MIXED PARTITIONS AND SPREADS OF PROJECTIVE SPACES

by

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A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

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There are places I'll remember all my life though some have changed...

Lennon/McCartney

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ABSTRACT

This thesis deals primarily with different types of partitions of projective spaces, namely *spreads* and *mixed partitions*. It is well known that there is a close relationship between spreads of odd dimensional projective spaces and finite affine translation planes. Hence, more knowledge of spreads will hopefully one day lead to a better understanding of finite affine translation planes.

We discuss spreads constructed via a method described in [19]. This method is carefully explained in Chapter 2 using geometric techniques to better understand the algebraic construction given in [19]. The basic method starts with a mixed partition consisting of linear spaces together with Baer subspaces, and then lifts these spaces to a spread in a higher dimensional space. It is possible to construct the same translation plane directly from the associated mixed partition, although the construction of a translation plane from the associated spread is better known.

Chapter 3 provides the reader with some classical examples of mixed partitions. These partitions are constructed via group theoretic techniques and are shown to generate the Desarguesian affine plane. Automorphism groups are discussed in Chapter 4. This work is used to determine properties of the collineation groups of such "geometrically lifted" spreads. We discuss the algebraic kernel of the affine planes arising from these spreads, and we prove some general properties about automorphisms of mixed partitions and their associated spreads.

In Chapter 5 we will generalize the lifting method given in [19]. From this,

we prove a result about $(n-1)$-spreads of $\mathcal{PG}(2n-1,q)$ generating affine translation planes which are *not* n-dimensional over their kernel. This will lead to a general theory about distinct spreads lying in projective spaces of different dimension which generate isomorphic translation planes. This result, together with a result of Lüneburg [27], provides a complete unifying theory for spreads which generate isomorphic translation planes.

Finally, Chapter 6 will provide the reader with some concrete examples of mixed partitions showing that this method is indeed useful for finding spreads. The author gives concrete constructions of several infinite families, and discusses the translation planes they construct via the Bose/Andrè model.

Chapter 1

INTRODUCTION

1.1 History

Geometry has always been a central part of mathematics. From the time we are in grade school we are taught the language of geometry. We are taught about points and lines, and we learn words like triangle and square. Moreover, we naturally view the world in *perspective*. That is, we view geometric objects by the way they look from our personal standpoint. The first attempt at recreating perspective in artwork appeared in the sixteenth century. This is when we first see artists viewing objects as emanating from a point at infinity and using this model to recreate 3-dimensional objects on a 2-dimensional surface.

As a mathematical discipline, projective geometry is concerned with those geometrical properties which are invariant under perspective, or, in mathematical terms, *projection*. Hence, any notion of measure, length, angle, area, congruence, similarity, etc., has no place in projective geometry. The basics of projective geometry began early, being applied unknowingly by the ancient Greek and Roman architects as well as early Egyptians and Mayans. However, the first formal mathematical treatment of projective geometry must be attributed to the French architect Gérard Desargues (1593-1661) in a book entitled *An Attempt to Deal with the Intersection of a Cone with a Plane*. It is in this book that Desargues presents the famous *Desargues' Theorem* which has become a central theorem in modern projective geometry. The theorem basically says that if two triangles are in perspective

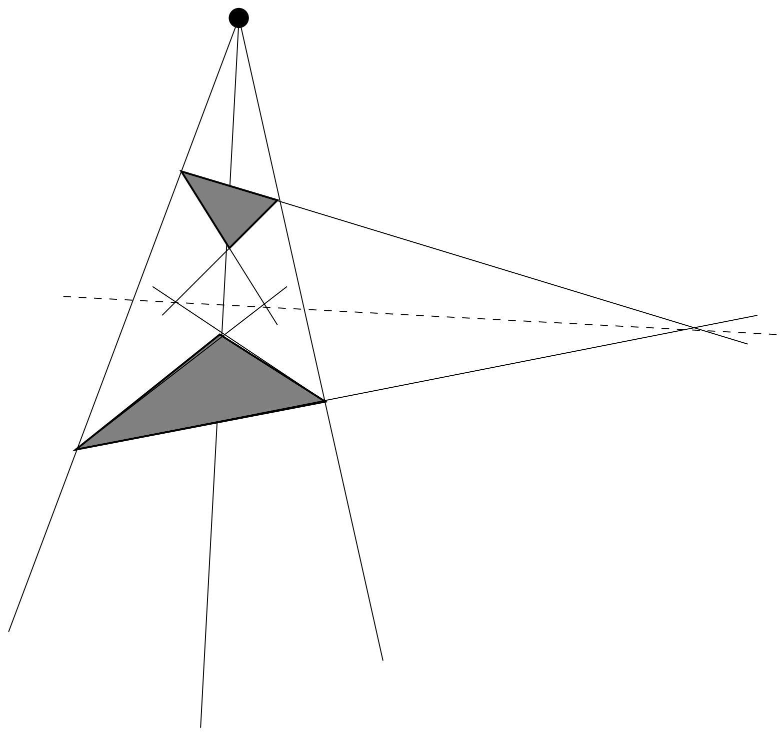

Figure 1.1: Desargues' Theorem

from a point, then the "meets" of their corresponding sides are collinear (see Figure 1.1).

It was in the mid 1600s that Desargues met the young Blaise Pascal and encouraged him to apply the new methods of projective geometry to the study of conic sections. In 1640, at the age of only 16, Pascal published a short work which included his famed theorem of the *hexagrammum mysticum*, now known simply as *Pascal's Theorem*. It states that the intersections of the cross lines determined by 6 points on a conic are collinear (see Figure 1.2). Pascal actually did not give a proof of his theorem. Rather, he said that it proves to be true for a circle and so, by "projection and section", must also be true for all conics.

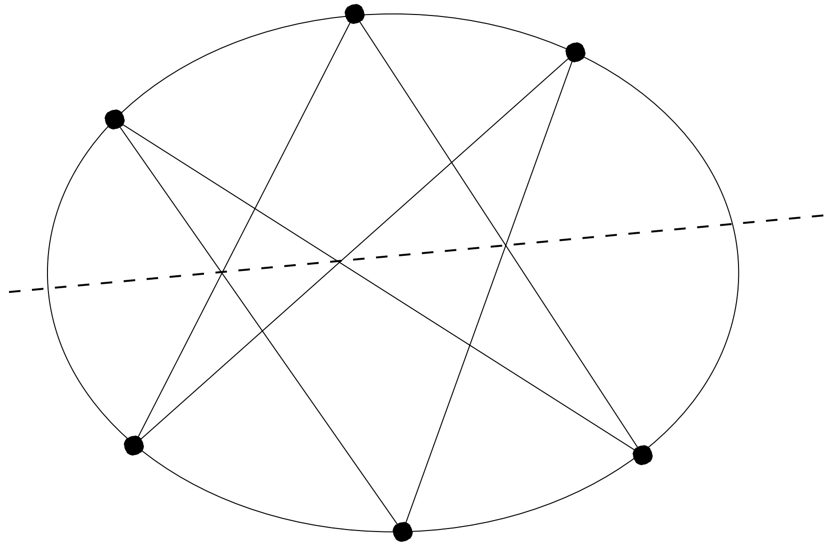

Figure 1.2: Pascal's Theorem

Unfortunately, not much attention was paid to Desargues' and Pascal's discoveries until a century later when Gaspard Monge published his own lectures on descriptive geometry. Monge certainly presented the basic concepts of projective geometry. However, these concepts became a recognized branch of geometry only after Monge's most talented student, Jean Victor Poncelet (1788-1867), devoted his time to the further development of Desargues' work. Poncelet's volume, *A Treatment of the Projection of Figures*, is probably the first recognized textbook of projective geometry. There was a strong revival of geometry in the nineteenth century. The discovery of non-Euclidean geometry provided a powerful stimulus to new thinking, and the work of Poncelet in projective geometry served as one of the outcomes.

Projective geometry has not been dormant since. Steady contributions have been made throughout the nineteenth and twentieth centuries. Finite geometry earned a respectable place in the area of discrete mathematics, thanks in some part to the contributions of Beniamino Segre (1903-1977). Segre's contributions to geometry are many, but he is remembered in particular for his study of geometries over fields other than the complex numbers. By 1955, Segre was working mostly with geometries over finite fields and was producing results which are now classified as combinatorics rather than geometry. Segre was the first to discuss the possibility of characterizing geometric objects using only their combinatorical properties. This is commonly referred to as *Segre's point of view.*

Throughout this thesis, we will attempt to use the geometry as much as possible. When necessary, algebra and algebraic geometry will play a significant role. However, the author has made significant efforts to keep the arguments as geometric as possible.

1.2 Finite Affine and Projective Planes

We wish to explore some of the structures of finite geometry in more detail. We start with one of the most basic structures, the *affine plane.*

Definition 1.2.1 *An* **affine plane** *is a set of points, together with a set of subsets of these points, called lines, such that*

1. *every two distinct points determine a unique line,*
2. *for every line l and every point P not on l, there exists a unique line m through P with no point in common with l,*
3. *there exist three noncollinear points.*

The last axiom is often referred to as a *non-degeneracy* axiom. A single point, or a set of two points incident with a common line, certainly satisfy the first two axioms. We do not, however, want to refer to such configurations as affine planes. Hence, we require at least three noncollinear points.

It should be noted that the term *affine* was first coined by the famous eighteenth century mathematician Leonard Euler. It is not hard to check that the real coordinate plane is one particular example of an affine plane. However, many more examples exist and can easily be constructed. In particular, there is a classical model for an affine plane. Let F be any field and let V be a two-dimensional vector space over F. We define our "points" to be all of the vectors in V and our "lines" to be all of the cosets of all 1-dimensional subspaces of V. Incidence is given by containment. It is not hard to show that this incidence structure forms an affine plane, and it is denoted by $\mathcal{AG}(2, F)$. Throughout this thesis, we will work in the finite case. That is, we will let F be a finite field. We use $GF(q)$ to denote the finite field with q elements, and, in this case, we write $\mathcal{AG}(2, F) = \mathcal{AG}(2, q)$.

Note that in the previous model of an affine plane using a vector space over a finite field, every line contains the same number of points, namely q, where q is the order of the finite field. It turns out that for any finite affine plane, the number of points on a line is always a constant. One can use this fact and the set of axioms to show the following:

Proposition 1.2.2 *Every finite affine plane has a unique associated natural number* $n \geq 2$ *such that*

1. *every line contains* n *points,*
2. *every point is concurrent with* $n + 1$ *lines,*
3. *there exist exactly* n^2 *points,*
4. *there exist exactly* $n^2 + n$ *lines.*

This unique natural number n *is called the* **order** *of the affine plane.*

With affine planes comes the notion of parallel lines. We will define two lines l and l' of an affine plane to be *parallel* if $l = l'$ or $l \cap l' = \emptyset$. One can easily show that the parallel relation forms an equivalence relation on the lines of any affine plane. We call the congruence classes *parallel classes.*

Standing on railroad tracks helps to better explain how we naturally view the world projectively. In the affine plane, we naturally have parallel lines. However, from the perspective of a man looking at the horizon, parallel lines seem to meet, just like the railroad tracks seem to meet. Thus, one naturally arrives at the notion of lines *meeting at infinity*. This can be laid out more specifically, and more mathematically, in what we call a *projective plane*.

Definition 1.2.3 *A* **projective plane** *is a set of points, together with a set of subsets of these points, called lines, such that*

1. *every two distinct points determine a unique line,*
2. *every two distinct lines meet in a unique point,*
3. *there exist four points, no three collinear.*

Again we have a non-degeneracy axiom, namely Axiom # 3. This guarantees that small configurations with only three (or less) points or only three lines are not considered projective planes.

Having given the definitions of both affine and projective planes, we should note that there is a tight connection between the two. Let l_∞ be any line of a projective plane π, and consider the incidence structure $\mathcal{A}$ obtained by deleting the line l_∞ and all of the points incident with it. Hence, the points of $\mathcal{A}$ are the points of π not on the line l_∞, and the lines of $\mathcal{A}$ are all of the lines of π except for the special line l_∞. Here, each line contains one less point. One can easily check that $\mathcal{A}$ forms an affine plane. Notice that the process of removing the line l_∞ creates parallel classes of lines since any two lines which meet at a common point of l_∞ no longer meet in $\mathcal{A}$. We write π^{l_∞} to denote this affine plane.

This process is, in fact, reversible. That is, given any affine plane, one can obtain a projective plane by first creating new points, one for each parallel class of lines. These points are commonly referred to as *points at infinity*, and any particular point at infinity is incident with all of the lines in exactly one parallel class. We then add the *line at infinity*, l_∞, which is incident with all of the points at infinity. This process is commonly referred to as *completing* the affine plane to a projective plane. It turns out that this completion process is unique.

Theorem 1.2.4 *Let $\mathcal{A}$ be an affine plane. Then there exists a unique projective plane π such that $\mathcal{A} \cong \pi^{l_\infty}$ for some line l_∞ in π.*

If there exists a map ϕ from a projective plane π to another projective plane π' which forms a bijection on the points and lines, and preserves incidence, then we say that π and π' are *isomorphic* and the map ϕ is called an *isomorphism*. In the above theorem, two non-isomorphic affine planes could potentially complete to isomorphic projective planes. In particular, if l and l' are two distinct lines of a projective plane π, it is possible that π^l and $\pi^{l'}$ are non-isomorphic.

There is also a classical model for obtaining projective planes. Let F be any field and let V be a 3-dimensional vector space over F. We define our "points" to be all of the 1-dimensional subspaces of V and our "lines" to be all of the 2-dimensional subspaces of V. Incidence is given by the natural containment. It is not hard to show that this incidence structure forms a projective plane, and is denoted by $\mathcal{PG}(2, F)$. When F is the finite field $GF(q)$, we write $\mathcal{PG}(2, F) = \mathcal{PG}(2, q)$.

Just as in Proposition 1.2.2, projective planes also have a uniquely defined *order*. As one might expect, the duality of the points and lines of a projective plane makes the counting more symmetric.

Proposition 1.2.5 *Every finite projective plane has a unique associated natural number $n \geq 2$ such that*

1. *every line contains $n + 1$ points,*

2. *every point is concurrent with $n + 1$ lines,*

3. *there exist exactly* $n^2 + n + 1$ *points,*

4. *there exist exactly* $n^2 + n + 1$ *lines.*

As before, this unique natural number n *is called the* **order** *of the plane.*

We now show that the classical projective plane, $\mathcal{PG}(2, q)$, has order q. Let V be a 3-dimensional vector space over the finite field $GF(q)$. The number of points on a line of $\mathcal{PG}(2, q)$ is the same as the number of 1-dimensional subspaces in a 2-dimensional subspace of V. There are $q^2 - 1$ non-zero vectors in such a 2-dimensional vector subspace, but the $q - 1$ non-zero scalar multiples of any vector all generate the same 1-dimensional subspace. We conclude that each line of $\mathcal{PG}(2, q)$ contains $\frac{q^2-1}{q-1} = q + 1$ points. Hence, from Proposition 1.2.5, $\mathcal{PG}(2, q)$ has order q.

1.3 Projective Geometries

Naturally, one can extend the notion of a projective plane to higher dimensions. Just like we define real n-dimensional space, we can define n-dimensional projective spaces. First, the axiomatic definition:

Definition 1.3.1 *A* **projective geometry** *is a set of points, together with a set of subsets of these points, called lines, such that*

1. *every two distinct points determine a unique line,*

2. *if* A, B, C *and* D *are four points such that the line* AB *intersects the line* CD*, then* AC *also intersects the line* BD*,*

3. *every line contains at least three points,*

4. *there exist at least two lines.*

Axiom #2 (see Figure 1.3) is a truly ingenious way of saying that any two lines of a plane meet, despite the fact that planes have not yet been defined. Some people

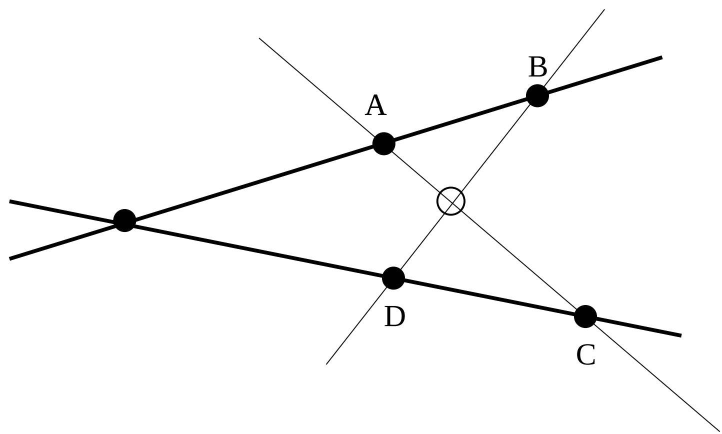

Figure 1.3: Axiom #2: coplanar lines meet

refer to this axiom as the axiom of Pasch, because the German geometer Moritz Pasch (1843-1930) used a similar picture. Other subspaces are defined similarly to the way they are defined for vector spaces. For instance, if P is a point not on a line l, one can construct a "plane" by taking the point set determined by the union of all of the lines joining P to a point of l. Similarly, a non-incident point-plane pair uniquely determines a "solid", and so forth. Basically, if $d+1$ is the smallest number of points needed to generate a certain projective subspace S, we say that S has projective dimension d. A rigorous mathematical treatment of geometric dimension would require much more work. We will see shortly that this detour is unnecessary.

Projective geometries which are not projective planes can be constructed in a similar fashion to that of the classical projective plane. That is, to create a classical projective geometry of dimension n, we start with an $(n+1)$-dimensional vector space V over a field F. We let the "points" be all of the 1-dimensional subspaces of V, the "lines" are all of the 2-dimensional subspaces, the "planes" are all of the 3-dimensional subspaces, and so on. The n-dimensional subspaces are commonly referred to as "hyperplanes". Incidence is again given by the natural containment. We denote this projective space as $\mathcal{PG}(n, F)$, and if F is finite of order q, we write $\mathcal{PG}(n, q)$. Through this model, one naturally arrives at the notion of *homogeneous coordinates* for projective points. Since a point P of $\mathcal{PG}(n, q)$ is given by a 1-dimensional subspace U of V, we say that any non-zero vector in U *induces* the point P. Hence, more than one vector can induce the same projective point. We will frequently *normalize* vectors to avoid this problem. That is, we can scalar multiply a non-zero vector $\mathbf{v}$ so that the first non-zero entry from the left (or right) in $\mathbf{v}$ is a 1. In this way, every projective point is induced by a unique normalized vector.

Isomorphism among higher dimensional projective spaces is defined the same way it is for projective planes. Hence, two projective spaces Σ and Σ' are said to be isomorphic if there exists a bijection from the points of Σ to the points of Σ' which preserves subspaces and incidence. One can only wonder how many projective geometries exist up to isomorphism. In a most remarkable result of Veblen and Young [32] around 1903, we have

Theorem 1.3.2 *Any projective geometry whose projective dimension is not 2 is classical.*

That is, every projective geometry which is not a projective plane can be modeled using a vector space. This gives us a natural method of viewing subspaces. If $\Sigma = \mathcal{PG}(n, q)$ is modeled by an $(n + 1)$ dimensional vector space V over $GF(q)$, we define an r**-space** of Σ to be the set of projective points induced by the non-zero vectors in some $(r + 1)$-dimensional subspace of V. Hence, the projective dimension is always one less than the vector space dimension. In particular, note that the projective line can be modeled using a 2-dimensional vector space. The only non-zero subspaces are the 1-dimensional subspaces representing the points of the projective line.

As mentioned above, when a projective geometry can be modeled using a vector space over a (skew) field, we will call the projective geometry *classical*. A result originally due to Hilbert [16], but also proven by Baer [2], says that a finite projective geometry is classical if and only if it satisfies the Theorem of Desargues mentioned in Section 1.1. Hence, we often refer to classical projective geometries as being *Desarguesian* and we use the terms "classical" and "Desarguesian" interchangably when working in the finite setting.

With the vector space model in mind, we can use the "Dimension Theorem" when working with higher dimensional geometries. That is, if S and T are subspaces of a projective geometry of dimension greater than 2, then

$$dim(S+T) = dim(S) + dim(T) - dim(S \cap T).$$

In particular, we can conclude that lines and hyperplanes must always meet in at least a point. This will prove useful in many of the constructions yet to come.

Of course, one naturally asks about the case when the dimension of a projective space is 2. Throughout the twentieth century, mathematicians have tackled the problem of constructing *non-classical* projective planes, those which cannot be modeled by a vector space over a (skew) field, through algebraic techniques. This is usually done by constructing a "weak" algebraic system, such as a quasifield, and then using it to supply coordinates (see [21] for examples). One large class of non-classical planes for which many examples have been constructed is the class of so-called *translation planes*.

1.4 Translation Planes

In any area of mathematics there is always the notion of *symmetry*. It is easy to look at one's tiled floor and find many symmetries. That is, shifting the tile up three inches and across four inches takes the pattern back to itself. Mathematicians use symmetry as a measure of the amount of structure in an object. This is one method we use to try to classify projective planes.

We define a *collineation* of a projective space Σ to be a bijection which sends points to points, lines to lines, and, in general, subspaces of dimension n to other subspaces of dimension n, and preserves incidence. The set of all collineations of a projective space Σ forms a group under composition, denoted by $Aut(\Sigma)$. We typically write S^α for the image of a subspace S under a collineation α. The full collineation group of the classical projective space $\mathcal{PG}(n, q)$ is completely determined.

Theorem 1.4.1 The Fundamental Theorem of Projective Geometry *Let $\Sigma = \mathcal{PG}(n, q)$ have underlying vector space V over $GF(q)$. Then $Aut(\Sigma) = P\Gamma L(n+1, q)$, where $P\Gamma L(n+1, q)$ is the group of all semi-linear transformations on the vector space V.*

We write $PGL(n + 1, q)$ to denote the linear portion of $P\Gamma L(n + 1, q)$. Notice that non-zero scalar multiples of the same non-singular matrix induce the same collineation in $\mathcal{PG}(n + 1, q)$. Hence, if we let Z_0 be the center of the general linear group $GL(n+1, q)$, then $PGL(n+1, q) \cong GL(n+1, q)/Z_0$.

Throughout this thesis, we will be particularly interested in projective planes. As was indicated in Theorem 1.3.2, projective planes are not necessarily classical, and so the full automorphism group of a projective plane can be quite different from $P\Gamma L(3, q)$. We need to examine collineations in a projective plane more carefully. If a collineation of a projective plane π fixes all the points on a line l of π, then the collineation is called a *perspectivity*, and the line of fixed points is called the *axis*. It is not hard to show that for any nonidentity perspectivity there must always be a uniquely determined fixed point, called the *center*, such that all lines passing through this point are fixed. Such a collineation with center V and axis l is called a (V, l)-*perspectivity* or a (V, l)-*central collineation*. When the center V is on the axis

l, we call such a collineation an *elation*. If V is not on l, we call such a collineation a *homology*.

It is not hard to show that a perspectivity with center V and axis l, if it exists, is uniquely determined by the image of any one point not equal to V and off l. In particular, one can show that in a classical projective plane of order q, there are exactly q elations and $q-1$ homologies with fixed center and axis. In non-classical projective planes, this might not be true.

Definition 1.4.2 *We say a projective plane π is* **(V,l)-transitive** *if for any two distinct points A and B with $VA = VB$, $A \neq V \neq B$, and $A \notin l, B \notin l$, there is a (V,l)-perspectivity α in $Aut(\pi)$ with $A^\alpha = B$.*

In other words, if a projective plane admits all possible perspectivities with center V and axis l, we say the plane is (V,l)-transitive.

Definition 1.4.3 *We say the line l in a projective plane π is a* **translation line** *if π is (V,l)-transitive for every point V on the line l.*

Now consider a projective plane π which contains a translation line l. In other words, π admits all possible elations with axis l, and we consider the affine plane π^l. Such an elation α of π induces an automorphism on the affine plane π^l which fixes a parallel class of lines (the class which corresponds to the center of α), is fixed-point free, and preserves parallelism. This is exactly what we call a *translation*.

Hence, when a projective plane π contains a translation line l, we call π a *translation plane*, and the affine plane π^l is called an *affine translation plane*. For non-classical finite translation planes, the translation line is always uniquely determined. If l is a translation line of π, it is not hard to see that the subgroup $\mathcal{T}$ of $Aut(\pi^l)$ containing all of the elations with axis l acts transitively on the the affine points. This subgroup $\mathcal{T}$ is called the *translation group*. One can show that the full automorphism group $Aut(\pi^l)$ is uniquely determined by $\mathcal{T}$ and the subgroup of $Aut(\pi^l)$ which stabilizes a particular affine point, taken to be the "origin". This subgroup is often called the *translation complement*. For translation planes of order q, the translation groups are all isomorphic. Hence, determining the translation complement of such a plane is often helpful in classifying the translation plane. It should be noted however that two non-isomorphic translation planes could have translation complements which are isomorphic.

A considerable amount of work has been done to try to classify, or at least categorize, translation planes. This appears to be a hefty task with no clear end in sight. Hence, many finite geometers have been working on classifying subsets of translation planes (flag-transitive planes, for instance). But nevertheless, the search for new translation planes has been ongoing for many decades. One of the more remarkable results of the early fifties shows that translation planes can be studied using higher dimensional projective spaces. First we introduce some notation.

Definition 1.4.4 *An* r**-spread** *of* $\Sigma = \mathcal{PG}(d, q)$ *is a collection of* r*-spaces of* Σ *which together partition the points of* Σ.

By counting points we see that an r-spread of $\mathcal{PG}(d, q)$ can exist only if $q^r + \cdots + q + 1$ divides $q^d + \cdots + q + 1$. This immediately implies $(r + 1)|(d + 1)$. As it turns out, this divisibility property is also a sufficient condition for the existence of spreads (see Corollary 4.17 of [17]). We will examine the construction of spreads in Chapter 3.

A few of the pioneering mathematicians in projective geometry proved that spreads can actually be used to construct translation planes. More specifically, it was proven independently by Bruck/Bose [7] and by Andrè [1] that any $(n - 1)$-spread of $\mathcal{PG}(2n - 1, q)$ could be used to construct a finite affine translation plane of order q^n. Moreover, it was shown in [7] that the converse is true. That is, every finite translation plane can be obtained via this construction. This method is commonly referred to as the *Bose/Andrè Model* for constructing translation planes.

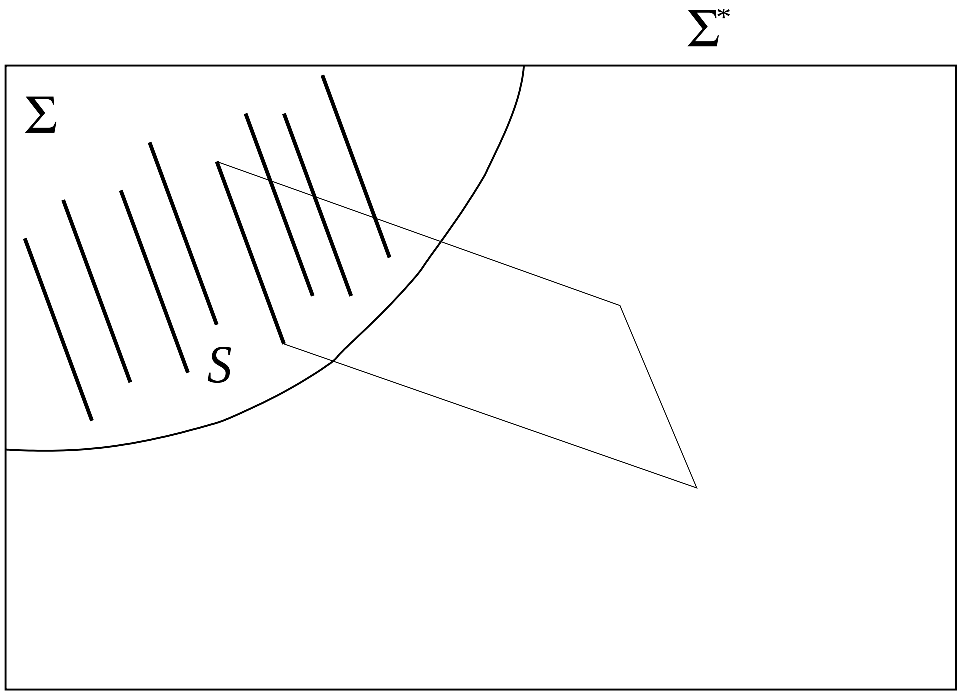

Figure 1.4: The Bose/Andrè model

1.4.1 The Bose/Andrè Model

Let $\Sigma = \mathcal{PG}(2n-1,q)$ and embed Σ in $\Sigma^* \cong \mathcal{PG}(2n,q)$. The hyperplane Σ is often referred to as the "hyperplane at infinity" in this model. Furthermore, let $\mathcal{S}$ be an $(n-1)$-spread of Σ. We define an incidence structure as follows. The "points" are the points of $\Sigma^* \setminus \Sigma$. The "lines" are the the n-dimensional projective spaces of Σ^* which meet the hyperplane Σ in an element of the spread $\mathcal{S}$ (see Figure 1.4). Incidence is given by containment. One can check that this incidence structure, typically denoted $\pi(\mathcal{S})$, forms a finite affine plane of order q^n, and it was proven in [7] that the plane is, in fact, a translation plane. One completes this affine plane to a projective plane by viewing the elements of $\mathcal{S}$ as the points at infinity, and $\mathcal{S}$ itself becomes the line at infinity.

Of course, the Desarguesian plane $\mathcal{PG}(2,q)$, being a translation plane, can be constructed via this method. This leads to a new definition.

Definition 1.4.5 *A* $(t-1)$**-regulus** *of* $\Sigma = \mathcal{PG}(2t-1, q)$ *is a collection of* $q+1$ $(t-1)$*-spaces of* Σ *with the property that any line meeting three of the* $(t-1)$*-spaces meets all of the* $(t-1)$*-spaces.*

A straight forward linear algebra argument shows the following.

Lemma 1.4.6 *Let* S_1*,* S_2*, and* S_3 *be three distinct, pairwise disjoint* $(t-1)$*-spaces of* $\mathcal{PG}(2t-1, q)$*. Then there exists a unique regulus containing* S_1*,* S_2*, and* S_3*.*

For $q > 2$, We will say that a spread $\mathcal{S}$ of $\mathcal{PG}(2t-1, q)$ is *regular* if for every three distinct elements of $\mathcal{S}$, the unique regulus determined by them is a subset of $\mathcal{S}$. When $q = 2$, every triple of pairwise skew lines forms a regulus by our definition. Hence, one needs a more careful definition of regular spreads in the case when $q = 2$. This definition is unimportant for our purposes and can be found in [18] for instance. It should be mentioned also that reguli in projective 3-space have another special property. The set of points covered by the lines of a regulus form the points of a *hyperbolic quadric* (see [17]). We will use the quadratic form associated with such a quadric for some of the arguments in later chapters. As it turns out, Bruck [7] proved that regular spreads are intimately connected with the classical planes.

Theorem 1.4.7 *The affine plane* $\pi(\mathcal{S})$ *is classical if and only if the spread* $\mathcal{S}$ *is regular.*

There is a special property of regular spreads which we will refer to several times in this thesis. If $\mathcal{S}$ is any spread of $\mathcal{PG}(2n-1, q)$, we will write $Aut(\mathcal{S})$ for the subgroup of $P\Gamma L(2n, q)$ which fixes $\mathcal{S}$. For regular spreads, there is always a cyclic subgroup of $Aut(\mathcal{S})$ which fixes every element of $\mathcal{S}$ and acts regularly on the points of any spread element. This special subgroup is often referred to as the *Bruck Kernel.*

As mentioned earlier, the translation complement of a translation plane can be a great aid in determining the type of translation plane arising from a given spread. When viewing a translation plane in the Bose/Andrè model as above, one can show that the translation complement is isomorphic to the direct product of $Aut(\mathcal{S})$ with the cyclic group of order $q-1$. We use this fact to try to classify planes in Chapter 6.

Another characteristic of translation planes should be mentioned here. For any translation plane π there is always an associated *kernel*. The definition of the kernel comes from the way coordinates are assigned to a plane (see [21], for instance). This is unimportant at the moment and would require an extreme detour. The important point is that for any projective plane π of order q, there is always a coordinatizing algebraic system R of order q for π. Moreover, there is always a uniquely defined finite field F inside R. This field is maximal in the sense that it is not contained in any larger subfield of R. The system R can then be thought of as a t-dimensional vector space over F, where t is the unique natural number such that $|R| = |F|^t$. In fact, F is the kernel of π, and we say the translation plane π is *t-dimensional over its kernel*. Hence, we have two quite different notions of dimension. There is the geometric dimension, which is always 2 in the case of translation planes, and there is the *algebraic dimension* which is determined by the kernel. It was shown in [7] that an $(n-1)$-spread of $\mathcal{PG}(2n-1, q)$ generates a translation plane from the Bose/Andrè model whose algebraic dimension is a divisor of n. Moreover, the kernel must contain $GF(q)$. Letting F^* denote the multiplicative group of the field F, the group $F^*/GF(q)^*$ is isomorphic to the subgroup Υ of $Aut(\mathcal{S})$ which fixes each spread element. In particular, $|F^*| = (q-1) \cdot |\Upsilon|$. As a result, we will frequently look at this subgroup of $Aut(\mathcal{S})$ in order to determine the algebraic dimension of the associated plane.

We should point out here that the multiplicative group of the kernel of a plane π is isomorphic to a subgroup of the translation complement of π. In fact, it is the subgroup of the translation complement which stabilizes each point on the line at infinity. But the translation complement is certainly a homology group since it fixes the origin and every point on the line at infinity. Moreover, every homology is a perspectivity, and it is well-known that every perspectivity is a linear collineation (see [21]). Hence, we know that the kernel is a linear subgroup of $Aut(\pi)$. We use this result when examining kernels in Chapter 6.

A famous result of Lüneburg [27] says that some spreads generate isomorphic translation planes.

Theorem 1.4.8 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two spreads of $\mathcal{PG}(2n-1, q)$. Then $\pi(\mathcal{S}_1) \cong \pi(\mathcal{S}_2)$ if and only if there is a collineation ϕ of $\mathcal{PG}(2n-1, q)$ such that $\mathcal{S}_1^\phi = \mathcal{S}_2$.*

Unfortunately, the theorem says nothing about spreads which lie in projective spaces of different dimension. It is possible that such spreads could also generate the same translation plane. We explore such spreads in Chapter 5.

1.4.2 Translation Planes Directly from Mixed Partitions

It is less well known that spreads are not the only type of partition which can be used to construct affine planes. We will discover in Chapter 2 that there is a close connection between certain types of spreads and another type of partition. Throughout this thesis, we will use the term *mixed partition* to mean a partition of an odd dimensional square order projective space, say $\mathcal{PG}(2n-1, q^2)$, into two types of objects, $(n-1)$-spaces and so-called Baer subspaces. More specifically, a mixed partition of $\Pi = \mathcal{PG}(2n-1, q^2)$ is a partition of the points of Π into Baer subspaces of dimension $2n-1$ (copies of $\mathcal{PG}(2n-1, q)$) and $\mathcal{PG}(n-1, q^2)$'s. We will use the term *Baer subspace* for any isomorphic copy of $\mathcal{PG}(2n-1, q)$ contained in $\mathcal{PG}(2n-1, q^2)$. So unless otherwise specified, the dimension of the Baer subspace is always assumed to be the same as the dimension of the space in which it is contained.

One standard method of constructing Baer subspaces is through the use of coordinates. Let $\Sigma = \mathcal{PG}(d, q^2)$ with underlying vector space V, and let $P_1, P_2, \ldots, P_{d+2}$ be $d+2$ points of Σ which are in *standard position*. That is, no $d+1$ of the points lie in the same $(d-1)$-dimensional subspace (i.e. a hyperplane). Moreover, for $1 \leq i \leq d+2$ let $\mathbf{v_i}$ be any vector of V which induces the point P_i. Then, since V has dimension $d+1$, one can find unique scalars k_i in $GF(q^2)$ such that

$$\sum_{i=1}^{d+1} k_i \mathbf{v_i} = \mathbf{v}_{d+2}.$$

We then consider all the points induced by vectors in the $GF(q)$-linear span of $\{k_i \mathbf{v_i} : 1 \leq i \leq d+1\}$. This collection of vectors certainly forms a $(d+1)$-dimensional vector space over $GF(q)$ and so induces an isomorphic copy of $\mathcal{PG}(d, q)$ inside Σ. We will use this method to construct Baer subspaces throughout this thesis.

It turns out that mixed partitions can be used to construct affine planes. The construction is alluded to in a paper by Freeman [15], and also developed in a paper of Bruen and Thas [10]. We give the construction now.

Let $\Pi = \mathcal{PG}(2n-1, q^2)$ and embed Π in $\Pi^* \cong \mathcal{PG}(2n, q^2)$. Again, the hyperplane Π is often referred to as the "hyperplane at infinity" in this model. Furthermore, let $\mathcal{P}$ be a mixed partition of Π (a partition containing Baer subspaces and $(n-1)$-spaces). We define a new incidence structure as follows. The "points" are the points of $\Pi^* \setminus \Pi$. There are two different types of "lines" in this incidence structure. The first type of line is given by any n-dimensional projective space of Π^* which meets the hyperplane Π in an $(n-1)$-space of the partition $\mathcal{P}$. This is exactly the way all the lines were defined in the Bose/Andrè Model. The second type of line is given by a Baer subspace of Π^* (an isomorphic copy of $\mathcal{PG}(2n, q)$) which meets the hyperplane Π in one of the Baer subspaces of the mixed partition $\mathcal{P}$ (see Figure 1.5). Incidence is given by containment. We let $\pi(\mathcal{P})$ represent this incidence structure.

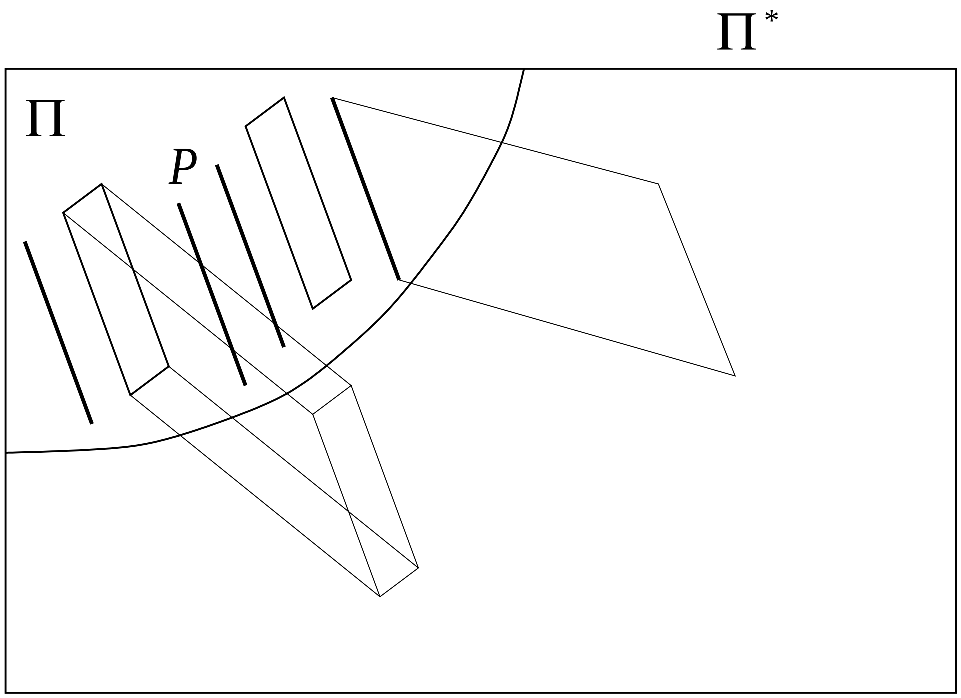

Figure 1.5: An affine plane directly from a mixed partition

Since this construction is not as well known as the Bose/Andrè construction, we provide proof that this incidence structure is, in fact, an affine plane of order q^{2n}. We start with the following lemma.

