagram F_4, see [44].

Using this special behaviour one can find embeddings of certain Moufang quadrangles and octagons. This is the starting point. Our goal is to find a 'nice' property of the embedding of the exceptional Moufang quadrangles in buildings of type F_4 that guarantees that *any* quadrangle embedded in a building of type F_4 with that property, is automatically a Moufang quadrangle. This property will be denoted by (OV) in Section 3.9. Roughly, we require that the points of the quadrangle are points of the building, the lines of the quadrangle are hyperlines of the building (with natural incidence), and (OV) says that any two noncollinear points of the quadrangle are never contained in a hyperline of the building. In other words, *collinearity in the quadrangle coincides with cohyperlinearity in the building.* This very natural property surprisingly is enough to characterize the Moufang quadrangles arising from buildings of type F_4.

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3.1 Some further definitions on generalized quadrangles

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized quadrangle and let x be an arbitrary point. The set of points of Γ collinear with x will be denoted by $x^\perp$. For a set $X \subseteq \mathcal{P}$, we denote by $X^\perp$ the set of points collinear with all points of X, and we abbreviate $(X^\perp)^\perp$ by $X^{\perp\perp}$. If y is a point opposite x, then $\{x, y\}^\perp$ is called the *perp* of the pair x, y. The *span* of the pair x, y is the set $\{x, y\}^{\perp\perp}$. If every span containing x is also a perp (of a pair of different points, needless to say), then the point x is called *regular*. Dually one defines *regular lines.* If x is a regular point, then the geometry $\Gamma^*_x = (x^\perp \setminus \{x\}, \{\{x, y\}^\perp : y \not\sim x\}, \in \text{ or } \ni)$ is a dual net (associated to x) (see Section 1.9.2) , i.e., it has the property that for every point $z \in x^\perp \setminus \{x\}$ and every *block* $B = \{x, y\}^\perp$, with y opposite x, there is a unique point $z' \in B$ not collinear with z (collinearity in Γ^*_x). If Γ^*_x is a dual affine plane, then we call x a *projective point*. The motivation for this terminology is that the geometry $\Gamma_x = (x^\perp, \{\{x,y\}^\perp : y \in \mathcal{P}\}, \in \text{ or } \ni)$ is then a projective plane, called the *perp-plane* in x. Projective points have nice properties. For instance, one can easily check that x is a projective point if and only if the geometry $(\mathcal{P} \setminus x^\perp, \{L \in \mathcal{L} : x \not\mathtt{I}\ L\} \cup \{\{x,y\}^{\perp\perp} : y \not\sim x\}, \mathtt{I} \text{ or } \in \text{ or } \ni)$ is a generalized quadrangle if and only if every pair of distinct perps contained in $x^\perp$ meet in a unique point (this construction is known as the *Payne construction*, see [25]).

Finally we introduce some notions concerning symmetry in generalized quadrangles. A point x of a generalized quadrangle is called a *center of symmetry* if it is regular and if the group of collineations fixing $x^\perp$ pointwise acts transitively on the set $\{x,y\}^{\perp\perp} \setminus \{x\}$, for some, and hence for every, point y opposite x. The dual notion is called an *axis of symmetry*.

3.2 Examples of generalized quadrangles

We introduce some classes of generalized quadrangles which will be of use later on.

3.2.1 Symplectic quadrangles

The prototype class of examples of generalized quadrangles is the class of *symplectic quadrangles*, which are defined as follows. Let ρ be a symplectic polarity in a 3-dimensional projective space $\mathsf{PG}(3,\mathbb{K})$ over a field $\mathbb{K}$. If $\mathcal{P}$ is the point set of $\mathsf{PG}(3,\mathbb{K})$, if $\mathcal{L}$ is the set of lines of $\mathsf{PG}(3,\mathbb{K})$ fixed by ρ, and if $\mathtt{I}$ denotes the incidence relation in $\mathsf{PG}(3,\mathbb{K})$, then $\mathsf{W}(\mathbb{K}) = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized quadrangle called the *symplectic quadrangle (over* $\mathbb{K}$*)*. All the points of $\mathsf{W}(\mathbb{K})$ are regular, even projective. Conversely, Schroth [31] proved that any generalized quadrangle all points of which are projective is isomorphic to a symplectic quadrangle. In fact, Theorem 6.2.1 of [54] asserts that, if all points of a generalized quadrangle Γ are regular and at least one point is projective, then all points are projective and Γ is a symplectic quadrangle. The first step in the proof is to show that if a point x of Γ is projective, then every opposite (regular) point is also projective. We record this step as a separate lemma for later reference.

Lemma 3.2.1 ([59]) *Let x, y be two opposite points of a generalized quadrangle Γ. If x is projective and y is regular, then y is projective too.*

The symplectic quadrangle has a lot of symmetry. All points of $\mathsf{W}(\mathbb{K})$ are centers of symmetry. Dually, all lines of $\mathsf{W}(\mathbb{K})$ are axes of symmetry if and only if $\mathbb{K}$ has characteristic 2. Also, $\mathsf{W}(\mathbb{K})$ is self-dual if and only if $\mathbb{K}$ is a perfect field with characteristic 2. Moreover, $\mathsf{W}(\mathbb{K})$ admits a polarity if and only if there exists a Tits automorphism $\theta : \mathbb{K} \longrightarrow \mathbb{K} : x \mapsto x^\theta$, so $(x^\theta)^\theta = x^2$, for all $x \in \mathbb{K}$ (see Section 1.9.1).

We now give a description of $\mathsf{W}(\mathbb{K})$ using coordinates (see [59]). Let $\mathsf{W}(\mathbb{K}) = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be the symplectic quadrangle over the field $\mathbb{K}$. Then we may take for $\mathcal{P}$ the following set:

$$\mathcal{P} = \{(\infty)\} \cup \{(a) : a \in \mathbb{K}\} \cup \{(k,b) : k,b \in \mathbb{K}\} \cup \{(a,l,a') : a,l,a' \in \mathbb{K}\},$$

and for $\mathcal{L}$ the set

$$\mathcal{L} = \{[\infty]\} \cup \{[k] : k \in \mathbb{K}\} \cup \{[a,l] : a,l \in \mathbb{K}\} \cup \{[k,b,k'] : k,b,k' \in \mathbb{K}\},$$

where ∞ is a symbol not contained in $\mathbb{K}$, and where incidence is given by

$$(a,l,a')\mathtt{I}[a,l]\mathtt{I}(a)\mathtt{I}[\infty]\mathtt{I}(\infty)\mathtt{I}[k]\mathtt{I}(k,b)\mathtt{I}[k,b,k'],$$

for all $a, a', b, k, k', l \in \mathbb{K}$, and

$$(a,l,a')\mathtt{I}[k,b,k'] \Longleftrightarrow \left\{ \begin{array}{lcl} a' & = & ak+b, \\ k' & = & a^2k + l - 2aa'. \end{array} \right.$$

We clearly see the asymmetry if the characteristic of $\mathbb{K}$ is unequal to 2. If, on the other hand, the characteristic of $\mathbb{K}$ is equal to 2, then the two above formulas are equivalent if squaring is an automorphism, i.e., the Frobenius is surjective, implying the field is perfect.

3.2.2 Mixed quadrangles

Mixed quadrangles are subquadrangles of the symplectic quadrangle $\mathsf{W}(\mathbb{K})$, for $\mathbb{K}$ an imperfect field with characteristic 2 (in the other case the only (thick) subquadrangles are symplectic quadrangles over subfields). Neither the point set nor the line set of these subquadrangles can be given by a nice set of equations in $\mathsf{PG}(3,\mathbb{K})$, because the corresponding collineation groups are not algebraic groups. The quickest and most elementary way to define the mixed quadrangles is using the coordinates of symplectic quadrangles introduced above.

So suppose $\mathbb{K}$ is imperfect and of characteristic 2, and let $\mathbb{K}^2$ be the subfield consisting of all squares. Let $\mathbb{K}'$ be a subfield with $\mathbb{K}^2 \subseteq \mathbb{K}' \subseteq \mathbb{K}$ and let L, L' be subspaces of $\mathbb{K}, \mathbb{K}'$

viewed as vector spaces over $\mathbb{K}', \mathbb{K}^2$, respectively, with $\mathbb{K}^2 \subseteq L'$ and $\mathbb{K}' \subseteq L$. We consider the description of $\mathsf{W}(\mathbb{K})$ with coordinates as above, and we now restrict the a, a', b to L and the k, k', l to L'. Then we obtain a subquadrangle that we denote by $\mathsf{W}(\mathbb{K}, \mathbb{K}; L, L')$ and call a *mixed quadrangle* (the terminology in [51] mentions *indifferent quadrangle*, but we prefer to name the geometries after the groups, as for the symplectic quadrangle). In order to have a unique definition, we also assume that L and L' generate $\mathbb{K}$ and $\mathbb{K}'$ as a ring. Note that $\mathsf{W}(\mathbb{K}) = \mathsf{W}(\mathbb{K}, \mathbb{K}; \mathbb{K}, \mathbb{K})$ and that $\mathsf{W}(\mathbb{K}, \mathbb{K}^2; \mathbb{K}, \mathbb{K}^2)$ is the dual of $\mathsf{W}(\mathbb{K})$ (and this dual is isomorphic to the generalized quadrangle arising from a nonsingular quadratic form of maximal Witt index in a five-dimensional vector space over $\mathbb{K}$).

It is convenient to also call $\mathsf{W}(\mathbb{K})$, with $\mathbb{K}$ perfect and of characteristic 2, a mixed quadrangle. In this case, we also write $\mathsf{W}(\mathbb{K}) = \mathsf{W}(\mathbb{K}, \mathbb{K}; \mathbb{K}, \mathbb{K})$.

In general, the dual of $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ is isomorphic to $\mathsf{W}(\mathbb{K}', \mathbb{K}^2; L', L^2)$; hence the class of mixed quadrangles is a self-dual one. Moreover, since all points of $\mathsf{W}(\mathbb{K})$ are regular, so are all points of every mixed quadrangle, and hence so are all lines of it. Notice that, applying duality twice, the subquadrangle $\mathsf{W}(\mathbb{K}^2, \mathbb{K}'^2; L^2, L'^2)$ of $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ is isomorphic to $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ itself.

Let us finally mention that all points of a mixed quadrangle are centers of symmetry, and all lines are axes of symmetry. Moreover, it follows from [37] and Theorem 21.10 in [51] that, if all lines of a generalized quadrangle Γ are axes of symmetry, and at least one point is regular, then Γ is a mixed quadrangle.

3.2.3 Suzuki quadrangles

It is well known, see Theorem 7.3.2 of [59], that a mixed quadrangle $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ admits a polarity if and only if $\mathbb{K}$ admits a Tits endomorphism $\theta : \mathbb{K} \longrightarrow \mathbb{K}$ and we can choose $\mathbb{K}', L, L'$ such that $\mathbb{K}' = \mathbb{K}^\theta$ and $L' = L^\theta$. Hence every polarity in $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ is the restriction of a polarity in $\mathsf{W}(\mathbb{K}, \mathbb{K}'; \mathbb{K}, \mathbb{K}')$. So the case of $L = \mathbb{K}$ is a kind of principal case. A self-polar mixed quadrangle shall be called a *Suzuki quadrangle*.

Remark 3.2.2 The mixed quadrangles and mixed hexagons have a similar algebraic background in the theory of mixed groups, and for this reason many properties are alike.

3.3 Dual nets

In Section 3.1, it was mentioned that one can associate a dual net to a regular point of a generalized quadrangle. We now take a closer look at dual nets in order to state the results.

Let $\Delta = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a dual net. Noncollinear points shall be called *parallel*, it is easy to see that parallelism is an equivalence relation in $\mathcal{P}$. Call the dual parallel classes of points *vertical lines* and introduce a new point ∞ incident with all vertical lines. This way we create a linear space $\overline{\Gamma} = (\overline{\mathcal{P}}, \overline{\mathcal{L}}, \overline{\mathtt{I}})$ (a *linear space* is a point-line geometry in which every pair of distinct points is incident with a unique line). A *triangle* is a set of three pairwise intersecting distinct elements of $\mathcal{L}$, but such that all three lines do not have a point in common. The 3 intersection points are also viewed as belonging to the triangle. Two triangles are said to be *in perspective from a point x* if there are three different lines through x of $\overline{\Gamma}$ each incident with a unique point of each triangle. Consider the following two conditions:

(LD) For every pair of triangles which are in perspective from the point ∞, and for which two pairs of corresponding sides meet on a vertical line V, the third pair of corresponding sides also meets on V.

(VY) If a line L meets two sides of a proper triangle in two distinct points, then L intersects the third side too.

If we want to fix and include the line V of (LD) in our assumptions, we more specifically say that the dual net satisfies (LD) *with respect to the vertical line V*.

The letters (LD) and (VY) stand for *Little Desargues* and *Veblen-Young*, respectively.

3.4 Results on mixed quadrangles

A famous conjecture says that every generalized quadrangle all elements of which are regular is isomorphic to a mixed quadrangle (in the form of a problem, this is Problem 8 in Appendix E of [59]). In the finite case, generalized quadrangles all of whose points are regular are not classified, unless one requires an additional condition on the corresponding dual nets, or on the parameters. In [39] the condition that these dual nets satisfy the Axiom of Veblen-Young does the job. In the present chapter we will classify all generalized quadrangles with a lot of regular points and lines, and for which the dual nets associated to the regular points satisfy the Axiom of Veblen-Young. Postponing a discussion of what 'a lot' precisely means to Section 3.5.3 (see Theorems 3.5.8 and 3.5.9), we here state the weakest form.

Main Result 3.4.1 *A generalized quadrangle* Γ *is isomorphic to some mixed quadrangle* $\mathsf{W}(\mathbb{K},\mathbb{K}';L,\mathbb{K}')$ *if and only if all points and lines of* Γ *are regular and the dual net associated to each regular point satisfies Condition* (VY)*.*

In order to include all mixed quadrangles, we have to appeal to Condition (LD).

Main Result 3.4.2 *A generalized quadrangle* Γ *is isomorphic to some mixed quadrangle* $\mathsf{W}(\mathbb{K},\mathbb{K}';L,L')$ *if and only if all points and lines of* Γ *are regular and the dual net associated to each regular point satisfies Condition* (LD)*.*

3.5 Proofs

General idea. — First we show that under certain assumptions Condition (LD) follows from Condition (VY). Then, using a flag consisting of a regular point and line, such that the point satisfies (LD), we construct collineations of the generalized quadrangle, making the line into an axis of symmetry. Enough axes of symmetry will then imply that the quadrangle is a mixed quadrangle.

3.5.1 Dual nets satisfying the axiom of Veblen-Young

Let $\Gamma = (\mathcal{P},\mathcal{L},\mathtt{I})$ be a dual net. As before, we call the dual parallel classes of points *vertical lines* and introduce a new point ∞ incident with all vertical lines. This way we created a linear space $\overline{\Gamma} = (\overline{\mathcal{P}},\overline{\mathcal{L}},\overline{\mathtt{I}})$. If two lines L, M intersect in this linear space, we write $L \sim M$. Let $\mathcal{V}$ be the set of all vertical lines. Our aim is to prove that Condition (LD) follows from Condition (VY), if there exists at least one pair of nonintersecting lines.

So henceforth we assume that Γ satisfies (VY), and that there are at least two nonintersecting lines in Γ. Clearly, the latter condition is equivalent with Γ being not a dual affine plane.

We begin with defining a projective plane for every pair of intersecting lines L, M. Indeed, let L, M be two intersecting lines in Γ, and let x be their intersection point. Then we consider the set of lines intersecting both L and M in two distinct points, together with the set of lines incident with x and meeting some line K that intersects L and M in two distinct points. We denote this set by $\mathcal{B}^*$. The point set $\mathcal{A}$ is defined to be the set of points incident with at least one element of $\mathcal{B}^*$, together with ∞. Now add all vertical lines to $\mathcal{B}^*$ by defining $\mathcal{B} = \mathcal{B}^* \cup \mathcal{V}$. If we denote the restriction of $\overline{\mathtt{I}}$ still by $\overline{\mathtt{I}}$ (slightly abusing notation), then we claim that $\Delta_{L,M} = (\mathcal{A}, \mathcal{B}, \overline{\mathtt{I}})$ is a projective plane.

Indeed, this is in fact a routine check. Let us first show that two distinct lines X, Y always meet. If at least one of X, Y belongs to $\mathcal{V}$, or if both X, Y are incident with x, then this is trivial. If none of X, Y is incident with x, then this follows directly from (VY), as by definition both of X and Y meet both of L and M. If X is incident with x, then it intersects some line K which also intersects both of L and M in distinct points. Since we may assume $K \neq Y$, we may also assume that Y, K, L form a proper triangle (as otherwise Y, K, M form one). Now (VY) implies that X meets Y.

Now we show that two distinct points $y, z \in \mathcal{A}$ are joined by exactly one line in $\mathcal{B}$. Indeed, we clearly may assume that neither y or z coincides with ∞, and that they are not incident with the same vertical line. Hence they are incident with a unique member $X \in \mathcal{L}$. We must show that $X \in \mathcal{B}^*$. By definition, $y\overline{\mathtt{I}}Y \in \mathcal{B}^*$ and $z\overline{\mathtt{I}}Z \in \mathcal{B}^*$. Suppose that $Y\mathtt{I}x$. Let $K \in \mathcal{B}^*$ be such that K intersects L, M, Y in three different points, and suppose that y is not incident with K. Choose an arbitrary point y' incident with K and not parallel to y. The line Y' joining y and y' meets both of L and M by (VY). We have shown that we may assume that Y is not incident with x, and hence neither Z. Moreover, using (VY), we can arrange that Y, Z do not meet on L or M (if they do then we may re-choose Y not incident with the intersections of Z with L and M). Then X meets two sides of both the triangles Y, Z, L and Y, Z, M in distinct points, and hence (VY) implies that X meets both of L and M. If X is not incident with x, then $X \in \mathcal{B}^*$ by definition; if $x\mathtt{I}X$, then with $K \in \{Y, Z\}$, we see that again $X \in \mathcal{B}^*$.

Clearly $\Delta_{L,M} = \Delta_{L',M'}$ for L', M' distinct nonvertical lines of $\Delta_{L,M}$. Hence if two projective planes of this form share two nonvertical lines, then they coincide.

If we now remove from $\Delta_{L,M}$ the point ∞ and the vertical lines, then we obtain a dual affine plane. Our assumptions and the existence and uniqueness of the projective plane constructed above now implies that the dual of Γ is a subplane covered net in the sense of Johnson [17]. It follows from the latter paper that we can identify $\mathcal{P}$ with the points of a projective space $\mathbf{P}$ minus a subspace W of codimension 2, and $\mathcal{L}$ can be identified with the lines of $\mathbf{P}$ that do not intersect W. Our hypothesis that Γ is not a dual affine plane implies that the dimension of $\mathbf{P}$ is at least 3, and hence it is a Desarguesian projective space.

Now if a pair of triangles is in perspective from ∞, and if two pairs of corresponding sides meet, then in $\mathbf{P}$, this means that the two triangles are also in perspective from a point (because two corresponding pairs of sides must lie in the same plane), and so by Desargues' theorem, also the third pair of corresponding sides meets, and this intersection point is collinear with the two others. This shows (LD).

Hence we have proved the following theorem.

Theorem 3.5.1 *A dual net which is not a dual affine plane satisfies* (VY) *only if it satisfies* (LD).

One of our crucial tools to characterize the mixed quadrangles is Property (LD) for the nets associated to the regular points of some generalized quadrangle Γ, which we now know to hold if (VY) is satisfied for these dual nets in the case they are not dual affine planes. In dual affine planes (VY) holds trivially, but (LD) is not necessarily true. A sufficient condition for (LD) is that the corresponding projective plane is a Moufang plane. And that is exactly what we are going to prove in the case that the generalized quadrangle contains 'enough' projective points.

3.5.2 Generalized quadrangles with a lot of projective points

In this section we concentrate on generalized quadrangles with a number of projective points. In fact, we only need one projective point and a set of regular points. More precisely, let Γ be a generalized quadrangle and let $\mathcal{O}$ be a set of regular points of Γ. We assume the following two conditions on $\mathcal{O}$.

(PP) At least one member of $\mathcal{O}$ is a projective point.

(TP) If x, y are opposite points of Γ, then $|\{x,y\}^\perp \cap \mathcal{O}| \neq 1$.

Our aim is to prove that, under these assumptions, all points of $\mathcal{O}$ are projective and every corresponding perp-plane is a Moufang projective plane. We will need the following characterization of Moufang projective planes by H. Van Maldeghem [60]. In a projective plane, a line L is called an *axis of transitivity* if the pointwise stabilizer of L acts transitively on the points not incident with L.

Theorem 3.5.2 ([60]) *A projective plane is a Moufang plane if and only if each line L is an axis of transitivity.*

Henceforth Γ is a generalized quadrangle with $\mathcal{O}$ a set of regular points of Γ satisfying (PP) and (TP).

We start with proving that all elements of $\mathcal{O}$ are projective.

Lemma 3.5.3 *Every element of $\mathcal{O}$ is a projective point of Γ.*

Proof. We know that there is at least one point $p \in \mathcal{O}$ which is projective. Let q be any other element of $\mathcal{O}$. If q is opposite p, then Lemma 3.2.1 implies that q is projective. Now suppose $q \sim p$. Let x, y be opposite points collinear to p such that x is incident with the line pq, but $x \neq q$. Then $p \in \{x, y\}^\perp$, implying by (TP) that some other element $p' \in \mathcal{O} \setminus \{p\}$ also belongs to $\{x, y\}^\perp$. Clearly, p' is opposite p and therefore is a projective point. But p' is also opposite q and hence Lemma 3.2.1 implies that q is projective.

The lemma is proved. $\square$

We now prove a lemma that will generate collineations of the perp-planes Γ_p, for $p \in \mathcal{O}$.

Lemma 3.5.4 *Let $p, q \in \mathcal{O}$, with p opposite q. Then the following function $\theta_{p,q}$ defines an isomorphism between Γ_p and Γ_q^D :*

- (i) *A point x of Γ_p is mapped to the block $x^{\theta_{p,q}}$ of Γ_q consisting of all the points collinear with both x and q.*
- (ii) *A block α of Γ_p is mapped to the point $\alpha^{\theta_{p,q}}$ of Γ_q collinear with q and with all points of α.*

Proof. First we show that $\theta_{p,q}$ is well defined by proving that for each block α of Γ_p, there is indeed a unique point $a \sim q$ collinear with all points of α. We may assume that $\alpha \neq \{p, q\}^\perp$, as otherwise $a = q$ is easily seen to be that unique point. Since Γ_p is projective, there is a unique point $r \in \{p, q\}^\perp \cap \alpha$. Now a is necessarily the unique point on the line rq which is collinear with any point of $\alpha \setminus \{r\}$.

The definition of $\theta_{p,q}$ now easily implies that, if $x \in \alpha$, with $x \sim p$ and α a block of Γ_p, then $\alpha^{\theta_{p,q}} \in x^{\theta_{p,q}}$. Also, the inverse mapping is apparently given by $\theta_{q,p}$, hence $\theta_{p,q}$ is bijective and so defines an isomorphism from Γ_p to the dual of Γ_q. $\square$

Note that we can write $x^{\theta_{p,q}} = \{q, x\}^\perp$ and $\alpha^{\theta_{p,q}} = \alpha^{\perp\perp} \cap q^\perp$, with $x \sim p$ and α a block of Γ_p.

We now consider three different points $p_1, p_2, p_3 \in \mathcal{O}$, with p_3 opposite both p_1 and p_2. By the previous lemma, we can combine θ_{p_1,p_3} and θ_{p_3,p_2} to an isomorphism $\phi := \theta_{p_1,p_3}\theta_{p_3,p_2}$ between Γ_{p_1} and Γ_{p_2}. Let us calculate the image of a point x of Γ_{p_1} under ϕ:

$$x^\phi = x^{\theta_{p_1,p_3}\theta_{p_3,p_2}} = (\{x, p_3\}^\perp)^{\theta_{p_3,p_2}} = \{x, p_3\}^{\perp\perp} \cap p_2^\perp.$$

If we apply this to a point a in $\{p_1,p_2\}^\perp$, then, since $a \in \{a,p_3\}^{\perp\perp} \cap p_2^\perp$, we see that $a^\phi = a$ (note the independence of p_3). We also have $p_1^\phi = \{p_1,p_3\}^{\perp\perp} \cap p_2^\perp$.

Now let p_3' be another point of $\mathcal{O}\backslash\{p_1,p_2\}$ opposite both p_1,p_2. We obtain a different isomorphism $\phi' := \theta_{p_1,p_3'}\theta_{p_3',p_2}$ between the two perp-planes Γ_{p_1} and Γ_{p_2}. This allows us to construct a collineation $\tau := \phi^{-1}\phi'$ of Γ_{p_2}. Using the independence mentioned in the above paragraph we see that $\{p_1,p_2\}^\perp$ is fixed pointwise under the action of τ. Choose points x,y in Γ_{p_2} different from p_2 and not contained in $\{p_1,p_2\}^\perp$. We can choose $p_3 \in \mathcal{O}$ in such a way that $p_1^\phi = x$ (this is possible since the span $\{p_1,x\}^{\perp\perp}$ contains at least two points of $\mathcal{O}$, and we can choose p_3 as one of these points different from p_1; then $p_1^\phi = \{p_1,p_3\}^{\perp\perp} \cap p_2^\perp = \{p_1,x\}^{\perp\perp} \cap p_2^\perp = x$). Analogously, we can choose $p_3' \in \mathcal{O}$ in such a way that $p_1^{\phi'} = y$. Combining this we obtain $x^\tau = x^{\phi^{-1}\phi'} = {p_1}^{\phi'} = y$.

Consequently, the pointwise stabilizer of $\{p_1,p_2\}^\perp$ in the collineation group of Γ_{p_2} acts transitively on all the other points of the plane, possibly except p_2. But if p_2 was fixed by this stabilizer, then the orbits of the other points would completely lie on lines through p_2, which is impossible by the transitivity already shown. So the pointwise stabilizer of $\{p_1,p_2\}^\perp$ is transitive on all points of the perp-plane Γ_2 except for the points of $\{p_1,p_2\}^\perp$ itself. Hence $\{p_1,p_2\}^\perp$ is an axis of transitivity in the projective plane Γ_{p_2}.

We can even do better.

Lemma 3.5.5 *Each block α of Γ_{p_2} is an axis of transitivity.*

Proof. Let α be a block of Γ_{p_2} not incident with p_2, so that α is a perp $\{p_2,x\}^\perp$ with x a point of Γ opposite p_2. The span $\{p_2,x\}^{\perp\perp}$ is a perp and contains p_2, hence it contains a second point $p_4 \in \mathcal{O}$. This implies $\alpha = \{p_2,p_4\}^\perp$ and the assertion follows from our previous discussion.

The blocks through p_2 can now be mapped to blocks not through p_2 by the pointwise stabilizers of the blocks not containing p_2. So the blocks through p_2 are also axes of transitivity. □

Now Theorem 3.5.2 implies that Γ_{p_2}, and hence all perp-planes of points in $\mathcal{O}$, are Moufang projective planes, and in particular satisfy Condition (LD).

Hence, in this section, we have shown the following theorem.

Theorem 3.5.6 *Let Γ be a generalized quadrangle and let $\mathcal{O}$ be a subset of regular points of Γ satisfying* (PP) *and* (TP). *Then all points of $\mathcal{O}$ are projective and all corresponding perp-planes are Moufang projective planes, and satisfy, in particular,* (LD).

3.5.3 Quadrangles with regular points satisfying (LD)

In this section, we will prove Main Result 3.4.1 and Main Result 3.4.2. They will follow from Theorem 3.5.1, Theorem 3.5.6 and the following lemma.

Lemma 3.5.7 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized quadrangle containing a flag $\{p, L\}$ consisting of a regular line L and a regular point p such that the dual net associated to p satisfies* (LD) *with respect to the vertical line defined by L. Then L is an axis of symmetry for Γ.*

Proof. First of all we notice that if there are only three lines through each point in Γ, then regularity of a point implies that there are also exactly three points on each line. Such a generalized quadrangle is always isomorphic to $\mathsf{W}(2)$, in which the assertion clearly holds. So we may assume that there are at least four lines through each point.

Let M be a line through p different from L. Let a, a' be two points incident with M but different from p. We will gradually construct a collineation θ mapping a to a' fixing L pointwise, and fixing all lines meeting L.

Lines intersecting L

For these lines N we set $N^\theta = N$.

