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A geometric approach to Mathon maximal arcs

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Preface

The most beautiful experience we can have is the mysterious. It is the fundamental emotion which stands at the cradle of true art and true science. Whoever does not know it and can no longer wonder, no longer marvel, is as good as dead, and his eyes are dimmed.

Albert Einstein

In October 2007 I started my work as a PhD student at the Department of Mathematics at Ghent University. The result of three and a half years of research is presented in this thesis.

In the first chapter some of the geometric background needed throughout this thesis is covered. We define several incidence geometries which are important in view of a good comprehension of the later chapters.

Chapter 2 covers the several known constructions of maximal arcs as well as some of their characterizations. We discuss these constructions chronologically while adding necessary information regarding the role of these geometric structures in this thesis. After having described Mathon's construction of maximal arcs we add some results concerning the conics and substructures of the arcs along with a geometric characterization of the Denniston arcs. Next we formulate a lemma of Aguglia, Giuzzi and Korchmáros which will prove to be of considerable interest in the following chapters. Finally, we end Chapter 2 by proving that a maximal arc, consisting of disjoint conics on a common nucleus, is always of Mathon type.

In april 2009 I enjoyed a first stay in California, where I visited UC San Diego, the university at which one of my supervisors, Stefaan De Winter, was working at the time. It was during these two months that we started working on the topic of “maximal arcs in projective planes”. The results in Chapter 3 are based on the findings we obtained in California. More specifically, we were able to prove some kind of generalization of the lemma of Aguglia, Giuzzi and Korchmáros which states that two conics on a common nucleus induce a unique degree-4 maximal arc of Denniston type. This generalization can in fact be seen as a geometric approach to Mathon’s construction of maximal arcs. After acquiring enough knowledge about this different approach we were quite sure it would be possible to solve the enumeration of the non-isomorphic Mathon 8-arcs in $\mathrm{PG}(2, 2^h)$, where h is prime. This is, in fact, the main result in Chapter 3. However, we did face a problem when considering the case $h = 7$ since the obtained number, using our formula, was not even an integer.

We were determined on solving the intriguing gap that was left in the case $\mathrm{PG}(2, 2^7)$ and wanted to study this particular family of maximal arcs more closely. Chapter 4 is dedicated to the maximal arcs of Mathon type that arise in $\mathrm{PG}(2, 2^7)$. This special class of Mathon arcs, which is described in detail throughout this chapter, admits a Singer group on the seven conics of these arcs. The explicit research concerning this class of arcs enabled us to count the total number of non-isomorphic Mathon maximal arcs of degree 8. Moreover, it turns out that the specific case we spotted in $\mathrm{PG}(2, 2^7)$ extends to two infinite families of Mathon arcs of degree 8 in $\mathrm{PG}(2, 2^k)$, k odd and divisible by 7, that maintain the nice property of admitting a Singer group.

During a talk I gave regarding the Singer arcs at a seminar at Ghent University J. A. Thas brought up the link between maximal arcs of Mathon type and partial flocks of the quadratic cone. This relation appeared to be of a rather algebraic nature. I consulted literature about this particular connection and considered the possibility of a more geometric link between the two incidence structures. During a scientific collaboration with my supervisors Frank De Clerck and Stefaan De Winter at Ohio University, we managed to complete a projection of a maximal arc of Mathon type onto a partial flock of the quadratic cone. Chapter 5 deals with this geometric connection. The established projection helped us when defining a composition on the planes associated to a partial flock and this allowed us to form an analogue of the synthetic version of Mathon’s Theorem, obtained in Chapter 3. Furthermore, we worked out some nice properties regarding the Denniston lines of a Mathon arc as well as a construction of such a maximal arc of Mathon type of degree $2d$, containing a Denniston arc of degree d, and provided that there is a solution to a certain given system of trace conditions.

Acknowledgements

It is obvious that I wouldn't have succeeded in completing this thesis if it wasn't for a few people.

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Next, I would like to thank my supervisors Frank De Clerck and Stefaan De Winter. Although they both have a very busy agenda, they still managed to find the time to answer my questions, provide new ideas, read my manuscripts and improve my mathematical writing skills. The memorable months we spent in San Diego, California, and Athens, Ohio, were wonderful in many ways.

I would also like to express my gratitude to Jacques Verstraete. He made sure that my stay in San Diego and at UCSD would be as comfortable as possible.

Many colleagues at the department helped me out in some way or another. Special thanks go to Jef Thas for pointing out the known link between maximal arcs of Mathon type and additive partial flocks. To Jan De Beule for assisting me during computer issues, and to Frédéric and Michiel, who I started my work as a PhD student with, for the pleasant times on many occasions.

Finally, I want to thank my family, mainly my parents, for giving me the opportunity to study, for their interest and for years of support on so many levels. Also thanks to my close friends, most of all Anita, for being there.

Thomas Maes

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Contents

Chapter 1

Introduction

This introductory chapter is meant as a quick overview of the geometric background needed throughout this thesis. We define a few geometric structures that will appear in later chapters and fix the notation we will use.

1.1 Incidence geometries

Definition 1.1.1. An *incidence geometry of rank n* is defined as an ordered set $(S, \mathrm{I}, \Delta, \sigma)$, where S is a non-empty set of *varieties*, I is a binary symmetric *incidence relation* between elements of S, Δ is a finite set of size n and σ is a surjective *type map* from S to Δ, such that no ordered pair of elements of S of the same type is in I.

A *point-line geometry*, or short *geometry*, is an incidence geometry of rank 2. Here the varieties of the two types are called *points* and *lines*. Such a geometry will also be denoted as a triple $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points and $\mathcal{B}$ is the set of lines. If $(p, b) \in \mathcal{P} \times \mathcal{B}$ or $(p, b) \in \mathcal{B} \times \mathcal{P}$ such that $p \,\mathrm{I}\, b$ we say that p is *incident with b*.

In most cases, the lines will be subsets of the point set $\mathcal{P}$ and the incidence relation I will then simply be the symmetrized containment.

The dual of a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the geometry $\mathcal{S}^D = (\mathcal{B}, \mathcal{P}, \mathrm{I})$.

We say that two points are *collinear* if they are contained in a line and, dually, we say that two lines are *concurrent* if and only if they have a non-empty intersection.

Collinear points x and y will be denoted by $x \sim y$, concurrent lines L and M will be denoted by $L \sim M$, while $x \not\sim y$ (resp. $L \not\sim M$) means that x and y are not collinear (resp. L and M are not concurrent). Note that $x \sim x$ and $L \sim L$.

1.1.1 Isomorphisms and automorphisms

Definition 1.1.2. An *isomorphism* from $(S, \mathrm{I}, \Delta, \sigma)$ to $(S', \mathrm{I}', \Delta, \sigma')$ is a bijection $\phi : S \to S'$ such that $\phi(x) \, \mathrm{I} \, \phi(y) \Leftrightarrow x \, \mathrm{I} \, y$ and $\sigma(x) = \sigma(y) \Leftrightarrow \sigma'(\phi(x)) = \sigma'(\phi(y))$, $\forall x, y \in S$. An *automorphism* of $(S, \mathrm{I}, \Delta, \sigma)$ is an isomorphism from $(S, \mathrm{I}, \Delta, \sigma)$ into itself.

Whenever there exists an isomorphism from an incidence geometry A to an incidence geometry B we will refer to A and B as *isomorphic*.

The automorphisms of an incidence geometry A form a group, called *the automorphism group*, and will be denoted by $\mathrm{Aut}(A)$. A subgroup of $\mathrm{Aut}(A)$ will be called *an* automorphism group.

1.1.2 Partial linear spaces

Definition 1.1.3. A finite geometry $\mathcal{S}$ is called a *partial linear space* if and only if two distinct points are incident with at most one line, or equivalently, if any two distinct lines are incident with at most one point. If every two distinct points are collinear, then we say that $\mathcal{S}$ is a *linear space*.

In this thesis we will only deal with partial linear spaces satisfying the next two properties.

- Each point is incident with $t + 1$ lines ($t \geq 1$).
- Each line is incident with $s + 1$ points ($s \geq 1$).

A partial linear space $\mathcal{S}$ satisfying these conditions is said to be of *order* (s, t), or, if $s = t$, of order s.

A *spread* of a partial linear space $\mathcal{S}$ is a set of lines of $\mathcal{S}$ partitioning the point set of $\mathcal{S}$.

1.2 Designs

Definition 1.2.1. Let t, k, v, λ be integers with $t < k < v$ and $\lambda > 0$. A $t-(v, k, \lambda)$ *design*, or t-*design* with parameters (v, k, λ), is a set S of v elements together with a set of k-subsets of S, called *blocks*, such that any t distinct elements of S are contained in exactly λ blocks.

A $t-(v, k, 1)$ design is also called a *Steiner system*.

Some useful known conditions between the parameters of a t-design are the following.

- The number b of blocks of a $t-(v, k, \lambda)$ design is given by

$$b = \lambda \binom{v}{t} \Big/ \binom{k}{t}.$$

- For $s \leq t$, a $t-(v, k, \lambda)$ design is also an $s-(v, k, \lambda_s)$ design, where

$$\lambda_s = \lambda \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}.$$

1.3 Projective spaces

Although we assume the reader to be familiar with the basics of projective geometry and finite classical polar spaces, we will recall some fundamental notions in order to fix notation and terminology.

Definition 1.3.1. The vector space of dimension $n+1$ over the finite field $\mathrm{GF}(q)$, with $q = p^h$, p prime and $h \geq 1$, is denoted by $\mathrm{V}(n+1, q)$. Let $D(\mathrm{V})$ be the set of proper, non-trivial subspaces of $\mathrm{V}(n+1, q)$ and define the incidence relation I as follows: $U \,\mathrm{I}\, W \Leftrightarrow U \subset W$ or $W \subset U$. The incidence geometry $(D(\mathrm{V}), \mathrm{I}, \Delta, \sigma)$ of rank n, with $\Delta = \{1, \ldots, n\}$, where σ maps each subspace onto its vectorial dimension, is called the *projective space* corresponding to

$\mathrm{V}(n+1, q)$. This projective space has *projective dimension* n and is denoted by $\mathrm{PG}(n, q)$.

In $\mathrm{V}(n+1, q)$, a subspace of dimension $i+1$, $i \geq -1$, is said to have projective dimension i and will be called an i-dimensional projective subspace. Since we will generally be working with projective spaces, we will simply use the term dimension rather than projective dimension. Subspaces of dimension $0, 1, 2$ and $n-1$ of $\mathrm{PG}(n, q)$ are called points, lines, planes and hyperplanes, respectively. The (-1)-dimensional projective space is the empty space.

We remark that the dual of a projective space $\mathrm{PG}(n, q)$ is again a $\mathrm{PG}(n, q)$. We say that $\mathrm{PG}(n, q)$ is *self-dual*.

Definition 1.3.2. The n-dimensional *affine space* $\mathrm{AG}(n, q)$ over the finite field $\mathrm{GF}(q)$ is obtained from the projective geometry $\mathrm{PG}(n, q)$ by designating a hyperplane H as being "at infinity" and deleting it, together with all the subspaces it contains.

A projective space can also be defined in an axiomatic way as an incidence geometry, called *axiomatic projective space*, satisfying the following axioms.

- For every two distinct points, there is exactly one line incident with both.
- If p_1, p_2, p_3 and p_4 are four distinct points, such that the lines p_1p_2 and p_3p_4 intersect, then the lines p_1p_3 and p_2p_4 intersect as well.
- Each line contains at least three points.

A *subspace* of an (axiomatic) projective space is a subset S of points, such that every line containing at least two points of S, is contained in S. The *axiomatic dimension* of a projective space is the largest number n for which there is a strictly increasing chain of subspaces satisfying $\emptyset \subset S_0 \subset \cdots \subset S_n = P$, where P denotes the full set of points.

It can be shown (see for instance [59]) that the only finite axiomatic projective spaces of dimension at least three are the projective spaces $\mathrm{PG}(n, q)$. We recall that projective spaces can also be defined over infinite (skew) fields, however, this is beyond the scope of this thesis.

An *axiomatic finite projective plane of order* q is actually a $2-(q^2+q+1, q+1, 1)$ design, and an *axiomatic finite affine plane of order* q, is a $2-(q^2, q, 1)$ design and conversely.

We say that a finite projective plane is *Desarguesian* if it is isomorphic to a $PG(2, q)$, for some prime power q. Several constructions of finite non-Desarguesian projective planes are known. The classification however is far from done (see for instance [36]) and in fact seems to be unfeasable with today's techniques.

Whenever we say that a projective subspace $PG(d, q)$ is *embedded* in a projective space $PG(n, q)$, $n > d$, we simply mean that $PG(d, q)$ is contained in $PG(n, q)$.

In Section 1.1.2 we have defined an isomorphism between two incidence geometries. In case the incidence geometries are projective spaces, an isomorpishm is called a *collineation*. Actually a collineation φ from $\Sigma_1 \cong PG(n, q)$ onto $\Sigma_2 \cong PG(n, q)$, $n \geq 2$, can be defined as a bijection between the points of Σ_1 and the points of Σ_2 such that three collinear points of Σ_1 are mapped onto three collinear points of Σ_2, which implies that incidence is preserved. If $\Sigma_1 = \Sigma_2$, then we will simply say that φ is *a collineation of* Σ_1. By the fundamental theorem of projective geometry, the *collineation group* of $PG(n, q)$ is the group $P\Gamma L(n + 1, q)$, induced by the non-singular semi-linear automorphisms of the vector space $V(n+1, q)$. Hence, every collineation φ of $PG(n, q)$ can be algebraically described as $\varphi : x \mapsto Ax^\sigma$, with A a non-singular $(n+1) \times (n+1)$ matrix and σ a field automorphism of $GF(q)$. We will often write $\varphi(\sigma, A)$. If $\sigma = 1$, then the collineation $\varphi(1, A) = \varphi(A)$ is called a *projectivity* and the group of all projectivities of $PG(n, q)$ is the *projective linear group* or *linear collineation group* $PGL(n + 1, q)$.

A collineation φ of $PG(n, q)$ is called a *perspectivity* if there exists a hyperplane H of $PG(n, q)$ and a point x, such that every point of H, and every hyperplane through x, is fixed by φ. The point x is called the *center* and H the *axis* of the perspectivity φ. If $x \in H$ then the perspectivity φ is called an *elation*, if $x \notin H$ then φ is called a *homolgy*.

A collineation from an n-dimensional projective space Σ onto its dual space Σ^D is called a *reciprocity* of Σ, and hence can also be denoted by $\varphi(\sigma, A)$. A reciprocity is also called a *correlation*, although some authors use this terminology only in the case $\sigma = 1$. If this correlation has order 2 it is called a *polarity*. The image of a subspace V under a polarity is denoted by $V^\perp$ and is called the *polar space of* V. If V is a subspace such that $V \subset V^\perp$, or $V^\perp \subset V$, then we say that V is *absolute*. If a subspace V is equal to its polar space $V^\perp$, then the subspace V is called *totally isotropic*.

We list the different types of polarities (σ, A) of $\mathrm{PG}(n,q)$ here.

- If $\sigma = 1$, q is odd and $A = A^T$, the polarity (σ, A) is called an *orthogonal* polarity.
- If $\sigma = 1$, q is even, $A = A^T$ and $a_{ii} \neq 0$ for some i, the polarity (σ, A) is called a *pseudo*-polarity.
- If $\sigma = 1$, $A = -A^T$ and $a_{ii} = 0$ for all i, i.e., every point is an absolute point, then n should be odd, and the polarity (σ, A) is called a *symplectic* polarity.
- If $\sigma \neq 1$, then $\sigma : x \mapsto x^{\sqrt{q}}$, with q a square, $A = A^{T^\sigma}$ and (σ, A) is called a *Hermitian* or *unitary* polarity.

1.4 Polar spaces

Since one of the known constructions of a maximal arc, the main topic of this thesis (see Section 1.6 for the definition), uses classical polar spaces, we cannot omit a modest introduction to these geometries. Much more information, as well as the theory of Hermitian forms and Hermitian varieties which will not be included here, can be found in for instance [12] and [35].

Polar spaces were first axiomatically introduced by Veldkamp [60]. Later on Tits perfected the theory [58].

Let

$$Q(X_0, \ldots, X_n) = \sum_{i,j=0,\, i\leq j}^{n} a_{ij} X_i X_j$$

be a quadratic form over $\mathrm{GF}(q)$. The associated *quadric* $\mathcal{Q}(n,q)$ in $\mathrm{PG}(n,q)$ is the set of points $p(x_0, \cdots, x_n)$ whose coordinates, with respect to a fixed basis, satisfy

$$Q(x_0, \ldots, x_n) = 0.$$

A quadric of $\mathrm{PG}(n,q)$ is called *singular* if there is a non-singular coordinate transformation that reduces the quadratic form to one in fewer variables. Otherwise, the quadric is called *non-singular*.

If $n = 2m$, all non-singular quadrics in $\mathrm{PG}(2m, q)$ are projectively equivalent to the quadric with equation $X_0^2 + X_1X_2 + \cdots + X_{2m-1}X_{2m} = 0$. These are the *parabolic* quadrics and are denoted by $Q(2m, q)$. In $\mathrm{PG}(2, q)$ a non-singular quadric is called a *conic*. It will soon become clear that conics play a pivotal role throughout this thesis.

If $n = 2m + 1$, a non-singular quadric in $\mathrm{PG}(2m + 1, q)$ is either projectively equivalent to the quadric with equation $X_0X_1 + \cdots + X_{2m}X_{2m+1} = 0$, in which case it is called *hyperbolic* and denoted by $Q^+(2m + 1, q)$, or it is projectively equivalent to the quadric with equation $f(X_0, X_1) + X_2X_3 + \cdots + X_{2m}X_{2m+1} = 0$, with f an irreducible homogeneous quadratic form over $\mathrm{GF}(q)$, in which case it is called *elliptic* and denoted by $Q^-(2m + 1, q)$.

Definition 1.4.1 ([58]). A *polar space of rank n*, $n > 2$, is a point set $\mathcal{P}$ together with a family of subsets of $\mathcal{P}$ called subspaces, satisfying the following axioms.

(i) A subspace, together with the subspaces it contains, is a d-dimensional projective space with $-1 \leq d \leq n - 1$.

(ii) The intersection of two subspaces is a subspace.

(iii) Given a subspace V of dimension $n - 1$ and a point $p \in \mathcal{P} \setminus V$, there is a unique subspace W of dimension $n - 1$ such that $p \in W$ and $V \cap W$ has dimension $n - 2$; W contains all points of V that are joined to p by a subspace of dimension 1, also called a *line*.

(iv) There exist two disjoint subspaces of dimension $n - 1$.

The quadrics defined above, together with the subspaces they contain, are examples of polar spaces.

The points of $\mathrm{PG}(n, q)$, $n \geq 3$ odd, and the totally isotropic subspaces of a non-singular symplectic polarity of $\mathrm{PG}(n, q)$, (together with all the (projective) subspaces they contain), form a *symplectic* polar space, which is denoted by $W(n, q)$.

These symplectic polar spaces, the quadrics and the Hermitian varieties are the *finite classical polar spaces.*

A subspace of maximum dimension contained in a quadric $\mathcal{Q}$ is called a *generator* of the quadric $\mathcal{Q}$. The dimension of such a generator is called the *projective* *index*. This projective index is equal to $n-1$ for $Q(2n,q)$ and $Q^-(2n+1,q)$ and is equal to n for $Q^+(2n+1,q)$.

1.5 Ovals and hyperovals

Definition 1.5.1. A *k-arc* $\mathcal{K}$ in $\mathrm{PG}(2,q)$ is a set of k points of which no three are collinear. It can easily be seen that $|\mathcal{K}| \le q+2$. A $(q+2)$-arc is called a *hyperoval* and can only exist if q is even. A $(q+1)$-arc in $\mathrm{PG}(2,q)$ is called an *oval*.

If $\mathcal{O}$ is an oval of $\mathrm{PG}(2,q)$, q even, it can be proven that the $q+1$ lines, intersecting $\mathcal{O}$ only in one point, are concurrent. This common point is called the *nucleus* of $\mathcal{O}$. An example of a hyperoval is a conic C together with its nucleus n. This is commonly known as a *regular hyperoval* (or *hyperconic*).

In a renowned theorem, Segre [45] proved, using some elegant arguments, that every oval in $\mathrm{PG}(2,q)$, with q odd, is a conic. In contrast with the case where q is odd, the study of hyperovals in $\mathrm{PG}(2,q)$, q even, is a deep and complex field and the subject of much research.

Known hyperovals

A hyperoval $\mathcal{O}$ in $\mathrm{PG}(2,q)$ ($q = 2^h, h > 1$) contains at least 4 points, no three of which are collinear. Without any restrictions we may assume that $\mathcal{O}$ passes through the four points $(1,0,0),(0,1,0),(0,0,1)$ and $(1,1,1)$, which implies that it is completely determined by its affine points $(x,y,1)$. We define $y = f(x)$ if and only if $(x,y,1)$ is a point of $\mathcal{O}$. It is easily seen that $f(x)$ is a permutation polynomial over $\mathrm{GF}(q)$ which is called an *o-polynomial*.

A lot of the known examples can be described by an o-polynomial of the form $f(x) = x^k$, also called a *monomial o-polynomial*. Define

$$D(h) = \{k \parallel x^k \text{ is an o-polynomial over } \mathrm{GF}(2^h)\}.$$

We summarize the known results on $D(h)$, which were observed by numerous authors ([14],[25],[34],[42]), in the following theorem.

Theorem 1.5.2. *If $k \in D(h)$ then $1/k, 1-k, 1/(1-k), k/(k-1)$ and $(k-1)/k$ (all taken modulo $2^h - 1$) are also elements of $D(h)$ and yield projectively equivalent hyperovals.*

We give a short description of the known elements in $D(h)$; the related hyperovals are called *monomial hyperovals*.

(i) It is clear that $2 \in D(h)$, for all h, and this gives us the regular hyperoval. Actually it is known that if $h \leq 3$, every hyperoval in $\mathrm{PG}(2, 2^h)$ is a regular hyperoval.

(ii) It was proved by Segre [46] that $2^i \in D(h)$ if and only if $\gcd(i, h) = 1$. These hyperovals are called *translation hyperovals* since they admit a group of translations acting transitively on the affine points of the hyperoval as an automorphism group. When $i \neq 1, h-1$, these hyperovals are not equivalent to regular hyperovals and examples exist for $h \geq 5$, but $h \neq 6$.

(iii) Another class of monomial hyperovals is given by $f(x) = x^6$, in the case where h is odd. These hyperovals, often called Segre hyperovals, were also discovered by Segre [47] in 1962, see also [48] for more details.

(iv) Let σ and γ be automorphisms of $\mathrm{GF}(2^h)$, h odd, such that $\gamma^4 \equiv \sigma^2 \equiv 2 \pmod{2^h - 1}$ then Glynn [25] proved that $\gamma + \sigma$ and $3\sigma + 4$ are elements of $D(h)$.

Remark 1.5.3. Although there are several hyperovals known that are not of the monomial type, it would take us too far to go into more detail. However, for all updated information on hyperovals, we refer to a nice electronic overview by Cherowitzo (see [13]).

The classification of hyperovals in small order planes

- As mentioned above, in $\mathrm{PG}(2, 2)$, $\mathrm{PG}(2, 4)$ and $\mathrm{PG}(2, 8)$, every hyperoval must be a regular hyperoval.
- Hall, as well as O'Keefe and Penttila, showed in [26] and [39], respectively, that there are two projectively distinct hyperovals in $\mathrm{PG}(2, 16)$: the regular hyperoval and the Lunelli-Sce hyperoval.

- In [44] Penttila an Royle showed that there are six projectively distinct hyperovals in $\mathrm{PG}(2,32)$: the regular hyperoval, the translation hyperoval, the Segre hyperoval, the Payne hyperoval, the Cherowitzo hyperoval and the O'Keefe-Penttila hyperoval.

1.6 Maximal arcs

This section contains some definitions as well as some basic properties concerning maximal arcs. The details about the actual known constructions are investigated in the next chapter.

Barlotti [7] introduced the idea of a maximal arc in a projective plane in 1956. Originally, maximal arcs were studied as a combinatorial extremal problem and appeared in the study of algebraic curves. Later it became clear that maximal arcs give rise to a great deal of interesting geometric structures. Some of these links are further investigated throughout this thesis.

Definition 1.6.1. A $\{k;d\}$-*arc* $\mathcal{K}$ in a finite projective plane of order q is a non-empty proper subset of k points such that some line of the plane meets $\mathcal{K}$ in d points, but no line meets $\mathcal{K}$ in more than d points. For given q and d, k can never exceed $q(d-1)+d$. If equality holds $\mathcal{K}$ is called a *maximal arc of degree* d, *a degree-*d *maximal arc*, *a* $\{q(d-1)+d;d\}$-*arc* or shorter, a d-*arc*.

Equivalently, a maximal arc can be defined as a non-empty, proper subset of points of a projective plane, such that every line meets the set in 0 or d points, for some d. If a line meets $\mathcal{K}$ it is said to be *secant* to $\mathcal{K}$, otherwise it is *external* to $\mathcal{K}$.

The following examples are trivial.

- Any single point of a projective plane of order q is a maximal $\{1;1\}$-arc in that plane.

- The set of points of an affine subplane of order q in a projective plane of order q is a $\{q^2;q\}$-arc.

For the rest of this thesis we will neglect these trivial examples.

Lemma 1.6.2 ([15]). *If $\mathcal{K}$ is a $\{q(d-1)+d; d\}$-arc in a projective plane π of order q, the set of lines external to $\mathcal{K}$ is a $\{q(q-d+1)/d; q/d\}$-arc in the dual plane.*

Proof. Let $\mathcal{K}^D$ be the set of points of the dual of π corresponding to the external lines of $\mathcal{K}$. We will show that every line of the dual plane meets $\mathcal{K}^D$ in 0 or in q/d points.

Let p be a point of $\mathcal{K}$. Then every line incident with p is secant to $\mathcal{K}$. It follows that the line of the dual plane corresponding to p does not meet $\mathcal{K}^D$.

On the other hand let p be a point not in $\mathcal{K}$. Since the secant lines incident with p partition the points of $\mathcal{K}$, it follows that there are $|\mathcal{K}|/d = q+1-\frac{q}{d}$ lines through p secant to $\mathcal{K}$, and thus q/d lines through p external to $\mathcal{K}$. We conclude that the line of the dual plane, corresponding to p, meets $\mathcal{K}^D$ in q/d points. □

It follows that a necessary condition for the existence of a $\{q(d-1)+d; d\}$-arc in a projective plane of order q is that d divides q. Denniston [20] showed that this necessary condition is sufficient in the Desarguesian projective plane $\mathrm{PG}(2,q)$ of order q when q is even (see Chapter 2).

Note that if π is a Desarguesian plane of order q which contains a maximal arc $\mathcal{K}$ of degree d, then it also contains a maximal arc of degree q/d, the so-called *dual maximal arc* of $\mathcal{K}$.

The following theorem, proved by Barlotti (see [7]), gives information on completing $\{k,d\}$-arcs to maximal arcs.

Theorem 1.6.3 ([7]). *If $\mathcal{K}$ is a $\{q(d-1)+d-1; d\}$-arc in a projective plane π of order q, it is incomplete and can be uniquely completed to a maximal $\{q(d-1)+d; d\}$-arc.*

For projective planes of odd order, no non-trivial constructions of maximal arcs were known, and so, through the years, several authors conjectured that non-trivial maximal arcs could not exist in $\mathrm{PG}(2,q)$, q odd. Cossu [15] proved that $\mathrm{PG}(2,9)$ has no $\{21;3\}$-arc and in 1974 J. A. Thas [52] showed that there are no maximal $\{2q+3;3\}$-arcs and no maximal $\{q(q-2)/3; q/3\}$-arcs in $\mathrm{PG}(2,q)$, $q>3$. Finally, Ball, Blokhuis and Mazzocca used a polynomial method to prove this more than 25-year-old conjecture.

Theorem 1.6.4 ([5]). *No non-trivial maximal arcs exist in* $\mathrm{PG}(2,q)$ *when* q *is odd.*

In [3] a second proof of this theorem was given by Ball and Blokhuis.

An overview of the constructions and characterizations of classical sets in $\mathrm{PG}(n,q)$ can be found in [18].

1.7 Generalized Quadrangles

Generalized quadrangles were introduced by Tits [56] and more information on the subject can be found in for instance [43] and [55].

Definition 1.7.1. A *generalized quadrangle* (GQ) is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t) satisfying the following axiom.

- If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$ holds.

For $x \in \mathcal{P}$, we put $x^\perp = \{y \in \mathcal{P} \parallel y \sim x\}$, and note that $x \in x^\perp$. For $x, y \in \mathcal{P}$, $x \neq y$, we have that $\{x,y\}^\perp = x^\perp \cap y^\perp$ and hence $|\{x,y\}^\perp| = s+1$ or $t+1$ according as $x \sim y$ or $x \not\sim y$. Further, $\{x,y\}^{\perp\perp} = \{u \in \mathcal{P} \| u \in z^\perp, z \in x^\perp \cap y^\perp\}$ and we have that $|\{x,y\}^{\perp\perp}| = s+1$ or $|\{x,y\}^{\perp\perp}| \leq t+1$ according as $x \sim y$ or $x \not\sim y$, respectively. If $x \not\sim y$ the set $\{x,y\}^{\perp\perp}$ is called the *hyperbolic line* defined by x and y.

Furthermore, we say that a pair of different points x, x' is *regular*, if $x \sim x'$, or if $x \not\sim x'$ and $|\{x,x'\}^{\perp\perp}| = t+1$. A point x is said to be regular provided $\{x,x'\}$ is regular for all $x' \in \mathcal{P}$, $x' \neq x$. Regularity for lines is defined dually.

We now introduce the notion of "ovoid" as defined by Tits in [57]. An ovoid $\mathcal{O}$ of $\mathrm{PG}(3,q)$ is a set of points of $\mathrm{PG}(3,q)$ no three of which are collinear and such that for any point of $\mathcal{O}$ the union of the lines which meet $\mathcal{O}$ only in that point, that is, the *tangent lines* at that point, is a set of lines of $\mathrm{PG}(2,q)$. If $\mathcal{O}$ is an ovoid in $\mathrm{PG}(3,q)$, its number of points is q^2+1. A celebrated theorem, independently proved by Barlotti [6] and Panella [41], shows us that every ovoid in $\mathrm{PG}(3,q)$, q odd or $q=4$, is an elliptic quadric.

To the contrary, in the even case, Tits [57] showed that for any $q = 2^{2e+1}$, with $e \geq 1$, there exists an ovoid which is not an elliptic quadric. These ovoids are called *Tits ovoids* or *Suzuki-Tits ovoids*, as the automorphism group is the (simple) Suzuki group $\mathrm{Sz}(2^{2e+1})$. So far, for even q, no other ovoids than the elliptic quadrics and the Tits ovoids are known.

If $\mathcal{O}$ is an ovoid in $\mathrm{PG}(3,q)$ any plane π of $\mathrm{PG}(3,q)$ intersects $\mathcal{O}$ in either one point or in an oval. If $|\pi \cap \mathcal{O}| = 1$ we say that π is a *tangent plane* of $\mathcal{O}$. At each of its points $\mathcal{O}$ has exactly one tangent plane. Finally, a nice result due to Brown [10] shows us that any ovoid $\mathcal{O}$ in $\mathrm{PG}(3,q)$ such that at least one plane intersects $\mathcal{O}$ in a conic, is an elliptic quadric.

We have a closer look at a few specific generalized quadrangles which will be linked later in this thesis. To denote the number of points and the number of lines of a GQ we will use v and b, respectively.

- The points of $\mathrm{PG}(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ, denoted by $\mathcal{W}(q)$, with parameters
$$s = t = q, v = b = (q+1)(q^2+1).$$
- Let $d = 2$ (respectively, $d = 3$) and let $\mathcal{O}$ be an oval (respectively, an ovoid) of $\mathrm{PG}(d,q)$. Furthermore, let $\mathrm{PG}(d,q) = H$ be embedded as a hyperplane in $\mathrm{PG}(d+1,q) = P$. Define points as

  (i) the points of $P \setminus H$,

  (ii) the hyperplanes X of P for which $|X \cap \mathcal{O}| = 1$, and

  (iii) one new symbol (∞).

  Lines are defined as

  (a) the lines of P which are not contained in H and meet $\mathcal{O}$ (necessarily in a unique point), and

  (b) the points of $\mathcal{O}$.

  The incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of P. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{O}$ in it. The point (∞) is incident with no line of type (a) and all lines of type (b).

  This GQ is denoted by $T_2(\mathcal{O})$ (respectively, $T_3(\mathcal{O})$). The parameters are
$$s = t = q, v = b = (q+1)(q^2+1)$$

and

$$s = q, t = q^2, v = (q+1)(q^3+1), b = (q^2+1)(q^3+1),$$

when $d = 2$ and $d = 3$, respectively.

- Let $\mathcal{O}$ be a hyperoval of $\mathrm{PG}(2,q)$, so q is even. Embed $\mathrm{PG}(2,q) = H$ as a hyperplane in $\mathrm{PG}(3,q) = P$. Define the points as the points of $P \setminus H$. The lines of the GQ are the lines of P not in H which meet $\mathcal{O}$ and the incidence is inherited from P. This GQ is denoted by $T_2^*(\mathcal{O})$ and has parameters

$$s = q-1, t = q+1, v = q^3, b = (q+2)q^2.$$

- The GQ *of Ahrens and Szekeres* $\mathrm{AS}(q)$.
For each odd prime power q there is a generalized quadrangle $\mathrm{AS}(q)$ of order $(q-1, q+1)$. The incidence structure $\mathrm{AS}(q) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ can be constructed as follows. Let the elements of $\mathcal{P}$ be the points of the affine 3-space $\mathrm{AG}(3,q)$ over $\mathrm{GF}(q)$. The elements of $\mathcal{B}$ are the following curves of $\mathrm{AG}(3,q)$.

$$\begin{array}{rl} (i) & x = \sigma,\ y = a,\ z = b \\ (ii) & x = a,\ y = \sigma,\ z = b \\ (iii) & x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma. \end{array}$$

Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence I is the natural one.

- The *Payne derived* GQ.
Let x be a regular point of a generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order q, $q > 1$. Define an incidence structure $\mathcal{P}(\mathcal{S}, x) = \mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ as follows. The point set $\mathcal{P}'$ is the set $\mathcal{P} \setminus x^\perp$. The lines of $\mathcal{B}'$ are of two types.

  · the lines of $\mathcal{B}$ which are not incident with x;

  · the hyperbolic lines $\{x, y\}^{\perp\perp}$ where $y \not\sim x$.

The incidence I' is the natural one. Then $\mathcal{S}'$ is a GQ of order $(q-1, q+1)$, which is called the Payne-derived GQ of $\mathcal{S}$ with respect to x.

It is known, see for instance [43], that AS(q) is a Payne-derived GQ. In fact, AS(q) $\cong \mathcal{P}(\mathcal{W}(q), x)$ (q odd). However, an actual map from one to the other does not seem to appear in the literature. We present such a map, together with a proof, in Appendix A.

1.8 Partial geometries

Bose [9] introduced partial geometries in 1963. As it turns out one can construct such geometries using maximal arcs. Some details are given here.

A partial geometry can be defined in the following way.

Definition 1.8.1. A *partial geometry* $\mathrm{pg}(s,t,\alpha)$ is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t) such that,

- if p is a point and L is a line not incident with p, there are exactly $\alpha > 0$ lines of $\mathcal{S}$ incident with p and concurrent with L.

If $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$, then one finds, using double counting arguments, that

$$v = \frac{(s+1)(st+\alpha)}{\alpha} \quad \text{and} \quad b = \frac{(t+1)(st+\alpha)}{\alpha}. \tag{1.1}$$

Notice that partial geometries are a generalization of the generalized quadrangles, since every partial geometry with $\alpha = 1$ is clearly a generalized quadrangle, and conversely.

1.9 Maximal arcs and partial geometries

Several partial geometries can be constructed using maximal arcs. This was proved by J. A. Thas ([51]) and for one class independently by Wallis [61] in 1974.

Let $\mathcal{K}$ be a maximal $\{q(d-1)+d; d\}$-arc, with $1 < d < q$, of a projective plane π of order q, not necessarily Desarguesian. Define the points of the incidence geometry $\mathcal{S}$ as the points of π which are not contained in $\mathcal{K}$. The lines of $\mathcal{S}$ are the lines of π which are incident with d points of $\mathcal{K}$, in other words, all the lines intersecting $\mathcal{K}$. The incidence is just the incidence of π. It is readily seen that the configuration $\mathcal{S}$, so defined, is a partial geometry $\mathrm{pg}(s,t,\alpha)$ with parameters:

- $s = q - d$
- $t = q - q/d$
- $\alpha = q - q/d - d + 1$
- $v = (q+1)(q-d+1)$
- $b = (q+1)(q-q/d+1)$.

Suppose that there exists a $\{q(d-1)+d; d\}$-arc $\mathcal{K}$, with $1 < d < q$, in the Desarguesian plane $\mathrm{PG}(2,q)$. Then we can define a second partial geometry denoted by $T_2^*(\mathcal{K})$ as follows.