Lemma 1.4.9 *Let $l \cong \mathcal{PG}(1, q^2)$ be the projective line of order q^2, and let $l_0 \cong \mathcal{PG}(1, q)$ be a Baer subline of l containing the point Q. Further, let P be a point of $l \setminus l_0$. Then, there is a unique Baer subline m through P such that $l_0 \cap m = Q$.*

Proof: It is well known that the points and Baer sublines of $\mathcal{PG}(1, q^2)$ form a $3-(q^2+1, q+1, 1)$ design (see [17]). Hence, the number of Baer sublines through both P and Q is $(q^2-1)/(q-1) = q+1$. But q of these lines meet the subline l_0 in a second point. Hence, there is a unique subline through P which meets l_0 in only the point Q. ∎

Theorem 1.4.10 *The incidence structure $\pi(\mathcal{P})$ described above forms an affine plane of order q^{2n}.*

Proof: First we show that every two points determine a unique line. We let U and V be two distinct points of $\pi(\mathcal{P})$. Then, the line l of Π^* determined by U and V meets the hyperplane Π in a unique point P. Let S be the unique element of $\mathcal{P}$ containing P. We have two cases. Either S is an $(n-1)$-space of Π, or S is a Baer subspace of Π. If S is an $(n-1)$-space, then the projective space spanned by S together with the line l is an n-space of Π and hence represents a line of $\pi(\mathcal{P})$. On the other hand, suppose S is a Baer subspace of Π. There is a unique Baer subline through the points U, V, and P. This Baer subline together with the space S generates a Baer subspace (of dimension $2n$) of Π^* which represents a line of $\pi(\mathcal{P})$. The space S and the Baer subline determined by P, U and V are both unique, which implies the uniqueness of the line of $\pi(\mathcal{P})$ through U and V.

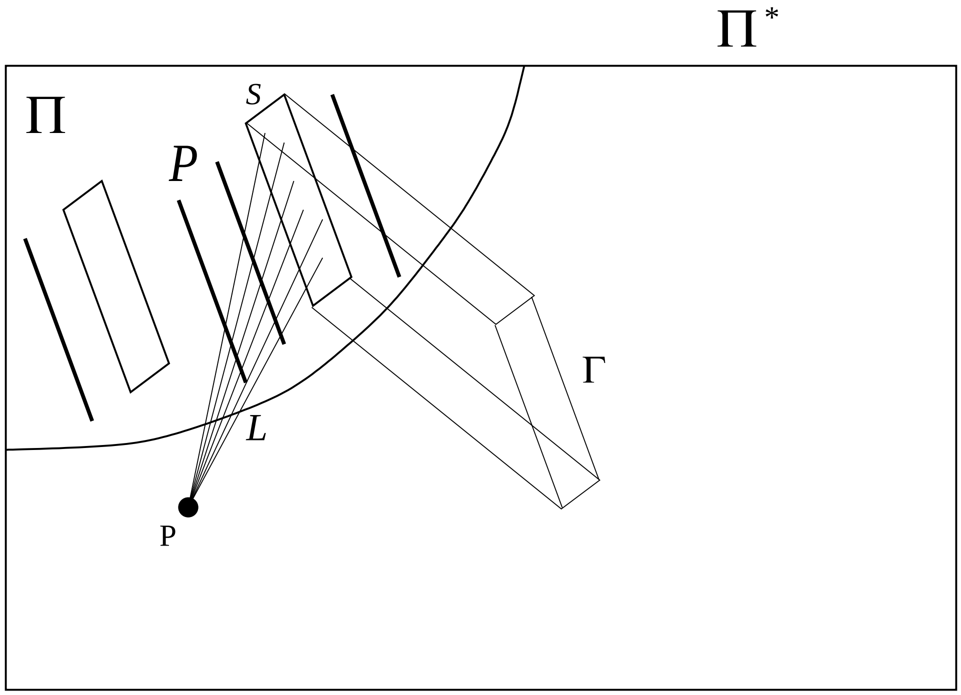

Figure 1.6: The set of lines $\mathcal{L}$

We now establish Axiom #2 of Definition 1.2.1. Let P be an affine point of $\pi(\mathcal{P})$ and let l be a line of $\pi(\mathcal{P})$ which does not contain P. We again have two cases. First suppose that the line l is represented by an n-space of Π^* which meets Π in an $(n-1)$-space S of $\mathcal{P}$. Then, the line m of $\pi(\mathcal{P})$ induced by the unique n-space of Π determined by the point P and the $(n-1)$-space S certainly contains P and is parallel to l. Here, the uniqueness of m follows from the uniqueness of the n-space representing m.

Now suppose that the line l is represented by a Baer $2n$-space Γ of Π^* meeting Π in a Baer subspace S of $\mathcal{P}$. Consider the lines in the set

$$\mathcal{L} = \{PX : X \in S\}$$

as shown in Figure 1.6. Every line through P which also meets Γ meets Γ in a point or in a Baer subline. No two lines through P could meet Γ in a Baer subline since then Γ would contain two coplanar lines which do not meet in a point of Γ.

The space Γ contains exactly

$$\frac{(q^{2n}+q^{2n-1}+\cdots+1)(q^{2n}+q^{2n-1}+\cdots+q)}{(q+1)q}$$

lines, and each of these lines extends to include q^2-q additional points in the space Π^*. As above, none of the lines of Γ can meet in a point outside of Γ. Therefore, the extensions of all of the lines of Γ cover exactly

$$\left(\frac{(q^{2n}+q^{2n-1}+\cdots+1)(q^{2n}+q^{2n-1}+\cdots+q)}{(q+1)q}\right)(q^2-q)$$

points of Π^*. But straightforward computation shows that this is

$$(q^{4n}+q^{4n-2}+\cdots+q^{2n+2})-(q^{2n-1}+q^{2n-3}+\cdots+q)$$

$$=(q^{4n}+q^{4n-2}+\cdots+q^2+1)-(q^{2n}+q^{2n-1}+\cdots+q+1),$$

exactly the number of points of $\Pi^*\setminus\Gamma$. Hence, every point of $\Pi^*\setminus\Gamma$ lies on a unique line which meets Γ in a Baer subline.

Now, since S is a hyperplane of Γ, any line of Γ must meet the space S in a point. Hence, there exists a unique line of $\mathcal{L}$ meeting Γ in a Baer subline. Let m be this unique line and let $Q=m\cap S$. Then, by Lemma 1.4.9, there is a unique Baer subline m_0 of m containing P and Q, but having no other point in common with the space Γ. Now consider the Baer $2n$-space Γ' of Π^* spanned by S and the Baer subline m_0. Every point of Γ' is on a line of $\mathcal{L}$. Since m is the only line of $\mathcal{L}$ which intersects Γ in a point of $\Gamma\setminus S$, and since $m_0\cap\Gamma=\{Q\}$, we see that $\Gamma\cap\Gamma'=S$. Therefore, Γ' induces a line of $\pi(\mathcal{P})$ which is parallel to l and contains the point P. This line is unique from the uniqueness of m_0.

Finally, it is clear that there exist at least 3 noncollinear points. Hence, we have constructed a finite affine plane. Since each line clearly contains q^{2n} points, the order of the plane is q^{2n}. ∎

We will see in Chapter 2 that the method given above and the Bose/Andrè method for constructing translation planes are closely connected. One special case of a mixed partition of $\mathcal{PG}(2n-1,q)$ is simply an $(n-1)$-spread. It should be noted that the construction method given here is exactly the same as the Bose/Andrè construction of Section 1.4.1 when one uses a spread rather than a *proper* mixed partition (one containing at least one Baer subspace).

1.5 Net Replacement and Derivation

In this last section of Chapter 1 we will look at a method whereby one can use a given translation plane to obtain another translation plane. This method, most generally known as *net replacement*, is discussed in detail in [29] and a special type of net replacement is described in detail in [24]. We start by describing the method and then applying it to our models discussed in the previous section.

For our purposes, a *replaceable net* is a set $\mathcal{N}$ of $(n-1)$-spaces of $\mathcal{PG}(2n-1,q)$, covering point set $\mathcal{C}$, such that there exists a different set $\mathcal{N}'$ of $(n-1)$-spaces, $\mathcal{N}\cap\mathcal{N}'=\emptyset$, which also covers $\mathcal{C}$. Such replaceable nets can be used to construct new translation planes from old translation planes. Suppose a spread $\mathcal{S}$ contains a replaceable net $\mathcal{N}$, and let $\mathcal{S}'$ be the spread $(\mathcal{S}\setminus\mathcal{N})\cup\mathcal{N}'$. Then the spread $\mathcal{S}'$ can be used to construct a translation plane $\pi(\mathcal{S}')$ which typically is not isomorphic to the plane $\pi(\mathcal{S})$. One special form of net replacement is more commonly called *derivation*, which we now discuss.

Let $\mathcal{A}$ be a finite affine plane of order n^2, and let l_∞ be the special line (i.e. the *line at infinity*) so that $\pi^{l_\infty}=\mathcal{A}$ for some projective plane π. Let S be a subset of $n+1$ points of l_∞ such that, for every two points X,Y of $\mathcal{A}$ for which the line through X and Y meets l_∞ in a point of S, there is a unique Baer subplane of π containing X,Y and the points of S. Such a set of points is called a *derivation set*. We define a new incidence structure $D(\mathcal{A})$ as follows. A point of $D(\mathcal{A})$ is simply a point of $\mathcal{A}$. A line of $D(\mathcal{A})$ is either a line of $\mathcal{A}$ meeting l_∞ in a point outside of

S, or is a Baer subplane of $\mathcal{A}$ meeting l_∞ in the set S. One can easily show that this incidence structure is in fact an affine plane of order n^2. The process is called *derivation* and the plane $D(\mathcal{A})$ is called the *derived plane.*

We now try to interpret derivation in the two models given in Section 1.4. For this, we refer the reader to Definition 1.4.5 of a regulus. When looking at 1-spreads of $\mathcal{PG}(3, q)$, reguli have a close connection with derivation. Referring to the Bose/Andrè Model of Section 1.4.1, embed $\Pi \cong \mathcal{PG}(3, q)$ into $\Pi^* = \mathcal{PG}(4, q)$, and let $\mathcal{S}$ be a spread of Π containing a regulus $\mathcal{R}$. Let $\pi(\mathcal{S})$ denote the finite affine plane obtained from this model. Moreover, let $\mathcal{R}^{opp}$ be the set of transversal lines of the regulus $\mathcal{R}$. One can easily show that $\mathcal{R}^{opp}$ is another regulus, called the *opposite regulus* of $\mathcal{R}$. It is well known that the plane obtained from the Bose/Andrè model using the spread $\mathcal{S}' = (\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{R}^{opp}$ is exactly the plane obtained by deriving $\pi(\mathcal{S})$. To see this, one needs to take a plane of Π^* which meets Π in a line of $\mathcal{S}'$ and interpret this point set in the plane $\pi(\mathcal{S}')$. The details can be found in [5], for instance. Hence, derivation is an example of net replacement.

Similarly, one can look at derivation in the Bose/Andrè model using higher dimensional spaces. Again, the general result can be found in [5]. We simply look at the special case of $\mathcal{PG}(7, q)$. Let $\mathcal{S}$ be a 3-spread of $\Sigma_0 \cong \mathcal{PG}(7, q)$ and, as in the Bose/Andrè model, embed Σ_0 into $\Sigma_0^* = \mathcal{PG}(8, q)$ as the hyperplane at infinity. Moreover, let $\pi(\mathcal{S})$ be the affine plane determine by $\mathcal{S}$. Then, we have:

Theorem 1.5.1 *A point set is an affine Baer subplane of $\pi(\mathcal{S})$ if and only if the corresponding point set of Σ_0^* is a 4-space not contained in Σ_0 which intersects Σ_0 in a 3-space meeting exactly $q^2 + 1$ elements of the spread $\mathcal{S}$.*

Hence, a Baer subplane of $\pi(\mathcal{S})$ is represented by a 4-space of Σ_0^* which meets Σ_0 in a solid S which intersects $q^2 + 1$ solids of the spread $\mathcal{S}$. Such a solid S acts as a transversal to $q^2 + 1$ elements of the spread $\mathcal{S}$. If these $q^2 + 1$ elements of the spread can be *covered* by q^2+1 pairwise disjoint transversal solids, then they would form a replaceable net.

We can say a little more about such a replaceable net of Σ_0. Let $\mathcal{N}$ be a set of q^2+1 pairwise disjoint solids of Σ_0 covering a point set $\mathcal{C}$, and let $\mathcal{N}'$ be a different set of q^2+1 pairwise disjoint solids of Σ_0 also covering $\mathcal{C}$.

Proposition 1.5.2 *Every solid of $\mathcal{N}$ meets every solid of $\mathcal{N}'$ in a line.*

Proof: Suppose that the solid S of $\mathcal{N}$ meets a solid of $\mathcal{N}'$ in a plane. Then, since lines and planes always meet in projective 3-space, S cannot meet any other solid of $\mathcal{N}'$ in a line. This implies that S must meet every other solid of $\mathcal{N}'$ in at most a point. But S contains q^3 additional points and there are only q^2 additional solids of $\mathcal{N}'$, a contradiction. Hence, no solid of $\mathcal{N}$ can meet a solid of $\mathcal{N}'$ in a plane. A similar argument now shows that no solid of $\mathcal{N}$ can meet a solid of $\mathcal{N}'$ in a single point. ■

We can now combine these results to obtain the following:

Theorem 1.5.3 *Let $\mathcal{N}$ be a set of q^2+1 pairwise disjoint solids of a spread $\mathcal{S}$ of Σ_0 and let $\mathcal{N}'$ be a different set of q^2+1 pairwise disjoint solids of Σ_0 such that every solid of $\mathcal{N}$ meets every solid of $\mathcal{N}'$ in a line. Then, letting $\mathcal{S}'=(\mathcal{S}\setminus\mathcal{N})\cup\mathcal{N}'$, $\pi(\mathcal{S}')\cong\mathcal{D}(\pi(\mathcal{S}))$.*

Proof: Since each element of $\mathcal{N}'$ meets each element of $\mathcal{N}$ in a line, the corresponding lines of $\pi(\mathcal{S}')$ are represented by 4-spaces of Σ meeting q^2+1 elements of $\pi(\mathcal{S})$ in a line. Hence, the result follows immediately from Theorem 1.5.1. ■

There are, of course, other types of replaceable nets. For instance, in Bruck [9], a family of *norm surfaces* are described. These surfaces are made up of $(q^{n-1}+\cdots+q+1)^2$ points of $\mathcal{PG}(2n-1,q)$ which are ruled by $(n-1)$-spaces. In fact, there are exactly n distinct ruling families and every point of the norm surface lies on exactly one member of each ruling family. If any such ruling family is part of a spread, there are clearly several possibilities for replacement.

As discussed in Section 1.4.2, one can also obtain an affine plane $\pi(\mathcal{P})$ from a mixed partition $\mathcal{P}$. We will look at some interpretations of derivation in this model as they arise. Additionally, we will see more examples of replaceable nets which *do not* correspond to derivation. These will be discussed as they are encountered. Our first step will be to look at the tight connection between mixed partitions and spreads.

Chapter 2

A GEOMETRIC LOOK AT THE HIRSCHFELD-THAS CONSTRUCTION

Having given a brief introduction to the study of translation planes, we are now ready to look at a method for construction spreads. More specifically, we will look at a method for constructing $(2n-1)$-spreads of $\mathcal{PG}(4n-1, q)$ using a mixed partition of $\mathcal{PG}(2n-1, q^2)$. The arguments given here are mostly geometric in nature as opposed to the algebraic methods which are given in [19].

2.1 The Theory in $\mathcal{PG}(4n-1, q^2)$

Let K be the finite field $GF(q^2)$ with primitive element β, and let F be its subfield $GF(q)$. For convenience, we name the element $\alpha = \beta^{q-1}$, whose order is $q+1$. Let $\Gamma = \mathcal{PG}(2n-1, q^2)$ with underlying vector space W over K, and let $\Sigma = \mathcal{PG}(4n-1, q^2)$ with underlying vector space V over K. Also, let $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$ be a Baer subspace of Σ in standard position. That is, Σ_0 consists of those points of Σ induced by vectors in V all of whose components lie in the subfield F of K.

We also establish some notation which will be used consistently throughout this work. We use homogeneous coordinates for the points of any projective space. Thus, non-zero vectors from V which are K-scalar multiples of one another induce the same projective point. We write $\mathbf{v} \sim \mathbf{w}$ to mean $\mathbf{v}$ and $\mathbf{w}$ are K-scalar multiples.

For much of the following we will be working in Γ and then embedding into Σ. We first show a nice way to do this embedding. Consider the K-linear transformation $\Delta : W \to V$ given by

$$\mathbf{v} \mapsto (\mathbf{v}, \alpha\mathbf{v}),$$

where $(\mathbf{v}, \alpha\mathbf{v})$ is the $4n$ dimensional vector whose first $2n$ coordinates are the coordinates of $\mathbf{v}$ and whose last $2n$ coordinates are the coordinates of $\alpha\mathbf{v}$. One can easily check that $ker(\Delta) = \{\mathbf{0}\}$ and hence Δ induces an embedding of Γ into Σ. By a slight abuse of notation, we also let Δ denote the induced embedding. For the construction we are leading up to, we need $\Gamma^\Delta \cap \Sigma_0 = \emptyset$. We show this now.

Lemma 2.1.1 *Under the embedding Δ, the image of Γ in Σ is disjoint from Σ_0.*

Proof: Let $\mathbf{v} = (v_1, v_2, ..., v_{2n}, \alpha v_1, \alpha v_2, ..., \alpha v_{2n})$ be a vector in V which induces a projective point of Γ^Δ. Then at least one of $v_1, v_2, ..., v_{2n}$ must be non-zero. By left normalizing $\mathbf{v}$ to get an equivalent projective point, the first non-zero coordinate becomes 1 and we get another coordinate to be α. But the coordinates of any normalized vector which induces a point of Σ_0 are all in F. Since $\alpha \notin F$, $\mathbf{v}$ cannot induce a point of Σ_0 and $\Gamma^\Delta \cap \Sigma_0 = \emptyset$. ∎

Definition 2.1.2 *With K as above, the map from K to K given by $k \mapsto k^q$ is an automorphism of the field K with fixed field F. We call this map the* **Frobenius automorphism**.

By letting the Frobenius automorphism act on components of a vector from V, we get a non-singular semi-linear transformation on V which therefore induces a collineation of Σ by Theorem 1.4.1. We also refer to the collineation as the Frobenius collineation (or map) of Σ. For the remainder of this section, we will use the term *conjugation* to mean application of the Frobenius map. We typically write P^q for the image of point $P \in \Sigma$ under the Frobenius map.

Lemma 2.1.3 *The points of Σ which are fixed under conjugation are precisely the points of Σ_0.*

Proof: Clearly the points of Σ_0 are fixed under conjugation. We show that no other point can be fixed.

Let P be a point of $\Sigma\backslash\Sigma_0$ and let $\mathbf{v} = (v_i)$ be any vector which induces the point P. Suppose that $P^q = P$. Then $\mathbf{v}^q \sim \mathbf{v}$ and there is a $k \in K$ such that $\mathbf{v}^q = k\mathbf{v}$. So,

$$\mathbf{v} = (\mathbf{v}^q)^q = (k\mathbf{v})^q = k^q\mathbf{v}^q = k^{q+1}\mathbf{v},$$

which means that the field element k must be a power of α. Let $k = \alpha^t$ for some t. Then $v_i^q = \alpha^t v_i = \beta^{t(q-1)} v_i$ for all i. But this means that $v_i = f_i\beta^t$, where $f_i \in F$, and $f_i = 0$ precisely when $v_i = 0$. Thus, $(v_i) = (\beta^t f_i) = \beta^t(f_i) \sim (f_i)$. Hence, the projective point P is induced by a vector which has all of its coordinates in F, and so lies in the Baer subspace Σ_0. ■

Lemma 2.1.4 *For any point $P \in \Sigma\backslash\Sigma_0$, the line $l_P = PP^q$ meets the Baer subspace Σ_0 in exactly $q+1$ points (i.e., in a Baer subline).*

Proof: Since $P \notin \Sigma_0$, $P^q \neq P$ by Lemma 2.1.3, and thus P and P^q determine a line. Let $\mathbf{v}$ be a vector which induces the point P, and consider the points on the line l_P induced by the vectors $\mathbf{v} + \alpha^i\mathbf{v}^q$ as i varies from 0 to q. Since $\alpha^q = \frac{1}{\alpha}$,

$$\left(\mathbf{v} + \alpha^i\mathbf{v}^q\right)^q = (\mathbf{v}^q + \alpha^{-i}\mathbf{v}) \sim (\alpha^i\mathbf{v}^q + \mathbf{v}),$$

and we see that the projective points induced by vectors of the form $\mathbf{v} + \alpha^i\mathbf{v}^q$ are all fixed by the Frobenius map. Hence they lie in Σ_0 by Lemma 2.1.3. Since no two of these points are the same, and there are $q+1$ choices for i, we get $q+1$ points of l_P lying in Σ_0. Clearly, l_P cannot contain more than $q+1$ points of Σ_0. ■

This next lemma appears quite elementary, yet will be instrumental in our ultimate construction. Let Π be any $(2n-1)$-space in Σ which is disjoint from Σ_0, for instance $\Pi = \Gamma^{\Delta}$. For the lemma, we restrict to $P \in \Pi$, however, the reader should note that all we really need is for P to be in some projective subspace that does not intersect Σ_0.

Lemma 2.1.5 *If $P \in \Pi$, then $P^q \notin \Pi$. Moreover, for any two distinct points, say P and Q of Π, the lines $l_P = PP^q$ and $l_Q = QQ^q$ do not intersect.*

Proof: Let $P \in \Pi$. If P^q were a point of Π, then the line l_P would be a line of Π and could not possibly meet the Baer subspace Σ_0, contradicting Lemma 2.1.4. Hence, $P^q \notin \Pi$, and as a result, $\Pi^q \cap \Pi = \emptyset$.

Now, consider the line PQ. Since P and Q are both in Π, the line PQ is also a line in Π. Similarly, the line P^qQ^q is a line of Π^q. Hence, these two lines do not meet. Suppose now that the lines l_P and l_Q meet. Then the lines determine a plane, which means that the four points P, Q, P^q, Q^q are coplanar. But then it follows that the lines PQ and P^qQ^q intersect, a contradiction. Hence, the lines l_P and l_Q do not intersect (see Figure 2.1). ■

Now, we fix a projective $(2n-1)$-space Π in Σ which does not intersect Σ_0, say $\Pi = \Gamma^{\Delta}$. Let $P_0, P_1, ..., P_{2n}$ be $2n+1$ points in general position in Π. In other words, no $2n$ of the P_i's lie in the same hyperplane (a $(2n-2)$-space). Then, by the above lemmas, $P_i^q \notin \Pi$ for each i and the lines $l_{P_i} = P_iP_i^q$ all intersect Σ_0 in a Baer subline. Let $\bar{l}_{P_i}$ represent $l_{P_i} \cap \Sigma_0$ for each i.

Lemma 2.1.6 *Any $2n$ of the sublines $\bar{l}_{P_i}$ generate Σ_0.*

Proof: Since the P_i's are in general position, their images under conjugation, P_i^q, are also in general position. Thus, any $2n$ of the P_i^q's generate the $(2n-1)$-space

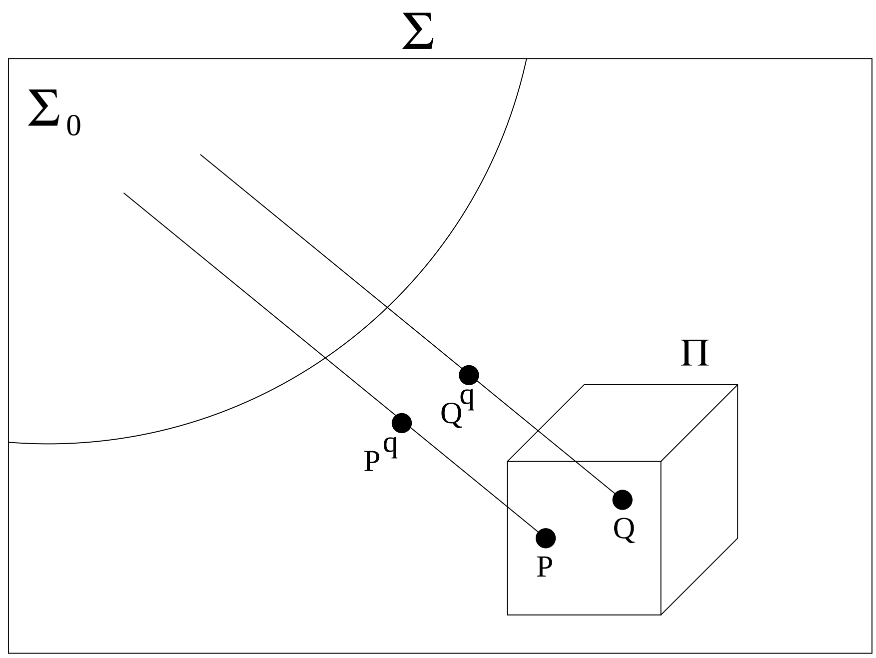

Figure 2.1: Lifting a point of Π

Π^q in Σ. From Lemma 2.1.5, Π^q cannot intersect Π, and, as shown in Lemma 2.1.3, Π^q cannot intersect Σ_0.

Since the $(2n-1)$-spaces Π and Π^q do not intersect, we immediately get that the vectors which induce the points $P_1, P_2, ..., P_{2n}, P_1^q, P_2^q ..., P_{2n}^q$ are all linearly independent. Hence, they generate all of the points of Σ. But then the lines l_{P_i}, for $i = 1, 2, ..., 2n$, generate Σ, and so the Baer sublines $\bar{l}_{P_i}$ generate Σ_0. We use the same argument for the other combinations of $2n$ distinct points. ■

Lemma 2.1.7 *Let $\mathcal{X} = \{\bar{l}_{P_i} : i = 0, 1, 2, \ldots, 2n\}$ be a set of $2n+1$ lines in Σ_0, as constructed earlier, with the property that any $2n$ of the lines generate Σ_0. Let Q_0 be any point on $\bar{l}_{P_0}$. Then, there exists exactly one Baer $(2n-1)$-space of Σ contained in Σ_0 which contains Q_0 and meets each of the $\bar{l}_{P_i}$'s in exactly one point.*

Proof: The proof of this statement will follow from the uniqueness of representation of a vector with respect to a given basis. All of the lines and the purported Baer $(2n-1)$-space in the statement of the lemma are contained in Σ_0. Hence, we can drop the "Baer" notation and view the lemma as a result about lines and $(2n-1)$-spaces of $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$. For $i = 1, 2, ..., 2n$, let $\mathbf{r}_i$ and $\mathbf{s}_i$ be two vectors which induce distinct projective points on the line $\bar{l}_{P_i}$. Since the lines of $\mathcal{X}$ generate Σ_0, the above $4n$ vectors are linearly independent and form a basis for the underlying vector space of Σ_0. Letting $\mathbf{t}$ be a vector which induces the projective point Q_0, we can express $\mathbf{t}$ in terms of this basis:

$$\sum_{i=1}^{2n} a_i\mathbf{r}_i + \sum_{i=1}^{2n} b_i\mathbf{s}_i = \mathbf{t}$$

for some $a_i, b_i \in F$. Since the vector $a_i\mathbf{r}_i + b_i\mathbf{s}_i$ induces a point on $\bar{l}_{P_i}$ for $i = 1, 2, ..., 2n$, the $(2n-1)$-space B generated by the $2n$ points induced by $a_i\mathbf{r}_i + b_i\mathbf{s}_i$, for $i = 1, 2, ..., 2n$, will meet each $\bar{l}_{P_i}$ in at least one point.

Suppose that the statement in the lemma is false. That is, either B meets one of the lines in more than one point, or there is another $(2n-1)$-space of Σ_0 through Q_0 that meets each line in a point. In either case, we draw the same conclusion. There must be a different set of $2n$ points, one on each of the $\bar{l}_{P_i}$'s, that are all in a $(2n-1)$-space B' of Σ_0 containing Q_0. Since these $2n$ points are on the $\bar{l}_{P_i}$'s, they will generate B'. Let $a_i'\mathbf{r}_i + b_i'\mathbf{s}_i$, for $i = 1, 2, ..., 2n$, be vectors which induce these $2n$ points. Of course, it is possible that $a_i' = a_i$ or $b_i' = b_i$ for some i. Then, since these points generate B',

$$\sum_{i=1}^{2n} c_i \left(a_i'\mathbf{r}_i + b_i'\mathbf{s}_i\right) = \mathbf{t}$$

for some $c_i \in F$. Hence,

$$\sum_{i=1}^{2n} c_i \left(a_i'\mathbf{r}_i + b_i'\mathbf{s}_i\right) = \sum_{i=1}^{2n} a_i\mathbf{r}_i + b_i\mathbf{s}_i$$

or

$$\sum_{i=1}^{2n} [(c_i a_i' - a_i)\mathbf{r}_i + (c_i b_i' - b_i)\mathbf{s}_i] = \mathbf{0}.$$

Since the $\mathbf{r}_i$'s and $\mathbf{s}_i$'s together form a basis for the underlying $4n$-dimensional vector space, the coefficients in the above equation must be zero. As a result,

$$c_i a_i' = a_i \text{ and } c_i b_i' = b_i$$

for all i, implying

$$a_i'\mathbf{r}_i + b_i'\mathbf{s}_i \sim c_i\,(a_i'\mathbf{r}_i + b_i'\mathbf{s}_i) = a_i\mathbf{r}_i + b_i\mathbf{s}_i.$$

Hence, $B = B'$ and B meets each $\bar{l}_{P_i}$ in a unique point. ■

We are now ready to begin the main construction. We first show how to lift a Baer subspace of Π.

Lemma 2.1.8 *Let B_0 be a Baer subspace of Π. Then, the Baer sublines $\bar{l}_P$, for each point P in B_0, form a set of transversal lines to a regulus in Σ_0.*

Proof: Let P_0 be any fixed point of Π and let $P_1, P_2, ..., P_{2n}$ be $2n$ other distinct points of Π such that $P_0, P_1, P_2, ..., P_{2n}$ are in general position. Then, since $q \geq 2$, we can find three distinct points Q_0, Q_1, and Q_2 on $\bar{l}_{P_0}$. Using Lemma 2.1.7, let Π_0, Π_1, and Π_2 be the three unique Baer $(2n-1)$-spaces through Q_0, Q_1 and Q_2 respectively that meet each of the l_{P_i}'s in exactly one point each. Then, by following the exact same argument as in Lemma 2.1.7, we show that these Baer spaces do not intersect. Hence, we can consider the regulus $\mathcal{R}$ in Σ_0 generated by these three $(2n-1)$-spaces of Σ_0. Now, each of the original lines $\bar{l}_{P_i}$ is a transversal to all of the $(2n-1)$-spaces in the regulus since it meets three of the them. Hence, all of the $(2n-1)$-spaces in the regulus meet all of the $\bar{l}_{P_i}$'s in exactly one point each. All we need to show is that for any other point P of B_0, the Baer subline $\bar{l}_P$ is also a transversal line to the $(2n-1)$-regulus $\mathcal{R}$.

As was shown in Lemma 2.1.4, for any point P, every point of the line $\bar{l}_P$ is induced by a vector of the form $\mathbf{v} + \alpha^j \mathbf{v}^q$ for some $j \in \{0, 1, ..., q\}$, where $\mathbf{v}$ is any vector which induces the point P. Let $\mathbf{v}_i$ be any vector which induces the point P_i for $i = 1, 2, \ldots, 2n$. For convenience, choose the $\mathbf{v}_i$'s so that

$$\mathbf{v}_0 = \mathbf{v}_1 + \mathbf{v}_2 + ... + \mathbf{v}_{2n}.$$

When these vectors are chosen this way, there is a clear representation for the $(2n-1)$-spaces of $\mathcal{R}$. Pick any point in $\bar{l}_{P_0}$ and let $\mathbf{v}_0 + \alpha^j \mathbf{v}_0^q$ be the vector which induces this point. Let Π_j be the solid through this point which meets each of the other $\bar{l}_{P_i}$'s in exactly one point. Then Π_j meets $\bar{l}_{P_i}$ in the point induced by vector

$$\mathbf{v}_i + \alpha^j \mathbf{v}_i^q$$

since

$$\sum_{k=1}^{2n} \left(\mathbf{v}_k + \alpha^j \mathbf{v}_k^q\right) = \sum_{k=1}^{2n} \mathbf{v}_k + \alpha^j \left(\sum_{k=1}^{2n} \mathbf{v}_k\right)^q = \mathbf{v}_0 + \alpha^j \mathbf{v}_0^q.$$

Now, let P, induced by vector $\mathbf{w}$, be any other point of Π. Then, there are scalars $a_i \in K$ so that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_{2n}\mathbf{v}_{2n}.$$

But then, for each j,

$$a_1(\mathbf{v}_1 + \alpha^j \mathbf{v}_1^q) + a_2(\mathbf{v}_2 + \alpha^j \mathbf{v}_2^q) + ... + a_{2n}(\mathbf{v}_{2n} + \alpha^j \mathbf{v}_{2n}^q)$$

$$= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_{2n}\mathbf{v}_{2n}) + \alpha^j (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_{2n}\mathbf{v}_{2n})^q$$

$$= \mathbf{w} + \alpha^j \mathbf{w}^q$$

which certainly induces a point on $\bar{l}_P$. Hence, the line l_P also meets the solid Π_j for each j, and we have shown that the lines $\bar{l}_P$, for every P in Π, form a ruling family for a regulus in Σ_0 (see Figure 2.2). ■

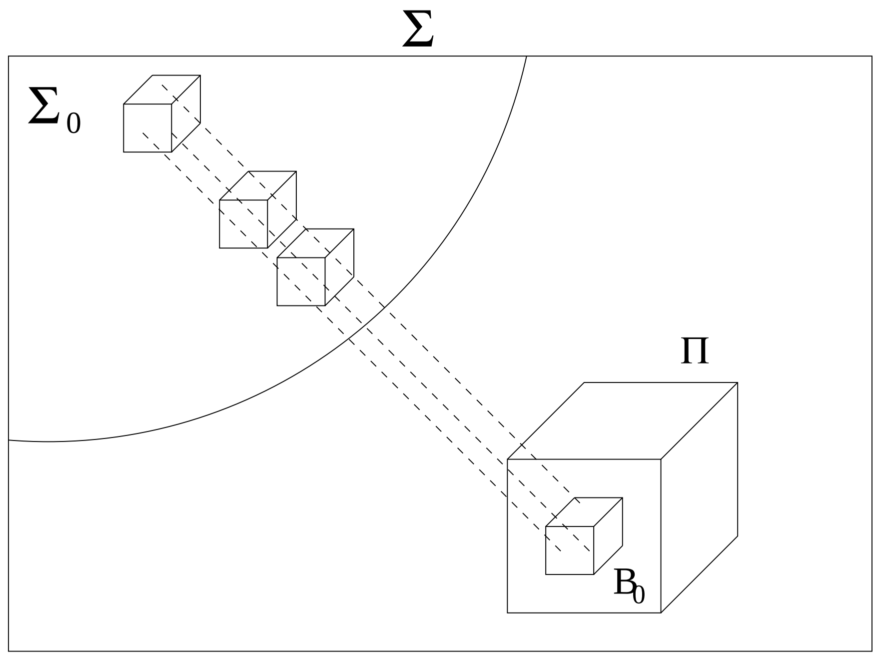

Figure 2.2: Lifting to a regulus

Hence, each Baer subspace, say B_0, lifts to a regulus of Σ_0. It also follows from the proof of Lemma 2.1.8 that we can write the members of this regulus as $B_0 + \alpha^i B_0^q$ as i varies between 0 and q. We will use this representation in later chapters. The last bit of theory we will need to complete the construction will come from the following lemma.

Lemma 2.1.9 *Let l be an $(n-1)$-space in Σ which does not meet Σ_0. Then the conjugate of l, l^q, does not intersect l, and hence l and l^q generate a $(2n-1)$-space. This $(2n-1)$-space meets Σ_0 in a Baer $(2n-1)$-space of Σ.*

Proof: Let P and Q be two distinct points on l. Then P^q and Q^q are on l^q, and the lines $l_P = PP^q$ and $l_Q = QQ^q$ do not intersect by Lemma 2.1.5. Hence, l and l^q do not meet and so generate a $(2n-1)$-space, say Π_l.

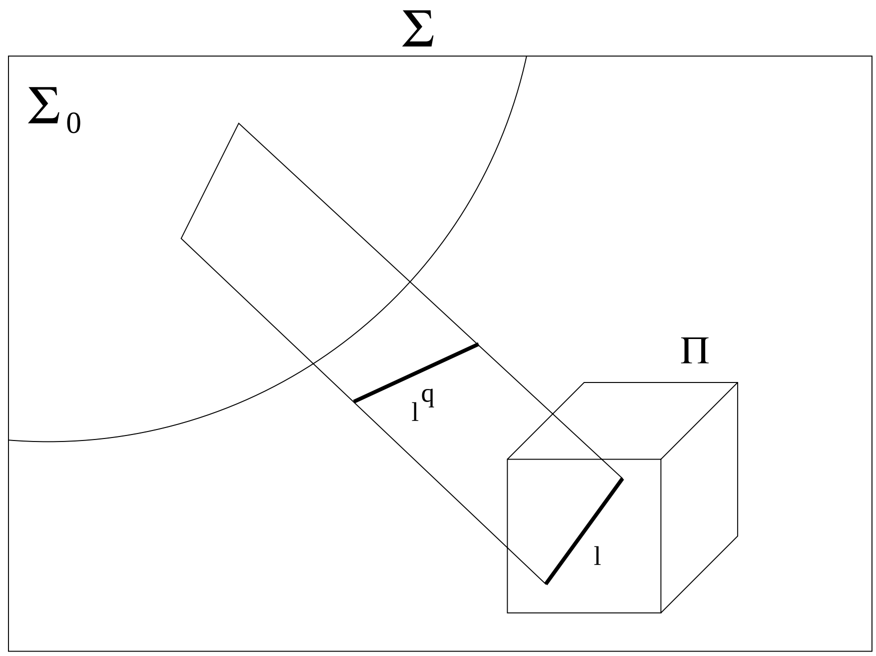

Figure 2.3: Lifting a line

For each point P on l, the line l_P meets Σ_0 in a Baer subline. Therefore, since these lines do not intersect, we get exactly $\left(q^{2(n-1)} + q^{2(n-2)} + ... + q^2 + 1\right)(q+1)$ points in $\Sigma_0 \cap \Pi_l$. A $(2n-1)$-space of Σ cannot meet Σ_0 in any more points, hence the points of $\Sigma_0 \cap \Pi_l$ must form a Baer $(2n-1)$-space. ■

The simplest case of Lemma 2.1.9 is when $n = 2$, and we have lines lifting to solids. This is pictured in Figure 2.3.

2.2 Creation of a Spread in $\mathcal{PG}(4n-1, q)$

We now explain carefully how to lift a mixed partition of $\mathcal{PG}(2n-1, q^2)$ to a spread of $\mathcal{PG}(4n-1, q)$. As before, let $\Sigma = \mathcal{PG}(4n-1, q^2)$, and let $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$ be the Baer subspace of Σ in standard position. Also, let $\Pi = \mathcal{PG}(2n-1, q^2)$ be a $(2n-1)$-dimensional projective space in Σ which does not intersect Σ_0. Such a $(2n-1)$-space always exists as shown in Lemma 2.1.1.

Let $\mathcal{P}$ be a mixed partition of Π which contains α Baer subspaces and β $(n-1)$-spaces. Then, by counting points, we get

$$\alpha(q^{2n-1} + ... + q + 1) + \beta(q^{2(n-1)} + ... + q^2 + 1) = q^{2(2n-1)} + q^{2(2n-2)} + ... + q^2 + 1$$

or

$$\alpha(q+1) + \beta = q^{2n} + 1.$$

The existence of such a partition is nontrivial (except when $\alpha = 0$ and $\beta = q^{2n}+1$), and will be explored in Chapter 6.

Before proving the main result, we briefly describe the "lifting" procedure. For each point P of Π, consider the line l_P generated by P and its conjugate point P^q. No two of these lines meet from Lemma 2.1.5, and all of these lines meet Σ_0 in a Baer subline. A simple counting argument shows that these Baer sublines actually partition the points of Σ_0. To create our desired $(2n-1)$-spread of Σ_0, we will work with these sublines in two ways. Some of them will be grouped together to form $(2n-1)$-spaces of Σ_0, and others will be grouped together to form the transversal lines to a regulus in Σ_0. For each such set of transversal lines, we will take the opposite ruling family of $(2n-1)$-spaces of Σ_0 to complete our spread. We will now show this in detail.