Points collinear to p not on L

Let N be a line through p different from both L and M, and let q be a point on N different from p; then we define the image of p under θ as follows. The perp α in Γ_p^* through a and q intersects L in a point b. Then q^θ is the intersection point of N with the perp through $a' = a^\theta$ and b. This way the image of a defines the image of a point q collinear with p, but not with a. We denote this as: $a \rightarrow q$. The image of a point c on M is defined by $q \rightarrow c$, for some point $q \sim p$ not collinear to c.

To show that θ is well defined, we have to prove that combining $a \rightarrow b$ with $b \rightarrow c$ (we will abbreviate this by $a \rightarrow b \rightarrow c$) where b is not collinear with either a or c, is independent of the choice of b. So suppose a, b, c and d are four points in $p^\perp$ not on L such that both b and d are not collinear with either a or c.

(i) If a is not collinear with c, then $a \to b \to c$ is equivalent with $a \to c$. Indeed, this follows directly from the condition (LD) applied to the triangles a, b, c and $a^\theta, b^\theta, c^\theta$ (where θ is defined using $a \to b \to c$). Similarly, $a \to d \to c$ is equivalent with $a \to c$ and the result follows.

(ii) Suppose that a is collinear with c. If b is not collinear with d then $a \to b \to c$ is equivalent with $a \to b \to d \to c$ which in its turn is equivalent with $a \to d \to c$. If b and d are collinear then we can choose a point e collinear with p but not with a or b and not on L (because there are at least four lines through a point in Γ). Then $a \to b \to c$ is equivalent with $a \to b \to e \to c$, $a \to e \to c$ and $a \to d \to c$ by using the previous arguments.

It is important to note that θ preserves the perps in Γ_p^*.

Lines and points opposite L or p

Let N be a line opposite L, and let $p \mathtt{I} A \mathtt{I} q \mathtt{I} N$. Then we define N^θ to be the unique line incident with q^θ in the (line) span containing L and N. The image of a point t incident with N is defined as the intersection point of N^θ with the unique line K through t intersecting L (these lines indeed intersect because of the regularity of L). The only thing left to show is that t^θ is well defined. If t is collinear with p then this is clear, so suppose $t \not\sim p$. The lines through t define a perp in Γ_p^*, which will be mapped to another perp by θ while fixing the intersection point r of K and L of the perp. The images of all the lines through t must meet K. Since they also must contain a point of the perp $\{p, t^\theta\}^\perp$, we see that they are all incident with t^θ. Hence t^θ is well defined. It is now also clear that θ and its inverse preserve incidence, and hence it is a symmetry. Since a and a' were basically arbitrary, it follows that L is an axis of symmetry, and the lemma follows. $\square$

We are now ready to prove slightly more general results than Main Results 3.4.1 and 3.4.2.

Theorem 3.5.8 *A generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a mixed quadrangle if and only if there is a subset $\mathcal{O} \subseteq \mathcal{P}$ of points and a nonempty subset $\mathcal{S} \subseteq \mathcal{L}$ of lines satisfying the following conditions.*

(i) *All points of $\mathcal{O}$ and all lines of $\mathcal{S}$ are regular.*

(ii) *Every (line) span containing a line of $\mathcal{S}$ contains at least two lines of $\mathcal{S}$.*

(*iii*) *Every element of* $\mathcal{S}$ *is incident with some element of* $\mathcal{O}$.

(*iv*) *The dual net associated to each regular point* x *of* $\mathcal{O}$ *satisfies* (LD) *with respect to a vertical line given by some element of* $\mathcal{S}$ *incident with* x.

In particular, if all elements of Γ *are regular and* (*iv*) *holds, then* Γ *is a mixed quadrangle.*

Proof. Fix a line L of $\mathcal{S}$. By (*iii*), there is a regular point p incident with L with the property that, by (*iv*), the associated dual net satisfies (LD). Lemma 3.5.7 implies that L is an axis of symmetry. Likewise, every element of $\mathcal{S}$ is an axis of symmetry. Let M be an arbitrary line opposite L. The span $\{L, M\}^{\perp\perp}$ contains some element $K \in \mathcal{S} \setminus \{L\}$, by (*ii*). Since L is an axis of symmetry, there is a collineation mapping K to M. Since K is an axis of symmetry, so is M. Hence all lines opposite L, and likewise all lines opposite K, are axes of symmetry. It is easy to see that for each element N of $\{L, K\}^{\perp}$ there is a line opposite all of L, K, N. We conclude that all lines of Γ are axes of symmetry. Since we have at least one regular point, we can conclude that Γ is a mixed quadrangle (see Section 3.2.2). $\square$

Theorem 3.5.9 *A generalized quadrangle* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ *is isomorphic to a mixed quadrangle* $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, \mathbb{K}')$ *if and only if there is a subset* $\mathcal{O} \subseteq \mathcal{P}$ *of points and a nonempty subset* $\mathcal{S} \subseteq \mathcal{L}$ *of lines satisfying the following conditions.*

(*i*) *All points of* $\mathcal{O}$ *and all lines of* $\mathcal{S}$ *are regular.*

(*ii*) *Every span containing a point of* $\mathcal{O}$ *contains at least two points of* $\mathcal{O}$.

(*ii*)′ *Every (line) span containing a line of* $\mathcal{S}$ *contains at least two lines of* $\mathcal{S}$.

(*iii*) *Every element of* $\mathcal{S}$ *is incident with some element of* $\mathcal{O}$.

(*iv*) *The dual net associated to each regular point of* $\mathcal{O}$ *satisfies* (VY).

In particular, if all elements of Γ *are regular and* (*iv*) *holds, then* Γ *is isomorphic to a mixed quadrangle* $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, \mathbb{K}')$.

Proof. If none of the points of $\mathcal{O}$ are projective, then Theorem 3.5.1 implies that, together with (*iv*), each dual net associated to a regular point of $\mathcal{O}$ satisfies (LD). From Theorem 3.5.8 we infer that Γ is isomorphic to a mixed quadrangle $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$. We now show that $L' = \mathbb{K}'$. Assume, by way of contradiction, that $L' \neq \mathbb{K}'$. Then we can choose elements $k, k' \in \mathbb{K}'$ such that $kk' \notin L'$. One easily calculates that in the coordinate representation of $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$, the perp $T_{a,a'} := \{(\infty), (a, l, a')\}^{\perp}$ consists of the point (a) together with the points $(x, ax + a')$, $x \in L'$. Now we consider the perps $T_{0,0} = \{(0)\} \cup \{(x, 0) : x \in L'\}$ and $T_{0,1} = \{(0)\} \cup \{(x, 1) : x \in L'\}$, which both meet the perps $T_{1,0} = \{(1)\} \cup \{(x, x) : x \in L'\}$ and $T_{(k^{-1}+1)^{-1}, k'(k^{-1}+1)^{-1}} = \{((k^{-1} + 1)^{-1})\} \cup \{(x, (k^{-1} + 1)^{-1}x + (k^{-1} + 1)^{-1}k') : x \in L'\}$. By (VY), the latter two perps must intersect. Hence there must exist $x \in L'$ such that

$$x = (k^{-1} + 1)^{-1}x + (k^{-1} + 1)^{-1}k',$$

which is equivalent with $kk' = x \in L'$, a contradiction.

If at least one point of $\mathcal{O}$ is projective, then by Theorem 3.5.6 and Assumption (ii), all points of $\mathcal{O}$ are projective, and all corresponding perp-planes are Moufang and satisfy (LD). Since they also satisfy (VY), the result now again follows from Theorem 3.5.8 and the computation performed in the previous paragraph. $\square$

3.6 Results on generalized Suzuki-Tits inversive planes

Let ρ be a polarity in a Suzuki quadrangle and let $\mathcal{O}$ be the set of its absolute points, which forms an ovoid of the Suzuki quadrangle - the so-called *Suzuki-Tits ovoid*. Viewed as a subset of points of $\mathsf{PG}(3, \mathbb{K})$, it is also an *ovoid* in the sense of Tits [43] (which is a set of points $\mathcal{O}$ in $\mathsf{PG}(3, \mathbb{K})$, such that for each point $p \in \mathcal{O}$ there is a plane for which the intersection with $\mathcal{O}$ only contains p, while all lines through p not in the plane intersect $\mathcal{O}$ in exactly two points).

First consider the case where the field $\mathbb{K}$ is perfect, so that the Suzuki quadrangle is in fact a symplectic quadrangle. With each ovoid of $\mathsf{PG}(3, \mathbb{K})$ corresponds an *inversive plane*, i.e. a rank 2 geometry consisting of a set of points and a set of circles, which are the intersections of planes in $\mathsf{PG}(3, \mathbb{K})$ with $\mathcal{O}$ containing more than one point, and provided with the natural incidence relation. It satisfies the following axioms.

[MP1'] Each 3 different points are contained in exactly one circle.

[MP2] For each circle C and each pair of points x, y with $x \in C$ and $y \notin C$, there exists an unique circle C' which contains y and touches C in x.

('*Touching*' circles are circles that meet in a unique point.) Another way to construct the circles would be taking for each nonabsolute point of the quadrangle the points of $\mathcal{O}$ collinear to it.

The inversive planes arising from the (perfect) Suzuki-Tits ovoids have been characterized by a set of axioms by H. Van Maldeghem in [58]. We will generalize this result below.

We now turn to the general case, not demanding perfectness anymore. Here we define the set of circles as follows. A *circle* is the set of points of $\mathcal{O}$ collinear to some point not contained in $\mathcal{O}$. If we denote the family of circles by $\mathcal{C}$, then we obtain a geometry $(\mathcal{O}, \mathcal{C}, \in \text{ or } \ni)$. These *generalized inversive planes* satisfy the following axioms.

[MP1] Each 3 different points are contained in at most one circle.

[MP2] For each circle C and for every pair of points $x, y \in \mathcal{P}$ with $x \in C$ and $y \notin C$, there exists a unique circle C' which contains y and touches C in x.

[CH1] There exist no 3 circles which are pairwise touching in different points.

[CH2] For each circle C and every pair of points $x, y \notin C$, we have the following three possibilities: no circle containing x, y touches C, one circle does, or all circles do.

Remark 3.6.1 The circles in the nonperfect case also can be realized as plane intersections, but not all plane intersections containing more than one point give rise to circles.

There are a lot of geometries that satisfy the above axioms. For instance every inversive plane obtained from an ovoid of a projective 3-space over a field with characteristic 2. In order to further distinguish the geometries corresponding to the polarities in Suzuki quadrangles, we use the observation that each circle C has a very special point, which we denote by ∂C and call the *gnarl* of the circle. Indeed, if C is the set of points of $\mathcal{O}$ collinear with the point $x \notin \mathcal{O}$, then there is a unique absolute line incident with x and hence a unique point ∂C of C such that the line joining ∂C with x is absolute. Alternatively, ∂C is the unique point of C incident with x^ρ.

The function ∂ has the following properties.

[ST1] For each pair of points x, y there exists a unique circle C which contains x and such that $\partial C = y$.

[ST2] For each circle C and point $x \notin C$, there is at most one circle C' which contains both of x and ∂C, and such that $\partial C' \in C$.

[TR] Let C be an arbitrary circle, and let $x, y \in C$ ($\partial C \neq x \neq y \neq \partial C$). Let D be a circle through $\partial C \neq \partial D$. For each circle E different from C, containing both x and ∂C, and intersecting D in two distinct points $\partial C, z$, we consider the circle E^* through z and touching C in ∂C. We also consider the circle E^{**} containing y, touching E in ∂C. Then $E^* \cap E^{**}$ is contained in a circle D' through ∂C which is essentially independent of E.

If $\mathbb{K}$ is perfect, we have an inversive plane, and this allows us to impose a stronger version of [MP1].

[MP1'] Each 3 different points are contained in exactly one circle.

Remark 3.6.2 As the terminology of gnarl suggests, generalized inversive planes are examples of the geometries defined by Moufang sets described in Section 1.8.1. The Moufang set in question acts on the Suzuki-Tits ovoid, and is called accordingly the *Suzuki-Tits Moufang set*. It is in some way the characteristic 2 counterpart of the Ree-Tits Moufang set.

The properties mentioned so far characterize the generalized inversive planes arising from polarities in mixed quadrangles.

Main Result 3.6.3 *Let $\mathcal{P}$ be a set and $\mathcal{C}$ a set of distinguished subsets of $\mathcal{P}$ all containing at least 3 elements. Also suppose there is a map $\partial : \mathcal{C} \rightarrow \mathcal{P}$ such that $\forall C \in \mathcal{C} : \partial C \in C$. We call the elements of $\mathcal{C}$ circles and if two of them have only one point in common, we say they touch at that point. Then $(\mathcal{P}, \mathcal{C}, \partial)$ satisfies the conditions* [MP1], [MP2], [CH1], [CH2], [ST1], [ST2] *and* [TR]*, if and only if $\mathcal{P}$ can be embedded in a self-polar mixed quadrangle $\mathsf{W}(\mathbb{K}, \mathbb{K}'; L, L')$ as the set of absolute points of a polarity ρ. The set $\mathcal{C}$ corresponds to the family of sets of absolute points collinear with a nonabsolute point, and the map ∂ maps a circle onto its gnarl, i.e., ∂C, with $C = x^\perp \cap \mathcal{P}$, is the unique point of $\mathcal{P}$ incident with x^ρ.*

If we want to restrict to self-polar mixed quadrangles of type $\mathsf{W}(\mathbb{K}, \mathbb{K}'; \mathbb{K}, \mathbb{K}')$, then we may introduce the following alternative axiom (where we call a set of points *cocircular* if they belong to a common circle).

[F] Let x be an arbitrary point, and let x_1, x_2, x_3 be three points pairwise cocircular with x, but not all cocircular with x. If a point y is cocircular with x and x_1, and also with x en x_2, but if y, x, x_1, x_2 are not cocircular, then y, x, x_3 are cocircular.

And we will show:

Main Result 3.6.4 *Let $\mathcal{P}$ a set and $\mathcal{C}$ a set of distinguished subsets of $\mathcal{P}$ all containing at least 3 elements. Also suppose there is a map $\partial : \mathcal{C} \to \mathcal{P}$ such that $\forall C \in \mathcal{C} : \partial C \in C$. We call the elements of $\mathcal{C}$ circles and if two of them have only one point in common, we say they touch at that point. Then $(\mathcal{P}, \mathcal{C}, \partial)$ satisfies the conditions* [MP1], [MP2], [CH1], [CH2], [ST1], [ST2] *and* [F]*, if and only if $\mathcal{P}$ can be embedded in a self-polar mixed quadrangle $\mathsf{W}(\mathbb{K}, \mathbb{K}'; \mathbb{K}, \mathbb{K}')$ as the set of absolute points of a polarity ρ. The set $\mathcal{C}$ corresponds to the family of sets of absolute points collinear with a nonabsolute point, and the map ∂ maps a circle onto its gnarl, i.e., ∂C, with $C = x^{\perp} \cap \mathcal{P}$, is the unique point of $\mathcal{P}$ incident with x^{ρ}.*

As mentioned before, if $\mathbb{K}$ is perfect, then this is an inversive plane which allows us to impose a stronger version of [MP1], which was denoted as [MP1']. Using this axiom instead of [MP1] allows us to improve upon the characterization given in [58], by deleting one axiom.

Main Result 3.6.5 *Let $\mathcal{P}$ a set and $\mathcal{C}$ a set of distinguished subsets of $\mathcal{P}$ all containing at least 3 elements. Also suppose there is a map $\partial : \mathcal{C} \to \mathcal{P}$ such that $\forall C \in \mathcal{C} : \partial C \in C$. We call the elements of $\mathcal{C}$ circles and if two of them have only one point in common, we say they touch at that point. Then $(\mathcal{P}, \mathcal{C}, \partial)$ satisfies the conditions* [MP1'], [MP2], [CH1], [CH2], [ST1] *and* [ST2]*, if and only if $\mathcal{P}$ can be embedded in a projective space $\mathsf{PG}(3, \mathbb{K})$, for some perfect field $\mathbb{K}$ of characteristic 2 admitting a Tits automorphism θ, such that $\mathcal{P}$ is the set of absolute points of a polarity of a certain symplectic quadrangle $\mathsf{W}(\mathbb{K})$ in $\mathsf{PG}(3, \mathbb{K})$ and the set of circles of $\mathcal{P}$ is equal to the set of plane sections of $\mathcal{P}$ in $\mathsf{PG}(3, \mathbb{K})$.*

3.7 Proofs

General idea. — Using the axioms, we construct a generalized quadrangle from the generalized inversive plane. Using the results from Section 3.5.3, we then show this is a mixed quadrangle, satisfying the desired properties.

In this section, we generalize the main theorem of [58] to all self-polar mixed quadrangles. It will turn out that we need exactly the more general form in the previous Main Results 3.4.1 and 3.4.2 in order to prove Main Results 3.6.3 and 3.6.4.

Let $\mathcal{P}$ be a set and $\mathcal{C}$ a distinguished set of subsets of $\mathcal{P}$ all containing at least 3 elements. Also we have a map $\partial : \mathcal{C} \to \mathcal{P}$ such that $\forall C \in \mathcal{C} : \partial C \in C$. We call the elements of $\mathcal{C}$

circles and if two of them have only one point in common, we say they *touch at that point*. The element ∂C of a circle C will be called the *gnarl* of C. We assume that $(\mathcal{P}, \mathcal{C}, \partial)$ satisfies the conditions [MP1], [MP2], [CH1], [CH2], [ST1], [ST2] and [TR].

First, we will prove some further properties using these axioms. All these lemmas are copies or reformulations of lemmas in [58], with similar proofs, although [MP1] and [ST2] here are slightly weaker than the corresponding axioms in [58]. We mention them without proof.

Lemma 3.7.1 *Suppose we have 3 different circles C, D and E. If C and E both touch D at some point x, then C touches E at x.*

Lemma 3.7.2 *For every circle C and every point x not contained in C there exists a unique circle D with $\partial D \in C$, $\partial C \neq \partial D$ and containing both of x and ∂C.*

Lemma 3.7.3 *If a circle C touches D at ∂D, then $\partial C = \partial D$.*

We now proceed with constructing a geometry $\Gamma = (\mathcal{P}^*, \mathcal{L}^*, \mathtt{I})$ out of $(\mathcal{P}, \mathcal{C}, \partial)$. This is also similar to the perfect case in [58], but since it is crucial for the rest, we repeat it here.

We identify both $\mathcal{P}^*$ and $\mathcal{L}^*$ with the union of $\mathcal{P}$ and $\mathcal{C}$. To avoid confusing the elements of $\mathcal{P}^*$ with those of $\mathcal{L}^*$, we put a subscript p or l to denote to which set it belongs, i.e., for all $x \in \mathcal{P}$ and all $C \in \mathcal{C}$, we have $x_p, C_p \in \mathcal{P}$ and $x_l, C_l \in L$. A point x_p, $x \in \mathcal{P}$, is incident with y_l, $y \in \mathcal{P}$, if and only if $x = y$. A point x_p, $x \in \mathcal{P}$, is incident with the line C_l, $C \in \mathcal{C}$, if and only if C_p is incident with x_l if and only if $\partial C = x$. Finally, the point C_p, $C \in \mathcal{C}$, is incident with D_l, $D \in \mathcal{C}$, if and only if $\partial C \in D$, $\partial D \in C$ and $\partial C \neq \partial D$. This new geometry Γ obviously admits a polarity $\rho : \mathcal{P}^* \leftrightarrow \mathcal{L}^* : C_p \mapsto C_l, x_p \mapsto x_l, C_l \mapsto C_p, x_l \mapsto x_p$. The absolute flags are of the form $\{x_p, x_l\}$ with $x \in \mathcal{P}$.

The following lemma tells us when two points are collinear in Γ.

Lemma 3.7.4 *For all $x, y \in \mathcal{P}$ and $C, D \in \mathcal{C}$, the following holds.*

(i) *The point x_p is collinear with the point y_p if and only if $x = y$.*

(ii) *The point x_p is collinear with the point C_p if and only if $x \in C$.*

(iii) *The point C_p is collinear with the point D_p if and only if C and D touch each other.*

Also, two different elements of $\mathcal{P}^*$ are incident with at most one element of $\mathcal{L}^*$.

Proof.

(*i*) Suppose $x_p \mathtt{I} C_l \mathtt{I} y_p$; then, by definition, $x = \partial C = y$.

(*ii*) If x_p is collinear with C_p, then $x_p \mathtt{I} x_l \mathtt{I} C_p$, or there is an $E \in \mathcal{C}$ such that $x_p \mathtt{I} E_l \mathtt{I} C_p$. In the first case we have $x = \partial C \in C$; in the second case $x = \partial D \in C$. Suppose now that $x \in C$. If $x = \partial C$, then $x_p \mathtt{I} x_l \mathtt{I} C_p$ and so x_p is collinear with C_p. If $x \neq \partial C$, then there is a unique circle D with gnarl x through δC by [ST1], so $x_p \mathtt{I} D_l \mathtt{I} C_p$.

(*iii*) If $C_p \mathtt{I} z_l \mathtt{I} D_p$, with $z \in \mathcal{P}$, then the claim follows from [ST1]. Suppose that $C_p \mathtt{I} E_l \mathtt{I} D_p$, with $E \in \mathcal{C}$. Then $\partial E \in C \cap D$, and since $D \neq C$, we have $\partial D \neq \partial C$. Clearly, also $\partial C \neq \partial E \neq \partial D$. Since $\partial C, \partial D \in E$, the result follows from [ST2].

Conversely, suppose C and D touch. If they touch at ∂C, then by Lemma 3.7.3, $\partial C = \partial D$ and $C_p \mathtt{I} (\partial C)_l \mathtt{I} D_p$. So we can assume that they touch at a point x different from ∂C and different from ∂D. Let E be the circle containing ∂D and so that $\partial E = x$, and assume by way of contradiction that $\partial C \notin E$. By Lemma 3.7.2 there exists a circle F containing ∂C and x, and with $\partial F \in E$. Our assumption implies $F \neq E$. We claim that either $D = F$ or F touches D at x. Indeed, if not, then D and F share some point $y \neq x$. Note that $y \notin E$ as otherwise F and D coincide with E, a contradiction. But then both D and F have their gnarl on E, contain the gnarl of E and contain a further point $y \notin E$. Lemma 3.7.2 implies that $D = F$. Our claim follows. Now by Lemma 3.7.1, F touches C at x, contradicting $\partial C \in F \cap C$. So we have that $C_p \mathtt{I} E_l \mathtt{I} D_p$. $\square$

Our goal now is to show that Γ is a Suzuki quadrangle. First we prove that Γ is a generalized quadrangle.

Lemma 3.7.5 *There are no three different, pairwise collinear points in $\mathcal{P}^*$ unless they are all incident with the same line.*

Proof. First suppose one of the points is of the form x_p with $x \in \mathcal{P}$; then the other points must be of the form C_p and D_p $(C, D \in \mathcal{C})$ with $x = C \cap D$. If $x = \partial C$, then $x = \partial D$ and all the points are incident with the line x_l. If $x \neq \partial C$, then $C_p \mathtt{I} E_l \mathtt{I} D_p$, with $E \in \mathcal{C}$ and hence $\partial E = x$. But then also $x_p \mathtt{I} E_l$.

Now suppose we have three points of the form C_p, D_p and E_p with $C, D, E \in \mathcal{C}$. By collinearity, the circles C, D and E all have to touch each other. Axiom [CH1] implies that they touch in one common point x. So C_p, D_p and E_p are all collinear with x_p. By the first part of the proof we obtain that C_p, D_p, x_p lie on one line F_l and C_p, E_p, x_p lie one line G_l ($F, G \in \mathcal{C}$). Both F_l and G_l contain C_p and x_p, so, by the last assertion of Lemma 3.7.4, C_p, D_p and E_p all are incident with $F_l = G_l$. $\square$

Lemma 3.7.6 *A point in $\mathcal{P}^*$ and a line in $\mathcal{L}^*$ lie at distance at most* 3 *from each other.*

Proof. We prove that for any point X and any line M not incident with X, there is a point on M collinear with X.

Case 1. First suppose $X = x_p$ and $M = y_l$, with $x, y \in \mathcal{P}, x \neq y$. Condition [ST1] tells us that there is a circle C with gnarl x trough y. Now C_p is collinear with y_p (by Lemma 3.7.4) and incident with x_p (since $\partial C = x$).

Case 2. Secondly suppose $X = x_p$ and $M = C_l$, with $x \in \mathcal{P}$, $C \in \mathcal{L}$, and $\partial C \neq x$. If $x \in C$ then the point D_p, with D the circle with gnarl x through ∂C, is incident with C_l and collinear with x_p.

If x is not on C, then by Lemma 3.7.2 there exists a circle D through x sharing two distinct points (namely, ∂C and ∂D) with C. The point D_p is now on C_l and collinear with x_p.

Case 3. Taking duality in account, there is one case left to consider, where $X = C_p$ and $M = D_l$, with $C, D \in \mathcal{L}$ and C_p not incident with D_l in Γ. The first possibility is that $\partial C = \partial D$. Then C_p is collinear with $(\partial C)_p$, which is incident with D_l.

Now suppose that $\partial C \neq \partial D \in C$. Then the point $(\partial D)_p$ is collinear with C_p and lies on D_l. The case where $\partial C \in D$ is the dual of the case just handled.

So we may assume that $\partial C \notin D, \partial D \notin C$. By Axiom [MP2] and the fact that a circle contains 3 or more points, there are at least two circles C_1 and C_2 with gnarl ∂D and touching C. By Axiom [CH1] these two circles have a second point $x \neq \partial D$ in common. Due to [CH2] all circles through x and ∂D touch C. So we can consider the circle E, guaranteed to exist by Lemma 3.7.2, which contains the two points $\partial D, x$, and has its gnarl on D. This circle E touches C, hence E_p is collinear with C_p and is incident with D_l. $\square$

Now we want to apply Theorem 3.5.9. Hence we have to find a suitable set of regular points and regular lines. We will consider the set of absolute points and absolute lines of Γ with respect to the polarity ρ mentioned above.

Lemma 3.7.7 *The absolute points and lines of* Γ *are regular.*

Proof. Because of the polarity ρ, we only need to prove that when three different points $\{U, V, W\}$ are collinear with two noncollinear points X, Y, with $X = x_p$ for some $x \in \mathcal{P}$, then each point collinear with U and V is also collinear with W.

Since U and V are two noncollinear points collinear with x_p, we may write, by Lemma 3.7.4, $U = C_p$, $V = D_p$, with $C, D \in \mathcal{C}$, $x \in C \cap D$, and with C and D not touching each other. The latter condition implies that C and D share an additional point $y \neq x$. Then y_p is collinear with both C_p and D_p. We set $W = E_p$, with $E \in \mathcal{C}$ and $x \in E$. If $Y = y_p$, then $y \in E$. The points collinear with C_p and D_p are, besides x_p and y_p, all points F_p, with F a circle touching both C and D. But by Condition [CH2], the circle E also touches F, so E_p is collinear with F_p.

If $Y \neq y_p$, then it is one of the F_p above, and the assertion follows anyway. $\square$

Note that the previous proof immediately implies the following lemma.

Lemma 3.7.8 *Every span of* Γ *containing an absolute point of* ρ *contains exactly two absolute points. Also the dual holds.*

In view of the two previous lemmas, it only remains to check Condition (*iv*) of Theorem 3.5.9 in order to prove that Γ is a mixed quadrangle. Therefore we have to look at the dual net corresponding to a regular point x_p, $x \in \mathcal{P}$. In view of the previous results, one can easily give the following description of the dual net $\Gamma^*_{x_p}$. The points are the circles containing x and the blocks are the points different from x, with incidence given by containment. The circles with gnarl x correspond to a class of parallel points given by the line $x_l = x_p^\rho$ of the quadrangle Γ. Then the following observations are immediate.

Lemma 3.7.9 (*i*) *With the above notation,* $(\mathcal{P}, \mathcal{C}, \partial)$ *satisfies Condition* [TR] *if and only if for each point* $x \in \mathcal{P}$*, the dual net* $\Gamma^*_{x_p}$ *satisfies Axiom* (LD) *with respect to the parallel class of points given by the line* x_l *of* Γ.

(*ii*) *With the above notation,* $(\mathcal{P}, \mathcal{C}, \partial)$ *satisfies Condition* [F] *if and only if for each point* $x \in \mathcal{P}$*, the dual net* $\Gamma^*_{x_p}$ *satisfies Axiom* (VY).