Let $\mathrm{PG}(2,q)$ be embedded as a plane H in $\mathrm{PG}(3,q) = P$. Define the points of the incidence geometry $T_2^*(\mathcal{K})$ as the points of $P \setminus H$. The lines of $T_2^*(\mathcal{K})$ are the lines of P that are not contained in H and meet $\mathcal{K}$, necessarily, in a unique point. The incidence is that of P. Again, one readily proves, that $T_2^*(\mathcal{K})$ is a partial geometry $\mathrm{pg}(s,t,\alpha)$ with parameters:

- $s = q - 1$
- $t = qd - q + d - 1$
- $\alpha = d - 1$
- $v = q^3$
- $b = q^2(qd - q + d)$.

This partial geometry is often called the *linear representation of the maximal arc $\mathcal{K}$*.

As the existence of the $\{qd-q+d; d\}$-arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$ implies the existence of a $\{q(q-d+1)/d; q/d\}$-arc $\mathcal{K}'$ in $\mathrm{PG}(2,q)$, it follows that there also exists a partial geometry $T_2^*(\mathcal{K}')$ with parameters:

- $s = q - 1$
- $t = q(q-d+1)/d - 1$

- $\alpha = q/d - 1$
- $v = q^3$
- $b = q^3(q - d + 1)/d.$

Chapter 2

Known constructions of maximal arcs

This second chapter is devoted to the several known constructions of maximal arcs as well as some of their characterizations. In 1969 Denniston [20] used a special pencil of conics to construct maximal arcs in Desarguesian planes of even order. Five years later, a second construction was found by J. A. Thas [51]. He used ovoids and spreads in the generalized quadrangle $\mathcal{W}(q)$ to construct maximal arcs of degree q in planes of order q^2. In 1980 it was again J. A. Thas [53], this time employing quadrics and spreads in projective spaces, who constructed degree q^{t-1} maximal arcs in symplectic translation planes of order q^t. Finally, in 2001 Mathon described a construction of maximal arcs using sets of conics on a common nucleus in $\mathrm{PG}(2, q)$. As was proven by Hamilton and Mathon [29], every maximal arc constructed in this way gives rise to an infinite class of maximal arcs.

In fact, the maximal arcs constructed by Mathon are a generalization of the maximal arcs introduced by Denniston. However, instead of starting with Mathon's construction and extracting the maximal arcs of Denniston we will, across this chapter, discuss the known constructions chronologically while adding some useful results related to these geometric structures. A lot of these results can also be found in [27].

2.1 Denniston maximal arcs

From now on let $q = 2^h$. The usual absolute trace map from the finite field $\mathrm{GF}(2^h)$ to $\mathrm{GF}(2)$, denoted by Tr, is defined as follows. For $x \in \mathrm{GF}(2^h)$,

$$\mathrm{Tr}(x) = x + x^2 + \cdots + x^{2^{h-1}}.$$

We represent the points of the Desarguesian projective plane $\mathrm{PG}(2, q)$ as triples (a, b, c) over the Galois field $\mathrm{GF}(q)$ and the lines as triples $[u, v, w]$ over $\mathrm{GF}(q)$. A point (a, b, c) is incident with a line $[u, v, w]$ if and only if $au + bv + cw = 0$.

Furthermore, let $\xi(\alpha) = \xi^2 + \alpha\xi + 1$, $\alpha \in \mathrm{GF}(q)$ be an irreducible polynomial over $\mathrm{GF}(q)$ and let $\mathcal{F}$ denote the set of conics

$$F_\lambda : x^2 + \alpha xy + y^2 + \lambda z^2 = 0, \ \ \lambda \in \mathrm{GF}(q). \tag{2.1}$$

We see that F_0 is a singular conic, i.e., the point $(0, 0, 1)$, and that every other conic in the pencil is non-degenerate and has nucleus F_0. Due to the fact that $\xi^2 + \alpha\xi + 1$ is an irreducible polynomial over $\mathrm{GF}(q)$ it is clear that the line $z = 0$ is external to all conics of $\mathcal{F}$. Furthermore, we see that this pencil, being the union of the point $F_0(0, 0, 1)$ and the $q - 1$ non-degenerate conics, is a partition of the points of the plane, not on the line $z = 0$. For the rest of this thesis the pencil above will be referred to as the *standard pencil* and the line $z = 0$, often denoted by F_∞, will be called the *line at infinity* or, when confusion could occur (see later in this chapter), the *Denniston line*.

In [1] Abatangelo and Larato showed that the standard pencil $\mathcal{F}$ is stabilized by a cyclic group of order $q + 1$. The orbits of the group are the conics of the pencil and the group is given by

$$C_{q+1} = \left\{ \begin{pmatrix} a + \alpha b & b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} : a^2 + \alpha ab + b^2 = 1 \right\}. \tag{2.2}$$

It turns out (see [1] or [40]) that all the cyclic subgroups of $\mathrm{PGL}(3, q)$ of order $q + 1$ are conjugate in $\mathrm{PGL}(3, q)$. This implies that, up to isomorphism, there is only one pencil of conics.

In 1969 Denniston [20] proved the following theorem. The maximal arcs arising from this theorem are called *maximal arcs of Denniston type* or *Denniston maximal arcs*.

Theorem 2.1.1 ([20]). *If A is an additive subgroup of order d of* $\mathrm{GF}(q)$*, then the union $\mathcal{K}$ of the points of all F_λ, with $\lambda \in A$, is a maximal $\{q(d-1)+d; d\}$-arc in* $\mathrm{PG}(2,q)$*.*

Proof. We need to show that every line in the plane meets $\mathcal{K}$ in either 0 or d points. However, since we know that C_{q+1} stabilizes $\mathcal{K}$ and acts regularly (or sharply transitively) on the points of the line at infinity, we only need to show that the lines incident with a given point on the line at infinity intersect $\mathcal{K}$ in 0 or d points.

Let p be the point $(1,1,0)$. We will prove that every line through p meets $\mathcal{K}$ in 0 or d points. These lines through p are $\{[1,1,n] : n \in \mathrm{GF}(q)\} \cup \{[0,0,1]\}$. Of course, the line $[0,0,1]$, which is the line at infinity of the pencil, is external to $\mathcal{K}$. The line $[1,1,0]$ contains the nucleus $(0,0,1)$. Any line containing the nucleus is tangent to each of the conics in the pencil. Hence, $[1,1,0]$ meets the set $\mathcal{K}$ in $|A| = d$ points.

We still need to consider the lines $[1,1,n], n \neq 0$. None of these lines meets the nucleus of the conics and so they all meet each of the conics in either 0 or 2 points . Since $[1,1,n]$ is incident with q points not on the line at infinity, it must meet $q/2$ of the conics of the pencil in 2 points, and $q/2$ of them in 0 points.

The points on a line $[1,1,n]$ are $\{(x, x+n, 1) : x \in \mathrm{GF}(q)\} \cup \{p\}$. So $[1,1,n], n \neq 0$, does not meet the conic F_λ, $\lambda \in \mathrm{GF}(q)$, if and only if the quadratic equation

$$x^2 + \alpha x(x+n) + (x+n)^2 + \lambda = 0$$

has no solutions for any $x \in \mathrm{GF}(q)$. This is equivalent to

$$\mathrm{Tr}\left(\frac{n^2+\lambda}{\alpha n^2}\right) = 1.$$

Since the trace map is additive and $\mathrm{Tr}(1/\alpha) = 1$, due to the fact that $\xi^2+\alpha\xi+1$ is irreducible, we find the condition

$$\mathrm{Tr}\left(\frac{\lambda}{\alpha n^2}\right) = 0.$$

From this condition we see that, if the line $[1,1,n]$ does not meet the conics F_{λ_1} and F_{λ_2}, it also does not meet the conic $F_{\lambda_1+\lambda_2}$. Hence, if we define the set

$$H_n = \{\lambda \in \mathrm{GF}(q) : [1,1,n] \cap F_\lambda = \emptyset\},$$

then H_n is an additive subgroup of index 2 in $\mathrm{GF}(q)$. Define $H'_n = \mathrm{GF}(q) \setminus H_n$, then H'_n is the set of $\lambda \in \mathrm{GF}(q)$ such that $[1,1,n]$ meets F_λ in 2 points. Now, if A is a subgroup of H_n, then of course $A \cap H'_n = \emptyset$ and so $[1,1,n] \cap \mathcal{K} = \emptyset$. If A is not a subgroup of H_n, then $\mathrm{GF}(q) = AH_n$ and so

$$\mathrm{GF}(q)/H_n = AH_n/H_n \cong A/(A \cap H_n).$$

It follows that, since $A \cap H_n$ has index 2 in A, $|A \cap H'_n| = |A|/2 = d/2$. Moreover, every $\lambda \in A \cap H'_n$ corresponds to $[1,1,n]$ meeting the conic F_λ in 2 points. This implies that $[1,1,n]$ intersects $\mathcal{K}$ in d points. □

We know that additive subgroups of size d of $\mathrm{GF}(q)$ exist for all d dividing q. Hence the previous theorem shows that the necessary condition for a maximal arc of degree d to exist, more specifically, that d divides q, is indeed sufficient in $\mathrm{PG}(2,q)$.

Abatangelo and Larato proved some other interesting results which characterize the maximal arcs of Denniston type. They show, for instance, that Denniston maximal arcs, introduced as above, are the only ones arising in this context.

Theorem 2.1.2 ([1]). *If A is a subset of $\mathrm{GF}(q)$ such that the union of the points of all F_λ, $\lambda \in A$, is a maximal arc, then A is a subgroup of the additive group of $\mathrm{GF}(q)$.*

Furthermore, they give a characterization of Denniston maximal arcs in terms of collineation groups.

Theorem 2.1.3 ([1]). *If a maximal arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$, q even, is invariant under a linear collineation group of $\mathrm{PG}(2,q)$ which is cyclic and has order $q+1$, then $\mathcal{K}$ is a maximal arc of Denniston type.*

Corollary 2.1.4. *The dual of a maximal arc of Denniston type is a maximal arc of Denniston type.*

In the next theorem the full collineation stabilizers of the Denniston maximal arcs are calculated. This is a result by Hamilton and Penttila.

Theorem 2.1.5 ([32]). *In $\mathrm{PG}(2,2^e)$, $e > 2$, let $\mathcal{K}$ be a degree-d Denniston maximal arc, $q = 2^e$, $2 < d < q/2$, with additive subgroup A. Define the group*

G acting on GF(2^e) *by*

$$G = \{x \mapsto ax^\sigma : a \in \mathrm{GF}(2^e)^*, \sigma \in \mathrm{Aut}\,\mathrm{GF}(2^{2e})\}.$$

Then the collineation stabilizer of $\mathcal{K}$ *is isomorphic to* $C_{2^e+1} \rtimes G_A$*, the semidirect product of a cyclic group of order* $(2^e + 1)$ *with the stabilizer of* A *in* G*.*

In that same paper ([32]) it was shown that there are, up to isomorphism, exactly two degree-4 maximal arcs of Denniston type in PG$(2, 16)$, while there is a unique degree-4 Denniston maximal arc, and hence a unique degree-8 Denniston maximal arc in the plane PG$(2, 32)$.

2.2 Maximal arcs constructed by Thas

J. A. Thas has given two constructions of maximal arcs. Before we are able to discuss these two construction we need to introduce some new notions. More precisely we will give the general construction of a *translation plane*, known as the André-Bruck-Bose construction (see for instance [11]).

Definition 2.2.1. A $(t-1)$-*spread* S of PG$(2t-1, q)$ is a set of projective spaces of dimension $t-1$ such that every point of PG$(2t-1, q)$ lies in exactly one element of S. Equivalently, it is a set of $q^t + 1$ pairwise disjoint $(t-1)$-dimensional projective spaces.

Now, let PG$(2t-1, q)$ be embedded as a hyperplane H in PG$(2t, q) = P$, and let S be a $(t-1)$-spread of H. We can construct a new incidence geometry π^S as follows.

The points of π^S are the points of $P \setminus H$, together with the elements of the spread. The lines of π^S are the t-dimensional subspaces of P which intersect H in a member of S; together with the line L_∞ whose points are the elements of the spread.

The incidence relation of π^S is inherited from P.

One easily proves that π^S is a projective plane, known as *translation plane* of order q^t, with translation line L_∞.

A spread that gives rise to a Desarguesian projective plane will be referred to as a *Desarguesian spread*.

Remark 2.2.2. Let x be an affine point of a translation plane, then the group of all homologies of the plane with axis L_∞ and center x is known to be isomorphic to the multiplicative group of a field (see for instance [37]). This field is known to be isomorphic to a structure known as the *kernel* of the translation plane. Every translation plane of order q^t with kernel containing $\mathrm{GF}(q)$ can be constructed by using the André-Bruck-Bose construction.

2.2.1 Thas maximal arcs of type I

In the beginning of this chapter we mentioned that, in 1974, Thas [51] used ovoids and spreads to construct maximal arcs of order q in planes of order q^2. In the following a construction of these maximal arcs is given as well as a characterization result. We will call these arcs *Thas maximal arcs of type I*.

It is well known, see for instance [19], that any ovoid of $\mathrm{PG}(3, q)$, q even, gives rise to a symplectic polarity of $\mathrm{PG}(3, q)$. The totally isotropic lines with respect to this polarity are the tangent lines to the ovoid. In fact, the points of $\mathrm{PG}(3, q)$ together with the totally isotropic lines form a generalized quadrangle $\mathcal{W}(q)$ ([43] and [50]). In this setting, the ovoid of $\mathrm{PG}(3, q)$ is an ovoid of $\mathcal{W}(q)$, and a spread of $\mathrm{PG}(3, q)$ of tangent lines to the ovoid is a spread of $\mathcal{W}(q)$.

Now, consider an ovoid $\mathcal{O}$ and a 1-spread W of $\mathrm{PG}(3, 2^m)$, $m > 0$, such that each line of W has one and only one point in common with $\mathcal{O}$. In other words, W belongs to the linear complex of lines defined by $\mathcal{O}$. Let $\mathrm{PG}(3, 2^m)$ be embedded as a hyperplane H in $\mathrm{PG}(4, 2^m) = P$ and let x be a point of $P \setminus H$.

Denote the set of the points of $P \setminus H$ which are collinear with x and a point of $\mathcal{O}$ by C. Remark that $x \in C$. In what follows we will show that the point set C is a maximal $\{2^{3m} - 2^{2m} + 2^m; 2^m\}$-arc of the translation plane π^W defined by the 1-spread W.

First of all we remark that $|C| = (2^{2m} + 1)(2^m - 1) + 1 = 2^{3m} - 2^{2m} + 2^m$ and that the line at infinity W of π^W has no point in common with C. Next, we consider a plane $\mathrm{PG}(2, 2^m)$, denoted by P', not contained in $H = \mathrm{PG}(3, 2^m)$, that contains a line L of the 1-spread W. We now have to distinguish three cases:

- If $x \in P'$, then evidently $|P' \cap C| = 2^m$.
- Suppose $x \notin P'$ and the 3-dimensional space xP' intersects H in the

tangent plane of $\mathcal{O}$ at the point y, where $\{y\} = L \cap \mathcal{O}$. Since a point $p \in P' \cap C$ would imply that the line xp is contained in P', clearly a contradiction, it follows that, in this case, $|P' \cap C| = 0$.

- Suppose $x \notin P'$ and the 3-dimensional space xP' contains a point $z \in \mathcal{O} \setminus \{y\}$, with $\{y\} = L \cap \mathcal{O}$. Since it is not a tangent plane at $\mathcal{O}$, the plane $xP' \cap H$ has exactly $2^m + 1$ points in common with $\mathcal{O}$. Each one of these points, except for the point y, give rise to a point in the intersection $P' \cap C$. It follows immediately that $|P' \cap C| = 2^m$.

From the above we conclude that the plane P' intersects the set C in just 2^m points or in none at all. In other words, the point set C is a maximal $\{2^{3m} - 2^{2m} + 2^m; 2^m\}$-arc of the projective plane π^W defined by the 1-spread W.

These maximal arcs were characterized by Hamilton and Penttila in the following theorem.

Theorem 2.2.3 ([31]). *Let π be a translation plane of order q^2. Then a nontrivial maximal arc $\mathcal{K}$ in π is a Thas maximal arc of type I if and only if it is stabilized by a homology of order $q - 1$ with axis the translation line of π.*

Remark 2.2.4. The known ovoids $\mathcal{O}$ of $\mathrm{PG}(3, 2^m)$ are the elliptic quadrics and the Tits ovoids. As mentioned already in Section 1.7, a Tits ovoid is only defined in a $\mathrm{PG}(3, 2^{2s+1})$, with $s \geq 1$. The known 1-spreads W of $\mathrm{PG}(3, 2^m)$ which belong to a linear complex of lines are the Desarguesian spreads and the Lüneburg spreads. The latter is also only defined in a $\mathrm{PG}(3, 2^{2s+1})$, with $s \geq 1$ (for more information see [50]). Hence if W is a Lüneburg spread the above construction of Thas yields maximal arcs in the Lüneburg translation plane. Assume on the other hand that W is a Desarguesian spread, then this construction of Thas provides us with a $\{2^{3m} - 2^{2m} + 2^m; 2^m\}$-arc of the Desarguesian plane $\pi = \mathrm{PG}(2, 2^{2m})$. However, it was remarked by Thas that this maximal arc is a Denniston maximal arc (see also [40] and [51]). If $\mathcal{O}$ is the Tits ovoid, hence $m = 2s + 1$, with $s \geq 1$, then up to isomorphism, there are two Thas maximal arcs of type I arising from the Tits ovoid and these maximal arcs are not of Denniston type (see [32]). Furthermore, it turns out (see for instance [29]) that the Thas maximal arcs of type I are isomorphic to their dual maximal arcs.

2.2.2 Thas maximal arcs of type II

Thas established a second construction of maximal arcs in 1980 ([53]). Using quadrics and spreads in projective spaces he managed to construct degree-q^{t-1} arcs in symplectic translation planes of order q^t. These maximal arcs will be called *Thas maximal arcs of type II*. In what follows, we discuss their construction and give some results concerning the existence and isomorphism problems.

Let q be even and let $Q^- := Q^-(2d-1, q)$ be an elliptic quadric of $\mathrm{PG}(2d-1, q)$, with $d \geq 2$, and let S^- be a spread of generators of Q^- ([54]). The quadric Q^- contains $(q^{d-1}-1)(q^d+1)/(q-1)$ points and the set S^- consists of q^d+1 subspaces of dimension $d-2$ which constitute a partition of Q^-. We remark that these subspaces are generators and that a $(d-1)$-spread of $\mathrm{PG}(2d-1, q)$ also has q^d+1 elements. Let us now consider a $(d-1)$-spread $S = \{P_1, P_2, \ldots\}$ of $\mathrm{PG}(2d-1, q)$ such that $\{P_1 \cap Q^-, P_2 \cap Q^-, \ldots\} = S^-$. Next, we embed $\mathrm{PG}(2d-1, q)$ as a hyperplane H in $\mathrm{PG}(2d, q) = P$, and we consider a point x of $P \setminus H$.

Analogous to the Thas maximal arcs of type I we denote the set of points which are collinear with x and a point of Q^- but are not contained in Q^- by C. Again, notice that $x \in C$. We will prove that C is a maximal $\{q^{2d-1} - q^d + q^{d-1}; q^{d-1}\}$-arc of the translation plane π^S of order q^d defined by S.

First of all we remark that

$$|C| = (q-1)\frac{(q^{d-1}-1)(q^d+1)}{(q-1)} + 1 = q^{2d-1} - q^d + q^{d-1}$$

and that the line at infinity S of π^S has no point in common with C. Now we consider a d-dimensional projective space $\mathrm{PG}(d, q)$, denoted by D, not contained in $H = \mathrm{PG}(2d-1, q)$, which contains an element P_i of S. Again, we need to distinguish a few cases:

- If $x \in D$, then $|D \cap C| = (q-1)\dfrac{q^{d-1}-1}{q-1} + 1 = q^{d-1}$.
- Suppose $x \notin D$ and that $xD \cap H := D'$, a d-dimensional projective space, is the polar space of $P_i \cap Q^- = D \cap Q^-$ with respect to Q^-. In that case we find that $|D \cap C| = 0$.
- Suppose $x \notin D$ and that $xD \cap H := D'$, a d-dimensional projective space, contains a point $z \in Q^- \setminus (P_i \cap Q^-)$. Since $D' \cap Q^- = Q'$ contains the

$(d-2)$-dimensional space $P_i \cap Q^-$ and since $P_i \cap Q' = P_i \cap Q^-$, the singular space of Q' is a hyperplane of $P_i \cap Q^-$. We can now consider Q' to be a cone with vertex the $(d-3)$-dimensional singular space and base a conic in the plane. This implies

$$\begin{aligned} |Q'| &= \frac{q^{d-2}-1}{q-1} + q + 1 + \frac{q^{d-2}-1}{q-1}(q+1)(q-1) \\ &= \frac{q^{d-2}-1}{q-1} + q^{d-1} + q^{d-2} \\ &= q^{d-1} + \frac{q^{d-1}-1}{q-1}. \end{aligned}$$

The expression $\frac{q^{d-1}-1}{q-1}$ is of course associated to the points of the $(d-2)$-dimensional space $P_i \cap Q^-$. Hence we find that $|D \cap C| = q^{d-1}$.

It follows that D intersects the set C in exactly q^{d-1} points or in none at all. We conclude that C is a maximal $\{q^{2d-1} - q^d + q^{d-1}; q^{d-1}\}$-arc of the projective plane π of order q^d defined by S.

Remark that, if $d = 2$ and replacing Q^- by an ovoid, the same construction gives us the Thas maximal arcs of type I.

It is clear from the above that if one can construct a $(d-1)$-spread S of $\mathrm{PG}(2d-1, q)$ intersecting Q^- in a spread S^- of generators, then a Thas maximal arc of type II in the plane defined by S can be constructed. If Q is a non-singular quadric of a projective space $\mathrm{PG}(2d, q)$, q even, then Q always has a spread S^* that consists of q^d+1 subspaces of Q of dimension $d-1$ ([21]). Let $\mathrm{PG}(2d-1, q)$ be a hyperplane for which $\mathrm{PG}(2d-1, q) \cap Q = Q^-$ is an elliptic quadric. Then S^* induces a spread S^- of Q^-. If n is the nucleus of Q, then n cannot be contained in $\mathrm{PG}(2d-1, q)$ since $\mathrm{PG}(2d-1, q)$ is not tangent to Q.

Now, we project Q from n onto $\mathrm{PG}(2d-1, q)$. The projection of S^* is a $(d-1)$-spread $S = \{P_1, P_2, \ldots\}$ of $\mathrm{PG}(2d-1, q)$, and moreover $\{P_1 \cap Q^-, P_2 \cap Q^-, \ldots\} = S^-$. It is well-known that the maximal totally isotropic subspaces of Q are projected onto the maximal totally isotropic subspaces of a symplectic polarity θ of $\mathrm{PG}(2d-1, q)$ ([21]). It follows that all elements of S are maximal totally isotropic with respect to θ. We say that S is a symplectic $(d-1)$-spread of $\mathrm{PG}(2d-1, q)$. Remark that θ is the symplectic polarity defined by the quadric Q^- of $\mathrm{PG}(2d-1, q)$. Furthermore, it turns out that, for q even, any two spreads S and S^-, satisfying the conditions of the beginning of this subsection, arise as just described. It follows that, for q even, the construction of S and S^- is reduced to the construction of a spread S^* of a non-singular quadric Q of $\mathrm{PG}(2d, q)$.

We conclude that it is possible to construct a maximal $\{q^{2d-1} - q^d + q^{d-1}; q^{d-1}\}$-arc in any translation plane of order q^d, q even, with $\mathrm{GF}(q)$ a subfield of the kernel for which the corresponding $(d-1)$-spread is symplectic. A lot of examples of symplectic spreads are constructed by Dye [21].

The following characterization result was proved by Hamilton and Penttila.

Theorem 2.2.5 ([32]). *The Thas maximal arcs of type II that occur in Desarguesian planes are of Denniston type.*

Remark 2.2.6. The above arguments, once S and S^- are constructed, also hold for q odd. However it was proved in [8] that given a non-degenerate elliptic quadric in the projective space $\mathrm{PG}(2d-1, q)$, q odd, there does not exist a spread of $\mathrm{PG}(2d-1, q)$ such that each element of the spread meets the quadric in a generator.

2.3 Maximal arcs of Mathon type

In [38], Mathon constructed maximal arcs in Desarguesian projective planes generalizing the previously known construction of Denniston [20]. In this section we present conditions for a set of conics on a common nucleus to form a maximal arc. Such sets can be defined recursively using a special composition of these conics. Mathon's construction gives rise to several infinite families of maximal arcs and will be of great importance throughout this thesis. We begin by describing his construction here.

From now on let $q = 2^h$ and recall that Tr denotes the usual absolute trace map from the finite field $\mathrm{GF}(q)$ onto $\mathrm{GF}(2)$. Analogous to Section 2.1, we represent the points of the Desarguesian projective plane $\mathrm{PG}(2, q)$ as triples (a, b, c) over $\mathrm{GF}(q)$, and the lines as triples $[u, v, w]$ over $\mathrm{GF}(q)$.

For $\alpha, \beta \in \mathrm{GF}(q)$ such that $\mathrm{Tr}(\alpha\beta) = 1$, and $\lambda \in \mathrm{GF}(q)$ we define $F_{\alpha,\beta,\lambda}$ to be the conic

$$F_{\alpha,\beta,\lambda} = \{(x, y, z) : \alpha x^2 + xy + \beta y^2 + \lambda z^2 = 0\}.$$

Remark that the condition $\mathrm{Tr}(\alpha\beta) = 1$ is equivalent to demanding that the quadratic polynomial $\alpha\xi^2 + \xi + \beta$ is irreducible over $\mathrm{GF}(q)$. Now, let $\mathcal{F}$ be the set of all such conics. It is clear that all the conics in $\mathcal{F}$ have the point $F_{\alpha,\beta,0} := F_0(0,0,1)$ as their nucleus and that, due to the trace condition, the *line at infinity*, i.e., the line $z = 0$, is external to all conics. Every other conic is non-degenerate.

Remark 2.3.1. We mentioned in Section 2.1 that confusion might occur regarding the notion "line at infinity". So far, we introduced the line at infinity of a Denniston maximal arc, which is the line external to all conics in the standard pencil that contains the Denniston maximal arc, and the line at infinity of a Mathon maximal arc, which is the line external to all conics in $\mathcal{F}$. Now, as we will see in the next chapters, maximal arcs of Mathon type contain several maximal arcs of Denniston type. In these cases it will be convenient to consider the lines at infinity of the corresponding Denniston maximal arcs. Whenever we do so these lines at infinity will be called the *Denniston lines* while the external line of the maximal arc of Mathon type is still called "the line at infinity".

For given $\lambda \neq \lambda'$, define a *composition*

$$F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'} = F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda\oplus\lambda'}$$

where the operator $\oplus$ is defined as follows:

$$\alpha \oplus \alpha' = \frac{\alpha\lambda + \alpha'\lambda'}{\lambda + \lambda'}, \quad \beta \oplus \beta' = \frac{\beta\lambda + \beta'\lambda'}{\lambda + \lambda'}, \quad \lambda \oplus \lambda' = \lambda + \lambda'. \tag{2.3}$$

Clearly, the operator $\oplus$ is commutative, and since $(\alpha\oplus\alpha')\oplus\alpha'' = \alpha\oplus(\alpha'\oplus\alpha'') = (\alpha\lambda + \alpha'\lambda' + \alpha''\lambda'')/(\lambda + \lambda' + \lambda'')$ it is also associative. Moreover, $F_{\alpha,\beta,\lambda} \oplus F_{\alpha,\beta,\lambda'} = F_{\alpha,\beta,\lambda+\lambda'}$ implies that $\oplus$ is idempotent in the first two parameters. The following lemma gives a condition for the disjointness of a composition of conics.

Lemma 2.3.2 ([38]). *Two non-degenerate conics $F_{\alpha,\beta,\lambda}$, $F_{\alpha',\beta',\lambda'}$, $\lambda \neq \lambda'$ and their composition $F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'}$ are mutually disjoint if* $\mathrm{Tr}((\alpha\oplus\alpha')(\beta\oplus\beta')) = 1$.

In order to state Mathon's Theorem we need the notion of a closure. Given some subset $\mathcal{C}$ of $\mathcal{F}$, we say $\mathcal{C}$ is *closed* if for every $F_{\alpha,\beta,\lambda} \neq F_{\alpha',\beta',\lambda'} \in \mathcal{C}$, $F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda\oplus\lambda'} \in \mathcal{C}$.

Lemma 2.3.3 ([38]). *Suppose that a set $\mathcal{C} \subset \mathcal{F}$ containing N conics is closed under composition. Let $F' = F_{\alpha',\beta',\lambda'} \in \mathcal{F} \setminus \mathcal{C}$ be a non-degenerate conic with* $\mathrm{Tr}(\alpha'\beta') = 1$ *and such that* $\mathrm{Tr}((\alpha \oplus \alpha')(\beta \oplus \beta')) = 1$ *for every $F_{\alpha,\beta,\lambda} \in \mathcal{C}$. Then the closure $\mathcal{C}' = \langle \mathcal{C} \cup \{F'\} \rangle$ contains $2N+1$ conics and $\mathcal{C}' = \{F, F', F \oplus F' : F \in \mathcal{C}\}$.*

In the next result by Mathon ([38]) it is shown that sets of conics, closed under composition, can be used to construct maximal arcs. These maximal arcs will be referred to as *maximal arcs of Mathon type* or *Mathon maximal arcs.*

Theorem 2.3.4 ([38]). *Suppose $\mathcal{C} \subset \mathcal{F}$ is a closed set of $2^d - 1$ conics in* $\mathrm{PG}(2, 2^m)$*, $1 \leq d \leq m$. Then the union of the points on the conics of $\mathcal{C}$ together with their common nucleus F_0 is a maximal $\{2^{m+d} - 2^m + 2^d; 2^d\}$-arc $\mathcal{K}$ in* $\mathrm{PG}(2, 2^m)$*.*

Proof. We will show that every line of $\mathrm{PG}(2, 2^m)$ meets $\mathcal{K}$ in 2^d points or in 0 points. We know that, since $\mathrm{Tr}(\alpha\beta) = 1$, the line at infinity F_∞ is external to $\mathcal{K}$. Of course, every other line of the plane meets F_∞ in one of its points, which are $(1, 0, 0)$ and $(a, 1, 0), a \in \mathrm{GF}(2^m)$, and thus belongs to the set of lines

$$\{[0, 1, 0], [0, b, 1], [1, a, b] : b \in \mathrm{GF}(2^m)\}.$$

The lines $[0, 1, 0]$ and $[1, a, 0]$ contain the common nucleus $F_0(0, 0, 1)$ and are therefore tangent to every conic in $\mathcal{C}$. It follows that they meet $\mathcal{K}$ in $|\mathcal{C}|+1 = 2^d$ points. Consequently, any of the remaining lines meets a conic $F \in \mathcal{C}$ in either 0 or 2 points. A line $[0, b, 1]$ or $[1, a, b]$, with $b \neq 0$, containing the points $(x, 1, b)$ or $(a + bx, 1, x)$, $x \in \mathrm{GF}(2^m)$, is disjoint from $F = F_{\alpha,\beta,\lambda}$ if and only if the quadratic equation $\alpha x^2 + x + \beta + \lambda b^2 = 0$ or $\alpha(a + bx)^2 + (a + bx) + \beta + \lambda x^2 = 0$ has no solutions in $\mathrm{GF}(2^m)$, respectively. In the first case we find the trace condition

$$\mathrm{Tr}[\alpha(\beta + \lambda b^2)] = \mathrm{Tr}[\alpha\beta + \alpha b^2\lambda] = 1.$$

This is equivalent to the condition $\mathrm{Tr}(\alpha b^2 \lambda) = 0$, since $\mathrm{Tr}(\alpha\beta) = 1$. The second case yields the condition

$$\mathrm{Tr}\left[\frac{(\alpha a^2 + a + \beta)(\alpha b^2 + \lambda)}{b^2}\right] = \mathrm{Tr}\left[\frac{(\alpha a^2 + a + \beta)\lambda}{b^2} + \frac{(\alpha a^2 + a + \beta)\alpha b^2}{b^2}\right] = 1.$$

Since $\mathrm{Tr}(\alpha^2 a^2 + a\alpha + \alpha\beta) = 1 + \mathrm{Tr}(\alpha a + \alpha^2 a^2) = 1 + \mathrm{Tr}(\alpha a) + \mathrm{Tr}(\alpha a)^2 = 1$ we find the trace condition $\mathrm{Tr}((\alpha a^2 + a + \beta)\lambda / b^2) = 0$. In other words, we can say that the condition of disjointness of a line $[0, b, 1]$ with $F_{\alpha,\beta,\lambda}$ is equivalent to the condition

$$\mathrm{Tr}(\alpha b^2 \lambda) = 0, \tag{2.4}$$

while the condition of disjointness of a line $[1, a, b]$ is equivalent to

$$\mathrm{Tr}\left(\frac{(\alpha a^2 + a + \beta)\lambda}{b^2}\right) = 0. \tag{2.5}$$

To simplify notation we will refer to both these conditions by $\mathrm{Tr}(\alpha, \beta, \lambda) = 0$. Using (2.3) it is easily verified that in the two cases

$$\mathrm{Tr}(\alpha \oplus \alpha', \beta \oplus \beta', \lambda \oplus \lambda') = \mathrm{Tr}(\alpha, \beta, \lambda) + \mathrm{Tr}(\alpha', \beta', \lambda'). \tag{2.6}$$

Hence, if a line L does not meet both F and F' then it also does not meet the conic $F \oplus F'$. Now, there are two ways for L to interact with the conics of $\mathcal{C}$. If $\mathrm{Tr}(\alpha, \beta, \lambda) = 0$ for all $F \in \mathcal{C}$ then L is of course disjoint from $\mathcal{K}$. On the other hand, if $\mathrm{Tr}(\alpha, \beta, \lambda) = 1$ for some $F \in \mathcal{C}$ then we can use the recursive doubling argument of Lemma 2.3.3 and the identity (2.6) to see that there are exactly 2^{d-1} conics in $\mathcal{C}$ with trace 1. Given Lemma 2.3.2 we know that the conics are mutually disjoint. It follows that the line L intersects $\mathcal{K}$ in 2^d points. □

As we mentioned above, Mathon's construction is actually a generalization of the previously known construction of Denniston. This can be seen as follows.

Choose $\alpha \in \mathrm{GF}(q)$ such that $\mathrm{Tr}(\alpha) = 1$. Let A be a subset of $\mathrm{GF}(q)^\star = \mathrm{GF}(q) \setminus \{0\}$ such that $A \cup \{0\}$ is closed under addition. Then the point set of the conics

$$\mathcal{K}_A = \{F_{\alpha,1,\lambda} : \lambda \in A\}$$

together with the nucleus $F_0(0, 0, 1)$ is the set of points of a maximal arc of degree $|A| + 1$ in $\mathrm{PG}(2, q)$. This construction is exactly the definition of a maximal arc of Denniston type. The conics in $\mathcal{K}_A$ are a subset of the standard pencil of conics which, in this notation, is given by

$$\{F_{\alpha,1,\lambda} : \lambda \in \mathrm{GF}(q)\}.$$

As we already know, this pencil partitions the points of the plane, not on the line $z = 0$ into $q - 1$ disjoint conics on the common nucleus $F_0(0, 0, 1)$. More generally, a pencil of conics may be obtained as follows. Suppose F_1

and F_2 are non-degenerate quadratic forms over $\mathrm{GF}(q)$ that have no common zeros, i.e., the conics that they define have no common points. Then the set of polynomials

$$\{\mu F_1 + \nu F_2 : \mu, \nu \in \mathrm{GF}(q), \mu, \nu \text{ not both zero}\}$$

determines $q+1$ quadratic forms: $q-1$ pairwise disjoint non-degenerate conics, an exterior line to those conics and a point that is the nucleus of all conics.

Next we consider a closed set of conics $\mathcal{C} = \{F_{\alpha,\beta,\lambda}\}$ with parameters α, β which are polynomials in λ. In other words, we will describe closed sets of conics using functions $p : A \to \mathrm{GF}(q)$ and $r : A \to \mathrm{GF}(q)$, with A the set of values that λ ranges over, such that the closed set of conics $\mathcal{C}$ is given by the equations

$$\{p(\lambda)x^2 + xy + r(\lambda)y^2 + \lambda z^2 = 0 : \lambda \in A\}. \tag{2.7}$$

More formally we get the following theorem.

Theorem 2.3.5 ([38]). *Let* $p(\lambda) = \sum_{i=0}^{d-1} a_i \lambda^{2^i - 1}$ *and* $r(\lambda) = \sum_{i=0}^{d-1} b_i \lambda^{2^i-1}$ *be polynomials with coefficients in* $\mathrm{GF}(2^m)$*. For an additive subgroup* A *of order* 2^d *in* $\mathrm{GF}(2^m)$ *let* $\mathcal{C} = \{F_{p(\lambda),r(\lambda),\lambda} : \lambda \in A \setminus \{0\}\} \subset \mathcal{F}$ *be a set of conics with common nucleus* F_0*. If* $\mathrm{Tr}(p(\lambda)r(\lambda)) = 1$ *for every* $\lambda \in A \setminus \{0\}$ *then the set of points of all conics in* $\mathcal{C}$ *together with* F_0 *forms a maximal* $\{2^{m+d} - 2^m + 2^d; 2^d\}$*-arc* $\mathcal{K}$ *in* $\mathrm{PG}(2, 2^m)$*. If both* $p(\lambda)$ *and* $r(\lambda)$ *have* $d \leq 2$ *then* $\mathcal{K}$ *is a Denniston maximal arc.*

From these preceding results it follows that a maximal arc of degree d of Mathon type contains Mathon sub-arcs of degree d' for all d' dividing d. Another important consequence of the previous findings is that a degree-4 Mathon arc is necessarily of Denniston type. These conclusions can be found in [38], where Mathon also used his construction to present several new infinite families of maximal arcs in $\mathrm{PG}(2, q)$.