Theorem 2.2.1 *The α Baer subspaces together with the β $(n-1)$-spaces of a mixed partition $\mathcal{P}$ of Π can be lifted to a $(2n-1)$-spread of $\mathcal{PG}(4n-1, q)$.*

Proof: Let B be a Baer subspace of our mixed partition and pick $2n+1$ points of B which are in general position. For each point, we construct the line determined by the point and its conjugate. From Lemma 2.1.4, these lines all meet Σ_0 in Baer sublines, and, from Lemma 2.1.8, they form a ruling family of a regulus. Moreover, "reversing" this regulus gives us a set of $q+1$ projective spaces of dimension $2n-1$ over F. Let T be the set of all such solids obtained from the reguli determined by all of the Baer subspaces of the mixed partition $\mathcal{P}$.

Now, consider an $(n-1)$-space of our mixed partition. By the method described in Lemma 2.1.9, each of these $(n-1)$-spaces lifts to a single $(2n-1)$-space of Σ_0. Let U be the set of all such solids obtained from $(n-1)$-spaces of $\mathcal{P}$.

We now claim that the set $T \cup U$ is a spread of Σ_0. Note that we obtain the appropriate number of solids since $\alpha(q+1)+\beta = q^{2n}+1$, which is the proper size for a $(2n-1)$-spread of $\mathcal{PG}(4n-1,q)$. What is left is to show that these solids do, in fact, partition the point set of Σ_0. We do this by simply showing that any two such $(2n-1)$-spaces are disjoint.

Certainly, no two $(2n-1)$-spaces in the same regulus can intersect. Also, since it was shown that no two lines of the form PP^q for some $P \in \Pi$ can intersect, it is clear that no two $(2n-1)$-spaces from different reguli can intersect. Consider the method by which the remaining β $(2n-1)$-spaces are generated. These $(2n-1)$-spaces are formed by unions of sublines $\bar{l}_P$. Since none of these lines can intersect by Lemma 2.1.5, it follows that no two of the $(2n-1)$-spaces can intersect. Hence, we have a spread. ■

We now recap the observation made in the discussion above. The members of our constructed spread come from working with the Baer sublines of the lines of the form PP^q where $P \in \Pi$. These sublines partition the point set of Σ_0 and hence form a 1-spread of Σ_0. With this model in mind, it should be clear that none of the members of the $(2n-1)$-spread can intersect. As a final note, we point out a very nice property of the 1-spread $\{\bar{l}_P : P \in \Pi\}$.

Definition 2.2.2 *A 1-spread $\mathcal{G}$ is* **geometric** *if for every pair of distinct lines l_1 and l_2 of $\mathcal{G}$, $\mathcal{G}$ induces a spread in the solid generated by l_1 and l_2.*

Lemma 2.2.3 *The 1-spread $\mathcal{G}_0$ of Σ_0 obtained from the lines $\bar{l}_P$ for each P in Π is geometric.*

Proof: Let $\bar{l}_{P_1}$ and $\bar{l}_{P_2}$ be two lines of $\mathcal{G}_0$, and let $\mathbf{T} = \langle \bar{l}_{P_1}, \bar{l}_{P_2} \rangle$ be the solid generated by them in Σ_0. Now, consider the line $L = P_1P_2$ of Π, and lift every point on L to its corresponding Baer subline in Σ_0. Then, as discussed earlier, L lifts to a solid in Σ_0, and this solid is exactly $\mathbf{T}$. Hence, the solid $\mathbf{T}$ has an induced spread from $\mathcal{G}_0$; namely, the lines $\bar{l}_Q$ for all Q on the line L. ■

The geometric 1-spread will be very useful in finding certain automorphisms of lifted spreads. We discuss this in detail in Chapter 4.

2.3 Associated Translation Planes

As we saw in Sections 1.4.1 and 1.4.2, both the mixed partition of $\mathcal{PG}(2n-1, q^2)$ and the $(2n-1)$-spread of $\mathcal{PG}(4n-1, q)$ can be used to construct affine planes of order q^{2n}. Moreover, we have just shown that mixed partitions of $\mathcal{PG}(2n-1, q^2)$ naturally give rise to $(2n-1)$-spreads of $\mathcal{PG}(4n-1, q)$. Let $\mathcal{P}$ be a mixed partition of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ and let $\mathcal{S}$ be its associated spread of $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$. Also, let $\pi(\mathcal{P})$ and $\pi(\mathcal{S})$ be their associated affine planes. We can now show that $\pi(\mathcal{P})$ and $\pi(\mathcal{S})$ are isomorphic.

Embed Π into $\Pi^* \cong \mathcal{PG}(2n, q^2)$ as the hyperplane containing all points whose first homogeneous coordinate is 0. Similarly, embed Σ_0 into $\Sigma_0^* \cong \mathcal{PG}(4n, q)$ as the hyperplane containing all points whose first homogeneous coordinate is 0. We use the Bose/Andrè model as discussed in Section 1.4.1 for $\pi(\mathcal{S})$, and we use the mixed partition model from Section 1.4.2 to model $\pi(\mathcal{P})$. Hence, we look for a map from $\Pi^* \setminus \Pi$ to $\Sigma_0^* \setminus \Sigma_0$ which will induce a bijection from the affine points of $\pi(\mathcal{P})$ to the affine points of $\pi(\mathcal{S})$. Let $Tr(x)$ be the trace function from K to F. That is,

$Tr(x) = x + x^q$. Also, recall the special field element $\alpha = \beta^{q-1}$, and consider the map ϕ defined on the points of $\Pi^* \setminus \Pi$ as follows:

$$(1, \mathbf{v}) \mapsto (1, Tr(\mathbf{v}), Tr(\alpha\mathbf{v}))\,.$$

Here we let $(1, \mathbf{v})$ be the $(2n+1)$-dimensional vector whose first coordinate is 1 and whose last $2n$ coordinates are given by the $2n$-dimensional vector $\mathbf{v}$. Also, $Tr(\mathbf{v}) = Tr\left((v_i)\right) = (Tr(v_1), Tr(v_2), \ldots Tr(v_{2n}))$.

Lemma 2.3.1 *The map ϕ induces a bijection between the points of $\pi(\mathcal{P})$ and $\pi(\mathcal{S})$.*

Proof: Since the domain and codomain of ϕ have the same cardinality, we prove the lemma by showing that the map is injective. Let $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$ be two points of $\pi(\mathcal{P})$, induced by vectors $(1, \mathbf{u})$ and $(1, \mathbf{v})$ respectively, and suppose

$$(1, Tr(\mathbf{u}), Tr(\alpha\mathbf{u})) = (1, Tr(\mathbf{v}), Tr(\alpha\mathbf{v}))\,.$$

Then, $Tr(\mathbf{u}) = Tr(\mathbf{v})$ and $Tr(\alpha\mathbf{u}) = Tr(\alpha\mathbf{v})$. Hence, if we let $\mathbf{w} = \mathbf{u} - \mathbf{v}$,

$$Tr(\mathbf{w}) = Tr(\alpha\mathbf{w}) = \mathbf{0}.$$

So we have a system of equations in $\mathbf{w}$ and $\mathbf{w}^q$, namely

$$\begin{aligned} \mathbf{w} + \mathbf{w}^q &= 0 \\ \alpha\mathbf{w} + \alpha^q\mathbf{w}^q &= 0. \end{aligned}$$

Scalar multiplying the first equation by α^q and subtracting gives us

$$\alpha^q\mathbf{w} - \alpha\mathbf{w} = \mathbf{0}.$$

If $\mathbf{w} \neq \mathbf{0}$, this implies $\alpha^q = \alpha$, and so $\alpha^{q-1} = 1$. Since the order of α is $q+1$, this is a contradiction. Hence, $\mathbf{w} = \mathbf{0}$ and $\mathbf{u} = \mathbf{v}$. Therefore, $P_{\mathbf{u}} = P_{\mathbf{v}}$ and the map is a bijection. ∎

Recall now that there are two types of lines of $\pi(\mathcal{P})$. The first type of line is represented by an isomorphic copy of $\mathcal{PG}(n, q^2)$ in Π^* meeting Π in one of the $\mathcal{PG}(n-1, q^2)$'s of the mixed partition $\mathcal{P}$. We call these lines *Type A* lines. The other type of line is represented by a Baer $2n$-subspace of Π^* meeting Π in one of the Baer subspaces of the mixed partition $\mathcal{P}$. We call these lines *Type B* lines. We will define the image of each type of line under the map ϕ, show that these definitions are well-defined, and then prove that the map on lines is also a bijection. The incidence preserving property will follow from the definitions of the images of points and lines.

First, let l be a Type A line of $\pi(\mathcal{P})$. Then l is induced by all vectors in the K-linear span of $(0, \mathbf{u}_1), (0, \mathbf{u}_2), \ldots, (0, \mathbf{u}_n)$ and $(1, \mathbf{v})$, where the vectors $(0, \mathbf{u}_i)$ generate the n-dimensional vector space which induces an $(n-1)$-space S in the mixed partition $\mathcal{P}$ and $(1, \mathbf{v})$ induces an affine point of $\pi(\mathcal{P})$. Again, we write $(0, \mathbf{u})$ for the $(2n+1)$-dimensional vector with first coordinate 0 and last $2n$ coordinates given by the $2n$-tuple $\mathbf{u}$. We will introduce some new notation in order to simplify writing the images of lines under ϕ. Define

$$\lambda_i(\mathbf{v}) = \mathbf{v} + \alpha^i \mathbf{v}^q$$

where $0 \leq i \leq q$. Here, the projective points induced by the vectors $\lambda_i(\mathbf{v})$, as i varies, are exactly the $q+1$ points on the Baer subline $\bar{l}_P$ of Σ_0 determined by the point P of Π induced by vector $\mathbf{v}$. We define the image l^ϕ to be the set of points induced by all vectors in the F-linear span of

$$\begin{gathered}(0, \lambda_{i_1}(\mathbf{u}_1), \lambda_{i_1}(\alpha \mathbf{u}_1)) \\ (0, \lambda_{i_2}(\mathbf{u}_2), \lambda_{i_2}(\alpha \mathbf{u}_2)) \\ \vdots \\ (0, \lambda_{i_n}(\mathbf{u}_n), \lambda_{i_n}(\alpha \mathbf{u}_n))\end{gathered}$$

and

$$(1, Tr(\mathbf{v}), Tr(\alpha \mathbf{v}))$$

where $i_1, i_2, \ldots i_n$ all vary independently between 0 and q. Note that for any particular j, the points induced by the vectors

$$\big(0, \lambda_{i_j}(\mathbf{u}_1), \lambda_{i_j}(\alpha\mathbf{u}_1)\big)$$

as i_j varies between 0 and q are exactly the points on one of the geometric spread lines described in Lemma 2.2.3, and the set of all such points (as j varies) represents an element of the spread $\mathcal{S}$ defined by the lifting of the $(n-1)$-space S of the mixed partition $\mathcal{P}$. Hence, we have given a formal algebraic method for describing the lifting process from earlier in the chapter.

The Type B lines are considerably more difficult to work with. This is mostly because a Baer subspace of $\mathcal{P}$ does not lift to a single spread element. Rather, it lifts to an entire regulus composed of $q+1$ spread elements. In order to work with the Type B lines, we need to look more carefully at coordinates.

Suppose that l is a Type B line of $\pi(\mathcal{P})$. Then l is induced by all of the non-zero vectors contained in some Baer $2n$-space B_0^* of Π^* which meets the hyperplane Π in a Baer subspace B_0 of $\mathcal{P}$. Recall from Section 1.4 that a d-dimensional Baer subspace is uniquely determined by $d+2$ points, no $d+1$ of them in the same $(d-1)$-space. For generators of B_0^* we take the points $U_1, U_2, \ldots, U_{2n}$ induced by vectors $(0, \mathbf{u}_1), (0, \mathbf{u}_2), \ldots, (0, \mathbf{u}_{2n})$ together with V_1 and V_2 induced by vectors $(1, \mathbf{v}_1)$ and $(1, \mathbf{v}_2)$. Then, there exist scalars $k_i \in K$ such that

$$\sum_{i=1}^{2n} k_i(0, \mathbf{u}_i) + (1, \mathbf{v}_1) = (1, \mathbf{v}_2),$$

and every element of B_0^* is induced by a vector in the F-linear span of $k_1(0, \mathbf{u}_1)$, $k_2(0, \mathbf{u}_2), \ldots, k_{2n}(0, \mathbf{u}_{2n})$ and $(1, \mathbf{v}_1)$. Note that the first coordinates imply that the coefficient of $(1, \mathbf{v}_1)$ must be a 1. In fact, without loss of generality, we assume that $k_i = 1$ for each i. That is, we simply scalar multiply each vector to meet this assumption. With this assumption, every element of B_0 is induced by a vector in the F-linear span of $(0, \mathbf{u}_1)$, $(0, \mathbf{u}_2), \ldots, (0, \mathbf{u}_{2n})$. Let U^* induced by $(0, \mathbf{u}^*)$ be the

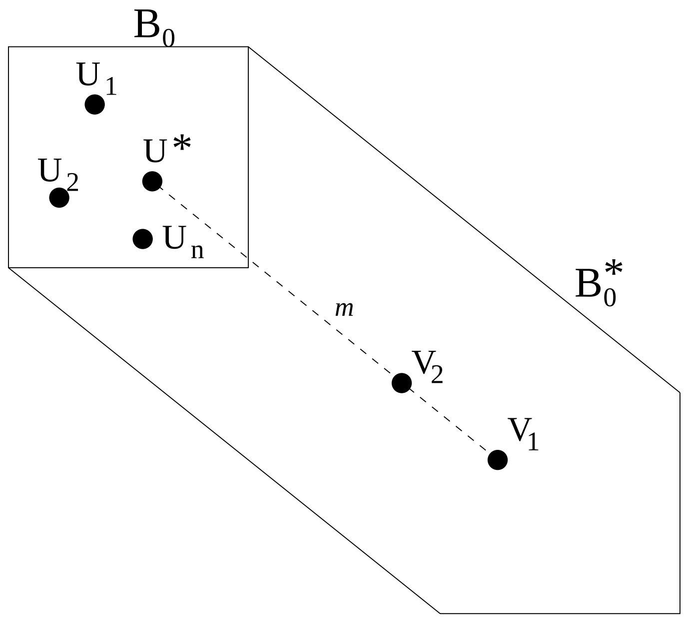

Figure 2.4: Defining the image of a Type B line

unique point of B_0 where the line m determined by the two points V_1 and V_2 meets B_0 (see Figure 2.4). Then, we can find scalars $f_i \in F$ such that

$$(0, \mathbf{u}^*) = \sum_{i=1}^{2n} f_i(0, \mathbf{u}_i).$$

A counting argument shows that there are exactly $q+1$ distinct Baer $2n$-spaces through B_0 and the point V_1, and there are exactly $q+1$ distinct Baer sublines of m through V_1 and U^*. We will now be able to use these $q+1$ Baer sublines to establish a representation for the Baer $2n$-spaces through B_0 and the point V_1. This will allow us to determine the exact spread element of $\mathcal{S}$ representing the point at infinity in the image of l under ϕ.

Consider the field elements $1, \beta, \beta^2, \ldots, \beta^q$ as distinct representatives for the multiplicative cosets of F^* in K^*. Then, for $0 \leq i \leq q$, the vectors

$$\beta^i(0, \mathbf{u}^*) + (1, \mathbf{v}_1)$$

all induce points on the line m and, moreover, each of these points together with V_1 and U^* determines a unique Baer subline of m through U^* and V_1. By finding the Baer subline containing the point V_2, we can determine the unique j so that the Baer subline of m through U^*, V_1 and the point induced by the vector

$$\beta^j(0, \mathbf{u}^*) + (1, \mathbf{v}_1)$$

contains the point V_2. Now recall that the Baer subspace B_0 lifts to a family of $q+1$ spread elements of Σ_0. As discussed in Lemma 2.1.8, these spread elements can be represented by

$$B_0 + \alpha^i B_0^q$$

for $0 \leq i \leq q$. We define the image of the line l under ϕ to be the points induced by vectors in the $2n$-space spanned by $B_0 + \alpha^j B_0^q$ and the image of V_1, where j is uniquely determined as above.

At this point it may seem as though the choice of the point V_1 is not arbitrary, and that changing the point V_1 may affect value of j. This, however, is not the case and we show this in detail now.

Lemma 2.3.2 *The image of a line under the map ϕ is well-defined.*

Proof: We first look at the Type A lines. Let l be a Type A line which is induced by all of the non-zero vectors contained in some n-space of Π^* which meets the hyperplane Π in an $(n-1)$-space of the mixed partition $\mathcal{P}$.

We suppose this element of the mixed partition contains the points induced by vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$. So, the line l is induced by vectors in the span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$, and $(1, \mathbf{u})$, where $(1, \mathbf{u})$ is a vector which induces some affine point. To show that ϕ is well defined on these lines, we need to show that the choice of $\mathbf{u}$ is arbitrary. That is, if the line induced by all non-zero vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_1)$ is the same as the line induced by all non-zero vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_2)$, then the images of these lines under ϕ are the same.

Suppose that we have two affine points induced by vectors $(1, \mathbf{u}_1)$ and $(1, \mathbf{u}_2)$ that lie on the same line which, in our model, meets the spread element determined by non-zero vectors in the K-linear span of $(0, \mathbf{s}_1), ..., (0, \mathbf{s}_n)$. Then there exist scalars $k_1, k_2, \ldots, k_n \in K$ such that

$$(1, \mathbf{u}_1) + k_1(0, \mathbf{s}_1) + k_2(0, \mathbf{s}_2) + \cdots + k_n(0, \mathbf{s}_n) = (1, \mathbf{u}_2),$$

which implies that

$$\mathbf{u}_1 + k_1\mathbf{s}_1 + k_2\mathbf{s}_2 + \cdots + k_n\mathbf{s}_n = \mathbf{u}_2.$$

If any of the k_i's is 0, then the argument will simplify. So, for generality, we assume that $k_i \neq 0$ for all i. Hence, we can find m_i so that $k_i^{q-1} = \alpha^{m_i}$ for each i. The image of the line induced by vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_1)$ certainly contains the points induced by the vectors

$$\begin{gathered}(0, \lambda_{m_1}(\mathbf{s}_1), \lambda_{m_1}(\alpha\mathbf{s}_1)) \\ (0, \lambda_{m_2}(\mathbf{s}_2), \lambda_{m_2}(\alpha\mathbf{s}_2)) \\ \vdots \\ (0, \lambda_{m_n}(\mathbf{s}_n), \lambda_{m_n}(\alpha\mathbf{s}_n))\end{gathered}$$

and

$$(1, Tr(\mathbf{u}_1), Tr(\alpha\mathbf{u}_1)).$$

But,

$$\begin{gathered}(0, \lambda_{m_i}(\mathbf{s}_i), \lambda_{m_i}(\alpha\mathbf{s}_i)) = \\ \left(0, \mathbf{s}_i + \alpha^{m_i}\mathbf{s}_i^q, \alpha\mathbf{s}_i + \alpha^{m_i-1}\mathbf{s}_i\right) \sim \\ (0, k_i\mathbf{s}_i + (k_i\mathbf{s}_i)^q, \alpha k_i\mathbf{s}_i + (\alpha k_i\mathbf{s}_i)^q)\end{gathered}$$

for each i. Hence, the first n vectors can be scalar multiplied by k_i respectively to obtain

$$(0, Tr(k_1\mathbf{s}_1), Tr(\alpha k_1\mathbf{s}_1))$$
$$(0, Tr(k_2\mathbf{s}_2), Tr(\alpha k_2\mathbf{s}_2))$$
$$\vdots$$
$$(0, Tr(k_n\mathbf{s}_n), Tr(\alpha k_n\mathbf{s}_n)).$$

Adding these vectors to

$$(1, Tr(\mathbf{u}_1), Tr(\alpha\mathbf{u}_1))$$

and recalling that the trace function is additive, we get

$$(1, Tr(\mathbf{u}_1 + k_1\mathbf{s}_1 + \cdots + k_n\mathbf{s}_n), Tr\left(\alpha(\mathbf{u}_1 + k_1\mathbf{s}_1 + \cdots + k_n\mathbf{s}_n)\right))$$
$$= (1, Tr(\mathbf{u}_2), Tr(\alpha\mathbf{u}_2)).$$

Hence, the image of the line induced by $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$ and $(1, \mathbf{u}_1)$ contains the image of the point induced by vector $(1, \mathbf{u}_2)$. Reversing the roles of $\mathbf{u}_1$ and $\mathbf{u}_2$ in this argument, we show that the image of the line induced by vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$, and $(1, \mathbf{u}_1)$ is the same as the image of the line induced by vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$, and $(1, \mathbf{u}_2)$. Hence, the map ϕ is well defined on Type A lines.

Now let l be a Type B line. Then l is induced by all of the non-zero vectors contained in some Baer $2n$-space B_0^* of Π^* which meets the hyperplane Π in a Baer subspace B_0 of $\mathcal{P}$. Let $U_1, U_2, \ldots, U_{2n}$, induced by vectors $(0, \mathbf{u}_1), (0, \mathbf{u}_2), \ldots, (0, \mathbf{u}_{2n})$, together with V_1 and V_2, induced by $(1, \mathbf{v}_1)$ and $(1, \mathbf{v}_2)$, be $2n + 2$ distinct points which determine this Baer $2n$-space B_0^* of $\mathcal{P}$. Here, the points U_i all lie in a Baer subspace of Π which is part of the mixed partition $\mathcal{P}$. Moreover, assume that the vectors $(0, \mathbf{u}_i)$ are properly scalar multiplied so that every point of B_0 is induced by a vector in the F-linear span of the vectors $(0, \mathbf{u}_i)$.

To show that the image of l under ϕ is well-defined, we need to show that the choice of points V_1 and V_2 is arbitrary. That is, if the line determined by the

U_i's, V_1 and V_2 is the same as the line determined by the U_i's, W_1 and W_2, then the images of these lines are also the same. Hence, we let V_1, V_2, W_1, and W_2 induced by vectors $(1, \mathbf{v}_1), (1, \mathbf{v}_2), (1, \mathbf{w}_1)$, and $(1, \mathbf{w}_2)$, respectively, be 4 such points. Then, there is a unique j such that $0 \leq j \leq q$ and

$$\beta^j \sum_{i=1}^{2n} f_i(0, \mathbf{u}_i) + (1, \mathbf{v}_1) = (1, \mathbf{v}_2).$$

Here, the vector $\sum_{i=1}^{2n} f_i(0, \mathbf{u}_i) = (0, \mathbf{u}^*)$ induces the point U^* as shown in Figure 2.4. Moreover, every point of B_0^* is in the F-linear span of

$$\left\{\beta^j(0, \mathbf{u}_i) : 0 \leq i \leq 2n\right\} \cup \left\{(1, \mathbf{v}_1)\right\}.$$

Hence, we can conclude that there are scalars $f_1^{(i)}$ and $f_2^{(i)}$ such that

$$\beta^j \sum_{i=1}^{2n} f_1^{(i)}(0, \mathbf{u}_i) + (1, \mathbf{v}_1) = (1, \mathbf{w}_1)$$

and

$$\beta^j \sum_{i=1}^{2n} f_2^{(i)}(0, \mathbf{u}_i) + (1, \mathbf{v}_1) = (1, \mathbf{w}_2).$$

To finish the argument, we show that the images of the points W_1 and W_2 are contained in the image of the line determined by the unique Baer $2n$-subspace B_0^* containing B_0 and the points V_1 and V_2.

Recalling the way images of Type B lines were defined earlier, the spread element representing the point at infinity on the line l is given by $B_0 + \alpha^j B_0^q$. Hence, the image of B_0^* under ϕ certainly contains the points induced by vectors

$$\begin{gathered}(0, \lambda_j(\mathbf{u}_1), \lambda_j(\alpha\mathbf{u}_1)) \\ (0, \lambda_j(\mathbf{u}_2), \lambda_j(\alpha\mathbf{u}_2)) \\ \vdots \\ (0, \lambda_j(\mathbf{u}_{2n}), \lambda_j(\alpha\mathbf{u}_{2n}))\end{gathered}$$

and

$$(1, Tr(\mathbf{v}_1), Tr(\alpha\mathbf{v}_1)).$$

Multiplying the first $2n$ vectors by β^j and noting that $\beta^j \cdot \alpha^j = \beta^{qj}$ gives us

$$(0, \beta^j \mathbf{u}_1 + (\beta^j \mathbf{u}_1)^q, \alpha\beta^j \mathbf{u}_1 + (\alpha\beta^j \mathbf{u}_1)^q)$$
$$(0, \beta^j \mathbf{u}_2 + (\beta^j \mathbf{u}_2)^q, \alpha\beta^j \mathbf{u}_2 + (\alpha\beta^j \mathbf{u}_2)^q)$$
$$\vdots$$
$$(0, \beta^j \mathbf{u}_{2n} + (\beta^j \mathbf{u}_{2n})^q, \alpha\beta^j \mathbf{u}_{2n} + (\alpha\beta^j \mathbf{u}_{2n})^q)\,,$$

and so all $2n+1$ vectors can be rewritten as

$$(0, Tr(\beta^j \mathbf{u}_1), Tr(\alpha\beta^j \mathbf{u}_1))$$
$$(0, Tr(\beta^j \mathbf{u}_2), Tr(\alpha\beta^j \mathbf{u}_2))$$
$$\vdots$$
$$(0, Tr(\beta^j \mathbf{u}_{2n}), Tr(\alpha\beta^j \mathbf{u}_{2n}))$$
$$(1, Tr(\mathbf{v}_1), Tr(\alpha \mathbf{v}_1))\,.$$

Since the trace function is addition, the F-linear span of these vectors certainly contains the vectors which induce the images of the projective points W_1 and W_2 under the map ϕ.

By reversing the roles of the V_i's and the W_i's, this shows that the images of lines as defined above are well-defined. ■

Lemma 2.3.3 *The image of a line of $\pi(\mathcal{P})$ under the map ϕ is a line of $\pi(\mathcal{S})$.*

Proof: To prove this statement we take an arbitrary line l of $\pi(\mathcal{P})$ and show that l^ϕ is a set of points of Σ_0^* which forms a $2n$-space meeting Σ_0 in an element of the spread $\mathcal{S}$. Let l, a Type A line, be the set of points induced by vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$, and $(1, \mathbf{u})$. To prove the lemma, we need to take any K-linear combination of these vectors (with the coefficient for $(1, \mathbf{u})$ being non-zero) and show that the image of the point induced by this vector, say $\mathbf{w}$, is in the image of the line l. Hence, we need to show that $\mathbf{w}$ is an F-linear combination of the vectors

$$\begin{gathered}(0, \lambda_{i_1}(\mathbf{s}_1), \lambda_{i_1}(\alpha\mathbf{s}_1)) \\ (0, \lambda_{i_2}(\mathbf{s}_2), \lambda_{i_2}(\alpha\mathbf{s}_2)) \\ \vdots \\ (0, \lambda_{i_n}(\mathbf{s}_n), \lambda_{i_n}(\alpha\mathbf{s}_n))\end{gathered}$$

and

$$(0, Tr(\mathbf{u}), Tr(\alpha\mathbf{u}))$$

where $0 \leq i_j \leq q$ for each j. But we have already shown that this is the case in the proof of Lemma 2.3.2. The argument is essentially the same for Type B lines. ∎

Lemma 2.3.4 *The map ϕ induces a bijection between the lines of $\pi(\mathcal{P})$ and $\pi(\mathcal{S})$.*

Proof: To prove that this map is a bijection, we simply show that the map on lines is onto. Let l be any line of $\pi(\mathcal{S})$ and let P and Q be two distinct points of l. Then by Lemma 2.3.1, there exist two distinct $2n$-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ such that $(1, Tr(\mathbf{u}), Tr(\alpha\mathbf{u}))$ and $(1, Tr(\mathbf{v}), Tr(\alpha\mathbf{v}))$ induce the projective points P and Q. Then, the two distinct points of $\pi(\mathcal{P})$ induced by $(1, \mathbf{u})$ and $(1, \mathbf{v})$ clearly determine a line m of $\pi(\mathcal{P})$ whose image under ϕ contains P and Q. Hence, $m^\phi = l$. ∎

Theorem 2.3.5 *The affine planes $\pi(\mathcal{P})$ and $\pi(\mathcal{S})$ are isomorphic.*

Proof: To prove the theorem, we show that ϕ is a bijection on the points and lines of the two planes, and that ϕ preserves incidence. The fact that ϕ is a bijection on the points is given by Lemma 2.3.1, and the bijection on the lines is given by Lemma 2.3.4. The incidence preserving property follows from the definition of the map. Hence $\pi(\mathcal{P})$ and $\pi(\mathcal{S})$ are isomorphic. ∎

In particular, Theorem 2.3.5 shows that the affine plane $\pi(\mathcal{P})$ constructed from a mixed partition $\mathcal{P}$ of $\mathcal{PG}(2n-1, q)$ is also a translation plane. In fact, $\pi(\mathcal{P})$

is isomorphic to $\pi(\mathcal{S})$, where $\mathcal{S}$ is the $(2n-1)$-spread of $\mathcal{PG}(4n-1,q)$ that is lifted from $\mathcal{P}$. Having thoroughly discussed the lifting process, we are now ready to look at some classical examples.

Chapter 3

THE CLASSICAL EXAMPLES

We are now ready to look at some special mixed partitions. In this chapter, we describe two different ways to construct a mixed partition which might be considered *classical*. The translation planes which they determine will be shown to be Desarguesian (or classical). Moreover, we will eventually show that these are the only partitions that lift to regular spreads, and hence generate Desarguesian planes. The proof of this fact requires much more knowledge of the associated automorphism groups. Hence, we save the proof for Chapter 4, where we give a thorough discussion of groups. For now, we give the two constructions.

3.1 The Regular Spread

The first classical example we will discuss is a mixed partition which contains no Baer subspaces. As was discussed in Section 1.4.2, the method of constructing translation planes directly from mixed partitions is exactly the same as the Bose/Andrè construction of Section 1.4.1 when one uses a spread for the mixed partition.

The classical example of a spread of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ can most easily be described by modeling projective spaces with finite fields. If we let $L = GF(q^{4n})$, and choose $H = GF(q^{2n})$ and $K = GF(q^2)$ as subfields of L, then L can be viewed as a $2n$-dimensional vector space over K. Similarly, H is an n-dimensional vector space over K. Hence, L serves as a model for Π and H models an $(n-1)$-dimensional subspace of Π. If we let η be a primitive element for L, then the subspaces

$$H, \eta H, \eta^2 H, ..., \eta^{q^{2n}} H$$

form a set of $q^{2n}+1$ distinct $(n-1)$-dimensional vector spaces, any two meeting only in the zero vector. Hence, they model a set of $q^{2n}+1$ skew $(n-1)$-spaces of Π and so form a spread. In fact, this model produces a spread $\mathcal{S}$ which is *regular* (the proof can be found in [8]). Thus, $\pi(\mathcal{S})$ is a Desarguesian affine plane by Theorem 1.4.7.

3.2 The "Classical" Mixed Partition

We will describe the other classical example of a mixed partition using group theory. The end result will be a partition of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ with exactly 2 distinct $(n-1)$-spaces and all of the remaining points partitioned into Baer subspaces. We will build the partition by considering one of its Baer subspaces $B_0 \cong \mathcal{PG}(2n-1, q)$. Consider B_0 modeled as $L = GF(q^{2n})$; that is, a $2n$-dimensional vector space over $F = GF(q)$, and let η be a primitive element of L. Then we can think of the points of B_0 as the field elements

$$1, \eta, \eta^2, ..., \eta^{q^{2n-1}+\cdots+q^2+q}.$$

Notice that we stop at $q^{2n-1}+\cdots+q^2+q$ since $\eta^{q^{2n-1}+\cdots+q^2+q+1}$ has order $q-1$, is therefore in F, and so represents the same projective point as the field element 1. Multiplication by η on these field elements induces a cyclic collineation which acts regularly on the points of $\mathcal{PG}(2n-1, q)$. This group is often called *Singer cycle*. We now try to find a matrix representation for this collineation.

To find the matrix representation for multiplication by η we need the companion matrix for the minimal polynomial for η. We can think of this polynomial as an irreducible polynomial of degree $2n$ over $GF(q)$, or as a product of two irreducible polynomials of degree n over $GF(q^2)$. Letting $P(x)$ be the minimal polynomial for η, we know that $P(x)$ has $\eta, \eta^q, \ldots, \eta^{q^{2n-1}}$ as its roots. Hence, the companion matrix for the minimal polynomial of η over $GF(q)$, which we will call A_q, is similar over $GF(q^{2n})$ to the matrix

$$\begin{bmatrix} \eta & & & \\ & \eta^q & & \\ & & \ddots & \\ & & & \eta^{q^{2n-1}} \end{bmatrix}.$$

Also, A_q is similar over $GF(q^2)$ to the block diagonal matrix

$$A_{q^2} = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & T \end{array}\right],$$

where S and T are $n \times n$ companion matrices for the minimal polynomials for η and η^q over $GF(q^2)$, respectively. Let $N \in GL(2n, q^2)$ be the matrix such that

$$N^{-1} A_q N = A_{q^2}.$$

The matrix A_q is an element of $GL(2n, q)$, but we think of A_q as an element of $GL(2n, q^2)$ and consider the induced action of A_q on the points of Π. Let V be the underlying vector space for the space Π, and let Θ_{A_q} and $\Theta_{A_{q^2}}$ be the collineations on Π induced by A_q and A_{q^2} respectively. Also, let $\langle \Theta_{A_q} \rangle$ and $\langle \Theta_{A_{q^2}} \rangle$ denote the cyclic groups generated by Θ_{A_q} and $\Theta_{A_{q^2}}$, respectively. We will make use of the following term.

Definition 3.2.1 *Let S and T be two geometric objects in a projective space $\mathcal{PG}(n, q)$. We say S and T are* **projectively equivalent** *if there exists a projectivity ϕ of $\mathcal{PG}(n, q)$ such that $S^\phi = T$.*

Lemma 3.2.2 *There exists a projectivity of Π mapping the point orbits under the group $\langle \Theta_{A_q} \rangle$ to the point orbits under $\langle \Theta_{A_{q^2}} \rangle$.*

Proof: Let $\mathcal{O}$ be the set of orbits of Π under the collineation group $\langle \Theta_{A_q} \rangle$, and let $O \in \mathcal{O}$. Then, for some vector $\mathbf{v}$, the orbit O is induced by vectors in $\{\mathbf{v}A_q^i : i = 0, 1, \ldots, q^{2n-1} + \cdots + q\}$. But $\{\mathbf{v}A_q^i\}$ is projectively equivalent to $\{\mathbf{v}A_q^i N\} = \{\mathbf{v}NA_{q^2}^i\}$, and $\{\mathbf{v}NA_{q^2}^i\}$ is the orbit containing the point induced by vector $\mathbf{v}N$ under the collineation group $\langle \Theta_{A_{q^2}} \rangle$. Since N induces a collineation on the points of Π, we have the result. In other words, the orbit partitions determined by the collineation groups $\langle \Theta_{A_q} \rangle$ and $\langle \Theta_{A_{q^2}} \rangle$ are projectively equivalent, and the collineation mapping one partition to the other is induced by the matrix N. ■

Because of Lemma 3.2.2, we can now work back and forth between the point orbits of $\langle \Theta_{A_q} \rangle$ and those of $\langle \Theta_{A_{q^2}} \rangle$. To start, we work with $\langle \Theta_{A_{q^2}} \rangle$ and examine the various orbits. Let $\mathbf{w}$ be a non-zero vector whose last n coordinates are all 0, and let $P_{\mathbf{w}}$ be the point of Π induced by vector $\mathbf{w}$. Then the orbit of $P_{\mathbf{w}}$ under $\langle \Theta_{A_{q^2}} \rangle$ will be completely determined by the $n \times n$ matrix S, which is a companion matrix for the minimal polynomial for η over $GF(q^2)$. Hence, $\langle \Theta_{A_{q^2}} \rangle$ is a Singer subgroup whose point orbit containing $P_{\mathbf{w}}$ is of length $q^{2n-2} + \cdots + q + 1$ and forms an $(n-1)$-space of Π. The same is true about non-zero vectors whose first n coordinates are all 0. Hence, $\langle \Theta_{A_{q^2}} \rangle$ creates two orbits (at least) of length $q^{2n-2} + \cdots + q + 1$ forming $(n-1)$-spaces of Π. We now examine the other orbits.

Since $\langle \Theta_{A_q} \rangle$ acting on $\mathcal{PG}(2n-1, q)$ forms a Singer cycle, we get immediately that one point orbit of $\langle \Theta_{A_q} \rangle$ on Π is the natural Baer subspace B_0; that is, the points induced by the vectors, all of whose homogeneous coordinates are in the subfield $GF(q)$. By Lemma 3.2.2, we get that one of the full point orbits of $\langle \Theta_{A_{q^2}} \rangle$, say O, is a Baer subspace as well. Fix a vector $\mathbf{v}$ which induces a point $P_{\mathbf{v}}$ in O. Then, since the points of O form a Baer subspace, we can find a set of $2n$ integers m_i such that

$$\left\{ \mathbf{v}A_{q^2}^{m_i} \right\}_{i=1}^{2n}$$

forms a basis for V.

Lemma 3.2.3 *Let* $\mathbf{w} \in V$ *be any vector whose first* n *coordinates are not all zero and whose last* n *coordinates are not all zero. Then the vectors in the set* $\left\{\mathbf{w}A_{q^2}^{m_i}\right\}_{i=1}^{2n}$ *form a basis for* V.

Proof: We only need to show that these vectors are linearly independent. Suppose

$$\sum_{i=1}^{2n} a_i \mathbf{w} A_{q^2}^{m_i} = \mathbf{0}.$$

We show that $a_i = 0$ for all i.

Write $\mathbf{w} = (\mathbf{x}, \mathbf{y})$ where $\mathbf{x}$ and $\mathbf{y}$ are n-dimensional vectors. Then we can write

$$\sum_{i=1}^{2n} a_i \mathbf{w} A_{q^2}^{m_i} = \left(\sum_{i=1}^{2n} a_i \mathbf{x} S^{m_i}, \sum_{i=1}^{2n} a_i \mathbf{y} T^{m_i} \right) = \mathbf{0}$$

or

$$\left(\mathbf{x} \sum_{i=1}^{2n} a_i S^{m_i}, \mathbf{y} \sum_{i=1}^{2n} a_i T^{m_i} \right) = \mathbf{0}.$$

Since neither $\mathbf{x}$ nor $\mathbf{y}$ is the zero vector, we can conclude that the matrices $\sum_{i=1}^{2n} a_i S^{m_i}$ and $\sum_{i=1}^{2n} a_i T^{m_i}$ are singular. But the matrix algebra generated by S (or T) over $GF(q^2)$ is isomorphic to the finite field $GF(q^2)$ (see [25], for instance). Therefore, the only singular matrix in this matrix algebra is the zero matrix. Referring back to the vector $\mathbf{v}$, we obtain

$$\mathbf{v} \sum_{i=1}^{2n} a_i A_{q^2}^{m_i} = \mathbf{0}$$

which means

$$\sum_{i=1}^{2n} a_i \mathbf{v} A_{q^2}^{m_i} = \mathbf{0}.$$

But the vectors in $\left\{\mathbf{v}A_{q^2}^{m_i}\right\}_{i=1}^{2n}$ are linearly independent. Hence, $a_i = 0$ for all i which proves the lemma. ■

Lemma 3.2.4 *Let* $\mathbf{w} \in V$ *be any vector whose first* n *coordinates are not all zero and whose last* n *coordinates are not all zero. Then the points in the orbit of the* *point induced by vector* $\mathbf{w}$ *under the collineation group* $\langle \Theta_{A_{q^2}} \rangle$ *form a Baer subspace of* Π.