Putting together the last four lemmas, Main Results 3.6.3 and 3.6.4 follow from Theorem 3.5.8 and 3.5.9, respectively.

If we substitute Condition [MP1] by Condition [MP1'], then the dual net $\Gamma^*_{x_p}$ is clearly a dual affine plane, so Axiom (VY), or the equivalent Condition [F], is trivially true. Whence Main Result 3.6.5 holds (the other direction of that theorem being contained in [58]).

3.8 Metasymplectic spaces

We use the following definition of metasymplectic spaces ([59, p. 79]): a *metasymplectic space* $\mathcal{M}$ is a rank 4 geometry with four types of elements, called *points*, *lines*, *planes* and *hyperlines*, and a (symmetric) incidence relation satisfying the four axioms listed below.

(M1) The residue of any flag of type {point, line} or {plane, hyperline} is a projective plane.

(M2) The residue of any flag of type {point, plane}, {line, hyperline} or {line, plane} is a generalized digon.

(M3) The residue of any flag of type {point, hyperline} is a generalized quadrangle.

(M4) Two distinct nonpoint elements have different sets of points incident with them.

Using (M1) to (M4), one can prove that the dual property of (M4) is satisfied as well, making the definition self-dual. The flag complexes of these metasymplectic spaces form the buildings of type F_4. Note that these axioms imply thickness because generalized polygons are thick by definition.

Remark 3.8.1 Instead of the notion 'hyperline', some authors use the term 'symplecton'.

3.8.1 Embeddings of quadrangles in the metasymplectic space

We consider embeddings of the following kind: given a metasymplectic space $\mathcal{M}$ together with a set $\mathcal{P}$ of points of $\mathcal{M}$ and a set $\mathcal{H}$ of hyperlines of $\mathcal{M}$, the incidence relation defined on them by taking the restriction of the incidence relation of $\mathcal{M}$, defines a generalized quadrangle Γ. We then say that the quadrangle Γ is *point-hyperline embedded* in $\mathcal{M}$.

Examples of such embeddings are constructed by Hendrik Van Maldeghem and Bernhard Mühlherr in [21]. There it is shown that the exceptional Moufang quadrangles of type F_4 and certain mixed quadrangles appear as fixed point structures of involutions of metasymplectic spaces over fields with characteristic 2. As the subquadrangles of a point-hyperline embedded quadrangle will also be point-hyperline embedded, orthogonal and symplectic quadrangles also appear. All these quadrangles are Moufang and share the property that no two points of the quadrangle are collinear in the metasymplectic space.

Embeddings will be denoted *improper* if all hyperlines in $\mathcal{H}$ incident with a certain point in $\mathcal{P}$ always share a line. By substituting each point with its associated line in this case, it follows that we can view the quadrangle embedded 'by' lines and hyperlines.

We now construct an example of an improper embedding. Let $\{p, L\}$ be an incident point-line pair of a metasymplectic space $\mathcal{M}$ which is defined over some field containing the finite field of four elements. The residue of this flag forms a projective plane, containing a sub projective plane isomorphic to $\mathsf{PG}(2,4)$. The symplectic quadrangle $\mathsf{W}(2)$ can be embedded in this plane (see [7]). Returning to our metasymplectic space $\mathcal{M}$, we have embedded $\mathsf{W}(2)$ in $\mathcal{M}$ 'by' planes and hyperlines. Now choose for each plane of this embedding a point incident with the plane, producing a point-hyperline embedding. If the field which defines the metasymplectic space is 'large enough', it is clear that the choices can be made such that no two collinear points of the quadrangle are collinear in the metasymplectic space.

Remark 3.8.2 All of the known embeddings such that no two points of the quadrangle are collinear in the metasymplectic space, occur in characteristic 2 or are improper. The existence of the known proper embeddings originates from an algebraic setting, however this algebraic setting does not yield such embeddings for odd characteristic. For this reason it could be conjectured that these only occur in characteristic 2. More about the underlying algebraic setting can be found in [59, App. C].

3.9 Results on embedded quadrangles in metasymplectic spaces

We now pose the inverse question: when is a point-hyperline embedded quadrangle Moufang?

Main Result 3.9.1 *Let Γ be a generalized quadrangle point-hyperline embedded in a metasymplectic space $\mathcal{M}$, with $\mathcal{P}$ the set of points and $\mathcal{H}$ the set of lines of the quadrangle. Then Γ will be either a Moufang quadrangle, or improperly embedded, if the following property holds:*

(OV) No 2 points of $\mathcal{P}$ in the same hyperline of $\mathcal{H}$ are collinear in $\mathcal{M}$.

Remark 3.9.2 It can be shown that the residue of a hyperline forms a polar space (see property (M9) in the next section). Condition (OV) then reformulates to: the points of $\mathcal{P}$ in the same hyperline of $\mathcal{H}$ form a partial ovoid of the corresponding polar space.

Remark 3.9.3 Note that our definition of generalized polygon asks that Γ is thick: if this would not be the case, counterexamples occur.

3.10 Proof

General idea. — We first investigate what the possibilities are for a single apartment of the generalized quadrangle to be embedded. Using this, we can show that the embedding is convex (see Section 1.1.3), or improper. Applying a result of H. Van Maldeghem and B. Mühlherr, this implies Main Result 3.9.1.

Suppose we have $\mathcal{M}, \Gamma, \mathcal{P}, \mathcal{H}$ as given in the statement of the above result. We do not require that the property (OV) holds yet.

If we refer to a point or line, we mean a point or line of the metasymplectic space, unless explicitly noted otherwise.

3.10.1 Further concepts and some lemmas about metasymplectic spaces

The following lemma can be found in [59, p. 80] - we will not reproduce the proof here.

Lemma 3.10.1 *We have the following properties:*

(M5) Let x and y be two points of $\mathcal{M}$. Then one of the following situations occurs:

- $x = y$.
- *There is a unique line incident with both x and y. In this case, we call x and y* collinear.
- *There is a unique hyperline incident with both x and y. In this case there is no line incident with both x and y, and we call x and y* cohyperlinear.
- *There is a unique point z collinear with both x and y. In this case we call x and y* almost opposite.

- *There is no point collinear with both x and y.*

(M6) The intersection of two hyperlines is either empty, or a point, or a plane.

(M7) Let x be a point and h a hyperline of $\mathcal{M}$. Then one of the following situations occurs:

- $x \in h$.
- *There is a unique line L in h such that x is collinear with all points of L. Every point y of h which is collinear with all points of L is cohyperlinear with x and the unique hyperline containing both also contains L. Every other point z of h is almost opposite x and the unique point collinear with both lies on L.*
- *There is a unique point u of h cohyperlinear with x, and the hyperline containing x and u only has u in common with h. All points v of h collinear with u are almost opposite x, and the point collinear with both doesn't lie in h. All points w of h cohyperlinear with u are opposite x.*

(M9) The residue of a hyperline forms a polar space. □

Note that the dual statements also hold. Property (M8) given in [59] is omitted as we will not need it here.

Let W be the spherical Coxeter group of type F_4; this is the group generated by symbols s_1, s_2, s_3, s_4 and identity element e, with relations $(s_i s_j)^{m_{ij}} = e$, and m_{ij} as given in the following matrix:

$$(m_{ij}) = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}$$

Two maximal flags of a metasymplectic space (which are chambers of the F_4-building) are s_1, s_2, s_3 or s_4-*adjacent* respectively, if those two flags differ in a point, line, plane or hyperline respectively.

We define the spherical Coxeter group $W_{\{1,2,3\}}$ to be the subgroup of W generated by s_1, s_2 and s_3, and analogously $W_{\{2,3,4\}}$ will be the subgroup generated by s_2, s_3 and s_4.

Lemma 3.10.2 *The following double cosets are written in such a way that the representative is of shortest length:*

- $W_{\{2,3,4\}}s_1s_2s_3s_2s_1W_{\{2,3,4\}}, W_{\{1,2,3\}}s_4s_3s_2s_3s_4W_{\{1,2,3\}}$;
- $W_{\{1,2,3\}}s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1W_{\{2,3,4\}}, W_{\{2,3,4\}}s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4W_{\{1,2,3\}}$;
- $W_{\{2,3,4\}}s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1W_{\{2,3,4\}}$,
  $W_{\{1,2,3\}}s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4W_{\{1,2,3\}}$;
- $W_{\{2,3,4\}}s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4W_{\{1,2,3\}}$,
  $W_{\{1,2,3\}}s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4s_1s_2s_3s_2s_1W_{\{2,3,4\}}$.

Proof. By long but straightforward calculations. □

The following important theorem by Bernhard Mühlherr and Hendrik Van Maldeghem ([22]) gives us more information about convex subbuildings (see Section 1.1.3 for a definition).

Theorem 3.10.3 *A convex subbuilding of a Moufang building is again a Moufang building.* □

Or applied to our case (F_4-buildings are always Moufang):

Corollary 3.10.4 *A convex point-hyperline embedded quadrangle Γ in a metasymplectic space $\mathcal{M}$ is Moufang.* □

3.10.2 Embedding apartments

First we investigate how the apartments of the quadrangle are embedded in $\mathcal{M}$. Let $\{p,h\},\{q,g\}$ ($p,q \in \mathcal{P}, h,g \in \mathcal{H}$) be 2 chambers of Γ such that $p \notin g, q \notin h$ and the hyperlines h and g intersect in a point or plane (these are the only possibilities due to (M6)). Collinearity and opposition will be used relative to the metasymplectic space $\mathcal{M}$ and not the quadrangle Γ, unless stated otherwise.

Lemma 3.10.5 *If h and g intersect in a point u, then one of the following holds:*

- *The points p and q are opposite and both are cohyperlinear with u.*
- *The points p and q are almost opposite and at least one point is collinear with u.*
- *The points p and q are cohyperlinear and both are collinear with u.*

- *The points p and q are collinear and both are collinear with u.*

Proof.

- If p and q are opposite then (M7) applied to the point p and hyperline g tells us that there is exactly one point of g cohyperlinear with p; therefore u will be this point. It now follows that p and q both are cohyperlinear with u.

- If p and q are almost opposite, then applying (M7) to p and g leaves us with two possibilities. If there is a unique point (this point will again be denoted with u) of g cohyperlinear with p, then q will be collinear with u. If on the other hand there is a unique line L in g of points collinear with p, then the possibility that u is cohyperlinear with p implies that u is collinear with all points of L and that h contains L. But h and g intersect in a point and do not have a line in common, so p is collinear with u.

- If p and q are cohyperlinear, then again applying (M7) to p and g implies that there is a line L in g of points collinear with p (the other possibility for cohyperlinearity would imply that $u = q$, which is ruled out). If u would be cohyperlinear with p, then h and g would intersect in a line as explained in the previous point, so p is collinear with u. Interchanging the roles of p and q gives that both points are collinear with u.

- In the last case where p is collinear with q, Property (M7) implies that p is collinear with all the points of a line L of g. If u would be cohyperlinear with p then the unique hyperline h containing u and p would also contain q, which is impossible. It follows that u is collinear with p and also with q. □

Lemma 3.10.6 *If h and g intersect in a plane π, then p and q are not opposite.*

Proof. Suppose p and q are opposite. Then p and q are on distance 3 from each other, but (M9) gives us that the points on distance 1 from p in π will be on a line of π, and the same holds for q. Two lines in a plane always have at least one point in common, so the distance between p and q is 2, resulting in a contradiction. □

Let the points p, q, r, s and hyperlines denoted by pq, qr, rs, sp define an apartment in Γ. If the points p and r are opposite then the two lemmas above imply that if two points of the apartment are collinear in Γ, they are cohyperlinear in $\mathcal{M}$. The hyperline pq intersects qr in a point - the same holds for sp and rs. The other mutual positions can be divided in 2 possibilities due to the third lemma:

- The hyperlines pq and sp intersect in a point. Then q and s are opposite and qr and rs also intersect in a point.
- The hyperlines pq and sp intersect in a plane. Then q and s are not opposite and qr and rs also intersect in a plane.

We now state a lemma which will be used to 'reduce' the quadrangle.

Lemma 3.10.7 *If each two points in a set X of points in $\mathcal{M}$ are collinear, then this set is contained in a plane.*

Proof. Let $x \in X$ be a point. If we take the residue of this point, we obtain a dual rank 3 polar space where the lines xy with $y \in X \backslash \{x\}$ form dual generators. All these generators intersect in lines of the polar space. If we would have a proper 'triangle' of these generators and lines, the lines would meet in a single point. Taking the residue again of this point, we would have a proper triangle in a quadrangle, which is impossible. So all the generators xy with $y \in X \backslash \{x\}$ share at least one line, and translating this back to $\mathcal{M}$ we obtain that all points are contained in a plane. □

3.10.3 Embedding quadrangles

Condition (OV)

From now on suppose that condition (OV) holds. Let Σ be an apartment of Γ. If two hyperlines of Σ which intersect in Γ share a point, then there has to be an opposite pair of points (in $\mathcal{M}$) in Γ, so according to the previous section the other two hyperlines in Σ must also intersect in a point. Because the projectivity group of a point of our quadrangle is 2-transitive on the (hyper)lines through that point, either any two hyperlines in $\mathcal{H}$ which intersect in Γ share a point, or all hyperlines in $\mathcal{H}$ which intersect in Γ share a plane.

In the second case we can replace each point $p \in \mathcal{P}$ with a line L_p such that all hyperlines of $\mathcal{H}$ through p contain that line (this is possible due to the dual of Lemma 3.10.7), so we obtain a quadrangle consisting of lines and hyperlines where no two lines which are collinear in the quadrangle are contained in one plane (otherwise the points corresponding to the two lines would be collinear in $\mathcal{M}$), so we are in the improper case.

In the first case we have that two points of $\mathcal{P}$ are cohyperlinear if they are collinear in Γ, and opposite if they are not. For hyperlines in $\mathcal{H}$ we have the dual properties. In the next section we will show convexity of quadrangles within $\mathcal{M}$ with such properties.

Convexity of quadrangles

In this section we prove that the embedded quadrangle Γ is convex in $\mathcal{M}$. Herefore we use that two points of $\mathcal{P}$ are cohyperlinear if they are collinear in Γ, and opposite if they are not, and the dual properties for hyperlines in $\mathcal{H}$, then

The next lemma gives us the needed building blocks for the rest of the proof of convexity.

Lemma 3.10.8 *Let h be a hyperline and p, q be two cohyperlinear points in h. If we have a chamber C containing p and h, then there is a shortest gallery with associated word $s_1s_2s_3s_2s_1$ from C to a chamber containing q.*

Proof. The residue of h will be a rank 3 polar space with p and q opposite points in it. The theory of buildings tells us that we can embed the flags $C\backslash\{h\}$ and $\{q\}$ of this polar space in an octahedron (this forms an apartment of the rank 3 polar space, see [28]). In this octahedron it is easily seen that there is a shortest gallery with associated word $s_1s_2s_3s_2s_1$ from C to a chamber containing q and h. Because this word is a shortest presentation of the corresponding element in the group W, this will be a shortest gallery. $\square$

Now let A and B be two flags of Γ. It is clear that there exists a shortest gallery γ_Γ in Γ between these flags starting from a chamber C in Γ containing A, to a chamber D containing B. Using the above lemma (and the dual statement) to 'lift' this gallery to a gallery $\gamma_\mathcal{M}$ in $\mathcal{M}$, we obtain galleries from each chamber containing C (now viewed as flags in $\mathcal{M}$) to a certain chamber containing D (viewed as a flag in $\mathcal{M}$) with words consisting of an alternating consecution of the 'building block' $s_1s_2s_3s_2s_1$ and the dual $s_4s_3s_2s_3s_4$. Lemma 3.10.2 implies that these are also shortest galleries between chambers containing A and chambers containing B in $\mathcal{M}$. Because the galleries can start from each chamber containing C, the product of simplex B with simplex A will be completely contained within C and so also within the subbuilding Γ, hence the embedded quadrangle Γ is convex. Corollary 3.10.4 now implies that the quadrangle Γ is Moufang.

Chapter 4

‘Rank three’ case, or two-dimensional $\mathbb{R}$-buildings

The results in this chapter are about $\mathbb{R}$-buildings, the first series of results are about two-dimensional $\mathbb{R}$-buildings, the others hold for general $\mathbb{R}$-buildings.

Polygons with valuation. — In 1986, Jacques Tits ([47]) classified the affine buildings of rank at least 4. In fact, he also included in his work the so-called *systèmes d'appartements*, or *apartment systems*. Later on people also called them *nondiscrete affine buildings* ([28]) or $\mathbb{R}$-*buildings*. Basically, these are building-like structures with one big difference: they are no longer simplicial. Easy examples are $\mathbb{R}$-trees (rank 2 case; these are trees that continuously branch), or the ‘buildings’ related to the ‘parahoric’ subgroups of a Chevalley group over a field with nondiscrete valuation. From the geometric point of view, the case of rank 3 — when the apartments are 2-dimensional — is very interesting since nonclassical phenomena occur there.

In [47] Tits associates to every *symmetric* apartment system a so-called *building at infinity*, which is a simplicial spherical building, see also [8]. The rank of this building at infinity is precisely the dimension of its apartments. Hence, in the 2-dimensional case, generalized polygons appear. When the apartment system is irreducible, then this polygon is not a digon. In the simplicial case, the only generalized polygons that occur are projective planes, generalized quadrangles and generalized hexagons.

In a series of rather long papers [52, 53, 54, 56, 15], Hendrik Van Maldeghem (jointly with Guy Hanssens in the last quoted paper) investigates in detail two classes of affine buildings (namely, those with projective planes and generalized quadrangles at infinity) and characterizes the corresponding spherical buildings at infinity. This leads to many new examples of such affine buildings, explicitly defined and with knowledge of the automorphism groups. Originally, the characterization made use of the notion of a *discrete valuation* on the algebraic structures that coordinatize projective planes and generalized quadrangles, but in later papers [55, 57], the valuation was defined directly on the geometry. The hope was that with such a direct definition, the case of type $\widetilde{\mathsf{G}}_2$, which was the only remaining case, would become treatable with much less effort. One of the reasons why it did *not* is that, although the paper [55] provides the exact condition for a generalized hexagon with valuation, the lack of symmetry in the formulae prevented from deducing a *general* formulae independent of the type, and hence from (1) further generalization to nondiscrete valuations, and (2) composing a type-free proof.

In the present chapter, we start such a type-free approach, which ought to eventually lead to a characterization of all irreducible 2-dimensional affine apartment systems. More in particular, we first show how any irreducible 2-dimensional affine apartment system gives rise to a generalized polygon with a specific valuation, by which we mean, with the terminology of [55], an explicitly defined weight sequence. One of the crucial observations to achieve this is to slightly modify, or re-scale, the valuation as defined from a rank 3 affine building as defined in [57]. Indeed, roughly speaking, the valuation between two elements as defined in [57] counted the graph theoretic distance between two vertices in the simplicial complex related to the affine building. The purpose was to end up with a natural number. But taking the Euclidean distance instead will put much more symmetry into the picture, and at the same time we will have a closed formula for the weight sequences. Also the nondiscrete case can clearly be included in a natural way. The fact that the discrete case enjoys a characterization as in [55] seems to be a happy coincidence in this viewpoint.

The other question now is, what can we say when we are given a generalized polygon with (nondiscrete) valuation? The first thing we obtain is that the only weight sequences (for a definition see below) that can occur are exactly the ones that occur for the valuations of generalized polygons at infinity of two-dimensional $\mathbb{R}$-buildings. Moreover, if $n = 3, 4$ we provide a detailed proof for the complete equivalence between generalized n-gons with real valuation and 2-dimensional $\mathbb{R}$-buildings. As an application we construct classes of explicit examples of such structures which are not of Bruhat-Tits type, and which include locally finite ones. These constructions are similar to the constructions due to Hendrik Van Maldeghem in the simplicial case, see [52, 53, 54, 56].

Remarkably, as a byproduct, we obtain that projective planes with valuation are equivalent with ultrametric planes in which all triangles satisfy the sine rule, for an appropriate though natural definition for angles between lines.

In the ideal case, one would like to prove the conjecture that the just mentioned equivalence holds for all $n \geq 3$. However, this seems to be out of reach for now. In our present approach, the complications in the proofs seem to grow exponentially with the girth. For $n = 5$, it is just feasible, but too long to include here. For $n = 6$, assuming discreteness allows for an alternative argument, as we shall see. Notice that our proofs for $n = 3, 4$ provide different arguments for the simplicial case, which are in fact drastically shorter and more direct than the original proofs of Hendrik Van Maldeghem. One does not need to go around the *Hjelmslev geometries* and the rather complicated axiomatization related to this (see e.g. [15]). These geometries were needed to define the vertices of the affine building. In the present approach, we do no longer have vertices, but the points of the apartment system are the different valuations that emerge from the given one. This simple idea, however, requires a lot of unavoidable technicalities to take care of. For example, it is already fairly technical to prove that the residue of an n-gon with valuation is again a generalized n-gon. We will do this explicitly for $n \leq 6$. It will be clear that similar methods should work in general, but our present approach fails for that. So, on the one hand, the present methods are significantly stronger than the old ones developed by Hendrik Van Maldeghem in the eighties, on the other hand, one needs an improvement of another magnitude to prove the full conjecture.

These results are joint work with Hendrik Van Maldeghem and are contained in two papers, both accepted for publication, one in *Adv. Geom.*, the other in *Pure Appl. Math. Q.*

Completeness of $\mathbb{R}$-buildings. — As already indicated in Section 1.8.2, there exist various results which hold for complete $\mathbb{R}$-buildings. All affine (discrete) buildings are complete, but this is not true for general $\mathbb{R}$-buildings. The question that now rises is: which $\mathbb{R}$-buildings are complete? Especially for those $\mathbb{R}$-buildings arising from Tits' classification ([47]) a full answer is something that should be aimed at.

In Section 4.10 we take the first step to such an answer. We prove that an $\mathbb{R}$-building is complete, if and only if all the $\mathbb{R}$-trees corresponding to its walls are complete. The next step (which we are currently researching) is then to determine which $\mathbb{R}$-trees are complete. This problem seems to be answerable in algebraic terms for those $\mathbb{R}$-trees coming from higher-dimensional $\mathbb{R}$-buildings.

Subbuildings of $\mathbb{R}$-buildings corresponding to fixbuildings at infinity. — Just like the result mentioned in the previous paragraph, this result is also a research in progress. The setting is the following: when an automorphism group acts on a spherical building, then the fixed structure is in 'most' cases again a (spherical) building. Such a statement is not true for ($\mathbb{R}$-)buildings which are not spherical, because there is no such thing as opposition in these cases.

Consider some affine building Λ with a group G acting on it, while the fixed structure in Λ is not necessarily again an affine building, the fixed structure in the spherical building Λ_∞ at infinity is most often again a spherical building Λ'_∞. If we now return to the finite part of the affine building, one might wonder if there exists an embedded affine building Λ' with Λ'_∞ at infinity.

In Section 4.14 we give a positive answer for some, but not all, cases using geometric methods. For one of the steps in the proof we generalize the notion of trees corresponding to walls and sector-panels (see Section 1.8.2). This generalization is not entirely unknown, but a proof doesn't seem to exist in the literature.

These results are joint work with Hendrik Van Maldeghem.

4.1 Two-dimensional $\mathbb{R}$-buildings

As mentioned in Section 1.8.2, the $\mathbb{R}$-buildings of dimension at least 3 are known. For the first series of results of this chapter we will only deal with the (unclassifiable) $\mathbb{R}$-buildings of dimension 2, i.e., $|S| = 2$ and $\overline{W}$ is the dihedral group of order $2n$, for some $n \in \mathbb{N}$, $n \geq 3$. So the building at infinity and the residues are (weak) generalized n-gons. The elements of the (weak) generalized polygon at infinity correspond to sector-panels of the $\mathbb{R}$-building. So one can discern two classes of sector-panels in the $\mathbb{R}$-building, one corresponding to the *points* P, the other to the *lines* L (the choice which type of sector-panels correspond to the points or lines can be chosen arbitrarily). Roman letters will be used for elements of the building at infinity, Greek letters for points of Λ.

Let x, y be two adjacent elements of Λ_∞ and $\alpha \in \Lambda$; then we denote the length (measured with the distance d) of the common part of the sector-panels x_α and y_α by $u_\alpha(x, y)$.

4.2 Polygons with valuation

Now we continue with defining generalized polygons with valuation. Let $\Gamma = (P, L, \mathtt{I})$ be a generalized n-gon with point set P and line set L, and let u be a function called *valuation* acting on both pairs of collinear points and pairs of concurrent lines, and images in $\mathbb{R}^+ \cup \{\infty\}$ (we use the natural order on this set with ∞ as largest element). Then we call (Γ, u) an *n-gon with (nondiscrete) valuation* and *weight sequence* $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots a_{2n-1}) \in (\mathbb{R}^+)^{2n-2}$ if the following conditions are met:

(U1) On each line there exists a pair of points p and q such that $u(p,q) = 0$, and dually for points.

(U2) $u(x,y) = \infty$ if and only if $x = y$.

(U3) $u(x,y) < u(y,z)$ implies $u(x,z) = u(x,y)$ if x, y and z are collinear points or concurrent lines.

(U4) Whenever $x_0 \mathtt{I} x_1 \mathtt{I} x_2 \mathtt{I} \ldots \mathtt{I} x_{2n} = x_0$, with $x_i \in P \cup L$, one has

$$\sum_{i=1}^{n-1} a_i u(x_{i-1}, x_{i+1}) = \sum_{i=n+1}^{2n-1} a_i u(x_{i-1}, x_{i+1}).$$

One direct implication of (U3) is that u is symmetric (by putting $x = z$). Also remark that this definition is self-dual, so whenever a statement is proven, we also have proven the dual statement. Finally, we note that, due to (U2), Axiom (U4) is trivially satisfied whenever the x_i, $0 \leq i \leq 2n$, form a degenerate apartment.

Remark 4.2.1 The difference with the definition in [55] is that in the current one, the type of the element x_0 is arbitrary, while in [55], x_0 was required to be a line. On the other hand, in [55], the image of u had to be natural or ∞. The main result of [55] says that, in this case, $n \in \{3, 4, 6\}$, the function u is also a valuation on the dual n-gon, and the weight sequences are uniquely determined up to duality. These weight sequences are, however, only self-dual if $n = 3$. Hence, only in the case $n = 3$, a valuation on an n-gon in the sense of [55] will be a valuation on an n-gon in the above sense. However, rescaling the valuation between lines by a factor $\sqrt{2}$ (multiplying or dividing according to the weight sequence) for $n = 4$ turns the valuation on a 4-gon in the sense of [55] into a valuation in the above sense. Similarly for 6-gons. Taking this rescaling into account, we see that the above definition is essentially a generalization of the definition in [55]. We will come back to this in more detail in Section 4.4.1, where we will show how our main results relate to the conjectures stated in [55] and [57].

If we speak about the valuation of a side or corner x in an ordinary n-gon Ω, we mean the valuation between (respectively) the two corners or sides incident with x in Ω. If we talk about the valuations in an ordinary n-gon, then we mean all the valuations of sides and corners. A path $(x_0, x_1, \ldots, x_m)$ is said to have valuation zero if $u(x_{i-1}, x_{i+1}) = 0$ for each $i \in \{1, 2, \ldots, m-1\}$. Because of (U2) such a path has to be nonstammering. We now show some preliminary lemmas which we will use to formulate one of the main results.

Lemma 4.2.2 *Given a line L and a point $p\mathtt{I}L$, then there exists a point $q\mathtt{I}L$ such that $u(p,q)=0$.*

Proof. Due to (U1) there exist two points $r, s\mathtt{I}L$ such that $u(r,s)=0$. Applying (U3) we obtain that either $u(p,r)=0$ or $u(p,s)=0$, and in each case we have found a suitable q. $\square$

Lemma 4.2.3 *Each path $(x_0, x_1, \ldots, x_m)$ with $m \leq n+1$ and valuation zero is contained in an ordinary n-gon Ω where all the valuations of corners and sides are zero.*

Proof. Using the previous lemma we can extend the path to a path $(x_0, x_1, \ldots, x_n, x_{n+1})$ with valuation zero. It is now easily seen that the other valuations in the unique ordinary n-gon containing this path are zero too by (U4). $\square$

In order to make notations easier, an ordinary n-gon with all valuations zero will be referred to as a *nonfolded n-gon*. If there are exactly two nonzero valuations in (necessarily) opposite elements x and y of an ordinary n-gon, then this ordinary n-gon will be referred to as a *simply folded n-gon folded along x (or y)*, and two elements in such an n-gon at the same distance from x (and hence also at the same distance from y) are said to be *folded together* in that n-gon. The Main Result 4.3.2 will imply that $a_1 = a_{n+1}$ and that the valuations in x and y are equal due to (U4).