A few examples

Next, let us have a quick look at two examples of maximal arcs that are induced by polynomials, as described in Theorem 2.3.5. For more information, see [38].

- The dual of the Lunelli-Sce hyperoval in $\mathrm{PG}(2,16)$ is a degree-8 maximal arc which is formed by
$$\{x^2 + xy + (w^{11} + w^{10}\lambda + \lambda^3)y^2 + \lambda z^2 : \lambda \in \langle 1, w, w^2\rangle \setminus \{0\}\},$$
where w is a primitive element in $\mathrm{GF}(16)$ satisfying $w^4 + w = 1$.
- In $\mathrm{GF}(32)$, let w be a primitive element satisfying $w^{18} + w = 1$. The set of 15 conics
$$\{p(\lambda)x^2 + xy + r(\lambda)y^2 + \lambda z^2 : \lambda \in \langle 1, w, w^7, w^9\rangle \setminus \{0\}\},$$
with $p(\lambda) = w^{25} + w^{16}\lambda + w^{10}\lambda^3 + w^{30}\lambda^7$ and $r(\lambda) = w^{27} + w^5\lambda + w^{11}\lambda^3 + w^3\lambda^7$, forms a maximal arc of degree 16 in $\mathrm{PG}(2,32)$ which is the dual of Cherowitzo's hyperoval.

Given a closed set of conics, the following theorem ([29]) can be used to construct more examples. It shows us that any closed set of conics is still a closed set of conics in an odd order extension of the underlying field.

Theorem 2.3.6 ([29]). *Let $\mathcal{C}$ be a closed set of conics in $\mathrm{PG}(2,q)$. Then the equations of the conics of $\mathcal{C}$ give a closed set of conics in $\mathrm{PG}(2,q^m)$, for any $m \geq 1$, m odd.*

It immediately follows from the theorem that given a degree-d maximal arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$ arisen from a closed set of conics, then there exist degree-d maximal arcs $\mathcal{K}_m$ in $\mathrm{PG}(2,q^m)$ for all odd positive integers m. Furthermore, the arc $\mathcal{K}_m$ contains $\mathcal{K}$ in the subplane $\mathrm{PG}(2,q)$ of $\mathrm{PG}(2,q^m)$.

The following theorem is also interesting in this context.

Theorem 2.3.7 ([29]). *Let $\mathcal{C}$ be a closed set of conics giving rise to a degree-d maximal arc $\mathcal{K}$, $8 \leq d < q/2$, in $\mathrm{PG}(2,q)$ that is not of Denniston type. Then there exist maximal arcs of degree r of Mathon type that are not of Denniston type in $\mathrm{PG}(2,q)$ for all $r \geq 8$, r dividing d.*

Concerning the conics and substructures within the maximal arcs of Mathon type the next theorems were proved.

Theorem 2.3.8 ([29]). *Let $\mathcal{K}$ be a degree-d maximal arc in $\mathrm{PG}(2,q)$, $d < q/2$, constructed from a closed set of conics $\mathcal{C}$ with nucleus F_0. Then the point set of $\mathcal{K}$ contains no non-degenerate conics apart from those of $\mathcal{C}$.*

An easy geometric characterization of the Denniston maximal arc, regarding its dual, is found here.

Theorem 2.3.9 ([29]). *Let $\mathcal{K}$ be a degree-d maximal arc in* $\mathrm{PG}(2, q)$*, constructed from a closed set of conics $\mathcal{C}$ with nucleus F_0. Then $\mathcal{K}$ is of Denniston type if and only if its dual contains a regular hyperoval.*

Corollary 2.3.10 ([29]). *The dual of a Mathon maximal arc, not of Denniston type, constructed from a closed set of conics cannot be constructed from a closed set of conics.*

On a side note, regarding the Thas maximal arcs of type I, it was proven that if they arise from a spread of tangent lines to a Tits ovoid, then they cannot be constructed from a closed set of conics ([29]).

There are various families of Mathon maximal arcs known that are not of Denniston type. Every Mathon arc that is not of Denniston type will be called a *proper Mathon arc*. Actually, the most difficult part in checking that a given subset of conics of $\mathcal{F}$ is a maximal arc lies in checking whether the trace condition of Lemma 2.3.2 holds. In Chapter 3 we will present a more geometric approach to these maximal arcs of Mathon type that helps us to cope with this problem.

At this stage it might be good to give an account of the known maximal arcs in Desarguesian projective planes of small order.

Maximal arcs in small Desarguesian planes

(i) The plane $\mathrm{PG}(2, 8)$ has up to isomorphism only one maximal arc of degree 4; it is of Denniston type and is the dual of the regular hyperoval.

(ii) The plane $\mathrm{PG}(2, 16)$ has up to isomorphism two maximal arcs of degree 8: the dual of the regular hyperoval which is of Denniston type, and the dual of the Lunelli-Sce hyperoval which is of proper Mathon type. It has two non-isomorphic maximal arcs of degree 4, both of Denniston type and both self-dual. Actually, as a consequence of a more general treatise on maximal arcs for small parameters, it has been proved in [4] that in $\mathrm{PG}(2, 16)$ no other maximal arcs exist than the known ones.

(iii) The plane $\mathrm{PG}(2, 32)$ has six non-isomorphic hyperovals and hence the same number of maximal arcs of degree 16. The dual of the regular hyperoval of course yields the Denniston maximal arc of degree 16. The dual of the Cherowitzo hyperoval is a proper Mathon arc in $\mathrm{PG}(2,32)$. None of the other hyperovals yield a Mathon maximal arc, this is due to private communication with Bamberg. As far as the other maximal arcs of Denniston type are concerned, there is one of degree 4 and its dual of degree 8. In his original paper ([38]), Mathon gives a construction of three maximal arcs of degree 8 (and hence of three maximal arcs of degree 4 that are not of Mathon type, but of "dual Mathon type"), which are not of Denniston type. In the next chapter we will prove that there are no other maximal arcs of Mathon type of degree 8.

The following important lemma was proved by Aguglia, Giuzzi and Korchmáros in [2]. It shows that two conics on the same nucleus can uniquely be extended to a maximal arc of degree 4 of Denniston type.

Lemma 2.3.11 ([2]). *Given any two disjoint conics C_1 and C_2 on a common nucleus. Then there is a unique degree-4 maximal arc of Denniston type containing $C_1 \cup C_2$.*

This lemma will be of critical importance throughout the upcoming chapters and it will be one of the key elements in a more geometric approach to the maximal arcs of Mathon type.

Using especially the previous lemma we were able to prove that a maximal arc consisting of conics on a common nucleus, always has to be of Mathon type.

Theorem 2.3.12. *A maximal arc $\mathcal{K}$ consisting of disjoint conics on a common nucleus is a maximal arc of Mathon type.*

Proof. If all compositions of all conics in $\mathcal{K}$ are contained in $\mathcal{K}$, then the maximal arc $\mathcal{K}$ is of Mathon type. Now, suppose dat $\mathcal{K}$ is not a maximal arc of Mathon type. This implies that there are two conics C_1 and C_2 in $\mathcal{K}$ of which the composition $C_1 \oplus C_2$ is not contained in $\mathcal{K}$. In other words, there is a point $p \in C_1 \oplus C_2$ that is not contained in $\mathcal{K}$, i.e., p is an external point. However, due to Lemma 2.3.11, we know that the three conics C_1, C_2 and $C_1 \oplus C_2$ induce a unique degree-4 maximal arc of Denniston type. Hence, every line incident with p should intersect either C_1 or C_2. This means that p is not an external point, contradiction. □

Chapter 3

Geometric approach to Mathon maximal arcs

In the previous chapter we introduced Mathon's construction of maximal arcs which in fact generalizes the construction of Denniston. Clearly, various families of Mathon maximal arcs are known that are not of Denniston type, the so called *proper Mathon arcs* (see Chapter 2). As mentioned before, the most difficult part in checking that a given subset of conics is a maximal arc lies in checking whether the trace condition of Lemma 2.3.2 holds. In Section 3.1 we will present a more geometric approach to these arcs that allows us to overcome this problem. Furthermore, this geometric approach will be the key to the main result of this chapter, which is the enumeration of the non-isomorphic Mathon 8-arcs in $\mathrm{PG}(2, 2^h)$, $h > 4$ and $h \neq 7$ prime.

The enumeration problem for Mathon arcs was first studied in [30], where bounds were derived for the number of isomorphism classes of Mathon arcs of "big" degree. The techniques of [30] however failed for small degree arcs, and the enumeration of such arcs was left as an open problem.

The results in this chapter are published in the paper [17].

3.1 A synthetic construction of Mathon arcs

First we recall Lemma 2.3.11 which states that any two disjoint conics C_1 and C_2 on a common nucleus induce a unique degree-4 maximal arc of Denniston type containing $C_1 \cup C_2$.

We intent to generalize this result to a synthetic version of Mathon's construction. In order to do so we need to make sure that there exists a line external to the given set of conics. More precisely, we need to prove that there is at least one line that can be used as the line at infinity of the constructed maximal arc of Mathon type.

Lemma 3.1.1. *Given a degree-d maximal arc M of Mathon type, $d < q/2$, consisting of $d-1$ conics on a common nucleus n, and a conic C disjoint from M with the same nucleus n, there exists a line external to $M \cup C$.*

Proof. First we count the number of secants to M. Since $(q+1)(q/d-1)+1$ is the number of external lines to M, the number of secants to M is equal to

$$q^2+q+1-((q+1)\left(\frac{q}{d}-1\right)+1) = \left(\frac{d-1}{d}\right)q^2+\left(\frac{2d-1}{d}\right)q+1.$$

Next we count the number of lines that intersect both M and C. At first we will disregard the $q+1$ tangents to C, they will be added at the end. Since the tangents to C are disregarded, a secant line l to both C and M must intersect C in 2 points and M in d points. This implies that the total number of secants to both M and C is equal to

$$\frac{1}{2}\left(\frac{(q+1)(d-1)+1}{d}-1\right)(q+1)+q+1 = \left(\frac{d-1}{2d}\right)q^2+\left(\frac{3d-1}{2d}\right)q+1.$$

We know that the number of lines intersecting C is $(q+1)q/2+q+1$. This means that the number of lines that intersect C but do not intersect M is

$$\frac{(q+1)q}{2}+q+1-\left(\left(\frac{d-1}{2d}\right)q^2+\left(\frac{3d-1}{2d}\right)q+1\right) = \frac{q^2}{2d}+\frac{q}{2d}.$$

Finally we are able to count the number of secants to $M \cup C$. We find

$$\begin{aligned}\left(\frac{d-1}{d}\right)q^2+\left(\frac{2d-1}{d}\right)q+1+\frac{q^2}{2d}+\frac{q}{2d} &= \left(\frac{2d-1}{2d}\right)q^2+\left(\frac{4d-1}{2d}\right)q+1\\ &< q^2+q+1.\end{aligned}$$

This proves that there exists an external line to $M \cup C$. □

Using Lemma 3.1.1 we are able to prove the following result, which can be seen as a synthetic version of Mathon's construction.

Theorem 3.1.2 (Synthetic version of Mathon's theorem). *Let M be a maximal arc of degree d, $d < q/2$, of Mathon type, consisting of $d-1$ conics on a common nucleus n, and let C_d be a conic disjoint from M with the same nucleus n, then there is a unique degree-$2d$ maximal arc of Mathon type containing $M \cup C_d$.*

Proof. Denote the $d-1$ conics in the maximal arc M by $C_1, C_2, C_3, ..., C_{d-1}$. Due to Lemma 3.1.1 we know there exists an external line r to $M \cup C_d$. We recoordinatize the plane $\mathrm{PG}(2,q)$ in such a way that the line r now has equation $z = 0$ and the common nucleus n has coordinates $(0,0,1)$. This provides us with the setting in which the conic C_i has equation $\alpha_i x^2 + xy + \beta_i y^2 + \lambda_i z^2 = 0$. Next we define $\overline{C_i} := \alpha_i \beta_i$. It is clear that $\mathrm{Tr}(\overline{C_i}) = 1$, $\forall i = 1, ..., d$. We can now construct the degree-$2d$ maximal arc containing $M \cup C_d$. Let $C_i \oplus C_d := C_{i+d}$ $\forall i = 1, ..., d-1$. The construction used in the proof of Lemma 2.3.11, which is based on Mathon, implies that $\mathrm{Tr}(\overline{C_{i+d}}) = 1$. Due to Lemma 2.3.2 it follows that C_i, C_d and C_{i+d} are mutually disjoint.
Next we need to check that the conics C_i and C_{j+d}, $\forall i, j = 1, ..., d-1$, are disjoint, i.e., $\mathrm{Tr}(\overline{C_i \oplus C_{j+d}}) = 1$. Let $C_i \oplus C_j = C_k$ be another conic which is defined in the closed set M, then

$$\begin{aligned}
\mathrm{Tr}(\overline{C_i \oplus C_{j+d}}) &= \mathrm{Tr}(\overline{C_i \oplus C_j \oplus C_d}) \\
&= \mathrm{Tr}(((\alpha_i \oplus \alpha_j) \oplus \alpha_d)((\beta_i \oplus \beta_j) \oplus \beta_d)) \\
&= \mathrm{Tr}((\alpha_k \oplus \alpha_d)(\beta_k \oplus \beta_d)) \\
&= \mathrm{Tr}(\overline{C_k \oplus C_d}) \\
&= \mathrm{Tr}(\overline{C_{k+d}}) \\
&= 1.
\end{aligned}$$

Also the conics C_{i+d}, $\forall i = 1, ..., d-1$, have to be mutually disjoint. This holds since

$$\begin{aligned}
\mathrm{Tr}(\overline{C_{i+d} \oplus C_{j+d}}) &= \mathrm{Tr}(\overline{C_i \oplus C_d \oplus C_j \oplus C_d}) \\
&= \mathrm{Tr}(\overline{C_i \oplus C_j}) \\
&= \mathrm{Tr}(\overline{C_k}) \\
&= 1,
\end{aligned}$$

where again $C_k = C_i \oplus C_j$ is a conic in the original degree-d maximal arc M of Mathon type. It now follows that $\bigcup_{i=1}^{2d-1} C_i$ is a closed set of conics on a common nucleus n which, due to Theorem 2.3.4, gives rise to a degree-$2d$ maximal arc of Mathon type. □

3.2 Denniston 4-arcs

In Chapter 2, Theorem 2.1.5, we saw how Hamilton and Penttila ([32]) determined the collineation stabiliser of a degree-d Denniston maximal arc.

In the next lemma we will show, using the same notation, that the order of G_A is 2 in $\mathrm{GF}(2^{2h+1})$, $2h+1$ prime and $2h+1 \neq 2, 3$, with A the additive subgroup used to construct a degree-4 Denniston maximal arc.

Lemma 3.2.1. *In* $\mathrm{PG}(2, 2^{2h+1})$*,* $2h+1$ *prime, and* $2h+1 \neq 3$*, let* $\mathcal{D}$ *be a degree-4 Denniston maximal arc defined by an additive subgroup* A*. Define the group* G *acting on* $\mathrm{GF}(2^{2h+1})$ *by*

$$G = \{x \mapsto ax^{\sigma} : a \in \mathrm{GF}(2^{2h+1})^*, \sigma \in \mathrm{Aut}\,\mathrm{GF}(2^{4h+2})\}.$$

Then $|G_A| = 2$*.*

Proof. First we remark that the plane $\mathrm{PG}(2, 2^{2h+1})$ can be coordinatized in such a way that the additive subgroup $A = \{0, 1, w, w+1\}$, with $w \in \mathrm{GF}(2^{2h+1}) \setminus \{0, 1\}$, is associated to the maximal arc $\mathcal{D}$ of Denniston type. We will denote the multiplicative order of the element $w \in A$ in $\mathrm{GF}(2^{2h+1})$ by $o(w)$.
Let $\varphi \in G_A$. Since $\varphi(0) = 0$ we can restrict the action of φ on A to its action on $\{1, w, w+1\}$. The action of φ on each element of $\{1, w, w+1\}$ has either order 1, 2 or 3.
First we suppose $\sigma = 1$.

- If $a = 1$ then $\varphi = \mathrm{id}$ in G.
- If $a \neq 1$ then the action of φ on 1 has either order 2 or 3.
    - · If the order is 2 then

$$\varphi(\varphi(1)) = a^2 = 1$$

which implies that $a = 1$, clearly a contradiction.

· If the order is 3 then

$$\varphi(\varphi(\varphi(1))) = a^3 = 1$$

which implies that $3|2^{2h+1}-1$. But since

$$2^{2h+1}-1 = 2^{2h}+2^{2h-1}+\cdots+1 = 2^{2h}+2^{2h-2}3+2^{2h-4}3+\cdots+2^2 3+3,$$

we again find a contradiction.

From now on suppose $\sigma \neq 1$.

(1.) Assume φ acts trivially on $\{1, w, w+1\}$. Then $\varphi(1) = 1$ implies $a = 1$. Furthermore $\varphi(w) = aw^\sigma = w^\sigma$. Since the action of φ on each element of $\{1, w, w+1\}$ has order 1 there has to follow that $w^\sigma = w$, which implies $w^{\sigma-1} = 1$. This means $o(w)|\sigma - 1$ but of course we know $o(w)|2^{2h+1} - 1$. Now suppose $\sigma = 2^l$, $l \in \mathbb{N}^*$. Note that $l < 4h + 2$. Then:

$$o(w)|\gcd(2^l - 1, 2^{2h+1} - 1),$$

which implies that

$$o(w)|2^{\gcd(l,2h+1)} - 1.$$

Now two possibilities can occur.

- If $l = 2h + 1$ then $\varphi : x \mapsto x^{2^{2h+1}}$, and so φ indeed acts trivially on A.
- If $l \neq 2h+1, 0$ then $\gcd(l, 2h+1) = 1$. It follows that $o(w) = 1$ and so $w = 1$, which is clearly a contradiction.

(2.) Assume the orbit on some element of $\{1, w, w+1\}$ has length 2 under the action of φ. We consider two cases.

(a) If $\varphi(1) = 1$ then of course $a = 1$ holds again. This implies $\varphi(w) = w^\sigma$ and $\varphi(w^\sigma) = w^{\sigma^2}$ but since the action of φ has order 2 it follows that $w^{\sigma^2} = w$, implying $w^{\sigma^2-1} = 1$. We find that $o(w)|\sigma^2 - 1$ and also $o(w)|2^{2h+1} - 1$. Using $\sigma = 2^l$ as we did above, we find, as $2 \not| \, 2h+1$ and $2h + 1$ is prime,

$$\begin{aligned} o(w)|\gcd(2^{2l} - 1, 2^{2h+1} - 1) &\Rightarrow o(w)|2^{\gcd(2l,2h+1)} - 1 \\ &\Rightarrow o(w)|2^{\gcd(l,2h+1)} - 1. \end{aligned}$$

Now the same two possibilities as in (1.) can occur, hence φ acts trivially on A, clearly a contradiction.

(b) Without loss of generality we can assume that $\varphi(1) = w$. In this case we find that $a = w$. Furthermore $\varphi(\varphi(1)) = \varphi(w) = w^{\sigma+1}$ and so $w^{\sigma+1} = 1$ since the action of φ has order 2. This implies $w^{\sigma^2-1} = 1$ which gives us $o(w)|\sigma^2 - 1$ and again we know $o(w)|2^{2h+1} - 1$. Using the same arguments as we did in (a), we see that

$$o(w)|2^{\gcd(l,2h+1)} - 1.$$

Again the two possibilities we encountered in (1.) can occur.

- If $l = 2h + 1$, then $\varphi : x \mapsto wx^{2^{2h+1}}$ and again

$$\varphi(\varphi(1)) = w^2 = 1,$$

  a contradiction.
- If $l \neq 2h + 1, 0$ then $\gcd(l, 2h + 1) = 1$. It follows that $o(w) = 1$ and so $w = 1$, a contradiction.

(3.) Now assume the orbit length is 3 under the action φ. Without loss of generality we can assume that $\varphi(1) = w$, then $a = w$. From this we find that $\varphi(\varphi(\varphi(1))) = w^{\sigma^2+\sigma+1}$, which of course has to be equal to 1. We deduce that $w^{\sigma^3-1} = 1$, implying that $o(w)|\sigma^3 - 1$ while $o(w)|2^{2h+1} - 1$ still holds. If we again set $\sigma = 2^l$, $l \in \mathbb{N}^*$ and $l < 4h + 2$, we find that $o(w)|2^{\gcd(l,2h+1)} - 1$, since $3 \not| \, 2h + 1$. Remark that in case $2h + 1 = 3$ the degree-4 maximal arc would be a dual hyperoval of $\mathrm{PG}(2, 8)$. The same two possibilities as in (1.) can occur.

- If $l = 2h + 1$, then $\varphi : x \mapsto wx^{2^{2h+1}}$ and again

$$\varphi(\varphi(\varphi(1))) = w^3 = 1,$$

  a contradiction.
- If $l \neq 2h + 1, 0$ then $\gcd(l, 2h + 1) = 1$. It follows that $o(w) = 1$ and so $w = 1$, a contradiction.

We have proven that φ either is $\mathrm{id} \in G$ or $\varphi : x \mapsto x^{2^{2h+1}}$, hence $|G_A| = 2$. □

Remark 3.2.2. We have just shown that if $q = 2^p$, p prime, $p \neq 2, 3$, then the full automorphism group G of a degree-4 Denniston arc has size $2(q+1)$ and is isomorphic to $C_{q+1} \rtimes C_2$. Let us have a closer look at the action of this group on the arc. It is well known ([1]) that in G there is a cyclic subgroup of order $q+1$ stabilizing all three conics of the arc and acting sharply transitively on the points of each of these conics. Furthermore this group stabilizes the line at infinity L of the pencil determined by the arc and acts sharply transitively on the points of this line. The group G also contains $q+1$ involutions. These involutions are exactly the $q+1$ elations with axis a line through the nucleus, and center the intersection of this line with the Denniston line L, stabilizing each of the three conics of the arc. There is exactly one such involution for each line through the nucleus.

In the following lemma we count the number of isomorphism classes of degree-4 maximal arcs of Denniston type.

Lemma 3.2.3. *The number of isomorphism classes of degree-*4 *maximal arcs of Denniston type in* $\mathrm{PG}(2, 2^{2h+1})$*,* $2h+1$ *prime,* $2h+1 \neq 3$ *is*

$$N = \frac{2^{2h}-1}{3(2h+1)}.$$

Proof. Since, by recoordinatizing the plane, we can always assume that a degree-4 maximal arc of Denniston type is contained in the standard pencil, it suffices to calculate the number of isomorphism classes of degree-4 maximal arcs in the standard pencil.

First of all we count the total number of degree-4 maximal arcs of Denniston type in the standard pencil. We have $(2^{2h+1}-1)$ choices to pick a first conic and $(2^{2h+1}-2)$ choices to pick a second conic. Since Lemma 2.3.11 states that there is a unique degree-4 maximal arc containing these 2 conics the total number of degree-4 maximal arcs in the standard pencil is

$$\frac{(2^{2h+1}-1)(2^{2h+1}-2)}{6}.$$

Let $\mathcal{D}$ be a degree-4 maximal arc of Denniston type. Due to Theorem 2.1.5 and Lemma 3.2.1 we know that

$$|\mathrm{Aut}(\mathcal{D})| = 2(2^{2h+1}+1).$$

Using this along with the fact that the order of the collineation stabiliser of the standard pencil is $2(2h+1)(2^{4h+2}-1)$ (see proof of Theorem 2.1.5), we can count the number of degree-4 maximal arcs of Denniston type that are isomorphic to $\mathcal{D}$. We obtain

$$\frac{2(2h+1)(2^{4h+2}-1)}{2(2^{2h+1}+1)} = (2h+1)(2^{2h+1}-1).$$

Finally the number of isomorphism classes of degree-4 maximal arcs of Denniston type in the pencil is

$$\frac{(2^{2h+1}-1)(2^{2^{2h+1}-2})}{6(2h+1)(2^{2h+1}-1)} = \frac{2^{2h}-1}{3(2h+1)}.$$

□

Lemma 3.2.4. *The number of degree-*4 *maximal arcs of Denniston type in the standard pencil in* $\mathrm{PG}(2, 2^{2h+1})$*,* $2h+1$ *prime,* $2h+1 \neq 3$ *which are isomorphic to a given one and contain a given conic* C *equals* $3(2h+1)$*.*

Proof. Let $\mathcal{D}$ be any degree-4 maximal arc. The result follows immediately from the facts that the standard pencil contains $(2h+1)(2^{2h+1}-1)$ isomorphic copies of $\mathcal{D}$, the standard pencil contains $2^{2h+1}-1$ conics, and $\mathcal{D}$ contains 3 conics, keeping in mind that $\mathrm{Aut}(\mathcal{D})$ acts as described in Remark 3.2.2. □

Remark 3.2.5. Before moving on to the next section we quickly show that the number N of isomorphism classes of degree-4 maximal arcs found in Lemma 3.2.3 is indeed an integer.

If p is an odd prime and $p \neq 3$, then we have to show that $3p \mid 2^{p-1}-1$. Since 3 and p are coprime we prove that both $3 \mid 2^{p-1}-1$ and $p \mid 2^{p-1}-1$ are satisfied.

It is easy to see that, putting $p-1 = 2h$,

$$3 \mid 2^{p-1}-1 \Leftrightarrow 3 \mid 4^h-1 \Leftrightarrow 4-1 \mid 4^h-1,$$

which proves the first part. On the other hand $p \mid 2^{p-1}-1$ is the so-called little theorem of Fermat.

3.3 Mathon 8-arcs

Let us first have a look at the geometric structure of a maximal 8-arc of Mathon type; this is based on [28]. Note that if $\mathcal{K}$ is a maximal arc constructed from a closed set of conics $\mathcal{C}$ on a common nucleus, then the point set of that arc contains no non-degenerate conics apart from those of $\mathcal{C}$ (see Theorem 2.3.8). From Lemma 2.3.11 it immediately follows that every Mathon 8-arc contains exactly seven Denniston 4-arcs, and each two of these seven 4-arcs have exactly one conic in common. One in fact easily sees that the structure with as point set the conics of $\mathcal{K}$, line set the degree-4 subarcs of Denniston type, and the natural incidence is isomorphic to $\mathrm{PG}(2,2)$, the Fano plane. In accordance with Chapter 2 the lines at infinity of each of the pencils determined by the degree-4 subarcs are called the *Denniston lines* of $\mathcal{K}$. If $\mathcal{K}$ is of Denniston type there is a unique such line, otherwise there are exactly seven distinct Denniston lines (see Theorem 2.2 of [28] and the remark preceding it). Suppose namely that two subarcs $\mathcal{K}_1$ and $\mathcal{K}_2$ would have the same Denniston line. Let C be the conic belonging to both $\mathcal{K}_1$ and $\mathcal{K}_2$. Since a conic and a line uniquely determine a pencil, it follows that $\mathcal{K}_1$ and $\mathcal{K}_2$ belong to the same pencil, yielding that $\mathcal{K}$ is of Denniston type. Note that it is essential here that any two of the degree-4 arcs have a conic in common. In [28] it is noticed that all known Mathon 8-arcs seem to have an involution stabilizing $\mathcal{K}$ and all of its conics. Theorem 2.3 of [28] gives a sufficient condition for such an involution to exist. In the next lemma we show that such an involution always exists.

Lemma 3.3.1. *Let $\mathcal{K}$ be a proper Mathon 8-arc. Then the seven Denniston lines of $\mathcal{K}$ are concurrent and there exists a unique involution stabilizing $\mathcal{K}$ and all conics contained in $\mathcal{K}$. This involution is the elation with center the point of intersection of the Denniston lines and axis the line containing the nucleus of $\mathcal{K}$ and the center.*

Proof. Denote the seven degree-4 Denniston subarcs of $\mathcal{K}$ by $\mathcal{D}_i$, $i = 1, \ldots, 7$. Let n be the nucleus of (the conics of) $\mathcal{K}$. Let L_i be the Denniston line of $\mathcal{D}_i$. Let c be the intersection of L_1 and L_2. Consider the unique involution ι with center c and axis nc that stabilizes the conic C that is the intersection of $\mathcal{D}_1$ and $\mathcal{D}_2$. It is well known that ι will stabilize all conics in $\mathcal{D}_1$ and $\mathcal{D}_2$ (see e.g. the proof of Theorem 2.1.5). Now let $\mathcal{D}_3$ be the unique third 4-arc that contains C. As $\mathcal{K}$ is uniquely determined by $\mathcal{D}_1$ and $\mathcal{D}_2$ (see Theorem 3.1.2) it follows that ι must stabilize $\mathcal{D}_3$. Hence it must stabilize the Denniston line of $\mathcal{D}_3$, implying that L_3 contains c. It now also follows that ι stabilizes all conics of $\mathcal{K}$ and that all Denniston lines have to be stabilized; we deduce that all Denniston lines are concurrent at c. □

Corollary 3.3.2. *Let $\mathcal{K}$ be a proper Mathon 8-arc in $\mathrm{PG}(2, 2^p)$, p prime, $p \neq 2, 3, 7$. Then $\mathrm{Aut}(\mathcal{K}) \cong C_2$.*

Proof. Let ϕ be a non-trivial automorphism of $\mathcal{K}$. Clearly ϕ has to fix the intersection point c of the Denniston lines of $\mathcal{K}$.

First suppose that ϕ stabilizes one of the degree-4 maximal subarcs of $\mathcal{K}$. From Remark 3.2.2 and the fact that $c^\phi = c$ it follows that ϕ is the unique involution ι described in the previous lemma.

So, suppose that ϕ does not stabilize any of the Denniston subarcs. Hence no orbit of ϕ on the subarcs has length 1. As there are seven subarcs, the set O of orbit lengths has to be one of the following: $\{7\}, \{5, 2\}, \{4, 3\}, \{3, 2\}$. Suppose $O = \{3, 2\}$. Then ϕ^2 stabilizes some subarc and hence has to be the involution ι. It follows that ϕ cannot have an orbit of length 3, contradiction. The cases $O = \{5, 2\}$ and $O = \{4, 3\}$ are excluded in an analogous way.

Hence ϕ cyclically permutes the seven subarcs. Suppose that ϕ would belong to $\mathrm{PGL}(3, 2^p)$. As ϕ fixes the line nc containing the nucleus and c, and 2^p is not divisible by 7, we see that ϕ must fix a second line through c. If ϕ would fix a third line through c it would fix all lines through c, a contradiction as ϕ cyclically permutes the Denniston lines of $\mathcal{K}$. Hence 7 divides $2^p - 1$, which implies that 3 divides p, a contradiction. Hence $\phi \in \mathrm{P\Gamma L}(3, 2^p) \setminus \mathrm{PGL}(3, 2^p)$. As 7 is prime it follows that 7 divides the prime p, yielding that $p = 7$, the final contradiction. □

In order to be able to count the number of isomorphism classes of degree-8 maximal arcs of Mathon type we need to know how many isomorphic images of a given degree-8 maximal Mathon arc there are. The following technical lemma will play a key role in our final calculations.

Lemma 3.3.3. *Let $\mathcal{K}$ be a proper Mathon 8-arc in $\mathrm{PG}(2, 2^{2h+1})$, $2h+1$ prime, and $h \neq 1, 3$. Then the number of degree-8 maximal arcs isomorphic to $\mathcal{K}$ that have one of their degree-4 maximal subarcs in the standard pencil, contain a fixed given conic C from the standard pencil and have the same intersection point for their Denniston lines is $21(2h + 1)$.*

Proof. Let C be a conic in the standard pencil. It is well known that $G := \mathrm{Aut}(C) \cong \mathrm{P\Gamma L}(2, 2^{2h+1})$. Hence $|G| = |\mathrm{P\Gamma L}(2, 2^{2h+1})| = (2h+1)(2^{2h+1}+1)(2^{4h+2} - 2^{2h+1})$, which is the number of group elements that stabilize C and its nucleus n. The group G acts transitively on the points not on C and distinct from n. From this we can deduce that

$$|G_{C,n,(0,1,0)}| = \frac{(2h+1)(2^{2h+1}+1)(2^{4h+2} - 2^{2h+1})}{(2^{2h+1}+1)(2^{2h+1}-1)} = (2h+1)2^{2h+1}.$$

The group $G_{C,n,(0,1,0)}$ acts transitively on the lines through $(0,1,0)$ that do not intersect C. Since $\frac{2^{2h+1}}{2}$ is the number of such lines, this implies that

$$|G_{C,[X=0],[Z=0]}| = \frac{|G_{C,n,(0,1,0)}|}{\frac{2^{2h+1}}{2}} = 4h+2.$$

Now suppose $\mathcal{K}$ is a proper Mathon arc of degree 8. Let $\mathcal{D}_i$, $i = 1\dots,7$ denote the seven 4-arcs of Denniston type contained in $\mathcal{K}$, and let $C_1 = C, \dots, C_7$ denote the seven conics of $\mathcal{K}$. Without loss of generality we may suppose that $\mathcal{D}_1$ belongs to the standard pencil and that C is the conic belonging to both $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. Furthermore we may assume that $(0,1,0)$ is the intersection point of the Denniston lines of $\mathcal{K}$. We want to count the number of isomorphic images of $\mathcal{K}$ that contain C, have a degree-4 subarc in the standard pencil, and that have $(0,1,0)$ as intersection point of the Denniston lines. Recall that $|\mathrm{Aut}(\mathcal{K})| = 2$. Let ϕ be an automorphism of the plane mapping $\mathcal{K}$ onto an isomorphic image of the desired type. First suppose ϕ stabilizes C and the standard pencil. From the above we know that there are $4h+2$ choices for ϕ. Also, there are exactly $4h+2$ choices for ϕ that would map the pencil determined by $\mathcal{D}_i$, $i = 2,3$, onto the standard pencil and stabilize C. We obtain $3(4h+2)$ choices for ϕ that stabilize C. Now let C_i, $i \neq 1$ be any other conic of $\mathcal{K}$. Suppose that $C_i^{\phi} = C$. As one of the three pencils determined by C_i and $\mathcal{K}$ has to be mapped onto the standard pencil, we see in an analogous way that there are $3(4h+2)$ choices for ϕ such that $C_i^{\phi} = C$. We obtain that in total there are $21(4h+2)$ choices for ϕ. It follows that there are exactly $21(2h+1)$ isomorphic images of $\mathcal{K}$ of the desired type. $\square$

Given a degree-4 maximal arc of Denniston type $\mathcal{D}_1$ in the standard pencil consisting of the conics C_1, C_k, C_{k+1}. Due to Lemma 2.3.11 each conic C disjoint from $\mathcal{D}_1$ together with C_1 will give rise to another degree-4 maximal arc of Denniston type which will be isomorphic to one of the degree-4 maximal arcs of Denniston type in the standard pencil. In what follows we will establish the trace conditions that express the disjointness of the conic C with respect to $\mathcal{D}_1$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be 2 non-isomorphic degree-4 maximal arcs of Denniston type. Without loss of generality we can assume that both arcs are contained in the standard pencil and that both contain a common conic C_1. Let the additive subgroups $\{0, 1, k, k + 1\}$ and $\{0, 1, l, l + 1\}$, with $k \neq l, l + 1$ and $k, l \in \mathrm{GF}(2^{2h+1}) \setminus \{0, 1\}$, be the ones associated to the maximal arcs $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. In other words we assume $\mathcal{D}_1$ consists of the conics $C_i, i = 1, k, k + 1$ given by the equation

$$C_i : x^2 + xy + y^2 + iz^2 = 0$$

and $\mathcal{D}_2$ consists of the conics $C_j, j = 1, l, l + 1$ given by

$$C_j : x^2 + xy + y^2 + jz^2 = 0.$$

Consider the automorphisms θ of $\mathrm{PG}(2, 2^{2h+1})$ determined by the matrix

$$\begin{pmatrix} \sqrt{\lambda}^{-\sigma} & 0 & 0 \\ t & \sqrt{\lambda}^{-\sigma} & 0 \\ \sqrt{\sqrt{\lambda}^{-\sigma} t + t^2} & 0 & 1 \end{pmatrix}, \tag{3.1}$$

and the field automorphism σ, with $\lambda = 1, l, l + 1$ and $t \in \mathrm{GF}(2^{2h+1})$. These automorphisms will map C_λ onto C_1 while $(0, 0, 1)^\theta = (0, 0, 1)$ and $(0, 1, 0)^\theta = (0, 1, 0)$. In fact all automorphisms of $\mathrm{PG}(2, 2^{2h+1})$ which fix $(0, 0, 1)$ and $(0, 1, 0)$ and map C_λ onto C_1 are of the form θ. There are 3 possibilities for θ that we have to take into account: $C_1^\theta = C_1$, $C_l^\theta = C_1$ and $C_{l+1}^\theta = C_1$. We will look at the case where C_l is mapped onto C_1 and examine what values for t satisfy the conditions

$$C_1^\theta \cap C_k = \emptyset$$

and

$$C_1^\theta \cap C_{k+1} = \emptyset.$$

Analogous results can be found in the cases $C_1^\theta = C_1$ and $C_{l+1}^\theta = C_1$. First we construct the image of C_1 under θ. It is clear that the point $(0, 1, 1)$, which is the intersection of C_1 and the x-axis, is mapped onto the point $(0, \sqrt{l}^{-\sigma}, 1)$.