Proof: We have shown that each orbit containing a point induced by a vector $\mathbf{w}$, whose first or last n coordinates are not all 0, contains a basis. Since $GL(2n, q^2)$ acts transitively on ordered bases, we let M be a matrix which maps the basis $\left\{\mathbf{v}A_{q^2}^{m_i}\right\}_{i=1}^{2n}$ to the basis $\left\{\mathbf{w}A_{q^2}^{m_i}\right\}_{i=1}^{2n}$. In particular, we assume that $\mathbf{v}A_{q^2}^{j}$ maps to $\mathbf{w}A_{q^2}^{j}$ for $j = m_1, m_2, \ldots, m_{2n}$. Let $P_{\mathbf{u}}$ be an arbitrary point in the orbit of $P_{\mathbf{v}}$ under $\langle \Theta_{A_{q^2}} \rangle$, say induced by the vector $\mathbf{u} = \mathbf{v}A_{q^2}^{k}$. Then, it is clear by linearity that the collineation induced by M maps the point $P_{\mathbf{u}}$ to the point induced by vector $\mathbf{w}A_{q^2}^{k}$. Hence, the orbit of the point induced by $\mathbf{w}$ is projectively equivalent to the orbit of the point induced by vector $\mathbf{v}$. Therefore, each full orbit of $\langle \Theta_{A_{q^2}} \rangle$ is a Baer subspace. ∎

Theorem 3.2.5 *The orbits of* $\langle \Theta_{A_{q^2}} \rangle$ *form a mixed partition of* $\mathcal{PG}(2n-1, q^2)$ *containing two copies of* $\mathcal{PG}(n-1, q^2)$ *and* $(q-1)(q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1)$ *Baer subspaces.*

Proof: By Lemmas 3.2.3 and 3.2.4, we know that there are exactly 2 distinct orbits forming $(n-1)$-spaces, and all remaining orbits are Baer subspaces. The total number of points of $\mathcal{PG}(2n-1, q^2)$ is

$$q^{4n-2} + q^{4n-4} + \cdots + q^2 + 1,$$

and so the total number of points covered by Baer subspaces is

$$\left(q^{4n-2} + q^{4n-4} + \cdots + q^{2n}\right) - \left(q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1\right)$$
$$= \left(q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1\right)\left(q^{2n} - 1\right)$$

which factors as

$$\left(q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1\right)(q-1)\left(q^{2n-1} + q^{2n-2} + \cdots + q + 1\right).$$

Hence, the number of Baer subspaces is exactly $(q-1)(q^{2n-2}+q^{2n-4}+\cdots+q^2+1)$.
■

We will use $\mathcal{P}_0$ to denote this mixed partition in subsequent chapters. We are now ready to examine some properties of lifted spreads which we can ascertain from the associated mixed partition.

Chapter 4

AUTOMORPHISM GROUPS

In this chapter, we cover some general results about group actions both on the mixed partition and on the lifted spread. The goal is to find as much information as possible about the full automorphism group of the spread, or equivalently, the translation complement. As discussed in Chapter 2, there is an embedding Δ of $\mathcal{PG}(2n-1, q^2)$ into $\mathcal{PG}(4n-1, q^2)$, namely

$$\mathbf{v} \mapsto (\mathbf{v}, \alpha\mathbf{v})$$

where $\mathcal{PG}(2n-1, q^2)$ maps to the space which we call Π. Recall that $\alpha = \beta^{q-1}$, where β is a primitive element for the field $GF(q^2)$. For the remainder of this chapter, we will assume that the space Π is the space which arises from this particular embedding. Also, we will use $\mathcal{S}$ to denote the spread which arises from the lifting of a mixed partition $\mathcal{P}$.

4.1 General Results

We start by examining a very natural group which arises from the lifting routine described in Chapter 2. For any regular spread (or more generally, any *normal* spread, see [26]) of $\mathcal{PG}(2n-1, q)$, there is always an associated *Bruck Kernel*. That is, a cyclic group which fixes each element of the regular spread and acts regularly on the points of any spread element. We have a similar group which arises from the construction given in Chapter 2. As discussed in Chapter 2, the Baer sublines coming from the lifted points of Π form a geometric 1-spread of Σ_0. This

1-spread will also admit a cyclic group which fixes each line. The group will act on the lifted spread by fixing each $(n-1)$-space coming from a lifted line, and permuting the $(n-1)$-spaces within a regulus coming from a lifted Baer subspace in a single orbit of length $q+1$. We will call this group κ and now develop a matrix representation for it.

Let

$$a_i = \frac{1-\alpha^{i+2}}{1-\alpha^2}$$
$$b_i = \frac{\alpha-\alpha^{i+1}}{1-\alpha^2}$$
$$d_i = \frac{\alpha^i-\alpha^2}{1-\alpha^2}$$

and let A_i be the $2n \times 2n$ matrix defined by

$$A_i = \begin{bmatrix} a_i & & & \\ & a_i & & \\ & & \ddots & \\ & & & a_i \end{bmatrix}.$$

We define B_i and D_i similarly, and consider the set of $4n \times 4n$ matrices

$$M_\kappa = \left\{ M_{\kappa,i} = \begin{bmatrix} A_i & B_i \\ -B_i & D_i \end{bmatrix} : i \in \{0,1,2,...,q\} \right\}.$$

Now, letting $\gamma = \beta^{\frac{q-1}{2}}$ and multiplying the i^{th} matrix, denoted by $M_{\kappa,i}$, in this set by γ^{-i} we can easily check that we obtain a matrix, all of whose entries are in F. Hence, these matrices can be viewed as elements of $PGL(4n,q)$.

Lemma 4.1.1 *The set of matrices M_κ forms a cyclic group of order $q+1$ under matrix multiplication.*

Proof: To show that this finite set forms a group, we simply show closure. The fact that these matrices are non-singular will follow from computations in the next lemma. By multiplying $M_{\kappa,i}$ and $M_{\kappa,j}$, we see that we only need to show that

$$a_i a_j - b_i b_j = a_{i+j},$$

$$a_i b_j + b_i d_j = b_{i+j},$$

and

$$-b_i b_j + d_i d_j = d_{i+j}.$$

With these identities in hand, we can easily see that first matrix $M_{\kappa,1}$ will be a generator for the cyclic group. To prove the first identity, note that

$$\begin{aligned}
&a_i a_j - b_i b_j \\
&= \left(\frac{1-\alpha^{i+2}}{1-\alpha^2}\right)\left(\frac{1-\alpha^{j+2}}{1-\alpha^2}\right) - \left(\frac{\alpha-\alpha^{i+1}}{1-\alpha^2}\right)\left(\frac{\alpha-\alpha^{j+1}}{1-\alpha^2}\right) \\
&= \frac{1}{(1-\alpha^2)^2}\left[\left(1-\alpha^{i+2}-\alpha^{j+2}+\alpha^{i+j+4}\right) - \left(\alpha^2-\alpha^{i+2}-\alpha^{j+2}+\alpha^{i+j+2}\right)\right] \\
&= \frac{1}{(1-\alpha^2)^2}\left[\left(1+\alpha^{i+j+4}\right) - \left(\alpha^2+\alpha^{i+j+2}\right)\right] \\
&= \frac{1}{(1-\alpha^2)^2}\left(1-\alpha^2\right)\left(1-\alpha^{i+j+2}\right) \\
&= \frac{1-\alpha^{i+j+2}}{1-\alpha^2} \\
&= a_{i+j}.
\end{aligned}$$

Next, observe that

$$\begin{aligned}
&a_i b_j + b_i d_j \\
&= \left(\frac{1-\alpha^{i+2}}{1-\alpha^2}\right)\left(\frac{\alpha-\alpha^{j+1}}{1-\alpha^2}\right) + \left(\frac{\alpha-\alpha^{i+1}}{1-\alpha^2}\right)\left(\frac{\alpha^j-\alpha^2}{1-\alpha^2}\right) \\
&= \frac{1}{(1-\alpha^2)^2}\left[(\alpha-\alpha^{j+1}-\alpha^{i+3}+\alpha^{i+j+3}) + (\alpha^{j+1}-\alpha^3-\alpha^{i+j+1}+\alpha^{i+3})\right] \\
&= \frac{1}{(1-\alpha^2)^2}\left[\alpha+\alpha^{i+j+3}-\alpha^3-\alpha^{i+j+1}\right] \\
&= \frac{1}{(1-\alpha^2)^2}(1-\alpha^2)(\alpha-\alpha^{i+j+1})
\end{aligned}$$

$$= \frac{\alpha - \alpha^{i+j+1}}{1-\alpha^2}$$
$$= b_{i+j}.$$

Finally,

$$-b_i b_j + d_i d_j$$
$$= -\left(\frac{\alpha - \alpha^{i+1}}{1-\alpha^2}\right)\left(\frac{\alpha - \alpha^{j+1}}{1-\alpha^2}\right) + \left(\frac{\alpha^i - \alpha^2}{1-\alpha^2}\right)\left(\frac{\alpha^j - \alpha^2}{1-\alpha^2}\right)$$
$$= \frac{1}{(1-\alpha^2)^2}\left[-(\alpha^2 - \alpha^{j+2} - \alpha^{i+2} + \alpha^{i+j+2}) + (\alpha^{i+j} - \alpha^{i+2} - \alpha^{j+2} + \alpha^4)\right]$$
$$= \frac{1}{(1-\alpha^2)^2}\left[-\alpha^2 - \alpha^{i+j+2} + \alpha^{i+j} + \alpha^4\right]$$
$$= \frac{1}{(1-\alpha^2)^2}(1-\alpha^2)(\alpha^{i+j} - \alpha^2)$$
$$= \frac{\alpha^{i+j} - \alpha^2}{1-\alpha^2}$$
$$= d_{i+j}.$$

Hence, $M_{\kappa,i}M_{\kappa,j} = M_{\kappa,i+j}$ and M_κ is closed. Again, to show M_κ is cyclic, we note that $M_{\kappa,1}$ will generate the group. In other words, $M_{\kappa,1}^n = M_{\kappa,n}$. Since $M_{\kappa,q+1} = I = M_{\kappa,0}$, M_κ forms a cyclic group of order $q+1$. ■

We will abuse notation and write M_κ for the group generated by the matrices in the set M_κ.

Lemma 4.1.2 *The cyclic group M_κ induces a collineation group κ of order $q+1$ acting on Σ_0 which stabilizes a lifted spread $\mathcal{S}$ by fixing each lifted $(n-1)$-space and moving the spread elements within the reguli coming from lifted Baer subspaces around in a single orbit.*

Proof: First we note that the group M_κ does not contain any scalar multiples of the identity matrix apart from $M_{\kappa,0} = I$. Hence, no two matrices of M_κ induce the same collineation on Σ_0. We will now show that the group acts regularly on the points of every line $\bar{l}_P$. More specifically, we show that for any point P, $P + \alpha^i P^q$ is moved to the point $P + \alpha^{i+j} P^q$ by application of the collineation κ_j induced by the matrix $M_{\kappa,j}$. Since every point of Σ_0 can be written in the form $P + \alpha^i P^q$ for a unique point P of Π and a unique i, where $0 \leq i \leq q$, this will also show that the matrices in M_κ are all non-singular.

Let $w + \alpha^i w^q$ be the first coordinate of a vector which induces the point $P + \alpha^i P^q$. Then, the $(2n+1)^{\text{st}}$ coordinate of $P + \alpha^i P^q$ is $\alpha w + \alpha^{i-1} w^q$ since $P \in \Pi$. We compute the first and $(2n+1)^{\text{st}}$ coordinates of a vector which induces the image point $(P + \alpha^i P^q)^{\kappa_j}$. This result can be extended to all of the remaining coordinates. Now, the first coordinate of the image $(P + \alpha^i P^q)^{\kappa_j}$ is

$$\begin{aligned}
&a_j(w + \alpha^i w^q) - b_j(\alpha w + \alpha^{i-1} w^q) \\
&= \frac{1-\alpha^{j+2}}{1-\alpha^2}(w + \alpha^i w^q) - \frac{\alpha - \alpha^{j+1}}{1-\alpha^2}(\alpha w + \alpha^{i-1} w^q) \\
&= w\left(\frac{1-\alpha^{j+2} - \alpha(\alpha - \alpha^{j+1})}{1-\alpha^2}\right) + \alpha^i w^q \left(\frac{1-\alpha^{j+2} - \frac{1}{\alpha}(\alpha - \alpha^{j+1})}{1-\alpha^2}\right) \\
&= w\left(\frac{1-\alpha^2}{1-\alpha^2}\right) + \alpha^i w^q \left(\frac{\alpha^j(1-\alpha^2)}{1-\alpha^2}\right) \\
&= w + \alpha^{i+j} w^q.
\end{aligned}$$

Also, the $(2n+1)^{\text{st}}$ coordinate of the image is

$$\begin{aligned}
&b_j(w + \alpha^i w^q) + d_j(\alpha w + \alpha^{i-1} w^q) \\
&= \frac{\alpha - \alpha^{j+1}}{1-\alpha^2}(w + \alpha^i w^q) + \frac{\alpha^j - \alpha^2}{1-\alpha^2}(\alpha w + \alpha^{i-1} w^q) \\
&= w\left(\frac{\alpha - \alpha^{j+1} + \alpha(\alpha^j - \alpha^2)}{1-\alpha^2}\right) + \alpha^i w^q \left(\frac{\alpha - \alpha^{j+1} + \frac{1}{\alpha}(\alpha^j - \alpha^2)}{1-\alpha^2}\right) \\
&= w\left(\frac{\alpha(1-\alpha^2)}{1-\alpha^2}\right) + \alpha^i w^q \left(\frac{\alpha^{j-1}(1-\alpha^2)}{1-\alpha^2}\right) \\
&= \alpha w + \alpha^{i+j-1} w^q.
\end{aligned}$$

Extending this result to the other coordinates, we have

$$(P + \alpha^i P^q)^{\kappa_j} = P + \alpha^{i+j} P^q$$

for any projective point P in Π, and the result follows. In other words, $\kappa_j : P + \alpha^i P^q \mapsto P + \alpha^{i+j} P^q$ for all points P of Π, where the exponents on α are read modulo $q+1$. ■

Corollary 4.1.3 *Let $\mathcal{P}$ be a mixed partition of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ which consists solely of $(n-1)$-spaces (i.e. no Baer subspaces), and let $\mathcal{S}$ be the $(2n-1)$-spread of $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$ which arises from the lifting of $\mathcal{P}$. Then the translation plane $\pi(\mathcal{S})$ determined by the spread $\mathcal{S}$ is at most n-dimensional over its kernel.*

Proof: The easiest proof follows from the correspondence established in Theorem 2.3.5. Since $\mathcal{P}$ is an $(n-1)$-spread in this case, the translation plane $\pi(\mathcal{P})$ is at most n-dimensional over its kernel, which implies that $\pi(\mathcal{S})$ is also at most n-dimensional over its kernel.

We can give another proof which relies only on the group relations. The group κ as described above is an automorphism of *any* lifted $(2n-1)$-spread, regardless of the mixed partition. Since κ fixes any $(2n-1)$-space of $\mathcal{S}$ which comes from a lifted $(n-1)$-space of $\mathcal{P}$, every element of $\mathcal{S}$ is fixed under κ. Hence, by the correspondence given in Section 1.4.1, the multiplicative group of the kernel of $\pi(\mathcal{S})$ contains at least $(q-1)(q+1) = q^2 - 1$ elements. As a result, $\pi(\mathcal{S})$ cannot be $2n$-dimensional over its kernel. Therefore, the largest possible dimension for $\pi(\mathcal{S})$ is n. ■

We are now ready to investigate the most important result of this chapter. That is, we can obtain a great deal of information about the automorphism group of a lifted spread simply by examining the automorphism group of the associated mixed partition.

Theorem 4.1.4 *Any linear automorphism of the mixed partition $\mathcal{P}$ will induce an automorphism on the lifted spread $\mathcal{S}$ by acting on the orbits of the group κ.*

Proof: Any automorphism of $\mathcal{P}$ must send Baer subspaces to Baer subspaces and $(n-1)$-spaces to $(n-1)$-spaces. Let $T = [t_{i,j}]$ be any $2n \times 2n$ matrix which induces an automorphism ϕ of $\mathcal{PG}(2n-1, q^2)$. Then, referring to the embedding Δ from Chapter 2, the $4n \times 4n$ matrix

$$\hat{T} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$$

will induce a collineation on Π in Σ which acts on the embedded mixed partition in the same way that ϕ acts on the mixed partition of $\mathcal{PG}(2n-1, q^2)$.

We abuse notation a bit and let ϕ be this automorphism acting on Π, the $(2n-1)$-space of Σ disjoint from Σ_0. Each point of Π corresponds to a unique Baer subline in Σ_0, that is, a κ-orbit in Σ_0. We look for a collineation Φ of Σ_0 which acts on these lines the same way that ϕ acts on the points of Π. In other words, if ϕ moves the $(n-1)$-space L_1 to the $(n-1)$-space L_2 in Π, then the corresponding automorphism of Σ_0, Φ, would move the spread element coming from the lifting of L_1 to the spread element coming from the lifting of L_2. Similarly, if ϕ moves the Baer subspace B_1 to another Baer subspace B_2, then Φ would move the entire regulus in Σ_0 determined by B_1 to the entire regulus determined by B_2.

Define the $4n \times 4n$ matrix M_T as follows:

$$M_T = (m_{i,j})$$

where

$$\begin{array}{ll} m_{i,j} \in F & \text{for } 1 \leq i, j \leq 4n \\ m_{i,j} + \alpha m_{i+2n,j} = t_{i,j} & \text{for } 1 \leq i, j \leq 2n \\ \frac{1}{\alpha} m_{i,j+2n} + m_{i+2n,j+2n} = t_{i,j} & \text{for } 1 \leq i, j \leq 2n. \end{array}$$

We can describe M_T in the following way. Write $T = T_1 + \alpha T_2 = \frac{1}{\alpha}T_3 + T_4$ where T_i has all of its entries in F for $i = 1, 2, 3, 4$. Then

$$M_T = \begin{bmatrix} T_1 & T_3 \\ T_2 & T_4 \end{bmatrix}.$$

We now define Φ to be the collineation acting on Σ induced by the matrix M_T. Notice that Φ leaves Σ_0 invariant. Using the notation from Chapter 2, we let l_P be the line determined by P and P^q, and we let $\bar{l}_P = l_P \cap \Sigma_0$. We now prove that Φ acts as desired. Namely, we show that the following diagram describes the action of ϕ and Φ:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P^{\phi} \\ \big\downarrow {\scriptstyle lift} & & \big\downarrow {\scriptstyle lift} \\ \bar{l}_P & \xrightarrow{\Phi} & \bar{l}_P^{\Phi} \end{array}$$

Let $(\mathbf{v}, \alpha\mathbf{v})$ be any vector which induces a point P in Π. Then the points on the Baer subline determined by P in Σ_0 are

$$P + \alpha^i P^q \text{ for } 0 \leq i \leq q.$$

The image of the point induced by $(\mathbf{v}, \alpha\mathbf{v})$ under ϕ is induced by the vector $(T\mathbf{v}, \alpha T\mathbf{v})$, which lifts to the Baer subline containing the points induced by vectors $(T\mathbf{v} + \alpha^i T^q \mathbf{v}^q, \alpha T\mathbf{v} + \alpha^{i-1} T^q \mathbf{v}^q)$, for $0 \leq i \leq q$. Moreover, the original point P induced by $(\mathbf{v}, \alpha\mathbf{v})$ lifts to the Baer subline induced by vectors $(\mathbf{v} + \alpha^i \mathbf{v}^q, \alpha\mathbf{v} + \alpha^{i-1}\mathbf{v}^q)$ for $0 \leq i \leq q$. Here we are using the fact that $\alpha^q = \alpha^{-1}$. But then, for any i,

$$\left(\mathbf{v} + \alpha^i \mathbf{v}^q, \alpha\mathbf{v} + \alpha^{i-1}\mathbf{v}^q\right) M_T$$

$$= \left((\mathbf{v} + \alpha^i \mathbf{v}^q)T_1 + (\alpha\mathbf{v} + \alpha^{i-1}\mathbf{v}^q)T_2, (\mathbf{v} + \alpha^i \mathbf{v}^q)T_3 + (\alpha\mathbf{v} + \alpha^{i-1}\mathbf{v}^q)T_4\right)$$
$$= \left(\mathbf{v}(T_1 + \alpha T_2) + \alpha^i \mathbf{v}^q \left(T_1 + \frac{1}{\alpha}T_2\right), \mathbf{v}(T_3 + \alpha T_4) + \alpha^i \mathbf{v}^q \left(T_3 + \frac{1}{\alpha}T_4\right)\right)$$
$$= \left(\mathbf{v}(T_1 + \alpha T_2) + \alpha^i \mathbf{v}^q (T_1 + \alpha T_2)^q, \alpha\mathbf{v}\left(\frac{1}{\alpha}T_3 + T_4\right) + \alpha^{i-1}\mathbf{v}^q \left(\frac{1}{\alpha}T_3 + T_4\right)^q\right)$$

$$= \left(\mathbf{v}T + \alpha^i \mathbf{v}^q T^q, \alpha \mathbf{v}T + \alpha^{i-1}\mathbf{v}^q T^q\right)$$

Hence, the collineation Φ moves the points on the Baer subline $\bar{l}_P$ to the points on the Baer subline $\bar{l}_{P\phi}$. In other words, $\bar{l}_P^\Phi = \bar{l}_{P\phi}$ for each point P in Π. Moreover, from the work above, we see that points at the i^{th} *level* stay at the i^{th} level under Φ; that is, if $P^\phi = Q$ for $P \in \Pi$, then $(P + \alpha^i P^q)^\Phi = (Q + \alpha^i Q^q)^\Phi$ for $0 \leq i \leq q$.

Assume now that ϕ leaves invariant the mixed partition $\mathcal{P}$ of Π, and let Φ be the induced collineation of Σ as above. Further, let S be any $(2n-1)$-space of the lifted spread $\mathcal{S}$ in Σ_0. If S comes from an $(n-1)$-space L of $\mathcal{P}$, then S is the union of Baer sublines $\bar{l}_P$ for all P on L. The image of these Baer sublines under Φ will be the Baer sublines $\bar{l}_{P\phi}$, whose union is the $(2n-1)$-space coming from the lifting of the $(n-1)$-space L^ϕ, which is another $(n-1)$-space of the mixed partition $\mathcal{P}$. Hence, the image of S under Φ is another element of $\mathcal{S}$. Now, let S be a $(2n-1)$-space in a regulus coming from a lifted Baer subspace B of $\mathcal{P}$. Then the ruling lines of the regulus are the lines $\bar{l}_P$ for all points P in B, and S meets each of these ruling lines in exactly one point. Now, the image under Φ of these Baer sublines will be the Baer sublines $\bar{l}_{P\phi}$, which form the set of transversal lines for another regulus, and the image of S, S^Φ, will meet each of these lines in exactly one point. But the only $(2n-1)$-spaces meeting each transversal line of a regulus in exactly one point are the elements of the regulus. Hence, S gets mapped under Φ to a $(2n-1)$-space in the regulus coming from the lifted Baer subspace B^ϕ of $\mathcal{P}$. This $(2n-1)$-space is also a member of $\mathcal{S}$.

Therefore, for any automorphism ϕ of the mixed partition of Π, we can find a *lifted* automorphism Φ whose action on the geometric 1-spread of Σ_0 is consistent with the action of ϕ on the points of Π. Since $(n-1)$-spaces move to $(n-1)$-spaces and Baer subspaces move to Baer subspaces under ϕ, this automorphism of the geometric 1-spread will preserve $\mathcal{S}$ as well, hence giving us an automorphism of $\mathcal{S}$. ■

We have shown that any linear automorphism of a mixed partition will necessarily induce an automorphism of the lifted spread. We can prove more about this relationship when the mixed partition is "proper". By *proper*, we mean that the mixed partition has at least one Baer subspace.

Lemma 4.1.5 *Let ϕ be an automorphism of a proper mixed partition $\mathcal{P}$ of Π and let Φ be the lifted automorphism of the associated spread $\mathcal{S}$ of Σ_0. Moreover, let S be a $(2n-1)$-space in a regulus which comes from the lifting of a Baer subspace of $\mathcal{P}$. Then, for any $k \in \kappa$, $S^{\Phi k} = S^{k\Phi}$.*

Proof: Let B be any Baer subspace of $\mathcal{P}$. Then, as discussed in Chapter 2, the elements of $\mathcal{S}$ which come from the lifting of B can be written as $B_i = B + \alpha^i B^q$ as i varies between 0 and q. Let $S = B_i$ for some fixed i. From the proof of Theorem 4.1.4, it is clear that

$$B_i^{\Phi} = \left(B + \alpha^i B^q\right)^{\Phi} = B^{\phi} + \alpha^i (B^{\phi})^q = (B^{\phi})_i.$$

Now, let k be any arbitrary element of κ. From Lemma 4.1.2, we know that k maps $B_i = B + \alpha^i B^q$ to $B + \alpha^j B^q = B_j$ for some j. In fact, we know that $P + \alpha^i P^q$ gets mapped to $P + \alpha^j P^q$ for *all* points P of Π. Hence,

$$B_i^{\Phi k} = (B^{\phi})_i^k = B_j^{\phi}$$

and

$$B_i^{k\Phi} = (B_j)^{\Phi} = B_j^{\phi},$$

which proves the lemma. ∎

With this lemma in hand, we can show that the group κ commutes with any other lifted linear automorphism group, say $\bar{G}$, of a lifted spread, as long as the associated mixed partition is proper. This will allow us to form the group $\kappa \times \bar{G}$ as a subgroup of $Aut(\mathcal{S})$.

Theorem 4.1.6 *Let G be a linear automorphism group of a proper mixed partition $\mathcal{P}$ of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ which lifts to the automorphism group $\bar{G}$ acting on the lifted spread $\mathcal{S}$ of $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$. Then $\bar{G}$ centralizes κ in $PGL(4n, q)$. Moreover, κ and $\bar{G}$ intersect in only the identity collineation.*

Proof: Let P be a point of the lifted spread $\mathcal{S}$ which lies in a $(2n-1)$-space S of a regulus which comes from the lifting of a Baer subspace of $\mathcal{P}$. Let l be the line of the geometric 1-spread associated with the lifting of $\mathcal{P}$ which contains the point P. Then, l is fixed under the group κ (in fact, l is a κ-orbit). Hence, letting g be any arbitrary element of the group $\bar{G}$, $P \in S \cap l$ implies $P^k \in S^k \cap l$, and so $P^{kg} \in S^{kg} \cap l^g$. On the other hand, $P \in S \cap l$ implies $P^g \in S^g \cap l^g$, and so $P^{gk} \in S^{gk} \cap l^g$. From Lemma 4.1.5, we conclude that $P^{kg} = P^{gk}$ for every point P in S. Since S was an arbitrary $(2n-1)$-space which comes from the lifting of a Baer subspace of $\mathcal{P}$, we know that there are at least $q+1$ such spaces S. Since $q+1 \geq 2$, and any 2 disjoint $(2n-1)$-spaces of Σ_0 will span all of Σ_0, we conclude that the collineation kg has the same action as gk on all of the points of Σ_0. Here we use the fact that g and k are both linear. Hence, $kg = gk$ for any $k \in \kappa$ and $g \in \bar{G}$.

The only automorphism of the group $\bar{G}$ which fixes each κ orbit is the identity. This follows from the way the lifting of an automorphism of $\mathcal{P}$ is defined in Theorem 4.1.4; that is, the group $\bar{G}$ acts on the κ orbits in the same way that the group G acts on the points of Π. Hence, κ and $\bar{G}$ intersect in only the identity collineation. ■

Our last theorem on the relationship between collineation groups of a mixed partition $\mathcal{P}$ and its associated lifted spread $\mathcal{S}$ will help in future kernel arguments. We have already shown in Theorem 4.1.4 that every collineation of $\mathcal{P}$ lifts to a collineation of $\mathcal{S}$. We now show that some special collineations of $\mathcal{S}$ *must* come from the lifting of a collineation of $\mathcal{P}$.

Theorem 4.1.7 *Let $\mathcal{P}$ be a proper mixed partition of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ and let $\mathcal{S}$ be its associated lifted spread of $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$. Suppose that ψ_0 is a linear collineation of Σ_0 leaving each element of $\mathcal{S}$ invariant. Then there exists a linear collineation of Π leaving every element of $\mathcal{P}$ invariant which lifts to the automorphism ψ_0.*

Proof: Using the same notation as in Chapter 2, we consider a Baer subspace B_0 of the mixed partition of Π. Let $P_0, P_1, \ldots, P_{2n}$ be $2n+1$ points in general position in B_0. Then, from Lemma 2.1.6, the $2n+1$ lines l_{P_i} have the property that any $2n$ of them will generate the whole space $\Sigma = \mathcal{PG}(4n-1, q^2)$. From this, it follows from Lemma 2.1.7 that if Q_0 is any point of l_{P_0}, then there is exactly one $(2n-1)$-space of Σ through Q_0, say Π_{Q_0}, which meets each of the other lines in exactly one point each. Hence, we can build a regulus $\mathcal{R}$ from these transversal lines and the associated $(2n-1)$-spaces through them. The transversal lines of this regulus will be all of the lines l_P as P varies over Π, and Π will be one of the elements of the regulus.

We now extend ψ_0 to an automorphism ψ of the whole space Σ and try to determine the action which ψ induces on Π. Note that this extension is not unique. We simply let ψ be *any* extension of ψ_0. Contained in the above regulus $\mathcal{R}$ is the *sub*-regulus $\mathcal{R}_0$ of Σ_0 which comes from the lifting of B_0. Since each $(2n-1)$-space in $\mathcal{S}$ is fixed under ψ by the hypothesis of the theorem, the images under ψ of the $2n+1$ lines l_{P_i} must contain sublines which are ruling lines of $\mathcal{R}_0$. In other words, if a regulus $\mathcal{R}_0$ is fixed under some collineation ψ, then the transversal lines of $\mathcal{R}_0$ are fixed under ψ as a set of lines. But, as in Lemma 2.1.7, the $(2n-1)$-spaces of $\mathcal{R}$ are uniquely determined by these lines. We conclude that $\mathcal{R}$ must also be fixed under ψ. It is well known (see [17]) that the linear automorphism group for such a regulus is isomorphic to $PGL(2, q^2) \times PGL(2n, q^2)$. Now, $q+1$ of the elements of $\mathcal{R}$, say $R_1, ..., R_{q+1}$, meet Σ_0 in Baer subspaces; these are the elements of $\mathcal{R}_0$ in

Σ_0. Since the elements of $\mathcal{R}_0$ are fixed, and the whole regulus $\mathcal{R}$ is fixed, each of $R_1, ..., R_{q+1}$ must also be fixed. Hence, $q+1$ of the elements of $\mathcal{R}$ are fixed under ψ. Since $q \geq 2$, we know that at least 3 of the elements of $\mathcal{R}$ are fixed under ψ. It follows that all of the elements of $\mathcal{R}$ are fixed under ψ. In particular, this means that Π is fixed under ψ. More importantly, we note that ψ acts as a collineation on the set of lines l_P for P in Π and, moreover, if $P^\psi = Q$, then $l_P^\psi = l_Q^\psi$.

Therefore, ψ induces an automorphism of Σ which fixes Π. So now we concentrate on the action which ψ induces on the points of Π. Since ψ restricted to Σ_0 fixes all of the elements of the spread $\mathcal{S}$, ψ must fix each $(n-1)$-space and each Baer subspace of the mixed partition of Π. Since the Baer sublines $\bar{l}_P$, for $P \in \Pi$, form the orbits of the group κ, we see that ψ acts on the κ orbits in the same fashion as a lifted automorphism of the mixed partition $\mathcal{P}$. Hence, any collineation of $\mathcal{S}$ which fixes all of the spread elements must come from a lifted automorphism of $\mathcal{P}$ fixing all of the elements of $\mathcal{P}$. ■

We note that the result above is only true about linear automorphisms. However, from the discussion on the kernel given in Section 1.4.1, we know that any automorphism which fixes each element of a spread must be linear.

The last general group theory result will also be useful in the computation of the kernel of some of the planes described in subsequent chapters. For this, we restrict to projective 3-space. It is well known (see [18]) that a 1-regulus and one additional skew line of $\mathcal{PG}(3, q^2)$ uniquely determine a regular spread. In addition, $PGL(4, q^2)$ is 3-transitive on lines. From these two results, we can prove the following:

Theorem 4.1.8 *Let $\mathcal{L}$ be a set of four distinct lines of a regular spread $\mathcal{S}$ of $\mathcal{PG}(3, q^2)$, which are not all contained in the same regulus, and suppose that all four lines are fixed line-wise by a linear collineation ϕ. Then ϕ is an element of the Bruck Kernel of the regular spread $\mathcal{S}$.*

Proof: Since $PGL(4, q^2)$ is 3-transitive on lines, we assume $\mathcal{L}$ contains the lines

$$l_\infty = \langle (0,0,1,0), (0,0,0,1) \rangle$$

$$l_0 = \langle (1,0,0,0), (0,1,0,0) \rangle$$

and

$$l_1 = \langle (1,0,0,1), (0,1,1,0) \rangle.$$

Let $\mathcal{R}$ be the unique regulus determined by these three lines. It is not hard to check that the quadratic form associated with this regulus is given by $x_0x_2 - x_1x_3 = 0$ when we write vectors as (x_0, x_1, x_2, x_3). Hence, the lines of this regulus, apart from l_∞, are given by

$$l_k = \langle (1,0,0,k), (0,1,k,0) \rangle$$

for $k \in K$. Also, by the conditions of the theorem, the fourth line of $\mathcal{L}$, say m, is not in $\mathcal{R}$.

Let $T = [t_{i,j}]$ be any matrix which induces ϕ. Then, since l_∞ is fixed, $t_{i,j} = 0$ for $i = 3, 4$ and $j = 1, 2$. Additionally, since l_0 is fixed, $t_{i,j} = 0$ for $i = 1, 2$ and $j = 3, 4$. Finally, since l_1 is fixed, we get that T has the form

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{bmatrix}$$

for some choice of a, b, c, d with $ad - bc \neq 0$. From the form of T, it should be clear that ϕ fixes all of the lines of $\mathcal{R}$. Since $\mathcal{S}$ is regular, $\mathcal{R} \subset \mathcal{S}$ and hence the line m of $\mathcal{S}$ must be skew to each line of $\mathcal{R}$. That is, the $q^2 + 1$ points of m are all distinct from the $(q^2+1)^2$ points covered by $\mathcal{R}$.

Let G be the group of all collineations induced by matrices of the above form. Now, let $\mathbf{v} = (v_0, v_1, v_2, v_3)$ be any vector which induces a projective point P not on any line of $\mathcal{R}$, and suppose that ϕ fixes the point P. Then

$$\mathbf{v}T = (av_0 + cv_1, bv_0 + dv_1, dv_2 + bv_3, cv_2 + av_3) = k\mathbf{v}$$

for some scalar $k \in GF(q)$. Hence,

$$av_0 + cv_1 = kv_0$$

and

$$cv_2 + av_3 = kv_3.$$

This system has a unique solution for a/k and c/k if and only if $v_0v_2 - v_1v_3 \neq 0$. This is certainly the case since, if not, then the vector v would satisfy the quadratic form, and hence the point P would be on one of the lines of $\mathcal{R}$. So, the unique solution must be the obvious solution: $a/k = 1$ and $c/k = 0$. Applying this result to the similar equations involving b and d, we get $b/k = 0$ and $d/k = 1$. Thus, we have shown that the G-stabilizer of any one point not on a line of $\mathcal{R}$ is simply the identity. The number of such points is $q^6 + q^4 + q^2 + 1 - (q^2 + 1)^2 = q^6 - q^2$, and the number of such matrices is

$$\frac{(q^4 - 1)(q^4 - q^2)}{q^2 - 1} = q^6 - q^2.$$

Since the group G has the same size as the number of points on which it acts, and any one point stabilizer is the identity, the group of all such matrices acts regularly on the points not covered by the lines of the regulus $\mathcal{R}$. Therefore, there are only $q^2 + 1$ elements of G which can fix the line m. These are exactly the elements of the associated Bruck Kernel. ■

This previous result will prove useful for the following reason. Suppose we have a mixed partition which contains four skew lines from a regular spread as described in the hypothesis of the theorem. Then, the only automorphism which could possibly fix each member of the mixed partition is an element of the associated Bruck Kernel. If we can rule out the nonidentity elements of the Bruck Kernel as possible automorphisms, we would show that the only automorphism fixing each member of the mixed partition is the identity. From Theorem 4.1.7, this would imply that the only automorphism fixing each member of the lifted spread is the identity. Hence, we would have a spread which produces a translation plane of order q^4 which is 4-dimensional over its kernel.

4.2 Regular Spreads

In this final section of the chapter, we will prove a classification result about regular spreads. We show that the only proper mixed partition of Π which lifts to a regular spread is the classical mixed partition discussed in Chapter 3.

Theorem 4.2.1 *Let $\mathcal{P}$ be a proper mixed partition of $\Pi \cong \mathcal{PG}(2n-1, q^2)$ that gives rise to a regular spread $\mathcal{S}$ of $\Sigma_0 \cong \mathcal{PG}(4n-1, q)$ via the geometric lifting of Chapter 2. Then $\mathcal{P}$ is projectively equivalent to the classical mixed partition $\mathcal{P}_0$ described in Section 3.2.*

Proof: Suppose that the partition $\mathcal{P}$ is proper. Then, by Theorem 4.1.7, we know that any automorphism which fixes each member of the spread $\mathcal{S}$ is a lifted automorphism of the mixed partition $\mathcal{P}$. Moreover, since $\mathcal{S}$ is regular, we know that the group Φ fixing each element of $\mathcal{S}$ is a cyclic group of order $q^{2n-1} + \cdots + q + 1$ which acts regularly on the points of any spread element. Applying Theorem 4.1.7, we know that there is a collineation group ϕ of Π which lifts to Φ, and this group ϕ is also a cyclic group of order $q^{2n-1} + \cdots + q + 1$. Let $P + \alpha^i P^q$, for some i, be a point of Σ_0 which lies in a $(2n-1)$-space which comes from the lifting of a Baer subspace in Π. From the proof of Theorem 4.1.4, we know that Φ acts on the point $P + \alpha^i P^q$ in the same way that ϕ acts on the point P. Since $(P + \alpha^i P^q)^\Phi$ is a $(2n-1)$-space of Σ_0, we know that P^ϕ is a Baer subspace B' of Π. Since ϕ acts regularly on the points of B', ϕ acts as a Singer cycle for B'. We now consider the other point orbits of ϕ on Π.

From Lemma 2.12 of [21] we know that any two Baer subspaces of Π are projectively equivalent. In particular, there exists a linear collineation ρ such that $B'^\rho = B_0$ where B_0 is the Baer subspace consisting of all the points induced by vectors all of whose homogeneous coordinates are in the subfield $GF(q)$. Now the group $\rho^{-1}\phi\rho$ acts as a Singer cycle for B_0. From Theorem 3.2.5 and its preceding lemmas, we know that the orbits of the group $\rho^{-1}\phi\rho$ form the classical mixed partition $\mathcal{P}_0$. It follows that $\mathcal{P}^\rho = \mathcal{P}_0$ and so $\mathcal{P}$ is projectively equivalent to the classical mixed partition $\mathcal{P}_0$. ■

After the discussion of the next chapter, we will see that there are only two types of mixed partitions which lift to regular spreads. The first, of course, is $\mathcal{P}_0$. The other type will arise when we look at lifting *non*-proper mixed partitions, namely spreads.

Chapter 5

EQUIVALENT SPREADS

As seen in Theorem 1.4.8, it is possible for two distinct spreads of $\mathcal{PG}(2n-1,q)$ to give rise to isomorphic translation planes. It is also possible for two spreads of projective spaces of distinct dimensions to give rise to isomorphic translation planes. In this chapter we will prove a relationship between $(n-1)$-spreads of $\mathcal{PG}(2n-1,q^r)$ and $(nr-1)$-spreads of $\mathcal{PG}(2nr-1,q)$ which give rise to isomorphic translation planes via the Bose/Andrè construction. We will start by following the construction given in Chapter 2, but with q^2 generalized to q^r for any $r \geq 2$.

5.1 Lifting a Spread

Let K be the finite field $GF(q^r)$ with primitive element β, and let F be its subfield $GF(q)$. Also, as before, we let $\alpha = \beta^{q-1}$. Let $\Gamma = \mathcal{PG}(2n-1,q^r)$ with underlying vector space W over K, and let $\Sigma = \mathcal{PG}(2nr-1,q^r)$ with underlying vector space V over K. Also, let $\Sigma_0 \cong \mathcal{PG}(2nr-1,q)$ be an r^{th}-root sub-$(2nr-1)$-space of Σ in standard position. That is, Σ_0 consists of those points of Σ induced by vectors in V all of whose components lie in the subfield F of K. For the remainder of this section we will use the term "subgeometry" to mean an r^{th}-root space. Also, note that our notation is consistent with that of Chapter 2. In Chapter 2 we were simply focusing on the special case when $r = 2$.

Just as in Chapter 2, we have a semi-linear map which arises from the Frobenius automorphism of the field K. We call this map *conjugation* and the images of a point under this map are called *conjugates*. This collineation has order r rather than 2, but still maintains many of the properties proven earlier.