Two opposite elements in Γ are said to be *residually opposite* if there is a shortest path between them with valuation zero. If this is the case, then by (U4) all shortest paths between both elements have valuation zero. If x is an element of Γ, then we denote with $[x]_{\mathrm{opp}}$ the set of residually opposite elements to x. This set is nonempty due to the previous lemma. We say that two elements x and y are *residually equivalent* if $[x]_{\mathrm{opp}} = [y]_{\mathrm{opp}}$. The equivalence class is denoted by $[x]=[y]$. It is clear that all elements of one equivalence class share the same type, so these classes can be referred to as *residual points* ($[P]$) or *residual lines* ($[B]$) depending on the type. A residual point $[p]$ is said to be incident with a residual line $[L]$ if there are $p' \in [p]$ and $L' \in [L]$ such that $p'\mathtt{I}L'$. We then write $[p]\mathtt{I}_r[L]$. The geometry $\Gamma_r([P],[B],I_r)$ is the *residue* defined by u. The distance d_r in the incidence graph of this geometry is called the *residual distance*.

Remark 4.2.4 Note that we already have defined a notion of residue, which was associated to a point of an $\mathbb{R}$-building, in Section 4.1. It follows from Main Result 4.3.1 in the next section, and from the definition of residues in $\mathbb{R}$-buildings, that for a generalized polygon with valuation defined by a point in a two-dimensional $\mathbb{R}$-building, the two notions are essentially the same. From the context it should be clear which one is meant.

4.3 Results on two-dimensional $\mathbb{R}$-buildings and polygons with valuation

Main Result 4.3.1 *Let $(\Lambda, \mathcal{F})$ be a two-dimensional $\mathbb{R}$-building and $\alpha \in \Lambda$. Then u_α as defined in Section 4.1 defines a valuation on the generalized n-gon at infinity Λ_∞, with weight sequence $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1})$, where $a_i = |\sin(i\pi/n)|$.*

For the other three main results, let (Γ, u) be a generalized n-gon with (nondiscrete) valuation and weight sequence $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1})$.

Main Result 4.3.2 *If u has nonzero values, then the weight sequence $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1})$ is a multiple of the weight sequence $(b_1, b_2, \ldots, b_{n-1}, b_{n+1}, \ldots, b_{2n-1})$ with $b_i = |\sin(i\pi/n)|$.*

Main Result 4.3.3 *If $3 \leq n \leq 6$, the residue defined by u is a (weak) generalized n-gon.*

Main Result 4.3.4 *If $n \in \{3, 4\}$, or if $n = 6$ and u is discrete, there exists a two-dimensional $\mathbb{R}$-building $(\Lambda, \mathcal{F})$ such that Γ is isomorphic to the generalized polygon at infinity of $(\Lambda, \mathcal{F})$ with valuation as in Main Result 4.3.1.*

4.4 Applications

We list some applications and corollaries of the main results.

4.4.1 The discrete case

Let (U4′) be the Condition (U4) with the additional requirement that $x_0 \in L$, and let $\Gamma = (P, L, \mathtt{I})$ be a generalized n-gon, $n \geq 3$. Suppose that (Γ, u) satisfies (U1), (U2), (U3) and (U4′), and suppose in addition that the image of u is in $\mathbb{N} \cup \{\infty\}$, the set of natural numbers, including 0, together with ∞. Then we say that (Γ, u) is a generalized polygon *with discrete valuation.* The main result of [55] says that, in this case, $n \in \{3, 4, 6\}$ and the weight sequence $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1})$ can be chosen as follows.

(WS3) If $n = 3$, then $(a_1, a_2, a_4, a_5) = (1, 1, 1, 1)$.

(WS4) If $n = 4$, then $(a_1, a_2, a_3, a_5, a_6, a_7) = (1, 1, 1, 1, 1, 1)$ or $(1, 2, 1, 1, 2, 1)$.

(WS6) If $n = 6$, then $(a_1, a_2, \ldots, a_5, a_7, \ldots, a_{11}) = (1, 1, 2, 1, 1, 1, 1, 2, 1, 1)$ or $(1, 3, 2, 3, 1, 1, 3, 2, 3, 1)$.

In the cases (WS4) and (WS6), where there are two possibilities, it is proved in [55] that the weight sequences are dual to one another, i.e., if (Γ, u) has one weight sequence, then, if Γ^D is the dual of Γ (obtained from Γ by interchanging the point set and the line set), then (Γ^D, u) is a polygon with discrete valuation with respect to the other weight sequence.

This gave birth to the conjecture that *a generalized hexagon Γ is 'isomorphic' to the building at infinity of some (thick) affine building of type $\widetilde{\mathsf{G}}_2$ if and only if there exists u such that (Γ, u) is a generalized hexagon with discrete valuation and with one of the two above weight sequences.* The Main Results 4.3.1 and 4.3.2 seem to be in contradiction with this, since, applied to discrete affine buildings of type $\widetilde{\mathsf{G}}_2$, there is only one weight sequence, namely

$$(a_1, a_2, \ldots, a_5, a_7, \ldots, a_{11}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$

and it does not consist of only natural numbers! But the above conjecture was evidenced by the situation for the types $\widetilde{\mathsf{A}}_2$ and $\widetilde{\mathsf{C}}_2$, where the valuation measured simplicial distance, and not Euclidean distance, as in the present approach. In the $\widetilde{\mathsf{G}}_2$ case, this means that, in view of the fact that the lengths of the panels (of a chamber) containing the special vertex (for terminology, see [47]) have ratio $1 : \sqrt{3}$, to go from the weight sequence of the present approach to the weight sequences of the discrete valuation, we must multiply the valuation on the point pairs with $\sqrt{3}$ (or do this with the valuation on line pairs), and then take a suitable multiple.

As explained earlier, one can do a similar procedure with type $\widetilde{\mathsf{C}}_2$, as is clear from the above.

4.4.2 Ultrametric projective planes

In this application we explore a surprising link between projective planes with valuations and some geometric conditions from Euclidean geometry.

Suppose (Γ, u) is a generalized triangle (or projective plane) with valuation. Choose $t \in \mathbb{R}$ with $t > 1$. We then can define a function $\mathsf{d}(p, q) = t^{-u(p,q)} \in [0, 1]$ on pairs of points, and a similar function $\angle(L, M) = \arcsin(t^{-u(L,M)}) \in [0, \pi/2]$ on pairs of lines.

Theorem 4.4.1 *A projective plane* Γ *with a distance function* d *on pairs of points valued in* $[0,1]$ *and an angle function* $\angle$ *on pairs of lines valued in* $[0,\pi/2]$*, is constructed from a projective plane with valuation as above, and hence is isomorphic to the building at infinity of some* $\mathbb{R}$*-building, if and only if the following conditions are fulfilled.*

(M1) d *is an ultrametric (this is a metric satisfying the stronger triangular inequality* $\mathsf{d}(p,q) \leq \max(\mathsf{d}(p,r),\mathsf{d}(r,q))$*).*

(M2) *Two lines have angle zero if and only if they are equal.*

(M3) *On each line there are two points on maximal distance 1 from each other.*

(M4) *Through each point there are two lines with a right* $(\pi/2)$ *angle.*

(M5) *The sine rule is fulfilled, i.e., if we have a triangle with lengths of the sides* A, B *and* C *and opposing angles* α, β *and* γ, *then*

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}.$$

The proof is postponed to Section 4.9.

4.4.3 Examples and constructions

$n = 3$

Here we rely on some results for the discrete case. Hendrik Van Maldeghem proved in [57] that the notion of a projective plane with valuation is equivalent to one of a planar ternary ring with valuation. Moreover he also investigated in [52] how the valuation behaves in planar ternary rings with extra algebraic properties (nearfields, quasifields, linear PTRs, etc.). In particular he proved the following result, the arguments of which can be copied verbatim in the nondiscrete case.

Proposition 4.4.2 *A* quasifield with valuation v*, which is a unary function with values in* $\mathbb{Z}\cup\{\infty\}$ *gives rise to a planar ternary ring with valuation (and so also to a projective plane with valuation, and an affine apartment system with a projective plane at infinity), if the following three conditions are fulfilled:*

(V1) $v(a) = \infty$ *if and only if* $a = 0$.

(V2) If $v(a) < v(b)$, then $v(a+b) = v(a)$.

(V3) $v(a_1 b - a_2 b) = v(a_1 - a_2) + v(b)$.

We now construct such quasifields (again inspired by results of Hendrik Van Maldeghem in [52], but now with the function v having values in $\mathbb{R} \cup \{\infty\}$). Let $K_{+,\cdot}$ be a field with a nondiscrete valuation v in the classical sense (which is in fact the above definition for quasifields applied to fields, so (V3) becomes $v(ab) = v(a) + v(b)$).

Remark 4.4.3 Notice that the classical affine apartment systems with a (Desarguesian) projective plane at infinity already appear here by taking quasifields with valuation which are (skew) fields.

Now let α be a field automorphism, with finite order, of K, preserving the valuation v. So α generates a finite group of automorphisms G. One can define the *norm map* $n : K \to K : a \mapsto \prod_{\alpha' \in G} a^{\alpha'}$. Notice that $v(n(a)) = |G| v(a)$. Let σ be a map from the image of the norm map n to G such that $\sigma(1)$ is the unit element of G, and so that $v(a) = v(b)$ implies $\sigma(n(a)) = \sigma(n(b))$.

It follows that one can construct an André quasifield $K_{+,\odot}$ by taking the elements of K with the addition of the field and a new multiplication $\odot : K \times K \to K : (a, b) \mapsto a \cdot b^{\sigma(n(a))}$. Moreover, we now show that this quasifield with the map v forms a quasifield with valuation. We only have to verify (V3) for the new multiplication. First remark that $v(a \odot b) = v(a \cdot b^{\sigma(n(a))}) = v(a) + v(b^{\sigma(n(a))}) = v(a) + v(b)$. The last step holds because α, and so all elements of G, preserve v.

We now calculate $v(a_1 \odot b - a_2 \odot b)$. There are two possibilities that can occur.

- $v(a_1) \neq v(a_2)$ - suppose without loss of generality that $v(a_1) < v(a_2)$. Then

$$\begin{aligned} v(a_1 \odot b - a_2 \odot b) &= v(a_1 \odot b) && (4.1)\\ &= v(a_1) + v(b) && (4.2)\\ &= v(a_1 - a_2) + v(b), && (4.3) \end{aligned}$$

  where the first step is true because $v(a_1 \odot b) = v(a_1) + v(b) < v(a_2) + v(b) = v(a_2 \odot b)$, (V2), and $v(-1) = 0$ (which easily follows from the definition of valuation).

- The other possibility is that $v(a_1) = v(a_2)$. Then
$$v(a_1 \odot b - a_2 \odot b) = v(a_1 \cdot b^{\sigma(n(a_1))} - a_2 \cdot b^{\sigma(n(a_2))}) \tag{4.4}$$
$$= v((a_1 - a_2) \cdot b^{\sigma(n(a_1))}) \tag{4.5}$$
$$= v(a_1 - a_2) + v(b^{\sigma(n(a_1))}) \tag{4.6}$$
$$= v(a_1 - a_2) + v(b), \tag{4.7}$$
where the second step holds because $v(a_1) = v(a_2)$ implies $\sigma(n(a_1)) = \sigma(n(a_2))$.

Combining both cases, we see that (V3) holds for the quasifield $K_{+,\odot}$ with valuation v.

We now provide some explicit examples of the above situation. Let k be any field, let M be a subset of $\mathbb{N}\backslash\{0\}$ generated multiplicatively by a certain set of primes. Now let K be the field of rational functions in t, but allowing all rational powers r/s of t with $s \in M$. If $k(t) = f(t)/g(t) \in K$ with $f(t)$ and $g(t)$ polynomials (also allowing powers of the form above), we then set $v(k(t))$ to be the minimal nonvanishing power of t in $f(t)$, minus the minimal nonvanishing power of t in $g(t)$. One verifies that K together with v forms a field with valuation.

- Let k be a finite field with characteristic p and M the set of integer powers of p. Then a suitable choice of α is the automorphism that maps $t^{\frac{r}{s}}$ to $(\frac{t^{1/s}}{1+t^{1/s}})^r$.
- Now let k be any field and M generated by all the odd primes (so M is the set of the odd nonnegative integers). Now one can set α to be the automorphism that maps $t^{\frac{r}{s}}$ to $(-t^{\frac{1}{s}})^r$.

All of these examples have a nonclassical projective plane at infinity, but have classical residues. In addition the residues of the $\mathbb{R}$-building are finite when k is finite.

There are also examples where one can choose one residue completely freely. For a given planar ternary ring R, one can define a 'positively valuated ternary ring' $R\{t\}$, similarly as in the discrete case, see [53]. Indeed, one considers the power series $\sum_{n\in N} a_n t^n$ in t where N is a set of positive integer multiples of a certain rational number (for different power series, this number may be different) and $a_n \in R$ for $n \in N$. Since any finite number of such power series can be thought of as belonging to the same discrete version of this construction, the ternary operation can be copied from [53], and also the proof of the fact that we have a positively valuated ternary ring. Now, in completely the same way as in the discrete case, one constructs a projective plane with (nondiscrete) valuation out of this. The residue defined by this valuation is precisely the projective plane coordinatized by R. To the best of our knowledge, these are the first examples of such nondiscrete apartment systems with an arbitrary (possibly finite) residue.

$n = 4$

The construction we will explain here is again inspired by an example for the discrete case by Hendrik Van Maldeghem in [54]. We will only sketch what the coordinatizing structure with valuation looks like. All proofs for the finite case still hold here (this is due to the fact that any finite number of elements in the coordinatizing structure can be 'embedded' in a coordinatizing structure of a discrete case). In particular, the reader can consult [57] for explicit formulae to derive the valuation of a generalized quadrangle from the valuation of the coordinatizing structure.

Consider the finite field $k = \mathsf{GF}(q)$ with $q = 2^h$. Let h_1 and h_2 be two natural numbers such that $q-1$ and $-1+2^{1+h_1+h_2}$ are relatively prime (for example $h=3$, $h_1=1$ and $h_2=0$). For $i=1,2$, let θ_i be raising to the power 2^{h_i}, forming automorphisms of this finite field. Now consider the field K of Laurent series $\sum_{n\in N} a_n t^n$ in t where N is a set of integer multiples of a certain rational number, bounded below (again, for different Laurent series, this number may be different), and $a_n \in k$ for $n \in N$. There is a natural valuation on this field defined by $v(\sum_{n\in N} a_n t^n) = m$, where m is the smallest element of N such that a_m is nonzero (well defined by the boundedness below). We define $v(0)$ to be ∞. One can extend θ_i for $i \in \{1,2\}$ to the field K by

$$(\sum_{n\in N} a_n t^n)^{\theta_i} = \sum_{n\in N} a_n^{\theta_i} t^n. \tag{4.8}$$

The coordinatizing structure is now given by:

$$Q_1(k,a,l,a') = (k^{\theta_1})^2.a + a', \tag{4.9}$$

$$Q_2(a,k,b,k') = a^{\theta_2}.k + k', \tag{4.10}$$

wih $k,l,k',a,b,a' \in K$ and v the natural valuation.

For more information about this example and coordinatizing structures, see [54]. One can show that this example defines a generalized quadrangle with valuation where both the quadrangle itself and its residue are nonclassical.

These are, to the best of our knowledge, the first explicitly defined examples of nondiscrete $\mathbb{R}$-buildings of this nature.

4.5 Proof of Main Result 4.3.1

General idea. — The hard part of the proof will be showing Axiom (U4). This will be proven by investigating how the sums occuring in (U4) change when we 'move around' the point α.

The first lemma deals with the exact shape of the intersection of two sectors with the same source, and sharing a sector-facet.

Lemma 4.5.1 *Let C and C' be two sectors with the same source τ which share a sector-facet F. Then the intersection of both is formed by the convex hull of F and the common part of the other two sector-facets of C and C'.*

Proof. Take any apartment Σ containing C (and so also τ). If Σ contains C' then there is nothing left to prove. If this is not the case then there is a unique apartment Σ'_∞ at infinity containing C' and sharing a half-apartment with Σ_∞. A remark in [24, p. 10] states that if two apartments share a half-apartment at infinity, they also do in the $\mathbb{R}$-building itself. This implies the exact form of the intersection. □

If $C \neq C'$, then such an intersection is called a *chimney* with source τ ([30]). We refer to the *width* of the chimney as the distance between the parallel walls bordering it.

Corollary 4.5.2 *Let r, s, t be elements of Λ_∞ such that $r\mathtt{I}s\mathtt{I}t$, and let τ be a point of Λ; then the width of the chimney defined by the intersection of the sector containing r_τ and s_τ, and the one containing s_τ and t_τ, equals* $\sin(\pi/n)u_\tau(r, t)$.

Proof. Directly from the definitions and the previous lemma. □

Now let α be an arbitrary point of Λ and consider the map u_α. The Axiom (U1) will be satisfied because given an element x at infinity there is always an apartment containing x_α where we then can find the needed element y adjacent to x such that $u_\alpha(x, y) = 0$. The second Axiom (U2) is satisfied trivially and (U3) follows from the convexity of sector-panels.

The main difficulty is (U4). Let x_0 and x_n be two opposite elements of Λ_∞ and $M := (x_0, x_1, \ldots, x_n) \in (P \cup B)^{n+1}$ such that $x_0\mathtt{I}x_1\mathtt{I}\ldots\mathtt{I}x_n$. We define the function

$$f : \mathbb{R}^+ \to \mathbb{R}^+ : l \mapsto \sum_{i=1}^{n-1} \sin(i\pi/n)u_\beta(x_{i-1}, x_{i+1}),$$

with $\beta \in (x_0)_\alpha$ at distance l from α. If we can prove that f only depends on x_0, x_n and α, then we have proven (U4) and Main Result 4.3.1 (in view of the fact that (U4) is trivially satisfied in degenerate apartments).

Before we go on we need the notion of 'distance in the residues'. Let x and y be elements of Λ_∞ and $\beta \in \Lambda$. Then we define the *residual distance* $\mathsf{d}_\beta(x, y)$ *at* β to be the distance between $[x]_\beta$ and $[y]_\beta$ as defined in the generalized n-gon $[\Lambda]_\beta$ (a point and an incident line are at distance 1, two collinear points are at distance 2, ...).

Remark 4.5.3 Similar to residues, the notion residual distance has been used already for a different object, see Remark 4.2.4 for more information about both notions. Again, from the context it should be clear what is meant.

The next lemma investigates the local behaviour of the valuations.

Lemma 4.5.4 *Let r, s, t be elements of Λ_∞ such that $r\mathtt{I}s\mathtt{I}t$, and β a point on $(x_0)_\alpha$ with $\mathsf{d}(\alpha, \beta) = l$. Then there exists some $\delta > 0$ such that for any β' on $(x_0)_\alpha$ with $\mathsf{d}(\alpha, \beta') \in [l, l + \delta]$, the following holds:*

$$u_{\beta'}(r, t) = u_\beta(r, t) + \epsilon \frac{\sin(\mathsf{d}_\beta(s, x_0)\pi/n)}{\sin(\pi/n)} \mathsf{d}(\beta, \beta'),$$

where ϵ is a constant equal to

$$\begin{cases} -1 & \text{if } \mathsf{d}_\beta(r, x_0) = \mathsf{d}_\beta(t, x_0) = \mathsf{d}_\beta(s, x_0) - 1, \\ 1 & \text{if } \mathsf{d}_\beta(r, x_0) = \mathsf{d}_\beta(t, x_0) = \mathsf{d}_\beta(s, x_0) + 1, \\ 0 & \text{if } \mathsf{d}_\beta(r, x_0) \neq \mathsf{d}_\beta(t, x_0). \end{cases}$$

Proof. Let C be the sector spanned by r_β and s_β and C' the one by s_β and t_β. Both these sectors have source β. Using Lemma 1.8.1, we can find apartments Σ and Σ' containing C and an element of the germ $[x_0]_\beta$, and C' and an element of the germ $[x_0]_\beta$, respectively. Let δ be the length of the part of $(x_0)_\beta$ included in $\Sigma \cap \Sigma'$. Obviously $\delta > 0$. Let β' be on $(x_0)_\alpha$ with $\mathsf{d}(\alpha, \beta') \in [l, l + \delta]$. The sectors $C_{\beta'}$ and $C'_{\beta'}$ with source β' now lie in the apartments Σ and Σ', respectively.

Using the intersection of both apartments one can easily calculate that the width of the chimney defined by r, s and t with source at β' is $\epsilon \sin(\mathsf{d}_\beta(s, x_0)\pi/n)\mathsf{d}(\beta, \beta')$ larger than the one with source β, with ϵ as in the table above. Using Corollary 4.5.2 we now obtain the desired result. $\square$

As an immediate consequence of the previous lemma, we see that f is right-continuous. Left-continuity (and because of this also continuity) can be proved analogously. Using a similar reasoning one can prove the following lemma and corollary, which we will need to prove a later result.

Lemma 4.5.5 *For any $\mathbb{R}$-building $(\Lambda, \mathcal{F})$ (not necessarily 2-dimensional), and C_∞ and D_∞ two adjacent chambers at infinity, only differing in the vertices x_∞ and y_∞, the function $u : \Lambda \to \mathbb{R}$ which maps a point α to the length of the common part of the sector-facets x_α and y_α, is uniformly continuous.* □

Corollary 4.5.6 *For an $\mathbb{R}$-building $(\Lambda, \mathcal{F})$, and two adjacent chambers C_∞ and D_∞ at infinity, the subset of points α of Λ for which $[C]_\alpha = [D]_\alpha$ is an open subset of the metric space defined on Λ.*

Proof. Directly from the above lemma. □

Applying Lemma 4.5.4 to the (finite number of) valuations occurring in the definition of f now implies that for every $l \in \mathbb{R}^+$ there exists some $\bar{\delta} > 0$ (the minimum occurring in the application to each valuation) and $a_l \in \mathbb{R}$ such that $f(l') = f(l) + a_l(l' - l)$ for every $l' \in [l, l + \bar{\delta}]$. The next step in our proof is to show that a_l only depends on x_0, x_n, α and l. One thing which is directly clear is that a_l only depends on the distances $\mathsf{d}_\beta(x_0, x_i)$ with $i \in \{1, 2, \ldots, n\}$, and on the point β on $(x_0)_\alpha$ with $d(\alpha, \beta) = l$. Because of this we can reduce this combinatorially as follows. Define the sequence $(y_0, y_1, \ldots, y_n)$, with $y_i := \mathsf{d}_\beta(x_0, x_i)$, $i \in \{0, 1, 2, \ldots, n\}$. This sequence consists of nonnegative integers such that two consecutive ones differ by exactly one, and the extremities y_0 (which equals 0) and y_n are constants. An entry different from the extremities with the property that both neighbours are strictly smaller will be called a *peak*; if both neighbours are strictly larger, then we call the entry a *valley*. The sequence will determine the a_l uniquely.

If two sequences produce the same a_l we will say that they are *equivalent*. We now show that each sequence is equivalent to the unique sequence with no valleys, which will be called the *standard sequence*. Therefore we look at the sum χ of all the y_i's. The number χ is clearly an integer and bounded. Consider any sequence different from the standard sequence; then it has at least one valley, say at the entry $y_j = m$. We now break the problem down to some different cases and show that in each case the given sequence is equivalent with one obtained from the first one by replacing y_j by $y_j + 2$. This equivalent sequence has a larger sum, and because this sum is an integer and is bounded by the sum obtained from the standard sequence, recursion implies that all sequences are equivalent to the standard sequence. Note that $j \geq 2$, so $j - 2$ is always well-defined.

In the following we will denote π/n by π_n for ease of notation.

(*i*) **Case** $(y_{j-2}, y_{j-1}, y_j, y_{j+1}, y_{j+2}) = (m+2, m+1, m, m+1, m)$**.** As indicated above, we show that this is equivalent with $(y_{j-2}, y_{j-1}, y'_j, y_{j+1}, y_{j+2}) = (m+2, m+1, m+2, m+$

$1, m)$. Indeed, using the expression for a_l from the definition of f and Lemma 4.5.4, we see that we must show

$$\begin{aligned}-\sin(j\pi_n)\sin(m\pi_n)+\sin((j+1)\pi_n)\sin((m+1)\pi_n) =\\ -\sin((j-1)\pi_n)\sin((m+1)\pi_n)+\sin(j\pi_n)\sin((m+2)\pi_n).\end{aligned}$$

Indeed, we perform the following elementary calculations.

$$\begin{aligned}&-\sin(j\pi_n)\sin(m\pi_n)+\sin((j+1)\pi_n)\sin((m+1)\pi_n)\\ &= 1/2(-\cos((j-m)\pi_n)+\cos((j+m)\pi_n)+\cos((j-m)\pi_n)-\cos((j+m+2)\pi_n))\\ &= 1/2(\cos((j+m)\pi_n)-\cos((j+m+2)\pi_n)),\end{aligned}$$

while

$$\begin{aligned}&-\sin((j-1)\pi_n)\sin((m+1)\pi_n)+\sin(j\pi_n)\sin((m+2)\pi_n)\\ &= 1/2(-\cos((j-m-2)\pi_n)+\cos((j+m)\pi_n)+\cos((j-m-2)\pi_n)\\ &\quad -\cos((j+m+2)\pi_n))\\ &= 1/2(\cos((j+m)\pi_n)-\cos((j+m+2)\pi_n)).\end{aligned}$$

It follows that the two sequences are equivalent.

(*ii*) **Case** $(y_{j-2}, y_{j-1}, y_j, y_{j+1}, y_{j+2}) = (m, m+1, m, m+1, m+2)$**.** This is analogous to the previous case.

(*iii*) **Case** $(y_{j-2}, y_{j-1}, y_j, y_{j+1}, y_{j+2}) = (m, m+1, m, m+1, m)$**.** Here, we show that this is equivalent with $(y_{j-2}, y_{j-1}, y'_j, y_{j+1}, y_{j+2}) = (m+2, m+1, m+2, m+1, m)$. Indeed, as before, we must show that

$$\begin{aligned}\sin((j-1)\pi_n)\sin((m+1)\pi_n)-\sin(j\pi_n)\sin(m\pi_n)+\sin((j+1)\pi_n)\sin((m+1)\pi_n)\\ = \sin(j\pi_n)\sin((m+2)\pi_n).\end{aligned}$$

This equality is the same as the one in Case (i), but with one term swapped from side. The same conclusion follows.

(*iv*) **Case** $(y_{j-2}, y_{j-1}, y_j, y_{j+1}, y_{j+2}) = (m+2, m+1, m, m+1, m+2)$**.** Here we must show that

$$\begin{aligned}-\sin(j\pi_n)\sin(m\pi_n) = &-\sin((j-1)\pi_n)\sin((m+1)\pi_n)\\ &+\sin(j\pi_n)\sin((m+2)\pi_n)-\sin((j+1)\pi_n)\sin((m+1)\pi_n).\end{aligned}$$

This equality is the same as in Case (iii) but with m substituted by $-m-2$. Again the same conclusion follows.

(*v*) **Case** $j = n - 1$**.** In this case we can reuse the previous arguments by adding an extra element $x_{n+1} \mathtt{I} x_n$ with corresponding $y_{n+1} := y_n \pm 1$, and extending f with an extra coefficient $\sin(n\pi/n) u_\beta(x_{n-1}, x_{n+1})$ (which is zero anyway due to $\sin \pi = 0$).

This proves that each sequence is equivalent to the standard sequence, and so that all sequences are equivalent and a_l only depends on x_0, x_n, α and l.

We now need an elementary result from analysis, which we prove for completeness' sake.