Next, we know that $(0, \sqrt{l}^{-\sigma}, 1) \neq (0, \sqrt{k}, 1)$, or equivalently, that $l^{-\sigma} \neq k$ since this would immediately imply that $C_1^\theta \cap C_k \neq \emptyset$, a contradiction. Analogously $(0, \sqrt{l}^{-\sigma}, 1) \neq (0, \sqrt{k+1}, 1)$, i.e., $l^{-\sigma} \neq k+1$, since in this case the contradiction $C_1^\theta \cap C_{k+1} \neq \emptyset$ would hold.

Furthermore we look at the image of a general point $(1, y, z)$ of C_1, $y, z \in \mathrm{GF}(2^{2h+1})$, where of course $1 + y + y^2 + z^2 = 0$ holds. We find

$$\begin{pmatrix} \sqrt{l}^{-\sigma} & 0 & 0 \\ t & \sqrt{l}^{-\sigma} & 0 \\ \sqrt{\sqrt{l}^{-\sigma} t + t^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ z \end{pmatrix}^{\sigma} = \begin{pmatrix} \sqrt{l}^{-\sigma} \\ t + \sqrt{l}^{-\sigma} y^{\sigma} \\ \sqrt{\sqrt{l}^{-\sigma} t + t^2} + z^{\sigma} \end{pmatrix},$$

with $\sigma \in \mathrm{Aut}(\mathrm{GF}(2^{2h+1}))$. The condition $C_1^\theta \cap C_k = \emptyset$ is satisfied if and only if the equation

$$l^{-\sigma} + \sqrt{l}^{-\sigma} t + l^{-\sigma} y^{\sigma} + t^2 + l^{-\sigma} y^{2\sigma} + k\sqrt{l}^{-\sigma} t + kt^2 + kz^{2\sigma} = 0$$

has no solutions in $\mathrm{GF}(2^{2h+1})$. Equivalently, since $1 + y^{\sigma} + y^{2\sigma} = z^{2\sigma}$, we find

$$\begin{aligned} &(l^{-\sigma} + k) z^{2\sigma} + \sqrt{l}^{-\sigma} t + t^2 + k\sqrt{l}^{-\sigma} t + kt^2 = 0 \\ \Leftrightarrow \quad & z^{2\sigma} = \frac{(1+k)t(\sqrt{l}^{-\sigma} + t)}{(l^{-\sigma} + k)}. \end{aligned}$$

Hence the conics C_1^θ and C_k will be disjoint if and only if the equation

$$1 + y^{\sigma} + (y^{\sigma})^2 + \frac{(1+k)t(\sqrt{l}^{-\sigma} + t)}{(l^{-\sigma} + k)} = 0.$$

has no solutions in y^{σ}, or equivalently if and only if

$$\mathrm{Tr}\Big[1 + \frac{(1+k)t(\sqrt{l}^{-\sigma} + t)}{(l^{-\sigma} + k)}\Big] = 1.$$

Since $\mathrm{Tr}(1) = 1$ in $\mathrm{GF}(2^{2h+1})$ we find the condition

$$\mathrm{Tr}\Big[\frac{(1+k)t(\sqrt{l}^{-\sigma} + t)}{(l^{-\sigma} + k)}\Big] = 0. \tag{3.2}$$

Analogously, the trace condition

$$\mathrm{Tr}\Big[\frac{kt(\sqrt{l}^{-\sigma}+t)}{(l^{-\sigma}+k+1)}\Big]=0 \tag{3.3}$$

is necessary and sufficient for $C_1^\theta \cap C_{k+1}=\emptyset$.

It is clear that also the conic C_{l+1}^θ has to be disjoint from both C_k and C_{k+1}. However, due to Lemma 2.3.11, we know that the two conics C_1 and C_1^θ give rise to a unique degree-4 maximal arc of Denniston type. The third conic contained in this 4-arc has to be C_{l+1}^θ, since we are actually looking at the image of $\mathcal{D}_2$ under θ. Using Theorem 3.1.2 we know that the degree-4 maximal arc $\mathcal{D}_1$ and the conic C_1^θ induce a unique degree-8 maximal arc in which of course all conics are mutually disjoint. Since $\mathcal{D}_2^\theta$ is contained in this 8-arc we can conclude that C_{l+1}^θ will be disjoint from all other conics in the 8-arc. This implies that the two trace conditions originating from the disjointness of C_{l+1}^θ will lead to the same values for t.

Next, consider a degree-4 maximal arc $\mathcal{D}$ in the degree-8 maximal arc. If $\theta_{t'}=\iota\theta_t$, where ι is the unique involution described in Lemma 3.3.1, fixing all conics in the 8-arc, then we know $\mathcal{D}^{\theta_t}=\mathcal{D}^{\theta_{t'}}$. Since $\theta_{t'}\neq\theta_t$, the values t and t' will of course be distinct. However, these t-values have to give rise to the same degree-4 arc $\mathcal{D}^{\theta_t}$. In other words, these t-values come in pairs, which means that two t-values induce one and the same line at infinity or equivalently, one and the same degree-4 maximal arc of Denniston type.

Suppose there would be a third value t'' inducing the same degree-4 arc of Denniston type. This means $\mathcal{D}^{\theta_t}=\mathcal{D}^{\theta_{t''}}$ or $\mathcal{D}^{\theta_t\theta_{t''}^{-1}}=\mathcal{D}$. Since t and t'' are presumed to be distinct, it follows that $\theta_t\theta_{t''}^{-1}=\iota$ which means that $\theta_t=\iota\theta_{t''}$ or equivalently $\iota\theta_t=\theta_{t''}$. We conclude that $\theta_{t''}=\theta_{t'}$ or $t''=t'$.

Remark 3.3.4. There are no restrictions on σ since $\mathcal{D}_1$ and $\mathcal{D}_2$ are non-isomorphic. On the other hand, consider $\mathcal{D}_1$ consisting of the conics C_1, C_k, C_{k+1} and the automorphism fixing the conic C_1. If in that case σ is the identity then the conics C_k and C_k^θ will intersect in the point $(0,\sqrt{k},1)$ on the x-axis. Analogously the conics C_{k+1} and C_{k+1}^θ intersect in $(0,\sqrt{k+1},1)$. This of course does not occur in disjoint conics.

Finally, we have enough tools to start counting the number of isomorphism classes of degree-8 Mathon arcs in $\mathrm{PG}(2,2^{2h+1})$, $2h+1\neq 7$ and prime.

Theorem 3.3.5. *The number of isomorphism classes of proper Mathon arcs of degree* 8 *in* $\mathrm{PG}(2, 2^{2h+1})$, $2h+1 \neq 7$ *and prime, is exactly*

$$\frac{N}{14}(2^{2h-2}-1)((6h+3)N-1),$$

where $N = (2^{2h}-1)/3(2h+1)$.

Proof. Let $\mathcal{D}^i$, $i = 1, ..., N$, be chosen fixed and representative of each isomorphism class of degree-4 maximal arcs of Denniston type in the standard pencil. Assume $\mathcal{D}^i$ consists of the conics C_1, C_2^i and C_3^i, $i = 1, ..., N$. First of all we want to calculate how many degree-8 maximal arcs of Mathon type contain one of the N degree-4 maximal arcs $\mathcal{D}^i$, say $\mathcal{D}^1$, have the x-axis as elation axis and the intersection point of the Denniston lines as elation centre.

- Assume $i \neq 1$.

  Let θ be an automorphism of $\mathrm{PG}(2, 2^{2h+1})$ as given by the matrix in (3.1). We need to count in how many ways we can map C_2^i onto C_1 such that both conditions

$$\begin{cases} C_1^\theta \cap C_2^1 = \emptyset \\ C_1^\theta \cap C_3^1 = \emptyset \end{cases}$$

  are satisfied. As seen above these conditions of disjointness are equivalent to the two trace conditions

$$\begin{cases} \mathrm{Tr}[A_1(\sigma)t + B_1(\sigma)t^2] = 0 \\ \mathrm{Tr}[A_2(\sigma)t + B_2(\sigma)t^2] = 0, \end{cases}$$

  where A_1, A_2, B_1 and B_2 are functions of σ. This can also be written as

$$\begin{cases} \mathrm{Tr}[(A_1(\sigma) + \sqrt{B_1(\sigma)})t] = 0 \\ \mathrm{Tr}[(A_2(\sigma) + \sqrt{B_2(\sigma)})t] = 0, \end{cases}$$

  which are two linear equations that correspond to two hyperplanes in the vector space $\mathrm{V}(2h+1, 2)$. Since $A_1(\sigma) + \sqrt{B_1(\sigma)} \neq A_2(\sigma) + \sqrt{B_2(\sigma)}$, which is easily checked by adding (3.2) and (3.3), the corresponding hyperplanes intersect in a $(2h-1)$-dimensional subspace. We conclude that there are $2^{2h-1} = \frac{2^{2h+1}}{4}$ solutions to the system of trace conditions above. This means that for every σ there are $\frac{2^{2h+1}}{4}$ solutions for t. However, since these t-values come in pairs we find, for every field automorphism σ, that there are $\frac{2^{2h+1}}{8}$ degree-4 maximal arcs. One of them will give rise to a degree-8 maximal arc of Denniston type and so there are

$$(2h+1)\Big(\frac{2^{2h+1}}{8} - 1\Big)$$

automorphisms θ that satisfy the needed conditions and induce a degree-8 maximal arc of Mathon type. One such automorphism leads to two conics disjoint from C_2^1 and C_3^1 and so we get

$$(2h+1)\Big(\frac{2^{2h+1}}{4} - 2\Big)$$

conics disjoint from C_2^1 and C_3^1.
In exactly the same way we can map C_3^i onto C_1 and also C_1 onto C_1. This gives us

$$3(2h+1)\Big(\frac{2^{2h+1}}{4} - 2\Big)$$

conics that expand $\mathcal{D}^1$ to a degree-8 maximal arc of Mathon type.

- Now assume $i = 1$.

In the cases where C_2^1 is mapped onto C_1 and C_3^1 is mapped onto C_1 we find again

$$(2h+1)\Big(\frac{2^{2h+1}}{4} - 2\Big)$$

conics to expand $\mathcal{D}^1$. If we consider the case where C_1 is fixed however, we have to make sure that σ is not the identity as seen in the remark above. And so in the case $i = 1$ we get

$$2(2h+1)\Big(\frac{2^{2h+1}}{4} - 2\Big) + 2h\Big(\frac{2^{2h+1}}{4} - 2\Big)$$

conics to expand $\mathcal{D}^1$.

As there are $N-1$ choices for $\mathcal{D}^i$, $i \neq 1$ there are in total

$$(N-1)(6h+3)\Big(\frac{2^{2h+1}}{4}-2\Big)+(6h+2)\Big(\frac{2^{2h+1}}{4}-2\Big)$$

such conics.

Suppose we counted one of these conics, say C, twice. Since, due to Lemma 2.3.11, this conic C induces a unique degree-4 maximal arc together with C_1 it would imply that C is the image of two conics contained in one of the N 4-arcs $\mathcal{D}^i$. However, this would give rise to an automorphism of the 4-arc that does not fix the conics, clearly a contradiction.

In other words, we can use each one of these conics to expand $\mathcal{D}^1$ to a degree-8 maximal arc of Mathon type. Moreover, since the four conics disjoint from $\mathcal{D}^1$ in a degree-8 maximal arc of Mathon type all give rise to this same degree-8 arc, we find

$$\frac{1}{4}\Big[(N-1)(6h+3)\Big(\frac{2^{2h+1}}{4}-2\Big)+(6h+2)\Big(\frac{2^{2h+1}}{4}-2\Big)\Big]$$

degree-8 maximal arcs of Mathon type that contain $\mathcal{D}^1$. Of course there were N choices for $\mathcal{D}^1$ and so there are

$$\frac{N}{4}\Big[(N-1)(6h+3)\Big(\frac{2^{2h+1}}{4}-2\Big)+(6h+2)\Big(\frac{2^{2h+1}}{4}-2\Big)\Big]$$

degree-8 maximal arcs of Mathon type that contain the degree-4 maximal arc $\mathcal{D}^i$. As a result of Lemma 3.3.3 we now find

$$\frac{N}{28}\Big[(N-1)(6h+3)\Big(\frac{2^{2h+1}}{4}-2\Big)+(6h+2)\Big(\frac{2^{2h+1}}{4}-2\Big)\Big]$$

non-isomorphic degree-8 maximal arcs of Mathon type in $\mathrm{PG}(2,2^{2h+1})$, $2h+1 \neq 7$. Remark that we divided by 7 as Lemma 3.2.4 and Lemma 3.3.3 state. This is due to the fact that we now fix an entire degree-4 maximal arc in the pencil, not only the conic C_1. □

Remark 3.3.6. If $2h+1=7$ the situation changes. Let ϕ be a non-trivial automorphism of $\mathcal{K}$.

If ϕ stabilizes one of the degree-4 maximal subarcs of $\mathcal{K}$ we have seen in the proof of Corollary 3.3.2 that ϕ must be the unique involution ι described in Lemma 3.3.1.

If ϕ does not stabilize any of the Denniston subarcs it turns out (see Chapter 4, Remark 4.1.1) that $|\mathrm{Aut}(\mathcal{K})| = 14$ and we can no longer benefit from the fact that $\mathrm{Aut}(\mathcal{K}) \cong C_2$, which implies that the previous counting arguments no longer hold.

This particular case is studied in detail in the next chapter. There, we will describe these specific maximal arcs, count them and finally show that they can be extended to infinite families.

We conclude this section with a lemma that assures us that the number of isomorphism classes of proper Mathon 8-arcs in $\mathrm{PG}(2, 2^{2h+1})$, $2h + 1 \neq 7$ and prime, really is an integer.

Lemma 3.3.7. *In* $\mathrm{GF}(2^{2h+1})$, $2h + 1 \neq 7$ *and prime,*

$$14 \mid N(2^{2h-2} - 1)((6h + 3)N - 1),$$

with $N = (2^{2h} - 1)/3(2h + 1)$, *holds.*

Proof. We know that $2h+1 \neq 7$ and prime. This implies that $7 \nmid 3(2h+1)$. It follows that we need to show that

$$14 \mid (2^{2h} - 1)(2^{2h-2} - 1)((6h + 3)(2^{2h} - 1) - 3(2h + 1)).$$

Since $(6h+3)(2^{2h} - 1) - 3(2h+1) = 3(2h+1)(2^{2h} - 2) = 6(2h+1)(2^{2h-1} - 1)$ the above is equivalent to

$$14 \mid 6(2h + 1)(2^{2h} - 1)(2^{2h-1} - 1)(2^{2h-2} - 1).$$

It remains to be shown that $7 \mid (2^{2h} - 1)(2^{2h-1} - 1)(2^{2h-2} - 1)$. Clearly 7 will divide $2^k - 1$, $k \in \mathbb{N}^*$, if and only if $k \equiv 0 \mod 3$. Since surely one of the exponents $2h$, $2h - 1$ or $2h - 2$ satisfies this condition we conclude that $14 \mid N(2^{2h-2} - 1)((6h + 3)N - 1)$. □

3.4 Maximal arcs in PG(2, 32)

In this section we will consider the case $\mathrm{PG}(2, 32)$. Due to a randomized computer search Mathon ([38]) found three non-isomorphic degree-8 maximal arcs in PG(2, 32). It now follows from Theorem 3.3.5 that there are exactly three such arcs. In this section we will describe these arcs and conclude with the actual equations of their conics as they were written down by Mathon in [38]. In [32] Hamilton and Penttila showed that there is, up to isomorphism, a unique degree-4 maximal arc of Denniston type in PG(2, 32). Let w be a primitive element in GF(32) satisfying $w^{18} + w = 1$. The three conics C_1, C_w and C_{w+1}, given by

$$\{x^2 + xy + y^2 + \lambda z^2 | \lambda \in \langle 1, w\rangle \setminus \{0\}\},$$

determine a degree-4 maximal arc of Denniston type $\mathcal{D}_1$ on the nucleus $(0, 0, 1)$. Due to the above, the number of isomorphism classes of degree-8 maximal arcs of Mathon type in PG(2, 32) is equal to the number of isomorphism classes of degree-8 maximal arcs of Mathon type that contain $\mathcal{D}_1$ while the intersection point $(0, 1, 0)$ of the Denniston lines is fixed. This means we need to count the number of conics with nucleus $(0, 0, 1)$ that are disjoint from $\mathcal{D}_1$ while fixing the point $(0, 1, 0)$. It is clear (Lemma 2.3.11) that every such conic, together with the conic C_1, determines a degree-4 maximal arc of Denniston type $\mathcal{D}_2$, which of course is isomorphic to $\mathcal{D}_1$. We now consider automorphisms θ of PG(2, 32) such that $(\mathcal{D}_1)^\theta$ contains C_1, $(0, 0, 1)^\theta = (0, 0, 1)$ and $(0, 1, 0)^\theta = (0, 1, 0)$. We need to take into account three possibilities for θ, more precisely: $C_1^\theta = C_1$, $C_w^\theta = C_1$ and $C_{w+1}^\theta = C_1$. First let us consider the automorphism θ given by

$$\theta : x \rightarrow Mx^\sigma,$$

with

$$M := \begin{pmatrix} w^{-9\sigma} & 0 & 0 \\ t & w^{-9\sigma} & 0 \\ \sqrt{w^{-9\sigma}t + t^2} & 0 & 1 \end{pmatrix},$$

where $\sigma \in \mathrm{Aut}(\mathrm{GF}(32))$ and $t \in \mathrm{GF}(32)$. This automorphism will indeed fix the points $(0, 0, 1)$ and $(0, 1, 0)$ while $C_{w+1}^\theta = C_1$. The trace conditions that satisfy the conditions of disjointness: $C_1^\theta \cap C_w = \emptyset$ and $C_1^\theta \cap C_{w+1} = \emptyset$, are

$$\begin{cases} \mathrm{Tr}\Big[\dfrac{w^{9\sigma}t(1+w)(1+w^{9\sigma}t)}{(1+w^{1+18\sigma})}\Big] = 0 \\ \mathrm{Tr}\Big[\dfrac{w^{9\sigma}t(1+w^{18})(1+w^{9\sigma}t)}{(1+w^{18+18\sigma})}\Big] = 0. \end{cases}$$

For all $\sigma \in \mathrm{Aut}(\mathrm{GF}(32))$ we find eight elements $t \in \mathrm{GF}(32)$ satisfying these conditions. More precisely, for every σ, we find the following t-values.

$$
\begin{array}{ll}
\sigma = 1: & t = 0, w^8, w^{22}, w^{21}, w^{11}, w^{30}, w^6, w^{15} \\
\sigma = 2: & t = 0, w^{13}, w^6, w^{28}, w^{29}, w^{22}, w^{18}, w^{15} \\
\sigma = 4: & t = 0, w^2, w, w^{19}, w^{10}, w^{22}, w^{17}, w^{26} \\
\sigma = 8: & t = 0, w^{21}, w^2, w^{13}, w^{18}, w^{16}, w^{11}, w^{15} \\
\sigma = 16: & t = 0, w^7, w^9, w^{12}, w^{29}, w^{14}, w^{17}, w^{11}
\end{array}
$$

These eight elements t are partitioned into pairs. For example if $\sigma = 1$ we find the pairs

$$(0, w^{22}), (w^8, w^{21}), (w^{11}, w^{30}), (w^6, w^{15}). \tag{3.4}$$

The case $C_w^\theta = C_1$ can be handled in an analogous way. The trace conditions now are

$$
\begin{cases}
\mathrm{Tr}\Big[\dfrac{w^{-15\sigma}t(1+w)(1+w^{-15\sigma}t)}{(1+w^{\sigma+1})}\Big] = 0 \\
\mathrm{Tr}\Big[\dfrac{w^{-15\sigma}t(1+w^{18})(1+w^{-15\sigma}t)}{(1+w^{\sigma+18})}\Big] = 0.
\end{cases}
$$

The t-values for every σ, which are again partitioned in pairs, are listed below.

$$
\begin{array}{ll}
\sigma = 1: & t = 0, w^{21}, w^{19}, w^{24}, 1, w^{25}, w^{11}, w^{15} \\
\sigma = 2: & t = 0, w^{20}, w^{30}, w^{24}, w^{10}, w^{14}, w^{18}, w^{23} \\
\sigma = 4: & t = 0, w^3, w^2, w^{20}, 1, w^{29}, w^5, w^8 \\
\sigma = 8: & t = 0, w^4, w^{12}, w^{24}, w^5, w^{22}, w^{27}, w^{16} \\
\sigma = 16: & t = 0, w^4, w^6, w^9, w^5, w^{22}, w^{23}, w^{15}
\end{array}
$$

Finally we have a look at the case where C_1 is fixed. In accordance to Remark 3.3.4 we must demand that $\sigma \neq 1$ otherwise the x-axis is fixed pointwise and it would be impossible for the conics C_w and C_{w+1} to obtain disjoint images. In the same way as seen above the conditions of disjointness result in the following system of trace conditions:

$$
\begin{cases}
\mathrm{Tr}\Big[\dfrac{t(1+w)(1+t)}{(w+w^\sigma)}\Big] = 0 \\
\mathrm{Tr}\Big[\dfrac{t(1+w^{18})(1+t)}{(w^{18}+w^\sigma)}\Big] = 0.
\end{cases}
$$

The t-values for every σ are:

$$\begin{array}{ll}\sigma = 2: & t = 0, w^7, w^6, w^{24}, 1, w^{22}, w^{27}, w^{15}\\ \sigma = 4: & t = 0, w^4, w^{12}, w^{24}, 1, w^{10}, w^{23}, w^{15}\\ \sigma = 8: & t = 0, w^2, w, w^{19}, 1, w^5, w^{18}, w^{11}\\ \sigma = 16: & t = 0, w^4, w^{12}, w^{24}, 1, w^{10}, w^{23}, w^{15}.\end{array}$$

Each one of these pairs (see (3.4)) give rise to a unique degree-4 maximal arc of Denniston type. This means that, for each one of them, we get two conics disjoint from $\mathcal{D}_1$. One of these degree-4 maximal arcs is contained in the pencil of $\mathcal{D}_1$ and so it leads to a degree-8 maximal arc of Denniston type. The other three induce proper Mathon arcs of degree 8.
Now we are able to count the conics that give rise to a maximal arc of Denniston type ("D-conics") as well as the conics that give rise to a maximal arc of Mathon type ("M-conics"). Remark that only four values for σ can be included in the case where C_1 is fixed since the identity leads to a contradiction.

	"D-conics"	"M-conics"
$C_{w+1}^{\theta} = C_1$	$5 \times 1 \times 2$	$5 \times 3 \times 2$
$C_w^{\theta} = C_1$	$5 \times 1 \times 2$	$5 \times 3 \times 2$
$C_1^{\theta} = C_1$	$4 \times 1 \times 2$	$4 \times 3 \times 2$
	28	84

It follows that we find 84 "M-conics". This means there are 21 degree-8 maximal arcs of Mathon type. As each of these proper Mathon arcs of degree 8 have an automorphism group of size 2 and contain exactly seven degree-4 maximal arcs of Denniston type, which are isomorphic to $\mathcal{D}_1$, we obtain three non-isomorphic proper Mathon arcs of degree 8.

On a more technical note we can calculate the equation of the conic C_1^{θ} using the matrix M and the matrix

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

associated to the equation $x^2 + xy + y^2 + z^2 = 0$ of C_1. Analogous results hold for the conics C_w^{θ} and C_{w+1}^{θ}. This will enable us to construct the entire degree-8 maximal arc using Theorem 3.1.2. We need to calculate the form

$M^{T^{-1}} A^\sigma M^{-1}$. Since $A = A^\sigma$ and

$$M^{-1} := \begin{pmatrix} w^{9\sigma} & 0 & 0 \\ tw^{18\sigma} & w^{9\sigma} & 0 \\ \sqrt{w^{-9\sigma}t + t^2} w^{9\sigma} & 0 & 1 \end{pmatrix}$$

we find that $(M^{-1})^T A^\sigma M^{-1}$ is equal to the matrix

$$\begin{pmatrix} w^{18\sigma} + tw^{27\sigma} + t^2 w^{36\sigma} + (w^{-9\sigma}t + t^2) w^{18\sigma} & w^{18\sigma} + tw^{27\sigma} & \sqrt{w^{-9\sigma}t + t^2} w^{9\sigma} \\ tw^{27\sigma} & w^{18\sigma} & 0 \\ \sqrt{w^{-9\sigma}t + t^2} w^{9\sigma} & 0 & 1 \end{pmatrix}.$$

This means that the equation of the conic C_1^θ is given by

$$(w^{18\sigma} + tw^{27\sigma} + t^2 w^{36\sigma} + (w^{-9\sigma}t + t^2)w^{18\sigma})x^2 + w^{18\sigma}xy + w^{18\sigma}y^2 + z^2 = 0,$$

with $t \in \mathrm{GF}(32)$ and $\sigma \in \mathrm{Aut}(\mathrm{GF}(32))$, which is equivalent to the equation

$$(1 + (1 + w^{18\sigma})w^{-9\sigma}t + (1 + w^{18\sigma})t^2)x^2 + xy + y^2 + w^{13\sigma}z^2 = 0. \qquad (3.5)$$

Let us now consider the case $\sigma = 4$ and $t = w^2$. We obtain

$$w^{12}x^2 + xy + y^2 + w^{21}z^2 = 0$$

as the equation of C_1^θ. If we multiply this equation by w^{19}, set $y = w^{12}y'$ and $z = w^8 z'$, we find

$$x^2 + xy' + w^{12}y'^2 + w^{25}z'^2 = 0,$$

which is equivalent to

$$x^2 + xy + w^{12}y^2 + w^{25}z^2 = 0.$$

Using Theorem 3.1.2 and Mathon's composition we can easily compose the remaining three conics of the degree-8 arc. Their equations are

$$\begin{array}{rcl} C_1 \oplus C_1^\theta & : & x^2 + xy + w^6y^2 + w^{21}z^2 = 0 \\ C_w \oplus C_1^\theta & : & x^2 + xy + w^{18}y^2 + w^{16}z^2 = 0 \\ C_{w+1} \oplus C_1^\theta & : & x^2 + xy + w^{20}y^2 + w^9z^2 = 0. \end{array}$$

This way we managed to construct the degree-8 maximal arc consisting of the conics $\{C_1, C_w, C_{w+1}, C_1^\theta, C_1 \oplus C_1^\theta, C_w \oplus C_1^\theta, C_{w+1} \oplus C_1^\theta\}$. In [38] Mathon found the three degree-8 maximal arcs (not of Denniston type) in PG(2, 32) formed by

$$\{x^2 + xy + (w^k + w^l\lambda + w^m\lambda^3)y^2 + \lambda z^2 | \lambda \in \langle 1, w, w^9\rangle \setminus \{0\}\},$$

with exponents $(k, l, m) = (12, 15, 4), (5, 25, 14)$, and $(6, 19, 8)$, respectively. The 8-arc constructed above is exactly the one of Mathon corresponding to the exponents $(k, l, m) = (6, 19, 8)$. The other two proper Mathon 8-arcs in GF(32) are found, for instance, in the following way.

Consider the case $\sigma = 1$ and $t = w^8$ and substitute these values in equation (3.5). We find

$$w^{17}x^2 + xy + y^2 + w^{13}z^2 = 0,$$

which is the equation of another conic $C_1'^{\theta}$ that induces a proper Mathon 8-arc. If we multiply this equation by w^{14} and set $y = w^{17}y'$, we get

$$x^2 + xy' + w^{17}y'^2 + w^{27}z'^2 = 0,$$

or equivalently,

$$x^2 + xy + w^{17}y^2 + w^{27}z^2 = 0.$$

Using the same arguments as before we can write down the equations of the remaining conics in the degree-8 arc. These are

$$\begin{aligned} C_1 \oplus C_1'^{\theta} &: \ x^2 + xy + w^8y^2 + w^6z^2 = 0 \\ C_w \oplus C_1'^{\theta} &: \ x^2 + xy + w^{26}y^2 + w^{29}z^2 = 0 \\ C_{w+1} \oplus C_1'^{\theta} &: \ x^2 + xy + w^{12}y^2 + w^3z^2 = 0. \end{aligned}$$

However, the obtained conics do not have the exact same equations as the conics in one of the two proper Mathon 8-arcs corresponding to the exponents $(k, l, m) = (12, 15, 4)$ or $(5, 25, 14)$. Therefore, in order to attain the wanted equations, consider the equation of the conic $C_w \oplus C_1'^{\theta}$ and set $z = w^9z'$. This gives us the equivalent equation

$$x^2 + xy + w^{26}y^2 + w^{16}z^2 = 0,$$

which is exactly the equation of the conic corresponding to the value $\lambda = w^{16}$ in the proper Mathon 8-arc with exponents $(12, 15, 4)$. Using the composition we find

$$\begin{aligned} C_1'^{\theta} &: \ x^2 + xy + w^{11}y^2 + w^{25}z^2 = 0 \\ C_1 \oplus C_1'^{\theta} &: \ x^2 + xy + w^{12}y^2 + w^{21}z^2 = 0 \\ C_{w+1} \oplus C_1'^{\theta} &: \ x^2 + xy + w^{10}y^2 + w^9z^2 = 0. \end{aligned}$$

The conics $\{C_1, C_w, C_{w+1}, C_1'^{\theta}, C_1 \oplus C_1'^{\theta}, C_w \oplus C_1'^{\theta}, C_{w+1} \oplus C_1'^{\theta}\}$ are the ones contained in the proper Mathon 8-arc with exponents $(12, 15, 4)$.

To obtain the last one of the three proper Mathon 8-arcs corresponding to $(5, 25, 14)$ take $\sigma = 2$ and $t = w^6$ and substitute these values in equation (3.5). We get

$$w^2x^2 + xy + y^2 + w^{26}z^2 = 0,$$

the equation of a conic $C_1''^{\theta}$, inducing a proper Mathon 8-arc. Multiplying this equation by w^{29} and setting $y = w^2y'$, gives us

$$x^2 + xy' + w^2y'^2 + w^{24}z'^2 = 0,$$

or equivalently,

$$x^2 + xy + w^2y^2 + w^{24}z^2 = 0.$$

Thus, the equation of the remaining conics of the 8-arc are

$$\begin{aligned} C_1 \oplus C_1''^{\theta} &: \; x^2 + xy + w^{13}y^2 + w^{15}z^2 = 0 \\ C_w \oplus C_1''^{\theta} &: \; x^2 + xy + w^9y^2 + w^{13}z^2 = 0 \\ C_{w+1} \oplus C_1''^{\theta} &: \; x^2 + xy + w^{24}y^2 + w^{14}z^2 = 0. \end{aligned}$$

Again, in order to attain the wanted equations, consider the equation of the conic $C_1 \oplus C_1''^{\theta}$ and set $z = w^{28}z'$. We get the equivalent equation

$$x^2 + xy + w^{13}y^2 + w^9z^2 = 0,$$

which is exactly the equation of the conic corresponding to the value $\lambda = w^9$ in the proper Mathon 8-arc with exponents $(5, 25, 14)$. Through composition we obtain

$$\begin{aligned} C_1''^{\theta} &: \; x^2 + xy + w^{22}y^2 + w^{16}z^2 = 0 \\ C_w \oplus C_1''^{\theta} &: \; x^2 + xy + w^3y^2 + w^{25}z^2 = 0 \\ C_{w+1} \oplus C_1''^{\theta} &: \; x^2 + xy + w^5y^2 + w^{21}z^2 = 0. \end{aligned}$$

The conics $\{C_1, C_w, C_{w+1}, C_1''^{\theta}, C_1 \oplus C_1''^{\theta}, C_w \oplus C_1''^{\theta}, C_{w+1} \oplus C_1''^{\theta}\}$ are exactly the ones contained in the proper Mathon 8-arc corresponding to the exponents $(5, 25, 14)$.

3.5 Final remarks

- It is clear that a maximal arc of Mathon type in $\mathrm{PG}(2,q)$ always contains a degree-4 arc of Denniston type. From this it follows that the dual of a maximal arc of Mathon type is actually the intersection of degree-$q/4$ Denniston arcs. Furthermore, such an intersection cannot be constructed from a closed set of conics (see Corollary 2.3.10).

- In Section 2.3 (Theorem 2.3.5) we have used certain λ-polynomials $\{p,r\}$ to map additive subgroups A of $\mathrm{GF}(2^m)$ to subsets of conics on a common nucleus which form maximal arcs in $\mathrm{PG}(2,2^m)$. At the end of his paper ([38]) Mathon asks the following question. What is the largest d of a proper Mathon arc of degree d generated by a $\{p,r\}$-map in $\mathrm{PG}(2,2^m)$? In a first paper by Fiedler, Leung and Xiang ([22]) the authors prove that there are always $\{p,1\}$-maps generating proper Mathon maximal arcs of degree $2^{(\lfloor \frac{m}{2} \rfloor+1)}$ in $\mathrm{PG}(2,2^m)$. In [23], a second paper by the same authors, Mathon's question is nearly completely answered. Specifically, the authors prove that, when $m \geq 5$ and $m \neq 9$, the largest d of a proper Mathon arc of degree 2^d in $\mathrm{PG}(2,2^m)$ generated by a $\{p,1\}$-map is $(\lfloor \frac{m}{2} \rfloor+1)$. For $\{p,r\}$-maps, they proved that, if $m \geq 7$ and $m \neq 9$, then the largest d of a proper Mathon maximal arc of degree 2^d in $\mathrm{PG}(2,2^m)$ generated by a $\{p,r\}$-map is either $\lfloor \frac{m}{2} \rfloor + 1$ or $\lfloor \frac{m}{2} \rfloor + 2$.

Chapter 4

Singer 8-arcs of Mathon type

In Chapter 3 we counted the number of non-isomorphic Mathon maximal arcs of degree 8 in $\mathrm{PG}(2, 2^h)$, $h \neq 7$ and prime. In this chapter we will show that in $\mathrm{PG}(2, 2^7)$ a special class of Mathon maximal arcs of degree 8 arises which admits a sharply transitive, or Singer, group on the seven conics of these arcs. We will give a detailed description of these arcs, and then count the total number of non-isomorphic Mathon maximal arcs of degree 8. Finally, we show that these special arcs found in $\mathrm{PG}(2, 2^7)$ extend to two infinite families of Mathon arcs of degree 8 in $\mathrm{PG}(2, 2^k)$, k odd and divisible by 7, while maintaining the nice property of admitting a Singer group.

The following results can be found in [16].

4.1 Introduction

First, recall Theorem 3.1.2, the synthetic version of Mathon's construction, which states that, given a degree-d maximal arc M of Mathon type and a conic C_d disjoint from M with the same nucleus n, there is a unique degree-$2d$ maximal arc of Mathon type containing $M \cup C_d$.

In the previous chapter this result was used to count the number of non-isomorphic maximal arcs of Mathon type of degree 8 in $\mathrm{PG}(2, 2^p)$, with p prime and $p \neq 2, 3, 7$. The fact that our count did not work for $p = 7$ suggested that something special might be going on in $\mathrm{PG}(2, 2^7)$. In what follows we will see that this is indeed the case, as we will show that this specific plane admits two maximal degree-8 arcs of Mathon type with a particularly interesting automorphism group that do not exist in any of the other planes $\mathrm{PG}(2, 2^p)$, p prime.

Let $\mathcal{K}$ be a proper Mathon 8-arc. As a result of Lemma 3.3.1 we know that there always exists an involution stabilizing $\mathcal{K}$ and all of its conics. In the specific case of $\mathrm{PG}(2, 2^p)$, p prime and $p \neq 2, 3, 7$, we proved in Corollary 3.3.2 that $\mathrm{Aut}(\mathcal{K}) \cong C_2$.

The properties above, Theorem 3.1.2 and a counting argument that will be exploited also later in this chapter made it possible to count the number of non-isomorphic degree-8 arcs of Mathon type in $\mathrm{PG}(2, 2^{2h+1})$, $2h + 1 \neq 7$ and prime. More precisely, we found that the number of non-isomorphic degree-8 maximal arcs of Mathon type in $\mathrm{PG}(2, 2^{2h+1})$, $2h+1 \neq 7$ and prime, is exactly

$$\frac{N}{14}(2^{2h-2} - 1)((6h + 3)N - 1),$$

where $N = (2^{2h} - 1)/3(2h + 1)$ is the number of non-isomorphic Denniston arcs of degree 4 in $\mathrm{PG}(2, 2^{2h+1})$.

Remark 4.1.1. As we mentioned above, this result does not hold for $2h+1 = 7$ (note that in that case the obtained number is not even an integer), because Corollary 3.3.2 fails for $2h + 1 = 7$. Next, we discuss in more detail why this happens.