Lemma 5.1.1 *The points of Σ which are fixed under conjugation are precisely the points of Σ_0.*

Proof: Clearly the points of Σ_0 are fixed under conjugation. We show that no other point can be fixed. Let P be a point of $\Sigma \backslash \Sigma_0$ and let $\mathbf{v} = (v_i)$ be some non-zero vector which induces the point P. If $P^q = P$, then $\mathbf{v}^q \sim \mathbf{v}$ and there is a $k \in K$ such that $\mathbf{v}^q = k\mathbf{v}$. Thus,

$$\mathbf{v}^{q^2} = (\mathbf{v}^q)^q = (k\mathbf{v})^q = k^q\mathbf{v}^q = k^{q+1}\mathbf{v}$$

and consequently,

$$\mathbf{v}^{q^3} = \left(\mathbf{v}^{q^2}\right)^q = \left(k^{q+1}\mathbf{v}\right)^q = k^{q^2+q}\mathbf{v}^q = k^{q^2+q+1}\mathbf{v}.$$

Continuing in this fashion, we obtain

$$\mathbf{v} = \mathbf{v}^{q^r} = k^{q^{r-1}+q^{r-2}+\cdots+q^2+q+1}\mathbf{v}.$$

This means that the field element k must be some power of α. Let $k = \alpha^t$ for some t. Then $v_i^q = \alpha^t v_i = \beta^{t(q-1)}v_i$ for all i. But this means that $v_i = f_i\beta^t$, where $f_i \in F$ and $f_i = 0$ precisely when $v_i = 0$. So, $(v_i) = (\beta^t f_i) = \beta^t(f_i) \sim (f_i)$. Hence, the projective point P is induced by a vector which has all of its coordinates in F, and so lies in the subgeometry Σ_0. ■

We will again need a representation for the space $\Pi \cong \mathcal{PG}(2n-1, q^r)$ which is disjoint from Σ_0. We define an embedding similar to the one in Chapter 2. Consider the K-linear transformation $\Theta : W \to V$ given by

$$\mathbf{v} \mapsto (\mathbf{v}, \alpha\mathbf{v}, \alpha^2\mathbf{v}, \ldots, \alpha^{r-1}\mathbf{v}),$$

where $V = \bigoplus_{i=1}^{r} W$. One can easily check that $ker(\Theta) = \{\overline{0}\}$ and hence Θ induces an embedding of Γ into Σ. By a slight abuse of notation, we also let Θ denote the induced embedding. For the construction we are leading up to, we need $\Gamma^{\Theta} \cap \Sigma_0 = \emptyset$. Using the same argument as before, this is clearly the case. Let Π be the image of Γ in Σ.

We are now in a position to generalize the construction of the geometric spread. In the case when $r = 2$, the Frobenius map is an order 2 collineation, so we only have one conjugate of a point. However, when $r > 2$, we have more than just one conjugate. We use all of the conjugates to construct a geometric spread of Σ_0. We start with some theory about the space Π.

Lemma 5.1.2 *The* $(2n-1)$*-spaces* $\Pi, \Pi^q, \Pi^{q^2}, \ldots, \Pi^{q^{r-1}}$ *together generate the entire* $(2nr-1)$*-space* Σ.

Proof: Consider the following set of $2n$ vectors which induce points of Π. Here, we use ";" to separate $2n$-tuples:

$$
\begin{gathered}
(1,0,0,\ldots,0;\alpha,0,0,\ldots,0;\alpha^2,0,0,\ldots,0;\ldots;\alpha^{r-1},0,0,\ldots 0)\\
(0,1,0,\ldots,0;0,\alpha,0,\ldots,0;0,\alpha^2,0,\ldots,0;\ldots;0,\alpha^{r-1},0,\ldots 0)\\
(0,0,1,\ldots,0;0,0,\alpha,\ldots,0;0,0,\alpha^2,\ldots,0;\ldots;0,0,\alpha^{r-1},\ldots 0)\\
\vdots\\
(0,0,0,\ldots,1;0,0,0,\ldots,\alpha;0,0,0,\ldots,\alpha^2;\ldots;0,0,0,\ldots \alpha^{r-1}).
\end{gathered}
$$

In other words, letting $\mathbf{e}_i$ be the vector with 1 in the i^{th} position and 0 everywhere else, we consider the points $(\mathbf{v}, \alpha\mathbf{v}, \alpha^2\mathbf{v}, \ldots, \alpha^{r-1}\mathbf{v})$ as $\mathbf{v}$ varies over the vectors $\mathbf{e}_i$ for $i \in \{1, 2, \ldots, 2n\}$. These $2n$ vectors are certainly linearly independent. We now consider the set of $2nr$ vectors consisting of all of the vectors above together with all of the conjugates of all of these vectors. The claim is that all of these vectors together form a linearly independent set. To show this, we take a linear combination of all $2nr$ vectors and set it equal to zero. Let $k_{i,j}$ be the coefficient in front of the j^{th} conjugate of the vector coming from $\mathbf{e}_i$. By looking at coordinates $i, 2n+i, 4n+i, \ldots, 2n(r-1)+i$, for $1 \leq i \leq 2n$, we get the following system of equations:

$$\begin{gathered}
k_{i,0} + k_{i,1} + k_{i,2} + \cdots + k_{i,r-1} = 0 \\
\alpha k_{i,0} + \alpha^{q} k_{i,1} + \alpha^{q^2} k_{i,2} + \cdots + \alpha^{q^{r-1}} k_{i,r-1} = 0 \\
\alpha^2 k_{i,0} + \alpha^{2q} k_{i,1} + \alpha^{2q^2} k_{i,2} + \cdots + \alpha^{2q^{r-1}} k_{i,r-1} = 0 \\
\vdots \\
\alpha^{r-1} k_{i,0} + \alpha^{(r-1)q} k_{i,1} + \alpha^{(r-1)q^2} k_{i,2} + \cdots + \alpha^{(r-1)q^{r-1}} k_{i,r-1} = 0.
\end{gathered}$$

This system of linear equations has a unique solution if the determinant of the coefficient matrix is non-zero. This matrix is given by

$$\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha & \alpha^{q} & \alpha^{q^2} & \cdots & \alpha^{q^{r-1}} \\
\alpha^2 & \alpha^{2q} & \alpha^{2q^2} & \cdots & \alpha^{2q^{r-1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha^{r-1} & \alpha^{(r-1)q} & \alpha^{(r-1)q^2} & \cdots & \alpha^{(r-1)q^{r-1}}
\end{bmatrix}.$$

Hence, we have a Vandermonde matrix whose second row consists of the roots of the minimal polynomial for α. The determinant of this matrix (see [14]) is given by

$$\prod_{x<y} \left(\alpha^y - \alpha^x\right),$$

where $0 \leq x < y \leq q$. The only power of α equal to 1 in this specified range is α^0, which immediately gives us that $\alpha^x \neq \alpha^y$ when $x \neq y$. Hence, this determinant is non-zero and the system has a unique solution. Therefore $k_{i,j} = 0$ for $0 \leq j \leq r-1$, implying that all of the $k_{i,j}$'s are 0, and the vectors are linearly independent.

Since these vectors form a linearly independent set, we immediately see that the projective spaces $\Pi, \Pi^q, \ldots, \Pi^{q^{r-1}}$ generate the entire space Σ. ■

Lemma 5.1.3 *For any point $P \in \Pi$, the points $P, P^q, P^{q^2}, \ldots, P^{q^{r-1}}$ generate an $(r-1)$-space of Σ which meets the subgeometry Σ_0 in a subgeometry of dimension $(r-1)$.*

Proof: The fact that the r points generate an $(r-1)$-space follows immediately from Lemma 5.1.2. We only need to show that the space meets Σ_0 in a subgeometry of dimension $(r-1)$.

Let P be any point of Π and let $\mathbf{v}$ be any vector which induces P. Consider the points P_i induced by the vectors $\lambda_i(\mathbf{v})$ defined as follows:

$$\lambda_i(\mathbf{v}) = \mathbf{v} + \alpha^i \mathbf{v}^q + \alpha^{i(q+1)}\mathbf{v}^{q^2} + \alpha^{i(q^2+q+1)}\mathbf{v}^{q^3} + \cdots + \alpha^{i(q^{r-2}+\cdots+q^2+q+1)}\mathbf{v}^{q^{r-1}}$$

for $0 \leq i \leq q^{r-1} + \cdots + q^2 + q + 1$. By applying the Frobenius map and multiplying by α^i, we see that $\alpha^i \lambda_i(\mathbf{v})^q = \lambda_i(\mathbf{v})$ and so $P_i^q = P_i$ for each i. Here we use the fact that the order of α is $q^{r-1} + \cdots + q^2 + q + 1$. Therefore, we have at least $q^{r-1} + \cdots + q^2 + q + 1$ points in the space spanned by P and all of its conjugates that also lie in the space Σ_0. Since an $(r-1)$-space of Σ could not possibly meet Σ_0 in any more points, we have the result. ■

For convenience, we let g_P be the intersection of the space spanned by $P, P^q, \ldots P^{q^{r-1}}$ with Σ_0. Note that in Chapter 2 we let $\bar{l}_P$ denote this intersection.

Corollary 5.1.4 *Let P and Q be two distinct points of Π. Then g_P and g_Q do not intersect.*

Proof: By Lemma 5.1.2, the points P, Q, and all of their conjugates form a linearly independent set. Hence, the space spanned by the conjugates of P and the space spanned by the conjugates of Q cannot meet. ■

We can now generate a geometric $(r-1)$-spread of Σ_0 the same way that we did in the $r = 2$ case from Chapter 2. For each point P of Π, we look at the subgeometry of dimension $(r-1)$, say g_P, generated by P and all of its conjugates. We let $\mathcal{G}_0$ denote the set of all such spaces.

Theorem 5.1.5 *The set of spaces $\mathcal{G}_0$ forms a geometric $(r-1)$-spread of $\Sigma_0 \cong \mathcal{PG}(2nr-1, q)$.*

Proof: The fact that the spaces g_P form a spread follows from a counting argument and by Lemma 5.1.3 and Corollary 5.1.4. We only need to show that $\mathcal{G}_0$ is geometric.

Let g_P and g_Q be two $(r-1)$-spaces of $\mathcal{G}_0$ and let $\mathcal{T}$ be the projective space of Σ_0 spanned by g_P and g_Q. In other words, $\mathcal{T}$ is the intersection of the $GF(q^r)$-span of g_P and g_Q with Σ_0. Let R be any point on the line determined by P and Q in Π and consider the space g_R (see Figure 5.1). We claim that g_R is contained in $\mathcal{T}$. To show this, let U be any point of g_R. Letting $\mathbf{u}$ be any vector which induces the point R, from the proof of Lemma 5.1.3 we know that U is induced by a vector of the form

$$\lambda_k(\mathbf{u}) = \mathbf{u} + \alpha^k \mathbf{u}^q + \alpha^{k(q+1)}\mathbf{u}^{q^2} + \cdots + \alpha^{k(q^{r-2}+\cdots+q+1)}\mathbf{u}^{q^{r-1}}$$

for some k with $0 \leq k \leq q^{r-1} + \cdots + q$. The vector $\mathbf{u}$ can be written as $\mathbf{v} + a\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are any two vectors which induce the points P and Q, respectively, and $a \in GF(q^r) \cup \{\infty\}$. We now find two points in g_P and g_Q which determine a line containing the point U.

We know that the points of g_P are induced by vectors of the form

$$\lambda_i(\mathbf{v}) = \mathbf{v} + \alpha^i \mathbf{v}^q + \alpha^{i(q+1)}\mathbf{v}^{q^2} + \cdots + \alpha^{i(q^{r-2}+\cdots+q+1)}\mathbf{v}^{q^{r-1}}$$

as i varies, and the points of g_Q are induced by vectors of the form

$$\lambda_j(\mathbf{w}) = \mathbf{w} + \alpha^j \mathbf{w}^q + \alpha^{j(q+1)}\mathbf{w}^{q^2} + \cdots + \alpha^{j(q^{r-2}+\cdots+q+1)}\mathbf{w}^{q^{r-1}}$$

as j varies. We let $i = k$ and, since a^{q-1} must be a power of α, we choose j so that $\alpha^j = \alpha^k a^{q-1}$. Substituting for i and j, we consider the line l of Σ_0 determined by

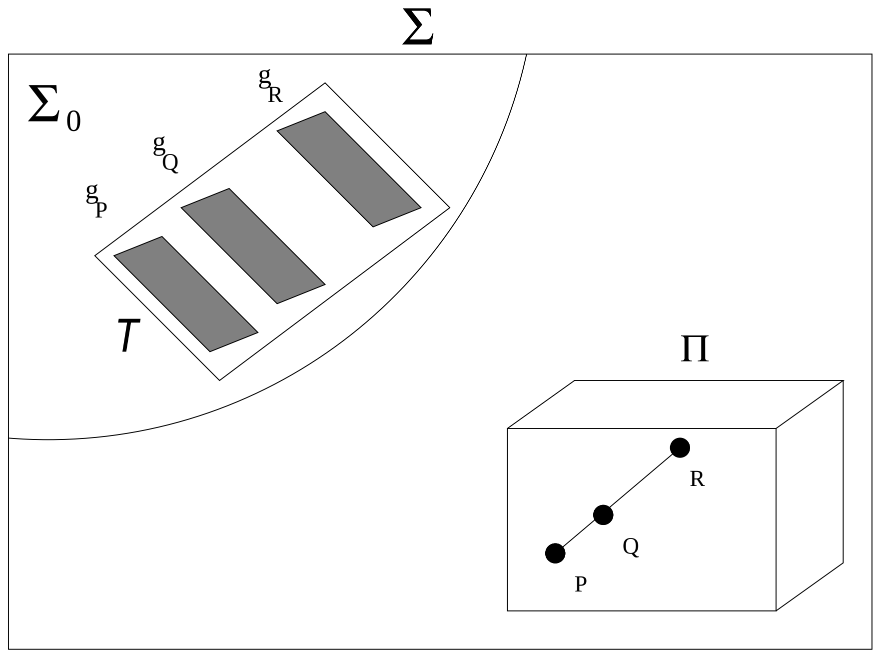

Figure 5.1: Creating the geometric spread

the points induced by vectors $\lambda_i(\mathbf{v})$ and $\lambda_j(\mathbf{w})$. That is, l is the intersection of the line L determined by the points induced by vectors $\lambda_i(\mathbf{v})$ and $\lambda_j(\mathbf{w})$ with the space Σ_0. We claim that this subline l contains the point U. One can easily check that

$$\lambda_i(\mathbf{v}) + a\lambda_j(\mathbf{w}) = \lambda_k(\mathbf{v} + a\mathbf{w}) = \lambda_k(\mathbf{u}).$$

Hence, the line l contains the point U, and since U was arbitrary, all of the points of g_R are in the space $\mathcal{T}$.

There are exactly $q^r + 1$ possible choices for a point R (including P and Q). Hence, the space $\mathcal{T}$, which has dimension $2r - 1$, contains $q^r + 1$ disjoint spaces from the set $\mathcal{G}_0$. Therefore, $\mathcal{T}$ has an induced spread from $\mathcal{G}_0$, and $\mathcal{G}_0$ is geometric. ■

Theorem 5.1.6 *Let $\mathcal{S}_{n-1}$ be an $(n-1)$-spread of $\Pi \cong \mathcal{PG}(2n-1, q^r)$. Then $\mathcal{S}_{n-1}$ can be lifted to an $(nr-1)$-spread $\mathcal{S}_{nr-1}$ of $\Sigma_0 \cong \mathcal{PG}(2nr-1, q)$.*

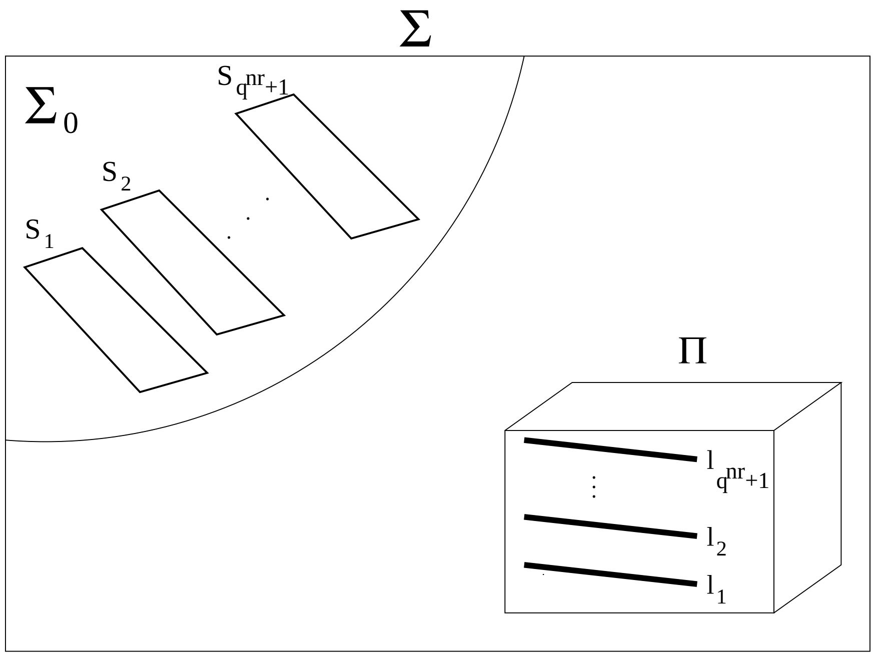

Figure 5.2: Lifting a spread

Proof: Let l_i be any spread element of $\mathcal{S}_{n-1}$ and consider the space spanned by l_i and all of its conjugates. By Lemma 5.1.2 these r copies of $\mathcal{PG}(n-1, q^r)$ generate an $(nr-1)$-space S_i of Σ and, by Lemma 5.1.3 and Corollary 5.1.4, this $(nr-1)$-space meets Σ_0 in an isomorphic copy of $\mathcal{PG}(nr-1, q)$.

We can now vary l_i over all the elements of $\mathcal{S}_{n-1}$. Each of the lifted spread elements consists of the union of spaces g_P as P varies over all points of l_i. Since none of the spaces g_P intersect, none of the spaces S_i intersect. There are exactly $q^{nr}+1$ spread elements of $\mathcal{S}_{n-1}$, and hence we obtain $q^{nr}+1$ disjoint copies of $\mathcal{PG}(nr-1, q)$ in Σ_0. That is, we have constructed a spread of Σ_0 (see Figure 5.2). ■

5.2 Equivalence of Spreads

We are now ready to prove the main result of this chapter. We will show that if one applies the construction process of Section 5.1 to a spread in the space Π, the lifted spread is *equivalent* to the original spread. By equivalent, we mean that the two spreads generate isomorphic translation planes via the Bose/Andrè model described in Section 1.4.1.

Let $\mathcal{S}_{n-1}$ be an $(n-1)$-spread of $\Pi \cong \mathcal{PG}(2n-1, q^r)$, and let $\mathcal{S}_{nr-1}$ be the lifted spread of $\Sigma_0 \cong \mathcal{PG}(2nr-1, q)$. In order to show the relationship between these spreads, we will need to construct the associated translation planes and exhibit an isomorphism between them. For this we refer to the Bose/Andrè model discussed in Section 1.4.1.

Embed Π in $\Pi^* \cong \mathcal{PG}(2n, q^r)$ as the "hyperplane at infinity" so that Π is the set of all points in Π^* which are induced by vectors whose first coordinate is 0. Similarly, let $\Sigma_0^* = \mathcal{PG}(2nr, q)$ and (using the same notation as in Chapter 2) embed Σ_0 in Σ_0^* as the "hyperplane at infinity", where again Σ_0 consists of those points induced by vectors whose first coordinate is 0.

Any spread element of Π is an $(n-1)$-space and therefore is generated by n points, say S_i for $0 \leq i \leq n-1$. For simplicity, we will say S_i is induced by the vector $(0, \mathbf{s}_i)$ where $\mathbf{s}_i$ represents a $2n$-tuple over $GF(q^r)$. Also, in order to simplify the representation of the image of spread elements, we recall the notation from the previous section. Let

$$\lambda_i(\mathbf{v}) = \mathbf{v} + \alpha^i \mathbf{v}^q + \alpha^{i(q+1)} \mathbf{v}^{q^2} + \cdots + \alpha^{i(q^{r-2}+\cdots+q+1)} \mathbf{v}^{q^{r-1}}$$

for any vector $\mathbf{v}$ and for any i. Now letting $\mathbf{v}$ be any linear combination of the $\mathbf{s}_i$'s, the map

$$\mathcal{L} : \mathcal{S}_{n-1} \to \mathcal{S}_{nr-1}$$

defined on the spread elements of $\mathcal{S}_{n-1}$ is given by the lifting process described in Lemma 5.1.3. In vector form,

$$(0, \mathbf{v}) \mapsto \left\{ \left(0, \lambda_i(\mathbf{v}), \lambda_i(\alpha\mathbf{v}), \ldots, \lambda_i(\alpha^{r-1}\mathbf{v})\right) : 0 \leq i \leq q^{r-1} + \cdots + q^2 + q \right\}.$$

Let $\pi(\mathcal{S}_{n-1})$ be the affine translation plane of order q^{nr} which arises from the spread $\mathcal{S}_{n-1}$, and let $\pi(\mathcal{S}_{nr-1})$ be the affine translation plane of the same order which arises from the spread $\mathcal{S}_{nr-1}$. We define a map

$$\phi : \pi(\mathcal{S}_{n-1}) \to \pi(\mathcal{S}_{nr-1})$$

as follows: If $P \in \pi(\mathcal{S}_{n-1})$ is represented by the vector $(1, v)$ (where v is a $2n$-tuple), then P^ϕ is the affine point of $\pi(\mathcal{S}_{nr-1})$ represented by the vector

$$\left(1, Tr(\mathbf{v}), Tr(\alpha\mathbf{v}), Tr(\alpha\mathbf{v}^2), \ldots, Tr(\alpha\mathbf{v}^{r-1})\right).$$

Here, we write $Tr(x)$ for the trace function from the field K to the field F, and $Tr(\mathbf{v}) = Tr\left((v_i)\right) = (Tr(v_1), Tr(v_2), \ldots, Tr(v_{2n}))$. Recall that every affine point of $\pi(\mathcal{S}_{n-1})$ and $\pi(\mathcal{S}_{nr-1})$ is represented by a vector whose first coordinate is 1.

Lemma 5.2.1 *The map ϕ defined above sets up a bijection between the points of $\pi(\mathcal{S}_{n-1})$ and $\pi(\mathcal{S}_{nr-1})$.*

Proof: Since the domain and codomain of ϕ have the same cardinality, we prove the lemma by showing that the map is injective. Let $P_\mathbf{u}$ and $P_\mathbf{v}$ be two points of $\pi(\mathcal{S}_{n-1})$, represented by vectors $(1, \mathbf{u})$ and $(1, \mathbf{v})$ respectively, and suppose

$$\left(1, Tr(\mathbf{u}), Tr(\alpha\mathbf{u}), \ldots, Tr(\alpha^{r-1}\mathbf{u})\right) = \left(1, Tr(\mathbf{v}), Tr(\alpha\mathbf{v}), \ldots, Tr(\alpha^{r-1}\mathbf{v})\right).$$

Then, $Tr(\mathbf{u}) = Tr(\mathbf{v})$, $Tr(\alpha\mathbf{u}) = Tr(\alpha\mathbf{v})$ and so on. Hence, if we let $\mathbf{w} = \mathbf{u} - \mathbf{v}$,

$$Tr(\mathbf{w}) = Tr(\alpha\mathbf{w}) = Tr(\alpha^2\mathbf{w}) = \cdots = Tr(\alpha^{r-1}\mathbf{w}) = 0.$$

So we have a system of equations in the conjugates of $\mathbf{w}$, namely

$$\mathbf{w} + \mathbf{w}^q + \mathbf{w}^{q^2} + \cdots + \mathbf{w}^{q^{r-1}} = 0$$
$$\alpha\mathbf{w} + \alpha^q\mathbf{w}^q + \alpha^{q^2}\mathbf{w}^{q^2} + \cdots + \alpha^{q^{r-1}}\mathbf{w}^{q^{r-1}} = 0$$
$$\alpha^2\mathbf{w} + \alpha^{2q}\mathbf{w}^q + \alpha^{2q^2}\mathbf{w}^{q^2} + \cdots + \alpha^{2q^{r-1}}\mathbf{w}^{q^{r-1}} = 0$$
$$\vdots$$
$$\alpha^{r-1}\mathbf{w} + \alpha^{(r-1)q}\mathbf{w}^q + \alpha^{(r-1)q^2}\mathbf{w}^{q^2} + \cdots + \alpha^{(r-1)q^{r-1}}\mathbf{w}^{q^{r-1}} = 0,$$

and we can apply the same technique as in Lemma 5.1.2. Thinking of $\mathbf{w}$ as the variable, this system of r equations in r variables (the conjugates of $\mathbf{w}$) has a unique solution since the coefficient matrix is a Vandermonde matrix with determinant not equal to zero. Hence, the unique solution must be $\mathbf{w}^{q^i} = 0$ for all i, and so $\mathbf{u} = \mathbf{v}$. Therefore, $P_{\mathbf{u}} = P_{\mathbf{v}}$ and the map is a bijection. ■

We will now define the image of a line under the map ϕ, show that this definition is well-defined, and prove that the image of a line is a line. The fact that ϕ preserves incidence will follow from the definitions of the images of points and lines. This will prove that ϕ is the desired isomorphism.

If l is the line of $\pi(\mathcal{S}_{n-1})$ induced by non-zero vectors in the K-linear span of $(0, \mathbf{s}_1), (0, \mathbf{s}_2), \ldots, (0, \mathbf{s}_n)$, and $(1, u)$, then we will define its image l^ϕ to be the set of all points induced by all non-zero vectors in the F-linear span of

$$(0, \lambda_{i_1}(\mathbf{s}_1), \lambda_{i_1}(\alpha\mathbf{s}_1), \ldots, \lambda_{i_1}(\alpha^{r-1}\mathbf{s}_1))$$
$$(0, \lambda_{i_2}(\mathbf{s}_2), \lambda_{i_2}(\alpha\mathbf{s}_2), \ldots, \lambda_{i_2}(\alpha^{r-1}\mathbf{s}_2))$$
$$\vdots$$
$$(0, \lambda_{i_n}(\mathbf{s}_n), \lambda_{i_n}(\alpha\mathbf{s}_n), \ldots, \lambda_{i_n}(\alpha^{r-1}\mathbf{s}_n))$$

and

$$\big(1, Tr(\mathbf{u}), Tr(\alpha\mathbf{u}), \cdots, Tr(\alpha^{r-1}\mathbf{u})\big)$$

where, for every j, i_j varies between 1 and $q^{r-1} + \cdots + q + 1$.

Lemma 5.2.2 *The action of ϕ on lines of $\pi(\mathcal{S}_{n-1})$ is well-defined.*

Proof: Suppose l is a line of $\pi(\mathcal{S}_{n-1})$. Then l is represented by an n-space of Π^* that meets Π in an element of the spread $\mathcal{S}_{n-1}$. We suppose this spread element is generated by the points induced by vectors in the K-linear span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$. So, the line l is induced by vectors in the span of $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$, and $(1, \mathbf{u})$, where $(1, \mathbf{u})$ is a vector which induces some affine point. To show that ϕ is well-defined on lines, we need to show that the choice of $\mathbf{u}$ is arbitrary in the following sense: if the line induced by $\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_1)\rangle$ is the same as the line induced by $\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_2)\rangle$, then the images of these lines under ϕ are the same.

Suppose that we have two affine points induced by vectors $(1, \mathbf{u}_1)$ and $(1, \mathbf{u}_2)$ that lie on the same line which, in the Bose/Andrè model, meets the spread element determined by $\langle(0, \mathbf{s}_1), ..., (0, \mathbf{s}_n)\rangle$. Then

$$\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_1)\rangle = \langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_2)\rangle$$

and there exists $k_1, k_2, \ldots, k_n \in K$ such that

$$(1, \mathbf{u}_1) + k_1(0, \mathbf{s}_1) + k_2(0, \mathbf{s}_2) + \cdots + k_n(0, \mathbf{s}_n) = (1, \mathbf{u}_2)$$

which implies that

$$\mathbf{u}_1 + k_1\mathbf{s}_1 + k_2\mathbf{s}_2 + \cdots + k_n\mathbf{s}_n = \mathbf{u}_2.$$

If any of the k_i's is 0, then the argument will simplify. So, for generality, we assume that $k_i \neq 0$ for all i. Hence, we can find m_i so that $k_i^{q-1} = \alpha^{m_i}$ for each i. The image of the line induced by vectors in $\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_1)\rangle$ certainly contains the points induced by the vectors

$$\begin{gathered}(0, \lambda_{m_1}(\mathbf{s}_1), \lambda_{m_1}(\alpha\mathbf{s}_1), \ldots, \lambda_{m_1}(\alpha^{r-1}\mathbf{s}_1)) \\ (0, \lambda_{m_2}(\mathbf{s}_2), \lambda_{m_2}(\alpha\mathbf{s}_2), \ldots, \lambda_{m_2}(\alpha^{r-1}\mathbf{s}_2)) \\ \vdots \\ (0, \lambda_{m_n}(\mathbf{s}_n), \lambda_{m_n}(\alpha\mathbf{s}_n), \ldots, \lambda_{m_n}(\alpha^{r-1}\mathbf{s}_n))\end{gathered}$$

and

$$\left(1, Tr(\mathbf{u}_1), Tr(\alpha\mathbf{u}_1), \cdots, Tr(\alpha^{r-1}\mathbf{u}_1)\right).$$

The first n vectors can be scalar multiplied by k_i, respectively, to obtain

$$\begin{gathered}(0, Tr(k_1\mathbf{s}_1), Tr(\alpha k_1\mathbf{s}_1), \ldots, Tr(\alpha^{r-1}k_1\mathbf{s}_1))\\ (0, Tr(k_2\mathbf{s}_2), Tr(\alpha k_2\mathbf{s}_2), \ldots, Tr(\alpha^{r-1}k_2\mathbf{s}_2))\\ \vdots\\ (0, Tr(k_n\mathbf{s}_n), Tr(\alpha k_n\mathbf{s}_n), \ldots, Tr(\alpha^{r-1}k_n\mathbf{s}_n)).\end{gathered}$$

Adding these vectors to the last vector

$$\big(1, Tr(\mathbf{u}_1), Tr(\alpha\mathbf{u}_1), \ldots, Tr(\alpha^{r-1}\mathbf{u}_1)\big)$$

and recalling that the trace function is additive, we obtain

$$\begin{aligned}&(1, Tr(\mathbf{u}_1 + k_1\mathbf{s}_1 + \cdots + k_n\mathbf{s}_n), Tr\left(\alpha(\mathbf{u}_1 + k_1\mathbf{s}_1 + \cdots + k_n\mathbf{s}_n)\right),\\ &\qquad \ldots, Tr\left(\alpha^{r-1}(\mathbf{u}_1 + k_1\mathbf{s}_1 + \cdots + k_n\mathbf{s}_n)\right))\\ &= \big(1, Tr(\mathbf{u}_2), Tr(\alpha\mathbf{u}_2), \ldots, Tr(\alpha^{r-1}\mathbf{u}_2)\big).\end{aligned}$$

Hence, the image of the line induced by $(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n)$ and $(1, \mathbf{u}_1)$ contains the image of the point induced by vector $(1, \mathbf{u}_2)$. Reversing the roles of $\mathbf{u}_1$ and $\mathbf{u}_2$ in this argument, we show that the image of the line induced by vectors in $\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_1)\rangle$ is the same as the image of the line induced by vectors in $\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u}_2)\rangle$. Hence, the map ϕ is well-defined on lines. ■

Lemma 5.2.3 *The image of a line under the map ϕ defined above is a line of $\pi(\mathcal{S}_{nr-1})$.*

Proof: To show that the image of a line l of $\pi(\mathcal{S}_{n-1})$ is a line of $\pi(\mathcal{S}_{nr-1})$, we need to show that l^ϕ is a set of points induced by all non-zero vectors in an nr-dimensional subspace of Σ_0^* which meets Σ_0 in an element of its $(nr-1)$-spread $\mathcal{S}_{nr-1}$. Let l be the set of points induced by vectors in $\langle(0, \mathbf{s}_1), \ldots, (0, \mathbf{s}_n), (1, \mathbf{u})\rangle$. To prove the lemma, we need to take any K-linear combination of these vectors (with the coefficient for $(1, \mathbf{u})$ being non-zero) and show that the image of the point induced by this vector is induced by some F-linear combination of the vectors

$$(0, \lambda_{i_1}(\mathbf{s}_1), \lambda_{i_1}(\alpha\mathbf{s}_1), \ldots, \lambda_{i_1}(\alpha^{r-1}\mathbf{s}_1))$$
$$(0, \lambda_{i_2}(\mathbf{s}_2), \lambda_{i_2}(\alpha\mathbf{s}_2), \ldots, \lambda_{i_2}(\alpha^{r-1}\mathbf{s}_2))$$
$$\vdots$$
$$(0, \lambda_{i_n}(\mathbf{s}_n), \lambda_{i_n}(\alpha\mathbf{s}_n), \ldots, \lambda_{i_n}(\alpha^{r-1}\mathbf{s}_n))$$

and

$$\big(1, Tr(\mathbf{u}), Tr(\alpha\mathbf{u}), \ldots, Tr(\alpha^{r-1}\mathbf{u})\big)$$

for some i_j's in $\{0, 1, \ldots, q\}$. But we have already shown that this is the case in the proof of Lemma 5.2.2. ■

Theorem 5.2.4 *The affine planes $\pi(\mathcal{S}_{n-1})$ and $\pi(\mathcal{S}_{nr-1})$ are isomorphic.*

Proof: To prove the theorem, we show that ϕ is a bijection on points and lines, and that ϕ preserves incidence. The fact that ϕ is a bijection on the points is given by Lemma 5.2.1, and the bijection on the lines follows from Lemma 5.2.1 and Lemma 5.2.3. The incidence preserving property follows from the definition of the map. Hence $\pi(\mathcal{S}_{n-1})$ and $\pi(\mathcal{S}_{nr-1})$ are isomorphic. ■

5.3 Constructing Equivalent Spreads

The most useful application of the previous section comes from the geometric $(r-1)$-spread of Theorem 5.1.5. This geometric spread will be the key to constructing equivalent spreads for translation planes of order q^m which are not m-dimensional over their kernel. We let $\mathcal{G}_0$ be the geometric spread of Σ_0 which arises from the lifting of the points in the projective space Π as defined before Lemma 5.1.2.

Let $\mathcal{S}_{m-1}$ be an $(m-1)$-spread of $\mathcal{PG}(2m-1, q)$, and suppose that $\mathcal{S}_{m-1}$ generates a translation plane $\pi(\mathcal{S}_{m-1})$ which is *not* m-dimensional over its kernel.

Let n be the dimension of $\pi(\mathcal{S}_{m-1})$. Then m can be written as nr for some $r > 1$, and we know that there is some $(n-1)$-spread $\mathcal{S}_{n-1}$ of $\mathcal{PG}(2n-1, q^r)$ generating a plane $\pi(\mathcal{S}_{n-1})$ which is isomorphic to $\pi(\mathcal{S}_{m-1})$. We now have a way to construct $\mathcal{S}_{n-1}$.

Lemma 5.3.1 *The space $\mathcal{PG}(2m-1, q)$ as described above has a unique geometric $(r-1)$-spread $\mathcal{G}$ with the property that every element of $\mathcal{G}$ is contained in some element of $\mathcal{S}_{m-1}$.*

Proof: Embed $\mathcal{PG}(2m-1, q)$ in $\Sigma = \mathcal{PG}(2m-1, q^r)$ as the canonical subgeometry of dimension $2m-1$. Let Σ_0 be the image of $\mathcal{PG}(2m-1, q)$ under this embedding. Hence, Σ_0 contains all of the points of Σ induced by non-zero vectors, all of whose homogeneous coordinates are in $GF(q)$. We again abuse notation and let $\mathcal{S}_{m-1}$ represent the embedded $\mathcal{S}_{m-1}$. Since $\pi(\mathcal{S}_{m-1})$ is n-dimensional, we know that there exists an $(n-1)$-spread $\mathcal{S}_{n-1}$ of $\Gamma = \mathcal{PG}(2n-1, q^r)$ which generates a plane which is isomorphic to $\pi(\mathcal{S}_{m-1})$. Take Γ and embed it in Σ in the manner described in Section 5.1. Now, using the method described in Section 5.1, lift the embedded spread, which we continue to call $\mathcal{S}_{n-1}$, to an $(m-1)$-spread $\mathcal{S}'$ of Σ_0. From Section 5.2, we know that $\pi(\mathcal{S}') \cong \pi(\mathcal{S}_{n-1})$. But $\pi(\mathcal{S}_{n-1}) \cong \pi(\mathcal{S}_{m-1})$ by the assumptions of the theorem. Hence, $\pi(\mathcal{S}') \cong \pi(\mathcal{S}_{m-1})$. Thus, by Theorem 1.4.8, we know that the spread $\mathcal{S}'$ can be mapped to the spread $\mathcal{S}_{m-1}$ via some collineation, say Ψ (i.e. $\mathcal{S}_{m-1} = \mathcal{S}'^{\Psi}$). The spread elements of $\mathcal{S}'$ are made up of a union of spaces from the geometric spread of Theorem 5.1.5. Applying Ψ to this geometric spread, we see that $\mathcal{S}_{m-1}$ also has an associated geometric spread with the desired property.

To show uniqueness, suppose there were two different geometric spreads with the desired property. Then, each of these spreads would have an associated cyclic group which fixes each of the spread elements of $\mathcal{S}_{m-1}$ (see [26]). Hence, the group H leaving each element of $\mathcal{S}_{m-1}$ invariant would contain two different cyclic subgroups of the same order. Since H is a cyclic group, this is a contradiction. ■

As shown previously, any geometric $(r-1)$-spread $\mathcal{G}$ of $\mathcal{PG}(nr-1,q)$ has a nontrivial subgroup of its full automorphism group which acts as the identity on each spread element. This group, which we will called κ, has projective order $q^{r-1}+\cdots+q+1$ and the orbits of κ are exactly the elements of $\mathcal{G}$. We can now apply this theory to construct equivalent spreads.

Algorithm for Constructing Equivalent Spreads

Let $\mathcal{S}_{m-1}$ be an $(m-1)$-spread of $\mathcal{PG}(2m-1,q)$ which generates a translation plane $\pi(\mathcal{S}_{m-1})$ which is not m-dimensional over its kernel. Let n be the dimension of $\pi(\mathcal{S}_{m-1})$ over its kernel. Embed $\mathcal{PG}(2m-1,q)$ in $\Sigma=\mathcal{PG}(2m-1,q^r)$ as the canonical subgeometry of dimension $(2m-1)$, and again let its image under the embedding be denoted by Σ_0. We continue to write $\mathcal{S}_{m-1}$ for the image of $\mathcal{S}_{m-1}$ under this embedding in Σ_0. By Lemma 5.3.1 the orbits of the group κ associated with $\mathcal{S}_{m-1}$ (i.e. the group leaving each element of $\mathcal{S}_{m-1}$ invariant) form a geometric spread $\mathcal{G}$ of Σ_0. Since all geometric spreads of Σ_0 are projectively equivalent [26], we can map $\mathcal{G}$ to $\mathcal{G}_0$ via some collineation ψ. We can now think of the spread $\mathcal{S}_0=\mathcal{S}_{m-1}^{\psi}$ as being in "standard position", and we can extend the elements of $\mathcal{S}_0$ to the whole space Σ. These extended spread elements will each meet the space $\Pi\cong\mathcal{PG}(2n-1,q^r)$ (as defined in Section 5.1) in an $(n-1)$-space, and together, these $(n-1)$-spaces will form a spread $\mathcal{S}_{n-1}$ of Π. By Theorem 5.2.4, $\mathcal{S}_{n-1}$ will generate the same translation plane as $\mathcal{S}_{m-1}$. Hence, $\mathcal{S}_{n-1}$ is equivalent to $\mathcal{S}_{m-1}$.

By combining this result with Theorem 1.4.8, we now have a unifying theory for spreads which generate isomorphic translation planes. In particular, we can finish the characterization of mixed partitions which lift to regular spreads.

Theorem 5.3.2 *Let $\mathcal{P}$ be a non-proper mixed partition (i.e. a spread) of $\mathcal{PG}(2n-1,q^2)$ that gives rise to a regular spread $\mathcal{S}$ via the geometric lifting of Chapter 2.*

Then $\mathcal{P}$ is a regular spread of Π.