Lemma 4.5.7 *If g is a continuous real function defined over $\mathbb{R}^+$ such that for every $l \in \mathbb{R}^+$ there is a δ for which $g(l') = g(l)$ for every $l' \in [l, l + \delta]$, then g is constant over $\mathbb{R}^+$.*

Proof. Define $\Psi := \{x \in \mathbb{R}^+ | (\exists \delta' > 0)(\forall x' \in [x - \delta', x + \delta'])(g(x) = g(x'))\}$ as the set of 'constant points'. If an interval lies completely in Ψ, then g is constant over that interval because the preimage of the image of an element in such an interval is both open (due to the definition of Ψ) and closed (because of the continuity of g) in the connected interval. It follows also from the continuity of g that this is also true for the closure of an interval lying completely in Ψ. If the set $\mathbb{R}^+ \backslash \Psi$ is nonempty, then it has an infimum t. Note that by assumption, there exists some $\delta > 0$ such that $[0, \delta[\subseteq \Psi$. Hence $t > 0$ and the interval $[0, t[$ lies completely in Ψ, implying that g is constant over $[0, t]$. But we also know that there exists a δ' such that g is constant over $[t, t + \delta']$, so $[0, t + \delta'[$ lies in Ψ. This contradicts the fact that t is an infimum. So $\Psi = \mathbb{R}^+$ and g is constant over $\mathbb{R}^+$. $\square$

Lemma 4.5.8 *There is an $l \in \mathbb{R}^+$ such that $f(l') = 0$ if $l' \geq l$.*

Proof. Let i be minimal with respect to the property $\mathsf{d}_\alpha(x_0, x_i) \neq i$. It is clear that, if $\beta \in (x_0)_\alpha$, then $\mathsf{d}_\beta(x_0, x_j) = j$ for $j < i$ (because the sectors spanned by x_0 till x_j with source α form a part of an apartment and contain those with source β). Suppose there is no $\beta \in (x_0)_\alpha$ such that also $\mathsf{d}_\beta(x_0, x_i) = i$. In such a case we have that the function

$$g : \mathbb{R}^+ \to \mathbb{R}^+ : l \mapsto u_\beta(x_{i-2}, x_i), \text{ with } \mathsf{d}(\beta, \alpha) = l,$$

is strictly positive for each $\beta \in (x_0)_\alpha$ (because a zero value would imply that $\mathsf{d}_\beta(x_0, x_i) = i$). As we know by Lemma 4.5.4, for every $l \in \mathbb{R}^+$ there is a δ such that

$$g(l') = g(l) - \frac{\sin((i-1)\pi_n)}{\sin \pi_n}(l' - l), \text{ for every } l' \in [l, l + \delta].$$

The function $g(l)+\frac{\sin((i-1)\pi_n)}{\sin\pi_n}l$ then complies to the statement of Lemma 4.5.7 and is constant. But this is impossible since for l large enough, this would imply that $g(l)$ is negative. Consequently g cannot be strictly positive, yielding that there is a $\beta \in (x_0)_\alpha$ such that also $\mathsf{d}_\beta(x_0,x_i)=i$.

Repeating this process a finite number of times will produce an l such that $\mathsf{d}_\beta(x_0,x_n)=n$ if $\mathsf{d}(\beta,\alpha)\geq l$. This implies that $u_\beta(x_{i-1},x_{i+1})$ is zero for each $i\in\{1,2,\dots,n-1\}$, which on its turn implies that $f(d(\beta,\alpha))=0$. □

Let us reiterate what we know about the function f defined over $\mathbb{R}^+$:

(O) For high enough values it is zero.

(C) The function is continuous.

(P) For every $l\in\mathbb{R}^+$ there is a $\bar{\delta}$ and an $a_l\in\mathbb{R}$ such that $f(l')=f(l)+a_l(l'-l)$ for every $l'\in[l,l+\bar{\delta}]$ where a_l depends only on l,x_0,x_n and α.

Lemma 4.5.9 *Two functions satisfying the three conditions* (O), (C) *and* (P) *(with the same a_l) are equal over $\mathbb{R}^+$.*

Proof. Because we know that f satisfies the above conditions, we can assume that one of the functions is f - let the other be f'. Consider $g=f'-f$; then g is continuous, is zero for high enough values, and for every $l\in\mathbb{R}^+$ there is a δ (the minimum of the two $\bar{\delta}$ related to f and f') such that $g(l')=g(l)$ for every $l'\in[l,l+\delta]$. Lemma 4.5.7 now implies that g is constant, and so zero over $\mathbb{R}^+$.

This implies that f and f' are equal. □

As a_l only depends on l,x_0,x_n and α, it is a direct corollary of the previous lemma that f only depends on x_0,x_n and α, which has previously been said to imply (U4). This completes the proof of Main Result 4.3.1.

4.6 Proof of Main Result 4.3.2

We start with a polygon Γ with valuation u, with weight sequence $(a_1,a_2,\dots,a_{n-1},a_{n+1},a_{n+2},\dots a_{2n-1})$, and such that u has nonzero values. Our proof is heavily inspired by a similar result for the discrete case in [55] by Hendrik Van Maldeghem. In fact, we will use some of the results (with the proofs remaining valid in the nondiscrete case) obtained there, directly in our proof. In particular, and to begin with, it is shown in 3.1 of [55] that the weight sequence of a given polygon with valuation having nonzero values is unique, up to a nonzero multiple. As is also exploited in [55], this has as consequence that the weight sequence is symmetric, i.e., $a_i = a_{n-i} = a_{n+i} = a_{2n-i}$ for $i \in \{1, 2, \ldots, n-1\}$.

Now let $(x_0, x_1, \ldots, x_{2n} = x_0)$ be any closed path of length $2n$ in Γ. Because of (U4) we know that

$$\sum_{i=1}^{n-1} a_i u(x_{i-1}, x_{i+1}) = \sum_{i=n+1}^{2n-1} a_i u(x_{i-1}, x_{i+1}),$$

and also that

$$\sum_{i=3}^{n+1} a_{i-2} u(x_{i-1}, x_{i+1}) = \sum_{i=n+3}^{2n+1} a_{i-2} u(x_{i-1}, x_{i+1}).$$

If one takes the sum of both equations, and simplifies the resulting expression using $a_1 = a_{n-1} = a_{n+1} = a_{2n-1}$, one obtains

$$\begin{aligned} & a_2 u(x_1, x_3) + \sum_{i=3}^{n-1} (a_i + a_{i-2}) u(x_{i-1}, x_{i+1}) + a_{n-2} u(x_{n-1}, x_{n+1}) \\ = {} & a_{n+2} u(x_{n+1}, x_{n+3}) + \sum_{i=n+3}^{2n-1} (a_i + a_{i-2}) u(x_{i-1}, x_{i+1}) + a_{2n-2} u(x_{2n-1}, x_{2n+1}). \end{aligned}$$

This implies that

$$(a_2, a_3 + a_1, a_4 + a_2, \ldots, a_{n-1} + a_{n-3}, a_{n-2}, a_{n+2}, a_{n+3} + a_{n+1}, \ldots, a_{2n-1} + a_{2n-3}, a_{2n-2})$$

is also a weight sequence. Hence there exists some positive real number k satisfying

$$\begin{cases} ka_1 = a_2, \\ ka_2 = a_3 + a_1, \\ ka_3 = a_4 + a_2, \\ \ldots \\ ka_{n-2} = a_{n-1} + a_{n-3}, \\ ka_{n-1} = a_{n-2}. \end{cases} \tag{4.11}$$

One notices, by taking the sum of all equations in the system of equations above, that

$$k \sum_{i=1}^{n-1} a_i = 2 \sum_{i=1}^{n-1} a_i - (a_1 + a_{n-1}).$$

This implies that $1 \leq k < 2$. As a consequence, we can find an $\alpha \in]0, \pi/3]$ such that $k = 2\cos\alpha$. Also remark that $a_j = ka_{j-1} - a_{j-2}$ for $j \in \{3, n-1\}$. If we formally set $a_0 = a_n = 0$, then this is also true for $j \in \{2, n\}$. Furthermore we can suppose that $a_1 = \sin\alpha$.

Lemma 4.6.1 *For $i \in \{0, 1, \ldots, n\}$ we have $a_i = \sin(i\alpha)$.*

Proof. We prove this using induction on i. It is clear that this holds for $i = 0$ and $i = 1$ (by assumption and by definition of α, respectively). So let $i \geq 2$ such that $a_j = \sin ja$ for $j < i$. Then we know that:

$$\begin{aligned} a_i &= ka_{i-1} - a_{i-2} \\ &= 2\cos\alpha \sin[(i-1)\alpha] - \sin[(i-2)\alpha] \\ &= \sin i\alpha \end{aligned}$$

The second equality follows from the induction hypothesis, the third from the trigonometric formula $\sin a + \sin b = 2\sin[(a+b)/2]\cos[(a-b)/2]$. □

Lemma 4.6.2 *$\alpha = \pi/n$.*

Proof. We have that $a_n = 0$, so $\sin n\alpha = 0$ by the previous lemma. This yields $\alpha = m\pi/n$, with $m \in \mathbb{N}_0$ smaller than or equal to $n/3$ (since $\alpha \in]0, \pi/3]$). At the same time we have $a_i > 0$ for $i \in \{1, \ldots, n-1\}$. Let t be the smallest integer greater than or equal to n/m. Because $n/m \leq t \leq 2n/m$ (by $n/m \geq 3$), it holds that $tm\pi/n \in [\pi, 2\pi]$, so $a_t \geq 0$. As t clearly is in $\{1, 2, \ldots, n\}$, we obtain that $t = n$, which implies that $m = 1$ (because $m \in \mathbb{N}_0$ and $n \geq 3$) and $\alpha = \pi/n$. □

Combining the two previous lemmas, we obtain:

Corollary 4.6.3 *For $i \in \{0, 1, \ldots, n\}$, $a_i = \sin(i\pi/n)$, and any other weight sequence of (Γ, u) is a multiple of this.*

Remark 4.6.4 It is easy to see that all $k \in \mathbb{R}$ satisfying Equation 4.11 are precisely the eigenvalues of the path graph P_{n-1} of length $n-2$, consisting of $n-1$ vertices. Moreover, since all a_i are positive, it is the unique eigenvalue for which the coordinates of the associated eigenvectors have constant sign. This observation can be used to give an alternative proof of the previous corollary. Doing so, one sees that $2\cos(\pi/n)$ is in fact the largest eigenvalue of P_{n-1}.

4.7 Proof of Main Result 4.3.3

By the proof of the previous main result one can suppose for the proof of the current and following main result that the weight sequence is given by $a_i = |\sin(i\pi/n)|/\sin(\pi/n)$. In particular, we have that $a_1 = 1$.

Let n be a natural number with $3 \leq n \leq 6$ for the rest of this section.

If x and y are opposite elements, let $\tau(x, y)$ be the sum $\sum_{i=1}^{n-1} a_i u(x_{i-1}, x_{i+1})$ where $(x_0 = x, x_1, \ldots, x_{n-1}, x_n = y)$ is a shortest path from x to y; (U4) guarantees independence of the chosen path.

Two elements x and y are said to be *t-residually equivalent*, if for each element z the following are equivalent:

- z is opposite x and $\tau(x, z) < t$;
- z is opposite y and $\tau(y, z) < t$.

Notice that when $t = 0$, this definition is trivially fulfilled.

Lemma 4.7.1 *Two adjacent elements x and y are $u(x, y)$-residually equivalent, but not t-residually equivalent with $t > u(x, y)$.*

Proof. Let z be an element opposite x with $\tau(x, z) < u(x, y)$. Consider the unique shortest path $(x_0 = x, x_1 = xy, x_2, \ldots, x_n = z)$ from x to z containing xy. Because $a_1 = 1$, it holds that $u(x, x_2) \leq \tau(x, z) < u(x, y)$, so $u(y, x_2) = u(x, x_2)$ by (U3). This implies that y and z are opposite and that $\tau(y, z) = \tau(x, z)$ (the last is easily seen when considering the path $(y, x_1, x_2, \ldots, x_n = z)$).

If $t > u(x, y)$, then consider a path $(x, xy, y = y_2, \ldots, y_n)$ where the path $(y_2, \ldots, y_n)$ has valuation zero (possible by Lemma 4.2.2). $\square$

Corollary 4.7.2 *If $x \mathtt{I} y \mathtt{I} z$, then $[x] = [z]$ if and only if $u(x, z) > 0$.*

Lemma 4.7.3 *Given a closed path Ψ, there are at least two sides having the same minimal valuation among all sides in Ψ.*

Proof. Let x and y be the two points on a side with minimal valuation, and suppose all other sides have valuation strictly larger than $u(x,y)$. Let t be the second smallest valuation among the sides in Ψ. By repeatedly using Lemma 4.7.1 and going from x to y in Ψ not using xy, one proves that x and y are t-residually equivalent, which contradicts Lemma 4.7.1. □

Lemma 4.7.4 *If two elements x and y are not residually equivalent, but if there exist $a\mathtt{I}x$ and $b\mathtt{I}y$ which are residually equivalent, then there is an element z residually opposite one element of $\{x,y\}$, but at distance $n-2$ from the other.*

Proof. Without loss of generality, one can suppose that there exists an element d which is residually opposite x, but not residually opposite y.

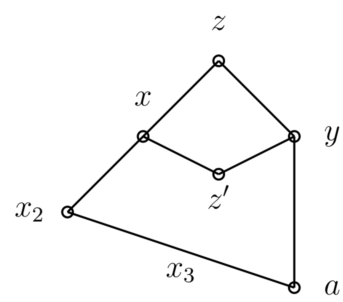
According to Lemma 4.2.2, there exists an element c incident with x such that $u(a,c)=0$. Let $(x=x_0, c=x_1, \ldots, x_{n-1}, d=x_n)$ be the unique shortest path from x to d containing c. The element x_{n-1} is residually opposite, and so also opposite, a and b. This implies that $\mathsf{d}(y,d)=n$ or $\mathsf{d}(y,d)=n-2$. In the second case we are done, so suppose we are in the first case. Let $(y=y_0, y_1, \ldots, y_{n-2}, y_{n-1}=x_{n-1}, y_n=d)$ be the unique shortest path from y to d containing x_{n-1}. Because the element x_{n-1} is residually opposite b, the path $(b, y=y_0, y_1, \ldots, y_{n-2}, y_{n-1}=x_{n-1})$ has valuation zero. As y is not residually opposite d, the valuation $u(y_{n-2},d)$ has to be non zero. So $x_{n-2}\neq y_{n-2}$ and $u(x_{n-2},y_{n-2})=0$. The element x_{n-2} will now be the desired element z, because it is residually opposite y, but at distance $n-2$ from x. □

Lemma 4.7.5 *Let Ω be a simply folded n-gon. If two elements x and y are folded together in Ω, then they are residually equivalent.*

Proof. Here we need to distinguish between the different possibilities for n. Let z be an element of Ω such that Ω is folded along z.

- $n=3$. For this case the result follows directly from Corollary 4.7.2.

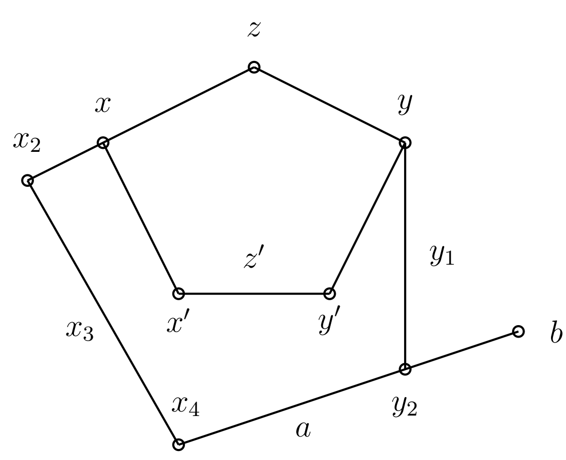
- $n=4$. Again using Corollary 4.7.2, one only needs to prove that the two elements of Ω at distance 2 from z are residually equivalent. Suppose this is not the case. Using the previous lemma, one can assume without loss of generality that there is an element a residually opposite x, but at distance 2 from y.



Let (x, xz, x_2, x_3, a) be the unique shortest path (which has valuation zero) from x to a containing xz. Let z' be the element opposite z in Ω. The element x_3 is residually opposite xz', and so also residually opposite yz' due to Corollary 4.7.2. This implies that the valuations $u(y, a)$ and $u(x_3, ay)$ are zero. But as also the valuations $u(xz, x_3)$ and $u(x_2, a)$ are zero, (U4) would imply that $u(xz, zy) = 0$, which is a contradiction.

- $n = 5$. Using Corollary 4.7.2 and the previous lemma, one can assume without loss of generality that x and y are at distance 2 from z, and that there exists an element a residually opposite x, but at distance 3 from y.

  Let $(x, xz, x_2, x_3, x_4, a)$ be the unique shortest path (which has valuation zero) from x to a containing xz, and let (y, y_1, y_2, a) be the shortest path from y to a. Choose an element $b\mathtt{I}a$ such that $u(b, x_4) = 0$ (this is possible due to Lemma 4.2.2). The element xz is residually opposite b, and so also yz. All of this implies that the path (yz, y, y_1, y_2, a, b) has valuation zero. A consequence is that $u(x_4, y_2) > 0$, otherwise we could have chosen b to be y_2, leading to a contradiction.

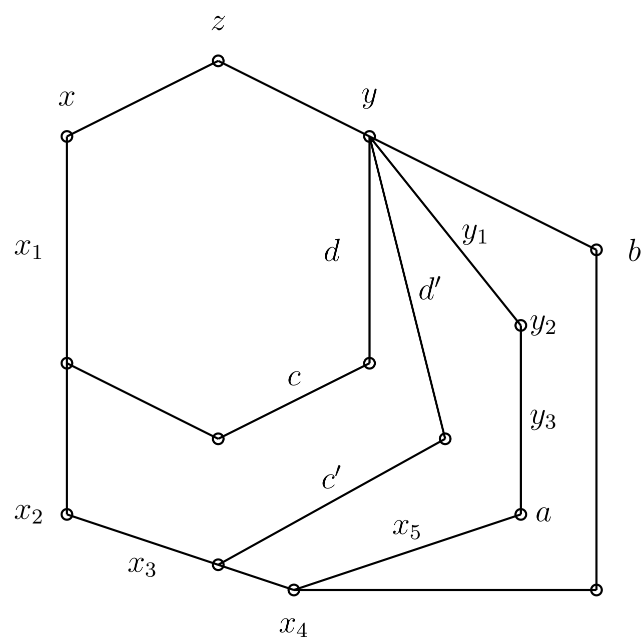


Let z' be the element opposite z in Ω, and let x', y' be the elements incident with z' closest to x and y respectively. Now x' and y' are both residually opposite x_3, implying that the unique shortest path from yy' to x_3 has valuation zero. If we look in the unique ordinary pentagon containing yy', x_3 and y_2, we see that the valuation of x_3 in this pentagon is nonzero because of (U4) and $u(x_4, y_2) > 0$. By (U3) we then obtain that the valuation of x_3 in the unique ordinary pentagon containing x_3, yy' and z is zero. This contradicts (U4) and the fact that the valuation of z in this pentagon is nonzero.

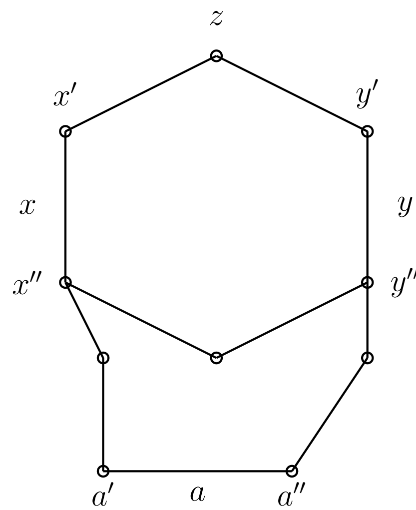
- $n = 6$. Apart from the case handled in Corollary 4.7.2, there are two cases to consider here.

  - The first case is when x and y lie at distance 2 or 4 from z; without loss of generality one can suppose this to be 2. Similarly to the previous cases, let a be an element residually opposite x, but at distance 4 from y. Let x_1 be the unique element of Ω at distance 1 from x and 3 from z. Now consider the unique shortest path $(x, x_1, x_2, x_3, x_4, x_5, a)$ from x to a containing x_1, and the unique shortest path (y, y_1, y_2, y_3, a) from y to a. Observe that $x_4 \in [z]_{\text{opp}}$. Let Ω' be the unique ordinary simply folded hexagon containing z, x_4, x and yz, and let b be the element opposite x_2 in this hexagon. By (U3), the unique ordinary hexagon containing y, b, y_1, and x_4 is nonfolded, so $u(y, b)$ is zero and $x_4 \in [y]_{\text{opp}}$.

    Let Ω'' be the unique ordinary hexagon containing z, y and x_3, and Ω''' the unique ordinary hexagon containing y, b and x_3. Let c and c' respectively be the elements opposite xz in the hexagons Ω and Ω'' respectively. Let d and d' be the projections of c and c', respectively, on y. The hexagon Ω''' is a simply folded hexagon folded along y (remember that $u(y, b)$ was zero). So $u(yz, d')$ is nonzero, and so $u(d, d')$ is zero. This implies that $c \in [c']_{\text{opp}}$, so also the element c'' opposite yz in Ω is in $[c']_{\text{opp}}$. Because the unique path from c'' to c' containing x_2 has valuation zero, also the path from xz to c' containing x has valuation zero. So $xz \in [c']_{\text{opp}}$, which gives $yz \in [c']_{\text{opp}}$, and this is a contradiction because yz and c' are at distance 4 from each other.



– The last case to handle is the case where x and y are at distance 3 from z. For the final time, consider an element $a \in [x]_{\mathrm{opp}}$ at distance 4 from y. Let x' and y' be the projections from z on x and y, respectively, and let x'' and y'' be the elements in Ω at distance 4 from z and 1 from x and y, respectively. Let a' be the projection of x'' on a; this element is residually opposite x', so it is also residually opposite y' (as shown in the previous case). The unique shortest path from y' to a' containing a (and because of this also y) now has valuation zero. Let a'' be the projection of y' on a. This element is residually opposite x'', but cannot be residually opposite y'' as it is only at distance 4 from y''. This contradicts the previous case applied to x'' and y''.



□

Lemma 4.7.6 *Let x, y be elements of Γ such that $[x]\mathtt{I}_r[y]$. Then there exists some $y' \in [y]$ such that $x\mathtt{I}y'$.*

Proof. Let F be the set of all flags containing an element of $[x]$ and one of $[y]$. Let $\{x', y'\}$ be a flag of F such that the sum d of distances of x' and y' to x is minimal. If $d = 1$, then $x' = x$ and $x\mathtt{I}y'$. So we may suppose that $d > 1$.

Suppose that the distance of x to y' is one bigger than the distance from x to x'. Let $(x_0 = x, x_1, \ldots, x_{j-1} = x', x_j = y')$ be the shortest path from x to y' containing x' ($j \leq n$). Let i be the smallest integer such that the subpath $(x_i, \ldots, x_{j-1}, x_j)$ has valuation zero. We have that $i \geq 1$ (because otherwise it is impossible that $x' \in [x]$) and $i \leq j - 1$. Using Lemma 4.2.2 we can extend this subpath to a path $(x_i, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{i+n})$ with valuation zero of length n. Consider the unique path $(x'_i = x_i, x'_{i+1} = x_{i-1}, \ldots, x'_{i+n} = x_{i+n})$ from x_i to x_{i+n} containing x_{i-1}. Then using (U4), we see that this path has valuation zero. These two paths together form an ordinary n-gon Ω, which is simply folded along x_i. The previous lemma implies that $x'_{j-1} \in [x]$ and $x'_j \in [y]$. But the sum of distances to x of these two incident elements is strictly less than d, contradicting the minimality of d.

The case where the distance of x to x' is one bigger than the distance from x to y is proven analogously. □

The diameter of our new geometry Γ_r is clearly n. In order to prove it is a (weak) generalized n-gon we have to show that there is no closed nonstammering path of length less than

$2n$. So suppose by way of contradiction that we have such a path $([x_0], [x_1], \ldots, [x_{2m}] = [x_0])$ with $2 \leq m < n$. The previous lemma allows us to lift the path into a (not necessarily closed) path $(x'_0, x'_1, \ldots, x'_{2m})$ such that $[x'_i] = [x_i]$.

Due to Corollary 4.7.2 and the fact that the original path was nonstammering, this path has valuation zero. If $2m < n$, we extend this path to a path $(x'_0, x'_1, \ldots, x'_{2m}, x'_{2m+1}, \ldots, x'_n)$ with valuation zero, of length n (this is possible by Lemma 4.2.2). In each case we have that x'_n is residually opposite x'_0, but not opposite, and so certainly not residually opposite x'_{2m}. Hence we have a contradiction and we have proven the Main Result 4.3.3.

4.8 Proof of Main Result 4.3.4

General idea. — Starting from one valuation u on Γ, we will construct more valuations. Each of these valuations will correspond to a point of our $\mathbb{R}$-building. We will use results from Section 4.5, which prove the current problem in the other direction. For example, in that section Lemma 4.5.4 tells us how a valuation should behave when we 'move' the point it is defined from. We will use this information to construct new valuations.

We now return to our case. Let (Γ, u) be a generalized n-gon with valuation, x an element of Γ, and $t \in \mathbb{R}^+$ a positive real number. We want to define a new valuation $u^{V(x,t)}$ with $V(x,t)$ an operator called the *translation operator* ($u^{V(x,t)}$ will be referred to as the *t-translation of u towards x*, and u is *t-translated towards x*).

How do we construct this new valuation? Remember that each element y has a certain residual distance $\mathsf{d}_r(x, y)$ from x in the residue Γ_r defined by the valuation u. We now 'predict' the *translated residual distance* $\mathsf{d}_r^{x,t}(y)$ from x to y when t-translating u, as it would be if we were indeed in an $\mathbb{R}$-building (we changed the notation of the residual distance to an unary function to stress the dependability of x, and the fact that we will only need distances from x). This function defined for $t \in [0, +\infty[$ will be right-continuous and piecewise constant. First thing one needs to assure here is that for two incident elements y, z, the translated residual distances $\mathsf{d}_r^{x,t}(y)$ and $\mathsf{d}_r^{x,t}(z)$ differ by only one. The definition of this function will be referred to as step (C1), the 'difference condition' as condition (C2).

Because we know how the (translated) residual distances would behave if we were in an $\mathbb{R}$-building, we can use Lemma 4.5.4 to predict how the translated individual valuations would behave if we were indeed in an affine apartment system (this is done by a trivial integration of a piecewise constant function). The set of all these individual valuations allows to construct a new 'valuation' $u^{V(x,t)}$ (we still need to verify this is really a valuation). On the third and fourth page of Section 4.5, it was shown that the weighted sum of the coefficients of t along the path $(x_0, \ldots, x_n)$ depends only on the residual distances of d_0 and d_n of x_0 and x_n respectively, under the assumption that $d_0 = x_0$. The argument in that section can be extended to show that this weighted sum depends also only on d_0 and d_n when d_0 is not zero, by applying the same idea as in Case (v) of that section if $j = 1$ is a valley. Because here the predicted individual valuations behave in the same way as they would in the affine apartment system case, this result can be applied here (also using the fact that for two incident elements the residual distances differ only by one) to guarantee that (U4) will be satisfied by $u^{V(x,t)}$. The condition (U2) is trivially satisfied. For more insight in how $u^{V(x,t)}$ is constructed, see the example in the section below.

For the other two conditions and positivity of the valuation, we will define and use the $\mathbb{R}$-trees associated to elements of Γ.

Choose a point x in a given tree. We can define a valuation v acting on the set of pairs (e, f) of ends (parallel classes of sectors) of this tree as the length of the intersection of the two half apartments with boundary x and ends e and f. The point x will be called the *base point of the valuation*.

One property of v is that for three arbitrary ends e, f, g the inequality $v(e, f) < v(f, g)$ implies $v(e, g) = v(e, f)$. Now, given any binary function w acting on a set E obeying this property, one can (re)construct a tree (if w is already a valuation of a tree, then we will obtain the same tree) by taking the set $\{(e, t)|e \in E, t \in \mathbb{R}^+\}$ and applying the equivalence relation

$$(e, t) \sim (f, s) \Leftrightarrow t = s \text{ and } t \leq w(e, f)$$

($e, f \in E$ and $s, t \in \mathbb{R}^+$). The base point of this tree is the equivalence class $\{(e, 0)|e \in E\} =: x$. The set of ends of this tree is in natural bijective correspondence with E and the valuation in this tree with base point x coincides with w. (This construction is a special case of the one of Alperin and Bass in [3].)

It is easily seen that this property is the same as (U3) when we restrict u to a point row or line pencil. So to each line L or point p of Γ we can associate a tree named $T(L)$ or $T(p)$ with a certain base point. The location of this base point will play a major role in the next sections. Other choices of base points yield other valuations of the tree.