Now, let $\mathcal{K}$ be a degree-8 maximal arc of Mathon type in $\mathrm{PG}(2, 2^7)$, and let ϕ be a non-trivial automorphism of $\mathcal{K}$. If ϕ stabilizes one of the degree-4 maximal subarcs of $\mathcal{K}$ we found out in the previous chapter (proof of Corollary 3.3.2) that ϕ must be the unique involution ι described in Lemma 3.3.1. Now suppose that ϕ does not stabilize any of the Denniston subarcs. Since 7 is the only possible orbit length of ϕ on these subarcs it follows that the order of $\langle\phi\rangle$ has to be a multiple of 7. Let the order of $\langle\phi\rangle$ be $k7$, with $k \in \mathbb{N}^\star$. In that case $|\langle\phi\rangle_{\mathcal{D}}| = k$, where $\mathcal{D}$ is any of the seven Denniston subarcs of degree 4. Furthermore, since $|\mathrm{Aut}(\mathcal{K})_{\mathcal{D}}| = 2$ (see Chapter 3) we find that $k = 2$. This means that $|\mathrm{Aut}(\mathcal{K})| = 14$. Hence, in $\mathrm{PG}(2, 2^7)$ a proper Mathon arc of degree 8 could have a full automorphism group of order 2 or of order 14. The latter type of arc would be of specific interest, especially because of the subgroup of order 7 cyclically permuting the conics of the arc.

This remark suggests the existence of two classes of degree-8 maximal arcs of Mathon type in $\mathrm{PG}(2, 2^7)$.

- The degree-8 maximal arcs of Mathon type that have a full automorphism group of order 2 will be referred to as *normal* 8-*arcs* (see Section 4.4).
- Those admitting a full automorphism group of order 14 will be called *Singer* 8-*arcs*.

Also, with a little abuse of definition, Denniston arcs of degree 8 that admit a group acting sharply transitively on their seven conics will be called *Singer* 8-*arcs* as well in the next two sections.

We will now move to a detailed analysis of Mathon maximal arcs of degree 8 in $\mathrm{PG}(2, 2^7)$, and prove the existence of two classes of Singer 8-arcs of Mathon type.

4.2 Necessary conditions for the existence of a Singer arc

Let $\mathcal{D}_1$ be a given degree-4 maximal arc of Denniston type in the standard pencil consisting of the conics C_1, C_w, C_{w+1} with nucleus $F_0(0, 0, 1)$. (Note that every degree-4 maximal arc of Denniston type is isomorphic to such arc.) Due to Lemma 2.3.11 each conic C disjoint from $\mathcal{D}_1$, and with nucleus F_0, will give rise to a degree-8 maximal arc of Mathon type (which might be of Denniston type).
Let the additive subgroup $\{0, 1, w, w + 1\}, w \in \mathrm{GF}(2^7) \setminus \{0, 1\}$, be the one associated to the maximal arc $\mathcal{D}_1$. In other words we assume that the conics C_1, C_w, C_{w+1} contained in $\mathcal{D}_1$ are given by the equation

$$C_i : x^2 + xy + y^2 + iz^2 = 0,$$

where $i = 1, w, w + 1$.

If we now assume that $\mathcal{K} := \langle \mathcal{D}_1, C \rangle$ is a Singer 8-arc, then necessarily all degree-4 arcs of Denniston type contained in it have to be isomorphic, as these seven Denniston arcs will be cyclically permuted by the Singer group (= the cyclic group of order 7 permuting the conics, and hence the arcs). This explains why we will consider images of $\mathcal{D}_1$ that have exactly one conic in common with $\mathcal{D}_1$.

Consider the automorphism $\theta_{t,\sigma}$ of $\mathrm{PG}(2, 2^7)$ given by

$$\theta_{t,\sigma} : x \mapsto Ax^\sigma$$

with

$$A := \begin{pmatrix} \sqrt{w}^{-\sigma} & 0 & 0 \\ t & \sqrt{w}^{-\sigma} & 0 \\ \sqrt{\sqrt{w}^{-\sigma}t + t^2} & 0 & 1 \end{pmatrix}, \tag{4.1}$$

$\sigma \in \mathrm{Aut}(\mathrm{GF}(2^7))$ and $t \in \mathrm{GF}(2^7)$. This automorphism will map C_w onto C_1 while $(0,0,1)^\theta = (0,0,1)$ and $(0,1,0)^\theta = (0,1,0)$. This latter restriction on the automorphism $\theta_{t,\sigma}$ can be made without loss of generality since the seven Denniston lines of a Singer 8-arc of Mathon type have to be concurrent; hence the above restriction simply chooses the line $x = 0$ to be the axis of the unique elation stabilizing our 8-arc under construction. Notice that we could equally well map C_{w+1} onto C_1 or consider any other combination of two of the conics C_1, C_w, C_{w+1} since our purpose is to find a Denniston arc of degree 4 intersecting $\mathcal{D}_1$ in exactly one conic and being isomorphic to $\mathcal{D}_1$. In the previous chapter we obtained two trace conditions that are equivalent to this property. It is clear that these trace conditions still have to hold. They can be written as:

$$\mathrm{Tr}\left[\frac{(1+w)t(\sqrt{w}^{-\sigma}+t)}{(w^{-\sigma}+w)}\right] = 0 \tag{4.2}$$

and

$$\mathrm{Tr}\left[\frac{wt(\sqrt{w}^{-\sigma}+t)}{(w^{-\sigma}+w+1)}\right] = 0. \tag{4.3}$$

In view of $|\mathrm{Aut}(\mathcal{K})_{\mathcal{D}}| = 2$ for any degree-4 Denniston arc contained in $\mathcal{K}$ (see, for instance, Remark 4.1.1), we see that the only automorphisms mapping $\mathcal{D}_1$ onto $\mathcal{D}_2 := \mathcal{D}_1^{\theta_{t,\sigma}}$, while fixing $(0,1,0)$ and stabilizing $z = 0$, are $\theta_{t,\sigma}$ and $\iota\theta_{t,\sigma}$ (where ι is the involution described in Lemma 3.3.1). Hence, if we assume $\mathcal{K}$ to be a Singer 8-arc of Mathon type, these two automorphisms of $\mathrm{PG}(2, 2^7)$ should belong to the automorphism group of $\mathcal{K}$. It is now natural to look at the action of powers of $\theta_{t,\sigma}$.

4.2.1 The action on the line $x = 0$

We will first concentrate on the action of this automorphism on the line $x = 0$.

If we want the Singer group to act on the seven conics of the maximal 8-arc, these conics will be cyclically permuted. However, since it is still a degree-8 arc of Mathon type their tangent points $(0, y_i, 1)$, $i = 1, \ldots, 7$, at the line $x = 0$ should not only be distinct, but furthermore, these points should, together with the nucleus, form an additive group $\{0, y_1, \ldots, y_7\}$ of order 8.

We also remark that, in this case, the field automorphism σ cannot be the identity. This follows from the proof of Corollary 3.3.2 in Chapter 3 where we proved that the non-trivial automorphism ϕ mentioned above cannot belong to $\mathrm{PGL}(3, 2^p)$.

First of all we will calculate these intersection points. The automorphism $\theta_{t,\sigma}$ acts on the points $(0, y, 1), y \in \mathrm{GF}(2^7)$, contained on the line $x = 0$ as follows:

$$\begin{pmatrix} \sqrt{w}^{-\sigma} & 0 & 0 \\ \frac{t}{\sqrt{\sqrt{w}^{-\sigma} t + t^2}} & \sqrt{w}^{-\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix}^{\sigma} = \begin{pmatrix} 0 \\ \sqrt{w}^{-\sigma} y^{\sigma} \\ 1 \end{pmatrix},$$

with $\sigma \in \mathrm{Aut}(\mathrm{GF}(2^7))$ and $t \in \mathrm{GF}(2^7)$. Notice that the point $(0, \sqrt{w}, 1)$ is indeed mapped onto the point $(0, 1, 1)$. In order to find all the intersection points, i.e., the images of the point $(0, 1, 1)$ under $\theta_{t,\sigma}$, $\theta_{t,\sigma}^2$, $\theta_{t,\sigma}^3$, $\ldots$, $\theta_{t,\sigma}^7$, we only need the element on position $(2, 2)$ of the matrices A, $A \cdot A^{\sigma}$, $A \cdot A^{\sigma} \cdot A^{\sigma^2}, \ldots$, $A \cdot A^{\sigma} \cdot A^{\sigma^2} \cdot A^{\sigma^3} \cdot A^{\sigma^4} \cdot A^{\sigma^5} \cdot A^{\sigma^6}$ (the dots in between the matrices indicate the usual matrix multiplication). These are, respectively,

$$\sqrt{w}^{-\sigma}, \sqrt{w}^{-(\sigma^2+\sigma)}, \sqrt{w}^{-(\sigma^3+\sigma^2+\sigma)}, \ldots, \sqrt{w}^{-(\sigma^6+\cdots+\sigma^2+\sigma)}.$$

This means we found that the seven intersection points are

$$(0, 1, 1), (0, \sqrt{w}^{-\sigma}, 1), (0, \sqrt{w}^{-(\sigma^2+\sigma)}, 1), \ldots, (0, \sqrt{w}^{-(\sigma^6+\cdots+\sigma^2+\sigma)}, 1).$$

Eventually we want to show that the set of elements

$$\{0, 1, \sqrt{w}^{-\sigma}, \sqrt{w}^{-(\sigma^2+\sigma)}, \sqrt{w}^{-(\sigma^3+\sigma^2+\sigma)}, \ldots, \sqrt{w}^{-(\sigma^6+\cdots+\sigma^2+\sigma)}\}$$

forms an additive group. Therefore we will start by proving the following lemma.

Lemma 4.2.1. *The set*

$$\{0, 1, \sqrt{w}^{-\sigma}, \sqrt{w}^{-(\sigma^2+\sigma)}, \sqrt{w}^{-(\sigma^3+\sigma^2+\sigma)}, \dots, \sqrt{w}^{-(\sigma^6+\cdots+\sigma^2+\sigma)}\}$$

of elements in $\mathrm{GF}(2^7)$*, with* σ *any non-trivial automorphism of* $\mathrm{GF}(2^7)$*, can be written as the set*

$$\{0, 1, x, x^3, x^7, x^{15}, x^{31}, x^{63}\},$$

where x *is one of the elements* $\sqrt{w}^{-(\sigma^i+\cdots+\sigma)}$.

Proof. Let $\sigma = 2^k$, $k = 1, \dots, 6$. Notice that there is exactly one integer $l \neq 0 \pmod 7$ such that $\sigma^l = 2^{kl} = 2$, and that different k yield different l. Set $x := \sqrt{w}^{-(\sigma+\cdots+\sigma^l)}$.

Now consider

$$x^3 = \sqrt{w}^{-3(\sigma+\sigma^2+\cdots+\sigma^l)}.$$

We need to show that

$$3(\sigma + \sigma^2 + \cdots + \sigma^l) = \sigma + \sigma^2 + \cdots + \sigma^{j_3} \tag{4.4}$$

for some j_3.

If $l = 1, 2, 3$ we see that $j_3 = 2, 4$ and 6 respectively, satisfies equation (4.4). If $l = 4$ we find

$$\sigma + \sigma^2 + \sigma^3 + \sigma^4 + \sigma^5 + \sigma^6 + \sigma^7 + \sigma^8$$

on the left hand side of (4.4).

Now, using $\sigma^7 = 1$, and the equality $\sigma^6 + \sigma^5 + \cdots + \sigma + 1 = 0$, we see that, if $l = 4$, $j_3 = 1$ is a solution to (4.4). In an analogous way we can compute the values of j_3 that satisfy equation (4.4) for $l = 5, 6$. In fact, using the same argument, we see that $j_3 \equiv 2l \mod 7$ is the unique solution (mod 7) to (4.4). In exactly the same way, we can also show that for each l there are unique solutions (mod 7) to the equations

$$k(\sigma + \sigma^2 + \cdots + \sigma^l) = \sigma + \sigma^2 + \cdots + \sigma^{j_k},$$

with $k = 7, 15, 31, 63$.

We get the following table.

l	j_3	j_7	j_{15}	j_{31}	j_{63}
1	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

(4.5)

Clearly each row contains every non-trivial power of σ exactly once. This implies that for every l, with the given choice of x, our set is indeed representable as

$$\{0, 1, x, x^3, x^7, x^{15}, x^{31}, x^{63}\}.$$

□

We remark that the polynomials $1 + x + x^7$ and $1 + x^3 + x^7$ are primitive over $\mathrm{GF}(2^7)$.

Lemma 4.2.2. *The set* $\{0, 1, x, x^3, x^7, x^{15}, x^{31}, x^{63}\}$ *of distinct elements of the finite field* $\mathrm{GF}(2^7)$ *is a subgroup of the additive group of* $\mathrm{GF}(2^7)$ *if and only if either* $1 + x = x^7$, *or* $1 + x^3 = x^7$.

Proof. Setting $1 + x = x^7$ we can easily construct the following Cayley table:

$+$	0	1	x	x^3	x^7	x^{15}	x^{31}	x^{63}
0	0	1	x	x^3	x^7	x^{15}	x^{31}	x^{63}
1	1	0	x^7	x^{63}	x	x^{31}	x^{15}	x^3
x	x	x^7	0	x^{15}	1	x^3	x^{63}	x^{31}
x^3	x^3	x^{63}	x^{15}	0	x^{31}	x	x^7	1
x^7	x^7	x	1	x^{31}	0	x^{63}	x^3	x^{15}
x^{15}	x^{15}	x^{31}	x^3	x	x^{63}	0	1	x^7
x^{31}	x^{31}	x^{15}	x^{63}	x^7	x^3	1	0	x
x^{63}	x^{63}	x^3	x^{31}	1	x^{15}	x^7	x	0

It is clear that all necessary conditions are satisfied. A similar Cayley table can be constructed in the case $1 + x^3 = x^7$, which is equivalent to the case $1+x = x^{31}$. We conclude this proof by showing that the remaining three cases $1 + x = x^3$, $1 + x = x^{15}$ and $1 + x = x^{63}$ do not determine a group under the addition in $\mathrm{GF}(2^7)$. Suppose that $1 + x = x^3$. Then $x^3 + x^7 = x^3(1 + x^4) =$

$x^3(1+x)^4 = x^{15}$ and $x^7 + x^{15} = x^7(1+x^8) = x^7(1+x)^8 = x^{31}$, which implies that $x^3 = x^{31}$, a contradiction. If $1+x = x^{15}$ then $x^7 + x^{15} = x^7(1+x)^8 = x^7x^{120} = x^{127} = 1$, implying $x = x^7$, a contradiction. Finally, if $1+x = x^{63}$, we have $x + x^3 = x(1+x)^2 = xx^{126} = 1$ which implies $x^3 = x^{63}$, again a contradiction. □

4.2.2 From Denniston 4-arc to Singer 8-arc

In the following Lemma we consider degree-4 maximal arcs of Denniston type, containing the conic $C_1 : x^2 + xy + y^2 + z^2 = 0$, in the standard pencil. We remind the reader that every degree-4 maximal arc of Denniston type is isomorphic to one in the standard pencil containing C_1.

Lemma 4.2.3. *The number of conics in the standard pencil of* $\mathrm{PG}(2, 2^7)$ *generating, together with* C_1*, a degree-*4 *Denniston arc of a given isomorphism type is exactly* 42*.*

Proof. Consider the standard pencil. Consider the conic $C_1 : x^2+xy+y^2+z^2 = 0$. This is the conic containing the point $(0, 1, 1)$ on the line $x = 0$. Furthermore the point $(0, 0, 1)$ is the nucleus of C_1, and $(0, 1, 0)$ is the intersection point of the lines $x = 0$ and $z = 0$. This means that so far three points on the line $x = 0$ are taken. The other 126 points on that line are contained in the 126 conics left in the standard pencil. Of course, every one of those conics together with C_1 gives rise to a unique degree-4 maximal arc of Denniston type. Since the third conic in such a 4-arc is determined we find that there are 63 degree-4 arcs of Denniston type in the pencil. We also know that there are exactly 3 isomorphism classes of degree-4 arcs of Denniston type in $\mathrm{PG}(2, 2^7)$, each of which has an automorphism group isomorphic to $C_{q+1} \rtimes C_2$ (see Lemma 3.2.1, Remark 3.2.2 and Lemma 3.2.3 of Chapter 3). It follows that there are 21 degree-4 arcs in each class, or equivalently, that there are 42 conics in the standard pencil generating, together with C_1, a degree-4 Denniston arc of a given isomorphism type. □

We already proved that the set of elements

$$\{0, 1, \sqrt{w}^{-\sigma}, \sqrt{w}^{-(\sigma^2+\sigma)}, \sqrt{w}^{-(\sigma^3+\sigma^2+\sigma)}, \dots, \sqrt{w}^{-(\sigma^6+\dots+\sigma^2+\sigma)}\}$$

can be written as $A = \{0, 1, x, x^3, x^7, x^{15}, x^{31}, x^{63}\}$, with x a function of w. Moreover we know that A forms a group under the addition in $\mathrm{GF}(2^7)$ if and

only if either $1+x = x^7$ or $1+x^3 = x^7$. It is clear that the set of solutions of the equation $1 + x = x^7$ and the set of solutions of the equation $1 + x^3 = x^7$ have to be disjoint, otherwise $x = x^3$, a contradiction. This implies that in each of the two cases we have seven possible values for x. For every given non-trivial field automorphism σ, each of these values of x yields a unique value of w. In other words, we have $2 \times 7 \times 6 = 84$ possible values of w, that is, we get 84 conics which, together with the automorphism $\theta_{t,\sigma}$, possibly give rise to a Singer 8-arc. Note that, since $1+x+x^7$ and $1+x^3+x^7$ are not conjugate under any field automorphism the 84 values of w are indeed distinct. Furthermore note that, in view of Lemma 4.2.1 and Lemma 4.2.2, the σ in $\theta_{t,\sigma}$ is uniquely determined once we have chosen a specific value of w out of these 84.
In view of Lemma 4.2.3 we can also conclude that these 84 conics together with C_1 determine exactly two isomorphism classes of degree-4 maximal arcs of Denniston type in the standard pencil. In other words, at most two of the three isomorphism types of degree-4 maximal arcs of Denniston type in $\mathrm{PG}(2, 2^7)$ can possibly be extended to a Singer 8-arc.

4.3 Necessary and sufficient condition

We start by proving a lemma that provides us with a necessary and sufficient condition in order for $\theta_{t,\sigma}$ to generate a Singer 8-arc.

Lemma 4.3.1. *Let $\mathcal{D} = \{C_1, C_2, C_3\}$ be a 4-arc of Denniston type in $\mathrm{PG}(2, 2^7)$. Let θ be an automorphism of $\mathrm{PG}(2, 2^7)$ with the properties that $C_2^\theta = C_1$, $C_4 := C_1^\theta$ is disjoint from C_1, C_2 and C_3, and C_4 has the same nucleus as C_1, C_2 and C_3. If $\mathcal{D}^{\theta^2}$ intersects both $\mathcal{D}$ and $\mathcal{D}^\theta$ in a conic, then $\mathcal{D}$ together with θ generate a Singer 8-arc, and consequently θ has order divisible by 7.*

Proof. Set $\mathcal{D} = \{C_1, C_2, C_3\}$, $\mathcal{D}^\theta = \{C_1, C_4, C_5\}$, with $C_2^\theta = C_1$, $C_1^\theta = C_4$ and $C_3^\theta = C_5$. So clearly $C_1 \oplus C_2 = C_3$ and $C_1 \oplus C_4 = C_5$. Let $\langle \mathcal{D}, C_4 \rangle$ denote the 8-arc generated by $\mathcal{D}$ and C_4.

As $C_1 \in D^\theta$, we see that $C_4 = C_1^\theta \in \mathcal{D}^{\theta^2}$. There are two possible cases (recall that $\mathcal{D}^{\theta^2}$ intersects $\mathcal{D}$ in a conic which has to be distinct from C_1):

(i) $\mathcal{D}^{\theta^2} = \{C_4, C_2, C_2 \oplus C_4 =: C_6\}$,

(ii) $\mathcal{D}^{\theta^2} = \{C_4, C_3, C_3 \oplus C_4 =: C_7\}$.

We discuss both cases separately. Note that all $\oplus$-additions and related computations are well defined by Lemma 2.3.11 and Theorem 3.1.2.

(i) **The case $\mathcal{D}^{\theta^2} = \{C_4, C_2, C_2 \oplus C_4 =: C_6\}$.**

From $C_4 \in \mathcal{D}^\theta$ it follows that $C_4^\theta \in \mathcal{D}^{\theta^2}$. Clearly $C_4^\theta \neq C_4$. We will show that C_4^θ cannot be C_2 either. This would clearly yield an automorphism of order a power of 3 stabilizing the 8-arc $\langle \mathcal{D}, C_4 \rangle$. Hence θ necessarily would stabilize one of the conics in this 8-arc. But then there has to be a line that is not fixed pointwise containing at least 3 fixpoints, and so $\theta \in \mathrm{P\Gamma L}(3, 2^7) \setminus \mathrm{PGL}(3, 2^7)$. Consequently 7 divides the order of θ, a contradiction. Hence $C_4^\theta = C_6$. As $C_2^\theta = C_1$ we obtain $\mathcal{D}^{\theta^3} = \{C_1, C_6, C_1 \oplus C_6 =: C_7\}$. We need to show that $\mathcal{D}^{\theta^3} \in \langle \mathcal{D}, C_4 \rangle$. But this is true since $C_7 = C_1 \oplus C_6 = C_1 \oplus C_2 \oplus C_4 = C_3 \oplus C_4 \in \langle \mathcal{D}, C_4 \rangle$.

Next we look at $\mathcal{D}^{\theta^4}$. Note that from the previous step it follows that $C_6^\theta = C_7$, and hence that $\mathcal{D}^{\theta^4} = \{C_4, C_7, C_4 \oplus C_7\}$. But $C_4 \oplus C_7 = C_4 \oplus C_3 \oplus C_4 = C_3$. And so $\mathcal{D}^{\theta^4} = \{C_4, C_7, C_3\} \in \langle \mathcal{D}, C_4 \rangle$.

Consequently, $\mathcal{D}^{\theta^5} = \{C_6, C_3, C_5\} \in \langle \mathcal{D}, C_4 \rangle$, $\mathcal{D}^{\theta^6} = \{C_7, C_5, C_2\} \in \langle \mathcal{D}, C_4 \rangle$, and $\mathcal{D}^{\theta^7} = \{C_3, C_2, C_1\} = \mathcal{D}$.

It is now also clear that the action of θ on the conics of $\langle \mathcal{D}, C_4 \rangle$ is described by $C_1 \to C_4 \to C_6 \to C_7 \to C_3 \to C_5 \to C_2 \to C_1$. Hence $\mathcal{D}$ and θ generate a unique Singer 8-arc, and θ has order divisible by 7.

(ii) **The case $\mathcal{D}^{\theta^2} = \{C_4, C_3, C_3 \oplus C_4 =: C_7\}$.**

First assume that $C_4^\theta = C_7$. But then $C_5^\theta = C_3$, from which $C_3^{\theta^2} = C_3$, yielding a contradiction as in the case $C_4^\theta = C_2$ above. Hence this case cannot occur, and consequently $C_4^\theta = C_3$.

We quickly see that $\mathcal{D}^{\theta^3} = \{C_3, C_5, C_3 \oplus C_5 =: C_6\}$. Now $C_6 = C_3 \oplus C_5 = C_3 \oplus C_1 \oplus C_4 = C_2 \oplus C_4$, and hence $\mathcal{D}^{\theta^3} \in \langle \mathcal{D}, C_4 \rangle$.

From $C_3^\theta = C_5$ and $C_5^\theta = C_7$ it follows that $\mathcal{D}^{\theta^4} = \{C_5, C_7, C_6^\theta = C_5 \oplus C_7\}$. But $C_6^\theta = C_5 \oplus C_7 = C_1 \oplus C_4 \oplus C_3 \oplus C_4 = C_2$, and so $\mathcal{D}^{\theta^4} \in \langle \mathcal{D}, C_4 \rangle$.

Consequently, $\mathcal{D}^{\theta^5} = \{C_7, C_6, C_1\} \in \langle \mathcal{D}, C_4 \rangle$, $\mathcal{D}^{\theta^6} = \{C_6, C_2, C_4\} \in \langle \mathcal{D}, C_4 \rangle$, and $\mathcal{D}^{\theta^7} = \{C_2, C_1, C_3\} = \mathcal{D}$.

It is now also clear that the action of θ on the conics of $\langle \mathcal{D}, C_4 \rangle$ is described by $C_1 \to C_4 \to C_3 \to C_5 \to C_7 \to C_6 \to C_2 \to C_1$. Hence $\mathcal{D}$ and θ generate a unique Singer 8-arc, and θ has order divisible by 7.

□

Remark 4.3.2. As mentioned in Section 4.2, if a Mathon arc is a Singer 8-arc, it can be constructed (or at least it is isomorphic to one that can be constructed) from a Denniston 4-arc $\mathcal{D}$ in the standard pencil containing the conic $C_1 : x^2 + xy + y^2 + z^2 = 0$ together with an appropriate automorphism $\theta_{t,\sigma}$. Such automorphism clearly has to satisfy the conditions of Lemma 4.3.1, and so Lemma 4.3.1 provides us with necessary and sufficient conditions on $\theta_{t,\sigma}$ in order to give rise to a Singer 8-arc. This means that theoretically the necessary subgroup condition analyzed in Lemma 4.2.1 and Lemma 4.2.2 would also follow from the above necessary and sufficient condition. However, we believe that first dealing with the subgroup condition as we did, makes the analysis of the above necessary and sufficient condition easier, and further provides insightful information on the Singer 8-arcs that will arise.

Remark 4.3.3. Let $\theta_{t,\sigma}$ be an automorphism as considered in (4.1). Suppose furthermore that $\theta_{t,\sigma}$ gives rise to a Singer 8-arc. As mentioned in Lemma 3.3.1 (see also Remark 4.1.1) there is a unique involution ι stabilizing all conics of the arc. Hence also $\theta'_{t,\sigma} := \iota\theta_{t,\sigma} = \theta_{t,\sigma}\iota$ will be an automorphism giving rise to the same Singer 8-arc as $\theta_{t,\sigma}$. This involution is easily seen to be induced by

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And consequently $\theta'_{t,\sigma} : x \mapsto EAx^\sigma$, where

$$EA = \begin{pmatrix} \sqrt{w}^{-\sigma} & 0 & 0 \\ t + \sqrt{w}^{-\sigma} & \sqrt{w}^{-\sigma} & 0 \\ \sqrt{\sqrt{w}^{-\sigma} t + t^2} & 0 & 1 \end{pmatrix}.$$

Thus $\theta'_{t,\sigma} = \theta_{t+\sqrt{w}^{-\sigma},\sigma}$. This implies that the t-values corresponding to a given Singer 8-arc come in pairs, t and $t + \sqrt{w}^{-\sigma}$.

Remark 4.3.4. In our analysis so far we have nowhere required that the hypothetical Singer 8-arc would be a proper Mathon arc. Hence, some of the Singer 8-arcs we will discover in what follows may well be arcs of Denniston type. It is however, given $\theta_{t,\sigma}$, easy to decide whether an arc is of Denniston or proper Mathon type. To be of Denniston type all conics of the arc should be contained in the standard pencil, and hence all degree-4 maximal arcs in the considered 8-arc should have the same Denniston line, namely $z = 0$. Consequently this line should be fixed by $\theta_{t,\sigma}$. This happens if and only if the element on position $(3, 1)$ of matrix A is equal to zero, or equivalently $\sqrt{w}^{-\sigma} t + t^2 = 0$. Hence, if and only if $t = 0$ or $t = \sqrt{w}^{-\sigma}$. In view of the previous remark, we see that both of these t-values will correspond to one and the same Denniston 8-arc.

We are now ready to start exploiting Lemma 4.3.1.

Assume that the same settings as presented in Section 4.2 hold, that is, the additive subgroup $\{0, 1, w, w+1\}, w \in \mathrm{GF}(2^7) \setminus \{0, 1\}$ is the one associated to the maximal arc $\mathcal{D}$. The conics with equation

$$x^2 + xy + y^2 + iz^2 = 0,$$

where $i = 1, w, w + 1$, that are contained in $\mathcal{D}$ are denoted by C_1, C_2 and C_3, respectively. Next, let $\theta = \theta_{t,\sigma}$ be an automorphism of $\mathrm{PG}(2, 2^7)$ as defined in (4.1).

Instead of choosing w to be one of the 84 values found in Section 4.2, we will instead fix $\sigma = 2$. In view of Lemma 4.2.1 we can do so without loss of generality. Once x is known, this will determine w uniquely.

Hence $\theta : p \mapsto Ap^2$, with

$$A = \begin{pmatrix} w^{-1} & 0 & 0 \\ t & w^{-1} & 0 \\ \sqrt{w^{-1}t + t^2} & 0 & 1 \end{pmatrix}.$$

Suppose that $\mathcal{D}^\theta = \{C_1, C_4, C_5\}$, with $C_2^\theta = C_1$, $C_1^\theta = C_4$ and $C_3^\theta = C_5$. Due to the proof of the previous lemma we need to consider two specific cases which possibly can lead to a Singer 8-arc. Either $C_1^{\theta^2} = C_3$ or $C_1^{\theta^2} = C_6$, where $C_6 := C_2 \oplus C_4$. We will investigate both cases separately.

Let $p = (x, y, 1)$ be a general point of the conic C_1. We know that the automorphism θ^2 is determined by the matrix $A.A^2$, which is given by

$$\begin{pmatrix} w^{-3} & 0 & 0 \\ w^{-2}t + w^{-1}t^2 & w^{-3} & 0 \\ w^{-2}\sqrt{w^{-1}t + t^2} + w^{-1}t + t^2 & 0 & 1 \end{pmatrix}.$$

Using this we are able to compute the point p^{θ^2}. This gives us

$$p^{\theta^2} = \begin{pmatrix} w^{-3}x^4 \\ (w^{-2}t + w^{-1}t^2)x^4 + w^{-3}y^4 \\ (w^{-2}\sqrt{w^{-1}t + t^2} + (w^{-1}t + t^2))x^4 + 1 \end{pmatrix}. \tag{4.6}$$

We start with the case $C_1^{\theta^2} = C_3$. This means that we want p^{θ^2} to be contained in $\mathcal{C}_3$. Expressing this condition yields the following equation:

$$\begin{gathered} w^{-6}x^8 + (w^{-3}x^4)((w^{-2}t + w^{-1}t^2)x^4 + w^{-3}y^4) + (w^{-4}t^2 + w^{-2}t^4)x^8 \\ +w^{-6}y^8 + (w+1)((w^{-4}(w^{-1}t + t^2) + w^{-2}t^2 + t^4)x^8 + 1) = 0. \end{gathered}$$

Using the fact that $x^2 + xy + y^2 + 1 = 0$, and so $x^8 + x^4y^4 + y^8 + 1 = 0$, we can simplify the previous equation to

$$w^{-6} + w^{-2}t^4x^8 + (w^{-4}t + w^{-3}t^2 + w^{-1}t^2 + wt^4 + w^{-4}t^2 + w^{-2}t^2 + t^4)x^8 + w + 1 = 0,$$

which is actually

$$\begin{aligned} ((w^{-2} + w + 1)t^4 + (w^{-4} + w^{-3} + w^{-2} + w^{-1})t^2 + w^{-4}t)x^8 \\ +w^{-6} + w + 1 = 0. \end{aligned} \tag{4.7}$$

As this should hold for any point p on C_1, this equation should be identically zero. This means that both the coefficients of x^8 and x^0, which are $(w^{-2}+w+1)t^4 + (w^{-4} + w^{-3} + w^{-2} + w^{-1})t^2 + w^{-4}t$ and $w^{-6} + w + 1$ respectively, have to be 0. First, we have a look at the condition

$$w^{-6} + w + 1 = 0.$$

With the notation used in Lemma 4.2.1 and the fact that $\sigma = 2$ we find that $l = 1$ which implies that $x = w^{-1}$. We find that

$$\begin{aligned} & w^{-6} + w + 1 = 0 \\ \Leftrightarrow\quad & x^6 + x^{-1} + 1 = 0 \\ \Leftrightarrow\quad & x^7 + x + 1 = 0. \end{aligned}$$

We conclude that $w^{-6}+w+1=0$ if and only if $x^7+x+1=0$. In other words, the case where $x^7+x+1=0$ is the only possible case that allows the coefficient of x^0 in (4.7) to be 0. Furthermore the identity

$$(w^{-2}+w+1)t^4+(w^{-4}+w^{-3}+w^{-2}+w^{-1})t^2+w^{-4}t=0$$

has to hold. Equivalently, we find

$$t(t+w^{-1})((w^{-2}+w+1)t^2+(w^{-3}+w^{-1}+1)t+w^{-3})=0.$$

This will provide us with four values for t which are $t=0$, $t=w^{-1}$, $t=w^{115}$ and $t=w^{39}$. As follows from Remark 4.3.3 and Remark 4.3.4 it is clear that the two solutions $t=0$ and $t=w^{-1}$ will lead to a degree-8 maximal arc of Denniston type. The two other values of t will extend the degree-4 maximal arc $\mathcal{D}$ to a unique Singer 8-arc.

Next, we move on to the second case $C_1^{\theta^2}=C_6$. We now aim for p^{θ^2} to be contained in $\mathcal{C}_6$. First of all we have to compute C_4 which is the image of C_1 under θ. We can do this using the matrix A and the matrix

$$B:=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},$$

which is associated to the equation $x^2+xy+y^2+z^2=0$ of the conic C_1. We need to calculate the form $(A^{-1})^TBA^{-1}$. Since

$$A^{-1}=\begin{pmatrix}w&0&0\\w^2t&w&0\\w\sqrt{w^{-1}t+t^2}&0&1\end{pmatrix},$$

we find that $(A^{-1})^TBA^{-1}$ is the matrix

$$\begin{pmatrix}w^2+(w^3+w)t+(w^4+w^2)t^2&w^2+w^3t&w\sqrt{w^{-1}t+t^2}\\w^3t&w^2&0\\w\sqrt{w^{-1}t+t^2}&0&1\end{pmatrix}.$$

It follows that the equation of the conic $C_1^{\theta}=C_4$ is given by

$$(w^2+(w^3+w)t+(w^4+w^2)t^2)x^2+w^2xy+w^2y^2+z^2=0.$$

Multiplying this equation by w^{-2} yields

$$C_4 : (1 + (w + w^{-1})t + (w^2 + 1)t^2)x^2 + xy + y^2 + w^{-2}z^2 = 0.$$

Since we know that $C_2 : x^2 + xy + y^2 + wz^2 = 0$ we can now determine the equation of the conic $C_6 := C_2 \oplus C_4$. We get

$$C_6 : \frac{(w + (1 + (w + w^{-1})t + (w^2 + 1)t^2)w^{-2})}{w + w^{-2}}x^2 + xy + y^2 + (w + w^{-2})z^2 = 0,$$

or equivalently,

$$\begin{aligned} C_6 : (w + (1 + (w + w^{-1})t + (w^2 + 1)t^2)w^{-2})x^2 + (w + w^{-2})xy \\ +(w + w^{-2})y^2 + (w^2 + w^{-4})z^2 = 0. \end{aligned}$$

Using (4.6) the condition $p^{\theta^2} \in C_6$ can be expressed in the following way:

$$\begin{aligned} &(w + (1 + (w + w^{-1})t + (w^2 + 1)t^2)w^{-2})(w^{-3}x^4)^2 \\ &\quad +(w + w^{-2})(w^{-3}x^4)(w^{-2}tx^4 + w^{-1}t^2x^4 + w^{-3}y^4) \\ &\quad +(w + w^{-2})((w^{-4}t^2 + w^{-2}t^4)x^8 + w^{-6}y^8) \\ &\quad +(w^2 + w^{-4})(w^{-4}(w^{-1}t + t^2) + w^{-2}t^2 + t^4)x^8 + 1) = 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} &((w^2 + w^{-1})t^4 + (w^{-2} + 1)t^2 + (w^{-4} + w^{-3})t + w^{-8} + w^{-5})x^8 \\ &\quad +(w^{-8} + w^{-5})x^4y^4 + (w^{-8} + w^{-5})y^8 + w^{-4} + w^2 = 0. \end{aligned}$$

Again, using the fact that $x^8 + x^4y^4 + y^8 + 1 = 0$ (since $x^2 + xy + y^2 + 1 = 0$), this equation can be simplified to

$$\begin{aligned} &((w^2 + w^{-1})t^4 + (w^{-2} + 1)t^2 + (w^{-4} + w^{-3})t)x^8 \\ &\quad +w^{-8} + w^{-5} + w^{-4} + w^2 = 0. \end{aligned} \tag{4.8}$$

Analogous to the first case this equation should be identically zero. We start by checking if the coefficient of x^0 can be equal to 0 and, since $x = w^{-1}$ with the notation of Lemma 4.2.1, we see that

$$\begin{aligned} & w^{-8} + w^{-5} + w^{-4} + w^2 = 0 \\ \Leftrightarrow\ & x^8 + x^5 + x^4 + x^{-2} = 0 \\ \Leftrightarrow\ & x^{10} + x^7 + x^6 + 1 = 0. \end{aligned}$$

Now assume that $x^7+x^3+1=0$. In this case we find that $x^{10}+x^7+x^6+1 = x^{10}+x^6+x^3 = x^3(x^7+x^3+1)=0$, exactly what we wanted. On the other hand, suppose that $x^7+x+1=0$ holds. This would imply that

$$\begin{aligned} x^{10}+x^7+x^6+1 &= x^7(x^3+1)+(x^3+1)^2 \\ &= (x^3+1)(x^7+x^3+1) \\ &= (x^3+1)(x^3+x) \\ &= x(x^3+1)(x^2+1). \end{aligned}$$

But since $x \neq 0$, $x^3 \neq 1$ and $x^2 \neq 1$ this can never be 0. In other words the case in which $x^7+x^3+1=0$ is the only possible case that allows the coefficient of x^0 to be 0. Finally we have a look at the identity

$$(w^2+w^{-1})t^4+(w^{-2}+1)t^2+(w^{-4}+w^{-3})t=0,$$

assuring us that also the coefficient of x^8 in (4.8) is 0. The four solutions satisfying this equation are $t=0$, $t=w^{-1}$, $t=w^{91}$ and $t=w^8$. Analogous to what we have seen above the two solutions $t=0$ and $t=w^{-1}$ yield a degree-8 maximal arc of Denniston type. The other two values for t that satisfy this equation will lead to a unique Singer 8-arc.