Proof: Because of the equivalence established in Theorem 5.2.4, we know that $\pi(\mathcal{P})$ must be a classical plane. Hence, by Theorem 1.4.7, $\mathcal{P}$ is a regular spread. ■

Combining this result with Theorem 4.2.1, we have a complete classification of mixed partitions which lift to regular spreads.

Chapter 6

EXAMPLES OF MIXED PARTITIONS AND THEIR ASSOCIATED TRANSLATION PLANES

In this final chapter we discuss the existence of mixed partitions. We start with a complete classification of mixed partitions in $\mathcal{PG}(3,4)$ and follow with several examples (some previously known) of infinite families of mixed partitions of $\mathcal{PG}(3,q^2)$. For each partition we will describe certain subgroups of the full automorphism group of the partition. Finally, at the end of the chapter, we will discuss the translation planes obtained from each of the partitions.

6.1 A Complete Classification for $\mathcal{PG}(3,4)$

To effectively search for mixed partitions of $\Pi = \mathcal{PG}(3,4)$, we use some transitivity properties of $G = P\Gamma L(4,q)$. For instance, it is well known that G acts 3-transitively on lines of Π. Additionally, G acts transitively on Baer subspaces. Since the Baer subspaces contain more points than the lines, we start trying to build our mixed partition with the Baer subspaces. Since $|\mathcal{PG}(3,4)| = 85$, $|\mathcal{PG}(3,2)| = 15$, and $|\mathcal{PG}(1,4)| = 5$, we obtain the numerical possibilities given in Table 6.1.

As the 1-spreads of $\mathcal{PG}(3,4)$ are well known (see [13]), we concentrate on constructing *proper* mixed partitions; that is, those containing at least one Baer subspace. We start by constructing a list of all Baer subspaces in Π. As the stabilizer in G of a given Baer subspace Π_0 acts transitively on the Baer subspaces disjoint from Π_0, we may start by fixing 2 disjoint Baer subspaces, say B_0 and B_1. We

Table 6.1: Possible mixed partition types for $\mathcal{PG}(3,4)$

# Baer subspaces	# Lines
5	2
4	5
3	8
2	11
1	14
0	17

then eliminate any Baer subspaces which have points in common with either of the two starters. From here, we can construct the set of 3-tuples, 4-tuples, and 5-tuples of disjoint Baer subspaces using the two starters and the list of disjoint Baer subspaces. Combining these tuples with the single Baer subspace B_0 and the pair $\{B_0, B_1\}$, we obtain a collection of all tuples of Baer subspaces which could be used to construct a mixed partition. Each tuple is then completed to a mixed partition in all possible ways by adjoining disjoint lines. This completion was done exhaustively using the set of all lines disjoint from the given tuple of Baer subspaces. The software package *Magma* [12] was then used to check for equivalences under $P\Gamma L(4,q)$ among the mixed partitions obtained. The 10 mutually inequivalent partitions found are summarized in Table 6.2.

First note the description of the partition types. In the case where there are five lines, these lines could potentially form a regulus. Hence, the number of transversal lines to these five lines is listed to illustrate that this occurs in two cases. The translation planes of order 16 were classified in [13], and their automorphism groups were determined by [30]. The plane in the table refers to the translation plane determined from the mixed partition (or equivalently, the lifted 3-spread of $\mathcal{PG}(7,2)$) via the method discussed in Section 1.4.2. The group size is the order of the full automorphism group of this associated translation plane, and the dimension

Table 6.2: Mixed partitions of $\mathcal{PG}(3,4)$

#	Partition Type	Plane	Size of Group	Dim
1	5 Subspaces, 2 lines	Desarguesian	$2^{14} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	1
2	4 Subspaces, 5 lines (1 transversal)	Semifield Plane with Kernel GF(2)	73728	4
3	4 Subspaces, 5 lines (5 transversals)	Derived Semifield Plane	55296	4
4	4 Subspaces, 5 lines (5 transversals)	Semifield Plane with Kernel GF(4)	442368	2
5	4 Subspaces, 5 lines (2 transversals)	Hall Plane	921600	2
6	3 Subspaces, 8 lines	Dempwolff Plane	92160	4
7	3 Subspaces, 8 lines	Derived Semifield Plane	55296	4
8	1 Subspace, 14 lines	L-R OR J-W	258048	4
9	1 Subspace, 14 lines	L-R OR J-W (different from above)	258048	4
10	1 Subspace, 14 lines	Derived Semifield Plane	55296	4

is the dimension of this plane over its kernel. Since the Johnson-Walker (J-W) plane and the Lorimer-Rahilly (L-R) plane have isomorphic automorphism groups, it was difficult to determine which plane arises from the two partitions with 1 Baer subspace which have the same automorphism group. However, it was possible to check using *Magma* that the planes determined by these partitions are non-isomorphic.

It is interesting to note that every translation plane of order 16 can be constructed from one of the mixed partitions in Table 6.2. Also, the derived semifield plane appears in the table three different times. Hence, three non-isomorphic mixed partitions generate the same translation plane. Finally, notice that there does not exist a mixed partition with exactly 2 Baer subspaces. This is the only configuration from Table 6.1 which does not exist.

Some of the partitions in Table 6.2 are discussed elsewhere in this thesis. For instance, the partition containing 5 Baer subspaces is the classical mixed partition discussed in Section 3.2. Hence, it generates the Desarguesian plane as shown in Section 4.2. We look at a construction in the next section by which we can obtain partition #5 from the classical mixed partition. Finally, partitions #3 and #4 will be generalized in Section 6.3.

6.2 Pseudo-Reguli and Partitions from the Classical Partition

Let $\mathcal{P}_0$ be the classical mixed partition of $\Pi = \mathcal{PG}(3, q^2)$ as discussed in Chapter 3. Recall that $\mathcal{P}_0$ contains 2 lines and $(q-1)(q^2+1)$ Baer subspaces. We will use this mixed partition to construct new mixed partitions. First, we will prove a few lemmas about how the lines of Π can meet $\mathcal{P}_0$.

Let l_0 and l_∞ be the two distinct lines of the partition $\mathcal{P}_0$. Recall that the partition $\mathcal{P}_0$ is made up of orbits of a group Θ of order q^3+q^2+q+1 which acts as a Singer cycle on the points of the Baer subspaces of $\mathcal{P}_0$. Let σ be a generator of Θ.

Lemma 6.2.1 *If a line l of Π meets each of the lines l_0 and l_∞ in a point, then l*

meets $q-1$ of the Baer subspaces of $\mathcal{P}_0$ in a Baer subline and is disjoint from the remaining $q^2(q-1)$ Baer subspaces.

Proof: Let l be a line of Π meeting each of l_0 and l_∞ in a point, say R_1 and R_2 respectively. Furthermore, suppose that l meets a Baer subspace B_0 of $\mathcal{P}_0$ in a unique point Q. Because of the structure of the group Θ (see Section 3.2), we know that $R_i^{\sigma^{q^2+1}} = R_i$ for each i. Hence $l^{\sigma^{q^2+1}} = l$, which implies that $Q^{\sigma^{q^2+1}} = Q$. But this contradicts the action of Θ on the points of B_0. ■

This small result leads to the first new type of partition. Let l be any line of Π which meets each of l_0 and l_∞ in a point and consider the orbit of l under the cyclic group $H = \langle \sigma^{q+1} \rangle$ of order q^2+1. The lines in this orbit each meet l_0 and l_∞ in a single point, and it is not hard to show that these lines induce a regular spread in $q-1$ of the Baer subspaces of $\mathcal{P}_0$. Replacing these Baer subspaces and the two lines l_0 and l_∞ with the lines of l^H, we generate a partition of Π containing q^2+1 lines and $q^2(q-1)$ Baer subspaces. We will call this new partition $\mathcal{P}_0'$.

We should note that the lines of l^H form a *pseudo-regulus*, which was originally defined in [15].

Definition 6.2.2 *Given a regular spread $\mathcal{S}$ of $\Pi_0 \cong \mathcal{PG}(3,q)$ embedded in $\Pi = \mathcal{PG}(3,q^2)$, let $\mathcal{F}$ be the partial spread of Π obtained by extending the lines of $\mathcal{S}$ to the space Π. This partial spread $\mathcal{F}$ is called a* **pseudo-regulus** *of Π.*

Theorem 6.2.3 *(Freeman, [15]) If $\mathcal{F}$ is a pseudo-regulus of $\mathcal{PG}(3,q^2)$, then $\mathcal{F}$ is contained in a spread of $\mathcal{PG}(3,q^2)$.*

With this result, other mixed partitions arise. It is shown in [15] that the point set covered by the lines of a pseudo-regulus can always be partitioned into $q-1$ disjoint (transversal) Baer subspaces and 2 transversal lines. Hence, by taking any spread containing a pseudo-regulus, one can replace the pseudo-regulus with the

$q-1$ transversal Baer subspaces and 2 transversal lines, yielding a mixed partition with q^4-q^2+2 lines and $q-1$ Baer subspaces.

Lemma 6.2.4 *If a line l of Π meets exactly one of the lines l_0 and l_∞ in a point, then l meets exactly q^2 of the Baer subspaces of $\mathcal{P}_0$ in a single point and is disjoint from the remaining $(q-1)(q^2+1)-q^2$ Baer subspaces.*

Proof: Without loss of generality, let l be a line of Π meeting l_0 in a point, say R, with $l \cap l_\infty = \emptyset$. Also, for contradiction, suppose that l meets a Baer subspace B of $\mathcal{P}_0$ in a Baer subline $\bar{l}$. Then, as before, $R^{\sigma^{q^2+1}} = R$, which means that the lines l and $l^{\sigma^{q^2+1}}$ share at least one common point. If $l \neq l^{\sigma^{q^2+1}}$, then $\bar{l}$ and $\bar{l}^{\sigma^{q^2+1}}$ are coplanar and, since they are both contained in B, they must intersect in a point of B. Hence, we have two distinct lines sharing two common points, a contradiction. Therefore, $\bar{l} = \bar{l}^{\sigma^{q^2+1}}$ which implies $l = l^{\sigma^{q^2+1}}$. The orbit of l under Θ could not be of any shorter length because of the action of Θ on the points of l_0. Therefore, the orbit l^Θ contains exactly q^2+1 lines.

The points of any Baer subspace in $\mathcal{P}_0$ form a Θ-orbit of length q^3+q^2+q+1. This means that l could not possibly meet a Baer subspace in a single point. Thus, l meets every Baer subspace of $\mathcal{P}_0$ in 0 or $q+1$ points. Therefore, $q^2+1 = 1+k(q+1)$ where k is the number of Baer subspaces which meet l in a Baer subline. This implies $(q+1)|q^2$, a contradiction. ■

Lemma 6.2.5 *If a line l of Π is disjoint from both l_0 and l_∞, then l meets at most one of the Baer subspaces of $\mathcal{P}_0$ in a Baer subline.*

Proof: We prove the contrapositive. Let l be a line of Π and suppose that l meets two distinct Baer subspaces of $\mathcal{P}_0$, say B_m and B_n, in Baer sublines m and n, respectively. Now consider the orbit of m under the group Θ. Either m^Θ is a full orbit of length q^3+q^2+q+1, or m^Θ forms a regular spread of B_m (see [4]). If m^Θ

is a full orbit, then there are two distinct lines of m^Θ which intersect, forcing the corresponding Baer sublines of B_n to be coplanar and, therefore, to intersect in a point of B_n. Since they have two points in common, this implies that the two lines are the same, contradicting their distinctness.

Hence, m^Θ must be a line-orbit of length q^2+1. Now suppose that l meets another Baer subspace different from B_m and B_n in a unique point R. Then R^Θ is a point orbit of length q^2+1, a contradiction. Hence, l must meet all Baer subspaces in 0 or $q+1$ points. By a simple counting argument, this implies that l meets each of l_0 and l_∞ in a unique point. ■

It is interesting to note that the conclusion in the proof of Lemma 6.2.5 is stronger than stated in the lemma. Hence, we could weaken the hypothesis by letting l be disjoint from *at least one* of the lines l_0 and l_∞. However, the case when l is disjoint from exactly one of the lines l_0 and l_∞ is covered in Lemma 6.2.4.

Putting all of the above lemmas together, we find that there are exactly four different intersection patterns of lines of Π (different from l_0 and l_∞) with the partition $\mathcal{P}_0$.

Type 1 Lines which meet both l_0 and l_∞ in a point, meet exactly $q-1$ of the Baer subspaces in a Baer subline, and are disjoint from the remaining $q^2(q-1)$ Baer subspaces

Type 2 Lines which meet exactly one of l_0 and l_∞ in a unique point, meet exactly q^2 Baer subspaces in a unique point, and are disjoint from the remaining $(q-1)(q^2+1)-q^2$ Baer subspaces

Type 3 Lines skew to both l_0 and l_∞ which meet exactly one Baer subspace in a Baer subline, exactly q^2-q Baer subspaces in a unique point, and are disjoint from the remaining q^3-2q^2+2q-2 Baer subspaces

Type 4 Lines skew to both l_0 and l_∞ which meet exactly q^2+1 Baer subspaces in a unique point, and are disjoint from the remaining $(q-2)(q^2+1)$ Baer subspaces

For the purposes of this section, we are particularly interested in the Type 1 and Type 4 lines. Each of these will give rise to a mixed partition. We start by counting the number of lines of each type. There are clearly $(q^2+1)^2$ Type 1 lines. The lines of Type 1 or Type 2, along with l_0 and l_∞, can be counted by inclusion/exclusion giving us

$$2\left[(q^2+1)(q^4+q^2)+1\right]-(q^2+1)^2=$$

$$2q^6+3q^4+1.$$

Hence there are

$$(2q^6+3q^4+1)-(q^2+1)^2-2=$$

$$2q^6+2q^4-2q^2-2=$$

$$2(q^2+1)^2(q^2-1)$$

Type 2 lines. The total number of Baer sublines of Π contained in a Baer subspace of $\mathcal{P}_0$ is $(q^3-1)(q^2+1)^2$, but $(q^2+1)^2(q-1)$ of these Baer sublines generate Type 1 lines. Hence, the total number of Type 3 lines is $q(q^2+1)^2(q^2-1)$. We can now count the number of Type 4 lines by subtracting the total number of Type 1, 2, and 3 lines (plus an extra 2 for l_0 and l_∞) from the total number of lines of Π, $(q^4+1)(q^4+q^2+1)$. This gives us exactly $q(q-1)(q^3-q^2-q-1)(q^3+q^2+q+1)$ Type 4 lines. A summary of the number of lines of the different types appears in Table 6.3.

Theorem 6.2.6 *There exists a mixed partition $\mathcal{P}_0''$ of Π with q^3+q^2+q+3 lines and $(q-2)(q^2+1)$ Baer subspaces.*

Proof: We construct such a mixed partition from the classical partition $\mathcal{P}_0$. From the discussion above, one can always find a line l of Π which meets exactly q^2+1 Baer subspaces of $\mathcal{P}_0$ in a unique point. By starting with $\mathcal{P}_0$ and replacing these

Table 6.3: Numbers of lines of different types

Line Type	Number
Type 1	$(q^2+1)^2$
Type 2	$2(q^2+1)^2(q^2-1)$
Type 3	$q(q^2+1)^2(q^2-1)$
Type 4	$q(q-1)(q^3-q^2-q-1)(q^3+q^2+q+1)$

q^2+1 Baer subspaces with the set of lines in l^{Θ}, we get the desired mixed partition. ■

6.3 The Existence of a *Regulus Type* Mixed Partition

Having given some basic constructions of mixed partitions, we are now ready to start looking at some new and more involved constructions. The idea is to generalize some of the examples in $\mathcal{PG}(3,4)$ found by computer. Two of the partitions in Table 6.2 are particularly interesting, namely #3 and #4. Here we see examples of mixed partitions containing exactly $q+1$ lines which form a regulus. One can only naturally wonder whether this partition is part of an infinite family.

6.3.1 Preliminaries

As above, we work in $\Pi = \mathcal{PG}(3,q^2)$. We let $F = GF(q)$, $K = GF(q^2)$, and let β be a primitive element of K. Also, as we have used throughout this thesis, we let $\alpha = \beta^{q-1}$, whose order is $q+1$. Our proofs in this section are almost all coordinate arguments. We start by recalling the method given in Chapter 1 for constructing Baer subspaces, but we restrict to the 3-dimensional case.

Lemma 6.3.1 *Let A,B,C,D, and E be five points in general position in* Π. *That is, the five points satisfy the property that no four of them lie in the same plane. Then, there is a unique Baer subspace of* Π *containing A,B,C,D, and E.*

Proof: The projective space Π can be modeled as a 4-dimensional vector space over K. Call this vector space V. We let $\mathbf{v}_A$ be any vector in V which induces the point A and define $\mathbf{v}_B, \mathbf{v}_C, \mathbf{v}_D$ and $\mathbf{v}_E$ similarly. Since no four of these points lie in the same plane, any four of the corresponding vectors are linearly independent. Since V is 4-dimensional over K, there are unique non-zero field elements $x_A, x_B, x_C, x_D \in K^*$, such that

$$x_A\mathbf{v}_A + x_B\mathbf{v}_B + x_C\mathbf{v}_C + x_D\mathbf{v}_D = \mathbf{v}_E.$$

Consider the projective points induced by vectors in the F-linear span

$$\langle x_A\mathbf{v}_A, x_B\mathbf{v}_B, x_C\mathbf{v}_C, x_D\mathbf{v}_D\rangle.$$

Clearly, $\mathbf{v}_E$ is in this span (letting all of the coefficients in a linear combination be 1). Hence, we have a 4-dimensional vector space over F which contains the vectors $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C, \mathbf{v}_D$, and $\mathbf{v}_E$. This vector space induces a desired Baer subspace of Π.

What is left is to show that no other Baer subspace contains these five points. This follows simply from the uniqueness of representation of vectors with respect to a given basis. Since $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C$ and $\mathbf{v}_D$ form a basis for V, there is exactly one linear combination of them which produces the vector $\mathbf{v}_E$. The only other linear combinations of $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C$ and $\mathbf{v}_D$ which would produce a vector which induces the point E are

$$kx_A\mathbf{v}_A + kx_B\mathbf{v}_B + kx_C\mathbf{v}_C + kx_D\mathbf{v}_D = k\mathbf{v}_E$$

for some $k \in K$. But the F-linear span

$$\langle kx_A\mathbf{v}_A, kx_B\mathbf{v}_B, kx_C\mathbf{v}_C, kx_D\mathbf{v}_D\rangle$$

produces the same set of projective points as the F-linear span

$$\langle x_A\mathbf{v}_A, x_B\mathbf{v}_B, x_C\mathbf{v}_C, x_D\mathbf{v}_D\rangle.$$

Hence, there is only one Baer subspace through these five points. ■

We will construct the Baer subspaces for our mixed partition in Section 6.3.2. Our method will be exactly what was described above. For now, we define the point set covered by the lines of our partition. Since these lines are supposed to form a regulus, their point set must be a hyperbolic quadric as discussed in Section 1.4.1. We give the quadratic form here. Let

$$\mathcal{Q} = \{\langle(x_0, x_1, x_2, x_3)\rangle : x_0x_2 - x_1x_3 = 0\}.$$

Lemma 6.3.2 *The quadric $\mathcal{Q}$ is a non-degenerate hyperbolic quadric.*

Proof: Clear by the general form for non-degenerate quadrics given in [17]. ■

Now consider a certain set of matrices. Let

$$G = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & b & 0 \\ a & 0 & 0 & b \end{bmatrix} : a \in K, b \in F^* \right\},$$

where F^* is the set of non-zero elements in the finite field F.

Lemma 6.3.3 *The set G induces a collineation group of order $q^2(q-1)$ which fixes the quadric $\mathcal{Q}$.*

Proof: The proof that G forms a group is a straightforward computation. We take two elements of the set G and multiply:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_1 & b_1 & 0 \\ a_1 & 0 & 0 & b_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_2 & b_2 & 0 \\ a_2 & 0 & 0 & b_2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_1 + b_1 a_2 & b_1 b_2 & 0 \\ a_1 + b_1 a_2 & 0 & 0 & b_1 b_2 \end{bmatrix}.$$

Since $a_1 + b_1 a_2 \in K$ and $b_1 b_2 \in F^*$, the set is closed, and we have a faithful matrix representation of the group G. To show that G fixes $\mathcal{Q}$, we take an arbitrary point of the quadric, say P, induced by vector (x_0, x_1, x_2, x_3), and show that P^g is also in the quadric for any $g \in G$. By multiplication, we get that P^g is induced by vector

$$(x_0 + ax_3, x_1 + ax_2, bx_2, bx_3).$$

Using the definition of the quadric $\mathcal{Q}$, the point P^g is in $\mathcal{Q}$ if and only if

$$(x_0 + ax_3)bx_2 = (x_1 + ax_2)bx_3.$$

This equation is equivalent to $x_0 bx_2 = x_1 bx_3$, or $x_0 x_2 = x_1 x_3$ since $b \neq 0$. This, however, is clear since the original point P is in $\mathcal{Q}$. ■

Hence, we have shown that the set G induces a collineation group which leaves invariant the quadric $\mathcal{Q}$. We slightly abuse notation from this point on and also call the induced collineation group G. This group has a nice orbit structure on the points off the quadric. In fact, the number of G-orbits off the quadric $\mathcal{Q}$ is exactly the same as the number of points in a Baer subspace.

Lemma 6.3.4 *The points of $\Pi \setminus \mathcal{Q}$ are partitioned into exactly $q^3 + q^2 + q + 1$ orbits of length $q^2(q-1)$ under the group G.*

Proof: We first show that there are no short orbits on the points off $\mathcal{Q}$ by showing that there are no non-trivial stabilizers and appealing the the orbit-stabilizer theorem. Let T, induced by vector (x_0, x_1, x_2, x_3), be an arbitrary point not on the quadric $\mathcal{Q}$. Then, T^g is induced by vector $(x_0 + ax_3, x_1 + ax_2, bx_2, bx_3)$, where a and b are the parameters in the matrix representation of g. We assume that $T = T^g$ and show that $a = 0$ and $b = 1$, therefore showing that the group element g is the identity. Since $T = T^g$, there is some $k \in K^*$ such that

$$\left\{ \begin{array}{c} kx_0 = x_0 + ax_3 \\ kx_1 = x_1 + ax_2 \\ kx_2 = bx_2 \\ kx_3 = bx_3 \end{array} \right\}.$$

Now, x_2 and x_3 cannot both be 0, otherwise $x_0x_2 = x_1x_3$ and the point T would be in $\mathcal{Q}$. Without loss of generality, assume $x_3 \neq 0$. Then, from the last equation, we get $k = b$. Note that if $x_2 \neq 0$, we get the same result from the third equation. Substituting into the first two equations, we get

$$\left\{ \begin{array}{c} bx_0 = x_0 + ax_3 \\ bx_1 = x_1 + ax_2 \end{array} \right\}.$$

If $b \neq 1$, we may solve simultaneously to obtain $x_1 = \frac{x_0x_2}{x_3}$, or $x_1x_3 = x_0x_2$.

But this is a contradiction since if $x_0x_2 - x_1x_3 = 0$, then T would be a point of the quadric $\mathcal{Q}$. Therefore, we must have $b = 1$. This implies $kx_2 = x_2$ and $kx_3 = x_3$. Since x_2 and x_3 cannot both be 0 (T outside $\mathcal{Q}$), $k = 1$. But then we necessarily have $a = 0$. Hence, the group G has no non-trivial point stabilizers on the points outside the quadric $\mathcal{Q}$. Therefore, since the group has order $q^2(q-1)$, each point orbit outside $\mathcal{Q}$ must have length $q^2(q-1)$. Since the number of points outside $\mathcal{Q}$ is $q^6 + q^4 + q^2 + 1 - (q^2+1)^2 = q^6 - q^2$, we see that we get exactly $q^3 + q^2 + q + 1$ orbits. ∎

This nice orbit structure suggests a method of producing a mixed partition. If we could find a Baer subspace, say B, disjoint from $\mathcal{Q}$, which shares exactly one point with each of the G-orbits off $\mathcal{Q}$, we could let G act on B. The G-orbit of B would consist of $q^2(q-1)$ pairwise disjoint Baer subspaces which, together with one of the ruling families of $\mathcal{Q}$, would create a mixed partition of Π. In order to find such a Baer subspace B, **we will restrict ourselves to the case when q is even for the remainder of Section 6.3**. To continue, we certainly need more information about the orbits of G off $\mathcal{Q}$. We start by giving a short lemma which will prove quite useful in future arguments.

Lemma 6.3.5 *For $i \in \{0, 1, 2, ..., q\}$, the only i such that $\alpha^i \in F$ is $i = 0$. In this case, $\alpha^i = 1$.*

Proof: Suppose $\alpha^i \in F$. Then since every non-zero element of F has order which divides $q-1$, $\alpha^{i(q-1)} = 1$. But α has order $q+1$. So that means $(q+1)|i(q-1)$. Now, q is even, so $q+1$ and $q-1$ do not have any factors in common. Hence, $(q+1)|i$. Because of the restrictions on i, the only possibility is that $i = 0$. ■

We are now in a position to get a complete description of the G-orbits off the quadric $\mathcal{Q}$.

Lemma 6.3.6 *The point orbits under G of the points outside $\mathcal{Q}$ are given by the points induced by the following sets of vectors:*

$$L_j = \{(t\alpha^j, r, 1, 0) : t \in F^*, r \in K\}$$

$$M_j = \{(r, t\alpha^j, 0, 1) : t \in F^*, r \in K\}$$

$$N_{i,j} = \{(r, t\alpha^j + r\beta^i, \beta^i, 1) : t \in F^*, r \in K\}$$

for $i \in \{0, 1, 2, ..., q^2 - 2\}$ and $j \in \{0, 1, 2, ..., q\}$.

Proof: To prove this lemma, consider the following $(q+1)(q^2+1)$ vectors, which induce distinct projective points: $(\alpha^j, 0, 1, 0), (0, \alpha^j, 0, 1)$, and $(0, \alpha^j, \beta^i, 1)$, where $i \in \{0, 1, 2, ..., q^2-2\}$ and $j \in \{0, 1, 2, ..., q\}$. We call the induced points L-type, M-type, and N-type points, respectively. First, we note that none of these vectors induce points in the hyperbolic quadric simply by checking the quadratic form on its coordinates. We now show that none of the induced points lie in the same G-orbit, and thus these $(q+1)(q^2+1)$ points form a system of distinct representatives for the G-orbits off $\mathcal{Q}$.

Let g be any element of G. Then g is induced by some matrix $M_{a,b}$ where a and b are the parameters described in the definition of G. First consider an L-type point. Let $\mathbf{v}_L = (\alpha^j, 0, 1, 0)$ be any vector which induces an L-type point and consider its image, $\mathbf{v}_L M_{a,b} = (\alpha^j, 0, 1, 0)M_{a,b} = (\alpha^j, a, b, 0)$, under the matrix $M_{a,b}$. Since the image under $M_{a,b}$ has last coordinate 0, $\mathbf{v}_L M_{a,b}$ cannot induce an M-type or an N-type point. Now, suppose that $\mathbf{v}_L M_{a,b}$ induces another L-type point. Then, there is some $k \in K$ and some $j' \in \{0, 1, 2, ..., q\}$ so that

$$(\alpha^j, a, b, 0) = k(\alpha^{j'}, 0, 1, 0).$$

This immediately tells us that $a = 0$ and $k = b$. But then $\alpha^j = b\alpha^{j'}$, which means that $\alpha^{j-j'} = b \in F$. By Lemma 6.3.5 and because of the restrictions on j and j', $j = j'$ and thus $b = 1$. Hence, $M_{a,b}$ must be the identity matrix. Therefore, all of the L-type points are in distinct G-orbits, and, moreover, no L-type point is in the same G-orbit as an M-type or an N-type point.

Now consider an M-type point. Let $\mathbf{v}_M = (0, \alpha^j, 0, 1)$ be any vector which induces an M-type point and consider its image $\mathbf{v}_M M_{a,b} = (0, \alpha^j, 0, 1)M_{a,b} = (a, \alpha^j, 0, b)$, under the matrix $M_{a,b}$. As discussed above, $\mathbf{v}_M M_{a,b}$ cannot induce an L-type point. Suppose that $\mathbf{v}_M M_{a,b}$ induces another M-type point. Then, as before, there is some $k \in K$ and some $j' \in \{0, 1, 2, ..., q\}$ so that

$$(a, \alpha^j, 0, b) = k(0, \alpha^{j'}, 0, 1)$$

This immediately tell us that $a = 0$ and $b = k$, which leads to the same conclusion as above that g must be the identity. Also $\mathbf{v}_M M_{a,b}$ cannot induce an N-type point since all N-type points are induced by vectors having non-zero entries in their third coordinate.

The only possibility left is that an N-type point maps to another N-type point under some group element g. If this were possible, then there must be some $k \in K$, some $i' \in \{0, 1, 2, ..., q^2 - 2\}$, and some $j' \in \{0, 1, 2, ..., q\}$ so that

$$(0, \alpha^j, \beta^i, 1)M_{a,b} = (a, \alpha^j + a\beta^i, b\beta^i, b) = k(0, \alpha^{j'}, \beta^{i'}, 1)$$

This implies that $a = 0$ and $b = k$, and again leads to the same conclusion. Hence, we have shown that all of the L-type, M-type, and N-type points lie in distinct G-orbits.

We can now find a representation for the points in any orbit. The points in an L-type orbit are induced by the vectors of the form $(\alpha^j, a, b, 0)$ as a varies over K and b varies over F^*. Similarly, the points in an orbit of an M-type point are induced by vectors of the form $(a, \alpha^j, 0, b)$, and the points in an orbit of an N-type point are induced by vectors of the form $(a, \alpha^j + a\beta^i, b\beta^i, b)$. By dividing by b, letting $r = \frac{a}{b}$ and $t = \frac{1}{b}$, we get the parametrization described in the statement of the lemma.

We can now do a quick count to check that all points are covered. There are $q + 1$ L-type points, $q + 1$ M-type points, and $(q^2 - 1)(q + 1)$ N-type points, for a total of $2(q + 1) + (q + 1)(q^2 - 1) = (q + 1)(q^2 + 1) = q^3 + q^2 + q + 1$ total points. This is exactly the number of G-orbits outside of $\mathcal{Q}$. Hence, we have found a representation for each G-orbit off the quadric $\mathcal{Q}$. ■

6.3.2 Finding a G-*perfect* Baer Subspace

Our goal now is to find a G-*perfect* Baer subspace. That is, we want to find a Baer subspace B that meets each of the G-orbits off the quadric $\mathcal{Q}$ in exactly one

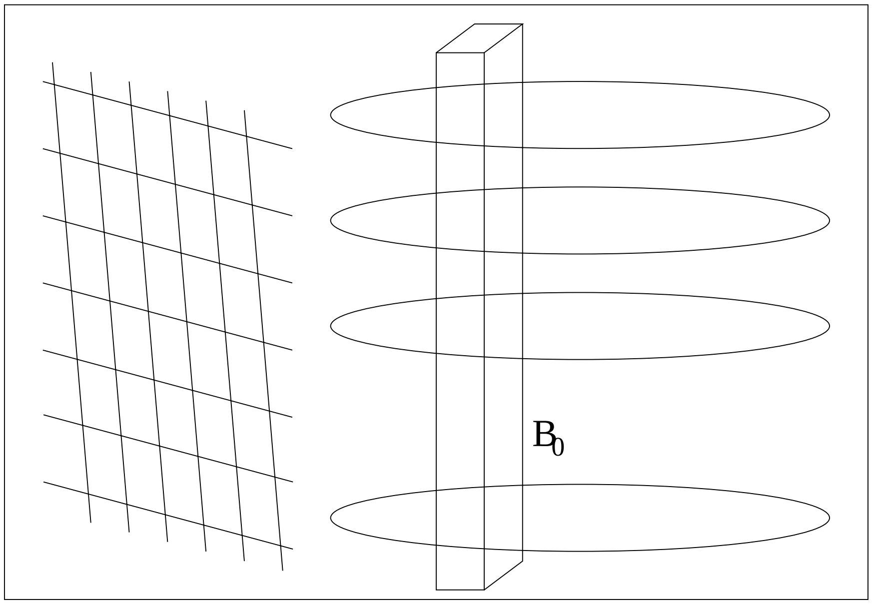

Figure 6.1: A *G-perfect* Baer subspace

point (see Figure 6.1). Using the software package *Magma*, it was possible to find such a Baer subspace for small even values of q, and moreover, to generalize the example to all even q. We will represent B by five points in general position whose linear span over F generates all of the points of B_0.

In order to simplify the computations, we choose a primitive element β of K such that $\mathrm{Tr}_{K/F}(\beta) = 1$. Therefore, $\beta^q + \beta = 1$ or equivalently, $\beta = \frac{1}{1+\beta^{q-1}} = \frac{1}{1+\alpha}$. The existence of such a primitive element is given in [11]. This extra trace condition on the primitive element β allows for many identities with the field element α. We will frequently use identities such as $\alpha\beta = 1 + \beta$, $\alpha\beta + \alpha^q\beta^q = \beta + \beta^q$ and so on in our computations.

Consider the following five vectors:

$$\mathbf{p}_1 = (\alpha^4, \alpha^3, 1, 0)$$

$$\mathbf{p}_2 = (\alpha, \alpha^2, 0, 1)$$

$$\mathbf{p}_3 = (\alpha^3, \alpha^2, 1, 0)$$

$$\mathbf{p}_4 = (1, \alpha, 0, 1)$$

$$\mathbf{p}_5 = \left(\frac{\alpha^3 + 1}{\alpha}, \frac{1}{\beta}, 1, 1\right)$$

Lemma 6.3.7 *The five vectors defined above satisfy the following:*

1. *Any four are linearly independent.*

2. $\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \alpha\beta\mathbf{p}_5$.

Proof: To prove that any four vectors are linearly independent, we first consider $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$. By looking at the last two coordinates of these vectors, one can easily check that no three of them are linearly dependent. Now consider the possibility that $a\mathbf{p}_1 + b\mathbf{p}_2 + c\mathbf{p}_3 = \mathbf{p}_4$ for some triple of scalars $a, b, c \in K$. We immediately get that $b = 1$ and $a = c$ by looking at the last two coordinates. But then, by looking at the first two coordinates,

$$a\alpha^4 + \alpha + a\alpha^3 = 1$$

and

$$a\alpha^3 + \alpha^2 + a\alpha^2 = \alpha,$$

which together means

$$\alpha(1 + \alpha^2) = 1 + \alpha^2.$$

Since $\alpha \neq 1$, it must be the case that $\alpha^2 = 1$ and since α has order $q + 1$, $(q + 1)|2$. But this is clearly false since $q \geq 2$.

Now consider $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_5\}$. Again, one can easily check that no three of these four vectors are linearly dependent. Now, suppose that $a\mathbf{p}_1 + b\mathbf{p}_2 + c\mathbf{p}_3 = \mathbf{p}_5$ for some triple of scalars $a, b, c \in K$. Then by looking at the last two coordinates, $b = 1$ and $a + c = 1$. Looking at the first two coordinates,

$$a\alpha^4 + \alpha + (a + 1)\alpha^3 = \frac{\alpha^3 + 1}{\alpha},$$

and

$$a\alpha^3 + \alpha^2 + (a+1)\alpha^2 = \frac{1}{\beta},$$

which together means that

$$\frac{\alpha}{\beta} + \frac{\alpha^3+1}{\alpha} = \alpha + \alpha^3.$$

Replacing $\frac{1}{\beta}$ with $\alpha + 1$, we get

$$\alpha(1+\alpha) + \frac{\alpha^3+1}{\alpha} = \alpha + \alpha^3$$

and hence,

$$\alpha^4 = 1.$$

But, as before, α has order $q+1$ and thus $(q+1)|4$ which is clearly false since $q+1$ is odd. Similar computations show that no four of the points are coplanar.

To show that, $\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \alpha\beta\mathbf{p}_5$, we simply do the computations on the left hand side of the equality.

First component:

$$\beta\alpha^4 + \beta\alpha + \alpha^3 + 1 = (\alpha\beta + 1)\left(\alpha^3 + 1\right) = \beta\left(\alpha^3 + 1\right).$$

Second component:

$$\beta\alpha^3 + \beta\alpha^2 + \alpha^2 + \alpha = \alpha\left(\alpha + 1\right)\left(\beta\alpha + 1\right) = \alpha\left(\alpha + 1\right)\beta = \alpha.$$

Third and fourth components:

$$\beta + 1 = \alpha\beta.$$

Hence, we get $\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \alpha\beta\mathbf{p}_5$. ■

We now appeal to the result given in Section 6.3.1. By Lemma 6.3.1, the five vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ and $\mathbf{p}_5$ induce five projective points which are in general position, and they determine a unique Baer subspace

$$\mathcal{B}_0 = \left\{\langle a\beta\mathbf{p}_1 + b\beta\mathbf{p}_2 + c\mathbf{p}_3 + d\mathbf{p}_4\rangle : a, b, c, d \in F, \text{ not all } 0\right\}.$$

We will show in the next section that $\mathcal{B}_0$ is precisely the Baer subspace we need. To establish the desired result, we need a series of technical lemmas which we now prove.

Lemma 6.3.8 *Any point of K^* can be written uniquely in the form $t\alpha^j$, where $t \in F^*$ and $j \in \{0, 1, 2, ..., q\}$.*

Proof: We let $A = \{(t, \alpha^j) : t \in F^*, j \in \{0, 1, 2, ..., q\}\}$ and consider the map $f : A \to K^*$ defined by $f(t, \alpha^j) = t\alpha^j$. Since the domain and the codomain of this function have the same cardinality, we only need to show that this function is injective.

Suppose that there is some element of K^*, say β^i, such that $t_1\alpha^{j_1} = \beta^i$ and $t_2\alpha^{j_2} = \beta^i$. Then $\alpha^{j_2 - j_1} = \frac{t_1}{t_2} \in F^*$. But, from Lemma 6.3.5, the only non-zero powers of α which are in F are multiples of $q+1$. This means that $(q+1)|(j_2 - j_1)$. Since $j_1, j_2 \in \{0, 1, 2, ..., q\}$, the only possibility is that $j_2 - j_1 = 0$, or $j_2 = j_1$. This implies that $t_1 = t_2$ and the result follows. ■

Lemma 6.3.9 *For $b, d \in F$, not both zero, $\frac{b\alpha\beta+d}{b\beta+d}$ is a power of α. Moreover, every power of α can be written this way for some choice of b and d.*

Proof: First note that $b\beta + d \neq 0$ since $\beta \notin F$. Raising $\frac{b\alpha\beta+d}{b\beta+d}$ to the $(q+1)^{\text{st}}$ power, we obtain

$$\begin{aligned}
\left(\frac{b\alpha\beta + d}{b\beta + d}\right)^{q+1} &= \left(\frac{b\alpha\beta + d}{b\beta + d}\right)^{q} \left(\frac{b\alpha\beta + d}{b\beta + d}\right) \\
&= \left(\frac{b\alpha^q\beta^q + d}{b\beta^q + d}\right) \left(\frac{b\alpha\beta + d}{b\beta + d}\right) \\
&= \frac{b^2\alpha^{q+1}\beta^{q+1} + bd(\alpha\beta + \alpha^q\beta^q) + d^2}{b^2\beta^{q+1} + bd(\beta + \beta^q) + d^2} \\
&= \frac{b^2\beta^{q+1} + bd(\beta^q + \beta) + d^2}{b^2\beta^{q+1} + bd(\beta + \beta^q) + d^2}
\end{aligned}$$

$$= \frac{b^2\beta^{q+1} + bd + d^2}{b^2\beta^{q+1} + bd + d^2}$$
$$= 1$$

Here, we strongly use the condition that $\beta^q + \beta = 1$ and $\alpha\beta = \beta^q$. The first part of the lemma now follows from the fact that $(q+1)^{\text{st}}$ roots of unity in K are precisely the various powers of α.