We now return to the problem of (U1), (U3) and positivity. Obviously, this will be solved if we can show that the change in valuations of elements incident with an element y of Γ is described by changing the base point in the tree $T(y)$. With an eye on the above lemma, we want to move the base point towards an end corresponding to an element $a \mathtt{I} y$ with

$d_r^{x,t}(a) = d_r^{x,t}(y) - 1$ over a length of $t \sin(d_r^{x,t}(y)\pi/n)/\sin(\pi/n)$, with t a certain translation length such that the translated residual distances of a and y stay the same. In order that the valuations obtained by this change of base point correspond to the predictions of the valuations using the above lemma, we need to verify three things.

- If the valuation of the pair consisting of a and another element $b\mathtt{I}y$ is going to decrease (equivalent with saying that $d_r^{x,t}(b) = d_r^{x,t}(y) - 1$ and $d_r^{x,t}(y) \neq n$), then this valuation corresponds to the predicted valuation using the displacement of the base point in the tree, if the two half-apartments with ends a and b and source the base point have more in common than only the base point, so $u^{V(x,t)}(a,b) > 0$. (We refer to this as condition (C3).)
- If the valuation of the pair consisting of a and another element $b\mathtt{I}y$ is going to stay the same (equivalent with saying that $d_r^{x,t}(b) = d_r^{x,t}(y) + 1$), then we have correspondence between the two predictions if the base point lies in the apartment with ends a and b, so $u^{V(x,t)}(a,b) = 0$. (This will be condition (C4).)
- Finally note that if the valuation is going to increase (two elements $b, c\mathtt{I}y$ with $d_r^{x,t}(b) = d_r^{x,t}(c) = d_r^{x,t}(y) + 1$), we would need that the base point lies on the intersection of the apartment with ends a and b, and the one with ends a and c (so $u^{V(x,t)}(a,b) = u^{V(x,t)}(a,c) = 0$). But this is already covered by (C4), so there is no extra condition needed.

In the next part of the proof (after the example), we consider each case seperately.

4.8.1 An example

We will illustrate with an example how $u^{V(x,t)}$ will be calculated in practice. Suppose we are in the $n = 3$ case, and that x is a point. Let us say we have two points x_1, x_2 different from x, and we want to define $u^{V(x,t)}(x_1, x_2)$. (For the (C1) used here we refer to the next section.)

Suppose $u(x, x_i) = t_i$ and suppose $u(x_1, x_2) = t_2$, with $t_1 > t_2 > 0$ (there are other cases, but let's rectrict to this one). The residual distances are all zero between these points. Let L be the line joining x_1 and x_2. Then ϵ in the formula of Lemma 4.5.4 equals -1. Here, we can take $\delta = t_2$ (so far, the residual distances to x do not change according to (C1)), and we obtain

$$u^{V(x,t)}(x_1, x_2) = t_2 - t \text{ for } t \leq t_2.$$

From then on, ϵ becomes zero until $t = t_1$, since the residual distance to x from x_1 differs from that to x_2; to x_2 it becomes 2 and to x_1 it is 0. Hence

$$u^{V(x,t)}(x_1, x_2) = 0 \text{ for } t_2 < t \leq t_1.$$

Note that, up to now, the residual distance from x to L was always 1, hence the quotient of the sines has always been 1. This is going to change in the next paragraph.

For $t \geq t_1$, ϵ equals 1, and the quotient of the sines is still 1, but only for $t \leq \tau(x, L)$ according to (C1), which is by definition bigger than t_1. Hence

$$u^{V(x,t)}(x_1, x_2) = t - t_1 \text{ for } t_1 < t \leq \tau(x, L).$$

At $t = \tau(x, L)$, the sine of $\mathsf{d}(x, L)\pi/3$ becomes 0, and so the valuation becomes constant again:

$$u^{V(x,t)}(x_1, x_2) = \tau(x, L) - t_1 \text{ for } \tau(x, L) < t.$$

4.8.2 $n = 3$

We define (C1) and check (C2), (C3) and (C4).

(C1)

- If $\mathsf{d}(x, y) = 0$, then $\mathsf{d}_r^{t,x}(y) = 0$ for $t \in [0, +\infty[$.
- If $\mathsf{d}(x, y) = 1$, then $\mathsf{d}_r^{t,x}(y) = 1$ for $t \in [0, +\infty[$.
- If $\mathsf{d}(x, y) = 2$, then
  - $\mathsf{d}_r^{t,x}(y) = 0$ for $t \in [0, u(x, y)[$,
  - $\mathsf{d}_r^{t,x}(y) = 2$ for $t \in [u(x, y), +\infty[$.
- If $\mathsf{d}(x, y) = 3$, then
  - $\mathsf{d}_r^{t,x}(y) = 1$ for $t \in [0, \tau(x, y)[$,
  - $\mathsf{d}_r^{t,x}(y) = 3$ for $t \in [\tau(x, y), +\infty[$.

(C2)

Let y and z be a pair of incident elements. Without loss of generality one can suppose that $\mathsf{d}(x,y)+1=\mathsf{d}(x,z)$. The only not completely trivial cases are where $\mathsf{d}(x,y)=2$ and $\mathsf{d}_r^{t,x}(y)=0$. This happens when $t\in[0,u(x,y)[$, so also $t<\tau(x,z)=u(x,y)+u(y,z)$, and so $\mathsf{d}_r^{t,x}(z)=1$. We conclude that (C2) is satisfied.

(C3)

Let again y be an element, with a,b two elements incident with y, such that $\mathsf{d}_r^{x,t}(a)+1=\mathsf{d}_r^{x,t}(b)+1=\mathsf{d}_r^{x,t}(y)$. The only cases for which we need to verify that $u^{V(x,t)}(a,b)>0$ are $\mathsf{d}_r^{x,t}(y)=1$ or 2.

- If $\mathsf{d}(x,y)=1$, then $\mathsf{d}_r^{x,t}(a)+1=\mathsf{d}_r^{x,t}(b)+1=\mathsf{d}_r^{t,x}(y)=1$. One can choose $a=x$, then $\mathsf{d}(x,b)=2$, so in this case $t\in[0,u(x,b)[$. The following now holds: $u^{V(x,t)}(a,b)=u(x,b)-t>0$.
- If $\mathsf{d}(x,y)=2$, then $\mathsf{d}_r^{t,x}(y)=2$ for $t\in[u(x,y),+\infty[$. Assume that $a=xy$ and $\mathsf{d}(x,b)=3$. This yields that $t\in[u(x,y),\tau(x,b)[=[u(x,y),u(x,y)+u(a,b)[$. One checks that $u^{V(x,t)}(a,b)=u(a,b)-t+u(x,y)>0$, so (C3) holds here.
- If $\mathsf{d}(x,y)=3$, then $\mathsf{d}_r^{t,x}(y)=1$ for $t\in[0,\tau(x,y)[$. This case is similar to the case $\mathsf{d}(x,y)=1$, but now using Lemma 4.7.3 instead of (U3).

(C4)

Let y be an element, with a,b two elements incident with y, such that $\mathsf{d}_r^{x,t}(a)+1=\mathsf{d}_r^{x,t}(b)-1=\mathsf{d}_r^{x,t}(y)$. We only need to verify that $u^{V(x,t)}(a,b)=0$ is when $\mathsf{d}_r^{x,t'}(b)<\mathsf{d}_r^{x,t}(b)$ for $t'<t$.

- If $\mathsf{d}(x,y)=1$, we again choose x to play the role of a. It is clear that the conditions then tell that $t=u(x,b)$, and $u^{V(x,t)}(x,b)=u(x,b)-t=0$.
- If $\mathsf{d}(x,y)=2$, then $\mathsf{d}_r^{t,x}(y)=2$ for $t\in[u(x,y),+\infty[$. We choose a to be the element xy. The element b lies at distance 3 from x because of this, and $t=\tau(x,b)$. Similarly to the (C3) case one checks that $u^{V(x,t)}(a,b)=u(a,b)-t+u(x,y)=0$.
- If $\mathsf{d}(x,y)=3$, then $\mathsf{d}_r^{t,x}(y)=1$ for $t\in[0,\tau(x,y)[$. This case is similar to the case $\mathsf{d}(x,y)=1$, but now using Lemma 4.7.3 instead of (U3).

4.8.3 $n = 4$

Before we check the conditions, we state some useful lemmas.

Lemma 4.8.1 *It is impossible to have an ordinary quadrangle Ω containing exactly two sides with nonzero valuations, such that opposite elements have the same valuation, but each two corners of a side have different valuations.*

Proof. Suppose that such a quadrangle Ω does exist. Then let p, q be corners of Ω such that $u(p, q) > 0$, and such that the valuation in p is bigger than the one in q. There exists an $r \mathtt{I} pq$ such that $u(p, r) = u(q, r) = 0$ (by Lemma 4.2.2 and (U3)). Let Ω', Ω'' be the ordinary quadrangles sharing a path of length 4 with Ω and containing r, p and r, q, respectively. Denote the element opposite pq in Ω by s. Let p', q' and r' be the projections of, respectively, p, q and r on s. Because the valuation in p is bigger than in q, (U4) applied in both Ω' and Ω'' yields $u(r', q') < u(r', p')$ (because these are the only two other different terms in applying (U4) in both quadrangles), so $u(r', q') = u(p', q') > 0$ by (U3).

The valuations of the elements r and r' in Ω' cannot be equal because the valuation of q in Ω' is strictly smaller than the valuation of q' in Ω'. So the two corners with smallest valuation in Ω' — guaranteed by (the dual of) Lemma 4.7.3 — have to be in the corners q and r'. Applying (U4) we obtain $u(q, q') + \sqrt{2}u(qq', qr) + u(q, r) = u(q', r') + \sqrt{2}u(r'q', r'r) + u(r, r')$, which implies that $u(q', r') = 0$, a contradiction. □

Lemma 4.8.2 *Let a, b be two opposite elements. Then there exist two paths (a, x_1, x_2, x_3, b) and (a, y_1, y_2, y_3, b) from a to b such that $u(a, x_2) = u(x_2, b)$, $u(a, y_2) = u(y_2, b)$ and $u(x_1, y_1) = 0$, if and only if for each path (a, z_1, z_2, z_3, b) the equality $u(a, z_2) = u(z_2, b)$ holds.*

Proof. The implication from right to left is trivial by (U1). So suppose the left part of the statement is satisfied.

First remark that (U4) tells us that $u(x_3, y_3) = 0$, so the situation is symmetric in a and b. Suppose that $u(a, z_2) < u(z_2, b)$; then without loss of generality we may assume that $u(x_1, z_1) = 0$ (by (U3)). But then $u(x_2, a) + \sqrt{2}u(x_1, z_1) + u(a, z_2) < u(x_2, b) + \sqrt{2}u(x_3, z_3) + u(b, z_2)$, which contradicts (U4). □

If for two opposite elements a and b the situation of the above lemma holds, then we say that those two points are *equidistant*.

Lemma 4.8.3 *If two opposite points x, y are not equidistant, then there exists a path (x, a, b, c, y) from x to y, such that $u(x, b) \geq u(b, y)$ and $u(a, c) = 0$.*

Proof. First note that, if for all paths (x, a', b', c', y) from x to y it would happen that $u(x, b') \leq u(b', y)$, then Condition (U4) or Lemma 4.8.2 is violated in a quadrangle defined by two paths (x, a', b', c', y) and (x, a'', b'', c'', y), where a' and a'' are chosen so that $u(a', a'') = 0$ (which is possible due to (U1)).

So we know the existence of a path (x, a', b', c', y) with $u(x, b') > u(b', y')$. If $u(a', c') = 0$, then we are finished, so assume this is not the case. Using Lemma 4.2.2, we can find $a'' \mathtt{I} x$ with $u(a', a'') = 0$. Let (x, a'', b'', c'', y) be the unique shortest path from x to y containing a''. Lemma 4.7.3 tells us that either $u(c', c'') = 0$ or $u(a'', c'') = 0$. If we are in the first case, then applying Lemma 4.7.3 again on the other type of elements in the ordinary quadrangle leads to a contradiction with Lemma 4.8.1. So $u(a'', c'') = 0$. Using (U4) one sees that (x, a'', b'', c'', y) is a path with the desired properties. □

We are now ready to check (C1), (C2), (C3) and (C4).

(C1)

- If $\mathsf{d}(x, y) = 0$, then $\mathsf{d}_r^{t,x}(y) = 0$ for $t \in [0, +\infty[$.
- If $\mathsf{d}(x, y) = 1$, then $\mathsf{d}_r^{t,x}(y) = 1$ for $t \in [0, +\infty[$.
- If $\mathsf{d}(x, y) = 2$, then
  - $\mathsf{d}_r^{t,x}(y) = 0$ for $t \in [0, u(x, y)[$,
  - $\mathsf{d}_r^{t,x}(y) = 2$ for $t \in [u(x, y), +\infty[$.
- If $\mathsf{d}(x, y) = 3$, with $x \mathtt{I} a \mathtt{I} b \mathtt{I} y$ then
  - $\mathsf{d}_r^{t,x}(y) = 1$ for $t \in [0, u(x, b) + u(a, y)/\sqrt{2}[$,
  - $\mathsf{d}_r^{t,x}(y) = 3$ for $t \in [u(x, b) + u(a, y)/\sqrt{2}, +\infty[$.
- If $\mathsf{d}(x, y) = 4$, then in the case that there exist a, b and c such that $x \mathtt{I} a \mathtt{I} b \mathtt{I} c \mathtt{I} y$, with $u(x, b) \neq u(b, y)$, let $k(x, y)$ be the minimum of both (this is independent of a, b and c due to Lemma 4.7.3). In the case that x and y are equidistant, we define $k(x, y)$ to be equal to $\tau(x, y)/2$. Then we have
  - $\mathsf{d}_r^{t,x}(y) = 0$ for $t \in [0, k(x, y)[$,

- $\mathsf{d}_r^{t,x}(y) = 2$ for $t \in [k(x,y), \tau(x,y) - k(x,y)[$.
- $\mathsf{d}_r^{t,x}(y) = 4$ for $t \in [\tau(x,y) - k(x,y), +\infty[$.

(C2)

Let y, z be a pair of incident elements. Without loss of generality one can suppose that $\mathsf{d}(x,y) + 1 = \mathsf{d}(x,z)$. There are three nontrivial cases.

- $\mathsf{d}(x,y) = 2$, with $\mathsf{d}_r^{t,x}(y) = 0$, and $\mathsf{d}_r^{t,x}(z) = 3$. This yields $t \in [0, u(x,y)[\cap [u(x,y) + u(xz,z)/\sqrt{2}, +\infty[$. The last intersection is clearly empty and so this case cannot occur.
- $\mathsf{d}(x,y) = 3$, with $\mathsf{d}_r^{t,x}(y) = 1$ and $\mathsf{d}_r^{t,x}(z) = 4$. Let $x\mathtt{I}a\mathtt{I}b\mathtt{I}y$. This situation occurs when $t \in [0, u(x,b) + u(a,y)/\sqrt{2}[\cap [\tau(x,z) - k(x,z), +\infty[$. As $k(x,z) \le \min(u(x,b), u(b,z)) + u(a,y))/\sqrt{2}$, the range for t is empty, so this case cannot occur either.
- $\mathsf{d}(x,y) = 3$, with $\mathsf{d}_r^{t,x}(y) = 3$ and $\mathsf{d}_r^{t,x}(z) = 0$. Let $x\mathtt{I}a\mathtt{I}b\mathtt{I}y$. This happens when $t \in [0, k(x,z)[\cap [u(x,b) + u(a,y)/\sqrt{2}, +\infty[$. Again the bound $k(x,z) \le \min(u(x,b), u(b,z)) + u(a,y))/\sqrt{2}$ leads to a contradiction.

(C3)

Let again y be an element, with a, b two elements incident with y, such that $\mathsf{d}_r^{x,t}(a) + 1 = \mathsf{d}_r^{x,t}(b) + 1 = \mathsf{d}_r^{x,t}(y)$.

- If $\mathsf{d}(x,y) = 1$, then $\mathsf{d}_r^{t,x}(y) = 1$ for $t \in [0, +\infty[$. Let a be the element x. then $\mathsf{d}(x,b) = 2$, so in this case $t \in [0, u(x,b)[$. The following now holds: $u^{V(x,t)}(a,b) = u(x,b) - t > 0$.
- If $\mathsf{d}(x,y) = 2$, then $\mathsf{d}_r^{t,x}(y) = 2$ for $t \in [u(x,y), +\infty[$. We may assume that $a = xy$ and $\mathsf{d}(x,b) = 3$. This yields that $t \in [u(x,y), u(x,y) + u(a,b)/\sqrt{2}[$. One checks that $u^{V(x,t)}(a,b) = u(a,b) - \sqrt{2}(t - u(x,y)) > 0$, so (C3) holds here.
- If $\mathsf{d}(x,y) = 3$, with $x\mathtt{I}p\mathtt{I}q\mathtt{I}y$ then
  - $\mathsf{d}_r^{t,x}(y) = 1$ for $t \in [0, u(x,q) + u(p,y)/\sqrt{2}[$. We distinguish two subcases.

  * If $u(x,q) > t$, then we choose $a = q$. The element b is then at distance 4 from x, with $\mathsf{d}_r^{t,x} = 0$, hence $t \in [0, k(x,b)[$. If $u(q,b) \leq t$, then $u(q,b) = k(x,b) \leq t$ which is impossible (remember $u(x,q) > t$). As $u^{V(x,t)}(q,b) = u(q,b) - t$, Condition (C3) is satisfied here.
  * The other subcase is where $u(x,q) \leq t$. Note that $\mathsf{d}_r^{t,x} = 2$, so $\mathsf{d}(x,b) = 4$. Since $u(x,q) \leq t$ and $t < k(x,b)$, we have $u(q,b) = u(x,q)$ and $u(p,y) > 0$. We construct a as follows: let r be an element incident with x such that $u(p,r) = 0$ and let s be an element incident with r such that $u(x,s) = 0$. The element a is the projection of s on y. Let c be the projection of b on r. Lemmas 4.7.3 and 4.8.1 yield $u(a,s) = u(y,as) = 0$, $u(r,as) = \tau(x,a)/\sqrt{2}$, a and x are equidistant (by Lemma 4.8.2), and $\mathsf{d}_r^{t,x}(x,a) = 0$. As $u^{V(x,t)}(a,b) = u(a,b) - t$, we have to prove that $u(a,b) \geq k(x,b)$ in order to prove (C3).

    Let Ω be the unique quadrangle containing b, y, s and r. If b and x are equidistant, then the valuation of b in Ω is zero, and (U4) implies $u(a,b) \geq u(r,as)/\sqrt{2} = k(x,b)$. Finally suppose that b and x are not equidistant; then Lemma 4.8.2 implies $u(x,s) \neq u(s,c)$, and so $u(x,s), u(s,c) \geq k(x,b)$ (by definition of $k(x,b)$). Applying (U4) in Ω tells us now that $u(a,b) \geq u(s,c) \geq k(x,b)$, which we needed to show.

- $\mathsf{d}_r^{t,x}(y) = 3$ for $t \in [u(x,q) + u(p,y)/\sqrt{2}, +\infty[$. Let a be q in this case. This implies that the element b will be at distance 4, while $\mathsf{d}_r^{x,t}(b) = 2$. So $t \in [k(x,b), \tau(x,b) - k(x,b)[$, which also means that b and x are not equidistant. Careful analysis reveals that $u^{V(x,t)}(a,b) = \tau(x,b) - k(x,b) - t$, which is strictly larger than zero because $\mathsf{d}_r^{x,t}(b) = 2$ implies that $t \in [k(x,b), \tau(x,b) - k(x,b)[$.

- If $\mathsf{d}(x,y) = 4$, then $\mathsf{d}_r^{t,x}(y) = 2$ for $t \in [k(x,y), \tau(x,y) - k(x,y)[$. Notice that x and y are not equidistant. Let (x,p,q,a,y) be a path as constructed in Lemma 4.8.3. This fixes our choice of a. Let (x,r,s,b,y) be the unique path from x to y containing b. One checks that $u^{V(x,t)}(a,b) = u(a,b) - \sqrt{2}(t - k(x,y)) = u(a,b) - \sqrt{2}(t - u(y,q))$. The value of t is strictly smaller than $u(x,s) + u(s,b)/\sqrt{2}$ (because $\mathsf{d}_r^{t,x}(b) = 1$). All we have to check is that $u^{V(x,t)}(a,b) \geq 0$ when $t = u(x,s) + u(s,b)/\sqrt{2}$. Using (U4), one proves that $u^{V(x,t)}(a,b) = u(p,r) \geq 0$ for this value of t.

This concludes the proof of (C3) in this case.

(C4)

In this case, the condition (C4) can be proved analogously as the proof of (C3).

4.8.4 $n = 6$ and the valuation is discrete

Here the discreteness allows us to define the translations in a much easier way using recursion. We start with a valuation u where the valuations of one type of elements are integer multiples of 3, while valuations of the other type are integer multiples of $\sqrt{3}$ (with proper rescaling, this is a consequence of the discreteness, see Section 4.4.1). The valuation u also defines a residual distance d_r. We use this as the constant translated residual distance $\mathsf{d}_r^{x,t}$ with $t \in [0, 1[$ or $[0, \sqrt{3}/2[$, depending on the type of x (notice that this implies (C1) and (C2)). The condition (C4) is satisfied because it is satisfied for $t = 0$, and because the valuations in question stay zero. The discreteness makes it so that because (C3) is satisfied for $t = 0$, it will also be satisfied for t in the ranges above (because the range is small enough such that the valuation in question cannot decrease to zero).

Let's clarify this with an example first. Suppose that x is an element such that the valuations of that type of element are integer multiples of $\sqrt{3}$, and let $k \in [0, \sqrt{3}/2]$. Applying what is said above, the displacement of the base point of the trees associated with an element y with residual distance $\mathsf{d}_r(x, y)$ to yield the valuation $u^{V(x,k)}$ will be as given in the following table; all displacements are towards an element which is in the residue closest to x:

$\mathsf{d}_r(x, y)$	0	1	2	3	4	5	6
Displacement of base point	none	k	$\sqrt{3}k$	$2k$	$\sqrt{3}k$	k	none

Note that k is small enough so that the displacements do not make the base points reach branching points of the trees, except for the maximal value $k = \sqrt{3}/2$ and $\mathsf{d}_r(x, y) = 3$. In order to satisfy (C3), branching points are not supposed to be crossed as valuations are not allowed to decrease to zero (which is what happens at branching points), except for the final point (for a k-translation, (C3) needs only to be checked for values t in $[0, k[$).

We can repeat the same procedure on the new valuations we obtain but with one major caveat: the valuations are no nice integer multiples anymore (because we can k-translate with k a real number in $[0, 1]$ or $[0, \sqrt{3}/2]$ depending on the situation). However, we can handle this as follows. Let W be a Coxeter group of type $\widetilde{\mathsf{G}}_2$ acting naturally on a Euclidean affine plane $\mathbf{A}$. Take a special vertex s. Notice that, with proper rescaling, the distances from s to all the walls of a parallel class of walls is exactly the image set of the valuations u of the elements incident with a certain type of elements. Let s' be a point of the plane $\mathbf{A}$ at distance k from s, on the same wall (with type the element we have translated to) as s. Due to Lemma 4.5.4 (or by looking at the example above), we can again identify distances from s' to all the walls of a parallel class with image sets of valuations $u^{V(x,k)}$ of certain elements as above. (We can no longer identify with a type of elements; there will be more classes of elements, due to the residue corresponding with $u^{V(x,k)}$ being a weak generalized hexagon.)

We can now l-translate $u^{V(x,k)}$ to an element y in the same way as above, with l small enough so that we do not 'cross' any walls with the corresponding displacement of the point in the plane. The displacement will now happen along the line at angle $d\pi/n$ with the line through s and s', with d the distance in the residue of $u^{V(x,k)}$ from x to y. One cannot cross the wall because we will have moved some base points of trees to branching points. Note however that 'arriving' at a wall is allowed, so one can get across that wall with the next translation.

This procedure allows us to repeat the construction, obtaining all subsequent translations of u we want.

We again clarify further with an example. Suppose x is as in the above example and let t be $\sqrt{3}/3$. Now suppose that y is an element which is at distance 2 from x in the residue of $u^{V(x,k)}$. With the above procedure it follows that we l-translate to y with $l \in [0, \sqrt{3}/3]$ (when $l = \sqrt{3}/3$, we arrive again in a special point of $\mathbf{A}$). Again we could make a table and confirm indeed that the base points reach branching points of the tree except for the maximal value $l = \sqrt{3}/3$.

4.8.5 What about $n = 5$ and the nondiscrete case for $n = 6$?

One could use similar techniques as for the cases $n = 3$ and $n = 4$ to investigate these cases. The things one would need to prove are mostly quantitative versions of the qualitative lemmas of the proof of Main Result 4.3.2. However extending the, already extensive, complexity of the case studies $n = 3$ and $n = 4$ to these higher cases, would probably require an extremely extensive case study and a massive number of pages. For this reason we choose to restrict ourselves to the already handled cases.

4.8.6 Some first observations

Now that we defined additional valuations, we need to show that they form the point set of an $\mathbb{R}$-building. We need some properties to do so.

Lemma 4.8.4 *The residual distance of x and y in the residue of $u^{V(x,t)}$ equals $\mathsf{d}_r^{t,x}(y)$.*

Proof. This follows from the way we defined (C1) for $n = 3$ and $n = 4$, and from the construction for the discrete case when $n = 6$. $\square$

Lemma 4.8.5 *If $\mathsf{d}_r^{x,t}(y) = n$, then $\mathsf{d}_r^{x,t'}(y) = n$ for every $t' \geq t$.*

Proof. The only case for which this is not directly clear is $n = 6$. Applying the previous lemma we see that in the residue of $u^{V(x,t)}$ the elements x and y are residually opposite and that each shortest path between both has valuation zero. Because of the way we defined $u^{V(x,t')}$, it follows that the path also has valuation zero for $u^{V(x,t')}$. This proves the lemma. $\square$

Corollary 4.8.6 *When translating towards x, the residual distance $\mathsf{d}_r^{x,t}(y)$ only increases, up to the point that $\mathsf{d}_r^{x,t}(y) = \mathsf{d}(x,y)$.*

Proof. Again we only need to prove this when $n = 6$. Because of the previous lemma and the fact that the residue is a weak generalized n-gon where each element is incident with at least 2 elements, we see that $\mathsf{d}_r^{x,t}(y)$ only increases. It increases to $\mathsf{d}(x,y)$ because if for an arbitrary element z we have $\mathsf{d}_r^{x,t}(z) = \mathsf{d}(x,z) < n$, then for an element $a\mathtt{I}z$ there exists $t' \geq t$ such that $\mathsf{d}_r^{x,t}(a) = \mathsf{d}(x,a)$ (this is due to the displacement of the base point of the tree associated to z, which happens at a constant rate towards the projection of x on z). Repeating this argument implies that $\mathsf{d}_r^{x,t}(y)$ will eventually become $\mathsf{d}(x,y)$. $\square$

4.8.7 Structural properties of the set of translated valuations

Let $\Lambda(u)$ be the set of all valuations obtained by translating u a finite number of times.

Lemma 4.8.7 *If we know the values of a valuation v on the pairs of elements incident with an element x, and we know that an element y is residually opposite x, then we know the values of v on the pairs of elements incident with y.*

Proof. Let $a, b\mathtt{I}y$; then (U4) in an n-gon containing a, b, x and y tells us that $v(a,b) = v(a',b')$, where a' and b' are the projections on x of a and b, respectively. $\square$

Lemma 4.8.8 *Let Ω be an n-gon in Γ, nonfolded for a valuation $v \in \Lambda(u)$, such that all values of v in the line pencils of the corners and points on the sides of Ω are known; then the values of v are known entirely.*

Proof. Let x be an element of Γ. Let y be an element of Ω with minimal distance k to x. Notice that $k < n$. If $k = 0$, then we know the valuations of pairs of elements incident with x, so suppose $k > 0$. Let z be the projection of x on y. Then there are two ordinary n-gons containing z and sharing a path of length n with Ω. By applying (U3), (U4) at least one of these two n-gons is nonfolded for the valuation v. Let Ω' be such an n-gon. The valuations in the line pencils of the corners and points on the sides of Ω' are known because of the previous lemma. The minimal distance from x to an element of Ω' is now strictly less than k. So by repeating the above argument one sees that one knows the value of v everywhere. □

Corollary 4.8.9 *If* $\mathsf{d}_r^{t',x}(y) = 0$ *for all* $t' \in [0, t[$, *then* $u^{V(x,t)} = u^{V(y,t)}$.

Proof. If $n = 6$, then this follows from the 'discrete' construction.