We can conclude the previous findings by saying that in both cases $x^7+x+1=0$ and $x^7+x^3+1=0$ the degree-4 maximal arc $\mathcal{D}$ can uniquely be extended to a Singer 8-arc.

We end this section by providing actual equations of the two Singer 8-arcs in $\mathrm{PG}(2,2^7)$. Let a be a primitive element of $\mathrm{GF}(2^7)$. Both Singer 8-arcs clearly can be given by the set

$$\{(x,y,z) \in \mathrm{PG}(2,2^7) | a^i x^2 + xy + y^2 + a^j z^2 = 0\} \cup \{(0,0,1)\}, \tag{4.9}$$

where there are seven ordered pairs (i,j), each corresponding to one of the conics of the arc. There are two cases. The unique, up to isomorphism, Singer 8-arc in the case where $1+a+a^7=0$ is the one with exponents

$$(i,j) = (0,0),(0,-1),(0,6),(16,2),(39,14),(93,62),(101,30). \tag{4.10}$$

In the other case where a satisfies $1+a^3+a^7=0$ the unique, up to isomorphism, Singer 8-arc is the one with exponents

$$(i,j) = (0,0),(0,-1),(0,30),(18,2),(12,62),(33,6),(43,14). \tag{4.11}$$

These values for (i, j) can be obtained by actually computing the morphism $\theta_{t,\sigma}$ using the above, and then letting this morphism act on C_1 . This can easily be done using a computer algebra package such as GAP ([24]).

4.3.1 Extra trace condition

We wind up this section (on a side note) with a quick look at the elements of the matrix

$$A^\star := A \cdot A^\sigma \cdot A^{\sigma^2} \cdot A^{\sigma^3} \cdot A^{\sigma^4} \cdot A^{\sigma^5} \cdot A^{\sigma^6},$$

with A the matrix (4.1), and the condition that it induces. Although Lemma 4.3.1 provides us with necessary and sufficient conditions in order to give rise to a Singer 8-arc it could be interesting to have a look at the elements of the matrix $A^\star$. This matrix is the one that occurs in the automorphism $\theta^7_{t,\sigma}$. To simplify notations we will rewrite A as follows:

$$A := \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & 1 \end{pmatrix}. \tag{4.12}$$

The diagonal elements on positions $(1, 1)$ and $(2, 2)$ of $A^\star$ are both

$$a^{\sigma^6+\sigma^5+\cdots+\sigma+1} = a^{\frac{\sigma^7-1}{\sigma-1}} = 1.$$

The element on position $(2, 1)$ of $A^\star$ is

$$a^{\sigma^6}(a^{\sigma^5}(a^{\sigma^4}(a^{\sigma^3}(a^{\sigma^2}(a^\sigma b + ab^\sigma) + a^{\sigma+1}b^{\sigma^2}) + a^{\sigma^2+\cdots+1}b^{\sigma^3}) + a^{\sigma^3+\cdots+1}b^{\sigma^4}) \\ +a^{\sigma^4+\cdots+1}b^{\sigma^5}) + a^{\sigma^5+\cdots+1}b^{\sigma^6}.$$

This is equivalent to

$$a^{\sigma^6+\cdots+\sigma}b + a^{\sigma^6+\cdots+\sigma^2+1}b^\sigma + a^{\sigma^6+\cdots+\sigma^3+\sigma+1}b^{\sigma^2} + \cdots + a^{\sigma^5+\cdots+1}b^{\sigma^6},$$

which can, of course, be written as

$$\frac{b}{a} + \frac{b^\sigma}{a^\sigma} + \frac{b^{\sigma^2}}{a^{\sigma^2}} + \frac{b^{\sigma^3}}{a^{\sigma^3}} + \frac{b^{\sigma^4}}{a^{\sigma^4}} + \frac{b^{\sigma^5}}{a^{\sigma^5}} + \frac{b^{\sigma^6}}{a^{\sigma^6}}.$$

This is exactly $\mathrm{Tr}(\frac{b}{a})$.

Finally we have a look at the element on position $(3,1)$ which is

$$a^{\sigma^6}(a^{\sigma^5}(a^{\sigma^4}(a^{\sigma^3}(a^{\sigma^2}(a^{\sigma}c+c^{\sigma})+c^{\sigma^2})+c^{\sigma^3})+c^{\sigma^4})+c^{\sigma^5})+c^{\sigma^6}.$$

Equivalently, we find

$$\begin{aligned}
& a^{\sigma^6+\cdots+\sigma}c+a^{\sigma^6+\cdots+\sigma^2}c^{\sigma}+a^{\sigma^6+\cdots+\sigma^3}c^{\sigma^2}+\cdots+a^{\sigma^6}c^{\sigma^5}+c^{\sigma^6} \\
= & \frac{c}{a}+\frac{c^{\sigma}}{a^{\sigma+1}}+\frac{c^{\sigma^2}}{a^{\sigma^2+\cdots+1}}+\frac{c^{\sigma^3}}{a^{\sigma^3+\cdots+1}}+\frac{c^{\sigma^4}}{a^{\sigma^4+\cdots+1}}+\frac{c^{\sigma^5}}{a^{\sigma^5+\cdots+1}}+\frac{c^{\sigma^6}}{a^{\sigma^6+\cdots+1}} \\
= & \frac{c}{(a^{\frac{1}{\sigma-1}})^{\sigma-1}}+\frac{c^{\sigma}}{(a^{\frac{1}{\sigma-1}})^{\sigma^2-1}}+\frac{c^{\sigma^2}}{(a^{\frac{1}{\sigma-1}})^{\sigma^3-1}}+\cdots+\frac{c^{\sigma^6}}{(a^{\frac{1}{\sigma-1}})^{\sigma^7-1}} \\
= & a^{\frac{1}{\sigma-1}}\left[\frac{c}{(a^{\frac{1}{\sigma-1}})^{\sigma}}+\frac{c^{\sigma}}{(a^{\frac{1}{\sigma-1}})^{\sigma^2}}+\frac{c^{\sigma^2}}{(a^{\frac{1}{\sigma-1}})^{\sigma^3}}+\cdots+\frac{c^{\sigma^6}}{(a^{\frac{1}{\sigma-1}})^{\sigma^7}}\right] \\
= & a^{\frac{1}{\sigma-1}}\mathrm{Tr}\left[\frac{c}{a^{\frac{\sigma}{\sigma-1}}}\right].
\end{aligned}$$

And so the matrix $A^{\star}$ is actually the matrix

$$A^{\star} := \begin{pmatrix} 1 & 0 & 0 \\ \mathrm{Tr}(\frac{b}{a}) & 1 & 0 \\ a^{\frac{1}{\sigma-1}}\mathrm{Tr}[\frac{c}{a^{\frac{\sigma}{\sigma-1}}}] & 0 & 1 \end{pmatrix}. \tag{4.13}$$

Since we want the Singer group to act on the seven conics of the maximal 8-arc we want these conics to be fixed under $\theta^7_{t,\sigma}$. In other words, the matrix $A^{\star}$ should induce the identity map or at least the unique involution fixing the conics. As the element on position $(2,1)$ of (4.13) is either 0 or 1 it does not impose an extra condition. More specifically, 0 would induce the identity while 1 would induce the involution. The element on position $(3,1)$ however has to be equal to 0, otherwise the matrix $A^{\star}$ would not induce an automorphism that fixes the seven conics.

It can be proved that, next to (4.2) and (4.3),

$$\mathrm{Tr}\left[\frac{\sqrt{\sqrt{w}^{-\sigma}t+t^2}}{(\sqrt{w}^{-\sigma})^{\frac{\sigma}{\sigma-1}}}\right]=0,$$

or equivalently,

$$\mathrm{Tr}\left[(\sqrt{w}^{\frac{\sigma^2}{\sigma-1}} + w^{\frac{\sigma^2}{\sigma-1}}\sqrt{w}^{-\sigma})t\right] = 0 \tag{4.14}$$

is a new trace condition in order to lead to the Singer 8-arcs since the three trace conditions (4.2), (4.3) and (4.14) are linearly independent. This can again be checked by using, for instance, GAP ([24]) as a calculator.

4.4 The count

In this section we will count the number of Singer 8-arcs and the number of normal 8-arcs in $\mathrm{PG}(2, 2^7)$. Since these are the only two classes of degree-8 arcs of Mathon type in $\mathrm{PG}(2, 2^7)$ it will lead to the total number of degree-8 maximal arcs of Mathon type in $\mathrm{PG}(2, 2^7)$. This will fill the hole that was left in Chapter 3. In the following lemma the number of Singer 8-arcs of proper Mathon type in $\mathrm{PG}(2, 2^7)$ is counted.

Lemma 4.4.1. *In* $\mathrm{PG}(2, 2^7)$ *there are, up to isomorphism, exactly two Singer* 8*-arcs.*

Proof. At the end of section 4.2 we concluded that at most two out of the three isomorphism types of degree-4 maximal arcs of Denniston type in $\mathrm{PG}(2, 2^7)$ could possibly be extended to a Singer 8-arc. In the previous section it became clear that both of these isomorphism classes induce a unique Singer 8-arc. □

Lemma 4.4.2. *The number of non-isomorphic normal* 8*-arcs in* $\mathrm{PG}(2, 2^7)$ *is* 199*.*

Proof. This proof is quite analogous to the proof of Theorem 3.3.5 in the previous chapter. Let $\mathcal{D}^1, \mathcal{D}^2$ and $\mathcal{D}^3$ be chosen fixed and representative of each of the three isomorphism classes of degree-4 maximal arcs of Denniston type in the standard pencil. Assume $\mathcal{D}^i$ consists of the conics C_1, C_2^i and C_3^i, $i = 1, 2, 3$. We will start by counting how many degree-8 maximal arcs of Mathon type contain one of the three degree-4 maximal arcs, say $\mathcal{D}^1$, have the line $x = 0$ as elation axis and $(0, 1, 0)$ as the intersection point of their lines at infinity. This is done as follows. Every conic disjoint from $\mathcal{D}^1$ and having the same nucleus as $\mathcal{D}^1$ will extend $\mathcal{D}^1$ in a unique way to a maximal arc of Mathon type (which may be Denniston). On the other hand, every Mathon arc of degree 8 that contains $\mathcal{D}^1$ will give rise to four such conics. Now each such conic, together with C_1, generates a unique maximal arc of degree 4 (of Denniston type) which has to be isomorphic to one of the $\mathcal{D}^i$, $i = 1, 2, 3$. Hence it will be sufficient to count in how many ways we can map $\mathcal{D}^i$, $i = 1, 2, 3$, on an isomorphic degree-4 arc which intersects $\mathcal{D}^1$ exactly in C_1 plus the nucleus, and which has a Denniston line containing $(0, 1, 0)$.

- Assume $i \neq 1$.

  Clearly we need an automorphism θ such that $(\mathcal{D}^i)^\theta$ satisfies the properties described above. Hence θ has to map one of the conics C_1 or C^i_j, $j = 2, 3$ onto C_1. It follows that θ is of the form (4.1), with w the value corresponding to C_1 or C^i_j, respectively. The above conditions will be satisfied iff the trace conditions (4.2) and (4.3) seen above, are satisfied. These two conditions can be written as

$$\begin{cases} \mathrm{Tr}[A_1(w,\sigma)t] = 0 \\ \mathrm{Tr}[A_2(w,\sigma)t] = 0, \end{cases}$$

  where A_1 and A_2 are both functions of w and σ. We actually obtain two linear equations that correspond to two hyperplanes in the vector space $V(7, 2)$. In Chapter 3 it was shown that these hyperplanes are distinct. This means that for every w and every field automorphism σ there are 2^5 solutions for t. As noticed in the previous chapter, and as seen in the previous section, these t-values always come in pairs. This implies that, for every w and σ, there are 2^4 degree-4 maximal arcs. One of them gives rise to a degree-8 maximal arc of Denniston type, and so there are

$$3 \cdot 7 \cdot (2^4 - 1)$$

  automorphisms θ that satisfy the needed conditions and induce a degree-8 maximal arc of proper Mathon type. Of course, one such automorphism leads to two conics disjoint from $\mathcal{D}^1$ and so we get

$$2 \cdot 3 \cdot 7 \cdot (2^4 - 1) = 630$$

  conics that extend $\mathcal{D}^1$ to a degree-8 maximal arc of Mathon type.

- Now assume $i = 1$.

  In the cases where C_2^1 is mapped onto C_1 and C_3^1 is mapped onto C_1 we also find
  $$2 \cdot 7.(2^4 - 1)$$
  conics to extend $\mathcal{D}^1$. If we now consider the case where C_1 is fixed however, we have to make sure that σ is not the identity since this would result in conics which are not disjoint (see Remark 3.3.4 in Chapter 3). And so when $i = 1$ we find
  $$2 \cdot 2 \cdot 7 \cdot (2^4 - 1) + 2 \cdot 6 \cdot (2^4 - 1) = 600$$
  conics to extend $\mathcal{D}^1$ in this case.

Finally, as there were two choices for $\mathcal{D}^i$, $i \neq 1$, there are in total
$$2 \cdot 630 + 600 = 1860$$
conics that extend $\mathcal{D}^1$ to a proper Mathon arc satisfying the desired properties. Of course, there were three choices for $\mathcal{D}^1$ and so we get
$$3 \cdot 1860 = 5580$$
conics that will extend some $\mathcal{D}^i$ to a degree-8 maximal arc of Mathon type. However, due to Lemma 4.4.1 we know that two out of the three isomorphism classes of degree-4 maximal arcs can be extended to a unique Singer 8-arc. This means that eight conics will extend some $\mathcal{D}^i$ to a Singer 8-arc which implies that there are actually
$$5580 - 8 = 5572$$
conics that will extend some $\mathcal{D}^i$ to a normal 8-arc. Since the four conics disjoint from $\mathcal{D}^i$ in such a normal 8-arc all give rise to the same 8-arc there are 1393 normal 8-arcs that contain some degree-4 maximal arc $\mathcal{D}^i$. Hence, the number of non-isomorphic normal 8-arcs is
$$1393/7 = 199.$$
The fact that we divide by 7 is a consequence of Corollary 3.3.2 and Lemma 3.3.1 in Chapter 3, which states that there is a unique isomorphism of the plane mapping a degree-4 Denniston arc in a normal Mathon 8-arc onto one of the $\mathcal{D}^i$. □

The total number of non-isomorphic degree-8 maximal arcs of Mathon type is now easily calculated and yields the following theorem.

Theorem 4.4.3. *The number of non-isomorphic degree-*8 *maximal arcs of proper Mathon type in* $\mathrm{PG}(2, 2^7)$ *is equal to* 201*, two of which are Singer* 8*-arcs, and* 199 *of which are normal* 8*-arcs.*

4.5 Bigger fields

It turns out that Singer 8-arcs also exist in bigger fields.

Consider $\mathrm{GF}(2^h)$, with $h = 7l$ and h odd. Let, for now, TR denote the usual absolute trace map from the finite field $\mathrm{GF}(2^h)$ onto GF(2) and let tr be the usual absolute trace map from the field $\mathrm{GF}(2^7)$ onto GF(2). Now, since h is odd and a multiple of 7 the equality $\mathrm{TR}(\alpha) = \mathrm{tr}(\alpha)$ will hold for every $\alpha \in \mathrm{GF}(2^7)$, subfield of $\mathrm{GF}(2^h)$. This implies that all conics from (4.9), as well with exponents (4.10) as with exponents (4.11), are exterior to the line $z = 0$. However, one has to be careful with the definition of Singer arc over these bigger fields. To see this, first consider the case where $l \neq 7k$ for some (odd) k. In this case consider the smallest positive t such that $lt \equiv 1 \pmod 7$. Then the field automorphism $\tau = 2^{lt}$ acts as squaring on the subfield $\mathrm{GF}(2^7)$ and has order 7 (over the big field). If we now replace the automorphism $\sigma = 2$ from Section 3 by τ, then we can easily see that we produce two Mathon 8-arcs that admit a cyclic group of order 7 acting sharply transitively on the seven conics of the arc. These are clearly Singer 8-arcs in the obvious sense.

However, if we consider the case where $l = 7k$ for some (odd) k, then there is no field automorphism of $\mathrm{GF}(2^l)$ that acts as squaring on the subfield $\mathrm{GF}(2^7)$ and has order 7 (over the big field). In this case every field automorphism that acts as squaring on the subfield $\mathrm{GF}(2^7)$ will have as order a proper power of 7. In this case the arcs we produce will only admit a cyclic group acting transitively on the seven conics of the arc, but not sharply transitively, as the elements of the unique cyclic subgroup of order 7 will stabilize all conics of the arc (but not fix them pointwise). One could define such arcs as *Singer* 8*-arcs of the second kind*.

4.6 Open questions

The following two questions seem now to be natural.

- Can we construct Singer arcs of degree bigger than 8, that is, are there for example degree-16 arcs that admit a (cyclic) automorphism group acting sharply transitively on their conics? If so, over which fields do these arcs exist? What about Singer arcs of the second kind?

- Proper 8-arcs of Mathon type have, considered conicwise, naturally the structure of the Fano plane. Furthermore, the Singer group of the Mathon 8-arcs of Singer type acts as a Singer group on this Fano plane. This Singer group is only a subgroup of the full automorphism group of the Fano plane. Are there fields over which there exist Mathon 8-arcs that admit the full automorphism group of the Fano plane (in its natural action on the conics of the arc)? If not, what is the largest subgroup of $\mathrm{PGL}(3, 2)$ that can occur, and over which fields does this happen? In such case one would of course not require that, if a conic is stabilized by an automorphism, it is fixed pointwise.

Chapter 5

Partial flocks of the quadratic cone yielding Mathon arcs

As a consequence of a question asked by J. A. Thas we started studying the link between maximal arcs of Mathon type and partial flocks of the quadratic cone ([33]). This link is of a rather algebraic nature. In this last chapter we establish, due to a specific projection, a geometric connection between these two structures. We also define a composition on the flock planes and use this to work out an analogue of the synthetic version of Mathon's Theorem (see Chapter 3). Finally we show how it is possible to construct a maximal arc of Mathon type of degree $2d$, containing a Denniston arc of degree d, and provided that there is a solution to a certain given system of trace conditions.

5.1 Partial flocks

This first section will serve as an introduction to partial flocks as well as a brief description of the algebraic link between partial flocks and maximal arcs of Mathon type as it was proved in [33]. Suppose that $\mathscr{K}$ is a quadratic cone in $\mathrm{PG}(3,q)$ with vertex x. A *partial flock* $\mathscr{F}$ of $\mathscr{K}$ is a set of disjoint (non-singular) conics on the cone $\mathscr{K}$. A partial flock is called *complete* if it is not contained in a larger partial flock. A *flock* $\mathscr{F}$ of $\mathscr{K}$ is a partial flock of size q. The planes containing the conics of the flock are called the *flock planes*. If all the flock planes of a partial flock have a line in common, then this partial flock is called *linear*. Flocks are related to some elation generalized quadrangles of order (q^2, q), line spreads of $\mathrm{PG}(3, q)$ and, when q is even, families of ovals in $\mathrm{PG}(2, q)$, called herds ([49]).

From now on, let the order q, of the field $\mathrm{GF}(q)$, be even and suppose that the cone $\mathscr{K}$ has equation $X_1X_3 = X_2^2$. The vertex is the point $x(1, 0, 0, 0)$ and does not belong to any plane of a (partial) flock $\mathscr{F}$. The conics of $\mathscr{F}$ are defined by k planes V_i, $i \in \{1, \ldots, k\}$, of which the equations can be written in the form

$$X_0 + f(t)X_1 + tX_2 + g(t)X_3 = 0, \tag{5.1}$$

with $t \in B$, where B is some subset of $\mathrm{GF}(q)$, and f and g are functions from B to $\mathrm{GF}(q)$. In order to form a partial flock the intersection of any two of the planes (5.1) with $\mathscr{K}$ has to be empty. This means that the system of equations

$$\begin{cases} X_0 + f(s)X_1 + sX_2 + g(s)X_3 = 0 \\ X_0 + f(t)X_1 + tX_2 + g(t)X_3 = 0 \\ X_1X_3 = X_2^2 \end{cases}$$

for any $s, t \in B$, with $s \neq t$, can have no solutions in $\mathrm{GF}(q)$. This is equivalent with the condition that the quadratic equation

$$(f(s) + f(t))X_1^2 + (s + t)X_1X_2 + (g(s) + g(t))X_2^2 = 0$$

has no solutions in $\mathrm{GF}(q)$. In other words, the k conics $V_i \cap \mathscr{K}$, $i \in \{1, \ldots, k\}$, form a partial flock of $\mathscr{K}$ if and only if

$$\mathrm{Tr}\Big[\frac{(f(s) + f(t))(g(s) + g(t))}{(s + t)^2}\Big] = 1, \ \ \forall s, t \in B, s \neq t. \tag{5.2}$$

We already know (see Chapter 2) that a closed set of conics $\mathcal{C}$, which can be used to construct maximal arcs of Mathon type, may be written in the form

$$\mathcal{C} = \{(x, y, z) : p(\lambda)x^2 + xy + r(\lambda)y^2 + \lambda z^2 = 0, \lambda \in A\},$$

where A is a subset of $\mathrm{GF}(q) \setminus \{0\}$ such that $A \cup \{0\}$ is closed under addition and p and r are functions from A to $\mathrm{GF}(q)$.

Hamilton and J. A. Thas proved in [33] that the functions p and r associated to $\mathcal{C}$ give rise to a partial flock in the following way. Set $B = A \cup \{0\}$ and define the functions f and g on B by $f(0) = g(0) = 0$ and $f(t) = tp(t), g(t) = tr(t)$ for $t \in A$. Since A, p and r define a closed set of conics we know that

$$\frac{sp(s) + tp(t)}{s+t} = p(s+t) \quad \text{and} \quad \frac{sr(s) + tr(t)}{s+t} = r(s+t), \tag{5.3}$$

for $s, t \in A$, with $s \neq t$. As $s + t \in A$ the trace condition for the closed set of conics gives us

$$\begin{aligned} 1 = \mathrm{Tr}[p(s+t)r(s+t)] &= \mathrm{Tr}\Big[\big(\frac{sp(s)+tp(t)}{s+t}\big)\big(\frac{sr(s)+tr(t)}{s+t}\big)\Big] \\ &= \mathrm{Tr}\Big[\frac{(f(s)+f(t))(g(s)+g(t))}{(s+t)^2}\Big]. \end{aligned}$$

This implies that f, g and B define a partial flock.

From (5.3) we know that $sp(s) + tp(t) = (s+t)p(s+t)$ and $sr(s) + tr(t) = (s+t)r(s+t)$, or equivalently that $f(s)+f(t) = f(s+t)$ and $g(s)+g(t) = g(s+t)$. In other words, the functions f and g arising from a closed set of conics are additive on B and also B is closed under addition. A partial flock with these properties will be called an *additive* partial flock.

Conversely, suppose an additive partial flock is given with functions f and g on an additive subgroup B of $\mathrm{GF}(q)$. Now define $A = B \setminus \{0\}$ and functions $p(t) = f(t)/t$ and $r(t) = g(t)/t$, $t \in A$. Since f and g are additive on B we see that

$$\frac{sp(s) + tp(t)}{s+t} = \frac{f(s) + f(t)}{s+t} = \frac{f(s+t)}{s+t} = p(s+t)$$

and hence the closure principle holds. Furthermore, it is clear that

$$\begin{aligned} \mathrm{Tr}[p(s+t)r(s+t)] &= \mathrm{Tr}\Big[\frac{f(s+t)g(s+t)}{(s+t)^2}\Big] \\ &= \mathrm{Tr}\Big[\frac{(f(s)+f(t))(g(s)+g(t))}{(s+t)^2}\Big] \\ &= 1. \end{aligned}$$

The above implies that, using the fact that the functions f and g are additive on B, the required trace and closure conditions on A in order to give rise to a closed set of conics, and hence a maximal arc of Mathon type in $\mathrm{PG}(2, q)$ are satisfied. Knowing all the above, the following theorem holds.

Theorem 5.1.1 ([33]). *A degree-d maximal arc of Mathon type gives rise to an additive partial flock of size d of the quadratic cone in* $\mathrm{PG}(3, q)$*, and conversely.*

In what follows we will show that a partial flock, corresponding to a maximal arc M of degree d of Mathon type, is linear, if and only if M is of Denniston type. This is also mentioned in [33] without an explicit proof.

Given a degree-d maximal arc M of Mathon type of which the corresponding partial flock is linear. Consider two random planes $X_0 + tp(t)X_1 + tX_2 + tr(t)X_3 = 0$ and $X_0 + sp(s)X_1 + sX_2 + sr(s)X_3 = 0$, $s, t \in A$ and $s \neq t$, that define two conics contained in the partial flock. Expressing that the intersection line of these two planes has to be contained in the plane $X_0 = 0$ (as $X_0 = 0$ is also a flock plane) is equivalent to imposing that the rank of the matrix

$$\begin{pmatrix} 1 & tp(t) & t & tr(t) \\ 1 & sp(s) & s & sr(s) \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is equal to two. This condition will be satisfied if and only if $p(s) = p(t)$ and $r(s) = r(t)$. Now say $p(s) = p(t) = a$ and $r(s) = r(t) = b$. It follows that the conics contained in M can be written as

$$ax^2 + xy + by^2 + \lambda z^2 = 0, \quad \lambda \in A,$$

and it follows that M is a maximal arc of Denniston type.

Conversely, if M is a maximal arc of Denniston type, then the corresponding partial flock is determined by planes with equation

$$X_0 + tX_1 + tX_2 + atX_3 = 0, \quad t \in B,$$

where a is a fixed element of $\mathrm{GF}(q)$. It is clear that all these planes have the line $X_1 + X_2 + aX_3 = 0$, $X_0 = 0$ in common, hence the corresponding partial flock is linear.

We conclude that a partial flock, corresponding to a maximal arc M of degree d of Mathon type, is linear if and only if M is of Denniston type.

5.2 Projection

As became clear in the previous section a maximal arc of degree d of Mathon type gives rise to an additive partial flock of size d of the quadratic cone in $\mathrm{PG}(3, q)$, and conversely. The link between these two geometric structures is of an algebraic nature and is based on the trace condition of Mathon's construction. The authors of [33] also remark in their paper that a closed set of conics of size $d - 1$ on a common nucleus in $\mathrm{PG}(2, q)$, q even, can be projected onto the quadratic cone and in this way induces a partial flock of the quadratic cone. However, this partial flock did not appear to have as many nice properties as the one arising from the algebraic approach.

In this section we will establish a more geometric link between the maximal arcs of Mathon type in $\mathrm{PG}(2, q)$ and additive partial flocks in $\mathrm{PG}(3, q)$. This is done by obtaining a geometric link between the partial flock arising from projection and the additive partial flock. We will see that the relation between the two partial flocks basically is an "inversion" on the nuclear line of the cone.

Before continuing we first provide a short lemma that guarantees that our projections are well defined.

Lemma 5.2.1. *Let $\mathscr{K}$ be a quadratic cone with vertex x in $\mathrm{PG}(3, q)$, let N be its nuclear line, and let π be any plane not through x. Denote $N \cap \pi$ by n, and let p be any point on N distinct from x and n. Then the projection from p of any conic C in π with nucleus n onto the cone $\mathscr{K}$ is a conic on $\mathscr{K}$.*

Proof. First note that every line through p intersects the cone in a unique point. Hence the projection of C results in $q + 1$ points on $\mathscr{K}$. We need to show they form a conic.

Consider any plane γ in $\mathrm{PG}(3, q)$ not containing x and not containing p. Then γ clearly intersects $\mathscr{K}$ in a conic, and if we project this conic from p onto π then we obtain a conic in π having n as its nucleus. In this way we obtain $q^2(q - 1)$ conics in π with nucleus n.

On the other hand, in $\mathrm{PG}(2, q)$ every conic with nucleus $(0, 0, 1)$ is of the form $\alpha X_0^2 + X_0X_1 + \beta X_1^2 + \lambda X_2^2 = 0$ with $\lambda \neq 0$ and α, β arbitrary elements of $\mathrm{GF}(q)$. Hence there are $q^2(q - 1)$ conics having a given point as their nucleus.

It follows that the conics with nucleus n in π are in one-to-one correspondence with the planes not through x or p. The lemma follows. □

Now let M be a degree-d maximal arc of Mathon type in the plane $\mathrm{PG}(2,q)$. Embed $\mathrm{PG}(2,q)$ in $\mathrm{PG}(3,q)$ and assume that $\mathrm{PG}(2,q)$ is the plane with equation $X_0 = 0$. To simplify the calculations ahead we will assume that the conics contained in M have equations

$$\alpha^2 X_1^2 + X_1 X_3 + \beta^2 X_3^2 + \lambda^2 X_2^2 = 0, \tag{5.4}$$

with α, β and λ elements of $\mathrm{GF}(q)$. Of course the quadratic polynomial $\alpha^2 x^2 + x + \beta^2$ has to be irreducible over $\mathrm{GF}(q)$ and this is satisfied if $\mathrm{Tr}(\alpha^2\beta^2) = \mathrm{Tr}(\alpha\beta) = 1$. Hence the change of notation does not alter the trace condition. In the plane $X_0 = 0$ all conics contained in M have nucleus $(0,1,0)$ and the line at infinity is the line $X_2 = 0$. These conics will sometimes be denoted by $C : (\alpha^2, \beta^2, \lambda^2)$ using only these specific coefficients.

Next, let $\mathscr{K}$ be a quadratic cone in $\mathrm{PG}(3,q)$. Suppose the cone $\mathscr{K}$ has equation $X_1 X_3 = X_2^2$. The vertex is the point $x(1,0,0,0)$ and does not belong to the plane $X_0 = 0$. Notice that the conic which is the intersection of $\mathscr{K}$ and the plane $X_0 = 0$ is not contained in M since the elements α and β cannot be zero. It is clear that the nuclear line N is the intersection of the planes $X_1 = 0$ and $X_3 = 0$ which is the line with points $(t,0,1,0), t \in \mathrm{GF}(q)$ and the vertex x. Notice that N intersects $X_0 = 0$ in the point $(0,0,1,0)$, the common nucleus of all conics in M.

Take the point $n(1,0,1,0)$ on the line N. We will project the elements in M onto the cone $\mathscr{K}$. This means that, for every conic C in M, we look for the plane V that intersects $\mathscr{K}$ exactly in the projection of C from the point n. In other words, the conic C contained in M is projected onto the conic which is the intersection of V and $\mathscr{K}$. Furthermore the line with equation $X_2 = 0$, $X_0 = 0$, which is the line at infinity of M, spans a plane with the point n. This plane will determine a conic on the cone $\mathscr{K}$ which will be the projection of the line at infinity of M. In the following lemmas we will determine the planes that induce the projections of each of the conics contained in M. Remark that these planes induce a partial flock on the quadratic cone $\mathscr{K}$ since all conics, as well as the line at infinity, of M are disjoint. However this partial flock is not additive, where the partial flock arising in [33] is an additive partial flock. In what follows we will look for the link between the partial flock induced by the projection and the additive partial flock induced by the planes in [33].

Lemma 5.2.2. *If* $\alpha^2 X_1^2 + X_1 X_3 + \beta^2 X_3^2 + \lambda^2 X_2^2 = 0$ *is the equation of a conic* C *in* M*. Then the plane that contains the projection of the conic* C *from* n *on the cone* $\mathscr{K}$ *has equation*

$$\lambda X_0 + \alpha X_1 + (\lambda + 1)X_2 + \beta X_3 = 0. \tag{5.5}$$

Proof. Take three points on C, for example $p(0, \lambda, \alpha, 0), q(0, 0, \beta, \lambda)$ and $r(0, \beta^2\lambda, \alpha\beta^2, \lambda)$. The line pn contains the points $(0, \lambda, \alpha, 0) + t(1, 0, 1, 0) = (t, \lambda, \alpha + t, 0), t \in \mathrm{GF}(q)$. If $t = \alpha$ the point satisfies $X_1 X_3 = X_2^2$ and it follows that the intersection of pn and $\mathscr{K}$ is the point $p'(\alpha, \lambda, 0, 0)$. In an analogous way we find that the lines qn and rn intersect the cone $\mathscr{K}$ in the points $q'(\beta, 0, 0, \lambda)$ and $r'(\beta\lambda + \alpha\beta^2, \beta^2\lambda, \beta\lambda, \lambda)$ respectively. These three points determine the plane that intersects $\mathscr{K}$ in the projection of C. This plane has an equation of the form $aX_0 + bX_1 + cX_2 + dX_3 = 0$. The conditions resulting from the requirement that the points p', q' and r' must be contained in this plane are

$$\begin{cases} \alpha a + \lambda b = 0 \\ \beta a + \lambda d = 0 \\ (\beta\lambda + \alpha\beta^2)a + \beta^2\lambda b + \beta\lambda c + \lambda d = 0. \end{cases}$$

We see that $(a, b, c, d) = (\lambda, \alpha, \lambda + 1, \beta)$ satisfies this system of equations and therefore the plane that determines the projection of C has equation

$$\lambda X_0 + \alpha X_1 + (\lambda + 1)X_2 + \beta X_3 = 0.$$

□

The planes obtained in Lemma 5.2.2 will be called *conic planes* since these planes are associated to the conics in M. It will become clear that they need to be distinguished from the planes associated to the Denniston lines and the line at infinity of a maximal arc of Mathon type (see Chapter 2). The latter will be called the *Denniston planes* and the *singular plane*, respectively.

Lemma 5.2.3. *If* $X_2 = 0$ *is the equation of the line at infinity of* M*. Then the singular plane has equation* $X_0 + X_2 = 0$*.*

Proof. It suffices to consider the span of $X_2 = 0$ and the point n. This is indeed the plane $X_0 + X_2 = 0$. □

Using Lemma 5.2.2 and Lemma 5.2.3 we can find all the planes that determine the projection of the maximal arc M onto the quadratic cone $\mathscr{K}$ and form a partial flock on $\mathscr{K}$. However these planes do not seem to correspond immediately to the planes found in [33]. In other words we are looking for the correspondence between the two partial flocks. In particular, applying our notation on the theory from [33] yields the planes

$$X_0 + \alpha^2\lambda^2 X_1 + \lambda^2 X_2 + \beta^2\lambda^2 X_3 = 0 \tag{5.6}$$

together with the plane $X_0 = 0$, which induce the additive partial flock.

So far, we have found the conic planes with equations $\lambda X_0 + \alpha X_1 + (\lambda+1)X_2 + \beta X_3 = 0$ and the singular plane $X_0 + X_2 = 0$. Now, consider the automorphism $\delta \in \mathrm{PGL}(4, q)$ given by

$$\begin{aligned} X_0 &\to X_0 + X_2, \\ X_i &\to X_i, \;\; i > 0, \end{aligned}$$

that fixes the cone $\mathscr{K}$. This will map the singular plane $X_0 + X_2 = 0$ to the plane $X_0 = 0$ and the image of the conic planes under δ is

$$\lambda X_0 + \alpha X_1 + X_2 + \beta X_3 = 0. \tag{5.7}$$

Remark 5.2.4. The point $n(1, 0, 1, 0)$ from which we project is in fact arbitrary. Suppose we had chosen any other point $(x, 0, 1, 0), x \neq 0$ on the nuclear line. In this case we can still find an automorphism that fixes the cone $\mathscr{K}$ and switches the singular plane and the plane $X_0 = 0$ in the following way. If we project from the point $(x, 0, 1, 0), x \neq 0$ we get conic planes with equations

$$\frac{\lambda}{x} X_0 + \alpha X_1 + (\lambda + 1)X_2 + \beta X_3 = 0 \tag{5.8}$$

and the singular plane $X_0 + xX_2 = 0$. Next, consider the automorphism $\delta' \in \mathrm{PGL}(4, q)$ given by

$$\begin{aligned} X_0 &\to xX_0 + xX_2, \\ X_i &\to X_i, \;\; i > 0. \end{aligned}$$

This δ' maps the singular plane to the plane $X_0 = 0$ and the planes (5.8) to the planes

$$\lambda X_0 + \alpha X_1 + X_2 + \beta X_3 = 0$$

while stabilizing the cone. Hence, the partial flock induced by the plane $X_0 = 0$ and the planes (5.7) is, up to isomorphism, the unique partial flock arising from projection. We will now establish a link between this partial flock and the additive partial flock arising from the maximal arc.

The planes found in (5.7) intersect the nuclear line in the points $(1, 0, \lambda, 0)$.