To show the second part, suppose that

$$\frac{b_1\alpha\beta + d_1}{b_1\beta + d_1} = \frac{b_2\alpha\beta + d_2}{b_2\beta + d_2}$$

and thus

$$b_1d_2\alpha\beta + d_1b_2\beta = b_2d_1\alpha\beta + d_2b_1\beta.$$

Cancelling β and regrouping, we get

$$(b_1d_2 + b_2d_1)\alpha = d_1b_2 + d_2b_1.$$

Since $\alpha \notin F$, necessarily $b_1d_2 = b_2d_1$. If $b_1 = 0$, then $d_1 \neq 0$ and thus $b_2 = 0$. Similarly for b_2. Hence, b_1 and b_2 are either both 0 or both non-zero. Thus we can write $b_1 = \lambda b_2$, for some $\lambda \in F^*$, and we obtain $\lambda b_2d_2 = b_2d_1$. Now, either $b_2 = 0$, which would force $b_1 = 0$, or $\lambda d_2 = d_1$. Note that if $b_2 = b_1 = 0$, then $d_1 \neq 0 \neq d_2$. Hence, in either case, we get that

$$\frac{b_1\alpha\beta + d_1}{b_1\beta + d_1} = \frac{b_2\alpha\beta + d_2}{b_2\beta + d_2} \Rightarrow (b_1, d_1) = \lambda(b_2, d_2)$$

for some $\lambda \in F^*$. Therefore, since there are $\frac{q^2-1}{q-1} = q+1$ different choices for ordered pairs (b, d) up to F^*-scalar multiples, there must be at least $q+1$ different values for $\frac{b\alpha\beta+d}{b\beta+d}$. Since there are exactly $q+1$ distinct powers of α, every power of α can be written this way. ■

The final lemma of this section seems a bit unusual at first, but it will prove to be incredibly useful in the proof of Theorem 6.3.11 in the subsequent section.

Lemma 6.3.10 *As before, let β be a primitive element of K whose trace from K to F is 1, and let $\alpha = \beta^{q-1}$. Then, the expression*

$$M_i = \alpha^2(\alpha^i\beta^i) + \alpha + \beta^i\left(\alpha^3(\alpha^i\beta^i) + 1\right)$$

is never 0 for $0 \leq i \leq q^2 - 2$.

Proof: We prove this lemma by considering the trace of $\frac{1}{\alpha}M_i$ from K to F. If M_i is 0, then this trace must also be zero. We show that this cannot be the case. Since $(\alpha\beta)^q = \beta$, $\beta^q = \alpha\beta$, and $\alpha^q = \frac{1}{\alpha}$, we see that

$$\frac{1}{\alpha}M_i = \alpha(\alpha^i\beta^i) + 1 + \beta^i\left(\alpha^2(\alpha^i\beta^i) + \frac{1}{\alpha}\right),$$

and hence

$$\left(\frac{1}{\alpha}M_i\right)^q = \frac{1}{\alpha}\beta^i + 1 + (\alpha\beta)^i\left(\frac{1}{\alpha^2}\beta^i + \alpha\right).$$

Adding $\frac{1}{\alpha}M_i + \left(\frac{1}{\alpha}M_i\right)^q$, we get

$$Tr\left(\frac{1}{\alpha}M_i\right) = \left(\alpha\beta^2\right)^i\left(\alpha^2 + \frac{1}{\alpha^2}\right) = \left(\alpha\beta^2\right)^i\left(\frac{\alpha^4+1}{\alpha^2}\right) = \left(\alpha\beta^2\right)^i\left(\frac{1}{\alpha^2\beta^4}\right) = \omega^{i-2},$$

where $\omega = \alpha\beta^2 = \beta^{q+1}$ is a primitive element of F. Hence, this trace is not 0, and thus $M_i \neq 0$. ■

6.3.3 Proving the Existence

We are now in a position to state the main result of this section. We will prove the existence of a mixed partition of the type described earlier; that is, a mixed partition containing $q^2 + 1$ lines which form a regulus and $q^2(q-1)$ Baer subspaces. This partition involves our previously described Baer subspace $\mathcal{B}_0$.

The goal in the following arguments will be to show that $\mathcal{B}_0$ meets all of the G-orbits off the quadric $\mathcal{Q}$ in exactly one point. This will be done indirectly and requires establishing a bijection between the points of $\mathcal{B}_0$ and each of the G-orbits. First we establish some notation.

Let

$$A = \{(a : b : c : d) : a, b, c, d \in F, a \text{ and } c \text{ not both } 0, b \text{ and } d \text{ not both } 0\}$$

and

$$I = \left\{(i, j) : i \in \{0, 1, 2, ..., q^2 - 2\}, j \in \{0, 1, 2, ..., q\}\right\}.$$

In the set A, the notation $(a : b : c : d)$ means that non-zero F-scalar multiples of vectors are considered equal. Hence $(a_1 : b_1 : c_1 : d_1) = (a_2 : b_2 : c_2 : d_2)$ in A if and only if there is a non-zero element $k \in F$ such that $(a_1, b_1, c_1, d_1) = k(a_2, b_2, c_2, d_2)$.

Using inclusion/exclusion and dividing out F-scalar multiples, we see that $|A| = \frac{q^4-q^2-q^2+1}{q-1} = (q + 1)(q^2 - 1)$. Also, $|I| = (q + 1)(q^2 - 1)$. Hence, to show that there is a bijection from A to I, we only need to find an injection.

Letting $\Lambda = \{\alpha^i : 0 \leq i \leq q\}$, we define

$$B = K^* \times \Lambda = \left\{(\beta^i, \alpha^k) : i \in \{0, 1, 2, ..., q^2 - 2\}, k \in \{0, 1, 2, ..., q\}\right\}.$$

Note that $|B| = (q+1)(q^2-1)$. Thus, using Lemma 6.3.9, we may define a function

$$\phi_1 : A \to B$$

via

$$\phi_1(a : b : c : d) = \left(\frac{a\beta + c}{b\beta + d}, \frac{b\alpha\beta + d}{b\beta + d}\right)$$

It is easy to see that ϕ_1 is well-defined. From Lemma 6.3.10, we know that the expression

$$\alpha^k M_i = \left(\alpha^2(\alpha^{i+k}\beta^i) + \alpha^{k+1}\right) + \beta^i \left(\alpha^3(\alpha^{i+k}\beta^i) + \alpha^k\right)$$

is not equal to 0. Hence, from Lemma 6.3.8, $\alpha^k M_i$ can be written uniquely in the form $t\alpha^j$ where $t \in F^*$ and $0 \leq j \leq q$. We now define ϕ_2

$$\phi_2 : B \to I$$

via

$$\phi_2(\beta^i, \alpha^k) = (i, j),$$

where j is the unique exponent of α such that

$$\exists t \in F^* \text{ with } t\alpha^j = \alpha^k M_i.$$

The map ϕ_1 is clearly surjective from Lemma 6.3.9 and thus bijective by a simple cardinality argument. Similarly, ϕ_2 is surjective by Lemma 6.3.8 and Lemma 6.3.9. Hence, ϕ_2 is bijective by the same cardinality argument. Thus, the composition $\phi = \phi_1 \circ \phi_2$ is a bijection from A to I.

We can now prove the main theorem which will leads to the existence of our desired mixed partition.

Theorem 6.3.11 *The Baer subspace $\mathcal{B}_0$ contains exactly one point from each of the G-orbits outside the hyperbolic quadric $\mathcal{Q}$.*

Proof: Recall the Baer subspace

$$\mathcal{B}_0 = \{\langle a\beta\mathbf{p}_1 + b\beta\mathbf{p}_2 + c\mathbf{p}_3 + d\mathbf{p}_4\rangle : a, b, c, d \in F, \text{ not all } 0\}$$

as defined in Section 6.3.2. We will identify the point $P = \langle a\beta\mathbf{p}_1 + b\beta\mathbf{p}_2 + c\mathbf{p}_3 + d\mathbf{p}_4\rangle$ with $(a : b : c : d)$ and then find the corresponding G-orbit in which the associated point lies. Moveover, we will show that every G-orbit contains a point from $\mathcal{B}_0$, thereby giving us the result. First we consider the tuples where $b = d = 0$. Then the associated projective point can be represented by the vector

$$\begin{aligned} a\beta\mathbf{p}_1 + c\mathbf{p}_3 &= \left(a\beta\alpha^4 + c\alpha^3, a\beta\alpha^3 + c\alpha^2, a\beta + c, 0\right) \\ &= \left(\alpha^3(a\beta\alpha + c), \alpha^2(a\beta\alpha + c), a\beta + c, 0\right). \end{aligned}$$

Since the zero vector does not induce a projective point, a and c cannot both be zero. Hence, since $\beta \notin F$, $a\beta + c \neq 0$, and dividing by $a\beta + c$ to right normalize the vector, we get

$$\left(\alpha^3 \left(\frac{a\beta\alpha + c}{a\beta + c}\right), \alpha^2 \left(\frac{a\beta\alpha + c}{a\beta + c}\right), 1, 0\right).$$

From Lemma 6.3.9, the first and second coordinates are both powers of α, and moreover, every power of α can be obtained for some choice of a and c. Letting i be such that

$$\alpha^i = \alpha^2 \left(\frac{a\beta\alpha + c}{a\beta + c}\right),$$

we obtain the following right-normalized vectors as a and c vary:

$$l_1 = \left\{(\alpha^{i+1}, \alpha^i, 1, 0) : i \in \{0, 1, ..., q\}\right\}.$$

Similarly, if we consider linear combinations where $a = c = 0$, we obtain

$$l_2 = \left\{(\alpha^i, \alpha^{i+1}, 0, 1) : i \in \{0, 1, ..., q\}\right\}.$$

Note that the vectors in l_1 and l_2 will induce points on two projective lines (they actually form Baer sublines), and these two lines meet each of the L-type and M-type orbits in exactly one point. What remains is to show that the other linear combinations of the $\mathbf{p}_i$'s will yield vectors which induce exactly one point from each of the N-type orbits.

To this end, we claim that for any element $(a : b : c : d)$ in A the linear combination $a\beta\mathbf{p}_1 + b\beta\mathbf{p}_2 + c\mathbf{p}_3 + d\mathbf{p}_4$ is an element of the $(i, j)^{\text{th}}$ orbit, $N_{i,j}$, where $(i, j) = (\phi_2 \circ \phi_1)(a : b : c : d)$. Note that

$$a\beta\mathbf{p}_1 + b\beta\mathbf{p}_2 + c\mathbf{p}_3 + d\mathbf{p}_4$$

$$= \left(a\beta\alpha^4 + b\beta\alpha + c\alpha^3 + d, a\beta\alpha^3 + b\beta\alpha^2 + c\alpha^2 + d\alpha, a\beta + c, b\beta + d\right).$$

Now $b\beta + d \neq 0$ as b and d are not both 0 and $\beta \notin F$. Hence we can right normalize to obtain the equivalent projective point

$$\left(\alpha^3\left(\frac{a\alpha\beta+c}{b\beta+d}\right)+\left(\frac{b\alpha\beta+d}{b\beta+d}\right),\alpha^2\left(\frac{a\alpha\beta+c}{b\beta+d}\right)+\alpha\left(\frac{b\alpha\beta+d}{b\beta+d}\right),\frac{a\beta+c}{b\beta+d},1\right).$$

Using $\alpha\beta = \beta^q$ and $b\beta^q + d \neq 0$ (again because $\beta^q \notin F$), we have

$$\left(\frac{a\alpha\beta+c}{b\beta+d}\right)=\left(\frac{a\beta^q+c}{b\beta^q+d}\right)\left(\frac{b\beta^q+d}{b\beta+d}\right)=\left(\frac{a\beta+c}{b\beta+d}\right)^q\left(\frac{b\alpha\beta+d}{b\beta+d}\right).$$

But now the bijection established by ϕ_1 gives us the unique i and k such that

$$\left(\frac{a\beta+c}{b\beta+d}\right)=\beta^i \text{ and } \left(\frac{b\alpha\beta+d}{b\beta+d}\right)=\alpha^k$$

and thus

$$\left(\frac{a\alpha\beta+c}{b\beta+d}\right)=\beta^{qi}\alpha^k=(\alpha\beta)^i\alpha^k.$$

Similar computations involving the various components allow us to rewrite the vector $\mathbf{p}$ which induces P as

$$\mathbf{p}=\left(\alpha^3(\alpha\beta)^i\alpha^k+\alpha^k,\alpha^2(\alpha\beta)^i\alpha^k+\alpha(\alpha^k),\beta^i,1\right)$$

$$=\left(\alpha^3(\alpha^{i+k}\beta^i)+\alpha^k,\alpha^2(\alpha^{i+k}\beta^i)+\alpha^{k+1},\beta^i,1\right).$$

Now, letting $r = \alpha^3(\alpha^{i+k}\beta^i) + \alpha^k$, we get

$$\mathbf{p}=\left(r,\alpha^2(\alpha^{i+k}\beta^i)+\alpha^{k+1},\beta^i,1\right)$$

$$=\left(r,\alpha^2(\alpha^{i+k}\beta^i)+\alpha^{k+1}+r\beta^i+r\beta^i,\beta^i,1\right)$$

$$=\left(r,\alpha^2(\alpha^{i+k}\beta^i)+\alpha^{k+1}+\beta^i\left(\alpha^3(\alpha^{i+k}\beta^i)+\alpha^k\right)+r\beta^i,\beta^i,1\right).$$

Recalling the definition of M_i from Lemma 6.3.10, we see that the second coordinate is $\alpha^k M_i + r\beta^i$. So our second bijection ϕ_2 gives us the unique j so that we may write coordinates for $\mathbf{p}$ as

$$\mathbf{p}=\left(r,t\alpha^j+r\beta^i,\beta^i,1\right)$$

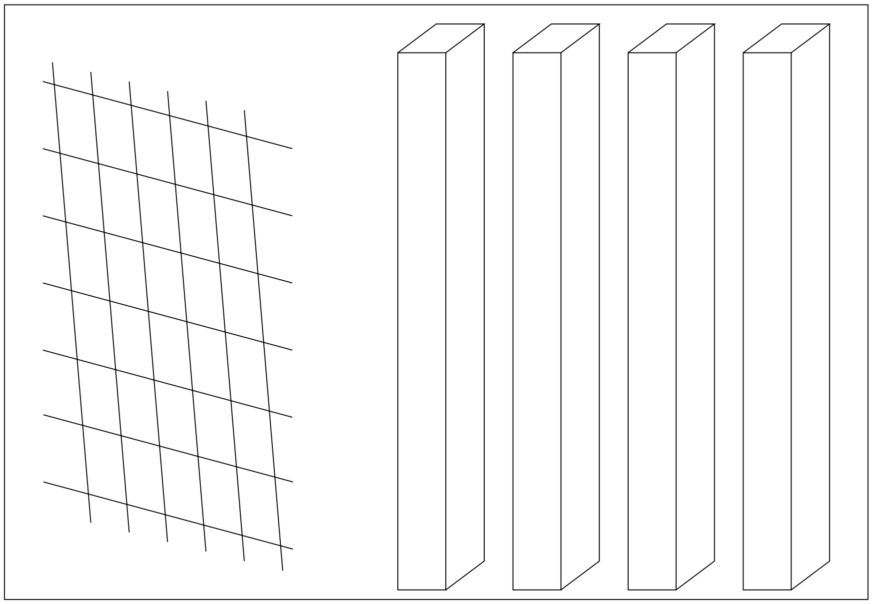

Figure 6.2: The partitions $\mathcal{P}_1$ and $\mathcal{P}_2$

where t is the unique element of F^* so that

$$t\alpha^j = \alpha^k M_i.$$

That is, $\mathbf{p} \in N_{i,j}$ and the result follows from the fact that $\phi = \phi_1 \circ \phi_2$ is a bijection from A to I. ■

Theorem 6.3.12 *Let $q = 2^k$ for some $k \geq 1$. Then there exists a mixed partition of the projective space $\mathcal{PG}(3, q^2)$ consisting of exactly $q^2 + 1$ lines and exactly $q^2(q-1)$ Baer subspaces.*

Proof: For the lines of our partition, we may use either ruling family of the hyperbolic quadric $\mathcal{Q}$. Now, consider the group G and the Baer subspace $\mathcal{B}_0$ as defined earlier. Note that $\mathcal{B}_0$ is disjoint from the hyperbolic quadric since, by Theorem 6.3.11, it meets each of the G-orbits outside $\mathcal{Q}$ in exactly one point and there are exactly $q^3 + q^2 + q + 1$ such orbits. Thus, the orbit of $\mathcal{B}_0$ under G is a collection of $q^2(q-1)$ mutually disjoint Baer subspaces, all of which are disjoint from $\mathcal{Q}$. Hence the Baer subspaces in this orbit together with the one of the sets of ruling lines from the hyperbolic quadric create a mixed partition of $\mathcal{PG}(3, q^2)$ of the desired type (see Figure 6.2). We will refer to these two mixed partitions as $\mathcal{P}_1$ and $\mathcal{P}_2$. ■

6.3.4 Automorphisms of $\mathcal{P}_1$ and $\mathcal{P}_2$

The first step in determining the translation planes associated with $\mathcal{P}_1$ and $\mathcal{P}_2$ is to exhibit certain subgroups of $Aut(\mathcal{P}_1)$ and $Aut(\mathcal{P}_2)$. We can exhibit two such groups here. First we establish some notation for the lines of each partition. As noted earlier, the hyperbolic quadric $\mathcal{Q}$ contains two sets of ruling lines. One can easily check that one such ruling family consists of the lines

$$l_k = \langle (0, 1, k, 0), (1, 0, 0, k) \rangle, k \in K$$

$$l_\infty = \langle (0, 0, 1, 0), (0, 0, 0, 1) \rangle,$$

and the other ruling family consists of the lines

$$l'_k = \langle (1, k, 0, 0), (0, 0, k, 1) \rangle, k \in K$$

$$l'_\infty = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle.$$

We will refer to the partition containing the first set of lines as $\mathcal{P}_1$ and the second one as $\mathcal{P}_2$. The reader should note that we have redefined l_0 and l_∞ from their definition in Section 6.2.

Consider the two matrices

$$M_\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{\alpha\beta} & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \frac{1}{\alpha\beta} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$M_\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}.$$

It is easy to see that the cyclic group generated by M_τ and the cyclic group generated by M_γ are both of order $q+1$. In fact,

$$M_\tau^i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{\alpha\beta^i} & \alpha^i & 0 & 0 \\ 0 & 0 & \alpha^i & \frac{1}{\alpha\beta^i} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$M_\gamma^j = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^j & 0 \\ 0 & 0 & 0 & \alpha^j \end{bmatrix}.$$

Moreover, one can easily check that M_τ and M_γ commute. Hence, we can consider the internal direct product of the cyclic groups generated by M_τ and M_γ. A general matrix in this product is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1+\alpha^i}{\alpha} & \alpha^i & 0 & 0 \\ 0 & 0 & \alpha^{i+j} & \alpha^j \frac{1+\alpha^i}{\alpha} \\ 0 & 0 & 0 & \alpha^j \end{bmatrix}.$$

Let τ and γ be the collineations of Π induced by M_τ and M_γ, respectively. Also, let G_0 be the collineation group generated by τ and γ, and consider the action of G_0 on the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$. We will now show that G_0 fixes the mixed partitions and so is a subgroup of $Aut(\mathcal{P}_1)$ and $Aut(\mathcal{P}_2)$.

We first show that G_0 fixes the Baer subspace $\mathcal{B}_0$. Recall that the points used to define $\mathcal{B}_0$ are induced by the vectors

$$\mathbf{p}_1 = (\alpha^4, \alpha^3, 1, 0),$$

$$\mathbf{p}_2 = (\alpha, \alpha^2, 0, 1),$$

$$\mathbf{p}_3 = (\alpha^3, \alpha^2, 1, 0),$$

$$\mathbf{p}_4 = (1, \alpha, 0, 1),$$

and

$$\mathbf{p}_5 = \left(\frac{\alpha^3+1}{\alpha}, \frac{1}{\beta}, 1, 1\right).$$

A straightforward computation shows that

$$\mathbf{p}_1 M_\tau = \left(\frac{1+\alpha^3}{\alpha\beta}\right)\beta\mathbf{p}_2 + \alpha\mathbf{p}_3 + \frac{1}{\beta^2}\mathbf{p}_4,$$

$$\mathbf{p}_2 M_\tau = \left(\frac{1}{\beta^2}\right)\beta\mathbf{p}_2 + \alpha\mathbf{p}_4,$$

$$\mathbf{p}_3 M_\tau = \left(\frac{1}{\beta}\right)\beta\mathbf{p}_1 + \left(\frac{1}{\alpha\beta^3}\right)\beta\mathbf{p}_2 + \frac{1}{\beta}\mathbf{p}_3 + \frac{1}{\beta}\mathbf{p}_4,$$

and

$$\mathbf{p}_4 M_\tau = \left(\frac{1}{\beta}\right)\beta\mathbf{p}_2.$$

By scalar multiplying these equations by β^2, β^2, β and 1, respectively, we see that the image under the collineation τ of every generating point in $\mathcal{B}_0$ stays in $\mathcal{B}_0$. This is enough to show that $\mathcal{B}_0$ is fixed under the collineation τ. Similarly,

$$\mathbf{p}_1 M_\gamma = \alpha\mathbf{p}_3,$$

$$\mathbf{p}_2 M_\gamma = \left(\frac{1}{\alpha\beta}\right)\beta\mathbf{p}_1,$$

$$\mathbf{p}_3 M_\gamma = \left(\frac{1}{\beta}\right)\beta\mathbf{p}_1 + \frac{1}{\beta}\mathbf{p}_3,$$

and

$$\mathbf{p}_4 M_\gamma = \left(\frac{1}{\beta}\right) \beta \mathbf{p}_2 + \frac{1}{\beta}\mathbf{p}_4.$$

which implies that $\mathcal{B}_0$ is fixed under the collineation γ.

Now recall the group G defined in Section 6.3.1. This group certainly is a subgroup of the full automorphism group of $\mathcal{P}_1$ and $\mathcal{P}_2$. Moreover, a straight forward computation shows that G_0 is contained in the normalizer of G in $PGL(4, q^2)$. We use this to show that G_0 fixes the collection of Baer subspaces in $\mathcal{P}_1$ and $\mathcal{P}_2$. Let g_2 be an arbitrary element of G_0. Then, since G acts transitively (in fact, regularly) on the Baer subspaces of $\mathcal{P}_1$ and $\mathcal{P}_2$, for any other Baer subspace, say $\mathcal{B}_k$, there is a group element $g \in G$ such that $\mathcal{B}_0^g = \mathcal{B}_k$. Since G_0 is contained in the normalizer of G in $PGL(4, q^2)$, there exists $g' \in G$ such that $g \cdot g_2 = g_2 \cdot g'$. Hence,

$$\mathcal{B}_k^{g_2} = (\mathcal{B}_0^g)^{g_2} = \mathcal{B}_0^{(g \cdot g_2)} = \mathcal{B}_0^{(g_2 \cdot g')} = (\mathcal{B}_0^{g_2})^{g'} = \mathcal{B}_0^{g'} = \mathcal{B}_{k'}$$

where $\mathcal{B}_{k'}$ is the image of $\mathcal{B}_k$ under the group element g', and $\mathcal{B}_0^{g_2} = \mathcal{B}_0$ since $\mathcal{B}_0$ is fixed under the group G_0 as shown above. Hence the Baer subspace of $\mathcal{P}_1$ and $\mathcal{P}_2$ are fixed under the group G_0.

Last, we look at the action of G_0 on the lines of $\mathcal{P}_1$ and $\mathcal{P}_2$. Since G_0 fixes each one of the Baer subspaces in $\mathcal{P}_1$ and $\mathcal{P}_2$, we know that G_0 must also fix the quadric $\mathcal{Q}$. It is well known (see [18]) that any collineation fixing a hyperbolic quadric of $\mathcal{PG}(3, q^2)$ must either leave invariant both of the associated reguli or interchange the two. Note that the lines

$$l_0 = \langle (0,1,0,0), (1,0,0,0) \rangle$$

and

$$l_0' = \langle (1,0,0,0), (0,0,0,1) \rangle$$

are clearly fixed under γ and τ. This implies that G_0 fixes the regulus in $\mathcal{P}_1$ and fixes the regulus in $\mathcal{P}_2$. Hence, we obtain the following.

Proposition 6.3.13 *The automorphism groups $Aut(\mathcal{P}_1)$ and $Aut(\mathcal{P}_2)$ both admit a subgroup of order $q^2(q-1)(q+1)^2$.*

Proof: One can see from the matrix definitions of G and G_0 that $G \cap G_0$ contains only the identity collineation. Since G_0 is a subgroup of the normalizer of G in $PGL(4, q^2)$, $Aut(\mathcal{P}_1)$ and $Aut(\mathcal{P}_2)$ both contain the subgroup $G \rtimes G_0$, which has order $q^2(q-1)(q+1)^2$. ■

Corollary 6.3.14 *The plane $\pi(\mathcal{P}_1)$ is at most 2-dimensional over its kernel.*

Proof: We have shown that the group generated by τ leaves each of the Baer subspaces of $\mathcal{P}_1$ invariant. A straightforward computation shows that the group generated by τ leaves each of the lines of $\mathcal{P}_1$ invariant as well. Hence, the group which fixes every element of $\mathcal{S}_1$ contains at least $q+1$ elements. From the discussion of the kernel given in Section 1.4.1, we know that the multiplicative group of the kernel contains at least $(q-1)(q+1)$ elements and must therefore contain $GF(q^2)$. Hence, $\pi(\mathcal{P}_1)$ is at most 2-dimensional over its kernel. ■

We can now determine the kernel of $\pi(\mathcal{P}_1)$ and $\pi(\mathcal{P}_2)$. As discussed in Section 1.4, the multiplicative group of the kernel of a translation plane is isomorphic to the direct product of the collineation group of Σ_0 leaving each element of the associated spread invariant (which is necessarily a linear subgroup) with the cyclic group of order $q-1$. From Theorem 4.1.7, we know that any such group arises from a collineation group of $\mathcal{PG}(3, q^2)$ which fixes each element of the mixed partition. We examine the conditions on a 4×4 matrix which induces a collineation acting on the space Π fixing each element of our mixed partition.

First examine $\mathcal{P}_1$. Let M_δ be a 4×4 matrix inducing a collineation δ of Π which fixes each element of $\mathcal{P}_1$. Then, in particular, δ fixes the lines l_0 and l_∞. This immediately implies that M_δ has the general form

$$M_\delta = \begin{bmatrix} m_{1,1} & m_{1,2} & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 & 0 \\ 0 & 0 & m_{3,3} & m_{3,4} \\ 0 & 0 & m_{4,3} & m_{4,4} \end{bmatrix}.$$

Since δ fixes the line l_1, we can obtain even more restrictions on the entries of M_δ, giving us the general form

$$M_\delta = \begin{bmatrix} m_{1,1} & m_{1,2} & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 & 0 \\ 0 & 0 & m_{2,2} & m_{2,1} \\ 0 & 0 & m_{1,2} & m_{1,1} \end{bmatrix}.$$

In order to restrict this matrix further, it is easier to work with the spread of $\mathcal{PG}(7,q)$. In other words, from Theorem 4.1.4, any linear automorphism δ of $\mathcal{P}_1$ lifts to an automorphism Δ acting on the lifted spread $\mathcal{S}_1$ associated with $\mathcal{P}_1$. Take the matrix M_δ inducing the automorphism δ and lift it to an automorphism acting on $\mathcal{S}_1$. We now use the fact that Δ leaves each element of $\mathcal{S}_1$ invariant to find more restrictions on the entries of M_δ. In fact, it can be shown that there are at most $q+1$ collineations of Σ_0 leaving invariant each element of $\mathcal{S}_1$. The details are given in the appendix, but we state the result here.

Theorem 6.3.15 *The translation plane constructed from the mixed partition $\mathcal{P}_1$ is 2-dimensional over its kernel.*

In a similar fashion, if each line of $\mathcal{P}_2$ is fixed by a collineation induced by a 4×4 matrix, we can immediately determine that the matix has the form

$$\begin{bmatrix} m_{1,1} & 0 & 0 & m_{1,4} \\ 0 & m_{1,1} & m_{1,4} & 0 \\ 0 & m_{4,1} & m_{4,4} & 0 \\ m_{4,1} & 0 & 0 & m_{4,4} \end{bmatrix}$$

Again, in order to restrict this matrix further, it is easier to work with the spread $\mathcal{S}_2$ of $\mathcal{PG}(7, q)$ associated with $\mathcal{P}_2$. This time it can be shown that the only collineation of Σ_0 leaving each element of $\mathcal{S}_2$ invariant is the identity. The details are given in the last part of the appendix, but we state the result here.

Theorem 6.3.16 *The translation plane constructed from the mixed partition $\mathcal{P}_2$ is 4-dimensional over its kernel.*

Having exhibited the subgroup $G \rtimes G_0$ of $Aut(\mathcal{P}_1)$ of order $q^2(q-1)(q+1)^2$, we now examine the point orbits induced by $G \rtimes G_0$ on the line at infinity of $\pi(\mathcal{P}_1)$. First note that the line

$$l_0 = \langle (0,1,0,0), (1,0,0,0) \rangle$$

is fixed by both G and G_0. Now consider the line

$$l_1 = \langle (0,1,1,0), (1,0,0,1) \rangle.$$

Recall that every element k of K^* can be written in the form $f \cdot \alpha^j$ for some $f \in F^*$ and some $i \in \{0, 1, \ldots, q\}$. Hence, by letting $a = 0$ and $b = f$ in the definition of G given at the beginning of Section 6.3.1 and choosing $i = 0$ and the appropriate value of j in the definition of G_0, we know that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix}$$

induces an element of $G \rtimes G_0$ for any $k \in K^*$. But the image of l_1 under the collineation induced by this matrix is l_k. Moreover, the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

also induces an element of $G \rtimes G_0$. Under the collineation induced by this matrix, the point induced by vector $(0, 1, 1, 0)$ maps to the point induced by $(0, 0, 1, 0)$, and the point induced by vector $(1, 0, 0, 1)$ maps to the point induced by $(0, 0, 0, 1)$. Hence, and the image of l_1 under this collineation is l_∞.

Recall from Theorem 4.1.6 that any collineation group of $\mathcal{S}_1$ centralizes the group κ in $PGL(4n, q)$. Let $H = \kappa \times (G \rtimes G_0)$.

Theorem 6.3.17 *The group H acting on $\mathcal{S}_1$ induces a collineation group of the plane $\pi(\mathcal{P}_1)$ which fixes a single point on the line at infinity, creates one orbit of length q^2, and creates one orbit of length $q^2(q^2 - 1)$.*

Proof: Recall that every line of $\mathcal{P}_1$ lifts to a single solid in the associated spread $\mathcal{S}_1$ of Σ_0 and that each of the $q^2(q-1)$ Baer subspaces of $\mathcal{P}_1$ lifts to a set of $q+1$ solids which form a regulus in Σ_0. The group $G \rtimes G_0$ acts transitively on the Baer subspaces of $\mathcal{P}_1$. Hence, this group lifts to a subgroup of $Aut(\mathcal{S}_1)$ which permutes the reguli coming from lifted Baer subspaces in a single orbit of length $q^2(q-1)$. But the group κ permutes the $q+1$ solids within a regulus coming from a lifted Baer subspace. Hence, the group H induces a subgroup of $Aut(\pi(\mathcal{P}_1))$ which creates an orbit of length $q^2(q^2-1)$ on the line at infinity of $\pi(\mathcal{P}_1)$.

The group κ fixes all of the solids of $\mathcal{S}_1$ which come from lifted lines of $\mathcal{P}_1$, and the group $G \rtimes G_0$ fixes one line of $\mathcal{P}_1$ and permutes the remaining q^2 lines in one orbit. Hence, the group H induces the desired action on the line at infinity of $\pi(\mathcal{P}_1)$. ■

The group H is also a subgroup of $Aut(\mathcal{S}_2)$. In this case, H induces a collineation group acting on $\pi(\mathcal{S}_2)$ which again creates a point orbit of length $q^2(q-1)$ on the line at infinity of $\pi(\mathcal{S}_2)$. The remaining point orbits on the line at infinity are not as easy to determine. We will see in Section 6.5 that this is not needed to determine the type of the translation plane generated by $\mathcal{P}_2$.

6.4 A Partition from a Regular Spread

The objective of this section will be to create a new type of mixed partition using a special group action. For the following, we will be working in $\Pi = \mathcal{PG}(3, q^2)$ where **we now assume** q **is odd**. As before, we let K be the finite field $GF(q^2)$, and let β be a primitive element of K. We let F be the subfield $GF(q)$, so that $\omega = \beta^{q+1}$ is a primitive element of F. We will also make use of the special element $\epsilon = \beta^{\frac{q+1}{2}}$, where one can easily show that $\epsilon^q = -\epsilon$.

Throughout this thesis we have always modeled a regular spread using finite fields. This model is very convenient for some applications, but quite lacking for others. When using fields to model odd dimensional projective spaces, there is no convenient way of representing a Baer subspace. The best way to model a Baer subspace is by using coordinates. For this new construction, we attempt to bridge the gap between these two models. We use a coordinate model for a regular spread which allows us to more easily look at Baer subspaces in the same model.

6.4.1 A Model for a Regular Spread

For our construction, we will take a regular spread in Π and find a Baer subspace that meets each line of the regular spread in at most one point. Note that the only possible intersection sizes are $0, 1$, or $q+1$ where the $q+1$ intersection size corresponds to a Baer subline. So, we are looking for a Baer subspace that does not meet any of the lines of the regular spread in a Baer subline (see Figure 6.3).

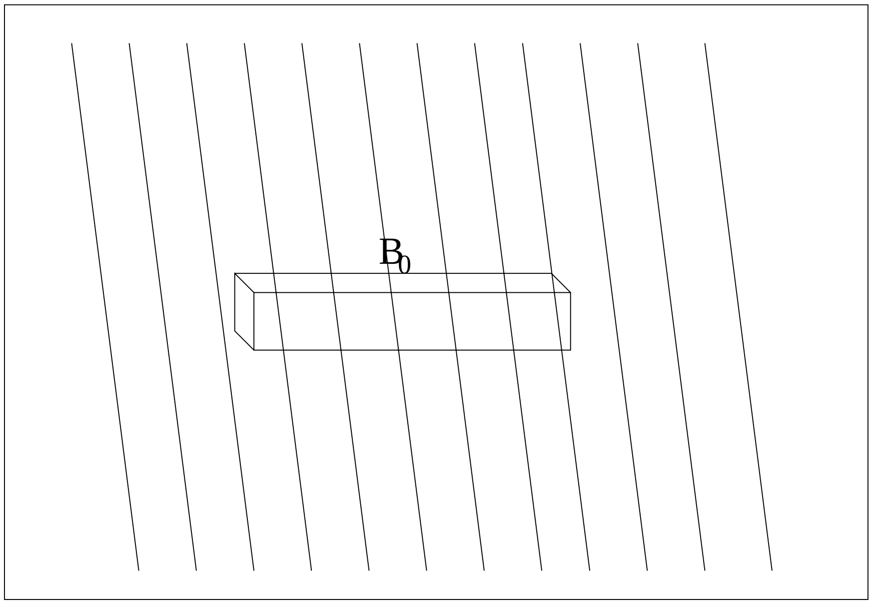

Figure 6.3: A "good" Baer subspace

We start with a representation of a regular spread. In Bruck [8] it is shown that the lines

$$\left\{ l_{(x,y)} = \langle (x, y, 1, 0), (\beta y, x, 0, 1) \rangle : x, y \in K \right\}$$

together with the extra line

$$l_\infty = \langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle$$

form a regular spread of Π. These lines are constructed from the ruling families of certain quadrics, and the coordinates given here are carefully determined in [3]. We construct our desired regular spread from this model where the basis for the underlying vector space is non-standard. Alternatively, we can think of the change of basis as the application of some collineation.

Let

$$M_\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\epsilon & 0 \\ 0 & 1 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix M_ϕ induces a collineation ϕ on Π. Applying this collineation to Π, we can write the images of the generators for the lines of our new regular spread as

$$\langle (x, y+1, -\epsilon(y-1), 0) \rangle$$

$$\langle (\beta y, x, -\epsilon x, 1) \rangle.$$

We will use this regular spread to construct a new mixed partition.

6.4.2 Proving the Existence

Let $\mathcal{S}^*$ be the spread obtained from the lines defined above, and let B_0 be the natural Baer subspace of Π, that is, the one whose homogeneous coordinates are in the subfield $GF(q)$.

Theorem 6.4.1 *Every line of $\mathcal{S}^*$ meets B_0 in at most one point.*

Proof: Consider the line $l^\phi_{(x,y)}$ of $\mathcal{S}^*$. An arbitrary point on this line can be written as

$$\langle (\beta y + \lambda x, x + \lambda(y+1), -\epsilon x - \epsilon\lambda(y-1), 1) \rangle$$

for some $\lambda \in K$, or

$$\langle (x, y+1, -\epsilon(y-1), 0) \rangle.$$

Hence, each line of the spread has just one point whose last coordinate is zero. So, if a line of $\mathcal{S}^*$ meets B_0 in a Baer subline, then that Baer subline would have at least two points whose last coordinates are non-zero. For contradiction, suppose the following two vectors induce points of $l^\phi_{(x,y)}$ which are also both in B_0:

$$\mathbf{q}_1 = (\beta y + \lambda_1 x, x + \lambda_1(y+1), -\epsilon x - \epsilon\lambda_1(y-1), 1)$$

and

$$\mathbf{q}_2 = (\beta y + \lambda_2 x, x + \lambda_2(y+1), -\epsilon x - \epsilon\lambda_2(y-1), 1)\,,$$

where $\lambda_1 \neq \lambda_2$. We will start by looking at 2 special cases. First, suppose that $y = 1$. Then

$$\mathbf{q}_1 = (\beta + \lambda_1 x, x + 2\lambda_1, -\epsilon x, 1)$$

and

$$\mathbf{q}_2 = (\beta + \lambda_2 x, x + 2\lambda_2, -\epsilon x, 1)\,.$$

Now, $\mathbf{q}_1 - \mathbf{q}_2$ must also induce a point of B_0 and, moreover, since $\mathbf{q}_1$ and $\mathbf{q}_2$ both have all of their coordinates in F, $\mathbf{q}_1 - \mathbf{q}_2$ must have all of its coordinates in F as well. But

$$\mathbf{q}_1 - \mathbf{q}_2 = (x(\lambda_1 - \lambda_2), 2(\lambda_1 - \lambda_2), 0, 0)$$

which means that $\lambda_1 - \lambda_2 \in F$. Since $\lambda_1 - \lambda_2 \neq 0$, we get $x \in F$ from the first coordinate. But from the third coordinate of $\mathbf{q}_1$, $\epsilon x \in F$. The only possibility is that $x = 0$. If $x = 0$, the first coordinate of $\mathbf{q}_1$ is β which is clearly not in F, a contradiction.

Now consider the second special case when $y = -1$. Then, as before,

$$\mathbf{q}_1 = (-\beta + \lambda_1 x, x, -\epsilon x + 2\epsilon\lambda_1, 1)$$

$$\mathbf{q}_2 = (-\beta + \lambda_2 x, x, -\epsilon x + 2\epsilon\lambda_2, 1)$$

and so

$$\mathbf{q}_1 - \mathbf{q}_2 = (x(\lambda_1 - \lambda_2), 0, 2\epsilon(\lambda_1 - \lambda_2), 0)\,.$$

Since $\mathbf{q}_1$ and $\mathbf{q}_2$ are both right normalized, we get from their second coordinates that $x \in F$. So suppose $x = 0$. Then the first coordinate of $\mathbf{q}_1$ is $-\beta$ which is clearly not in F and we get a contradiction. Therefore, $x \neq 0$. But then the first coordinate of $\mathbf{q}_1 - \mathbf{q}_2$ tells us that $\lambda_1 - \lambda_2 \in F$, which, from the third coordinate of

$\mathbf{q}_1 - \mathbf{q}_2$, tells us that $\epsilon \in F$ which is a contradiction. Hence, we can assume from this point on that $y \neq 1$ and $y \neq -1$.