In the other cases, let Ω be a nonfolded n-gon (for u) containing x. If we can prove that for each element z in Ω the relation $\mathsf{d}_r^{t',x}(z) = \mathsf{d}_r^{t',y}(z)$ holds for all $t' \in [0, t[$, then the displacements of the base points in the trees corresponding to the elements of Ω are the same, so by the previous lemma also $u^{V(x,t)} = u^{V(y,t)}$. Moreover, it suffices to prove this for z equal to x and equal to the element opposite x in Ω because of (C2).

If $z = x$, then note that, due to the symmetry of the definitions in (C1), $\mathsf{d}_r^{t',y}(x) = 0$ is equivalent with $\mathsf{d}_r^{t',x}(y) = 0$ for all $t' \in \mathbb{R}^+$, so also for $t' \in [0, t[$. So the result follows from the assumption.

If z is opposite x in Ω, note that due to the residual equivalency of x and y (by Lemma 4.8.4), we have that $\tau(x, z) = \tau(y, z) = 0$, and so $\mathsf{d}_r^{t',x}(z) = \mathsf{d}_r^{t',y}(z) = n$ for all $t' \in \mathbb{R}^+$. □

Remark 4.8.10 It should also be noted that at this point one can prove that the group of projectivities of a line L preserves the tree structure associated with L. This allows for a characterization due to Jacques Tits in the case $n = 3$, which was formulated without proof in [47].

4.8.8 Apartments

An apartment in our $\mathbb{R}$-building will consist of all valuations in $\Lambda(u)$ for which a given ordinary n-gon is nonfolded. Here, we investigate which valuations keep a given ordinary n-gon nonfolded. Later on, this will give us the affine structure of the apartments.

Let u be a valuation, and let Ω be a nonfolded n-gon in Γ containing an element x. Note that due to (U4) and multiple use of Lemma 4.2.2, each flag can be embedded in such a nonfolded n-gon, so results obtained here for single points or flags of Ω are true for all points or flags.

Using the definition of t-translation one easily obtains that a translation $V(x,t)$ moves the base point of the tree corresponding to an element y of Ω along the apartment of that tree with ends the two elements of Ω incident with y. The new base point lies at length $t\sin(\mathsf{d}(x,y)\pi/n)/\sin(\pi/n)$ towards the projection of x on y (note that when this projection is not defined, the length will be zero).

Consider the real affine real two-dimensional space $\mathbf{A}$. One can think of this as a (degenerate) affine apartment system with an ordinary n-gon at infinity. Identify this n-gon with Ω and let α be a point of $\mathbf{A}$. Now consider the point at distance t on the sector-panel with source α and direction x. We observe that for an element y of Ω at infinity, the distance component perpendicular to the direction to y of the original to the new point is $t\sin(\mathsf{d}(x,y)\pi/n)/\sin(\pi/n)$, which is exactly the same as above.

Note also that Ω is nonfolded for the valuation $u^{V(x,t)}$, and that the displacements of the base points in the aforementioned trees describe $u^{V(x,t)}$ completely when u is known, due to Lemma 4.8.8. So we can identify the points of $\mathbf{A}$ with the valuations obtained by translating u to elements of a certain nonfolded n-gon for u. This spawns a few direct consequences.

Corollary 4.8.11 *Let x be an element of Γ and let t and s be nonnegative real numbers. Then*

- $u^{V(x,t)V(x,s)} = u^{V(x,t+s)}$ *(local additivity).*
- $u^{V(x,t)V(y,s)} = u^{V(y,s)V(x,t)}$ *if $x\mathtt{I}y$ (local commutativity).*
- $u^{V(x,t)V(y,t)} = u$ *if $\tau_u(x,y)=0$ (reversibility).*
- *Let $(x_0,x_1,\ldots,x_i)$ (with $i\leq n$) be a path with valuation zero for some valuation u, and suppose that v is a valuation obtained from u by subsequently translating towards the respective elements of the path. Then there exists a $j\in\{1,\ldots,i\}$ and $t',s'\in\mathbb{R}^+$ such that $v=u^{V(x_{j-1},t')V(x_j,s')}$. In addition, the total sum of lengths of all the translations does not increase.*

Note that the reversibility statement also implies that, if $v\in\Lambda(u)$, then $\Lambda(v)=\Lambda(u)$.

4.8.9 Convexity

The next thing to investigate is how an ordinary n-gon Ω behaves with respect to translations towards elements outside Ω. This will allow us to prove the (convexity) condition (A2) later on.

Lemma 4.8.12 *Let Ω be an ordinary n-gon and x an element not residually equivalent with any of the elements of Ω. Then Ω cannot be a nonfolded n-gon for $u^{V(x,t)}$ with $t > 0$.*

Proof. Consider the closed path $(x_0, \ldots, x_{2n} = x_0)$ that Ω forms. There is an $i \in \{1, \ldots, 2n\}$ such that the residual distances from x to x_{i-1} and x_{i+1}, are both larger than the residual distance from x to x_i. We excluded that x_i is residually equivalent to x, so the right derivative (with respect to t) of the valuation $u^{V(x,t)}(x_{i-1}, x_{i+1})$ is positive in a certain interval (for t) containing 0, where the residual distances to x in the path are constant. This implies that Ω is not nonfolded for t in this interval but different from zero. We also know that we can partition $[0, +\infty[$ in a finite set of intervals with constant residual distances to x in the path, so repeating the above argument proves the lemma. $\square$

Lemma 4.8.13 *Let $\{p, L\}$ be a flag in Γ, let l, m be positive real numbers, and let Ω be a nonfolded n-gon. Then, if Ω is nonfolded for the valuation $u^{V(p,l)V(L,m)}$, it is also nonfolded for the valuations $u^{V(p,l')V(L,m')}$, for all $l' \in [0, l]$ and $m' \in [0, m]$. Moreover, there is a point p' and line L' in Ω such that $u^{V(p,l')V(L,m')} = u^{V(p',l')V(L',m')}$ for all $l' \in [0, l]$ and $m' \in [0, m]$.*

Proof. For the first assertion, note that, using Corollary 4.8.6, it follows that if we are translating to a certain flag $\{p, L\}$, we can first 'use up' that much of the translations to p and L (note that these commute) such that we only end up with valuations to elements not residually equivalent to an element of the ordinary n-gon. If we now translate further than this, the apartment loses its nonfoldedness and never regains it, due to Lemma 4.8.12. So if for $u^{V(p,l)V(L,m)}$ the n-gon Ω is still nonfolded, it has to be that p and L remain residually equivalent to elements of the n-gon for the whole translation. So if we translate 'less' ($u^{V(p,l')V(L,m')}$ with $l' \in [0, l]$ and $m' \in [0, m]$), Ω will still be nonfolded.

The second assertion now follows from Lemma 4.8.12 and Corollary 4.8.9 (the elements p and L stay residually equivalent with the same pair of incident elements of the n-gon for the whole translation because of Corollary 4.8.6). $\square$

4.8.10 Existence of apartments containing two valuations

Lemma 4.8.14 *Let u be a valuation, and $v, w \in \Lambda(u)$. Then there exists a point p and line $L\mathtt{I}p$ in Γ, and nonnegative real numbers k and l such that $w = v^{V(p,k)V(L,l)}$.*

Proof. First remark that $w \in \Lambda(u) = \Lambda(v)$. So w can be obtained from v with a series of i translations. We prove with induction that this series of translations can be reduced into the desired form.

If $i \leq 1$ this is trivial. If $i > 1$ we can reduce the last $i-1$ translations into the desired form, so we have that $w = v^{V(x,k)V(y,l)V(z,m)}$ with $y\mathtt{I}z$ and $k, l, m \in \mathbb{R}^+$ (note that the last two translations commute).

We now start a second induction on $j = \max(\mathsf{d}(x,y), \mathsf{d}(x,z))$. If this is 1, then we are done because of Corollary 4.8.11. So suppose that $j > 1$, and that we can reduce to the desired form if the maximum is strictly less than j. Without loss of generality, assume the maximum in the definition is reached for $\mathsf{d}(x,z)$. Let t be the smallest real positive number such that the residual distance between x and z in $v^{V(x,t)}$ equals the actual distance in Γ. There exists an element x' such that $\mathsf{d}(x',z) < \mathsf{d}(x,z)$ and x' is residually equivalent with x for $v^{V(x,t')}$, with $t' < t$ (the existence of such an x' will be clarified below).

If $k \leq t$, then $w = v^{V(x,k)V(y,l)V(z,m)} = v^{V(x',k)V(y,l)V(z,m)}$, and so we are done in this case by the second induction hypothesis. If $k > t$, then

$$w = v^{V(x,k)V(y,l)V(z,m)} = \left(v^{V(x,t)}\right)^{V(x,k-t)V(y,l)V(z,m)}.$$

By the definition of t, there exists a nonfolded n-gon for the valuation $v^{V(x,t)}$ containing x, y and z. This implies that the last three translations can be reduced into the desired form of two translations towards two incident elements in the path from x to z (by Corollary 4.8.11). If both of these translations are not towards z, then we are done due to the second induction hypothesis. If this is not the case then $w = \left(v^{V(x,t)}\right)^{V(y,l')V(z,m')} = \left(v^{V(x',t)}\right)^{V(y,l')V(z,m')}$ for certain l' and m', which is again reducable due to the second induction hypothesis.

All that is left to do is clarify the existence of the element x' above. We will only point out which elements should be chosen as x', the verification of the conditions is easily done. We can assume that $\mathsf{d}(x,z) \geq 2$.

- $n = 3$
  - $\mathsf{d}(x,z) = 2$; here we set $x' = z$.

- $\mathsf{d}(x,z) = 3$; here we take $x' \mathtt{I} z$, such that $u(xx', z) = 0$. The existence of such an x' follows from applying Lemma 4.7.3 on a triangle containing x, z and two elements incident with x, constructed by (U1).

• $n = 4$

- $\mathsf{d}(x,z) = 2$; here we set $x' = z$.
- $\mathsf{d}(x,z) = 3$; let (x,a,b,z) be the unique path of lenght 3 from x to z. If $u(a,z) = 0$, we let x' be b. If this is not the case then let c be an element incident with x and such that $u(a,c) = 0$. Next construct an element d incident with c such that $u(x,d) = 0$. The last two constructions are possible by Lemma 4.2.2. Finally x' will be the projection of d on z. Note that x and x' are equidistant due to Lemmas 4.8.1 and 4.8.2.
- $\mathsf{d}(x,z) = 4$; if x and z are equidistant, we let x' be z. Otherwise, using Lemma 4.8.3, we can construct a path (x,a,b,c,z) such that $u(x,b) \geq u(b,z)$ and $u(a,c) = 0$. Here we let x' be the element b.

• $n = 6$ and discrete. In this case the existence is guaranteed by the discreteness and Lemma 4.7.6. □

Corollary 4.8.15 *If we reduce $v^{V(p,l)V(L,m)V(p',l')V(L',m')}$ to an expression of the form $v^{V(p'',l'')V(L'',m'')}$, then $l'' + m'' \leq l + m + l' + m'$.*

Proof. All the reductions in the proof of the above lemma use Corollary 4.8.11, which does not increase the sum of the lengths of the translations. □

Lemma 4.8.16 *For each pair of valuations $v, w \in \Lambda(u)$ there is an ordinary n-gon Ω in Γ which is nonfolded for both v and w.*

Proof. Due to the previous lemma there exists a point p and line $L \mathtt{I} p$ in Γ, $l, m \in \mathbb{R}^+$ such that $w = v^{V(p,l)V(L,m)}$. Let Ω be an ordinary n-gon in Γ containing p and L such that Ω is nonfolded for v (these exist because of Lemma 4.2.3). Because both p and L lie in Ω, translations towards p and L produce valuations for which Ω remains nonfolded. In particular this holds for $w = v^{V(p,l)V(L,m)}$. □

4.8.11 Building the affine apartment system

We end by putting all the pieces together to form an affine apartment system. Let $\Lambda(u)$ be the set of points. Remember that if $v \in \Lambda(u)$, then $\Lambda(u) = \Lambda(v)$.

Let Ω be an ordinary n-gon of Γ. Consider the set $A(\Omega)$ of all the valuations in $\Lambda(u)$ for which this n-gon is nonfolded. Suppose that two valuations v_1 and v_2 are in this set. Lemma 4.8.14 tells us that there exists a flag $\{p, L\}$ in Γ and $k, l \in \mathbb{R}^+$ such that $v_2 = v_1^{V(p,k)V(L,l)}$. As Ω is nonfolded for both v_1 and v_2, Lemma 4.8.13 implies that there exists a flag $\{p', L'\}$ in Ω such that $v_2 = v_1^{V(p',k)V(L',l)}$. We can conclude that all the valuations in the set $A(\Omega)$ can be obtained out of each other by translating towards elements of Ω. This is exactly the set of valuations which has been studied in Corollary 4.8.11. In the reasoning before the statement of this corollary it was seen that the valuations can be interpreted as points of $\mathbf{A}$. The sector with source $v \in \Lambda(u)$ and direction the flag $\{p, L\}$ will be the set $\{v^{V(p,k)V(L,l)} | k, l \in \mathbb{R}^+\}$.

This allows us to define a chart $f_{\Omega,v,p,L}$, for a $v \in \Lambda(u)$, and Ω a nonfolded n-gon, containing a flag $\{p, L\}$ (the chart is defined such that a chosen fixed sector of $\mathbf{A}$ is mapped to the sector with source v and direction $\{p, l\}$). Let $\mathcal{F}$ be the collection of all these charts. Condition (A1) can now easily seen to be true.

The second condition to check is (A2). Let $f = f_{\Omega,v,p,L}$ and $f' = f_{\Omega',v',p',L'}$ be two charts in $\mathcal{F}$. Let $X = f^{-1}(f'(\mathbf{A}))$. The points (or valuations) which are in the image of both charts, are those valuations for which both Ω and Ω' are nonfolded. Let v'' be a valuation for which this is the case (if there is not such a v'', the condition (A2) is trivially satisfied). Lemma 4.8.13 implies that X is star convex for $f^{-1}(v'')$. Because v'' is arbitrary in $f(X)$, one obtains that X is convex. That X is also closed follows from the fact that translations change the valuations continuously.

Next thing we need to show is the existence of a $w \in W$ such that $f|_X = f' \circ w|_X$. Consider both X and the similar set $X' = f'^{-1}(f(\mathbf{A}))$. In order to prove the existence of such a w we need to prove that X can be mapped onto X' by some $w \in W$. The map $\phi = f'^{-1} \circ f$ is bijective from X to X'. Let x_1 and x_2 be elements of X. Then their images under f are two valuations v_1 and v_2. Because they lie in the same apartment $A(\Omega)$, there is a flag $\{q, M\}$ in Ω and $k, l \in \mathbb{R}^+$ such that $v_2 = v_1^{V(q,k)V(M,l)}$. But as these two valuations are also in $A(\Omega')$, we know by Lemma 4.8.13 that there exists a flag $\{q', M'\}$ in Ω' such that $v_2 = v_1^{V(q',k)V(M',l)}$. Since the lengths of the translations and the type of elements towards the translations happen is invariant, it follows that ϕ is distance preserving and preserves the type of the directions at infinity of $\mathbf{A}$. This implies the existence of the needed w.

Condition (A3) is satisfied because of Lemma 4.8.16.

Now, (A4) can be shown to be true as follows: suppose we have two sectors related to two flags $\{p, L\}$ and $\{q, M\}$ of Γ. These can be embedded in an ordinary n-gon Ω. The apartment $A(\Omega)$ contains sectors with directions $\{p, L\}$ and $\{q, M\}$. This only leaves us to prove that two sectors related to the same flag always intersect in a subsector. This last assumption is true because if we have two ordinary n-gons Ω and Ω' containing p and L, it follows from Corollary 4.8.6 that there exist $l, m \in \mathbb{R}^+$ such that for each $l' \geq l, m' \geq m$ the valuation $u^{V(p,l')V(L,m')}$ takes only the value zero in both Ω and Ω'. The set of these valuations forms the desired subsector.

For (A5) we have three ordinary n-gons Ω, Ω' and Ω'', each pair sharing a path of length n. From (U3) and (U4) we deduce that, if for a valuation $v \in \Lambda(u)$ the ordinary n-gon Ω is nonfolded, then at least one of Ω' and Ω'' is nonfolded for v, too. This means that every point of $A(\Omega)$ belongs to $A(\Omega')$ or to $A(\Omega'')$, or to both. Since it is easy to see that the intersection of two apartments is closed, the sets $A(\Omega) \cap A(\Omega')$ and $A(\Omega) \cap A(\Omega'')$ are not disjoint, proving (A5).

It remains to prove that the 'distance' function d defined on pairs of valuations by (A1), (A2) and (A3) is indeed a distance function. (For two valuations v and $v^{V(p,k)V(L,l)}$, the distance between both is defined as the length of the third side of a triangle in a Euclidean plane, where two sides have length k and l, and with the angle between both sides π/n.) However, by re-reading the proof in [24, §1] of the equivalence of the various definitions for affine apartment systems, one sees that the weaker inequality $\mathsf{d}(u, v) \leq 2(\mathsf{d}(u, w)+\mathsf{d}(w, v))$ also suffices. This inequality is a direct consequence of Corollary 4.8.15.

So we conclude that the set of points $\Lambda(u)$, endowed with the set of apartments

$$\{A(\Omega) \mid \Omega \text{ is an ordinary } n\text{-gon of } \Gamma\},$$

forms a 2-dimensional affine apartment system with the generalized n-gon Γ at infinity.

All that is left to show is that the construction of Main Result 4.3.1 applied to the affine apartment system defined on $\Lambda(u)$ and the point defined by the valuation u, gives us back the valuation u on Γ. One has to prove that, if x and y are adjacent, the corresponding sector-panels with source u share a line segment of length $u(x, y)$. This follows from Corollary 4.8.9 and the fact that, if x and y are adjacent, one has $\mathsf{d}_r^{t,x}(y) = 0$ if and only if $t \in [0, u(x, y)[$.

This concludes the proof of Main Result 4.3.4.

4.9 Proof of Application 4.4.1

Suppose we have a projective plane Γ and a real number $t \in \mathbb{R}^+ \setminus \{0\}$. Also suppose we are either given a valuation u, or two functions d and $\angle$ satisfying the conditions listed in Theorem 4.4.1. Use the identities $\mathsf{d}(p,q) = t^{-u(p,q)}$ and $\angle L, M = \arcsin(t^{-u(L,M)})$ to reconstruct the other function(s).

It is easily seen that Condition (U2) for valuations corresponds to Condition (M2) and the part "$\mathsf{d}(p,q) = 0 \Leftrightarrow p = q$" of Condition (M1).

If we have three points p, q and r, then

$$u(p,q) \geq \min(u(p,r), u(r,q)) \Leftrightarrow \mathsf{d}(p,q) \leq \max(\mathsf{d}(p,r), \mathsf{d}(r,q)).$$

The left hand side is satisfied for a valuation because of (U3) and Lemma 4.7.3; the right hand side is satisfied for a distance because of (M1). So Condition (U3) for points on a line is equivalent with the inequality part of (M1).

Condition (U1) for valuations is equivalent with Conditions (M3) and (M4).

Also Condition (U4) corresponds directly to the sine rule Condition (M5).

The only part that needs a closer look is how Condition (U3) for valuations follows from Conditions (M1) up to (M5) (and the already proven Conditions (U1), (U2), (U3) for points on a line, and (U4)). Let L, M and N be three lines through a point p. By (U1), there exist two lines Y and Z through p such that $u(Y,Z) = 0$. Since (U1) and (U3) hold for points on a line, Lemma 4.2.2 also holds. So there exist $q \mathtt{I} Y$ and $r \mathtt{I} Z$ with $u(p,q) = u(p,r) = 0$. We now have for the line qr that $\tau(p, qr) = 0$ by (U4). (Note that τ is well-defined because (U4) holds.)

Let l, m and n be the respective projections of L, M and N on the line qr . Using (U4) we see that $u(L,M) = u(l,m)$, $u(M,N) = u(m,n)$ and $u(L,n) = u(l,n)$. So Condition (U3) for the three lines L, M and N follows directly from the same Condition (U3) for the three points l, m and n.

4.10 A condition on the completeness of $\mathbb{R}$-buildings

As we have discussed in Section 1.8.2, the completeness of the metric space formed by an $\mathbb{R}$-building allows us to apply various results for complete CAT(0)-spaces. While all discrete $\mathbb{R}$-buildings are complete, this statement is not true for arbitrary $\mathbb{R}$-buildings. With the next theorem we want to provide a tool to determine when a certain $\mathbb{R}$-building forms a complete metric space and when not.

Main Result 4.10.1 *The metric space* (Λ, d) *defined by an* $\mathbb{R}$*-building* $(\Lambda, \mathcal{F})$ *is complete if and only if all of the metric spaces defined by the trees associated to its walls are complete.*

The question as to whether a certain $\mathbb{R}$-tree forms a complete metric space seems an easier question, which will hopefully be resolved for $\mathbb{R}$-trees from $\mathbb{R}$-buildings of dimension three and higher, using algebraic methods.

Remark 4.10.2 This result is related to a result of Bruhat and Tits ([8]) where they use the additional assumption that the building at infinity is Moufang.

4.11 Proof

First assume that the metric space (Λ, d) is complete, and let m be a wall of the spherical building at infinity. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the tree $T(m)$. The union of the apartments of the $\mathbb{R}$-building which at infinity contain m form a subset $K \subset \Lambda$ isometric to the direct product of the metric space formed by $T(m)$ and $\mathbb{R}$ (see Section 1.8.2).

Using this subset K, we can 'lift' the Cauchy sequence $(\alpha_n)_{n\in\mathbb{N}}$ to a Cauchy sequence $(\beta_n)_{n\in\mathbb{N}}$ in $K \subset \Lambda$. As the metric space (Λ, d) is complete, this sequence converges to some point $\beta \in \Lambda$. Our goal is to prove that the point β lies in K, implying that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges. For this we have to prove that β lies in an apartment which at infinity contains the wall m. Let S_∞ and S'_∞ be two opposite maximal sector-panels of m; if we can prove that the germs of sector-panels $[S]_\beta$ and $[S']_\beta$ in the residue at β are still opposite, we are done. Equivalent with this last statement is that for a shortest gallery from a chamber C_∞ containing S_∞ to a chamber C'_∞ containing S'_∞, the corresponding gallery from $[C]_\beta$ to $[C']_\beta$ always is nonstammering. As this is the case for each point of the sequence $(\beta_n)_{n\in\mathbb{N}}$, Corollary 4.5.6 implies that this is also the case for β. So we have proven that the metric space defined by the $\mathbb{R}$-tree $T(m)$ is complete.

We are now left with the other direction to prove. Assume that all the metric spaces defined by the trees corresponding to walls at infinity are complete. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the metric space (Λ, d). Let $(\bar{\Lambda}, \bar{\mathsf{d}})$ be the metric completion of (Λ, d). In this larger metric space the Cauchy sequence $(\alpha_n)_{n\in\mathbb{N}}$ does converge to some point α. Choose some chamber C_∞ at infinity and consider the sequence of sectors $(C_{\alpha_n})_{n\in\mathbb{N}}$.

Lemma 4.11.1 *Let* C_β *and* C_γ *be two sectors with sources* β *and* γ *respectively, and having the same direction* C_∞*. Then there exists a constant* $k \in \mathbb{R}^+$ *depending on the type* *of the $\mathbb{R}$-building, such that there exists a point δ for which the sector C_δ is a subsector of both C_β and C_γ, and $\mathsf{d}(\beta,\delta), \mathsf{d}(\gamma,\delta) \leq k\mathsf{d}(\beta,\gamma)$.*

Proof. Embed the sector C_β in an apartment Σ, and the sector C_γ in an apartment Σ'. Let δ be the point of $C_\beta \cap C_\gamma$ closest to β (possible because this intersection is a closed subset of Σ due to Condition (A2)). The sector C_δ is a subsector of both C_β and C_γ.

Let D_∞ and D'_∞ be the chambers opposite C_∞ in respectively Σ_∞ and Σ'_∞. Note that $\beta \in D_\delta$ and $\gamma \in D'_\delta$. Due to the way we defined δ, we have that $D_\delta \cap D'_\delta = \{\delta\}$. Consider the retraction r on the apartment Σ centered at the germ of D_δ (see [24, Prop. 1.17]). This retraction maps the sector D'_δ to some sector D''_δ in Σ, only sharing its source δ with the sector D_δ. As $r(\gamma)$ lies in D''_δ, it follows that there exists some constant k such that $\mathsf{d}(\beta,\delta), \mathsf{d}(r(\gamma),\delta) \leq k\mathsf{d}(\beta, r(\gamma))$. Because the retraction does not change distances to δ, and does not increase the other distances, this implies the desired result. $\square$

Corollary 4.11.2 *There exists a constant k', such that for each sector C_β, and $l \in \mathbb{R}^+$, there exists a point $\delta \in C_\beta$ with $\mathsf{d}(\beta,\delta) = k'l$, such that for each point γ at distance at most l from β, the sector C_δ is a subsector of C_γ.*

Proof. All the sectors C_ρ with $\mathsf{d}(\rho,\beta) < t$, $t \in \mathbb{R}^+$ and $\rho \in C_\beta$, contain a common point τ which lies at a distance $k''t$ from β, with k'' some constant. The result then follows from applying the above lemma. $\square$

Using Corollary 4.11.2 one can find a sequence of points $(\beta_n)_{n\in\mathbb{N}}$ in Λ which also converges to the point α, and such that if $i < j$, then the sector C_{β_i} is a subsector of C_{β_j}. In the completion $\bar{\Lambda}$ we obtain a subset isometric to a sector, where the 'source' is α (by applying Corollaire 2.11 from [24] and its preceding text). Note that the interior (as it would be in an apartment) lies in Λ.

Let S_∞ be a sector-panel of C_∞. The sequence $(S_{\beta_n})_{n\in\mathbb{N}}$ forms a Cauchy sequence in the tree $T(S_\infty)$, contained in a half-line. Using the completeness of this tree, we can extend this half-line to an apartment, and find a sector C'_∞ such that $[C]_{\beta_n} \neq [C']_{\beta_n}$ for all $n \in \mathbb{N}$. Regarding the limit situation in $\bar{\Lambda}$, one obtains a subset isometric to two sectors with the same 'source' α and sharing a 'sector-panel'. Note again that the interior (as it would be in an apartment) lies in Λ because the geodesic in $\bar{\Lambda}$ between two points of Λ lies again in Λ (due to the above corollary).

Repeating the algorithm one can obtain a subset K of $\bar{\Lambda}$ isometric to a half-apartment with α on its 'wall' M, and such that all points of K not in M lie in Λ. Considering the complete wall-tree $T(m)$ where m is the direction at infinity of the walls in K parallel to M, we see that K has to lie completely in Λ, proving that $\alpha \in \Lambda$, and completeness.

4.12 Generalizations of $\mathbb{R}$-trees related to walls and panels at infinity

As already mentioned in Section 1.8.2, we will generalize the notion of trees associated to walls and panels at infinity in this section.

We let $(\Lambda, \mathcal{F})$ be an $\mathbb{R}$-building with set of apartments $\mathcal{A}$. Let Λ_∞ be the corresponding building at infinity with set of apartments $\mathcal{A}_\infty$ (in one-to-one correspondence with the elements of $\mathcal{A}$). Let S_∞ be a certain simplex at infinity (with a corresponding sector-facet S); then its residue at infinity is a (possible weak) spherical building $(\Lambda_\infty)_{S_\infty}$.

One can repeat the two constructions from Section 1.8.2, but now replacing the sector-panel π by the sector-facet S_∞, and the wall m by the smallest convex subcomplex B of the sector-facet S_∞ and some opposite sector-facet S'_∞ (a set in the $\mathbb{R}$-building with this subcomplex B at infinity will be referred to as a *subspace*).

These two constructions yield injections from $\mathbb{R}^m$ (where m is the rank of the residue of S_∞) into sets $T(S_\infty)$ and $T(B)$. We now claim that the following two constructions both yield $\mathbb{R}$-buildings with as building at infinity $(\Lambda_\infty)_{S_\infty}$, forming a generalization of Section 1.8.2.

4.13 Proof

Before proving that these two constructions yield $\mathbb{R}$-buildings we show that they are equivalent. For this we need a few lemmas. When we use the notion of subsector-facet, we only mean sector-facets which are subsets of the other sector-facet, and having the same rank.