Next, consider the inversion ι on the nuclear line defined by

$$\begin{aligned}(1, 0, y, 0) &\mapsto (1, 0, 1/y, 0), \quad y \neq 0\\(1, 0, 0, 0) &\mapsto (0, 0, 1, 0),\\(0, 0, 1, 0) &\mapsto (1, 0, 0, 0).\end{aligned}$$

Then ι induces an involution, fixing the point $(1, 0, 1, 0)$, on the points of the nuclear line. We can now use ι to construct a map ϕ on each plane V that doesn't intersect neither one of the points $(1, 0, 0, 0)$ nor $(0, 0, 1, 0)$. Each one of these planes intersects the plane $X_0 = 0$ in a unique line L and the nuclear line in a point $(1, 0, y, 0)$, $y \neq 0$. Define the map ϕ on the planes V as follows:

$$\phi(V) = \langle L, \iota(1, 0, y, 0)\rangle = \langle L, (1, 0, 1/y, 0)\rangle.$$

Applying ϕ on the planes (5.7) results in the planes

$$X_0 + \alpha\lambda X_1 + \lambda X_2 + \beta\lambda X_3 = 0.$$

Remark 5.2.5. The correspondence deduced above can also be seen by considering the automorphisms $\theta_\lambda \in \mathrm{PGL}(4, q)$ given by the matrix

$$M_\lambda := \begin{pmatrix}\lambda^2 & 0 & \lambda^2 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

and applying θ_λ on the original conic planes (5.5) and the singular plane $X_0 + X_2 = 0$. Remark that, since θ_λ depends on λ, with every conic plane, there corresponds a unique automorphism θ_λ.

Finally, we apply the automorphism $\kappa \in \mathrm{PGL}(4, q)$ given by

$$(a, b, c, d) \mapsto (a^2, b^2, c^2, d^2),$$

that fixes the cone $\mathscr{K}$. This yields

$$X_0 + \alpha^2\lambda^2 X_1 + \lambda^2 X_2 + \beta^2\lambda^2 X_3 = 0,$$

and so, the conic planes found in (5.5) are isomorphic to the planes given in (5.6).

We can summarize as follows. Given a maximal arc of Mathon type in the plane $X_0 = 0$ in $\mathrm{PG}(3, q)$ with common nucleus $(0, 1, 0)$ and $X_2 = 0$ as line at infinity, i.e., given the coefficients $(\alpha^2, \beta^2, \lambda^2)$. Projection from the point $n(1, 0, 1, 0)$ on the nuclear line onto the cone $\mathscr{K}$ gives rise to a partial flock equivalent to the one with flock planes (5.7) and $X_0 = 0$. This partial flock is not yet additive. Applying the simple map ϕ, arising from an inversion on the nuclear line, to these planes, and then the automorphism κ gives us the planes (5.6) found in [33], i.e., an additive partial flock. Of course all the above works in both ways.

5.3 Plane composition

It is natural to wonder about the relation between these conic planes and the singular (or Denniston) planes and to check whether the equations of these planes can be calculated directly. We already know (see Lemma 2.3.11 in Chapter 2) that, given any two disjoint conics on a common nucleus in a plane, there is a unique third disjoint conic on the same nucleus such that the three conics form a degree-4 maximal arc of Denniston type. This result can be translated to a result concerning conic planes.

We start by introducing a standard equation for planes not containing the point $n(1, 0, 1, 0)$. A plane with an equation of the form

$$aX_0 + bX_1 + (a + 1)X_2 + cX_3 = 0, \quad a, b, c \in \mathrm{GF}(q) \tag{5.9}$$

is said to have a *standard equation*. This equation is unique when the coefficients of X_0 and X_2 are distinct, that is when the plane does not contain n.

Lemma 5.3.1. *Given any two planes in* $\mathrm{PG}(3, q)$*, not containing* n *or* x*, and intersecting the cone* $\mathscr{K}$ *disjointly, there is a unique third plane such that the projection of the intersection of these three planes with* $\mathscr{K}$ *from* n *onto the plane* $X_0 = 0$ *induces a degree-4 maximal arc of Denniston type.*

Proof. Given two conic planes and their standard equation $V : \lambda X_0 + \alpha X_1 + (\lambda+1)X_2 + \beta X_3 = 0$ and $W : \lambda' X_0 + \alpha' X_1 + (\lambda'+1)X_2 + \beta' X_3 = 0$. These conic planes V and W are associated to the conics $C_1 : (\alpha^2, \beta^2, \lambda^2)$ and $C_2 : (\alpha'^2, \beta'^2, \lambda'^2)$ in $X_0 = 0$. Using Lemma 2.3.11 we know that the conic

$$C_1 \oplus C_2 : \Big(\frac{\alpha^2\lambda^2 + \alpha'^2\lambda'^2}{\lambda^2 + \lambda'^2}, \frac{\beta^2\lambda^2 + \beta'^2\lambda'^2}{\lambda^2 + \lambda'^2}, \lambda^2 + \lambda'^2\Big)$$

is the unique conic inducing a degree-4 maximal arc of Denniston type containing both C_1 and C_2. The unique conic plane corresponding to $C_1 \oplus C_2$ has equation

$$V \oplus W : (\lambda + \lambda')X_0 + \frac{\alpha\lambda + \alpha'\lambda'}{\lambda + \lambda'}X_1 + (\lambda + \lambda' + 1)X_2 + \frac{\beta\lambda + \beta'\lambda'}{\lambda + \lambda'}X_3 = 0.$$

□

Notice that the partial flock associated to a maximal arc of Denniston type should be linear. One easily checks that the three planes in the above lemma have a line in common. Also note that the coefficients in the standard equation of the plane $V \oplus W$ are obtained using a Mathon composition.

We know that, if the equation of the conic plane associated to a conic $C : (\alpha^2, \beta^2, \lambda^2)$ is given by $\lambda X_0 + \alpha X_1 + (\lambda + 1)X_2 + \beta X_3 = 0$ this equation is standard. Once the conic plane is set in standard notation we can use the following lemma to determine the singular plane associated to a degree-4 maximal arc of Denniston type.

Lemma 5.3.2. *Given two conic planes V and W in $\mathrm{PG}(3,q)$, the singular plane inducing the line at infinity of the unique degree-4 maximal arc of Denniston type induced by V and W can be found by the sum of the equations of V and W.*

Proof. The conic planes $V : \lambda X_0 + \alpha X_1 + (\lambda + 1)X_2 + \beta X_3 = 0$ and $W : \lambda' X_0 + \alpha' X_1 + (\lambda' + 1)X_2 + \beta' X_3 = 0$ are associated to the two conics $C_1 : (\alpha^2, \beta^2, \lambda^2)$ and $C_2 : (\alpha'^2, \beta'^2, \lambda'^2)$ in $X_0 = 0$. We are looking for the singular conic in the pencil

$$\{\mu C_1 + \nu C_2 : \mu, \nu \in \mathrm{GF}(q), \mu, \nu \neq 0\}.$$

Since both conics C_1 and C_2 have 1 as coefficient of the term X_1X_3 the singular conic in the pencil above can be found by simply taking the sum of both conics, i.e., $\mu = \nu = 1$. This gives us $(\alpha^2 + \alpha'^2)X_1^2 + (\lambda^2 + \lambda'^2)X_2^2 + (\beta^2 + \beta'^2)X_3^2 = 0$ which is equivalent to

$$(\alpha + \alpha')X_1 + (\lambda + \lambda')X_2 + (\beta + \beta')X_3 = 0, \tag{5.10}$$

yielding the equation of the line at infinity of the unique degree-4 maximal arc of Denniston type induced by V and W in the plane $X_0 = 0$. Taking the sum of the equations of the two conic planes V and W gives us the plane with equation

$$(\lambda + \lambda')X_0 + (\alpha + \alpha')X_1 + (\lambda + \lambda')X_2 + (\beta + \beta')X_3 = 0.$$

Intersecting that plane with the plane $X_0 = 0$ results in the same equation of the line at infinity. □

Remark that if we take the sum of the conic planes V and $V \oplus W$ in the proof of Lemma 5.3.1 we do not exactly find equation (5.10). However, we do find the same line at infinity. By multiplying that equation by the right scalar we can always attain equation (5.10). It is clear that we obtain a different singular plane if the equations of the conic planes are not standard. In that case Lemma 5.3.2 does not work.

Next we consider the intersections of each of the planes in the partial flock, i.e., the conic planes and the singular plane, with the nuclear line N. We know that N consists of the points $(t, 0, 1, 0), t \in \mathrm{GF}(q)$ and the vertex $x(1, 0, 0, 0)$. Since the singular plane should always induce a line at infinity on the plane $X_0 = 0$ in the projection from $n(1, 0, 1, 0)$ we know that this singular plane intersects the nuclear line in the point n. Furthermore, suppose the planes V, W and $V \oplus W$, as seen in the proof of Lemma 5.3.1, are the three conic planes associated to a random degree-4 maximal arc of Denniston type. Their intersections with the nuclear line gives us the points $(\lambda+1, 0, \lambda, 0)$, $(\lambda'+1, 0, \lambda', 0)$ and $(\lambda + \lambda' + 1, 0, \lambda + \lambda', 0)$, respectively. If, to these three points, we add the vertex $x(1, 0, 0, 0)$ we see that, in the X_2-component, the elements of the additive group of order 4 that induce the Denniston 4-arc are given.

5.4 Analogue of the synthetic theorem

In Chapter 3 we obtained a synthetic version of Mathon's theorem. Mainly by using Mathon's composition and Lemma 2.3.11 it became clear that, given a degree-d maximal arc M of Mathon type and a conic C disjoint from M, there is a unique maximal arc of degree $2d$ of Mathon type containing $M \cup C$. With the tools given above it is possible to translate this theorem to a theorem concerning partial flocks. First we will extend the additive linear partial flock of size 4 corresponding to a degree-4 maximal arc of Denniston type.

Theorem 5.4.1. *Given an additive linear partial flock $\mathscr{F}$ of size 4 and given a plane V', not containing the point $n'(0,0,1,0)$ or x, and such that V' intersects $\mathscr{K}$ in a conic disjoint from the elements of $\mathscr{F}$. Then there is a unique additive partial flock of size 8 containing the conics determined by V' and the four planes defining $\mathscr{F}$.*

Proof. This follows immediately from the analysis in the previous sections, Theorem 5.3.2 and Theorem 3.1.2 . □

Remark 5.4.2. If the plane V' in the previous theorem contains the intersection line of the four planes $V'_1, \ldots, V'_4$ defining $\mathscr{F}$, the partial flock of size 8 will be linear, thus inducing a degree-8 maximal arc of Denniston type.

The previous theorem can be generalized to maximal arcs of Mathon type in the following way.

Theorem 5.4.3. *Given an additive partial flock $\mathscr{F}$ of size d and given a plane V', not containing the point n', and such that V' intersects $\mathscr{K}$ in a conic disjoint from the elements of $\mathscr{F}$. Then there is a unique additive partial flock of size $2d$ containing the conics determined by V' and the d planes defining $\mathscr{F}$.*

Proof. The proof is analogous to the proof of Theorem 5.4.1. □

Using Lemma 5.3.2 and the equation of the singular planes we can deduce some properties concerning the Denniston lines.

Lemma 5.4.4. *Given a degree-$2d$ maximal arc M of Mathon type that contains a degree-d maximal arc D of Denniston type. Then all Denniston lines of M are concurrent.*

Proof. After projection from the point $n(1,0,1,0)$ the maximal arc D gives rise to a linear partial flock on the cone $\mathscr{K}$. In other words, all the planes inducing this partial flock intersect in a common line L. Using Theorem 5.4.3 we can choose a suitable plane V to construct the partial flock of size $2d$ that corresponds to the degree-$2d$ maximal arc M. However, as was noticed in Remark 5.4.2, this plane V cannot contain the common line L and hence V must intersect L in a point p in $\mathrm{PG}(3, q)$. Furthermore, using Lemma 5.3.2, since all Denniston planes actually are linear combinations of V and the conic planes in the linear partial flock corresponding to D, it is clear that p will be contained in all Denniston planes. Finally, after projection from n on the plane $X_0 = 0$, we see that all the Denniston lines must be concurrent as they all contain the projection of p. □

Remark 5.4.5. Note that the above lemma provides an alternative proof for the fact that the Denniston lines of a degree-8 maximal arc of proper Mathon type are concurrent. (See Chapter 3.)

Another property regarding the Denniston lines concerns the coefficients α and β in the equation of the conics given by (5.4).

Lemma 5.4.6. *The Denniston lines of a maximal arc of Mathon type are concurrent if the coefficient α or β is a constant.*

Proof. Suppose that $\alpha \in \mathrm{GF}(q)$ is a constant in the equation of the conics contained in a maximal arc of Mathon type as given in (5.4). In this case let $V : \lambda X_0 + \alpha X_1 + (\lambda+1)X_2 + \beta X_3 = 0$ and $W : \lambda' X_0 + \alpha X_1 + (\lambda'+1)X_2 + \beta' X_3 = 0$ be two random conic planes. Using Lemma 5.3.2 we know that the singular plane induced by V and W has equation

$$(\lambda + \lambda')X_0 + (\lambda + \lambda')X_2 + (\beta + \beta')X_3 = 0.$$

It is clear that the point $(0, 1, 0, 0)$ is always contained in this plane. This implies that all Denniston lines are concurrent. An analogous argument holds if β is a constant. □

5.5 Additive group

Consider an additive group G of order $2d$. In this section we will discuss how, under certain circumstances, it is possible to construct a degree-$2d$ maximal arc M of Mathon type (having G as its related additive group), and containing a degree-d maximal arc of Denniston type. Consider the plane $X_0 = 0$ and let $G := \{0, 1, \lambda_1, \lambda_2, \ldots, \lambda_{2d-2}\}$. Now we take an additive subgroup $H := \{0, 1, \lambda_1, \ldots, \lambda_{d-2}\}$ of order d of G and we choose the line $X_2 = 0$ as a line at infinity. The elements of H give rise to a degree-d maximal arc D of Denniston type in the standard pencil determined by the nucleus $(0, 1, 0)$, the line $X_2 = 0$ at infinity and the conic $X_1^2 + X_1X_3 + X_3^2 + X_2^2 = 0$ induced by the element 1 in H. In other words, D consists of the conics

$$C_{\lambda^2} : X_1^2 + X_1X_3 + X_3^2 + \lambda^2 X_2^2 = 0,$$

with $\lambda^2 = 1, \lambda_1^2, \ldots, \lambda_{d-2}^2$. Every conic C_{λ^2} has nucleus $(0, 1, 0)$ and the line at infinity of D is the line $X_2 = 0$.

Next, we choose an element in G that is not contained in H, say λ_d. It is clear that $H \cup \{\lambda_d\}$ generates G. Because we are trying to construct a degree-$2d$ maximal arc M of Mathon type that contains D we need, using Theorem 3.1.2, a conic $C : \alpha^2 X_1^2 + X_1X_3 + \beta^2 X_3^2 + \lambda_d^2 X_2^2 = 0$, with $\alpha, \beta \in \mathrm{GF}(q)$, disjoint from D on the same nucleus $(0, 1, 0)$. However, since M contains the degree-d maximal arc D we can assume without loss of generality, using Lemma 5.4.4 and Lemma 5.4.6, that $\alpha = 1$. It follows that if we can find a suitable element β we will be able to construct the entire maximal arc M.

We know that the two conics C_1 and C uniquely determine a third conic $C_1 \oplus C$ in order to form a degree-4 maximal arc of Denniston type. This 4-arc has a unique line at infinity L, which is also uniquely determined by C_1 and C (use for instance Lemma 5.3.2), moreover, C_1 and L induce the conic C. This implies that it suffices to determine L in order to find C and thus M.

Since $\alpha = 1$ we can assume that L has an equation of the form

$$\rho X_2 + X_3 = 0, \quad X_0 = 0,$$

$\rho \in \mathrm{GF}(q)$. The singular plane S associated to L has an equation of the form $AX_0 + \rho X_2 + X_3 = 0$. As we know that this plane has to contain the point $n(1, 0, 1, 0)$ we find that $A = \rho$. Hence S has equation

$$\rho X_0 + \rho X_2 + X_3 = 0. \tag{5.11}$$

Furthermore, the conic plane that determines C_1 has equation $X_0+X_1+X_3=0$ and the conic plane that determines C has equation $\lambda_d X_0+X_1+(\lambda_d+1)X_2+\beta X_3=0$. Since these two equations are standard, their sum also provides us with the equation of the associated singular plane S. We find that S must have the equation $(\lambda_d+1)X_0+(\lambda_d+1)X_2+(\beta+1)X_3=0$, or equivalently

$$\frac{\lambda_d+1}{\beta+1}X_0+\frac{\lambda_d+1}{\beta+1}X_2+X_3=0. \tag{5.12}$$

From (5.11) and (5.12) we see that $\rho=(\lambda_d+1)/(\beta+1)$, or equivalently

$$\beta=\frac{\lambda_d+1}{\rho}+1. \tag{5.13}$$

Next, we need to express that the conic C, with β as given in (5.13), is disjoint from the maximal arc D. The elements ρ that satisfy these conditions of disjointness will provide us with the proper elements β that induce C and thus the maximal arc M.

The conic C is given by the equation

$$X_1^2+X_1X_3+\Big(\frac{\lambda_d+1}{\rho}+1\Big)^2X_3^2+\lambda_d^2X_2^2=0$$

and has to be disjoint from the conics $C_{\lambda^2}: X_1^2+X_1X_3+X_3^2+\lambda^2X_2^2=0$, with $\lambda^2=1,\lambda_1^2,\ldots,\lambda_{d-2}^2$ that form the degree-d maximal arc D. Suppose that the point $p(x_1,x_2,x_3)$ in $X_0=0$ is a point of C_{λ^2}. In that case we know that $x_1^2+x_1x_3+x_3^2+\lambda^2x_2^2=0$ holds. If p is also contained in C we find that

$$x_1^2+x_1x_3+\Big(\frac{(\lambda_d+1)^2}{\rho^2}+1\Big)x_3^2+(\lambda_d^2+\lambda^2)x_2^2+\lambda^2x_2^2=0,$$

which is now of course equivalent to

$$\frac{\lambda_d+1}{\rho}x_3+(\lambda_d+\lambda)x_2=0.$$

It follows that

$$x_2=\frac{(\lambda_d+1)}{\rho(\lambda_d+\lambda)}x_3.$$

If we now substitute this in the equation of the conic C_{λ^2} we find

$$X_1^2+X_1X_3+X_3^2+\lambda^2\frac{(\lambda_d+1)^2}{\rho^2(\lambda_d+\lambda)^2}X_3^2=0.$$

Since we want the conics C_{λ^2} and C to be disjoint we want the quadratic equation

$$x^2 + x + \left(1 + \lambda^2 \frac{(\lambda_d + 1)^2}{\rho^2(\lambda_d + \lambda)^2}\right) = 0$$

to have no solutions over GF(q). This will be the case if and only if

$$\mathrm{Tr}\left[1 + \lambda^2 \frac{(\lambda_d + 1)^2}{\rho^2(\lambda_d + \lambda)^2}\right] = 1. \tag{5.14}$$

Distinguishing the cases $q = 2^h$, h odd and h even, we can simplify condition (5.14) further. If h is odd we know that $\mathrm{Tr}[1] = 1$ and condition (5.14) is equivalent to

$$\mathrm{Tr}\left[\frac{\lambda(\lambda_d + 1)}{\rho(\lambda_d + \lambda)}\right] = 0. \tag{5.15}$$

On the other hand, if h is even then $\mathrm{Tr}[1] = 0$ and we analogously find

$$\mathrm{Tr}\left[\frac{\lambda(\lambda_d + 1)}{\rho(\lambda_d + \lambda)}\right] = 1. \tag{5.16}$$

We conclude that all elements ρ that satisfy condition (5.14) give rise to a suitable element β as given in (5.13). Substituting this β in the equation $X_1^2 + X_1X_3 + \beta^2X_3^2 + \lambda_d^2X_2^2 = 0$, where we assumed $\alpha = 1$ as seen above, gives us a conic C which is disjoint from the degree-d maximal arc D and therefore induces a degree-$2d$ maximal arc of Mathon type where the coefficients of the term X_2^2 are the squares of the elements in $G \setminus \{0\}$.

Hence, as soon as the above system of trace conditions has a non-trivial solution we can construct a proper maximal degree-$2d$ arc of Mathon type, containing a degree-d maximal arc of Denniston type. In a worst case scenario all the trace conditions could be linearly independent (over GF(2)). In such case, with $q = 2^h$ we are guaranteed of the existence of a Mathon maximal arc of degree $2^{\lfloor \log_2(h) \rfloor + 1}$ having the prescribed additive group, containing a maximal arc of Denniston type of degree $2^{\lfloor \log_2(h) \rfloor}$. So in general one should be able to analyze the linear (in)dependence of the trace conditions. Though we do not believe that in general they are all independent, the analysis of dependence seems to be a hard problem, and an interesting topic for future research.

Appendix A

Isomorphism between $\mathrm{AS}(q)$ and the Payne derivation of $\mathcal{W}(q)$

The two generalized quadrangles mentioned in the title were considered in Chapter 1 (Section 1.7). We already brought up that $\mathrm{AS}(q)$ is actually a Payne-derived GQ, more precisely, $\mathrm{AS}(q) \cong \mathcal{P}(\mathcal{W}(q), x)$, with q odd. In this appendix we present an actual map between the two incidence geometries. The reason we were so interested in describing an actual map is rooted in a (failed) attempt to construct new partial geometries. Let us first explain this in some more detail. Let $\mathcal{K}$ be a maximal arc of Mathon type in $\mathrm{PG}(2, q)$, q even. Then one can construct the partial geometry $T_2^*(\mathcal{K})$. Now, since $\mathcal{K}$ is a union of conics on the same nucleus, it is obvious that $T_2^*(\mathcal{K})$ is the union of GQs $T_2^*(\mathcal{O})$ sharing (only) a spread of symmetry (see below). Of course if q is odd there are no (non-trivial) maximal arcs in $\mathrm{PG}(2, q)$, but nevertheless there is a GQ, $\mathrm{AS}(q)$, with parameters $(q-1, q+1)$, that is, the parameters that a GQ arising from a hyperoval would have. Furthermore, $\mathrm{AS}(q)$, has a unique spread of symmetry. It was our aim to see if it is possible to construct, for odd q, partial geometries with the parameters of a $T_2^*(\mathcal{K})$ by taking unions of several $\mathrm{AS}(q)$ sharing (only) their spread of symmetry. As in $\mathcal{P}(\mathcal{W}(q), x)$ lines are lines of $\mathrm{PG}(3, q)$, which are easier to deal with than the original curves describing $\mathrm{AS}(q)$, we believed it was useful to have an easy algebraic tool to go from $\mathrm{AS}(q)$ to $\mathcal{P}(\mathcal{W}(q), x)$ and back. That is the main reason for having constructed the explicit isomorphism described in this appendix. Unfortunately, it never has been useful in constructing new partial geometries.

A.1 A closer look at both geometries

Before we introduce the above mentioned map we should have a closer look at both generalized quadrangles.

- The generalized quadrangle of Ahrens and Szekeres AS(q).

  Recall the definition of the generalized quadrangle $\mathrm{AS}(q) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, q an odd prime, of order $(q-1, q+1)$ as seen in Chapter 1. For the convenience of the reader we restate the elements of $\mathcal{B}$, which are the following curves of $\mathrm{AG}(3, q)$:

$$\begin{array}{rl} (i) & x = \sigma,\ y = a,\ z = b \\ (ii) & x = a,\ y = \sigma,\ z = b \\ (iii) & x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma. \end{array}$$

  Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence I is the natural one.

  It is clear that $|\mathcal{P}| = q^3$, that $|\mathcal{B}| = q^2(q+2)$, and that each element of $\mathcal{B}$ is incident with q elements of $\mathcal{P}$. For each value of c there are q^2 curves of type (iii), and these curves have no point in common. For suppose the curves corresponding to (a, b, c) and (a', b', c) intersect. Then for some σ we have $c\sigma^2 - b\sigma + a = c\sigma^2 - b'\sigma + a'$ and $-2c\sigma + b' = -2c\sigma + b$. This clearly implies $b = b'$ and $a = a'$. Similarly, no two curves of type (i) (or type (ii)) intersect.

  Thus we have $q+2$ families of disjoint curves, q^2 curves in each family and q points on each curve, and each point of $\mathcal{P}$ is incident with exactly $q+2$ elements of $\mathcal{B}$, one from each family.

  Let $\pi_\infty : u = 0$ be the plane at infinity of $\mathrm{AG}(3, q)$. Now consider the curves of type (i). The equations can be written as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} + \sigma \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

  It is clear that these curves are lines and that they all have the same projective point $(1, 0, 0, 0)$ at infinity.

Next, let $S = \{L_1, ..., L_{1+st}\}$ be a spread of a generalized quadrangle Q of order (s, t) and let G_S be the group of automorphisms of Q that fixes each line of S.

In order to continue we need the definition of a spread of symmetry.

Definition A.1.1. A *spread of symmetry* is a spread S satisfying the following property: for every $K, L \in S$ and every two lines M and N meeting K and L, there exists an automorphism $\theta \in G_S$ such that $M^\theta = N$. If Q is a grid, then S is a spread of symmetry if and only if $|G_S| = s+1$.

The following Lemma shows us that there is a spread of symmetry in $\mathrm{AS}(q)$ and where it is situated.

Lemma A.1.2. *The lines of type* (i) *of* $\mathrm{AS}(q)$ *form the spread of symmetry.*

Proof. We need to find an automorphism group of the GQ that fixes the lines of type (i) linewise and acts regularly on the points of these lines. Consider the affine maps $\theta_k : x' = x + k,\ y' = y,\ z' = z,\ k \in \mathrm{GF}(q)$. These q maps clearly form a group of automorphisms of $\mathrm{AS}(q)$ acting in the desired way. □

We move on by having a look at the curves of type (iii) in the case $c = 0$. This gives us the following equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} -b \\ 0 \\ 1 \end{pmatrix}.$$

It is clear that the q^2 lines given by these equations determine q points $(-b, 0, 1, 0), b \in \mathrm{GF}(q)$, in the plane at infinity π_∞. Together with the point $(1, 0, 0, 0)$ these q points form a line L in π_∞. We obtain the following situation: through every point $(-b, 0, 1, 0)$ on L there are q lines of $\mathrm{AS}(q)$ lying in a plane. This plane has L as its line at infinity and is completely determined by the elements b and $c(= 0)$ of $\mathrm{GF}(q)$. The q values for a then determine each of the q lines.

In the general case $(c \neq 0)$ the equations of the curves of type (iii) are equivalent to

$$\begin{cases} x = cz^2 - bz + a \\ y = -2cz + b\,. \end{cases} \tag{A.1}$$

This system of equations gives rise to a conic in a plane of AG$(3, q)$. Remark that $(1, 0, 0, 0)$ is the point at infinity of each of these conics.

Given any point $p = (x, y, z)$ in AG$(3, q)$ we can always choose a curve of type (iii) through p such that the coordinates of p can be written in the following way:

$$\begin{aligned} x &= c_p z^2 - b_p z + a_p \\ y &= -2c_p z + b_p, \end{aligned}$$

with $a_p, b_p, c_p \neq 0 \in \mathrm{GF}(q)$ the elements that determine the chosen conic, i.e., the curve of type (iii). Note that a_p, b_p and c_p are not uniquely determined by p.

- The Payne derived GQ.

  Using the notation from Chapter 1 we have that $\mathcal{S}' = \mathcal{P}(\mathcal{S}, x) = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a GQ of order $(q-1, q+1)$, which is called the Payne-derived GQ of $\mathcal{S}$ with respect to x. In what follows we will assume $\mathcal{S}$ to be the generalized quadrangle $\mathcal{W}(q)$ in its natural embedding in PG$(3, q)$ with the associated symplectic polarity ρ determined by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

  We know that each point of $\mathcal{W}(q)$ is regular and so we will consider $\mathcal{P}(\mathcal{W}(q), (1, 0, 0, 0)) := \overline{\mathcal{P}}$, the Payne derived of $\mathcal{W}(q)$ with respect to the point $(1, 0, 0, 0)$.

  Remark A.1.3. Since we consider the Payne derived $\overline{\mathcal{P}}$ it is clear that in this case the spread of symmetry is the spread consisting of the lines through the point $(1, 0, 0, 0)$.

A.2 An actual map between AS(q) and $\overline{\mathcal{P}}$

Finally, we present a map from AS(q) to $\overline{\mathcal{P}}$, q odd, and prove that it is indeed an isomorphism between the two geometries.

Theorem A.2.1. *Let $p = (x, y, z)$ be a point of* AG(3, q)*, with $x = c_pz^2 - b_pz + a_p, y = -2c_pz + b_p, c_p \neq 0$. Then the map $\alpha : p \mapsto p^\alpha = (x - c_pz^2 + b_pz/2, y, z/2)$ is an isomorphism between* AS(q) *and* $\overline{\mathcal{P}}$.

Proof. First of all remark that the coordinates (x, y, z) of the point p are written as given in (A.1). It is clear that p is a certain point of some conic in AG(3, q) which is a line of type (iii) in AS(q). To prove that the map α is well-defined we need to show that it does not depend on the conic we chose to represent p. Suppose that p can also be written as follows.

$$\begin{aligned} x &= \overline{c_p}z'^2 - \overline{b_p}z' + \overline{a_p} \\ y &= -2\overline{c_p}z' + \overline{b_p} \\ z &= z' \end{aligned}$$

with $\overline{a_p}, \overline{b_p}, \overline{c_p} \in \mathrm{GF}(q)$. Clearly $z = z'$. It immediately follows that $-2c_pz + b_p = -2\overline{c_p}z + \overline{b_p}$. If we multiply this last equation by z and then divide it by 2 we obtain

$$-c_pz^2 + \frac{b_pz}{2} = -\overline{c_p}z^2 + \frac{\overline{b_p}z}{2}.$$

This proves that the map α is well-defined.

Now suppose that $(x, y, z)^\alpha = (x', y', z')^\alpha$ or equivalently that $(x - c_pz^2 + b_pz/2, y, z/2) = (x' - \overline{c_p}z'^2 + \overline{b_p}z'/2, y', z'/2)$. It follows immediately that $z = z', y = y'$ and so $-2c_pz + b_p = -2\overline{c_p}z + \overline{b_p}$. Using the same arguments we did above, we find $x = x'$. This implies that α is injective, hence bijective.

It remains to be shown that α preserves collinearity. Notice that, in order to prove that α is an isomorphism, it is indeed sufficient to show that α preserves collinearity. This follows from the fact that three distinct points on a line in that case have to be mapped onto three distinct points on a line, since a GQ does not contain triangles.

We start by considering a curve of type (i). Let (σ_1, a, b) and (σ_2, a, b) determine two points on such a line. The images of these points under the map α are $(\sigma_1 - c_pb^2 + b_pb/2, a, b/2)$ and $(\sigma_2 - c'_pb^2 + b'_pb/2, a, b/2)$ respectively, with $b_p, b'_p, c_p, c'_p \in \mathrm{GF}(q)$. The line of AG(3, q) containing these two points can be written in the following way:

$$(\sigma_1 - c_pb^2 + b_pb/2, a, b/2) + t(\sigma_2 - \sigma_1 - b^2(c'_p - c_p) + b/2(b'_p - b_p), 0, 0),$$

in which t ranges over GF(q). It is clear that the point at infinity of this line is $(1,0,0,0)$. This proves that α maps the spread of symmetry of AS(q) to the spread of symmetry of the Payne-derived GQ $\overline{\mathcal{P}}$.

Next we need to show that α maps a curve of type (iii) to an absolute line with respect to the symplectic polarity ρ. Consider two points $p_1 = (c_p\sigma_1^2 - b_p\sigma_1 + a_p, -2c_p\sigma_1 + b_p, \sigma_1)$ and $p_2 = (c_p\sigma_2^2 - b_p\sigma_2 + a_p, -2c_p\sigma_2 + b_p, \sigma_2)$ on such a curve, with $a_p, b_p, c_p \in \mathrm{GF}(q)$. It suffices to show that p_2^α lies in $(p_1^\alpha)^\rho$. Since $p_2^\alpha = (-b_p\sigma_2/2 + a_p, -2c_p\sigma_2 + b_p, \sigma_2/2)$ and the affine part of $(p_1^\alpha)^\rho$ is the hyperplane in AG($3,q$) with equation

$$-X + \frac{\sigma_1}{2}Y + (2c_p\sigma_1 - b_p)Z - \frac{b_p\sigma_1}{2} + a_p = 0$$

it is easy to see that the condition holds.

Finally we will prove that also a curve of type (ii) is mapped onto an absolute line. Consider two distinct points p_1 and p_2 on such a curve. Then there are $a, b \in \mathrm{GF}(q)$ and $\sigma, \sigma' \in \mathrm{GF}(q)$ such that $p_1 = (a, \sigma, b)$ and $p_2 = (a, \sigma', b)$. Now it is clear that in fact $(x, y, z)^\alpha = (x + \frac{yz}{2}, y, \frac{z}{2})$. Hence $p_2^\alpha = (a + \frac{\sigma' b}{2}, \sigma', \frac{b}{2})$, and the affine part of $(p_1^\alpha)^\rho$ is the hyperplane in AG($3,q$) with equation

$$-X + \frac{b}{2}Y - \sigma Z + a + \frac{\sigma b}{2} = 0.$$

It is now clear that $p_2^\alpha \in (p_1^\alpha)^\rho$. This proves that the line $\langle p_1^\alpha, p_2^\alpha \rangle$ is absolute, concluding our proof. □

Appendix B

Nederlandstalige samenvatting

In deze Nederlandstalige samenvatting zullen we de resultaten uit deze thesis bondig op een rijtje zetten. Hierbij is het niet de bedoeling om in detail te treden, bijgevolg zullen in deze samenvatting geen bewijzen worden opgenomen. Indien meer informatie gewenst is, verwijzen we graag naar de Engelstalige tekst, waarvan we hier de structuur zullen aanhouden.

B.1 Inleiding

In dit inleidende hoofdstuk wordt het algemeen wiskundig kader geschetst waarin deze thesis zich bevindt. Enkele fundamentele, meetkundige begrippen worden opgefrist en de notaties worden vastgelegd. We gaan ervan uit dat de lezer voldoende vertrouwd is met de basisbegrippen uit de projectieve meetkunde en verwijzen voor een vollediger inleiding naar Chapter 1 van de Engelstalige tekst. Bij wijze van inleiding herhalen we hier de definitie van een maximale boog, het centrale thema van deze verhandeling.

Definitie B.1.1. Een $\{k;d\}$-boog $\mathcal{K}$ in een eindig projectief vlak van de orde q is een niet-ledige deelverzameling van k punten zodat een rechte van het vlak de verzameling $\mathcal{K}$ snijdt in d punten, maar geen enkele rechte $\mathcal{K}$ snijdt in meer dan d punten. Gegeven q en d dan is k nooit groter dan $q(d-1)+d$. Wanneer hier de gelijkheid geldt noemt men de verzameling $\mathcal{K}$ een *maximale boog van graad* d, een $\{q(d-1)+d;d\}$*-boog* of korter nog, een d*-boog.*

Een equivalente definitie van zo'n maximale boog $\mathcal{K}$ is een niet-ledige verzameling punten in het vlak, zodanig dat elke rechte $\mathcal{K}$ snijdt in 0 of in d punten. Een rechte die $\mathcal{K}$ snijdt wordt een *secant* genoemd terwijl een rechte die disjunct is aan $\mathcal{K}$ een *externe* rechte wordt genoemd.

Het is duidelijk dat zowel een punt als een affien vlak voorbeelden zijn van maximale bogen. Deze twee voorbeelden zullen we echter triviaal noemen.

Lemma B.1.1. *Als $\mathcal{K}$ een $\{q(d-1)+d; d\}$-boog is in een projectief vlak π van de orde q, dan vormen de externe rechten aan $\mathcal{K}$ een $\{q(q-d+1)/d; q/d\}$-boog in het duale vlak.*

Bijgevolg is een nodige voorwaarde voor het bestaan van een $\{q(d-1)+d; d\}$-boog in een projectief vlak van de orde q dat d een deler is van q. Denniston [20] toonde aan dat deze voorwaarde voldoende is in het Desarguesiaanse projectieve vlak $\mathrm{PG}(2,q)$, met q even (Chapter 2).

Merk op dat, indien π een Desarguesiaans vlak is van orde q dat een maximale boog $\mathcal{K}$ van graad d bevat, π ook een maximale boog van graad q/d bevat. Deze boog heten we de *duale maximale boog* van $\mathcal{K}$.

Ball, Blokhuis en Mazzocca gebruikten polynomiale methoden om het volgende, meer dan 25 jaar oude vermoeden, te bewijzen.

Stelling B.1.2 ([5]). *Er bestaan geen niet-triviale maximale bogen in het projectieve vlak $\mathrm{PG}(2,q)$ wanneer q oneven is.*

B.2 Gekende constructies van maximale bogen

Het tweede hoofdstuk is volledig gewijd aan de reeds gekende constructies van maximale bogen alsook aan een aantal karakterisaties daaromtrent. In 1969 gebruikte Denniston [20] een speciale waaier van kegelsneden om op die manier een maximale boog te construeren in de Desarguesiaanse vlakken van even orde. Vijf jaar later was het J. A. Thas [51] die, d.m.v ovoïden en spreads in veralgemeende vierhoeken $\mathcal{W}(q)$, maximale bogen van graad q vormde in vlakken van orde q^2. In 1980 was het opnieuw J. A. Thas [53] die een nieuwe maximale boog van graad q^{t-1} introduceerde in symplectische translatievlakken van de orde q^t. Hij deed dit aan de hand van kwadrieken en spreads in projectieve ruimten. Tenslotte definieerde Mathon [38] in 2001 een compositie op een verzameling kegelsneden met gemeenschappelijke kern. Op basis hiervan ontstaan maximale bogen in $PG(2, q)$ die, zoals onder meer bewezen werd door Hamilton en Mathon [29], nieuwe families van maximale bogen induceren.