Going back to our original forms for $\mathbf{q}_1$ and $\mathbf{q}_2$, we get that

$$\mathbf{q}_1 - \mathbf{q}_2 = (x(\lambda_1 - \lambda_2), (y+1)(\lambda_1 - \lambda_2), -\epsilon(y-1)(\lambda_1 - \lambda_2), 0)$$

also induces a point of B_0. Since $y \neq 1$, we can right normalize this vector and get the homogeneous coordinates for the associated point to be

$$\left(\frac{x}{-\epsilon(y-1)}, \frac{y+1}{-\epsilon(y-1)}, 1, 0\right).$$

So, $\frac{x}{-\epsilon(y-1)} \in F$ and $\frac{y+1}{-\epsilon(y-1)} \in F$. In particular, note that $y \neq 0$, and the Frobenius map acts as the identity on these values. Hence,

$$\left(\frac{y+1}{-\epsilon(y-1)}\right)^q = \frac{y+1}{-\epsilon(y-1)}$$

which implies

$$\frac{y^q+1}{\epsilon(y^q-1)} = \frac{y+1}{-\epsilon(y-1)}.$$

Cross multiplying gives us

$$-\epsilon(y-1)(y^q+1) = \epsilon(y^q-1)(y+1)$$

$$-y^{q+1} - y + y^q + 1 = y^{q+1} + y^q - y - 1$$

$$2y^{q+1} = 2$$

$$y^q = \frac{1}{y}.$$

Now, since $\frac{x}{-\epsilon(y-1)} \in F$, we can use the Frobenius map again to get

$$\left(\frac{x}{-\epsilon(y-1)}\right)^q = \frac{x}{-\epsilon(y-1)}$$

which implies that

$$\frac{x^q}{\epsilon(y^q-1)} = \frac{x}{-\epsilon(y-1)}.$$

Cancelling the ϵ and cross multiplying gives us

$$x^q(-y+1) = x(y^q - 1).$$

Finally, substituting $\frac{1}{y}$ for y^q,

$$x^q(1-y) = \frac{x}{y}(1-y)$$

and since $y \neq 1$,

$$x^q y = x.$$

With these two powerful identities, $y^q = \frac{1}{y}$ and $x^q y = x$, we can now find a contradiction. Since, $\mathbf{q}_1$ and $\mathbf{q}_2$ are both normalized, $x + \lambda_i(y+1) \in F$ for $i = 1, 2$. Hence, the vector $\mathbf{q} = [x + \lambda_2(y+1)]\mathbf{q}_1 - [x + \lambda_1(y+1)]\mathbf{q}_2$ has all of its coordinates in F. Now,

$$\mathbf{q} = \big(\beta y(y+1)(\lambda_2 - \lambda_1) + x^2(\lambda_1 - \lambda_2), 0,$$
$$-\epsilon x(\lambda_1 - \lambda_2)(y-1) - \epsilon x(\lambda_2 - \lambda_1)(y+1), (\lambda_2 - \lambda_1)(y+1)\big)$$

and since $y \neq -1$, we can right normalize to get

$$\mathbf{q} = \left(\beta y + \frac{-x^2}{y+1}, 0, \frac{\epsilon x(y-1)}{y+1} - \epsilon x, 1\right).$$

Hence,

$$\epsilon\left(\frac{x(y-1)}{y+1} - x\right) \in F.$$

Again using the Frobenius map,

$$\left(\epsilon\left(\frac{x(y-1)}{y+1} - x\right)\right)^q = \epsilon\left(\frac{x(y-1)}{y+1} - x\right)$$

which implies

$$-\epsilon\left(\frac{x^q(y^q-1)}{y^q+1} - x^q\right) = \epsilon\left(\frac{x(y-1)}{y+1} - x\right).$$

Making the appropriate substitutions,

$$-\left(\frac{\frac{x}{y}(\frac{1}{y}-1)}{\frac{1}{y}+1} - \frac{x}{y}\right) = \left(\frac{x(y-1)}{y+1} - x\right),$$

and simplifying gives us

$$\frac{-x(\frac{1}{y}-1)}{1+y}+\frac{x}{y}=\frac{x(y-1)}{1+y}-x.$$

Multiplying by $1+y$, we get

$$-x\left(\frac{1}{y}-1\right)+\frac{x}{y}(1+y)=x(y-1)-x(1+y)$$

or

$$\frac{-x}{y}+x+\frac{x}{y}+x=xy-x-x-xy.$$

Finally, by cancelling terms, we obtain $2x=-2x$. Since q is odd, $x=0$.

Now, since $x=0$, we can rewrite $\mathbf{q}_1$ as

$$\mathbf{q}_1=(\beta y,\lambda_1(y+1),-\epsilon\lambda_1(y-1),1)$$

which implies that $\beta y\in F$. But $\beta y\in F\Rightarrow\frac{\beta^q}{y}=\beta y\Rightarrow y^2=\beta^{q-1}$. So

$$y=\beta^{\frac{q-1}{2}}$$

which means that

$$1=y^{q+1}=\beta^{\frac{q^2-1}{2}}=-1$$

which is a contradiction. Hence, in all cases we get a contradiction if we assume that two points of a line of $\mathcal{S}^*$ lie in B_0. ■

We now carefully explain how to use this Baer subspace to generate a new type of mixed partition of Π.

Theorem 6.4.2 *In $\Pi=\mathcal{PG}(3,q^2)$, q odd, there exists a mixed partition $\mathcal{P}_S$ with q^2+1 Baer subspaces and $q^4-q^3-q^2-q$ lines.*

Proof: We start with the regular spread $\mathcal{S}^*$ described above. By the previous theorem, we know that the natural Baer subspace meets each line of $\mathcal{S}^*$ in at most

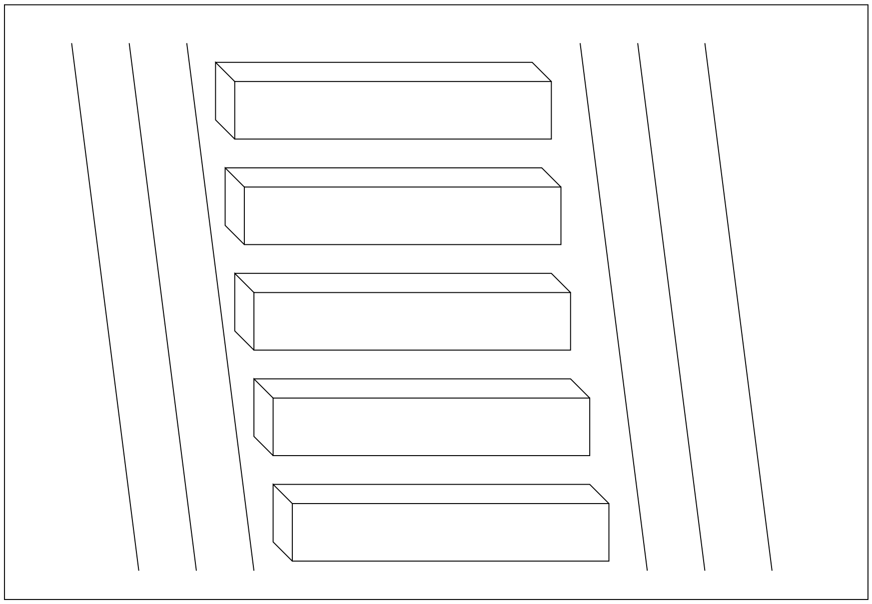

Figure 6.4: A partition from a regular spread

one point. Every regular spread has an associated Bruck Kernel [8]; that is, a cyclic group of order q^2+1 which acts regularly on the points of each line of the regular spread. Let ξ denote the Bruck Kernel associated with $\mathcal{S}^*$. By applying ξ to B_0, we get an orbit of q^2+1 Baer subspaces, pairwise disjoint by the regularity, that cover the points on the lines of $\mathcal{S}^*$ which intersect B_0 in exactly one point. These q^2+1 Baer subspaces together with the lines of $\mathcal{S}^*$ which do not intersect B_0 will form the desired mixed partition which we call $\mathcal{P}_S$ (see Figure 6.4). ■

6.4.3 Automorphisms of $\mathcal{P}_S$

We now examine the kernel of the translation plane which arises from the mixed partition described above.

Theorem 6.4.3 *The mixed partition $\mathcal{P}_S$ constructed above generates a translation plane which is 4-dimensional over its kernel.*

Proof: We appeal to the results from Chapter 4. Let $\mathcal{S}_S$ be the spread of Σ_0 which arises from the lifting of $\mathcal{P}_S$. Since $\mathcal{P}_S$ is proper, we know from Theorem 4.1.7 that any element of the full automorphism group of the lifted spread $\mathcal{S}_S$ which fixes each element of $\mathcal{S}_S$ must be a lifted automorphism of the mixed partition. There are exactly $q^4 - q^3 - q^2 - q$ lines in our mixed partition. A short induction argument shows that $q^4 - q^3 - q^2 - q > q + 1$ for any $q \geq 3$. Hence, there are more than $q + 1$ lines in our partition, which implies that the partition contains four lines not all contained in the same regulus. It follows, by Theorem 4.1.8, that the only automorphism of Π which fixes all of the lines of $\mathcal{P}_S$ is an element of the Bruck Kernel ξ for the associated regular spread $\mathcal{S}^*$. But ξ permutes the Baer subspaces in a cyclic orbit of length $q^2 + 1$. Hence, the only automorphism which fixes each member of the mixed partition is the identity. By Theorem 4.1.7, this implies that the only automorphism of $\mathcal{S}_S$ which fixes every element of $\mathcal{S}_S$ is the identity and so the plane $\pi(\mathcal{P}_S)$ must be 4-dimensional over its kernel. ■

We can also say a word about the automorphism group of the mixed partition $\mathcal{P}_S$. As noted earlier, the Bruck Kernel of the regular spread used to construct $\mathcal{P}_S$ certainly is a subgroup of the full automorphism group of $\mathcal{P}_S$. This group fixes each of the lines of $\mathcal{P}_S$ and acts regularly on the Baer subspaces of $\mathcal{P}_S$. In practice, for small values of q, there seems to be another cyclic group which acts on $\mathcal{P}_S$. This work is still in progress.

6.5 Associated Translation Planes

In this final section of Chapter 6, we will attempt to determine the type of translation plane constructed by each of the mixed partitions previously described. This process is, in general, quite difficult, and much of this work is still in progress. Our method will be to obtain information about the translation complement of such a plane by looking at automorphisms of the associated mixed partitions and appealing to Theorem 4.1.4.

We first look at the two partitions $\mathcal{P}_0'$ and $\mathcal{P}_0''$ constructed from the classical mixed partition $\mathcal{P}_0$. Recall that the partition $\mathcal{P}_0'$ is constructed by replacing 2 lines and $q-1$ Baer subspaces of $\mathcal{P}_0$ with q^2+1 lines. From the lifting process of Chapter 2, note that these $q-1$ Baer subspaces and 2 lines of $\mathcal{P}_0$ correspond to a set $\mathcal{N}$ of q^2+1 solids of $\mathcal{PG}(7,q)$. Moreover, the q^2+1 replacement lines used in $\mathcal{P}_0'$ correspond to a set $\mathcal{N}'$ of q^2+1 different solids of $\mathcal{PG}(7,q)$. It is not hard to see that each solid of $\mathcal{N}'$ intersects each solid of $\mathcal{N}$ in exactly $q+1$ points which necessarily form a line. Hence, we have replaced a set $\mathcal{N}$ of q^2+1 solids with a set $\mathcal{N}'$ of q^2+1 different solids such that any solid of $\mathcal{N}$ meets any solid of $\mathcal{N}'$ in a line. This is an example of net replacement as discussed in Section 1.5. From Theorem 1.5.1, this replacement corresponds to derivation in the associated plane. Since the derived Desarguesian plane is a Hall plane (see [21], for instance), we deduce that our mixed partition $\mathcal{P}_0'$ containing q^2+1 lines and $q^2(q-1)$ Baer subspaces generates a Hall plane. Note that this mixed partition corresponds to partition #5 in Table 6.2 when $q=2$.

Now consider $\mathcal{P}_0''$. Here we are replacing q^2+1 Baer subspaces of $\mathcal{P}_0$ with q^3+q^2+q+1 lines. This replacement corresponds to replacing a set $\mathcal{N}$ of q^3+q^2+q+1 solids with a set $\mathcal{N}'$ of q^3+q^2+q+1 different solids in the associated spread of Σ_0. From the lifting process one sees that every solid of $\mathcal{N}$ meets every solid of $\mathcal{N}'$ in a unique point. This is another example of net replacement as described in Section 1.5.

The group Θ whose orbits form the members of the classical partition $\mathcal{P}_0$ will certainly act on the new partition $\mathcal{P}_0''$. Let Ψ_Θ be the automorphism group of Σ_0 which comes from the lifting of Θ via Theorem 4.1.4. The members of the net $\mathcal{N}'$ are all contained in the same orbit of length q^3+q^2+q+1 under Ψ_Θ since Θ acts regularly on the points of each Baer subspace of $\mathcal{P}_0$. Moreover, Ψ_Θ fixes every other element of the lifted spread since Θ fixes each element of $\mathcal{P}_0$. Thus, we have a net with very similar properties to those of an Andrè net described in [22]. Hence, it seems likely (see [23]) that the mixed partition $\mathcal{P}_0''$ generates a generalized Andrè plane.

We now turn to the partition $\mathcal{P}_1$. The orbit structure on the line at infinity of $\pi(\mathcal{P}_1)$ given in Theorem 6.3.17 is not necessarily the orbit structure of the full automorphism group of $\pi(\mathcal{P}_1)$. That is, there may be automorphisms of the spread coming from $\mathcal{P}_1$ which are not lifted automorphisms of the mixed partition $\mathcal{P}_1$. Through additional calculations using *Magma* for small values of q, it appears as though the full translation complement of $\pi(\mathcal{P}_1)$ has one fixed point and one orbit of length q^4 on the line at infinity. We recall a theorem from [21].

Theorem 6.5.1 *The projective plane π is a semifield plane if and only if π is $((\infty),[\infty])$, $((0),[\infty])$, and $((\infty),[0])$-transitive.*

In particular, this says that the full automorphism group of a semifield plane fixes one point on the line at infinity and admits one orbit of length q^4 on the line at infinity. Computations using *Magma* for small q together with this theorem indicate that the plane $\pi(\mathcal{P}_1)$ is likely to be a semifield plane. This seems to be confirmed by the work of Johnson in [23].

The process of reversing the regulus of $\mathcal{P}_1$ to get $\mathcal{P}_2$ results in q^2+1 solids of the associated spread of $\mathcal{PG}(7,q)$ being replaced with q^2+1 different solids. As discussed in Section 1.5, this reversal process is equivalent to derivation, telling us that the plane $\pi(\mathcal{P}_2)$ is a derived semifield plane, assuming that $\pi(\mathcal{P}_1)$ is a semifield plane as above.

Finally, consider the partition $\mathcal{P}_S$. Just as we saw at the beginning of this chapter, the construction of this new family of mixed partitions comes from a type of replacement. Here we are replacing q^3+q^2+q+1 lines of a regular spread with q^2+1 Baer subspaces. In the associated spread of $\mathcal{PG}(7,q)$, this amounts to replacing a set $\mathcal{N}$ of q^3+q^2+q+1 solids of the spread with a set $\mathcal{N}'$ of q^3+q^2+q+1 different solids such that any solid of $\mathcal{N}$ meets any solid of $\mathcal{N}'$ in a unique point. One can see this immediately from the lifting process. Once again the set $\mathcal{N}$ appears to be an Andrè net (see [23]), and the translation plane coming from this mixed partition is believed to be a generalized Andrè plane. Proving the existence of an abelian group of order $(q^3+q^2+q+1)^2$ acting on the spread $\mathcal{S}_S$ will confirm that these planes are indeed generalized Andrè planes by the following theorem given in Johnson [22].

Theorem 6.5.2 *Let π be a translation plane of order q^n, $n>2$, and kernel containing $GF(q)$. If π admits an abelian collineation group G of order $((q^n-1)/(q-1))^2$ in the linear translation complement, then π is a generalized Andrè plane. Furthermore, if n is prime or $(n,q-1)=1$, then π is an Andrè plane.*

Chapter 7

CONCLUSION

We have seen that translation planes can be studied using spreads and mixed partitions. These different partitions are closely related as was seen in Chapter 2. We examined a method by which one could use a mixed partition to construct a translation plane directly, and we proved that this method is equivalent to the construction from the associated spread using the Bose/Andrè model.

The mixed partitions which give rise to regular spreads, and hence, Desarguesian affine planes, were completely determined. We examined the group theoretic relationships between mixed partitions and their associated spreads. The work in Chapter 4 laid the foundation for examining the automorphism groups of the spreads constructed in Chapter 6.

The relationship between "equivalent spreads" given in Chapter 5 is probably the most significant theoretical result of this thesis. A result of Lüneburg gives the relationship between spreads of the same dimensional space which generate isomorphic translation planes. We now know the relationship between spreads of *different* dimensional spaces which generate isomorphic translation planes. These two results together provide a unifying theory for spreads which generate isomorphic translation planes.

Finally, the work in Chapter 6 provides some specific examples of mixed partitions and associated spreads. This is an area with obvious growth potential. The examples of Chapter 6 are all in $\mathcal{PG}(3, q^2)$. That is, we constructed infinite families by allowing q to vary over all prime powers (some examples had q restricted to even or odd values), but no examples were generalized for higher dimensions. Because of the numerous examples in 3-space, the author certainly believes that such higher dimensional examples exist, but are simply hard to find.

Many of the examples for small q found in this thesis were discovered over long periods of time using the software package *Magma*. For instance, all mixed partitions of $\mathcal{PG}(3,4)$ were discovered by an exhaustive search. Then each partition was analyzed for special properties (like the existence of a regulus in the partition $\mathcal{P}_1$ of Section 6.3). Once such a property was discovered, this property was used to try to construct a similar partition in $\mathcal{PG}(3,9)$ or $\mathcal{PG}(3,16)$, for instance. This frequently required searching for special lines or Baer subspaces with certain properties. It is this type of searching which can take days, weeks, or longer. Once partitions are discovered for several values of q, we then try to use the examples to construct a general partition of $\mathcal{PG}(3,q^2)$ for an infinite number of values of q.

Increasing the dimension from 3 to some higher odd dimension forces the computer to work much harder and so far has produced no results. Also, some of the basic structures, like reguli, have quite different properties in $\mathcal{PG}(3,q^2)$ than they do in $\mathcal{PG}(2n-1,q^2)$, for $n \geq 3$. As a result, the partitions like $\mathcal{P}_1$ are not easy to generalize to higher dimensional spaces. Hence, to find mixed partitions of higher dimensional spaces, one needs to start with a clear idea of where and how to look.

The classification of a translation plane from its associated mixed partition is another area which needs to be explored. As we pointed out in Section 6.5, determining the type of translation plane generated from a mixed partition can be quite difficult. There are many known classification results, but these results usually require detailed knowledge of the group acting on the associated spread. It would be interesting to find geometric properties a of spread $\mathcal{S}$ (or a mixed partition) which would help determine the type of translation plane generated by $\mathcal{S}$.

There is also the possibility that other types of "mixed" partitions can be used to construct translation planes. For instance, if we use r^{th}-root subspaces of $\mathcal{PG}(2n-1, q^r)$ rather than Baer subspaces of $\mathcal{PG}(2n-1, q^2)$ to build a mixed partition, there may be a technique similar to the one in Section 1.4.2 which can be used to construct translation planes. The existence of such a technique has been explored by the author and will be further examined after the completion of this thesis.

It might also be interesting to see how the theory developed here can apply to other areas. In recent years the study of flag-transitive affine planes has become quite popular. It would be interesting to examine the kinds of mixed partitions which give rise to such planes. This may eventually lead to better classification results or maybe some new examples.

Appendix

KERNEL ARGUMENTS

In this appendix, we will find the kernel of the translation planes $\pi(\mathcal{P}_1)$ and $\pi(\mathcal{P}_2)$ as defined in Section 6.3. These arguments are quite technical and involve a large amount of matrix and field computations. Although most terminology is repeated here, the reader is referred back to Section 6.3 for more details about the mixed partitions and their construction.

In order the find the kernel of the translation planes associated with $\mathcal{P}_1$ and $\mathcal{P}_2$, we need to take a closer look at the Baer subspaces in the partitions. We start by examining some specific coordinates for the "lifting" of $\mathcal{B}_0$. Recall that the mixed partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ are defined for q even. We use this property in some of the computations.

A.1 Lifting the Baer Subspace $\mathcal{B}_0$

Recall the Baer subspace $\mathcal{B}_0$ of Π which lies in the mixed partitions $\mathcal{P}_1$ and $\mathcal{P}_2$. The points induced by the vectors

$$\mathbf{p}_1 = (\alpha^4, \alpha^3, 1, 0),$$

$$\mathbf{p}_2 = (\alpha, \alpha^2, 0, 1),$$

$$\mathbf{p}_3 = (\alpha^3, \alpha^2, 1, 0),$$

$$\mathbf{p}_4 = (1, \alpha, 0, 1),$$

and

$$\mathbf{p}_5 = \left(\frac{\alpha^3+1}{\alpha}, \frac{1}{\beta}, 1, 1\right)$$

generate $\mathcal{B}_0$, and the vectors have the property that

$$\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \beta^q\mathbf{p}_5.$$

Using the embedding of Γ into Σ from Chapter 2, the vectors which induce the images of the points induced by the above vectors are given by

$$\mathbf{p}_1 = (\alpha^4, \alpha^3, 1, 0, \alpha^5, \alpha^4, \alpha, 0)$$

$$\mathbf{p}_2 = (\alpha, \alpha^2, 0, 1, \alpha^2, \alpha^3, 0, \alpha)$$

$$\mathbf{p}_3 = (\alpha^3, \alpha^2, 1, 0, \alpha^4, \alpha^3, \alpha, 0)$$

$$\mathbf{p}_4 = (1, \alpha, 0, 1, \alpha, \alpha^2, 0, \alpha)$$

$$\mathbf{p}_5 = \left(\frac{\alpha^3+1}{\alpha}, \frac{1}{\beta}, 1, 1, \alpha^3+1, \frac{\alpha}{\beta}, \alpha, \alpha\right).$$

Note that we abuse notation and continue to use $\mathbf{p}_i$ to denote the embedded $\mathbf{p}_i$.

The points on the line $l_{\mathbf{p}_i}$ in Σ_0 are induced by all vectors in the set

$$\left\{\alpha^j\mathbf{p}_i^q + \mathbf{p}_i : j \in \{0, 1, 2, ..., q\}\right\}.$$

We temporarily fix j and consider the following vectors:

$$\alpha^{j+1}\mathbf{p}_1^q + \mathbf{p}_1 = \hat{\mathbf{p}}_{1,j}$$

$$\alpha^{j+1}\mathbf{p}_2^q + \mathbf{p}_2 = \hat{\mathbf{p}}_{2,j}$$

$$\alpha^{j}\mathbf{p}_3^q + \mathbf{p}_3 = \hat{\mathbf{p}}_{3,j}$$

$$\alpha^{j}\mathbf{p}_4^q + \mathbf{p}_4 = \hat{\mathbf{p}}_{4,j}$$

$$\alpha^{j}(\beta^q\mathbf{p}_5)^q + (\beta^q\mathbf{p}_5) = \hat{\mathbf{p}}_{5,j}$$

Then each of the $\hat{\mathbf{p}}_i$'s induces a point of Σ_0. Since

$$\beta\hat{\mathbf{p}}_{1,j} + \beta\hat{\mathbf{p}}_{2,j} + \hat{\mathbf{p}}_{3,j} + \hat{\mathbf{p}}_{4,j}$$

$$
\begin{aligned}
&= (\beta\alpha^{j+1}\mathbf{p}_1^q + \beta\mathbf{p}_1) + (\beta\alpha^{j+1}\mathbf{p}_2^q + \beta\mathbf{p}_2) + (\alpha^j\mathbf{p}_3^q + \mathbf{p}_3) + (\alpha^j\mathbf{p}_4^q + \mathbf{p}_4) \\
&= (\beta\alpha^{j+1}\mathbf{p}_1^q + \beta\alpha^{j+1}\mathbf{p}_2^q + \alpha^j\mathbf{p}_3^q + \alpha^j\mathbf{p}_4^q) + (\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
&= \alpha^j\,(\beta\alpha\mathbf{p}_1^q + \beta\alpha\mathbf{p}_2^q + \mathbf{p}_3^q + \mathbf{p}_4^q) + (\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
&= \alpha^j\,(\beta^q\mathbf{p}_1^q + \beta^q\mathbf{p}_2^q + \mathbf{p}_3^q + \mathbf{p}_4^q) + (\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
&= \alpha^j\,(\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)^q + (\beta\mathbf{p}_1 + \beta\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
&= \alpha^j(\beta^q\mathbf{p}_5)^q + (\beta^q\mathbf{p}_5) \\
&= \hat{\mathbf{p}}_{5,j}
\end{aligned}
$$

we immediately get that these vectors generate a Baer subspace $\hat{B}_j$. By Lemma 2.1.7, there is a unique solid of Σ_0 through the point induced by $\hat{\mathbf{p}}_{5,j}$ that intersects each of the lines $l_{\mathbf{p}_i}$ in a point. Hence, we have found generators for one of the 3-spaces, namely $\hat{B}_j$, in the regulus of Σ_0 defined by $\mathcal{B}_0$. To get the other solids in the regulus, simply let j vary, $0 \leq j \leq q$. This gives us the $q+1$ solids in the regulus determined by $\mathcal{B}_0$.

A.2 The Kernel of $\pi(\mathcal{P}_1)$

Let $\mathcal{S}_1$ be the 3-spread of Σ_0 constructed by lifting the mixed partition $\mathcal{P}_1$ as described in Chapter 2. We look for the kernel of $\pi(\mathcal{P}_1)$ by considering restrictions on a matrix which induces a collineation of Σ_0 fixing each element of the spread $\mathcal{S}_1$. Let T_ϕ be an 8×8 matrix all of whose entries are in F, and let ϕ be the collineation induced by T_ϕ on the space Σ_0. Moreover, assume that ϕ fixes each member of the spread $\mathcal{S}_1$. From the work in Section 6.3.4 and the group relations of Section 4.1, we can show that T_ϕ has the following general form.

$$T_\phi = \begin{bmatrix} a & b & 0 & 0 & e & f & 0 & 0 \\ c & d & 0 & 0 & g & h & 0 & 0 \\ 0 & 0 & d & c & 0 & 0 & h & g \\ 0 & 0 & b & a & 0 & 0 & f & e \\ e & f & 0 & 0 & a+\frac{1}{\omega}e & b+\frac{1}{\omega}f & 0 & 0 \\ g & h & 0 & 0 & c+\frac{1}{\omega}g & d+\frac{1}{\omega}h & 0 & 0 \\ 0 & 0 & h & g & 0 & 0 & d+\frac{1}{\omega}h & c+\frac{1}{\omega}g \\ 0 & 0 & f & e & 0 & 0 & b+\frac{1}{\omega}f & a+\frac{1}{\omega}e \end{bmatrix}$$

where a, b, c and d are arbitrary elements of F. This is obtained by assuming that ϕ fixes each solid of $\mathcal{S}_1$ which arises from a lifted line of the mixed partition $\mathcal{P}_1$. To complete our analysis of the kernel, we must assume that ϕ fixes the solid $\hat{B}_j$. To do this, consider the two vectors

$$\hat{\mathbf{p}}_{3,j} = \left(1+\omega^2, \omega, 0, 0, \frac{1}{\omega}, 1+\omega^2, \omega^2, 0\right)$$

and

$$\hat{\mathbf{p}}_{4,j} = \left(0, 1, 0, 0, 1, \frac{1}{\omega}, 0, 1, \right).$$

which are generators of $\hat{B}_j$. The images of the induced points under the collineation ϕ are induced by the vectors

$$\begin{aligned} \hat{\mathbf{p}}_3 T_\phi = \Bigg(& (a+g)(1+\omega^2) + c\omega + e\frac{1}{\omega}, (b+h)(1+\omega^2) + d\omega + f\frac{1}{\omega}, \omega^2 h, \omega^2 g, \\ & (e+c+\frac{1}{\omega}g)(1+\omega^2) + g\omega + \left(a + \frac{1}{\omega}e\right)\frac{1}{\omega}, \\ & (f+d+\frac{1}{\omega}h)(1+\omega^2) + h\omega + \left(b+\frac{1}{\omega}f\right)\frac{1}{\omega}, \omega^2 d + \omega h, \omega^2 c + \omega g \Bigg) \end{aligned}$$

and

$$\begin{aligned} \hat{\mathbf{p}}_4 T_\phi = \Bigg(& c + e + \frac{1}{\omega}g, d + f + \frac{1}{\omega}h, f, e, g + \frac{1}{\omega}e + a + \frac{1}{\omega}c + \frac{1}{\omega^2}g, \\ & h + \frac{1}{\omega}f + b + \frac{1}{\omega}d + \frac{1}{\omega^2}h, b + \frac{1}{\omega}f, a + \frac{1}{\omega}e \Bigg). \end{aligned}$$

To ensure that these images under ϕ are still in $\hat{B}_j$, we use the defining equations for $\hat{B}_j$, or equivalently, the basis vectors for the orthogonal complement of the vector space representing $\hat{B}_j$ inside the vector space V of dimension 8 over $GF(q)$ which we use to model Σ_0. These vectors are given by

$$\mathbf{o}_1 = (1, \omega, 0, 1, \omega, 0, 0, 0),$$

$$\mathbf{o}_2 = (0, 1, 1, \omega, 0, \omega, \omega, 0),$$
$$\mathbf{o}_3 = \left(\omega, 0, 0, \frac{1}{\eta}, 0, w, 0, 1\right),$$

and

$$\mathbf{o}_4 = \left(1, \frac{1}{\omega}, \omega, 1, \omega, 1, 0, \omega\right).$$

We now check that these vectors are linearly independent and span the above orthogonal complement. To show that these are linearly independent, consider a linear combination of these vectors which yields the zero vector, and let a, b, c and d be the respective coefficients. Then by looking at the seventh coordinate, b must be 0. By looking at the third coordinate, we see that d is 0. But now the eighth coordinate makes $c = 0$, which imediately implies that $a = 0$ as well. Hence, these vectors are linearly independent. To show that they span the orthogonal complement of the vector space which induces $\hat{B}_j$, one can easily check that the appropriate dot products are all 0.

In order to achieve more restrictions on the entries of the matrix T_ϕ, we dot each of the above vectors with the vectors $\hat{\mathbf{p}}_3 T_\phi$ and $\hat{\mathbf{p}}_4 T_\phi$ to achieve the following system of linear equations:

	dot product		equation
A:	$\hat{\mathbf{p}}_4 T_\phi \cdot \mathbf{o}_1$	$0 =$	$e + h + \omega(d + f + g + a)$
B:	$\hat{\mathbf{p}}_4 T_\phi \cdot \mathbf{o}_2$	$0 =$	$e + h$
C:	$\hat{\mathbf{p}}_4 T_\phi \cdot \mathbf{o}_3$	$0 =$	$\omega(c + b) + \frac{1}{\omega}h + (g + f + d + a) + \omega(e + h)$
D:	$\hat{\mathbf{p}}_4 T_\phi \cdot \mathbf{o}_4$	$0 =$	$\omega(f + g) + h + b$
E:	$\hat{\mathbf{p}}_3 T_\phi \cdot \mathbf{o}_1$	$0 =$	$\omega^3(b + h + e + c) + \omega^2(a + d) + \omega(b + h + e) + f$
F:	$\hat{\mathbf{p}}_3 T_\phi \cdot \mathbf{o}_2$	$0 =$	$\omega^3(g + f) + \omega^2(b + h) + \omega f$
G:	$\hat{\mathbf{p}}_3 T_\phi \cdot \mathbf{o}_3$	$0 =$	$\omega(1 + \omega^2)(a + g + f + d) + h + e + b + \frac{1}{\omega}f$

First we note that D and F together imply that $f = 0$, and we can eliminate F since it adds no new information. We can also cancel ω from E giving us

$$
\begin{aligned}
A:&\quad 0 = e + h + \omega(d + g + a)\\
B:&\quad 0 = e + h\\
C:&\quad 0 = \omega(c + b) + \tfrac{1}{\omega}h + (g + d + a) + \omega(e + h)\\
D:&\quad 0 = \omega g + h + b\\
E:&\quad 0 = \omega^2(b + h + e + c) + \omega(a + d) + b + h + e\\
G:&\quad 0 = \omega(1 + \omega^2)(a + g + d) + h + e + b
\end{aligned}
$$

Plugging B into A, C, E and G, and then using A in C and G reduces the system. In particular, we get from G that $b = 0$. Hence, the system is reduced to

$$
\begin{aligned}
A:&\quad 0 = d + g + a\\
B:&\quad 0 = e + h\\
C:&\quad 0 = \omega c + \tfrac{1}{\omega}h\\
D:&\quad 0 = \omega g + h\\
E:&\quad 0 = \omega^2 c + \omega(a + d)
\end{aligned}
$$

We can now deduce that all of the variables can be written in terms of a and c.

Hence, we can write

$$T_\phi = \begin{bmatrix} a & 0 & 0 & 0 & \omega^2 c & 0 & 0 & 0 \\ c & a+\omega c & 0 & 0 & \omega c & \omega^2 c & 0 & 0 \\ 0 & 0 & a+\omega c & c & 0 & 0 & \omega^2 c & \omega c \\ 0 & 0 & 0 & a & 0 & 0 & 0 & \omega^2 c \\ \omega^2 c & 0 & 0 & 0 & a+\omega c & 0 & 0 & 0 \\ \omega c & \omega^2 c & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & \omega^2 c & \omega c & 0 & 0 & a & 0 \\ 0 & 0 & 0 & \omega^2 c & 0 & 0 & 0 & a+\omega c \end{bmatrix}.$$

We have already shown in Corollary 6.3.14 that the kernel of $\pi(\mathcal{P}_1)$ is at least $GF(q^2)$ by showing that the kernel of $\pi(\mathcal{P}_1)$ has a subgroup isomorphic to the group generated by τ. But we have just shown that the kernel can contain, at most, collineations induced by matrices of the form given above. There are only $q^2 - 1$ such matrices (by letting a and c vary, but not both 0). Note that all scalar multiples of the identity matrix are in this list (let $c = 0$). So the number of collineations produced is exactly $q + 1$. Hence, every matrix of the form given above induces a collineation in the kernel of $\pi(\mathcal{P}_1)$ and we have a plane which is exactly 2-dimensional over its kernel. We restate Theorem 6.3.15.

Theorem A.2.1 *The translation plane $\pi(\mathcal{P}_1)$ is 2-dimensional over its kernel.*

A.3 The Kernel of $\pi(\mathcal{P}_2)$

By following the same basic procedure as above, we can determine that a given 8×8 matrix which induces a collineation fixing each solid coming from a line of $\mathcal{P}_2$ has the general form

$$T_\phi = \begin{bmatrix} a & 0 & 0 & b & e & 0 & 0 & f \\ 0 & a & b & 0 & 0 & e & f & 0 \\ 0 & c & d & 0 & 0 & g & h & 0 \\ c & 0 & 0 & d & g & 0 & 0 & h \\ e & 0 & 0 & f & a+\frac{1}{\omega}e & 0 & 0 & b+\frac{1}{\omega}f \\ 0 & e & f & 0 & 0 & a+\frac{1}{\omega}e & b+\frac{1}{\omega}f & 0 \\ 0 & g & h & 0 & 0 & c+\frac{1}{\omega}g & d+\frac{1}{\omega}h & 0 \\ g & 0 & 0 & h & c+\frac{1}{\omega}g & 0 & 0 & d+\frac{1}{\omega}h \end{bmatrix},$$

where a, b, c and d are arbitrary elements of F. To complete our analysis of the kernel, we again assume that ϕ, the collineation induced by the matrix T_ϕ, fixes the solid $\hat{B}_j$. Consider the two vectors which generate points of $\hat{B}_j$ given by

$$\hat{\mathbf{p}}_{3,j} = \left(1+\omega^2, \omega, 0, 0, \frac{1}{\omega}, 1+\omega^2, \omega^2, 0\right),$$

and

$$\hat{\mathbf{p}}_{4,j} = \left(0, 1, 0, 0, 1, \frac{1}{\omega}, 0, 1, \right).$$

The images of the induced points under the collineation ϕ are induced by the vectors

$$\begin{aligned}\hat{\mathbf{p}}_{3,j} = \Bigg(& a(1+\omega^2) + \frac{1}{\omega}e, \omega a + (1+\omega^2)e + \omega^2 g, \omega b + (1+\omega^2)f + \omega^2 h, b(1+\omega^2) + \frac{1}{\omega}f, \\ & e(1+\omega^2) + \frac{1}{\omega}\left(a + \frac{1}{\omega}e\right), \omega e + (1+\omega^2)\left(a+\frac{1}{\omega}e\right) + \omega^2\left(c + \frac{1}{\omega}g\right), \\ & \omega f + (1+\omega^2)\left(b + \frac{1}{\omega}f\right) + \omega^2\left(d+\frac{1}{\omega}h\right), f(1+\omega^2) + \frac{1}{\omega}\left(b+\frac{1}{\omega}f\right)\Bigg)\end{aligned}$$

and

$$\begin{aligned}\hat{\mathbf{p}}_{4,j} = \Bigg(& e+g, a+\frac{1}{\omega}e, b+\frac{1}{\omega}f, f+h, a+\frac{1}{\omega}e + c + \frac{1}{\omega}g, \\ & e + \frac{1}{\omega}\left(a+\frac{1}{\omega}e\right), f+\frac{1}{\omega}\left(b+\frac{1}{\omega}f\right), b+\frac{1}{\omega}f + d + \frac{1}{\omega}h\Bigg).\end{aligned}$$

To ensure that these images under ϕ are still in $\hat{B}_j$, we recall the basis vectors for the orthogonal complement of the vector space which induces $\hat{B}_j$, namely

$$(1, \omega, 0, 1, \omega, 0, 0, 0),$$

$$(0, 1, 1, \omega, 0, \omega, \omega, 0),$$

$$\left(\omega, 0, 0, \frac{1}{\omega}, 0, w, 0, 1\right),$$

and

$$\left(1, \frac{1}{\omega}, \omega, 1, \omega, 1, 0, \omega\right).$$

To achieve more restrictions on the entries of T_ϕ, we dot the images given previously with the above basis vectors of the orthogonal complement. We end up with the following system of linear equations:

	dot product		equation
A:	$v_2 \cdot \mathbf{o}_1$	$0 =$	$e + f + h + \omega c$
B:	$v_2 \cdot \mathbf{o}_2$	$0 =$	$\frac{1}{\omega}e + \omega h + \omega e + \frac{1}{\omega}g$
C:	$v_2 \cdot \mathbf{o}_3$	$0 =$	$\omega g + a + \frac{1}{\omega}g + b + d$
D:	$v_2 \cdot \mathbf{o}_4$	$0 =$	$\frac{1}{\omega^2}e + \omega a + e + \omega c + \frac{1}{\omega^2}g + f + \omega d$
E:	$v_1 \cdot \mathbf{o}_1$	$0 =$	$\omega^3 g + b + \omega^2 b + \frac{1}{\omega}f$
F:	$v_1 \cdot \mathbf{o}_2$	$0 =$	$\omega^3 a + \omega^2 e + \omega^3 c + \omega^2 f + \omega b + f + \omega^3 d$
G:	$v_1 \cdot \mathbf{o}_3$	$0 =$	$\omega b + \omega^3 c + \omega^2 g + f + \omega^2 f$

Adding together F and G and dividing by ω^2 gives us $\omega a + e + \omega d + g = 0$. But $\frac{1}{\omega}B + D + A$ reduces to $\omega a + \omega d + e = 0$. Hence, $g = 0$. Adding $\frac{1}{\omega}B + A$ yields the equation

$$H : \frac{1}{\omega^2}e + f + \omega c = 0.$$

We consider $\omega^2 H + \omega^3 C + F + \omega E$. This reduces to $\omega^2 e + e = 0$. The only possibility is that $e = 0$. Hence, $g = e = 0$, and we can rewrite the system as

$$
\begin{aligned}
A\colon\quad 0 &= f + h + \omega c \\
B\colon\quad 0 &= \omega h \\
C\colon\quad 0 &= a + b + d \\
D\colon\quad 0 &= \omega a + \omega c + f + \omega d \\
E\colon\quad 0 &= b + \omega^2 b + \tfrac{1}{\omega} f \\
F\colon\quad 0 &= \omega^3 a + \omega^3 c + \omega^2 f + \omega b + f + \omega^3 d \\
G\colon\quad 0 &= \omega b + \omega^3 c + f + \omega^2 f \\
H\colon\quad 0 &= f + \omega c
\end{aligned}
$$

Hence, by equation B, $h = 0$ the thus $f = \omega c$ by A. Equation D then implies $a = d$ and from C we obtain $b = 0$. But then E implies $f = 0$, which in turn implies that $c = 0$ from H. Hence, every variable is 0, except for a and d which are equal, and the general form for T_ϕ is

$$
T_\phi = \begin{bmatrix}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{bmatrix}.
$$

Therefore, we have shown that the only linear collineation of Σ_0 which fixes each element of the spread $\mathcal{S}_2$ is the identity. Hence, the kernel of the translation plane $\pi(\mathcal{P}_2)$ is isomorphic to $GF(q)$. We restate Theorem 6.3.16.

Theorem A.3.1 *The translation plane $\pi(\mathcal{P}_2)$ is 4-dimensional over its kernel.*

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