Lemma 4.13.1 *Let S_α and S_β be two sector-facets with the same direction S_∞ and sources $\alpha, \beta \in \Lambda$. Then there exists an apartment containing subsector-facets of both.*

Proof. We embed S_∞ in a chamber at infinity C_∞. It follows from (A4) that the two corresponding sectors C_α and C_β have a common subsector C_γ with source γ (note that there is no uniqueness here). Let C'_∞ be a chamber at infinity containing S_∞ and adjacent to C_∞, and such that the germs of the sectors C_γ and C'_γ are different, and containing a subsector-facet of S_α (to verify the existence of such a C'_∞, consider any apartment containing C_α). Analogously we choose a C''_∞ containing a subsector-facet of S_β, with the additional requirement that C''_∞ is adjacent with C'_∞.

Consider the two sectors C'_γ and C''_γ. Let Σ be any apartment containing C'_γ. Because C''_γ is adjacent to this sector, there exists an apartment Σ' sharing a half-apartment with Σ and containing C''_γ. There is also a third apartment Σ'' sharing half-apartments with both of the previous apartments. As each pair of points in the union of all three apartments Σ, Σ' and Σ'', lies in at least one of these apartments (by (A5)), we have that at least one of these apartments contains subsector-facets of both S_α and S_β. □

Lemma 4.13.2 *Let S_α be a secor-facet, and S'_∞ a sector-facet at infinity opposite to S_∞. Then there exists a unique subspace containing both S'_∞ at infinity and a subsector-facet of S_∞.*

Proof. Let B be some minimal subspace containing both S'_∞ and S_∞ at infinity. Let $\beta \in B$ be a point. By the above lemma there exists an apartment Σ containing subsectors of S_α and S_β. In particular there exists a sector C_γ with source γ on S_β containing a subsector of S_α. The germ of this sector is opposite to some germ of a sector D_γ containing S'_γ. It is clear that the apartment defined by C_∞ and D_∞ contains a desired subspace. Unicity is trivial. □

The above lemma makes clear that the sets of points of the two constructions are in one-to-one correspondence with each other. An apartment from the second construction (using sector-facets) is easily seen to imply an apartment in the first construction (using subspaces). Conversely, for the first construction, one sees that all apartments containing S_∞ and a subspace in the residue of S_∞ at infinity, correspond to one apartment of the second construction, establishing a one-to-one correspondence.

We now verify (A1)-(A4) and the triangular inequality ($\Delta \leq$) for both constructions. The above implies that we can choose which one of both constructions to verify the condition for.

(A1),(A2) Directly from the corresponding conditions of the original building and the second construction.

(A3) From Lemma 4.13.1 using the first construction.

(A4) Notice that sectors in the second construction are in fact sectors of the original building, so (A4) for the original building gives us a sector which can be seen to be a sector of the first construction.

(A5) From the second construction and (A5) for the original building.

($\triangle \leq$) From the second construction and the triangular inequality for the original building.

As the conditions are all verified, we have proved that these constructions yield $\mathbb{R}$-buildings.

4.14 Subbuildings corresponding to fixbuildings at infinity

Let $(\Lambda, \mathcal{F})$ be some affine building with an automorphism group G acting on it, fixing at least one point (when G is finite this is implied by the Bruhat-Tits Theorem 1.8.4). This group G also acts on the spherical building Λ_∞ at infinity. Suppose that the group G acts type-preservingly on the spherical building Λ_∞, such that the fixed simplicial complex Λ'_∞ forms a building, and such that for each fixed simplex S_∞, there also exists an opposite fixed simplex S'_∞.

We do not demand that this new building is of the same rank as the original building. While the fixed structure of an automorphism group G of a spherical building is in 'most' cases again a building, this is no longer the case for general buildings, in particular for the fixed structure of G in the affine building Λ.

In this section, we will try to show that, despite the fact that the fixed structure is not necessarily an affine building, in many (but not all) cases this fixed structure does contain an affine building $(\Lambda', \mathcal{F}')$ with the fixed subbuilding Λ'_∞ as spherical building at infinity. A list of some cases where the construction works is listed at the end of the next section.

Remark 4.14.1 As one can notice from the notations used, we will consider affine buildings as being discrete $\mathbb{R}$-buildings. It appears that the proof can be extended to the nondiscrete case by replacing the induction argument, and considering the completion $\bar{\Lambda}$ of the metric space (Λ, d) when it is not complete.

4.15 Proof

Let S_∞ and S'_∞ be maximal fixed and opposite simplices at infinity, and let B be the unique apartment of Λ'_∞ containing both.

Lemma 4.15.1 *There exists at least one fixed subspace of the $\mathbb{R}$-building such that the corresponding structure at infinity is B.*

Proof. By applying Lemma 4.13.2, knowing that there is at least one fixed point and keeping in mind that if a point and some simplex at infinity is fixed, then also all the points on the unique sector-facet with this source and direction are fixed. □

By the above lemma we know that there is at least one fixed subspace with B at infinity; now consider all such fixed subspaces. All these subspaces form a set F of points of the $\mathbb{R}$-building $T(B)$. As the original $\mathbb{R}$-building is discrete, this $\mathbb{R}$-building will also be discrete, and because of this also complete. The set F is nonempty and bounded (because of the maximality of S_∞ and S'_∞), and has for this reason a unique center due to Theorem 1.8.4. With this unique center corresponds a fixed subspace of the original $\mathbb{R}$-building with B at infinity. We will call this subspace the *middle fixed subspace corresponding to B*. These will form the apartments of the new $\mathbb{R}$-building Λ'. Using the associated structure at infinity, one can define charts on them such that both Conditions (A1) and (A2) are satisfied (for proving (A2) keep in mind that the original $\mathbb{R}$-building $(\Lambda, \mathcal{F})$ satisfies (A2)).

Remark that we can perform a similar construction to obtain a center using fixed asymptotic classes instead of fixed subspaces (these two notions are in bijective correspondence due to Lemma 4.13.2). If we look at things this way it follows that if two apartments of Λ' share a maximal fixed simplex at infinity, then the corresponding sector-facets in both apartments are asymptotic, or equivalently, they share a subsector-facet. Condition (A4) now follows from applying the fact that two chambers of a building lie in a common apartment of the spherical building Λ'_∞.

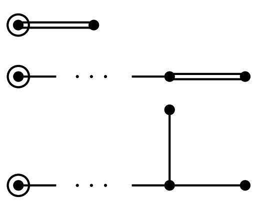
The same reasoning combined with convexity shows that if two apartments of Λ' share a half-apartment at infinity, then the apartments themselves share a half-apartment. The next condition we handle, and this is the part where the extra assumptions come in, is Condition (A5). Assume that there exist three apartments of Λ' pairwise sharing a half-apartment, while the intersection of all three is nonempty. Such a configuration we will call a *triangle configuration*. Using the generalization of the 'trees corresponding to walls construction' from Section 4.12, one can obtain a *triangle configuration* of subspaces isometric to the real affine line. Because the sum of the angles of a triangle in a CAT(0)-space is less or equal than π, the triangle formed by these subspaces satisfies it too. If vertices of the appropriate type (the residues of rank one at infinity) lie at angles strictly more than $\pi/3$ (considering apartments as spheres), then the configuration is impossible, and Condition (A5) has to satisfied.

The triangle inequality is trivially satisfied as it is satisfied for the original $\mathbb{R}$-building. The only condition one still has to verify is Condition (A3). Note that due to discreteness and the apartments we defined, the structure Λ' is a chamber complex. Suppose C, D and D' are three chambers, where C and D lie in an apartment Σ of Λ', while D' lies in an apartment Σ' of Λ'. Using a lemma with a similar statement and proof as Lemma 1.8.1, it follows easily that C and D' lie in a common apartment of Λ'. Repeating this construction proves (A3).

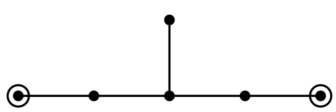
As we have proven Conditions (A1)-(A5) and the triangle inequality, the set Λ' forms indeed an $\mathbb{R}$-building. It is easily seen that Λ'_∞ is the building at infinity of Λ'.

We end with listing some diagrams of embeddings for which we verified the aforementioned condition on the angles (this list is not exhaustive). The diagram itself depicts the type of the building Λ_∞, the encircled nodes show how the fixbuilding Λ'_∞ is embedded. We group these diagrams per type of the embedded building Λ'_∞.

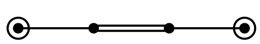
- A_1:



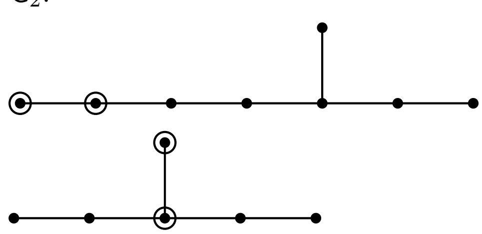
- A_2:



- C_2:



- G_2:



Bijlage A

Nederlandstalige samenvatting

A.1 Inleiding

De titel van deze thesis luidt: 'Een studie van gebouwen van lage rang'. De theorie der gebouwen is ontwikkeld in de vroege jaren 60 door Jacques Tits. Het doel hiervan was om een meetkundig instrument te verschaffen om de belangrijkste klassen van enkelvoudige groepen te bestuderen, namelijk de enkelvoudige algebraïsche groepen, de klassieke groepen, de groepen van gemengd type en de Frobenius-gedraaide Chevalley groepen.

Waarom nu van lage rang? Jacques Tits bewees twee belangrijke classificaties van bepaalde klassen van gebouwen. Die van de sferische gebouwen van rang minstens 3 in 1974 ([44]), en die van de affiene gebouwen van rang minstens 4 in 1986 ([47]). Als men echter de sferische gebouwen van rang 2 en de affiene gebouwen van rang 3 bekijkt, dan is een classificatie onmogelijk. Deze gevallen verliezen hierdoor echter niet hun belangrijkheid, omdat ze nog steeds sterke meetkundige eigenschappen hebben en door de extra vrijheid een veel rijker gedrag vertonen.

Wij hebben verscheidene karakteriseringen en constructies van zulke gebouwen van lage rang bekomen - deze zijn terug te vinden in Sectie A.2.

A.1.1 Simpliciale complexen

Een *simpliciaal complex* S gedefinieerd op een verzameling X is een verzameling van deelverzamelingen van X zodanig dat als een bepaalde deelverzameling een element is van S, dan ook elke deelverzameling ervan. De elementen van X noemt men de *punten*, die van S de *simplexen*.

Een *maximaal simplex* is een simplex niet bevat in een groter simplex. Twee maximale simplexen zijn *adjacent* als hun doorsnede een simplex is met één punt minder dan de 2 maximale simplexen.

Een simpliciaal complex noemt men een *kamercomplex* als men elke twee maximale simplexen kan verbinden met een keten van adjacente maximale simplexen. De maximale simplexen noemt men in dit geval *kamers*. Deze definitie impliceert ook dat elke twee kamers even groot zijn. De *panelen* zijn dan de op één na grootste simplexen.

Een kamercomplex noemt men *dun* als elk paneel in juist twee kamers ligt, en *dik* als het altijd in minstens drie kamers ligt.

A.1.2 Gebouwen

Gebouwen zijn de dikke kamercomplexen S waarvoor er een verzameling $\mathcal{A}$ van dunne deel-kamercomplexen bestaat (*appartementen* genoemd), zodat aan volgende twee voorwaarden voldaan is.

- Elke twee kamers liggen in een appartement.
- Voor elke twee appartementen A en B bestaat er een isomorfisme van A naar B dat de doorsnede elementsgewijs vasthoudt.

De orde van de kamers noemt de *rang* van het gebouw.

A.1.3 Interessante gevallen

Rang 1

Een rang 1 gebouw is een verzameling punten X ($|X| \geq 3$) waarbij de appartementen de puntenparen zijn. Om deze gevallen meer structuur en betekenis te geven, definieert men *Moufangverzamelingen*. Hierbij veronderstelt men voor elk element $x \in X$ een groep (de *wortelgroep* genaamd) die regulier werkt op de overige punten van X. Ook eist men dat de groep $G^\dagger$ (de *kleine projectieve groep*) voortgebracht door alle wortelgroepen de verzameling van alle wortelgroepen normaliseert.

Rang 2

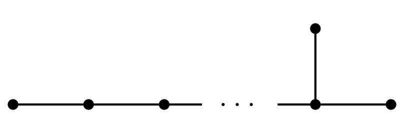
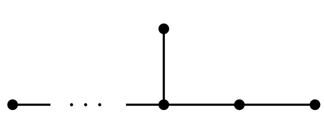
De rang twee gevallen kan men opsplitsen in twee categorieën. De eerste categorie is waar de appartementen oneindig zijn; deze gebouwen komen overeen met de boomgrafen zonder eindpunten, en waarbij elke top minstens 3 buren heeft. De appartmenten zijn hier oneindige lijngrafen.

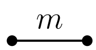
De tweede categorie (met eindige appartementen) komt overeen met de bipartiete grafen met maximale afstand n en minimale cykels van lengte $2n$. Meestal kiest men één van deze verzamelingen en associeert men daarmee *punten*, met de andere *rechten*, en men zegt dat een punt en een rechte *incident* zijn als de bijhorende toppen adjacent zijn. Op deze manier bekomt men een (rang 2) meetkunde die men een *veralgemeende n-hoek* noemt (meestal kortweg n-hoek als er geen verwarring kan optreden). De appartementen komen in de graaf overeen met $2n$-hoeken, en in de veralgemeende n-hoek met n-hoeken (wat de naamgeving verklaart).

Een *dualiteit* van een veralgemeende veelhoek is een automorfisme van het bijhorende gebouw dat punten op rechten afbeeldt en vice versa. Men noemt een dualiteit een *polariteit* als ze van orde 2 is. Een punt (rechte) van de veralgemeende veelhoek is *absoluut* als het (ze) incident is met zijn (haar) beeld.

Sferische gebouwen

Sferische gebouwen zijn gebouwen waarbij de appartementen eindige kamercomplexen zijn. De naam komt van het feit dat men in dit geval de appartementen kan opvatten als betegelingen van sferen. De mogelijke appartementen kan men classificeren als volgt (zonder verder te specificeren wat de diagrammen betekenen):

- A_n: $(n \geq 1)$
- C_n: $(n \geq 2)$
- D_n:  $(n \geq 4)$
- E_n:  $(n = 6, 7, 8)$

- F_4:
- $\mathsf{I}_2(m)$: 

 $(m \geq 5)$

De sferische gebouwen van rang minstens 3 zijn geclassificeerd door Jacques Tits ([44]). Men kan aantonen dat ze *Moufang* zijn (wat een bepaalde groep-theoretische voorwaarde is). Ruwweg komen ze overeen met de volgende groepen:

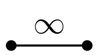
- klassieke groepen,
- algebraïsche groepen,
- gemengde groepen.

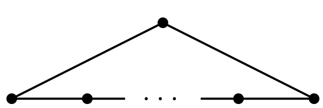
Voor sferische gebouwen van rang 2 (de veralgemeende veelhoeken) bestaan er zogenaamde vrije constructies waardoor een classificatie onmogelijk is.

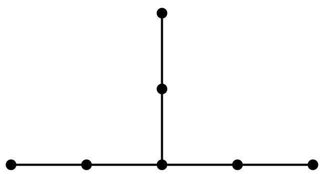
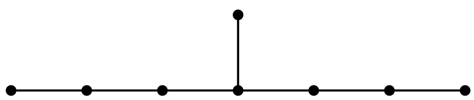
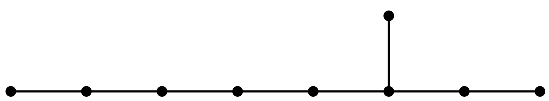
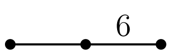
De gebouwen van type A_n corresponderen met n-dimensionale projectieve ruimtes.

Affiene gebouwen

Affiene gebouwen zijn gebouwen waarbij de appartementen betegelingen zijn van affiene Euclidische ruimtes. De *dimensie* van de affiene ruimte is de rang van het gebouw min 1. Ook hier kan men de appartementen classificeren (opnieuw zonder er dieper op in te gaan):

- $\widetilde{\mathsf{A}}_1$: 

- $\widetilde{\mathsf{A}}_n$:  $(n \geq 2)$
- $\widetilde{\mathsf{B}}_n$:  $(n \geq 3)$
- $\widetilde{\mathsf{C}}_n$: $(n \geq 2)$
- $\widetilde{\mathsf{D}}_n$:  $(n \geq 4)$

- $\widetilde{\mathsf{E}}_6$: 
- $\widetilde{\mathsf{E}}_7$: 
- $\widetilde{\mathsf{E}}_8$: 
- $\widetilde{\mathsf{F}}_4$:
- $\widetilde{\mathsf{G}}_2$: 

Met een affien gebouw kan men een zogenaamd *gebouw op oneindig* associëren, en dit is dan een sferisch gebouw waarbij de rang 1 lager is dan de rang van het oorspronkelijk gebouw. Gebruik makende van deze constructie en de classificatie van de sferische gebouwen van rang minstens 3, kon Jacques Tits de affiene gebouwen van rang minstens 4 classificeren ([47]).

Deze classificatie was niet beperkt tot de affiene gebouwen, maar omvatte ook de $\mathbb{R}$-gebouwen met dimensie minstens 3, die niet-discrete veralgemeningen zijn van affiene gebouwen. Deze structuren hebben ook affiene ruimtes als appartementen, en tevens een sferisch gebouw op oneindig.

Voor het rang 3 (of equivalent dimensie 2) geval bestaan opnieuw vrije constructies (door Mark Ronan [27]), en is classificatie dus uitgesloten.

De rang 2 gevallen komen overeen met de bomen uit de voorgaande sectie. De 1-dimensionale $\mathbb{R}$-gebouwen zijn de zogenaamde $\mathbb{R}$-*bomen*, die niet-discrete veralgemeningen zijn van bomen.

A.2 Resultaten

De resultaten kan men ruwweg opdelen in drie categorieën.

A.2.1 Rang 1: Moufangverzamelingen

Merk op dat de definitie van een Moufangverzameling een puur groep-theoretische definitie is, in tegenstelling tot de definitie van gebouwen. Als de wortelgroepen niet commutatief zijn, kan men echter toch een rang k meetkunde definiëren op de punten van de Moufangverzameling, waarbij k de nilpotentieklasse is van de wortelgroepen. De fundamentele vraag is dan: is de automorfismegroep van de Moufangverzameling gelijk aan die van de meetkunde?

In dit hoofdstuk bestuderen we de *Ree-Tits Moufangverzamelingen*. De punten van zo een Moufangverzameling zijn de absolute punten van een polariteit van de *Ree zeshoek*. De wortelgroepen zijn van nilpotentie klasse 3, wat zeldzaam is want op één andere, recent ontdekte, klasse ([23]) na, zijn de wortelgroepen van alle andere gekende Moufangverzamelingen van lagere nilpotentieklasse.

Men bekomt dus voor Ree-Tits Moufangverzamelingen rang 3 meetkundes, *Ree meetkundes* genaamd, waarvan we de elementen *punten*, *cirkels* en *sferen* noemen. We zijn er in geslaagd aan te tonen dat de automorfismegroep van deze meetkunde (en van deelmeetkundes waarbij men enkel punten en cirkels, of punten en sferen beschouwt), inderdaad de automorfismegroep van de Moufangverzameling is.

Een interessant gevolg hiervan is dat als een automorfisme van de Ree zeshoek de absolute punten stabiliseert, ook de absolute rechten gestabiliseerd worden.

Deze resultaten zijn bekomen in samenwerking met Fabienne Haot en Hendrik Van Maldeghem.

A.2.2 Rang 2: Veralgemeende vierhoeken

Gemengde vierhoeken. — Een paar niet-collineaire punten p en q van een veralgemeende vierhoek noemt men *regulier*, als voor elk punt r dat collineair is met twee punten die beide collineair zijn met zowel p en q, alle punten collineair met zowel p en q ook collineair zijn met r. Een punt p is *regulier* als alle mogelijke niet-collineaire puntenparen met p erin regulier zijn. Met een regulier punt kan men een bepaalde meetkunde - een *duaal net* - associëren. Analoog definieert men reguliere rechten.

Eén bepaalde klasse van Moufang veralgemeende vierhoeken is de klasse van *gemengde vierhoeken*. Dit zijn de enige gekende vierhoeken waarvan alle punten en rechten *regulier* zijn. Men vermoedt dat deze de enige zijn.

Wij bewezen een zwakkere versie van dit vermoeden. Ruwweg tonen we aan dat als een veralgemeende vierhoek 'genoeg' reguliere punten en rechten bevat, en als de duale netten corresponderend met de reguliere punten voldoen aan het *Axioma van Veblen en Young* (een rechte die twee zijden van een driehoek snijdt, maar niet in een hoekpunt, snijdt ook de derde zijde) de vierhoek een gemengde vierhoek is.

Deze resultaten en die uit de volgende paragraaf zijn bekomen in samenwerking met Van Maldeghem.

Veralgemeende Suzuki-Tits inversieve vlakken. — Bepaalde gemengde vierhoeken laten polariteiten toe. De absolute punten hiervan kan men opnieuw opvatten als een Moufangverzameling, en de bijhorende meetkundes noemt men *veralgemeende Suzuki-Tits inversieve vlakken*. Als een toepassing op de karakterisering van gemengde vierhoeken hebben we een karakterisering voor (perfecte) Suzuki-Tits inversieve vlakken van Hendrik Van Maldeghem ([61]) uitgebreid naar het niet-perfecte geval, en de oorspronkelijke karakterisering voor het perfecte geval vereenvoudigd.

Inbeddingen van veralgemeende vierhoeken in gebouwen van type F_4. — De eerste voorbeelden van veralgemeende veelhoeken onstonden bijna allemaal als *inbeddingen* in projectieve ruimtes (die corresponderen met gebouwen van type A_n). Hierbij zijn de punten van de veelhoek punten van de projectieve ruimte, en de rechten van de veelhoek rechten van de projectieve ruimte waarbij de incidentie de natuurlijke is. Als deze inbedding aan bepaalde 'mooie' voorwaarden voldoet (bv. alle rechten van de veelhoek door een punt liggen in een bepaalde deelruimte), dan erft de veralgemeende veelhoek symmetrie-eigenschappen over van de projectieve ruimte, waardoor men classificaties en karakteriseringen van bepaalde Moufangvierhoeken kan opstellen.

Echter niet alle Moufang veralgemeende veelhoeken kan men 'mooi' inbedden in een projectieve ruimte. Bijvoorbeeld de zogenaamde *exceptionele veralgemeende vierhoek van type* F_4 is niet op deze manier inbedbaar in een projectieve ruimte, maar wel in een gebouw van type F_4. Deze gebouwen kan men opvatten als rang 4 meetkundes van *punten*, *rechten*, *vlakken* en *hyperrechten*, *metasymplectische ruimtes* genoemd. De vierhoeken kan men dan inbedden door middel van punten en hyperrechten. Ook bv. de gemengde vierhoeken kan men op deze manier inbedden.

Wij hebben aangetoond dat als een veralgemeende vierhoek ingebed is in een metasymplectische ruimte door middel van punten en hyperrechten, waarbij twee punten op dezelfde rechte in de vierhoek nooit op één rechte liggen van de metasymplectische ruimte, dan ofwel de vierhoek Moufang is, ofwel dat de inbedding 'ontaard' is.

A.2.3 Rang 3: 2-Dimensionale $\mathbb{R}$-gebouwen

Veelhoeken met valuatie. — Zoals reeds vermeld hebben affiene gebouwen (en hun veralgemening als $\mathbb{R}$-gebouwen) een sferisch gebouw op oneindig. Als de rang van dit gebouw op oneindig minstens drie is kan men het gebouw op oneindig en het affien gebouw (of $\mathbb{R}$-gebouw) zelf classificeren. Als de rang van het gebouw op oneindig echter 2 is (dus een veralgemeende veelhoek), is een classificatie onmogelijk.

Men kan zich wel afvragen welke veralgemeende veelhoeken gebouwen op oneindig zijn van een $\mathbb{R}$-gebouw. Hendrik Van Maldeghem voerde voor dit doel *veralgemeende veelhoeken met (discrete) valuatie* in ([55]), en bewees dat een veralgemeende n-hoek met $n \in \{3,4\}$ een discrete valuatie toelaat als en slechts als de veelhoek het gebouw op oneindig is van een (discreet) affien gebouw van type $\widetilde{\mathsf{A}}_2$ of type $\widetilde{\mathsf{C}}_2$.

Wij hebben de definitie van veelhoek met valuatie uitgebreid naar het niet-discrete geval, als volgt:

Zij $\Gamma = (P, L, \mathtt{I})$ een veralgemeende n-hoek met punten P, rechten L en incidentie $\mathtt{I}$, en zij u een functie, de *valuatie* genaamd, werkend op de paren collineaire punten en paren snijdende rechten, waarbij de beelden in $\mathbb{R}^+ \cup \{\infty\}$ liggen. Dan noemen we (Γ, u) een *n-hoek met (niet-discrete) valuatie* en *gewichtreeks* $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1}) \in (\mathbb{R}^+)^{2n-2}$ als de volgende condities voldaan zijn.

(U1) Op elke rechte ligt er een paar punten p en q zodat $u(p,q) = 0$, en analoog voor rechten door een punt.

(U2) $u(x,y) = \infty$ als en slechts als $x = y$.

(U3) $u(x,y) < u(y,z)$ impliceert dat $u(x,z) = u(x,y)$ als x, y en z collineaire punten of snijdende rechten zijn.

(U4) Telkens als $x_0 \mathtt{I} x_1 \mathtt{I} x_2 \mathtt{I} \ldots \mathtt{I} x_{2n} = x_0$, met $x_i \in P \cup L$, heeft men:

$$\sum_{i=1}^{n-1} a_i u(x_{i-1}, x_{i+1}) = \sum_{i=n+1}^{2n-1} a_i u(x_{i-1}, x_{i+1}).$$

Wij zijn er in geslaagd aan te tonen dat enerzijds de veelhoek op oneindig van een 2-dimensionaal gebouw altijd een veelhoek met valuatie is, en anderzijds dat een n-hoek (met $n = 3, 4$) met valuatie het gebouw op oneindig is van een 2-dimensionaal gebouw. Het resterende discrete geval, zeshoeken met discrete valuatie en gebouwen van type $\widetilde{\mathsf{G}}_2$, hebben we ook opgelost. Tevens is aangetoond dat er voor elke n maar één mogelijke gewichtreeks is (op veelvouden en valuaties die overal waarde nul hebben na).

Als toepassing van deze karakterisering hebben we verscheidene veelhoeken met valuatie (en dus ook de bijhorende $\mathbb{R}$-gebouwen) geconstrueerd.

Deze resultaten zijn bekomen in samenwerking met Hendrik Van Maldeghem.

Volledigheid van $\mathbb{R}$-gebouwen. — Een metrische ruimte noemt men *volledig* als elke Cauchyrij convergeert. Net zoals we een appartement van een $\mathbb{R}$-gebouw kunnen opvatten als een affiene Euclidische ruimte, kunnen we een $\mathbb{R}$-gebouw opvatten als een metrische ruimte bestaande uit aan elkaar gevoegde affiene Euclidische ruimtes. Als deze metrische ruimte volledig is, dan zijn er bepaalde resultaten van toepassing, bv. de Bruhat-Tits fixpuntstelling. Een (discreet) affien gebouw levert altijd een volledige metrische ruimte op, een niet-discreet $\mathbb{R}$-gebouw niet altijd.

Het doel is nu na te gaan welke $\mathbb{R}$-gebouwen precies volledig zijn. In deze thesis nemen we een stap in de richting van een antwoord. We herleiden de vraag tot de vraag welke $\mathbb{R}$-bomen er volledig zijn. Deze nieuwe vraag hopen we dan algebraïsch te kunnen beantwoorden.

Deelgebouwen van $\mathbb{R}$-gebouwen corresponderende met fixgebouwen op oneindig. — Als een groep werkt op een sferisch gebouw, dan is in de meeste gevallen de fixstructuur opnieuw een (sferisch) gebouw. Voor algemene gebouwen (en dus ook affiene gebouwen) geldt dit niet.

Veronderstel dat een groep G op een affien gebouw Λ werkt; alhoewel de fixstructuur van G in Λ niet noodzakelijk terug een gebouw is, is de fixstructuur in het sferisch gebouw Λ_∞ op oneindig dit meestal wel. Wij bewezen nu dat de fixstructuur in het $\mathbb{R}$-gebouw in bepaalde gevallen wel een deelgebouw bevat met de fixstructuur in Λ_∞ als gebouw op oneindig.

Deze resultaten zijn bekomen in samenwerking met Hendrik Van Maldeghem.

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