Eigenlijk zijn de maximale bogen die geconstrueerd werden door Mathon een veralgemening van de maximale bogen van Denniston. In de Engelstalige versie hebben we, tijdens het voorstellen van de verschillende constructies, de chronologische volgorde aangehouden. In deze beknopte samenvatting zullen we vertrekken van de constructie van Mathon en vervolgens aanduiden hoe de maximale bogen van Denniston terug te vinden zijn in deze veralgemening. Bovendien laten we hier de twee constructies van J. A. Thas buiten beschouwing gezien het vervolg van deze thesis vooral gebasseerd is op de maximale bogen van Mathon (en Denniston).

Maximale bogen van Mathontype

We veronderstellen vanaf nu telkens dat $q = 2^h$. In $PG(2, q)$, gerepresenteerd door homogene coördinaten over $GF(q)$, stellen we de punten voor door (a, b, c) en de rechten door $[u, v, w]$. Een punt (a, b, c) is incident met een rechte $[u, v, w]$ als en slechts als $au + bv + cw = 0$. Verder zullen we de gebruikelijke, absolute trace-afbeelding van $GF(q)$ naar $GF(2)$ noteren als Tr, zodat, voor $x \in GF(q)$,

$$\mathrm{Tr}(x) = x + x^2 + x^{2^2} + \cdots + x^{2^{h-1}}.$$

Voor $\alpha, \beta \in GF(q)$ waarvoor geldt dat $\mathrm{Tr}(\alpha\beta) = 1$, en voor $\lambda \in GF(q)$ definiëren we $F_{\alpha,\beta,\lambda}$ als de kegelsnede

$$F_{\alpha,\beta,\lambda} = \{(x, y, z) : \alpha x^2 + xy + \beta y^2 + \lambda z^2 = 0\}.$$

Merk op dat de tracevoorwaarde $\mathrm{Tr}(\alpha\beta) = 1$ equivalent is met de eis dat het kwadratisch polynoom $\alpha\xi^2 + \xi + \beta$ irreducibel is over $GF(q)$. De verzameling van alle dergelijke kegelsneden zullen we voortaan voorstellen door $\mathcal{F}$. Het is duidelijk dat alle kegelsneden uit $\mathcal{F}$ het punt $F_{\alpha,\beta,0} := F_0(0, 0, 1)$ als kern bevatten en dat, wegens de tracevoorwaarde, de rechte $z = 0$ extern is t.o.v. alle kegelsneden. Deze rechte wordt bijgevolg de *rechte op oneindig* genoemd. Alle overige kegelsneden zijn niet-ontaard.

Voor gegeven $\lambda \neq \lambda'$ definiëren we de *compositie*

$$F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'} = F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda\oplus\lambda'}$$

waarbij de operator $\oplus$ als volgt gedefiniëerd is:

$$\alpha \oplus \alpha' = \frac{\alpha\lambda + \alpha'\lambda'}{\lambda + \lambda'}, \quad \beta + \beta' = \frac{\beta\lambda + \beta'\lambda'}{\lambda + \lambda'}, \quad \lambda \oplus \lambda' = \lambda + \lambda'. \tag{B.1}$$

Lemma B.2.1 ([38]). *Twee niet-ontaarde kegelsneden $F_{\alpha,\beta,\lambda}, F_{\alpha',\beta',\lambda'}, \lambda \neq \lambda'$ en hun compositie $F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'}$ zijn onderling disjunct als*

$$\mathrm{Tr}((\alpha \oplus \alpha')(\beta \oplus \beta')) = 1.$$

Een deelverzameling $\mathcal{C} \subset \mathcal{F}$ wordt *gesloten* genoemd als voor elke $F_{\alpha,\beta,\lambda} \neq F_{\alpha',\beta',\lambda'} \in \mathcal{C}$, geldt dat $F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda\oplus\lambda'} \in \mathcal{C}$.

In Mathon zijn volgende resultaat ([38]) zien we hoe gesloten verzamelingen van kegelsneden kunnen dienen om maximale bogen te vormen. Een dergelijke maximale boog wordt een *maximale boog van Mathontype* genoemd, of korter, een *Mathon maximale boog.*

Stelling B.2.1 ([38]). *Veronderstel dat $\mathcal{C} \subset \mathcal{F}$ een gesloten verzameling is van $2^d - 1$ kegelsneden in $\mathrm{PG}(2, 2^m)$, $1 \leq d \leq m$. De unie van de punten op de kegelsneden van $\mathcal{C}$ en hun gemeenschappelijke kern F_0 vormen een maximale $\{2^{m+d} - d^m + 2^d; 2^d\}$-boog $\mathcal{K}$ in $\mathrm{PG}(2, 2^m)$.*

Zoals al aangegeven werd in het begin van deze sectie kan de constructie van Mathon gezien worden als een veralgemening van de constructie van Denniston. Beschouw daartoe de volgende setting.

Kies een $\alpha \in \mathrm{GF}(q)$ waarvoor $\mathrm{Tr}(\alpha) = 1$. Veronderstel dat A een deelverzameling is van $\mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}$ zodanig dat $A \cup \{0\}$ gesloten is onder de optelling. In dit geval zal de puntenverzameling van kegelsneden

$$\mathcal{K}_A = \{F_{\alpha,1,\lambda} : \lambda \in A\}$$

samen met de gemeenschappelijke kern F_0 de puntenverzameling zijn van een maximale boog van graad $|A| + 1$ in $\mathrm{PG}(2, q)$. Deze constructie is precies de definitie van een *maximale boog van Dennistontype*, of korter, *Denniston maximale boog.* De kegelsneden in $\mathcal{K}_A$ zijn een deelverzameling van de *standaard kegelsnedenwaaier*, meer bepaald, de verzameling

$$\{F_{\alpha,1,\lambda} : \lambda \in \mathrm{GF}(q)\}.$$

Deze waaier vormt een partitie onder de punten van het vlak, disjunct van de rechte $z = 0$, bestaande uit $q - 1$ disjuncte kegelsneden op de gemeenschappelijke kern $F_0(0, 0, 1)$. De rechte $z = 0$, disjunct aan alle kegelsneden, zullen we de *rechte op oneindig* noemen van de Denniston maximale boog. Dit is dus eigenlijk de rechte op oneindig van de waaier die de Dennistonboog bevat.

Opmerking B.2.2. Vermits we zowel de rechte op oneindig van een Denniston maximale boog als de rechte op oneindig van een Mathon maximale boog geïntroduceerd hebben bestaat de kans dat er wat verwarring optreedt i.v.m. deze term. Zoals verder ook zal blijken bevatten Mathon maximale bogen verschillende Denniston maximale bogen. Wanneer we dan de rechte op oneindig beschouwen van een Denniston maximale boog, als deelboog van een Mathon maximale boog, zullen we deze rechte de *Dennistonrechte* noemen van de beschouwde Dennistonboog. De externe rechte van de Mathonboog blijft simpelweg de "rechte op oneindig".

Een gesloten verzameling kegelsneden $\mathcal{C} = \{F_{\alpha,\beta,\lambda}\}$ kan ook beschreven worden waarbij de parameters α en β veeltermen zijn in λ. We kunnen dus zo een gesloten verzameling beschrijven aan de hand van functies $p : A \to \mathrm{GF}(q)$ en $r : A \to \mathrm{GF}(q)$, waarbij λ de waarden uit de verzameling A doorloopt. De vergelijkingen die dan de gesloten verzameling $\mathcal{C}$ bepalen worden dan gegeven door

$$\{p(\lambda)x^2 + xy + r(\lambda)y^2 + \lambda z^2 = 0 : \lambda \in A\}. \qquad \text{(B.2)}$$

Uit de resultaten in [38] volgt dat een maximale boog van Mathontype van graad d Mathon deelbogen bevat van graad d' voor elke deler d' van d. Bovendien blijkt dat elke Mathonboog van graad 4 een Dennistonboog is. In hetzelfde artikel gebruikt Mathon zijn constructie ook om een aantal nieuwe, oneindige families van maximale bogen in $\mathrm{PG}(2, q)$ voor te stellen.

Er zijn verscheidene families van Mathonbogen gekend die niet van het Dennistontype zijn. Een Mathon maximale boog die niet van Dennistontype is zullen we voortaan vaak een *ware Mathonboog* noemen. Het is echter nogal omslachtig om, tijdens het nagaan of een gegeven deelverzameling kegelsneden van $\mathcal{F}$ een maximale boog is, telkens opnieuw de tracevoorwaarde uit Lemma B.2.1 te controleren. In de volgende sectie zullen we een meer meetkundige interpretatie geven van de maximale bogen van Mathontype die ons zal helpen bij dit probleem.

Een uiterst handig instrument hierbij is het volgende lemma van Aguglia, Giuzzi en Korchmáros (zie [2]). Zij toonden aan dat twee kegelsneden met een gemeenschappelijke kern op unieke wijze uitbreidbaar zijn.

Lemma B.2.2. *Gegeven twee kegelsneden* C_1 *en* C_2 *met een gemeenschappelijke kern, er bestaat dan een unieke Denniston maximale boog van graad* 4 *die* $C_1 \cup C_2$ *bevat.*

Bovendien konden we van bovenstaand lemma gebruik maken om de volgende stelling te bewijzen.

Stelling B.2.3. *Een maximale boog* $\mathcal{K}$*, bestaande uit disjuncte kegelsneden met een gemeenschappelijk kern, is altijd van Mathontype.*

B.3 Meetkundige interpretatie van Mathon maximale bogen

Zoals gezegd proberen we in deze sectie een eerder meetkundige interpretatie te geven van de Mathonbogen. Deze nieuwe aanpak zal het ook mogelijk maken om het aantal niet-isomorfe Mathon 8-bogen te tellen in $\mathrm{PG}(2, 2^h), h > 4$ en $h \neq 7$ priem.

Het tellen van Mathonbogen werd reeds bestudeerd in [30], waar grenzen werden berekend voor het aantal isomorfieklassen van Mathonbogen van "hoge" graad. De gebruikte technieken faalden echter voor bogen van kleinere graad en het tellen van dergelijke bogen werd als een open probleem gepostuleerd.

De resultaten uit deze sectie zijn terug te vinden in [17].

B.3.1 Een synthetische constructie van Mathonbogen

De meetkundige interpretatie waarvan sprake zal eigenlijk neerkomen op een soort veralgemening van Lemma B.2.2. Na aangetoond te hebben dat we, gegeven een Mathon maximale boog van graad d en een disjuncte kegelsnede C met dezelfde kern, steeds een externe rechte kunnen vinden die dienst zal doen als rechte op oneindig van de nieuwe boog, kunnen we de volgende stelling bewijzen.

Stelling B.3.1 (Synthetische versie van Mathons stelling). *Zij M een Mathon maximale boog van graad d, $d < q/2$, bestaande uit $d-1$ kegelsneden met een gemeenschappelijke kern n en zij C_d een kegelsnede disjunct van M met dezelfde kern n. Er bestaat dan een unieke Mathon maximale boog van graad $2d$ die $M \cup C_d$ bevat.*

Als $q = 2^p$, $p \neq 2, 3$ priem, is het mogelijk om aan te tonen dat de automorfismegroep G van een Denniston maximale boog van graad 4 orde $2(q+1)$ heeft en isomorf is met $C_{q+1} \rtimes C_2$. Deze groep G bevat een cyclische deelgroep van de orde $q+1$ die alle drie de kegelsneden fixeert en scherp transitief werkt op de punten van elk van die kegelsneden. Bovendien fixeert deze groep de rechte L op oneindig van de 4-boog en werkt die scherp transitief op de punten van L. De groep G bevat ook $q+1$ involuties. Deze zijn precies de $q+1$ elaties, met als as een rechte door de kern, en als centrum de doorsnede van deze rechte met de rechte L, die elk van de drie kegelsneden van de 4-boog fixeren. Elke rechte door de kern induceert precies één dergelijke involutie.

In het volgende lemma tellen we het aantal isomorfieklassen van Denniston maximale bogen van graad 4.

Lemma B.3.1. *Het aantal isomorfieklassen van Denniston maximale bogen van graad 4 in* $\mathrm{PG}(2, 2^{2h+1})$*, $2h+1$ priem en $2h+1 \neq 3$, is precies*

$$N = \frac{2^{2h}-1}{3(2h+1)}.$$

Lemma B.3.2. *Het aantal Denniston maximale bogen van graad 4 in de standaard kegelsnedenwaaier in* $\mathrm{PG}(2, 2^{2h+1})$*, $2h+1$ priem en $2h+1 \neq 3$, die isomorf zijn met een gegeven Denniston 4-boog en bovendien een gegeven kegelsnede C bevatten is gelijk aan* $3(2h+1)$*.*

B.3.2 Mathon maximale bogen van graad 8

Vooraleer we het telprobleem i.v.m. de niet-isomorfe 8-bogen aanpakken is het cruciaal om de meetkundige structuur van een Mathon maximale boog $\mathcal{K}$ van graad 8 te bestuderen. Wegens Lemma B.2.2 weten we dat elke Mathon 8-boog precies zeven Denniston 4-bogen bevat, en dat elke twee van deze 4-bogen precies één kegelsnede gemeen hebben. Meer nog, men kan inzien dat de meetkundige structuur met als puntenverzameling de kegelsneden van $\mathcal{K}$, als rechtenverzameling de 4-bogen van Dennistontype en de natuurlijke incidentie, isomorf is met $\mathrm{PG}(2,2)$.

Indien $\mathcal{K}$ van Dennistontype is vallen de zeven Dennistonrechten, corresponderend met de zeven deelbogen, samen. Indien $\mathcal{K}$ een ware Mathonboog is zijn er precies zeven verschillende Dennistonrechten en geldt de volgende stelling.

Lemma B.3.3. *Stel dat $\mathcal{K}$ een ware Mathon 8-boog is, dan zijn de zeven Dennistonrechten van $\mathcal{K}$ concurrent en bestaat er een unieke involutie die $\mathcal{K}$, en alle kegelsneden die $\mathcal{K}$ bevat, fixeert. Deze involutie is de elatie met als centrum de doorsnede van de Dennistonrechten en als as de rechte door de kern van $\mathcal{K}$ en het centrum.*

Gevolg B.3.2. *Veronderstel dat $\mathcal{K}$ een ware Mathon 8-boog is in $\mathrm{PG}(2,2^p)$, p priem en $p \neq 2,3,7$, dan is $\mathrm{Aut}(\mathcal{K}) \cong C_2$.*

Vermits we het aantal isomorfieklassen willen tellen van Mathon maximale bogen van graad 8 is het noodzakelijk om na te gaan hoeveel isomorfe beelden er zijn van een gegeven Mathon 8-boog. Het volgende lemma is van essentieel belang voor ons eindresultaat.

Lemma B.3.4. *Veronderstel dat $\mathcal{K}$ een ware Mathon 8-boog is in $\mathrm{PG}(2,2^{2h+1})$, $2h+1$ priem en $h \neq 1,3$, dan is het aantal maximale bogen van graad 8 die isomorf zijn met $\mathcal{K}$, die één van hun deelbogen van graad 4 in de standaardwaaier hebben en die hetzelfde punt hebben als doorsnede van de Dennistonrechten gelijk aan $21(2h+1)$.*

Gegeven een Denniston maximale boog $\mathcal{D}$ van graad 4 in de standaard kegelsnedenwaaier. Veronderstel dat $\mathcal{D}$ bestaat uit de kegelsneden C_1, C_k, C_{k+1}. Wegens Lemma B.2.2 weten we dat elke kegelsnede C, disjunct van $\mathcal{D}$, samen met de kegelsnede C_1 een andere Denniston 4-boog genereert die isomorf is met één van de Denniston maximale bogen van graad 4 uit de standaardwaaier. Wanneer we op zoek gaan naar de voorwaarden die verzekeren dat de kegelsnede C disjunct is van de 4-boog $\mathcal{D}$ vinden we twee tracevoorwaarden van de vorm

$$\mathrm{Tr}[A(\sigma)t + B(\sigma)t^2] = 0,$$

waarbij A en B functies zijn van het veldautomorfisme σ en waarbij $t \in \mathrm{GF}(2^{2h+1})$.

Tenslotte beschikken we over voldoende instrumenten om de telling van het aantal isomorfieklassen van Mathon 8-bogen aan te vatten.

Stelling B.3.3. *Het aantal isomorfieklassen van ware Mathon* 8*-bogen in het vlak* $\mathrm{PG}(2, 2^{2h+1})$*,* $2h+1 \neq 7$ *en* $2h+1$ *priem, is precies*

$$\frac{N}{14}(2^{2h-2}-1)((6h+3)N-1),$$

met $N = (2^{2h}-1)/3(2h+1)$*.*

Het geval $2h+1=7$ is vrij apart en wordt uitgebreid besproken in het vierde hoofdstuk van deze thesis (Chapter 4).

B.3.3 Maximale bogen in PG(2, 32)

In PG(2, 32) vond Mathon via computerberekeningen drie niet-isomorfe ware Mathon maximale bogen van graad 8. Hij vermoedde dan ook dat dit de enige ware Mathon 8-bogen zijn in dit vlak. Wegens het voorgaande kunnen wij nu bevestigen dat er inderdaad precies drie dergelijke maximale bogen te vinden zijn in PG(2, 32). Het is bovendien mogelijk om, steunend op bovenstaande theorie en gebruik makend van de specifieke tracevoorwaarden $\mathrm{Tr}[A(\sigma)t + B(\sigma)t^2] = 0$, de verschillende t-waarden te berekenen die aanleiding geven tot ware Mathon 8-bogen. Bijgevolg zijn we ook in staat om, zonder computerhulp, de vergelijkingen van de drie ware Mathonbogen op te stellen.

B.4 Singer 8-bogen van Mathontype

In deze sectie tonen we aan dat er in $\mathrm{PG}(2, 2^7)$ een bijzondere klasse van maximale 8-bogen van Mathontype opduiken die de actie van een scherp transitieve groep, of Singergroep, toelaat op de zeven kegelsneden van dergelijke 8-bogen. Deze worden in dit hoofdstuk in detail beschreven waardoor we uiteindelijk ook in staat zijn om het aantal niet-isomorfe Mathon 8-bogen te tellen. Het zal ook blijken dat deze speciale bogen, gevonden in $\mathrm{PG}(2, 2^7)$, uitbreidbaar zijn naar twee oneindige families van Mathon 8-bogen in $\mathrm{PG}(2, 2^k)$, k oneven en deelbaar door 7, die nog steeds de Singeractie toelaten op hun kegelsneden.

Deze resultaten zijn terug te vinden in [16].

B.4.1 Inleiding

Zoals in vorige sectie reeds aangegeven werd blijkt de telling uit Stelling B.3.3 niet te werken in het geval $2h + 1 = 7$. In dit geval bekomen we immers niet eens een geheel getal. De reden hiervoor is het falen van Gevolg B.3.2 in dit bijzonder geval. Het komt er namelijk op neer dat een ware Mathon boog van graad 8 in $\mathrm{PG}(2, 2^7)$ een automorfismegroep kan hebben van zowel orde 2 als orde 14. De bogen van het laatste type blijken bijzonder interessant te zijn, vooral omwille van de deelgroep van orde 7 die de kegelsneden van de boog op cylcische wijze permuteert.

Dit voorgaande suggereert het bestaan van twee klassen van Mathon 8-bogen in $\mathrm{PG}(2, 2^7)$.

- De Mathon maximale bogen van graad 8 die een automorfismegroep bevatten van orde 2. Deze zullen ook de *normale* 8-*bogen* worden genoemd.
- De 8-bogen die een automorfismegroep van orde 14 bevatten. Deze zullen de *Singer* 8-*bogen* worden genoemd.

In de Engelstalige tekst volgt nu een gedetailleerde analyse van de Mathon maximale bogen van graad 8 in $\mathrm{PG}(2, 2^7)$ waarin het bestaan van twee klassen Singer 8-bogen van Mathontype aangetoond wordt. In deze ontleding kiezen we de rechte $x = 0$ als as van de unieke elatie die de 8-boog fixeert.

Indien we eisen dat de Singergroep op de zeven kegelsneden van de maximale 8-boog werkt zullen deze kegelsneden cyclisch gepermuteerd worden. Vermits het bovendien nog steeds om een Mathon 8-boog gaat moeten hun raakpunten $(0, y_i, 1)$, $i = 1, \ldots, 7$, met de rechte $x = 0$ uiteraard niet alleen verschillend zijn, maar moeten ze bovendien samen met de kern aanleiding geven tot een additieve groep $\{0, y_1, \ldots, y_7\}$ van orde 8.

Eerst en vooral is het mogelijk om aan te tonen dat $\{0, y_1, \ldots, y_7\}$ steeds kan geschreven worden als de verzameling $\{0, 1, x, x^3, x^7, x^{15}, x^{31}, x^{63}\}$, waarbij x één van de elementen y_i is, de tweede coördinaat van de snijpunten. Vervolgens bewijzen we dat $\{0, 1, x, x^3, x^7, x^{15}, x^{31}, x^{63}\}$ inderdaad een deelgroep is van de additieve groep van $\mathrm{GF}(2^7)$ als en slechts als voldaan is aan $1 + x = x^7$ of $1 + x^3 = x^7$. Aan de hand hiervan zal uiteindelijk blijken dat ten hoogste twee van de drie isomorfietypes van Denniston 4-bogen in $\mathrm{PG}(2, 2^7)$ mogelijk uitbreidbaar zijn tot een Singer 8-boog.

B.4.2 Nodige en voldoende voorwaarde

Een nodige en voldoende voorwaarde opdat een Singer 8-boog gegenereerd zou worden vinden we in het volgende lemma.

Lemma B.4.1. *Stel dat* $\mathcal{D} = \{C_1, C_2, C_3\}$ *een Denniston* 4*-boog is in het vlak* $\mathrm{PG}(2, 2^7)$*. Zij* θ *een automorfisme van* $\mathrm{PG}(2, 2^7)$ *met de eigenschappen dat* $C_2^\theta = C_1$*, dat* $C_4 := C_1^\theta$ *disjunct is van* C_1, C_2 *en dat* C_3*, en* C_4 *dezelfde kern hebben als* C_1, C_2 *en* C_3*. Als* $\mathcal{D}^{\theta^2}$ *zowel* $\mathcal{D}$ *als* $\mathcal{D}^\theta$ *snijdt in een kegelsnede, dan induceert* $\mathcal{D}$ *samen met* θ *een Singer* 8*-boog en bijgevolg is de orde van* θ *deelbaar door* 7*.*

Dit lemma kunnen we benutten om in te zien dat, zowel in het geval $x^7+x+1 = 0$ als in het geval $x^7 + x^3 + 1 = 0$, een Denniston 4-boog $\mathcal{D}$ op unieke wijze kan uitgebreid worden naar een Singer 8-boog. Ook hier is het mogelijk om de expliciete gedaante van beide Singer 8-bogen weer te geven (zie Section 4.3 in de Engelstalige tekst).

B.4.3 De telling in $\mathrm{PG}(2, 2^7)$

Het bovenstaande indachtig zullen we nu het aantal Singer 8-bogen en het aantal normale 8-bogen in $\mathrm{PG}(2, 2^7)$ tellen. Vermits dit de enige twee klassen van Mathon 8-bogen zijn in dit vlak halen we hieruit het totaal aantal maximale bogen van Mathontype van graad 8 in $\mathrm{PG}(2, 2^7)$. Uit de vorige twee subsecties kunnen we volgende lemma's afleiden.

Lemma B.4.2. *In* $\mathrm{PG}(2, 2^7)$ *zijn er, op isomorfisme na, precies twee Singer* 8*-bogen.*

Lemma B.4.3. *Het aantal niet-isomorfe normale* 8*-bogen in* $\mathrm{PG}(2, 2^7)$ *is* 199*.*

Het totaal aantal niet-isomorfe Mathon 8-bogen volgt nu onmiddellijk.

Stelling B.4.1. *Het aantal niet-isomorfe ware Mathon* 8*-bogen in* $\mathrm{PG}(2, 2^7)$ *is gelijk aan* 201*, twee daarvan zijn Singer* 8*-bogen, de overige* 199 *zijn normale* 8*-bogen.*

B.4.4 Grotere velden

De Singer 8-bogen bestaan ook in grotere velden. Beschouw daartoe $\mathrm{GF}(2^h)$, met $h = 7l$ en h oneven. Veronderstel even dat TR de notatie is voor de trace-afbeelding van $\mathrm{GF}(2^h)$ naar $\mathrm{GF}(2)$ en dat tr de notatie is voor de trace-afbeelding van $\mathrm{GF}(2^7)$ naar $\mathrm{GF}(2)$. Vermits h oneven is en een veelvoud van 7 zal $\mathrm{TR}(\alpha) = \mathrm{tr}(\alpha)$ voor elke $\alpha \in \mathrm{GF}(2^7)$, deelveld van $\mathrm{GF}(2^h)$. Alle bekomen kegelsneden zullen dus steeds de rechte $z = 0$ als externe rechte hebben. Er treden nu twee verschillende gevallen op.

- Indien $l \neq 7k$ voor een zekere (oneven) k bekomen we de Singer 8-bogen zoals hierboven beschreven.
- Indien $l = 7k$ voor een zekere (oneven) k bekomen we maximale 8-bogen waarop een cyclische groep transitief werkt op de zeven kegelsneden maar niet scherp transitief. Men zou dergelijke 8-bogen *Singer* 8-*bogen van het tweede type* kunnen noemen.

B.5 Mathon maximale bogen en partiële flocks van de kwadratische kegel

In de eerste sectie van het laatste hoofdstuk van de thesis wordt kennis gemaakt met partiële flocks en wordt de algebraïsche link tussen partiële flocks en maximale bogen van Mathontype uitgelegd. Deze link werd beschreven door Hamilton en J. A. Thas in [33]. Uiteindelijk zal het de bedoeling zijn om, aan de hand van projectie, een meetkundig verband te bekomen tussen partiële flocks en Mathon maximale bogen. Dit zal ons o.a. in staat stellen om, na het definiëren van een compositie op de flockvlakken, een analogon te formuleren van de synthetische versie van de stelling van Mathon en, gegeven een gepaste additieve groep G, een Mathon maximale boog te construeren met G als bijhorende additieve groep.

B.5.1 Partiële flocks

Veronderstel dat $\mathscr{K}$ een kwadratische kegel is in $\mathrm{PG}(3, q)$ met top x. Een *partiële flock* $\mathscr{F}$ van $\mathscr{K}$ is een verzameling (niet-singuliere) kegelsneden op de kegel $\mathscr{K}$. De vlakken die de kegelsneden van de flock bevatten worden vaak *flockvlakken* genoemd. Wanneer alle flockvlakken van een partiële flock snijden in eenzelfde rechte noemen we de partiële flock *lineair*.

Veronderstel dat de orde q van het veld $\mathrm{GF}(q)$ even is en dat de kegel $\mathscr{K}$ vergelijking $X_1X_3 = X_2^2$ heeft. De top is dan het punt $x(1,0,0,0)$ en behoort tot geen enkel vlak van de partiële flock. De kegelsneden van $\mathscr{F}$ worden gedefinieerd door de k vlakken V_i, $i \in \{1,\ldots,k\}$, waarvan de vergelijkingen kunnen geschreven worden als

$$X_0 + f(t)X_1 + tX_2 + g(t)X_3 = 0,$$

met $t \in B$, waarbij B een zekere deelverzameling is van $\mathrm{GF}(q)$, en f en g functies van B naar $\mathrm{GF}(q)$.

We weten reeds uit Sectie B.2 dat een gesloten verzameling $\mathcal{C}$ van kegelsneden kan weergegeven worden aan de hand van functies p en r van A naar $\mathrm{GF}(q)$.

Hamilton en J. A. Thas toonden in [33] aan dat de functies p en r, geassocieerd met $\mathcal{C}$, aanleiding geven tot een partiële flock. Zij $B = A \cup \{0\}$ en definiëer de functies f en g op B als volgt: $f(0) = g(0) = 0$ en $f(t) = tp(t), g(t) = tr(t)$, met $t \in A$. Op die manieren bepalen f, g en B een partiële flock. Er geldt bovendien dat B gesloten is onder de optelling en dat deze functies f en g additief zijn op B. Een partiële flock met deze eigenschap wordt *additief* genoemd.

Ook het omgekeerde resultaat is geldig en ze vonden volgende stelling.

Stelling B.5.1 ([33]). *Een Mathon maximale boog van graad d geeft aanleiding tot een additieve, partiële flock van orde d van de kwadratische kegel in* $\mathrm{PG}(3,q)$*, en omgekeerd.*

Bovendien kan men ook aantonen dat een partiële flock, corresponderend met een Mathon maximale boog M, lineair is als en slechts als M van Denniston-type is.

B.5.2 Projectie

De link tussen partiële flocks en Mathon maximale bogen, zoals hierboven aangegeven, is van een nogal algebraïsche aard. In [33] werd reeds opgemerkt dat een gesloten verzameling van $d-1$ kegelsneden met een gemeenschappelijke kern in $\mathrm{PG}(2,q)$, q even, geprojecteerd kan worden op de kwadratische kegel en op die manier een partiële flock induceert. Deze gevonden partiële flock leek echter niet dezelfde, mooie eigenschappen te bevatten als de flock afkomstig van de algebraïsche link.

In deze sectie zullen we een eerder meetkundige link voorstellen tussen beide meetkundige structuren. We zullen m.a.w. een meetkundige link zoeken tussen de partiële flock, ontstaan door projectie, en de additieve partiële flock. Bovendien zal blijken dat de connectie tussen beiden eigenlijk afkomstig is van een "inversie" op de kernrechte van de kegel.

Veronderstel dat M een Mathon maximale boog is van graad d in $\mathrm{PG}(2,q)$ en stel dat $\mathrm{PG}(2,q)$ het vlak is met vergelijking $X_0 = 0$ ingebed in $\mathrm{PG}(3,q)$. Om de notatie te vereenvoudigen zullen we aannemen dat de kegelsneden uit M een vergelijking hebben van de vorm

$$\alpha^2 X_1^2 + X_1X_3 + \beta^2 X_3^2 + \lambda^2 X_2^2 = 0,$$

met α, β en λ elementen van $\mathrm{GF}(q)$. Bijgevolg hebben alle kegelsneden van M in het vlak $X_0 = 0$ het punt $(0,1,0)$ als kern en de rechte $X_2 = 0$ als rechte op oneindig.

Zij $\mathscr{K}$ opnieuw de kwadratische kegel in $\mathrm{PG}(3,q)$ met vergelijking $X_1X_3 = X_2^2$. Beschouw het punt $n(1,0,1,0)$ op de kernrechte N. Vervolgens projecteren we de elementen uit M op de kegel $\mathscr{K}$. We zoeken m.a.w. voor elke kegelsnede C uit M het vlak dat $\mathscr{K}$ precies snijdt in de projectie van C vanuit het punt n. De projectie van de rechte op oneindig $X_2 = 0, X_0 = 0$ van M is de doorsnede van $\mathscr{K}$ en het vlak opgespannen door die rechte en het punt n. In volgende lemma's bepalen we deze vlakken die de projecties van de kegelsneden van M induceren.

Lemma B.5.1. *Als $\alpha^2 X_1^2 + X_1X_3 + \beta^2 X_3^2 + \lambda^2 X_2^2 = 0$ de vergelijking is van een kegelsnede C in M, dan heeft het vlak die de projectie van C bepaalt vanuit n op de kegel $\mathscr{K}$ de vergelijking*

$$\lambda X_0 + \alpha X_1 + (\lambda + 1)X_2 + \beta X_3 = 0. \tag{B.3}$$

Vermits deze vlakken corresponderen met de kegelsneden van M zullen ze *kegelsnedenvlakken* genoemd worden. Deze vlakken zullen uiteraard moeten onderscheiden worden van de vlakken afkomstig van de Dennistonrechten en de rechte op oneindig van een Mathon maximale boog. Deze laatste zullen respectievelijk *Dennistonvlakken* en *singuliere vlakken* worden genoemd.

Lemma B.5.2. *Als* $X_2 = 0$ *de vergelijking is van de rechte op oneindig van* M*, dan heeft het singulier vlak de vergelijking* $X_0 + X_2 = 0$.

Aan de hand van de voorgaande twee lemma's zijn we in staat om alle vlakken te bepalen die de projectie van M op $\mathscr{K}$ induceren. Deze vlakken vormen een partiële flock die echter nog niet additief is.

Het verband tussen de geprojecteerde flock en de additieve flock kan als volgt samengevat worden. Gegeven een Mathon maximale boog in het vlak $X_0 = 0$ in $\mathrm{PG}(3, q)$ met als gemeenschappelijke kern het punt $(0, 1, 0)$ en de rechte $X_2 = 0$ als rechte op oneindig. Projectie op de kegel $\mathscr{K}$ vanuit het punt $n(1, 0, 1, 0)$ op de kernrechte induceert een partiële flock equivalent met een partiële flock ontstaan uit de flockvlakken $\lambda X_0 + \alpha X_1 + X_2 + \beta X_3 = 0$ en $X_0 = 0$. Deze partiële flock is echter nog niet additief. Passen we nu op deze vlakken een afbeelding toe, afkomstig van een inversie op de kernrechte, en daarna een gepast automorfisme, vinden we steeds de vlakken uit [33], m.a.w. een additieve partiële flock. Dit geldt uiteraard in beide richtingen.

B.5.3 Vlakkencompositie

Men kan zich nu de vraag stellen wat de relatie is tussen de kegelsnedenvlakken en de singuliere vlakken en of het enigszins mogelijk is om de vergelijking van deze vlakken onderling te berekenen. In een volgend lemma zullen we de eigenschap uit Lemma B.2.2 vertalen naar een resultaat betreffende kegelsnedenvlakken.

Eerst introduceren we een standaardvergelijking voor vlakken die het punt $n(1, 0, 1, 0)$ niet bevatten. Een vlak met een vergelijking van de vorm

$$aX_0 + bX_1 + (a + 1)X_2 + cX_3 = 0, \quad a, b, c \in \mathrm{GF}(q)$$

noemen we een vlak met *standaardvergelijking*. Deze vergelijking is uniek wanneer de coëfficiënten van X_0 en X_2 verschillend zijn, of nog, wanneer het vlak niet incident is met het punt n.

Lemma B.5.3. *Gegeven twee willekeurige vlakken in* $\mathrm{PG}(3, q)$ *die* n *noch* x *bevatten en een disjuncte doorsnede hebben met de kegel* $\mathscr{K}$*. Er bestaat dan een uniek derde vlak zodanig dat de projectie vanuit* n *op het vlak* $X_0 = 0$ *van de doorsnede van deze drie vlakken met* $\mathscr{K}$ *een Denniston* 4*-boog induceert.*

Eénmaal de vergelijking van een kegelsnedenvlak standaard is kunnen we volgend lemma gebruiken om het singulier vlak, geassocieerd aan een Denniston maximale boog van graad 4 te bepalen.

Lemma B.5.4. *Zij V en W twee kegelsnedenvlakken in* $\mathrm{PG}(3, q)$. *Het singulier vlak die de rechte op oneindig induceert van de unieke Denniston* 4*-boog bepaald door V en W kan gevonden worden door de som te nemen van de vergelijkingen van V en W.*

B.5.4 Analogon van de synthetische stelling van Mathon

De synthetische versie van de stelling van Mathon kan eveneens vertaald worden naar een resultaat omtrent partiële flocks.

Stelling B.5.2. *Gegeven een additieve partiële flock $\mathscr{F}$ van orde d en gegeven een vlak V' die het punt $n'(0, 0, 1, 0)$ niet bevat en zodanig dat V' de kegel $\mathscr{K}$ snijdt in een kegelsnede disjunct van de elementen van $\mathscr{F}$. Er bestaat dan een unieke additieve partiële flock van orde $2d$ die de kegelsneden bevat bepaald door V' en de d vlakken die $\mathscr{F}$ induceren.*

Gebruik makend van Lemma B.5.4 en de vergelijking van de singuliere vlakken kunnen we enkele eigenschappen afleiden i.v.m. de Dennistonrechten.

Lemma B.5.5. *Gegeven een Mathon maximale boog M van graad $2d$ die een Denniston maximale boog D van graad d bevat, dan zijn alle Dennistonrechten van M concurrent.*

Lemma B.5.6. *De Dennistonrechten van een Mathon maximale boog zijn concurrent als de coëfficiënt α of β constant is.*

B.5.5 Additieve groep

Beschouw een additieve groep G van orde $2d$. In deze sectie van de thesis bespreken we hoe het mogelijk is om, onder bepaalde omstandigheden, een Mathon maximale boog M van graad $2d$ te construeren waarvan G de corresponderende additieve groep is en zo dat M een Denniston d-boog bevat. Vermits deze uiteenzetting vrij technisch is verwijzen we de lezer graag door naar Sectie 5.5 van de Engelstalige tekst.

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